

## Integer polyhedra and TDI

A lecture note on CS 598CSC: Combinatorial Optimization given by Chandra Chekuri from illinois.edu can be found from the link given below.

The note gives some insights on the relation between TDIness and TUM, say

$Ax \leq b$  may be TDI even if  $A$  is not TUM, or in other words, the TDI-ness of a linear system is a weaker sufficient condition for integrality of  $\{x : Ax \leq b\}$  and moreover guarantees that the dual is integral whenever the prime objective vector is integral.

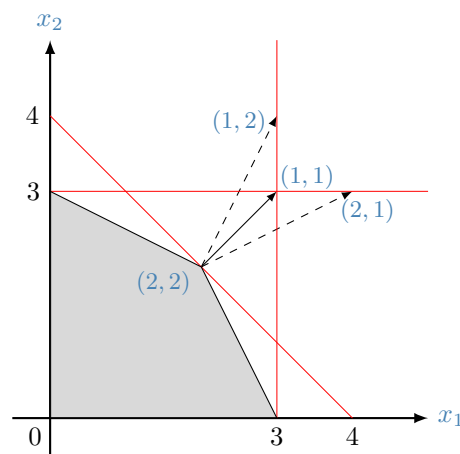


Figure 7.1: The cone  $C$  and the polytope  $Z$

Note that TDI is a property of a system of inequalities, *not* a property of the corresponding polyhedron. Systems

$$\left\{ x \in \mathbb{R}^2 : \begin{array}{l} x_1 \geq 0, \ x_2 \geq 0 \\ x_1 + 2x_2 \leq 6 \\ 2x_1 + x_2 \leq 6 \end{array} \right\} \quad \text{and} \quad \left\{ x \in \mathbb{R}^2 : \begin{array}{l} x_1 \geq 0, \ x_2 \geq 0 \\ x_1 + 2x_2 \leq 6 \\ 2x_1 + x_2 \leq 6 \\ \textcolor{red}{x_1 + x_2 \leq 4} \\ \textcolor{red}{x_1 \leq 3, \ x_2 \leq 3} \end{array} \right\}$$

define the same polyhedron  $P = \text{conv}\{(0, 0), (3, 0), (2, 2), (0, 3)\}$ . But the first system, however, is not TDI. For example, if we take the cost vector  $c^T$  to be  $(1, 1)$ , then the primal maximum is achieved by  $(2, 2)$  with value 4. But  $(1, 1)$  cannot be expressed as a linear integer combination of  $(1, 2)$  and  $(2, 1)$ , the normal vectors of the tight constraints at point  $(2, 2)$ . Therefore, there is no integral dual optimum and the first system is not TDI.

Nevertheless, after extending the first system with some redundant constraints, we have an additional

normal vector at  $(2, 2)$ , namely,  $(1, 1)$ . And now  $(1, 1)$  is an integer combination of the normal vectors at  $(2, 2)$ . Moreover, the second system is in fact TDI.

And this example also shows that a TDI-system usually contains more constraints than necessary for defining the polyhedron.

A deeper look into this example actually gives necessary for a system to be TDI. We explain this in general context. Consider the problem  $\max\{c^T x \mid Ax \leq b\}$  with  $c$  integral, and assume it has finite optimum  $\delta$ . Then it is achieved by some vector  $x^*$  in the face  $F$  defined by the intersection of  $x \mid Ax \leq b$  with the hyperplane  $c^T x = \delta$ . For simplicity assume that the face  $F$  is an extreme point of the polyhedron and let  $A'x = b'$  be the set of all inequalities in  $Ax \leq b$  that are tight at  $x^*$ . The dual is  $\min\{y^T b \mid y \geq 0, y^T A = c\}$ . By LP duality theory,  $c$  can be expressed as a non-negative combination of the row vectors of  $A'$  (in other words  $c$  is in the cone of the row vector of  $A'$ ). As entries of  $y$  corresponding to non-tight constraints in  $A$  must be 0. TDI of  $Ax \leq b$  requires that there is an integer solution to  $yA' = c, y \geq 0$  for any integral  $c$ . (Geometrically, any integral  $c$  in the cone generated by row vectors of  $A'$  can be expressed as a non-negative combination of row vectors of  $A'$ .) Recall that every rational cone admits a Hilbert basis, of which each integral vector in this cone can be expressed as the non-negative integer combination. This observation motivates the following theorem, which reveals the relation between TDI and Hilbert bases.

