Baffour Ba Series

CORE

MATHEMATICS

for

Schools and Colleges

By:

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PREFACE

Baffour Ba Series, is a series of Mathematics, covering the contents of Senior High Schools and Colleges sylabuses.

In this book, each chapter is broken down into short, manageable sections intended to completely alleviate or reduce to a large extent, the canker that is termed in Ghanaian context as ***“Math’s Phobia”*** by providing in its content, concise notes and commentaries with well – illustrated diagrams on each topic and subtopic of the Senior High Schools and Colleges Syllabuses.

The book reflects the authors experience that students work better from work examples than abstract discussion of principles, hence the provision of numerous and comprehensive range of worked examples. A numbered, step-by-step approach to problem solving, trial test and challenge problems intended to cater for gifted students are greatly featured in this book. Not left out in this book is ***“exercises”*** on each subtopic which contains an extensive selection of Multiple- choice, Fill-ins, True or False, Essay – type questions similar in standard to the WASSCE questions. Tackling these exercises is no doubt, an excellent form of revision. Answers to all exercises are also provided to help students assess their level of progress.

Taking into accounts, full analysis of the pattern and level of difficulty of examination questions, the last part of the book is ***“some solved past questions”***,and ***“Trial Objective Test”*** with answers within its contents**.**

ACKNOWLEDGEMENTS

A myriad of services including spiritual, moral, inspirational, motivational and financial support, typing, editing, printing, designing and what have you, were deeply rendered by friends, family members, students, staff members, some co - tutors of J.H.S, S.H.S and Colleges of Education to make this work a success. To mention a few: Mrs. Joyce Mensah, Mr. Kwabena Agyeman (More Questions), Mr. Asamoah Koduah Thick, Mr. Stephen Amoateng, Mr. Frank Owusu, Mr. Kwaku Boateng, Mr. Augustine Boadi, Mr. Isaac Owusu Mensah, Mr. Owusu Blessing, Mr. K.D. Bosiako, Mr. Adjei Domfeh Romeo, Mr. Bernard Kusi Knambers, Mr. Attakorah Jacob, Sampson Brobbey, George Osei Mensah (Kin – George), Cherish Asamoah Baffour – Ba, Godbless Agyei Asamoah Baffour Ba Snr, Godbless Agyei Asamoah Baffour Ba Jnr, Grace Adwoa Pomaa, Benjamin Osei – Mensah (Benjii), Appiah Collins, Maybel Owusu Ansah, Antwi Gloria, Mr. Opoku John, Department of Mathematics, AGASS, Department of Mathematics, Okess .

I also wishto acknowledge the fact that the book is not absolutely free from errors of typing, grammar and inaccuracy. These occurred as a result of oversight but not ignorance and incompetence on the part of the author and editors. These should therefore, not undermine the credibility of the book, for they say *"to err is human"*. However, your comments, corrections, suggestions and criticism are warmly welcomed for consideration and rectification in the next edition.

Baffour Asamoah
(*Baffour Ba*)

TABLE OF CONTENTS

1 SETS AND OPERATIONS ON SETS

Definition of a Set
A set is the collection of objects based on a well-defined property. Thus, we can have table, chair, pencil and book as objects made of wood. Similarly, we can have Monday, Tuesday… Sunday, as set of days of the week.

Describing a Set
A set may be described by;
a. Listing the members of the set. For example, the set of even numbers from 2 to 8 = $\{2, 4, 6, 8\}$
b. Stating the property, for example, {even numbers less than 10}
c. Using the set builder notation. For example, $\{x : 2 \le x \le 10\}$, where x is an even number.

Exercises 1.1
A. Given that A = {1, 2, 3, 4, 5} list the elements of the following Sets.
1. $\{x^2 : x \in A\}$ 2. $\{\frac{1}{x} : x \in A\}$
3. $\{2x : x \in A\}$ 4. $\{4x + 1 : x \in A\}$

B. List the elements of the following set;
1. $\{x: 9 < x < 21\}$
2. $\{x: 4 \le x < 36$, where x is a multiple of $4\}$
3. $\{x: -5 \le x \le 5\}$
4. $\{x: 9 < x < 21\}$

C. Describe the following sets in words;
1. $A = \{2, 4, 6, 8…100\}$
2. $C = \{a, e, i, o, u\}$
3. $D = \{6, 12, 18…42\}$
4. $E = \{2, 3, 5, 7, 11, 13, 17, 19\}$

D. Describe with set builder notation:
1. A = {Even numbers between 10 and 20}
2. B = {Integers from -2 to 5}
3. $C = \{-3, -2, -1, 0, 1, 2, 3\}$

Representation of Sets
A set is always represented by a capital letter and its members are represented by small letters.

e.g. $A = \{m, e, n\}$

Set A Members (elements)

The sign **"∈"** is read as **"*belongs to*"** and the sign **"∉"** is read as ***"does not belong to"***. Thus, we can write $m \in$ A and $n \in A$ because 'm' and 'n' are members of set A. On the other hand, we can write $b \notin A$ and $d \notin A$ because "b" and "d" are not members of set A.

Worked Examples
Put in ∉ or ∈ in the following, given that $A = \{x : 1 < x < 10\}$ and $B = \{x : x \ge -5\}$.
(i) -2 A (ii) 2 A (iii) -10 A

Solution
i. $-2 \notin A$ ii. $2 \in A$ iii. $-10 \notin A$

Exercises 1.2
A. Put in ∈ or ∉ in the following
1. 6 ……… {Perfect squares}
2. 27……… {3, 6, 9………99}
3. 10 ……… $\{x : x > 8\}$
4. 19 ……… {Prime numbers}

B. Rewrite the following using set symbols:
1. 64 is a perfect square
2. 7 is a prime number.
3. $\frac{5}{2}$ is not an integer.
4. 41 is not an odd number less than 25.

Types of Set

1. Null set or empty set

It is a set with no element or member. For example, there is no triangle with 5 sides. This set is therefore an empty set. It is denoted by { } or ∅.

2. Unit set

It is a set that has only one element or member. For e.g. A = {x: $3 < x < 5$} = {4}.

3. Finite set

It is a set that can take all its members. Precisely, a finite set is a set that comes to an end. For e.g, A = {even numbers between 2 and10}
⇒A = {4, 6, 8}

4. Infinite set

It is s set that cannot take all its members. Precisely, an infinite is a sets does not come to an end. For example: B = {even numbers}
⇒B = {2, 4, 6, 8, 10 . . .}

Exercises 1.3

A. Identify the empty and unit sets.

1. {Odd factors of 16}
2. {Even numbers between 16 and 18}
3. {Prime numbers between 16 and 19}
4. {A number that is odd and even}

B. Identify the finite and infinite sets;

1. {Multiples of 3}   2. {1, 2, 3…….100}
3. {Factors of 12}   4. {Even prime numbers}

Relationship between Sets

Two or more sets can be related as; subsets, equal sets or equivalent sets.

Subsets

A set "M" is said to be the subset of another set "N" if all the elements of "M" can be found in " N".

Subset is denoted by ⊂ or ⊃. For example, if N = {1, 2, 3, 4, 5, 6} and M = {1, 2, 3}, then M is a subset of N, because all the elements of M can be found in N. This is written as M⊃N or N⊂M.

Worked Examples

1. If Y = {house, tree} and X = {cat, house, tree. Find the relationship between Y and X.

Solution

Since all the elements in Y can be found in X, we say Y is a subset of X, i.e. Y⊂X

2. If P = {2, 4, 8} and Q = {even counting numbers less than 12}. What is the relationship between P and Q?

Solution

P = {2, 4, 8} and Q = {2, 4, 6, 8, 10}
Since all the elements of P can be found in Q, P is a subset of Q. That is P⊂Q

3. P = {x : $20 < x < 30$, where x is odd} and Q = {23, 29}. Establish a relationship between sets P and Q.

Solution

P = {21, 23, 25, 27, 29} and Q = {23, 29}
Since all the members of Q can be found in P, we say Q is a subset of P. i.e. Q⊂P

Listing the Subsets of a Set

The empty set and the set itself are the first two subsets of every set. The other subsets include each element of the set and a combination of the elements in twos, threes etc until each and every element combines with each other and all other elements.

Note: ***The number of subsets in a set can be found by the formula: 2^n where n is the number of elements in the set***

Worked Examples

1. If $M = \{x, y\}$. List all the subsets of M.

Solution

Subsets of $M = \{\ \}, \{x, y\}, \{x\}$ and $\{y\}$

Number of subsets = 4

2. List all the subsets of $B = \{a, b, c\}$.

Solution

Subsets of $B = \{\ \}, \{a, b, c\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}$ and $\{b, c\}$.

Number of subsets = 8

3. Find the number of subsets in the set $A = \{1, 2, 3, 4\}$

Solution

Number of subsets = 2^n, where $n = 4$

Number of subsets = $2^4 = 2 \times 2 \times 2 \times 2 = 16$

Exercises 1.4

A. Find the number of subsets in a set that contains the following number of elements;

(1) 5 (2) 7 (3) 8 (4) 16

B. 1. If $A = \{1, 2, 3\}$, $B = \{3, 2\}$and $C = \{3\}$ determine which of the sets is a subset of the other.

2. Write the pair of subsets of the set $A = \{1, 2, 3, 4, 6, 12\}$

3. How many pair of subsets are there in the set of factors of 18.

Challenge Problem

C.Use True or False for the following:

If $P = \{x : 40 < x < 50$, where x is prime$\}$, $Q = \{47, 43\}$, $R = \{41, 47\}$, $A = \{1, 2, 3\}$, $B = \{3, 2\}$, $C = \{3\}$ and $D = \{2, 1, 3\}$, then:

1. $B \subset A$ 2. $A \subset B$ 3. $C \subset A$ 4. $C \subset B$

Equal Sets

Two or more sets are said to be equal if they have identical or the same elements. For e.g, if $A = \{1, 2, 3\}$ and $B = \{3, 1, 2\}$, then $A = B$ because A and B have the same elements.

Worked Examples

1. If $P = \{1, 2, 3, 8, 10\}$, $Q = \{8, 1, x, 3, 2\}$ and $P = Q$. What is the value of x?

Solution

$P = \{1, 2, 3, 8, 10\}$ and $Q = \{8, 1, x, 3, 2\}$

By matching or comparing the two sets, $x = 10$ because 10 is the only element that is not in set Q.

Exercises 1.5

Which of the pair of sets are equal?

1. $\{5, 7, 8, 10\}$ and $\{e, f, g, h\}$.
2. {even primes} and {2}.
3. {4, 8, 12…} and {multiples of 4}.
4. {multiples of 2} and {even numbers}.
5. $A = \{x : x$ is an integer, and $-1 \le x \le 4\}$ and $B = \{-1, 0, 1, 2, 3, 4\}$.

Equivalent Sets

Two or more sets are said to be equivalent if they have the same number of elements. For e.g, If $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, then A and B are equivalent sets because there are three elements in each set.

Exercises 1.6

Which of the pair of sets are equivalent?

1. P ={ factors of 24}and Q ={factors of 30}

2. P = {2, 3, 4, 5} and Q = {64, 2, 16, 27, 12}
3. P = {prime factors of 24}
Q = {prime factors of 30}
4. A= {x : 1 < x < 10, where x is even}
B ={x : 1< x < 10, where x is odd}

Intersection of Sets

The intersection of sets A and B is the set of elements that can be found in both sets. It is denoted by ∩. Thus, A intersection B is written as A∩B.

Worked Examples

1. If A = {5, 6, 7, 8, 9} and B ={9, 10, 11}. Find:
i. A ∩ B
ii. What type of set is A ∩ B?

Solution

A = {5, 6, 7, 8, 9} and B = {9, 10, 11}
i. A ∩ B = {9}   ii. A ∩ B is a unit set.

2. Find the intersection of the two sets:
K = {prime numbers less than 20}
L = {odd numbers less than 18}

Solution

K = {2, 3, 5, 7, 11, 13, 17, 19}
L = {1, 3, 5, 7, 9, 11, 13, 15, 17}
K∩L = {3, 5, 7, 11, 13, 17}

3. If X = {prime numbers less than 12} and
Y = {odd numbers less than 13}. Find X ∩ Y.

Solution

X = {2, 3, 5, 7, 11} and Y = {1, 3, 5,7, 9, 11}
X∩Y = {3, 5, 7, 11}

4. P = {prime numbers less than 20}
Q = {odd numbers less than 10}.
Find P∩Q.

Solution

P = {2, 3, 5, 7, 11, 13, 17, 19}
Q = {1, 3, 5, 7, 9}
P∩Q = {3, 5, 7}

Exercices 1.7

A. 1. Given that M = {1, 2, 3, 4, 5….20}
Q = {3, 4, 5, 6, 7, 8} and R = {2, 3, 5, 7}, where Q and R are subsets of M, find:
i. Q∩R   ii. What type of set is Q∩R?

2. If A = {1, 2, 4, 7}, B = {2, 6, 8} and
C = {4, 5, 6, 7, 8}. Find the following:
i. A ∩ B   ii. B ∩ C   iii.A ∩ C

3. If A = {1, 2, 3, 4,}, B = {1, 2, 5, 6} and C = {3, 5, 7}, list all the members of the following sets:
i. A ∩ (B∩C)   ii. A ∪ (B∩C)

4. Let F = {1, 2, 3….}, G = {4, 14, 24} and H = {1, 3, 5, 7}. Find the following sets:
i) F ∩ G   ii) G ∩ H   iii) F ∩ H

B. Find the intersection of the pair of sets.

1. A = {Even numbers less than 12}
B = {Prime numbers less than 10}
2. M = {Prime factors of 30}
N = {Prime factors of 20}
3. P = {Multiples of 12} and Q = {Factors of 24}
4. K = {Factors of 6} and L = {Factors of 12}
5. Given that P = {Factors of 20},
Q = {Factors of 16}
R = {Multiples of 5 less than 20}
a. List all the elements of P, Q and R
b. find:
i. P∩Q   iii. P∩R iii. Q∩R iv. Q∩P
c. What type of set is P∩Q?
d. What is the relationship between?
i. P∩Q and Q∩P   ii. P∩Q and P

Union of Sets

The union of two or more sets is a combination of all the elements of the involving sets in a single set. Union is denoted by ∪. For example, If $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, then A union B, written as: $A \cup B = \{1, 2, 3, 4, 5, 6\}$

Worked Examples:

1. Given that:
A = {x : 1 < x < 10, where x is odd},
B = { x : x is a factor of 12} and
C = {multiples of 4 less than 10}
(a) List all the elements of A, B and C.
(b) Find:
i. A ∩ B ii. B ∩ C iii. A ∩ B ∩ C
iv. A U C v. B U C vi. A U B
c. What type of set is?
i. A∩B? ii. A∩C? iii. A∪C?
d. What is the relationship between A∩B and B∩C ?

Solution

a. A = {3, 5, 7, 9}, B = {1, 2, 3, 4, 6, 12} and C = {4, 8}

b. i. A∩B = {3} ii. B∩C = {4}
iii. A∩ B ∩ C = { }
iv. A ∪ C = { 3, 4, 5, 7, 8, 9 }
v. B ∪ C = {1, 2, 3, 4, 6, 8, 12}
vi. A ∪ B = { 1, 2, 3, 4, 6, 7, 9, 12 }

c. i. A∩B = {3} is a unit set
ii. A∩C = { } is an empty set.
iii. A∪C = {3, 4, 5, 7, 8, 9} is a finite set.

d. A∩ B = { 3 } and B ∩ C = { 4 }, therefore, they are equivalent sets.

2. Find A∪B, if A = {factors of 24} and
B = {factors of 8}

Solution

A = {1, 2, 3, 4, 6, 8, 12, 24} and B = {1, 2, 4, 8}
Therefore A∪B = {1, 2, 3, 4, 6, 8, 12, 24}

3. If X = {x : x < 13, where x is prime}
Y = {x : x < 13, where x is odd}.
i. List the elements of X and Y.
ii. List the members of X∪Y.

Solution

i. X = {2, 3, 5, 7, 11} and Y = {1, 3, 5, 7, 9, 11}
Therefore, X∪Y = {1, 2, 3, 4, 5, 7, 9, 11}

4. List the elements of each set.
B = {whole numbers from 20 to 25}
D = {factors of 63} and find the members of:
i. B∩D ii. B∪D

Solution

B = {20, 21, 22, 23, 24, 25}
D = {1, 3, 7, 9, 21, 63}
i. B∩D = {21}
ii. BUD ={1,3,7,9,20,21,22,23, 24, 25,63}

5. If P = {7, 11, 13} and Q = {7, 9, 11, 13}. Find P∪Q.

Solution

P = {7, 11, 13} and Q = {7, 9, 11, 13}.
Therefore, P∪Q = {7, 9, 11, 13}

Solution

A = {1, 2, 3, 4, 5} and B = {3, 4, 6}
i. A∪B = {1, 2, 3, 4, 5, 6}
ii. Number of elements in A∪B = n (A∪B) = 6

Exercises 1.8

Given that E = {7, 9, 1}, F = {1, 5, 7, 9} and G = {1, 7, 13}. Find the following sets:
1. E∪F 2. E∪G 3. F∪G 4. E∪ (F∪G)

Properties of Sets Operations

1. Commutative property

a. $A \cap B = B \cap A$ b. $A \cup B = B \cup A$

Consider the sets A = {1, 3, 6, 7, 8, 9} and B = {2, 3, 7, 8, 10}

$A \cup B$ ={1, 2, 3, 6, 7, 8, 9, 10}

$B \cup A$ = {1, 2, 3, 6, 7, 8, 9, 10}

$A \cup B = B \cup A$.

Hence, we say the union of sets is ***commutative***

Similarly, $A \cap B$ = {3, 7, 8}, $B \cap A$ = {3, 7, 8}

$A \cap B = B \cap A$

Hence, we say the intersection of sets is ***commutative***

2. Associative property

a. $(A \cap B) \cap C = A \cap (B \cap C)$

b. $(A \cup B) \cup C = A \cup (B \cup C)$

Consider the sets A = {1, 5, 6, 7}, B = {2, 5, 7, 6} and C = {1, 7, 8, 10}

(AUB)UC = {1, 2, 5, 6, 7}U{1, 7, 8, 10}

= ***{1, 2, 5, 6, 7, 8, 10}***

AU(BUC) = {1, 5, 6,7}U{1, 2, 6, 7, 8,10}

= {1, 2, 5, 6, 7, 8, 10}

(AUB)UC = AU(BUC)

Hence for any three sets A, B and C, ***(AUB)UC = AU(BUC)***. The operation of union of sets is ***associative.***

Similarly, $(A \cap B) \cap C$ = {5, 6, 7} ∩{1,7, 8, 10}

= {7}

$A \cap (B \cap C)$ = {1, 5, 6, 7} ∩ {7}= {7}

$(A \cap B) \cap C = A \cap (B \cap C)$

Hence for any three sets A, B and C,
$(A \cap B) \cap C = A \cap (B \cap C)$. The operation of intersection of sets is ***associative.***

3. Distributive property

a. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

b. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Consider the sets, A = {1, 2, 5, 8, 9}, B = {2, 5, 8, 9, 10} and C = {5, 9, 12, 13}

$A \cup (B \cap C)$ = {1, 2, 5, 8, 9} U {5, 9}

= **{1, 2, 5, 8, 9}**

$(A \cup B) \cap (A \cup C)$

= {1, 2, 5, 8, 9, 10} ∩{1, 2, 5, 8, 9, 12, 13}

= {1, 2, 5, 8, 9}

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Hence for any three sets A, B and C,
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
The union is said to be ***distributive over intersection.***

In a similar manner, ***$A \cap (B \cup C)$***
= {1, 2, 5, 8, 9}∩{2, 5, 8, 9, 12, 13}**= { 2, 5, 8, 9}**

$(A \cap B) \cup (A \cap C)$

= {2, 5, 8, 9} ∪ {5, 9}

= {2, 5, 8, 9}

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Hence for any three sets *A*, *B* and *C*,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. The intersection is said to be ***distributive*** over the union.

Exercises 1.9

1. Given A ={1, 3, 4, 5} B = {1, 3, 5, 8} and C = {3, 5, 9, 10}, verify that:
(AUB)UC = AU(BUC)

2. In A = 2, 4, 5, 7, 10}, B = {2, 3, 5, 8} and C = {2, 5, 10, 12}, show that:
$(A \cap B) \cap C = A \cap (B \cap C)$

3. Verify that A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C) for the sets A = {10, 12, 15}, B = {9, 10, 11, 12, 13} and C = {8, 12, 15}

4. For the sets A = {2, 3, 5, 8}, B = {2, 5, 10, 12} and C = {2, 4, 5, 7, 10},verify the relation A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

Venn Diagrams

Sets can be represented in diagrams called Venn diagrams. In doing so, the universal set and the complement is very necessary.

Universal Set

It is the set within which all other subsets can be found. It is denoted by ∈ **or μ**. Students are restricted to work within the universal set and not go beyond it.

Complement of a Set

It is the set of elements that can be found in the universal set but not in a particular set. It is denoted by the symbol ^{I}. For instance, if **μ** = {1, 2, 3, 4, 5} and *A* = {1, 2, 3}, then the complement of *A*, written as: A^I = {4, 5}.

Worked Examples

Given that U = {whole numbers from 1 to 10}, *A* = {first five even numbers} and *B* = {first five odd numbers}, if *A* and *B* are subsets of U, find:

i. A^I ii. B^I

Solution

U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

i. *A* = {2, 4, 6, 8, 10} A^I = {1, 3, 5, 7, 9}

ii. *B* = {1, 3, 5, 7, 9 }. B^I = {2, 4, 6, 8}

Exercises 1.10

Given that: U = {x : $1 \leq x \leq 12$},

A = {x : x is numbers less than 10}

B = {x : x is multiples of 3} and

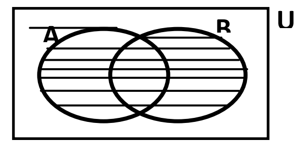
C = {x : x is factors of 16}

a. List all the elements of U, A, B and C

b. Find: (i) A^I (ii) B^I (iii) C^I

Illustration of Diagrams

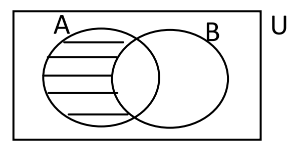
A. Joint Sets



The shaded regions represent **AUB**

The non-shaded region represents **$(AUB)^1$**

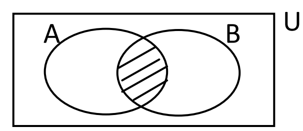
2.



The shaded portion represents:

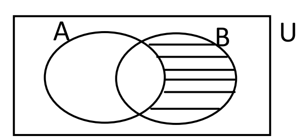
$A \cap B^1$ = A only

3.



The shaded portion represents **A∩B** and the non-shaded portions represent **$(A\cap B)^1$**

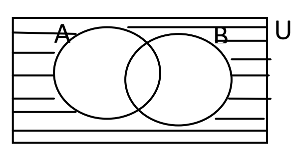
4.



The shaded portion represents:

$A^1 \cap B$ = B only

5.

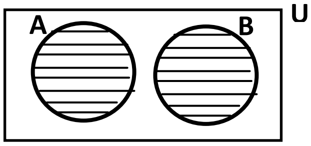


The shaded portion represents:

$A^1 \cap B^1 = (AUB)^1$. That is outside A or B or both.

B. Disjoint sets

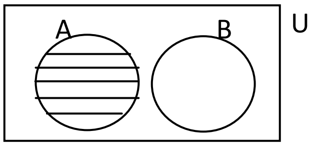
1.



The shaded regions represent **AUB**

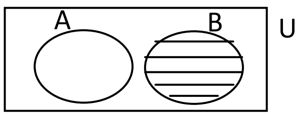
The non-shaded region represents **(AUB)^{1}**

2.



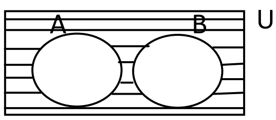
The shaded portion represents: **A∩B^{1} = A only**

3.



The shaded portion represents: **A^{1}∩B = B only**

4.

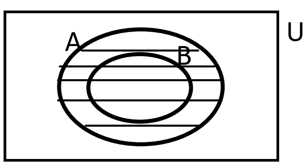


The shaded portion represents:

A^{1}∩B^{1} = (AUB)^{1}. That is outside A or B or both.

C. Subsets (B ⊂ A)

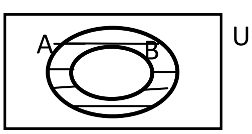
1.



The shaded regions represent **AUB**

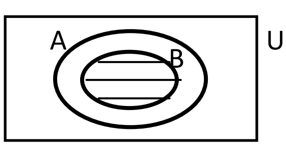
The non-shaded region represents **(AUB)^{1}**

2.



The shaded portion represents: **A∩B^{1} = A only**

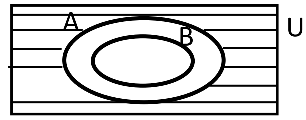
3.



The shaded portion represents **A∩B** and **B only**.

The non-shaded portions represent **(A∩B)^{1}**

4.

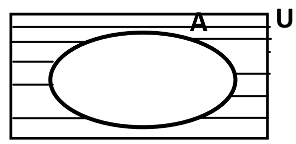


The shaded portion represents:

A^{1}∩B^{1} = (AUB)^{1}. That is outside A or B or both

D. *Single Sets*

1.



The shaded region is **A^{I}.**

The non-shaded region is **A.**

Worked Examples

1. P and Q are subsets of the universal,

U = {x: 1 ≤ x ≤ 10}

P = {x: x is even numbers}

Q = {x: x is odd numbers}

i. List all the elements of U, P and Q

ii. Represent U, P and Q in a diagram

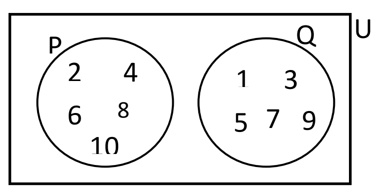
iii. Find the number of subsets in Q.

Solution

i. U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

P = {2, 4, 6, 8, 10} Q = {1, 3, 5, 7, 9}

ii. P ∩ Q = { } disjoint diagram



iii. n(Q) = 5

Number of subsets = 2^n, but n = 5

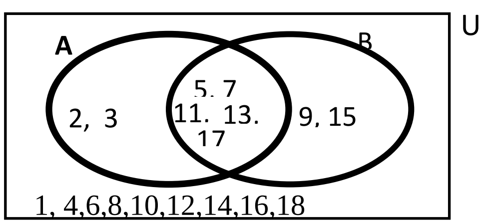
2^5 = 32 subsets

2. Given that: $\cup = \{x: 1 \leq x \leq 18\}$
$A = \{x : x$ is prime numbers$\}$ and $B = \{x : x$ is odd numbers greater than 3$\}$
a. If A and B are subsets of the universal set, U, list all the elements of A and B.
b. Find the intersection of sets A and B
c. i Illustrate U, A and B on a Venn diagram.
ii. Shade the region for prime factors of 18 on the Venn diagram.

Solution

a. $\cup = \{1, 2, 3, 4........18\}$
$A = \{2, 3, 5, 7, 11, 13, 17\}$
$B = \{5, 7, 9, 11, 13, 17\}$

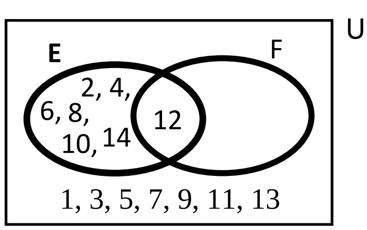
b i. $A \cap B = \{5, 7, 11, 13, 17\}$
ii. $A \cup B = \{2, 3, 5, 7, 11, 13, 15, 17\}$



3. a. E and F are subsets of the universal set,
$\cup = \{x : 1 \leq x < 15$, where x is a positive integer$\}$,
$E = \{x : 1 < x < 15$, where $x \in$ even numbers$\}$ and
$F = \{x : 9 < x < 15, x \in$ multiples of 4$\}$.
i. List the elements of U, E and F.
ii. Draw a Venn diagram to show the sets, U, E and F.

Solution

i. $\cup = \{1, 2, 3,4,5,6,7,8,9,10,11,12,13,14\}$
$E = \{2, 4, 6, 8, 10, 12, 14\}$ and $F = \{12\}$
ii.

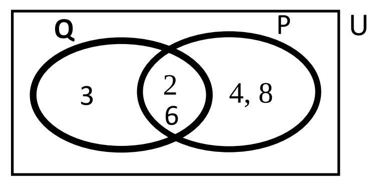


4. M is the set consisting of all positive integers between 1 and 10. P and Q are subsets of M such that $P = \{$factors of 6$\}$ and $Q = \{$multiples of 2$\}$
i. List the element of M, P and Q.
ii. Represent M, P and Q on a Venn diagram.
iii. Find $P \cap Q$.

Solution

i. $M = \{2, 3, 4, 5, 6, 7, 8, 9\}$
$P = \{2, 3, 6\}$ and $Q = \{2, 4, 6, 8\}$

ii. $P \cap Q = \{2, 6\}$



iii. $P \cap Q = \{2, 6\}$

Exercises 1.11

1. If $U = \{x : 5 < x < 15, x \in$ integers$\}$,
$A = \{x : x$ is factors of 14$\}$ and $B = \{x : x$ is multiples of 3$\}$, where A and B are subsets of $\cup$, show this information on a Venn diagram.

2. Given that $\cup = \{x : x$ is factors of 24$\}$, $P = \{x : x$ is odd numbers$\}$ and $Q = \{x : x$ is prime numbers$\}$, where P and Q are subsets of U
a. List all the elements of $\cup$, P and Q.
b. Represent $\cup$, P, and Q in a Venn diagram.

3. Given that $\cup = \{x : 18 < x \leq 25\}$,
$A = \{x : x$ is multiples of 2$\}$ and
$B = \{x : x$ is multiples of 3$\}$, where A and B are subsets of $\cup$
i. List the elements of A and B.
ii. Show the information in a Venn diagram.
iii. List the members of $(A \cup B)^1$
iv. What type of set is B only?

4. Given that U = {x: x is factors of 36}, M = {x: x is factors of 10} and N = {x: x is multiples of 3 to 12} where M and N are subsets of U.
i. Show U, M and N in a Venn diagram.
ii. From the diagram, list the elements of {M∪N} and N only.

5. a. Show the following sets in a Venn diagram; μ = {x : $1 \leq x \leq 10$}, A = {x : $1\leq x \leq 5$} and B = {x : $3 < x < 9$}
b. Show by shading the following regions;
i. $A\cap B^1$ ii. $A^1\cap B$

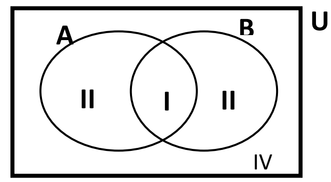
6. A and B are subsets of a universal set, μ = {x : $1 \leq x \leq 18$} such that A = { x : x is even numbers} and B = {x : x is multiples of 3}
i. List the elements of the sets, μ, A, B, (A∩B), (A∪B) and $(A\cup B)^1$
ii. Illustrate the information on a Venn diagram.

7. Given that μ is the set of positive integers less than 100 and the set *A* and *B* are subsets of μ. *A* is the set of multiples of 5 and *B* is the set of multiples of 7;
i. List the elements of *A*, *B* and A∩*B*.
ii. Describe in words the elements of the set A∩B.
iii. Write down the values of $n(A)$, $n(B)$, $n(A\cap B)$.
Show that; $n(AUB) = n(A) + n(B) - n(A\cap B)$

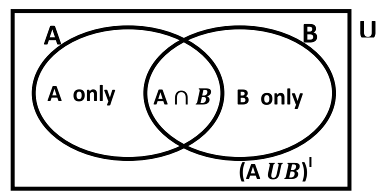
Two Set Problems

Two set problem arises when a number of people are made to choose between only two items. The choice could be for one item only, or both items or none of the items. The problem occurs whenever the choice made is either more than or less than the number of people making the choice. For instance, 15 students were made to choose between English and French, as their favorite language. 9 chose English, 5 chose French and 4 did not choose any of the languages. It is seen that the sum of the choices is 9 + 5 + 4 = 18, which is more than 15 (the number of students), creating a problem and impression of an overlap somewhere. Cases of this nature are called ***two set problems*** and are solved by first representing them in Venn diagrams.

Regions of the Venn Diagram



II + I + III + IV = U



$\boldsymbol{n(A) - n(A\cap B) + n(A\cap B) + n(B) - n(A\cap B) + n(A\cup B)^1 = U}$

$\boldsymbol{n(A)}$ **only +** $\boldsymbol{n(A\cap B) + n(B)}$ **only +** $\boldsymbol{n(A\cup B)^I = U}$

Exactly one = $\boldsymbol{n}$ **(A) only +** $\boldsymbol{n}$**(B) only**

The first circle of the Venn diagram must be represented by the capital letter of the first item and the second circle, by the capital letter of the second item. For e.g. 'A' for agriculture and 'B' for business as shown above.

Region I or (A∩B) is represented by a variable usually, ***x,*** when its value is unknown. When the value is given, put it in that region.

Region II or A only is represented by the number of people who opted for it, ***n(A),*** minus region II, ***x,*** (whether region II's value is given or not). That is: ***n(A) – x***. If ***n(A) only*** is given, do not subtract ***x*,** because ***n(A)– x = n(A) only***, meaning ***x*** has been subtracted already.

Region III or B only must be represented by the number of people who opted for it, ***n(B),*** minus region II, ***x***, (whether region II's value is given or not). That is: ***n(B) – x*** . If ***n(B) only*** is given, do not subtract ***x*,** because *n**(B) – x = n(B) only***, meaning ***x*** has been subtracted already

Region IV or $(\mathbf{A\cup B})^{\mathbf{1}}$ represents the number of people who refuse to make a choice between the two items. If the question goes along with the phrase *"each person chose at least one item",* then $(AUB)^1$ or region IV is zero and cannot be represented on the Venn diagram.

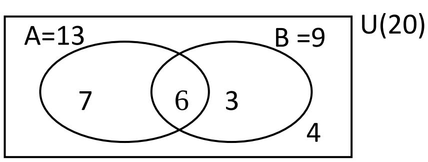
The sum of the numbers of all the regions must be equal to the total number of people.

$\mathbf{U} = n(\mathbf{A})\ \mathbf{only} + n(\mathbf{A\cap B}) + n(\mathbf{B})\ \mathbf{only} + n(\mathbf{A\cup B})^{\mathbf{I}}$

Note: *Venn diagrams for two set problems must always intersect (Joint set)**.** Before the diagram is drawn, prepare a data that defines the variables and the values of the variables used.*

Worked Examples

1. The number of students who offer Agriculture or Business or both or none at a certain college is represented in the diagram below;



How many students offer;

i. Agriculture but not Business?

ii. Business but not Agriculture?

iii. Neither Agriculture nor Business?

iv. Exactly two subject?

v. Exactly one subject?

vi. Agriculture or Business or both?

Solution

i. Agriculture but not Business = 7

ii. Business but not Agriculture = 3

iii. Neither Agriculture nor Business = 4

iv. Exactly two subject = 6

v. Exactly one subject = 7 + 3 = 10

vi. Agriculture or Business or both

= 7 + 6 + 3 = 16

2. In a class of 60 students, 40 play Football and 25 play Hockey. Each student play at least one game

a. Represent this information in a Venn diagram.

b. How many students play both games?

c. How many students play Football only?

d. How many students play Hockey only?

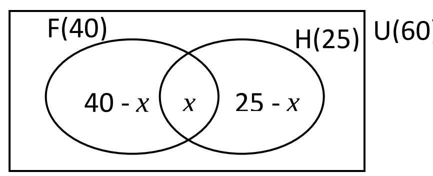
Solution

a. Let ∪ = {Students in the class},

F = {students who play football},

H = {students who play hockey}

$n(\cup) = 60$, $n(\text{F}) = 40$, $n(\text{H}) = 25$, $n(\text{F}\cap \text{H}) = x$



b. $40 – x + x + 25 – x = 60$

$65 – x = 60$

$x = 65 – 60 = 5$ students

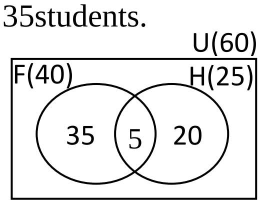
c. n(F) only = $40 – x$. But $x = 5$

n(F) only = 40 – 5 = 35students.

iv. n(H) only

= $25 – x$, but $x = 5$

= 25 – 5 = 20 students



Therefore, 20 students played Hockey only.

3. There are 50 pupils in a class. Out of this number, $\frac{1}{10}$ speaks French only and $\frac{4}{5}$ of the remainder speak both French and English. If the rest speak English only; Find the number of students who speak

i. Both French and English. ii. English only.

iii. Draw a Venn diagram to illustrate the above information.

Solution

Let ∪ = {pupils in the class},

E = {pupils who speak English}

F = {pupils who speak French}

$n(\cup) = 50$, n (F) only = $\frac{1}{10} \times 50 = 5$

Remainder = 50 – 5 = 45.

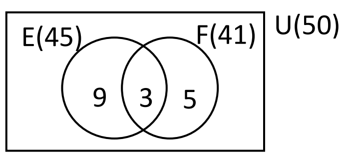
$n(E \cap F) = \frac{4}{5} \times 45 = 36$

ii. The rest

= 50 – (5 + 36)

= 50 – 41= 9

n (E) only = 9



4. In a survey, 100 students were to choose between two new drinks, Rola and Koka. 61 liked Rola, 8 liked both drinks and 10 like neither. How many liked;

a. Koka b. Koka but not Rola

Solution

Let **∪** = {students in the class},

R = {students who like Rola}

K = {students who like Koka}

$n(V) = 100$ $n(R) = 61$ $n(K) = x$

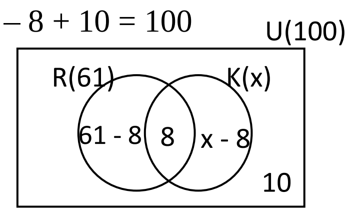
$n(R \cap K) = 8$ $n(R \cup K)^1 = 10$

$61 - 8 + 8 + x - 8 + 10 = 100$

$63 + x = 100$

$x = 100 - 63$

$x = 37$



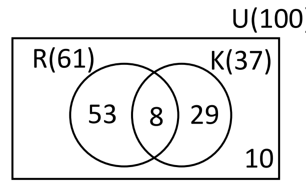
ii. Kola but not Rola

= Kola only

= $x - 8$,

= 37 – 8

= 29



5. In a class of 50 students, $\frac{1}{5}$ of them study Biology only, $\frac{3}{10}$ of them study Chemistry only and 10% of them study neither subject.

i. Represent this information in a Venn diagram

ii. How many students study both subjects?

iii. How many students study Chemistry?

Solution

i. Let U represent the pupils in the class,

B represents pupils who study Biology,

C represents pupils, who study Chemistry,

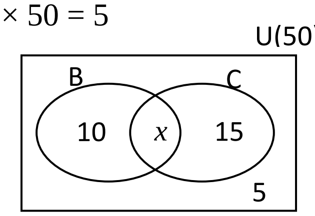
$n(U) = 50$,

$n(B)$ only $= \frac{1}{5} \times 50 = 10$

$n(C)$ only $= \frac{3}{10} \times 50 = 15$

$n(BUC)^1 = \frac{10}{100} \times 50 = 5$

$n(B \cap C) = x$

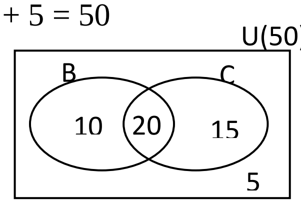


ii. $10 + x + 15 + 5 = 50$

$x + 30 = 50$

$x = 50 - 30$

$x = 20$



iii. C = x + 15 = 20 + 15 = 35

6. A class was asked to choose between Benz and Toyota or both or none. 39% of them chose Benz, 52% of them chose Toyota and 23% of them chose neither Benz nor Toyota. Show this in a Venn diagram and find the percentage of students who chose both brands of cars.

Solution

Let ∪ = {students in the class},

B = {students who chose Benz}

T = {students who chose Toyota}

$n(B\cup T) = 100\%$, $n(B) = 39\%$, $n(T) = 52\%$

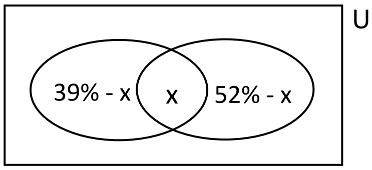
$n(B\cup T)^1 = 23\%$, $n(B\cap T) = x$

$39\% - x + x + 52\% - x + 23\% = 100\%$

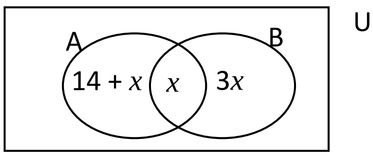
$114\% - x = 100\%$

$x = 114\% - 100\%$

$x = 14\%$



7. *A* and *B* are two sets and the numbers of elements are shown in the diagram below;



Given that $n(A) = n(B)$, calculate:

i. x ii. $n(A\cup B)$

Solution

i. $14 + x + x = x + 3x$

$14 + 2x = 4x$

$14 = 2x$

$x = 7$

ii. $n(A\cup B)$

$= n(A)$ only $+ n(B)$ only $+ n(A\cap B)$

But $n(A)$ only $= 14 + x = 14 + 7 = 21$

$n(B)$ only $= 3x = 3(7) = 21$

$n(A\cap B) = 7$

$\Rightarrow n(A\cup B) = 21 + 21 + 7 = 49$

Exercises 1.12

1. A number of students wrote an examination in physics and chemistry and had the following records; 60% passed in physics, 40% passed in chemistry and 15% passed in neither subject.

i. Show this in a Venn diagram.

ii. Find the percentage of students who passed both subjects?

iii. What percentage of students passed in at least one subject?

2. There are 30 students in a class. 17 of them belong to the hockey team, 9 belong to the football team and 6 belong to both teams. How many students do not belong to any of the teams?

3. A survey was conducted among some people to find out their favorite network as far as Airtel and Vodafone were concern. 44% opted for Airtel only, 16% chose both networks and 17% did not opt for any of the networks. Ifeach student opted for at least one network;

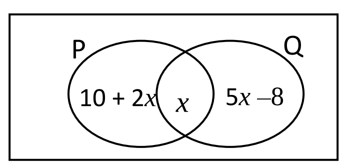
i. what percentage of the people opted for Vodafone only?

ii. what percentage of the people opted for exactly one network?

4. In a class of 35 boys, A is the set of boys who take athletics and C is the set who play cricket. $n(A) = 15$, $n(C) = 16$, $n(A\cap C) = 5$. Using the whole class as the Universal set, draw a Venn diagram and mark the numbers in their appropriate regions.

ii. How many boys take neither athletics nor cricket?

5. P and Q are two sets and the numbers of elements are shown in the diagram below;



Given that $n(P) = n(Q)$, calculate:

i. $n(P\cap Q)$ ii. $n(Q)$ iii. $n(P\cup Q)$

6. In an examination, x pupils take the history paper and $3x$ pupils take the mathematics paper. Given that 6 pupils take both papers, illustrate the data on a Venn diagram indicating the number of pupils in each region. If the number of pupils taking the examination is 46, find x.

7. In an examination, x pupils scored less than 51 marks, $2x$ pupils scored more than 49 marks and 4 pupils scored exactly 50 marks. Illustrate the information on a labeled Venn diagram. Given that the total number of pupils taking the examination was 32, find x

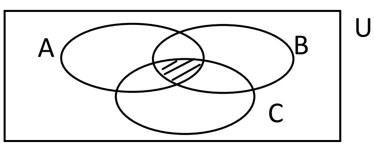
Three Set Problems

Three set problem arises when a number of people are made to choose between three items. The choice could be for one item only, two items only, all the three items or none of the three items.

Diagrams for Three Sets

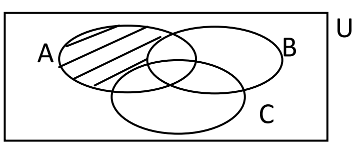
A. Shading One Region

1.



The shaded region is $A\cap B\cap C$

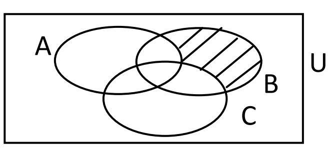
2.



The shaded region represents **A only**

$= A\cap B^1\cap C^1 = A\cap(B\cap C)^1$

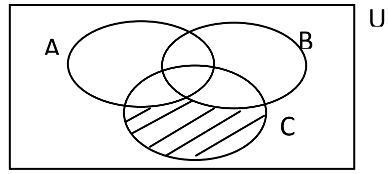
3.



The shaded region represents **B only**

$= A^1\cap C^1\cap B = (A\cup C)^1 \cap B$

4.

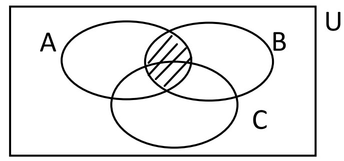


The shaded region represents ***C only***

$= A^1\cap B^1\cap C = (AUB)^1\cap C$

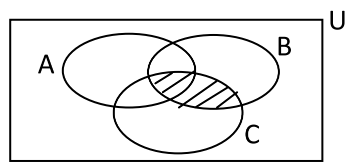
B. Shading Two Regions

1.



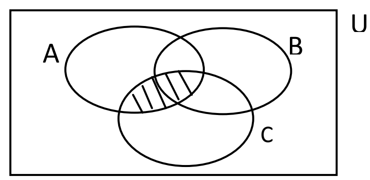
The shaded region represents **A∩B**

2.



The shaded region represents **B∩C**

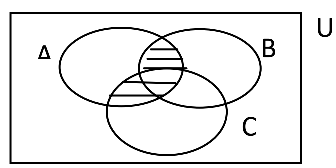
3.



The shaded region represents **A∩C**

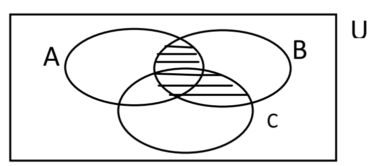
C. Shading Three Regions

1.



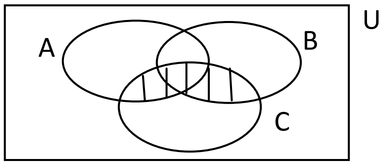
The shaded region represents $A\cap(B\cup C)$

2.



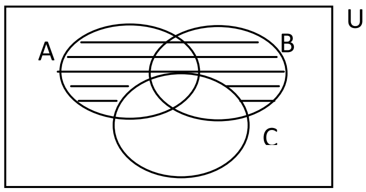
The shaded region represents $B\cap(A\cup C)$

3.



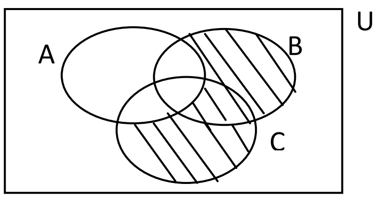
The shaded region represents $C\cap(A\cup B)$

4.



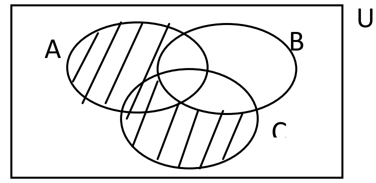
The shaded region represents $(A\cup B)\cap C^1$

5.



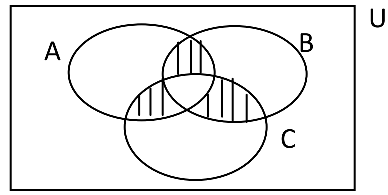
The shaded region represents $(B\cup C)\cap A^1$

6.



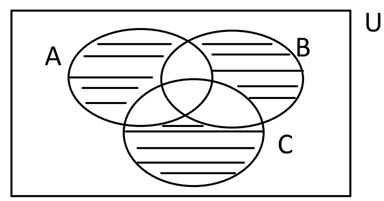
The shaded region represents $(A\cup C)\cap B^1$

7.



The shaded regions represent exactly two items

8.



The shaded regions represent exactly one item

Solving Problems Involving Three Overlapping Sets

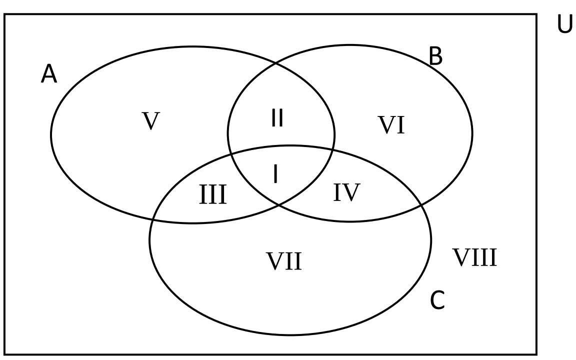


Fig. I

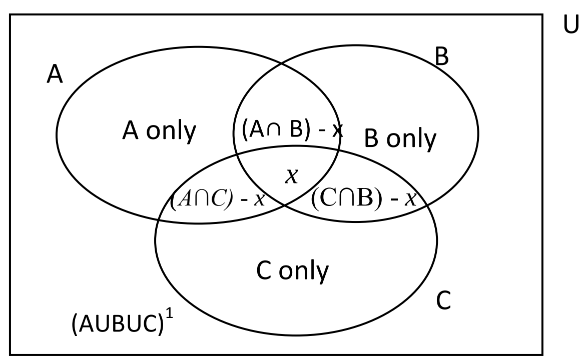


Fig. II

For three sets A, B, and C,

1. a. $n(AUBUC)$ = (I + II + III + IV + V + VI + VII)

= A only + B only + C only + (A∩B) only + (B∩C) only + (A∩C) only + (A∩B∩C), if the values of exactly two items only and exactly one item are given

b. $n(AUBUC) = n(A) + n(B) + n(C) – n(A\cap B) – n(B\cap C) – n(A\cap C) + n(A\cap B\cap C)$, if each person like all the three items (without complement)

c. $n(AUBUC) = (I + II + III + IV + V + VI + VII + VIII)$

$= n(A) + n(B) + n(C) – n(A\cap B) – n(B\cap C) – n(A\cap C) + n(A\cap B\cap C) + n(AUBUC)^1$, if some people do not like all the three items (with complement)

2. Number of people in exactly one set: = (V + VI + VII)

$= n(A) + n(B) + n(C) – 2n(A\cap B) – 2n(A\cap C) – 2n(B\cap C) + 3n(A\cap B\cap C)$

3. Number of people in exactly two of the sets = (II + III + IV)

$= n(A\cap B) + n(A\cap C) + n(B\cap C) – 3n(A\cap B\cap C)$

4. Number of people in exactly three of the sets = $n(A\cap B\cap C)$ = I

5. $n(A\cap B)$ only = $(A\cap B)\cap C^1$ = II

6. $n(A\cap C)$ only = $(A\cap C) \cap B^1$ = III

7. $n(B\cap C)$ only = $(B\cap C) \cap A^1$ = IV

8. Number of people in set A only = $A \cap (B^1\cap C^1)$ = V

9. Number of people in set B only = $B \cap (A^1\cap C^1)$ = VI

10. Number of people in set C only = $C \cap (A^1\cap B^1)$ = VII

11. Number of people who do not like any of the three = $n(A\cup B\cup C)^1$ = VIII

12. Number of people in two or more sets = (II + III + IV + I) = (at least 2 sets) :

$= n(A\cap B) + n(A\cap C) + n(B\cap C) – 2n(A\cap B\cap C)$

13. $n(A)$ = V+ I + II + III

14. n (B) = I + II + IV + VI

15. $n(C)$ = I + III + IV + VII

16. $n(A\cap B)$ = I + II

17. $n(A\cap C)$ = I + III

18. $n(B\cap C)$ = I + IV

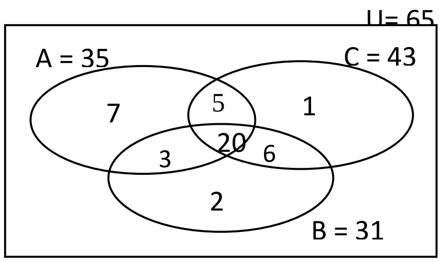
Note: Questions must be carefully read in order to place values at their respective regions in the diagram. Any region without a given value must be represented by a preferred variable.

Type 1

It involves the situation whereby the values of all the regions of the Venn diagram is given to answer some related questions.

Worked Examples

1. The number of students at Asaaman S.H.S 1A, offering the various combinations of Arts, Biology and Chemistry is shown on the diagram below;



How many students study:

i. Art and Biology but not Chemistry

ii. Biology but not Art

iii. Chemistry but neither Art nor Biology

iv. Biology and Chemistry

v. Art or Chemistry (or both)?

vi. Art or Chemistry but not both?

vii. Neither Arts nor Biology nor Chemistry

vii. How many students are in the class?

Solution

i. n(Art and Biology but not Chemistry) = 3

ii. n(Biology but not Art) = 2 + 6 = 8

iii. n(Chemistry but neither Art nor Biology) = 12

iv. n(Biology and Chemistry) = 20 + 6 = 26

v. n(Art or Chemistry or both)

= 3 + 7 + 20 + 5 + 6 + 12 = 53

vi. n(Art or Chemistry (but not both)

= 3 + 7 + 6 + 12 = 28

vii. Neither Arts nor Biology nor Chemistry = 10

viii. Number of students in the class;

= 7 + 3 + 5 + 20 + 2 + 6 + 12 + 10 = 65

Type 2

It involves finding the value of the intersection of the three sets. This can be done by either using the formula, the diagram or the cover – up method as shown in the examples below. Take note of the fact that for all three sets A, B and C,

1. n(AUBUC) = $n(A) + n(B) + n(C) - n(A\cap B) - n(B\cap C) - n(A\cap C) + n(A\cap B\cap C)$, if each person like all the three items (No complement)

2. n(AUBUC) = $n(A) + n(B) + n(C) - n(A\cap B) - n(B\cap C) - n(A\cap C) + n(A\cap B\cap C) + n(AUBUC)^1$, if some people do not like all the three items (Complement)

Worked Examples

1. In a survey of 200 students of a school, it was found that 120 study Mathematics, 90 study Physics and 70 study Chemistry, 40 study Mathematics and Physics, 30 study Physics and Chemistry, 50 study Chemistry and Mathematics and 20 study none of these subjects.

a. Find the number of students who study all three subjects.

b. How many students study;

i. Physics only? ii. Chemistry only?

Solution

Let U = {Students in the school}

M = {Mathematics students}

P = {Physics students}

C = {Chemistry students}

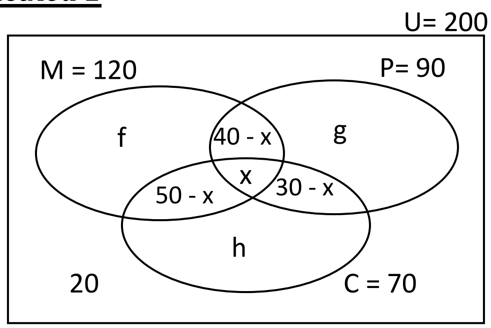
n(MUPUC) = 200, n(M) = 120,

n(P) = 90, n(C) = 70,

$(MUPUC)^1 = 20$, $n(M\cap P\cap C) = x$

$n(M\cap P) = 40$, $n(P\cap C) = 30$, $n(M\cap C) = 50$,

Method 1



For three sets M, P, and C, $n(\text{MUPUC})$
$= n(\text{M}) + n(\text{P}) + n(\text{C}) - n(\text{M}\cap\text{P}) - n(\text{P}\cap\text{C})$
$- n(\text{M}\cap\text{C}) + n(\text{M}\cap\text{P}\cap\text{C}) + n(\text{MUPUC})^1$
By substitution,
$200 = 120 + 90 + 70 - 40 - 30 - 50 + x + 20$
$200 = 180 + x$
$x = 200 - 180 = 20$

Method II
For set M,
$f + 40 - x + x + 50 - x = 120$
$f = 30 + x$ …………… (1)

For set P,
$g + 40 - x + x + 30 - x = 90$
$g = 20 + x$ …………….(2)

For set C,
$h + 50 - x + x + 40 - x = 70$
$h = -10 + x$ …………… (3)

For three sets M, P and C, $n(\text{MUPUC})$
$= (\text{I} + \text{II} + \text{III} + \text{IV} + \text{V} + \text{VI} + \text{VII} + \text{VIII})$
$= x + 40 - x + 50 - x + 30 - x + f + g + h + 20$
$200 = x + 40 - x + 50 - x + 30 - x + 30 + x + 20 + x - 10 + x + 20$
$200 = 180 + x$
$x = 200 - 180 = 20$

Method III: Cover up method
On the diagram, cover up one of the circles and add the value of that covered circle to the sum of the values of the other regions uncovered and equate them to the value of the universal set.

Covering set C, we have
$70 + f + 40 - x + g + 20 = 200$
But $f = 30 + x$ and $g = 20 + x$
By substitution,
$70 + 30 + x + 40 - x + 20 + x + 20 = 200$
$180 + x = 200$
$x = 200 - 180 = 20$

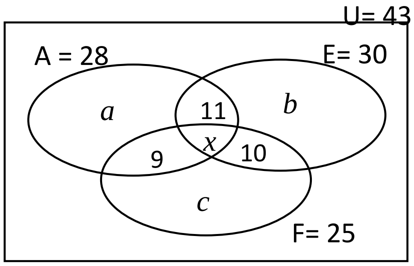
b. i. $n(\text{P})$ only $= g = 20 + x$
$= 20 + 20 = 40$

ii. $n(\text{C})$ only $= h = -10 + x$
$= -10 + 20 = 10$

2. A class of 43 students was asked to choose between offering Arabic or English or French or all the three languages. 28 of them chose Arabic, 30 chose English, 25 chose French, 11 chose Arabic and English only, 9 chose Arabic and French only and 10 chose English and French only. If each student chose at least one language;
i. illustrate this information on a Venn diagram
ii. how many students chose exactly three languages?
iii. how many students chose exactly two languages?
iv. find the number of students who chose Arabic only or English only or French only.

Solution
Let A represents students who chose Arabic, E represents people who chose English and F represents people chose French
$n(\text{AUEUF}) = 63$, $n(\text{A}) = 28$, $n(\text{E}) = 30$,
$n(\text{F}) = 25$, $n(\text{A}\cap\text{E})$ only $= 11$,
$n(\text{A}\cap\text{F})$ only $= 9$, $n(\text{E}\cap\text{F})$ only $= 10$,
$n(\text{A}\cap\text{E}\cap\text{F}) = x$

i.



ii. ***Method 1***

For set A,

$a + 9 + x + 11 = 28$

$a = 8 - x$ ………. .(1)

For set E'

$b + 11\ 10 + x = 30$

$b = 9 - x$ ………..(2)

For set F,

$c + 9 + 10 + x = 25$

$c = 6 - x$ ………..(3)

But a = A only, b = E only and c = F only

For three sets A, E, and F without complement, n(AUEUF) = A only + E only + F only + (A∩E) only + (A∩F) only + (E∩F) only + n(A∩E∩F)

$43 = 8 - x + 9 - x + 6 - x + 11 + 9 + 10 + x$

$43 = 53 - 2x$

$2x = 10$

$x = 5$

Method II

Covering up set F,

$25 + 8 - x + 11 + 9 - x = 43$

$53 - 2x = 43$

$2x = 10$

$x = 5$

iii. Exactly two languages,

$= 9 + 11 + 10 = 30$

iv. Arabic only or English only or French only

$= a + b + c$

$= 8 - x + 9 - x + 6 - x$, (But $x = 5$)

$= 8 - 5 + 9 - 5 + 6 - 5 = 8$

Type 3

This involves the situation where the values of all the regions are given to determine the value of the complement (None of the three)

Worked Examples

In a survey of 120 college students, it was found that 47 like Voltic mineral water, 52 like Silver mineral water, 58 like Deep mineral water, 25 like both Voltic mineral and Silver mineral water, 18 like both Silver mineral and Deep mineral water and 21 like both Voltic mineral and Deep mineral water. If 13 of the students like all the three brands of water;

i. Illustrate this information on a Venn diagram

ii. How many students like none of the three brands of water?

Solution

Note : Because the question did not state that every student like at least one brand of water, it implies that there is a complement of the 3 sets.

Let U = college students

V = students who like Voltic mineral water,

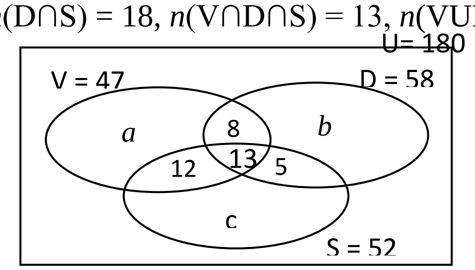
E = students who like Deep mineral water

S = students who like Silver mineral water

n(VUDUS) = 120, n(V) = 47, n(D) = 58,

n(S) = 52, n(V∩S) = 25, n(V∩D) = 21,

n(D∩S) = 18, n(V∩D∩S) = 13, $n(\text{VUDUS})^1 = x$



For three sets V, D, and S, $n(V \cup D \cup S)$
$= n(V) + n(D) + n(S) - n(V \cap D) - n(D \cap S)$
$- n(V \cap S) + n(V \cap D \cap S) + n(V \cup D \cup S)^1$

By substitution,
$120 = 47 + 58 + 52 - 21 - 18 - 25 + 13 + x$
$120 = 106 + x$
$x = 120 - 106 = 14$

Method 2
For set V,
$a + 12 + 13 + 8 = 47$
$a = 47 - 33$
$a = 14$ …………(1)

For set D,
$b + 8 + 13 + 5 = 58$
$b = 58 - 26$
$b = 32$………….(2)

For set S,
$c + 12 + 13 + 5 = 52$
$c = 52 - 30$
$c = 22$………..(3)

For three sets V, D and S,
$n(V \cup D \cup S)$
$= (I + II + III + IV + V + VI + VII + VIII)$
$= 12 + 8 + 13 + 5 + a + b + c + x$
By substitution,
$120 = 12 + 8 + 13 + 5 + 14 + 32 + 22 + x$
$120 = 106 + x$
$x = 120 - 106 = 14$

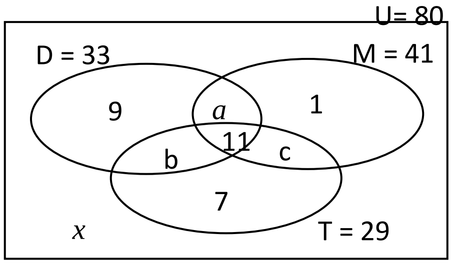
Method 3
Covering up set S,
$52 + a + 8 + b + x = 120$
But $a = 14$ and $b = 32$
$52 + 14 + 8 + 32 + x = 120$
$106 + x = 120$
$x = 120 - 106 = 14$

Type 4
It involves finding the values for two or more regions simultaneously.
I. Draw the Venn diagram and fill the given values in their appropriate regions
II. Represent regions without values with any preferred variables
III. For each circle, derive an equation for each of the variable represented
IV. Solve the equations for the values of the variables
V. Draw a new diagram and substitute the values of the variables

Worked Examples
1. A survey conducted at Maase Hospital revealed that out of the number of 80 patients admitted in June last year, 33 suffered from diarrhea, 41 suffered from malaria, 29 suffered from typhoid, 9 suffered from diarrhea only, 12 suffered from malaria only, 7 suffered from typhoid only and 11 were sick of all the three diseases.
a. Represent this information in a Venn diagram
b. Use your diagram to find the number of patients who suffered from;
i. diarrhea and malaria only?
ii. diarrhea and typhoid only?
iii. malaria and typhoid only?
iv. none of the three diseases?

Solution
U = {patients at the hospital}
D = {diarrhoea patients}
M = {malaria patients}
T = {typhoid patients}
$n(D) = 33$, $n(M) = 41$, $n(T) = 29$,
$n(D)$ only = 9, $n(M)$ only = 12, $n(T)$ only = 7, $n(D \cap M \cap T) = 11$, $n(D \cup M \cup T)^1 = x$,
$n(D \cap M)$ only = a

n(D∩T) only = b n(M ∩T) only = c



From set D

$a + b + 9 + 11 = 33$

$a + b + 20 = 33$

$a + b = 13$……………. (1)

From set M,

$a + c + 11 + 12 = 41$

$a + c = 41 – 23$

$a + c = 18$……………..(2)

From set T,

$b + c + 11 + 7 = 29$

$b + c = 29 – 18$

$b + c = 11$………………(3)

eqn (2) – eqn (1)

$c – b = 5$ ……………… (4)

eqn (3) + eqn (4)

$2c = 16$

$c = 8$

Put $c = 8$ in eqn (2)

$a + 8 = 16$

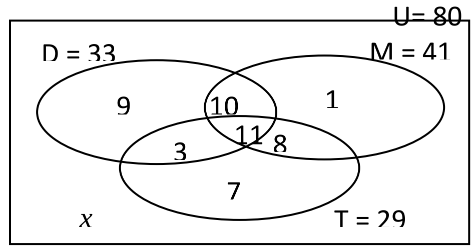
$a = 18 – 8 = 10$

Put $a = 10$ in eqn (1)

$10 + b = 13$

$b = 13 – 10 = 3$

Therefore, $a = 10$, $b = 3$ and $c = 8$



b. i. n(D∩M) only = 10

ii. n(D∩T) only = 3

iii. n(M∩T) only = 8

iv. None of the three diseases,

$80 = 9 + 10 + 11 + 3 + 8 + 12 + 7 + x$

$80 = 60 + x$

$x = 80 – 60$

$x = 20$

2. In a class of 50 students, their choice of the colors Red, Yellow and Green were as follows; 32 like Red, 28 like Yellow, 24 like Green, 10 like Red only, 11 like both Red and Yellow colors and 4 like exactly three of the colors. If each student like at least one of the colors, find the number of students like;

i. exactly one color , ii. exactly two colors.

Solution

U = {students in the class}

R = {students who like red color}

Y = {students who like yellow color}

G = {students who like green color}

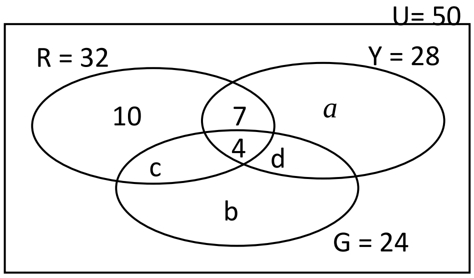
n(U) = 50, n(R) = 32 , n(Y) = 28, n(G) = 24,

n(R) only = 10, n(Y) only = a,

n(G) only = b, n(R∩Y∩R) = 4,

n(R∩Y) only = 7 n(R∩G) only = c

n(Y∩G) only = d



For set R,
$10 + 7 + 4 + c = 32$
$c = 32 – 21$
c = 11…………………(1)

For set Y,
$7 + 4 + a + d = 28$
$a + d = 28 – 11$
$a + d = 17$…………… (2)

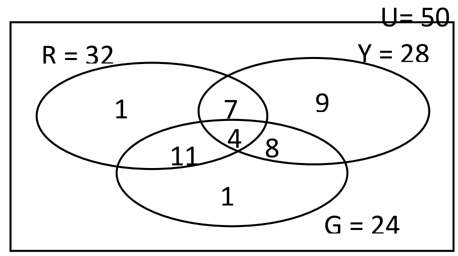
For set G,
$4 + c + b + d = 24$
$4 + 11 + b + d = 24$ (Put $c = 11$)
$b + d = 24 – 15$
$b + d = 9$………………….(3)

For set U,
$10 + 7 + 4 + a + b + c + d = 50$

But $c = 11$ and $b + d = 9$
$10 + 7 + 4 + a + 11 + 9 = 50$ (*By substitution*)
$a = 50 – 41 = 9$

Put $a = 9$ in eqn (2)
$9 + d = 17$
$d = 17 – 9 = 8$

Put $d = 8$ in eqn (3)
$b + 8 = 9$
$b = 9 – 8$
$b = 1$
Therefore, $a = 9$, $b = 1$, $c = 11$ and $d = 8$



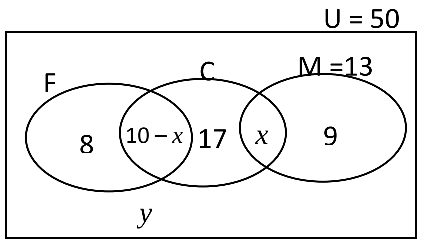
i. Exactly one color;
= n(R) only + n(Y) only + n(G) only
= 10 + 9 + 1 = 20

ii. Exactly two colors;
n(R∩Y) only + n(R∩G) only + n(Y∩G) only
= 7 + 11 + 8 = 26

Type 5
It involves the situation where two sets intersects with each other but not all the three sets.

Worked Examples
In a school, 50 students were asked their preference for three brands of soft drinks namely; Fanta, coke and malt. None liked all the three brands, 8 liked only Fanta, 17 liked only coke and 9 liked only malt. 10 liked two brands, 13 liked malt and none liked Fanta and malt
i. Illustrate the information on a Venn diagram
ii. How many students liked;
a. fanta, b. coke, c. at least one soft drink,
d. one soft drink, e. none of the drinks.

Solution
U = {Students}
F = {students who like Fanta}
C = {students who like coke}
M = {students who liked malt}
n(U) = 50, n(F) only = 8, n (C) only = 17
n(M) only = 9, n(M) = 13, $n(C \cap M) = x$
$n(F \cap C) = 10 – x$ $n(F \cup C \cup M)^1 = y$
$n(F \cap C \cap M)$ = { }



For set M,

$x + 9 = 13$

$x = 13 – 9 = 4$

$n(F∩C) = 10 – x$

But $x = 4$

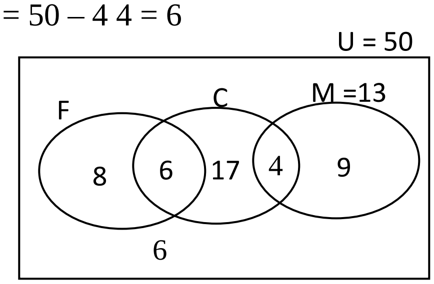
$n(F∩C) = 10 – 4 = 6$

n(F) only + n(F∩C) + n (C) only + n(C∩M) + n(M) only + n(FUCUM)^1= U

$\Rightarrow 8 + 6 + 17 + 4 + 9 + y = 50$

$44 + y = 50$

$y = 50 – 4\ 4 = 6$



ii.a. Number of students who liked Fanta

= 8 + 6 = 14

b. Number of students who liked coke;

= 6 + 17 + 4 = 27

c. Number of students who liked at least one soft drink ; = 8 + 6 + 17 + 4 + 9 = 44

d. Number of students who liked one soft drink;

= 8 + 17 + 9 = 34

e. none of the drinks = 6

Some Solved Past Questions

1. 50 children had a choice of beans, plantain and rice, of which 21 of them took beans, 24 took plantain and 18 took rice. 3 took beans only, 9 took plantain only and 2 took rice only. 5 took all the three items of food.

i. Draw a Venn diagram to illustrate this information

ii. Use your diagram to find the number of children who took:

a. Plantain and beans only;
b. Rice and beans;
c. None of the three items of food.

Solution

i. Let U = {Number of children}

B = {childrenwholikebeans}

P = {Childrenwholike plantain}

R = {children who like rice}

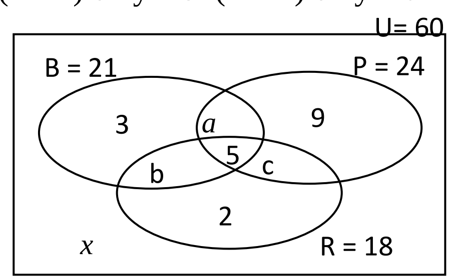
n(U) = 60, n(B) = 21, n(P) = 24, n(R) = 18,

n(B) only = 3, n(P) only = 9,

n(R) only = 2, n(B∩P∩R) = 5,

n(BUPUR)^1 = x, n(B∩P) only = a

n(B∩R) only = $b$$n$(P ∩R) only = c



ii. From set B

$a + b + 5 + 3 = 21$

$a + b = 21 – 5 – 3$

$a + b = 13$……………. (1)

From set P,

$a + c + 5 + 9 = 24$

$a + c = 24 – 5 – 9$

$a + c = 10$……………..(2)

From set R,

$b + c + 5 + 2 = 18$

$b + c = 18 – 5 – 2$

$b + c = 11$………………(3)

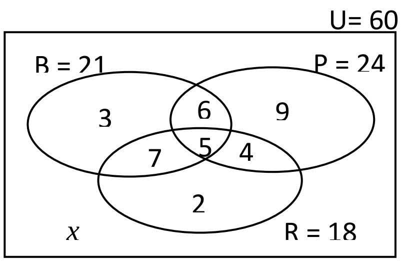
eqn (1) – eqn (2);
$b - c = 3$ ……………… (4)

eqn (3) + eqn (4);
$2b = 14$
$b = 7$

Put $b = 7$ in eqn (1);
$a + 7 = 13$
$a = 13 - 7 = 6$

Put $a = 6$ in eqn (2);
$6 + c = 10$
$c = 10 - 6 = 4$

Therefore, $a = 6$, $b = 7$ and $c = 4$



For set U;
$3 + 6 + 7 + 5 + 9 + 4 + 2 + x = 50$
$x = 50 - 36 = 14$

a. Plantain and beans only:
$n(P \cap B)$ only = 6

b. Rice and beans,
$n(R \cap B) = 5 + 2 = 7$

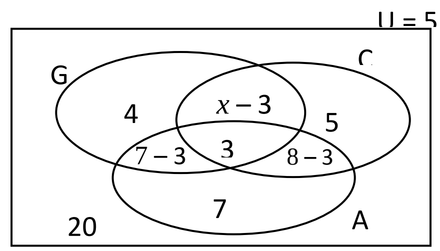
c. None of the three items of food:
$n(BUPUR)^1 = x = 14$

2. In a class of 55 students, some study at least one of the following subjects: General science, Commerce and Accounts. 20 students study none of them. The following table gives further details of the subjects studied:

General Science only	4
Commerce only	5
Accounts only	7
All three subjects	3
Gen. Sci& Accounts	7
Commerce & Accounts	8

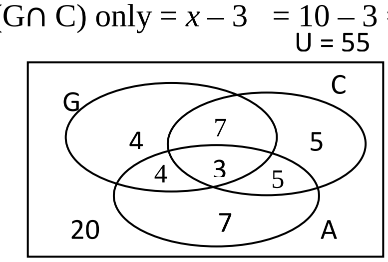
a. Illustrate the above information in a Venn diagram.
b. Find the number of students who studied:
i. General science or accounts or both, but not commerce,
ii. Commerce.

Solution
a. General science and accounts = x



$4 + (7 - 3) + 3 + (x - 3) + 7 + (8 - 3) + 5 + 20 = 55$
$4 + 4 + x + 7 + 5 + 5 + 20 = 55$
$x + 45 = 55$
$x = 55 - 45 = 10$

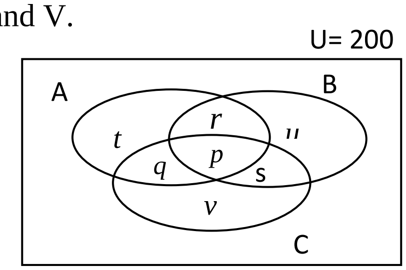
$\Rightarrow n(G \cap C)$ only $= x - 3 \quad = 10 - 3 = 7$



b. i. Number of students who studied General science or accounts or both, but not commerce
$= 4 + 4 + 7 = 15$ students

ii. Number of students who studied Commerce;
= 7 + 3 + 5 + 5 = 20 students

3. A, B and C are three intersecting sets. Seven regions of the Venn diagram are P, Q, R, S, T, U and V.



Find; a. i. $A \cap B^1$ ii. $A \cap (B \cup C)$

b. Write an expression for each of the regions in terms of A, B and C

i. P ii. Q iii. T

Solution

a. i. $A \cap B^1 = \{t\}$ ii. $A \cap (B \cup C) = \{r, p, q\}$

b. i. P = $(A \cap B \cap C)$ ii. Q = $(A \cap C)$ only

iii. T = A only

Exercises 1.13

1. A, B and C are sets. $n(A) = 17$, $n(B) = 29$ and $n(C) = 14$. Also, $n(A \cup B \cup C) = 45$, $n(A \cap B \cap C^1) = 6$, $n(B \cap C \cap A^1) = 1$and $n(A \cap C \cap B^1) = 4$. With the aid of a Venn diagram, find $n(A \cap B \cap C)$.

2. A school has rugby 15, a cricket 11 and a swimming 8. In all the three teams are 3 boys, 9 are in the rugby team only and 5 are in both the rugby and cricket teams; 2 boys are in the cricket team only. Show these facts in a Venn diagram, and deduce the number of boys who represent the school in the cricket team only.

3. A survey of the members of a university hall revealed the following statistics; 60% liked football, 50% liked hockey, 50% liked athletics, 30% liked football and hockey, 20% liked hockey and athletics, 30% liked football and athletics and 10% liked all the three sports;

a. What percentage of the house liked football and hockey, but not athletics?

b. What percentage liked exactly two out of the three sports?

c. What percentage did not like any of the three sports?

4. In a survey of 60 people, it was found that 25 people read newspaper H, 26 read newspaper T, 26 read newspaper I, 9 read both H and I, 11 read both H and T, 8 read both T and I, 3 read all three newspapers. Find the number of people who read at least one of the newspapers

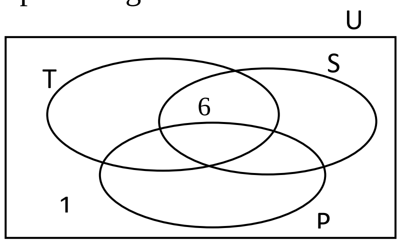
5. In a survey of 200 students of a school, it was found that 120 study mathematics, 90 study physics and 70 study chemistry, 40 study mathematics and physics, 30 study physics and chemistry, 50 study chemistry and mathematics and 20 study none of these subjects. Find the number of students who study all three subjects.

6. A class of 26 boys were each required to have a certain text book in English, French and Mathematics. 19 boys had the English book, 23 the French and 15 the mathematics. 16 had both English and French books, 14 had French and mathematics and 13 had mathematics and English. How many boys in the class possessed all the three books?

7. In a class of 48 students, it is known that 24 of them do Arts, 20 do Chemistry and 22 do Biology. All the students do at least three subjects while 7 do Art and Biology, 6 do Art and Chemistry but not Biology and 8 do Chemistry and Biology. Three do all the three subjects. How

many of them do Chemistry only or Biology only or Art only?

8. The universal set μ is the set of positive integers less than 13. The subsets are defined as follows; T = {multiples of 2}, S = {multiples of 6} P = {prime numbers}. Copy the diagram below and write the numbers 1, 2, 3,…,12 in the appropriate regions



9. In a secondary school, there are 124 students in form two. Of these 86 play table tennis, 84 play football and 94 play volley ball: 30 play table tennis and volley ball, 34 play volley ball and football and 42 play table tennis and football. Each student play at least one of the three games and *x* student plays all the three games.
a. Display these facts in a Venn diagram.
b. Write down an equation in *x* and hence find *x*

10. Out of a group of 20 persons, 7 like classical music, 12 like soul music, and 10 like highlife music; furthermore, 3 like both classical and highlife music, 2 like both classical and soul music and 2 like all the three kinds music. Draw a Venn diagram and find how many of the twenty persons like soul and highlife music but not classical music. (Assume all the 20 persons like at least one of the three kinds of music)

11. A examination was held for the filling of vacancies in three branches A, B and C of a certain service. There were 75 candidates, all of whom were asked to name the branch or branches in which they were willing to accept a vacancy, if it were offered. The result of this enquiry was as follows: 3 candidates would accept A only, 3 B only and 3 C only, There were all together 15 students who would not accept A, 10 who would not accept B and 34 who would not accept C. With the aid of a Venn diagram, or otherwise, Find how many were prepared to accept a vacancy in any of the three branches.

12. In a certain school, there are 118 boys in form three. Of these, 56 play table tennis, 67 play football and 44 play hockey. 23 play table tennis and football, 18 play football and hockey and 20 play hockey and table tennis. Everybody plays at least one game and *n* boys play all the three games
i. Express these facts in a Venn diagram
ii. Find the value of *n* from the diagram

13. In a class of 50 students, 27 study French, 24 study History and 30 study Geography. Each student studies at aleast one of the three subjects. 5 study all three subjects whilst 11 study French and History, 8 study History and Geography but not French and 12 study French and Geography.
i. Display these information on a Venn diagram
ii. How many students study only one of the three subjects?
iii. How many study exactly two of the three subjects?

2 REAL NUMBER SYSTEM

The Real Number System

The following set of numbers constitute the real number system:

1. Whole numbers (W): The system of whole numbers consists of numbers in the set {0, 1, 2, 3…}

2. Natural numbers (N): The system of natural numbers consists of numbers in the set {1, 2, 3…}. They are also called Counting numbers

3. Integers (Z): The system of integers consists of numbers in the set; {… -3, -2, -1, 0, 1, 2, 3…}. This is made up of both negative and positive whole numbers including zero.

4. Rational numbers (Q): The system of rational numbers consists of the set of both positive and negative fractions. In other words, they are numbers that can be expressed in the form, $\frac{a}{b}$, where a and b are integers but $b \neq 0$.

5. Irrational numbers (H): The system of irrational numbers consists of numbers that cannot be expressed in the form $\frac{a}{b}$. This is because their decimals are non-terminating. For example, π = 3.13857… is said to be an irrational number because the digits are neither repeating successively nor terminating. Other examples of irrational numbers include; $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$.

Note: The root sign is not a characteristic of all irrational numbers because numbers like $\sqrt{4}$, $\sqrt{9}$, etc are not irrational but rational numbers.

6. Real numbers (R):The system of real numbers consists of the set of both rational and irrational numbers. This is represented by the set;

R = {… -2, -1, $\frac{-2}{3}$, 0 , $\frac{1}{5}$, 1, $\sqrt{4}$…}

Relating the Number Systems

At this point, the set of real numbers can be defined to include negative and positive integers, zero, positive and negative fractions as well as decimals and irrational numbers, which can be represented by points on the number line.

The numbers 1, 2, 3… are called ***counting numbers or natural numbers or positive integers.***

The numbers 0, 1, 2, 3… are called ***whole numbers.***

If the set of rational numbers are represented by **Q**, the set of integers by **Z**, the set of whole numbers by **W** and the set of natural numbers by **N**, then;

Q = $\{\ldots -5, -6, -1\frac{1}{4}, \ 0, \ 1, \ 2, \ 7\frac{1}{6}, \ 11.4 \ldots\}$

Z = $\{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$

W = $\{0, 1, 2, \ 3 \ldots\}$

N = $\{1, 2, 3 \ldots\}$.

From the above, the following observations can be made;

I. The set of natural numbers (N) is both a subset of whole numbers (W) and integers (Z)

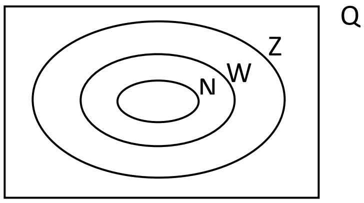
II. The set of whole numbers (W) is also a subset of integers (Z).

III. The set of integers (Z) is a subset of rational numbers (Q),

These are summarized as follows;

N⊂W, N⊂ Z and N⊂ Q

Also, W⊂Z, W⊂Q and Z⊂Q These relationships are shown in the diagram below;



The rectangle represents the set of rational numbers Q. Since N, W and Z are inside the rectangle, they are all said to be the subset of Q. Likewise, each set of numbers that is contained in another is said to be a subset

Worked Examples
Find the following sets
a. N ∩Z   b. QUZ

Solution
a. N = {1, 2, 3…} and $W = \{0, 1, 2, 3 ...\}$.
Therefore, N∩W = {1,2,3 ... } = N

b. Z = $\{... -2, -1, 0, 1, 2 ...\}$ and
Q = $\left\{-2, -1, -1\frac{1}{2}, \; 0, 1, \; 1\frac{1}{2}, 2 ...\right\}$
Since Z⊂Q, then QUZ = Q

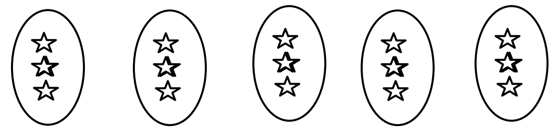
Exercises 2.1
A. Find the following;
1. NUZ   2. Q∩N   3.W∩Z   4. W∩N
5. Q∩Z   6. WUN   7. NUW   8. ZUW

B. Clasify each as sets N, W, Z and Q
1. -2.75   2. 0   3. $\frac{22}{7}$ 4. $6\frac{1}{5}$ 5. 17

Common Fractions
A fraction is defined as part of a whole or group. It consists of two parts: ***numerator*** and ***denominator***. Thus, if a unit is divided into***'b'*** equal parts and ***'a'*** part out of it is taken, we have a fraction of $\frac{\boldsymbol{a}}{\boldsymbol{b}}$, where ***a*** is the numerator and ***b*** is the denominator.

It follows that if a unit is divided into 8 equal parts and 3 out of it is taken, we have a fraction of $\frac{3}{8}$, where 3 is the numerator, and 8is the denominator and the remaining fraction is $\frac{5}{8}$

Similarly, when 15 stars are put into groups of 3, we have 5 groups of 3 as shown below;



When 2 groups of 3 are selected out of the 5 groups, we have a fraction of $\frac{2}{5}$ and the remaining fraction is $\frac{3}{5}$.

Exercises 2.2
1. In a test marked out of 10, Romeo scored $\frac{7}{10}$. What fraction of the test did he get wrong?

2. Some oranges were shared between Cain and Abel. If Abel received $\frac{9}{20}$ of the oranges, what fraction of the oranges wasreceived by Cain?

3. In a 12-round boxing bout, Bukom Banku knocked out his Mexican opponent on round 8
i. Express this as a fraction in its lowest term.
ii. What fraction of the bout was not fought?

Types of Fractions
The various types of fractions are as follows.
1. Proper fractionIt is a fraction whose numerator is less than the denominator .That is in the form $\frac{a}{b}$, where $a < b$ or $b > a$. Examples are; $\frac{2}{5}$, $\frac{3}{8}$, $\frac{9}{10}$ ...

2. Improper fraction
It is a fraction whose numerator is greater than the denominator. That is, $\frac{a}{b}$ where $a > b$, or
$b < a$. Examples are; $\frac{5}{2}$, $\frac{8}{3}$, $\frac{10}{9}$, $\frac{4}{1}$, $\frac{21}{8}$ …

3. Mixed fraction

It is a combination of a whole number and a proper fraction i.e. $A\frac{b}{c}$, where c > *b* or *b* < *c*

e. g. $1\frac{3}{4}$, $15\frac{1}{2}$, $3\frac{1}{4}$ *etc*

4. Decimal fraction

It is a fraction whose denominator is a power of ten. Decimal fractions can be written with a whole number, making it simpler to do calculations. Eg are; $\frac{43}{100} = 0.43$, $\frac{51}{1000} = 0.051\ldots$

5. Like fractions

They are two or more fractions with the same (equal) denominator. That is: $\frac{a}{b}$ and $\frac{c}{b}$. Examples are the pairs; $\frac{6}{4}$ and $\frac{1}{6}$, $\frac{2}{5}$ and $\frac{3}{5}$ etc.

6. Unlike Fractions

They are two or more fractions with different (unequal) denominators. That is; $\frac{a}{b}$ and $\frac{c}{d}$. Examples are; $\frac{5}{7}$ and $\frac{3}{4}$, $\frac{4}{7}$ and $\frac{11}{21}$,

7. A unit fraction

It is a fraction that has a numerator of 1.

Examples are; $\frac{1}{7}$, $\frac{1}{10}$, $\frac{1}{15}$ etc

Decimals as Common Fractions

To write a decimal fraction as improper fraction;

I. Identify the number of decimal places;

II. Remove the decimal point to make it a whole number;

III. Divide the whole by ten exponents the number of decimal places . For example, in 0.25, there are 2 decimal places. If the decimal point is removed, we have a whole of 25. Then divide 25 by ten exponent the number of decimal places which is 2 and express the fraction in its lowest form. That is; $0.25 = \frac{25}{10^2} = \frac{25}{100} = \frac{1}{4}$

Worked Examples

1. Express 0.625 as a fraction to its lowest term

Solution

0.625 (3 d. p) = $\frac{625}{10^3} = \frac{625}{1000} = \frac{5}{8}$

2. Write 0.55 as a fraction in its lowest term

Solution

0.55 (2 d. p)

$\frac{55}{10^2} = \frac{55}{100} = \frac{11}{20}$

3. Change 0.42 as a fraction in its lowest term.

Solution

$0.42 = \frac{42}{10^2} = \frac{42}{100} = \frac{21}{50}$

Exercises 2.4

Write the following decimals as fractions

1. 0.75	2. 1.5	3. 0.025
4. 0.15	5. 2.22	6. 0.035

Proper Fractions as Decimal Fractions

When the denominator of a fraction cannot be expressed as ten or a power of ten, then long division is required. For example, $\frac{3}{7}$ can be represented on the long division table as ; $7\sqrt{3}$ to mean 3 ÷ 7. Since 7 is greater than 3, it cannot divide 3 to give a whole number but a value less than one (i.e. zero point)

$$7\overline{\smash{)}3}\quad 0.$$

- Suffix 0 to the 3 to get 30.

$$7\overline{\smash{)}30}\quad 0.$$

- Continue with the division .i.e. 7 into 30 to get 4.

$$7\,\overline{)\,3\,0}\quad 0.4$$

- *multiply the quotient (answer) by 7 and write the product under 30 and subtract.*

$$\begin{array}{r} 0.4 \\ 7\,\overline{)\,3\,0} \\ -28 \\ \hline 2 \end{array}$$

For all remainders, suffix 0 and divide by the divisor

$$\begin{array}{r} 0.485\ldots \\ 7\,\overline{)\,3\,0} \\ -28 \\ \hline 20 \\ 14 \\ \hline 60 \\ 56 \\ \hline 40 \end{array}$$

Worked Examples

Express the following as decimal fraction.

1. $\frac{5}{16}$ = 0.3125

$$\begin{array}{r} 16\,\overline{)\,5\,0} \\ -48 \\ \hline 20 \\ -16 \\ \hline 40 \\ -3\,2 \\ \hline 80 \\ -80 \\ \hline \end{array}$$

$\frac{5}{16}$ = 0.3125

Mixed Fractions as Improper Fractions

A mixed fraction is a fraction written in the form $\boldsymbol{A}\,\frac{b}{c}$, where $c > b$. In $\mathbf{A}\,\frac{b}{c}$, **A** is a whole number, ***b*** is the numerator and ***c*** is the denominator.

To write a mixed fraction as an improper fraction, that is: $\boldsymbol{A}\,\frac{b}{c}$ in the form $\frac{d}{e}$;

I. Multiply the denominator by the whole number. i.e. $\mathbf{A} \times \boldsymbol{c}$

II. Add the numerator to the product obtained.

i. **e.** $(\boldsymbol{A} \times \boldsymbol{c}) + \boldsymbol{b}$

III. Write a division of the results by the denominator and simplify the numerator of the new result. That is; $\frac{(\mathbf{A}\times\mathbf{c})+\mathbf{b}}{\mathbf{c}} = \frac{(\mathbf{Ac})+\mathbf{b}}{\mathbf{c}}$

Worked Examples

Express as improper fractions;

1. $5\,\frac{2}{3}$ 2. $6\,\frac{3}{4}$

Solution

1. $5\,\frac{2}{3} = \frac{(3\times5)+2}{3} = \frac{15+2}{3} = \frac{17}{3}$
2. $6\,\frac{3}{4} = \frac{(4\times6)+3}{5} = \frac{24+3}{5} = \frac{27}{4}$

Exercises 2.5

A. Write the following as decimals;

1. $3\,\frac{2}{5}$ 2. $10\,\frac{1}{4}$ 3. $11\frac{1}{2}$ 4. $15\frac{2}{3}$ 5. $3\frac{3}{5}$

Improper Fractions as Mixed Fractions

Fractions of the form, $\frac{e}{c}$, where $e > c$ are called ***improper fractions***.

Improper fractions can be written as mixed fractions of the form $\boldsymbol{A}\,\frac{R}{C}$. That is: $\frac{e}{c} = \boldsymbol{A} + \frac{R}{C}$.

Worked Examples

Write the following as mixed fractions;

1. $\frac{17}{5}$ 2. $\frac{33}{7}$

Solution

1. $\frac{17}{5} = 3R\,\frac{2}{5} = 3 + \frac{2}{5} = 3\,\frac{2}{5}$
2. $\frac{33}{7} = 4\text{ R }\frac{5}{7} = 4 + \frac{5}{7} = 4\,\frac{5}{7}$

Exercises 2.6

Write the following as mixed fractions

1. $\frac{17}{3}$ 2. $\frac{40}{6}$ 3. $\frac{100}{15}$ 4. $\frac{53}{7}$ 5. $\frac{21}{5}$ 6. $\frac{7}{5}$

Terminating and Recurring Decimals

A decimal is said to be terminating when it comes to an end. For example, 0.75, 3.4, 0.625, 0.25, etc

A decimal is said to be recurring when it repeats digits and does not come to an end. Eg. 0.3333…, 0.181818…, 3.142142… *etc*

Recurring decimals are terminated by placing a dot on the digit that repeats. For example, 0.333…can be written as $0.\dot{3}$, whereas 4.181818… can also be written as $4.\dot{1}\dot{8}$. for short.

If more than two digits repeat, place a dot on the first and last digits that repeat or sometimes a line over the pattern. For example, 0.657657… is terminated as $0.\dot{6}5\dot{7}$.

Exercises 2.7

Express the following as a decimal

1. $\frac{9}{5}$ 2. $\frac{2}{25}$ 3. $\frac{8}{5}$ 4. $\frac{10}{7}$ 5. $\frac{25}{4}$

Equivalent Fractions

Two or more fractions that are written in different forms but have the same value are called ***equivalent fractions***. For instance, if $\frac{5}{15} = \frac{1}{3}$ and $\frac{10}{30} = \frac{1}{3}$, then $\frac{5}{15}$ and $\frac{10}{30}$ are said to be equivalent fractions, because they have the same value.

Equivalent fractions are formed by multiplying the same number by the numerator and denominator of a fraction. That is; $\boldsymbol{\frac{a}{b} = \frac{a \times n}{b \times n}}$, where *n* is a natural number.

Similarly, equivalent fractions can be formed by dividing the same number by the numerator and the denominator. That is: $\boldsymbol{\frac{c}{d} = \frac{c \div m}{d \div m}}$, where *m* is a natural number.

