LECTURE 7:
Integer polyhedra and TDI

Polyhedral Theory in CO and ILP Mar. 2015

7.1 TDI

A lecture note on CS 598CSC: Combinatorial Optimization given by Chandra Chekuri from illinois.edu can be found from the link given below.

https://courses.engr.illinois.edu/cs598csc/sp2010/lectures/lecture12.pdf

The note gives some insights on the relation between TDIness and TUM, say

if A is TUM, $Ax \leq b$ is TDI for all b, or

$Ax \leq b$ may be TDI even if A is not TUM, or in other words, the TDIness of a linear system is a weaker sufficent condition for integrality of $\{x : Ax \leq b\}$ and moreover guarantees that the dual is integral whenever the prime objective vector is integral.

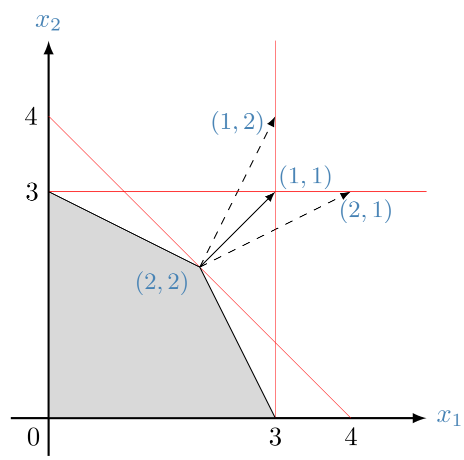

Figure 7.1: The cone C and the polytope Z

Note that TDI is a property of a system of inequalities, *not* a property of the corresponding polyhedron. Systems

$$\left\{ x \in \mathbb{R}^2 \ : \ \begin{array}{l} x_1 \geq 0, \ x_2 \geq 0 \\ x_1 + 2x_2 \leq 6 \\ 2x_1 + x_2 \leq 6 \end{array} \right\} \quad \text{and} \quad \left\{ x \in \mathbb{R}^2 \ : \ \begin{array}{l} x_1 \geq 0, \ x_2 \geq 0 \\ x_1 + 2x_2 \leq 6 \\ 2x_1 + x_2 \leq 6 \\ x_1 + x_2 \leq 4 \\ x_1 \leq 3, \ x_2 \leq 3 \end{array} \right\}$$

define the same polyhedron $P = \text{conv}\{(0,0),(3,0),(2,2),(0,3)\}$. But the first system, however, is not TDI. For example, if we take the cost vector c^T to be $(1,1)$, then the primal maximum is achieved by $(2,2)$ with value 4. But $(1,1)$ cannot be expressed as a linear integer combination of $(1,2)$ and $(2,1)$, the normal vectors of the tight constraints at point $(2,2)$. Therefore, there is no integral dual optimum and the first system is not TDI.

Nevertheless, after extending the first system with some redundant constraints, we have an additional normal vector at $(2, 2)$, namely, $(1, 1)$. And now $(1, 1)$ is an integer combination of the normal vectors at $(2, 2)$. Moreover, the second system is in fact TDI.

And this example also shows that a TDI-system usually contains more constraints than necessary for defining the polyhedron.

A deeper look into this example actually gives necessary for a system to be TDI. We explain this in general context. Consider the problem $\max\{c^T x | Ax \leq b\}$ with c integral, and assume it has finite optimum δ. Then it is achieved by some vector x^* in the face F defined by the intersection of $x|Ax \leq b$ with the hyperplane $c^T x = \delta$. For simplicity assume that the face F is an extreme point of the polyhedron and let $A'x = b'$ be the set of all inequalities in $Ax \leq b$ that are tight at x^*. The dual is $\min\{y^T b | y \geq 0, y^T A = c\}$. By LP duality theory, c can be expressed as a non-negative combination of the row vectors of A'(, in other words c is in the cone of the row vector of A'). As entries of y corresponding to non-tight constraints in A must be 0. TDI of $Ax \leq b$ requires that there is an integer solution to $yA' = c, y \geq 0$ for any integral c. (Geometrically, any integral c in the cone generated by row vectors of A' can be expressed as a non-negative combination of row vectors of A'.) Recall that every rational cone admits a Hilbert basis, of which each integral vector in this cone can be expressed as the non-negative integer combination. This observation motivates the following theorem, which reveals the the realtion between TDI and Hilbert bases.

7.2 TDI system and Hilbert bases

Theorem 7.1 *The rational system $Ax \leq b$ is TDI if and only if for each face F of the polyhedron $P := \{x | Ax \leq b\}$, the rows of A which are tight in F form a Hilbert basis.*

Proof.

□

Remark: In fact we showed in the proof that we can restrict F in the theorem above to minimal face.

minimal TDI

Theorem 7.2 *For each rational polyhedron P there exists a TDI-system $Ax \leq b$ with A integral and $P = \{x | Ax \leq b\}$. And b can be chosen to be integral if and only if P is integral. Moreover, if P is full-dimensional, there exists a unique minimal TDI-system.*

Proof. □

The example above raises the question of whether one can take any rational system $Ax \leq b$ and make it TDI by adding sufficiently many redundant inequalities. Indeed that is possible, and this is based on the theorem that "every rational polyhedral cone has a finite integral Hilbert basis"

We use our previous example to demonstrate this. In our previous example, a Hilbert basis for the cone (the dual cone associated with vertex $(2, 2)$) defined by the vectors $(1, 2)$ and $(2, 1)$ is given by the set of vectors $H = \{(1, 1), (1, 2), (2, 1)\}$. We can get the additional vector $(1, 1)$ by adding the redundant constraint $x_1 + x_2 \leq 4$ in the first system.

Successively considering the dual cones corresponding to the vertices $(0, 0)$, $(3, 0)$ and $(0, 3)$, one can show that the linear system ?????? is TDI. For example, the cone corresponding to the vertex $(3, 0)$ has a Hilbert basis $\{(2, 1), (1, 0), (0, -1)\}$.

References

[CW87] D. Coppersmith and S. Winograd, "Matrix multiplication via arithmetic progressions," *Proceedings of the 19th ACM Symposium on Theory of Computing*, 1987, pp. 1–6.

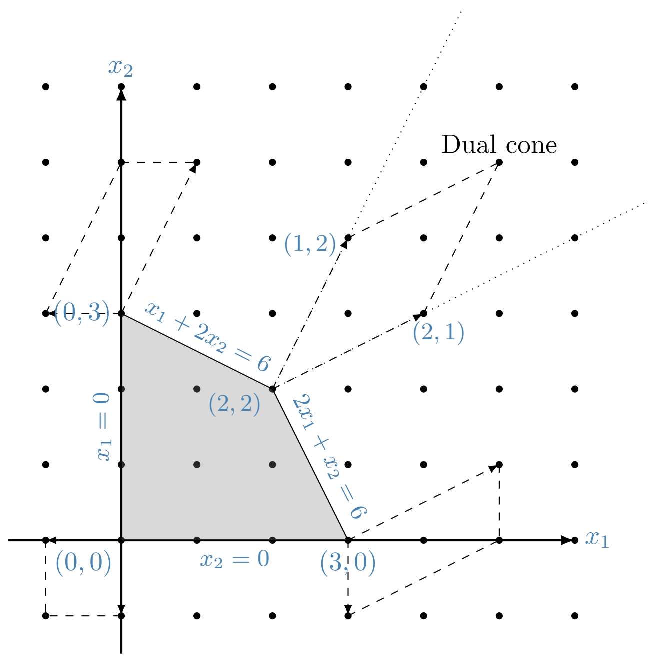

Figure 7.2: The polyhedron and its dual cones