## 7.2 TDI system and Hilbert bases

**Theorem 7.1** *The rational system  $Ax \leq b$  is TDI if and only if for each face  $F$  of the polyhedron  $P := \{x \mid Ax \leq b\}$ , the rows of  $A$  which are tight in  $F$  form a Hilbert basis.*

*Proof.*

□

**Remark:** In fact we showed in the proof that we can restrict  $F$  in the theorem above to minimal face.  
*minimal TDI*

**Theorem 7.2** *For each rational polyhedron  $P$  there exists a TDI-system  $Ax \leq b$  with  $A$  integral and  $P = \{x \mid Ax \leq b\}$ . And  $b$  can be chosen to be integral if and only if  $P$  is integral. Moreover, if  $P$  is full-dimensional, there exists a unique minimal TDI-system.*

*Proof.*

□

The example above raises the question of whether one can take any rational system  $Ax \leq b$  and make it TDI by adding sufficiently many redundant inequalities. Indeed that is possible, and this is based on the theorem that “every rational polyhedral cone has a finite integral Hilbert basis”

We use our previous example to demonstrate this. In our previous example, a Hilbert basis for the cone (the dual cone associated with vertex  $(2, 2)$ ) defined by the vectors  $(1, 2)$  and  $(2, 1)$  is given by the set of vectors  $H = \{(1, 1), (1, 2), (2, 1)\}$ . We can get the additional vector  $(1, 1)$  by adding the redundant constraint  $x_1 + x_2 \leq 4$  in the first system.

Successively considering the dual cones corresponding to the vertices  $(0, 0)$ ,  $(3, 0)$  and  $(0, 3)$ , one can show that the linear system ????? is TDI. For example, the cone corresponding to the vertex  $(3, 0)$  has a Hilbert basis  $\{(2, 1), (1, 0), (0, -1)\}$ .

## References

- [CW87] D. COPPERSMITH and S. WINOGRAD, “Matrix multiplication via arithmetic progressions,” *Proceedings of the 19th ACM Symposium on Theory of Computing*, 1987, pp. 1–6.

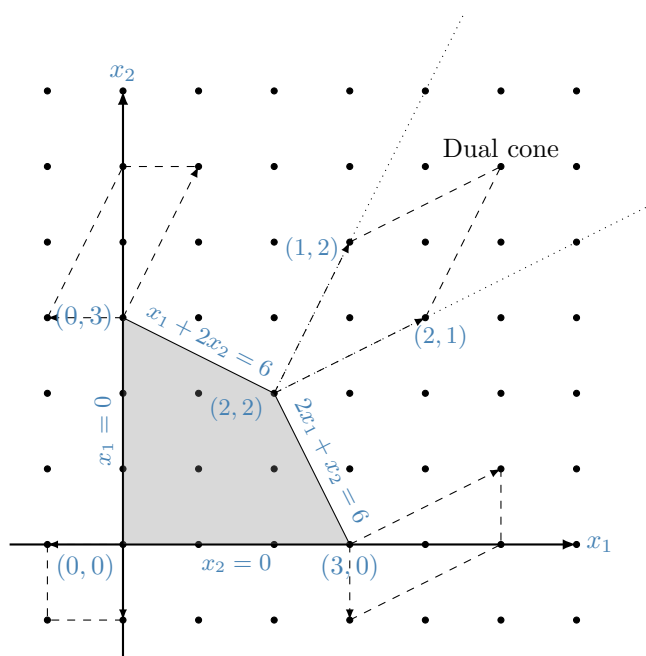


Figure 7.2: The polyhedron and its dual cones