Worked Examples

Write the next four immediate equivalents fractionsfor the following fractions;

i. $\frac{5}{4}$ ii. $\frac{3}{7}$

Solutions

i. $\frac{5}{4} = \frac{5\times2}{4\times2} = \frac{5\times3}{4\times3} = \frac{5\times4}{4\times4} = \frac{10}{8} = \frac{15}{12} = \frac{20}{16} = \frac{25}{20}$

ii. $\frac{3}{7} = \frac{3\times2}{7\times2} = \frac{3\times3}{7\times3} = \frac{3\times4}{7\times4} = \frac{3\times5}{7\times5} = \frac{6}{14} = \frac{9}{21} = \frac{12}{28} = \frac{15}{35}$

Application of Equivalent Fractions

Two or more equivalent fractions are said to have the same value and can be equated as such. Thus, if one of them contains a variable, the value of the variable can be found as follows:

I. Simply equate the equivalent fractions;

II. Find the cross products;

III. Solve for the value of the variable.

Worked Examples

1. If 1 : *x* is equivalent to $6\frac{1}{4}$: 25, find *x*.

Solution

$\frac{(6 \times 4)+1}{4} = \frac{25}{4}$ $\left(\begin{array}{c}\text{Change the mixed fraction}\\ \text{to an improper fraction.}\end{array}\right)$

$\Rightarrow \frac{1}{x} = \frac{\frac{25}{4}}{25}$

$\frac{1}{x} = \frac{1}{4}$,

x = 4

2. If $\frac{9}{21}$ is equivalent to $\frac{1+x}{7}$, find the value of *x* .

Solution

$\frac{9}{21} = \frac{1+x}{7}$

$21(1 + x) = 7 \times 9$ **(By cross multiplication)**
$21 + 21x = 63$ **(Expansion)**
$21x = 63 - 21$
$21x = 42$
$x = 2$

3. If $\frac{1}{x} = 1\frac{1}{2}$, find x

Solution

$1\frac{1}{2} = \frac{(2 \times 1) + 1}{2} = \frac{3}{2}$ (Change mixed fraction to improper fraction.)

$\frac{1}{x} = \frac{3}{2}$

$3x = 2$ **(By cross multiplication)**

$x = \frac{2}{3}$

Exercises 2.8

A. Write four equivalent fractions:

1. $\frac{5}{6}$ 2. $\frac{9}{4}$ 3. $\frac{5}{3}$ 4. $\frac{7}{11}$

B. Find the missing numbers;

1. $\frac{2}{3} = \frac{x}{9}$ 2. $\frac{3}{4} = \frac{x}{16}$ 3. $\frac{4k}{9} = 12$

4. $\frac{2}{5} = \frac{10}{x}$ 5. $\frac{3}{4} = \frac{x}{16}$ 6. $\frac{x}{4} = 2$

Comparing Fractions

Two or more fractions are compared by putting in the symbols; $>$ (greater than) or $<$ (less than). The act is intended to identify the fraction that is greater or lesser than the other.

Worked Examples

1. Compare $\frac{2}{5}$ and $\frac{3}{2}$

Solution

I. Write equivalent fractions for each.

$\frac{2}{5} = \frac{4}{10}, \frac{3}{15}, \frac{8}{20}, \frac{10}{25} \ldots$

$\frac{3}{2} = \frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \frac{15}{10} \ldots$

II. Select the fractions with common denominator as substitutes to the original fractions. That is $\frac{2}{5} = \mathbf{\frac{4}{10}}$ and $\frac{3}{2} = \mathbf{\frac{15}{10}}$

III. Compare the substitute fractions and select the one with the larger numerator as the greater fraction. $\Rightarrow \frac{4}{10} < \frac{15}{10}$

IV. Conclude that $\frac{2}{5}$ is less than $\frac{3}{2}$. $\Rightarrow \frac{2}{5} < \frac{3}{2}$

2. Compare $\frac{3}{5}$ and $\frac{6}{11}$

Solution

$\frac{3}{5} = \frac{6}{10}, \frac{9}{15}, \frac{12}{20}, \frac{15}{25}, \frac{18}{30}, \frac{21}{35}, \frac{24}{40}, \frac{27}{45}, \frac{30}{50}, \frac{33}{55} \ldots$

$\frac{6}{11} = \frac{12}{22}, \frac{18}{33}, \frac{24}{44}, \frac{30}{55}, \frac{36}{66}, \frac{42}{77} \ldots$

$\frac{33}{55} > \frac{30}{55}$

$\therefore \frac{3}{5} > \frac{6}{11}$

Comparing Unit Fractions

To compare two or more fractions each with numerator 1 (unit fractions), the fraction with smaller denominator is the greatest. For example, in $\frac{1}{2}$ and $\frac{1}{3}$, $\frac{1}{2}$ is greater than $\frac{1}{3}$, written as: $\frac{1}{2} > \frac{1}{3}$

Exercises 2.9

Put in $<$, $>$ or $=$;

1. $\frac{1}{11}, \frac{1}{8}$ 2. $\frac{1}{4}, \frac{1}{10}$ 3. $\frac{1}{3}, \frac{4}{3}$ 4. $\frac{3}{4}, \frac{5}{4}$ 5. $\frac{5}{2}, \frac{1}{8}$

Ordering Fractions

Two or more fractions are ordered by arranging them either in ascending (increasing) order or descending (decreasing) order.

The two common methods that can be used to order fractions are;

1. The L.C.M Method

Rewrite the fractions to have a common denominator and order the numerators accordingly

2. Multiplication by 100%

Multiply each fraction by 100% and round the product or answer to the nearest whole number. Then order the whole numbers in accordance with the fractions.

Worked Examples

1. Arrange the fractions; $\frac{3}{4}, \frac{2}{3}, \frac{3}{5}$ in order of ascendency.

Solution

Method 1

Find the L.C.M. of all the denominators and workout as follows,

$\frac{3}{4}$, $\frac{2}{3}$, $\frac{3}{5} = \frac{45 \quad , \quad 40 , \quad 36}{60}$

Comparing the numerators, the ascending order becomes $\frac{36}{60}, \frac{40}{60}, \frac{45}{60} = \frac{3}{5}, \frac{2}{3}, \frac{3}{4}$

Method 2

Multiply each fraction by 100 and round off the answer to the nearest whole number.

$\frac{3}{4} \times 100\% = 75\%$

$\frac{2}{3} \times 100\ \% = 67\%$

$\frac{3}{5} \times 100\% = 60\%$

Comparing the product, the ascending order is

60, 67, 75 = $\frac{3}{5}$, $\frac{2}{3}$, $\frac{3}{4}$

2. Arrange the following fractions in descending order $\frac{9}{16}, \frac{5}{8}$, 0.62

Solution

Multiply each fraction by 100%

$\frac{9}{16} \times 100\% = 56\%$

$\frac{5}{8} \times 100\% = 63\ \%$

$0.62 \times 100\% = 62\%$

Descending order = 63, 62, 56 = $\frac{5}{8}$, 0.62, $\frac{9}{16}$

Exercises 2.10

A. Arrange in ascending order:

1. $\frac{3}{4}, \frac{2}{3}, \frac{3}{5}$ 2. $4\frac{1}{4}, 4\frac{1}{2}, 4\frac{1}{3}$ 3. $\frac{3}{4}, \frac{2}{3}, \frac{4}{5}$

4. $\frac{7}{12}, \frac{3}{5}, \frac{7}{15}, \frac{3}{4}$ 5. $\frac{4}{13}, \frac{4}{7}, \frac{4}{9}$ 6. $\frac{1}{3}, \frac{2}{5}, \frac{7}{8}, \frac{1}{10}$

B. Arrange in descending order;

1. $\frac{1}{2}$, $\frac{17}{20}$, $\frac{3}{4}, \frac{8}{5}$ 2. $\frac{5}{6}$, $\frac{4}{5}$, $\frac{4}{7}$

3. $\frac{2}{3}, \frac{3}{5}, \frac{4}{7}$ 4. 0.32, $\frac{2}{5}$, 27%, $\frac{1}{3}$

C.1. Mr. Johnny used $\frac{6}{13}$ of a full tank of petrol to travel to Kumasi and $\frac{5}{11}$ of the full tank of petrol to travel to Accra. Which journey required more petrol?

2. Mrs. Aku uses $\frac{7}{12}$ of a bag of flour formeat pies and $\frac{9}{16}$ of the same size bag of flour for cakes. For which item does she uses less flour?

Addition and Subtraction of Fractions

To add and subtract fractions, consider whether they have like or unlike denominators. Like denominators refer to the same denominator, e.g. $\frac{3}{5}$ and $\frac{2}{5}$ and unlike denominators refer to different denominators, e.g. $\frac{3}{5}$ and $\frac{5}{2}$.

Addition of Fractions with Like Denominators

This is done by adding the numerators of the respective fractions and maintaining the common denominator. i. e $\frac{b}{a} + \frac{c}{a} = \frac{b+c}{a}$

Worked Examples

Perform the following addition;

1. $\frac{5}{12}+\frac{6}{12}$ 2. $\frac{5}{14}+\frac{8}{14}$

Solution

1. $\frac{5}{12}+\frac{6}{12}=\frac{5+6}{12}=\frac{11}{12}$

2. $\frac{5}{14}+\frac{8}{14}=\frac{5+8}{14}=\frac{13}{14}$

Subtraction of Fractions with Like Denominators

This is done by subtracting the numerators of the respective fractions and maintaining the common denominator. i. e $\frac{b}{a}-\frac{c}{a}=\frac{b-c}{a}$

Worked Examples

Perform the following subtractions.

1. $\frac{7}{12}-\frac{4}{12}=\frac{7-4}{12}=\frac{3}{12}=\frac{1}{4}$
2. $\frac{9}{20}-\frac{6}{20}-\frac{1}{20}=\frac{9-6-1}{20}=\frac{2}{20}=\frac{1}{10}$

Addition and Subtraction of Fractions with Unlike Denominators

<u>Method 1</u>

To add or subtract two or more fractions with unlike denominators, convert the given fraction to fractions with the same denominator and add or subtract accordingly.

<u>Method 2</u>

I. Make use of the facts that:

1. $\frac{c}{a}+\frac{d}{b}=\frac{cb+ad}{ab}$ 2. $\frac{c}{a}-\frac{d}{b}=\frac{cb-ad}{ab}$

II. Simplify the final answer where possible.

Worked Examples

1. $\frac{7}{2}+\frac{5}{3}$

Solution

<u>Method 1</u>

L.C.M of 2 and 3 = 6

$\frac{7}{2}=\frac{7\times 3}{2\times 3}=\frac{21}{6}$

$\frac{5}{3}=\frac{5\times 2}{3\times 2}=\frac{10}{6}$

$\Rightarrow \frac{21}{6}+\frac{10}{6}=\frac{21+10}{6}=\frac{31}{6}$

<u>Method 2</u>

$\frac{7}{2}+\frac{5}{3}=\frac{21+10}{6}=\frac{31}{6}$

Multiplication of Fractions

To multiply two or more fractions;

I. Change all mixed fractions if any.

II. Multiply numerators and denominators respectively.

III. Reduce the product to the lowest terms, if possible. i.e. $\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d}=\frac{ac}{bd}$

Worked Examples

1. Perform $3\frac{1}{2}\times 2\frac{1}{5}$

Solution

$3\frac{1}{2}\times 2\frac{1}{5}=\frac{7}{2}\times\frac{11}{5}$

$\Rightarrow \frac{7}{2}\times\frac{11}{5}=\frac{7\times 11}{2\times 5}=\frac{77}{10}$

2. Perform $\frac{2}{5}\times\frac{15}{8}$

Solution

$\frac{2}{5}\times\frac{15}{8}=\frac{2\times 15}{5\times 8}=\frac{30}{40}=\frac{3}{4}$

3. Find $\frac{3}{4}$ of $\frac{1}{3}$

Solution

$\frac{3}{4}\times\frac{1}{3}=\frac{3\times 1}{4\times 3}=\frac{3}{12}=\frac{1}{4}$

Exercises 2.13

Perform the following;

1. $8\frac{5}{7} \times 2\frac{7}{10}$ 1. $\frac{2}{3}$ of $\frac{2}{5}$ 3. $\frac{3}{4}$ of $\frac{3}{5}$

Division of Fractions

Division of fractions, is performed by going through the following steps:

I. Change the division to multiplication sign.

II. Find the reciprocal of the second fraction.

III. Multiply the respective numerators and denominators. This is summarized as:

$$\boxed{\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}}$$

Worked Examples

Perform the following;

1. $\frac{3}{4} \div \frac{5}{6}$

Solution

<u>Method 1</u>

$\frac{3}{4} \div \frac{5}{6}$ $\left(\begin{matrix}\boldsymbol{Change\ \div to\ \times and}\\ \boldsymbol{reciprocate\ the\ second\ fraction}\end{matrix}\right)$

$\frac{3}{4} \times \frac{6}{5} = \frac{18}{20} = \frac{9}{10}$

<u>Method 2</u>

$\frac{3}{4} \div \frac{5}{6}$ $\left(\begin{matrix}\boldsymbol{Multiply\ both\ numerators\ and}\\ \boldsymbol{denominators\ by\ LCM = 12}\end{matrix}\right)$

$= \frac{\frac{3}{4}\times 12}{\frac{5}{6}\times 12} = \frac{3\times 3}{5\times 2} = \frac{9}{10}$

2. $1\frac{3}{4} \div 3\frac{1}{5}$

Solution

$1\frac{3}{4} \times 3\frac{1}{5} = \frac{7}{4} \times \frac{16}{5} = \frac{7\times 16}{4\times 5} = \frac{112}{20} = \frac{28}{5}$

Complex Fractions

Complex fractions are usually fractional expressions that involve a combination of two or more fractions, two or more operators and a bracket**.**

To simplify or evaluate complex fractions, apply ***BODMAS/BEDMAS*** principles which defines the order of carrying out the simplification or evaluation as:

Bracket → of / Exponent → Division → Multiplication → Addition → Subtraction

Before ***BODMAS*** is applied, change all mixed fractions to improper fractions and write the final answer as a mixed fraction if possible.

Worked Examples

1. With out using tables or calculators evaluate: $37\frac{1}{2} \div \frac{5}{9}\ of \left(\frac{4}{7} + \frac{1}{5}\right) - 80\frac{1}{3}$

Solution

$37\frac{1}{2} \div \frac{5}{9}\ of \left(\frac{4}{7} + \frac{1}{5}\right) - 80\frac{1}{3}$

$\frac{75}{2} \div \frac{5}{9}\ of \left(\frac{4}{7} + \frac{1}{5}\right) - \frac{241}{3}$ (*change mixed fractions*)

$\frac{75}{2} \div \frac{5}{9}\ of \left(\frac{20+7}{35}\right) - \frac{241}{3}$ (*Bracket first)*

$\frac{75}{2} \div \frac{5}{9}\ of\ \frac{27}{35} - \frac{241}{3}$

$\frac{75}{2} \div \frac{5}{9} \times \frac{27}{35} - \frac{241}{3} = \frac{75}{2} \div 1 \times \frac{3}{7} - \frac{241}{3}$ (*of*)

$= \frac{75}{2} \div \frac{3}{7} - \frac{241}{3}$

$\frac{75}{2} \times \frac{7}{3} - \frac{241}{3} = \frac{25}{2} \times \frac{7}{1} - \frac{241}{3} = \frac{175}{2} - \frac{241}{3}$ $(Division)$

$\frac{525-482}{6} = \frac{43}{6} = 7\frac{1}{6}$ *(Subtraction)*

2. $\frac{2}{3}$ of $6\frac{3}{4} \div (2\frac{4}{15} - 1\frac{2}{3})$

Solution

$\frac{2}{3}\ of \frac{27}{4} \div \left(\frac{34}{15} - \frac{5}{3}\right)$

$= \frac{2}{3}\ of\ \frac{27}{4} \div \frac{34-25}{15}$

$= \frac{2}{3}\, of\ \frac{27}{4} \div \frac{3}{5}$

$= \frac{2}{3} \times \frac{27}{4} \div \frac{3}{5}$

$= \frac{1}{1} \times \frac{9}{2} \div \frac{3}{5} = \frac{9}{2} \times \frac{5}{3} = \frac{3}{2} \times \frac{5}{1} = \frac{15}{2} = 7\frac{1}{2}$

3. Simplify $\frac{\frac{3}{4}\left(3\frac{3}{8} + 1\frac{5}{6}\right)}{2\frac{1}{8} - 1\frac{1}{2}}$, without using tables or calculators

Solution

$\frac{\frac{3}{4}\left(3\frac{3}{8} + 1\frac{5}{6}\right)}{2\frac{1}{8} - 1\frac{1}{2}} = \frac{\frac{3}{4}\left(\frac{27}{8} + \frac{11}{6}\right)}{\frac{17}{8} - \frac{3}{2}}$

Consider the numerator;

$\frac{3}{4}\left(\frac{27}{8} + \frac{11}{6}\right) = \frac{3}{4} \times \left(\frac{81+44}{24}\right) = \frac{3}{4} \times \frac{125}{24} = \frac{1}{4} \times \frac{125}{8} = \frac{125}{32}$

Consider the denorminator,

$\frac{17}{8} - \frac{3}{2} = \frac{17-12}{8} = \frac{5}{8}$

$\Rightarrow \frac{\frac{3}{4}\left(\frac{27}{8} + \frac{11}{6}\right)}{\frac{17}{8} - \frac{3}{2}} = \frac{125/32}{5/8} = \frac{125}{32} \div \frac{5}{8} = \frac{125}{32} \times \frac{8}{5} = \frac{25}{4} = 6\,\frac{1}{4}$

4. Evaluate $\frac{2\frac{7}{8} \times 1\frac{1}{5}}{8 - 2\frac{1}{4}}$, without using tables or calculators.

Solution

$\frac{2\frac{7}{8} \times 1\frac{1}{5}}{8 - 2\frac{1}{4}} = \frac{\frac{23}{8} \times \frac{6}{5}}{8 - \frac{9}{4}}$

Consider the numerator,

$\frac{23}{8} \times \frac{6}{5} = \frac{23}{4} \times \frac{3}{5} = \frac{69}{20}$

Consider the denorminator

$8 - \frac{9}{4} = \frac{32-9}{4} = \frac{23}{4}$

$\Rightarrow \frac{\frac{23}{8} \times \frac{6}{5}}{8 - \frac{9}{4}} = \frac{69/20}{23/4} = \frac{69}{20} \div \frac{23}{4} = \frac{69}{20} \times \frac{4}{23} = \frac{3}{5} \times \frac{1}{1} = \frac{3}{5}$

5. Simplify $\frac{\frac{3}{4} - \frac{7}{8} + \frac{1}{2}}{\frac{3}{4}\, of \left(\frac{7}{8} - \frac{1}{2}\right)}$, without calculators.

Solution

Consider the numerator:

$\frac{3}{4} - \frac{7}{8} + \frac{1}{2} = \frac{6 - 7 + 4}{8} = \frac{3}{8}$

Consider the denorminator:

$\frac{3}{4}\ of\ \left(\frac{7}{8} - \frac{1}{2}\right)$

$= \frac{3}{4}\ of\ \frac{7-4}{8} = \frac{3}{4}\ of\ \frac{3}{8} = \frac{3}{4} \times \frac{3}{8} = \frac{9}{32}$

$\Rightarrow \frac{\frac{3}{4} - \frac{7}{8} + \frac{1}{2}}{\frac{3}{4}\, of \left(\frac{7}{8} - \frac{1}{2}\right)}$

$= \frac{3/8}{9/32} = \frac{3}{8} \div \frac{9}{32} = \frac{3}{8} \times \frac{32}{9} = \frac{1}{1} \times \frac{4}{3} = \frac{4}{3} = 1\frac{1}{3}$

Exercises 2.15

Simplify the following.

1. $\left(\frac{2}{3} - \frac{1}{2}\right)\ of\ \left(\frac{1}{6} \div \frac{1}{2} \times \frac{2}{3} + \frac{3}{4}\right)$
2. $\left(2\frac{1}{4} - 1\frac{5}{8}\right) + \frac{4}{3}\ of\ \frac{2}{9}$
3. $\left(3\frac{1}{2} + 1\frac{1}{4}\right) \div \left(3\frac{1}{2} - 1\frac{1}{4}\right)$
4. $\left(4\frac{3}{4} - 1\frac{5}{6}\right) \div 1\frac{1}{24} \times \left(1\frac{2}{3} + 2\frac{1}{2}\right)$
5. $\frac{3\frac{4}{7} - 1\frac{1}{3} \div 2\frac{2}{5}}{\frac{13}{3} - 1\frac{6}{7}}$
6. $\frac{\frac{3}{4} - \frac{7}{8} + \frac{1}{2}}{\frac{3}{4}\, of \left(\frac{7}{8} - \frac{1}{2}\right)}$

Spending in Terms of Fraction

(Application of Addition and Subtraction of Fractions)

When part(s) of a whole is used, the parts put together are equal to the whole. In other words, since fraction is part of a whole, fractions of the same whole sum up to one. For example, $\frac{1}{6} + \frac{2}{6} + \frac{3}{6} = 1$.

Thus, ***Fraction spent* + *Fraction left* = 1**….(1)

From equation (1), it can be deduced that;

1. Fraction spent* = 1 – *Fraction left

2. Fraction left* = 1− *Fraction spent

In general, if the fraction spent out of a given whole is $\frac{a}{b}$, then the fraction left or remaining fraction = $\mathbf{1}-\frac{a}{b}=\frac{b}{b}-\frac{a}{b}=\frac{b-a}{b}$

Worked Examples

1. A boy spends $\frac{2}{7}$ of his pocket money on transport and $\frac{1}{5}$ of the money on sweets.
i. What fraction of his pocket money did he spend on transport and sweets?
ii. Calculate the fraction of his money left

Solution

i. Fraction spent on transport and sweets
$=\frac{2}{7}+\frac{1}{5}=\frac{10+7}{35}=\frac{17}{35}$

ii. Fraction left = 1 – Fraction spent
$=\mathbf{1}-\frac{17}{35}=\frac{35}{35}-\frac{17}{35}=\frac{35-17}{35}=\frac{18}{35}$

2. A student spent $\frac{17}{35}$ of his pocket money on transport and fruits. He spent $\frac{5}{6}$ of the remainder on sweets. What fraction of his pocket money is left?

Solution

Fraction Spent + Fraction Left = 1

Therefore, *fraction left = 1− fraction spent*

$=1-\frac{17}{35}=\frac{35}{35}-\frac{17}{35}=\frac{35-17}{35}=\frac{18}{35}$

$\frac{5}{6}$ of the remainder $=\frac{5}{6}\times\frac{18}{35}=\frac{3}{7}$

Fraction left $=\frac{18}{35}-\frac{3}{7}=\frac{18-15}{35}=\frac{3}{35}$

3. Kwaku travelled from Accra to Kumasi. He travelled $\frac{3}{10}$ of the journey by Lorry, $\frac{2}{5}$ of the journey by taxi and the rest by train. What fraction of the journey did he travel by train?

Solution

Fraction travelled by lorry and taxi
$=\frac{3}{10}+\frac{2}{5}=\frac{3+4}{10}=\frac{7}{10}$

Fraction travelled by train,
$=1-\frac{7}{10}=\frac{10}{10}-\frac{7}{10}=\frac{10-7}{10}=\frac{3}{10}$

4. Amina spent $\frac{17}{21}$ of her pocket money on transport and food. If she spend $\frac{2}{7}$ on transport only, what fraction did he spend on food?

Solution

Let the fraction spent on food = x and fraction spent on transport only = $\frac{2}{7}$

Fraction spent on transport and Food = $\frac{17}{21}$

$\frac{2}{7}+x=\frac{17}{21}$

$x=\frac{17}{21}-\frac{2}{7}=\frac{17-6}{21}=\frac{11}{21}$

Therefore, the fraction spent on food = $\frac{11}{21}$

5. A tank can hold 240 liters of water. How much water is in the tank when it is $\frac{4}{5}$ full?

Solution

If $\frac{5}{5}$ or 1 = 240 liters,

$\frac{4}{5}=\frac{4}{5}\times 240$ liters

= 192 liters

Therefore, 192 liters of water will be in the tank when it is $\frac{4}{5}$ full.

Exercises 2.16

A. 1. Keziah shared Gh¢2,700.00 between his two children. The older child received $\frac{4}{9}$ of the amount. How much did the younger child receive?

2. Tobia did $\frac{2}{10}$ of his homework in the first 30 minutes and $\frac{1}{10}$ of the homework in the next 30 minutes. What fraction of the homework was left for him to do?

3. Lily fetched a bucket of water and used $\frac{2}{11}$ of the water for scrubbing the bath house, $\frac{4}{11}$ of the water for bathing and the rest for drinking. What fraction of the water did Lily leave for drinking?

4. Mr. Green has a piece of farm plot. He decided to use $\frac{4}{7}$ of the land to cultivate cassava, $\frac{1}{6}$ for pineapple production and the rest for fish pond. Find the fraction of the land he reserved for fish production.

Finding the Total Quantity given the Fraction(s) Spent and the Remaining Quantity

To find the total quantity given the fraction spent and the remaining quantity, make use of any of the following methods;

<u>Method I</u>

I. Calculate the remaining fraction.

II. Make use of the fact that the remaining fraction is equal to the remaining quantity.

i.e.* *Fraction left = Quantity left or

Remaining fraction = Remaining quantity

III**.** By simple proportion, find the value of the whole (total quantity) which is 1. That is:

If ***fraction left = quantity left,***

$$\Rightarrow \mathbf{1} = \frac{quantity\ left}{fraction\ left} = \frac{remaining\ \ quantity}{remaining\ \ fraction}$$

<u>Method 2</u>

I. Represent the total quantity by a variable, say x

II. Write an equation of the sum of fractions of the total quantity and the remaining quantity and equate it to the total quantity, x. That is,

Total quantity = fraction/sum of fraction(s)of the total quantity + remaining quantity

For instance, if the fractions of the total quantity spent are $\frac{a}{b}$, $\frac{c}{d}$ and the remaining quantity is m , then the total quantity,

$$\boldsymbol{x = \frac{a}{b}x + \frac{c}{d}x + m}$$

Solve for x to get the total quantity.

Worked Examples

1. Mr. Brown withdrew some money from the bank. He gave $\frac{1}{2}$ of it to his son and $\frac{1}{3}$ to his daughter. If he had Gh¢500.00 left, how much did he take from the bank?

Solution

<u>Method 1</u>

Fraction Spent + *Fraction Left* = 1

But fraction spent = $\frac{1}{2} + \frac{1}{3} = \frac{3+2}{6} = \frac{5}{6}$

⇒*Fraction left* = 1− *Fraction spent*

$= 1 - \frac{5}{6} = \frac{6}{6} - \frac{5}{6} = \frac{6-5}{6} = \frac{1}{6}$

But remaining fraction = Remaining amount.

⇒ if $\frac{1}{6}$ = Gh¢500.00

⇒1 = 6 × Gh¢500 = Gh¢3,000.00

The amount withdrawn was Gh¢3,000.00

<u>Method 2</u>

Let the amount withdrawn be x

$\frac{1}{2}x + \frac{1}{3}x + 500 = x$

Multiply through by L.C.M = 6

$6 \times \frac{1}{2}x + 6 \times \frac{1}{3}x + 6 \times 500 = 6x$

⇒$3x + 2x + 3000 = 6x$

$3000 = 6x - 3x - 2x$

$3000 = x$ *or* $x = 3000$

The amount withdrawn was Gh¢3,000.00

2. Mr. Brown spent $\frac{1}{4}$ of his monthly salary on

rent, $\frac{2}{5}$ on food and $\frac{1}{6}$ on books. If he still had Gh¢55.00 left, what was his monthly salary?

Solution

Let the monthly salary of the man be x

$\frac{1}{4}x + \frac{2}{5}x + \frac{1}{6}x + 55 = x$

Multiply through by L.C.M = 60

$60 \times \frac{1}{4}x + 60 \times \frac{2}{5}x + 60 \times \frac{1}{6}x + 60 \times 55 = 60x$

$15x + 12(2)\,x + 10x + 3300 = 60x$

$3300 = 60\,x - 15x - 24x - 10x$

$3300 = 11x$

$x = 300$

The monthly salary was Gh¢300.00

3. After reading $\frac{1}{5}$and $\frac{2}{3}$ of a story book on the first and second days respectively, Tamar had 10 pages of her story bookunread. Find the total number of pages of her story book.

Solution

Let the total pages of the story book be x.

$\frac{1}{5}x + \frac{2}{3}x + 10 = x$

$15 \times \frac{1}{5}x + 15 \times \frac{2}{3}x + 15 \times 10 = 15x$

$3x + 5(2)\,x + 150 = 15x$

$150 = 15x - 3x - 10x$

$150 = 2x$

$x = 75$ pages

4. A barrel is $\frac{4}{5}$ full of water and 190 liters ofwater is drawn from it leaving it $\frac{1}{6}$ full.Find the capacity of the barrel.

Solution

Let x be the fraction of water drawn

$\frac{4}{5} - x = \frac{1}{6}$

$x = \frac{4}{5} - \frac{1}{6} = \frac{24-5}{30} = \frac{19}{30}$

Fraction of water drawn = $\frac{19}{30}$

Amount of water drawn = 190 liters

But fraction of water drawn is equal to amount of water drawn.

$\Rightarrow \frac{19}{30}$ = 190 liters

Capacity of barrel = $\frac{4}{5} + \frac{1}{5} = 1$

If $\frac{19}{30}$ = 190 liters,

$\Rightarrow \frac{4}{5} = \frac{190 \times {}^{4}/_{5}}{{}^{19}/_{30}}$

$= \frac{{}^{760}/_{5}}{{}^{19}/_{30}} = \frac{760}{5} \div \frac{19}{30} = \frac{760}{5} \times \frac{30}{19}$ = 240 liters

If $\frac{4}{5}$ = 240 liters,

$\Rightarrow \frac{1}{5} = \frac{240 \times {}^{1}/_{5}}{{}^{4}/_{5}}$

$= \frac{{}^{240}/_{5}}{{}^{4}/_{5}} = \frac{240}{5} \div \frac{4}{5} = \frac{240}{5} \times \frac{5}{4}$ = 60 liters

Capacity of barrel = $\frac{4}{5} + \frac{1}{5}$

= $240l + 60l = 300$*liters*

Exercises 2.17

1. Mr. White purchased some bottles of mineral water for his son. After a week, he used $\frac{4}{7}$ of the bottles leaving 6 bottles.

i. Find the fraction of the bottles of mineral water that remain after a week.

ii. How many bottles of mineral water did he purchased that week?

2. Jerry read $\frac{3}{9}$ of a story book which contained 24 pages.

i. What fraction of the story book was not read?

ii. How many pages of the story book were not read?

3. Adwoa came with three different vegetables from the market. Out of those items, $\frac{7}{16}$ were tomatoes, $\frac{3}{8}$ were eggs and the remaining, pepper.
i. What fraction of the items was pepper?
ii. If she went to the market with Gh¢4,800.00, calculate how much money she spent on each item.

4. Lazarus spent $\frac{1}{3}$ of his pocket money on food, $\frac{2}{7}$ on transportation and the rest on books,.
i. What fraction of the money did he spent on books? Ans = $\frac{11}{21}$
ii. If he had Gh¢66.00 left after spending on food and books, how much money did Lazarus have in his pocket? Ans: 231

5. Araba used $\frac{1}{4}$ of his money to purchase a school bag and $\frac{1}{3}$ of the money to purchase a school uniform. Find her total money if she has Gh¢25.00 left after her purchases.

Decimal Fractions

A decimal is a fraction with power of ten as denominator. For example, $\frac{7}{10}$, $\frac{27}{100}$, $\frac{23}{100}$...
Study the pattern below carefully,

$\frac{7}{10} = \frac{7}{10^1} = 0.7$ (1d.p),
$\frac{27}{100} = \frac{27}{10^2} = 0.27$ (2d.p),
$\frac{23}{1000} = \frac{23}{10^3} = 0.023$ (3d.p)

It is observed that the number of decimal places depend on the power of ten of the denominator.
The decimal point separates the whole number part on its right from the fraction (or less than a whole part) on its left. For instance, in 8.75, the digit 8 on the left of the decimal point is a whole number whilst 75 on the right of the decimal point is less than one.

Fractions as Decimal Fractions

To write fractions with power of ten as decimals, go through the following steps;
I. Identify the power of ten within the fraction
II. Recognize the numerator as a whole number.
III. Move the decimal point at the right of the numerator to the left according to the power of 10 to obtain the decimal fraction.

Worked Examples

1. Write $\frac{8}{100}$ as a decimal.

Solution

$\frac{8}{100} = \frac{8}{10^2}$.
The power of 10 $= 2$
Therefore $\frac{8}{100} = 0.08$

2. What is the decimal for $\frac{25}{100}$?

Solution

$\frac{25}{100} = \frac{25}{10^2} = 0.25$

Exercises 2.18

Without using Calculculators, express each as a decimal.

1. $\frac{11}{1000}$ 2. $\frac{60}{1000}$ 3. $\frac{30}{10000}$ 4. $\frac{9}{1000}$

Decimal Places

The number of digits at the right of a decimal point in a numeral is called the ***decimal place*** of that numeral. For example, 0.25 has two decimal places because, there are two digits after the decimal point.

Worked Examples

Identify the number of decimal places
1) 5.3154 2) 5 3) $\frac{25}{100}$

Solution

1. 5. 3154 (4d.p) 2. 5 (0 d.p) 3. $\frac{25}{100}$ = 0.25(2d.p)

Exercises 2.19

Identify the number of decimal place(s)

1. $\frac{88}{10}$ 2. $\frac{5}{1000}$ 3. $\frac{55}{1000}$ 4. $\frac{2.2}{10}$ 5. $\frac{6.5}{100}$

Common Fractions as Decimal Fractions

To express a fraction as a decimal, multiply the numerator and the denominator by a common natural number such that the denominator becomes a power of 10. The power of 10 indicates the number of decimal places the numerator should be expressed.

Worked Examples

Convert the following to a decimal fraction;

1. $\frac{2}{25}$ 2. $\frac{5}{20}$

Solutions

1. $\frac{2}{25} = \frac{2 \times 4}{25 \times 4} = \frac{8}{100} = \frac{8}{10^2}$ = 0.08

2. $\frac{5}{20} = \frac{5 \times 5}{20 \times 5} = \frac{25}{100} = \frac{25}{10^2}$ = 0.25

Exercises 2.20

Express as decimals

1. $\frac{11}{50}$ 2. $\frac{16}{250}$ 3. $\frac{25}{500}$ 4. $\frac{22}{200}$

Decimal Fractions as Common Fractions

To convert a decimal fraction to a common fraction;

I. identify the number of decimal place(s) in the decimal fraction.

II. Remove the decimal point to make it a whole number.

III. Divide the whole number by ten exponents the number of decimal places.

Worked Examples

1. Express 0.625 as a fraction in its lowest term.

Solution

0.625 (3 d .p) = $\frac{625}{10^3} = \frac{625}{1000} = \frac{5}{8}$

2. Write 0.011 as a fraction

Solution

0.011 (3 d.p) = $\frac{11}{10^3} = \frac{11}{1000}$

3. Write 0.55 as a fraction in its lowest term.

Solution

0.55 (2 d.p) = $\frac{55}{10^2} = \frac{55}{100} = \frac{11}{20}$

Exercises 2.21

Write the following as common fractions

1) 0. 14 2) 0.025 3) 0.01 4) 0.035

5) 2.4 6) 0.11 7) 0.002 8) 1.15

Ordering Decimal Fractions

It is the act of arranging decimals in either ascending or descending order. The steps are as follows:

I. Express the given decimals as a fraction with a common denominator.

II. By comparison, identify the fraction with the highest numerator, followed by the next, in that order.

III. Arrange the given decimals in either ascending or descending order.

Worked Examples

1. Arrange 0.63, 0.59, 0.6, 0.5 in ascending order.

Solution

Highest decimal place = 2

Express each decimal as a fraction with a denorminator of "10 exponent 2"

0.63 (2 d.p) $= \frac{63}{100}$,   0.59 (2 d.p) = $\frac{59}{100}$

0.6 (1d.p) $= \frac{60}{100}$,   0.5 (1d.p) = $\frac{50}{100}$

$\frac{50}{100} < \frac{59}{100} < \frac{60}{100} < \frac{63}{100}$ *(Comparing numerators)*

In ascending order : 0.5, 0.59, 0.6, 0.63

2. Write the decimals fractions in descending order: 0.167, 0.25, 0.5, and 0.33

Solution

Highest d.p. = 3

⇒Express each decimal as a fraction with a denorminator of "10 exponent 3".

0.167 = $\frac{167}{1000}$,   0.25 = $\frac{250}{1000}$

0.5 = $\frac{500}{1000}$,   0.33 = $\frac{330}{1000}$

$\frac{500}{1000} > \frac{330}{1000} > \frac{250}{1000} > \frac{167}{1000}$

In descending order; 0.5, 0.33, 0.25, 0.167

Exercises 2.22

A. Write in ascending order;

1) 0.51, 0.1, 0.02, 0.05   2) 2.5, 0.5, 0. 25

3) 0.17, 1.7, 0.0017, 10   4) 0.33, 0.81, 0.4, 4.4

B. Write in descending order;

1. 3.5, 0.63, 3.3, 0.33
2. 0.69, 0.52, 2.8, 6.3
3. 2.19, 3.47, 0.02, 2.83

Addition and Subtraction of Decimals

To add and subtract decimal fractions, align or arrange the digits in a vertical column or on the place value chart, such that the decimal points are in line with each other.

Worked Examples

1. Add 3.50 + 2.83

Solution

$$\begin{array}{r} 3.50 \\ +\ 2.83 \\ \hline 6.33 \\ \hline \end{array}$$

2. Find 2.83 + 15.69

Solution

$$\begin{array}{r} 2.83 \\ +\ 15.69 \\ \hline 18.52 \\ \hline \end{array}$$

Multiplication of Decimals

To multiply two or more decimals, follow the steps below;

I. Find the sum of decimal places

II. Remove the decimal point and multiply the whole numbers.

III. After getting the product, move the decimal point to the left with respect to the sum of the decimal places to get the product of the decimals.

Worked Examples

1. Multiply 3.25 by 0.5

Solution

Method 1

3.25 × 0.5

Sum of d.p = 2 + 1 = 3

325 × 5 = 1625

Express 1625 to 3 d.p = 1. 625

∴ 3.25 × 0 .5 = 1.625 (3 d. p)

Method 2

3.25 × 0.5

$= \frac{325}{100} \times \frac{5}{10} = \frac{1625}{1000}$ = 1.625

2. Find 6.801×5.13

Solution

6.801×5.13

Sum of d. p = 3 + 2 = 5

$6801 \times 513 = 3488913$

$\therefore 6.801 \times 5.13 = 34.88913$

Exercises 2.24

Perform the following:

1. $0.7 \times 4.5 \times 1.01$ 2. $0.4 \times 0.12 \times 0.6$

Division of Decimals

To divide two decimals;

I. Change all the decimals to fractions.

II. Then change the division sign to multiplication.

III. Find the product of the first fraction and the reciprocal of the second fraction.

Worked Examples

1. Divide 0.85 by 0.25

Solution

$0.85 \div 0.25 = \frac{85}{100} \div \frac{25}{100} = \frac{85}{100} \times \frac{100}{25} = 3.4$

2. Perform $0.64 \div 0.2$

Solution

$\frac{64}{100} \div \frac{2}{10} = \frac{64}{100} \times \frac{10}{2} = 3.2$

Standard Forms

It is the process of writing a number as aproduct of a decimal and a power of ten. That is $a.b \times 10^n$ where $1 \leq a < 10$ and n is given by the number of places the point is displaced from the standard position. For example, 124 can be written as; 1.24×10^2 and this is called ***the standard form*** of 124.

A number written in standard form must have the following features:

1. a decimal fraction,
2. a multiplication sign,
3. a positive or negative power of ten.

Rules

I. Identify the position or location of the decimal point

II. Relocate, or move or place the decimal point between the first two natural numbers(standard position).

III. Each movement of the decimal point to left is 10 and each movement of the decimal point to the right is 10^{-1}

IV. After completion, ignore zeros before the first counting number and after the last counting number.

Worked Examples

1. Write 38200 in standard form.

Solution

The standard form of 38200 = 3.82×10^4

2. Write 0.00053 in standard form.

Solution

Place the decimal point between 5 and 3 by moving 3 places to the right;

$0.00053 = 5.3 \times 10^{-1} \times 10^{-1} \times 10^{-1} \times 10^{-1}$

$= 5.3 \times 10^{-4}$

3. Write 84.36 in standard form.

Solution

$84.36 = 8.436 \times 10$

4. Write 61.38×10^2 in standard form.

Solution

61. 38 = $6.138 \times 10 \times 10^2 = 6.138 \times 10^3$

Exercises 2.26

A. Write the following in standard form

1) 0.0014 2) 0.00003975 3) 349.3
4) 98000 5) 342.5×10^2 6) 4.3×10^{-3}

B. Express the square root of each of the following in the form: $a \times 10^n$

1. 0.0009 2. 9×10^4
3. 0.36×10^{-8} 4. 1.6×10^{-3}

Numerals for Standard Forms

When writing numerals for standard forms, consider the following:

I. If the power of ten is positive, move the decimal point to the right according to the value of the power of ten.

II. If the power of ten is negative, move the decimal point to the left according to the value of the power of ten.

III. After the movement(s), write all the numerals but occupy all empty space (s) with zero (s) to get the numeral for the standard form.

Worked Examples

Write the numerals for the standard forms:

1) 9.06×10^3 2) 1.25×10^{-4} 3) $14.4 \times 10^{-2} \times 10^5$

Solutions

1. In 9.06×10^3,

i. Power of ten = 3,

ii. Move decimal point, 3 times to the right

iii. Fill empty space with zero.

$\Rightarrow 9.06 \times 10^3 = 9060$

2. In 1.25×10^{-4},

i. Power of ten = −4,

ii. Move decimal point 4 times to the left

iii. Fill empty spaces with zeros.

$\Rightarrow 1.25 \times 10^{-4} = 0.000125$

Exercises 2.27

Write the numerals for the following:

1.342.5 × 10^2 2. 117.3×10^{-3}
3. 5.3×10^{-4} 4. 71.803×10^5
5. $402 \times 10^{-5} \times 10^8$ 6. $9 \times 10^{-2} \times 10^4$

Addition and Subtraction of Standard Forms

To add or subtract standard form numbers:

I. Rewrite each standard form number as an ordinary number

II. Add or subtract these ordinary numbers

III. Convert the results to standard form

Work Examples

1. Calculate $3.2 \times 10^4 + 9.7 \times 10^3$

Solution

$3.2 \times 10^4 = 3.2 \times 10,000 = 32,000$

$9.7 \times 10^3 = 9.7 \times 1,000 = 9,700$

$32,000 + 9,700 = 41,700$

$41,700 = 4.17 \times 10^4$

$\therefore 3.2 \times 10^4 + 9.7 \times 10^3 = 4.17 \times 10^4$

2. Add 4.25×10^{-3} and 8.3×10^{-4}

Solution

$4.25 \times 10^{-3} = 0.00425$

$8.3 \times 10^{-4} = 0.00083$

$0.00425 + 0.00083 = 0.00508$

$\Rightarrow 0.00508 = 5.08 \times 10^{-3}$

$\therefore 4.25 \times 10^{-3} + 8.3 \times 10^{-4} = 5.08 \times 10^{-3}$

3. Calculate $5.5 \times 10^5 - 6.95 \times 10^4$

Solution

$5.5 \times 10^5 = 5.5 \times 100,000 = 550,000$

$6.95 \times 10^4 = 6.95 \times 10{,}000 = 69{,}500$
$550{,}000 - 69{,}500 = 480{,}500$
$480{,}500 = 4.805 \times 10^5$
$\therefore 5.5 \times 10^5 - 6.95 \times 10^4 = 4.805 \times 10^5$

Exercises 2.28

A. Perform the following:

1.5 × 10^5 + 3 × 10^4

2. $4.5 \times 10^{-3} + 3.5 \times 10^{-2}$
3. $7.5 \times 10^5 + 2.5 \times 10^3$
4. $7.5 \times 10^{-3} + 2.5 \times 10^{-3}$

B. Perform the following;

1. $8 \times 10^4 - 6.95 \times 10^3$
2. $5.4 \times 10^3 - 9.95 \times 10^2$
3. $1 \times 10^{-3} - 1 \times 10^{-5}$
4. $7.5 \times 10^{-3} - 2.5 \times 10^{-3}$

Multiplying Standard Form Numbers

Numbers in standard forms are multiplied by finding the product of the whole numbers (mantissas) and adding the exponents. That is :
$a^m \times a^n = a^{m+n}$

Worked Examples

1. Calculate $2 \times 10^2 \times 3 \times 10^4$

Solution

$2 \times 10^2 \times 3 \times 10^4$
$= (2 \times 3) \times (10^2 \times 10^4)$
$= 6 \times 10^{2+4}$
$= 6 \times 10^6$

2. Multiply 3×10^5 by 2.5×10^{-2}

Solution

$3 \times 10^5 \times 2.5 \times 10^{-2}$
$(3 \times 2.5) \times (10^5 \times 10^{-2})$
$= 7.5 \times 10^{5-2}$
$= 7.5 \times 10^3$

Exercises 2.29

Perform the following:

1. $2 \times 10^6 \times 6 \times 10^{-2}$
2. $6 \times 10^{-2} \times 3 \times 10^2$
3. $6.3 \times 10^4 \times 4 \times 10^{-2}$
4. $1.2 \times 10^5 \times 5.2 \times 10^4$

Application of Standard Forms

Sometimes, students are required to perform an operation and express, write or leave the answer in standard form. Students must therefore, make use of the knowledge of writing numbers in standard form, as well as writing the numerals for standard forms.

Worked Examples

1. Simplify $\frac{0.6 \times 32 \times 0.004}{1.2 \times 0.008 \times 0.16}$ and leave your answer in standard form

Solution

$\frac{0.6 \times 32 \times 0.004}{1.2 \times 0.008 \times 0.16}$

$= \frac{6 \times 10^{-1} \times 32 \times 4 \times 10^{-3}}{12 \times 10^{-1} \times 8 \times 10^{-3} \times 16 \times 10^{-2}}$

$= \frac{6 \times 32 \times 4 \times 10^{-1} \times 10^{-3}}{12 \times 8 \times 16 \times 10^{-1} \times 10^{-3} \times 10^{-2}}$

$= \frac{1 \times 10^{-4}}{2 \times 10^{-6}}$

$= \frac{1}{2} \times 10^{-4+6} = 0.5 \times 10^2 = 5 \times 10^1$

2. Evaluate $\frac{(0.00042 \times 10^{-6})(15000)}{(5000 \times 10^2)\,(0.0021 \times 10^8)}$, without using calculation.

Solution

$= \frac{(0.00042 \times 10^{-6})(15000)}{(5000 \times 10^2)\,(0.0021 \times 10^8)}$

$= \frac{(42 \times 10^{-5} \times 10^{-6})(15000)}{(5000 \times 10^2)\,(21 \times 10^{-4} \times 10^8)}$

$= \frac{2 \times 3 \times 10^{-5} \times 10^{-6}}{10^{2-4+8}}$

$= \frac{6 \times 10^{-11}}{10^{2-4+8}}$

$= \frac{6 \times 10^{-11}}{10^{6}} = 6 \times 10^{-11-6} = 6 \times 10^{-17}$

3. Without using calculators, evaluate $\sqrt{\frac{0.0048 \times 0.81 \times 10^{-7}}{0.027 \times 0.04 \times 10^{6}}}$, leaving your answer in standard form.

Solution

$\sqrt{\frac{0.0048 \times 0.81 \times 10^{-7}}{0.027 \times 0.04 \times 10^{6}}}$

$= \left(\frac{48 \times 10^{-4} \times 81 \times 10^{-2} \times 10^{-7}}{27 \times 10^{-3} \times 4 \times 10^{-2} \times 10^{6}}\right)^{\frac{1}{2}}$

$= \left(\frac{12 \times 10^{-4} \times 3 \times 10^{-2} \times 10^{-7}}{10^{-3} \times 10^{-2} \times 10^{6}}\right)^{\frac{1}{2}}$

$= \left(\frac{36 \times 10^{-6} \times 10^{-7}}{10^{-5} \times 10^{6}}\right)^{\frac{1}{2}}$

$= \left(\frac{36 \times 10^{-13}}{10^{1}}\right)^{\frac{1}{2}}$

$= (36 \times 10^{-13-1})^{\frac{1}{2}}$

$= (36 \times 10^{-14})^{\frac{1}{2}}$

$= (36)^{\frac{1}{2}} \times (10^{-14})^{\frac{1}{2}}$

$= (6^2)^{\frac{1}{2}} \times (10^{-14})^{\frac{1}{2}} = 6 \times 10^{-7}$

4. Given that $a = 5.0 \times 10^2$, $a = 12.0 \times 10^2$ and $c = 100$, evaluate without using tables or calculators $\sqrt{\frac{a^2 + b^2}{c}}$ and leave the answer in standard form.

Solution

$\sqrt{\frac{a^2 + b^2}{c}}$

$= \sqrt{\frac{(5 \times 10^2)^2 + (12 \times 10^2)^2}{100}}$

$= \sqrt{\frac{5^2 \times 10^4 + 12^2 \times 10^4}{100}}$

$= \sqrt{\frac{25 \times 10^4 + 144 \times 10^4}{100}}$

$= \left(\frac{(25 + 144) \times 10^4}{100}\right)^{\frac{1}{2}}$

$= \left(\frac{169 \times 10^4}{10^2}\right)^{\frac{1}{2}}$

$= (169 \times 10^{4-2})^{\frac{1}{2}}$

$= (13^2)^{\frac{1}{2}} \times (10^2)^{\frac{1}{2}} = 13 \times 10 = 1.3 \times 10^2$

Exercises 2.30

A**. Without using calculators simplify and leave the answer in standard form:**

1. $\frac{7.2 \times 10^5 \times 2.7 \times 10^8}{0.009 \times 10^{10} \times 0.0012 \times 10^4}$
2. $\frac{0.048 \times 10^7 \times 0.00056 \times 10^7}{0.016 \times 10^{-2} \times 0.00014 \times 10^9}$
3. $\frac{(0.002 \times 10^{-2})^2 \times 10^7 \times (0.03 \times 10^2)^2 \times 10^{-4}}{0.0012}$

C. Simplify without using calculators and leave your answer in standard form:

1. $\sqrt{\frac{(0.04)^2 \times 10^4 \times 6300 \times 10^6}{0.7 \times 10^{-1} \times 0.09 \times 10^8}}$
2. $\sqrt{\frac{120 \times 10^{-5} \times (1.4)^2 \times 10^{9}}{1.5 \times 10^{-6} \times 0.04 \times 0.2 \times 10^8}}$
3. $\sqrt[3]{\frac{0.0216 \times 10^{-3} \times 0.024 \times 10^{-4} \times 0.018 \times 10^{-5}}{36000 \times 10^{-6} \times 0.96 \times 10^{-3}}}$
4. $\sqrt[3]{\frac{0.0018 \times 10^{-6} \times 0.0072 \times 10^{-1} \times 0.002}{400 \times 10^{-7} \times 0.03 \times 10^{-2}}}$

D. 1. Given that $m = 13.0 \times 10^2$, $n = 5.0 \times 10^2$ and $r = 10$, evaluate without using tables or calculators $\sqrt{\frac{m^2 - n^2}{r^2}}$, leaving your answer in standard form.

2. Given that $a = 4.0 \times 10^{-2}$, $b = 4.0 \times 10^{-2}$ and $c = 1.0 \times 10^{-4}$, evaluate without using tables or calculators $\frac{a^3 + b^3}{c^2}$, leaving your answer in standard form.

Approximation

Measures of length, mass, time, area, population, moneyetc, should always be given to a reasonable degree of approximation, especially when they cannot be stated with exactness or precision.

There are three main ways by which approximations may be done. By rounding off to:
1. a nearest appropriate unit;
2. a given number of decimal places;
3. a given number of significant figures.

Rounding to the Nearest Multiples of Ten

In a given numeral, the place values are identified from the left of a decimal point as; ***Ones (Nearest whole number), Tens, Hundreds, Thousands, Ten thousand, Hundred thousand, One million etc***. To the right of the decimal point, the place values are identified as; ***Tenth, Hundredth, Thousandth, Ten thousandth etc***

Rules for Rounding – off a Number

Rounding a number to a given place, that is: nearest ten or multiples of ten depends on the following figure on the right (the next immediate digit to the right).
1. If the next figure is greater than 5, increase the round - off figure by 1.
2. If the next figure is 5, round off to the nearest even number .
3. If the next figure is less than 5, leave the round – off figure as it is.

In each case, the digits after the round off figure are replaced by zeros.

Worked Examples

Round – off 1,908,653.727 to the nearest;
i. whole number ii. Tens iii. hundreds
iv. Thousands v. ten thousands
vi. hundred thousands vii. one million
viii. hundredth ix. tenth

Solution

i. To the nearest whole number = 1,908,654
ii. Nearest ten = 1,908,600
iii. Nearest hundred = 1,908,700
iv. Nearest thousand = 1,909,000
v. Ten thousand = 1,910,000
vi. Hundred thousand = 1,900,000
vii. One million = 2,000,000
viii. Hundredth = 1,908,653.730
ix. Tenth = 1,908,653.700

Exercises 2.31

1. **Round 51,624,362 to the nearest:**
i. tens ii. Thousands iii. hundred thousands
iv. Hundreds v. ten thousands vi. one million

2. Round 342.0532 to the nearest;
a. Tenth b. Whole number
c. Hundredth d. Ten thousandth

Writing to a Given Number of Decimal Places

I. Identify the number of decimal places in the given decimal fraction.
II. Identify the retain digit(s) that represents the decimal places to which you are approximating
III. Consider the next immediate digit on the right of the retain digit(s);
a. If that digit is less than 5(0 to 4), maintain the retain digit(s) as the answer.
b. If that digit is more than 4 (5 to 9), increase the last retain digit(s) by 1 and nothing more.

Worked Examples

Round off 5.20735 to:
i. 4 decimal places ii. 3 decimal places
iii. 2 decimal places iv. 1 decimal place

Solution

i. 5.20735 = 5.2074 to 4 d. p.

ii. 5.20735 = 5.20735 to 3 d. p.
iii. 5.20735 = 5.21 to 2 d. p.

2. Find 4.13 × 1.15 and correct your answer to 2 decimal places.

Solution
4.13 × 1.15
Sum of d .p = 2 + 2 = 4
413 ×115 = 47495
⇒4.13 × 1.15 = 4.7495 (4 d. p) = 4.75 (2 d. p)

Exercises 2.32
Correct to the decimal places indicated:

1. 0.05431 (2d.p)	2. 20.556 (2d.p)
3. 432.97 (1d.p)	4. 40.4563 (2d.p)
5. 5.9730 (3d.p)	6. 7.9994 (2d.p)

Significant Figures
Another convenient way to indicate the degree of approximation is by means of the number of figures used. Thus, we say 21.3 has 3 significant figures and that 54 has 2 significant figures.
All digits are significant but in the case of zero, it is insignificant when it used to indicate the position of a decimal point. In other words, zero(s) immediately after the decimal point is not significant.

Finding the Number of Significant Figures
To identify the number of significant figures in a given numeral observe the following guidelines:
Steps:
1. Count from left to right, the first counting number, ignoring the zeros to locate the significant figures. For e.g. 0.0045672 and 63051 have 5 significant figures as shown below;

a.
0. 0 0 4 5 6 7 2
↓↓↓↓↓
1 2 3 4 5 (5 S.F)

b.
6 3 0 5 1
↓↓↓↓↓
1 2 3 4 5 (5 S.F)

2. All non-zero digits are significant. For example, 261542 has 6 significant figures.

3. All zeros between non-zero digits are significant. For e.g. 0.4002 has 4 significant figures.

4. Leading zeros to the left of the first non-zero digits are not significant; such zeros merely indicate the position of the decimal point: e.g. 0.013 has 2 significant figures and 0.00407 has 3 significant figures

5. Trailing zeroes that are also to the right of a decimal point in a number are significant:
0.0230 has 3 significant figures,
0.4200 has 4 significant figures.

6. When a number ends in zeroes that are not to the right of a decimal point, the zeros are ***not necessarily*** significant: e.g. 190 may be 2 or 3 significant figures, 50600 may be 3, 4, or 5 significant figures.

Writing a Number to a Given Number of Significant Figures
Observe of the following rules when correcting a number to a given number of significant figures:
I. Identify the retain digits representing the given number of significant figures to be converted to. For e.g. to convert 6305 (4 s.f) to 3 s.f, the retain digits are 630

II. Consider the digit to the immediate right of the last retain digit(s):
a. If it is 5 or more, (i.e. 5, 6, 7, 8, 9), increase the last retain digit by 1.

b. If it is less than 5 (that is 0, 1, 2, 3, 4), maintain the last retain digit. In other words, the last retain digit remains the same.

III. All numbers after the retain digits are replaced by zero(s) to maintain the value of the given number, where necessary. This rule is applicable to only decimal whole numbers like 45271. For e.g: n45271 (5 s.f) = 45300(3.s.f)

Worked Examples

1. Correct 0.00479 to 2 significant figures.

Solution

Zeros before non-zero digit(s) are not significant.
Significant figures = 47
Retain digits = 0.0047.
The next immediate digit = 9 (greater than 4) Last retain digit + 1 = (7 + 1) = 8.
Therefore, 0.00479 (3 s.f) = 0.0048(2 s.f)

2. Correct 62049 to 3 significant figures.

Solution

Retain digits = 620
The next immediate digit = 4 (less than 5)
Last retain + zero = (0 + 0) = 0
Replaced the rest of the digits after the last retain digit by zeros.
Therefore, 62049 (5 s.f) = 62000 (3 s.f)

Note : *The 3 significant figures in the above example (62000, 3 s.f.) are 620. This zero is significant because it occupies the position as last significant figure (3^{rd} digit of 3.s.f)*

3. Correct 2719 to two significant figures.

Solution

Retain digits = 27
The next immediate digit = 1, (less than 5)
Last retain digit + zero = 7 + 0 = 7
Replaced the rest of the digits with zeros.
∴ 2719 = 2700(2 s.f)

4. Correct 143.6943 to 4 significant figures.

Solution

Retain digits = 143.6
The next immediate digit = 9, (greater than 4)
Last retain digit + one = 6 + 1 = 7
Replace the remaining digits by zeros
∴ 143.6943 = 143.7(4 s.f)

Exercises 2.33

A. Correct to the significant figures indicated;

1. 466901 to 4.s.f 4. 65004 to 3.s.f
2. 107422 to 2.s.f 5. 17612761 to 5.s.f

B. Write to the indicated significant figures:

1. 0.005799 to 3.s.f 2. 0.099506 to 2.s.f
3. 55.61489 to 5.s.f 4. 176.904 to 3.s.f

Rational Numbers

Any number that can be expressed in the form, $\frac{a}{b}$, where $b \neq 0$ and a and b are integers is called ***a rational number***. For e.g, $\frac{1}{4}$, - 4, $3\frac{1}{2}$, 5.2, 0.3
Rational numbers can also be described as numbers with terminating or repeating decimal representation. For e.g, $\frac{1}{4}$ = 0.25 and $\frac{1}{3}$ = 0.3333…

A mixed number can also be described as a rational number. That is: $\mathbf{A\frac{b}{c} = \frac{A\times C+b}{C}}$. For

example, $2\frac{1}{3} = \frac{3\times 2+1}{3}$ = 2.33

Exercises 2.36

A. Write in the form $\frac{a}{b}$, b ≠ 0

1)12.6 2) − 0.04 3) − 3.3 4) $3\frac{1}{3}$ 5) $5\frac{2}{10}$ 6)$12\frac{3}{4}$

Rational Numbers on a Number Line

All rational numbers can be represented on a number line. Locate positive integers on the right of 0 and negative integers on the left of 0 on the number line.

I. Representing Rational Numbers of the Form; $\frac{a}{b}$ or $\frac{-a}{b}$, $b \neq 0$, on a Number Line

Draw a number line and divide each unit into the number of times of the denominator (b) . Count from zero, to the value of the numerator (a) to locate the position of the fraction.

Worked Example

1. Represent $\frac{3}{7}$ and $\frac{11}{7}$ on the same number line;

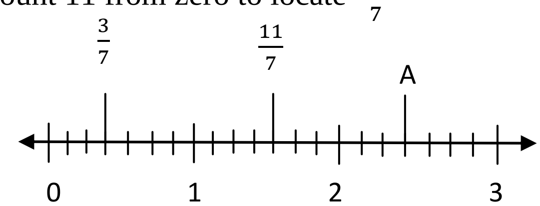
Solution

To locate $\frac{3}{7}$ and $\frac{11}{7}$ on the number line;

a. Divide each unit of the number line into 7 equal parts

b. Count 3 from 0 to locate $\frac{3}{7}$

c. Count 11 from zero to locate $\frac{11}{7}$



What rational number represents A?

III. Representing Rational Number of the Form $A\frac{b}{c}$, $c \neq 0$, on Number Line

Draw a number line and divide each unit into ***c*** equal parts. Locate ***A*** on the number line from 0, and if it positive, count ***b*** times from ***A*** to the right and if it is negative, count ***b*** times from ***A*** to the left to locate the mixed fraction on the number line.

Worked Examples

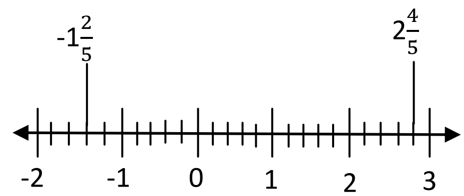
1. Represent $-1\frac{2}{5}$ and $2\frac{4}{5}$ on a the same number line.

Solution

a. Divide each unit of the number line into 5 equal parts

b. Count 2 from -1 to the left to locate $-1\frac{2}{5}$

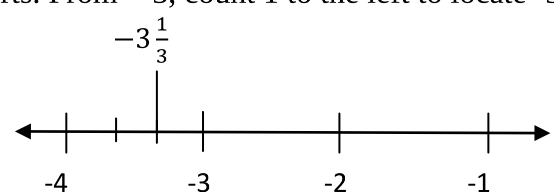
c. Count 4 from 2 to the right to locate $2\frac{4}{5}$ as shown below;



2. Represent $-3\frac{1}{3}$ on a number line.

Solution

Locating -3 and dividing each unit into 3 equal parts. From - 3, count 1 to the left to locate $-3\frac{1}{3}$



Decimal in the form A.B on a Number Line

I. Draw a number line and divide each unit into 10 equal segments.

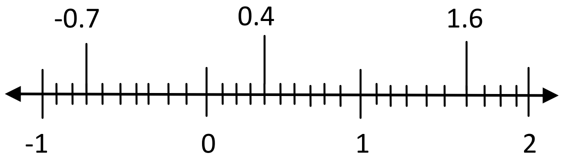
II. Identify the whole number (***A***) on the number line and from ***A***, count ***B*** times to the right (if ***A*** is positive)

III. From ***A***, count ***B*** times to the left (if *A* is negative).

Worked Examples

Represent the following on the same number line 0.4, 1.6, and - 0.7

Solution



Exercises 2.37

A. On separate number lines, represent the following group of rational numbers;

1. $\frac{4}{5}, \frac{11}{5}, \frac{7}{5}$ 3. $\frac{-1}{4}, \frac{-9}{4}, \frac{-3}{4}$

2. $\frac{-2}{6}, \frac{-5}{6}, \frac{-7}{6}$ 4. $1\frac{1}{3}$, $3\frac{2}{3}$, $4\frac{5}{3}$

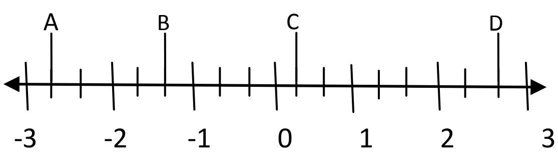
B. Locate the following on a number line;

2. $-2\frac{1}{3}$ 5. 6.3 8. $-3\frac{4}{7}$

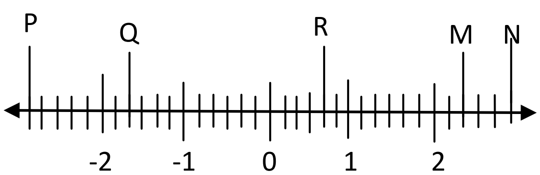
3. - 4.8 6. $-6\frac{3}{4}$ 9. -3.5

C. On the following number lines, identify the rational numbers represented by letters

1.



2.



Terminating and Non –Terminating Decimals

Consider the following examples,

1. 19 ÷ 8 = 2.373 2. 6 ÷ 5 = 1.2

3. 15 ÷ 11 = 1.3636… 4. 1÷ 3 = 0.3333…

In examples 1 and 2, the digits after the decimal point come to an end. Such decimals are called ***Terminating decimals.*** On the other hand, in examples 3 and 4, the digits after the decimal point keep repeating themselves and are therefore called ***repeating or recurring or non-terminating decimals.***

Recurring decimals are written in the short form by placing a dot on the digit(s) that repeats. From example, 3, 15÷11=$1.\dot{3}\,\dot{6}$ (Read as *"one point three, six, dot, dot"*) and from example 4, 1÷ 3 = $0.\dot{3}$ (*Read as "zero point three, dot"*)

Exercises 2.38

Express as recurring decimals;

1. $\frac{2}{7}$ 2. $\frac{4}{9}$ 3. $\frac{3}{11}$ 4. $\frac{4}{3}$

Irrational Numbers

Consider $\pi = \frac{22}{7}$ = 3. 142857143…

It is crystal clear that it is non- terminating but non-repeating decimal. That is to say that although the digits are not terminating, they are not repeating successively as well. Such numbers are called ***irrational numbers***. Other examples of irrational numbers are $\sqrt{5}$, $\sqrt{2}$ and $\sqrt{3}$

Note*: It is not the root sign that determines whether a number is irrational or not. This is because* $\sqrt{25} = 5$, $\sqrt{4} = 2$, *so* $\sqrt{25}$ *and* $\sqrt{4}$ *are rational numbers.*

Exercises 2.39

A. 1. Give 3 examples of irrational numbers.

2. Decimals that do not come to an end are called…

3. Decimals that repeat digits are called…

4. Numbers that do not repeat digits but do not come to an end are called…..

B. Identify the type of decimal, whether terminating or non-terminating

1. $\frac{14}{3}$ 2. $\frac{7}{3}$ 3. $\frac{11}{9}$ 4. $\frac{3}{10}$ 5. $\frac{6}{5}$

Repeating Decimals as Fractions

To write repeating decimals as fractions;

I. Represent the decimal by a variable say x, and name it equation (1).
II. Identify the number of digits that repeat.
If one digit repeats, multiply eqn (1) by 10.
If two digits repeat, multiply eqn (1) by 100.
If three digits repeat, multiply eqn (1) by 1000 in that order to obtain eqn (2)
III. Find eqn (2) − eqn (1) and simplify to get the variable as representing the fraction.

Worked Examples

1. Write $0.\dot{3}$ as a fraction

Solution

Let $x = 0.333$…............. eqn (1)
Because only one digit is repeating, multiply

eqn 1 by 10.
eqn (1) × 10
$10x = 3.33$…................. eqn (2)

eqn (2) – eqn (1)
$9x = 3$
$x = \frac{1}{3}$

2. Express 0.424242… as a fraction

Solution

Let $x = 0.424242$…......... eqn (1)

eqn (1) × 100
$100\,x = 0.42.4242$…....... eqn (2)
eqn (2) – eqn (1);

$99x = 42$
$x = \frac{14}{33}$

3. Express 0.13333… as a fraction

Solution

Let $x = 0.13333$ ………..(1)
eqn (1) × 10
$10x = 1.3333$ …………..(2)
eqn (1) × 100
$100x = 13.333$ …………(3)

eqn (3) – eqn (2);
$90x = 12$
$x = \frac{2}{15}$

Exercises 2.40

Express the decimals as fractions;

1) $0.\dot{6}$ 2) $4.\dot{2}\dot{4}$ 3) $0.\dot{1}\dot{2}$
4) $0.\dot{6}5\dot{7}$ 5) $1.\dot{1}\dot{5}$ 6) $3.1\dot{4}\dot{3}$

Properties of Operations on Rational Numbers

1. Commutative Property of addition

Addition is said to be commutative on the set of rational numbers, if for any two rational numbers, $\frac{a}{b}$ and $\frac{c}{d}$, $b \neq 0$ and $d \neq 0$, $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$

2. Commutative Property of Multiplication

Multiplication is said to be commutative on the set of rational numbers if for any two rational numbers, $\frac{a}{b}$ and $\frac{c}{d}$, $b \neq 0$ and $d \neq 0$, $\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$

3. Associative Property of Addition

Addition is said to be associative on rational numbers if for any three rational numbers, $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$, $b \neq 0$, $d \neq 0$ and $f \neq 0$;
$\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right)$

4. Associative Property of Multiplication

Multiplication is said to be associative on rational numbers if for any three rational numbers, $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$, $b \neq 0$, $d \neq 0$ and $f \neq 0$;
$\left(\frac{a}{b} \times \frac{c}{d}\right) \times \frac{e}{f} = \frac{a}{b} \times \left(\frac{c}{d} \times \frac{e}{f}\right)$

5. Distributive Property of Multiplication over addition

For all rational numbers, $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$, $b \neq 0$, $d \neq 0$ and $f \neq 0$;

$$\frac{a}{b}\left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} \times \frac{c}{d}\right) + \left(\frac{a}{b} \times \frac{e}{f}\right)$$

6. Distributive Property of Multiplication over Subtraction

For all rational numbers, $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$, $b \neq 0$, $d \neq 0$ and $f \neq 0$;

$$\frac{a}{b}\left(\frac{c}{d} - \frac{e}{f}\right) = \left(\frac{a}{b} \times \frac{c}{d}\right) - \left(\frac{a}{b} \times \frac{e}{f}\right)$$

Exercises 2.41

Given that $\frac{2}{5} \times \left(\frac{3}{4} \times 3.2\right) = \left(\frac{a}{b} \times \frac{3}{4}\right) \times 3.2$, find:

i. the values of *a* and *b*

ii. What property of multiplication

Binary Operations

A binary operation is any symbol or sign that is used to combine two elements of a set according to some clearly defined rule.

Apart from the usual operations, +, – ,÷, ×, ∪, ∩, there are other signs such as Δ, ◘, ◙, ◊, ○, *, etc that are used to combine two elements of a set according to some rule. For example, in

a * *b* = 2*a* – *b*, *a* and *b* are combined by the operation * under the rule 2*a* – *b*. Therefore, the symbol, * is called a ***binary operation***

Worked Examples

1. A binary operation is defined over the set of real numbers, R, as *a* * *b* = *a* – 3*b*. Find:

i) 5 * 2 ii) 2 * 5

Solution

a **b* = *a* – 3*b*

i. 5 * 2 = 5 – 3(2)

5 * 2 = 5 – 6 = - 1

ii. 2 * 5 = 2 – 3(5)

2 * 5 = 2 – 15 = - 13

2. A binary operation * is defined over R, the set of real numbers by *a***b* = 3*a* + *b* – 1. Find *a***b* if;

i. *a* = 2 and *b* = -3 ii.*a* = -1 and *b* = 4

Solution

i. *a* * *b* = 3*a* + *b* – 1

2 *-3 = 3(2) + (-3) – 1

2 *-3 = 6 – 3 – 1 = 2

ii. *a* = -1 and *b* = 4

-1 * 4 = 3(-1) + 4 – 1

-1 * 4 = -3 + 4 – l = 0

3. The operation ◙ is defined on the set of real numbers R, by *m* ◙ *n* = $\sqrt{mn}$, where the positive square root is taken. Find: 10 ◙ (4 ◙ 25)

Solution

m ◙ *n* = $\sqrt{mn}$,

10 ◙ (4 ◙ 25)

4 ◙ 25 = $\sqrt{4 \times 25}$ = $\sqrt{100}$ = 10 (***Bracket first)***

10 ◙ 10 = $\sqrt{10 \times 10}$ = $\sqrt{100}$ = 10

3. A binary operation ○ is defined on the set Q of rational numbers by: *x*○*y* = *x* + *y* + 2*xy*.

a. Evaluate $\frac{3}{4} \circ \frac{1}{2}$ and $\frac{1}{2} \circ \frac{3}{4}$

b. Find the value of *x* given that *x* ○ 3 = 24

Solution

a. $\frac{3}{4} \circ \frac{1}{2} = \frac{3}{4} + \frac{1}{2} + 2\,(\frac{3}{4} \times \frac{1}{2}) = \frac{3}{4} + \frac{1}{2} + \frac{3}{4} = 2$

$\frac{1}{2} \circ \frac{3}{4} = \frac{1}{2} + \frac{3}{4} + 2\left(\frac{1}{2} \times \frac{3}{4}\right) = \frac{1}{2} + \frac{3}{4} + \frac{3}{4} = 2$

b. $x \circ 3 = x + 3 + 2\,(3x)$
$x \circ 3 = 3 + x + 6x$
$x \circ 3 = 7x + 3$ **(But $x \circ 3 = 24$)**
$7x + 3 = 24$
$7x = 21$
$x = 3$

4. The operation ∇ is defined on the set of real number s by $a \nabla b = a + b + 2ab$, where $a, b \in R$
i. a. Calculate $2 \nabla 3$ and find $(2\nabla 3)\nabla 5$;
b. Find the truth set of $a\nabla 7 = (a\nabla 5) + (a\nabla 2)$

Solution
i. $a\nabla b = a + b + 2ab$
$2 \nabla 3 = 2 + 3 + 2(2)(3) = 17$

$(2\nabla 3)\nabla 5 = 17\nabla 5$
$= 17 + 5 + 2(17)(5) = 192$

ii. $a\nabla 7 = (a\nabla 5) + (a\nabla 2)$
$a\nabla 7 = a + 7 + 2(a)(7)$
$= 15a + 7$

$a\nabla 5 = a + 5 + 2(a)(5)$
$= 11a + 5$
$a\nabla 2 = a + 2 + 2\,(a)\,(2)$
$= 5a + 2$

$a\nabla 7 = (a\nabla 5) + (a\nabla 2)$
$\Rightarrow 15a + 7 = 11a + 5 + 5a + 2$
$15a - 11a - 5a = 7 - 7$
$-a = 0, \quad a = 0$

5. The operation * is defined on the set of real numbers by $p * q = p^2 + q^2 - 2pq$, where $p, q \in R$
a. i. Evaluate $\sqrt{3} * \frac{1}{\sqrt{12}}$ and $\frac{1}{\sqrt{12}} * \sqrt{3}$
ii. Use your results in (i) to evaluate;
$\left(\sqrt{3} * \frac{1}{\sqrt{12}}\right) * \left(\sqrt{3} * \frac{1}{\sqrt{12}}\right)$

Solution
a. $p * q = p^2 + q^2 - 2pq$
$\sqrt{3} * \frac{1}{\sqrt{12}} = \left(\sqrt{3}\right)^2 + \left(\frac{1}{\sqrt{12}}\right)^2 - 2\left(\sqrt{3}\right)\left(\frac{1}{\sqrt{12}}\right)$
$= 3 + \frac{1}{12} - 2\left(\sqrt{3}\right)\left(\frac{1}{\sqrt{3} \times \sqrt{4}}\right)$
$= 3 + \frac{1}{12} - 2\left(\frac{1}{2}\right)$
$= 3 + \frac{1}{12} - 1 = \frac{25}{12}$

$\frac{1}{\sqrt{12}} * \sqrt{3} = \left(\frac{1}{\sqrt{12}}\right)^2 + \left(\sqrt{3}\right)^2 - 2\left(\frac{1}{\sqrt{12}}\right)\left(\sqrt{3}\right)$
$= \frac{1}{12} + 3 - 2\left(\frac{1}{\sqrt{3} \times \sqrt{4}}\right)\left(\sqrt{3}\right)$
$= \frac{1}{12} + 3 - 1 = \frac{25}{12}$

ii. $\left(\sqrt{3} * \frac{1}{\sqrt{12}}\right) * \left(\sqrt{3} * \frac{1}{\sqrt{12}}\right)$
$\frac{49}{12} * \frac{49}{12} = \left(\frac{25}{12}\right)^2 + \left(\frac{25}{12}\right)^2 - 2\left(\frac{25}{12}\right)\left(\frac{25}{12}\right) = 0$

6. The operation * is defined on the set of real numbers by $m*n = \frac{m-n}{n}$, $n \neq 0$
a. Evaluate $3 * (5 * 2)$
b. Find the truth set of :
i. $8 * k = 12 * 3$ ii. $k * 8 = 12 * 3$

Solution
a. $3 * (5 * 2)$
$m * n = \frac{m-n}{n}$
$(5 * 2) = \frac{5-2}{2} = \frac{3}{2}$
$\Rightarrow 3 * (5 * 2) = 3 * \frac{3}{2} = \frac{3 - {}^{3}/_{2}}{{}^{3}/_{2}} = \frac{{}^{3}/_{2}}{{}^{3}/_{2}} = 1$

b. i. $8 * k = 12 * 3$
$\frac{8-k}{k} = \frac{12-3}{3}$
$\frac{8-k}{k} = \frac{9}{3}$
$\frac{8-k}{k} = 3$
$8 - k = 3k$
$8 = 3k + k$
$8 = 4k$
$k = \frac{8}{4} = 2$

ii. $k * 8 = 12 * 3$

$\frac{k-8}{8} = \frac{12-3}{3}$

$\frac{k-8}{8} = \frac{9}{3}$

$\frac{k-8}{8} = 3$

$k - 8 = 3 \times 8$

$k - 8 = 24$

$k = 24 + 8$

$k = 23$

7. Two binary operations are defined as follows: $p * q = \frac{1}{p} + \frac{1}{q}$ and $p \oplus q = \frac{1}{p} - \frac{1}{q}$

a. If $p = \frac{3}{5}$ and $q = \frac{1}{3}$, evaluate ;

i. $p * q$ ii. $p \oplus q$ iii. $\frac{p*q}{\mathrm{p} \oplus \mathrm{q}}$

b. If $p * q = \frac{1}{\mathrm{p} \oplus \mathrm{q}}$, evaluate $p^2 \oplus q^2$

Solution

a. i. $p * q$

$p * q = \frac{1}{p} + \frac{1}{q}$

$\frac{3}{5} * \frac{1}{3} = \frac{1}{{}^3/_5} + \frac{1}{{}^1/_3} = \frac{5}{3} + \frac{3}{1} = \frac{5+9}{3} = \frac{14}{3}$

ii. $p \oplus q$

$p \oplus q = \frac{1}{p} - \frac{1}{q}$

$\frac{3}{5} \oplus \frac{1}{3} = \frac{1}{{}^3/_5} - \frac{1}{{}^1/_3} = \frac{5}{3} - \frac{3}{1} = \frac{5-9}{3} = -\frac{4}{3}$

iii. $\frac{p*q}{\mathrm{p} \oplus \mathrm{q}} = \frac{{}^{14}/_3}{{}^{-4}/_3} = -\frac{7}{2}$

b. From $p * q = \frac{1}{\mathrm{p} \oplus \mathrm{q}}$,

$\frac{1}{p} + \frac{1}{q} = \frac{1}{{}^1/_p - {}^1/_q}$ ***(By substitution)***

$\frac{p+q}{pq} = \frac{1}{{}^{q-p}/_{pq}} = \frac{1}{q-p} \times \frac{pq}{1} = \frac{pq}{q-p}$

$\frac{p+q}{pq} = \frac{pq}{q-p}$

$(p + q)(q - p) = (pq)(pq)$ ***(cross multiplication)***

$p^2 - q^2 = p^2q^2$…………..(1)

But $p \oplus q = \frac{1}{p} - \frac{1}{q}$

$p^2 \oplus q^2 = \frac{1}{p^2} - \frac{1}{q^2} = \frac{q^2-p^2}{p^2q^2}$

From eqn (1),

$p^2 - q^2 = p^2q^2$

$p^2 \oplus q^2 = \frac{p^2q^2}{p^2q^2} = 1$

Exercises 2.42

1. A binary operation * is defined on the set real numbers by $x*y = x + y + xy$. Evaluate:

a. 3 * 5 b. -2 * -2

2. The operation ◙ is defined over the set of real numbers, R, by a ◙ $b = a + b^2$. Find:

a. (3 ◙ 2) ◙ 5 b. k if 3 ◙ k = 7

3.A binary operation * is defined on R, the set real numbers by $p*q = p^2 + 2pq$. Find;

i. 3 * 5 ii. 5 * 3

iii. (3 * 5) * (5 *3) iv. p if $p*4 = 9$

4. Given that u Δ v = $\frac{uv}{u+v}$, $u + v \neq 0$, where u, $v \in$ R, evaluate. (4 Δ 7) Δ (7 Δ 5)

5. If * means 'square the first number and divide by the second', find the values of;

a. 9 * 6 b. $\frac{1}{3} * \frac{2}{3}$ c. 1.2 * 0.018

6. The operation ∼ is defined by $m \sim n = \frac{1}{2}(m - n)$

i. a. Form a table of the operation ∼ on the set {2, 4, 6}.

b. Is the operation ∼ commutative? Give a reason.

ii. If $m = 2$, $n = 4$ and $p = 6$, evaluate;

a. $m \sim (n + p)$ b. $(m \sim n) + (m \sim p)$

7. Given that $a * b$ denotes $\frac{a+b}{a}$, evaluate:

i. 2 * 1 ii. 1 * 2 iii. (1 * 2) * (2 * 1)

8. Given that $a * b$ denotes $a^2 - b^2$, evaluate:
i. $2\frac{1}{2} * 1\frac{1}{2}$ ii. Find a when $a * 7 = 1 * a$

Table of Binary Operation
Given a well-defined binary operation, a table of valuescan be constructed for a given set of values.
I. Identify the definition of the binary and the given set of values
II. If there is ***n*** number of elements in the set, prepare ***n*** × ***n*** square table
III. Place the operator at the top left corner of the table
IV. Occupy the first row and column of the table with the elements of the given set as shown below;

*	n_1	n_2	n_3
n_1			
n_2			
n_3			

IV. Operate each element of the first row against each element of the second row under the binary definition and record your answers in the cells or boxes until it is completed.

Worked Examples
1. The operation ◊ is defined on the set R of real numbers by $x \lozenge y = x^2 + y^2 - 4$
a. Copy and complete the table below for the operation ◊ on the set M = {1, 2, 3, 4}

◊	1	2	3	4
1	2	1	6	
2		4	9	16
3	6	9		21
4	13			28

b. Use the table to;
i. evaluate 2 ◊ 4 and 4 ◊ 2
ii. what can you say about 2 ◊ 4 and 4 ◊ 2?
iii. find (1 ◊ 2) ◊ 4

Solution
a. M = {1, 2, 3, 4}
$x \lozenge y = x^2 + y^2 - 4$
$1 \lozenge 4 = 1^2 + 4^2 - 4 = 13$
$2 \lozenge 1 = 2^2 + 1^2 - 4 = 1$
$2 \lozenge 3 = 2^2 + 3^2 - 4 = 9$

$3 \lozenge 3 = 3^2 + 3^2 - 4 = 14$
$4 \lozenge 2 = 4^2 + 2^2 - 4 = 16$
$4 \lozenge 3 = 4^2 + 3^2 - 4 = 21$

◊	1	2	3	4
1	2	1	6	13
2	1	4	9	16
3	6	9	14	21
4	13	16	21	28

b. From the table;
i. 2 ◊ 4 = 16 and 4 ◊ 2 = 16

ii. 2 ◊ 4 = 4 ◊ 2 = 16.
The operation ◊ is commutative.

iii. From the table (1◊ 2) = 1
(1 ◊ 2) ◊ 4 = 1 ◊ 4 = 13

2. An operation is defined on the set of integers Z, by, $x * y = x + y + 3xy$, where x and $y \in Z$.
i. Construct a table for this operation on the set, $S = \{-1, 0, 1, 2\}$
ii. Find from your table, a number $b \in S$, such that $c * b = c$ for all $c \in S$.

Solution
1.$x * y = x + y + 3xy$
$S = \{-1, 0, 1, 2\}$

$x * y = x + y + 3xy$

-1 * -1 = -1 – 1 + 3(-1)(-1) = 1

-1 * 0 = -1 + 0 + 3(-1)(0) = -1

-1 * 1 = -1 + 1 + 3(-1)(1) = -3

-1 * 2 = -1 + 2 + 3(-1)(2) = -5

0 * -1= 0 – 1 + 3(0)(-1) = -1

0 * 0 = 0 + 0 + 3(0)(0) = 0

0 * 1 = 0 + 1 + 3(0)(1) = 1

0 * 2 = 0 + 2 + 3(0)(2) = 2

1 * -1= 1 – 1 + 3(1)(-1) = -3

1 * 0 = 1 + 0 + 3(1)(0) = 1

1 * 1 = 1 + 1 + 3(1)(1) = 5

1 * 2 = 1 + 2 + 3(1)(2) = 9

2 * -1= 2 – 1 + 3(2)(-1) = -5

2 * 0 = 2 + 0 + 3(2)(0) = 2

2 * 1 = 2 + 1 + 3(2)(1) = 9

2 * 2 = 2 + 2 + 3(2)(2) = 16

*	-1	0	1	2
-1	1	-2	-3	-5
0	-1	0	1	2
1	-3	1	5	9
2	-5	2	9	16

ii. $c * b = c$

1 * 0 = 1

2 * 0 = 2

Therefore $b = 0$

Exercises 2.43

1. The binary operation * is defined by $m * n = mn – m$. Copy and complete the table below for * on the set {3, 4, 5, 6}

*	3	4	5	6
3	6			
4				
5				
6			24	

b. Find from the table:

i. 4 * 5 ii. $x * x = 20$

2. i. Copy and complete the table below forthe operation* defined by $a * b = b^2 – ab$ on the set N = {1, 2, 2.3, 3.5, 4.7}

*	1	2	2.3	3.5	4.7
2					
2.3					
3.5					
4.7					

ii. Use the table to evaluate 2.3 * 3.5

Properties of Binary Operation

The closure Property

Given that ***a*** and ***b*** are members of set **S** and the operation * is defined over **S**. If $a * b$ always gives an answer which is also in S, then the set S is said to be closed with respect to *. That is:

a * b∈S, a, b∈S

Worked Examples

The binary operation * is defined on the set A = {1, 2, 3, 4} by $x * y = 3x – xy$. Show whether * is closed under A in each of the following;

i. 1 * 2 ii. 3 * 4

Solution

i. $x * y = 3x – xy$.

But $x = 1$ and $y = 2$

1 * 2 = 3(1) – (1)(2)

= 3 – 2 = 1

1 * 2 = 1 ∈ A, therefore set A is closed under the operation*

ii. $x * y = 3x – xy$

3 * 4 = 3(3) – (3)(4)

= 9 – 12 = - 3

3 * 4 = - 3 ∉ A, therefore set A is not closed under the operation*

Exercises 2.44

The binary operation* is defined on the set M = {5, 6, 7, 8, 9}by $x * y = xy - 4y$. Show whether * is closed under M in each of the following;

i. $5 * 6$ ii. $5 * 8$ iii. $7 * 9$ iv. $6 * 9$

The Commutative Property

Let * be any binary operation defined over the set S. We say the operation * is commutative if for any two numbers ***a*** and ***b*** belonging to ***S***,

$\boldsymbol{a * b = b * a}$

Worked Examples

1. Let $a * b = a + 2b$ be defined over the set of real numbers R. Verify whether or not, * is commutative

Solution

Method 1

$a * b = a + 2b$

Let $a = 3$ and $b = 4$

$3 * 4 = 3 + 2(4) = 3 + 8 = 11$

Let $a = 4$ and $b = 3$

$4 * 3 = 4 + 2(3) = 4 + 6 = 10$

$3 * 4 \neq 4 * 3$

Therefore the operation is not commutative

Method 2

For the operation to be commutative

$a * b = b * a$

$a * b = a + 2b$

$b * a = b + 2a$

$a + 2b \neq b + 2a$

$a * b \neq b * a$.

Therefore, the operation is not commutative

2. If $a * b = a + b + ab$, show whether the operation * is commutative or not.

Solution

The operation * is commutative if;

$a * b = b * a$

$a * b = a + b + ab$

$b * a = b + a + ba$

$a + b + ab = b + a + ba$

Therefore $a * b = b * a$

The operation * is commutative

Exercises 2.45

1. The binary operation Δ is defined over the set of real numbers as $m \Delta n = \frac{3m+2n}{m}$. Find:

a. $2 \Delta 3$ b. $4 \Delta 8$

2. The binary operation ○ is defined on the set of real numbers by $x \circ y = x^2 - y$. Find:

i. $3 \circ 5$ ii. $5 \circ 3$

iii. What can you say about i and ii?

3. If $a * b = ab + a + b$, solve the $a * 3 = 19$

4. The operation * on R is defined by $a * b = (a + b)^2 - a^2 - b^2$

a. Simplify the right hand side and calculate:

$-5 * (-1)$ and $\frac{1}{3} * 0$

b. Prove that the operation * is commutative.

5. The operation * on the integers a and b is defined by $a * b = ab + b$. Find x given that $4 * x = x * 4$

The Commutative Property on a Table

A binary operation, * defined over a set S, is said to be commutative if;

1. A table of values constructed is symmetrical about the leading diagonal.

Consider the table below;

*	*e*	*f*	*g*	*h*
e	*e*	*f*	*g*	*h*
f	*f*	*f*	*h*	*h*
g	*g*	*h*	*g*	*h*
h	*h*	*h*	*h*	*h*

It can be seen that the table is symmetrical about the leading diagonal. That is to say that each set of entries is a “reflection” of the other in the leading diagonal. Therefore the operation* is said to be commutative.

2. Test two values on the table by interchanging their positions to see if the operation gives the same answer. For e.g. from the above table:

$e * f = f$

$f * e = f$

$\Rightarrow e * f = f * e = f.$

The operation* is said to be commutative

Worked Examples

The operation * is defined over the set M ={1, 2, 3, 4} as $a * b = a + b - 4$. Construct a table of values and show whether the operation * is commutative or not.

Solution

$a * b = a + b - 4,$

Set M = { 1, 2, 3, 4}

*	1	2	3	4
1	-2	-1	0	1
2	-1	0	1	3
3	0	1	2	3
4	1	2	3	4

The operation * is commutative because the table is symmetrical about the leading diagonal .

Exercises 2.46

The binary operation ○ is defined as:

$x \circ y = x + y + 2xy$**.**

a. Find the values of t, k, r and s in the table for the operation ○ on the set P = {1, 2, 3, 4}

○	1	2	3	4
1	4	7	10	13
2	*t*	12	17	*s*
3	10	*k*	*r*	31
4	13	22	31	40

b. Show whether set P is closed or not with respect to the operation ○

c. Is the operation ○ is commutative or not?

The Associative Property

If a, b and $c$are members of a set of real numbers R, then the operation * defined over R is associative if: ***(a * b) * c = a *(b * c)***

Worked Examples

The operation ◙ is defined over the set of real numbers, R, by a ◙ $b = ab + a + b$. Show whether ◙ is associative or not.

Solution

Let a, b and c ∈ R, then

a ◙ (*b* ◙ *c*) = (*a* ◙ *b*) ◙ *c*

But *a* ◙ *b* = $ab + a + b$

L. H. S: (*a* ◙ *b*) ◙ *c* = *a* ◙ ($bc + b + c$)

Let *b* ◙ *c* = $bc + b + c = m$

a ◙ (*b* ◙ *c*) = *a* ◙ *m* = $am + a + m$

By substitution,

a ◙ (*b* ◙ c) = $a(bc + b + c) + a + (bc + b + c)$

$= abc + ab + ac + a + bc + b + c$

$= a + b + c + ab + bc + ac + abc$

R. H.S: *a* ◙ (*b* ◙ *c*) = ($ab + a + b$) ◙ *c*

Let $ab + a + b = n$

⇒ (*a* ◙ *b*) ◙ *c* = *n* ◙ c

By substitution,

$(a ◙ b) ◙ c = (ab + a + b)c + (ab + a + b) + c$
$= abc + ac + bc + ab + a + b + c$
$= a + b + c + ab + bc + ac + abc$

Comparing L.H.S results to R.H.S results, it is seen that $a ◙ (b ◙ c) = (a ◙ b) ◙ c$

Therefore, the operation ◙ is associative

Exercises 2.47

If $a * b = ab + a + b$, solve the equations $a * 3 = 19$ and $(a * 3) + (2 * a) = 4m$

The Distributive Property

If***a***, ***b*** and ***c*** are three members of the set of real numbers, R and the binary operations * and ◘ are defined over S, then * is distributive over ◘ if $\boldsymbol{a * (b ◘ c) = (a * b) ◘ (a * c)}$

However, if $\boldsymbol{a * (b ◘ c) \neq (a * b) ◘ (a * c)}$, then * is not distributive over ◘

Worked Examples

Two binary operations * and Δ are defined as $a * b = 2xy$ and $a \Delta b = a + b + 2$ for all $a, b \in$ R.

Evaluate: i. $4 * (5 \Delta 6)$ ii. $(4 * 5) \Delta (4 * 6)$

iii. What can you say about (i) and (ii)

Solution

i. $a * b = 2xy$ and $a \Delta b = a + b + 2$

In $4 * (5 \Delta 6)$, taking the bracket first,

$(5 \Delta 6) = 5 + 6 + 2 = 13$

$4 * 13 = 2(4)(13) = 104$

ii. $(4 * 5) \Delta (4 * 6)$,

$(4 * 5) = 2(4)(5) = 40$ *(First bracket)*

$4 * 6 = 2(4)(6) = 48$ *(Second bracket)*

$\Rightarrow (4 * 5) \Delta (4 * 6) = 40 \Delta 48$

$40 \Delta 48 = 40 + 48 + 2 = 90$

iii. The operation * is not distributive over Δ because $4 * (5 \Delta 6) \neq (4 * 5) \Delta (4 * 6)$

Exercises 2.48

1. Two binary operations * and ○ are defined over the set, R, of real numbers by $x * y = xy$ and $x \circ y = x + y$. Find:

i. $x * (y \circ z)$ ii. $(x * y) \circ (x * z)$

iii. Show whether the operation * is distributive over ○ or not

2. Let U = {1, 2, 3… 4}, A = {2, 3, 5, 7, 8} B = {1, 3, 5, 9} and C = {4, 9, 10} where A, B and C are subsets of U. Find:

i. A ∩ (BUC), ii. (A∩B) U (A∩C),

iii. What can you say about the operations ∩ and U over the set?

3 ALGEBRAIC EXPRESSIONS

Definition

Any mathematical statement written without an equal sign is called ***an expression***. An algebraic expression is therefore a mathematical statement that contains numbers and variables and / or an operator, but not an equal sign. For e.g, $2m - n$, $2p$, $6y + 3x$ etc.

