LECTURE 8:
The Matching Polytope

Polyhedra methods in combinatorial optimization July 2013

8.1 characteristic vectors

For any $T \subseteq S$, the characteristic vector of T is the $\{0,1\}^S$-vector satisfying

$$x_e^T = \begin{cases} 1 & e \in T \\ 0 & \text{otherwise} \end{cases}$$

Let $G = (V, E)$ be a graph. The perfect matching polytope $P_{pm}(G)$ of G is the convex hull of the characteristic vectors of perfect matching of G, and the matching polytope $P_m(G)$ of G is the convex hull of the characteristic vectors of matchings of G.

Clearly, any characteristic vectors of a (perfect) matching is a vertex of the (perfect) matching polytope. The origin and standard unit vectors of $\mathbb{R}^E$ are characteristic vectors of matching. So $\dim(P_m(G)) = |E|$. The dimension of $P_{pm}(G)$ is obviously smaller than $|E|$, as all vertices are contained in the hyperplane $\{x|\mathbf{1}^T x = |V|/2\}$.

Now we already have the V-representation for (perfect) matching polytope, but we are more interested in the H-representation. Since inequality constraints enable us to solve some questions initiated by (based on) matching polytope by using linear programming. The following description is due to Jack Edmonds (1965). It sometimes says that this result initiated the use of polyhedral method in combinatorial optimization.

As usual, we start with bipartite graph which is much easier to handle. Notice that any vector x of the perfect matching polytope $P_{pm}(G)$ satisfies:

$$x_e \geq 0, \text{ for } e \in E \tag{8.1a}$$
$$x(\delta(v)) = 1, \text{ for } v \in V \tag{8.1b}$$

where... The first condition (8.1a) is trivial. (8.1b) comes from the fact that each vertex x is a convex combination of all vectices of $P_{pm}G$, i.e., $x(\delta(v)) = \sum \lambda y(\delta(v)) = \sum \lambda = 1$, where y stands for vertices of $P_{pm}(G)$.

References

[CW87] D. Coppersmith and S. Winograd, "Matrix multiplication via arithmetic progressions," *Proceedings of the 19th ACM Symposium on Theory of Computing*, 1987, pp. 1–6.