Attack and Defence

There are n prospective targets for an air raid, with strategic values a_i, i=1... n. They are sorted according to their values so that $a_1 > a_2 ... > a_n$. Only one target will be attacked and the defender has only one anti-aircraft system. If an undefended target is raided its destruction is assured. If defended, it will be destroyed with probability 1-p, the parameter p in the interval [0,1] being a measure of the effectiveness of the defense. Which target should be defended, which target raided? Assuming possible losses to attacking aircraft to be negligible, the conflict can be modeled as a zero-sum game: If an undefended or defended
target i is raided, the attacker, player 1, receives payoff a_i or $(1-p)\,a_i$, respectively. The defender's respective payoffs are just the egatives of these, $-a_i$ and $-(1-p)\,a_i$, so that the players' payoffs always sum to zero. Here is the bimatrix game for three targets:

```
<< Wolfram/bimatrix.m
```

```
AA[i_, j_] := If[i == j, (1-p) a_i, a_i];
A = Array[AA, {3, 3}];
BimatrixF [A, -A]
```

	S_1	S_2	S_3
R_1	$-(1-p)\,a_1$ $(1-p)\,a_1$	$-a_1$ a_1	$-a_1$ a_1
R_2	$-a_2$ a_2	$-(1-p)\,a_2$ $(1-p)\,a_2$	$-a_2$ a_2
R_3	$-a_3$ a_3	$-a_3$ a_3	$-(1-p)\,a_3$ $(1-p)\,a_3$

where R_i = defend the ith target, S_j = attack the jth target. The function **NashEquilibria** can return solutions to bimatrix games which, like this one, are in symbolic form by taking advantage of the sophisticated Mathematica environment. As one might expect, though, it can only do so if it is passed a numerical example with which it can ``mimic'' the solutions algebraically. This is done via the option **Symbolic->s**, where **s** is a list of substitutions with representative numerical values for each of the symbols appearing in the bimatrix. Thus with the substitutions

```
s = Join[{p → 1/2}, Table[a_i → 10 - i, {i, 3}]]
```

$$\left\{p \to \frac{1}{2},\ a_1 \to 9,\ a_2 \to 8,\ a_3 \to 7\right\}$$

we can ask for the Nash equilibria in symbolic form as follows:

```
eq = NashEquilibria[A, -A, Symbolic -> s] // TableForm
```

$$\begin{array}{cccc}
\begin{matrix} \frac{a_2 a_3}{a_2 a_3 + a_1 (a_2 + a_3)} \\ \frac{a_1 a_3}{a_2 a_3 + a_1 (a_2 + a_3)} \\ \frac{a_1 a_2}{a_2 a_3 + a_1 (a_2 + a_3)} \end{matrix} & -\frac{(-3+p)\, a_1 a_2 a_3}{a_2 a_3 + a_1 (a_2 + a_3)} & \begin{matrix} \frac{(-2+p)\, a_2 a_3 + a_1 (a_2 + a_3)}{p\,(a_2 a_3 + a_1 (a_2 + a_3))} \\ \frac{a_2 a_3 + a_1 (a_2 + (-2+p)\, a_3)}{p\,(a_2 a_3 + a_1 (a_2 + a_3))} \\ \frac{a_2 a_3 + a_1 ((-2+p)\, a_2 + a_3)}{p\,(a_2 a_3 + a_1 (a_2 + a_3))} \end{matrix} & \frac{(-3+p)\, a_1 a_2 a_3}{a_2 a_3 + a_1 (a_2 + a_3)}
\end{array}$$

The postfixed TableForm is merely for readability. We see that the equilibrium strategies are completely mixed, and also that the algebra is a bit messy. Things get much nicerlooking if we make the substitution $a_i = 1/b_i$:

```
eq /. a_i_ → 1/b_i // Simplify // TableForm
```

$$\begin{array}{llll}
\frac{b_1}{b_1+b_2+b_3} & & \frac{(-2+p)\,b_1+b_2+b_3}{p\,(b_1+b_2+b_3)} & \\
\frac{b_2}{b_1+b_2+b_3} & \frac{3-p}{b_1+b_2+b_3} & \frac{b_1+(-2+p)\,b_2+b_3}{p\,(b_1+b_2+b_3)} & \frac{-3+p}{b_1+b_2+b_3} \\
\frac{b_3}{b_1+b_2+b_3} & & \frac{b_1+b_2+(-2+p)\,b_3}{p\,(b_1+b_2+b_3)} &
\end{array}$$

One guesses immediately that the equilibrium payoffs, for any value of n, are

$$H_1(P, Q) = (n-p)\Big/\sum_{j=1}^{n} 1/a_i = -H_2(P, Q) = \mathrm{H}$$

and that the equilibrium strategies are

$$P = H(1/a_1 \; \ldots \; 1/a_n)/(n-p),\; Q = (1 - H/a_1 \; \ldots \; 1 - H/a_n)/p$$