Like Terms and Unlike Terms

Likes terms are algebraic terms that have the same variable factors. For example, $5x$ and $10x$, $2m$ and $15m$, $3y$ and $8y$etc

Unlike terms are algebraic terms that have different variable factors. For example, $5x$ and $10y$, $3a$ and $8b$*etc*

Worked Examples

Group the like terms in the following:

1. $4a^2 + 3ab - 2a^2 + ab$
2. $2a + 4b^2 - 3a + 3b^2 - b^2$
3. $3p + 5q + 2p - q$

Solutions

1. $4a^2 + 3ab - 2a^2 + ab$
$= 4a^2 - 2a^2 + 3ab + ab$

2. $2a + 4b^2 - 3a + 3b^2 - b^2$
$= 2a - 3a + 4b^2 + 3b^2 - b^2$

3. $3p + 5q + 2p - q = 3p + 2p + 5q - q$

4. Re - group $4x^2 + 3x^2y - xy^2 + 2x^2y$

Solution

$4x^2 + 3x^2y - xy^2 + 2x^2$
$= 4x^2 - xy^2 + 3x^2y + 2x^2y$

Exercises 3.1

Group the following into like terms and unlike terms;2y, 6m, $12a$, $10m$, $13x$, $5x$, $5y$, $2k$, $15n$, $4p$, $7q$, $8r$, $9a$, $10v$, $3r$, $25x$ and $54u$

Simplifying Algebraic Expressions

The process of adding, subtracting, multiplying and dividing two or more algebraic expressions is called ***simplification.***

Addition and Subtraction of Algebra

Like terms can be added and subtracted. For instance, 3pens can be added to 7pens to get 10pens. That is, $3p + 7p = 10p$. Also 3pens can be taken away from 7pens to get 4pens. That is: $7p - 3p = 4p$

Unlike terms on the other hand, cannot be added nor subtracted. This is explained by the fact that 3pens cannot be added to nor subtracted from 7erasers. That is:

1. 3pens + 7erasers = 3pens + 7erasers
2. 7 erasers – 3pens = 7erasers – 3pens.

Thus under addition and subtraction of unlike terms, the variable terms remain the same.

Worked Examples

Simplify the following:

(1) $16y + 4y + 2y$ (1) $12a - 8a$
(3) $25t - 14t - t$ (4) $17x + 6y$

Solutions

(1) $16y + 4y + 2y = 22y$ (2) $12a - 8a = 4a$
(3) $25t - 14t - t = 10t$ (4) $17x + 6y = 17x + 6y$

Exercises 3.2

A. Simplify the following:

1. $1.32a + 7.68a$ 2) $10k - k$ 3. $13m + 51n$
4. 9s – 4s 5. $8m - 3m - m$ 6. $a - 3$

B. Simplify the following;

1. $16m + 6n - 18m + 4n$
2. $20y - 32x - 36n + 8y$
3. $3x + (x + 5x)$
4. $(9y - 11y) + 10y$

C. $A = 3x + 4y$ and $B = 2x - 5y$. Express the following in terms of x and y, in their simplest forms:
1. $A + B$ 2. $2A + 3B$ 3. $A - B$ 4. $3A - 2B$

Grouping Algebraic Expressions

Like and unlike terms of algebraic expression are regrouped to make addition and subtraction possible and simple.

In regrouping, the operation sign is part of the algebraic term and therefore must be lifted alongside. For example, in regrouping $2x + 6y + 3x - 2y$ we get;
$2x + 3x + 6y - 2y = 5x + 4y$

Worked Examples

Simplify the following;
1. $8m - 4n + 3m + 10n$

Solution

$8m - 4n + 3m + 10n$
$= 8m + 3m + 10n - 4n$
$= 11m + 6n$

2. $5x - 8y - x - 2y$

Solution

$5x - 8y - x - 2y$
$= 5x - x - 8y - 2y$
$= 4x - 10y$

3. Simplify $4p + 6p^2 - 2p + 2p^2$

Solution

$4p + 6p^2 - 2p + 2p^2$
$= 6p^2 + 2p^2 + 4p - 2p$
$= 8p^2 + 2p$

Exercises 3.3

A. Simplify the following:

1. $5p - 6a - 7p + 10a$
2. $8x^2 - 15a - 20x^2 + 3a$
3. $5w + 7p^2 - 4w + 3p^2$
4. $3x^2 + 6xy - 3y^2 + 4x^2 - 8xy + 2y^2$

B. Re-group the following and simplify:

1.$4a + 3b + 3a + 2b$ 4. $3x^2 - 4x^2 - 2x^3 + 4x^3 - x^2$
2. $p + 3q - p + 29$
3. $3h + 8 - 2h + 2$ 5. $7m + 6n + 3m + 2n$

Multiplication of Algebraic Expressions

This is done by expanding and regrouping the terms of the expression.
Reminder: $a^m \times a^n = a^{m+n}$

Worked Examples

1. Perform $3ab \times 2a$

Solution

$3ab \times 2a$
$= \mathbf{3} \times a \times b \times 2 \times a$
$= 3 \times 2 \times a \times a \times b = 6 \times a^2 \times b = 6a^2b$

2. Find $3a^2 \times 2ab \times 4bc$

Solution

$3a^2 \times 2ab \times 4bc$
$= (3 \times 2 \times 4)\, a^{2+1}b^{1+1}c = 24a^3b^2c$

3. Simplify $(3a^2b^3)(4a^3b)$

Solution

$3a^2b^3 \times 4a^3b = (3 \times 4)\ a^{2+3}b^{3+1} = 12a^5b^4$

Exercises 3.4

A. Simplify the following;

1. $5a^2b \times 3ab^2$ 2. $10m^2n \times 8m^4n^3$

3. $10ab^3 \times 2a^2b$ 4. $3qb^2 \times 4pq$

B. Simplify the following:

1. $\frac{6x}{11} \times \frac{5x}{4}$ 2. $\frac{1}{2}a^2b \times \frac{1}{3}ab$ 3. $\frac{5}{2}t + \frac{5}{2}t$

4. $\frac{2}{9}a^2b \times (-ab^3)$ 5. $\frac{13m}{16} \times -8$ 6. $(8x^2y^3)\ (\frac{3}{8}xy^4)$

Division of Algebraic Expressions

This is done by expanding the dividend and the divisor and dividing the like terms.

Reminder: $a^m \div a^n = a^{m-n}$

Worked Examples

Simplify the following;

1. $6a^3b^2 \div 3ab$

Solution

$6a^3b^2 \div 3ab$

$= \frac{6a^3b^2}{3ab} = \frac{6 \times a \times a \times a \times b \times b}{3 \times a \times b} = 2a^2b$

2. $35x^5y^2 \div 5x^2y$

Solution

$35x^5y^2 \div 5x^2y$

$\frac{35x^5y^2}{5x^2\,y} = (35 \div 5)\ x^{5-2}y^{2-1} = 7x^3y$

Exercises 3.5

Simplify;

1) $18k^3j^7 \div 3j^2k$ 2) $99a^3b^2m^4 \div 3a^3bm^3$

3) $48x^{11}y^6 \div 6x^3y^3$ 4) $3x^5 \div 27x^3y$

Algebraic Fractions

A. Addition and Subtraction

I. Involving Monomial Denominators

For all operations involving algebraic fractions, follow the same rule as in arithmetic.

To add or subtract algebraic fractions involving monomial denominators, observe the following;

1. Find the L.C.M of the denominators.
2. Express each fraction in terms of the L.C.M. and simplify.

Worked Examples

1. Simplify $\frac{2a}{3} - \frac{a-b}{2}$

Solution

$\frac{2a}{3} - \frac{a-b}{2}$

$= \frac{4a-3(a-b)}{6} = \frac{4a-3a-3b}{6} = \frac{a+3b}{6}$

2. Simplify $\frac{2a+4b}{3} - \frac{3(a-b)}{2}$

Solution

$\frac{2a+4b}{3} - \frac{3(a-b)}{2}$

$= \frac{2(2a+4b)-9(a-b)}{6} = \frac{4a+8b-9a+9b}{6} = \frac{-5a+17b}{6}$

II. Involving Binomial Denominators

To add or subtract algebraic fractions involving binomial denominators:

1. Factorize denominators (if necessary).
2. Find the L.C.M of the denominators.
3. Express each fraction in terms of the L.C.M. and simplify.

Work Examples

1. Simplify $\frac{3}{x+5} - \frac{2}{x+3}$

Solution

$\frac{3}{x+5} - \frac{2}{x+3}$

$= \frac{3(x+3)-2(x+5)}{(x+5)(x+3)} = \frac{3x+9-2x-10}{(x+5)(x+3)} = \frac{x-1}{(x+5)(x+3)}$

B. Multiplication and Division

I. Involving Binomial Denominators

Before attempting to simplify when multiplying or dividing algebraic fractions, factorize where possible and divide top and down by common factors. The contents of a bracket should be considered as a single term.

Work Examples

1. Simplify $\frac{2(x-3)}{(x+4)(x-3)}$

Solution

$\frac{2(x-3)}{(x+4)(x-3)} = \frac{2}{(x+4)}$

2. Simplify $\frac{3a^2-5a-12}{a^2-3a}$

Solution

$\frac{3a^2-5a-12}{a^2-3a} = \frac{(a-3)(3a+4)}{a(a-3)} = \frac{3a+4}{a}$

3. Simplify $\frac{a^2-1}{a^2+2a+1}$

Solution

$\frac{a^2-1}{a^2+2a+1} = \frac{(a+1)(a-1)}{(a+1)(a+1)} = \frac{a-1}{a+1}$

II. Involving Monomial Denominators

I. Factorize denominators and numerators if necessary

II. Cross out common factors and leave remaining factors as answer

Worked Examples

1. Find the product of $\frac{5}{y}$ and $\frac{1}{10y^2}$

Solution

$\frac{5}{y} \times \frac{1}{10y^2} = \frac{5}{10y^3} = \frac{1}{2y^3}$

2. Multiply $\frac{5}{6}$ by $\frac{2}{x^2}$

Solution

$\frac{5}{6} \times \frac{2}{x^2} = \frac{5\times 2}{6\times x^2} = \frac{10}{6x^2} = \frac{5}{3x^2}$

3. Perform $\frac{x}{y} \div \frac{y}{x}$

Solution

$\frac{x}{y} \div \frac{y}{x} = \frac{x}{y} \times \frac{x}{y} = \frac{x^2}{y^2}$

Exercises 3.6

A. Simplify the following;

1. $\frac{1}{x} + \frac{3}{x^3}$ 2. $\frac{3}{a} + \frac{5}{b}$ 3. $\frac{x}{p^2} - \frac{y}{pq}$

4. $\frac{3}{4x} + \frac{1}{3y} - \frac{2}{xy}$ 5. $\frac{3-x}{6} - \frac{x}{4}$ 6. $\frac{2}{3} \times \frac{5}{x}$

B. Simplify;

1. $\frac{3a^2b}{9a} \times \frac{5a^2b^3}{5a^2}$ 2. $\frac{3x^2 5y^3}{25x^2y} \times \frac{24y^4x^3}{6xy^2}$

3. $\frac{(-3x)^2(-5y)^3}{25(-x)^2(-y)}$ 4. $\frac{(2x)^3(-4y)^2}{8(-x)^3(-y)^4}$

C. Simplify:

1. $\frac{a^2-16}{a-4}$ 2. $\frac{x^2+5}{x^2+7x+10}$ 3. $\frac{a}{a-b} - \frac{a}{a+b}$

4. $\frac{1}{x+1} - \frac{1}{x^2+3x+2}$ 5. $\frac{1}{x} - \frac{1}{y}$ 6. $\frac{1}{b^2} - \frac{1}{b}$

7. $\frac{1}{1-a} - \frac{2}{2+a}$ 8. $\frac{1}{a-b} + \frac{1}{a-b}$ 9. $\frac{5x(x+5)}{x^2+7x+10}$

D. Simplify the expressions:

1. $\frac{ax+a}{a}$ 2. $\frac{x^2-4}{4x-8}$ 3. $\frac{x}{x^2+2x}$

4. $\frac{mn-n^2}{m^2-n^2}$ 5. $\frac{a^2-ax}{ax-a^2}$ 6. $\frac{3+3a}{6+6b}$

E. Simplify the expressions:

1. $\frac{2y-1}{2} - \frac{3y-2}{3}$ 2. $\frac{3}{x-5} - \frac{3}{x-2}$

3. $\frac{1}{x-1} - \frac{1}{x+1}$ 4. $\frac{1}{x-1} - \frac{1}{x^2-1}$

5. $\frac{x+2}{2} - \frac{2}{x+2}$ 6. $\frac{3a^2-15a+12}{3a^2-12a}$

7. $4 + \frac{1}{x} - \frac{1}{x^2}$ 8. $2 + \frac{1}{2x} + \frac{4}{5x}$

Expansion of Algebra

It is the method of removing brackets from an expression. In this process, the number or variable outside the bracket multiplies each factor or term inside the bracket. For example, the expansion of $2(a + b)$ is carried out as;

$2(a + b) = 2\times a + 2 \times b = 2a + 2b$

Worked Examples

1. Remove the brackets in $a – 2(b – 3c)$

Solution

$a – 2(b – 3c) = a – 2b + 6c$

2. Expand and simplify:

$3(6b – 9a) + 7(6a – 5b)$

Solution

$3(6b – 9a) + 7(6a – 5b)$

$= 18b – 27a + 42a – 35b$

$= 18b – 35b – 27a + 42a$

$= -17b + 15a$

3. Simplify $6(7a + 4) – 3(8a + 9)$

Solution

$6(7a + 4) – 3(8a + 9)$

$= 42a + 24 – 24a – 27$

$= 42a – 24a + 24 – 27$

$= 18a – 3$

4. Simplify $(5m + 3n) – (2m – n)$

Solution

$(5m + 3n) – (2m – n)$

$= 5m + 3n – 2m + n$

$= 5m – 2m + 3n + n$

$= 3m + 4n$

Exercises 3.9

A. Expand and simplify the following:

1. $4(2r – 3a) – 5(6r + a)$ 2. $7(2x – 5) + 15x$
3. $6(2p – 1) + 4(3 + 5p)$ 4. $2c\,(4c – 2) + 5r + 7)$

C. Expand and simplify the following;

1. $3\,(x + 2y) – 5(2x + 4y)$
2. $4(3a + 6b) – 3(2a + 4b)$
3. $5(2c + 4d) + 2(3c + 2d)$
4. $- 6(2 + 3s) + 4(3r – 5s)$

Binomial Expansion

A binomial is an algebraic expression with two terms.

Binomial expansion involves the act of removing the brackets on binomials. The process is also called ***multiplication of binomials***. For example, the expansion of $(a + 4)(a + 5)$ is the same as the product or multiplication of $(a + 4)$ and $(a + 5)$

$\Rightarrow\ (a + 4) \times (a + 5)$

The expansion is done by disintegrating the first bracket and multiplying each of its terms by the whole of the terms in the second bracket or vice-versa. That is:

$(a + 4)\ (a + 5)$

$= a(a + 5) + 4(a + 5)$

$= a^2 + 5a + 4a + 20$

$= a^2 + 9a + 20$

Worked Examples

Expand the following:

1. $(p - 3)(p - 8)$

Solution

$(p - 3)(p - 8)$

$p(p - 8) - 3(p - 8)$

$p^2 - 8p - 3p + 24$

$p^2 - 11p + 24$

2. Remove the brackets in $(p – 7)\,(p + 6)$

Solution

$(p-7)(p+6)$

$= p(p+6) - 7(p+6)$

$= p^2 + 6p - 7p - 42$

$= p^2 - p - 42$

3. Multiply $(2a + b)$ by $(a + 2b)$

Solution

$(2a + b)(a + 2b)$

$= 2a(a + 2b) + b(a + 2b)$

$= 2a^2 + 4ab + ab + 2b^2$

$= 2a^2 + 5ab + 2b^2$

4. Simplify the expression:

$(3x - y)(3x + y) - (3x + 2y)(3x - 2y)$

Solution

$(3x - y)(3x + y) - (3x + 2y)(3x - 2y)$

$= 3x(3x + y) - y(3x + y) - [3x(3x - 2y) + 2y(3x - 2y)]$

$= [9x^2 + 3xy - 3xy - y^2] - [9x^2 - 6xy + 6xy - 4y^2]$

$= 9x^2 - y^2 - [9x^2 - 4y^2]$

$= 9x^2 - y^2 - 9x^2 + 4y^2$

$= 9x^2 - 9x^2 - y^2 + 4y^2 = 3y^2$

5. Expand $(x - \sqrt{2})(x + \sqrt{2})$

Solution

$x(x + \sqrt{2}) - \sqrt{2}(x + \sqrt{2})$

$= x^2 + \sqrt{2}x - \sqrt{2}x - (\sqrt{2})^2$

$= x^2 - 2$

Exercises 3.10

A. Expand the following;

1. $(m-5)(m-9)$	4. $(p-6)(q+6)$
2. $(5a+3)(2a+2)$	5. $(3y+4)(y+3)$
3. $(3a + c)(4b + d)$	6. $(2x - y)(x + t)$

B**. Expand and simplify;**

1. $(3y - 4)(4y - 5) + (2y - 3)(y - 5)$
2. $(3m + n)(4m - n) + 2m - 3n)(m + 2n)$
3. $(a - 2b)(a + 3b) - (2a + b)(2a - b)$
4. $(6p - q)(p - 2q) - (3p - 2q)(2p - q)$
5. $(3p + 4q)(3p - 4q) - (p + q)(9p - 16q)$

Factorization of Algebraic Expressions

Factorization is the process of writing expressions in brackets. This is done by bringing out the highest common factor and/ or the common variable outside the bracket, whilst the rest of the factors are put into bracket. For example, $4x + 6$ is factorized as: $(2 \times 2x) + (2 \times 3) = 2(2x + 3)$

Similarly, $12x + 4xy$ is factorized as :

$12x + 4xy = (4 \times 3x) + (4 \times xy)$

Common factor = 4

Common variable = x

$12x + 4xy = 4x(3 + y)$

Worked Examples

1. Factorize the following:

i. $2a - 2a^2$	ii. $5ax - 15a^3x^2$
iii. $10x^3y^2 - 25x^4y$	**iv.** $2m^2 - 6mk$
v. $p^2 + pq$	*vi.* $7d + 14d^4 - 7d^3$

Solution

i. $2a - 2a^2$	$= 2a(1 - a)$
ii. $5ax - 15a^3x^2$	$= 5ax(1 - 3a^2x)$
iii. $10x^3y^2 - 25x^4y$	$= 5x^3y(2y - 5x)$
iv. $2m^2 - 6mk$	$= 2m(m - 3k)$
v. $p^2 + pq$	$= p(p + q)$
vi. $7d + 14d^4 - 7d^3$	$= 7d(1 + 2d^3 - d^2)$

2. Factorize $5xy + 10ny$

Solution

$5xy + 10ny = 5y(x + 2n)$

3. Factorize $3r^2s - 9rs^2$

Solution

$3r^2s - 9rs^2 = 3rs(r - 3s)$

4. Factorize $26x - 13y$

Solution

$26x - 13y = 13(2x - y)$

5. Factorize $a^2b - 2ab^2 - 3b^3$

Solution

$a^2b - 2ab^2 - 3b^3 = b(a^2 - 2ab - 3b^2)$

Exercises 3.11

A. Factorize the following:

1. $12yt^2 - 16y^2t$ 2. $21pq^2 - 49p^3q$ 3. $6pr + 42p^2r$
4. $16rs + 40rs^3$ 5. $27k^2m - 9m$ 6. $15cd^2 + 25c^2d$

B. Factorize the following:

1. $4x^2y - 6xy^2 + 8xyt$ 2. $3abc + 99c^2 - 15a^2c^2$
3. $12k^2m + 18kmn + km^2$ 4. $16p^2q^2 - 20pqr - 8pq$

Factorization by Grouping

Algebraic expressions having four terms are factorized by grouping. Observe the following steps when factorizing a four term algebraic expression;

I. Group the terms such that each pair can have a common factor or common variable.

II. Factorize each group separately to have the same factor in the bracket.

III. Bracket the factors outside the bracket and multiply it by one of the common factors in the brackets to complete factorization.

Worked Examples

1. Factorize $mp + np - mt - nt$

Solution

$mp + np - mt - nt$

$= (mp + np) - (mt - nt)$

$= p(m + n) - t(m + n)$

$= (p - t)(m + n)$

2. Find the factors of $ax + by + bx + ay$

Solution

$ax + by + bx + ay$

$= ax + ay + bx + by$

$= (ax + ay) + (bx + by)$

$= a(x + y) + b(x + y)$

$= (a + b)(x + y)$

3. Factorize $3mx + 2nx - 3my - 2yn$

Solution

$3mx + 2nx - 3my - 2yn$

$= (3mx + 2nx) - (3my - 2yn)$

$= x(3m + 2n) - y(3m + 2n)$

$= (3m + 2n)(x - y)$

4. Factorize $3a^2 + 2ab - 12ac - 8bc$ completely.

Solution

$3a^2 + 2ab - 12ac - 8bc$

$= (3a^2 + 2ab) - (12ac - 8bc)$

$= a(3a + 2b) - 4(3a + 2b)$

$= (a - 4c)(3a + 2b)$

Factorisation by Re- grouping

Worked Examples

1. Factorise $4wy + 3xz - 6wz - 2xy$

Solution

$4wy + 3xz - 6wz - 2xy$

$= (4wy - 6wz) - (2xy + 3xz)$ ***(re-qrouping)***

$= 2w(2y - 3z) - x(2y - 3z)$

$= (2w - x)(2y - 3z)$

2. Factorize $6 + xy - 3x - 2y$

Solution

$6 + xy - 3x - 2y$

$= (xy - 3x) - (2y + 6)$ ***(re-qrouping)***

$= x(y - 3) - 2(y - 3)$

$= (x - 2)(y - 3)$

Exercises 3.12

A. Factorize the following:

1. $2ap + aq - bq - 2bp$
2. $x^2 - ax + bx - ab$
3. $xy - 3xc + 2qy - 6qc$
4. $2pr - 4ps + qr - 2qs$
5. $pr + 3ps - 2qr - 6qs$
6. $4xy - 8y^2 - 8xd - 16yd$

B. Factorize the following:

1. $2yz + 5z - 8y - 20$
2. $2cd - 2ce + d^2 - e^2$
3. $3a + 3b + a^2 - b^2$
4. $3fd - 3fe - d^2 + e^2$

Quadratic Expressions of the form:
$\boldsymbol{ax^2 + bx + c}$

Any mathematical expression of the form: $ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$ is called a ***quadratic expression***. Thus, in $\boldsymbol{ax^2 + bx + c}$,

a is called the co-efficient of x^2,

b is called the co-efficient of x ,

c is called the constant term.

Factors of Quadratic Expressions

***1. Expressions of the Form**: $\boldsymbol{x^2 + bx + c}$*

To find the factors of expressions of the form: $x^2 + bx + c$, find all the factors of the constant term, c, such that the product of a pair of the factors equal to the constant term, c, and the sum of that same pair of factors equal the co-efficient of x . Simply put,

x^2 – (sum of roots)x + (product of roots)

$$\boldsymbol{x^2 + \underset{\substack{\downarrow \\ m + n}}{b\,x} + \underset{\substack{\downarrow \\ m \times n}}{c} \ldots\ldots \ (1)}$$

Where ***m*** and ***n*** are pair of factors of ***c,*** and

$bx = mx + nx$ …….(2)

Put eqn (2) in eqn (1)

$x^2 + mx + nx + c$

Once, four terms are obtained, divide the expression into two with a bracket leaving an operator in-between.

$\Rightarrow (x^2 + mx) + (nx + mn)$

Factorize completely to get;

$x(x + m) + n(x + m)$

$(x + n)(x + m)$

Worked Examples

1. Factorize $x^2 + 5x + 6$

Solution

In $x^2 + 5x + 6$, $a = 1$, $b = 5$ and $c = 6$

Find all the factors of the constant term, 6

6 = (1, 6), (2, 3), (-1, -6) and (-2, -3)

Select the pair of factors that sum up to the co-efficient of x, which is 5

$$x^2 + \underset{\substack{\downarrow \\ 2 + 3}}{5\,x} + \underset{\substack{\downarrow \\ 2 \times 3}}{6}$$

But $5x = 2x + 3x$

Re-write the expression to obtain four terms as, $x^2 + 2x + 3x + 6$

Divide the expression into two with a bracket leaving an operation sign in- between

$(x^2 + 2x) + (3x + 6)$

$x(x + 2) + 3(x + 2)$

$(x + 3)(x + 2)$

2. Factorize completely $x^2 + 8x + 15$

Solution

$x^2 + 8x + 15$

Factors of 15= (1, 15), (3, 5), (-1, -15) (-3, -5)
$8x = 3x + 5x$, $15 = 3 \times 5$
$x^2 + 3x + 5x + 15$
$(x^2 + 3x) + (5x + 15)$
$x(x + 3) + 5(x + 3)$
$(x + 3)(x + 5)$

2. Expressions of the form: $x^2 - bx + c$

When the coefficient of x is negative and the constant term is positive, i.e. $x^2 - bx + c$, find only the negative factors of the constant term, such that;

$c = (-m \times -n)$ and $b = (-m + -n)$ and follow the usual process. This is illustrated as follows:

$x^2 - mx - nx + (-n \times -m)$
$(x^2 - mx) - (nx + -n \times -m)$
$x(x - m) - n(x - m)$
$(x - n)(x - m)$

Worked Examples

1. Find the factors of $x^2 - 5x + 6$

Solution

$x^2 - 5x + 6$
Negative factors of 6 = (-1, -6) (-2, -3)
$-5x = -2x + -3x$, $6 = -2\times-3$
$x^2 - 2x - 3x + 6$
$(x^2 - 2x) - (3x + 6)$
$x(x - 2) - 3(x - 2)$
$(x - 2)(x - 3)$
Factors of $x^2 - 5x + 6 = (x - 2)(x - 3)$

2. Express $x^2 - 7x + 10$ as factors

Solution

$x^2 - 7x + 10$
Negative factors of 10 = (-1, -10) (-2, -5)
$-7x = -2x + -5x$, $10 = -2\times-5$
$x^2 - 2x - 5x + 10$
$(x^2 - 2x) - (5x + 10)$
$x(x - 2) - 5(x - 2)$
$(x - 2)(x - 5)$
Factors of $x^2 - 7x + 10 = (x - 2)(x - 5)$

3. Expressions of the form: $x^2 + bx - c$

When the co-efficient of x is positive and the constant term is negative,i.e.$x^2 + bx - c$, find the pair of factors of the constant term ($-c$), such that one of the pair is negative and the other one, positive

$\Rightarrow -c = (-m \times n)$, $b = (-m + n)$

The form: $x^2 + bx - c$ is factorized as:

$x^2 - mx + nx - mn$
$(x^2 - mx) + (nx - mn)$
$x(x - m) + n(x - m)$
$(x + n)(x - m)$

Worked Examples

1. Factorize $x^2 + 5x - 14$ completely

Solution

$x^2 + 5x - 14$
Factors of -14 = (-1, 14) (-2, 7),
$5x = -2x + 7x$, $-14 = -2\times7$
$x^2 - 2x + 7x - 14$
$(x^2 - 2x) + (7x - 14)$
$x(x - 2) + 7(x - 2)$
$(x - 2)(x + 7)$
Factors of $x^2 + 7x - 12 = (x - 2)(x + 7)$

2. Factorize $x^2 + 7x - 18$ completely

Solution

$x^2 + 7x - 18$

Factors of -18 = (-1, 18), (-2, 9) and (-3, 6)

$7x = -2x + 9x$, $-18 = -2\times9$
$x^2 - 2x + 9x - 18$
$= (x^2 - 2x) + (9x - 18)$
$= x(x - 2) + 9(x - 2)$
$= (x - 2)(x + 9)$

4. Expressions of the form: $x^2 - bx - c$

When the co-efficient of x is negative and the constant term is negative, i. e.$x^2 - bx - c$, find the pair of factors of the constant term ($-c$), such that one of the pair is positive and the other one, negative.

$\Rightarrow -c = (m \times -n)$, $b = (m - n)$. Of the pair, the first one is always bigger than the other.

Worked examples

Factorize the following quadratic expression

1. $x^2 - 8x - 9$ 2. $x^2 - 2x - 24$

Solutions

1. $x^2 - 8x - 9$

Factors of -9 = (1, -9) (-3, 3)

$-8x = x - 9x$, $-9 = 1\times-9$

$x^2 + x - 9x - 9$

$(x^2 + x) - (9x - 9)$

$x(x + 1) - 9(x + 1)$

$(x + 1)(x - 9)$

Factors of $x^2 - 8x - 9 = (x + 1)(x - 9)$

2. $x^2 - 2x - 24$

Factors of -24 = (4, -6)

$-2x = 4x - 6x$, $-24 = 4\times-6$

$x^2 + 4x - 6x - 24$

$(x^2 + 4) - (6x - 24)$

$x(x + 4) - 6(x + 4)$

$(x + 4)(x - 6)$

Factors of $x^2 - 2x - 24 = (x + 4)(x - 6)$

Exercises 3.13

A. Factorize the following:

1. $x^2 + 10x + 21$ 3. $x^2 + 12x + 27$
2. $x^2 + 11x + 30$ 4. $x^2 + 18x + 72$

B. Factorize completely:

1. $t^2 + 6t - 16$ 3. $m^2 + 10m - 24$
2. $m^2 + 10m - 39$ 4. $m^2 + 5m - 14$

C. Completely factorize the following:

1. $x^2 - 12x + 35$ 3. $x^2 - 15x + 44$
2. $x^2 - 19x + 48$ 4. $x^2 - 18x + 56$

D. Find the factors of the following:

1. $x^2 - 7x - 44$ 3. $x^2 - 5x - 50$
2. $x^2 - 7x - 30$ 4. $x^2 - x - 20$

5. Factors of Expressions of the Form:
$ax^2 + bx + c$, where $x \neq 1$ or $x > 1$

To factorize expressions of the form;

$ax^2 + bx + c$, where $x\neq1$ or $x>1$,

<u>Type I</u>

I. Find the product of the co-efficient of $\boldsymbol{x^2}$ and the constant term **(c) i. e. ($a\times c$)**

II. Find all the pair of factors of ***ac***

III. Find the pair of factors of ***ac***that sum up to the co-efficient of ***x***.

i.e. $ax^2 + bx + c$

$\boldsymbol{a \times c} = m \times n$, $\boldsymbol{bx} = \boldsymbol{m}x + \boldsymbol{n}x$

IV. Substitute $bx = mx + nx$in $ax^2 + bx + c$to obtain four terms of the expression.

$\Rightarrow ax^2 + mx + nx + c$

V. Factorize carefully by method of grouping to obtain the factors of the expression.

Note:

1. If b and c are positive, the pair must be negatives.

2. If b is positive and c is negative, negate the smallest factor of the pair.

3. If b and c are negative, negate the bigger factor of the pair.

4. If only b is negative, negate both pair of factors.

Worked Examples

Factorize the following completely

1. $3x^2 + 4x + 1$ 2. $5x^2 + 7x + 2$
3. $4x^2 + 4x - 3$ 4. $6x^2 - 5x - 6$

Solutions

1. $3x^2 + 4x + 1$

$ax^2 + bx + c$

$a = 3$ and $b = 1$

$\therefore a \times c = 3 \times 1 = 3$

Factors of 3 = (1, 3) and $4x = x + 3x$. Substitute in the expression to obtain four terms;

$\Rightarrow 3x^2 + x + 3x + 1$

$(3x^2 + x) + (3x + 1)$

$x(3x + 1) + 1(3x + 1)$

$(x + 1)(3x + 1)$

Factors of $3x^2 + 4x + 1 = (x + 1)(3x + 1)$

2. $5x^2 + 7x + 2$

$5 \times 2 = 10$

Factors of 10 = (2, 5) and $7x = 2x + 5x$.

$5x^2 + 2x + 5x + 2$

$(5x^2 + 2x) + (5x + 2)$

$x(5x + 2) + 1(5x + 2)$

$(x + 1)(5x + 2)$

Factors of $5x^2 + 7x + 2 = (x + 1)(5x + 2)$

3. $4x^2 + 4x - 3$

$4 \times -3 = -12$

Factors of -12 = (-2, 6) and $4x = -2x + 6x$.

$4x^2 - 2x + 6x - 3$

$(4x^2 - 2x) + (6x - 3)$

$2x(2x - 1) + 3(2x - 1)$

$(2x + 3)(2x - 1)$

Factors of $3x^2 + 4x - 3 = (2x + 3)(2x - 1)$

4. $6x^2 - 5x - 6$

$-6 \times 6 = -36$

Factors of –36 = (4, -9) and $-5x = 4x - 9x$.

$6x^2 + 4x - 9x - 6$

$(6x^2 + 4x) - (9x - 6)$

$2x(3x + 2) - 3(3x + 2)$

$(2x - 3)(3x + 2)$

Factors of $6x^2 - 5x - 6 = (3x + 2)(2x - 3)$

Exercises 3.14

A. Factorize the following;

1. $3x^2 + 11x + 6$ 3. $5x^2 - 12x + 4$
2. $6x^2 + 7x - 5$ 4. $7x^2 + 9x + 2$

B. Find the factors of the following;

1. $4x^2 - 21x + 20$ 4. $5x^2 - 17x + 6$
2. $12x^2 - 7x + 1$ 5. $2x^2 - 7x + 3$
3. $3x^2 - 20x + 12$ 6. $6x^2 + 11x - 2$

Type 2

Follow the same process as that of type 1

Worked Examples

Factorize the following:

1. $10x^2 - 9xy + 2y^2$ **2.** $5x^2 + 18xy + 9y^2$

Solution

$10x^2 - 9xy + 2y^2$

$10 \times 2 = 20$

Pair of factors of 20 that sum up to – 9

= (-5, - 4). $\Rightarrow -9xy = -5xy - 4xy$

Put $-9xy = -5xy - 4xy$ in $10x^2 - 9xy + 2y^2 \Rightarrow 10x^2 - 5xy - 4xy + 2y^2$

$(10x^2 - 5xy) - (4xy + 2y^2)$

$5x(2x - y) - 2y(2x - y)$

$(5x - 2y)(2x - y)$

2. Factorize $5x^2 + 18xy + 9y^2$

Solution

$5x^2 + 18xy + 9y^2$

$5 \times 9 = 45$

Pair of factors of 45 that sum up to 18 = (3, 15)

$\Rightarrow 18xy = 3xy + 15xy$
$5x^2 + 3xy + 15xy + 9y^2$
$(5x^2 + 3xy) + (15xy + 9y^2)$
$x(5x + 3y) + 3y(5x + 3y)$
$(x + 3y)(5x + 3y)$

Exercises 3.15
Factorize the following:

1. $2x^2 + 5xy + 3y^2$	2. $6x^2 - 8xy + 2y^2$
3. $14x^2 + 19xy - 3y^2$	4. $3x^2 - 7xy + 4y^2$
5. $5x^2 - 25xy - 30y^2$	6. $a^2 + ab - 6b^2$
7. $a^2 + 12ab - 45b^2$	8. $3r^2 + 7rs - 20s^2$
9. $a^2 + ab - 6b^2$	10. $2p^2 - 3pq - 2q^2$

Factors of Perfect Squares of the form:
$x^2 + bx + c$

In $x^2 + bx + c$, if the pair of factors of the constant term, c, is of the same kind and that same kind of factors sum up to the co-efficient of x, we say the expression is a ***perfect square***.

i. e. $x^2 + \underset{\underset{m + m}{\downarrow}}{b}x + \underset{\underset{m \times m}{\downarrow}}{c}$

Such that $c = m \times m$ and $bx = mx + mx$.

For all perfect squares,

1. $x^2 + bx + c$
$= (x + m)(x + m) = (x + m)^2$

2. $x^2 - bx + c$
$= (x - m)(x - m) = (x - m)^2$
(Where x is the root of x^2 and m is the root of c)

Worked Examples

1. Factorize $x^2 + 6x + 9$

Solution
$x^2 + 6x + 9$
Factors of 9 = (1, 9) and (3, 3)
But $6x = 3x + 3x$ and $9 = 3 \times 3$
$x^2 + 3x + 3x + 9$
$(x^2 + 3x) + (3x + 9)$
$x(x + 3) + 3(x + 3)$
$(x + 3)(x + 3) = (x + 3)^2$
$\therefore$ the factors of $x^2 + 6x + 9 = (x + 3)^2$

2. What are the factors of $x^2 - 10x + 25$?

Solution
$x^2 - 10x + 25$
Factors of 9 = (1, 25), (-5, -5), (-1,-25) (5, 5)
But $-10x = -5x - 5x$ and $25 = -5 \times -5$
$x^2 - 5x - 5x + 25$
$(x^2 - 5x) - (5x + 25)$
$x(x - 5) - 5(x - 5)$
$(x - 5)(x - 5) = (x - 5)^2$
Therefore factors of $x^2 - 10x + 25 = (x - 5)^2$

B. Factors of Perfect Squares of the form: $ax^2 + bx + c$

For perfect expressions of the form;
$ax^2 + bx + c$, where $x \neq 1$ or $x > 1$,

Method I

I. Find the product of the co-efficient of ***x^2*** and the constant term, ***c*. i. e. (a × c)**
II. Write all the pair of factors of (***ac***)
III. Find the pair of factors of the same kind of **ac,**that sum up to the co-efficient of ***x***.
i.e. $ax^2 + bx + c$
$a \times c = m \times m$, $bx = mx + mx$
IV. Substitute the pairs in place of bx to obtain four terms i.e. $bx = mx + mx$, so $ax^2 + mx + mx + c$
V. Factorize the four term expression by method of grouping to obtain the answer in the form $\left[\left(\sqrt{ax^2}\right) + \sqrt{c}\right]^2$

Method II

I. Try to find the square root of ***ax^2***and the constant term **c**

II. If possible:

a. express the answer in the form $\left[\left(\sqrt{ax^2}\right) + \sqrt{c}\right]^2$ on condition that the coefficient of x is positive

b. express the answer in the form $\left[\left(\sqrt{ax^2}\right) - \sqrt{c}\right]^2$ on condition that the coefficient of x is negative

Worked Examples

1. Factorize $4x^2 + 12x + 9$

Solution

<u>Method 1</u>

$4x^2 + 12x + 9$

$4 \times 9 = 36$

Factors of 36 = (1, 36) (2, 18), (3, 12), (6, 6)

But $12x = 6x + 6x$ and $36 = 6 \times 6$

$4x^2 + 6x + 6x + 9$

$(4x^2 + 6x) + (6x + 9)$

$2x(2x + 3) + 3(2x + 3)$

$(2x + 3)(2x + 3) = (2x + 3)^2$

<u>Method II</u>

$4x^2 + 12x + 9$

$\left(\sqrt{4x^2}\right) = 2x$ and $\left(\sqrt{9}\right) = 3$

Coefficient of x is positive

$(2x + 3)(2x + 3) = (2x + 3)^2$

2. $25x^2 – 20x + 4$

Solution

<u>Method 1</u>

$25x^2 – 20x + 4$

$25 \times 4 = 100$

Factors of 100 = (–10, –10)

But $– 20x = –10x –10x$ and $100 = –10 \times –10$

$25x^2 – 20x + 4$

$25x^2 – 10x – 10x + 4$

$(25x^2 – 10x) – (10x + 4)$

$5x(5x – 2) – 2(5x – 2)$

$(5x – 2)(5x – 2) = (5x – 2)^2$

<u>Method II</u>

$25x^2 – 20x + 4$

$\left(\sqrt{25x^2}\right) = 5x$ and $\left(\sqrt{4}\right) = 2$

Coefficient of x is negative

$(5x – 2)(5x – 2) = (5x - 2)^2$

Exercises 3.16

A. Factorize the following completely;

1. $x^2 + 10x + 25$	3. $x^2 – 32x + 256$
2. $x^2 + 40x + 400$	4. $x^2 – 26x + 169$

B. Find the factors of the following;

1. $9x^2 – 12x + 4$	2. $4x^2 – 20x + 25$
3. $4x^2 – 36x + 81$	4. $16x^2 –16x + 4$

Unknowns of a Perfect Quadratic Expression

A. Given the perfect square $ax^2 + bx + c$ to find the value of b,

I. Write the pair of common factors for $ax^2 + bx + c$

II. Simplify the common factors and compare

B. Given the perfect square $ax^2 + bx + c$ to find the value of the constant term c,

I.Use the fact that :

x^2 – *(sum of roots)x + (product of roots)*

II. Identify the sum of roots and multiply to get the product of roots.

III. Then make comparison.

Worked Examples

1. Find the integer k if $4a^2 + ka + 81$ is a perfect square

Solution

If $4a^2 + ka + 81$ is a perfect square,

$\Rightarrow 4a^2 + ka + 81 = (2a + 9)(2a + 9)$

But $(2a + 9)(2a + 9)$
$= 2a(2a + 9) + 9(2a + 9)$
$= 4a^2 + 18a + 18a + 81$

$4a^2 + 36a + 81 = 4a^2 + ka + 81$
$\Rightarrow 36a = ka$
$k = 36$

2. Find the number k given that $a^2 - 12a + k$ is a perfect square.

Solution
For all quadratic expressions:
$x^2 -$ *(sum of roots)*x + *(product of roots)*
If $a^2 - 12a + k$ is a perfect square
$\Rightarrow a^2 - 12a + k = a^2 - 6a - 6a + (6 \times 6)$
$a^2 - 12a + k = a^2 - 12a + 36$
By comparison, $k = 36$

Factors of Difference of Two Squares

Type I: $(a^2 - b^2)$

Factors of expressions of the form: $a^2 - b^2 = (a + b)(a - b)$. This process is called ***difference of two squares.*** The process is not applicable when the operator is "addition" but "subtraction"
1. $a^2 + b^2 \neq (a - b)(a + b)$
2. $a^2 + b^2 \neq (a + b)(a + b)$

Worked Examples
1. Factorize $4m^2 - n^2$

Solution
$4m^2 - n^2 = (2m)^2 - n^2$
$(2m)^2 - n^2 = (2m - n)(2m - n)$
Factors of $4m^2 - n^2 = (2m + n)(2m - n)$

2. What are the factors of $x^2 - 9y^2$?

Solution
$x^2 - 9y^2$, but $9 = 3^2$
$x^2 - 3^2y^2 = x^2 - (3y)^2$
$x^2 - (3y)^2 = (x + 3y)(x - 3y)$
$\Rightarrow$Factors of $x^2 - 9y^2 = (x + 3y)(x - 3y)$

3. Completely factorize $9p^2 - 16$

Solution
$9p^2 - 16$,
But $9 = 3^2$ and $16 = 4^2$
$3^2p^2 - 4^2 = (3p)^2 - 4^2$
$(3p)^2 - 4^2 = (3p + 4)(3p - 4)$
$\Rightarrow$ Factors of $9p^2 - 16 = (3p + 4)(3p - 4)$

4. Factorize $9 - (x + 1)^2$

Solution
$9 - (x + 1)^2$
$= 3^2 - (x + 1)^2$
$= \{(3) + (x + 1)\}\{(3) - (x + 1)\}$
$= (3 + x + 1)(3 - x - 1) = (4 + x)(2 - x)$

5. $(a + 3b)^2 - (2a - b)^2$
$= \{(a + 3b) + (2a - b)\}\{a + 3b) - (2a - b)\}$
$= (a + 3b + 2a - b)(a + 3b - 2a + b)$
$= (3a + 2b)(4b - a)$

Exercises 3.17
A**. Find the factors of the following;**
1. $81 - (5y)^2$ 2. $(8p)^2 - 9$ 3. $121 - 100m^2$
4. $16 - (3a)^2$ 5. $9y^2 - 4$ 6. $9x^2t^2 - 1$

B. Fcatorize the following:
1. $(a + b)^2 - 4r^2$
2. $(x + y)^2 - 9m^2$
3. $(n - 2p)^2 - 64q^2$
4. $(3t + 5u)^2 - (2t - 3u)^2$
6. $q^2 - (p - 2n)^2$
7. $(x + y)^2 - 4$
8. $4r^2 - (s - r)^2$

Type 2
Expressions containing one quadratic

Here, there is always a quadratic expression (usually perfect square) within the given expression

I. Identify the quadratic expression within the given expression and the remaining term.

II. Find the factors of the quadratic expression

III. Form a difference of two squares with the common factors obtained from the quadratic expression and the remaining term of the given expression.

IV. Simplify the difference of two squares

$\Rightarrow$: $\boldsymbol{a^2 - b^2 = (a + b)(a - b)}$

Note: *The expression can also contain two quadratic expressions*

Worked Examples

1. Factorize $x^2 - 6x + 9 - 4y^2$

Solution

Quadratic expression = $x^2 - 6x + 9$

Remaining term = $-4y^2$

Factors of quadratic expression

$x^2 - 6x + 9 = (x - 3)(x - 3) = (x - 3)^2$

$\Rightarrow x^2 - 6x + 9 - 4y^2$

$= (x - 3)^2 - 4y^2$

$= (x - 3)^2 - (2y)^2$

$= (x - 3 - 2y)(x - 3 + 2y)$

$= (x - 2y - 3)(x + 2y - 3)$

2. Factorize $m^2 - 2mn + n^2 - 9r^2$, completely:

Solution

$m^2 - 2mn + n^2 - 9r^2$

Quadratic expression = $m^2 - 2mn + n^2$

Remaining term = $-9r^2$

Factors of $m^2 - 2mn + n^2$

$(m^2 - mn) - (mn + n^2)$

$= m(m - n) - n(m - n)$

$= (m - n)^2$

$\Rightarrow m^2 - 2mn + n^2 - 9r^2$

$= (m - n)^2 - 9r^2$

$= (m - n)^2 - (3r)^2$

$= (m - n - 3r)(m - n + 3r)$

3. Factorise $a^2 - b^2 - 10b - 25$

Solution

$a^2 - b^2 - 10b - 25$

$= a^2 - [b^2 + 10b + 25]$

$= a^2 - [b^2 + 5b + 5b + 25]$

$= a^2 - [(b^2 + 5b) + (5b + 25)]$

$= a^2 - [b(b + 5) + 5(b + 5)]$

$= a^2 - [(b + 5)(b + 5)]$

$= a^2 - (b + 5)^2$

$= [a + (b + 5)]\ [a - (b + 5)]$

$= (a + b + 5)(a - b - 5)$

Exercises 3.18 A

Factorize the following;

1. $x^2 + 2xy + y^2 - 4$
2. $a^2 - 12a + 36 - 16b^2$
3. $4a^2 + 36a + 81 - 25b^2$
4. $4 - 20b + 25b^2 - a^2$
5. $a^2 - 4ab + 4b^2 - x^2 + 6x + 9$

Challenge Problems

Factorize the following:

1. $x^2 - 6x + 9 - 4y^2$ 2. $a^2 - b^2 + 12b - 36$

Other Applications

Finding the values of the variable(s) in a quadratic expression given the factors of the expression

I. Identify the given expression

II. Identify the factors of the given expression

III. Expand and simplify the factors to obtain an equivalent expression

IV. Compare the two equivalent expressions to obtain the values of the required variable(s)

Worked Examples

If $(x – 3)$ and $(2x + 3)$ are factors of $2x^2 + mx – n$, find the value of $(m + n)$

Solution

$(x – 3)\ (2x + 3) = 2x^2 + mx – n$

$x(2x + 3) – 3(2x + 3)$

$\Rightarrow 2x^2 + 3x – 6x – 9 = 2x^2 + mx – n$

$2x^2 – 3x – 9 = 2x^2 + mx – n$

Comparing L.H.S and R.H.S

$m = – 3$ and $n = – 9$

$(m + n) = – 3 + (-9) = -12$

Exercises 3.19

1. If $(x – 5)$ and $(x + 2)$ are factors of $x^2 + kx – 10$, find the value of k.

2. If $x^2 + mx + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2$, find the value of m

An Undefined Algebraic Fraction

An algebraic expression of the form, $\frac{a}{b}$, also called algebraic fraction, is said to be undefined (does not exist), if the denominator is equal to zero.

$\Rightarrow$; $\frac{a}{b}$, $b = 0$

To find the value(s) for which an algebraic fraction is undefined,

I. Equate the denominator of the expression to zero

II. Solve for the value of the involving variable.

III. The value(s) of the variable makes the expression undefined. This means that the variable can take all values except the value that makes the expression undefined

Worked Examples

1. Find the value of x for which $\frac{2}{1-3x}$ is undefined.

Solution

For $\frac{2x}{1-3x}$ to be undefined,

$1 – 3x = 0$ *(Equate the denominator to zero)*

$1 = 3x$, (*Solve for x)*

$x = \frac{1}{3}$.

$\therefore$ The expression is undefined when $x = \frac{1}{3}$

2. For which values of x is the expression $\frac{1-x}{(x-4)(x+2)}$ undefined.

Solution

For $\frac{1-x}{(x-4)(x+2)}$ to be undefined,

$(x – 4)(x + 2) = 0$

$x – 4 = 0$ or $x + 2 = 0$

$x = 4$ or $x = – 2$

3. Determine the values of x for which the expression $\frac{3x+5}{x^2+5x+6}$ is undefined.

Solution

For $\frac{3x+5}{x^2+5x+6}$ to be undefined,

$x^2 + 5x + 6 = 0$

$(x + 3)(x + 2) = 0$

$x + 3 = 0$ or $x + 2 = 0$

$x = -3$ or $x = - 2$.

$\therefore$ The undefined values are $x = -3$ or $x = -2$

4. For what values of y is $\frac{1}{y} + \frac{1}{y-1}$ undefined?

Solution

For $\frac{1}{y} + \frac{1}{y-1}$ to be undefined

$y(y – 1) = 0$

$y = 0$ or $y - 1 = 0$
$y = 0$ or $y = 1$

5. Find the values of x for which $\frac{2x\,(2x - 4)}{4x^2 - 16}$ is undefined.

Solution

$\frac{2x\,(2x - 4)}{4x^2 - 16} = \frac{2x\,(2x - 4)}{(2x + 4)(2x - 4)} = \frac{2x}{2x + 4}$

Now,
$2x + 4 = 0$
$2x = -4$
$x = -2$

Exercises 3.7

Find the value or values of the variable for which the expression is undefined;

1. $\frac{x - 16}{x^2 - 3x}$ 2. $\frac{x^2 + 25}{x^2 + 7x + 10}$ 3. $\frac{5x(x-5)}{(x-5)(x-2)}$
4. $\frac{3x}{2x - 4}$ 5. $\frac{2x + 5}{\sqrt{x - 4}}$ 6. $\frac{x^2 - 9}{x^2 - 3x + 2}$

A Zero Algebraic Fraction

An algebraic expression of the form, $\frac{a}{b}$ also called algebraic fraction, is zero, if the numerator is equal to zero. That is; $\frac{a}{b}$, $a = 0$

To find the value (s) for which an algebraic fraction is zero or the zero(s) of an algebraic fraction:

I. Equate the numerator of the expression to zero.
II. Solve for the value of the involving variable.
III. The value(s) of the variable makes the expression zero. This value(s) is called the zeros of the expression, meaning, if the variable takes that value, the entire expression is zero.

Worked Examples

1. Find the value of x for which $\frac{x - 2x}{6x(x-4)}$ is zero.

Solution

For $\frac{x - 2x^2}{6x(x - 4)}$ to be equal to zero;

$\frac{x\,(1 - 2x)}{6x(x-4)}$ (factotize and set the numerator to zero)

$1 - 2x = 0$ (solve for x)
$1 = 2x$
$x = \frac{1}{2}$

2. Find the value of x for which the expression $\frac{x^2 + 7x - 44}{x + 1}$ is zero.

Solution

For $\frac{x^2 + 7x - 44}{x + 1}$ to be zero,

$x^2 + 7x - 44 = 0$
$x^2 + 11x - 4x - 44 = 0$
$(x^2 + 11x) - (4x - 44) = 0$
$x\,(x + 11) - 4(x + 11) = 0$
$(x - 4)\,(x + 11) = 0$
$x - 4 = 0$ or $x + 11 = 0$
$x = 4$ or $x = -11$
$\therefore$ The expression is zero when $x = 4$ or $x = -11$

Exercises 3.8

Find the zeros of the following:

1. $\frac{x^2 - 9}{x^2 - 3x}$ 2. $\frac{2x^2 - 32}{x^2 + 7x + 10}$ 3. $\frac{-5x(x+5)}{(x-5)(x-2)}$
4. $\frac{3(x-1)}{2(x-4)}$ 5. $\frac{2x + 5}{(x+1)(x-4)}$ 6. $\frac{x^2 - 7x + 12}{3x + 2}$

Algebraic Substitution

It is the act of replacing variables with their specific values in a given expression. The process is also called ***evaluation of expressions.***

To evaluate algebraic expressions, substitute each variable with its value/number and perform the operations included.

Worked Examples

1. Evaluate $6z + 4x$, if $x = 3$ and $z = 2$

Solution

Replace x with 3 and z with 2 to evaluate the expression as shown below;

$6z + 4x = 6(2) + 4(3) = 12 + 12 = 24$

2. Evaluate the expression $4x + (7 – z) – 6y$ for $x = 2$, $z = 4$, $y = 5$;

Solution

Substitute $x = 2$, $z = 4$, $y = 5$ in

$4x + (7 – z) – 6y$

$= 4(2) + (7 – 4) – 6(5) = 8 + 3 – 30 = -19$

3.If $p = -2$ and $q = 3$, evaluate $\frac{p-2q}{p+2q} - \frac{q-p}{q+p}$

Solution

$\frac{p-2q}{p+2q} - \frac{q-p}{q+p}$

$= \frac{-2-2(3)}{-2+2(3)} - \frac{3-(-2)}{3+(-2)} = \frac{-8}{4} - \frac{5}{1} = \frac{-8-20}{4} = \frac{-28}{4} = -7$

4. Without using calculator, evaluate $\frac{2x-y}{z+2} - \frac{z+2y}{x}$, where $x = 2$, $y = -3$ and z = 4

Solution

$\frac{2x-y}{z+2} - \frac{z+2y}{x}$

$= \frac{2(2)-(-3)}{4+2} - \frac{4+2(-3)}{2} = \frac{7}{6} - \frac{-2}{2} = \frac{7-(-6)}{6} = \frac{13}{6}$

Exercises 3.20

A. 1. Evaluate $2xy – 8x + 3y – 12$, when $x = 5$ and $y = 7$.

2. If $m = 3$ and $n = -3$, evaluate $\frac{1}{2}(3m – n)$.

3. Given that $a = 2$ and $b = 3$, evaluate;

$(2a + b)(a – 2b)$

4. Factorize $6a^2 – ab – b^2$, and hence find the value of the expression when $a = 4\frac{1}{2}$ and $b = 6\frac{1}{2}$

5. If $x^2 + y^2 = 73$ and $xy = 12$, find the value of $(x + y)^2$

B. If $x = –1$, $y = –2$, $z = 3$, $t = 0$ evaluate:

1. $x^2 + y^2 + z^2$
3. $x^2 – 2xy + 5xyz$
2. zy^2x
4. $\frac{(xyt)}{z}$

4 SURDS

Meaning of Surds

On a calculator $\sqrt{2} \approx 1.414213562\ldots$, a decimal that does not terminate nor recur, hence cannot be expressed in the form $\frac{a}{b}$ (as a fraction of two integers). Such a root is called a ***surd***. A surd is therefore, the roots of rational numbers that cannot be expressed as rational numbers. Examples of surds are: $\sqrt{3}$, $\sqrt{5}$, $\sqrt[3]{0.35}$, $\sqrt[3]{6}$, $\sqrt[3]{21}$

In general, a number which cannot be expressed as fraction of two integers is called an ***irrational number.***

Most irrational numbers are not surds as it is in π.

Note that the presence of a root sign does not necessarily means that we are dealing with surds. Thus, $\sqrt{49}$ and $\sqrt[3]{1.728}$ are not surds because $7^2 = 49$ and $1.2^3 = 1.728$, and so $\sqrt{49} = 7$ and $\sqrt[3]{1.728} = 1.2$. That is $\sqrt{49}$, and $\sqrt[3]{1.728}$ are rational numbers .

Exercises 4.1

Which of the following are surds?

1. $\sqrt[3]{1}$ 2. $\sqrt{12}$ 3. $\sqrt{0.4}$ 4. $\sqrt{0.04}$
5. $\sqrt{400}$ 6. $\sqrt[3]{10}$ 7. $\sqrt{121}$ 8. $\sqrt[3]{54}$

Properties of Surds

1. $(\sqrt{a})^2 = \sqrt{a} \times \sqrt{a} = a$
   e.g. $\sqrt{7} \times \sqrt{7} = (\sqrt{7})^2 = 7$
2. $(\sqrt{ab}) = \sqrt{a} \times \sqrt{b}$
   e.g. $(\sqrt{5 \times 3}) = \sqrt{5} \times \sqrt{3} = \sqrt{15}$
3. $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ e.g. $\frac{\sqrt{10}}{\sqrt{5}} = \sqrt{\frac{10}{5}} = \sqrt{2}$
4. $a\sqrt{b} = a \times \sqrt{b} = \sqrt{b} \times a$
   e.g. $3 \times \sqrt{2} = \sqrt{2} \times 3 = 3\sqrt{2}$
5. $a\sqrt{b} \times c\sqrt{d} = a \times c \times \sqrt{b} \times \sqrt{d} = ac\sqrt{bd}$
   e.g. $2\sqrt{3} \times 4\sqrt{7} = 2 \times 4\sqrt{3} \times \sqrt{7} = 8\sqrt{21}$

Simplifying Surds

When a surd is expressed as product of a rational number and a surd, that is in the form $a\sqrt{b}$ or $\sqrt{b}$, where b is a prime number, the surd is said to be in its simplest form. E.g. $2\sqrt{3}$, $\sqrt{7}$. This means that $\sqrt{18}$ and $\sqrt{20}$ are not in their simplest form.

Rules

I. Break down the number in the root sign into products of two factors, such that one of the factors will be a perfect square (4, 9, 16, 25, 49…) and the other, a non – perfect square. e.g. $\sqrt{18} = \sqrt{9 \times 2}$

II. Distribute the root sign for each factor.
e.g. $18 = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2}$

III. Simplify the perfect square and leave the surd.
e. g.$18 = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3 \times \sqrt{2} = 3\sqrt{2}$

Worked Examples

1. Simplify $\sqrt{20}$

Solution

1. $\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$

2. Simplify $\sqrt{128}$

Solution

$\sqrt{128} = \sqrt{64} \times \sqrt{2} = 8\sqrt{2}$

3. Simplify $\sqrt{2940}$

Solution

$\sqrt{2940} = \sqrt{4} \times \sqrt{49} \times \sqrt{15}$
$= 2 \times 7 \times \sqrt{15} = 14\sqrt{15}$

2. Simplify $5\sqrt{8}$

Solution

$5\sqrt{8} = 5 \times \sqrt{4 \times 2} = 5 \times 2\sqrt{2} = 10\sqrt{2}$

Exercises 4.2

A. Express each in its simplest form;

1. $\sqrt{200}$ 2. $\sqrt{300}$ 3. $\sqrt{484}$
4. $\sqrt{272}$ 5. $\sqrt{819}$ 6. $\sqrt{725}$

Surds of the Form a$\sqrt{b}$ as a Single Square Roots, $\sqrt{c}$

Given a surd of the form **a**$\sqrt{b}$

I. Find the square of the coefficient of the square root. i.e. a^2

II. Put the square of the coefficient of the square root in a square root and multiply by the number in the square root. i.e. $\sqrt{a^2} \times \sqrt{b}$

III. Express the product as a single square root. $\sqrt{a^2} \times \sqrt{b} = \sqrt{c}$

Worked Examples

1. Express 6 $\sqrt{5}$ as a simple square root

Solution

$6\sqrt{5} = \sqrt{36} \times \sqrt{5} = \sqrt{36 \times 5} = \sqrt{180}$

Exercises 4.3

A. Express the following as square roots

1. $\frac{\sqrt{3}}{3}$ 2. $\frac{\sqrt{2}}{2\sqrt{3}}$ 3. $4\sqrt{5}$ 4. $\frac{2}{\sqrt{6}}$ 5. $\frac{\sqrt{2}}{2}$

B. Square the following;

1. $\frac{1}{2}\sqrt{2}$ 2. $(\sqrt{3} \times \sqrt{7})$ 3. $\frac{3\sqrt{a}}{\sqrt{(2b)}}$ 4. $\frac{1}{2\sqrt{p}}$

Addition and Subtraction of Surds

Two or more surds with a common factor or with the same number in the root signs are called ***like surds***. E.g. $5\sqrt{2}$ and $4\sqrt{2}$. Like surds can be added and subtracted.

On the other hand, two or more surds without a common factor or without the same number in the root sign are called ***unlike surds***. E.g. $5\sqrt{3}$ and $4\sqrt{2}$. Unlike surds cannot be added nor subtracted.

Worked Examples

1. Simplify $5\sqrt{3} - 7\sqrt{3} + 4\sqrt{3}$

Solution

$5\sqrt{3} - 7\sqrt{3} + 4\sqrt{3} = (5 - 7 + 4)\sqrt{3} = 2\sqrt{3}$

2. Simplify $\sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8}$

Solution

$\sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8}$

$= \sqrt{25} \times \sqrt{2} + \sqrt{2} - 2 \times \sqrt{9} \times \sqrt{2} + \sqrt{4} \times \sqrt{2}$

$= 5 \times \sqrt{2} + \sqrt{2} - 2 \times 3 \times \sqrt{2} + 2 \times \sqrt{2}$

$= 8\sqrt{2} - 6\sqrt{2}$

$= 2\sqrt{2}$

3. Express $3\sqrt{75} - \sqrt{12}$ in the form a$\sqrt{b}$

Solution

$3\sqrt{75} - \sqrt{12}$

$= 3\sqrt{25 \times 3} - \sqrt{4 \times 3}$

$= 3 \times \sqrt{25} \times \sqrt{3} - \sqrt{4} \times \sqrt{3}$

$= 3 \times 5\sqrt{3} - 2\sqrt{3}$

$= 15\sqrt{3} - 2\sqrt{3} = (15 - 2)\sqrt{3} = 13\sqrt{3}$

4. Given that $\sqrt{45} + \sqrt{20} = m\sqrt{5}$, find the value of m

Solution

$\sqrt{45} + \sqrt{20} = m\sqrt{5}$

$= \sqrt{9 \times 5} + \sqrt{4 \times 5}$

$= \sqrt{9} \times \sqrt{5} + \sqrt{4} \times \sqrt{5}$

$= 3\sqrt{5} + 2\sqrt{5}$

$5\sqrt{5} = m\sqrt{5}$
Therefore $m = 5$

Exercises 4.4

A. Simplify each of the following;

1. $9\sqrt{26} - 5\sqrt{26}$
2. $\sqrt{5} + 2\sqrt{5} - 3\sqrt{5}$
3. $12\sqrt{21} - 7\sqrt{21}$
4. $\sqrt{27} - \sqrt{3}$

B. Simplify:

1. $\sqrt{8} + \sqrt{18} - 2\sqrt{2}$
2. $\sqrt{75} + 2\sqrt{12} - \sqrt{27}$
3. $\sqrt{28} + \sqrt{175} - \sqrt{63}$6.
4. $\sqrt{80} + 6\sqrt{45}$
5. $\sqrt{1000} - \sqrt{40} - \sqrt{90}$
6. $\sqrt{512} + \sqrt{128} + \sqrt{32}$
7. $\sqrt{24} - 3\sqrt{6} - \sqrt{216} + \sqrt{294}$

Multiplication of Surds

A. Multiplying a Fraction by a Surd

Simplify the surd and cross out common factors if possible.

Worked Examples

1. Simplify $\frac{1}{5}\sqrt{60}$

Solution

$\frac{1}{5}\sqrt{60}$
$= \frac{1}{5} \times \sqrt{4 \times 15} = \frac{1}{5} \times \sqrt{4} \times \sqrt{15}$
$= \frac{1}{5} \times 2 \times \sqrt{15} = \mathbf{\frac{2\sqrt{15}}{5}}$

2. Simplify $\frac{5}{2}\sqrt{40}$

Solution

$\frac{5}{2}\sqrt{40}$
$= \frac{5}{2} \times \sqrt{4 \times 10}$
$= \frac{5}{2} \times \sqrt{4} \times \sqrt{10} = \frac{5}{2} \times 2 \times \sqrt{10} = 5\sqrt{10}$

3. Simplify $\frac{3}{4}\sqrt{36}$

Solution

$\frac{3}{4}\sqrt{36} = \frac{3}{4} \times \sqrt{4} \times \sqrt{9} = \frac{3}{4} \times 2 \times 3 = \frac{9}{2}$

Exercises 4.5

Simplify the following;

1. $\frac{1}{4}\sqrt{54}$
2. $\frac{1}{7}\sqrt{196}$
3. $\frac{4}{3}\sqrt{48}$
4. $\frac{\sqrt{88}}{4}$
5. $\frac{\sqrt{100}}{2}$
6. $\frac{\sqrt{24}}{6}$

B. Multiplying a surd by a Surd

To multiply two given surds:

<u>Method</u>

I. Break each surd into a product of two factors such that one will be a perfect square and the other, a non perfect square, if possible
II. Simplify by grouping like terms.

<u>Method 2</u>

I. Find the product of the given surds to obtain a single surd.
II. Break the single surd obtained into a product of two factors such that one will be a perfect square and the other, a non-perfect square, if possible.
III. Simplify by grouping like terms.

Worked Examples

1. Simplify $\sqrt{12} \times \sqrt{8}$

Solution

<u>Method 1</u>

$\sqrt{12} \times \sqrt{8}$
$= \sqrt{96} = \sqrt{16 \times 6} = \sqrt{16} \times \sqrt{6} = 4\sqrt{6}$

<u>Method 2</u>

$\sqrt{12} \times \sqrt{8} = \sqrt{4 \times 3} \times \sqrt{4 \times 2}$
$= \sqrt{4} \times \sqrt{3} \times \sqrt{4} \times \sqrt{2}$
$= 2 \times \sqrt{3} \times 2 \times \sqrt{2}$
$= 2 \times 2 \times \sqrt{3} \times \sqrt{2} = 4\sqrt{6}$

2. Simplify $\sqrt{48} \times \sqrt{80}$

Solution

$\sqrt{48} \times \sqrt{80} = \sqrt{16 \times 3} \times \sqrt{16 \times 5}$
$= \sqrt{16} \times \sqrt{3} \times \sqrt{16} \times \sqrt{5}$
$= 4\sqrt{3} \times 4\sqrt{5}$
$= 4 \times 4 \times \sqrt{3} \times \sqrt{5}$
$= 16\sqrt{15}$

3. Simplify $\sqrt{54} \times \sqrt{125}$

Solution

$\sqrt{54} \times \sqrt{125} = \sqrt{9 \times 6} \times \sqrt{25 \times 5}$
$= \sqrt{9} \times \sqrt{6} \times \sqrt{25} \times \sqrt{5}$
$= 3\sqrt{6} \times 5\sqrt{5}$
$= 3 \times 5 \times \sqrt{6} \times \sqrt{5}$
$= 15\sqrt{30}$

Exercises 4.6
Simplify the following:

1. $3\sqrt{3} \times \sqrt{75}$ 2. $\sqrt{3} \times \sqrt{12}$ 3. $3\sqrt{2} \times 2\sqrt{3}$
4. $5\sqrt{2} \times 4\sqrt{2}$ 5. $\sqrt{10} \times \sqrt{20}$ 6. $5\sqrt{5} \times \sqrt{20}$

C. Expansion of Surds
Reminders

1. $a^2 - b^2 = (a + b)(a - b)$
2. $(a - b)^2 = a^2 - 2ab + b^2$
3. $(a + b)^2 = a^2 + 2ab + b^2$
4. $(a + b)(c + d) = a(c + d) + b(c + d)$
5. $a(b + c) = ab + ac$

Type 1: Expansion of the form
a(b + c) = ab + ac

Worked Examples

1. Simplify $\sqrt{2}\left(\sqrt{8} - \frac{2}{\sqrt{8}}\right)$

Solution

$\sqrt{2}\left(\sqrt{8} - \frac{2}{\sqrt{8}}\right)$
$= \sqrt{16} - \frac{2\sqrt{2}}{\sqrt{4 \times 2}} = \sqrt{16} - 1 = 4 - 1 = 3$

2. Simplify $\sqrt{5}\left(\sqrt{5} + \frac{3}{2\sqrt{5}}\right)$

Solution

$\sqrt{5}\left(\sqrt{5} + \frac{3}{2\sqrt{5}}\right)$
$= \sqrt{25} + \sqrt{5} \times \frac{3}{2\sqrt{5}} = 5 + \frac{3}{2} = 5\frac{3}{2} = \frac{13}{2}$

3. Simplify $\sqrt{3}\left(\sqrt{12} + \frac{3}{\sqrt{12}}\right)$

Solution

$\sqrt{3}\left(\sqrt{12} + \frac{3}{\sqrt{12}}\right)$
$= \sqrt{36} + \sqrt{3} \times \frac{3}{\sqrt{12}}$
$= \sqrt{36} + \sqrt{3} \times \frac{3}{\sqrt{4} \times \sqrt{3}}$
$= 6 + \frac{3}{2} = 6\frac{3}{2} = \frac{15}{2}$

4. Evaluate $\sqrt{7}\left(3\sqrt{7} + \frac{6}{\sqrt{7}}\right)$

Solution

$\sqrt{7}\left(3\sqrt{7} + \frac{6}{\sqrt{7}}\right)$
$= 3 \times 7 + \sqrt{7} \times \frac{6}{\sqrt{7}} = 21 + 6 = 27$

5. Express $\sqrt{15}\left(\sqrt{27} - \frac{2}{\sqrt{3}}\right)$ in the form $p\sqrt{q}$, where p and q are real numbers.

Solution

$\sqrt{15}\left(\sqrt{27} - \frac{2}{\sqrt{3}}\right)$
$= \sqrt{15}\left(\sqrt{9} \times \sqrt{3} - \frac{2}{\sqrt{3}}\right)$
$= \sqrt{15}\left(3\sqrt{3} - \frac{2}{\sqrt{3}}\right)$
$= 3\sqrt{45} - \sqrt{15} \times \frac{2}{\sqrt{3}}$

$= 3 \times \sqrt{9} \times \sqrt{5} - 2\sqrt{5}$
$= 3 \times 3 \times \sqrt{5} - 2\sqrt{5}$
$= 9\sqrt{5} - 2\sqrt{5}$
$= 7\sqrt{5}$

Exercises 4.7
A. Simplify the following;
1. $4\sqrt{3}(7\sqrt{3} - 5\sqrt{6})$ 2. $3\sqrt{2}(4\sqrt{2} - 6\sqrt{3})$
3. $5\sqrt{13}(2\sqrt{26} - \sqrt{39})$ 4. $3\sqrt{2}(4\sqrt{2} - 6\sqrt{3})$

B. Simplify:
1. $\sqrt{2}\left(3\sqrt{2} + \frac{5}{\sqrt{2}}\right)$ 2. $\sqrt{3}\left(5\sqrt{3} + \frac{8}{\sqrt{3}}\right)$
3. $\sqrt{10}\left(3\sqrt{5} - \frac{9}{\sqrt{2}}\right)$ 4. $\sqrt{3}\left(5\sqrt{3} + \frac{8}{\sqrt{3}}\right)$
5. $3\sqrt{21}\left(8\sqrt{7} + \frac{7}{\sqrt{3}}\right)$ 6. $\sqrt{3}\left(5\sqrt{3} + \frac{8}{\sqrt{3}}\right)$

Type 2: Expansion of the form
(a + b) (c + d) = a(c + d) + b (c + d)

Worked Examples
1. Simplify $(\sqrt{6} - \sqrt{2})(\sqrt{6} + 3\sqrt{2})$

Solution
$(\sqrt{6} - \sqrt{2})(\sqrt{6} + 3\sqrt{2})$
$= \sqrt{6}(\sqrt{6} + 3\sqrt{2}) - \sqrt{2}(\sqrt{6} + 3\sqrt{2})$
$= 6 + 3\sqrt{12} - \sqrt{12} - 6$
$= 3\sqrt{12} - \sqrt{12}$
$= 2\sqrt{12} = 2 \times \sqrt{4} \times \sqrt{3} = 2 \times 2 \times \sqrt{3} = 4\sqrt{3}$

2. Simplify $(1 - \sqrt{3})\left(\frac{1}{3} + \sqrt{3}\right)$, giving your answer in the form P + Q$\sqrt{3}$

Solution
$(1 - \sqrt{3})\left(\frac{1}{3} + \sqrt{3}\right)$,
$= 1\left(\frac{1}{3} + \sqrt{3}\right) - \sqrt{3}\left(\frac{1}{3} + \sqrt{3}\right)$
$= \frac{1}{3} + \sqrt{3} - \frac{\sqrt{3}}{3} - 3$
$= \frac{1}{3} - 3 + \sqrt{3} - \frac{\sqrt{3}}{3}$
$= \frac{1-9}{3} + \frac{3\sqrt{3} - \sqrt{3}}{3}$
$= -\frac{8}{3} + \frac{2\sqrt{3}}{3}$
P = $-\frac{8}{3}$ and Q = $\frac{2}{3}$ (Not necessary)

3. Simplify $(6 + 2\sqrt{5})\left(3 - \frac{2}{\sqrt{5}}\right)$

Solution
$(6 + 2\sqrt{5})\left(3 - \frac{2}{\sqrt{5}}\right)$
$= 6\left(3 - \frac{2}{\sqrt{5}}\right) + 2\sqrt{5}\left(3 - \frac{2}{\sqrt{5}}\right)$
$= 18 - \frac{12}{\sqrt{5}} + 6\sqrt{5} - \frac{4\sqrt{5}}{\sqrt{5}}$
$= 18 - 4 - \frac{12}{\sqrt{5}} + 6\sqrt{5}$
$= 14 - \frac{12}{\sqrt{5}} + 6\sqrt{5}$
$= 14 - \frac{12}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} + 6\sqrt{5}$
$= 14 - \frac{12\sqrt{5}}{5} + 6\sqrt{5}$

4. Simplify $(3\sqrt{5} - 5\sqrt{7})(12\sqrt{15} + 9\sqrt{14})$

Solution
$= (3\sqrt{5} - 5\sqrt{7})(12\sqrt{15} + 9\sqrt{14})$
$= 3\sqrt{5}(12\sqrt{15} + 9\sqrt{14}) - 5\sqrt{7}(12\sqrt{15} + 9\sqrt{14})$
$= 36\sqrt{75} + 27\sqrt{70} - 60\sqrt{105} - 45\sqrt{98}$
$=36\sqrt{25 \times 3} + 27\sqrt{70} - 60\sqrt{105} - 45\sqrt{2 \times 49}$
$= 36\times5\sqrt{3} + 27\sqrt{70} - 60\sqrt{105} - 45\times7\sqrt{2}$
$= 180\sqrt{3} + 27\sqrt{70} - 60\sqrt{105} - 315\sqrt{2}$

Type 3: Expansion of the form:
$\boldsymbol{(a - b)^2 = a^2 - 2ab + b^2}$

Worked Examples
1. Simplify; $(5\sqrt{7} - 2\sqrt{5})^2$
Solution

$(5\sqrt{7} - 2\sqrt{5})^2$
$= (5\sqrt{7})^2 - 2(5\sqrt{7})(2\sqrt{5}) + (2\sqrt{5})^2$
$= 25\sqrt{49} - 20\sqrt{35} + 4\sqrt{25}$
$= 25(7) - 20\sqrt{35} + 4(5)$
$= 175 + 20 - 20\sqrt{35} = 195 - 20\sqrt{35}$

2. Express $(3\sqrt{2} - 2\sqrt{3})^2$ in the form $a + b\sqrt{6}$, where a and b are integers.

Solution
$(3\sqrt{2} - 2\sqrt{3})^2$
$= (3\sqrt{2})^2 - 2(3\sqrt{2})(2\sqrt{3}) + (2\sqrt{3})^2$
$= 9\sqrt{4} - 12\sqrt{6} + 4\sqrt{9}$
$= 9 \times 2 - 12\sqrt{6} + 4 \times 3$
$= 18 - 12\sqrt{6} + 12$
$= 30 - 12\sqrt{6}$

Exercises 4.8
Simplify the following:
1. $(\sqrt{5} - \sqrt{3})^2$ 2. $(\sqrt{6} - \sqrt{3})^2$
3. $(4\sqrt{2} - 3\sqrt{3})^2$ 4. $6\sqrt{4} - 2\sqrt{5})^2$

Type 4: Expansion of the form
$(a + b)^2 = a^2 + 2ab + b^2$

Worked Examples
Simplify $(\sqrt{5} + \sqrt{2})^2$

Solution
$(\sqrt{5} + \sqrt{2})^2$
$= (\sqrt{5})^2 + 2(\sqrt{5})(\sqrt{2}) + (\sqrt{2})^2$
$= 5 + 2(\sqrt{10}) + 2$
$= 7 + 2\sqrt{10}$

Exercises 4.9
Simplify the following:
1. $(\sqrt{2} + 1)^2$ 2. $(\sqrt{3} + \sqrt{2})^2$ 3. $(3\sqrt{2} + 2\sqrt{2})^2$

Type 5: Expansions of the form:
$a^2 - b^2 = (a + b)(a - b)$

Worked Examples
Find the product of $(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2})$

Solution
$(a + b)(a - b) = a^2 - b^2$
$(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2})$
$= \sqrt{5}^2 - \sqrt{2}^2 = 5 - 2 = 3$

Exercises
A. Simplify:
1. $(\sqrt{5} + 2)(\sqrt{5} - 2)$
2. $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})$
3. $(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})$
4. $(\sqrt{2} + \sqrt{5})(\sqrt{2} - \sqrt{5})$

B.1. Given that $x = 1 + \sqrt{2}$ and $y = 1 - \sqrt{2}$ simplify; i. $5x + 5y$ ii. $2xy$ iii. $x^2 + y^2$
2. Given that $p = \sqrt{5} + \sqrt{3}$ and $q = \sqrt{5} - \sqrt{3}$, simplify; i. $2p - 2q$ ii. $4pq$ iii. $p^2 - q^2$
3. Evaluate $p^2 - 2p$, if
i. $p = 2\sqrt{2}$ ii. $p = \sqrt{5} + 1$
4. Given $q = 3 + \sqrt{2}$ and $\bar{q} = 3 - \sqrt{2}$, find $q + \bar{q}$ and $q\bar{q}$

Some Solved Past Questions
1. $\sqrt{50{,}000} - \sqrt{125} + 5\sqrt{5}(\sqrt{5} - 5)$, leaving your answer in the from $a + b\sqrt{5}$,

Solution
$= \sqrt{10{,}000 \times 5} - \sqrt{25 \times 5} + 5 \times 5 - 25\sqrt{5}$
$= 100\sqrt{5} - 5\sqrt{5} + 25 - 25\sqrt{5}$
$= 100\sqrt{5} - 5\sqrt{5} + 25 - 25\sqrt{5}$
$= 100\sqrt{5} - 5\sqrt{5} - 25\sqrt{5} + 25$
$= 70\sqrt{5} + 25 = 25 + 70\sqrt{5}$

2. Simplify $\frac{3}{2}\sqrt{128} - \sqrt{50}$, leaving your answer in surd form.

Solution

$\frac{3}{2}\sqrt{128} - \sqrt{50}$

$= \frac{3}{2} \times \sqrt{64} \times \sqrt{2} - \sqrt{25} \times \sqrt{2}$

$= \frac{3}{2} \times 8\sqrt{2} - 5\sqrt{2}$

$= 3 \times 4\sqrt{2} - 5\sqrt{2}$

$= 12\sqrt{2} - 5\sqrt{2}$

$= 7\sqrt{2}$

3. Without using tables or calculator, simplify $\sqrt{50} - 2\sqrt{2}(2\sqrt{2} - 5) - 5\sqrt{2}$

Solution

$\sqrt{50} - 2\sqrt{2}(2\sqrt{2} - 5) - 5\sqrt{2}$

$= \sqrt{25} \times \sqrt{2} - 4 \times 2 + 10\sqrt{2} - 5\sqrt{2}$

$= 5\sqrt{2} - 8 + 10\sqrt{2} - 5\sqrt{2}$

$= 10\sqrt{2} - 8$

Substitution

It is the act of replacing the given value of a surd in an equation. For instance, given the value of $\sqrt{a}$

I. Break the surd (mother surd) into a product of two surds such that one of them will be $\sqrt{a}$

II. Substitute the value of the given surd and simplify to complete the work.

Worked Examples

1. Simplify $\sqrt{5.12}$, to five significant figures, given that $\sqrt{2} = 1.4142$

Solution

$\sqrt{5.12} = \sqrt{512 \times 10^{-2}}$

$= \sqrt{512} \times \sqrt{10^{-2}} = \sqrt{64} \times \sqrt{8} \times \sqrt{\frac{1}{100}}$

$= \sqrt{64} \times \sqrt{4} \times \sqrt{2} \times \frac{\sqrt{1}}{\sqrt{100}}$

$= 8 \times 2 \times \sqrt{2} \times \frac{1}{10}$

$= 16 \times \frac{1}{10} \times \sqrt{2}$

$= \frac{8}{5}\sqrt{2} = \frac{8}{5}(1.4142) = 2.2627$

2. Without using tables or calculator, evaluate $3\sqrt{7}(7 - 2\sqrt{7})$, if $\sqrt{7} = 2.646$

Solution

$3\sqrt{7}(7 - 2\sqrt{7})$

$= 21\sqrt{7} - 6 \times 7$

$= 21\sqrt{7} - 42$

$= 21(2.646) - 42$ **(Put $\sqrt{7} = 2.646$)**

$= 13.566$

3. Without using tables or calculator, evaluate $2\sqrt{5}(6 - 2\sqrt{5})$, if $\sqrt{5} = 2.236$

Solution

$2\sqrt{5}(6 - 2\sqrt{5})$,

$= 12\sqrt{5} - 4 \times 5$

$= 12\sqrt{5} - 20$

$= 12(2.236) - 20 = 6.832$

4. Without using tables or calculator, evaluate $3\sqrt{7}(7 - 2\sqrt{7})$, if $\sqrt{7} = 2.646$

Solution

$3\sqrt{7}(7 - 2\sqrt{7})$

$21\sqrt{7} - 6 \times 7$

$21\sqrt{7} - 42$

$\Rightarrow 21(2.646) - 42 = 13.566$

5. Without using tables or calculators evaluate $2\sqrt{5}(6 - 2\sqrt{5})$, if $\sqrt{5} = 2.236$

Solution

$2\sqrt{5}\,(6 - 2\sqrt{5})$, if $\sqrt{5} = 2.236$

$= 12\sqrt{5} - 4 \times 5$

$= 12\sqrt{5} - 20$

$\Rightarrow 12(2.236) - 20$ **(Put $\sqrt{5}$ = 2.236)**

$= 6.832$

Exercises 4.11

Given that $\sqrt{2}$ = 1.414 and $\sqrt{3}$ = 1.732, evaluate to three significant figures;

1. $\sqrt{648}$
2. $\sqrt{0.0675}$

Rationalizing the Denominators of Surds

Rationalization is the process of removing irrational numbers or surds from the denominator of a fractional surd.

*A. **Expressions of the form:** $\frac{a}{\sqrt{b}}$*

To rationalize the denominators of expressions of the form: $\frac{a}{\sqrt{b}}$, multiply the denominator of the fraction by the numerator and denominator of the fraction. ***Reminder:*** $\sqrt{a} \times \sqrt{a} = a$

Worked Examples

1. Simplify $\frac{6}{\sqrt{2}}$ given that $\sqrt{2} = 1.4142$

Solution

$\frac{6}{\sqrt{2}} = \frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$

But $\sqrt{2} = 1.4142$,

$3\sqrt{2} = 3 \times 1.4142 = 4.243$ (4 s. f)

2. Simplify $\sqrt{\frac{3}{2}}$

Solution

$\sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{2}$

3. Simplify $\frac{3\sqrt{2}}{5\sqrt{3}}$

Solution

$\frac{3\sqrt{2}}{5\sqrt{3}} = \frac{3\sqrt{2}}{5\sqrt{3}} \times \frac{5\sqrt{3}}{5\sqrt{3}} = \frac{3 \times 5 \times \sqrt{2} \times \sqrt{3}}{5 \times 5 \times 3} = \frac{15\sqrt{6}}{75} = \frac{\sqrt{6}}{5}$

3. Given that $\sqrt{2} = 1.4142$, find correct to 5 significant figures, the value of $3 - \frac{3}{\sqrt{2}}$

Solution

$3 - \frac{3}{\sqrt{2}} = 3 - \frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = 3 - \frac{3\sqrt{2}}{2}$

$3 - \frac{3\sqrt{2}}{2} = 3 - \frac{3 \times 1.4142}{2}$ **(Put $\sqrt{2}$ = 1.414214)**

$= 3 - 2.1213 = 0.87870$ (5. s.f)

4. Given that $\tan 22.5^0 = \frac{1}{\sqrt{2}+1}$ and that $\sqrt{2} = 1.4142$, calculate $\tan 22.5^0$ to four significant figures

Solution

$\tan 22.5^0 = \frac{1}{\sqrt{2}+1}$

But $\frac{1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{\sqrt{2}-1}{2-1} = \sqrt{2} - 1$

$\Rightarrow \tan 22.5^0 = \sqrt{2} - 1$, **(Put $\sqrt{2}$ = 1.4142)**

$\tan 22.5^0 = 1.4142 - 1 = 0.4142$ (4 s.f)

Some Solved Past Questions

1. Given that $\sqrt{5} = 2.236068$, find the value of $\frac{1}{\sqrt{5}}$ to five significant figures.

Solution

$\frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

$\frac{\sqrt{5}}{5} = \frac{2.236068}{5} = 0.44721$(5 s. f.)

2. Given that $\sqrt{2}$ = 1.414214, correct to five significant figures the value of $\left(3 - \frac{1}{\sqrt{2}}\right)$

Solution

$\left(3 - \frac{1}{\sqrt{2}}\right) = 3 - \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= 3 - \frac{\sqrt{2}}{2}$ **(Put $\sqrt{2}$ = 1.414214)**

$= 3 - \frac{1.414214}{2}$ = 2.2928 (5.s.f.)

3. Express $\frac{1+\sqrt{2}}{\sqrt{2}}$ in the form $p + q\sqrt{2}$, where $p, q \in$ R

Solution

$\frac{1+\sqrt{2}}{\sqrt{2}} = \frac{1+\sqrt{2}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{2}(1+\sqrt{2})}{2} = \frac{\sqrt{2}+2}{2} = \frac{\sqrt{2}}{2} + \frac{2}{2} = \frac{\sqrt{2}}{2} + 1$

4. Express $\frac{\sqrt{2}+\sqrt{5}}{\sqrt{10}}$ in the form $a\sqrt{5} + b\sqrt{2}$, where $a, b \in$ R

Solution

$\frac{\sqrt{2}+\sqrt{5}}{\sqrt{10}} = \frac{\sqrt{2}+\sqrt{5}}{\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}}$

$= \frac{\sqrt{10}(\sqrt{2}+\sqrt{5})}{10}$

$= \frac{\sqrt{20}+\sqrt{50}}{10}$

$= \frac{\sqrt{4}\times\sqrt{5}}{10} + \frac{\sqrt{25}\times\sqrt{2}}{10}$

$= \frac{2\sqrt{5}}{10} + \frac{5\sqrt{2}}{10}$

$= \frac{\sqrt{5}}{5} + \frac{\sqrt{2}}{2}$

$\Rightarrow a = \frac{1}{5}$ and $b = \frac{1}{2}$ (Not necessary)

Exercises 4.12

A. Rationalize the denominators:

1. $\frac{3}{\sqrt{6}}$ 3. $\frac{\sqrt{5}}{\sqrt{20}}$ 5. $\frac{10}{\sqrt{12}}$ 6. $\frac{\sqrt{4}}{\sqrt{3}}$ 7. $\frac{9}{4\sqrt{6}}$

B. Given that $\sqrt{2}$ = 1.414 and $\sqrt{3}$ = 1.732, evaluate to four significant figures;

1. $(3+\sqrt{2})^2$ 2. $\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{8}}$

3. $\frac{1}{2\sqrt{2}}$ 4. tan $75^0 = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

B. Expressions of the form: $\frac{c}{a+\sqrt{b}}$

To rationalize the denominators of expressions of the form: $\frac{c}{a+\sqrt{b}}$,

I. Multiply the conjugate of the denominator of the given fraction by the numerator and denominator of the given fraction.

II. Simplify numerators and denominators, where possible

Note; $a + \sqrt{b}$ and $a - \sqrt{b}$, where a is rational and $\sqrt{b}$ is a surd, are ***conjugate compound surds,*** also called ***conjugate surds.*** Thus,

1. $a + \sqrt{b}$ is the conjugate of $a - \sqrt{b}$

2. $a - \sqrt{b}$ is the conjugate of $a + \sqrt{b}$

3. Therefore, $\frac{1}{a+\sqrt{b}} = \frac{1}{a+\sqrt{b}} \times \frac{a-\sqrt{b}}{a-\sqrt{b}}$,

Worked Examples

1. Rationalize the denominator of $\frac{4}{\sqrt{3}-1}$

Solution

$\frac{4}{\sqrt{3}-1} = \frac{4}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$

$= \frac{4(\sqrt{3}+1)}{3-1} = \frac{4(\sqrt{3}+1)}{2} = 2(\sqrt{3}+1)$

2. Simplify $\frac{1-\sqrt{2}}{1+\sqrt{2}}$

Solution

$\frac{1-\sqrt{2}}{1+\sqrt{2}} = \frac{1-\sqrt{2}}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}}$

$= \frac{1-2\sqrt{2}+2}{1-2} = \frac{3-2\sqrt{2}}{-1} = 2\sqrt{2} - 3$

3. Rationalize the denominator of $\frac{1}{3-\sqrt{2}}$

Solution

$\frac{1}{3-\sqrt{2}} = \frac{1}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}}$

$= \frac{3+\sqrt{2}}{(3-\sqrt{2})(3+\sqrt{2})}$

$= \frac{3+\sqrt{2}}{9-2}$

$= \frac{3+\sqrt{2}}{7}$

Some Solved Past Questions

1. If $\left(\sqrt{3} - \frac{2}{\sqrt{3}}\right) = p\sqrt{3}$, find p

Solution

$\sqrt{3} - \frac{2}{\sqrt{3}}$

$= \sqrt{3} - \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \sqrt{3} - \frac{2\sqrt{3}}{3} = \frac{3\sqrt{3} - 2\sqrt{3}}{3} = \frac{\sqrt{3}}{3}$

$\Rightarrow \frac{\sqrt{3}}{3} = p\sqrt{3}, \ p = \frac{1}{3}$

2. Simplify $2\sqrt{175} - \sqrt{576} + \frac{35}{\sqrt{7}}$, leaving your answer in surd form

Solution

$2\sqrt{175} - \sqrt{576} + \frac{35}{\sqrt{7}}$

$= 2\sqrt{175} - \sqrt{16} \times \sqrt{36} + \frac{35}{\sqrt{7}}$

$= \sqrt{700} - \sqrt{16} \times \sqrt{36} + \frac{35}{\sqrt{7}}$

$= \sqrt{100} \times \sqrt{7} - \sqrt{16} \times \sqrt{36} + \frac{35}{\sqrt{7}}$

$= 10\sqrt{7} - 4 \times 6 + \frac{35}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$

$= 10\sqrt{7} - 24 + \frac{35\sqrt{7}}{7}$

$= 10\sqrt{7} + 5\sqrt{7} - 24$

$\mathbf{= 15\sqrt{7} - 24}$

Exercises 4.13

Rationalizing the denominator;

1. $\frac{2-\sqrt{3}}{2+\sqrt{3}}$ 2. $\frac{1}{\sqrt{3}-\sqrt{2}}$ 3. $\frac{2}{\sqrt{5}+\sqrt{3}}$

4. $\frac{9}{\sqrt{5}-\sqrt{2}}$ 5. $\frac{2}{\sqrt{7}+\sqrt{5}}$ 6. $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

05 NUMBER BASES

Base Ten Numerals (Decimals)

The decimal system or the base ten numeration system uses the following digits; 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

Writing Base Ten Numerals

Writing base ten numerals is a matter of making a sequential combination of the digits; 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. For instance, to begin the writings, the first digit precedes or combines with itself and the rest of the digits to obtain; 00, 01, 02, 03, 04, 05, 06, 07, 08, 09…

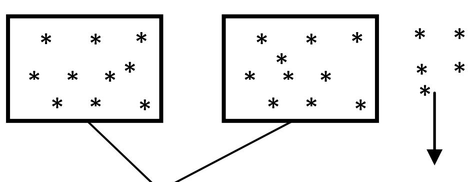
To proceed, the next digits (which is 1) combines (precedes) with all the base ten digits to obtain: 10, 11, 12, 13, 14, 15, 16, 17, …

To proceed, the next digit (which is 2) combines (precedes) with all the digits to obtain; 20, 21, 22, 23, 24, 25, 26, 27, 28, 29…

The process is followed sequentially to write all base ten numerals.

Basis of Base Ten Numerals

The base ten numeration system is based on groupings in tens. For e.g., the stars below can be grouped in tens as follows



2- groups often 5

This 2 – groups of tens and 5 units are written as 25. Similarly, 8 – groups of ten and 6 units are written as 86, giving the sequence tens and units.

Every 10 – groups of tens equals hundred (100) producing a sequence of Hundreds, Tens and Units.

10 – groups of hundreds equals thousands (1000) which in turn generates a sequence of Thousands, Hundreds, Tens and Units.

10 – groups of thousands equals ten thousands (10,000) which generates the sequence :
Ten thousands ←Thousands ← Hundreds ← Tens ← Units (Ones)

Worked Examples

A. Group the following base ten numerals into ten thousands, thousands, hundreds, tens and units

1) 312 2) 8 3) 4052 4) 51309

Solution

1. 312 = 3 - Hundreds, 1- Tens and 2- ones
2. 8 = 8 - Ones
3. 4052 = 4 -Thousands, 0- Hundreds, 5 -Tens, 2- ones.
4. 51309 = 5 - Ten thousands, 1 -Thousands, 3 - Hundreds, 0 - Tens and 9 - Units

B. Write the numerals for the following;

1. Seventy – two thousand, six hundred and thirty – three.
2. Forty – five thousand three hundred and twenty - three

Solution

1. Seventy – two thousand, six hundred and thirty – three

= 72(1000) + 6(100) + 3(10) + 3(1)
= 72000 + 600 + 30 + 3 = 72633

2. Forty - five thousand, three hundred and twenty- three
= 45(1000) + 3(100) + 2(10) + 3(1)
= 45000 + 300 + 20 + 3 = 45323

Exercises 5.1

A. **Write the numerals for the following;**

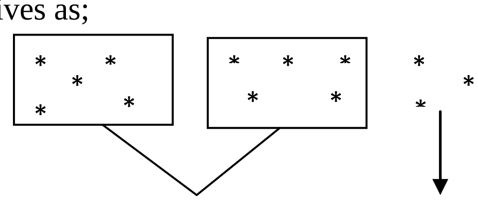
1. Sixty – four thousand, five hundred and twelve
2. Seven hundred and eighty –nine thousand five hundred and fifty – six
4. Nine hundred and sixty – nine million, sixty - four thousand, five hundred and thirty
5. Two hundred thousand and fifty – seven

B. Write the following numerals in words

1) 54381 2) 6,897,678 3) 12, 675, 220

Base Five Numeration

Base five (5) is based on groupings in fives. For example, the following stars can be grouped in fives as;



2 3
Groups of fives and units
(less than 5)

This can be written as 23_{five} and read as *"Two - Three base five"*

Note: The base depicts the manner of groupings.

