LECTURE 11:
Equivalence of some theorems in combinatorics

Lecture Notes on combinatorics Mar. 2013

Crossreference

Dilworth

König Theorem

Theorem 11.1 (Dilworth's Theorem) *The minimum number of chains into which the elements of a partially ordered set $P = (X, \prec)$ can be partitioned is equal to the maximum number of elements in an anti-chain of P.*

partition = disjoint covering induction proof

Dual of Dilworths Theorem

Dilworth Theorem via König Theorem

Proof. To prove Dilworth's theorem for a partially ordered set P with n elements, using König's theorem, define a bipartite graph $G[U, V]$ where $U = V = X$ and where (u, v) is an edge in G when $u \prec v$ in P.

First, we show that any matching M of $G[U, V]$ yields a chain partition of P. Let Z be the family of chains formed by including u and v in the same chain whenever there is an edge (u, v) in M. In particular, when $M = \emptyset$, each chain of Z is a singleton element in X. Obviously, Z is proper chain partition of P and $|Z| = n - |M|$.

Then we prove that any vertex cover C gives rise to a anti-chain with $n - |C|$. Let A be the set of elements of X that are not in C. Clearly, $|A| = n - |C|$. Moreover, A is an anti-chain, since an comparable pair in A will lead to an edge of $G[U, V]$ uncovered by C.

Since $|M| \leq |C|$, we have $n - |C| \leq n - |M|$. By König's theorem, there are a maximum matching M^* and minimum vertex cover C^* in $G[U, V]$ such that $|M^*| = |C^*|$. So $n - |C^*| = n - |M^*|$. □

References

[CW87] D. Coppersmith and S. Winograd, "Matrix multiplication via arithmetic progressions," *Proceedings of the 19th ACM Symposium on Theory of Computing*, 1987, pp. 1–6.