Writing Base Five Numerals

The base five numeration system uses the digits: 0, 1, 2, 3, 4.

Writing base five, numerals is a matter of combining the digits in a carefully, orderly manner.

To begin, the first digit, (0) combines with itself and the other digits to obtain:00, 01, 02, 03, 04

The next digit, 1, takes its turn to combine with or precede the other digits to obtain the sequence: 10, 11, 12, 13, 14

To proceed, the next digit 2, takes its turn to combine with or precede the other digits to obtain: 20, 21, 22, 23, 24
When the process is duly and carefully followed, as many base five numerals can be written

Exercises 5.2

1. Write the base five numerals up to 400.

Place Value of Base Five Numerals

The place value of base five numerals begins from the right as;
5^0, 5^1, 5^2, 5^3 etc
5^0 = units (ones)
5^1 = fives (5)
5^2 = twenty – fives (25)
5^3 = one hundred and twenty fives (125)
Therefore, the place value of the digits in 341_{five} can be identified as;

3	4	1
↓	↓	↓
5^2	5^1	5^0
(Twenty-fives)	(Fives)	(One or unit)

Worked Examples

Find the place value of the digit "2" in the following base five (5) numerals.
1) 3112_{five} 2) 23031_{five} 3) 1024_{five}

Solution

$$\begin{array}{lcccc} 1. & 3 & 1 & 1 & 2 \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ & 5^3 & 5^2 & 5^1 & 5^0 \end{array}$$

The place value of 2 in 3112_5 is 5^0 or ones

$$\begin{array}{lccccc} 2. & 2 & 3 & 0 & 3 & 1 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 5^4 & 5^3 & 5^2 & 5^1 & 5^0 \end{array}$$

The place value of 2 in 23031_5 is 5^4 or six hundred and twenty fives (625)

$$\begin{array}{lcccc} 3. & 1 & 0 & 2 & 4 \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ & 5^3 & 5^2 & 5^1 & 5^0 \end{array}$$

The place value of 2 in 1024_{five} is 5 or fives

Exercises 5.3

A. Identify the place value of the digit "3" in the following base five numerals;

1) 3041_{five} 2) 4034_{five} 3) 11003_{five}

B. What is the place value of the digits written against the numerals;

1. 132 (3) 2. 341 (1) 3. 4321 (3)
4. 1140 (4) 5. 208541 (0) 6. 275041 (2)

Addition in Base Five and other Bases

1. Positional method

One's ability to perform operations in base five and other bases depends on one's ability to write the numerals of that base, atleastthe first 40 numerals. The sum of the numbers (in base ten) is used as a position to identify the numeral that occupies that particular position as far as the numerals of that base is concern. For e.g. to perform 14 + 3 + 2 in base five, add the numbers to get 19. Then write the base five numerals and identify the number that occupies the 19th position as the answer. i.e. 0, 1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, 22, 23, 24, 30, 31, 32, 33, <u>**34**</u>, 40, 41, 42, 43, 44…

Therefore, $14 + 3 + 2 = 34_{five}$

Similarly, to perform 14 + 3 + 2 in base eight, find the sum of the numbers as 19. Then write the base eight numerals and identify the 19th numeral as the answer. i.e. 0, 1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 20, 21, 22, <u>**23**</u>, 24, 25,26, 28…

Therefore, $14 + 3 + 2 = 23_{eight}$

2. The remainder principle

In this method, the sum of the numbers is divided by the base in which the operation is performed to get *a whole number* and *remainder* as answer. For e.g. 14 + 3 + 2 = 19, in base five, 19 ÷ 5 = 3R4 = 34_{five}, .

Similarly, in base six, 19 ÷ 6 = 3R1 = 31_{six} and in base eight, 19 ÷ 8 = 2R3 = 23_{eight}

Worked Examples

1. Add 344 and 24 in base five

Solution

<u>Method 1</u>

Set out the work as follows;

$$\begin{array}{r} 344_{\ five} \\ +\ 24_{\ five} \\ \hline \\ \hline \end{array}$$

I. Add 4 and 4 to obtain 8. Locate the number that occupies the 8th position on the base five numeration and that is 13.

0, 1, 2, 3, 4, 10, 11, 12, <u>***13***</u> so write 3, leaving a remainder of 1

$$\begin{array}{r} 344_{five} \\ +\ 24_{five} \\ \hline 3 \\ \hline \end{array}$$

II. Add 4, 2 and 1 (the remainder) to obtain 7. Locate the 7th number of the base five numerals as 12 so write 2, remainder 1 That is;

$$\begin{array}{r} 344_{five} \\ +\ 24_{five} \\ \hline 23_{five} \\ \hline \end{array}$$

III. Add 3 and 1(the remainder) to get 4. The 4th numeral is 4, so record it to complete the

$$\begin{array}{r} 344_{five} \\ +\ 24_{five} \\ \hline 423_{five} \\ \hline \end{array}$$

Method 2

4 + 4 = 8, 8 ÷ 5 = 1R3 = 13, so write 3 R 1

4 + 2 + 1(R) = 7, 7 ÷ 5 = 1R 2 = 12, so record 2 leaving a remainder of 1

3 + 1(remainder) = 4, 4 ÷ 5 = 0 R 4 = 4, so record 4 to complete the work;

$$\begin{array}{r} 344_{five} \\ +\ 24_{five} \\ \hline 423_{five} \\ \hline \end{array}$$

2. Perform $4103_{five} + 2422_{five}$

Solution

Set out the work in vertical form or arrangement as follows;

$$\begin{array}{r} 4103_{five} \\ +\ 2422_{five} \\ \hline 12030_{five} \\ \hline \end{array}$$

3. Find the missing numeral in 2031 – **** = 343 if the operation was performed in base five

Solution

2031 – **** = 343

**** = 2031 – 343

$$\begin{array}{r} 2031_{five} \\ -343_{five} \\ \hline 1133_{five} \\ \hline \end{array}$$

Therefore, the missing numeral is 1133_{five}

Exercises 5.4

A. Perform the following in base five

1. $2344_{five} + 304_{five}$
2. $3001_{five} + 444_{five} + 2333_{five}$
3. $4441_{five} + 1441_{five} + 4214_{five}$
4. $4230_{five} + 2421_{five} + 4404_{five}$
5. $322_{five} + 122_{five} + 420_{five}$

B. Find the missing numerals;

1. $$\begin{array}{r} 411_{five} \\ -\ {*}{*}{*}_{five} \\ \hline 134_{five} \\ \hline \end{array}$$

2. $$\begin{array}{r} 1012_{five} \\ -\ {*}{*}{*}{*}_{five} \\ \hline 414_{five} \\ \hline \end{array}$$

Subtraction in Base Five

To perform subtraction in base five and other bases successfully, always set out the work in the vertical form or on the place value chart as we did for the addition in base five.

Take note of the fact that when a bigger numeral is subtracted from a smaller one, the need arises to borrow from the next immediate digit on the left with possessions.

Generally, in base *n*, a borrowed digit is *n*. Thus, in base five, each borrowed digit is 5. When properly added, subtraction becomes possible.

Worked Examples

1. Perform $321_{five} - 24_{five}$

Solution

I. Set out the work as follows;

$$\begin{array}{r} 321_{five} \\ -\ 24_{five} \\ \hline \\ \hline \end{array}$$

II.1 – 4, but 4 >1, so borrow 1 (five) from 2 and add to 1 to make 6. Subtract 4 from 6 to get 2.

That is:

$$\begin{array}{r} 321 \\ -\,24 \\ \hline 2 \\ \hline \end{array}$$

III. Proceed by taking note of the fact that the digit "2" is 1 (because, 1(five) has been borrowed already), so, 1 – 2, but 2 > 1, so borrow 1(five) from 3 and add to 1 to get 6. Now 6 – 2 = 4

$$\begin{array}{r} 321_{five} \\ -\,24_{five} \\ \hline 42_{five} \\ \hline \end{array}$$

iv. The next column number 3 is left with 2, (because 1 has been borrowed already) so record 2 to complete the work.

$$\begin{array}{r} 321_{five} \\ -\,24_{five} \\ \hline 242_{five} \\ \hline \end{array}$$

2. Find i. $304_{five} - 43_{five}$ ii) $1010_{five} - 323_{five}$

Solutions

i. $$\begin{array}{r} 304_{five} \\ -\,43_{five} \\ \hline 211_{five} \\ \hline \end{array}$$

ii. $$\begin{array}{r} 1010_{five} \\ -\,323_{five} \\ \hline 132_{five} \\ \hline \end{array}$$

Exercises 5.5

A. Perform the following;

1) $4002_{five} - 312_{five}$ 2) $3121_{five} - 2042_{five}$

3) $104_{five} - 42_{five}$ 4) $4000_{five} - 444_{five}$

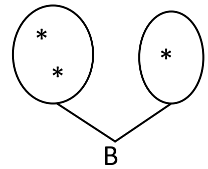
B. Find the missing addend in base five;

1) 432 + …. = 1000 2)…… + 442 = 4343

3) …. + 242 = 1020 4) 231 + ….. = 1104

Base Two (Binary) Numerals

The base two numeration system involves groupings in twos. For e.g., the following stars can be grouped in twos as:



This gives rise to 1 group of twos and 1 unit (less than two) written in base two as 11_2, and read as *"one- one base two"*

Writing Base Two Numerals

The digits for base two numeration system are 0, 1 only.

In writing base two numerals, make use of the previous knowledge of combining the digits as we did when writing the base ten and five numerals.

To start, 0 precede itself to give, 00 and precedes 1 to give 01. So we have: 00, 01,

The digit 1, takes its turn to precede 0 to give 10 and precede itself to give 11. So we have; 00, 01, 10, 11.

To continue, watch carefully, that all the other numerals precedes 0 and 1 only to generate all the other base two numerals. Watch it as;

10 precede 0 to give 100.

10 precedes 1 to give 101

Now we have: 0, 1, 10, 11, 100, and 101

The numeral that takes turn is 11. 11 precedes 0 to give 110, 11 precedes 1 to give 111 so we have; 0, 1, 10, 11, 100, 101, 110, 111,

The number that takes turns is 100. 100 precedes 0 to give 1000, 100 precedes 1 to give 1001 giving the sequence; 0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001…

As long as each numeral takes turn to precede 0 and 1, all the base two numerals can be generated.

Addition in Base Two

Write the base two numerals as follows:

0, 1, 10, 11, 100, 101, 110, 111 … to serve as guide in performing addition in base two. For example, to add 11 and 10 in base two , set out the work as shown below:

$$\begin{array}{r} 11 \\ +\underline{10} \\ \hline \end{array}$$

Add 1 and 0 to obtain 1

$$\begin{array}{r} 11 \\ +\underline{10} \\ \underline{1} \end{array}$$

Continue by adding 1 and 1 to get 2; then count from 0 to locate the number that occupies the 2nd position. That is; 10 and write it to get the final answer as;

$$\begin{array}{r} 11_{two} \\ +\underline{10_{two}} \\ \underline{101_{two}} \end{array}$$

Worked Examples

Perform the following in base two.

1. 11 + 101   2. 10 +10
3. 1001 + 11   4. 11 + 11

Solution

Set out the work in vertical form as follows;

1. $\begin{array}{r} 11_{two} \\ +\underline{101_{two}} \\ \underline{1000_{two}} \end{array}$ 2. $\begin{array}{r} 10_{two} \\ +\underline{10_{two}} \\ \underline{100_{two}} \end{array}$

3. $\begin{array}{r} 1001_{two} \\ +\underline{11_{two}} \\ \underline{1100_{two}} \end{array}$ 4. $\begin{array}{r} 11_{two} \\ +\underline{11_{two}} \\ \underline{110_{two}} \end{array}$

Exercises 5.6

Perform the following in base two;

1. 101 + 110   3. 110 + 111 + 10
2. 11010 + 10110   4. 11101 + 11011

Subtraction in Base Two

Similar to subtraction in base five, when the need arises to borrow, do so from the next immediate number on the left with possession.

Be mindful of the fact that any number borrowed in base two is 2. When the borrowed digit is added to wherever it is required, subtraction becomes possible. For e.g. to perform 10_{two} – 1_{two}, set out the work as follows,

$$\begin{array}{r} 10_{two} \\ -\underline{1_{two}} \\ \hline \end{array}$$

1 has to be subtracted from 0, but 1 > 0, so borrow from the next number on the left. Remember that the value of the borrowed number is 2, added to 0 to obtain 2 subtraction is now possible, 2 – 1 = 1. That is;

$$\begin{array}{r} 10 \\ -\underline{1} \\ 1_2 \end{array}$$

Worked Examples

1. $110_{two} – 11_{two}$ 2. $101_{two} – 1_{two}$
3. $101_{two} – 11_{two}$ 4. $1101_{two} – 110_{two}$

Solution

Set out the work in the vertical form as follows:

1. $\begin{array}{r} 110_{two} \\ -\underline{11_{two}} \\ \underline{11_{two}} \end{array}$ 2. $\begin{array}{r} 101_{two} \\ -\underline{1_{two}} \\ \underline{100_{two}} \end{array}$

3. $\begin{array}{r} 101_{two} \\ -\underline{11_{two}} \\ \underline{10_{two}} \end{array}$ 4. $\begin{array}{r} 1101_{two} \\ -\underline{110_{two}} \\ \underline{111_{two}} \end{array}$

Exercises 5.7

Perform the following in base two;

1. 110 – 101   2. 1010101 – 111111

3. 1101 – 111 4. 11011 – 10110

5. 1101 – 1001 6. 10100 – 1101

Other Bases Less than Ten

1. Decimal or denary system (base ten) counting is done in ones, tens, hundreds, thousands,etc and the digits used in writing are: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

2. Base five system, counting is done in ones, fives, twenty – fives, one twenty – fives, etc and the digits used in writing are: 0,1, 2,3, 4

3. Base two system (binary), counting is done in ones, twos, fours, eights etc and the digits used are in writing are: 0, 1

Base three system (ternary), counting is done in ones, threes, nines, twenty – sevens, etc and the digits used in writing are 0, 1, 2

Base eight system (octal), counting is done in ones, eights, sixty – fours etc and the digits used in writing are 0, 1, 2, 3, 4, 5, 6, 7

Multiplication of Bases

To multiply two or more numerals of the same base

<u>Method 1</u>

I. Write the numerals of that base (as many as you can)

II. Multiply the numerals (in base ten) and identify the product as a position.

III. Locate the numeral that occupies that position.

<u>Method II</u>

Use the *"remainder principle"* by going through the following processes:

I. Find the product of the numbers.

II. Divide each product by the base in which the operation is being performed.

III. Record the remainders as answers.

Worked Examples

1. Perform $234_{five} \times 4_{five}$, and leave the answer in base five.

Solution

<u>Method 1</u>

Base five numerals: 0, 1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, 22, 23, 24, 30, 31, 32, 33, 34, 40…..

Set out the work as follows;

$$\begin{array}{r} 234_{five} \\ \times\ 4_{five} \\ \hline \end{array}$$

I. Multiply 4 and 4 to get 16. The 16th number of the base five numerals is 31 so write 1 leaving a remainder of 3

$$\begin{array}{r} 234_{five} \\ \times\ 4 \\ \hline 1 \end{array}$$

II. Multiply 4 and 3 to get 12 and add 3(the remainder) to get 15. Identify the 15th numeral of base five as 30. Record 0, leaving a remainder of 3

$$\begin{array}{r} 234_{five} \\ \times\ 4 \\ \hline 01 \end{array}$$

III. Multiply 4 and 2 to get 8 and add 3(the remainder) to get 11. The 11th numeral of base five is 21 so record it to complete the work

$$\begin{array}{r} 234_{five} \\ \times\ 4 \\ \hline 2101_{five} \end{array}$$

Method 2

Set out the work as follows;

$$\begin{array}{r} 234_{five} \\ \times\ 4_{five} \\ \hline \end{array}$$

$4 \times 4 = 16$, $16 \div 5 = 3$ R $1 = 31$

Write (R) 1 as the answer, leaving the quotient 3

$$\begin{array}{r} 234_{five} \\ \times\ 4_{five} \\ \hline 1 \\ \hline \end{array}$$

$4 \times 3 = 12 + 3$ (quotient) $= 15$,

$15 \div 5 = 3R0 = 30$

Write 0 (R) as the answer, leaving the quotient, 3

$$\begin{array}{r} 234_{five} \\ \times\ 4_{five} \\ \hline 01 \\ \hline \end{array}$$

$4 \times 2 = 8 + 3$(quotient) $= 11$,

$11 \div 5 = 2$ R $1 = 21$

Write 21 to complete the work as shown

$$\begin{array}{r} 234_{five} \\ \times\ 4_{five} \\ \hline 2101 \\ \hline \end{array}$$

Exercises 5.8

A. Calculate the following;

1. $3214_{five} \times 3_{five}$ 2. $212_{five} \times 14_{five}$
3. $104_{six} \times 1530_{six}$ 4. $231_{four} \times 33_{four}$
5. $47_{eight} \times 21_{eight}$ 6. $123_{eight} \times 46_{eight}$

B. Find the perimeters of squares with sides of length:

a. 101_{two}cm b. 11011_{two}cm c. 1111_{two}cm

C. Find the areas of rectangles with lengths and breadths

1. 101_{two} and 101_{two} 2. 1010_{two} and 11_{two}
3. 101_{two} and 11_{two} 4. 1101_{two} and 110_{two}

Changing to Base Ten

a. Numerals of the form: A B C_x

To change a numeral of the form**A B C_x** to base ten, where x is positive and $x \neq 0$ follow the steps below;

I. Identify the base of the numeral as $\boldsymbol{x}$

II. Multiply the identified base by each individual digit of the numeral as shown;

$\mathbf{(A\times x)\ (B \times x)\ (C\times x)}$

III. Find the sum of the product of the base and the individual digit as shown below;

$\mathbf{(A \times x) + (B \times x) + (C\times x)}$

IV. Assign exponents on the identified base from the right to the left beginning from 0. That is; $\mathbf{(A \times x^2) + (B \times x^1) + (C \times x^0)}$

Simplify the equation, bearing in mind that

$x^0 = 1$ and $x^1 = x$

Worked Examples

1. Change 132_{five} to a number in base ten

Solution

A B $C_x = (A \times x^2) + (B \times x^1) + (C \times x^0)$

132_{five}

$= (1 \times 5^2) + (3 \times 5^1) + (2 \times 5^0)$

$= 25 + 15 + 2 = 42_{ten}$

2. Convert 2342_{five} to a base ten numeral

Solution

2342_{five}

$= (2 \times 5^3) + (3 \times 5^2) + (4 \times 5^1) + (2 \times 5^0)$

$= (2 \times 125) + (3 \times 25) + (4 \times 5) + (2 \times 1)$

$= 250 + 75 + 20 + 2 = 347_{ten}$

3. Change 10111_{two} to base ten

Solution

$= (1 \times 2^4) + (0 \times 2^3) + (1 \times 2^2) + (1 \times 2^1) + (1 \times 2^0)$

$= (1 \times 16) + (0 \times 8) + (1\times 4) + (1\times2) + (1\times1)$

$= 16 + 0 + 4 + 2 + 1 = 23_{\text{ten}}$

4. Convert 233_{four} as a base ten numeral.

Solution

233_{four}

$= (2 \times 4^2) + (3 \times 4^1) + (3 \times 4^0)$

$= (2 \times 16) + (3 \times 4) + (3 \times 1)$

$= 32 + 12 + 3 = 47_{\text{ten}}$

b. Numerals of the form: AB.CDE$_x$

To convert numerals of the form: AB **.** CDE$_x$ to base ten, where x is positive and $x \neq 0$, follow the steps below;

I. Multiply the base by each digit :

(A× x) (B × x) (C× x) (D× x) (E× x)

II. Find the sum of the product of the

(A× x) + (B × x) + (C× x) + (D × x)+ (E× x)

III. Assign exponents on the base (x) from the **point** to the left increasing from 0 and from the **point** to right decreasing from 0.

$(A \times x^1) + (B \times x^0) + (C\times x^{-1}) + (D \times x^{-2}) + (E \times x^{-3})$

IV. Simplify the equation bearing in mind that $x^0 = 1$ and $x^1 = x$

Worked Example

1. Convert 341.25_{six} to a decimal numeral.

Solution

341.25_{six}

$= (3 \times 6^2) + (4 \times 6^1) + (1 \times 6^0) + (2 \times 6^{-1}) + (5 \times 6^{-2})$

$= (3 \times 36) + (4 \times 6) + (1 \times 1) + (2 \times 0.17) + (5 \times 0.03)$

$= 108 + 24 + 1 + 0.33 + 0.14 = 133.47$

Exercises 5.9

A. Convert the following to base ten;

1. 4133_{five} 2. 3420_{five} 3. 1042_{five} 4. 13022_{five}

B. Change the following to base ten;

1. 1011_{two} 2. 11011_{two} 3. 1100_{two} 4. 100101_{two}

C. Convert the following to base ten;

1) 2122_{three} 2) 103.51_{eight} 3) 12000_{three}

4) 22.112_{three} 5) 126_{eight} 6) 700_{eight}

7. If $29_{\text{ten}} = x_{\text{eight}} = y_{\text{six}} = z_{\text{five}} = w_{\text{three}}$, find x, y, z and w

8. If $28_{\text{nine}} = 35_a = 101_b = 122_c$, find the values of a, b and c

D. Perform the following in base three;

1. 102 + 212 2. 2102 + 21 3. 2102 – 1021

4. 1212 – 1121 5. 221 × 21 6. 1000 ÷ 121

E. Perform in base eight:

1) 123 + 25 2) 256 + 127 3) 235 – 172

4) 1000 – 77 5) 32 × 6 6) 346 ×5

F. In which bases have the following calculations been done?

1. 12 + 3 = 21 2. 12 × 3 = 41 3. 231 + 132 = 413

4. 12 – 3 = 6 5. 12 × 3 = 41 6. 432 – 234 = 165

G. Change the following to base ten:

1. 12.34_{five}
2. 414.105_{six}
3. 3001.232_{four}
4. 10110.1001_{two}

Changing from Base Ten to Other Bases

The method of repeated division is used to change a base ten numeral to other bases. The steps are:

I. Identify the base ten numerals (decimal) and the base being converted to

II. Prepare a three column table with the base being converted to, in column 1, the decimal numeral in column two and Remainder (R) in column 3.

III. Use the base being converted to (divisor), to divide decimal numeral (dividend) and the answer (quotient) repeatedly, until you get a number that is less than itself (divisor).

IV. Carefully record the quotient in column 2 and the corresponding remainder (R) in column 3, after each division.

V. Write the numbers at the column of the remainder from the bottom upward as the answer.

Worked Examples

1. Write 259_{ten} as a number in base eight.

Solution

8	259	R
8	32	3
8	4	0
		4

$\Rightarrow 259_{ten} = 403_{eight}$

2. Change 19 to a binary numeral.

Solution

2	19	R
2	9	1
2	4	1
2	2	0
2	1	0
	0	1

$\Rightarrow 19 = 10011_2$

3. Change 53_{ten} to a number in base four

Solution

4	53	R
4	13	1
4	3	1
4	0	3

$\Rightarrow 53_{ten} = 311_{four}$

Exercises 5.10

A. Write as a number in base five;

1)162 2) 102 3) 167 4) 244 5) 30

B. Convert to a number in base two;

1) 59_{ten} 2) 77_{ten} 3)102_{ten} 4)66_{ten}

C. Change the following:

1. 119_{ten} to a base three numeral
2. 205_{ten} to a base three numeral.
3. 336_{ten} to a number in base four
4. 175_{ten} to a base six numeral.

Conversion Between Non Decimal Bases

It involves changing from one base to another, other than base ten. For e.g. to change 123_{four} to a number in base five, there is the need to change 123_{four} to a number in base ten, and then change further from the base ten numeral to base five.

Worked Examples

1. Convert 432_{five} to a number in base three.

Solution

Change 432_{five} to base ten

$= (4 \times 5^2) + (3 \times 5^1) + (2 \times 5^0)$

$= (4 \times 25) + (3 \times 5) + (2 \times 1)$

$= 100 + 15 + 2 = 117_{ten}$

Change 117_{ten} to base three

3	117	R
3	39	0
3	13	0
3	4	1
3	1	1
	0	1

$\Rightarrow 432_{five} = 11100_{three}$

2. Express 231_{four} as a number in base six

Solution

231_{four} to base ten

$= (2 \times 4^2) + (3\times 4^1) + (1\times 4^0)$

$= (2 \times 16) + (3 \times 4) + (1 \times 1)$

$= 32 + 12 + 1 = 45_{ten}$

Then Change 45_{ten} to base 6

6	45	R
6	7	3
6	1	1
	0	1

$\Rightarrow 231_{four} = 113_{six}$

Exercises 5.11

A. Convert the following;

1. 232_{five} to a base seven numeral.
2. 1011_{two} to a number in base four.
3. 131_{five} to binary numeral.
4. 133_{six} to a number in base three.

C. Change the following to base five

1. 1110_{two} 2. 101111_{two} 3. 10101_{two}

D. Write as a number in the bases indicated.

1. 422_{five} to base six 3. 1102_{three} as base four
2. 113_{four} as base three 4. 314_{five} as base three

Base Twelve

The Latin word for twelve is ***duodecim*** giving rise to the duodecimal numeration system.

The duodecimals system require twelve digits but we have ten already, so we invent two more namely *t* and *e* to write and count;

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, *t*, *e*

10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 1*t*, 1*e*

20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 2*t*, 2*e*..

The numeral $5t2e_{twelve}$ is read as *"five, tee, two, ee, base twelve"* and it is converted to base ten as;

$(5 \times 12^3) + (10 \times 12^2) + (2 \times 12^1) + (11 \times 12^0)$

Worked Examples

1. Convert $3t4_{twelve}$ to base ten

Solution

$3t4_{twelve}$

$= (3 \times 12^2) + (10\times 12^1) + (4 \times 12^0)$

$= 432 + 120 + 4 = 556_{ten}$

2. Convert 659_{ten} to base twelve

Solution

12	659	R
12	54	11(*e*)
12	4	6
	0	4

$\Rightarrow 659_{ten} = 46e$

3. Convert 546_{eight} to a base twelve numeral.

Solution

Change 546_{eight} to a number in base ten

$= (5 \times 8^2) + (4 \times 8^1) + (6 \times 8^0)$

$= (5 \times 64) + (4 \times 8) + (6 \times 1)$

$= 320 + 32 + 6 = 358$

Now change 358 to base 12

12	358	R
12	29	10
12	2	5
	0	2

$358_{ten} = 25t_{twelve}$

$\Rightarrow 546_{eight} = 25t_{twelve}$

Exercises 5.12

A. Convert from base twelve to base ten

1) 53 2) 90 3) 8*t* 4) *ett* 5) 2*t*9*e*

B. Convert from base ten to base twelve

1. 27 2) 100 3) 180 4) 1000 5) 3587

C. Calculate the following in base twelve

1. $42e + 9tt$ 2. $t894 + e97e$ 3. $357 - 319$

4. 896×3 5. $tet \times 7$ 6. $5tt1 \times e$

Tables for Number Bases

Table of values are usually constructed under addition $\oplus$ and multiplication $\otimes$ for a given base.

Given base n, the table is constructed using the numbers 0 to $(n - 1)$. Thus, for base 5, the set of values are {0, 1, 2, 3, 4} and for base 6, the range of values are 0 , 1, 2, 3, 4, 5. For large bases, a set of values may be given for operation.

On the table, the range of values for the base n or the 0 to $n - 1$, values occupy the first column and the first row, whilst the operator, whether + or × is circled to look like $\oplus$ or $\otimes$ respectively, and placed at the left top corner of the table as shown below:

$\oplus$	0		
0			

$\otimes$	0		
0			

Thereafter;

I. Perform the operation.

II. Divide the answer by the base.

III. Record only the remainder in the boxes or cells to complete the table.

Worked Examples

1. a. Draw an addition table for base six on the set P = 1, 2, 3, 4, 5

b. Use the table to find $(2 \oplus 3) \oplus 5$

c. Use the table to solve;

i. $5 \oplus x = 12$ ii. $x \oplus x = 12$

d. Why would you say addition is commutative on the table?

Solution

a.

$\oplus$	1	2	3	4	5
1	2	3	4	5	10
2	3	4	5	10	11
3	4	5	10	11	12
4	5	10	11	12	13
5	10	11	12	13	14

b.From the table;

$(2 \oplus 3) \oplus 5 = 5 \oplus 5 = 14$

c. From the table;

i. $5 \oplus x = 12$

$5 \oplus 3 = 12$

Therefore $\{x : x = 3\}$

ii. $x \oplus x = 12$

$4 \oplus 4 = 12$

Therefore$\{x : x = 4\}$

d. It is commutative because the principal diagonal is symmetrical.

2. a. Copy and complete the table below for addition base eight.

$\oplus$	1	3	5	7
1	2	4	6	
3		6	10	12
5	6	10		
7		12	14	16

b. Use the table to find the truth set of;

i. $7 \oplus x = 12$ ii. $m \oplus m = 12$

Solution

a. $1 \oplus 7 = 8$,   8 in base 8 = 10

$5 \oplus 5 = 10$,   10 in base 8 = 12

$5 \oplus 7 = 12$,   12 in base 8 = 14

$7 \oplus 1 = 8$,   8 in base 8 = 10

$\oplus$	1	3	5	7
1	2	4	6	10
3	4	6	10	12
5	6	10	12	14
7	10	12	14	16

b. From the table;

i. $7 \oplus x = 12$

$7 \oplus 3 = 12$

$\Rightarrow x = 3$

ii. $m \oplus m = 12$

$5 \oplus 5 = 12$

$\Rightarrow m = 12$

Exercises 5.13

1. a. Draw addition table for base seven on the set P = $\{1, 2, 3, 4, 5, 6\}$

b. Use the table to find; $(2 \oplus 3) \oplus (4 \oplus 2)$

c. Find the value of n if;

i. $n \oplus n = 11$ ii. $3 \oplus 4 = 2 \oplus n$

2. i. Copy and complete the table below for multiplication in base five.

$\otimes$	1	2	3	4
1	1	2		4
2	2			13
3	3			22
4	4	13	22	

b. Use the table to find:

i. $(2 \otimes 2) \otimes 4$ ii. $3 \otimes 3$

c. Use the table to find the truth set of;

i. $(1 \otimes m) \otimes m = 14$ ii. $m \otimes m = 31$

Ordering Numerals in Different Bases

To order two or more numerals in different bases, convert all the numerals to base ten and order them accordingly**.**

Worked Examples

Arrange the following in descending order of magnitude; 103_{four}, 1011_{two} and 43_{five}

Solution

Change all the numerals to base ten

$103_{four} = (1 \times 4^2) + (0 \times 4^1) + (3 \times 4^0)$

$= 16 + 0 + 3 = 19$

$1011_{two} = (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (1 \times 2^0)$

$= 8 + 0 + 2 + 1 = 11$

$43_{five} = (4 \times 5^1) + (3 \times 5^0) = 20 + 3 = 23$

$\Rightarrow 23 > 19 > 11$,

43_{five}, 103_{four} and 1011_{two} in descending order.

Exercises 5.14

A. Arrange in ascending order;

1. 3011_{four}, 1110_{two}, 221_{three}
2. 13_{six}, 40_{five}, 10110_{two}
3. 230_{five}, 332_{six}, 232_{four}, 2202_{three}

B. Arrange in descending order:

1. 244_{five}, 55_{six}, 2002_{three}, 14_{ten}
2. 11011_{two}, 11_{five}, 101_{three}
3. 403_{six}, 11_{eight}, 2201_{three}

Division of Bases

To perform division of numerals of the same base or different bases other than base ten,

I. Change the involving numerals to base ten.

II. Perform the division on the base ten numerals.

III. Change the answer from base ten to the required base.

Worked Examples

1. Perform $2134_{five} \div 3_{five}$, leaving the answer in base five

Solution

$2134_{five} \div 3_{five}$,

Change the numerals to base ten

$\frac{2134_{five}}{3_{five}} = \frac{(2 \times 5^3) + (1 \times 5^2) + (3 \times 5^1) + (4 \times 5^0)}{3}$

$= \frac{250+25+15+4}{3} = \frac{294}{3} = 98$

Change 98 to a base five numeral

5	98	R
5	19	3
5	3	4
5	0	3

$98 = 343_{five}$

Therefore, $2134_{five} \div 3_{five}$, $= 343_{five}$

2. Perform $201_{four} \div 23_{four}$

Solution

$201_{four} \div 23_{four}$

Change the numerals to base ten

$\frac{201_{four}}{23_{four}} = \frac{(2 \times 4^2) + (0 \times 4^1) + (1 \times 4^0)}{(2 \times 4^1) + (3 \times 4^0)}$

$= \frac{32+1}{8+3} = \frac{33}{11} = 3$

Change 3 to base four to get 3

Therefore, $201_{four} \div 23_{four} = 3_{four}$

Equations Involving Number Bases

Here, a numeral with unknown base (variable) is equated to another numeral with a known base.

To solve such problems, express each side of the equation as a base ten numeral, and solve for the value of the variable.

Worked Examples

1. If $42_n = 12_{ten}$, find the value of n

Solution

$(4 \times n^1) + (2 \times n^0) = 14$

$(4 \times n) + (2 \times 1) = 14$

$4n \quad + \quad 2 \quad = 14$

$4n = 14 - 2$

$4n = 12$

$n = 3$

2. If $102_n = 51$, find the value n.

Solution

$(1 \times n^2) + (0 \times n^1) + (2 \times n^0) = 51$

$n^2 + 0 + 2 = 51$

$n^2 + 2 = 51$

$n^2 = 51 - 2$

$n^2 = 49$

$n = \sqrt{49} = 7$

3. What is the value of A if $3A1_{five} = 91$

Solution

$(3 \times 5^2) + (A \times 5^1) + (1 \times 5^0) = 91$

$(3 \times 25) + (A \times 5) + (1 \times 1) = 91$

$75 \quad + \quad 5A \quad + \quad 1 \quad = 91$

$5A + 76 = 91$

$5A = 91 - 76$

$5A = 15$

$A = \frac{15}{5} = 3$

6. Find the value of x if $142_x = 113_{four}$

Solution

$142_x = 113_{four}$,

Change each side of the equation to a decimal

$(1 \times x^2) + (4 \times x^1) + (2 \times x^0) = (1 \times 4^2) + (1 \times 4^1) + (3 \times 4^0)$

$x^2 + 4x + 2 = 16 + 4 + 3$

$x^2 + 4x + 2 = 23$

$x^2 + 4x + 2 - 23 = 0$

$x^2 + 4x - 21 = 0$

$(x^2 + 7x) - (3x - 21) = 0$

$x(x + 7) - 3(x + 7) = 0$

$(x – 3)(x + 7) = 0$
$x – 3 = 0$ or $x + 7 = 0$
$x = 3$ or $x = –7$, but $x \neq -7$
Therefore $x = 3$

7. Given that $4(13)_n = 54_n$, find the value of the base n.

Solution
Change each side to base ten;
$4(13)_n = 54_n$
$4[(1 \times n^1) + (3 \times n^0)] = (5 \times n^1) + (4 \times n^0)$
$4(n + 3) = 5n + 4$
$4n + 12 = 5n + 4$
$12 – 4 = 5n – 4n$
$8 = n$

8. Find the base x such that $365_7 + 43_x = 217$

Solution
$365_7 + 43_x = 217$
$(3 \times 7^2) + (6 \times 7^1) + (5 \times 7^0) + (4 \times x^1) + (3 \times x^0) = 217$
$147 + 42 + 5 + 4x + 3 = 217$
$4x + 197 = 217$
$4x = 217 – 197$
$4x = 20$
$x = 5$

9. If $123_{five} – 42_x = 12$, find the base x

Solution
$123_{five} – 42_x = 12$
$(1 \times 5^2) + (2 \times 5^1) + (3 \times 5^0) – (4 \times x^1) + (2 \times x^0) = 12$
$25 + 10 + 3 – [4x + 2] = 12$
$25 + 10 + 3 – 2 – 4x = 12$
$36 – 4x = 12$
$- 4x = 12 – 36$
$- 4x = - 24$
$x = 6$

Exercises 5.15
A. Find the value of the variables
1. $63_x = 45$ 2. $43_n = 23$ 3. $34_n = 19_{ten}$
4. $12_n = 5$ 5. $11_n = 3$ 6. $33_n = 21_{ten}$

B. Find the value of the variables:
1. $62 = 2M2_{five}$ 2. $K30_{five} = 90$ 3. $3P_{five} = 19$
4. $M4_{five} = 14$ 5. $10K_{two} = 5$ 6. $2K3_{four} = 39$

C. Find the value of the variable;
1. $2(25)_x = 54_x$ 2. $3(12)_x = 2(22)_x$
3. $1100_{three} = 121_x$ 4. $302_x = 122_{six}$

D. Determine the base;
1. $52_x + 153_{seven} = 134$ 3. $414_{six} + 33_x = 251_{eight}$
2. $416_{seven} – 72_x = 151$ 4. $221_y + 302_{four} = 75$

Word Problems
Given a word problem involving number bases,
I. Write a mathematical equation.
II. Express each side of the equation as a number in base ten.
III. Solve for the value of the involving variable, if any.

Worked Examples
1. A number is written as 37 in base x. Twice the number is written as 75 in base x. Find the value of x.

Solution
$2(37)_x = 75_x$
$2[(3 \times x^1) + (7 \times x^0)] = (7 \times x^1) + (5 \times x^0)$
$2(3x + 7) = 7x + 5$
$6x + 14 = 7x + 5$
$6x – 7x = 5 – 14$
$- x = - 9$ $x = 9$

2. If $123_y = 83$, obtain an equation in y, hence find the value of y.

Solution

$123_y = 83$,

$(1 \times y^2) + (2 \times y^1) + (3 \times y^0) = 83$

$y^2 + 2y + 3 = 83$

$y^2 + 2y - 80 = 0$

$(y^2 + 10y) - (8y - 80) = 0$

$y(y + 10) - 8(y + 10) = 0$

$(y - 8)(y + 10) = 0$

$\Rightarrow y - 8 = 0$ or $y + 10 = 0$

$y = 8$ or $y = -10$

$\Rightarrow y = 8$, ignore the negative answer, $y \neq -10$

3. Evaluate $\sqrt{61_8}$ as a number in the decimal system.

Solution

$\sqrt{61_8} = \sqrt{(6 \times 8^1) + (1 \times 8^0)}$

$= \sqrt{48 + 1} = \sqrt{49} = 7$

4. Find the base two number that is equivalent to 49_{seven}.

Solution

Change 49_7 to base ten;

$= (4 \times 7^1) + (9 \times 7^0) = 28 + 9 = 37$

Express 37 as a number in base two;

2	37	R
2	18	1
2	9	0
2	4	1
2	2	0
2	1	0
	0	1

$49_{seven} = 100101_{two}$

5. If $27_x = 32_y$, find the

i. smallest possible replacements for x and y.

ii. largest possible replacements for x and y.

iii. possible replacements for x and y such that $x = y$

Solution

$27_x = 32_y$,

Express each side of the equation in base ten

$(2 \times x) + (7 \times x^0) = (3 \times y) + (2 \times y^0)$

$2x + 7 = 3y + 2$

Since each side is in base ten, substitute the base ten values: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

L.H.S	**R.H.S**
$2x + 7$	$3y + 2$
$2(0) + 7 = 7$	$3(0) + 2 = 2$
$2(1) + 7 = 9$	$3(1) + 2 = 5$
$2(2) + 7 = 11$	$3(2) + 2 = 8$
$2(3) + 7 = 13$	**$3(3) + 2 = 11$**
$2(4) + 7 = 15$	$3(4) + 2 = 14$
$2(5) + 7 = 17$	**$3(5) + 2 = 17$**
$2(6) + 7 = 19$	$3(6) + 2 = 20$
$2(7) + 7 = 21$	**$3(7) + 2 = 23$**
$2(8) + 7 = 23$	$3(8) + 2 = 27$
$2(9) + 7 = 25$	$3(9) + 2 = 30$

From the investigations,

$2x + 7 = 2(8) + 7 = 23, \quad \Rightarrow x = 8$

$3y + 2 = 3(7) + 2 = 23, \quad \Rightarrow y = 7$

$2x + 7 = 2(2) + 7 = 11, \quad \Rightarrow x = 2$

$3y + 2 = 3(3) + 2 = 11, \quad \Rightarrow y = 3$

$2x + 7 = 2(5) + 7 = 17, \quad \Rightarrow x = 5$

$3y + 2 = 3(5) + 2 = 17, \quad \Rightarrow y = 5$

Smallest possible replacements for x and y *is*

$x = 2$ and $y = 3$

Largest possible replacements for x and y *is*

$x = 8$ and $y = 7$

Possible replacements for x and y such that

$x = y$ *is* $x = 5$ and $y = 5$

Exercises 5.16

1. Make up tables for addition and multiplication of ternary (base three) numerals.

2. The length and breadth of a rectangle are 1101_{two} cm and 11_{two} cm. Find its perimeter and area as numbers in the base two.

3. Given that 12 is the equivalence of twice of 14_y , what is the possible scale of y.

4. If $120_x = 21_y$, find the smallest and largest possible replacements for x and y.

5. Show that 121_m, where m is a natural number greater than 2, must be a square number.

Challenge Problems

1. Show that no whole number replacements exist for $43_x = 26_y$

2. Copy and complete the following table. Each row represents the same number expressed in different ways.

Base	**12**	**10**	**8**	**5**	**3**	**2**
		27				
	2e					
						11101

3. Multiply the binary number 101 and 1011 and express your answer in the scales of ten.

4. A square has a side of length 120_{three} mm. Find its perimeter and area.

5. Show that 43 – 34 is a multiple of 7, if the numbers are in the scale of eight, and is a multiple of 5, and is a multiple of 5 if the numbers are in the scale of six. Check if it is also true for 52 – 25, and make a general statement.

Number Bases and Binary Combined

To solve problems involving a combination of binary operation and number bases:

I. Identify the binary operation and the given base.

II. Perform the binary operation taking note that answer obtained is in base ten.

III. Convert the answer to the given base.

IV. Complete the table of values if any, with the answers obtained from the conversion.

Worked Examples

1. The operation Δ is defined on the set of real numbers by $a \Delta b = \frac{4a-b}{b}$ in base four. Evaluate:

i. 9 Δ 3 ii. 84 Δ (10 Δ 5)

Solution

i. $a \Delta b = \frac{4a-b}{b}$

$9 \Delta 3 = \frac{4(9)-3}{3} = \frac{36-3}{3} = \frac{33}{3} = 11$

Change 11 to base four;

11 ÷ 4 = 2 R 3

$11 = 23_{four}$

Alternatively,

4	11	R
4	2	3
	0	2

$11 = 23_{four}$

ii. $a \Delta b = \frac{4a-b}{b}$

84 Δ (10 Δ 5)

$(10 \Delta 5) = \frac{4(10)-5}{5} = \frac{40-5}{5} = \frac{35}{5} = 7$

84 Δ (10 Δ 5) = 8 Δ 7

$84 \Delta 7 = \frac{4(84)-7}{7} = \frac{336-7}{7} = \frac{329}{7} = 47$

Change 47 to base four;

4	47	R
4	11	3
4	2	3
	0	2

$47 = 233_{four}$

2. If $a * b = 2ab - 2$, is the definition of the operation * in base 8;
a. Draw a table for * on the set { 2, 3, 4, 5, 6}
b. Use the table to evaluate the following;
i. 6 * 5 ii. 2 * 2 * 2 iii. (2 * 2) * 4
c. From the table, find the truth set of:
i. $n * n = 36$ ii. $n * (2 * 2) = 106$

Solution

a.

*	2	3	4	5	6
2	6	12	16	22	26
3	12	20	26	34	42
4	16	26	36	46	56
5	22	34	46	60	72
6	26	42	56	72	106

b. From the table;
i. 6 * 5 = 60
ii. 2 * 2 * 2 = 26
iii. (2 * 2) * 4 = 56

c. From the table;
i. $n * n = 36$
4 * 4 = 36
$\{n : n = 4\}$

ii. $n * (2 * 2) = 106$
6 * (2* 2) = 106
$\{n : n = 6\}$

Exercises 5.17

1. The operation * is defined by $m * n = \frac{5mn}{m+n}$.
Evaluate the following:
i. 12 *18 in base 8 ii. (3 * 2) * 9 in base 6

2. a. Copy and complete the table below for * defined by $m * n = 7mn - m$ in base five for the set {1, 2, 3, 4}

*	1	2	3	4
1			40	102
2	22	101		
3	33	124	220	
4	44		310	

b. Use the table to evaluate the following:
i. 2 * 4 ii. 4 * 4 iii. 1 * 2
c. Use the table to solve the following;
i. $x * x = 413$ ii. $2 * x = 130$

3. a. Complete the table below for ◙ defined by x ◙ $y = x + y - 2$ in base nine.

◙	3	4	5	6	7
3	4	5	6		8
4	5		7	8	
5	6		8	10	11
6	7		10		12
7	8	10		12	

b. From the table, evaluate the following;
i. 4 ◙ (4 ◙ 4) ii. (3 ◙ 6) ◙ (5 ◙ 4)
c. From the table, find the truth set of the following;
i. n ◙ n = 11 ii. (n ◙ 3) ◙ 6 = 8

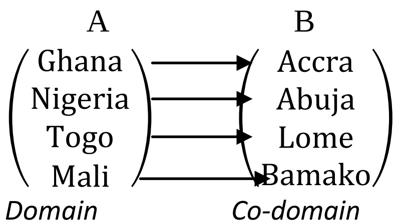
6 RELATIONS AND FUNCTIONS

Definition of Relations

A relation is a formula or statement (called rule) which enables one to match elements of two sets. Among the two sets, the first set is called ***domain*** and the second set is called ***co-domain.***

Relations can be described by means of;
1. Set of ordered pairs.
2. The solution set of open sentences. e.g *x is the square of y.*
3. Arrow diagrams and Cartesian graphs.

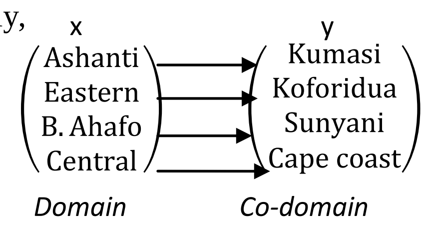
Study the diagram below;



Set A which contains the elements, Ghana, Nigeria, Togo, Mali occupies the first position and is called the ***domain*** and set B with elements Accra, Abuja, Lome, Bamako, occupying the second position is called the ***co-domain***.

The domain and the co-domain are always related such as we have in the diagram above;Set A consist of countries and set B consist of capital towns. Thus, the association between the two sets is described by the relation *"is the capital of"*.

Similarly,



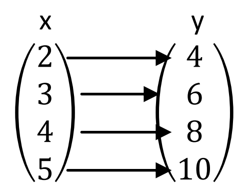
x represents the Domain and *y* represents the Co-domain.

In the diagram above, set *x* contains *"regions"* and set *y* contains *"regional capitals"*. As such, the two sets can be described by the relation ***"is the regional capital of".***

Worked Examples

Identify the relation that exist between the following sets

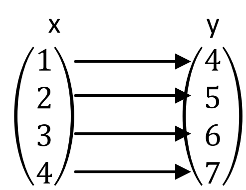
1.



Solution

If the diagram is studied carefully, you will realize that each element of the domain multiplied by 2, equals the corresponding image in the co-domain. Therefore, the relation ***"is twice of"*** befits the description of both sets

2.



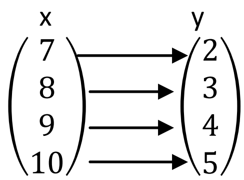
Solution

Study the diagram carefully. Observe that;

$1 + 3 = 4$, $2 + 3 = 5$, $3 + 3 = 6$, $4 + 3 = 7$

Therefore, the relation is ***"is 3 more than"***

3.



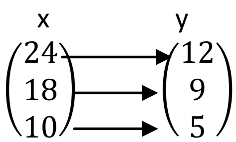
Solution

Observe that

7 – 5 = 2, 8 – 5 = 3, 9 – 5 = 4, 10 – 5 = 5

Therefore, the relation is ***"5 less than"***

4.



Solution

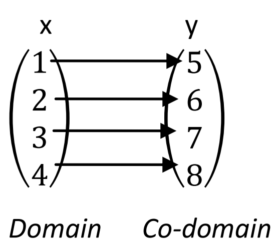
Observe that

$\frac{1}{2} \times 24 = 12$, $\frac{1}{2} \times 18 = 9$, $\frac{1}{2} \times 10 = 5$, Therefore, the relation is ***"is half of"***

Types of Relations

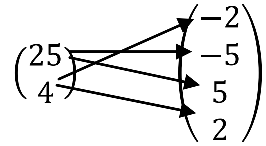
1. One – to – one Relation

It is the type of relation in which each element of the domain matches to only one element of the co-domain. This is represented as in the diagram below for the relation *"is increased by four"*



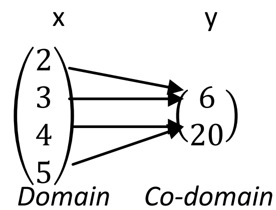
2. One – to – many Relations

It is the type of relation in which one or more element(s) of the domain matches to many elements in the co-domain. This is diagrammatically represented as shown below for the relation *"is square root of"*;



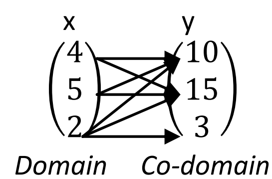
3. Many – to – one Relation

It is the type of relation in which more than one element of the domain match to one element of the co-domain. This is represented in the diagram below for the relation *"is a product of"*



4. *Many – to – many Relations*

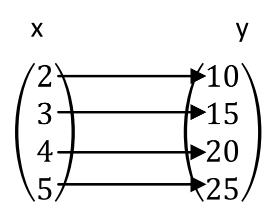
It is the type of relation in which more than one member of the domain match to many members of the co-domain as shown below for the relation, "*is less than"*



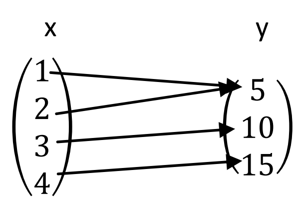
Exercises 6.1

Identify the type of relation:

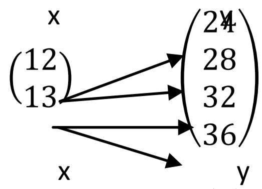
1.



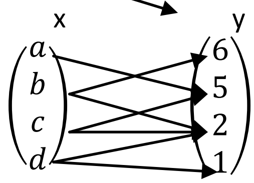
2.



3.

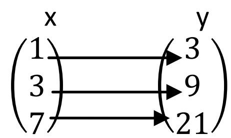


4.

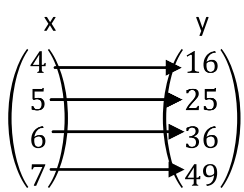


B. Identify the rule of relations;

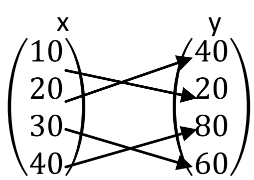
1.



2.



3.



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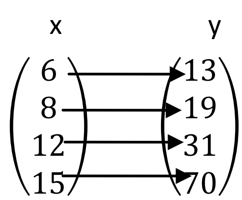


Unknown Domain or Co-domain

The three main components of a relation are domain , codomain and the rule. Given any two of them, the third one can be found.

***When the codomain is unknown*:**

i. Express the rule as mathematical equation involving *x* and *y*.

ii. Plug in the equation, the given values of the domain (*x*) to obtain the set of values of the codomain (*y*)

When the domain is unknown:

i. Express the rule as mathematical equation involving *x* and *y*.

ii. Change the subject to *x* and put in, the given values of the codomain (*y*) to obtain the set of values of the domain (*x*)

Note: It is not always possible to express the rules as a mathematical equation.

Worked Examples

1. The rule of a relation is *"is one-third of"*. Given the domain, {12, 15, 18, 24}, find the co-domain. Draw a diagram of the relation.

Solution

Rule =*"is one-third of"*

Domain ={12, 15, 18, 24},

According to the rule, $y = (\frac{1}{3}x)$,

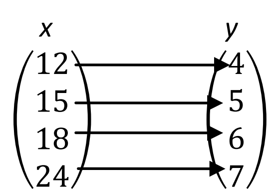
when $x = 12$, $y = \frac{1}{3} \times 12 = 4$,

when $x = 15$, $y = \frac{1}{3} \times 15 = 5$,

when $x = 18$, $y = \frac{1}{3} \times 18 = 6$,

when $x = 24$, $y = \frac{1}{3} \times 24 = 7$,

The co-domain = {4, 5, 6, 7} respectively.



2. The rule of a relation is *"is four less than twice"*. Given the codomain {6, 8, 18, 18, 22}, find the domain. Draw a diagram of the relation.

Solution

Rule : *"is four less than twice"*

Codomain = $\{6, 8, 10, 18, 22\}$

Domain = ***Unknown***

From the rule, $y = 2x - 4$

$x = \frac{y+4}{2}$ (Change subject to x)

When $y = 6$, $x = \frac{6+4}{2} = 5$

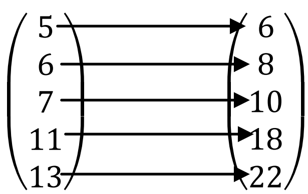
When $y = 8$, $x = \frac{8+4}{2} = 6$

When $y = 10$, $x = \frac{10+4}{2} = 7$

When $y = 18$, $x = \frac{18+4}{2} = 11$

When $y = 22$, $x = \frac{22+4}{2} = 13$

Domain = {5, 6, 7, 11, 13}



Exercises 6.2

1. Given the domain $\{2, 4, 6, 8, 10\}$, find the co-domain, if the rule of the relation is *"five more than thrice"*. Represent the relation in a diagram.

2. Given the co domain $\{13, 14, 15, 17\}$ for the relation "*is seven more than half of*";
i. find the respective elements of the domain,
ii. show the diagram of the relation

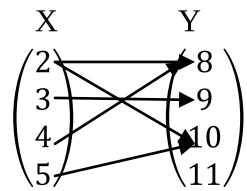
3. The co-domain of a relation is {25, 49, 81, 100}. If the rule of the relation is *"is a square of"*, determine the domain of the relation, for all positive integers and draw a diagram of the relation.

The Range of a Relation

The set of numbers in the co-domain that has corresponding members in the domain is called the ***range.***

The range is always in the co-domain (is a subset of the co-domain).

Consider the diagram below



The range = $\{8, 9, 10\}$.

Worked Examples

1. The rule of a relation is *"is seven more than twice*", the domain is $\{1, 2, 3, 4, 5\}$ and the co-domain is {8, 9, 10, 11, 12, 13, 14, 15, 17, 20} what is the range of the relation?

Solution

The rule : "is 7 more than twice"

$y = 2x + 7$

Domain = $\{1, 2, 3, 4, 5\}$

Let x represent the elements of the domain

From the rule, $y = (2x + 7)$,

When $x = 1$, $y = 2(1) + 7 = 9$

When $x = 2$, $y = 2(2) +7 = 11$

When $x = 3$, $y = 2(3) + 7 = 13$

When $x = 4$, $y = 2(4) + 7 = 15$

When $x = 5$, $y = 2(5) + 7 = 17$

Range of the relation = $\{9, 11, 13, 15, 17\}$

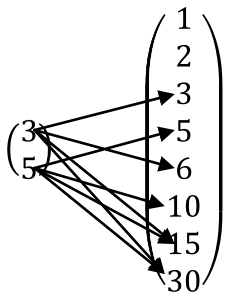
2. Given the domain = prime factors of 15} and the co domain = {factors of 30}, draw an arrow diagram for the rule "is a multiple of" and find the range of the relation.

Solution

Domain = {3, 5}

Codomain = {1, 2, 3, 5, 6, 10, 15, 30}

Rule : "is a multiple of"



Range = {3, 5, 6, 10, 15, 30}

Exercises 6.3

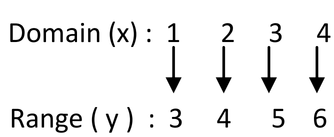
1. For the domain = {prime factors of 30} and the co domain = {factors of 12}. Draw an arrow diagram for the rule "is increased by one" and find the range of the relation.

2. Domain = {prime factors of 24} and the co domain = {factors of 36}. Draw an arrow diagram for the rule "is square of" and find the range of the relation.

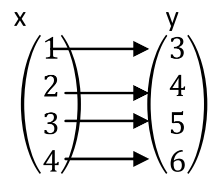
3. Domain = {factors of 12 that are multiples of 3} and the co domain = {odd factors of 63}.
i. Draw an arrow diagram for the rule "is divisible by" and find the range of the relation.

Relations as Ordered Pair

In relations, each elements of the domain has a corresponding element in the range. When each element in the domain is paired with its respective element of the range, we have ordered pairs: For instance, if the domain = $\{1, 2, 3, 4\}$ and range =$\{3, 4, 5, 6\}$ satisfy the rule "is 2 more than", then it can be represented diagrammatically as;



OR



Since 1 → 3, 2 → 4, 3 → 5, and 4 → 6, the ordered pairs formed are:(1, 3), (2, 4),(3, 5) and (4, 6). As ordered pairs, the first number is an element of the domain and the second number, the corresponding element in the range.

Worked Examples

1. Which of the following ordered pairs do not satisfy the rule of a relation *"is twice as"*:
(2, 4), (5, 10), (11, 22), (10, 5)

Solution

Rule is *"is twice as"* $y = 2x$
Domain = $\{2, 5, 11, 10\}$
From the rule,
When $x = 2$, $y = 2(2) = 4$
Therefore, (2, 4) satisfies the rule
When $x = 5$, $y = 2(5) = 10$
Therefore, (5, 10) satisfies the rule
When $x = 11$, $y = 2(11) = 22$
Therefore (11, 22) satisfies the rule
When $x = 10$, $y = 2(10) = 20$
Therefore (5, 10) does not satisfies the rule

2. Which pair of ordered pairs is either true or false for the rule "is the square of" (1, 1), (4, 16), (5, 20), (7, 49)

Solution

Rule is "*is the square of*"

$y = x^2$

When $x = 1$, $y = (1^2) = 1$

Therefore, (1, 1) is true
When $x = 4$, $y = (4^2) = 16$

Therefore (4, 16) is true
When $x = 5$, $y = (5^2) = 25$
Therefore (5, 20) is false
When $x = 7$, $y = (7^2) = 49$
Therefore (7, 49) is true

Exercises 6.4

A. Show which of the following ordered pairs do not satisfy the given rule of the relation in each case

1. Rule: *is one-third of*", (9, 3), (12, 4), (15, 6), (21, 7)
2. Rule "*is a multiple of*" (15, 5), (3, 9), (12, 36), (7, 49)
3. Rule: "*is nine plus half of*" (2, 10), (4, 11), (3, 10), (6, 12), (10, 14).

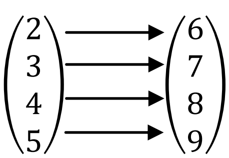
Mapping

A mapping is a special type of relation in which each member of the domain matches to a member in the co-domain. In other words, mapping is an association between two sets, say A and B, such that each element of Set A is associated with a unique element of set B.

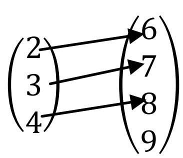
In a mapping, elements of the domain are represented by x and the elements of the co-domain are represented by y.

Consider the diagrams below;

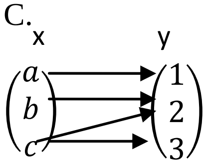
A.



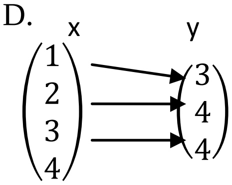
B.



C.



D.



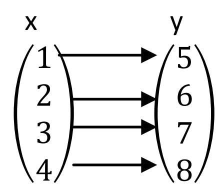
In diagrams A, and B, each element of the domain matches to an element in the co-domain. These are called ***mappings***. Thus, for a relation to be referred to as a mapping, all elements of the domain must be assigned.

On the other hand, in diagram C, ***c*** in the domain has two images in the co- domain so it is not a mapping but a relation. Similarly, in diagram D, 4 in the domain do not have a corresponding element in the co-domain. This relation is not a mapping. It follows that all mappings are relations but not all relations are mappings.

Types of Mapping

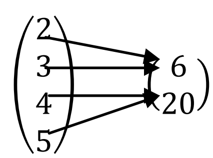
1. One – to – one mapping

It is the type of mapping in which each element of the domain matches to only one element of the co-domain and vice – versa. It is also called *one – to - correspondence*. This is represented as;



2. Many – to – one mapping

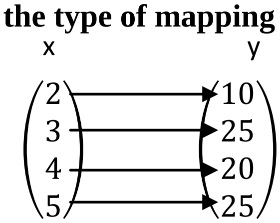
It is the type of relation in which more than one element of the domain match to one element of the co-domain. This is represented by the diagram below;



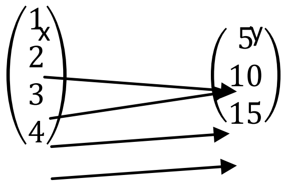
Exercises 6.5

Identify whether it is a mapping or/and a relation and the type of mapping / relation;

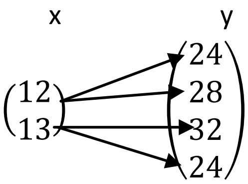
1.



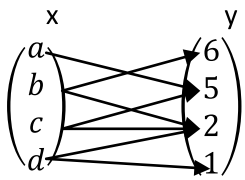
2.



3.



4.



Rule of a Mapping

The rule of a mapping is the statement or formula that shows how the domain and co-domain are related. Before finding the rule of a mapping, identify whether it is:

1. Linear mapping or linear reciprocal mapping
2. Exponential mapping or exponential reciprocal mapping
3. Quadratic mappings or quadratic reciprocal mappings

Rule of Linear Mappings and Linear Reciprocal Mappings

A linear mapping and its reciprocal mapping are kinds of mapping in which the elements of the domain have a common difference and the elements of the co-domain also have a common difference.

Method 1: (Gradient Approach)

All linear mappings and its reciprocals follow the rule; $\boldsymbol{y = mx + c}$ or $\boldsymbol{x \to mx + c}$ where $\boldsymbol{m}$ is the gradient, $\boldsymbol{c}$ is the y – intercept and $\boldsymbol{m}$ and $\boldsymbol{c}$ are constants.

I. Determine the value of m (the gradient) by the formula; $\boldsymbol{m = \frac{y_2 - y_1}{x_2 - x_1}}$

Pick two values of the domain (x) and label them as x_1 and x_2 and label their corresponding y values as $y_1$and $y_2$and use it to determine the gradient, m

II. Substitute the value of m in $y = mx + c$

III. Find the value of c by substituting any ordered pairs of x and y in $y = mx + c$

IV. Finally, substitute the values of m and c in $y = mx + c$ to get the rule of the mapping.

For linear reciprocal mappings, work without the constant numerator, k, to determine $y = mx + c$ and write the rule as; $\boldsymbol{y = \frac{k}{m\,x+c}}$ **or** $\boldsymbol{x \to \frac{k}{m\,x+c}}$

Method 2: (Simultaneous Approach)

I. Write two linear equations of the form $y = mx + c$, using the values of x and y of the mapping.

II. Solve the two equations simultaneously to determine the values of m and c.

III. Substitute the values of m and c in $y = mx + c$ to determine the rule of the mapping

For linear reciprocal mappings, work without the constant numerator, k, to determine $y = mx + c$ and write the rule as;

$y = \frac{k}{mx + c}$ or $x \to \frac{k}{mx + c}$

Worked Examples

1. What is the rule for this mapping?

x	1	2	3	4	55
↓	↓	↓	↓	↓	↓
y	1	3	5	7	9

Solution

Method 1

Difference in x – values = 1

Difference in y – values = 2

This means that the mapping is linear. Thus,

$y = mx + c$.

But $m = \frac{y_2 - y_1}{x_2 - x_1}$

If $x_1 = 3$ and $x_2 = 4$, $y_1 = 5$ and $y_2 = 7$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-5}{4-3} = 2$

Put $m = 2$ in $y = mx + c$ to obtain $y = 2x + c$

When $x = 3$, $y = 5$, substitute in $y = 2x + c$

$5 = 2(3) + c$

$5 = 6 + c$

$5 - 6 = c$

$-1 = c$ or $c = -1$

Put $c = -1$ in $y = 2x + c$ to obtain $y = 2x - 1$

The rule is: $y = 2x - 1$ or $x \rightarrow 2x - 1$

Method 2

$y = mx + c$

When $x = 2$, $y = 3$

$3 = 2m + c$……………….(1)

When $x = 3$, $y = 5$

$5 = 3m + c$ ……………….(2)

eqn (2) – eqn (1);

$2 = m$

Put $m = 2$ in eqn (1);

$3 = 2(2) + c$

$3 = 4 + c$

$3 - 4 = c$

$c = -1$

Substituting $m = 2$ and $c = -1$ in $y = mx + c$.

The rule is: $y = 2x - 1$ or $x \rightarrow 2x - 1$

2. In the mapping below, find the rule

x	3	6	9	12	15
↓	↓	↓	↓	↓	↓
y	4	2	0	-2	-4

Solution

$y = mx + c$

When $x = 3$, $y = 4$

$4 = 3m + c$……………….(1)

When $x = 6$, $y = 2$

$2 = 6m + c$ ……………….(2)

eqn (1) – eqn (2);

$2 = -3m$

$m = -\frac{2}{3}$

Put $m = -\frac{2}{3}$ in eqn (1);

$4 = 3(-\frac{2}{3}) + c$

$4 = -2 + c$

$4 + 2 = c$

$c = 6$

Substitute $m = -\frac{2}{3}$ and $c = 6$ in $y = mx + c$, to obtain $y = -\frac{2}{3}x + 6$ or $x \rightarrow -\frac{2}{3}x + 6$

3. Determine the rule of the mapping:

x	2	3	4	5
↓	↓	↓	↓	↓
y	$\frac{1}{13}$	$\frac{1}{17}$	$\frac{1}{21}$	$\frac{1}{25}$

Solution

For all linear reciprocal mappings, work without the constant numerator, $k = 1$

$y = mx + c$

When $x = 2$, $y = 13$

$13 = 2m + c$…………… (1)

When $x = 1$, $y = 9$

$9 = m + c$ ……………….(2)

eqn (1) – eqn (2);

$4 = m$

Put $m = 4$ in equation (1)

$13 = 2(4) + c$

$13 = 8 + c$

$13 - 8 = c$

$c = 5$

Substitute $m = 4$ and $c = 5$ in $y = mx + c$, to get $y = 4x + 5$.

The rule of the mapping is expressed in the form $y = \frac{k}{m\,x+c}$ or $x \to \frac{k}{m\;x+c}$

Substitute $m = 4$, $c = 5$ and $k = 1$.

The rule is: $y = \frac{1}{4x+5}$ or $x \to \frac{1}{4x+5}$

4. Determine the rule of the mapping below;

x	-5	0	5	10	15
↓	↓	↓	↓	↓	↓
y	$\frac{2}{-5}$	$\frac{2}{-3}$	$\frac{2}{-1}$	2	$\frac{2}{3}$

Solution

From the mapping, $k = 2$

$y = mx + c$

When $x = -5$, $y = -5$

$-5 = -5m + c$…………… (1)

When $x = 5$, $y = -1$

$-1 = 5m + c$ ……………….(2)

eqn (1) + eqn (2)

$-6 = 2c$

$c = -3$

Put $c = -3$ in equation (1)

$-5 = -5m - 3$

$-5 + 3 = -5m$

$-2 = -5m$

$m = \frac{2}{5}$

Substitute $m = \frac{2}{5}$ and $c = -3$ in $y = mx + c$, to get

$y = \frac{2}{5}x - 3$

For linear reciprocal mappings, $y = \frac{k}{ax+b}$

But $m = \frac{2}{5}$, $c = -3$ and $k = 2$.

$\Rightarrow y = \frac{2}{{}^{2}/_{5}x - 3} = \frac{2}{\frac{2x-15}{5}} = \frac{2 \times 5}{2x - 15}$

The rule is : $y = \frac{10}{2x-15}$ or $x \to \frac{10}{2x-15}$

Exercises 6.6

Determine the rule of the mappings;

1.

x	1	2	3	4	5
↓	↓	↓	↓	↓	↓
y	12	9	6	3	0

2.

x	-5	- 3	-1	1	3	5
↓	↓	↓	↓	↓	↓	↓
y	-5	-0	5	10	15	20

3.

x	-5	-4	-3	-2	-1
↓	↓	↓	↓	↓	↓
y	2	5	8	11	14

4.

x	1	2	3	4	5
↓	↓	↓	↓	↓	↓
y	$\frac{1}{8}$	$\frac{1}{11}$	$\frac{1}{14}$	$\frac{1}{17}$	$\frac{1}{20}$

5.

x	3	2	1	0	-1	-2
↓	↓	↓	↓	↓	↓	↓
y	-3	$\frac{3}{-4}$	$\frac{3}{-7}$	$\frac{3}{-10}$	$\frac{3}{-13}$	$\frac{3}{-16}$

6.

x	-6	-3	0	3	6
↓	↓	↓	↓	↓	↓
y	$\frac{2}{5}$	$\frac{2}{7}$	$\frac{2}{9}$	$\frac{2}{11}$	$\frac{2}{13}$

7.

x	0	1	2	3	4
↓	↓	↓	↓	↓	↓
y	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$

8.

x	-1	0	1	2	3
↓	↓	↓	↓	↓	↓
y	5	3	1	-1	-3

9.

x	1	2	3	4	5
↓	↓	↓	↓	↓	↓
y	-6	-8	-10	-12	-14

Rule of Exponential Mappings and Exponential Reciprocal Mappings

An exponential mapping is a kind of mapping in which the elements of the co-domain differs from each other by a common ratio, usually denoted by "*r*"

If the elements of the co – domain are fractions with a constant numerator, whilst the denominators differ by a common ratio, the mapping is said to be an exponential reciprocal mapping.

All exponential mappings follow the rule;

Rule 1

$\boldsymbol{y = r^x}$ **or** $\boldsymbol{x \rightarrow r^x}$

where r is the common ratio.

Rule 2

$\boldsymbol{y = ar^{x-b}}$ **or** $\boldsymbol{x \rightarrow ar^{x-b}}$

Where r is the common ratio and $(a, b) = (y_1, x_1)$

All exponential reciprocal mappings follow the two rules;

Rule 1

$\boldsymbol{y = \frac{k}{r^x}}$ **or** $\boldsymbol{x \rightarrow \frac{k}{r^x}}$ **……… (2)**

Where k is the common numerator and r is the common ratio of the elements of the co – domain.

Rule 2

$\boldsymbol{y = \frac{k}{ar^{x-b}}}$ **or** $\boldsymbol{x \rightarrow \frac{k}{ar^{x-b}}}$

Where $\boldsymbol{k}$ is the constant numerator, $\boldsymbol{r}$ is the common ratio of the values of the co – domain and $(a, b) = (y_1, x_1)$.

Worked Examples

1. Find the rule of the mapping;

x	0	1	2	3	4
↓	↓	↓	↓	↓	↓
y	1	3	9	27	81

Solution

Method 1

For exponential mappings $y = r^x$

But $r = 3 \div 1 = 3$

Put $r = 3$ in $y = r^x$

$\Rightarrow y = 3^x$

The rule is $y = 3^x$ or $x \rightarrow 3^x$

Method 2

From the mapping, $r = 3 \div 1 = 3$ and $a = 1$ and $b = 0$ substitute in $y = ar^{x-b}$

$\Rightarrow y = (1)(3)^{x-0} = 3^x$

The rule is $y = 3^x$ or $x \rightarrow 3^x$

2.

x	1	2	3	4
↓	↓	↓	↓	↓
y	4	16	64	256

Solution

Method 1

For exponential mappings, $y = r^x$

But $r = 4 \div 1 = 4$

Put $r = 4$ in $y = r^x$

$\Rightarrow y = 4^x$

The rule of the mapping is $y = 4^x$ or $x \rightarrow 4^x$

Method 2

From the mapping, $r = \frac{4}{1} = 4$ and $a = 4$ and $b = 1$ substitute in $y = ar^{x-b}$

$\Rightarrow y = (4)(4)^{x-1}$

The rule is $y = (4)4^{x-1}$or $y \rightarrow (4)(4)^{x-1}$

3. Determine the rule of the mapping

$$\begin{array}{ccccc} x & 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & \frac{3}{5} & \frac{3}{25} & \frac{3}{125} & \frac{3}{625} \end{array}$$

Solution

Method 1

For exponential reciprocal mappings,

$y = \frac{k}{r^x}$

k =3 (common numerator) and $r = \frac{25}{5} = 5$

Put k =1 and r = 5 in $y = \frac{k}{r^x}$

$\Rightarrow y = \frac{3}{5^x}$

The rule for the mapping is $y = \frac{3}{5^x}$

Method 2

From the mapping, $k = 3$, $r = 25 \div 5 = 5$,

$a = 5$ and $b = 0$, substitute in $y = \frac{k}{ar^{x-b}}$

$\Rightarrow y = \frac{3}{(5)(5)^{x-1}}$ or $x \rightarrow \frac{3}{(5)(5)^{x-1}}$

Exercises 6.7

Determine the rule of the mappings;

1.
$$\begin{array}{cccccc} x & 3 & 4 & 5 & 6 & 7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & 8 & 16 & 32 & 64 & 128 \end{array}$$

2.
$$\begin{array}{cccccc} x & 0 & 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \end{array}$$

3.
$$\begin{array}{ccccc} x & 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & 2 & 4 & 8 & 16 \end{array}$$

4.
$$\begin{array}{ccccc} x & 1 & 3 & 5 & 7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & \frac{1}{2} & \frac{1}{32} & \frac{1}{512} & \frac{1}{8192} \end{array}$$

5.
$$\begin{array}{ccccc} x & -3 & -1 & 1 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & 27 & 3 & \frac{1}{3} & \frac{1}{27} \end{array}$$

The Rule of Quadratic Mappings

A mapping is said to be of quadratic form if the first difference of the values of the co- domain is not constant but the second difference. The rule of quadratic mappings is in the form: $y = ax^2 + bx + c$

To determine the rule of quadratic mapping pick any three points (x, y), (x_1,y_1) and (x_2,y_2) and substitute each point into the relation $y = ax^2 + bx + c$ to get three equations and solve for the values of a, b and c simultaneously.

Worked Examples

1. What is the rule for the mapping?

$$\begin{array}{cccccc} x & 0 & 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & 2 & 6 & 12 & 20 & 30 \end{array}$$

Solution

First difference = 6 – 2 = **4**, 12 – 6 = **6**,
20 – 12 = **8**, 30 – 20 = **10**

Second difference = 6 – 4 = **2**, 8 – 6 = **2**
10 – 8 = **2**

Because the second difference gives a common difference, d = **2,** it is a quadratic mapping following the rule;

$y = ax^2 + bx + c$ or $x \rightarrow ax^2 + bx + c$

For any 3 points (1, 6), (2, 12) and (3, 20)

When $x = 1$, $y = 6$, by substitution

$6 = a(1)^2 + b(1) + c$

$6 = a + b + c$………………..(1)

When x = 2, y = 12, *by substitution*

$12 = a(2)^2 + b(2) + c$

$12 = 4a + 2b + c$…………….(2)

When x = 3, y = 20

$20 = a(3)^2 + b(3) + c$

$20 = 9a + 3b + c$ …………..(3)

The three equations are:

$a + b + c = 6$……………….(1)

$4a + 2b + c = 12$…………..(2)

$9a + 3b + c = 20$…………..(3)

eqn (2) – eqn (1)

$(4a – a) + (2b – b) + (c – c) = (12 – 6)$

$3a + b = 6$………………….(4)

eqn (3) – eqn (2)

$(9a – 4a) + (3b – 2b) + (c – c) = (20 – 12)$

$5a + b = 8$…………………(5)

eqn (5) – (4)

$(5a – 3a) + (b – b) = (8 – 6)$

$2a = 2$

$a = 1$

Put a = 1 into eqn (5)

$5 (1) + b = 8$

$b = 8 – 5 = 3$

Put a = 1 and b = 3 into eqn (1)

$1 + 3 + c = 6$

$4 + c = 6$

$c = 6 – 4 = 2$

Substitute a = 1, b = 3 and c = 2 in

$y = ax^2 + bx + c$ to get $y = x^2 + 3x + 2$

The rule is: $y = x^2 + 3x + 2$ or $x \rightarrow x^2 + 3x + 2$

2. Find the rule of the mapping below:

$$\begin{array}{cccccc} x & 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & 0 & 11 & 28 & 51 & 80 \end{array}$$

Solution

First difference = 11 – 0 = **11**, 28 – 11= **17**, 51 – 28 = **23**, 80 – 51 = **29**

Second difference = 17 – 11 = **6**, 23 – 17 = **6,** 29 – 23 = **6**

Because the second difference gives a common difference, d = **6,** it is a quadratic mapping following the rule;

$y = ax^2 + bx + c$ or $x \rightarrow ax^2 + bx + c$

For any 3 points (1, 0), (2, 11) and (3, 28)

When x = 1, y = 0, *by substitution*

$0 = a(1)^2 + b(1) + c$

$0 = a + b + c$………………..(1)

When x = 2, y = 11, *by substitution*

$11 = a(2)^2 + b(2) + c$

$11 = 4a + 2b + c$…………….(2)

When x = 3, y = 28

$28 = a(3)^2 + b(3) + c$

$28 = 9a + 3b + c$ ………….. (3)

The three equations are:

$a + b + c = 0$………………. (1)

$4a + 2b + c = 11$………….. (2)

$9a + 3b + c = 28$………… .(3)

eqn (2) – eqn (1);

$(4a – a) + (2b – b) + (c – c) = (11 – 0)$

$3a + b = 11$………………… .(4)

eqn (3) – eqn (2);

$(9a – 4a) + (3b – 2b) + (c – c) = (28 – 11)$

$5a + b = 17$………………… (5)

eqn (5) – (4) ;

$(5a - 3a) + (b - b) = (17 - 11)$

$2a = 6$

$a = 3$

Put $a = 3$ into eqn (5)

$5(3) + b = 17$

$15 + b = 17$

$b = 17 - 15 = 2$

Put $a = 3$ and $b = 2$ in eqn (1)

$3 + 2 + c = 0$

$5 + c = 0$

$c = -5$

Substitute $a = 3$, $b = 2$ and $c = -5$ in

$y = ax^2 + bx + c$ to get $y = 3x^2 + 2x - 5$

The rule is: $y = 3x^2 + 2x - 5$ or $x \rightarrow 3x^2 + 2x - 5$

Exercises 6.8

Find the rule of the mappings;

1.
$$\begin{array}{cccccc} x & 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & 3 & 8 & 15 & 24 & 35 \end{array}$$

2.
$$\begin{array}{cccccc} x & -3 & -1 & 1 & 3 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & 0 & -6 & 4 & 30 & 72 \end{array}$$

3.
$$\begin{array}{cccccc} x & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & 4 & 14 & 30 & 52 & 80 \end{array}$$

4.
$$\begin{array}{cccccc} x & -4 & -3 & -2 & -1 & 0 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & -73 & -41 & -19 & -7 & -5 \end{array}$$

5.
$$\begin{array}{cccccc} x & -2 & 0 & 2 & 4 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & -10 & -6 & -2 & 10 & 30 \end{array}$$

The Image Under A Given Mapping

Sometimes, mappings are given with some missing images. For e.g., in the mapping below, the image of 8 is given as m;

$$\begin{array}{ccccccc} x & 0 & 1 & 2 & 3 & 4\ldots\ldots & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & 5 & 7 & 9 & 11 & 13 & m \end{array}$$

To find the image of8 or the value of m under the mapping:

I. Find the rule of the mapping.

II. Use the rule of the mapping to find the value of y, when $x = 8$.

III. The value of y is the image of 8.

For example, the rule of the mapping above can be deduced as $y = 2x + 5$;

When $x = 8$, $y = 2(8) + 5$

$y = 16 + 5 = 21$

Since $8 \rightarrow 21$, $m = 21$

Worked Examples

1. In the mapping below, determine the image of 16

$$\begin{array}{ccccccc} x & 2 & 4 & 6 & 8\ldots & 16 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ y & 5 & 6 & 7 & 8\ldots & k \end{array}$$

Solution

For linear mappings, $y = mx + c$

Let $x_1 = 2$, $y_1 = 5$, $x_2 = 4$ and $y_2 = 6$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-5}{4-2} = \frac{1}{2}$

Put $m = \frac{1}{2}$ in $y = mx + c$ to obtain $y = \frac{1}{2}x + c$

When $x = 2$, $y = 5$, put in $y = \frac{1}{2}x + c$

$5 = \frac{1}{2}(2) + c$

$5 = 1 + c$

$5 - 1 = c$

$c = 4$

Put $c = 4$ in $y = \frac{1}{2}x + c$ to get $y = \frac{1}{2}x + 4$

The rule is; $y = \frac{1}{2}x + 4$ or $x \rightarrow \frac{1}{2}x + 4$

To find the image of 16 under that mapping,

Put $x = 16$ in $y = \frac{1}{2}x + 4$

$y = \frac{1}{2}(16) + 4$

$y = 8 + 4$

$y = 12$, so $k = 12$

Therefore, the image of 16 is 12

<u>Method II</u>

The image of 16 under the mapping is;

$16 \rightarrow \frac{1}{2}(16) + 4$

$16 \rightarrow 8 + 4$

$16 \rightarrow 12$, $k = 12$

Therefore, the image of 16 is 12

Exercises 6.9

In the mappings below, find the image of the values without images;

1.

x	9	12	15	18 …	33
y	4	6	8	10 …	k

2.

x	1	2	3	4	32
y	6	14	28	48	k

3.

x	1	2	3	4 …	9
y	5	15	45	135…	k

4.

-70	1	2	3	4	6	x
m	$-\frac{3}{4}$	-1	$-\frac{3}{2}$	-3	3	y

Word Problems

Given the rule of a mapping to find the image of a number under the mapping, substitute the number in place of x in the rule and workout to get the image of the number. That is; if $\boldsymbol{y = mx + c}$, then the value of $\boldsymbol{y}$ when $\boldsymbol{x = a}$, is the image of $\boldsymbol{a}$.

Worked Examples

1. Find the image of -4 under a mapping defined by the rule $x \rightarrow 5x + 3$

Solution

In $x \rightarrow 5x + 3$

When $x = -4$, $-4 \rightarrow 5(-4) + 3$

$-4 = -20 + 3$

$-4 \rightarrow -17$

The image of - 4 under the mapping is -17

2. A mapping is defined by $x \rightarrow 4x + 1$, find the images of $\frac{1}{2}$, $\frac{-1}{4}$ and $\frac{-1}{2}$ under the mapping.

Solution

i. $\frac{1}{2} \rightarrow 4\left(\frac{1}{2}\right) + 1 = 2 + 1 = 3$

ii. $\frac{-1}{4} \rightarrow 4\left(\frac{-1}{4}\right) + 1 = -1 + 1 = 0$

iii. $\frac{-1}{2} \rightarrow 4\left(\frac{-1}{2}\right) + 1 = -2 + 1 = -1$

3. Find the value of m in the mapping below;

x	-2	-1	0	2	3
y	m	-1	1	5	7

Solution

Since $7 - 5 = 2$

$-1 - m = 2$

$-1 - 2 = m$,

$-3 = m$ or $m = -3$

4. Find the image of -4 under the mapping:
$x \rightarrow 2x^2 + 5x - 3$

Solution

In $x \rightarrow 2x^2 + 5x - 3$, when $x = -4$

$-4 \rightarrow 2(-4)^2 + 5(-4) - 3$

$-4 \rightarrow 2(16) - 20 - 3$

$-4 \rightarrow 32 - 20 - 3$

$-4 \rightarrow 9$

The image of – 4 is 9

Exercises 6.10

1. A mapping is defined by the rule: $x \rightarrow 5x^2 - 3x - 2$. Find the images of the following under the mapping;

i. - 1 ii. - 4 iii. -7

2. Find the image of the following under the mapping defined by the rule: $n \rightarrow 12n + 4$

i. $\frac{5}{4}$ ii. $\frac{1}{12}$ iii. $\frac{-2}{3}$

3. Find the images of the domain {1, 3, 8} under a mapping defined by the rule, $x \rightarrow \frac{1}{5x+2}$. Show this mapping in a diagram and state the type of mapping.

Functions

A function is a special kind of relation in which each element of the first set called domain is paired with one and only one element of the second set called ***range***

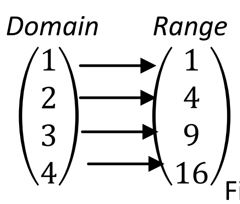


Fig. I

A member of the range which corresponds to a certain member of the domain is called the ***image*** of that member. For e.g. in the diagram above, 1 is the image of 1, 4 is the image of 2, 9 is the image of 3 and 16 is the image of 4.

In a function, two distinct members of the domain can have the same image, as seen in fig. II below; but not the other way round.

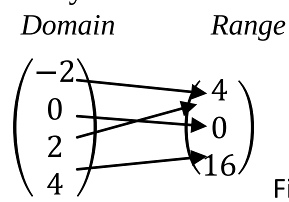


Fig. II

The Function Notation

The usual letters used to denote functions are *f*, *g*, *h* and their corresponding capital letters, but other letters may be used as well. For e.g., the statement *'f is function which maps x onto* x^3' is expressed in function notation as $f(x) = x^3$ or
$f : x \rightarrow x^3$

Types of Function

1. One – to - one function

When each member of the range has exactly one corresponding member of the domain the function is called *a one – to – one function* as in fig. I for $f : x \rightarrow x^2$

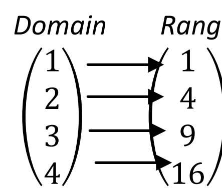


Fig. I

2. Many – to – one function

When a member of the range has two or more corresponding members in the domain the function is called *many – to – one function*, as in fig. II for $f : x \rightarrow x^2$

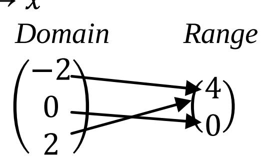


Fig. II

The Image under a given Function

For any set of ordered pairs such as (x, y) and (a, b), $f(x) = y$ and $f(a) = b$, means that the value of the function at x is y and the value of the function at a is b respectively.

In $f(x) = y$ or $f : x \to y$, ***the image of x is y.*** In other words, $f : a \to f(a)$, $f(a)$ is called the ***image*** of a under f, or the value of a at f. The set of values of $f(a)$ is called the ***range*** of the function, f

Worked Examples

1. Given that $f(x) = x^2 - x$, find $f(-10)$

Solution

If $f(x) = x^2 - x$,

$f(-10) = (-10)^2 - (-10)$

$= 100 + 10$

$= 110$

2. The domain of the function $g(x) = 5x + 1$ is {0, 1, 2, 3, 4, 5} find its range.

Solution

If $g(x) = 5x + 1$, then

$g(0) = 5(0) + 1 = 1$

$g(1) = 5(1) + 1 = 5 + 1 = 6$

$g(2) = 5(2) + 1 = 10 + 1 = 11$

$g(3) = 5(3) + 1 = 15 + 1 = 16$

$g(4) = 5(4) + 1 = 20 + 1 = 21$

$g(5) = 5(5) + 1 = 25 + 1 = 26$

The range of the function is:

$R = \{6, 11, 16, 21, 26\}$

3. If $f(x) = 2x - 3$ and $g(t) = \frac{t}{2} + 5$, find:

i. $f(4)$ ii. $g(-30)$ iii. $g(-30) - f(4)$

iv. $f(4) + g(-30)$ v. $\frac{g(-30)}{f(4)}$

Solution

If $f(x) = 2x - 3$ and $g(t) = \frac{t}{2} + 5$

i. $f(4) = 2(4) - 3 = 8 - 3 = 5$

ii. $g(-30) = \frac{-30}{2} + 5 = -15 + 5 = -10$

iii. $g(-30) - f(4) = -10 - 5 = -15$

iv. $f(4) + g(-30) = 5 + (-10) = 5 - 10 = -5$

v. $\frac{g(-30)}{f(4)} = \frac{-10}{5} = -2$

4. Find the element whose image under the function, $f : x \to 2x + 3$ is -9

Solution

(x, -9) under $f : x \to 2x + 3$ means

$f(x) = -9$

$2x + 3 = -9$

$2x = -9 - 3$

$2x = -12$

$x = -6$

5. A function f is defined as $f(x) = 3x^2 - 5x$

i. Evaluate $f(-3)$

ii. Find the values of x for which $f(x) = -\frac{3}{4}$

Solution

i. $f(x) = 3x^2 - 5x$

$f(-3) = 3(-3)^2 - 5(-3)$

$f(-3) = 3(9) + 15 = 42$

ii. If $f(x) = -\frac{3}{4}$

$\Rightarrow 3x^2 - 5x = -\frac{3}{4}$

$(4)3x^2 - (4)5x = -3$

$12x^2 - 20x = -3$

$12x^2 - 20x = -3$

$12x^2 - 20x + 3 = 0$

$12x^2 - 2x - 18x + 3 = 0$

$(12x^2 - 2x) - (18x + 3) = 0$

$2x(6x - 1) - 3(6x - 1) = 0$

$2x - 3 = 0$ or $6x - 1 = 0$

$x = \frac{2}{3}$ or $x = \frac{1}{6}$

6. The function f and g are defined as $f : x \rightarrow x - 2$ and $g : x \rightarrow 2x^2 - 1$. Solve:

i. $f(x) = g\left(\frac{1}{2}\right)$ ii. $f(x) + g(x) = 0$

Solution

i. $f : x \rightarrow x - 2$ and $g : x \rightarrow 2x^2 - 1$

$f(x) = g\left(\frac{1}{2}\right)$

$x - 2 = 2\left(\frac{1}{2}\right)^2 - 1$

$x - 2 = 2\left(\frac{1}{4}\right) - 1$

$x - 2 = \frac{1}{2} - 1$

$2x - 4 = 1 - 2$

$2x = -1 + 4$

$2x = 3$

$x = 1.5$

ii. $f(x) + g(x) = 0$

$(x - 2) + (2x^2 - 1) = 0$

$x + 2x^2 - 2 - 1 = 0$

$2x^2 + x - 3 = 0$

$2x^2 + 3x - 2x - 3 = 0$

$(2x^2 + 3x) - (2x - 3) = 0$

$x(2x + 3) - 1(2x + 3) = 0$

$x - 1 = 0$ or $2x + 3 = 0$

$x = 1$ or $2x = -3$

$x = 1$ or $x = -\frac{3}{2}$

Some solved Past Questions

1. The set P = {-2, -1, 0, 1, 2} maps onto Q by the function $f(x) = x^2 - 2$, where $x \in$ P.

i. Find the elements of Q.

ii. Draw a diagram showing the mapping between P and Q.

Solution

i. P = {-2, -1, 0, 1, 2}

$f(x) = x^2 - 2$,

$f(-2) = (-2)^2 - 2 = 2$

$f(-1) = (-1)^2 - 2 = -1$

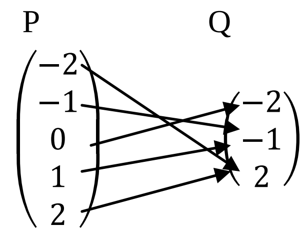
$f(0) = (0)^2 - 2 = -2$

$f(1) = (1)^2 - 2 = -1$

$f(2) = (2)^2 - 2 = 2$

Q = {-2, -1, 2}

ii.



2. A function f is defined as $f : x \rightarrow 3x^2 - 5x$

i. Evaluate $f(-3)$

ii. Find the values of x for which $f(x) = -\frac{3}{4}$

Solution

i. $f : x = 3x^2 - 5x$

$f(-3) = 3(-3)^2 - 5(-3)$

$f(-3) = 18 + 15$

$f(-3) = 32$

ii. $f(x) = -\frac{3}{4}$

$\Rightarrow 3x^2 - 5x = -\frac{3}{4}$

$12x^2 - 20x = -3$

$12x^2 - 20x + 3 = 0$

$12x^2 - 2x - 18x + 3 = 0$

$(12x^2 - 2x) - (18x + 3) = 0$

$2x(6x - 1) - 3(6x - 1) = 0$

$\Rightarrow (2x - 3)(6x - 1) = 0$

$2x - 3 = 0$ or $6x - 1 = 0$

$x = \frac{3}{2}$ or $x = \frac{1}{6}$

Exercises 6.11

A. Given that $g(x) = 1 + x^3$ and $f(x) = 2x^2$, find the values of;

1. $g(5)$ 2. $g\left(\frac{3}{4}\right)$ 3. $g(-2)$

4. $f\left(\frac{3}{4}\right)$ 5. $f(2) - g(-2)$ 6. $g(3) + f\left(\frac{1}{2}\right)$

B. 1. A function f is defined by $f(x) = \frac{6}{x^2 - 1}$, $x \in R$, but $x \neq \pm 1$

a. Calculate the values of f at 2, -3 and $\frac{1}{2}$

b. What element of the domain has the image 2?

2. A function f is defined by $f(x) = 3x - 1$, where $x \in R$.

a. What is the image of -3?

b. What is the value of $f\left(\frac{1}{3}\right)$?

c. Find the element of the domain whose image is 20

d. If $f(x) = -19$, find x

3. A function g is defined by $g : x \rightarrow x^2 + 1$, $x \in$ R,

a. Calculate the value of;

i. $g(-1)$ ii. g at 2

b. Find the image of zero under g

c. If $g(r) = 101$, find r

4. A function, f is defined by the $f(x) = \frac{x^2 + 8}{x - 4}$ ($x \neq$ 4), find the values of the function when:

i. x is 6 ii. x is - 4 iii. x is 0 iv. x is 1

5. Given that $f(x) = \frac{k}{x-1} + \frac{6}{x-2}$ and that $f(5) = 8$, calculate the value of: i. $k$ii. $f(4)$

6. A function t is defined by $t(x) = a(x + 2)$, $x \in$ R, a being a real number. If the image of 1 under t is -30, calculate the value of $t(4)$

7. If $f(x) = \frac{x}{ax + 2}$, find a such that $f(3) = 7$

8. Find the value of x which gives 6 for the value of the function defined by $y = \frac{x}{x - 1}$

9. A function f is defined by $f : x \rightarrow 8x^2 - 1$. Find the image of $2^{1/2}$ and 2^{-1} under f

10. Given that $f(x) = 2x^2 - 11x + k$ and that $f(3) = 0$, find the value of k and hence the value of $f(-1)$

11. Given that the function $f(x)$ is defined by $f(x) = (x - 2)(x - 3)$, solve $f(x) = 6$

12. Given that $f(x) = (x + 2)(x - 3)$, find:

i. $f(15)$

ii. the value of x for which $f(x) = 0$

iii. the range of values of x for which $f(x) < 0$

13. A function f is defined by $f : x \rightarrow 3^x - 3, x \in$ R.

i. Evaluate $f(0)$, $f(-1)$ and $f\left(\frac{1}{2}\right)$

ii. Find a such that $f(a) = 6$

14. The set P = {-2, -1, 0, 1, 2} maps onto Q by the function $f(x) = x^2 - 2$, where $x \in$ P

i. Find the elements of Q

ii. Draw a diagram showing the mapping between P and Q

C. Find the range of the function;

1. $f: x \rightarrow \frac{3}{5}x - 13$, Domain = $\{10,15,20,25\}$

2. $f : x \rightarrow x^2 + 2x - 8$, D = $\{0, 1, 2, 3, 4\}$

3. $f : x \rightarrow 20 + \frac{x}{3}$, D = $\{0, 12, 21, 30, 36\}$

Finding the Constants of a Function Given the Value of the Function at a Point

Given the values of a function at two or more given points, the constants of the functions represented by variables are found as follows:

I. Substitute the value of the function at the given points and equate to the value of the function at these points.

II. Obtain two or more equations of the same function and name them accordingly.

III. Solve the equations simultaneously to obtain the values of the variables (constants).

Worked Examples

1. Given that $f(x) = px + q$, find the values of p and q, if $f(2) = 4$ and $f(4) = 10$

Solution

$f(x) = px + q$,

$f(2) = 4$

$\Rightarrow f(2) = 2p + q = 4$

$2p + q = 4$ ……………(1)

$f(4) = 10$

$\Rightarrow f(4) = 4p + q = 10$

$4p + q = 10$ …………..(2)

eqn (2) – eqn (1)

$2p = 6$

$p = 3$

Substitute $p = 3$ in eqn (1)

$2(3) + q = 4$

$6 + q = 4$

$q = 4 – 6 = -2$

The function is $f(x) = 3x – 2$

2. Given that $f(x) = ax^2 – bx$, find the values of a and b, if $f(3) = 18$ and $f(-2) = - 4$

Solution

$f(x) = ax^2 – bx$

$f(3) = a(3)^2 – 3b$, but $f(3) = 18$

$\Rightarrow 9a – 3b = 18$ …………….(1)

$f(-2) = a(-2)^2 – (-2)b$, but $f(-2) = - 4$

$\Rightarrow 4a + 2b = - 4$ …………….(2)

eqn (1) × 2

$18a – 6b = 36$ …………….(3)

eqn (2) × 3

$12a + 6b = - 12$…………….(4)

eqn (3) + eqn(4)

$30a = 24$

$a = \frac{4}{5}$

Put $a = \frac{4}{5}$ in *eqn* (1)

$9\left(\frac{4}{5}\right) – 3b = 18$

$36 – 15b = 90$

$- 15b = 90 – 36$

$- 15b = 54$

$b = - \frac{18}{5}$

$f(x) = \frac{4}{5}x^2 + \frac{18}{5}x$

Exercises 6.12

1. A function h is defined by $h(x) = ax + b$, *where* $x \in$ R, and a and b are real numbers. If $h(0) = – 8$ and $h(6) = 22$, find the values a and b, and hence calculate $h(–1)$
2. $f(x)$ is defined by $f(x) = bx^2 + cx$, where b and c are constants. Given that $f(1) = 8$ and $f(2) = 22$, calculate the values of b and c. Hence solve the equation $f(x) = 2$
3. $f(x) = \frac{ax + b}{x - 1}$. Find a and b if $f(-1) = 2.5$ and $f(2) = 1$
4. $f(x) = \frac{bx + 2}{x^2 + b}$. Find b if (-2, 1) is an ordered pair of the function
5. Given that the function $f : x \rightarrow ax^2 – 2bx – 4$ maps $1 \rightarrow 3$, and $3 \rightarrow 35$, show that $a = 3$ and find the value of b

An Undefined Function

A function is said to be undefined when the denominator of the function is zero. i.e. $\frac{a}{b}$, $b = 0$. Therefore, for a function to be defined the denominator must not be equal to zero. i.e. $\frac{a}{b}$, $b \neq 0$

To determine the value for which a function is undefined,

I. Equate the denominator of the function to zero.

II. Solve for the value of the variable.

III. The value of the variable makes the function undefined.

IV. If the value of the variable is *a*, then we state $x \neq a$.

Worked Examples

1. Determine the value of x for which the function $f : x \rightarrow \frac{x^2-3}{2x+4}$ is undefined.

Solution

For $f : x \rightarrow \frac{x^2-3}{2x+4}$ to be undefined,

$\Rightarrow 2x + 4 = 0$,

$2x = -4$ (Solve for x)

$x = -2$

The function is undefined if $x = -2$

2. What values of x makes the function

$f : x \rightarrow \frac{2x}{(x+2)(x-3)}$ undefined?

Solution

For $f : x \rightarrow \frac{2x}{(x+2)(x-3)}$ to be defined,

$\Rightarrow (x + 2)(x - 3) = 0$,

$x + 2 = 0$ or $x - 3 = 0$

The values of x that makes the function undefined is $x = -2$ or $x = 3$

3. For what value of x is the function $f(x) = \frac{x}{3x-2}$ undefined?

Solution

For $f : x \rightarrow \frac{x}{3x-2}$ to be undefined,

$\Rightarrow 3x - 2 = 0$,

$3x = 2$

$x = \frac{2}{3}$

The function is undefined if $x = \frac{2}{3}$

Exercises 6.13

State the value(s) of *x* for which the function is undefined;

1. $f(x) = \frac{2+x}{x-3}$
2. $f(x) = \frac{5x}{x^2-9}$
3. $f(x) = \frac{2x-5}{(x-5)(x-2)}$
4. $f(x) = \frac{x+3}{x^2-7x+12}$

The Zeros of a Function

The zeros of a function is the value(s) of the variable that makes the numerator of the function zero, hence the whole function equals to zero. i. e. $f(x) = \frac{a}{b}$, $a = 0 \Rightarrow f(x) = 0$

To determine the zeros of a function:

I. Equate the numerator of the function to zero

II. Solve for the value of the variable.

III. The value(s) of the variable is/are the zeros of the function.

IV. If the value of the variable is *a*, then we state $x \neq a$

Worked Examples

1. Determine the zeros of the function:

$f : x \rightarrow \frac{x^2-36}{2x+4}$

Solution

$f : x \rightarrow \frac{x^2-36}{2x+4}$

$x^2 - 36 = 0$

$(x - 6)(x + 6) = 0$

$x = 6$ or $x = -6$

2. For what value of x is the function, $f(x) = \frac{1-5x}{2x}$ equal to zero

Solution

If $f(x) = \frac{1-5x}{2x}$ equal to zero

$\Rightarrow 1 - 5x = 0$

$1 = 5x$

$x = \frac{1}{5}$

3. Find the zeros $f(x) = \frac{x^2+7x+10}{5x-2}$

Solution

$f(x) = \frac{x^2+7x+10}{5x-2}$

$x^2 + 7x + 10 = 0$

$(x^2 + 5x) + (2x + 10) = 0$

$x(x + 5) + 2(x + 5) = 0$

$(x + 2)(x + 5) = 0$

$x + 2 = 0$ or $x + 5 = 0$

$x = -2$ or $x = -5$

4. What are the zeros of $f : x \rightarrow 3x + 2$?

Solution

$3x + 2 = 0$

$3x = -2$

$x = -\frac{2}{3}$

Exercises 6.14

State the zeros of the functions

1. $f : x \rightarrow \frac{5x+2}{x-5}$
2. $f : x \rightarrow \frac{(x-2)(x+5)}{2x-4}$
3. $f : x \rightarrow \frac{3x^2-27}{1-7x}$
4. $f : x \rightarrow \frac{x^2-7x+12}{x-5}$
5. $f : x \rightarrow \frac{1}{3}x - 2$
6. $f : x \rightarrow \frac{2}{3}x + 10$

The Inverse of a Function

The inverse rule of a given mapping is found by:

I. Interchange the positions of x and y in the function

II. Let y equal f^{-1}, the inverse of the function

III. Make y the subject of the relation, to get the inverse of the function.

Worked Examples

1. What is the inverse rule of the function, $f : x \rightarrow 3x + 2$?

Solution

Let $f(x) = y$

$\Rightarrow y = 3x + 2$

In $y = 3x + 2$,

Switch the positions of x *and* y,

$x = 3y + 2$.

Make y the subject of $x = 3y + 2$,

$x - 2 = 3y$

$\frac{x-2}{3} = \frac{3y}{3}$

$y = \frac{x-2}{3}$

The inverse function is:

$y = \frac{x-2}{3}$ or $f^{-1} : x \rightarrow \frac{x-2}{3}$

2. What is the rule for the inverse of the function $f : x \rightarrow \frac{x+2}{x-5}$, $x \neq 5$

Solution

Let $y = \frac{x+2}{x-5}$, $x \neq 5$

switch the positions of x *and* y

$x = \frac{y+2}{y-5}$, $x \neq 5$

$x(y - 5) = y + 2$

$xy - 5x = y + 2$

$xy - y = 2 + 5x$

$y(x - 1) = 2 + 5x$

$y = \frac{2+5x}{x-1}$

The inverse rule of the function is:

$y = \frac{2+5x}{x-1}$, $x \neq 1$ or $f^{-1}: x \rightarrow \frac{2+5x}{x-1}$, $x \neq 1$

Exercises 6.15

A. Find the inverse rule:

1. $f : x \rightarrow 5x + 13$
2. $f : x \rightarrow 10 + 4x$
3. $f : x \rightarrow 5x - 8$
4. $f : x \rightarrow 9 - 6x$

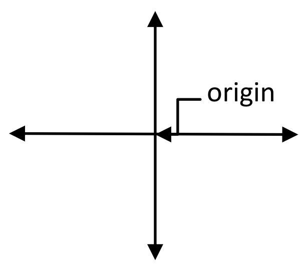
B. Determine the inverse rule:

1. $f : x \rightarrow 11x + \frac{2}{3}$ 2. $f : x \rightarrow \frac{4}{3x-5}, x \neq \frac{5}{3}$

3. $f : x \rightarrow \frac{1}{6}x - 1$ 4. $f : x \rightarrow \frac{1-4x}{x-2}, x \neq 2$

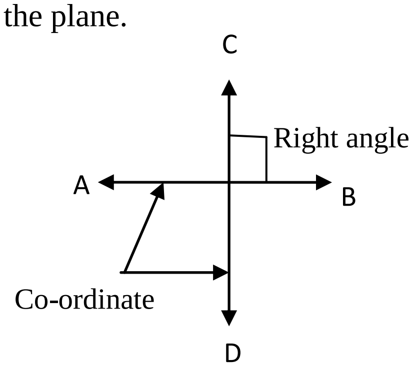
The Number Plane or Cartesian Plane

A number plane, also known as ***Cartesian plane*** is formed when a vertical line intersects a horizontal line. The point of intersection is called the ***origin***, usually denoted by O.



Co-ordinate Axes

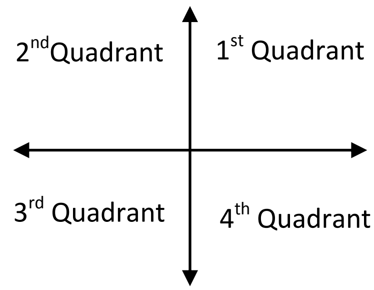
The lines that intersect at right angles forming the number plane are called the ***co- ordinate axes*** of the plane.



AB and *CD* are called ***co-ordinate axes of the plane***

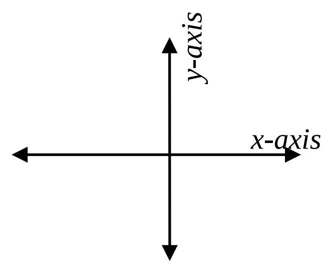
Quadrants

The co-ordinate axes divide the number plane into four segments in section called ***Quadrants***. The quadrants are viewed in the anticlockwise direction as shown in the diagram below;



Naming the Axes

The vertical co-ordinate axes is called the ***y – axis*** and the horizontal co-ordinate axis is called the ***x – axis***.

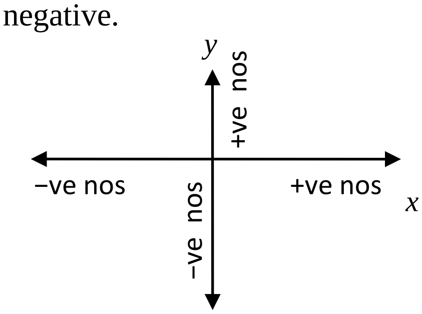


Numbering the Axes

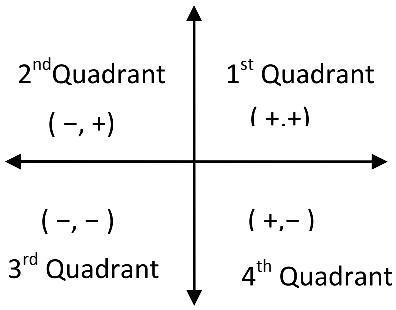
The *x* and *y* axes are divided into equal segment by a given scale. The point at which the *x* - axis intersects with the *y* - axis is called the ***origin*** and is labeled '0' (zero).

The numbers above the *x* – axis are positive. The numbers below the *x* – axis are negative.

The numbers to the right of the *y* – axis are positive and the numbers to the left of *y* – axis are negative.



The diagram below illustrates the four main ideas:



Scale of a Graph

The scale of a graph is the ratio of the space/distance/ interval between the lines to the space/distance/ interval between the numbers.

On the graph sheet, each line is separated from the other by a distance of 0.1cm.

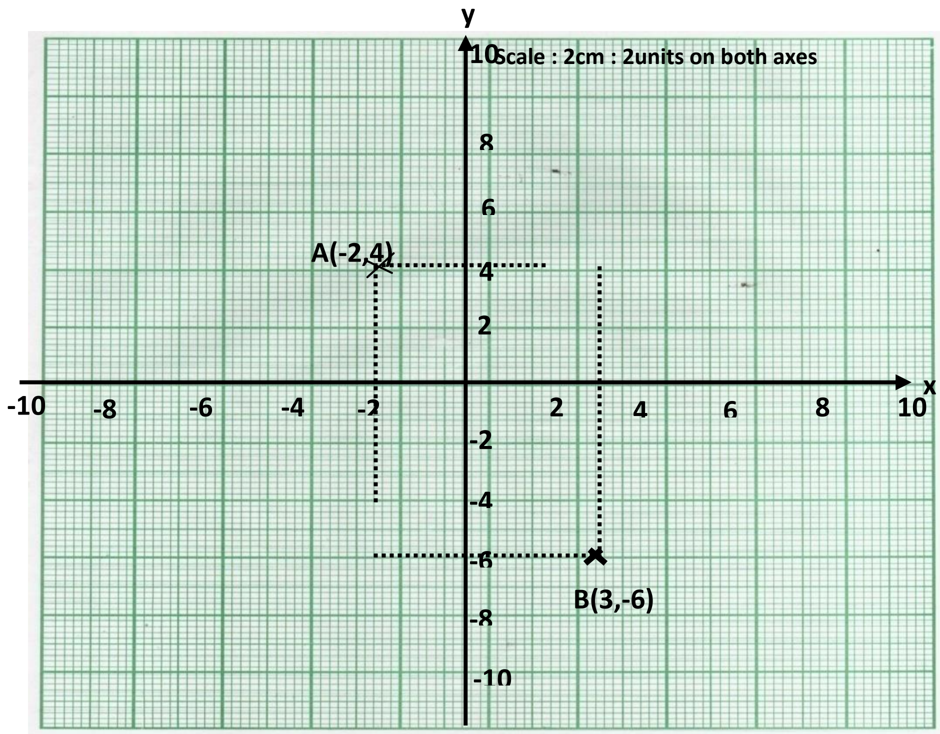
Each axis of the graph is numbered according to a given scale. Thus, a scale of ***a cm : b units*** on a particular axis implies that starting from the origin, mark an equal space or interval of ***a cm*** on that axis and ***b units*** implies that starting from the origin, number on each of the marked intervals of that axis, leaving ***b*** space(s) or intervals between the numbers. The first number on the positive axes is ***b*** *for* ***b units.***

Plotting Points on the Number Plane

Each point on the number plane can be described by an ordered pair in which the first element is the x – coordinate and the second element, the y – coordinate. It is always a good idea to start at the origin (0, 0) when plotting points on the number plane..

The ordered pair A (-2, 4), is found in the coordinate system when you move 2 steps to the left on the x – axis and 4 steps upwards on the y – axis

Similarly, the ordered pair A (3, –6), is found in the coordinate system when you move 3 steps to the right on the x – axis and 6 steps downwards on the y – axis



Exercises 6.16

Draw a number plane on a graph sheet for -10 $\leq x \leq 10$ and values and -12 $\leq y \leq 12$ and plot the following points on it:

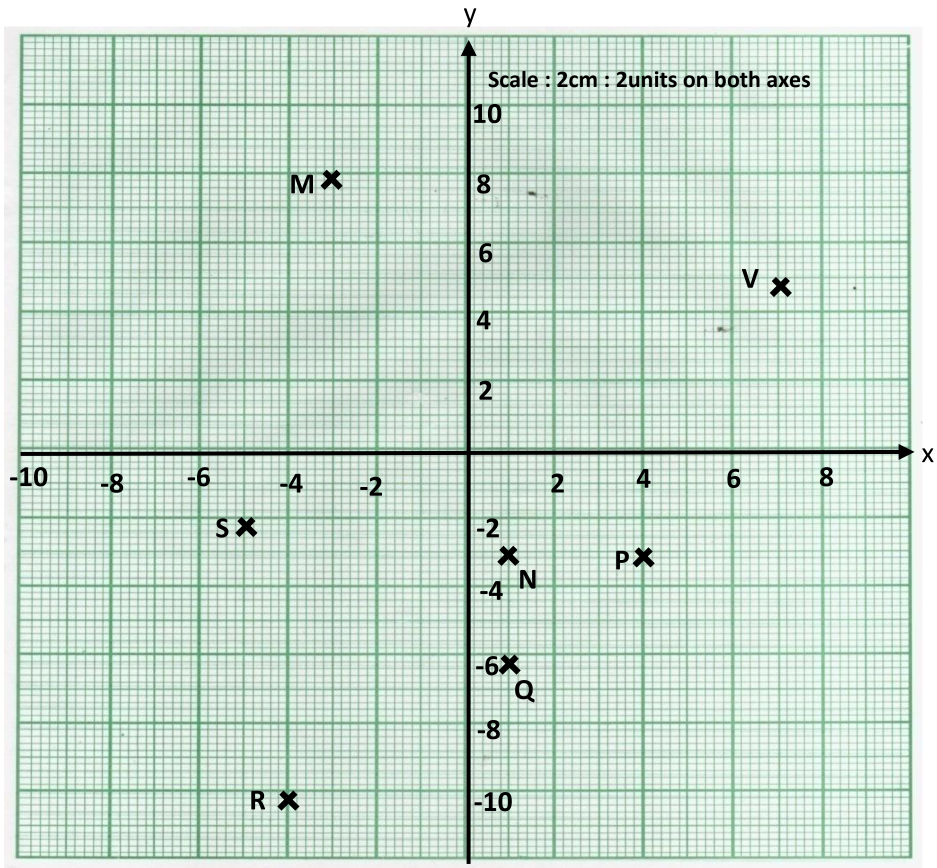
1. C($\frac{1}{2}$,3) 2. P(1,7) 3. Q($3\frac{1}{2}$, $6\frac{1}{2}$)

4. R(-5, 2) 5. S(4, 5) 6. T(-5,-5)

7. V(0,-3) 8. U(4,-7) 9. W(-2,-1)

Identifying the Coordinate of a Point on a Number Plane

To find out the coordinate of a point in the coordinate system, begin at that point and follow a vertical line either up or down to the x – axis. There is your x – coordinate. And do the same but following a horizontal line to find the y – coordinate. Write the point as ordered pair (x, y)



In the diagram above. the coordinate of the various points are P(4,-3), Q(1,-6), N(1,-3), M(-3,8), S(-5,-2), R(-4,-10) and V(7, 5)

Graph of a Linear Function

The word **"*linear*"** refers to relations without product of a variable and a variable. A linear function is therefore a function in which the degree of exponent of the variable is not more than one.

A function is said to be linear if it written in the form $f(x) = mx + c$ or $f : x \to mx + c$, where m and c are constants. Graphs obtained from linear functions are called ***graphs of linear function***

Drawing a Linear Graph

To draw the graph of a linear function,

I. Represent the function by y and the elements of the domain by x. That is $f(x) = y$

II. Substitute the values of x into the function to obtain the values of y.

III. Prepare a two column table of values for x and y as shown below:

x						
y						

IV. Plot the points described by the table of values on a standard graph sheet using a given or appropriate scale.

V. Join the points with a rule to form ***astraight line.***

Worked Examples

1. i. Make a table of values for $f : x \rightarrow 3x - 4$ for the domain {1, 2, 3, 4, 5}
ii. Plot the points on a standard graph using a scale of 2cm to 1 unit on x –axis and 2cm to 2 units on y – axis and join the points with a ruler to form a straight line

Solution

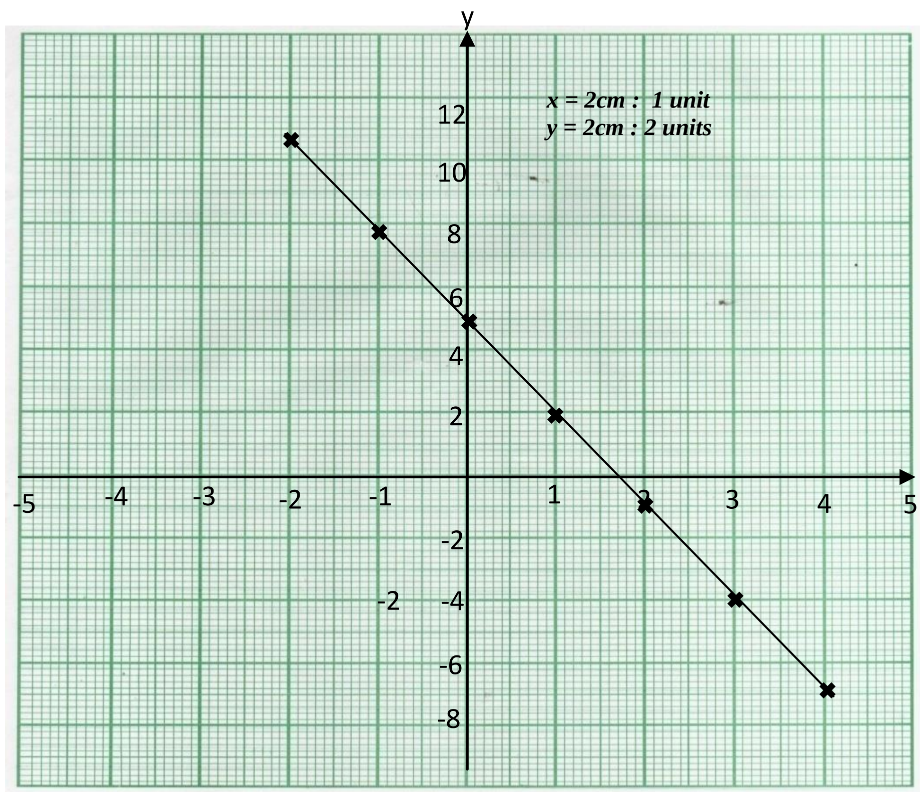
i. Let $f(x) = y$

$y = 3x - 4$, Domain {1, 2, 3, 4, 5}

When $x = 1$ $\quad 1 \rightarrow 3(1) - 4 = -1$
When $x = 2$ $\quad 2 \rightarrow 3(2) - 4 = 2$
When $x = 3$ $\quad 3 \rightarrow 3(3) - 4 = 5$
When $x = 4$ $\quad 4 \rightarrow 3(4) - 4 = 8$
When $x = 5$ $\quad 5 \rightarrow 3(5) - 4 = 11$

Table of values is:

x	1	2	3	4	5
$3x - 4$	-1	2	5	8	11



2. i. Make a table of values for the function $y = 5 - 3x$ for the domain { -2, -1, 0, 1, 2, 3, 4} and plot the set of points on a graph sheet using a scale of 2cm to 1 unit on x – axis and 2cm to 2 units on y – axis. Join the points with a rule to form a straight line.

Solution

$f : x \rightarrow 5 - 3x,$

Let $f(x) = y$

$\Rightarrow y = 5x - 3$

Domain$\{-2, -1, 0, 1, 2, 3, 4\}$

When $x = -2$, $y = 5 - 3(-2) = 11$

When $x = -1$, $y = 5 - 3(-1) = 8$

When $x = 0$, $y = 5 - 3(0) = 5$

When $x = 1$, $y = 5 - 3(1) = 2$

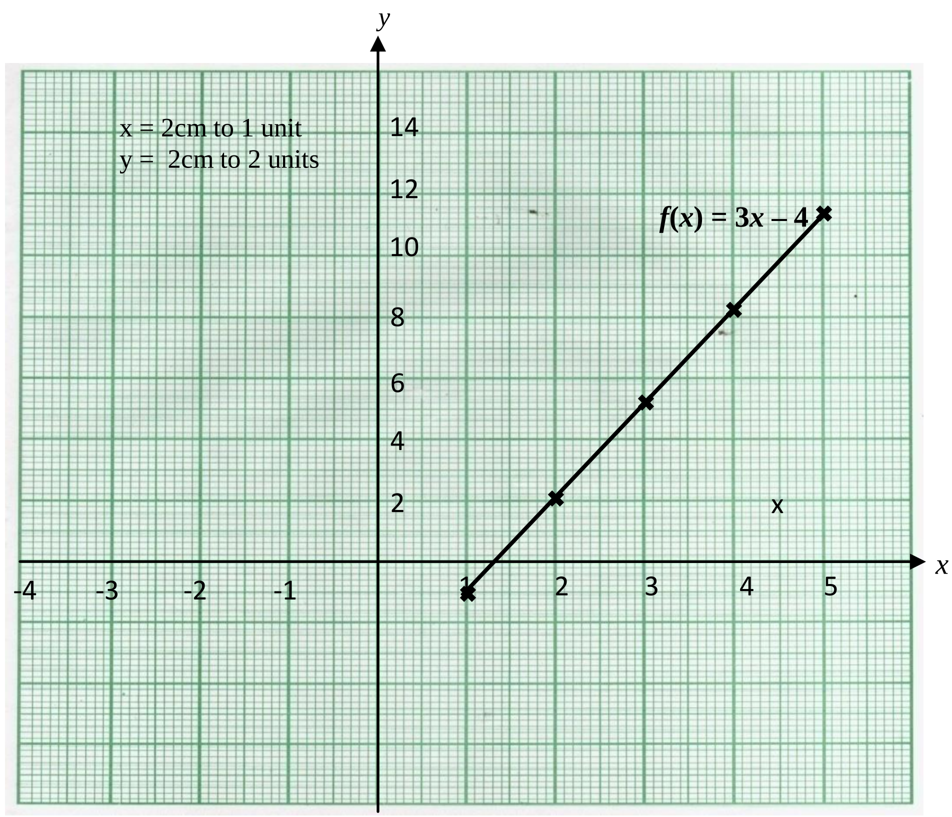
When $x = 2$, $y = 5 - 3(2) = -1$

When $x = 3$, $y = 5 - 3(3) = -4$

When $x = 4$, $y = 5 - 3(4) = -7$

$\therefore$ Table of values is:

x	-2	-1	0	1	2	3	4
y	11	8	5	2	-1	- 4	-7



Exercises 6.18

1. Draw a graph of $y = \frac{3}{5}x - 13$, for the domain $= \{-5, 0, 5, 15, 25, 30\}$, using a scale of 2cm to 5 units on the x – axis and 2cm to 2 units on the y – axis

2. Draw the graph of the relation $y = 2x + 1$ and $y = x + 2$ on the same graph sheet for the range $-3 \leq x \leq -3$, using a scale of 2cm to 2 units on both axes.

3. Copy and complete the table below for the relation $x + y = 180$;

x	0	30	60	90	120	150	180
y	180			90			0

ii. Using a scale of 2cm to 20 units on both axes, mark both axes from 0 to 180.
iii. Plot all the points on the graph sheet and join them with a ruler to form a straight line.
iv. Use your graph to find:
a. y when $x = 100$ *b.* x when $y = 70$

4. Copy and complete the table below for the relation $y = 2x + 3$

x	0	1	2	3	4	5	6
y		5	7				15

i. Using a scale of 2cm to 1 unit on x – axis and 2cm to 2 units on y – axis on a graph sheet, mark the x – axis from 0 to 5 and y – axis from 0 to 12.
ii. Plot on the graph sheet, the ordered pairs (x, y) from the table.
iii. With the help of a ruler, draw a straight line through all the points.
iv. From your graph, find y when $x = 3.5$

5. i. Copy and complete the table of the values for the relations; $y_1 = 2x + 5$ and $y_2 = 3 - 2x$ for x from - 4 to 3.

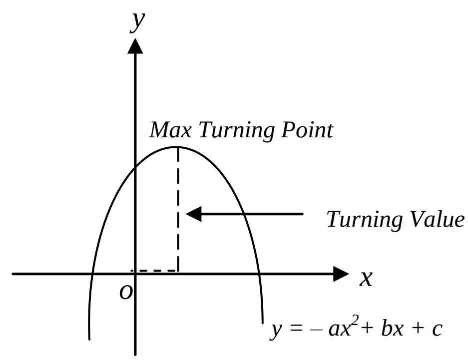
x	-4	-3	-2	-1	0	1	2	3
$y_1 = 2x + 5$	-3			3		7		4
$y_2 = 3 - 2x$	11	9		5				

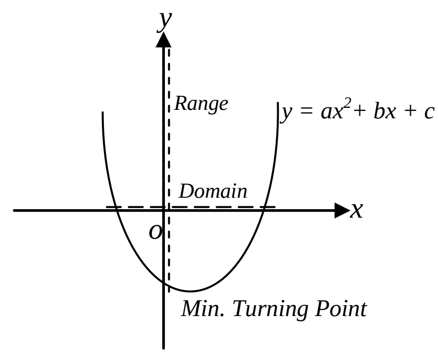
ii. Plot the graph of each relation on the same graph sheet using a scale of 2cm to 1 unit on the x – axis and 2cm to 2 units on the y – axis.
iii. Find the coordinates of the point of intersection of y_1 and y_2.
iv. Determine the shape of $y_1 = 2x + 5$.

Graph of a Quadratic Function

A function f, for which $f(x) = ax^2 + bx + c$, where $a, b, c \in R$ and $a \neq 0$, is called a ***quadratic function***

The Cartesian graph of every quadratic function is a ***Parabola of U or ∩ - shape***





The Graph of a Quadratic Function

To draw the graph of a quadratic function, make sure the function is in the form:
$f(x) = ax^2 + bx + c,\ a \neq 0$

Method I

I. Prepare a table of values for the given domain (x) of the function to obtain the values of the co – domain (y) of the function as shown below:

x	x_1	x_2	x_3	x_4
ax^2				
$+ bx$				
$+ c$				
y	y_1	y_2	y_3	y_4

II. Plot the points (x, y) on a graph sheet, using a given scale, or a convenience scale and join the points to make a free hand sketch of the required parabola

Method II

I. Substitute the values of the domain (x) into the function to obtain the values of the co – domain (y) and prepare a two column table of values for the ordered pairs (x, y) as shown below:

x			
y			

II. Plot the points (x, y) on a graph sheet, using a given scale, or a convenience scale and join the points to make a free hand sketch of the required parabola

Worked Examples

1. Prepare a table of values for $f(x) = x^2 + 1$ and the domain = {-1, 0, 1, 2, 3}

Solution

Method I

$f(x) = x^2 + 1$

Let $f(x) = y$

$\Rightarrow y = x^2 + 1$

x	-1	0	1	2	3
x^2	1	0	1	4	9
+ 1	1	1	1	1	1
y	2	1	2	5	10

Method II

$f(x) = x^2 + 1$

Let $f(x) = y$

$y = x^2 + 1$

Domain = {-1, 0, 1, 2, 3}

When $x = -1$, $-1 \rightarrow (-1)^2 + 1 = 2$

When $x = 0$, $0 \rightarrow (0)^2 + 1 = 1$

When $x = 1$, $1 \rightarrow (1)^2 + 1 = 2$

When $x = 2$, $2 \rightarrow (2)^2 + 1 = 5$

When $x = 3$, $3 \rightarrow (3)^2 + 1 = 10$

$\therefore$ Table of values is

x	-1	0	1	2	3
$x^2 + 1$	2	1	2	5	10

2. For the function $f(x) = x^2 + 2x - 8$ prepare a table of values for $-3 \leq x \leq 3$

Solution

Method 1

$f(x) = x^2 + 2x - 8$

Let $f(x) = y$

$\Rightarrow y = x^2 + 2x - 8$

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$2x$	-6	-4	-2	0	2	4	6
-8	-8	-8	-8	-8	-8	-8	-8
y	-5	-8	-9	-8	-5	0	7

Method II

$f(x) = x^2 + 2x - 8$

Let $f(x) = y$

$\Rightarrow y = x^2 + 2x - 8$

When $x = -3$, $y = (-3)^2 + 2(-3) - 8 = -5$

When $x = -2$, $y = (-2)^2 + 2(-2) - 8 = -8$

When $x = -1$, $y = (-1)^2 + 2(-1) - 8 = -9$

When $x = 0$, $y = (0)^2 + 2(0) - 8 = -8$

When $x = 1$, $y = (1)^2 + 2(1) - 8 = -5$

When $x = 2$, $y = (2)^2 + 2(2) - 8 = 0$

When $x = 3$, $y = (3)^2 + 2(3) - 8 = 7$

$\therefore$ The table of values is

x	-3	-2	-1	0	1	2	3
y	-5	-8	-9	-8	-5	0	7

Exercises 6.19

Prepare a table of values for the domain

1. $x^2 - 4x + 3 = 0$, D = {-4. -2, 0, 2, 4, 6}
2. $- x^2 + 2x + 3 = 0$ D = -5, -4, -3, -2, -1, 0}
3. $x^2 - 4x + 3 = 0$ D = {-3, 0, 3, 6, 9}

Completing a Given Table of Values

To complete a given table of values for a given quadratic function,

I. Identify the function

II. Substitute the values of the domain without images into the function

III. Workout for the images of the domain as described by the function

IV. Copy the table and fill in all gaps with their respective images to complete the table

Worked Examples

1. Copy and complete the table of values for
$f(x) = 3x^2 - 5x + 4$ for $-3 \leq x \leq 3$

x	-3	-2	-1	0	1	2	3
$3x^2$		12	3		3	12	27
$-5x+ 4$			9		-1		-11
$f(x)$	46	26		4			

Solution

In $3x^2$, when $x = -3$, $3(-3)^2 = 27$

When $x = 0$, $3(0)^2 = 0$

In $-5x + 4$, when $x = -3$, $-5(-3) + 4 = 19$

When $x = -2$, $-5(-2) + 4 = -6$

When $x = 0$, $-5(0) + 4 = 4$

When $x = 2$, $-5(2) + 4 = -6$

$f(-1) = 3 + 9 = 12$ $\quad f(1) = 3 - 1 = 2$

$f(2) = 12 - 6 = 6$ $\quad f(3) = 27 - 11 = 16$

x	-3	-2	-1	0	1	2	3
$3x^2$	27	12	3	0	3	12	27
$-5x+ 4$	19	14	9	4	-1	-6	-11
$f(x)$	46	26	12	4	2	6	16

2. Copy and complete the table below for
$f(x) = x^2 - 5x + 2$ for $-2 \leq x \leq 4$

x	-2	-1	0	1	2	3	4
$f(x)$	16		2			-4	-2

Solution

$f(x) = x^2 - 5x + 2$

When $x = -1$

$f(-1) = (-1)^2 - 5(-1) + 2 = 8$

When $x = 1$

$f(1) = (1)^2 - 5(1) + 2 = -2$

When $x = 2$,

$f(2) = (2)^2 - 5(2) + 2 = - 8$

x	-2	-1	0	1	2	3	4
$f(x)$	16	8	2	-2	-8	-4	-2

Exercises 6.20

1. The following is an incomplete table of values for the relation $y = \frac{x + 5}{x^2 + 1}$

x	-3	-2	-1	0	0.2	0.5	1	2	3
y	0.2				5			1.4	

Copy and complete the table

2. Copy and complete the table below for the relation $x^2 - 12x + 3 = 0$

x	-1.5	-1.0	-0.5	0	0.5	1.0
y				3.00		

3. Copy and complete the following table of values for the relation $y = x^2 - 3x + 7$

x	0.5	1	2	3	4	5.0
y				7.00		17.00

4. Copy and complete the table of values for the relation y = $x^2 - 5x - 2$ in the interval $-1 \leq x \leq 6$

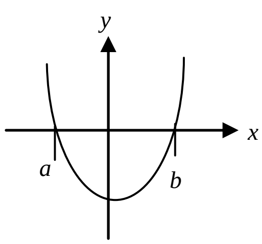
x	-1	0	1	2	3	4	5
y	4			-8		-6	

The Roots or Zeros of Quadratic Function

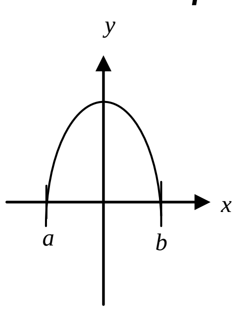
The roots of a quadratic function is the value(s) of x for which the function $f(x) = 0$ or $y = 0$. The root of the quadratic function is also called the ***zerosof the function***.

From the graph, the roots of a function is determine with cognizance to the nature of the parabola in relation to the x – axis

I. Whether U or ∩ – shaped, the points at which the parabola cuts the x – axis is the roots or zeros of the function as shown below,

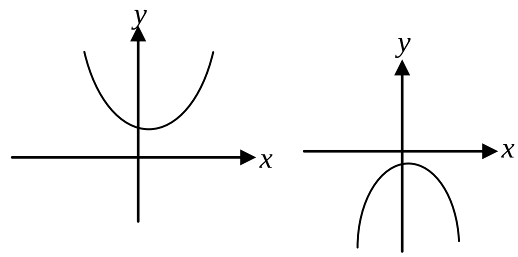


The zeros or truth set of the function is x = a or x = b



The zeros or truth set of the function is x = a or x = b

II. Whether U or ∩ – shaped, if the parabola does not cut the x – axis, the function is said to have no roots or no zeros as shown below;



In both diagrams, the functions have no roots or no zeros

Equation of Axes or Line of Symmetry

Parabolas can be described as being symmetrical, meaning that a line can be drawn through a parabola, to divide it into two equal parts, creating mirror images of each other. The straight line bisecting the parabola is called ***a line of symmetry***.

For all quadratic functions of the form: $f(x) = ax^2 + bx + c$, $a \neq 0$, the line of symmetry has the equation, $\boldsymbol{x = -\frac{b}{2a}}$

The Minimum and Maximum Points

The vertex of a parabola is the point of intersection of the line of symmetry and the parabola itself. The vertex is the turning point: either maximum (highest) or minimum (lowest) point of the parabola.

When the parabola is U – shaped, it is said to have *a minimum or least turning point* and when it is ∩ – shaped, it is said to have *a maximum or greatest turning point*

Values of *x* and *y* at the Turning Point

For all functions of the form:

$f(x) = ax^2 + bx + c$, $a \neq 0$, $f(x) = y$, at the turning point, $x = -\frac{b}{2a}$.

To get the value of y at the turning point, substitute $x = -\frac{b}{2a}$ *in* $y = ax^2 + bx + c$

The value of y is called the ***maximum or minimum value***, depending on the nature of the parabola.

Worked Examples

1. What is the value of x and y at the turning point of $f(x) = 2x^2 - 8x + 3$

Solution

In $f : x \rightarrow 2x^2 - 8x + 3$, $a = 2$ and $b = -8$

At the turning point, $x = -\frac{b}{2a} = -\frac{(-8)}{2 \times 2} = 4$

Let $f(x) = y$

$\Rightarrow y = 2x^2 - 8x + 3$

Put $x = 4$ in $y = 2x^2 - 8x + 3$

$y = 2(4)^2 - 8(4) + 3 = 3$

$\therefore$At the turning point of $2x^2 - 8x + 3 = 0$,

$x = 4$ and $y = 3$

The Range of a Function from the Graph

The ***"range"*** is the interval from the least value of y to the greatest value of y. When stating the range, the lower value (L.v) is written first, followed by the greatest value (G.v):

i. Using the brackets = [L.v, G.v]

ii. Using an inequality = L.v $\leq f(x) \leq$ G.v

iii. Listing from the lowest value to the greatest value e.g. Range = $\{a, b, c, d\}$

The Domain of a Function from the Graph

The ***domain*** is the interval from the least value of x to the greatest value of x. When stating the domain, the lower value (L.v) is written first, followed by the greatest value (G.v):

i. Using the brackets = [L.v, G.v]

ii. Using an inequality = L.v $\leq f(x) \leq$ G.v

iii. Listing from the lowest value to the greatest value e.g. Domain = $\{a, b, c, d\}$

Worked Examples

1. Draw the graph of $f(x) = 3x^2 - 5x + 4$ for the domain $\{x: -3 \leq x \leq 4\}$ using a scale of 2 cm : 1 unit on x – axis and 2cm : 5 units on y – axis. Use your graph to estimate:

a. the coordinates of the turning point of the graph,

b. the maximum or minimum value of the function,

c. the zeros of f,

d. the range of f for the given domain,

e. the equation of the axis of symmetry of the parabola.

Solution

<u>Method I</u>

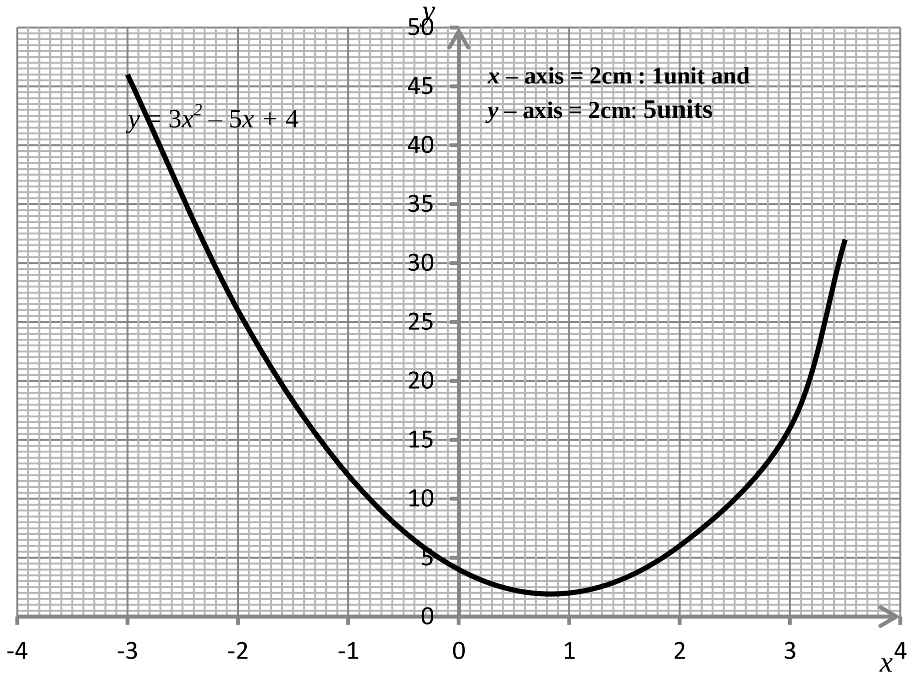
$f(x) = 3x^2 - 5x + 4$

Let $f(x) = y$

$\Rightarrow y = 3x^2 - 5x + 4$

$x = \{-3, -2, -1, 0, 1, 2, 3, 4\}$

x	-3	-2	-1	0	1	2	3	4
$3x^2$	27	12	3	0	3	12	27	48
$-5x$	15	10	5	0	-5	-10	-15	-20
+ 4	4	4	4	4	4	4	4	4
y	46	26	12	4	2	6	16	32



a. The coordinates of the turning point

At the turning point, $x = -\frac{b}{2a}$.

But in $y = 3x^2 - 5x + 4$, *a = 3, b = -5*

By substitution,

$x = -\frac{-5}{2(3)} = -\frac{-5}{6} = \frac{5}{6} = 0.8$

Put $x = 0.83$ in $y = 3x^2 - 5x + 4$,

$y = 3(0.8)^2 - 5(0.8) + 4$

$y = 1.92 - 4 + 4$

$y = 1.92$

Therefore, the turning point is (0.8, 2)

b. The function has a minimum value because it is U – shaped. The minimum value of $y = 3x^2 - 5x + 4$ is 2.

c. The zeros of *f* is the value(s) of *x* that makes *y* = 0. The curve does not cut the *x* – axis. Therefore, *f* has no zeros.

d. The range of *f* for the given domain is

$2 \leq y \leq 46$.

e. The equation of the axis of symmetry :

$x = -\frac{b}{2a}$.

In $y = 3x^2 - 5x + 4$, *a = 3, b = -5*

$x = -\frac{-5}{2(3)} = -\frac{-5}{6} = \frac{5}{6} = 0.8$

The equation of the axis of symmetry is $x = 0.8$

Exercises 6.21

1. a. Draw the graph of $y = x(x - 4)$ for $x =$ -1, 0, 1, 2, 3, 4, 5 with scales of 2cm to 1 unit on the *x* - axis and 2cm to 2 units on the *y* – axis

b. Find from the graph:

i. the coordinates of the turning point of *y*,

ii. the maximum or minimum value of *y*,

iii. the equation of the axis of symmetry of *y*,

2.a. On the same graph sheet, draw the graphs $f(x) = x^2 + 3$ and $g(x) = 3x + 1$ on a scale of 2cm to 1 unit on *x* – axis and 2cm : 5 units on *y* – axis, for $-4 \leq x \leq 4$

b. From the graph, find:

i. the coordinates of *a* and*b* themeeting points of *f(x)* and *g(x)*

ii. the coordinates of the turning point of *f(x)*

iii. the maximum or minimum value of the function *f*

iv. the zeros of *f*

v. the range of *f* for the given domain

vi. the equation of the axis of symmetry of the parabola

3. a. Draw the graphs of $f(x) = 5x - x^2$ and $g(x) = 8 - x$ on the same axes for the domain $-1 \leq x \leq 6$, and the scales 2cm to 1 unit on x – axis and 2*cm*: 5 units on y – axis

b. From the graph, find:

i. P, the coordinates of the point of intersection of *f (x)* and *g (x)*

ii. the coordinates of the turning point of *f*

iii. the maximum or minimum value of the function *f*,

iv. the zeros of *f*,

v. the range of *f* for the given domain,

vi. the equation of the axis of symmetry of the parabola.

4. a. On the same graph sheet, draw the graphs $f(x) = 6 - x - x^2$ and $g(x) = x + 3$ on the same scales of 2cm to 1 unit on x – axis and 2cm : 5 units on y – axis, for the domain $-4 \leq x \leq 3$

b. If *f(x)* and *g(x)* meet at the points *a* and *b*, find the coordinates of *a* and *b*

Sketching the Graph of a Function

To sketch the graph of a quadratic function,

I. Find the intercept on the y – axis, noting that at the y – axis, $x = 0$

II. Find the intercept on the x – axis, noting that at the x – axis, $y = 0$

III. Draw a curve on the x –y plane to pass through the points of intersection of x and y axis

Worked Example

Given the parabola, $y = 2x^2 - 5x - 3$, find:

i. the coordinates of the point of intersection with the y –axis,

ii. the coordinates of the point of intersection with the x –axis,

iii. the equation of the axis of symmetry,

iv. the coordinates of the vertex,

v. Sketch the parabola.

Solution

i. At the y – axis, $x = 0$

$y = 2x^2 - 5x - 3$

$y = 2(0)^2 - 5(0) - 3 = -3$

ii. At the x – axis, $y = 0$

When $y = 0$

$2x^2 - 5x - 3 = 0$

$(2x^2 - 6x) - (x - 3) = 0$

$2x(x - 3) + 1\,(x - 3) = 0$

$(2x + 1)\,(x - 3) = 0$

$2x = -1$ or $x = 3$

$x = -\frac{1}{2}$ or $x = 3$

The curve intersects the x – axis at $x = -0.5$ or $x = 3$

iii. Equation of the axis of symmetry; $x = -\frac{b}{2a}$

From $y = 2x^2 - 5x - 3$, $a = 2$ and $b = -5$

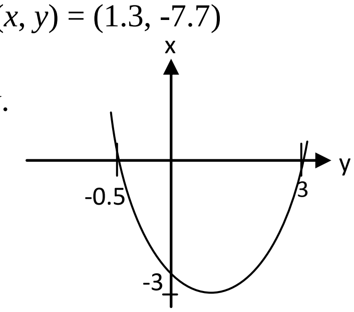
$x = -\frac{(-5)}{2(2)} = \frac{5}{4} = 1.25$

iv. Coordinates of the vertex

When $x = 1.25$, $y = (1.25)^2 - 5(1.25) - 3 = -7.7$

$(x, y) = (1.3, -7.7)$

v.



Exercises 6.22

Given the following parabolas,

1. $y = 2x^2 - x - 15$ 3. $y = (2x + 3)(2x - 5)$
2. $y = (x + 4)(x - 5)$ 4. $y = (3 - x)(x + 2)$

find:

i. the coordinates of the point of intersection with the y – axis,

ii. the coordinates of the point of intersection with the x – axis,

iii. the equation of the axis of symmetry,

iv. the coordinates of the vertex,

v. Sketch the parabola.

The Mid – point of a Straight Line

If A (x_1, y_1) and B(x_2, y_2) are any two points, then the coordinate of the midpoint, M of the straight line joining the points A and B, such that AM = MB is calculated as; $\mathbf{M} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Worked Examples

1. Find the midpoint of the straight line joining the points A(2, 1) and B (6, 5)

Solution

A(2, 1) and B (6, 5)

Let $x_1 = 2$, $y_1 = 1$, $x_2 = 6$ and $y_2 = 5$

Mid – point M = $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$$M = \left(\frac{2+6}{2}, \frac{1+5}{2}\right) = (4, 3)$$

2. Find the coordinates of the mid – point of the straight line joining E(-6,-1) and F(3, -4)

Solution

E(-6, -1) and F(3, -4)

Let $x_1 = -6$, $y_1 = -1$, $x_2 = 3$ and $y_2 = -4$

Mid – point M = $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

By substitution,

$$M = \left(\frac{-6+3}{2}, \frac{-1+(-4)}{2}\right) = \left(-\frac{3}{2}, -\frac{5}{2}\right)$$

Exercises 6.23

Find the coordinates of the mid – point of the lines joining the pair of points:

1. C (- 4, -2) and D(-10, -10)
2. M(2, -3) and N (4, 3)
3. S(- 6, 1) and T (6, 6)
4. M(0, -2) and N(- 4, 0)

The Coordinate of the End point of a Line Given the Mid-Point and One End point

I. Identify the given end point, the given midpoint and the unknown end point

II. Represent the unknown end point by the ordered pair (x, y).

II. Add the respective x and y components of the end points divide each sum by 2 and equate to the respective x and y component of the midpoint.

III. Form two separate equations and solve for the values of x and y to obtain the coordinates of the unknown end point.

Worked Examples

1. $M(4, 7)$ is the midpoint of the line joining $A(x, y)$ and B (6, 11). Find the coordinates of the point represented by x and y

Solution

M(4, 7), A(x, y) and B (6, 11)

Midpoint of $AB = M$

$\left(\frac{x+6}{2}, \frac{y+11}{2}\right) = (4, 7)$

$\frac{x+6}{2} = 4$……………(1)

$\frac{y+11}{2} = 7$…………..(2)

From eqn(1)

$x + 6 = 4 \times 2$

$x + 6 = 8$

$x = 8 - 6$
$x = 2$

From eqn (2)
$y + 11 = 7 \times 2$
$y + 11 = 14$
$y = 14 - 11 = 3$
$x = 2$ and $y = 3$

Therefore the coordinates of A is (2, 3)

Exercises 6.24

1. The point M(-2, -1) is the mid – point of $\overline{PQ}$. If the coordinate of P is (4, -5), find the coordinates of Q.

2. The point (5, 15) is the mid – point of $\overline{AB}$. If A is the point (3, 14), what is the coordinates of B?

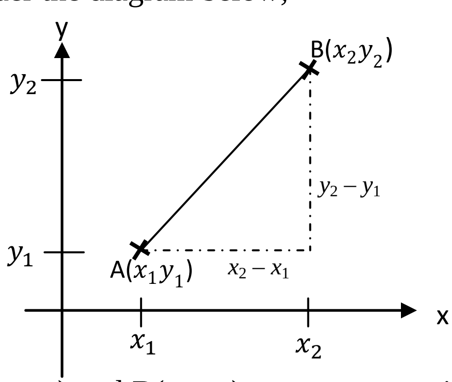
3. The point $\left(9\frac{1}{2}, 7\right)$ is the mid – point of $\overline{CD}$. If C is the point (11, 9), what is the coordinate of D?

4. If the coordinates of P is (-4, 7) and the coordinates of the midpoint of PQ is (3, 2), what is the coordinate of Q?

The Magnitude of a Straight Line

The magnitude of a straight line is also called the length or distance or modulus of a line.

Consider the diagram below;



If A (x_1, y_1) and B(x_2, y_2) are any two points, then by applying Pythagoras theorem, the magnitude or distance or length of the straight line joining the points A and B, is calculated as;

$$|AB|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$|\boldsymbol{AB}| = \sqrt{(\boldsymbol{x_2 - x_1})^2 + (\boldsymbol{y_2 - y_1})^2}$$

Worked Examples

1. Find the length of the line joining A (1, 3) and B (4, 7)

Solution

Let (1, 3) B (4, 7)
$\downarrow \ \downarrow \qquad \downarrow \ \downarrow$
$x_1 \ \ y_2 \qquad x_2 \ \ y_2$

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(4 - 1)^2 + (7 - 3)^2}$$
$$= \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5 \text{ units}$$

2. Find the length or magnitude of the line joining the points P(-5, -4) and Q(8, 7).

Solution

P $(-5, -4)$ Q $(8, 7)$
Let $x_1 = -5$ $y_1 = -4$ $x_2 = 8$, $y_2 = 7$,
By substitution,

$$/PQ/ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$/PQ/ = \sqrt{(8 - (-5)^2 + (7 - (-4))^2}$$
$$/PQ/ = \sqrt{13^2 + 11^2}$$
$$/PQ/ = \sqrt{169 + 121} = \sqrt{290} = 17.03 \textit{ units}$$

3. Find the distance of the point (8, -6) from the origin.

Solution

The coordinates of the origin is (0, 0),
The distance of the point (8, - 6) from the origin is:

$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(8-0)^2 + (-6-0)^2}$

$= \sqrt{8^2 + (-6)^2}$

$= \sqrt{64+36} = \sqrt{100}$ = 10 units

Exercises 6.25

A. Find the lengths of the straight lines joining the following pair of points.

1. A (3, 2) and B (7, 5)
2. M(0, -2) and N(- 4 0)
3. U(-1, -3) and V(-3, - 6)
4. S(-5, 6) and T (8, 9)

B.1. Find the length of the line joining K (-1, 2) and L (5, 9).

2. If P (2, 7) and Q (2, 3) are two points in the OXY plane. Find $|PQ|$.

3. Find the distance of the point (-15, 8) from the origin.

Challenge Problems

1. Show that the lines L : $x + y + 4 = 0$. K : $9x - 5y - 20 = 0$ and M : $5x - 9y + 20$, form an isosceles triangle.

2. A is the point (4, 0) and B is (0, 4) . A point C is such that CA = CB

i. Write down the coordinates of four possible positions of C.

ii. What is the equation of the locus of C?

iii. When < ACB = 40^0, state the sizes of the other two angles of Δ ACB iv. What are the coordinates of C if A, C, B do not form a triangle.

3. A($2t$, 0) and B(0, $-t$) are two points. If $|AB| = \sqrt{20}$, find the two values of t

4. P(1, 6), Q(-3, -1) and R(2, k) are three points. If $|PQ| = |PR|$, find the two values of k.

Determining the Type of Triangle Using the Distance Formula

Given the vertices of a triangle as A, B and C, the type of triangle is verified by finding the magnitude of AB, the magnitude of BC and the magnitude of AC. Thereafter, observe the following:

I. If the magnitudes of the three sides are equal, then Δ ABC is an equilateral triangle.

II. If the magnitudes of two sides are equal, then ΔABC is an isosceles triangle.

III. If the magnitudes of the three sides are unequal, then Δ ABC is a scalene triangle.

IV. If the magnitudes of the three sides form a Pythagorean triples, the three points form the vertices of a right – angled triangle.

Worked Examples

1. A, B and C are the points (3, 5), (7, 2) and (3, -1) respectively. What type of triangle is Δ ABC?

Solution

A(3, 5) B(7, 2) and C(3, -1)

$/AB/ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$/AB/ = \sqrt{(7-3)^2 + (2-5)^2}$

$/AB/ = \sqrt{4^2 + (-3)^2}$

$/AB/ = \sqrt{16+9} = \sqrt{25}$ = 5 units

$/AC/ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$/AC/ = \sqrt{(3-3)^2 + (-1-5)^2}$

$/AC/ = \sqrt{0^2 + (-6)^2}$

$/AC/ = \sqrt{0+36} = \sqrt{36} = 6units$

/*BC*/ = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1\ )^2}$
/*BC*/ = $\sqrt{(3 - 7)^2 + (-1 - 2)^2}$
/*BC*/= $\sqrt{(-4)^2 + (-3)^2}$
/*BC*/ = $\sqrt{16 + 9}$ = $\sqrt{25}$ = 5*units*
Since $|AB| = |BC|$ = 5, Δ*ABC* is isosceles

2. The triangle *PQR* has vertices at P(-3, 4), Q(3, 4) and R(0, 9). Show that Δ *PQR* is no other triangle than an equilateral.

Solution
P(-3, 4), Q(3, 4) and C(0, 9).
/*PQ*/ = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1\ )^2}$
/*PQ*/ = $\sqrt{(3 + 3)^2 + (4 - 4)^2}$
/*PQ*/ = $\sqrt{6^2 + (0)^2}$ = $\sqrt{36}$ = 6 units

/*QR*/ = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1\ )^2}$
/*QR*/ = $\sqrt{(0 + 3)^2 + (9 - 4)^2}$
/*QR*/ = $\sqrt{3^2 + (5)^2}$
/*QR*/ = $\sqrt{9 + 25}$ = $\sqrt{34}$ = 5 .8 ≈ 6 units

/*PR*/ = $\sqrt{\left(x_2 - x_1\right)^2 + (y_2 - y_1\ )^2}$
/*PR*/ = $\sqrt{(0 + 3)^2 + (9 - 4)^2}$
/*PR*/ = $\sqrt{3^2 + (5)^2}$ = $\sqrt{9 + 25}$ = $\sqrt{34}$
/*PR*/ = 5 .8 ≈ 6 units
$|PQ|$ = $|QR| = |PR|$ = 6 units
Therefore, Δ *PQR* is an equilateral triangle

3. What type of triangle is Δ *UVW* with vertices *U*(-2, 2), *V*(3, 2) and *W*(- 4, 5)?

Solution
U(-2, 2), V(3, 2) and W(- 4, 5)
/*UV*/ = $\sqrt{\left(x_2 - x_1\right)^2 + (y_2 - y_1\ )^2}$
/*UV*/ = $\sqrt{(3 + 2)^2 + (2 - 2)^2}$
/*UV*/ = $\sqrt{5^2 + (0)^2}$ = $\sqrt{25}$ = 5 *units*

/*VW*/ = $\sqrt{\left(x_2 - x_1\right)^2 + (y_2 - y_1\ )^2}$
/*VW*/ = $\sqrt{(-4 - 3)^2 + (5 - 2)^2}$
/*VW*/ = $\sqrt{(-7)^2 + (3)^2}$
/*VW*/ = $\sqrt{49 + 9}$ = $\sqrt{58}$ = 7.6 *units*

/*UW*/ = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1\ )^2}$
/*UW*/ = $\sqrt{(-2 + 4)^2 + (2 - 5)^2}$
/*UW*/ = $\sqrt{(2)^2 + (-3)^2}$
/*UW*/ = $\sqrt{4 + 9}$ = $\sqrt{13}$ = *3.6 units*
$|UV| \neq |VW| \neq |VW|$. Therefore, ΔUVW is a scalene triangle

Exercises 6.26
1. *P*, *Q* and *R* are the points (5, -3), (-6, 1) and (1, 8) respectively. Show that the triangle *PQR* is an isosceles triangle and find the mid - point of the base.

2. The three points of a triangle are at *A*(1,1), *B*(4, 5) and *C*(5, -2). Find the lengths of the side of the triangle and show that it is isosceles. What else can you say about the triangle?

3. Given the points *A*(4, 4), *B*(- 4, 1) and C(1, -4), show that triangle ABC is not any other triangle than an isosceles triangle.

4. The coordinates of the vertices A, B, C of the triangle ABC are (–3, 7), (2, 19) (10, 7) respectively, prove that the triangle is isosceles.
5. The three points of a triangle are at A(1, 1), B(4, 5) and C (1, 6). What type of triangle is Δ ABC?

Three Points Forming a Right Triangle

To determine whether three given points form the vertices of a right triangle;

I. Find the distance between each pair of points using the magnitude formula.

II. If the distances between the points form Pythagorean triples, the three points form the vertices of a right triangle.

Worked Examples

Determine whether the following points (– 8, 1), (-2, 9) and (10, 0) form the vertices of a right triangle

Solution

Name the points asA(-8, 1), B (-2, 9) and C(10, 0)

$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1 \ )^2}$

$= \sqrt{(-2 + 8)^2 + (9-1)^2}$

$= \sqrt{6^2 + 8^2} = \sqrt{100}$ = 10 units

$|BC| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1 \ )^2}$

$= \sqrt{(10 + 2)^2 + (0-9)^2}$

$= \sqrt{12^2 + (-9)^2}$

$= \sqrt{225}$ = 15units

$|AC| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1 \ )^2}$

$= \sqrt{(10 + 8)^2 + (0-1)^2}$

$= \sqrt{18^2 + (-1)^2} = \sqrt{325}$

$\approx$18.02 units

Check to see if the lengths,10, 15 and 18.02 form Pythagoras triples:

$10^2 + 15^2 = 325$

$(18.02)^2 = 325$

$\Rightarrow 10^2 + 15^2 = (18.02)^2$

The points form the vertices of a right triangle.

Method 2

$|AB| = \sqrt{100}$, $|BC| = \sqrt{225}$, $|AC| = \sqrt{325}$

By Pythagoras theorem

$\left(\sqrt{100}\right)^2 + \left(\sqrt{225}\right)^2 = \left(\sqrt{325}\right)^2$

100 + 225 = 325

325 = 325

The points form the vertices of a right triangle

Exercises 6.27

Three points that form the vertices of a triangle are given below. Show whether any of them form a right triangle;

1. (5, 2), (0, -3), (4, -4)
2. (7, 0), (-1, 0), (7, 4)
3. (-4, 3), (-7, -1), (3, -2)
4. (-3, 2) , (-1, 5), (-6, 4)

The Gradient of a Line

The ***gradient of a straight line,*** also known as the ***slope of a straight line*** is a measure of the steepness of the line. Steepness means, how an angle is falling or rising. The gradient is usually represented by the variable ***m***.

The gradient of a line can be determined by the following means:

1. from two given points,
2. from any linear relation,
3. from a linear graph.

Gradient of a Line from Two Points

If two points are given as A(x_1, y_1) and B(x_2, y_2), then the gradient of the line *AB*, is determine by the formula;

$m = \frac{y_2 - y_1}{x_2 - x_1}$ **or** $m = \frac{y_1 - y_2}{x_1 - x_2}$, where ***m*** is the gradient

Worked Examples

E is the point (4, 2) and F is the point (2, 1). Calculate the gradient of the straight line EF.

Solution

E (4, 2) and F (2, 1)

Let $x_1 = 4$, $y_1 = 2$, $x_2 = 2$ and $y_2 = 1$

Gradient of EF

$m = \frac{\mathbf{y_2 - y_1}}{\mathbf{x_2 - x_1}} = \frac{1-2}{2-4} = \frac{-1}{-2} = \frac{1}{2}$

2. The gradient of a line passing through the the points P(6, 7) and Q(x, 8) is $\frac{1}{3}$. Find the value of x.

Solution

P(6, 7), Q(x, 8), $m = \frac{1}{3}$

Let $x_1 = 6$, $y_1 = 7$, $x_2 = x$ and $y_2 = 8$

Gradient of PQ;

$m = \frac{\mathbf{y_2 - y_1}}{\mathbf{x_2 - x_1}} = \frac{8-7}{x-6} = \frac{1}{3}$

$\frac{8-7}{x-6} = \frac{1}{3}$

$\frac{1}{x-6} = \frac{1}{3}$

$\Rightarrow x - 6 = 3$

$x = 3 + 6 = 9$

Exercises 6.28

Find the gradient of the line which passes through the pair of points;

1. (-7, - 10) and (-5,-2)
2. (-3, - 8) and (- 4,-1)
3. (-12, - 6) and (- 2,-1)
4. (9, 14) and (- 2, 10)

Gradient of a Line given its Equation

To find the gradient of a line given its equation:

<u>Method 1</u>

Get y on its own such that the equation will be of the form, $y = mx + c$, where the gradient is m or the coefficient of x. The gradient can also be identified simply as the number in front of x.

$y = mx + c$

y = (slope)x + (where the line cuts the y – axis)

<u>Method 2</u>

If the equation of the line is in the form:

$ax + by + c = 0$, then the gradient $m = -\frac{a}{b}$

In words, Slope, $m = -\frac{\text{number in front of } x}{\text{number in front of } y}$

Note:

1. when using this method, make sure that every term is on the left hand side of the equation

2. The equation of a line through the origin, with gradient m, is y = mx

Worked Examples

1. What is the gradient of $y = -\frac{3}{4}x - 10$?

Solution

$y = \frac{-3}{4}x - 10$ compared to $y = mx + c$,

$m = \frac{-3}{4}$ or in the equation $y = \frac{-3}{4}x - 10$,

The number attached to x is $-\frac{3}{4}$. $\Rightarrow m = -\frac{3}{4}$

2. Find the gradient of line $4x + 2y - 5 = 0$.

Solution

$4x + 2y - 5 = 0$ compared to $ax + by + c = 0$

$a = 4$ and $b = 2$.

$m = -\frac{a}{b} = -\frac{4}{2} = -2$

3. Find the gradient of $5y = 8 + 4x$.

Solution

Re – arrange $5y = 8 + 4x$ to take the form:

$ax + by + c = 0$. $\Rightarrow -4x + 5y - 8 = 0$

By comparison, $a = -4$ and $b = 5$

$m = -\frac{a}{b} = -\frac{-4}{5} = \frac{4}{5}$

4. Find the gradient of the line $7x + 4y + 2 = 0$, and its intercepts on the x – axis and y – axis.

Solution

Express $7x + 4y + 2 = 0$ in the form:

$y = mx + c$ by making y the subject,

$4y = -7x - 2$

$y = -\frac{7}{4}x - \frac{2}{4}$

$y = -\frac{7}{4}x - \frac{1}{2}$ compared to $y = mx + c$

The gradient , $m = -\frac{7}{4}$ and the intercept on the y – axis, $c = -\frac{1}{2}$

To find the intercept on the x – axis, substitute $y = 0$ in $7x + 4y + 2 = 0$ and solve for x.

$7x + 4(0) + 2 = 0$

$7x + 2 = 0$

$7x = -2$

$x = -\frac{2}{7}$. The intercept on the x – axis is $-\frac{2}{7}$

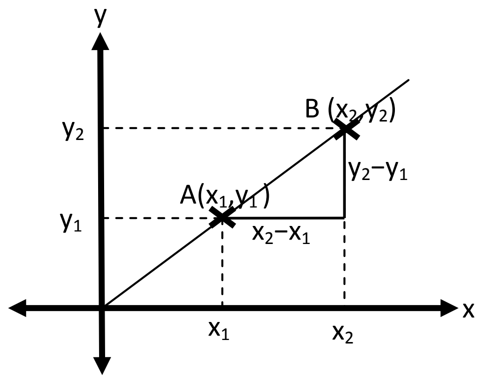
Exercises 6.29

What is the gradient of the straight lines;

1. $x - y - 1 = 0$	5. $2y - 6x - 1 = 0$
2. $3y - 7x + 6 = 0$	6. $\frac{3}{4}y = -12x - 6$
3. $\frac{1}{2}y - 8x + 1 = 0$	7. $\frac{2}{3}y + \frac{3}{5}x - 2 = 0$
4. $\frac{6}{5}x - \frac{2}{5}y = -3$	8. $\frac{7}{2}x + 7y + 5 = 0$

Gradient of a Line from a Linear Graph

Study the diagram below carefully;



To find the gradient of the line AB;

I. Choose any two points on the line AB.

II. Draw a triangle to join the two points as shown in the diagram above.

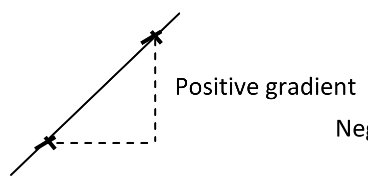
III. Calculate the gradient by,

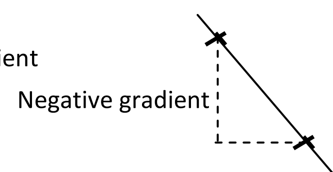
$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{y}{x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Note:

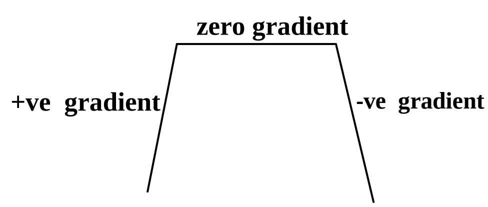
If the line rises from left to right, the gradient is positive.

If the line falls from left to right, the gradient is negative.





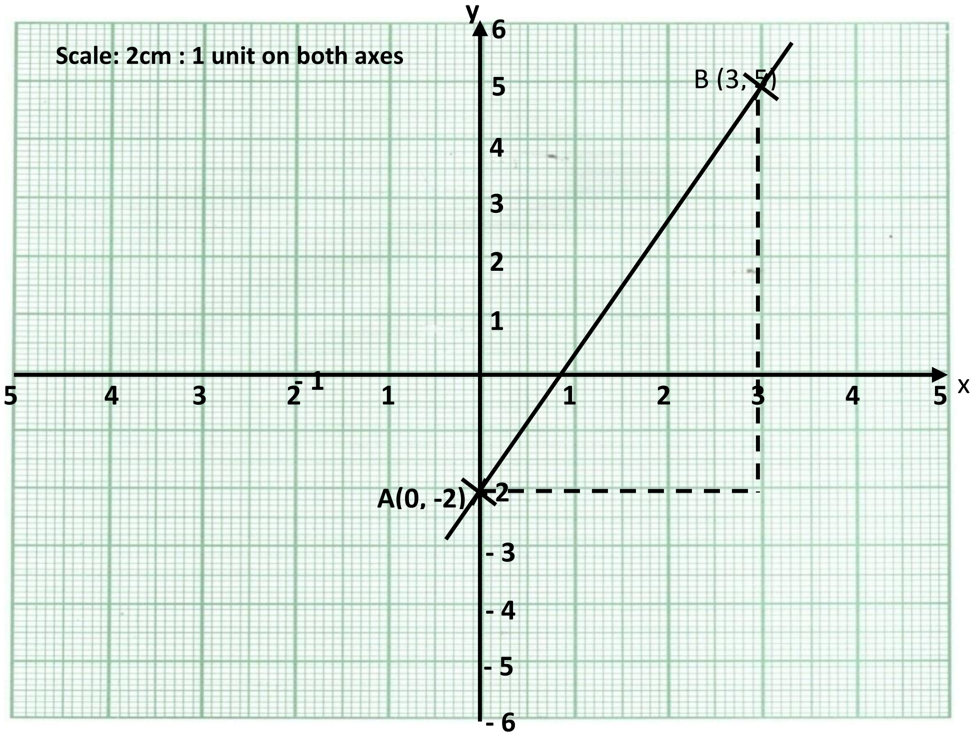
Summary



Worked Examples

1. Using a scale of 2cm to 1 unit on both axes, plot the points A $(0, -2)$ and B $(3, 5)$ on a graph sheet. Join the points with a ruler to form a straight line and find the gradient of the line.

Solution

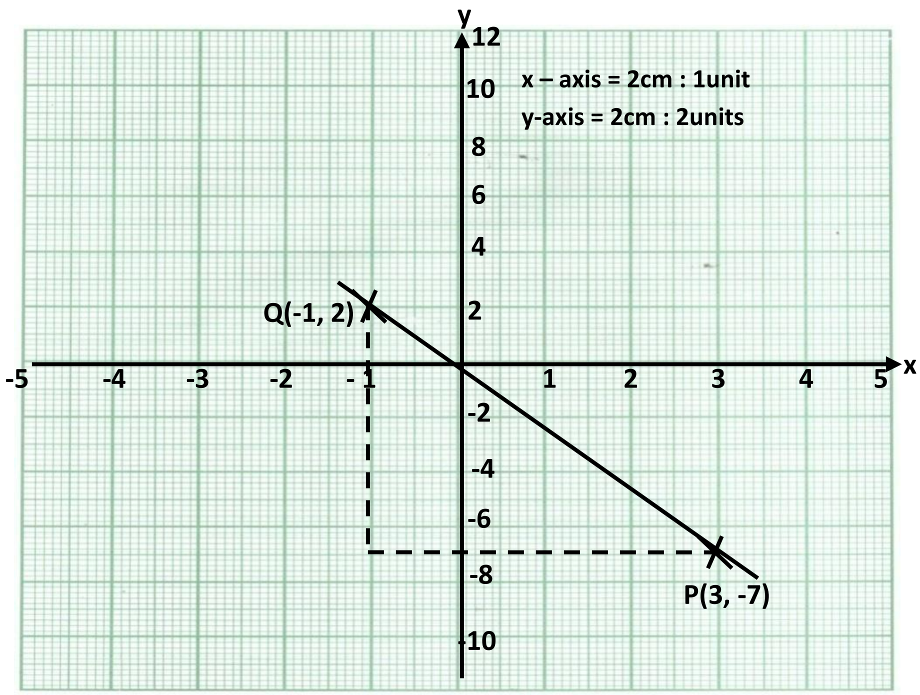


Gradient (AB) $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-2}{3-0} = \frac{3}{3}$. Therefore, m= 1

2. i. Using a scale of 2cm to1 unit on the x – axis and 2cm to 2units on y – axis, plot the points P(3,-7) and Q(-1,2). Join P and Q with a ruler

ii. Find the gradient of the line PQ

Solution



From the graph, $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - -7}{-1-3} = \frac{9}{-4}$

Exercises 6.30

A. 1. i. Using a scale of 2cm to 1 unit on both axes of a number plane, plot the points P (4, 5) and Q (−2, 3).

ii. Join the points with a ruler to form a straight line.

iii. Determine the gradient of the line.

2. i. On a graph sheet, plot the points$C$ (2, −3) and D (−4, 3), using a scale of 2cm to 1 unit on both axes.

ii. Use a ruler to join the points to form a straight line.

iii. Determine the slope of the line.

3. i. Using a scale of 2cm to 1 unit on both axes of a graph sheet, plot the points Q (1, −1), R (−1, 1), S (−2, 2) and T (− 4, 4)

ii. Draw a straight line through the points with the help of a ruler.

iii. Find the slope of the line.

iv. From the graph, determine the value of y when $x = -3$.

4. i. Using a scale of 2cm to 1 unit on both axes, locate the following points; A (0, 2), B $(1\frac{1}{2}, 1\frac{1}{2})$ and C (3, −3) on a graph sheet.

ii Connect the points with a straight line using a ruler.

iii Find the slope of the line.

5. i. Using a scale of 2cm to 1 unit on x – axis and 2cm to 2 units on y – axis, mark on a graph sheet, the x – axis from – 5 to 5 and y – axis from −12 to 12.

ii. Plot the points $E(3, -6)$, $F(0, 2)$, $G(1\frac{1}{2}, 6)$ and $H(-6, 10)$ and draw a straight line through them.

iii Determine the slope of the line.
iv. Use your graph to find:
a. y when $x = 2.5$ *b*. x when $y = -2$.

6. i. On a graph sheet, with a scale of 2cm to 2 units on both axes, mark the x – axis from -10 to 10 and the y – axis from -12 to 12.
ii. Plot the set of ordered pairs: (9, 11), (3, 3), (8, 0) and join the points with a rule to form a straight line.
iii. Determine the gradient of the line drawn.
iv . From your graph, find:
a. x when $y = 9$, b. y when $x = -3$.

7. i. Using a scale of 2cm to 2 units on both axes, draw a straight line through the points M (-5, 5) and N (4, 10)
ii. Determine the gradient of line MN.
iii. Find the co – ordinate of the point at which the line cuts the x – axis.

8. i. Copy and complete the table below for the function $f(x) = \frac{1}{3}x - 5$

x	-9	-3	0	3	6
$f(x)$	-8		-5		

ii Plot the values of x against $f(x)$ on a graph sheet, using a scale of 2cm to 2 units on both axes.
iii. What is the slope of the line?
iv. Use your graph to find :
a. x when $y = -6$ b. y when $x = -7$

9. i. Copy and complete the table of the values for the relations; $y_1 = 2x + 5$ and $y_2 = 3 - 2x$ for x from -4 to 3

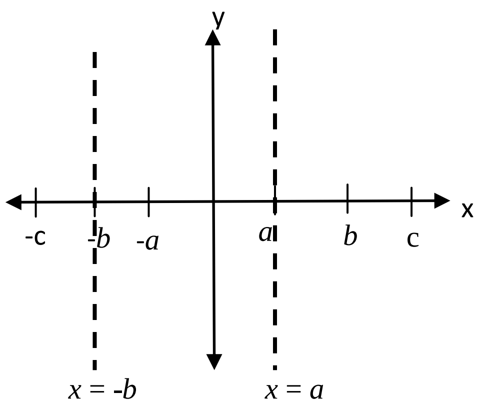
x	-4	-3	-2	-1	0	1	2	3
$y_1 = 2x + 5$	-3			3		7		4
$y_2 = 3 - 2x$	11	9		5				

ii. Plot the graph of each relation on the same graph sheet using a scale of 2cm to 1 unit on the x – axis and 2cm to 2 units on the y – axis.
iii. Find the coordinates of the point of intersection of y_1 and y_2.
iv. Determine the slope of $y_1 = 2x + 5$.

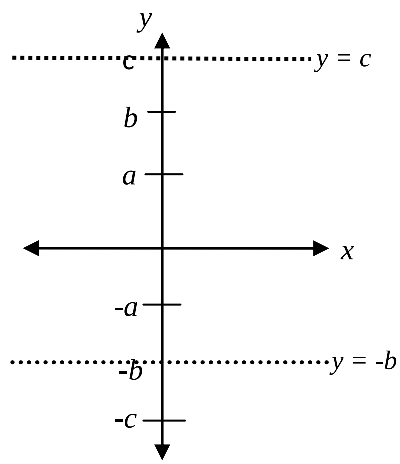
Drawing a Line at a Given Point on the x and y – axes

Sometimes students are required to draw a line at a given point on the x or y – axis. If the point is on the x – axis, say $x = \boldsymbol{a}$, draw a vertical/perpendicular line to pass through the point, $\boldsymbol{a}$, on the x – axis.
If the point is on the y – axis, say $y = \boldsymbol{c}$, draw a horizontal line to pass through the point, $\boldsymbol{c}$ on the y – axis.
These are illustrated in the diagrams below;

On the x – axis



On the y- axis



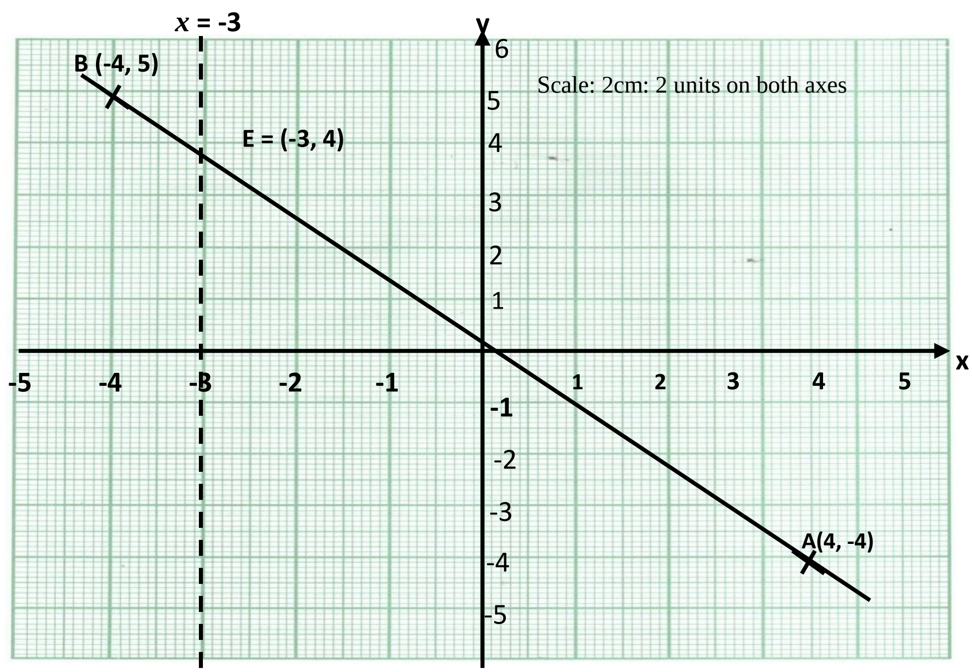
Worked Examples

1. i. Using a scale of 2cm to 1 unit on both axes, plot the points A (4, −4), B (−4, 5) and join the AB with a ruler.

ii On the same graph sheet, draw the line $x = -3$ to meet line AB at E.

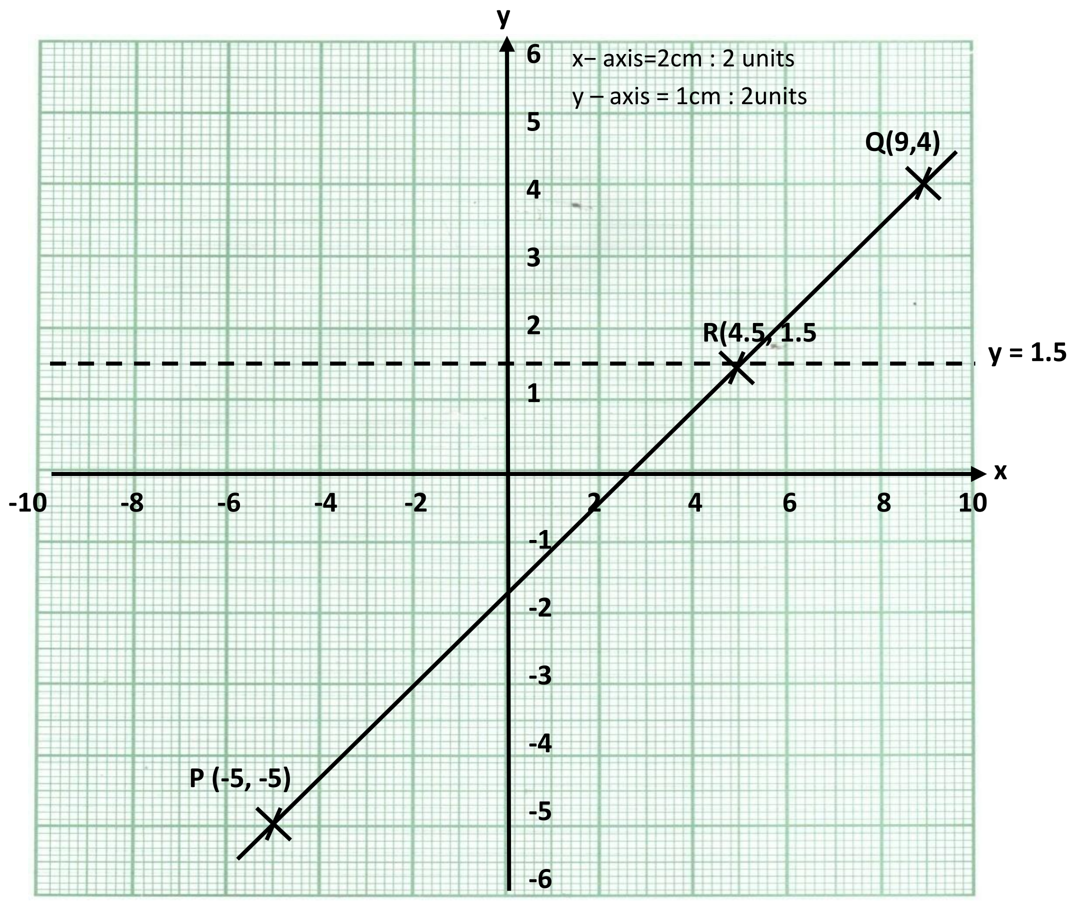
iii. Write the co – ordinate of E

Solution



2. i. Using a scale of 2cm to 2 units on the x – axis and 2cm to 1 unit on the y – axis, plot the points; P (-5, -5) and Q (9, 4). Draw a straight line to join PQ.

ii. Draw the line $y = 1.5$ to meet line PQ at R. Indicate the coordinate of R



Exercises 6.31

1. Using a scale of 2cm to 2 units on both axes, plot the points A (0, 10), B (−6, −2), C (4, 3) and D (−3, −11). Use a ruler to join the points A to B and C to D.

ii. Draw the line $x = -2$ to meet AB at P and CD at Q.

iii. Use a protractor to measure angles BPQ and PQC.

iv. What common name is given to the angles BPQ and PQC?

v. State the relationship between lines AB and CD.

2. i. Using a scale of 2cm to 1 unit on both axes, plot the points A (2, 3) and B (−3, 4) and join them with a long straight line.

ii On the same graph sheet, plot the point C (4, 2) and D (−2, 3) and join them with a long straight line extended through C to meet line AB. Name the meeting point M.

Measure the angle between the lines through AB and CD.

iii. Find the co – ordinates of the point M.

iv. On the same graph, draw the line $x = -2.5$.

3. i. Plot on a graph sheet, the points A (0, 5) and B (4, 5) using a scale of 2cm to 1 unit on both axes.

ii Measure the acute angle the line AB makes with the x – axis using a protractor

Collinear Points

Three or more points are said to be collinear when they lie on the same straight line. Two or more points that lie on the same straight line have the same gradient. Therefore, to show whether or not, two or more points are collinear:

I. Find the gradient of the lines

II. If the gradients are equal, conclude that they are collinear, if not, conclude that they are not collinear.

Worked Examples

Show that the three points (0, 0), (3, 5) and (21, 35) are collinear.

Solution

Let the points be $A(0, 0)$, $B(3, 5)$ and $C(21, 35)$

Gradient of $AB = \frac{y_2-y_1}{x_2-x_1} = \frac{5-0}{3-0} = \frac{5}{3}$

Gradient of $BC = \frac{y_2-y_1}{x_2-x_1} = \frac{35-5}{21-3} = \frac{30}{18} = \frac{5}{3}$

The gradient of AB = the gradient of BC

$\therefore$ A(0, 0), B(3, 5) and C(21, 35) are collinear.

2. The points (2, -3), (3, -1) and (4, k) lie on a straight line. Find k

Solution

Let the points be $A(2, -3)$, $B(3, -1)$ and $C(4, k)$

If A, B and C are collinear, then gradient of AB is equal to gradient of BC

Gradient of $AB = \frac{y_2-y_1}{x_2-x_1} = \frac{(-1)-(-3)}{3-2} = \frac{2}{1}$

Gradient of $BC = \frac{y_2-y_1}{x_2-x_1} = \frac{k-(-1)}{4-3} = \frac{k+1}{1}$

$\frac{2}{1} = \frac{k+1}{1}$

$2 = k + 1$

$2 - 1 = k$

$k = 1$

Exercises 6.32

A. Show that the three given points in each case are collinear.

1. (2, -3), (3, 1) (5, 9)
2. (- 3, 4), (1, 2) (7, - 1)
3. (- 3, 1), (1, 2) (9, 4)
4. (1, 2), (0, - 1) (- 2, - 1)
5. (6, 1), (3, 3), (- 3, 7)

Perpendicular Lines

Two lines are said to be perpendicular when the product of their gradient is – 1.That is if L_1 is perpendicular to L_2, then: $m_1 \times m_2 = -1$. In other words, for any two lines L_1 and L_2, if the gradient of one is m, the gradient of the other is $\frac{-1}{m}$.

To prove whether or not two lines and are perpendicular:

1. Find the slope of each line.
2. Multiply both slopes.
3. a. If the product in step 2 is -1, the lines are perpendicular.

   b. If the product is ***not*** – 1, the lines are ***not*** perpendicular.

Note:

Knowing the slope/gradient of a line, the slope/gradient of a line perpendicular to it is found by turning the known gradient upside down and changing its sign.

For e.g. if a line has a gradient of $-\frac{3}{5}$, then the gradient of a line perpendicular to it is $\frac{5}{3}$

Worked examples

1. Write down the gradients of lines perpendicular to the lines of gradients 3, $\frac{1}{4}$, – 6 and $-\frac{2}{3}$ respectively

Solution

Let m_1 = 3, $m_2 = \frac{1}{4}$, m_3= – 6 and $m_4 = -\frac{2}{3}$

The gradient of the line perpendicular to:

$m_1 = -\frac{1}{m_1} = -\frac{1}{3}$

$m_2 = -\frac{1}{m_2} = -\frac{1}{1/4} = -1 \times \frac{4}{1} = -4$

$m_3 = -\frac{1}{m_3} = -\frac{1}{(-6)} = \frac{1}{6}$

$m_4 = -\frac{1}{m_4} = -\frac{1}{-(2/3)} = -1 \times \frac{3}{-2} = \frac{3}{2}$

2. Given *A*(– 1, – 1), *B* (0, 4) *P*(– 4, 3), *Q*(6, 1), find whether or not *AB* is perpendicular to *PQ*

Solution

A(- 1, - 1), *B* (0, 4) P(- 4 , 3), Q(6, 1),

Let the gradient of AB be m_1

$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-(-1)}{0-(-1)} = \frac{5}{1} = 5$

Let the gradient of PQ be m_2

$m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-3}{6-(-4)} = \frac{-2}{10} = -\frac{1}{5}$

$m_1 \times m_2 = 5 \times -\frac{1}{5} = -1$

$\therefore$ *AB* is perpendicular to *PQ*

3. Given Q(1, 4), R(6, 6), S(2, -1) and T(12, 3), show whether or not, $\overline{QR}$ is perpendicular to $\overline{ST}$

Solution

Q(1, 4), R(6, 6), S(2, -1) and T(12, 3)

Let the gradient of QR be m_1

$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-4}{6-1} = \frac{2}{5}$

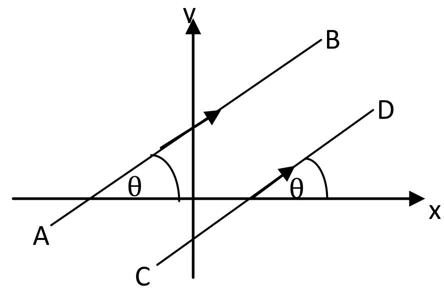
Let the gradient of ST be m_2

$m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-(-1)}{12-2} = \frac{4}{10} = \frac{2}{5}$

$m_1 \times m_2 = \frac{2}{5} \times \frac{2}{5} = \frac{4}{25} \neq -1$

$\therefore \overline{QR}$ is not perpendicular to $\overline{ST}$

Parallel Lines



From the diagram above, line AB is parallel to line BC. Since parallel lines make equal corresponding angles θ with the *x* – axis, parallel lines are said to have equal gradients (and vice versa)

In general, if two or more lines have the same or equal gradients, they are said to be parallel. That is , if L_1 is parallel to L_2 then $m_1 = m_2$

To prove whether or not two lines are parallel, do the following:

1. Find the slope of each line
2. *a*. If the slopes are the same, the lines are parallel
   b. If the slopes are not the same, the lines are not parallel

Worked Examples

If A(0,3), B(7,2), P(6, -1) and Q(-1, -2) show whether or not $\overline{AB}$ is parallel to $\overline{PQ}$

Solution

A(0,3) and B(7,2), P(6, -1) and Q(-1, -2)

Let the gradient of AB be m_1

$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-3}{7-0} = -\frac{1}{7}$

Let the gradient of PQ be m_2

$m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-(-1)}{-1-6} = \frac{-1}{-7} = \frac{1}{7}$

$$m_1 \neq m_2$$

Therefore, $\overline{AB}$ is not parallel to $\overline{PQ}$

2. Given that $Q(1, 4)$, $R(6, 6)$, $S(2, -1)$ and $T(12, 3)$, show whether or not $\overline{QR}$ is parallel to $\overline{ST}$.

Solution

$Q(1, 4)$, $R(6, 6)$, $S(2, -1)$ and $T(12, 3)$

Let the gradient of QR be m_1

$m_1 = \frac{\mathbf{y_2 - y_1}}{\mathbf{x_2 - x_1}} = \frac{6-4}{6-1} = \frac{2}{5}$

Let the gradient of ST be m_2

$m_2 = \frac{\mathbf{y_2 - y_1}}{\mathbf{x_2 - x_1}} = \frac{3-(-1)}{12-2} = \frac{4}{10} = \frac{2}{5}$

$m_1 = m_2 = \frac{2}{5}$

Therefore, $\overline{QR}$ is parallel to $\overline{ST}$

3. Show that the two lines $3x + 5y - 8 = 0$ and $5x - 3y - 11 = 0$ are perpendicular.

Solution

Let $L_1 = 3x + 5y - 8 = 0$ and $L_2 = 5x - 3y - 11 = 0$

From $L_1 = 3x + 5y - 8 = 0$,

$m_1 = \frac{-a}{b} = \frac{-3}{5}$

From $L_2 = 5x - 3y - 11 = 0$,

$m_2 = \frac{-a}{b} = \frac{-5}{-3} = \frac{5}{3}$

If L_1 is perpendicular to L_2, then $m_1 \times m_2 = -1$

$\frac{-3}{5} \times \frac{5}{3} = -1$

Exercises 6.33

A. Write the gradient of the line perpendicular to the lines with the following gradients

1. 4 2. $-\frac{1}{7}$ 3. -12 4. $\frac{7}{3}$ 5. $-\frac{2}{9}$

B. In the following cases, show whether or not AB is parallel or perpendicular to PQ

1. A(4, 3) B(8, 4) P(7, 1) Q(6, 5)
2. A(-2, 0) B(1, 9) P(2, 5) Q(6, 17)
3. A(8, -5) B(11, -3) P(1, 1) Q(-3, 5)
4. A(4, 3) B(-7, 3) P(5, 2) Q(5, -1)

C. Two points on L_1 and two points on L_2 are given. Determine whether L_1 and L_2 are parallel, perpendicular or neither

1. L_1(-2, 0) and (0, 6), L_2(-3, -4) and (0, 5)
2. L_1(-2, 0) and (0, 1), L_2(0, 0) and (-4, 4)
3. L_1(6, 3) and (8, 7), L_2(7, 2) and (6, 0)
4. L_1(1, 10) and (-1, 7), L_2(0, 3) and (1, 5)

D.1. Show that the two lines $3x - 2y + 8 = 0$ and $2x + 3y - 1 = 0$ are perpendicular.

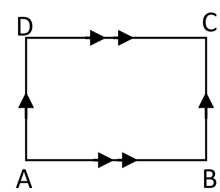
2. If the line $ax + 3y = 0$ is perpendicular to the line $3x - 5y - 4 = 0$, find the value of a.

3. $P(2, -3)$, $Q(3, 1)$ and $R(-1, k)$ are three points. If PQ is perpendicular to QR, find the value of k.

Application to Parallelograms

Given the vertices of a quadrilateral ABCD, it can be verified whether quadrilateral ABCD is a rectangle, a square or a parallogram.

First make a sketch of the figure as shown below:



Take note of the fact that for all parallelograms (rectangles and squares) with vertices *ABCD*;

I. *AB* is parallel to *CD* , meaning the gradient of *AB* is equal to that of *CD*.

II. *AD* is parallel to BC, meaning the gradient of *AD* is equal to that of *BC*.

Worked Examples

Show that $A(6, -2)$, $B(-2, -4)$, $C(0, 2)$ and $D(8, 4)$ are vertices of a parallelogram.

Solution

Method 1

$A(6, -2)$, $B(-2, -4)$, $C(0, 2)$ and $D(8, 4)$

If *ABCD* is a parallelogram, then;

gradient of *AB* = gradient of *DC*

Gradient of *AB*;

$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4-(-2)}{-2-6} = \frac{-2}{-8} = \frac{1}{4}$

Gradient of *DC*;

$m_{DC} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-4}{0-8} = \frac{-2}{-8} = \frac{1}{4}$

$m_{AB} = m_{DC}$

Therefore, *ABCD* is a parallogram

Method 2

A(6, - 2), B(- 2, - 4), C(0, 2) and D(8, 4)

If *ABCD* is a parallelogram, then;

gradient of *AD* = gradient of *BC*

Gradient of *AD*;

$m_{AD} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-(-2)}{8-6} = \frac{6}{2} = 3$

Gradient of *BC*;

$m_{BC} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-(-4)}{0-(-2)} = \frac{6}{2} = 3$

$m_{AD} = m_{BC}$

Therefore, *ABCD* is a parallogram

Exercises 6.33B

1. Show that $A(-3, 1)$, $B(1, 2)$, $C(0, -1)$ and $D(-4, -2)$ are vertices of a parallelogram.

2. Show that $P(1, 7)$, $Q(7, 5)$, $R(6, 2)$ and $S(0, 4)$ are vertices of a rectangle.

3. A quadrilateral has vertices at $P(2, 1)$, $Q(6, 3)$, $R(5, 5)$, $S(1, 3)$. Show that *PQRS* is a parallelogram. Find the length of the diagonals *PR* and *QS*.

4. A quadrilateral has vertices at $A(0, 0)$, $B(7, 24)$, $C(22, 44)$ $D(15, 20)$. Find the length of the sides. What type of quadrilateral is *ABCD*?

The Equation of a Straight Line

The general form of the equation of a straight line is $ax + by + c = 0$. On the other hand, the intercept form: $y = mx + c$ is equally acceptable.

The equation of a straight line can be determined from:

1. Any two given points on the line
2. The gradient of the line and a point on the line
3. The gradient of the line and the intercept on the y – axis

Equation of a Line given any Two Points

If A(x_1, y_1) and B(x_2, y_2) are any two points on a straight line or any two points joined by a straight line then, the equation of the line is determined by any of the following methods:

Method 1

I. Find the gradient using the formula,

$m = \frac{y_2 - y_1}{x_2 - x_1}$**……….(1)**

II. Substitute the value of m and (x_1, y_1) or m *and* (x_2, y_2) in the formulas below respectively:

$y - y_1 = m(x - x_1)$ or $y - y_2 = m(x - x_2)$

III. After substitution, express the equation $y - y_1 = m(x - x_1)$ or $y - y_2 = m(x - x_2)$ in the form: $ax + by + c = 0$ or $y = mx + c$

Method 2

I. Find the gradient using the formula,

$m = \frac{y_2 - y_1}{x_2 - x_1}$**……….(1)**

II. Using a point on the line, either *(x_1, y_1)* or *(x_2, y_2)*, substitute the values in $y = mx + c$ to determine the value of c (the intercept on y – axis)

III. Substitute the values of m and c in $y = mx + c$ to obtain the equation of the line

Worked Examples

1. Find the equation of the straight line joining the points (– 5, 2) and (3, – 4)

Solution

In (– 5, 2) and (3, – 4), x_1= -5, y_1 = 2,

x_2= 3 and y_2= – 4

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4\ -2}{3-(-5)} = -\frac{3}{4}$

$y - y_1 = m(x - x_1)$

$y - 2 = -\frac{3}{4}(x - (-5))$ ***(By substitution)***

$4(y - 2) = -3(x + 5)$

$4y - 8 = -3x - 15$ ***(Expansion)***

$4y + 3x - 8 + 15 = 0$

$3x + 4y + 7 = 0$

The equation of the line is $3x + 4y + 7 = 0$

<u>Method 2</u>

In (– 5, 2) and (3, – 4), x_1= -5, y_1 = 2,

x_2= 3 and y_2= – 4

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4\ -2}{3-(-5)} = -\frac{3}{4}$

$y - y_2 = m(x - x_2)$

$y - (-4) = -\frac{3}{4}(x - 3)$ ***(By substitution)***

$y + 4 = -\frac{3}{4}(x - 3)$

$4(y + 4) = -3(x - 3)$ ***(Multiply through by 4)***

$4y + 16 = -3x + 9$

$4y + 3x + 16 - 9 = 0$

$3x + 4y + 7 = 0$

<u>Method 3</u>

In (– 5, 2) and (3, – 4), x_1= -5, y_1 = 2,

x_2= 3 and y_2= – 4

Express the equation in the form $y = mx + c$,

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4\ -2}{3-(-5)} = -\frac{3}{4}$

Substitute $m = -\frac{3}{4}$ in $y = mx + c$

$y = -\frac{3}{4}x + c$

Using the point (- 5, 2) **(Because it lies on the line).**

When $x = -5$, $y = 2$, put in $y = -\frac{3}{4}x + c$

$2 = -\frac{3}{4}(-5) + c$

$4 \times 2 = -3(-5) + 4c$ ***(Multiply through by 4)***

$8 = 15 + 4c$

$8 - 15 = 4c$

$-7 = 4c$

$c = -\frac{7}{4}$

Substitute $c = -\frac{7}{4}$ in $y = -\frac{3}{4}x + c$

$y = -\frac{3}{4}x - \frac{7}{4}$ or $y = -\frac{3x-7}{4}$

The equation is $y = -\frac{3}{4}x - \frac{7}{4}$ or $y = -\frac{3x-7}{4}$

2. Find the equation of the line joining the points M(6, 3) and N(5, 8)

Solution

In M(6, 3) and N(5, 8), x_1= 6, y_1 = 3,

x_2= 5 and y_2= 8

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-3}{5-6} = -5$

$y - y_1 = m(x - x_1)$

$y - 3 = -5(x - 6)$ ***(By substitution)***

$y - 3 = -5x + 30$

$y + 5x - 3 - 30 = 0$

$5x + y - 33 = 0$

The equation of the lineis $5x + y - 33 = 0$

3. Find the equation of the line which passes through the points (2, -3) and (1, 3)

Solution

In (2, -3) and (1, 3), x_1= 2, y_1 = -3,

x_2= 1 and y_2= 3

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - -3}{1 - 2} = - 6$

$y - y_1 = m(x - x_1)$

$y - (-3) = -6(x - 2)$

$y + 3 = -6x + 12$

$y + 6x + 3 - 12 = 0$

$6x + y - 9 = 0$

The equation of the lineis $6x + y - 9 = 0$

Exercises 6.34

A. Find the equation of the straight line joining the following pair of points:

1. (3, 2) and (7, - 3)
2. (- 3, 4) and (8, 1)
3. (- 1, - 4) and (4, -3)
4. (- 2, 5) and (3, - 7)

B. 1. Find the equation of the straight line joining A(10, 0) and B(0, –7)

2. i. Write down the equation of a line L, joining the origin to the point A(4, 2).
ii. What is the gradient of this line?

3. The points (– 4, 2) and (1, –3) lie on the line with $y = mx + c$. Find m and c, and hence the equation of the line .

4. Given that the line $y = mx + c$ has a gradient of -1, and passes through the point (2, 3), find the values of m and c

5. The points A, B and C have co-ordinates (7, 0), (3, -3) (-3, 3) respectively. Find the coordinates of D, E, F, the mid-points of BC, CA, AB respectively.

The Equation of a Line given the Gradient and a Point on the Line

Given the gradient of a line (m) and a point (x, y) through which the line passes, the equation of the line is found as follows:

<u>Method I</u>

I. Substitute the value of the gradient (m) and the values of the point (x, y) in the relation $y = mx + c$, to determine the value of c (intercept on the y – axis)

II. Substitute the value of m and c in the equation $y = mx + c$ to determine the equation of the straight line.

<u>Method II</u>

I. Substitute the value of the gradient (m) and the values of the point (x_1, y_1) in the relation,

$y - y_1 = m(x - x_1)$.

II. Expand and re- arrange the equation in the form $ax + by + c = 0$ or $y = mx + c$, to obtain the equation of the line.

Worked Examples

1. Find the equation of a line with gradient $\frac{3}{2}$ which passes through the point (5, - 2)

Solution

<u>Method I</u>

$m = \frac{3}{2}$ and (x, y) = (5, – 2)

Substitute in $y = mx + c$

$-2 = \frac{3}{2}(5) + c$

$-2 \times 2 = 3(5) + 2c$

$-4 = 15 + 2c$

$-4 - 15 = 2c$

$2c = -19$

$c = -\frac{19}{2}$

Put $c = -\frac{19}{2}$ and $m = \frac{3}{2}$ in $y = mx + c$

$y = \frac{3}{2}x - \frac{19}{2}$

The equation of the line is $y = \frac{3}{2}x - \frac{19}{2}$

Method II

$m = \frac{3}{2}$ and $(x_1, y_1) = (5, -2)$

Substitute in $y - y_1 = m(x - x_1)$

$y - (-2) = \frac{3}{2}(x - 5)$

$y + 2 = \frac{3}{2}(x - 5)$

$2(y + 2) = 3(x - 5)$

$2y + 4 = 3x - 15$

$3x - 2y - 15 - 4 = 0$

$3x - 2y - 19 = 0$

The equation is $3x - 2y - 19 = 0$

2. The gradient of a line is 4. If the line passes through the point (1, 3), find the equation of the line.

Solution

$m = 4$, $(x_1, y_1) = (1, 3)$

Substitute in $y - y_1 = m(x - x_1)$

$y - 3 = 4(x - 1)$

$y - 3 = 4x - 4$

$4x - y - 4 - 3 = 0$

$4x - y - 7 = 0$

The equation of the line is $4x - y - 7 = 0$

3. Find the equation of a straight line that passes through (7, 5) with gradient $-\frac{3}{4}$

Solution

$m = -\frac{3}{4}$ and $(x_1, y_1) = (7, 5)$

Substitute in $y - y_1 = m(x - x_1)$

$y - 5 = -\frac{3}{4}(x - 7)$

$4(y - 5) = -3(x - 7)$

$4y - 20 = -3x + 21$

$4y + 3x - 20 - 21 = 0$

$3x + 4y - 41 = 0$

The equation of the line is $3x + 4y - 41 = 0$

Exercises 6.35

A. Write down the equations of the lines through the following points, and having the following gradients: express your answers in the form *ax + by + c = 0*, where *a*, *b* and *c* are integers

1. (- 2, 5), 3 2. (4, 3), $-\frac{1}{3}$ 3. (0, -3), 2

4. (2, - 5), $\frac{1}{3}$ 5. - 5, (- 3, -7) 6. (0, 4), $-\frac{1}{2}$

Challenge Problems

1. *PQRS* is a parallelogram in which the opposite vertices are *P*(2, 1) and *R*(4, 4). If the slope of *PQ* is $\frac{1}{3}$ and the slope of *PS* = -2, find:

i. the equation of *PQ*

ii. the equation of *QR*

iii. hence, or otherwise, find the coordinates of *Q* and *S*

2. i. A line of gradient $-\frac{4}{3}$ is drawn through the origin O. Write down the equation of this line.

ii. A point *A* is chosen on the line with first coordinate 3. If *B* is the point (5, 0), show that the triangle is an isosceles and calculate it area

iii. If C is the point (3, 4), calculate the area of *OABC*

The Equation of a Line given the Gradient and the Intercept on the y – axis

If a straight line cuts the *y* – axis at the point (0, *c*), the distance of this point from the origin is called the ***intercept*** on the *y* – axis.

The equation of a straight line of gradient *m*, making an intercept *c* on the *y* – axis is: $\boldsymbol{y = mx + c}$

Given the values of m and c, the equation of the straight line is found by substituting them in
$y = mx + c$

Worked Examples

1. A straight line which has a gradient of 7 cuts the y – axis at – 2. Find the equation of the line

Solution

Gradient, $m = 7$,
Intercept on y - axis, c = – 2
Substitute in $y = mx + c$ to get $y = 7x – 2$
The equation of the line is $y = 7x – 2$

2. Find the equation of a straight line with gradient $-\frac{5}{3}$ which cuts the axis at the point (0, 8)

Solution

$m = -\frac{5}{3}$. In (0, 8), the line cuts the x – axis at (0, 8). Hence $c = 8$.
$y = -\frac{5}{3}x + 8$.
The equation of the line is $y = -\frac{5}{3}x + 8$

Exercises 6.36

Find the equations of the straight lines of given gradients cutting the y – axis at the named points:

1. $m = 3$, (0, 2)	2. $m = -\frac{1}{5}$, (0, 4)
3. $m = 4$, (0, 6)	4. $m = \frac{2}{7}$, (0, - 4)
5. $m = -3$, (0, -2)	6. $m = \frac{1}{5}$, (0, 7)

The Equation of a Line through the Point (x_1, y_1) and Parallel to another Line,

Two or more lines are said to be parallel if they have the same gradient. That is if the gradient of a line is m, then the gradient of the line parallel to it is m.

The equation of the straight line parallel to another line of gradient m, passing through the point (x_1,y_1) is found by making y the subject of the formula: $\boldsymbol{y - y_1 = m(x - x_1)}$

Worked Examples

1. What is the equation of the line parallel to the line $2x + 7y – 8 = 0$, which passes through the point (– 1, 3)?

Solution

<u>Method 1</u>

In $2x + 7y – 8 = 0$
$m = -\frac{a}{b} = -\frac{2}{7}$
The line parallel to $2x + 7y – 8 = 0$ has a gradient of $-\frac{2}{7}$ and passes through (- 1 , 3)
$\Rightarrow m = -\frac{2}{7}$, $x_1 = -1$, $y_1 = 3$

Substitute in $y – y_1 = m(x - x_1)$
$y – 3 = -\frac{2}{7}(x – (–1)$
$y – 3 = -\frac{2}{7}(x + 1)$
$7(y – 3) = – 2(x + 1)$
$7y – 21 = – 2x – 2$
$7y + 2x – 21 + 2 = 0$
$2x + 7y – 19 = 0$
The equation is $2x + 7y – 19 = 0$

<u>Method 2</u>

Make y the subject of $2x + 7y – 8 = 0$
$7y = – 2x + 8$
$y = -\frac{2}{7}x + \frac{8}{7}$ compared to $y = mx + c$, $m = -\frac{2}{7}$

The line parallel to $2x + 7y – 8 = 0$ has a gradient of $-\frac{2}{7}$ and passes through (- 1 , 3)
Substitute in $y = mx + c$
$3 = -\frac{2}{7}$ (- 1) + c
$3 \times 7 = - 2 (-1) + 7c$
$21 = 2 + 7c$

$21 - 2 = 7c$
$19 = 7c$
$c = \frac{19}{7}$

Put $m = -\frac{2}{7}$ and $c = \frac{19}{7}$ in $y = mx + c$
$y = -\frac{2}{7}x + \frac{19}{7}$
The equation is $y = -\frac{2}{7}x + \frac{19}{7}$

The Equation of a Line which Passes through the Point (x_1, y_1) and Perpendicular to another Line

Two lines are said to be perpendicular if the product of their gradient is –1. That is, if the gradient of a line is m, then the gradient of the line perpendicular to it is $-\frac{1}{m}$

The equation of the straight line perpendicular to another line of gradient m, passing through the point (x_1, y_1) is found by making y the subject of the formula: $\boldsymbol{y - y_1 = -\frac{1}{m}(x - x_1)}$ after substituting the values of m and (x_1, y_1)

Worked Examples

1. Find the equation of the line perpendicular to the line $5x - 2y - 11 = 0$, which passes through the point (2, - 3).

Solution

In $5x - 2y - 11 = 0$
$m = -\frac{a}{b} = \frac{5}{2}$
The line perpendicular to the line $5x - 2y - 11 = 0$, has a gradient of $-\frac{2}{5}$ and passes through (2, - 3)
$\Rightarrow m = -\frac{2}{5}, x = 2, y = -3$
Substitute in $y - y_1 = m(x - x_1)$
$y - (-3) = -\frac{2}{5}(x - 2)$
$y + 3 = -\frac{2}{5}(x - 2)$
$5(y + 3) = -2(x - 2)$
$5y + 15 = -2x + 4$
$5y + 2x + 15 - 4 = 0$
$2x + 5y - 11 = 0$
The equation is $2x + 5y - 11 = 0$

Method 2

Make y the subject of $5x - 2y - 11 = 0$
$-2y = -5x + 11$
$y = \frac{-5}{-2}x - \frac{11}{2}$ compared to $y = mx + c$, $m = \frac{5}{2}$

The line perpendicular to the line $5x - 2y - 11 = 0$, has a gradient of $-\frac{2}{5}$ and passes through (2, -3)
$\Rightarrow m = -\frac{2}{5}$, $x = 2$, $y = -3$
Substitute in $y = mx + c$
$-3 = -\frac{2}{5}(2) + c$
$-3 \times 5 = -2(2) + 5c$
$-15 = -4 + 5c$
$-15 + 4 = 5c$
$-11 = 5c$
$c = -\frac{11}{5}$

Put $m = -\frac{2}{5}$ and c $= -\frac{11}{5}$ in $y = mx + c$
$y = -\frac{2}{5}x - \frac{11}{5}$
The equation is $y = -\frac{2}{5}x - \frac{11}{5}$

2. Find the equation of the perpendicular bisector of AB, where A and B are the points
(- 4, 8) and (0, – 2).

Solution

$A(-4, 8)$ and $B(0, -2)$
$x_1 = -4$, $y_1 = 8$, $x_2 = 0$ and $y_2 = -2$
Perpendicular bisector of AB passes through the mid – point of AB
$\text{M} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-4 + 0}{2}, \frac{8 + (-2)}{2}\right) = (-2, 3)$

The perpendicular bisector passes through the point (– 2, 3).

Gradient of AB

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2\ -8}{0-(-4)} = \frac{-10}{4} = -\frac{5}{2}$

Gradient of the perpendicular bisector of AB = $\frac{\mathbf{2}}{\mathbf{5}}$

Equation of the perpendicular bisector of AB which passes through (- 2, 3)

$y - y_1 = m(x - x_1)$

But $m = \frac{2}{5}$, $x_1 = -2$, and $y_1 = 3$

$y - 3 = \frac{2}{5}(x - (-2)$

$5(y - 3) = 2(x + 2)$

$5y - 15 = 2x + 4$

$5y - 2x - 15 - 4 = 0$

$5y - 2x - 19 = 0$

The equation is $-2x + 5y - 19 = 0$

Exercises 6.37

A. Find the equation of the straight line:

1. which passes through (- 3, - 2), perpendicular to $4x + 3y - 5 = 0$
2. which passes through (4, 0), perpendicular to $x + 7y + 4 = 0$
3. which passes through (3, – 4), perpendicular to the line $5x - 2y = 3$
4. which passes through (– 2, 3), parallel to $5x - 2y - 1 = 0$

B. 1. Find the equation of the straight lines through the point (3, –2) which are:

i. parallel to the line $2y + 5x = 17$

ii. Perpendicular to the line $2y + 5x = 17$

2. Find the equations of the lines passing through the point (4, –2) respectively;

i. parallel and

ii. perpendicular to the line $2x - 3y - 4 = 0$

3. Find the equation of a straight line through P(7, 5) perpendicular to the straight line AB whose equation is $3x + 4y - 16 = 0$

4. If the distance of the point (x, y) from the origin is always equal to the distance between (3, -9) and (-1, - 6), find an equation connecting x and y.

Verifying that a Point Belongs to a Line

To verify that a point lies on a line or belongs to a line:

I. Substitute the coordinates of the point (x, y) into the equation of the line.

II. If the coordinates satisfy the equation, then the point is on the line.

III. If the coordinates do not satisfy the equation, the point is not on the line .

Worked Examples

1. Show whether the point (1, 13) lie on the line $y = 6x + 7$.

Solution

Point $(x, y) = (1, 13)$

When $x = 1$,

$y = 6x + 7$

$y = 6(1) + 7 = 13$

$\therefore$ (1, 13) lie on the line $y = 6x + 7$

2. Show whether the point (13, 30) lies on the line $y = 2x + 2$

Solution

Point $(x, y) = (13, 30)$

When $x = 13$

$y = 2x + 2$

$y = 2(13) + 2 = 28$

When $x = 13$, $y \neq 30$

$\therefore$ (13, 30) does not lie on $y = 2x + 2$

3. Show whether the point (2, 3) belongs to the line $5x - 2y - 4 = 0$

Solution

For the point (2, 3), $x = 2$and $y = 3$

$5x - 2y - 4 = 0$

$5(2) - 2(3) - 4 = 0$

$0 = 0$

L.H.S = R. H. S

The point (2, 3) belongs to the line $5x - 2y - 4 = 0$

4. Given that the point (4, 2) lies on the line $y = 2x + c$, find c

Solution

$y = 2x + c$

$(x,y) = (4, 2)$

$2 = 2(4) + c$

$2 = 8 + c$

$2 - 8 = c$

$c = - 6$

Exercises 6.38

A. Which of the point belongs to its line

1. $y = 2x + 3$, (6, 16) 3. $y = 3x - 5$, (- 1, -8)
2. $3x + 4y = 1$, (-1, 1) 4. $5x - 2y + 7 = 0$, (1, -1)

Challenge Problems

1. A (2, -2) and B (4, 4) are two points

i. Find $|AB|$

ii. Find the coordinates of m, the mid – point of $\overline{AB}$

iii. Find the slope and the equation of the line AB.

iv. The point (–2, r) is on the line AB, find the value of r

v. The perpendicular bisector of the line segment AB cuts the x – axis at the point $p(h, 0)$ and the y – axis at the point $q(0, k)$, find the values of h and k.

2. The coordinates of the points (p, 5) and (-1, q) satisfy the equation $y = 2x - 1$. Calculate the values of p and q.

Point of Intersection of Two Lines

To find the point of intersection of two lines, solve the equations of the lines simultaneously by any preferred method unless stated. The solution set is the point of intersection of the lines.

Worked Example

Find the points of intersection of the lines $2x - 3y = 6$ and $4x + y = 19$

Solution

$2x - 3y = 6$ ……………(1)

$4x + y = 19$…………….(2)

eqn (1) × 2

$4x - 6y = 12$……………(3)

eqn (2) – eqn (3);

$7y = 7$

$y = 1$

Put $y = 1$ in eqn (1);

$2x - 3(1) = 6$

$2x = 6 + 3$

$2x = 9$

$x = \frac{9}{2}$

$x = 4.5$

∴ The point of intersection is (4.5, 1)

Exercises 6.39

A. Find the points of intersection of the following pair of straight lines:

1. $y = 3x + 2$ and $2x + 3y = 17$
2. $5x + 3y = 2$ and $x - y = 6$
3. $y = 5x + 2$ and $y = 3x - 1$
4. $5x + 7y + 29 = 0$ and $11x - 3y - 65 = 0$

Challenge Problems

1. L : $2x - 5y - 9 = 0$ and K : $3x - 2y - 8 = 0$ are two lines. L intersects K at the point q.

i. Find the coordinates of q

ii. Find the equation of the line M such that L is perpendicular and $q \in M$

iii. Show that the point r (4, -6) is on M

2. $P = \{(x, y) : 4x + 3y = -7\}$, $Q = \{(x, y) : 5x - 9y = -13\}$ and $R = \{(x, y) : 14x - 15y = -33\}$, $x, y \in R$. Prove that $P \cap Q \subset R$

3. $P = \{(x, y) : 2x + 3y + 5 = 0\}$ and $Q = \{(x, y) : 5x - 4y + 1 = 0\}$, $x, y \in R$. Find $P \cap Q$

7 PLANE GEOMETRY I

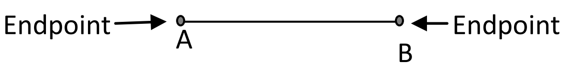
Lines and Planes

Definitions:

A Point: is an idea associated with position. It is symbolized by a dot (.) and represents specific location. It has neither size nor shape

A Line: is an infinite set of points which extends indefinitely in two directions

A Line Segment : is a set of points in a line consisting of twodistinct ends. It represents a collection of points inside the endpoints and it is named by its end points



A Ray: is a line segment that has only one defined end point and one side that extends endlessly away from the end point. A ray is named by its end point and by the other point on the line.

A Plane: is a flat surface which has length and width only. A plane therefore has two dimensions, length and width; no thickness. e.g. A floor of a football field

Exercises 7.1

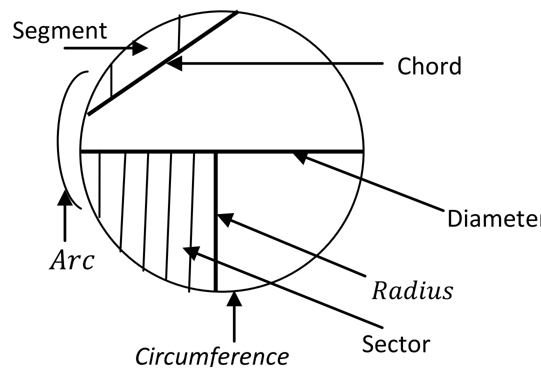
Fill in the blank spaces with the correct response

1. An idea associated with position is called………
2. An infinite set of points extending indefinitely in two dimensions is called …
3. A set of points in a line consisting of two distinct end points is called ……
4. A flat surface which has length and width only is called ………
5. A line that starts from a point and end at infinity is called…….

The Circle

A circle is a set of points in a plane which are at the same distance from a fixed point. The fixed point is called the ***centre*** of the circle and the set of points forms the circumference of the circle.

Parts of a Circle



Circumference: It is the distance around a circular region. It is also known as the length or perimeter of a circle.

Diameter: It is a straight line that divides a circle into two equal parts

Semi - circle: It is half a circle

Chord: It is s a straight line that connects any two points on a circle.

Arc: It is a portion on the circumference of a circle

Segment: It is the area bounded by an arc and a chord.

Radius : A line drwn from the center of a circle to touch any part of the circumference. The plural is radii

Sector: It is area bounded by two radii and an arc

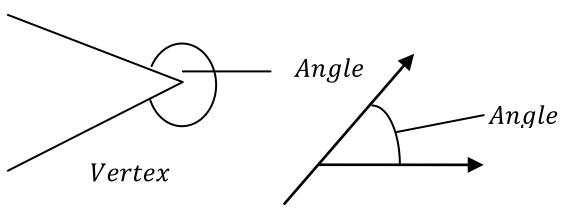
Exercises 7.2

Complete the each with the correct answer

1. Half a circle is called …
2. A straight line drawn from the center of a circle to touch any point on the circumference is called …
3. The distance around a circle is called…
4. Any straight line that passes throught the center of a circle, touching the circumference at both ends is called …

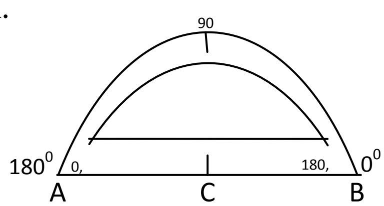
Angles

An angle is formed when two straight lines meet at a point. The point where the two straight lines meet is simply called *Vertex*.



Measurement of Angles

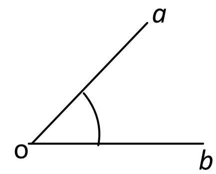
The instrument used to measure angles is called a **protractor**. The unit of measure is the degree (^{0}). The scale on the protractor is divided into degrees numbered from 0^0 to 180^0 starting from either end.



Using the Protractor

(A) Angles opening to the right.

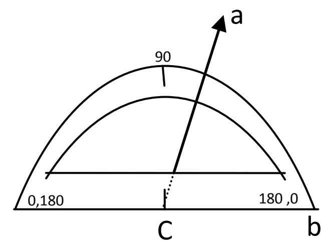
For e.g.



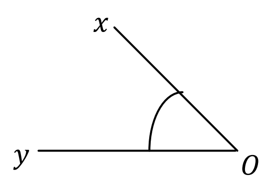
I. Place the point "C" of the protractor on point "o" of the angle.

II. Align the line segment "CB" of the protractor with arm "ob" of angle a o b so that CB falls exactly on "ob". The arm "o a" of angle aob points to the number of degrees the angle measures on the protractor.

III. In order to determine the size of angles opening to the right, the inner set of measurement is used.



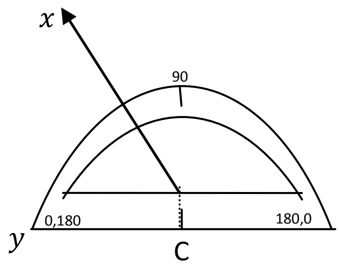
(B) Angles opening to the left.



I. Place the point "C" of the protractor on the point O of the angle.

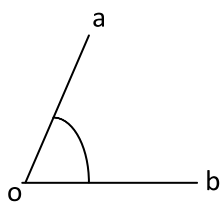
II. Line up $\overline{\mathbf{AC}}$ of the protractor with arm "$\overline{\mathbf{oy}}$" of angle **xop** so that AC falls exactly on arm "**oy**" of ***xoy.*** At this stage, the **ox** of angle *xoy* will point to the number of degrees the angle measures on the protractor.

III. In order to determine the size of angles opening to the left, use the outer values of measure.



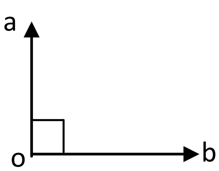
Types of Angles

1. Acute angle**:** Any angle whose measure is less than 90^0.



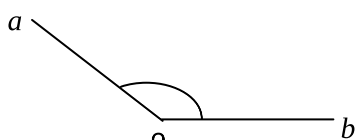
$0^0 < \theta < 90^0$

2. ***Right Angle*****:** Angle whose measure is exactly 90^0.



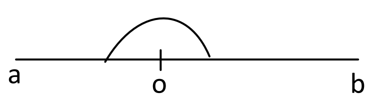
$\theta = 90^0$

3. ***Obtuse Angle*****:** Any angle whose measure is greater than 90^0 but less than 180^0



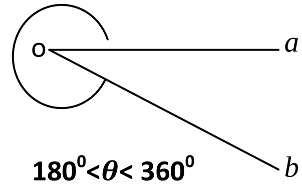
$90^0 < \theta < 180^0$

4. Straight Angle: Any angle whose measure is 180^0



$\theta = 180^0$

5. Reflex Angle: Any angle whose measure is greater than 180^0 but less than 360^0.



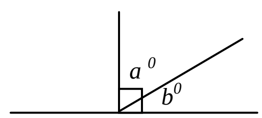
$180^0 < \theta < 360^0$

The values of the type of angles are summarized below:

0^0 < an acute angle < a right angle(90^0) < a straight angle (180^0) < a reflex angle < a complete turn (360^0)

Pair of Angles

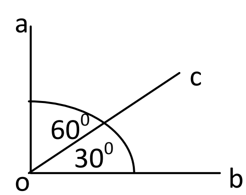
1. Complementary Angles: They are any two angles that sum up to 90^0.



$$a^0 + b^0 = 90^0$$

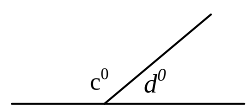
Complementary angles

For example, in the diagram below,:



$60^0 + 30^0 = 90^0$, 60^0 and 30^0 are complementary angles

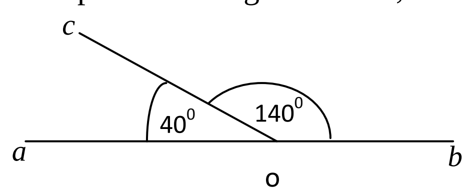
2. Supplementary Angles: They are any two angles that sum up to 180^0.



$$c^0 + d^0 = 180^0$$

Supplementary angles

For example in the diagram below;

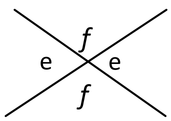


40^0 and 140^0 are said to be supplementary angles because $40^0 + 140^0 = 180^0$

3. Vertically opposite angle

When two lines cross each other, vertically opposite angles are formed. Vertically opposite angles are equal

Consider the diagram below;



i. ***Angle e = Angle e***
ii. ***Angle f = Angle f***

Worked Examples

The sizes of two angles are $(100 - x)^0$ and $(3x + 50)^0$. Calculate x if these two angles are supplementary

Solution

If $(100 - x)^0$ and $(3x + 50)^0$ are supplementary
$\Rightarrow (100 - x)^0 + (3x + 50)^0 = 180^0$
$100^0 - x^0 + 3x^0 + 50^0 = 180^0$
$2x^0 + 150^0 = 180^0$
$2x^0 = 180^0 - 150^0$
$2x^0 = 30^0$
$x = 15^0$

Exercises 7.4

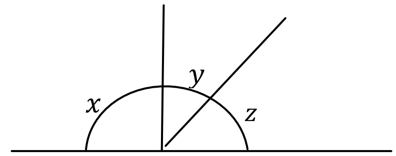
A. Fill in the spaces correctly:

1. Two straight lines meet at a point to form a figure that is called …
2. The supplementary angle of 48^0 is.................
3. The complement angle of 65^0 is ……………
4. Any angle that measures 90^0 is called……….
5. Two right angles equal to …………………..

Properties of Angles

1. ***Angles formed on a straight line sum up to 180^0.***

Consider the figure below;

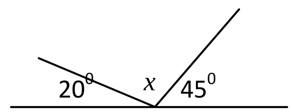


$\boldsymbol{x + y + z = 180^0}$

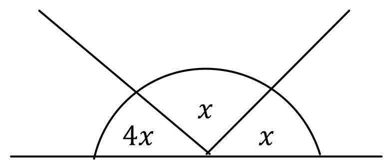
Worked Examples

Find the value of x in the diagrams below

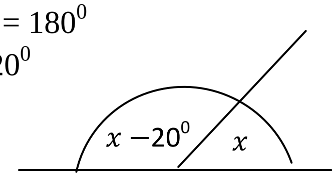
1. $x + 20^0 + 45^0 = 180^0$ (< s on a straight line)
$x = 180^0 - 20^0 - 45^0$
$x = 115^0$



2. $4x + x + x = 180^0$ (< s on a straight line)
$6x = 180^0$
$\frac{6x}{6} = \frac{180}{6}$,
$x = 30^0$

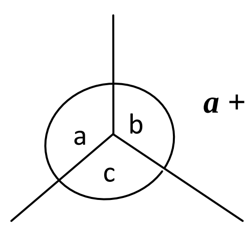


3. $\boldsymbol{x} - 20^0 + x = 180^0$
$2x = 180^0 + 20^0$
$2x = 200^0$
$\frac{2x}{2} = \frac{200}{2}$,
$x = 100^0$



2. ***Angles formed in a circle add up to 360^0***

Consider the figure below:

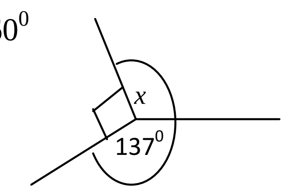


$\boldsymbol{a + b + c = 360^0}$

Worked Examples

Find the angles marked with letters

i. $x + 90^0 + 137^0 = 360^0$
$x = 360^0 - 90^0 - 137^0$
$x = 133^0$

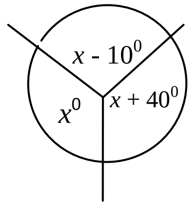


ii. $x - 10^0 + x + 40^0 = 360^0$

$2x = 360^0 + 10^0 - 40^0$

$2x = 330^0$

$x = 165^0$

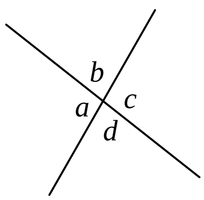


Angles Connected with Parallel Lines

1. *Vertically Opposite Angles*

When two straight lines intersect, the opposite angles formed are called ***vertically opposite angles.***

Vertically opposite angles are equal to each other. For example, in the figure below;



i. Angle *a* = Angle *c* **(Vertically opp. Angles)**

ii. Angle *b* = Angle *d* **(Vertically opp. Angles)**

Worked Examples

1. Given that $(12x - 100)^0$ and $(9x + 20)^0$ are vertically opposite angles. Calculate:

i. the value of x.

ii. the value of $(12x - 9x)^0$

iii. What is the supplementary angle of $(12x - 9x)^0$

Solution

i. $(12x - 100)^0$ and $(9x + 20)^0$ are vertically opposite angles

$(12x - 100)^0 = (9x + 20)^0$

$12x - 9x = 20^0 + 100^0$

$3x = 120^0$

$x = 40$

ii. $12x - 9x$

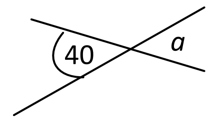
$= 12(40^0) - 9\ (40^0)$

$= 480^0 - 360^0 = 120^0$

iii. $180^0 - 120^0 = 60^0$

The supplementary angle of $12x - 9x = 60^0$

2. Find the angles marked with letters:

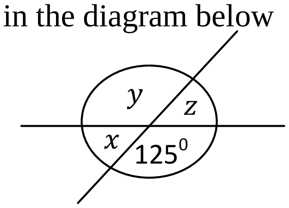


Solution

Angles a and 40^0 are vertically opposite angles.

Therefore $a = 40^0$

3. Find the values of the angles marked with letters in the diagram below



Solution

$z + 125^0 = 180^0$ (angles on straight line)

$z = 180^0 - 125^0 = 55^0$

$z = x$ (vertically opposite angles)

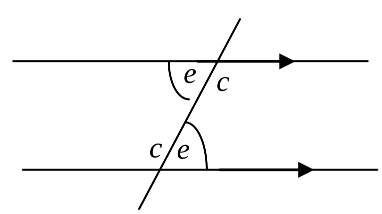
$x = 55^0$

$y = 125^0$ (vertically opposite angles)

2. *Alternate angles*

They are angles that are formed at the corners of a figure. They are also called Z or N or Σ angles. Alternate angles are equal.

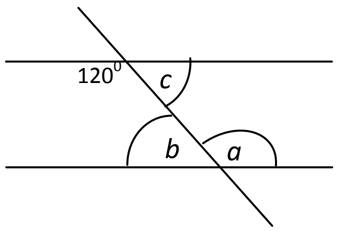
Consider the diagram below:



i. Angle *c* = Angle *c* **(Alternate angles)**
ii. Angle *e* = Angle *e* **(Alternate angles)**

Worked Example

Find a, b, c and d in the diagram below;



Solution

$a = 120^0$ (Alternate angles)

$\mathbf{c} + 120^0 = 180^0$ (Straight angles)

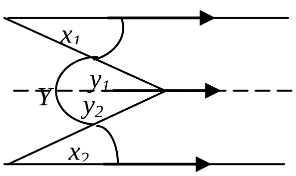
$c = 180^0 - 120^0$

$c = 60^0$

$c = b = 60^0$ (Alternate angles)

Special Alternate Angles

Consider the diagrams below:



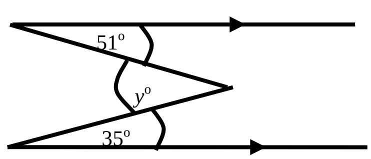
From the diagram above x_1 and y_1 are alternate angles. Therefore, $\boldsymbol{x_1 = y_1}$.

Similarly, $\boldsymbol{x_2}$ and $\boldsymbol{y_2}$ are alternate angles. Therefore $\boldsymbol{x_2 = y_2}$.

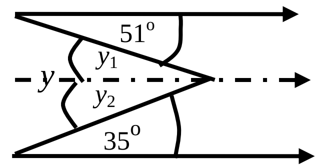
$\Rightarrow \boldsymbol{y_1 + y_2 = x_1 + x_2}$ and $\boldsymbol{y_1 + y_2 = y}$

Worked Examples

Find the value of y in the figure below:



Solution



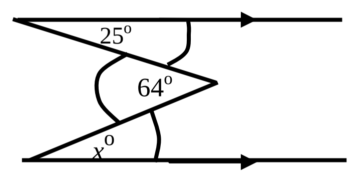
y_1 and 51^o are alternate angles so $y_1 = 51^o$

y_2 and 35^o are alternate angles so $y_2 = 35^o$

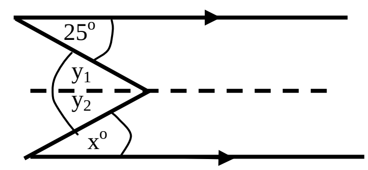
But $y = y_1 + y_2$

$y = 51^o + 35^o = 86^o$

2. In the diagram below, find the value of angle x.



Solution



$y_1 + y_2 = 64^o$

But $y_1 = 25^o$ (Alternate angles)

$25^o + y_2 = 64^o$

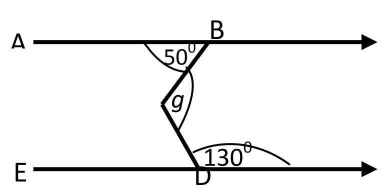
$y_2 = 64^o - 25^o = 39^o$

$y_2 = x^o$ (Alternate angles)

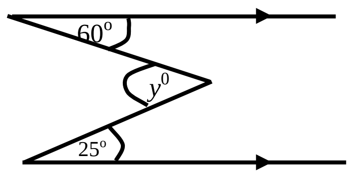
$x^o = 39^o$

Exercises 7.5

1. In the figure below, if $\overrightarrow{AB}$ is parallel to $\overrightarrow{ED}$, find the value of angle g

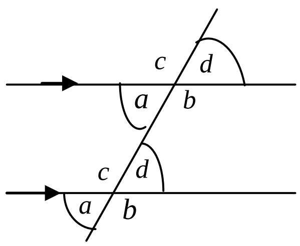


2. In the figure below, find the value of y



3. Corresponding angles

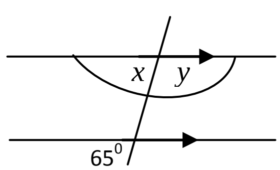
Consider the diagram below:



Angle *a* corresponds with angle *a*
Angle *b* corresponds with angle *b*
Angle *c* corresponds with angle *c*
Angle *d* corresponds with angle *d*
But corresponding angles are equal
⇒*a* = *a* , *b* = *b*, *c* = *c* and *d* = *d*

Worked Examples

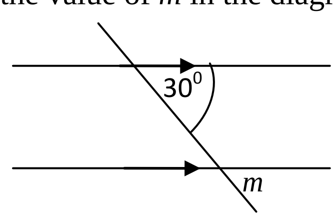
1. Find x and y in the diagram below?



Solution

$x = 65^0$ (Corresponding angles)
$x + y = 180^0$ (Angles on a straight line)
$65^0 + y = 180^0$
$y = 115^0$

2. Find the value of m in the diagram below

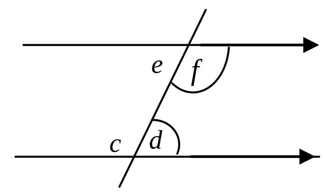


Solution

m and n are corresponding angles
$m = 30^0$

4. *Co-interior Angles*

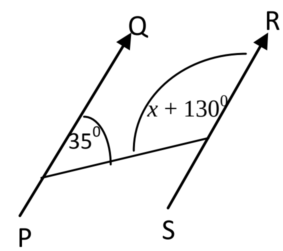
In the diagram below:



d and f are co – interior angles as well as e and c.
Co – interior angles are also called ***supplementary angles*** because they sum up to 180^0.
i. $d + f = 180^0$
ii $e + c = 180^0$

Worked Examples

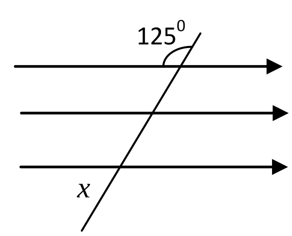
1. In the diagram, PQ is parallel to SR. Find the value of x



Solution

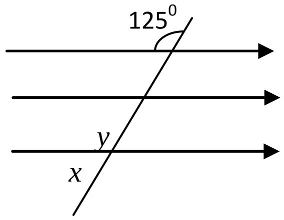
$(x + 130^0)$ and 35^0 are interior opposite angles. $(x + 130^0) + 35^0 = 180^0$
$x + 130^0 + 35^0 = 180^0$
$x = 180^0 – 130^0 – 35^0 = 15^0$

2. A straight line intersects three parallel lines as shown in the diagram below. Find the value of x



Solution

Name the angles as shown below;



$y = 125^0$ *(Corresponding angles)*

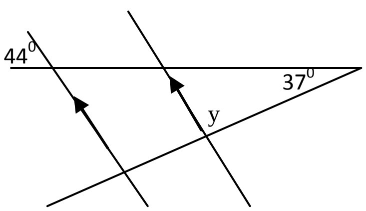
$x + y = 180^0$ *(Angles on a straight line)*

$x + 125^0 = 180^0$

$x = 180^0 – 125^0$

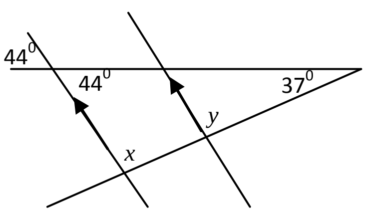
$x = 55^0$

3. In the figure below, find the value of the angle named *y*



Solution

Name the angles as shown below;



$x + 44^0 + 37^0 = 180^0$

$x + 81^0 = 180^0$

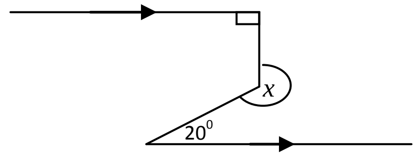
$x = 180^0 – 81^0$

$x = 99^0$

$x = y$ *(corresponding angles)*

But $x = 99^0$, $\Rightarrow y = 99^0$

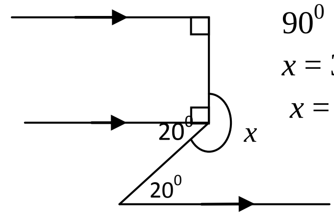
4. Find the value of *x* in the figure below;



Solution

Draw a parallel line througg angle *x* and name

the angles as shown below:

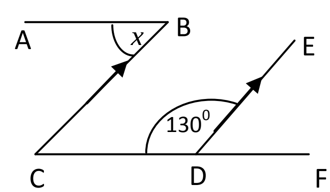


$90^0 + 20^0 + x = 360^0$

$x = 360^0 – 90^0 – 20^0$

$x = 250^0$

5. Find the value of angle *x*.



Solution

$x = <$ BCD (*Alternate angles*)

$<$ BCD $+ 130^0 = 180^0$ (*Interior opp.<s*)

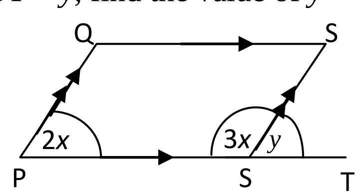
$<$ BCD $= 180^0 – 130^0$

$<$ BCD $= 50^0$

But $x = <$ BCD $= 50^0$

Therefore $x = 50^0$

8. In the diagram below, PQRS is a parallelogram. If < QPS = 2*x*, < RSP = 3*x* and < RST = *y*, find the value of *y*



Solution

2*x* and *y* are corresponding angles

$\Rightarrow y = 2x$

$3x + y = 180^0$ (< s on a straight line)

$3x + 2x = 180^0$ (But $y = 2x$)

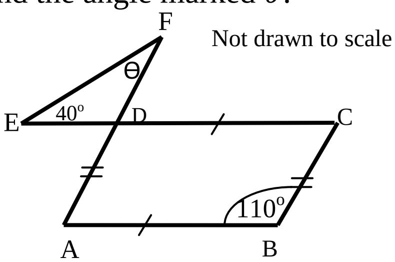
$5x = 180^0$

$x = 36^0$

$\Rightarrow y = 2x = 2(36^0)$

$y = 72^0$

9. Find the angle marked θ.



Solution

Angle ADC = 110°.

(Opposite interior angle of a parallelogram)

Angle EDF = 110° (opposite angle)

$\theta + 40^o + 110^o = 180^o$

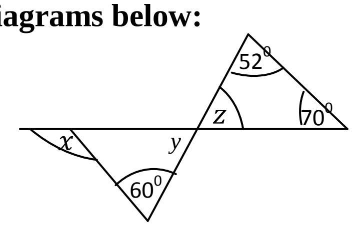
$\theta + 150^o = 180^o$

$\theta = 180^0 - 150^0 = 30^0$

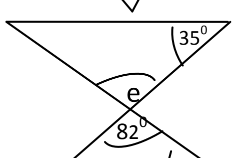
Exercises 7.6

A. Find the value of the variables: by variables in the diagrams below:

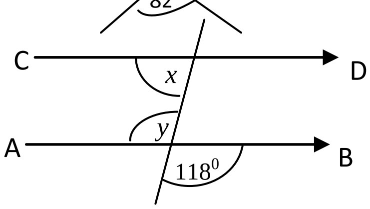
1.



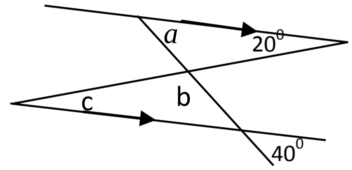
2.



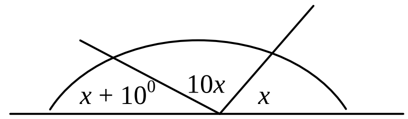
3.



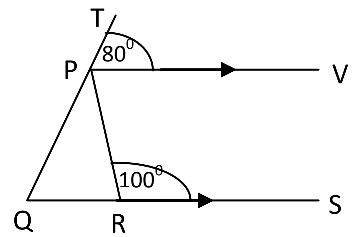
4.



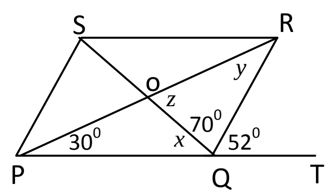
5.



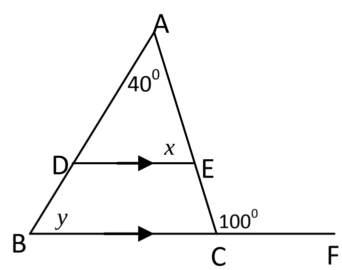
B. 1. In the figure below,< PRS = 100^0, < TPV = 80^0 and PV is parallel to QS. Explain why ΔPQR is isosceles



2. In the figure below, PQRS is a parallelogram. Mark in the sizes of the angles x , y and z



3. In the figure below, *DE// BC*. Mark in the sizes of angles x and y



4. *ABCD* is a parallelogram in which angle A = 72^0 and *AB* is equal in length to diagonal *BD*. Calculate the sizes of all the angle in the figure

Triangles

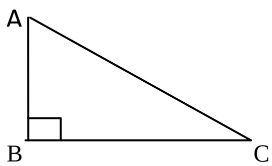
A triangle is a plane figure bounded by three straight lines. A triangle has three interior angles that sum up to 180^0. The area of a triangle,

$A = \frac{1}{2}$ ***base*** **×** ***height***

Types of Triangles

I. Right – angled Triangle

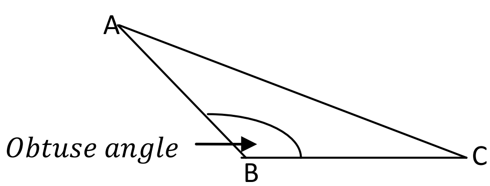
It is a triangle in which one angle is a right angle or 90^0



Note:
The symbol ∟ represents a right angle or 90^0

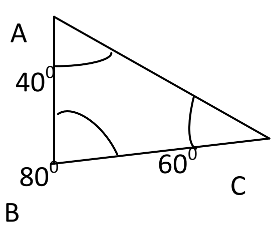
II. Obtuse – angled Triangle

It is a triangle in which one angle is an obtuse angle (greater than 90^0, but less than 180^0)



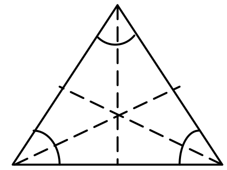
III. Acute - angled Triangle

It is a triangle in which each interior anlge is less than 90^0



IV. Equilateral Triangle

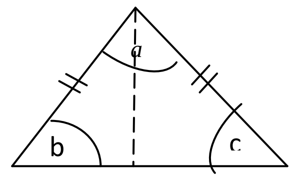
It is a triangle in which all the three sides are equal.



In an equilateral triangle, each angle is 60^0

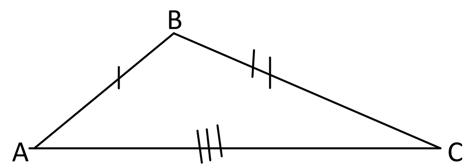
V. Isosceles Triangle

It is a triangle in which two base angles and corresponding sides are equal



VI. Scalene Triangle

It is a triangle in which all the three sides and angles are unequal



Congruent Triangles

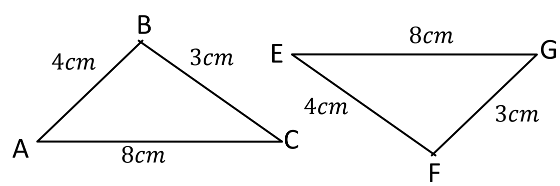
They are two or more triangles that have the same shape or the same size and angles. In other words, triangles are congruent if:

I. 3 sides = 3 sides

II. 2 sides, included angle = 2 sides, included angle

III. 2 angles, one side = 2 angles, corresponding side

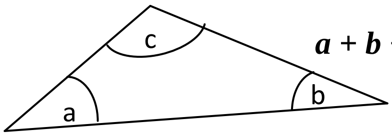
Example of a congruent triangle is shown below



ΔABC is congruent to Δ EFG because they have equal corresponding sides

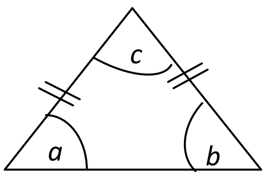
Properties of a Triangle

1. The sum of angles in a triangle is equal to 180^0



$\boldsymbol{a + b + c = 180^0}$

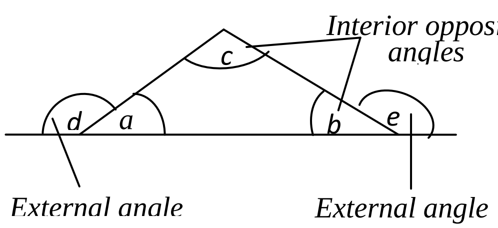
2. For all Isosceles triangles, the base angles are equal. The base angles are the angles that face the two equal sides



$a = b$ (base angles of an isosceles Δ)

The Exterior Angle Theorem

The exterior angle theorem of triangles states that the exterior angle of a triangle is equal to the sum of the two interior angles opposite to it (exterior angle).



From the figure above, the theorem is summarized as: $\boldsymbol{d = c + b}$ **and** $\boldsymbol{e = a + c}$

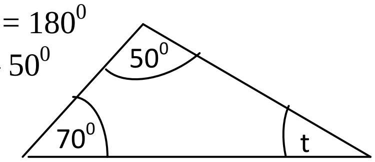
Worked Examples

Find the angles marked with letters:

1. $t + 70^0 + 50^0 = 180^0$

$t = 180^0 - 70^0 - 50^0$

$t = 60^0$

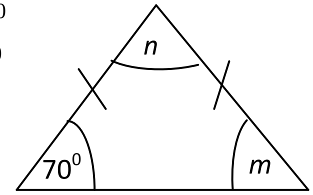


2. $m = 70^0$ (base angles of an isosceles Δ)

$n + 70^0 + 70^0 = 180^0$

$n = 180^0 - 70^0 - 70^0$

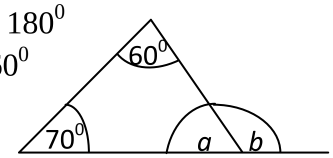
$n = 40^0$



3. $a + 70^0 + 60^0 = 180^0$

$a = 180^0 - 70^0 - 60^0$

$a = 50^0$



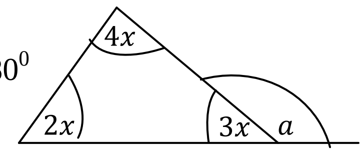
$b = 70^0 + 60^0$ (interior opposite angles)

$b = 130^0$

5. $2x + 4x + 3x = 180^0$

$9x = 180$

$x = 20^0$



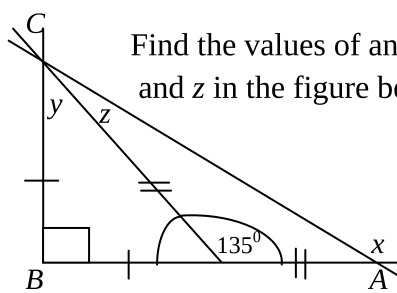
$a = 2x + 4x$ (Interior opposite angles)

$a = 2(20^0) + 4(20^0)$ (Put $x = 20^0$)

$a = 40^0 + 80^0$

$a = 120^0$

6. Find the values of angles x, y and z in the figure below;



Solution

Name the vertices and the angles as shown below;

Δ CDA is an isosceles triangle

Therefore $n = z$ *(Base <s of a triangle)*

$n + n + 135^0 = 180^0$

$2n + 135^0 = 180^0$

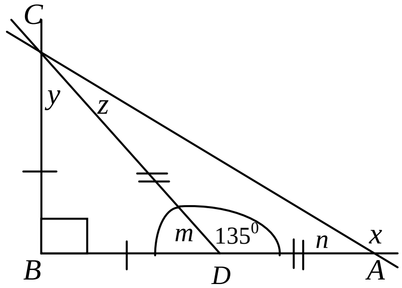
$2n = 180^0 - 135^0$

$2n = 45^0$

$n = \frac{45^0}{2}$

$n = 22.5^0$

$\Rightarrow z = 22.5^0$



$n + x = 180^0$ (< *s on a straight line*)

But $n = 22.5^0$

$22.5^0 + x = 180^0$

$x = 180^0 – 22.5^0 = 157.5^0$

$m + 135^0 = 180^0$

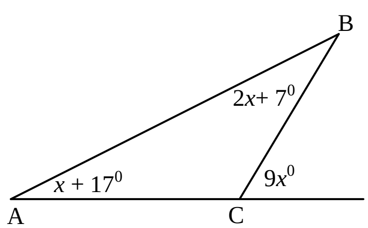
$m = 180^0 – 135^0 = 45^0$

Δ *CBD* is an isosceles triangle

$\Rightarrow m = y$ (*Base < s of a triangle*)

$y = 45^0$

7. In the figure below, *AC* is protruded to *D*. Find the value of *x* and < *ACB*



Solution

$x + 17^0 + 2x + 7^0 = 9x^0$ (Exterior < theorem)

$17^0 + 7^0 = 9x^0 – x^0 – 2x^0$

$24^0 = 6x^0$

$x^0 = \frac{24^0}{6} = 4^0$

< $ACB + 9x^0 = 180^0$ (< s on a straight line)

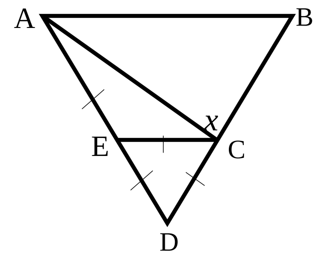
< $ACB + 9(4^0) = 180^0$ (But $x = 4^0$)

< $ACB + 36^0 = 180^0$

< $ACB = 180^0 – 36^0 = 144^0$

Some Solved Past Questions

1. In the diagram below, |AE| = |ED| = |DC| = |CE|



Calculate the size of the angle marked *x*.

Solution

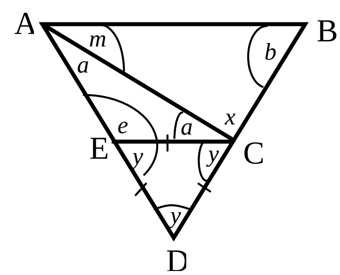
Name the angles as shown below

$y + y + y = 180^0$ **(Equilateral triangle)**

$3y = 180^o$

$y = \frac{180^0}{3}$

$y = 60^o$



$y + e = 180^o$ **(Angles on a straight line)**

$60^o + e^o = 180^o$

$e^o = 180^o – 60^o = 120^o$

$a + a + e^o = 180^o$ *(< in an isosceles Δ)*

$2a + 120 = 180^o$ (But $e = 120^o$)

$2a = 180^o – 120^o$

$2a = 60^o$

$a = 30^o$

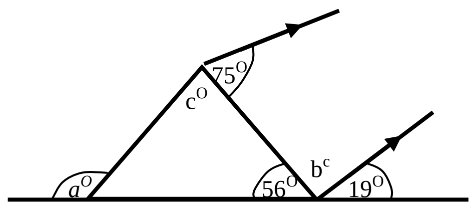
$y + a + x = 180^o$ *(Angles on a straight line)*

But $y = 60^o$ and $a = 30^o$

$60^o + 30^o + x = 180$

$x = 180^o – 60^o – 30^o = 90^o$

2. In the diagram below, find the value of angles *a*, *b* and *c*.



Solution

$b^o + 56^0 + 56^0 = 180^o$

(Base < s of an isosceles triangle)

$b = 180^o - 56^o - 56^o = 68^o$

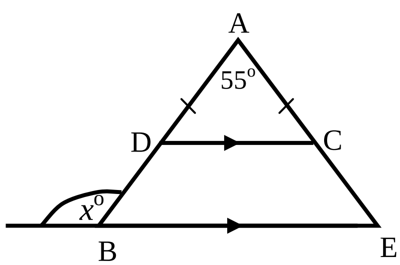
$a^o + 56^o = 180^o$*(< on a straight line)*

$a = 180^o - 56^o = 124^o$

$c + 56^o + 19^o = 180^o$*(<s on a straight line)*

$c = 180^o - 56^o - 19^o = 105^0$

3. Find the value of the angle marked x in the diagram below:



Solution

Name the angles as shown below;

$a + a + 55^o = 180^0$

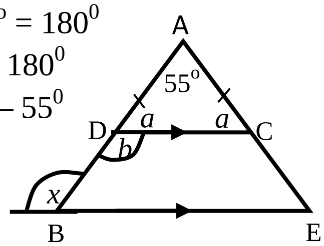
$2a^o + 55 = 180^0$

$2a = 180^0 - 55^0$

$2a = 125^o$

$2a = 125^o$

$a = 62.5^0$



$a + b = 180^o$ (*angles on a straight line*)

$62.5^0 + b = 180^o$

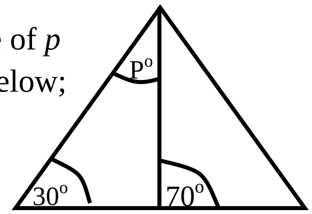
$b = 180^o - 62.5^0 = 117.5^0$

b and x are alternate angles

$\Rightarrow b = x$. But $b = 117.5^0$.

Therefore $x = 117.5^0$

4. Find the value of p in the diagram below;



Solution

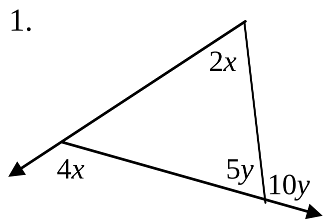
$p + 30^o = 70^o$*(external angle theorem)*

$p = 70^o - 30^o = 40^o$

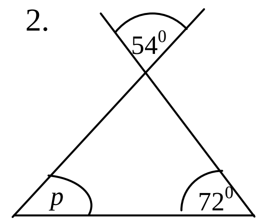
Exercises 7.7

A. Find the values of the angles marked with variables in the figures below:

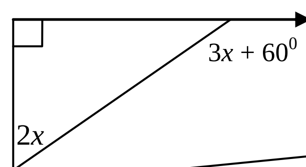
1.



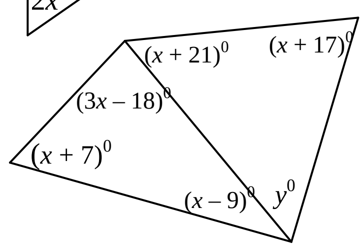
2.



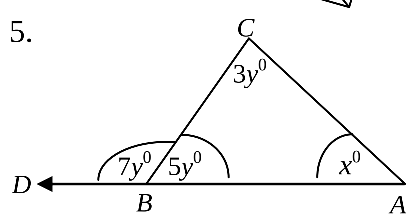
3.



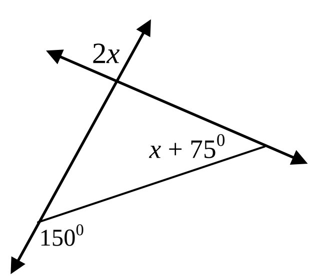
4.



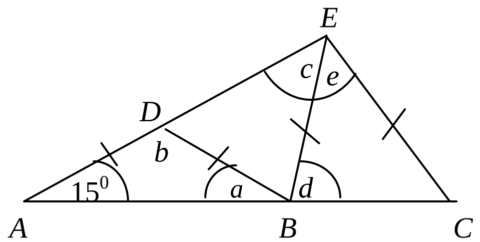
5.



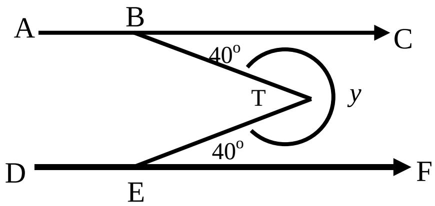
6.



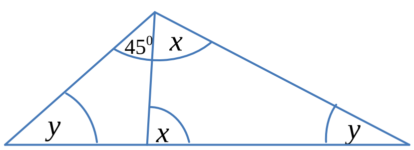
7.



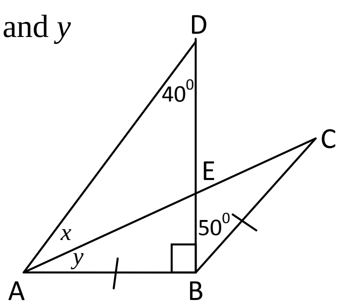
B. 1. Find the value of the reflex angle marked *y* in the diagram below:



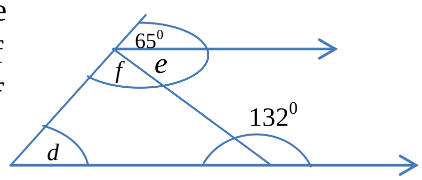
2. In the figure below, find the values of angles *x* and *y*



3. In the figure below *ABC* is an isosceles triangle. Triangle *ABD* has a right angle at *B*. < ADB = 40^0 , < CBE = 50^0. Work out for the sizes of angles *x* and *y*



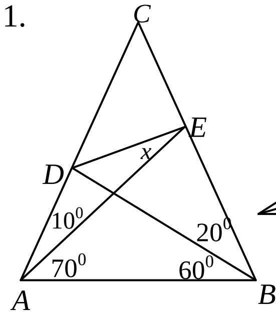
4. Determine the values of angles *d, e, f*



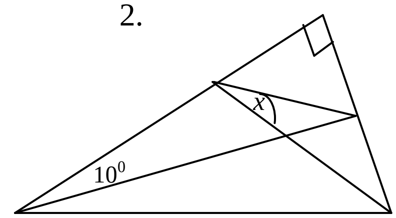
Challenge Problems

In the diagrams below, determine the value of *x*

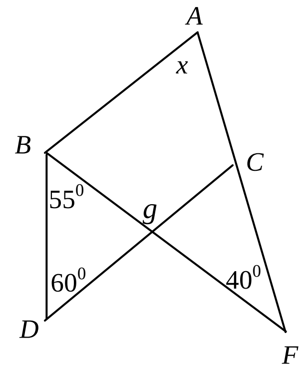
1.



2.



3. In the figure below, *AB* is parallel to *DC*.



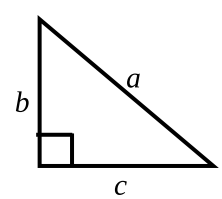
The Right – Angled Triangle

Any triangle which has one right – angle is called a ***right-angled triangle***.This means that in a right-angled triangle, an angle is 90^0

The Pythagoras Theorem
(Hypotenuse Rule)

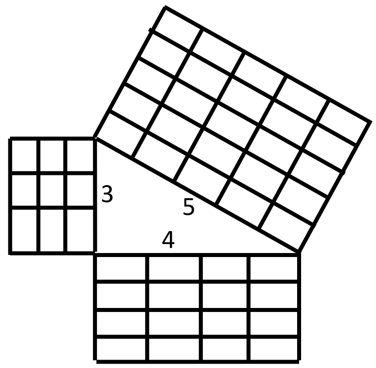
It states that the square on the hypotenuse of a right- angled triangle is equal to the sum of the lengths of the squares on the other two sides

This is illustrated in the diagram below:



$$\boxed{a^2 = b^2 + c^2}$$

This is illustrated in the diagram below;

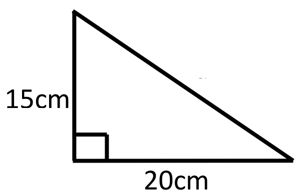


By Pythagoras theorem, $5^2 = 3^2 + 4^2$

Worked Examples

Find the length of the side marked with letters in the triangle below:

1.



Solution

$a^2 = 15^2 + 20^2$

$a^2 = 225 + 400$

$a^2 = 625$,

$a = \sqrt{625} =$ 25cm.

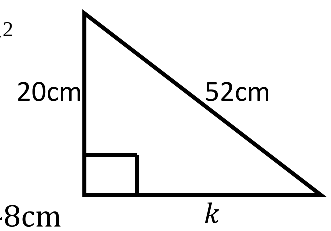
2. $52^2 = 20^2 + k^2$

$k^2 = 52^2 - 20^2$

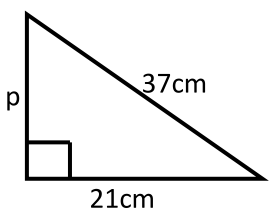
$k^2 = 2704 - 400$

$k^2 = 2304$

$k = \sqrt{2304} = 48$cm



3.



Solution

$37^2 = p^2 + 21^2$

$p^2 = 37^2 - 21^2$

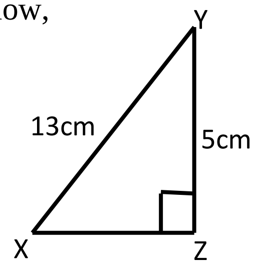
$P^2 = 1369 - 441$

$P^2 = 928$ cm

$P = \sqrt{928}$cm

$P = 30.5$cm (3 s. f)

4.In the diagram below, what is the length of *XZ*?



Solution

From Pythagoras theorem,

$13^2 = /XZ/^2 + 5^2$

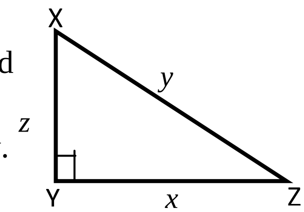
$/XZ/^2 = 13^2 - 5^2$

$/XZ/^2 = 169 - 25$

$/XZ/^2 = 144$

$/XZ/ = \sqrt{144} = 12$cm

5. XYZ is right angled triangle, with length of sides shown below.



Express z in terms of x and y

Solution

From Pythagoras theorem,

$y^2 = x^2 + z^2$

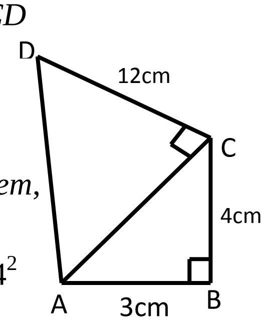
$z^2 = y^2 - x^2$

$z = \sqrt{y^2 - x^2}$

6. In the quadrilateral *ABCD* below, /*AB*/ = 3cm. /*BC*/ = 4cm, /*CD*/ = 12cm, angle *ABC* = 90^0 and angle *ACD* = 90^0. Calculate:

i. the perimeter of *ABCD*

ii. the area of *ABCD*



Solution

i. *By Pythagoras theorem,*

$/AC/^2 = /AB/^2 + /BC/^2$

$/AC/ = \sqrt{3^2 + 4^2}\ 3^2 + 4^2$

$/AC/ = \sqrt{25} = 5$cm

Again by Pythagoras theorem,

$/AD/^2 = /AC/^2 + /CD/^2$

But $/AC/$ = 5cm and $/CD/$ = 12cm

$/AD/^2 = 5^2 + 12^2$

$/AD/^2 = 25 + 144$

$/AD/^2 = 169$

$/AD/ = \sqrt{169} = 13$cm

Perimeter of ABCD

$= /AB/ + /BC/ + /CD/ + /DA/$

= 3cm + 4cm + 12cm + 13cm

= 32cm

ii. Area of *ABCD*;

= area of $\Delta\, ABC$ + area of $\Delta\, ACD$

Area of $\Delta\, ABC = \frac{1}{2}bh$

$= \frac{1}{2} \times /AB/ \times /BC/$

$= \frac{1}{2} \times 3 \times 4$

$= 6\text{cm}^2$

Area of triangle *ACD;*

$= \frac{1}{2} \times /CD/ \times /AC/$

$= \frac{1}{2} \times 12 \times 5$

$= 30cm^2$

Area of *ABCD*;

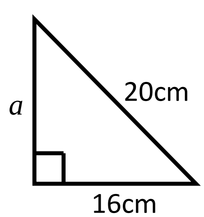
$= 6\text{cm}^2 + 30\text{cm}^2$

$= 36\text{cm}^2$

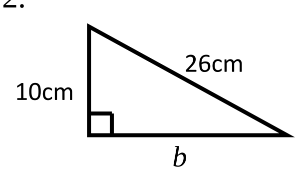
Exercises 7.8

A. Find the unknown lengths:

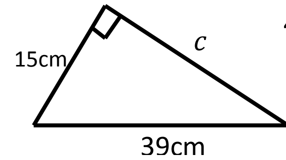
1.



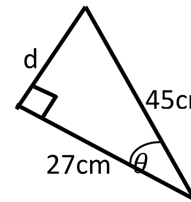
2.



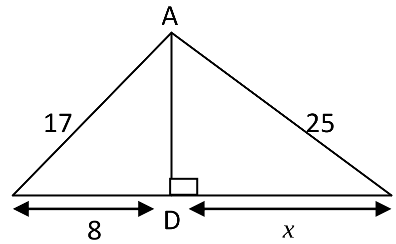
3.



4.



5.



Pythagorean Triples

The Pythagorean triples consist of a set of 3 positive integers that obey the Pythagoras theorem. This means that the sum of the squares of two of them equals the square of the other number. For e.g. {5, 12, 13} are Pythagorean triples because $5^2 + 12^2 = 13^2$. Similarly, {3, 4, 5} are Pythagorean triples because $3^2 + 4^2 = 5^2$.

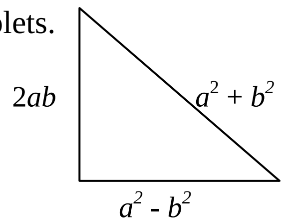
To investigate whether a given set of numbers are Pythagorean triples, equate the sum of the squares of the two smaller integers to the square of the bigger integer. In other words, equate the sum of the squares of the first two integers to the square of the third integer. For example, {3, 4, 5} is investigated as; $3^2 + 4^2 = 5^2$ and not $3^2 = 4^2 + 5^2$

The following sets of integers are Pythagorean triples.

1. {3, 4, 5}, since $5^2 = 3^2 + 4^2$
2. {5, 12, 13}, since $13^2 = 12^2 + 5^2$
3. {8, 15, 17}, since $17^2 = 15^2 + 8^2$
4. {7, 24, 25}, since $25^2 = 7^2 + 24^2$
5. {6, 8, 10}, since $10^2 = 8^2 + 10^2$

For any two positive integers, *a* and *b*, where

$a > b$, the three sides of the right angled – triangle can be expressed in terms of a and b to generate Pythagorean triplets.



Worked Examples

1. Show that the numbers 9, 12 and 15 represents the sides of a right-angled triangle.

Solution

Let a = 15cm, b = 9cm and c =12 cm

By Pythagoras theorem,

$a^2 = b^2 + c^2$

$15^2 = 9^2 + 12^2$

$225 = 81 + 144$

$225 = 225$

The triangle is a right-angled triangle.

2. A triangle has sides 3, 4 and 5 units. Show that it is a right-angled triangle.

Solution

Let a = 5, b = 4 and c = 3,

By Pythagoras theorem,

$a^2 = b^2 + c^2$

$5^2 = 4^2 + 3^2$

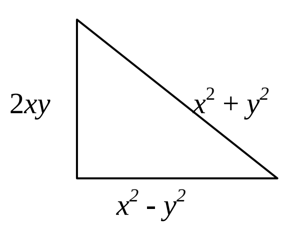
$25 = 16 + 9$

$25 = 25$

⇒The triangle is a right-angled triangle

3.Find the Pythagorean triplets, if $x = 3$ and $y = 2$.

Solution



$x^2 = 3^2 = 9$

$y^2 = 2^2 = 4$

$x^2 - y^2 = 9 – 4 = 5$

$2xy = 2(3)(2) = 12$

$x^2 + y^2 = 9 + 4 = 13$

Therefore the triplets are 5, 12 and 13

Exercises 7.9

A. Identify the set of numbers that form the sides of a right angled triangle.

1. {9, 12, 15} 2. {8, 15, 17} 3. {7, 24, 25} 4. {12, 15, 19} 5. {5, 8, 17} 6. {7, 8, 15}

B. 1. A triangle has sides AB, BC and CA measuring 14, 48 and 50 units.

i. Prove that the triangle is right – angled, and calculate its area. Ans: A = 336 sq. units

ii. Calculate the length of the altitude from B to CA

2. The points A, B and C have coordinates (-3, 1), (2, -1) and (4, 2) respectively. Show that the angle ABC is a right angle. Find the coordinates of D if :

i. $ABCD$ is a rectangle,

ii. $ABCD$ is a parallelogram,

iii. D is the centre of triangle ABC.

3. Prove that the triangle with sides PQ = 6cm, PR = 2.5cm and QR = 6.5cn is a right - angled

4. Mr. Green tells you that a right angled triangle has a hypotenuse of 13 and a leg of 5. If he asks you to find the other leg of the triangle without using a paper and pencil, what will be your answer?

Generating the Pythagoras Triples

If only one side of a right – angled triangle is known, then Pythagorean triples can be generated using the sides:

$\{ \boldsymbol{a}, [\frac{1}{2}(\boldsymbol{a}^2 - 1)], [\frac{1}{2}(\boldsymbol{a}^2 + 1)]\}$

Pythagorean triples are generated by the formula:

$[\frac{1}{2}(\boldsymbol{a}^2 + 1)]^2 = [\frac{1}{2}(\boldsymbol{a}^2 - 1)]^2 + \boldsymbol{a},$

where a is an odd integer greater than one.

Proof

$(x + 1)^2 – (x – 1)^2 = x^2 + 2x + 1 – (x^2 – 2x + 1)$

$(x + 1)^2 – (x – 1)^2 = x^2 + 2x + 1 – x^2 + 2x - 1)$

$(x + 1)^2 – (x – 1)^2 = 4x$

$(x + 1)^2 = (x – 1)^2 + 4x$

$\frac{1}{4}(x +1)^2 = \frac{1}{4}(x – 1)^2 + \frac{4x}{4}$ *(Divide through by 4)*

$[\frac{1}{2}(x +1)]^2 = [\frac{1}{2}(x - 1)]^2 + x$

Substitute $x = a^2$ to obtain:

$\mathbf{[\frac{1}{2}(a^2 +1)]^2 = [\frac{1}{2}(a^2 – 1)]^2 + a^2}$ **………(1)**

This identity can be used to generate Pythagorean triples ***a***, an odd integer greater than 1. For example, when $a = 3$,

$[\frac{1}{2}(3^2 +1)]^2 = [\frac{1}{2}(3^2 – 1)]^2 + a^2$

$[\frac{1}{2}(10)]^2 = [\frac{1}{2}(8)]^2 + a^2$

$5^2 = 4^2 + 3^2$

Worked Examples

1. The width of a rectangle is 7cm. What are the sizes of the length and diagonals if they are whole numbers in centimeters?

Solution

Applying Pythagorean triples for odd numbers: $[\frac{1}{2}(a^2 +1)]^2 = [\frac{1}{2}(a^2 – 1)]^2 + a^2$

When $a = 7$,

$[\frac{1}{2}(7^2 +1)]^2 = [\frac{1}{2}(7^2 – 1)]^2 + 15^2$

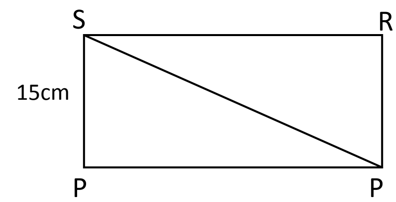
$[\frac{1}{2}(50)]^2 = [\frac{1}{2}(48)]^2 + 15^2$

$25^2 = 24^2 + 7^2$

Therefore, the sizes of the length and diagonal are 24 and 25 respectively

2. A Student was asked to prepare a rectangular model in the form of two joined right angled triangles with its shortest length being 15cm. Find the length of the other two sides and the area of the rectangle if the sides are all integers.

Solution



Applying Pythagorean triples for odd numbers: $[\frac{1}{2}(a^2 +1)]^2 = [\frac{1}{2}(a^2 – 1)]^2 + a^2$

When $a = 15$,

$[\frac{1}{2}(15^2 +1)]^2 = [\frac{1}{2}(15^2 – 1)]^2 + 15^2$

$[\frac{1}{2}(226)]^2 = [\frac{1}{2}(224)]^2 + 15^2$

$113^2 = 112^2 + 15^2$

⇒length = 112cm, breadth = 15cm and diagonal = 113cm

Area = Length × Breadth

= 112 cm × 15 cm

= 1680 cm^2

Exercises 7.10

Generate Pythagoras triples using the following pair of positive integers:

1. 3 and 2 2. 5 and 2 3. 4 and 1

Challenge Problems

If three sides of a triangle are $x^2 – y^2$, $2xy$ and $x^2 + y^2$, show that the triangle is a right – angled triangle

Application of Pythagoras Theorem

Some practical problems as well as real life situations are solved by application of Pythagoras theorem once a right – angled triangle is formed or produced. For e.g. when a ladder on a vertical wall. The lengths of the involving sides are calculated by applying the theory of Pythagoras. This is just a tip of the iceberg. Some other applications are discussed below:

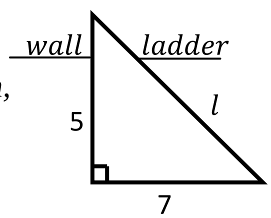
A. Placing a Ladder against a Wall

When a ladder is placed against a vertical wall, the foot of the ladder and the ground floor meet at a right angle. Knowing either how high the ladder is or the height of the wall or the distance between the legs, an unknown side can be calculated using the Pythagoras theorem.

Worked Examples

1. A ladder leans against a vertical wall of length 5m. The distance between the foot of the ladder and the wall is 7m. Find the length of the ladder.

Solution



By Pythagoras theorem,

$l^2 = 5^2 + 7^2$

$l^2 = 25 + 49$

$l^2 = 74,$

$l = \sqrt{74}$ m = 8.60m (3 s. f.)

2. A ladder leans against a vertical wall of length 9cm. If the length of the ladder is 12cm, find the distance between the foot of the ladder and that of the wall.

Solution

By Pythagoras theorem,

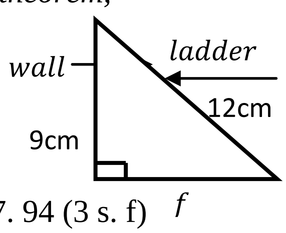
$12^2 = 9^2 + f^2$

$144 = 81 + f^2$

$144 - 81 = f^2$

$63 = f^2,$

$f = \sqrt{63}cm$ = 7. 94 (3 s. f)



3. A ladder which is 17m high is placed against a vertical wall. If the distance separating the foot of the ladder and the wall is 8m, what is the height of the wall?

Solution

Let the height of the wall be x

By Pythagoras theorem,

$17^2 = 8^2 + x^2$

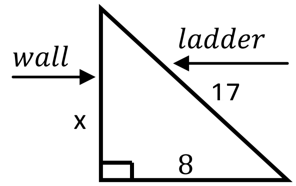
$289 = 64 + x^2$

$289 - 64 = x^2$

$225 = x^2,$

$x = \sqrt{225}m$

$x = 15m$



Exercises 7.11

1. A ladder leans against a vertical wall of height 12m. If the foot of the ladder is 5m away from the wall, calculate the length of the ladder.

2. A ladder leans against a vertical wall of height 16m. If the foot of the ladder is 8m away from the wall, calculate the length of the ladder.

3. A ladder is 8m long. The foot of the ladder is 3m away from the base of the wall. How far up the wall is the top of the ladder?

4. If a ladder to a slide is 8 feet and the ground from the ladder to the slide is 4 feet, then how far down then how far down will the child slide?

5. A 16 ft ladder leans against a wall with its base 4 ft from the wall. How far off the floor is the top of the ladder?

Challenge Problems

1. A ladder 16m long is placed so that its foot is 3m from a building. How much further must the foot of the ladder be moved from the building in order to lower the top of the ladder by 2m?

2. i. The greatest length of an extending ladder is 10m. Calculate the greatest distance up a vertical wall the ladder can reach when the foot of the ladder is 6m from the foot of the wall.

ii. When the ladder is adjusted to 8.5m, it reaches a point 7.5 m above the ground. Calculate the distance of the foot of the ladder
from the foot of the wall.

Length of the Diagonals of a Plane figure

I. Make a sketch of the figure

II. Represent the unknown side (diagonal) by any preferred variable

III. Apply Pythagoras theorem to find the length of the diagonal

B. The Diagonals of a Rectangle and a Square

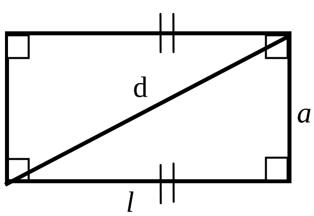
The sides of a rectangle meet at a right angle. A diagonal drawn from one corner to another divides the rectangle into two equal triangles. Given the dimensions of the rectangle, the length of the diagonal can be calculated by application of Pythagoras theorem. Similarly, given the length of the diagonal and how long or how wide the rectangle is, the other or unknown side can be calculated

From Pythagoras theorem,

$\boldsymbol{d^2 = l^2 + a^2}$

$\boldsymbol{l^2 = d^2 - a^2}$

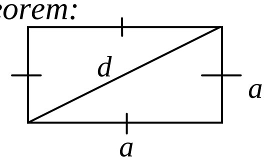
$\boldsymbol{a^2 = d^2 - l^2}$



For a square of side a, as shown below;

Using Pythagoras theorem:

I. The length of the diagonal, d is calculated as:



$d^2 = a^2 + a^2 = 2a^2$

$\boldsymbol{d = \sqrt{2(a^2)}}$

II. The length of a side, a, is calculated as:

$\boldsymbol{a = \sqrt{\frac{(d^2)}{2}}}$

Worked Examples

1. A rectangle has length of 8cm and a breadth of 6cm. How long is its diagonal?

Solution

From Pythagoras theorem,

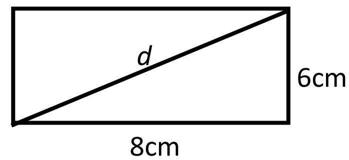
$d^2 = 8^2 + 6^2$

$d^2 = 64 + 36$

$d^2 = 100$

$d = \sqrt{100}$

$d = 10$cm



2. Find the length of the diagonal of a rectangle which is 8cm long and 5cm wide

Solution

Let x be the length of the unknown side

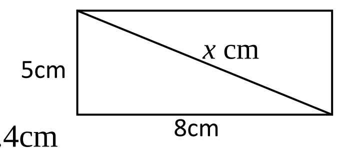
From Pythagoras theorem,

$x^2 = 5^2 + 8^2$

$d^2 = 25 + 64$

$d^2 = 89$

$d = \sqrt{89} = 9.4$cm



3. The diagonal of a rectangle is 20cm long. If the length of the rectangle is 17cm, how long is the breadth?

Solution

Let x represent the breadth of the rectangle

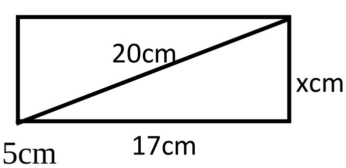
By Pythagoras theorem,

$20^2 = 17^2 + x^2$

$x^2 = 20^2 - 17^2$

$x^2 = 111$

$x = \sqrt{111} = 10.5$cm

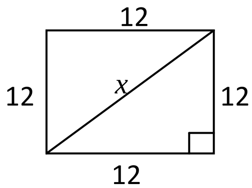


4. A square has a side 12cm. What is the length of the diagonal of the square?

Solution

Let the diagonal of the square be x

$x^2 = 12^2 + 12^2$
$x^2 = 288$
$x = \sqrt{288}$
$x = 16.97$ cm



Exercises 7.12

1. The length and breadth of a rectangle is 20 cm and 11cm respectively. How long is the diagonal of the rectangle?

2. The diagonal of a rectangle is 61cm long. If the breadth is 11cm, find its length.

3. The diagonal of a rectangle is 12 inches and its width is 6 inches. Find its length.

4. A rectangle which is 8cm by 6cm is divided into two right – angled triangles. What are the lengths of the bases and the altitudes of the two different isosceles triangles that can be formed from these triangles?

5. Find the area and perimeter of a square whose diagonal is 12cm long.

6. The length of a rectangle is 1 cm longer than its width. If the diagonal of the rectangle is 4cm, what are the dimensions of the rectangle?

Challenge Problems

1. In ΔABC, $AB = AC = 12$cm and $BC = 8$cm. Express the length of the altitude from A to BC as a surd in its simplest form.

2. An equilateral triangle has each side $2a$ meters long. Find the length of an altitude of the triangle, and hence find the area of the triangle in terms of a

3. The sides of a rectangular floor are x m and $(x + 7)$ m. The diagonal is $(x + 8)$ m. Calculate:
i. the value of x **Ans = 5**
ii. the area of the floor. **Ans : 60cm^2**

C. The Diagonals of a Rhombus

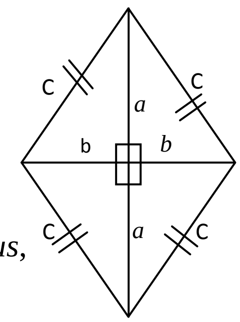
The diagonals of a rhombus bisect each other at 90^0. Given the length of the diagonals, the side of the rhombus can be calculated by applying the Pythagoras theorem as shown below;

By Pythagoras theorem,

$\boldsymbol{c^2 = a^2 + b^2}$

$\boldsymbol{a^2 = c^2 - b^2}$

$\boldsymbol{b^2 = c^2 - a^2}$



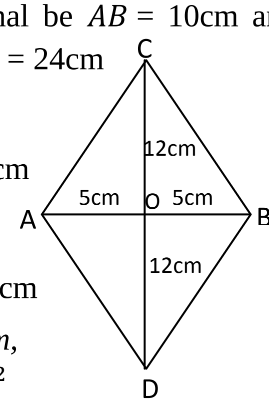
Perimeter of the rhombus,
$\boldsymbol{P = c + c + c + c = 4c}$

Worked Examples

1. The length of the diagonals of a rhombus are 10cm and 24cm. Find:
i. the side of the rhombus
i. the perimeter of the rhombus

Solution

Let the shorter diagonal be AB = 10cm and the longer diagonal be CD = 24cm



$|OB| = \frac{1}{2}|AB|$
$= \frac{1}{2} \times 10\text{cm} = 5\text{cm}$

$|OC| = \frac{1}{2}|CD|$
$= \frac{1}{2} \times 24\text{cm} = 12\text{cm}$

By Pythagoras theorem,
$|CB|^2 = |OC|^2 + |OB|^2$

$|CB|^2 = 12^2 + 5^2$
$|CB| = \sqrt{144 + 25} = 13\text{cm}$

ii. *Perimeter of rhombus*
P = $|AC| + |CB| + |BD| + |DA|$
$|AC| = |CB| = |BD| = |DA| = 13$cm
P = 13 + 13 + 13 + 13 = 52cm

Alternatively,

P = 4 × 13cm = 52cm

Exercises 7.13

1. The length of the diagonals of a rhombus is 48cm and 14cm. Find the perimeter of the rhombus.

2. The sides of a rhombus are 8cm long. One of its diagonal is 12cm. How long is the other diagonal?

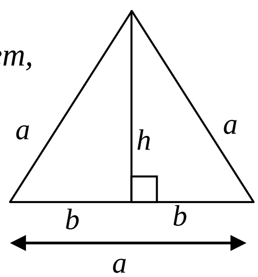
3. Find the length of a side of a rhombus whose diagonals are 15cm and 22cm.

D. Triangles

In an equilateral triangle, the lengths of the three sides are equal. The diagonal bisects the base at a right angle. Given the length of a side, the altitude (height) of an equilateral triangle can be calculated by the use of Pythagoras theorem.

By Pythagoras theorem,

$\boldsymbol{a^2 = h^2 + b^2}$

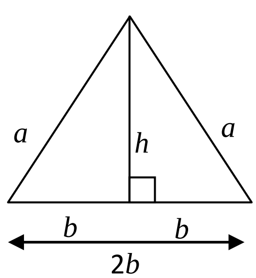


Similarly, in an isosceles triangle, the diagonal touches the base and divides it into at a right angle. Given the height and the two equal sides, the third side can be found by applying Pythagoras theorem. This is shown in the diagram below;

By Pythagoras theorem,

$\boldsymbol{h^2 = a^2 + b^2}$

$\boldsymbol{b^2 = h^2 - a^2}$



In general, if the triangle is isosceles, then drop a perpendicular before you apply the Pythagoras theorem.

Worked Examples

1. PQR is an equilateral triangle of side 14cm. Find the length of the perpendicular from P to QR and hence, find the area of the triangle.

Solution

Let the perpendicular from P to QR be h

By Pythagoras theorem,

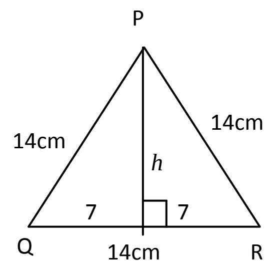
$14^2 = 7^2 + h^2$

$h^2 = 14^2 - 7^2$

$h^2 = 147$

$h = \sqrt{147}$

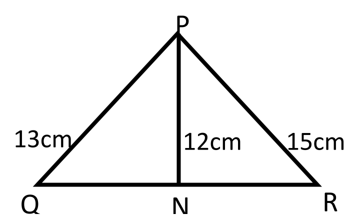
$h = 12.12$cm



A = $\frac{1}{2}bh$, but b = 14cm and h = 12cm

A = $\frac{1}{2}$ × 14 × 12 = 84cm^2

2. Calculate the length QR in the triangle PQR



Solution

$/QN/^2 = 13^2 - 12^2$

169 – 144 = 25

$/QN/ = \sqrt{25} = 5$

$/NR/^2 = 15^2 - 12^2$

$/NR/ = \sqrt{81} = 9$cm

But $/QR/ = /QN/ + /NR/$

= 5 + 9 = 14cm

3. A triangle has sides 17cm, 17cm, 16cm. Calculate the area of the triangle

Solution

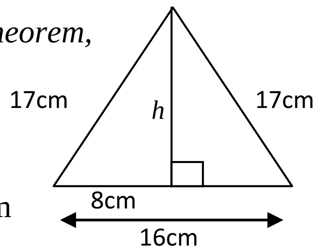
By Pythagoras theorem,

$17^2 = 8^2 + h^2$

$h^2 = 17^2 - 8^2$

$h^2 = 289 - 64$

$h = \sqrt{225} = 15\text{cm}$



$A = \frac{1}{2}bh$,

Substitute b = 16cm and h = 25cm

$A = \frac{1}{2} \times 16 \times 15 = 120\text{cm}^2$

4. The triangle *ABC* has *AB* = *AC* = 7 cm, and BC = 8cm. Find the area of the triangle, giving your answer to 3 significant figures.

Solution

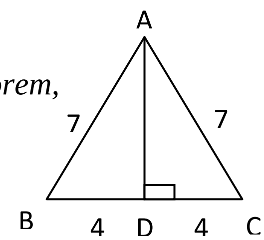
By Pythagoras theorem,

$7^2 = 4^2 + /AD/^2$

$/AD/^2 = 7^2 - 4^2$

$/AD/ = \sqrt{7^2 - 4^2}$

$/AD/ = \sqrt{33} = 5.7446$

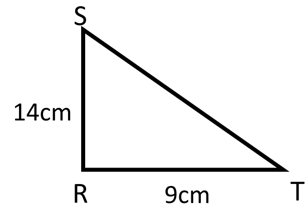


Area of a triangle = $\frac{1}{2} \times \text{BC} \times \text{AD}$

$= \frac{1}{2} \times 8 \times 5.7 = 23.0\text{cm}^3$

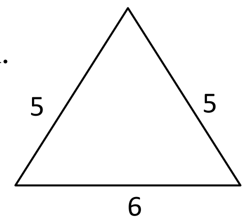
Exercises 7.14

1. The length of the side of an equilateral triangle is 30cm. Find the height of the triangle.
2. ∆PQR is a triangle with *PQ* = *QR* and *PR* = 6m. If the height of the triangle is 7m, find *PQ*
3. ∆ *LMN* has *LM* = *LN*, *MN* = 12cm and area is 300cm^2. Find the height of the triangle and hence find *LM*.
4. Find the perimeter of the triangle below correct to three significant figures?

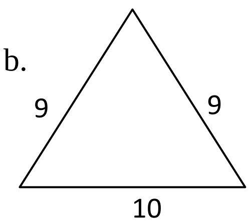


5. Find the altitude of each triangle;

a.



b.



D. Involving Quadratic Equations

Given the value of a side of a right – angled triangle, the next side as a variable and the third side as an increment or decrement in the variable, the actual dimensions (values of the variable) can be calculated.

I. Apply Pythagoras theorem,

II. Expand the involving brackets,

III. Equate the equation to zero,

IV. Solve the quadratic equation by using the quadratic method, ignoring negative answers,

V. Substitute the value of the variable to obtained the dimensions of the right – angled triangle.

Worked Examples

The length of one leg of a right – angled triangle is 2cm more than the other. If the length of the hypotenuse is 6cm, what are the lengths of the two legs?

Solution

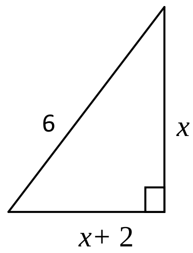
Draw a sketch of the problem, labeling the known and unknown lengths. If one leg is represented by x, the other is represented by $x + 2$

Use Pythagoras theorem to form the equation,

$x^2 + (x + 2)^2 = 6^2$

$x^2 + x^2 + 4x + 4 - 36 = 0$

$2x^2 + 4x - 32 = 0$



$a = 2, b = 4$ *and* $c = -32$ $\left(\begin{matrix} Using \\ quadratic\ formula \end{matrix}\right)$

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-2 \pm \sqrt{2^2-4(1)(-16)}}{2(1)} = -1 + \sqrt{17}$

$\Rightarrow x = -1 + \sqrt{17} = 3.123$

If $x = 3.123$, then $x + 2 = 3.123 + 2 = 5.123$

The lengths of the legs are approximately 3.123cm and 5.123cm.

2. The length of the three sides of a right – angled triangle form a set of consecutive even integers when arranged from least to greatest. If the second largest side has a length of x, form an equation and hence, solve for the three sides.

Solution

The 3 sides of the right – angled triangle when arranged from the least to the greatest are:

$(x - 2)$, x and $(x + 2)$

$(x - 2)^2 + x^2 = (x + 2)^2$

$(x - 2)(x - 2) + x^2 = (x + 2)(x + 2)$

$x(x - 2) - 2(x - 2) + x^2 = x(x + 2) + 2(x + 2)$

$x^2 - 2x - 2x + 4 + x^2 = x^2 + 2x + 2x + 4$

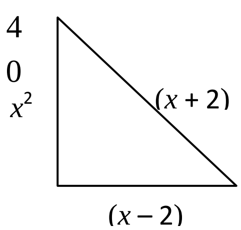
$x^2 - 4x + 4 + x^2 = x^2 + 4x + 4$

$2x^2 - x^2 - 4x - 4x + 4 - 4 = 0$

$x^2 - 8x = 0$

$x(x - 8) = 0$

$\Rightarrow x = 0$ or $x = 8$



When $x = 0$, the first side is negative. That is: $0 - 2 = -2$ so ignore the answer $x = 0$

When $x = 8$,

The 1st side is $8 - 2 = 6$units,

The second side is 8units,

The third side is : $8 + 2 = 10$units

Exercises 7.15

1. The hypotenuse of a right – angled triangle is $(2x + 3)$ cm long, and the other two sides have lengths x cm and $(x + 7)$ cm. Find x, and calculate the area of the triangle.

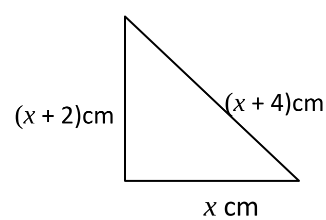
2. The sides of a right – angled triangle in ascending order of magnitude are 8cm, $(x - 2)$ cm and x cm. Find x.

3. The lengths in cm of the sides of a right – angled triangle are x, $(x + 2)$ and $(x + 1)$, where $x > 0$, find x.

4. The length of one leg of a right triangle is 1cm more than the other. If the length of the hypotenuse is 3cm, what are the lengths of the legs?

5. The length of one leg of a right triangle is 3cm more than the other. If the length of the hypotenuse is 8m, what are the lengths of the legs?

6. Find the lengths of the sides of the right – angled triangle



Solving Other Applications

Worked Examples

1. How long must a guywire be to reach from the top of a 30ft pole to a point on the ground 20ft from the base of the pole?

Solution

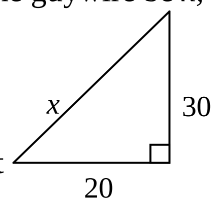
Let the length of the guywire be x,

$x^2 = 30^2 + 20^2$

$x^2 = 900 + 400$

$x^2 = 1,300$

$x = \sqrt{1,300} = 36$ ft



2. In a right – angled triangle, the hypotenuse is 39cm and the ratio of the other two sides is 12: 5. Find the sides.

Solution

Let the sides be (12x) cm, (5x)cm and 39 cm

By Pythagoras theorem,

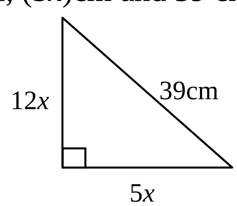
$39^2 = (12x)^2 + (5x^2)$

$1521 = 144x^2 + 25x^2$

$1521 = 169x^2$

$x^2 = 9$

$x = \sqrt{9} = 3cm$



$\Rightarrow 12x = 12(3) = 36cm$

$5x = 5(3) = 15cm$

The sides of the triangle are 15*cm*, 36*cm*, 39*cm*

4. A man standing 40m away from a tower notices that the distances from the top and bottom of a flagstaff on top of the tower are 50m and 45m respectively. Find the height of the flagstaff.

Solution

Let h and x be the heights of the flagstaff and that of the tower respectively as shown below;

From ΔABC,

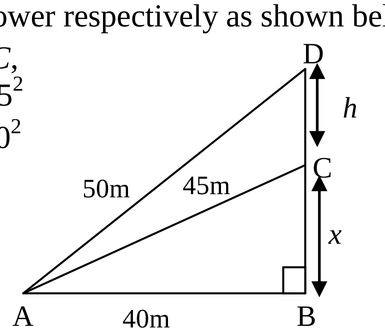
$40^2 + x^2 = 45^2$

$x^2 = 45^2 - 40^2$

$x^2 = 425$

$x = \sqrt{425}$

$x = 20.6$m



From ΔABD

$(h + x)^2 + 40^2 = 50^2$

$(h + x)^2 = 50^2 - 40^2$

$(h + x)^2 = 900$

$h + x = \sqrt{900}$

$h + x = 30$,

But $x = 20.6$

$h + 20.6 = 30$

$h = 30 - 20.6 = 9.4m$

The height of the flagstaff is 9.4m

5. Two points A and C are on the same level ground as the foot of a pole B. The distance between A and C is 70m and A and C are on the opposite sides of the vertical pole. The distances from the top of the pole, D, to A and C are 45m and 59m respectively. Find :

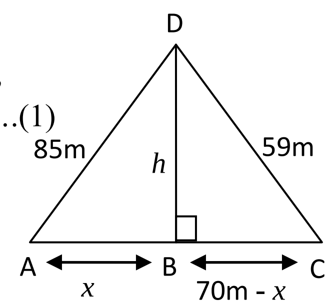
i. the distance between the foot of the pole, B, and the point A,

ii. the height, BD of the pole.

Solution

i. From ΔABD,

$x^2 + h^2 = 45$…….(1)



From ΔBCD,

$(70 - x)^2 + h^2 = 59^2$ ……….(2)

eqn (2) – eqn (1);

$(70 - x)^2 - x^2 = 59^2 - 45^2$

$70^2 - 140x + x^2 - x^2 = 1456$

$-140x = 1456 - 70^2$

$-140x = -3444$

$x = 24.6$ m

The distance between B and A = 24.6m

ii. $x^2 + h^2 = 45^2$

$\Rightarrow (24.6)^2 + h^2 = 45^2$ (But $x = 24.6$)

$h^2 = 45^2 - (24.6)^2$

$h = \sqrt{1,419.84}$

$h = 37.7$m

The height, BD of the pole is 37.7m

6. Two points *A* and *C*, are on the same level ground as the foot of the pole, *B*. The distance between *A* and *C* is 40m and *A* and C are on the same sides of the vertical pole. The distances from the top of the pole, *D* to *A* are 53m and 85m respectively. Find correct to one decimal place;

i. the distance between the foot of the pole, *B* and the point *A*,

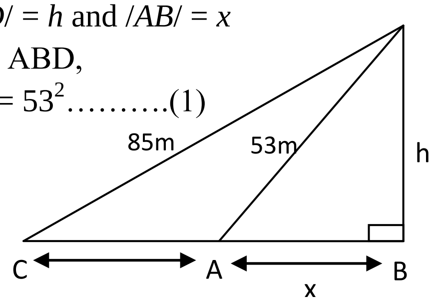
ii. the height, *BD*, of the pole.

Solution

Let $/BD/ = h$ and $/AB/ = x$

From Δ ABD,

$x^2 + h^2 = 53^2$……….(1)



From Δ BCD,

$(40 + x)^2 + h^2 = 85^2$ ……….(2)

eqn (2) – eqn (1)

$(40 + x)^2 - x^2 = 85^2 - 53^2$

$40^2 + 80x + x^2 - x^2 = 4,416$

$80x = 4,416 - 40^2$

$80x = 2,816$

$x = 35.2$ m

The distance between, *B* and *A* = 35.2m

ii. From Δ ABD

$x^2 + h^2 = 53^2$

$\Rightarrow (35.2)^2 + h^2 = 53^2$ (But $x = 35.2$)

$h^2 = 53^2 - (35.2)^2$

$h = \sqrt{1,569.96}$

$h = 39.6$m

The height, *BD* of the pole is 37.7m

7. Tom and Jerry meet at a corner. Tom turns 90^0 left and walks 9 paces; Jerry continues straight and walks 12 paces. Find the distance between the two of them.

Solution

Let the distance between Tom and Jerry be *x*

By Pythagoras theorem,

$x^2 = 9^2 + 12^2$

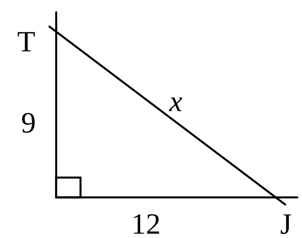
$x^2 = 81 + 144$

$x^2 = 225$

$x = \pm\sqrt{225}$

$x = \sqrt{225}$

$x = 15$ paces



The distance between Tom and Jerry is 15 paces

Exercises 7.16

1. How long must a guywire be to run from the top of a 20ft to point on the ground 8ft from the base of the pole?

2. If the hypotenus of a right – angled isosceles triangle is 6m, what is the length of each of the other side’s?

3. Two foggers run 8 miles north and then 5 miles west. What is the shortest distance, to the nearest tenth of a mile they must travel to return to their starting point?

4. A kite is flying so that it is 55 feet high, and its above the point 75feet from the flyer. How long is the string of the flyer?

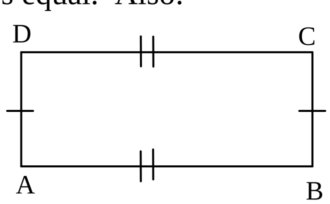
5. A right – angled triangle with sides *A*, *B* and C with respective sides *a, b, c* has the following measurements side *A* = 3cm, side *B* = 4cm. What is the length of side *C*?

6. A man starting from point *A*, walks 5km due east and then 4km due north to point *B*. Calculate to one decimal place, the distance from *A* to *B* direct.
ii. From *B* he walks 5km north and then 4km west to C, calculate to one decimal place:
a. the distance from *B* to *C*,
b. the distance from *A* to *C*.

7. Two points *A* and *C* are on the same level ground as the foot of a pole *B*. The distance between *A* and *C* is 16m and *A* and *C* are on the same side of the vertical pole. The distances from the top of the pole, *D*, to *A* and *C* are 40m and 50m respectively. Find :
i. the distance between the foot of the pole, *B*, and the point *A*,
ii. the height, *BD* of the pole.

A Quadrilateral
It is a plane figure bounded by four straight lines. Examples of quadrilaterals are Rectangles, Squares, Rhombuses, Kites, Trapeziums and Parallelograms. The interior angles of a quadrilateral sum up to 360^0

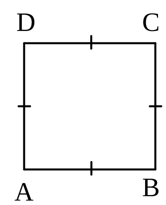
Types and Properties of Quadrilaterals
Rectangle : It is a quadrilateral with opposite pair of sides equal. Also:



I. Each interior angle is a right angle.
II. Each pair of opposite sides is parallel.
III. It has two diagonals that are not symmetrical.
IV. It fits its outline in two ways.
IV. Area, A = L × B
$A = |AB| \times |BC|$

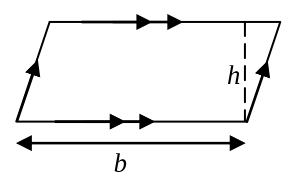
Square: It is a plane figure (special rectangle) with four sides equal. In the figure below, $|AB|=|BC|=|CD|=|DA|$

Also:



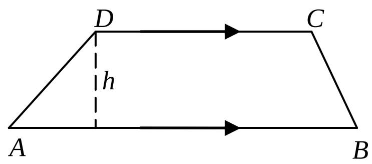
I. Each interior angle is a right angle.
II. Each pair of opposite sides is parallel.
III. It has two diagonals that are symmetrical.
IV. It fits its outline in four ways.
V. Area, A = L × L
$A = |AB| \times |AB|$

Parallelogram : It is any quadrilateral with both pair of opposite sides equal and parallel. Also:



I. Opposite angles are equal.
II. Diagonals bisect each other.
III. It has half turn of symmetry.
IV. It can be formed by two pairs of parallel lines.
V. Area = base × perpendicular height
$A = bh$

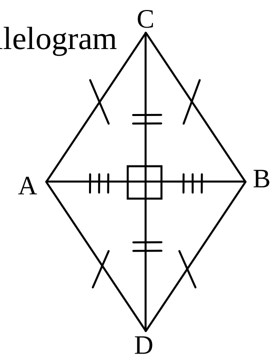
Trapezium: It is any quadrilateral with one pair of opposite sides parallel. Also;



I. It fits it outline in one way.

II. Area, A = $\frac{1}{2}(|AB| + |DC|)h$.

Rhombus: It is a parallelogram with four sides equal.
Also:



I. It consist of two congruent isosceles triangle, base to base.

II. It fits it outline in four ways.

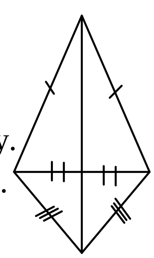
III. Its diagonals are axes of symmetry.

III. Its diagonals bisect at right angles.

IV. Area = $\frac{1}{2}$ × (Product of diagonals)

$$A = \frac{1}{2} \times (|AB| \times |CD|)$$

Kite: It consists of two isosceles triangles with equal bases. Also;



I. It fits its outline in two ways.

II. One diagonal is an axis of symmetry.

III. Each pair of adjacent sides is equal.

IV. Its diagonals bisect at right angles.

Unknown Angle of a Quadrilateral

The sum of interior angles of a quadrilateral is 360^0.

To calculate the value of an unknown angle in a quadrialateral;

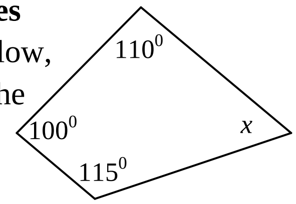
I. Represent the unknown angle by any preferred variable.

II. Sum up the angles and equate to 360^0.

III. Workout for the value of the variable.

Worked Examples

1. In the figure below, find the value of the angle marked *y*



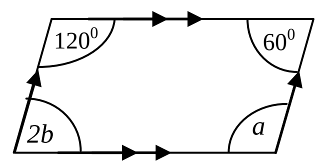
Solution

$x + 110^0 + 100^0 + 115^0 = 360^0$

$x = 360^0 - 110^0 - 100^0 - 115^0$

$x = 360^0 - 325^0 = 35^0$

2. Find the values of the letters in the diagram below;



Solution

For all parallelograms, opposite angles are equal. Therefore, from the diagram, $a = 120^0$ and $2b = 60^0$

$2b = 60^0$

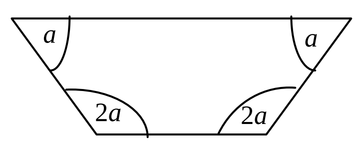
$b = 30^0$

$\Rightarrow a = 120^0$ and $b = 30^0$

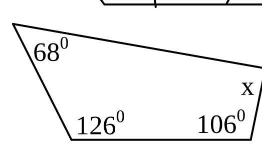
Exercises 7.17

A. Find the values of the unknown angles

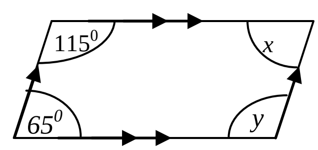
1.



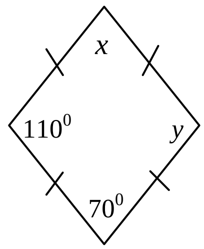
2.



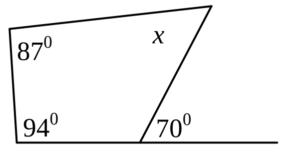
3.



4.



5.

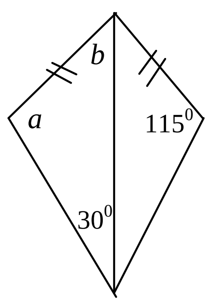


B. 1. The interior angles of a quadrilateral are $5x^0$, $3x^0$, 30^0 and $7x^0$. Find the value of x and the value of each interior angle of the quadrilateral.

2. A parallelogram has an interior angle of 95^0. Find the values of the other three interior angles.

3. Find the value of x and the value of each angle of a quadrilateral whose values are given as $\frac{1}{2}x$, $\frac{3}{2}x$, $\frac{2}{3}x$ and x.

4. In the diagram below, *ABCD* is a kite.



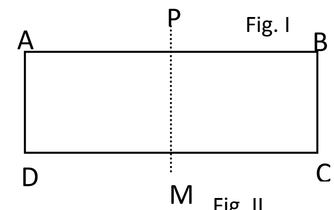
Calculate the value of the angles *a* and *b.*

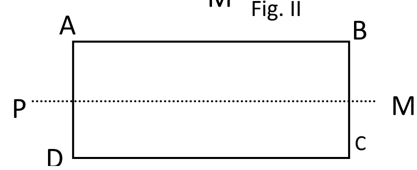
Symmetrical Objects and Line of Symmetry

Symmetrical objects are objects that can be folded or divided into two equal halves such that one-half fits exactly on the other half. For example, it is possible to draw a line (mirror) through a rectangle and an isosceles triangle to divide them into two halves such that one-half fits exactly on the other half. In these cases, the rectangle and the isosceles triangle are said to be ***Symmetrical*.**

A line that divides a figure into two equal halves is called a ***line of symmetry***. It is also called a ***line of fold.***

Consider the figures below;



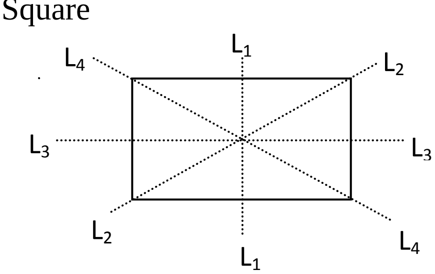


In both figures, PM is the line of symmetry. This implies that in fig. I, *APMD = PBCM* and in fig. II, *ABMP = PMDC*

Line of Symmetry of Some Figures:

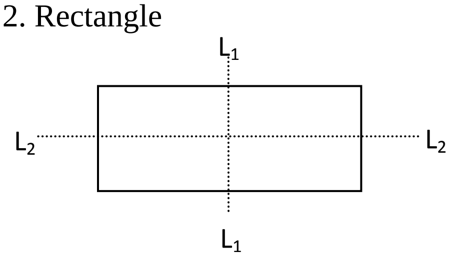
Rectangle, Square, Equilateral triangle, kite…

1. Square



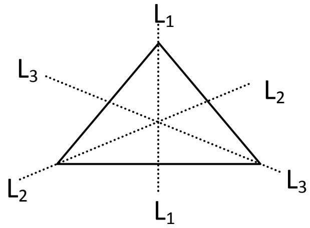
Line of symmetry of a square = 4

2. Rectangle



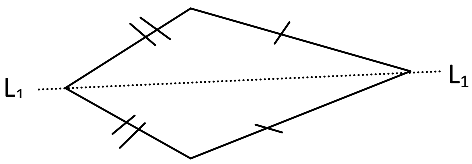
Line of Symmetry of a rectangle = 2

3. Equilateral triangle



Line of symmetry of an equilateral triangle is 3.

4. Kite.



Line of symmetry of a kite = 1

Exercises 7.18

1. Name any five figures that are symmetrical
2. Draw each of the following and indicate the lines of symmetry;

i. Rectangle ii. Rhombus
iii. Parallelogram iv. Scalene triangle
v. Circle vi. Isosceles

3. Copy and complete the table below:

Object	Line of Symmetry
Square	
Rectangle	
Isosceles triangle	
Rhombus	
Parallelogram	
Kite	
Circle	
Equilateral triangle	

.

Polygons

A polygon is any plane figure bounded by three or more straight lines. The following are names of polygons of three to ten sides.

Number of sides	Name
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
7	Heptagon
8	Octagon
9	Nonagon
10	Decagon

Sum of Interior Angles of a Polygon

Investigations

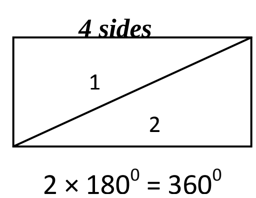
<u>Reminder</u>: The sum of interior angles of a triangle is 180^0.

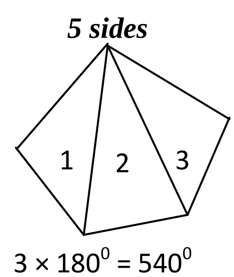
To investigate the sum of interior angles of a polygon given the number of sides :

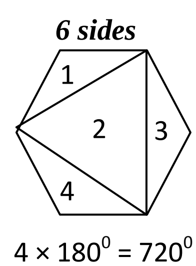
I. Draw the polygon.

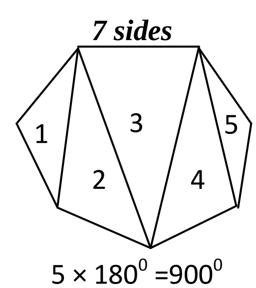
II. Find the total number of triangles in the Polygon by drawing diagonal lines, making sure that no line crosses another.

III. Multiply the number of triangles in thepolygon by 180^0 (the sum of interior angles of a triangle) to obtain the sum of interior angles, θ, of the polygon.

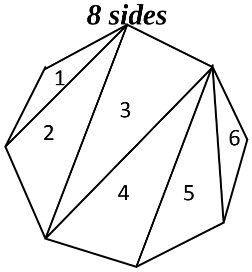






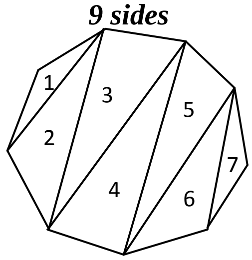


8 sides



$6 \times 180^0 = 1080^0$

9 sides



$7 \times 180^0 = 1260^0$

Summary

Number of sides	Number of Triangles	Sum of Interior Angles (θ)
3	(3 – 2) = 1	$1 \times 180^0 = 180^0$
4	(4– 2) = 2	$2 \times 180^0 = 360^0$
5	(5 − 2) = 3	$3 \times 180^0 = 540^0$
6	(6 – 2) = 4	$4 \times 180^0 = 720^0$
7	(7 − 2) = 5	$5 \times 180^0 = 940^0$
8	(8 − 2) = 6	$6 \times 180^0 = 1020^0$
9	(9 − 2) = 7	$7 \times 180^0 = 1260^0$
10	(10 − 2) = 8	$8 \times 180^0 = 1440^0$
n	$(n - 2)$	$(n - 2) \times 180^0$

From the table, it can be seen that the number of triangles is two less than the number of sides of the polygon. The conclusion is that;

For any polygon with n sides, the sum of interior angles, $\boldsymbol{\theta} = (\boldsymbol{n - 2}) \times \mathbf{180^0}$

Worked Examples

Find the sum of interior angles of a polygon with thefollowing sides;

1. 15 2. 12 3. 20

Solution

1. 15 sides

Solution

Sum of interior angles = (n – 2) 180^0

Substitute n = 15

⇒ Sum of interior angles,

= (15 – 2) 180^0

= 13 × 180^0

= $2{,}340^0$

2. 12 sides

Solution

Sum of interior angles = (n – 2) 180^0

Substitute n = 12

⇒ Sum of interior angles,

= (12 – 2) 180^0

= 10 × 180^0

= 1800^0

3. 20 sides

Solution

Sum of interior angles = (n – 2) 180^0

Substitute n = 20

⇒ Sum of interior angles

= (20 – 2) 180^0

= 18 × 180^0

= $3{,}240^0$

Exercises 7.19

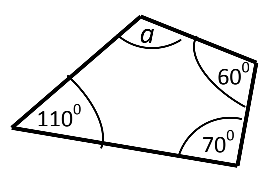
A. Find the sum of interior angles of a polygon with the following sides:

(1) 14 (2) 22 (3) 42

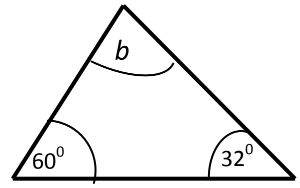
(4) 35 (5) 17 (6) 50

B**. Find the value of the unknown angles;**

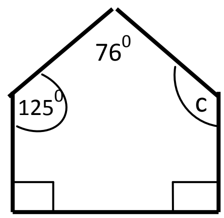
1.



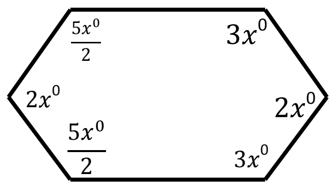
2.



3.



4.



Finding the Number of Sides of a Polygon

Given the sum of interior of a polygon as;

θ = $\boldsymbol{(n - 2) \times 180^0}$, the number of sides, n, of the polygon can be found by making n the subject of the formula as shown:

In θ = $\boldsymbol{(n - 2) \times 180^0}$

$\theta = 180^0 n - 360^0$

$\theta + 360^0 = 180^0 n$

$\boldsymbol{n = \frac{\theta + 360^0}{180^0}}$ **, w***here θ is the sum of interior <s*

Worked Examples

1. The sum of interior angles of a polygon is 900^0. Find the number of sides of the polygon

Solution

$\theta = 900^0 n = ?$

$n = \frac{\theta + 360^0}{180^0} = \frac{900^0 + 360^0}{180^0} = \frac{1260^0}{180^0} = 7$ sides

2. Find the number of sides of a polygon with whose sum of interior angles is $1,980^0$.

Solution

$\theta = 1980^0 n = ?$

$n = \frac{\theta + 360^0}{180^0} = \frac{1980^0 + 360^0}{180^0} = \frac{2340^0}{180^0} = 13$ sides

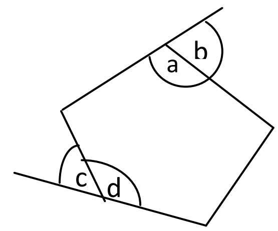
Exercises 7.20

A. Find the number of sides of a polygon with the following sums of interior angles:

1. 1080^0 2. 1440^0 3. 2880^0
4. 3600^0 5. 3060^0 6. 4860^0

Interior and Exterior Angles of a Polygon

Consider the polygon below;



Angles a and b are pair of inerior angles and exterior angles respectively, likewise angles c and d.

The sum of the pair of interior and exterior angles are supplementary (equal to 180^0) because they are angles formed on a straight line. Therefore, from the diagram above;

$a + b = 180^0$ and $c + d = 180^0$

It is conclusive that for all polygons;

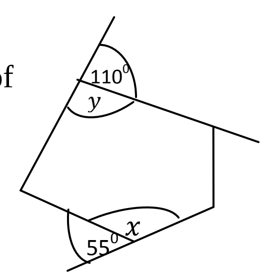
1. ***Interior angle + exterior angle = 180^0***

2. ***Exterior angle = 180^0 – interior angle***

3. ***Interior angle = 180^0 – Exterior angle***

Worked Examples

1. What is the value of the angles marked x and y in the figure below?



Solution

$x + 55^0 = 180^0$ (angles on a straight line)

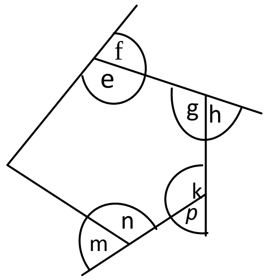
$x = 180^0 - 55^0 = 125^0$

$y + 110^0 = 180^0$(< s on a straight line)

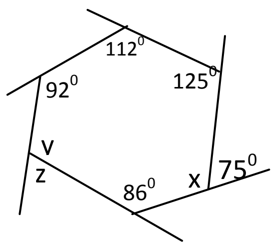
$y = 180^0 - 110^0 = 70^0$

Exercises 7.21

A. List the pair of exterior and interior angles in the polygon below;



B. In the hexagon below, determine the size of the interior and exterior angles.



A Regular Polygon

A regular polygon is a polygon that has allthe sides and angles equal. Examples are equilateral triangle, rhombus and a square.

For all regular polygons of $\boldsymbol{n}$ sides, it implies that there are $\boldsymbol{n}$ interior angles and $\boldsymbol{n}$ exterior angles.

If the sum of all interior angles of a regular polygon is $\theta = (n - 2) \times 180^0$, then each interior angle, $\theta = \frac{(\boldsymbol{n} - 2) \times \mathbf{180^0}}{\boldsymbol{n}}$

From $\theta = \frac{(\boldsymbol{n} - 2) \times \mathbf{180^0}}{\boldsymbol{n}}$**,** the number of sides

$$\boldsymbol{n} = \frac{\mathbf{360^0}}{\mathbf{180^0} - \boldsymbol{\theta}} \text{ (\textit{for regular polygons})}$$

Worked Examples

1. A regular polygon has 12 sides. What is the size of each interior angle?

Solution

Method 1

For a regular polygon, each interior angle

$\theta = \frac{(n - 2) \times 180^0}{n} = \frac{(12 - 2) \times 180^0}{12} = \frac{10 \times 180^0}{12} = 150^0$

Method 2

Because it is a regular polygon, all the 12 sides are equal. n =12

Exterior angle, $\theta = \frac{360^0}{n} = \frac{360^0}{12} = 30^0$

Interior angle + exterior angle $= 180^0$

$\Rightarrow$Interior angle $= 180^0$ – Exterior angle

$= 180^0 - 30^0$

$= 150^0$

2. A regular polygon has 9 sides. Find:

i. each interior angle of the polygon,

ii. the Sum of the interior angles of the polygon.

Solution

i. For a regular polygon each interior angle,

$\theta = \frac{(n - 2) \times 180^0}{n}$, but $n = 9$

$\theta = \frac{(9 - 2) \times 180^0}{9} = \frac{7 \times 180^0}{9} = 140^0$

ii. Sum of interior angles,

$\theta = (n - 2) \times 180^0$, but $n = 9$

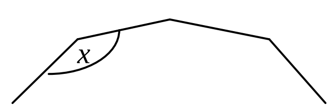
$\theta = (9 - 2) \times 180^0 = 7 \times 180^0 = 1{,}260^0$

Exercises 7.22

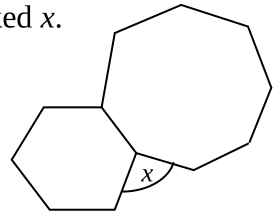
A. Find the value of each interior angle of a regular polygon with the following sides;

1. 13 2. 14 3. 17 4. 19 5. 21

B. 1. The diagram below shows part of a regular 10 – sided polygon. Workout the size of the angle marked *x*.

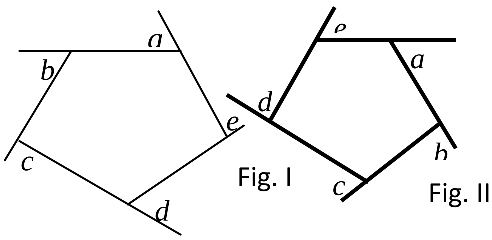


2. The diagram below shows a regular polygon and a regular hexagon. Calculate the size of the angle marked x.

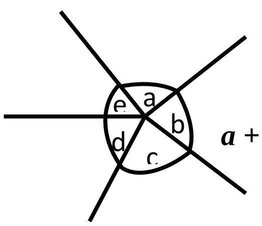


Sum of Exterior Angles of a Polygon

Consider the diagrams below:



When all the exterior angles are cut and moved to a fixed point, they form a complete turn of 360^0 as shown below;

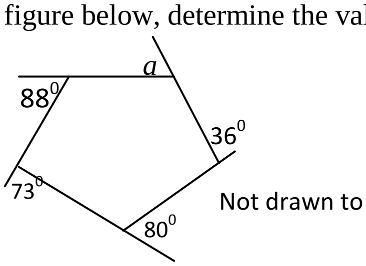


$\boldsymbol{a + b + c + d + e = 360^0}$

In general, for every polygon (regular or irregular), the sum of exterior angles is equal to 360^0. ⇒: θ **= 360⁰**

Worked Examples

1. In the figure below, determine the value of a



Solution

Exterior <s of a polygon sum up to 360^0

$a^0 + 88^0 + 73^0 + 80^0 + 36^0 = 360^0$

$a^0 = 360^0 - 88^0 - 73^0 - 80^0 - 36^0$

$a^0 = 83^0$

2. The exterior angles of a pentagon are 72^0, $3x^0$, 52^0, 36^0, and 50^0 . Determine;

i. the value of angle x

ii. the value of angle exterior angle $3x$

iii. the value of the interior angle of $3x$

Solution

i. Sum of exterior < s of a polygon = 360^0

$72^0 + 3x^0 + 52^0 + 36^0 + 50^0 = 360^0$

$3x = 360 - 72 - 52 - 36 - 50$

$3x = 150^0$

$x = 50^0$

ii. The value of angle $3x = 3 \times 50^0 = 150$

iii. Let the interior angle of $3x$ be y

$3x + y = 180^0$, but $3x = 150^0$

$150^0 + y = 180^0$

$y = 180^0 - 150^0 = 30^0$

3. The exterior angles of a polygon are $2x^0$, $(x - 20)^0$, x^0, $(3x + 10)^0$, $(x + 15)^0$ and $(2x + 5)^0$. Find the value of x

Solution

Exterior angles of a polygon = 360^0

$\Rightarrow 2x^0 + (x - 20)^0 + x^0 + (3x + 10)^0 + (x + 15)^0 + (2x + 5)^0 = 360^0$

$2x^0 + x^0 - 20^0 + x^0 + 3x^0 + 10^0 + x^0 + 15^0 + 2x^0 + 5^0 = 360^0$

$2x^0 + x^0 + x^0 + 3x^0 + x^0 + 2x^0 - 20^0 + 10^0 + 15^0 + 5^0 = 360^0$

$10x^0 + 10^0 = 360^0$

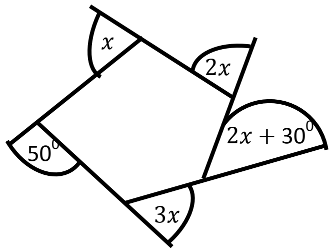
$10x^0 = 360^0 - 10^0$

$10x^0 = 350 = 35^0$

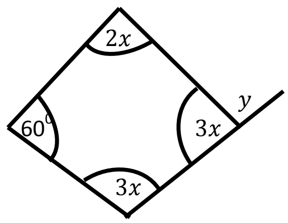
Exercises 7.23

A. Find the value of the angles marked *x*

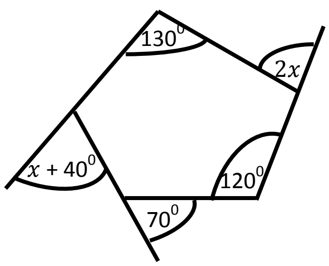
1.



2.



3.



Finding the Number of Sides, n, of a Regular Polygon Given its Exterior Angle,θ

The exterior angle, θ, of a regular polygon is related to the number of sides, *n*, by the formula: $\theta = \frac{\mathbf{360^0}}{\boldsymbol{n}}$. Therefore, knowing the value of each exterior angle θ, the number of sides, *n* is calculated by the formula; $\boldsymbol{n} = \frac{\mathbf{360^0}}{\boldsymbol{\theta}}$

Worked Examples

1. If each exterior angle of a regular polygon is 30^0, find the number of sides of the polygon.

Solution

$\theta = 30^0$ and n = ?

$n = \frac{360^0}{\theta} = \frac{360^0}{30} = 12$ sides

2. A regular polygon has an interior angle of 172^0.

i. Find its exterior angle.

ii. How many sides has the polygon?

Solution

i. Interior angle + exterior angle = 180^0

Exterior angle = 180^0 – interior angle

But interior angle = 172^0

Exterior angle = $180^0 - 172^0 = 8^0$

ii. sum of exterior angles = 360^0

But the exterior angle is 8^0

$\Rightarrow 8^0 = \frac{360^0}{n}$

Therefore $n = \frac{360^0}{8^0} = 45$ sides

3. Each interior angle of a regular polygon is 160^0, how many sides has it?

Solution

Interior angle + Exterior angle = 180^0

Exterior angle = 180^0 – Interior angle

But interior angle = 160^0

Exterior angle = $180^0 - 160^0 = 20^0$

Exterior angle = $\frac{360^0}{n}$,

Substitute exterior angle = 20^0

$\Rightarrow 20^0 = \frac{360^0}{n}$

$n = \frac{360^0}{20^0} = 18$ sides

4. Find the size of the interior angle of a regular polygon with 5 sides.

Solution

Exterior angle = $\frac{360^0}{n}$,

Substitute $n = 5$

Exterior angle = $\frac{360^0}{5} = 72^0$,

Exterior angle + Interior angle = 180^0

$\Rightarrow$Interior angle = 180^0 – exterior angle

$= 180^0 - 72^0 = 108^0$

5. Four interior angles of a hexagon are 130^0, 160^0, 112^0 and 80^0. If the remaining angles are equal, find the size of each of them.

Solution

Let each of the remaining angles be x

$\Rightarrow$sum of interior angles

$= 130^0 + 160^0 + 112^0 + 80^0 + x + x = 720^0$

$2x = 720^0 - 130^0 - 160^0 - 112^0 - 80^0$

$2x = 238^0$

$x = 119^0$

Therefore each remaining angle is 119^0

6. A regular polygon is inscribed in a circle of radius 14cm. If each interior angle is 120^0, find:

i. the value of each exterior angle,

ii. the area of the polygon.

Solution

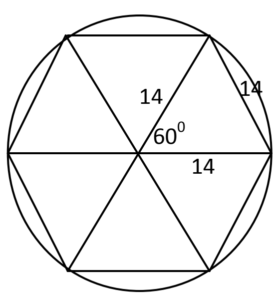
Let the exterior angle be A

Interior angle + Exterior angle = 180^0

$120^0 + A = 180^0$

$A = 180^0 - 120^0$

$A = 60^0$



ii. From the diagram,

Area of Δ

$= \frac{1}{2}(14)(14)\sin 60^0 = 84.87\text{ cm}^2$

Area of the polygon

$6 \times 84.87 = 509.2\text{ cm}^2$

Solved Past Question

Three interior angles of a polygon are 160^0 each. If the other interior angles are 120^0 each, find the number of sides of the polygon.

Solution

Let the interior angles of the polygon be n

$3(160^0) + (n - 3)120^0 = (n - 2)\, 180^0$

$480^0 + 120^0 n - 360^0 = 180^0\, n - 360^0$

$120^0 n - 180^0 n = -360^0 + 360^0 - 480^0$

$-60^0\, n = -\, 480^0$

$n = \frac{-480^0}{-60^0} = 8$ sides

Exercises 7.24

A. 1. The exterior angle of a regular polygon is 30^0, find the number of sides of the polygon.

2. The interior angle of a regular polygon is twice its exterior angle. Find the number of sides of the polygon.

3. A regular polygon has 12 sides. Find:

(i) its exterior and interior angles,

(ii) the sum of the interior angles.

8 EQUATIONS AND INEQUALITIES

Change of subject or Formula

A formula is an equation which expresses one symbol (called the subject of the formula) in terms of other symbols. For e.g, $v = u + at$.

Change of subject is the process whereby a variable is made to stand alone either at the left side or the right side of an equal sign in a formula or an equation. For instance, v is said to be the subject of $v = u + at$

The subject of a formula is changed in order to arrange the formula in a more suitable form for the required calculation

In dealing with change of subject, we categorize it into the following types;

1. Involving multiplication
2. Involving addition and subtraction
3. Involving division or fraction
4. Involving bracket and factorization
5. Involving squared variables
6. Involving squared root.

Involving Multiplication

When the equation involves multiplication, divide both sides of the equation by the variable or variables standing with or attached to the one to be made the subject.For example, to make h the subject of the relation $k = mgh$, divide both sides of the equation by mg to get h standing alone as shown below;

$\frac{k}{mg} = \frac{mgh}{mg}$,

$\Rightarrow h = \frac{k}{mg}$

Worked Examples

1. Make l the subject of $A = lb$

Solution

$A = l\,b$

$\frac{A}{b} = \frac{l\,b}{b}$,

$l = \frac{A}{b}$

2. Make h the subject of the relation $k = mgh$

Solution

$k = mgh$

$\frac{k}{mg} = \frac{mgh}{mg}$,

$h = \frac{k}{mg}$

Involving Addition and Subtraction

If it involves addition (+) or subtraction (–), the sign changes to the opposite sign when transferred to the other side of the equation. For example, to make u the subject of $v = u + at$, subtract at from both sides of the equation. That is :

$v - at = u + at - at$

$\Rightarrow v - at = u$ or $u = v - at$

Worked Examples

1. Make c the subject of $y = mx + c$

Solution

$y = mx + c$

$y - mx = c$

$\therefore c = y - mx$

2. Make p the subject of $p - 2l = w$.

Solution

$p - 2l = w$

$p = w + 2l$

3. Make x the subject of the $y = mx + c$.

Solution

$y = mx + c$

$y - c = mx$

$\frac{y-c}{m} = \frac{mx}{m}$

$x = \frac{y-c}{m}$

4. Make t the subject of the relation $v = u + at$.

Solution

$v = u + at$

$v - u = at$

$\frac{v-u}{a} = \frac{at}{a}$,

$t = \frac{v-u}{a}$

Involving Division (Fraction)

If it involves division or fraction, multiply through by the denominator or multiply the denominator by both sides of the equation and apply the guidelines under addition, subtraction or multiplication, where applicable.

Worked Examples

1. Make m the subject of $k = \frac{1}{2}mv^2$

Solution

$k = \frac{1}{2}mv^2$

$2 \times k = 2 \times \frac{1}{2}mv^2$

$2k = mv^2$

$\frac{2k}{v^2} = \frac{mv^2}{v^2}$,

$m = \frac{2k}{v^2}$

2. Make P the subject of $I = \frac{PTR}{100}$

Solution

$I = \frac{PTR}{100}$

$100 \times I = 100 \times \frac{PTR}{100}$

$100I = PTR$

$\frac{100I}{TR} = \frac{PTR}{TR}$,

$P = \frac{100I}{TR}$

3. Make b the subject of the relation,

$\frac{1}{a} = \frac{1}{b} + c$

Solution

$\frac{1}{a} = \frac{1}{b} + c$

$\frac{1}{a} - \frac{c}{1} = \frac{1}{b}$

$\frac{1-ac}{a} = \frac{1}{b}$

$\frac{1-ac}{a} = \frac{1}{b}$ (Reciprocate)

$b = \frac{a}{1-ac}$

4. Given that $\frac{1}{y} = \frac{1}{c} + \frac{1}{x}$, find the expression that is equal to y.

Solution

$\frac{1}{y} = \frac{1}{c} + \frac{1}{x}$

$\frac{1}{y} = \frac{x+c}{cx}$

$\frac{1}{y} = \frac{x+c}{cx}$ (Reciprocate)

$y = \frac{cx}{x+c}$

5. Make q the subject of the relation;

$w = \frac{n-q}{q}$

Solution

$w = \frac{n-q}{q}$

$q \times w = \frac{n-q}{q} \times q$

$qw = n - q$

$qw + q = n$ *(Group like terms)*

$q(w+1) = n$ *(Factorization)*

$\frac{q(w+1)}{q} = \frac{n}{q}$

$\frac{q(w+1)}{q} = \frac{n}{q}$

$w+1 = \frac{n}{q}$,

$q = \frac{n}{w+1}$

Involving Brackets

When the variable to be made the subject is in bracket, first remove the bracket by expansion, and continue the process until that variable stands alone either in front or at the back of the equation.

Worked Examples

1. Make a the subject of the relation;

P = 2 (a + b)

Solution

$P = 2(a+b)$

$P = 2a + 2b$ *(Expansion)*

$P - 2b = 2a$

$\frac{p-2b}{2} = \frac{2a}{2}$

$a = \frac{p-2b}{2}$

2. Make$d$ the subject of $S = n(a + d)$

Solution

$S = n(a+d)$

$S = na + nd$ *(Expansion)*

$S - na = nd$

$\frac{S-na}{n} = \frac{nd}{n}$, *(Divide through by n)*

$d = \frac{S-na}{n}$

3. If $R = \frac{1}{2}(a+b)k$. Find k in terms of R, a and b

Solution

$R = \frac{1}{2}(a+b)k$

$2 \times R = 2 \times \frac{1}{2}(a+b)k$ *Multiply both sides by 2*

$2R = (a+b)k$.

$\frac{2R}{(a+b)} = \frac{(a+b)k}{(a+b)}$,

$\frac{2R}{(a+b)} = k$

Involving Factorization

When the variable being made the subject appears more than once, identify the variables as like terms and group them at one side of the equation. Then factorize the group of like terms and follow the due process.

Worked Examples

1. Make p the subject of $mp - a = x - p$

Solution

$mp - a = x - p$

$mp + p = x + a$ *(Group like terms)*

$p(m+1) = x + a$ *(Factorize like terms)*

$P = \frac{x+a}{m+1}$

2. Make t the subject of $S = ut + \frac{1}{2}at$

Solution

$S = ut + \frac{1}{2}at$

Multiply through by 2

$2 \times S = 2 \times ut + 2 \times \frac{1}{2}at$

$2S = 2ut + at$

$2S = t(2u + a)$

$\frac{2S}{2u+a} = \frac{t(2u+a)}{2u+a}$,

$\frac{2S}{2u+a} = t$

3. In the formula: $d = \frac{S - na}{n}$, express n in terms of s, d and a

Solution

$\mathrm{d} = \frac{\mathrm{S - na}}{\mathrm{n}}$,

$n \times d = \frac{S - na}{n} \times n$ *(Multiply both sides by n)*

$nd = s - na$

$nd + na = s$

$n(d + a) = s$

$n = \frac{s}{d + a}$

Exercises 8.1

Make the variable in bracket the subject

A. 1. $C = 2\pi r$ (r) 2. $v = \pi r^2 h$ (h)

3. $V = IR$ (R) 4. $V = \frac{1}{3}\pi r^2 h (h)$

B. Express the letters in bracket in terms of the others in each of the following;

1. $A = 4(r - y)$ (y) 3. $A = \frac{1}{2}h(a + b)$ (a)

2. $V = 2a + h$ (a) 4. $S = \frac{n}{2}(1 - d)$ (d)

C. Make the variable in bracket the subject

1. $C = 2\pi r + rh$ (r)
2. $mt + n = mp + q$ (m)
3. $ax - x + 1 - b = 0$ (x)
4. $ax = bx + c$ (x)

D. Change the subject of the following formulas to the variables in bracket;

1. $t = \frac{1}{p} + \frac{1}{q}$. (q) 2. $y = 1 + \frac{1}{x}$ (x)

3. $l^2 = \frac{4\pi^2 \mathrm{T}}{\mathrm{g}}$ (T) 4. $\mathrm{P} = \frac{2m+1}{m}$ (m)

5. $y = \frac{x}{ax+b}$ (x) 6. $\frac{1}{y} = \frac{1}{x} + \frac{2}{u}$ (u)

E. Change the subject in each of the following formula to the variable stated;

1. $s = u + \frac{1}{2}at^2$ (a) 2. $2n + 5 = 7a$. (n)

3. $v = u + at$ (t) 4. $v^2 = u^2 + 2as$ (s)

5. $m = 3(2d + 1)$ (d) 6. $y = 2x + q$ (x)

F. Make the letter in the bracket the subject of each of the following formulas.

1. $\mathrm{C} = 2\pi r$ (r) 2. $\mathrm{C} = 2\pi rh$ (r)

3. $\mathrm{T} = \frac{PRT}{100}$ (R) 4. $\mathrm{V} = \pi \mathrm{r}^2 h$ (h)

Involving Squared Variable and Square Root (√)

1. If the variable to be made the subject of a formula is squared, a square root is introduced at both sides of the equation at the last stage of the work to make the squared variable a single variable.

2. On the other hand, if the subject variable contains a square root or is in a square root, square both sides of the equation to eliminate the square root, leaving the variable alone. This is done at the last stage of the process.

Worked Examples

Make t the subject of the relation $r = \sqrt{\frac{t - 3}{9 - t}}$

Solution

$r = \sqrt{\frac{t - 3}{9 - t}}$

$r^2 = \left(\sqrt{\frac{t - 3}{9 - t}}\right)^2$

$r^2 = \frac{t - 3}{9 - t}$

$(9 - t)r^2 = t - 3$ (Cross product)

$9r^2 - tr^2 = t - 3$

$9r^2 + 3 = t + tr^2$

$9r^2 + 3 = t(1 + r^2)$

$\frac{9r^2 + 3}{1 + r^2} = t$

Involving Powers or Exponent n, n > 2

When the variable being made the subject has a power of n, where $n > 2$, introduce the nth root at both sides of the equation at the last stage of the work to remove the nth power.

Worked Example

Change the subject of $V = 4\pi r^3$ to r

Solution

$V = 4\pi r^3$

$\frac{V}{4\pi} = \frac{4\pi r^3}{4\pi}$ *(Divide both sides by 4π)*

$\sqrt[3]{r^3} = \sqrt[3]{\frac{V}{4\pi}}$,

$r = \sqrt[3]{\frac{V}{4\pi}}$

Exercises 8.2

A. Make the letter in the bracket the subject of the following formula;

1. $v = \frac{1}{3}\pi r^2 h$ (r)
2. $k = \frac{1}{2}mv^2$ (v)
3. $S = \frac{2\pi}{r^2}$ (r)
4. $s = 4\pi r^2$ (r)
5. $T^2 = \frac{49l}{g}$ (T)
6. $2\pi fL = \frac{T}{fC}$ (f)

B. Make the letter in the bracket the subject of each formula.

1. $T = \frac{\sqrt{l}}{g}$ (l)
2. $S = u + \frac{1}{2}at^2$ (t)
3. $V = \frac{4}{3}\pi r^3$ (r)
4. $v = \sqrt{u} + at$ (u)
5. $T = 2\pi\sqrt{\frac{l}{g}}$ (l)
6. $T = 2\pi\sqrt{\frac{l}{g}}$ (g)

C. Solve for the variable in bracket:

1. $n = p - 3\sqrt{t + c}$ (t)
2. $M - m\sqrt{ct} = r$ (t)
3. $\sqrt{m - n} = \sqrt{3t}$ (n)
4. $\sqrt{2t + s} = \sqrt{s - t}$ (t)

Challenge problems

1. Make k the subject of $\frac{1}{n} = \sqrt{\frac{k^2+a^2}{hg}}$
2. If $pv^{3/2} = k$, express v in terms of p and k.
3. Given $x = y + \sqrt{y^2 + z^2}$, make z the subject of the formula.
4. Change the subject of $S = \frac{a(1-r^2)}{1-r}$ to
   i. a ii. r
5. Make x the subject of each of the following *a.* $px + qy + r = 0$ *b.* $x^2 + y^2 = r^2$ *c.* $\frac{x+a}{b} = \frac{c}{d}$
6. If $r = \frac{\sqrt{(1-a^2}}{a}$, express a in terms of r.

Substitution

It is the act of putting values in place of variables in a formula. To work out substitution successfully, make the variable without substitute value the subject of the formula and substitute the given values into the new equation.

Worked Examples

1. If $y = 2x + c$, find y when $x = 6$ and $c = 8$

Solution

$y = 2x + c$

Substitute $x = 6$ and $c = 8$

$\Rightarrow y = 2(6) + 8$

$y = 12 + 8 = 20$

2. If $q = u\,t + \frac{1}{2}at$, find q when $u = 20$, $t = 10$ and $f = 15$.

Solution

$q = ut + \frac{1}{2}a\,t$

By substituting, u = 20, t = 10, and f = 15.

$q = 20\,(10) + \frac{1}{2}(15)\,(10)$

$q = 200 + \frac{1}{2} \times 150 = 275$

3. Given that m = 2 and $n = \frac{3}{4}$, find the values of;

(i) $m^2(n-1)$ (ii) $n^2 - \frac{3}{m}$

Solution

i. *Substitute m=2 and n =* $\frac{3}{4}$, *in* $m^2(n-1)$

$\Rightarrow (-2)^2(\frac{3}{4} - 1) = 4\left(\frac{3}{4} - \frac{1}{1}\right) = 4\left(\frac{3-4}{4}\right) = 4\left(\frac{-1}{4}\right) = -1$

ii. *Substitute m* = -2 *and n* = $\frac{3}{4}$, *in* $n^2 - \frac{3}{\mathrm{m}}$

$\Rightarrow \left(\frac{3}{4}\right)^2 - \frac{3}{-2} = \frac{9}{16} - \frac{3}{-2} = \frac{9}{16} + \frac{3}{2} = \frac{33}{16}$

4. Given that C = $\frac{5}{9}$(F – 32)

i. Find F when C = 40

ii. If C = − 40, find F.

Solution

i. Make F the subject of C = $\frac{5}{9}$(F−32)

by multiplying both sides by 9,

$\Rightarrow 9 \times C = 9 \times \frac{5}{9}(F - 32)$

9 C = 5 (F – 32)

9C = 5F – 160 (*expansion*)

9C + 160 = 5F

$\frac{9C+160}{5} = \frac{5F}{5}$,

$F = \frac{9C+160}{5}$

When C = 40,

$F = \frac{9(40)+160}{5} = 104$

ii. When C = −40, $F = \frac{9(-40)+160}{5}$

$F = \frac{-360+160}{5} = \frac{-200}{5} = -40$

5. Given that $f = \frac{v\,w}{v+u}$, find the value of v if f = 20 and u = 5.

Solution

$f = \frac{vw}{v+u}$,

$(v + u)f = \frac{vw}{v+u}(v + u)$

$vf + uf = vu$ (*expansion)*

$vu - vf = uf$ (*re-grouping)*

$v(u - f) = uf$ (*factorization)*

$v = \frac{uf}{u-f}$

Put f = 20 and u = 5 in $v = \frac{uf}{u-f}$

$\Rightarrow v = \frac{5(20)}{5-20} = \frac{100}{-15} = \frac{20}{-3}$

6. In the relation, $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$, if R_1 = 1 and R_2 = 3, find R.

Solution

Make R the subject first:

$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$

$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_2 + R_1}{R_1R_2}$

$\frac{1}{R} = \frac{R_2 + R_1}{R_1R_2}$ (*Reciprocate)*

$R = \frac{R_1R_2}{R_2 + R_1}$

Put R_1 = 1 and R_2 = 3 in $R = \frac{R_1R_2}{R_2 + R_1}$

$\Rightarrow R = \frac{1 \times 3}{3 + 1} = \frac{3}{4}$

7. If $\frac{1}{R} = \frac{1}{6} + \frac{1}{9}$, find the value R.

Solution

$\frac{1}{R} = \frac{1}{6} + \frac{1}{9} = \frac{3+2}{18} = \frac{5}{18}$,

$\frac{1}{R} = \frac{5}{18}$ (Reciprocate)

$R = \frac{18}{5} = 3.6$

Exercises 8.3

A. Find the value of the letters in brackets

1. $p = \frac{3m-n}{m+t}$, (p) , m =12, n = 11, t = - 4

2. $T^2 = 49\frac{l}{g}$ (g), T = 15, l = 45

3. $\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$, ($f$) v = 4 and u = 12

4. s = $ut + \frac{1}{2}at^2$ (a), s = 40, u = 3 and t = 2

5. Given that $\frac{1}{R} = \frac{1}{T} + \frac{1}{t}$, T = 4 and t = 8, find R

6. Given that $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$, find k_2 when k_1= 2 and k = 1.

7. From the formula, F = $\frac{m\,(v^2 - u^2)}{2gx}$

i. find the numerical value of F, when m = 37.6, v = 0.74, u = 0.59, x = 0.83, g = 32.2

ii. Express v in terms of the other letters

B. 1. The perimeter of a regular hexagon is P = 6x. Change the subject of to$x$, and hence find x when P = 96.

2. The road resistance to a car is R = kv^2. Change the subject to v, and find v when k = 4 and R = 324.

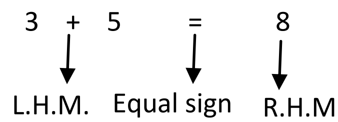
3. The rate at which water flows from a valve is v = $14\sqrt{h}$. Change the subject to h, and find h when v = 28.

4. In nuclear physics the formula E = kT occurs. If E = 4.14×10^{-14} and T = 300, find k.

Equations

An equation is a mathematical statement in which one expression is equal to another. It has two sides separated by an equal sign. It is therefore characterized by the following:

1. a left hand member (L.H.M),
2. an equal sign,
3. a right hand member (R.H.M)"



Variables

A variable is any symbol that is used to represent a number. Examples of variable are a, b, x, y… and examples of equations with variables are 3 + m = 5 and 2x – 3 = 15

The value(s) of a variable that satisfies or makes a mathematical statement true is called ***the truth set*** or ***the solution set.***

To find the truth set of an equation,

I. Ensure that the variable stands alone either in front or at the back of the equal sign. Thus, any positive or negative integer at the same side of variable is transposed (transferred) to the opposite side of the equal sign to assume the opposite sign

II. Simplify both sides of the equation to obtain the value of the variable

III. The results may be stated in three different ways:

a. the solution of the equation is x = a.

b. a is the root/truth/solution set of the equation.

c. {x : x = a} is the truth set of the equation.

General Rules

I. Remove all brackets by expansion.

II. If there are fractions, multiply the L.C.M. of the denominators by each term of the equation.

III. If there is only one fraction, multiply through by the denominator of the fraction.

IV. Transpose / group like terms and simplify.

V. Solve by division, where necessary.

Type 1: Involving Multiplication

Divide the coefficient of the variable by both sides of the equation to obtain the value of the variable.

Worked Examples

1. Solve $5x = 20$

Solution

$\frac{5x}{5} = \frac{20}{5}$,

$x = 4$

2. Find the solution set of $3m = 48$

Solution

$3m = 48$

$\frac{3m}{3} = \frac{48}{3}$,

$m = 16$

Type 2: Involving Division/Fraction

When the equation involves a fraction:

I. Identify the denominator of the fraction

II. Multiply the denominator by each term of the equation.

III. Divide each term of the equation by the coefficient of the variable to obtain the value of the variable. This step is also called ***solution by division.***

Worked Examples

Solve the following equations:

1. $\frac{3}{4}m = 21$

Solution

$\frac{3}{4}m = 21$

$4 \times \frac{3}{4}m = 21 \times 4$

$4m = 84$

$m = 41$

2. $\frac{4a}{3} = -12$

Solution

$\frac{4a}{3} = -12$

$3 \times \frac{4a}{3} = -12 \times 3$ (*Multiply both sides by* 3)

$4a = -36$

$a = -9$

Type 3: Involving all Procedure (s)

Here the principle for addition or subtraction is applied first, followed by that of division and /or multiplication.

Worked examples

Solve the following equations;

1. $2x + 5 = 15$

Solution

$2x + 5 = 15$

$2x = 15 - 5$ (*Transpose + 5 to become – 5)*

$2x = 10$

$\frac{2x}{2} = \frac{10}{2}$,

$x = 5$

2. $\frac{3}{2}m - 6 = 9$

Solution

$\frac{3}{2}m - 6 = 9$

$\frac{3}{2}m = 9 + 6$

$\frac{3}{2}m = 15$

$2 \times \frac{3}{2}m = 15 \times 2$

$3m = 30$

$\frac{3m}{3} = \frac{30}{3}$,

$m = 10$

Exercises 8.4

Find the truth set :

1. $50k + 10 = 160$ 4. $5(x + 2) = 50$
2. $5y - 3 = 21$
3. $20m + 5 = 45$ 5. $3(3x - 1) - 4 = 11$

Involving Two or More Variables

It is the type of equation which contains two variables of the same kind. For example;

$3x - 2 = 2x + 5$

To solve equations involving two variables of the same kind:

I. Group the like terms or the variable factors at one side of the equation.

II. Simplify both sides of the equation to obtain the value of the variable.

Worked Examples

Solve the following equations;

1. $3x - 2 = 2x + 5$

Solution

$3x - 2 = 2x + 5$

$3x - 2x = 5 + 2$ *(Group like terms)*

$x = 7$

2. $8y - 3 = 2y + 15$

Solution

$8y - 3 = 2y + 15$

$8y - 2y = 15 + 3$ *(Group like terms)*

$6y = 18$

$\frac{6y}{6} = \frac{18}{6}$ (*Divide both sides by* 6)

$y = 3$

Exercises 8.5

Solve the following equations:

1. $18a = 3a - 75$
2. $5x - 9 = 3x + 7$
3. $28 - 15a = 84 - 8a$
4. $4a + 7 + 2a = 11a - 3$
5. $6m + 3 - 8m = -7 - 4m$
6. $2y + 3 = 16 - 2y + 3$

Fractional Equations

It is a mathematical equation that involves two or more fractions.

Fractional equations are solved as follows:

I. Eliminate all fraction (s) by multiplying each term of the equation by the L.C.M. of the involving fraction(s);

II. Group like terms, if any;

III. Workout for the value of the involving variable .

Worked Examples

Solve the following equations;

(1) $\frac{2y}{9} = \frac{4}{3}$

Solution

$\frac{2y}{9} = \frac{4}{3}$,

$9 \times \frac{2y}{9} = 9 \times \frac{4}{3}$ *Multiply both sides by L.C.M = 9*

$2y = 12$

$y = \frac{12}{2} = 6$

(2) $\frac{x}{3} + \frac{x}{4} = 7$

Solution

$\frac{x}{3} + \frac{x}{4} = 7$

$12 \times \frac{x}{3} + \frac{x}{4} \times 12 = 12 \times 7 \left(\begin{matrix} Multiply\ each \\ term\ by\ L.C.M \end{matrix}\right)$

$4x + 3x = 84$

$7x = 84$

$x = \frac{84}{7}$

$x = 12$

3. $\frac{x}{4} - \frac{1}{6} = \frac{7\,(x-2)}{12}$

Solution

$\frac{x}{4} - \frac{1}{6} = \frac{7(\mathrm{x}-2)}{12}$

$\frac{x}{4} - \frac{1}{6} = \frac{7\mathrm{x}}{12} - \frac{14}{12}$ (*Remove brackets by expansion)*

$12 \times \frac{x}{4} - 12 \times \frac{1}{6} = 12 \times \frac{7x}{12} - 12 \times \frac{14}{12}$ $\left(\begin{matrix} \textit{Multiply each} \\ \textit{term by } L.C.M \end{matrix}\right)$

$3x - 2 = 7x - 14$

$3x - 7x = -14 + 2$

$-4x = -12$

$x = \frac{-12}{-4} = 3$

4. Solve $\frac{1}{5} - \frac{x}{4} = \frac{7\,(x+3)}{10}$

Solution

$\frac{1}{5} - \frac{x}{4} = \frac{7\,(x+3)}{10}$

$\frac{1}{5} - \frac{x}{4} = \frac{7x+21}{10}$

$20 \times \frac{1}{5} - 20 \times \frac{x}{4} = 20 \times \frac{(7x+21)}{10}$

$4 - 5x = 14x + 42$

$-5x - 14x = 42 - 4$

$-19x = 38$

$x = \frac{38}{-19}$

5. Solve;

$\frac{1}{4}(x-6) - \frac{1}{3}(x+5) = \frac{1}{5}(x-13)$

Solution

$\frac{1}{4}(x-6) - \frac{1}{3}(x+5) = \frac{1}{5}(x-13)$

$\frac{x}{4} - \frac{6}{4} - \frac{x}{3} + \frac{5}{3} = \frac{x}{5} - \frac{13}{5}$

$60 \times \frac{x}{4} - 60 \times \frac{6}{4} - 60 \times \frac{x}{3} + 60 \times \frac{5}{3}$

$= 60 \times \frac{x}{5} - 60 \times \frac{13}{5}$

$15x - 90 - 20x - 100 = 12x - 156$

$15x - 20x - 12x = -156 + 90 + 100$

$-17x = 34$

$x = \frac{34}{-17}$ = - 2

6. Solve $\frac{1}{2}\left(\frac{x}{3} + 1\right) = \frac{x}{4} - 2$

Solution

$\frac{1}{2}\left(\frac{x}{3} + 1\right) = \frac{x}{4} - 2$

$\frac{x}{6} + \frac{1}{2} = \frac{x}{4} - 2$

$12 \times \frac{x}{6} + 12 \times \frac{1}{2} = 12 \times \frac{x}{4} - 12 \times 2$

$2x + 6 = 3x - 24$

$6 + 24 = 3x - 2x$

$x = 30$

Some Solved Past Questions

1. Solve $\frac{\mathrm{x}}{4} + \frac{3}{5} = \frac{3\mathrm{x}}{5} - 2$

Solution

$20 \times \frac{\mathrm{x}}{4} + 20 \times \frac{3}{5} = 20 \times \frac{3\mathrm{x}}{5} - 20 \times 2$

$5x + 4\,(3) = 4(3x) - 40$

$5x + 12 = 12x - 40$

$5x - 12x = -40 - 12$

$-7x = -52$

$x = \frac{52}{7}$

2. Solve the equation: $\frac{2\mathrm{x}-1}{3} - \frac{\mathrm{x}-2}{4} = 1$

Solution

$\frac{2x-1}{3} - \frac{x-2}{4} = 1$

$12 \times \frac{2\mathrm{x}-1}{3} - 12 \times \frac{\mathrm{x}-2}{4} = 12 \times 1$

$8x - 4 - 3x + 6 = 12$

$8x - 3x = 12 + 4 - 6$

$5x = 10$

$x = \frac{10}{5} = 2$

3. Find the truth set of $\frac{2}{3}(3y - 1) - (y + 2) = \frac{1}{3}$

Solution

$\frac{2}{3}(3y - 1) - (y + 2) = \frac{1}{3}$

$\frac{2}{3}(3y - 1) - (y + 2) = \frac{1}{3}$

$2(3y - 1) - 3(y + 2) = 1$

$6y - 2 - 3y - 6 = 1$

$6y - 3y = 1 + 2 + 6$

$3y = 9$

$y = 3$

4. Solve $5(a - 5) - \frac{1}{2}(2a + 6) = 4$

Solution

$5(a - 5) - \frac{1}{2}(2a + 6) = 4$

$2 \times 5(a - 5) - 2 \times \frac{1}{2}(2a + 6) = 2 \times 4$

$10(a - 5) - 2a - 6 = 8$

$10a - 50 - 2a - 6 = 8$

$10a - 2a = 8 + 50 + 6$

$8a = 64$

$a = 8$

5. Solve the equation;

$\frac{1}{3}(x + 3) - 2(x - 5) = 4\frac{1}{3}$

Solution

$\frac{1}{3}(x + 3) - 2(x - 5) = 4\frac{1}{3}$

$3 \times \frac{1}{3}(x + 3) - 3 \times 2(x - 5) = 3 \times \frac{13}{3}$

$(x + 3) - 3 \times 2(x - 5) = 3 \times \frac{13}{3}$

$x + 3 - 6(x - 5) = 13$

$x - 6x = 13 - 3 - 30$

$-5x = -20$

$x = 4$

Exercises 8.6

A. Solve each of the following:

1. $2x - 3(x - 2) = \frac{1}{2}(x + 3) - 2$
2. $\frac{2a + 3}{4} - \frac{3a + 2}{12} = 1$
3. $\frac{x}{2} - \frac{2x}{3} + \frac{1}{4} = \frac{2}{3}(6 - 3x) - \frac{1}{12}$
4. $\frac{5y}{2} - \frac{5}{3} = \frac{1}{4}(3y + 1) + y$
5. $\frac{a - 2}{2} = 2 - \frac{a - 6}{10}$
6. $\frac{2}{3}(x - 2) - \frac{3}{2}(2x - 3) = 1 - \frac{1}{2}(x + 1)$

B. Solve the following equations:

1. $\frac{3}{5}(2y - 1) = \frac{1}{3}(y + 6)$
2. $\frac{1}{2}(a + 4) - \frac{1}{3}(a - 3) = 1$
3. $\frac{1}{2}(x + 3) - 2(x - 1) = -14$
4. $\frac{3}{4}(x + 3) - \frac{1}{3}(2x - 3) = \frac{5}{6}$
5. $\frac{1}{3}(2x - 5) - \frac{2}{7}\left(\frac{1}{3}x - 4\right) = 1$
6. $x\left(\frac{1}{2} + \frac{1}{3}\right) = 10\left(\frac{1}{3} + \frac{1}{4}\right)$
7. $\frac{1}{2}(5x - 4) = x + 1\frac{1}{2}$
8. $\frac{2}{3}(x - 3) = \frac{5}{6}(x + 6)$

Word Problem Involving Equations

In word problems involving equations, the relationship between certain numbers are stated in words. It is necessary to state the problem using mathematical equations for easy solution.

Hints on Forming Word Problems

a. Let x represent any unknown number

b. m more than the number $= x + m$

c. m less than the number $= x - m$

d. A number is decreased by $m = x - m$

e. A number is increased by $m = x + m$

f. Half a number $= \frac{1}{2}x$

g. Twice a number = $2x$

h. Twice the results of *m* more than a number = $2(x + m)$

i. Twice the results of *m* less than a number

$= 2(x – m)$

j. The use of the phrases " the results is",

"add/sum up to" "is" implies the equal sign

Steps for Solving Word Problems

I. Carefully read the problem.

II. Choose any preferred variable to represent the number required number.

III. Write an equation for the problem.

IV. Solve the equation.

V. Check your results with the words of the problem.

Type 1 : Involving Numbers

Worked Examples

1. When a certain number is decreased by five, the result is twelve. What is the number?

Solution

I. Let x represents the number

II. x increased by 5 is written as $x + 5$

III. the results is 12 , implies $x + 5 =12$

IV. Solve $x + 5 = 12$ to get the value of x

$\Rightarrow x + 5 = 12$

$x = 12 – 5 = 7$

2. Twice a certain number increased by eight is twenty. What is the number?

Solution

I. Twice a certain mumber is written as $2x$

II. plus 8 is written as $2x + 8$

III. the results is 20 implies $2x + 8 = 20$

$2x + 8 = 20$

$2x = 20 - 8$

$2x = 12$

$x = \frac{12}{2} = 6$

3. One – third of a number is decreased by five and the result is sixteen. Write an algebraic equation and hence find the number.

Solution

$\frac{1}{3}x – 5 = 16$

$\frac{1}{3}x = 16 + 5$

$\frac{x}{3} = 21$

$3 \times \frac{x}{3} = 3 \times 21$

$x = 63$

4. Three times a certain number increased by seven is 40. Express this in an equation and find the number.

Solution

Let x represents the number

$3x + 7 = 40$

$3x = 40 - 7$

$3x = 33$

$\frac{3x}{3} = \frac{33}{3}$,

$x = 11$

5. When five times a certain number is increased by three, the result is twice the same number increased by twelve. Find the number.

Solution

Let x represent the number

$5x + 3 = 2x + 12$

$5x - 2x = 12 - 3$

$3x = 9$

$\frac{3x}{3} = \frac{9}{3}$,

$x = 3$

6. When x is divided by 13, the quotient is 9 and the remainder is 3. Find the value of x.

Solution

$\frac{x}{13} = 9 + \frac{3}{13}$

$\frac{x}{13} \times 13 = 9 \times 13 + \frac{3}{13} \times 13$

$x = 117 + 3 = 120$

Some Solved Past Questions

1. When a certain number is subtracted from ten and the result is multiplied by two, the final result is four. Find the number.

Solution

Let x be the number

$2\ (10 - x) = 4$

$20 - 2x = 4$

$- 2x = 4 - 20$

$- 2x = - 16$

$x = 8$

2. A train fair for a school child is half the fare of a teacher. The total fare for 120 children and 15 teachers for an excursion is Gh¢180.00

i. Find the fare of a child

ii. How many children will go on the excursion with 20 teachers for a total fare of Gh¢240.00?

Solution

i. Let the fare for a teacher be x

The fare for a school child = $\frac{1}{2}x$

The fare for 15 teachers will be $15x$ and that of 120 children = $(120)\ \frac{1}{2}x = 60x$

The total fare for 120 children and 15 teachers is Gh¢180

$\Rightarrow 60x + 15x = 180$

$75\ x = 180$

$x = \frac{180}{75} = 2.4$

Therefore. the fare for a teacher is Gh¢2.40

But the fare of a child is half the fare of a teacher $= \frac{1}{2}\ (2.4) = 1.2$

Therefore, the fare of a child is Gh¢1.20

ii. Let the number of students be y

$1.2y + 20(2.4) = 240$

$1.2y + 48 = 240$

$1.2y = 240 - 48$

$1.2y = 192$

$y = 160$ students

Therefore 160 students can go on the excursion

Type 2: Involving Ages

A. Age Problems Involving a Single person

If the problem involves a single person, the solution is similar to the real number problems. Carefully read the question to determine the relationship between the numbers.

Worked Examples

1. Five years ago, Esi's age was half the age she will be in ten years. How old is she now?

Solution

Let Esi's present age = x

Esi's age five years ago = $x - 5$

Esi's age in ten years = $x + 8$

Half her age in ten years = $\frac{1}{2}\ (x + 10)$

Five years ago, Esi's age was half the age she will be in ten years;

$\Rightarrow x - 5 = \frac{1}{2}\ (x + 10)$

$2(x - 5) = x + 10$ (Solving for x)

$2x - 10 = x + 10$

$2x - x = 10 + 10$

$x = 20$

Esi is 20 years now.

A. Age Problems

If the problem involves the ages of two or more people, then using a table will be a good idea. This helps to organize the information to write the equations.

Worked Examples

1. John is twice as old as his friend Peter. Peter is 5 years older Alice. In 5 years, John will be three times as old as Alice. How old is Peter now?

Solution

Let x be Peter's age now.

	Age now	Age in 5 years
John	$2x$	$2x + 5$
Peter	x	$x + 5$
Alice	$x – 5$	$x – 5 + 5$

In five years , John will be three times as old as Alice;

$\Rightarrow 2x + 5 = 3(x – 5 + 5)$

$2x + 5 = 3x$

$2x – 3x = -5$

$-x = -5$

$x = 5$

Peter is now 5 years old.

2. John's father is five times older than John and John is twice as old as his sister Alice. In two years time, the sum of their ages will be 58. How old is John now?

Solution

	Age now	Age in two years
Father	$5(2x)$	$5(2x) + 2$
John	$2x$	$2x + 2$
Alice	x	$x + 2$

In two years time, the sum of their ages will be 58.

$\Rightarrow 5(2x) + 2 + 2x + 2 + x + 2 = 58$

$10x + 2 + 2x + 2 + x + 2 = 58$

$13x = 58 – 2 – 2 – 2$

$13x = 52$

$x = 4$

John is $2(x) = 2(4) =$ 8 years old.

Solved Past Questions

1. The sum of the ages of two brothers Kofi and Kwaku is 35. Kofi's age is two thirds of Kwaku's age. Find their ages.

Solution

Kwaku's age = x years,

Kofi = $\frac{2x}{3}$ years

Sum of their ages = 35

$x + \frac{2x}{3} = 35$

$3 \times x + 3 \times \frac{2x}{3} = 3 \times 35$

$3x + 2x = 105$

$5x = 105$

$x = 21$

Kwaku's age = 21 years

Kofi's age = $\frac{2}{3} \times 21 = 14$years

2. Kojo is n years old now;

i. How old was he 5 years ago?

ii. How old will he be 10 years from now.

iii. If his age in 10 years ago will be four times his age 5 years ago, how old is he now?

Solution

Let Kofi's age be n.

Then 5 years ago, his age was ($n – 5$) years.

10 years time, his age will be ($n +10$) years.

$n + 10 = 4(n – 5)$

$n + 10 = 4n – 20$

$n - 4n = -20 - 10$
$-3n = -30$
$n = 10$ years

3. Abanga's age in the next 10 years will be four times his age 5 years ago. How old is Abanga now?

Solution
Let x be Abanga's age
$x + 10 = 4(x - 5)$
$x + 10 = 4x - 20$
$x - 4x = -20 - 10$
$-3x = -30$
$x = 10$ years

4. Awa is m years old now and Fatou is y years older than Awa. If $(x - 5)$ years ago, Fatou was twice as old as Awa, express x in terms of y and m.

Solution
Awa's age = m
Fatuo's age = $m + y$
$(x - 5)$ years ago
Awa's age was = $m - (x - 5)$
$= m - x + 5$
Fatuo's age = $m + y - (x - 5)$
$= m + y - x + 5$
Fatou was twice as old as Awa,
$m + y - x + 5 = 2(m - x + 5)$
$m + y - x + 5 = 2m - 2x + 10$
$-x + 2x = 2m - m - y + 10 - 5$
$x = m - y + 10$

Exercises 8.7
A. Write an equation and solve it.
1. Six times a certain number decreased by two is four times the same number increased by twelve. Find the number.

2. When a certain number is increased by five, the result is six times the same number decreased by ten. What is the number?

3. If $\frac{3}{5}$ of a numbers is 4 more than $\frac{1}{2}$ the number, what is the number? Ans 40

4. I am thinking of a certain number. If I multiply by 6, add -18 to the product and multiply by one – third of the result, I get -2. What number am I thinking of?

5. One number is greater than another by fifteen. If five times the larger number minus twice the smaller number is three, what are the numbers?

6. A certain rational number is halved, and 23 is added to the results. The same number is doubled, and 21 is added to the results. If the two numbers obtained are equal, find the rational number.

7. A boy was asked to subtract seven from a certain number and divide the results by five. Instead, he subtracted five and divided the results by seven. His answer was four less than it should have been. What was the given number?

C**.** 1. A father is now twice as old as his son. If the sum of their ages ten years ago was fifty – two, find their present ages.

2. A boy is three years younger than his sister. If his age three years ago was two – thirds of her age at that time, what are their present ages?

3. A man is x years old and his son is 30 years younger. In 2 years, the father's age will be twice the son's age. Form an equation, and hence find the present ages of the father and son.

4. Ben is eight years older than Sarah. Ten years ago, Ben was twice as old as Sarah. Find the present ages of Ben and Sarah. Ans **10, 18**

5. Mary is three times older than her son. In 12 years, Mary's age will be one year less than twice her sons age. How old is each now? **Ans 11, 33**

6. Sally is three times as old as John. Eight years from now, Sally will be twice as old as John. How old is John? **Ans 8, 24**

7. Kim is 6 years more than twice Tims age. Two years ago, Kim was three times as old as Tim. How old was Kim two years ago? **10, 24**
8. Leah is two less than three times Rahel's age. Three years from now, Leah will be seven more than twice Rahels' age . How old will Rahel be in three years from now?

9. Becca is twice as old as Susan and George is 9 years older than Susan. Three years ago, Becca was 9 less than three times Susan's age. How old is George now?

10. Lauren is three less than twice Andy's age. Four years from now, Sam will be two more than twice Andy's age. Five years ago, Sam was three times Andy's age. How old was Lauren five years ago?

Challenge Problems
1. A man has a daughter and a son. The son is three years older than the daughter. In one year, the man will be six times as old as the daughter is now. In ten years, the man would be fourteen years older than the combined ages of his children at that time. What is the present age of the man. Ans: 7, 9, 10

2. Adam is about to embark on a journey on a narrow country lane that covers 32km and decides to go x km/h. On second thoughts, he calculates that if he increases the speed by 4km/h, his journey time can be cut down by 4 hours. find x. Ans 4km/h

3. A butcher bought some pheasants for Gh¢100.00. Had each cost Gh ¢1 less, he would have bought 5 more. How many pheasants did he buy? Ans: 20

4. A group of student went to a restaurant for a meal. When the bill of Gh¢175.00 was brought by a waiter, two of the cheeky ones from the group just sneaked off before the bill was paid, which resulted in the payment of extra Gh¢10.00 by each remaining student. How many students were in the group as well? Ans 7

5. The reciprocal of the sum of reciprocals of two numbers is 6. The sum of the numbers is 25. Find the numbers. Ans 10, 15

6. Andy has more money than Joe. If Andy gave Joe Gh¢20.00, they would have the same amount. If Joe gave Andy Gh¢22.00, Andy would then have twice as much as Joe. How much does each one actually have? Ans 106, 146

7. Noah wants to share a certain amount of money with 10 people. However, at te last miniute, he is thinking about decreasing the amount by 20 so he can keep 20 for himself and share the money with only 5 people. How much money is Noah trying to share if each person still gets the same amount. Ans 44

Consecutive Numbers
Consecutive numbers are two or more numbers

that follow each other. For example, 1, 2, 3 or 9, 10, 11 etc. For all consecutive numbers, if the first number is represented by the variable x, then second number is $x + 1$, the third number is $x + 2$, and the fourth number is $x + 3$, in that order.

For the sum of consecutive odd or even numbers, the first number is $\mathbf{x}$, the second number is $\mathbf{x + 2}$, the third number is $\mathbf{x + 4}$, the fourth is $\mathbf{x + 6}$, in that order.

Worked Examples

1. The sum of three consecutive numbers is 18. Find the numbers.

Solution

Let the first number = x

$\Rightarrow$ the second number = $x + 1$,

$\Rightarrow$ the third number = $x + 2$

The sum of the three numbers is 18

$\Rightarrow x + (x + 1) + (x + 2) = 18$

$x + x + 1 + x + 2 = 18$

$3x + 3 = 18$

$3x = 18 - 3$

$3x = 15$

$x = 5$

$\Rightarrow$the first number, $x = 5$

$\Rightarrow$ the second number, $x + 1 = 5 + 1 = 6$

$\Rightarrow$third number, $x + 2 = 5 + 2 = 7$

The three consecutive numbers are 5, 6, 7.

2. The sum of three consecutive even numbers is 18. Find the numbers.

Solution

Let the first even number = x

$\Rightarrow$the second even number = $x + 2$,

$\Rightarrow$ the third even number = $x + 4$

The sum of the three even numbers is 18

$x + (x + 2) + (x + 4) = 18$

$x + x + 2 + x + 4 = 18$

$3x + 6 = 18$

$3x = 18 - 6$

$3x = 12$

$x = 4$

4, 4 + 2 and 4 + 4 = 4, 6, 8

The 3 consecutive even numbers are 4, 6 and 8.

3. The sum of three consecutive odd numbers is 27. What are the numbers?

Solution

Let the first odd number be x.

$\Rightarrow$second odd number will be $x + 2$,

$\Rightarrow$ the third odd number will be $x + 4$,

The sum of the three even numbers is 27

$\Rightarrow x + (x + 2) + (x + 4) = 27$

$x + x + 2 + x + 4 = 27$

$3x + 6 = 27$

$3x = 27 - 6$

$3x = 21$

$x = 7$

The three consecutive odd numbers are: 7, 9, 11

4. Five times the smallest of three consecutive odd integers is seven more than twice the largest. Find the largest integer.

Solution

Let the three consecutive odd integers be:

x, $x + 2$, and $x + 4$ respectively.

Five times the smallest is seven more than twice the largest;

$\Rightarrow 5(x) = 7 + 2(x + 4)$

$5x = 7 + 2x + 8$

$5x - 2x = 7 + 8$

$3x = 15$

$x = 5$

The three consecutive odd numbers are:

5, 5 + 2, and 5 + 4 = 5, 7, 9

The largest integer is 9.

Involving Product

Worked Examples

1. Find two consecutive odd integers whose product is 99.

Solution

Let the two consecutive odd integers be:
x, and $x + 2$ respectively.
$x(x + 2) = 99$
$x^2 + 2x = 99$
$x^2 + 2x - 99 = 0$
$(x - 9)(x + 11) = 0$ (by factorization)
$(x - 9) = 0$ or $(x + 11) = 0$
$x = 9$ or $x = -11$

When $x = 9$,
The 1st odd integer is 9 and the 2nd is $(9 + 2) = 11$.

When $x = -11$,
The 1st odd integer is -11 and the 2nd is -9.

Thus, the two consecutive odd consecutive intgers are 9 and 11 or -11 and -9.

2. Find two consecutive even integers whose product is 224.

Solution

Let the two consecutive even integers be:
x and $x + 2$ respectively.
$x(x + 2) = 224$
$x^2 + 2x = 224$
$x^2 + 2x - 224 = 0$
$(x - 14)(x + 16) = 0$
$(x - 14) = 0$ or $(x + 16) = 0$
$x = 14$ or $x = -16$

When $x = 14$, the second is $(14 + 2) = 16$.
When $x = -16$, the second is $(-16 + 2) = -14$.

Thus, the two consecutive odd integers are:
14 and 16 or -16 and - 14.

Solved Past Questions

1. Find three consecutive odd integers such that the sum of the last two is fifteen less than five times the first.

Solution

Let the three consecutive odd integers be x,
$x + 2$ and $x + 4$,
The sum of the last two is fifteen less than five times the first;
$\Rightarrow x + 2 + x + 4 = 5x - 15$
$2x + 6 = 5x - 15$
$6 + 15 = 5x - 2x$
$21 = 3x$
$x = \frac{21}{3} = 7$
The consecutive odd integers are:
7, 7 + 2, 7 + 4 = 7, 9, 11

Exercises 8.8

1. Five times the second of three consecutive even intergers is six more than twice the sum of the first and third integers. Find the middle even integers.

2. Three times the second of three consecutive even integers is twelve less than twice the sum of the first and third integers. Find the largest even integer.

3. Find two consecutive integers such that five times the first equals ten more than three times the second.
4. Four times the smallest of three consecutive integers is three more than three times the largest. Find the middle integer.

5. Five times the second of three consecutive odd integers is twelve less than twice the sum of the

first and third integers. Find the largest odd integer.

6. A man has four sons each of whom (except the youngest) is 2 years older than his next younger brother. If the sum of the boys ages is 52 years, how old is the eldest son?

7. Find five consecutive even integers if the sum of the first and fifth is two less than three times the fourth.

B. Write an equation for each and solve

1. Find four consecutive odd integers if the sum of the first and fourth is three less than three times the second.

2. Find four consecutive integers if the sum of the second and fourth is 48.

3. Find three consecutive integers such that the sum of the first and third is – 34.

4. Find two consecutive odd integers if twice the larger, increased by the smaller, equals 85.

5. Find three consecutive even integers if their sum, decreased by the third equals – 22.

6. The ages in years of three brothers are consecutive. The sum of their ages is 39 decreasd by the age of the youngest. What are their ages?

7. The sum of three consecutive even integers is 50 more than the third integer. Find the three integers

8. The sum of three consecutive even integers is equal to 9 less than 4 times the least of the integers. Find these integers.

9. When the sum of four consecutive even numbers is divided by 7, the result is 4. Find the integers.

10. Find four consecutive multiples of five whose sum is 90.

C. Write an equation for each and solve

1. Find three consecutive multiples of 3 if the sum of the first and third is 12.

2. Find three consecutive integers if twice the middle integer is equal to the sum of the first and third.

3. Find four consecutive integers such that the sum of the two largest subtracted from three times the sum of the two smallest is 70.

4. Find four consecutive odd integers such that the sum of the two smallest added to four times the smallest is 92.

5. Half of the smaller of two consecutive even integers is equal to two more than the larger integer. Find the numbers

6. The sum of reciprocal of two consecutive integers is $\frac{13}{42}$. Find the integers. Ans: 6, 7

Inequalities

An inequality is a mathematical statement formed by relating two expressions to each other by using any of the following signs or symbols;

$<$ *less than*

$>$ *greater than*

$\leq$ *less than or equal to*

$\geq$ *greater than or equal to*

For example, $3x > 9$, $6 + x < 20$, $\frac{2}{3}x \leq 10$, etc

Solving Inequalities

All the steps involved in solving equations are applicable in solving inequalities except that when we multiply or divide both side of an inequality by a negative number, the direction of the inequality sign reverses. That is;

$<$ changes to $>$ and vice – versa whilst $\leq$ changes to $\geq$ and vice – versa.

Worked Examples

1. $\frac{3}{2}x + 9 \geq 18$

Solution

$\frac{3}{2}x + 9 \geq 18$

$\frac{3}{2}x \geq 18 - 9$

$\frac{3}{2}x \geq 9$

$2 \times \frac{3}{2}x \geq 2 \times 9$

$3x \geq 18$

$\frac{3x}{3} \geq \frac{18}{3}$,

$x \geq 6$

2. Solve $-2x \leq 10 + 3x$

Solution

$-2x \leq 10 + 3x$

$-2x - 3x \leq 10$

$-5x \leq 10$

$\frac{-5x}{-5} \leq \frac{10}{-5}$

$x \geq -2$ (*sign reverses*)

3. Solve $13x - 15 \geq 33 + 25x$

Solution

$13x - 25x \geq 33 + 15$

$-12x \geq 48$

$\frac{-12x}{-12} \geq \frac{48}{-12}$,

$x \leq -4$

4. Solve $\frac{x}{-2} + 4 > 17$

Solution

$\frac{x}{-2} + 4 > 17$

$\frac{x}{-2} > 17 - 4$

$\frac{x}{-2} > 13$

$\frac{x}{-2} \times -2 > 13 \times -2$

$x < -26$ (sign reverses)

5. Solve $\frac{1}{3}x - \frac{1}{4}(x + 2) \geq 3x - 1\frac{1}{3}$

Solution

$\frac{1}{3}x - \frac{1}{4}(x + 2) \geq 3x - 1\frac{1}{3}$

$\frac{1}{3}x - \frac{x}{4} - \frac{2}{4} \geq 3x - \frac{4}{3}$

$12 \times \frac{1}{3}x - 12 \times \frac{x}{4} - 12 \times \frac{2}{4} \geq 12 \times 3x - 12 \times \frac{4}{3}$

$4x - 3x - 6 \geq 36x - 16$

$4x - 3x - 36x \geq -16 + 6$

$-35x \geq -10$

$x \leq \frac{-10}{-35}$ *(division by a negative number)*

$x \leq \frac{2}{7}$

Exercises 8.9

A. Determine the solution set

1. $3x - 13 > 26$
2. $2x - 7 < 32 - x$
3. $\frac{x}{-4} + 9 \geq -5$
4. $14 \leq 8 - 2x$
5. $14 > 8 - 2x$
6. $8 - 7x < 11$

B. Determine the solution set

1) $-(2 + x) < 3 + (-7)$

2) $3x > (8 + x) - 2$

3) $\frac{2}{3} - \frac{1}{3}x \geq \frac{4}{3}x + \frac{2}{5}$

4) $\frac{3}{4}x + \frac{1}{2} \leq \frac{17}{16}$

5) $\frac{4}{5} - \frac{1}{5}x > \frac{2}{7}$

6. If $x = \{1, 3, 5, 7, 9, 11, 13, 15\}$, find the truth set of $x - 3 > 10$.

Inequalities on a Number Line

The truth set of inequalities can be represented on a number line by the use of a directed shaded circle or directed none shaded circle.

For all inequality signs:
The shaded circle is used for $\leq$ or $\geq$ signs and the none shaded circle is used for < or $>$ signs. The circle, either shaded or non-shaded is made to stand on top of the number that represents the truth set.

I. If the sign is $> or\ \geq$, the arrow is directed to the right.

II. If the sign is $< or\ \leq$ the arrow is directed to the left.

III. If the condition is given on the set of integers (Z), maintain the direction of the arrow and put a dot on the integers representing the solution set as shown in example 4 below.

Worked Examples

Solve the inequalities and represent the answer on a number line.

1. $x + 17 < 20$

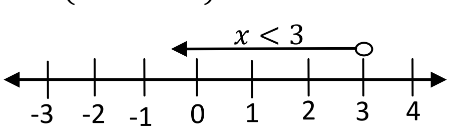
Solution

$x + 17 < 20$

$x < 20 - 17$

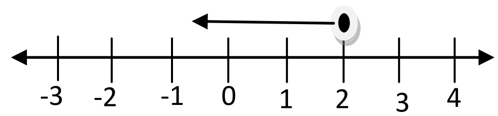
$x < 3$

Stand on the number that represents the truth set using non – shaded circle because the sign is $<$ (less than)



Arrow the non-shaded circle to the left and write the truth set on it.

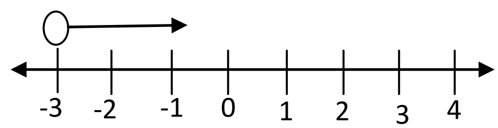
2. Identify the inequality shown in the number line below;



Solution

Because the circle is shaded and directed to the left, it represents the inequality symbol ≤. The shaded circle stands on 2 on the number line. Therefore $x \leq 2$

3. What inequality is represented on the number line below?



Solution

None shaded circle directed to the right represents >. Since the none shaded circle stands on -3, the inequality is $x > -3$

4. Solve $5 - 2x \geq x + 2$, where x is an integer and illustrate your answer on a number line.

Solution

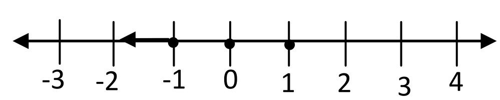
$5 - 2x \geq x + 2$

$5 - 2 \geq x + 2x$

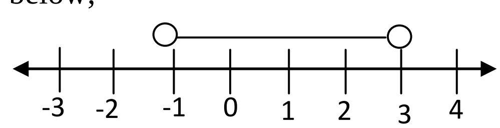
$5 - 2 \geq 3x$

$3 \geq 3x$

$1 \geq x$ or $x \leq 1$



5. Determine the inequality on the number line below;



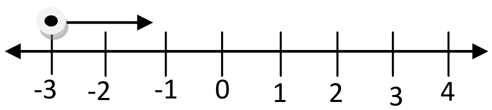
Solution

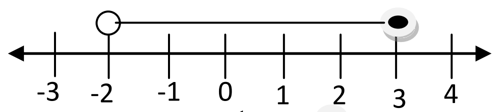
The non shaded circle on -1, directed to the right represent: $x > -1$ or $-1 < x$

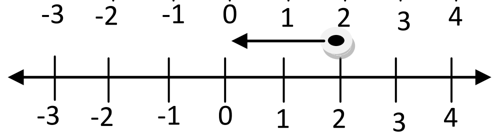
The non shaded circle is on 3, directed to the left so the truth set is $x < 3$. The two inequalities put together $= -1 < x < 3$

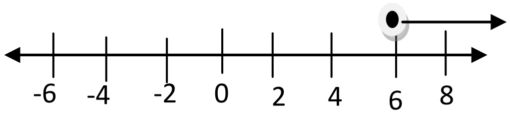
Exercises 8.10

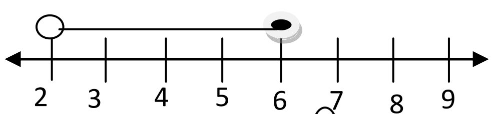
A. Identify the inequality on the number lines

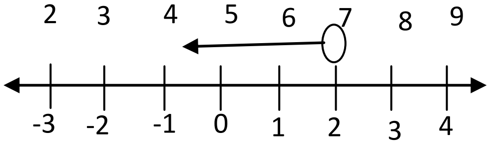
1.


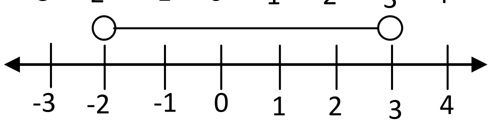
2.


3.


4.


5.


6.


7.


B. List the members of the following sets

1. $\{2 < x < 10\}$ 2. $-10 \le x \le 5$
3. $21 < y \le 27$ 4. $3 < x < 5$
5. $2 \le x \le 5$ 6. $-2 \le x < 2$

C. Solve the following and represent your answer on a number line.

1. $\frac{3}{5}(x-1) + 2 \le \frac{x}{2}$ 2. $\frac{2(x-1)}{3} - \frac{4x-5}{5} < 1$
3. $\frac{2}{3}(2x+5) < 8\frac{2}{3}$ 4. $\frac{1}{3}(x-1) - \frac{1}{2}(x-3) \le 1\frac{1}{4}$
5. $\frac{x-3}{3} \le \frac{1}{2} + x$ 6. $\frac{1}{3}(x-9) > x + 3$

Word Problems (Linear Inequalities)

Very often, problems involving linear inequalities are given in words. It is advisable to construct an open sentence and then find the truth set.

Steps

1. Read the problem carefully.
2. Assign a variable to represent the unknown.
3. Determine which inequality is required
4. Wite an inequality based on the given information.
5. Solve the inequality to find the answer.
6. Check your answer.

Worked Examples

1. A man, 40 years old is more than twice as old as his son. How old is the son?

Solution

Let the son's age be x

$\Rightarrow 40 > 2x$

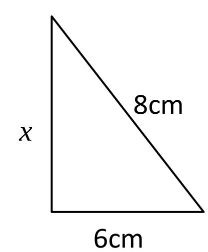
$\frac{2x}{2} < \frac{40}{2}$,

$x < 20$

Therefore, the son is less than 20 years.

2. Two sides of a triangle have lengths 6cm and 8cm. What is the length of the third side?

Solution



$6 + 8 > x$

$14 > x$

Also, $6 + x > 8$

$x > 8 - 6$

$x > 2$

Again, $8 + x > 6$

$x > 6 - 8$
$x > -2$,
But $x > -2$ satisfies the first two inequalities. Thus, $2 < x < 14$, meaning the third side has a length between 2cm and 14cm.

3. Every number of set of integers is greater than 40 and is less than 44. What are the members of the set?

Solution
Let x be a member of the set, then $x > 40$ and $x < 44$. The two can be put together as;
$40 < x < 44$.Therefore, the truth set of this inequality is {41, 42, 43}

4. You want to rent a car for a rental cost of Gh¢50.00 plus Gh¢30.00 per day, or a van for Gh¢80.00 plus Gh¢25.00 per day. For how many days at most is it cheaper to rent a car?

Solution
Let d = days required,
Car = $50 + 30d$, Van = $80 + 25d$
At most means "not more than, so $\leq$ is the sign. The inequality is ;
$50 + 30d \leq 80 + 25d$
$5d \leq 30$
$d \leq 6$

Check
Car = $50 + 30d = 50 + 30(6)$ = Gh¢230.00
Van = $80 + 25d = 80 + 25(6)$ = Gh¢230.00
The rental cost are the same for 6 days.

For less than 6days,
Car = $50 + 30d = 50 + 30(5)$ = Gh¢200.00
Van = $80 + 25d = 80 + 25(5)$ = Gh¢205.00
It is cheaper to rent the car for less than 6 days.

Or more than 6 days;
Car = $50 + 30d = 50 + 30(7)$ = Gh¢260.00
Van = $80 + 25d = 80 + 25(7)$ = Gh¢255.00
It is cheaper to rent the van for more than 6 days.

Exercises 8.11
Write an open sentence, and solve
1. Two sides of a triangle have lengths 10 cm and 14 cm. What is the length of the third side?

2. The smallest number in a set of integers is 572 and the largest is 999. What are the possible numbers of the set?

3. Your score in the 4 mathematics test were 71, 74, 75 and 86. What is the lowest score you need in your next test in order to have an average score of at least 80.

4. The sum of two consecutive integers is less than 79. Find the integer with the greatest sum.

Compound Inequalities
When two simple inequalities are connected by the word **"and"** or the word **"or"** a compound inequality is formed. For example, $x < 6$ and
$x < 9$, $2x - 9 \leq 5$ or $-4x \geq -12$

Compound Inequalities Connected by "or"
A compound inequality connected by the word "or" is true if one or the other or both of the simple inequalities are true. It is false only if both simple inequalities are false.
To determine whether such statement is true or false;
a. check if the first statement is true or false
b. check if the second statement is true or false.

Draw conclusions base on the following:
1. if a is true and b is true, the inequality is ***true;***

2. if a is true and b is false, the inequality is ***true;***
3. if a is false and b is true, the inequality is ***true;***
4. if a is false and b is false, the inequality is ***false.***

Worked Examples

Determine whether each compound inequalities is true or false.

1. $2 < 3$ or $2 > 7$ 2. $4 < 3$ or $4 \leq 7$

Solution

1. $2 < 3$ or $2 > 7$

Let a represents $2 < 3$ and b represents $2 > 7$.
$2 < 3$ is true, $2 > 7$ is false.
(a is true, b is false)
The statement $2 < 3$ or $2 > 7$ is true.

2. $4 < 3$ or $4 \leq 7$

Let a represents $4 < 3$ and b represents $4 \leq 7$.
$4 < 3$ is false, $4 \leq 7$ is false.
(a is false, b is false)
The statement $4 < 3$ or $4 \leq 7$ is false.

Exercises 8.12

Show whether the following is true or false;

1. $3 < 5$ or $3 < 10$ 2. $4 < 8$ or $4 > 2$
3. $4 \leq - 4$ or $0 \leq 0$ 4. $6 < 5$ or $-4 > -3$

Compound Inequalities Connected By "and"

A compound inequality connected by the word "and" is true if and only if both simple inequalities are true.

To determine whether such statement is true or false;

a. check if the first statement is true or false
b. check if the second statement is true or false.

Draw conclusions base on the following:

1. if a is true and b is true, the inequality is ***true;***
2. if a is true and b is false, the inequality is ***false;***
3. if a is false and b is true, the inequality is ***false;***
4. if a is false and b is false, the inequality is ***false.***

Worked Examples

Determine whether each compound inequality is true or false;

1. $5 > 2$ and $5 < 7$ 2. $6 < 5$ and $6 > 2$

Solution

1.$5 > 2$ and $5 < 7$

Let a represents $5 > 2$ and b represents $5 < 7$.
$5 > 2$ is true, $5 < 7$ is true.
(a is true, b is true)
The statement $5 > 2$ and $5 < 7$ is true.

2. $6 < 5$ and $6 > 2$

Let a represents $6 < 5$ and b represents $6 > 2$.
$6 < 5$ is false, $6 > 2$ is true.
(a is false, b is true)
The statement $6 < 5$ and $6 > 2$ is false.

Exercises 8.13

Determine whether each is true or false;

1. $-6 < 5$ and $-6 > -3$ 2. $1 < 5$ and $1 > -3$
3. $4 \leq 4$ and $-4 > -3$ 4. $3 < 5$ and $3 \leq 10$

Graphing Compound Inequalities

a. Using the connective "or"

The solution set to the compound inequality using the connective "or" is the union of the solution sets to each of the simple inequalities.

If A and B are the set of numbers, then the union of A and B is the set of all numbers that are in either A or B, denoted as $A \cup B$.

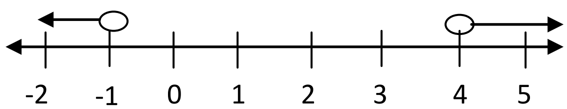
To represent such inequalities on a number line, graph each inequality separately on the same number line.

Worked Examples

1. Illustrate the solution set of the compound inequalities $x > 4$ or $x < -1$ on a number line.

Solution

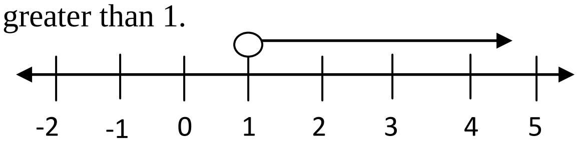
The union of these two sets is the set of numbers less than -1 (excluding -1) and set of numbers greater than 4 (excluding 4)



2. Show the solution set of the compound inequality $x > 3$ or $x > 1$ on a number line.

Solution

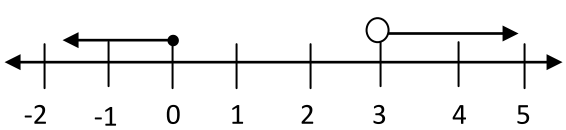
The union of these two sets is the set of numbers greater than 1.



3. Sketch the graph of the solution set of the compound inequality $x \leq 0$ or $x > 3$ on a number line.

Solution

The union of these two sets is the set of numbers less than or equal to 0 (incuding 0) and numbers greater than 3 (excluding 3).



b. Using the connective "and"

The solution set to the compound inequality using the connective "and" is the intersection of the solution sets to each of the simple inequalities.

If A and B are the set of numbers, then the intersection of A and B is the set of all numbers that are in both A and B, denoted as A∩B.

To represent such inequalities on a number line;

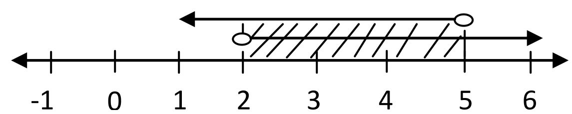
I. graph each inequality separately on the same number line,

II. identify the overlapping interval as the solution set of the compound inequality.

Worked Examples

1. Graph the solution set to the compound inequality $x > 2$ and $x < 5$ and list the intersection.

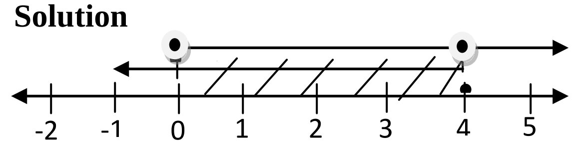
Solution



The intersection of these two solution set is the portion of the number line that is shaded, just the part between 2 and 5, excluding 2 and 5. That is $\{x : x = 3, 4\}$

2. Illustrate the solution set of the compound inequalities $x \leq 4$ and $x \geq 0$ on a number line.

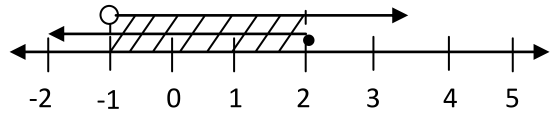
Solution



The intersection of these two solution set is the portion of the number line that is shaded, just the part from 0 to 4, inxcluding 0 and 4. That is $\{x : x = 0,1, 2, 3, 4\}$

3. Illustrate the solution set of the compound inequalities $x \leq 2$ and $x > -1$ on a number line.

Solution



The intersection of these two solution set is the portion of the number line that is shaded, just the part from 0 to 2, excluding -1 but including 2. That is $\{x : x = 0, 1, 2\}$

Exercises 8.14

Illustrate the following on a number line and list the members in each case:

A. 1. $x > -1$ or $x < 3$
2. $x \leq 6$ or $x > -2$
3. $x \geq 2$ or $x \geq 5$
4. $x < 7$ or $x > 0$
5. $x \leq 6$ or $x > 9$
6. $x > 3$ or $x < -3$

B. 1. $x > -1$ and $x < 4$
2. $x \leq 3$ and $x \leq 0$
3. $x > -2$ and $x \leq 4$
4. $x \leq 6$ and $x > 9$
5. $x \geq 4$ and $x \leq -4$
6. $x \geq 6$ and $x \leq 1$

C. 1. $\{x : x > 1\} \cap \{x : x < 4\}$
2. $\{x : x > -3\} \cap \{x : x < 3\}$
3. $\{x : x < 2\} \cap \{x : x > 5\}$
4. $\{x : x^2 > 9\}$
5. $\{x : x^2 < 2\} \cup \{x : x > 5\}$

Solving Compound Inequalities

An inequality may be read from left to right or from right to left. For example, $2 < x$ is read in the usual way as "2 is less than x". Reading from the right, we say "x is greater than 2" . The meaning is clearer when the variable is read first.

For compound inequality, another notation is used. For example, $x > 3$ and $x < 6$ is commonly written as $3 < x < 6$. This is read from the left to right as "3 less than x is less than 6." Reading the variable first, we say "x is greater than 3 and less than 6." So x is between 3 and 6, and reading x makes it more understandable.

Compound inequalities are solved by following the steps below:

Method 1

I. Form two inequalities such that each one contains one of the given inequality.
II. Solve each inequalities separately to obtain the truth set of each.
III. Put the solution sets together as one inequality, using set builder notation.

Method 2: (solving the original equation)

I. Group like terms, by transposition such that the variable factor remains at the middle.
II. Divide through by the coefficient of x.
III. Obtain the solution set in the interval notation.

- Use "**[**" and "**]**" if reading from the left, the first inequality is $\leq$ and the second is $\leq$.
- Use "(" and ")" if reading from the left, the first inequality is < and the second is < .
- Use "[" and ")" if reading from the left, the first inequality is $\leq$ and the second is < or vice versa

Worked Examples

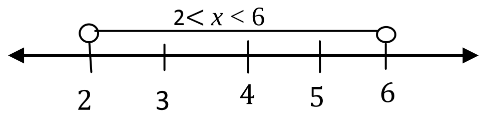
Find the solution set of $5 < 3x - 1 < 17$ and represent your answer ona number line.

Solution

Method 1

$5 < 3x - 1 < 17$
$\Rightarrow 5 < 3x - 1$ and $3x - 1 < 17$
$5 + 1 < 3x$ and $3x < 17 + 1$
$6 < 3x$ and $3x < 18$
$\frac{6}{3} < x$ and $x < \frac{18}{3}$
$2 < x$ and $x < 6$

The solution set is { x : 2 < x < 6}



Method 2

Solving in the original form;

$5 < 3x - 1 < 17$

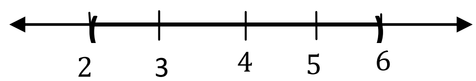
$5 + 1 < 3x < 17 + 1$ (Add 1 to both sides)

$6 < 3x < 18$

$\frac{6}{3} < \frac{3x}{3} < \frac{18}{3}$ (Divide throught by 3)

$2 < x < 6$

The solution set is (2, 6)



2. Solve the inequality $-2 \leq 2x - 3 < 7$ and graph the solution set.

Solution

$-2 \leq 2x - 3 < 7$

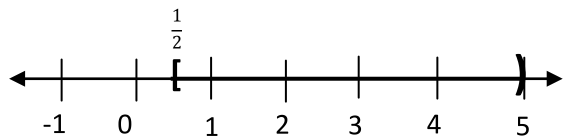
$-2 + 3 \leq 2x - 3 + 3 < 7 + 3$

$1 \leq 2x < 10$

$\frac{1}{2} \leq \frac{2x}{2} < \frac{10}{2}$

$\frac{1}{2} \leq x < 5$

The solution set is [$\frac{1}{2}$, 5)



Exercises 8.15

A. Solve and graph each inequality.

1. $5 < 2x - 3 < 1$
2. $-1 < 5 - 3x \leq 14$
3. $-1 < 3 - 2x < 9$
4. $-2 < 3x + 1 < 10$
5. $-1 \leq 3 - 2x < 11$
6. $3 \leq 3 - 5(x - 3) \leq 8$

B. Solve and grapheach inequality.

1. $2 \leq 4 - \frac{1}{2}(x - 8) \leq 10$
2. $-3 < \frac{2x-1}{3} < 7$
3. $0 \leq \frac{3-2x}{2} < 7$
4. $-2 < \frac{1-3x}{-2} < 7$
5. $-3 < \frac{3m+1}{2} \leq 5$

Meaning of Bearings

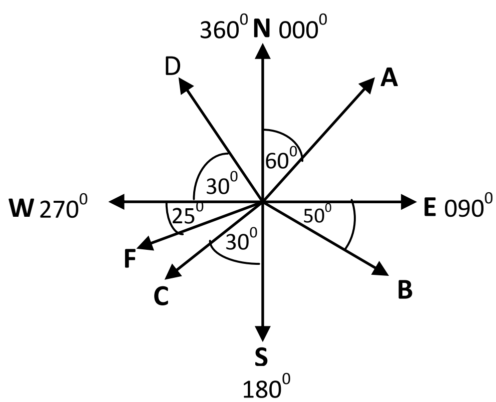
It is the measurement of direction using angles. This is done with the help ofthe cardinal points; ***North, South, East and West.***

There are two ways of measuring the bearing of a point from another point (o).

1. Using the North-pole and measuring in the clock wise direction.
2. Using the North or South Pole only and measuring eastward or westward.

Measuring in the Clockwise Direction from the North - pole

By this method, measurement is taken in the clockwise direction from the North Pole. The bearing is then written in 3 – digits. Thus, angles less than 100 are preceded by zero(s).



From the diagram, the bearings of the point A, B, C, D and F from the center, O, is

A = 060^0 B = $90^0 + 50^0 = 140^0$

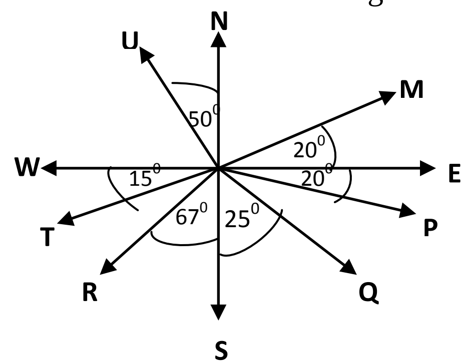
C = $90^0 + 90^0 + 30^0 = 210^0$

D = $90^0 + 90^0 + 90^0 + 30^0 = 300^0$

F = $90^0 + 90^0 + 30^0 + 35^0 = 245^0$

Worked Examples

Find the bearings of M, N, P, Q, R, S, T, U and W from the center O in the diagram below;



NOT DRAWN TO SCALE

Solution

The Bearings from the center, O, are:

M = 070^0

P = $90^0 + 20^0 = 110^0$

Q = $90^0 + 20^0 + 45^0 = 155^0$

R = $90^0 + 90^0 + 67^0 = 247^0$

S = $90^0 + 90^0 = 180^0$

T = $90^0 + 90^0 + 67^0 + 8^0 = 255^0$

U = $90^0 + 90^0 + 90^0 + 40^0 = 310^0$

W = $90^0 + 90^0 + 90^0 = 270^0$

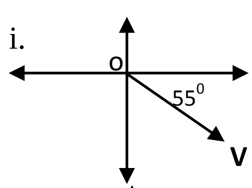
2. Represent the following bearings in a diagram:

i. V = 145^0 ii. K = 075^0 iii. B = 240^0

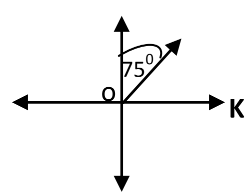
iv. M = 115^0 v. I = 204^0 vi. A = 330^0

Solutions

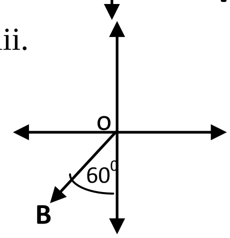
i.



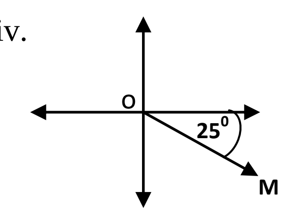
ii.



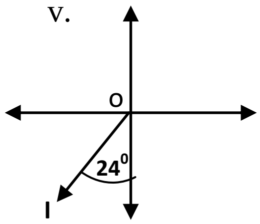
iii.



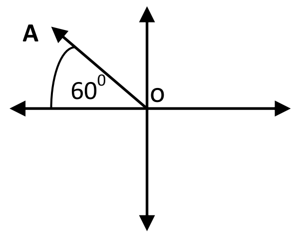
iv.



v.

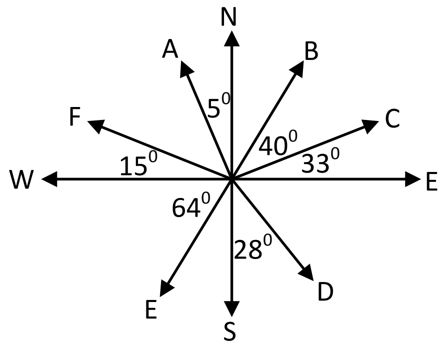


vi.



Exercises 9.1

A. Find the bearings of A, B, C, D, E and F in the diagram below;



B. Represent each bearings on a separate diagram;

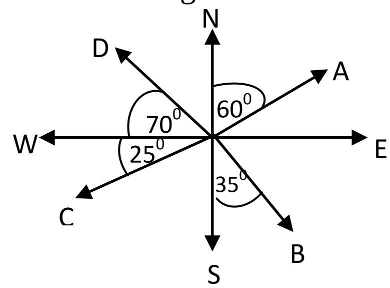
1. A = 213^0 2. D = 203^0 3. B = 122^0
4. E = 290^0 5. C = 048^0 6. F = 155^0

Using the North and South Poles and Measuring Towards the East or West

STEPS:

Here the acute angle formed with the North or South Pole is measured in the Eastward or Westward direction. Thus, measurement is taken in: North – West (N – W), North-East (N – E) and South - East (S− E), South - West (S − W) but not the other way round. The bearing is written by either the North (N) pole or South (S) pole first, followed by the acute angle formed with it and finally, the East (E) or South (S) indications.

Consider the diagram below:

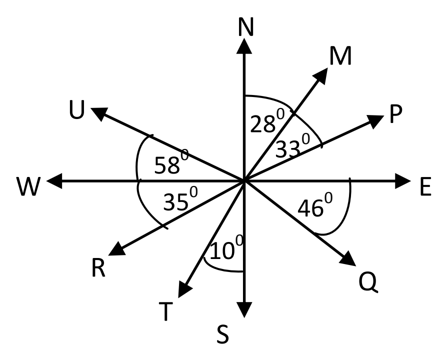


The bearings of A, B, C and D from the center, O, are found by finding the acute angle each of them makes with either the North – pole or South – pole.

1. Since A makes an angle of 60^0 with the North and lies between the North and East, the bearing of A is N 60^0 E.
2. Since B makes an angle of 35^0 with the South Pole and lies between South and East, the bearing of B is S 35^0 E.
3. Similarly, the bearing of C is found by calculating the angle C makes with the South. That is $90^0 – 25^0 = 65^0$. Because C lies between South and West, its bearing is on S 65^0 W.
4. Likewise, the bearing of D is found by finding the angle it forms with the North Pole. That is $90^0 − 70^0 = 20^0$. Because D is located between North and West, it is said to be on a bearing of N 20^0 W.

Worked Examples

A. Find the bearings of M, P, U, Q, R, S, and T from the center O, in the diagram below;



The bearings of M, P, U, Q, R, S, T, from the center O are:

M = N 28^0 E, P = N 61^0 E, U = N 32^0 W

Q = S 44^0 E, T = S 10^0 W, R = S 55^0 W

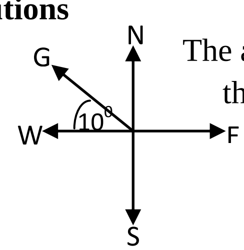
B. Show the following bearings from thecenter O, on a diagram and write the alternative bearings of each;

1. G = 280^0, 2. H =205^0,

3. K = 248^0, 4. L = 055^0,

Solutions

1.



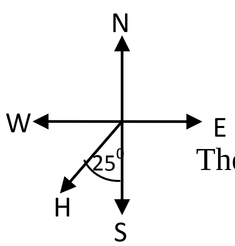
The angle G makes with the North-pole;

= $90^0 - 10^0$

= 80^0.

The alternate bearing of G = N 80^0 W.

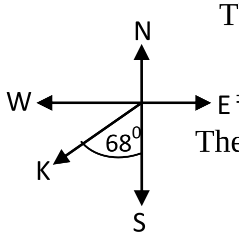
2.



The angle H makes with the South – pole = 25^0.

The alternate bearing of H = S 25^0 W

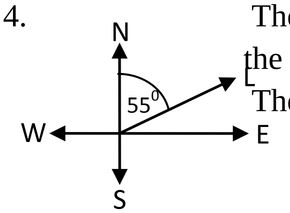
3.



The angle, k makes with the South – pole = 68^0.

The alternate bearing of k = S 68^0 W

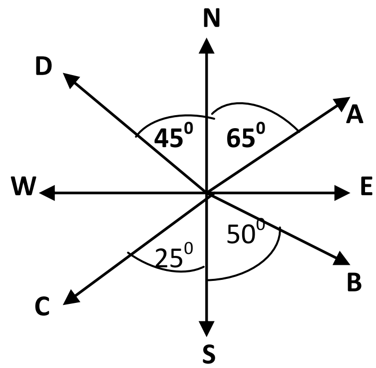
4.



The angle L makes with the North-pole = 55^0.

The alternate bearing of L = N 55^0 E

C. In the diagram below, determine the bearings of A, B, C and D from the center,



Solution

Bearings of A, B, C and D from the center,O, is:

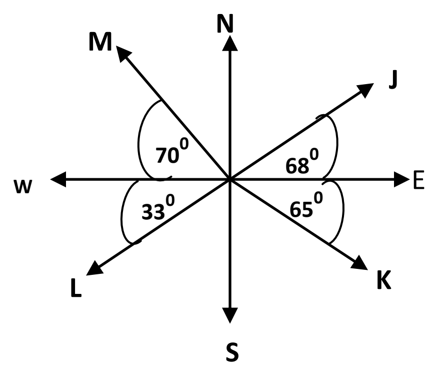
A = North 65^0 East = N 65^0 E

B = South 50^0 East = S 50^0E

C = South 25^0 West = S 25^0W

D = North 45^0 West = N 45^0W

4. Find the bearings of J, K, L and M from the center.



Solution

Bearing of J from O = (90^0– 68^0) = 22^0 from the North- pole towards east = N 22^0 E

Bearing of K from O = (90^0 – 65^0) = 25^0 with the south-pole towards east = S 25^0 E

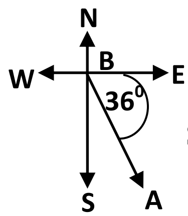
Bearing of L from O = (90^0 – 33^0) = 57^0 with the south-pole towards west = S 57^0 W

Bearing of M from O = (90^0 – 70^0) = 20^0 with the north-pole towards west = N 20^0 W

Some Solved Past Questions

1) The bearing of town A from town B is 126^0. Show this in a diagram.

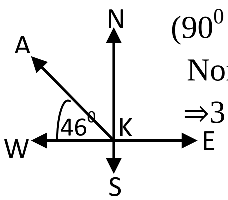
Solution



$(90^0 - 36^0) = 54^0$ with South Pole towards East,
$\Rightarrow 126^0 = S\ 54^0\ E$

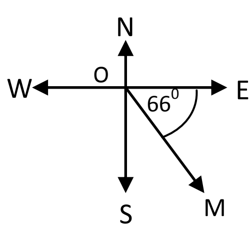
2. The bearing of Asuofua from Kejetia is 316^0. Represent this in a diagram and give an alternative name for this bearing.

Solution



$(90^0 - 46^0) = 44^0$ with North Pole towards West,
$\Rightarrow 316^0 = N\ 44^0\ W$

3. Identify the bearing represented in the diagram below:

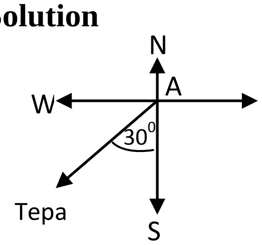


Solution

Bearing of M from O
$= 90^0 + 66^0$
$= 156^0$

4. Tepa is located $S30^0W$ of Abuakwa. Represent this in a diagram and find the bearing of Tepa from Abuakwa.

Solution



Bearing of Tepa from Sunyani;
$= 90^0 + 90^0 + 30^0$
$= 210^0$

Exercises 9.2

A. Write the alternate bearings of the following;

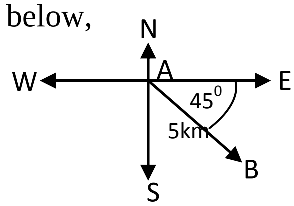
1) $S33^0\ W$ 2) $N25^0\ W$ 3) $N\ 60^0\ W$
4) $N15^0\ E$ 5) $S73^0\ E$ 6) $S\ 44^0\ W$

B. Write the alternate bearings:

1) 230^0 2) 038^0 3) 155^0
4) 105^0 5) 303^0 6) 355^0

Distance Bearing

When bearing is measured in the form ($a$km, b^0), where a is the distance and b^0 is the direction, it is called ***distance bearing.*** It is written with distance, followed by the direction, which is in the form *(distance, direction)*. For example, in the diagram below,

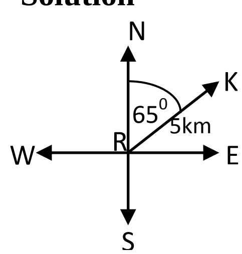


The distance bearing of AB = (5km, 135^0)

Worked Examples

1. A ship sails from port R on a bearing 065^0 to port K in a distance of 5km. Show this in a diagram and write the distance bearing of RK

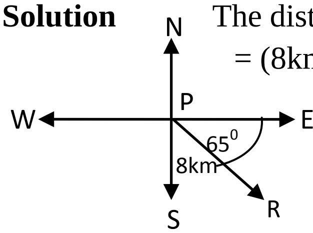
Solution



The distance bearing of RK= (5km, 065^0)

2. A village R is 8km from another village P on a bearing of 155^0 Show this in a diagram and write RP in distance bearing.

Solution



The distance bearing of RP
= (8km, 155^0)

Exercises 9.3

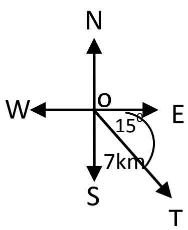
A.1. Represent the following bearings on the same diagram;

i. A = $S20^0$ W, ii. B = $S20^0$ E,

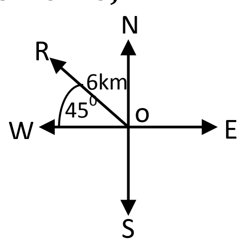
iii. C = $N39^0$ W iv. D = S 53^0 W

B. Write the bearings from o;

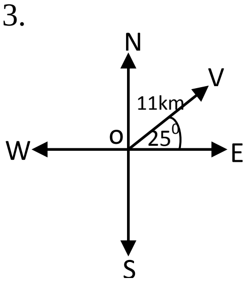
1.



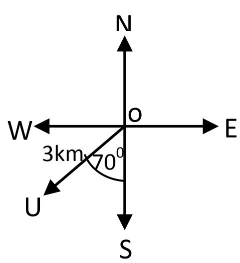
2.



3.

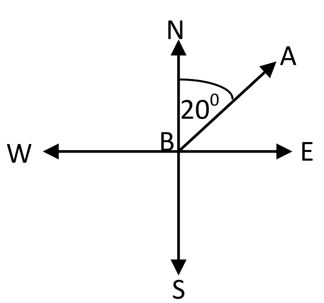


4.

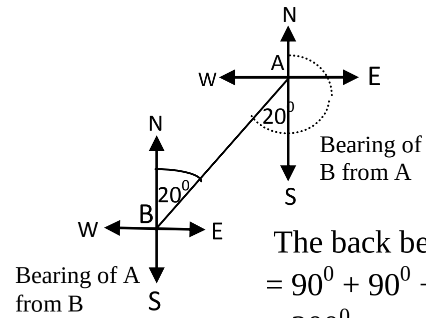


Back Bearings or Opposite Bearings.

The back bearing of say A from B is the bearing of B from A. For example, the bearing of A 20^0 from B is represented in the diagram below;



The back bearing which is the bearing of B from A is found by using A as the turning point on the same diagram and alternating the angle 20^0. Measurement is then taken from the North Pole in the clockwise direction.

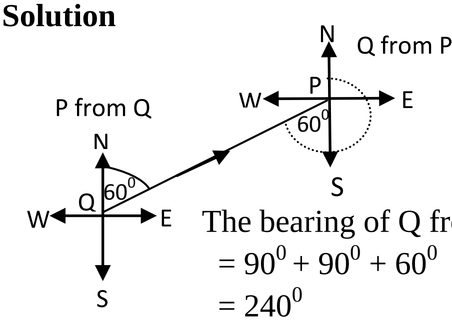


The back bearing;
= $90^0 + 90^0 + 20^0$
= 200^0

Worked Examples

1.The bearing of P from Q is 060^0. Find the bearing of Q from P.

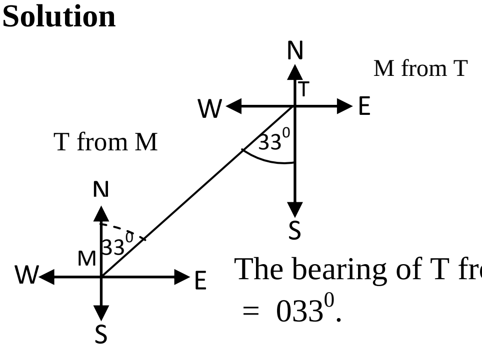
Solution



The bearing of Q from P;
= $90^0 + 90^0 + 60^0$
= 240^0

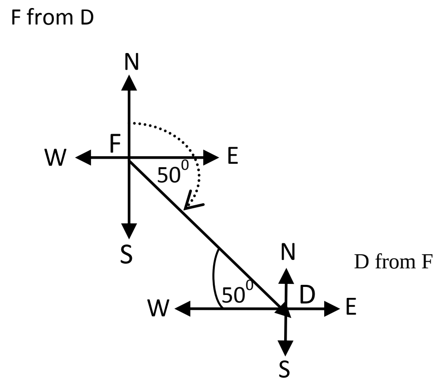
2. The bearing of a point M from T is 213^0. Find the bearing of T from M.

Solution



The bearing of T from M
= 033^0.

3. The bearing of D from F is 320^0. Find the bearing of F from D.



Solution

Bearing of F from D

$= 90^0 + 50^0$

$= 140^0$

NOTE

I. For all bearing less than or equal to 180^0, the back bearing is determine by adding 180^0 to the given bearing (angle). That is : $\theta + 180^0$, where $0 \leq \theta \leq 180^0$. For example, if the bearing of P from Q is 060^0, the bearing of Q from P is: $180^0 + 60^0 = 240^0$

II. For all bearings greater than 180^0, the back-bearing is determined by subtracting 180^0 from the given bearing (angle). That is:

$\theta - 180^0$, where $180^0 < \theta < 360^0$. For e.g., if the bearing of A from B is 205^0, then the bearing of B from A= $205^0 - 180^0 = 025^0$.

Exercises 9.4

A. 1. The bearing of P from Q is 045^0. What is the bearing of Q from P?

2. The bearing of town M from town K is 325^0. Find the bearing of the town K from town M.

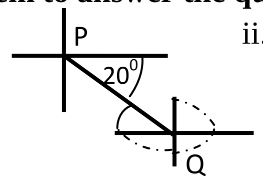
3. What is the bearing of R from T if the bearing of T from R is 300^0?

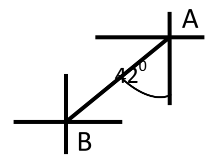
4. The bearing of Asuofua from Kokoben is 125^0, what is the bearing of Kokoben from Asuofua?

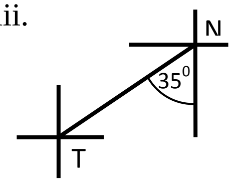
B.What is the back bearing of the following?

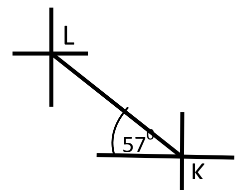
1. Bole from Bamboi = $S63^0E$
2. Konongo fromOdumase = 220^0
3. Keta from Krachie = 315^0
4. Secondi from Takoradi = $S27^0W$.

C. Study the diagrams below carefully and use them to answer the questions that follow;

i. 

ii. 

iii. 

iv. 

i. The back bearing of Q from P is…

ii. The back bearing of B from A is…

iii. The back bearing of T from N is…

iv. The back bearing of L from K is……

Other Applications

Bearings, Pythagoras and Trigonometry;

To solve problems involving bearings, Pythagoras and trigonometry;

I. Make a sketch of the problem.

II. If one of the angles of the triangle formed is a right angle (90^0), use Pythagoras theorem to find the length of the third side given two sides.

III. Knowing the three sides of the triangle, calculate the values of the other two angles in the triangle using any of the trigonometric ratios: *SOH, CAH, TOA*

IV. To find the bearing of say A from B, is to find the total angle turned through in the clockwise direction from the north pole of B to the direction of A.

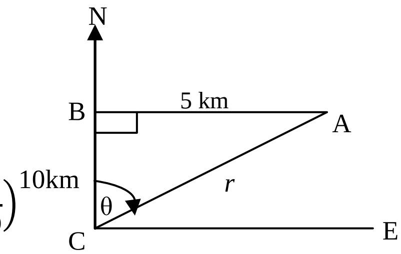
Type 1 (**Worked Examples)**

1. Ata's house is 5 km due east of Bortey's house. If Cudjoe's house is 10 km due south of Bortey house, find, correct to 1decimal place, the bearing and distance of Ata's house from cudjoe's house.

Solution

Let A, B, and C represent the positions of Ata's, Bortey's and Cudjoe's houses respectively.

Let (r, θ) be the position of Ata's house from Cudjoes's.



From ΔABC,

$\tan \theta = \frac{5}{10}$

$\theta = tan^{-1}\left(\frac{5}{10}\right)$

$\theta = 26.57^0$

By Pythagoras theorem,

$r^2 = 10^2 + 5^2$

$r^2 = 100 + 25$

$r^2 = 125$

$r = \sqrt{125} = 11.2$

The bearing of Ata's house from Cudjoes's house = (11.2km, 027^0)

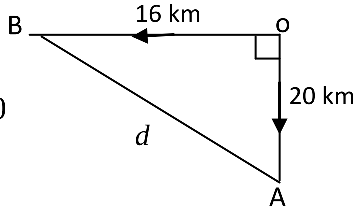
2. Two buses A and B are at a station. If bus A moves 20km due south and bus B moves 16km due west,

i. how far apart are the two buses,

ii. what is the bearing of bus B from bus A.

Solution

i. Let the position of the station be o and the distance between the buses be d



$d^2 = 16^2 + 20^2$

$d^2 = 256 + 400$

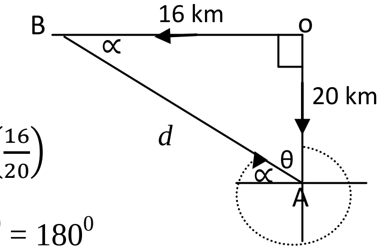
$d^2 = 656$

$d = \sqrt{656}$

$d = 26$ km

ii.

From the Δ



$\tan \theta = \frac{16}{20}$

$\theta = tan^{-1}\left(\frac{16}{20}\right)$

$\theta = 39^0$

$\propto + \theta + 90^0 = 180^0$

$\propto + 39^0 + 90^0 = 180^0$

$\propto = 180^0 - 90^0 - 39^0$

$\propto = 51^0$

Bearing of B from A;

$= 90^0 + 90^0 + 90^0 + \propto$

$= 90^0 + 90^0 + 90^0 + 51^0$

$= 321^0$

3. Salifu walks 500m due north, then 250m due east and finally 500m on a bearing of 055^0,

i. Sketch a diagram to illustrate Salifu's movement;

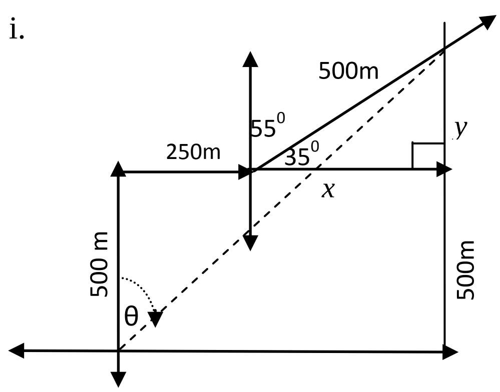
ii. Calculate to the nearest whole number, how far north Salifu has moved from the starting point;

iii. Calculate to the nearest whole number, how far east he has moved from the starting point;

iv. Calculate the bearing of Salifu's final position from the starting point.

Solution

i.



ii. How far north salifu has moved from the starting point = y + 500

But from the diagram,

sin $35^0 = \frac{y}{500}$

y = 500 sin 35^0

y = 287m

287m + 500m = 787m

iii. how far east he has moved from the starting point= x + 250

But from the diagram,

cos $35^0 = \frac{x}{500}$

x = 500 cos 35^0

x = 410m

410m + 250m = 660m

iv. Let the bearing of salifu's final position from the starting point be θ

tan θ $= \frac{660}{787}$

θ $= \tan^{-1}\left(\frac{660}{787}\right)$= 040^0 (Nearest whole number)

Trial Test

1. A man walks 500m due north, and then 350m due west. What is his bearing from the starting points? A = 325^0

2. Baru stands at the center of a football field. He first takes 10 steps north, then 5 steps east and finally 10 steps on a bearing of 045^0

a. how far east is Baru's final position from the center? A = 12 steps

b. how far north is Baru's final position from the center? A = 17 steps

c. Find the bearing and distance of Baru's final position from the center.A= (21s, 035^0)

3. A car is driven 10km on a bearing of 300^0 find how far the cars final position is:

a. north of its initial position,

b. west of its initial position. a = 8.7km, b = 5km

4. An aeroplane flies 150 km due south then 150km on a bearing of 045^0. With respet to its initial position, find:

a. how far south the plane is;

b. how far east the plane is;

c. how far awy and its bearing.

Ans: a. 44km, b. 106km, c. (115km, 115^0)

Type 2

(Two or More Bearings on the Same Diagram)

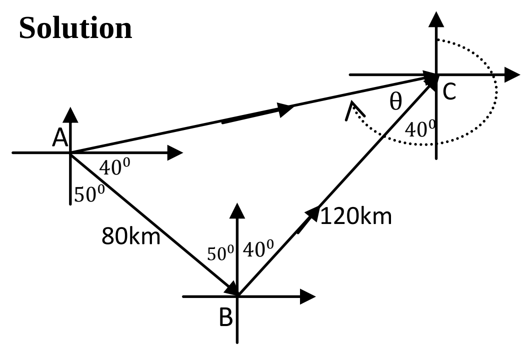
Worked Examples

1. An aircraft flew from port A on a bearing of 130^0 to another port B 80km apart. It then flew from port B on a bearing of 040^0 to port C, a distance of 120 km. Calculate:

i. the distance from port A to port C, to the nearest kilometer

ii. the bearing of port A from port C, to the nearest degree

Solution



From the diagram,

ABC is a right – angled triangle

By Pythagoras theorem,

$|AC|^2 = |AB|^2 + |BC|^2$

$|AC|^2 = 80^2 + 120^2$

$|AC|^2$ = 6400 + 14400

$|AC|^2$ = 20,800

$|AC| = \sqrt{20,800}$

$|AC|$ = 144km (Nearest kilometer)

ii. From the diagram,

tan θ = $\frac{80}{120}$

θ = $\tan^{-1}\left(\frac{80}{120}\right)$ = 34^0(Nearest degree)

∴ The bearing of port A from port B,

= 90^0 + 90^0 + 40^0 + 34^0 = 254^0

2. Three schools, *P*, *Q* and *R* are situated as follows: *Q* is on a bearing of 5km, 143^0 from *P* and *R* is on the bearing 11 km, $053^0$from *P*. Find:

i. the distance between school *Q* and School *R*, to the nearest kilometer,

ii. the bearing of school *Q* from school *R* to the nearest degree,

iii. T is a school situated in – between *Q* and *R,* such that *PT* is perpendicular to *QR*;

a. find the distance of *T* from *R,*

b. how far is school *T* from school *P*, to the nearest kilometer?

Solution

i. *From the diagram,*

PQR is a right – angled triangle

By Pythagoras theorem,

$|QR|^2$ = $|PQ|^2$ + $|PR|^2$

$|QR|^2$ = 5^2 + 11^2

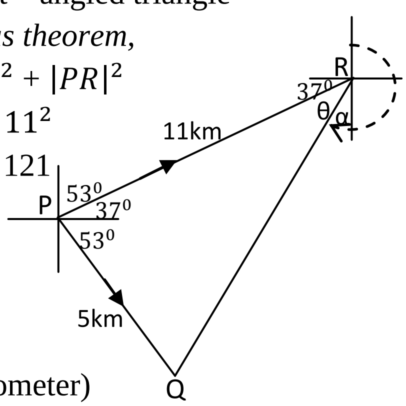
$|QR|^2$ = 25 + 121

$|QR|^2$ = 146

$|QR|$ = $\sqrt{146}$

$|QR|$ = 12km

(Nearest kilometer)



ii. *From the diagram,*

tan θ = $\frac{5}{11}$

θ = $\tan^{-1}\left(\frac{5}{11}\right)$= 24^0 (to the nearest degree)

$37^0 + 24^0 + \alpha = 90^0$

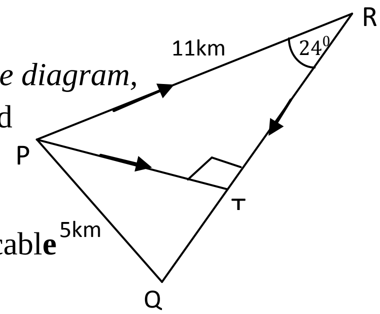
α = 90^0– 61^0 = 29^0

∴ The bearing of school Q from R,

= 90^0 + 90^0 +α

= 90^0 + 90^0 + 29^0

= 209^0



iii. *a. From the diagram,*

H = 11km and

A = *TR*,

CAH is applicable

cos 24^0 = $\frac{A}{H}$

cos 24^0= $\frac{|TR|}{11}$

$|TR|$ = 11 cos 24^0= 10km

∴ The distance of school T from school R is10km

b. By Pythagoras theorem,

$|PR|^2$ = $|PT|^2$ + $|TR|^2$

$|PR|^2$ – $|TR|^2$= $|PT|^2$

11^2 – 10^2 = $|PT|^2$

$|PT|^2$ = 121 – 100

$|PT|^2$ = 21

$|PT|$ = $\sqrt{21}$

$|PT|$ = 5km

∴ School *T* is 5km away from school *B*

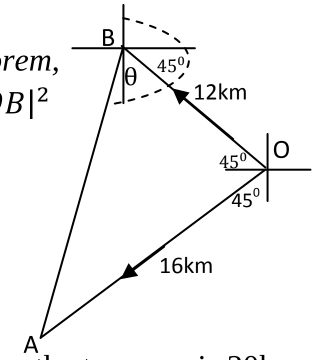
3. Two cars A and B moved from a station at the same time. Car A moved on the bearing 225^0 from the station and Car B moved on a bearing $315^0$from the station. If 3 hours later, the distance of A from the station was 16km and the distance of B from the station was 12km, find;

i. the distance between the two cars after 3 hours

ii. the bearing of car A from car B after 3 hours, to the nearest degree

Solution

From the diagram, Δ AOB is a right – angled triangle



By Pythagoras theorem,

$|AB|^2 = |AO|^2 + |OB|^2$

$|AB|^2 = 12^2 + 16^2$

$|AB|^2 = 144 + 256$

$|AB|^2 = 400$

$|AB| = \sqrt{400}$

$|AB| = 20\text{km}$

The distance between the two cars is 20km

ii. *From the diagram,*

Bearing of A from B = $90^0 + 45^0 + \theta$

$\tan\theta = \left(\frac{16}{12}\right)$

$\theta = \tan^{-1}\left(\frac{16}{12}\right) = 53^0$

Bearing of *A* from *B*;

$= 90^0 + 45^0 + \theta$

$= 90^0 + 45^0 + 53^0$

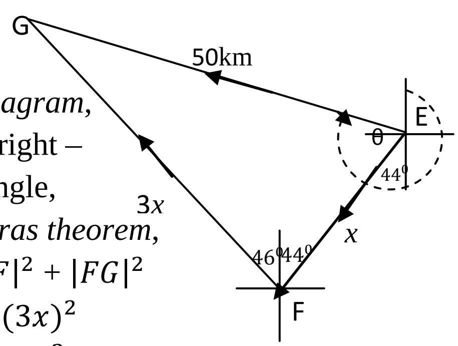
$= 188^0$

4. A ship sailed from port *E* to port *F*, *x* km away on a bearing of 224^0. From there, it sailed to port *G*, three times the distance from *E* to *F* on a bearing of 314^0. If the distance from *E* to *G* is 50km, find to the nearest whole number;

i. the distance from *F* to *G*

ii. the bearing of *G* from *E*

Solution



From the diagram,

Δ EFG is a right – angled triangle,

By Pythagoras theorem,

$|EG|^2 = |EF|^2 + |FG|^2$

$50^2 = x^2 + (3x)^2$

$2500 = x^2 + 9x^2$

$2500 = 10x^2$

$x^2 = 250$

$x = \sqrt{250} = 16\text{km}$

$|FG| = 3x$, but $x = 16$

$|FG| = 3 \times 16 = 48\text{km}$

The distance from F to G is 48km.

ii. *From the diagram,*

$\tan\theta = \frac{O}{A} = \frac{48}{16}$

$\theta = \tan^{-1}\left(\frac{48}{16}\right) = 72^0$

The bearing of G from E,

$= 90^0 + 90^0 + 44^0 + \theta$

$= 90^0 + 90^0 + 44^0 + 72^0 = 296^0$

Exercises 9.5

1. With respect to a fixed point, O, a point P is at (12km, 035^0) and another point Q is at (16km,125^0). Find the bearing and the distance of P from Q.

2. Two buses start off from a ststion, O. The first bus ends its journey at station P(39km, 050^0) from O. If the final stop of the second bus is Q(25km, 320^0) from O, find the position of Q from P.

3. A ship sailed from port *P* on a bearing of 113^0 to port *Q*, 32km apart. It then sailed from port *Q* to another port *R* on a bearing of 023^0, a distance of 54km. Calculate:

i. the distance from port *P* to port *R*, to the nearest kilometer,

ii. the bearing of port *P* from port *R*, to the nearest degree,

4. The location of three schools *A*, *B* and *C* are as follows: *B* is on a bearing of 150^0 from *A* and *C*

is on bearing of 240^0 from *A*.If *A* and *B* are 13*km* apart and *A* and *C* are 19*km* apart, calculate;

i. how far *B* is located from *C*, to the nearest kilometer.;

ii. the bearing of *C* from *B*, to the nearest degree.

5. Two aircrafts, *P* and *Q* flew from the same port at the same time. Craft P flew at 26*km/h* on a bearing of 345^0 and craft *Q* flew at 41*km/h* on a bearing of 075^0. Find after three hours;

i. the distance between the two crafts, to the nearest kilometer.

ii. the bearing of *P* from *Q*, to the nearest degree.

6. a. Bebu village is *x* km from Kejetia on a bearing 336^0. Adumasa village is 4*x* km from Bebu on a bearing 246^0. If the distance from Kejetia to Adumasa is 120km, calculate;

i. the distance from Bebu to Adumasa, to the nearest kilometer.

ii. the bearing of Adumasa from Kejetia, to the nearest degree.

b. Senfi is a village located on Kejetia – Adumasa road, which is perpendicular to Bebu. Find the distance of Senfi from Adumasa to the nearest kilometer.

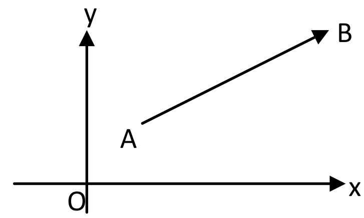
Vectors

A vector is any quantity that has magnitude and direction. This means that its measurement requires two or more numbers. Examples of vector quantities are weight, Velocity and displacement.

On the other hand, any quantity that has magnitude only is called a scalar quantity. Examples of scalar quantities are mass, speed and distance etc.

Representation of Vectors

A free vector is a vector that does not start from the origin as shown below;

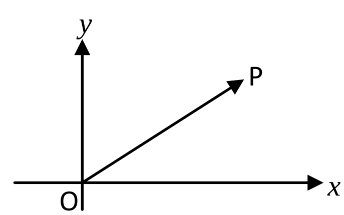


Free vectors are represented by a pair of capital letters with an arrow over their top.

The arrow indicates the direction of movement. Thus, movement from A to B is written as $\overrightarrow{AB}$ and movement from B to A is written as $\overrightarrow{BA}$ or $\overleftarrow{AB}$. These are called ***free vectors***. For example, if $\overrightarrow{AB}$ = $\binom{x}{y}$, then *x* - component is on top and*y* - component is at the bottom.

Position Vectors

It is a vector that has the origin as a starting point.



The vector $\overrightarrow{OP}$ indicates a movement from the origin O, to a pointP. *p*(small letter) is called the position vector of the point OP.Thus, if A = (*x*, *y*), then $\overrightarrow{OA} = a = \binom{x}{y}$.

Similarly, if B is the point (*h*, *k*), then $\overrightarrow{OB}$ represents the position vector *b* of B, and *b* = $\binom{h}{k}$

Column Vector

It is a vector written in vertical form $\binom{x}{y}$. For example, $\overrightarrow{AB}$ = $\binom{3}{2}$ and $\overrightarrow{EF}$ = $\binom{4}{5}$

Inverse or Negative Vector

If $\overrightarrow{AB} = \binom{x}{y}$, then the inverse of $\overrightarrow{AB}$, written as $\overrightarrow{BA} = \binom{-x}{-y}$

Worked Examples

1. If $\overrightarrow{MN} = \binom{3}{5}$. Find the inverse of $\overrightarrow{MN}$.

Solution

$\overrightarrow{NM} = -\binom{3}{5} = \binom{-3}{-5}$.

2. If $\overrightarrow{AB} = \binom{-12}{3}$, find $\overrightarrow{BA}$.

Solution

$\overrightarrow{BA} = -\binom{-12}{3} = \binom{12}{-3}$

Exercises 9.6

Find inverse of the following vectors.

(1) $\overrightarrow{AB} = \binom{3}{2}$ (2) $\overrightarrow{KL} = \binom{-6}{4}$

(3) $\overrightarrow{MN} = \binom{10}{-7}$ (4) $\overrightarrow{BC} = \binom{-1}{-11}$

Equal Vectors

Two or more vectors are said to be equal if their *x* and *y* components are the same. For example, if $\overrightarrow{AB} = \binom{-7}{2}$ and $\overrightarrow{MN} = \binom{-7}{2}$, then $\boldsymbol{AB} = \boldsymbol{MN}$

Worked Examples

1. If $\overrightarrow{AB} = \binom{2}{3}$ and $\overrightarrow{CD} = \binom{2}{3}$, what is the relationship between the two vectors?

Solution

$\overrightarrow{AB} = \overrightarrow{CD} = \binom{2}{3}$. Therefore, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal vectors.

Zero Vector

It is a vector that has no component. This means that both *x* and *y* components are zero.

For example, $\overrightarrow{AB} = \binom{0}{0}$

Perpendicular Vectors

Two vectors *a* and *b* are perpendicular if;

1. The product of their gradients is – 1.
2. *a* is the rotation of *b* or vice – versa through an angle of 90^0 or 270^0clockwise or anti – clockwise about the origin.
3. If their dot product is zero i. e. $a \times b = 0$

Worked Examples

Find a vector that is perpendicular to $a = (8, -4)$

Solution

Method 1

Let the vector perpendicular to $a = (8, -4)$ be $b = (x, y)$.

Then $a \times b = 0$

$\Rightarrow 8x + (-4y) = 0$

$\Rightarrow 8x - 4y = 0$

When $x = 1$ and $y = 2$

$8(1) - 4(2) = 0$

Therefore $b = (1, 2)$ is perpendicular to *a*

When $x = 2$ and $y = 4$

$8(2) - 4(4) = 0$

Therefore $b = (2, 4)$ is perpendicular to *a*.

When $x = -1$ and $y = -2$

$8(-1) - 4(-2) = 0$

Therefore $b = (-1, -2)$ is perpendicular to *a*.

The vectors perpendicular to $a = (8, -4)$ are (1, 2), (2, 4) (-1, -2) .

Method 2

$a = (8, -4)$

Let the vector perpendicular to $a = (8, -4)$ be $b = (x, y)$.

This means that $m_a \times m_b = -1$

But $m_a = \frac{-4}{8}$ and $m_b = \frac{y}{x}$

$\frac{-1}{2} \times \frac{y}{x}$ = -1
$\frac{-1}{2} \times \frac{2}{1}$ = -1
$\Rightarrow \frac{y}{x} = \frac{2}{1}$ so x = 1 and y = 2
The vector perpendicular to a = (8, - 4) is
b = (1, 2)

Exercises 9.7
Find at least one vector that is perpendicular to each of the following:
1. a = (12, 3) 2. b = (2, -5) 3. a = (-6, -7)
4. a = (- 4, 10) 5. a = (-3, -5) 6. a = (8, 2)

Parallel Vectors
Two or more vectors are parallel if they are multiples of each other. For eg, if a = (2, 4),:
⇒-1a = -1(-2, -4)
$\frac{1}{2}a = \frac{1}{2}$ (2, 4) = (1, 2)
2a = 2(2, 4) = (4, 8)
The vectors (2, 4), (-2, -4), (1, 2), (4, 8) are said to be parallel.

Worked Example
State whether the given pair of vectors are parallel or not.
1. $m = \left(-\frac{1}{2}, -4\right)$ $n = (-2, 16)$

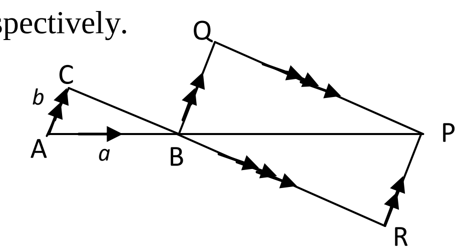
Solution
For two vectors x and y to be parallel, x = ky where k is a scalar.
For y coordinate, - 4 × - 4 = 16 T
For x coordinate, $-\frac{1}{2}$ × - 4 = 2 ≠ - 2 F
Not parallel vectors

Method 2
For two vectors to be parallel, their cross product is equal.
$\binom{-1/2}{-4} = \binom{-2}{16}$
- 4 × - 2 = 8
$-\frac{1}{2}$ × 16 = - 8
8 ≠ - 8
The vectors are not parallel

2. $m = \left(\frac{1}{2}, 4\right)$ $n = (2, 16)$

Solution
For two vectors to be parallel, their cross product is equal
$\binom{1/2}{4} = \binom{2}{16}$
4 × 2 = 8
$\frac{1}{2}$ × 16 = 8
8 ≠ 8
The vectors are parallel

Exercises 9.8
A. Given that a = (6, -9), b = (10, 7), c = (- 4, -12) and d = (8, 1), e = (- 6, 9) and f = (2, 6)
i. Name two vectors that are parallel.
ii. Find any three vectors parallel to vectors a, b and c.

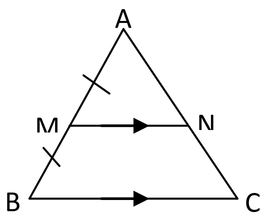
B. 1. A is the point (4, –1), B is (11, 5) and C is (6, 2). Find the coordinates of D, such that: i. $\overrightarrow{CD} = 2\overrightarrow{AB}$ ii. $\overrightarrow{CD} = 2\overrightarrow{BA}$
2. In the figure below, BQ and RP are parallel; QP and BR are parallel. $\overrightarrow{BQ} = 2\overrightarrow{AC}$ and $\overrightarrow{BP} = 2\overrightarrow{AB}$. $\overrightarrow{AB}$ and $\overrightarrow{AC}$ represent vector ***a*** and ***b*** respectively.


i. Express in terms of a and b, the vector represented by:

a. $\overrightarrow{BC}$ b. $\overrightarrow{BQ}$ c. $\overrightarrow{BP}$ d. $\overrightarrow{BR}$

ii. Why are C, B, R collinear?

2. In the figure below, M is the midpoint of AB, and MN//BC. $\overrightarrow{AB}$ represents vector ***c***, and $\overrightarrow{AC}$ represents vector ***b***



i. Show that $\overrightarrow{MN}$ represents $\frac{1}{2}(b-c)$

ii. What can you say about N?

Magnitude or Length of a Vector

If $\overrightarrow{AB} = \binom{x}{y}$, then the magnitude or length of $\overrightarrow{AB}$ is written as: $/\boldsymbol{\overrightarrow{AB}}/ = \sqrt{\boldsymbol{x^2+y^2}}$

Worked Examples

1. If $\overrightarrow{MN} = \binom{12}{5}$, find the length of $\overrightarrow{MN}$.

Solution

$|\overrightarrow{MN}| = \sqrt{12^2+5^2}$

$= \sqrt{144+25} = \sqrt{169}$ = 13units

2. Find $|\overrightarrow{CD}|$ if $\overrightarrow{CD} = \binom{-6}{-2}$

Solution

$|\overrightarrow{CD}| = \sqrt{-6^2+-2^2} = \sqrt{36+4} = \sqrt{40}$ units

3. Find the length of $\overrightarrow{CD}$ if $\overrightarrow{CD} = \binom{-3}{-4}$

Solution

$|\overrightarrow{CD}| = \sqrt{-3^2+-4^2}$

$= \sqrt{9+16} = \sqrt{25}$ = 5 units

4. What is the magnitude of $\overrightarrow{PR} = \binom{-4}{0}$?

Solution

$|\overrightarrow{PR}| = \sqrt{(-4)^2+0^2}$

$= \sqrt{16+0} = \sqrt{16} =$ 4 units

5. If $x = \binom{-2}{4}$ and $y = \binom{-3}{1}$, find correct to one decimal place /*x* + *y*/.

Solution

$x = \binom{-2}{4}$ and $y = \binom{-3}{1}$,

$x + y = \binom{-2}{4} + \binom{-3}{1} = \binom{-5}{5}$

$/x+y/ = \sqrt{5^2+(-5)^2}$

$= \sqrt{25+25} = \sqrt{50}$ = 7.1(1 d. p)

Exercises 9.9

Find the length of the following vectors;

1. $\overrightarrow{AB} = \binom{0}{6}$ 2. $\overrightarrow{MN} = \binom{5}{6}$ 3. $\overrightarrow{PR} = \binom{4}{7}$
4. $\overrightarrow{QR} = \binom{-5}{0}$ 5. $\overrightarrow{ST} = \binom{8}{-3}$ 6. $\overrightarrow{UV} = \binom{8}{6}$

Challenge Problem

ABCD is a square of side 1 unit. $\overrightarrow{AB}$ represents vector ***u,*** $\overrightarrow{BC}$ represents ***v***. Name line segments representing **– u, – v,** ***u*** **+** ***v*** and ***u*** **–** ***v*** **.** Calculate $|\boldsymbol{u}| + |\boldsymbol{v}|$ and $|\boldsymbol{u}+\boldsymbol{v}|$

Scalar Multiplication

It is the act of multiplying a number by the *x* and *y* components of a vector.

If $a = \binom{x}{y}$, then $k \times a = \binom{K \times x}{K \times y}$ also written as***ka*** **=** $\boldsymbol{\binom{Kx}{Ky}}$**.*k*** is called ***a scalar*** and ***ka***is called ***a scalarmultiplication.***

Worked Examples

1. Given that *a* = (3, 2) and *b* = (-4, 3). Find:

(i) 3*a* (ii) -5*b*

Solution

(i) $3a = \begin{pmatrix} 3\times 3 \\ 3\times 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$

(ii) $-5b = \begin{pmatrix} -5\times -4 \\ -5\times 3 \end{pmatrix} = \begin{pmatrix} 20 \\ -15 \end{pmatrix}$

2. Given that $c = (4, 7)$ and $d = (3, 6)$ Find:
a. $2c$ b. $3d$ c. $2d$ d. $-4d$

Solutions

(a) $2c = \begin{pmatrix} 2\times 4 \\ 2\times 7 \end{pmatrix} = \begin{pmatrix} 8 \\ 14 \end{pmatrix}$

(b) $3d = \begin{pmatrix} 3\times 3 \\ 3\times 6 \end{pmatrix} = \begin{pmatrix} 9 \\ 18 \end{pmatrix}$

(c) $2d = \begin{pmatrix} 2\times 3 \\ 2\times 6 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$

(d) $-4d = \begin{pmatrix} -4\times 3 \\ -4\times 6 \end{pmatrix} = \begin{pmatrix} -12 \\ -24 \end{pmatrix}$

Exercises 9.10

If $p = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, $t = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$, $r = \begin{pmatrix} -7 \\ 4 \end{pmatrix}$ and $u = \begin{pmatrix} -6 \\ -10 \end{pmatrix}$, find:

1) $-2r$ 2) $-\frac{1}{2}p$ 3) $-3t$ 4) $\frac{1}{3}t$

Addition of Vectors

Two or more vectors can be added by adding their respective x and y components. That is, if $A = (x_1, y_1)$ and $B = (x_2, y_2)$, then :

$\Rightarrow \overrightarrow{OA} + \overrightarrow{OB} = a + b = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$

Worked Examples

(1) If $m = \begin{pmatrix} -6 \\ 10 \end{pmatrix}$ and $n = \begin{pmatrix} -7 \\ -4 \end{pmatrix}$. find $m + n$.

Solution

$m + n = \begin{pmatrix} -6 \\ 10 \end{pmatrix} + \begin{pmatrix} -7 \\ -4 \end{pmatrix} = \begin{pmatrix} -6 \; -7 \\ 10 - 4 \end{pmatrix} = \begin{pmatrix} -13 \\ 6 \end{pmatrix}$

(2) Find the sum of s and t, if $s = \begin{pmatrix} -12 \\ 15 \end{pmatrix}$ and $t = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$

Solution

$s + t = \begin{pmatrix} -12 \\ 15 \end{pmatrix} + \begin{pmatrix} 0 \\ -5 \end{pmatrix} = \begin{pmatrix} -12 + 0 \\ 15 + -5 \end{pmatrix} = \begin{pmatrix} -12 \\ 10 \end{pmatrix}$

5. If $r = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $s = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, calculate, $6(r + 2s)$

Solution

$6(r + 2s)$

$= 6[\begin{pmatrix} 3 \\ 1 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 1 \end{pmatrix}]$

$= 6[\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -4 \\ 2 \end{pmatrix}]$

$= 6[\begin{pmatrix} 3-4 \\ 1+2 \end{pmatrix}]$

$= 6\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6\times -1 \\ 6\times 3 \end{pmatrix} = \begin{pmatrix} -6 \\ 18 \end{pmatrix}$

6. Given that $u = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$, find:
$\frac{1}{3}(u + \frac{1}{2}v)$.

Solution

$\frac{1}{3}(u + \frac{1}{2}v)$

$= \frac{1}{3}[\begin{pmatrix} -2 \\ 3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ 6 \end{pmatrix}]$

$= \frac{1}{3}[\begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2/2 \\ 6/2 \end{pmatrix}]$

$= \frac{1}{3}[\begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}]$

$= \frac{1}{3}[\begin{pmatrix} -2+1 \\ 3+3 \end{pmatrix}]$

$= \frac{1}{3}[\begin{pmatrix} -1 \\ 6 \end{pmatrix}] = \begin{pmatrix} -1/3 \\ 6/3 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 2 \end{pmatrix}$

Exercises 9.11

Given that $a = \begin{pmatrix} 3 \\ 12 \end{pmatrix}$, $b = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, $c = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$, find:

1. $\frac{1}{3}a + c$ 2. $2a + 3b$ 3. $c + b$
4. $c + 4b$ 5. $4a + 2c$ 6. $\frac{1}{2}b + \frac{1}{3}a$

Subtraction of Vectors

The difference between two or more vectors is found by subtracting the respective x and y components of the vector. That is: if $A = (x_1, y_1)$ and $B = (x_2, y_2)$ then:

$\overrightarrow{OA} - \overrightarrow{OB} = a - b = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}$

Worked Examples

(1) If $a = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, find $a - b$ and $b - a$.

Solution

$a - b = \binom{5}{7} - \binom{3}{4} = \binom{5-3}{7-4} = \binom{2}{3}$

$b - a = \binom{3}{4} - \binom{5}{7} = \binom{3-5}{4-7} = \binom{-2}{-3}$

2. If $x = \binom{-3}{2}$ and $y = \binom{4}{-1}$, find $3x - y$.

Solution

$x = \binom{-3}{2}$ and $y = \binom{4}{-1}$

$3x - y = 3\binom{-3}{2} - \binom{4}{-1}$

$= \binom{-9}{6} - \binom{4}{-1} = \binom{-9-4}{6+1} = \binom{-13}{7}$

3. If $r = \binom{2}{-1}$ and s $= \binom{-3}{2}$, calculate $2r - 3s$.

Solution

$r = \binom{2}{-1}$ and $s = \binom{-3}{2}$

$2r - 3s = 2\binom{2}{-1} - 3\binom{-3}{2}$

$= \binom{4}{-2} - \binom{-9}{-6} = \binom{4+9}{-2+6} = \binom{13}{4}$

4. Simplify; $\frac{1}{3}\binom{3}{6} + \frac{1}{2}\binom{2}{-4} + \frac{1}{4}\binom{-8}{0}$

Solution

$\frac{\mathbf{1}}{\mathbf{3}}\binom{3}{6} + \frac{\mathbf{1}}{\mathbf{2}}\binom{2}{-4} + \frac{\mathbf{1}}{\mathbf{4}}\binom{-8}{0}$

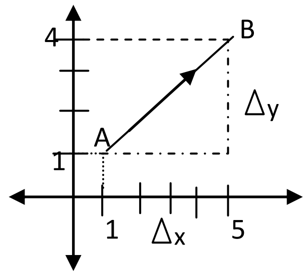
$= \begin{pmatrix} 3 \times 1/3 \\ 6 \times 1/3 \end{pmatrix} + \begin{pmatrix} 2 \times 1/2 \\ -4 \times 1/2 \end{pmatrix} + \begin{pmatrix} -8 \times 1/4 \\ 0 \times 1/4 \end{pmatrix}$

$= \binom{1}{2} + \binom{1}{-2} + \binom{-2}{0} = \binom{0}{0}$

Exercises 9.12

1. If $u = \binom{1}{3}$ and $v = \binom{4}{-1}$, find $2u + 3v$.
2. If $a = \binom{2}{4}$ and $b = \binom{-3}{2}$, find $2a - b$.
3. If $r = \binom{-2}{3}$ and s $= \binom{-4}{-1}$, find $r + 2s$
4. Find $p + 2q + r$, if $p = \binom{4}{5}$, $q = \binom{0}{-2}$ and $r = \binom{-3}{7}$,
5. Given the vectors $r = \binom{3}{-5}$, $p = \binom{-7}{-5}$ and $q = 2p - r$, find q.

Relating Free Vectors and Position Vectors



$\overrightarrow{AB} = \binom{5}{4} - \binom{1}{1}$

$= \binom{5-1}{4-1}$

$= \binom{4}{3}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{AB} = b - a$

For all free vectors such as $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AC}$, $\overrightarrow{CB}$ etc, the following generalizations can be made:

(i) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ (ii) $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

(iii) $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$ (iv) $\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC}$

Worked Examples

Given that $a = (3, 6)$, $b = (-7, 2)$, $c = (2, 5)$ and $d = \binom{-4}{-9}$. Find:

(1) $\overrightarrow{AC}$ (2) $\overrightarrow{AB}$ (3) $\overrightarrow{AD}$

(4) $\overrightarrow{CD}$ (5) $\overrightarrow{BD} + \overrightarrow{BC}$

Solution

(1) $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= \binom{2}{5} - \binom{3}{6} = \binom{2-3}{5-6} = \binom{-1}{-1}$

(2) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \binom{-7}{2} - \binom{3}{6} = \binom{-7-3}{2-6} = \binom{-10}{-4}$

(3) $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$= \binom{-4}{-9} - \binom{3}{6} = \binom{-4-3}{-9-6} = \binom{-7}{-15}$

(4) $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$= \binom{-4}{-9} - \binom{2}{5} = \binom{-4-2}{-9-5} = \binom{-6}{-14}$

(5) $\overrightarrow{BD} + \overrightarrow{BC} = \overrightarrow{OD} - \overrightarrow{OB} + \overrightarrow{OC} - \overrightarrow{OB}$

$= \binom{-4}{-9} - \binom{-7}{2} + \binom{2}{5} - \binom{-7}{2}$

$= \binom{3}{-11} + \binom{9}{3} = \binom{12}{-8}$

Exercises 9.13

1. Find $\overrightarrow{PQ}$ for the points P(-2, 3), Q(4, 7)

2. A is (2, 5), B is (6, 12), C is (-3, 10) and D is (8, 19). Convert in component forms the vectors: $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{DA}$ and $\overrightarrow{DB}$

3. Given P (-7, 1), Q (-3, 6), R (7, 2) and S (3, -3)

i. Find $\overrightarrow{PS}$ and $\overrightarrow{QR}$.

ii. Calculate the lengths of $\overrightarrow{PS}$ and $\overrightarrow{PQ}$.

4. The point A is (2, 3) and $\overrightarrow{AB}$ is$\binom{4}{-9}$. What are the co-ordinates of B?

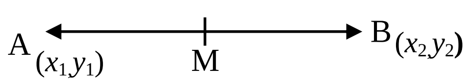
5. If P, Q and R are the points (2, 1), (-2, 3) and (1, 4) respectively, find the vector:

$\overrightarrow{OP} - \overrightarrow{PQ} + \overrightarrow{QR}$

Mid – Point of a Vector

If M is the midpoint of AB, then;

$\boldsymbol{m} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$. If A is the point ($x_1$, y_1) and B is the point (x_2,y_2) in the OXY plane, then the midpoint, *m*, of AB follows that *MA* = *MB*.



$$\mathbf{M} = [\tfrac{1}{2}(\boldsymbol{x_1} + \boldsymbol{x_2}), \tfrac{1}{2}(\boldsymbol{y_1} + \boldsymbol{y_2})]$$

In summary, use any of the following methods to find the midpoint of a line.

1. $\mathbf{M} = [\frac{1}{2}(\boldsymbol{x_1} + \boldsymbol{x_2}), \frac{1}{2}(\boldsymbol{y_1} + \boldsymbol{y_2})]$

2. $\mathbf{M} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

Worked Examples

If A (3, 4) and B (7, 12) are points in the OXY plane, find, C, the midpoint of AB.

Solution

Method 1

A (3, 4) and B = (7, 12)

M = $[\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)]$

M = $[\frac{1}{2}(3 + 7), \frac{1}{2}(4 + 12)]$

M = $[\frac{1}{2}(10), \frac{1}{2}(16)]$

M = $(5, 8)$

Therefore the mid – point is (5, 8)

Method 2

A (3, 4) and B = (7, 12)

M = $\frac{1}{2}(a + b)$

M = $\frac{1}{2}[\binom{3}{4} + \binom{7}{12}] = \frac{1}{2}\binom{10}{16} = \begin{pmatrix} 1/2 \times 10 \\ 1/2 \times 16 \end{pmatrix} = \binom{5}{8}$

Exercises 9.14

1. Given that M = (6, 5), N = (14, 11) and P = (2, 3), find C, the mid points of:

i. $\overrightarrow{MN}$ ii. $\overrightarrow{PN}$ iii. $\overrightarrow{PM}$

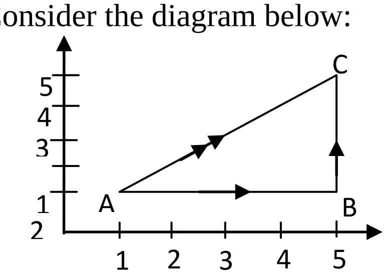
2. O is the origin, A is (3, –1) and B is (5, 5). Given that $\overrightarrow{OC} = \overrightarrow{AB}$,

i. find the coordinates of C.

ii. find also the midpoint of AC and the length of AC.

The Triangular Law of Vectors

Consider the diagram below:



From the diagram, $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$. $\boldsymbol{\overrightarrow{AC}}$ is said to be the ***resultant vector of*** $\boldsymbol{\overrightarrow{AB}}$ and $\boldsymbol{\overrightarrow{BC}}$.

$\Rightarrow \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

Similarly, if $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$ then, $\overrightarrow{PR}$ is the resultant vector of $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.

Worked Examples

1. Given that $\overrightarrow{AB} = \binom{8}{5}$ and $\overrightarrow{BC} = \binom{3}{5}$, find the resultant vector.

Solution

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= \binom{8}{5} + \binom{3}{5} = \binom{11}{10}$

2. If $\overrightarrow{PQ} = \binom{10}{9}$ and $\overrightarrow{QR} = \binom{6}{4}$. Find $\overrightarrow{PR}$.

Solution

$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$

$= \binom{10}{9} + \binom{6}{4} = \binom{16}{13}$

3. Find $\overrightarrow{MR}$ if $\overrightarrow{MN} = \binom{16}{8}$ and $\overrightarrow{NR} = \binom{-11}{-5}$

Solution

$\overrightarrow{MR} = \overrightarrow{MN} + \overrightarrow{NR}$

$\overrightarrow{MR} = \binom{16}{8} + \binom{-11}{-5} = \binom{5}{3}$

4. If $\overrightarrow{AB} = \binom{5}{4}$ and $\overrightarrow{AC} = \binom{12}{9}$, find $\overrightarrow{BC}$

Solution

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$\therefore \overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$

$\therefore \overrightarrow{BC} = \binom{12}{9} - \binom{5}{4} = \binom{12-5}{9-4} = \binom{7}{5}$

5. P, Q, R and T are points in the Cartesian plane. The coordinates of P and Q are (4, 1) and (-3, 2) respectively. $\overrightarrow{PT} = \binom{-1}{2}$, $\overrightarrow{QR} = \binom{2}{-3}$. Find $\overrightarrow{TR}$

Solution

P(4, 1), Q(-3, 2), $\overrightarrow{PT} = \binom{-1}{2}$ and $\overrightarrow{QR} = \binom{2}{-3}$

$\overrightarrow{TR} = r - t$

But $\overrightarrow{QR} = r - q$, *by substitution*

$\binom{2}{-3} = r - \binom{-3}{2}$

$\binom{2}{-3} + \binom{-3}{2} = r$

$\binom{2-3}{-3+2} = r$

$\binom{-1}{-1} = r$

$\overrightarrow{PT} = t - p$, *by substitution*

$\binom{-1}{2} = t - \binom{4}{1}$

$\binom{-1}{2} + \binom{4}{1} = t$

$\binom{-1+4}{2+1} = t$

$\binom{3}{3} = t$

$\overrightarrow{TR} = r - t$, but $r = \binom{-1}{-1}$ and $t = \binom{3}{3}$

$\overrightarrow{TR} = \binom{-1}{-1} - \binom{3}{3} = \binom{-4}{-4}$

6. ABCD is a parallelogram, with vertices A(x, y), B(5, 7), C (4, 3) and D (1, 2).

i. Find $\overrightarrow{AB}$ and $\overrightarrow{DC}$ and hence find the values of x and y.

ii. Calculate the magnitude of $\overrightarrow{AC}$.

Solution

i. $\overrightarrow{AB} = b - a$

$\overrightarrow{AB} = \binom{5}{7} - \binom{x}{y} = \binom{5-x}{7-y}$

$\overrightarrow{DC} = c - d$

$\overrightarrow{DC} = \binom{4}{3} - \binom{1}{2} = \binom{4-1}{3-2} = \binom{3}{1}$

For parallelogram $ABCD$, $\overrightarrow{AB} = \overrightarrow{DC}$

$\binom{5-x}{7-y} = \binom{3}{1}$

$\Rightarrow 5 - x = 3$ and $7 - y = 1$

$5 - 3 = x$ and $7 - 1 = y$

$x = 2$ and $y = 6$

ii. A (2, 6) and C (4, 3)

$\overrightarrow{AC} = c - a$

$\overrightarrow{AC} = \binom{4}{3} - \binom{2}{6} = \binom{2}{-3}$

$/\overrightarrow{AC}/ = \sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13} = 3.6$ units

7. The vertices of a triangle *ABC* are A(1, -3), B(7, 5) and C(-3, 5)

i. Express $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ as column vectors.

ii. Show that the triangle *ABC* is isosceles.

iii. Find the coordinate of the midpoint of $\overrightarrow{AC}$

iv. Find the equation of $\overrightarrow{AB}$

Solution

i. A(1, -3), $a = \binom{1}{-3}$, B(7, 5), $b = \binom{7}{5}$and C(-3, 5), $c = \binom{-3}{5}$

$\overrightarrow{AB} = b - a = \binom{7}{5} - \binom{1}{-3} = \binom{6}{8}$

$\overrightarrow{BC} = c - b = \binom{-3}{5} - \binom{7}{5} = \binom{-10}{0}$

$\overrightarrow{AC} = c - a = \binom{-3}{5} - \binom{1}{-3} = \binom{-4}{-8}$

ii. To show that triangle *ABC* is isosceles find; $/\overrightarrow{AB}/$, $/\overrightarrow{BC}/$ and $\overrightarrow{AC}$

$/\overrightarrow{AB}/ = \sqrt{6^2 + 8^2} = \sqrt{36+64} = 10$ units

$/\overrightarrow{BC}/ = \sqrt{10^2 + 0^2} = \sqrt{100+0} = 10$ units

$/\overrightarrow{AC}/ = \sqrt{(-4)^2 + (-8)^2} = \sqrt{16+64} = 8.9$ units

Since $/\overrightarrow{AB}/ = /\overrightarrow{BC}/$, ΔABC is isosceles

Some Solved Past Questions

1. P(6, 4), Q(-2, -2) and R(4, -6) are the vertices of triangle *PQR*.

i. Determine the coordinates of M and S, the mid points of $\overline{PQ}$ and $\overline{PR}$ respectively.

ii. Find $\overrightarrow{QR}$ and $\overrightarrow{MS}$.

iii. State the relationship between $\overrightarrow{QR}$ and $\overrightarrow{MS}$.

iv. Find the equation of $\overline{MS}$.

Solution

i. P(6, 4), Q(-2, -2) and R(4, -6)

$M = \frac{1}{2}(p + q)$

$M = \frac{1}{2}\left[\binom{6}{4} + \binom{-2}{-2}\right]$

$M = \frac{1}{2}\binom{4}{2} = \frac{1}{2}\begin{pmatrix}1/2 \times 4\\ 1/2 \times 2\end{pmatrix} = \binom{2}{1}$

$S = \frac{1}{2}(p + r)$

$S = \frac{1}{2}\left[\binom{6}{4} + \binom{4}{-6}\right]$

$S = \frac{1}{2}\binom{10}{-2} = \begin{pmatrix}1/2 \times 10\\ 1/2 \times -2\end{pmatrix} = \binom{5}{-1}$

ii. $\overrightarrow{QR} = r - q$

$\overrightarrow{QR} = \binom{4}{-6} - \binom{-2}{-2} = \binom{6}{-4}$

$\overrightarrow{MS} = s - m$

$\overrightarrow{MS} = \binom{5}{-1} - \binom{2}{1} = \binom{3}{-2}$

iii. $\overrightarrow{QR} = \binom{6}{-4}$ and $\overrightarrow{MS} = \binom{3}{-2}$

$2\overrightarrow{MS} = \overrightarrow{QR}$

$\overrightarrow{QR}$ and $\overrightarrow{MS}$ are parallel vectors

iv. Equation of $\overline{MS}$

$M = \binom{2}{1}$ and $S = \binom{5}{-1}$.

Gradient of $\overline{MS} = \frac{-1-1}{5-2} = -\frac{2}{3}$

$y - y_1 = m(x - x_1)$

$(x_1, y_1) = (2, 1)$

$y - 1 = -\frac{2}{3}(x - 2)$

$3(y - 1) = -2(x - 2)$

$3y - 3 = -2x + 4$

$2x + 3y - 3 - 4 = 0$

$2x + 3y - 7 = 0$

The equation of $\overline{MS}$ is $2x + 3y - 7 = 0$

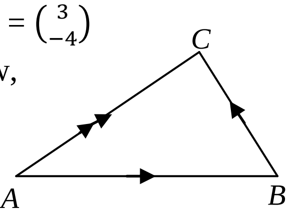
2. In ΔABC, $\overrightarrow{AB} = \binom{-2}{6}$ and $\overrightarrow{AC} = \binom{3}{-4}$. If M is

the mid point of $\overrightarrow{AB}$, express $\overrightarrow{CM}$ as a column vector.

Solution

$\overrightarrow{AB} = \begin{pmatrix}-2\\6\end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix}3\\-4\end{pmatrix}$

From triangular law,



$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$

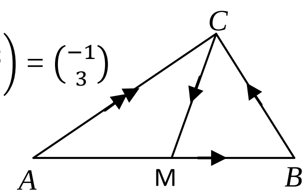
$\overrightarrow{BC} = \begin{pmatrix}3\\-4\end{pmatrix} - \begin{pmatrix}-2\\6\end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix}3+2\\-4-6\end{pmatrix} = \begin{pmatrix}5\\-10\end{pmatrix}$

If M is the mid point of $\overrightarrow{AB}$,

M = $\frac{1}{2}\overrightarrow{AB}$

M = $\frac{1}{2}\begin{pmatrix}-2\\6\end{pmatrix} = \begin{pmatrix}-2/2\\6/2\end{pmatrix} = \begin{pmatrix}-1\\3\end{pmatrix}$



From ΔMBC,

$\overrightarrow{MB} + \overrightarrow{BC} = \overrightarrow{CM}$

$\begin{pmatrix}-1\\3\end{pmatrix} + \begin{pmatrix}5\\-10\end{pmatrix} = \overrightarrow{CM}$

$\overrightarrow{CM} = \begin{pmatrix}4\\-7\end{pmatrix}$

Exercises 9.15

1. Given that $\overrightarrow{AB} = \begin{pmatrix}8\\6\end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix}7\\6\end{pmatrix}$

Find: (i) $\overrightarrow{CA}$ (ii) $\frac{1}{3}\overrightarrow{AC}$ (iii) $|\frac{1}{3}\overrightarrow{AC}|$

2. If A = (-6, -9), B = (8, 5) and C = (0, 3), find $\overrightarrow{AB} + \overrightarrow{BC}$.

3. O is the origin and *A* and *B* are the points (4, 5) and (–1, 3) respectively.

i. Express $2a - 3b$ in component form.

ii. What are the components of $\overrightarrow{AB}$?

iii. What are the components of $\overrightarrow{OP}$, where P is the midpoint of $\overrightarrow{AB}$?

iv. Calculate the length of AB, leaving your answer in surd form.

4. A and B are the points (–1, 1) and (3, 4) respectively,

i. write down the components of the position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

ii. write down the components of the vector represented by $\overrightarrow{\mathbf{AB}}$.

iii. calculate the magnitude of $\boldsymbol{b} - \boldsymbol{a}$

5. *A* (4, 7) is the vertex of triangle *ABC.*

$\overrightarrow{BA} = \begin{pmatrix}5\\3\end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix}4\\-3\end{pmatrix}$.

a. find the co-ordinates of *B* and *C.*

b. if *M* is the midpoint of the line $\overrightarrow{BC}$, find $\overrightarrow{AM}$.

6. *A* (-2, 3), *B* (2, -1), *C* (5, 0) and *D* (*x*, *y*) are the vertices of the parallelogram *ABCD.*

a. Find $\overrightarrow{AB}$ and $\overrightarrow{DC}$. Hence find the coordinates *D.*

b. Calculate, correct to one decimal place, $|\overrightarrow{DB}|$.

7. The points *P*, *Q*, *R*, and *S* are vertices of parallelogram in the Cartesian plane. The co-ordinates of *P* and *R* are (-8, 2) and (5, -2) respectively and $\overrightarrow{QR} = \begin{pmatrix}3\\-4\end{pmatrix}$. Find:

i. the co-ordinates of *Q* and *S*,

ii. the magnitude of *PR.*

8. a. *A*, *B*, *C* and *D* are four points such that *A* (-3, 2), *C* (6, 3), $\overrightarrow{AB} = \begin{pmatrix}5\\4\end{pmatrix}$ and $\overrightarrow{CD} = \begin{pmatrix}-5\\-4\end{pmatrix}$.

Calculate:

i. the coordinates of *B* and *D*,

ii. the vectors $\overrightarrow{BC}$ and $\overrightarrow{AD}$,

b. what is the relationship between $\overrightarrow{BC}$ and $\overrightarrow{AD}$?

9. The vertices of a triangle are *P* (1,-3), Q(7, 5) *R* (-3, 5).

i. Express $\overrightarrow{PQ}, \overrightarrow{QR}$ and $\overrightarrow{PR}$ as column vectors.

ii. Show that triangle *PQR* is isosceles.

10. *PQRS* is quadrilateral with *P* (2, 2), *S* (4, 4) and *R* (6, 4). If $\overrightarrow{PQ} = 4\overrightarrow{SR}$, find the coordinates of *Q*.

Challenge Problem

1. P(-1, -2), Q(5, *k*), R(8, 2) and S(*h*, 1) are the four vertices of the parallelogram PQRS. Find the values of *h* and *k*.

2. P(-1, 5), Q(-2, 1), R(3, -2) and S(*a*, *b*)are the four vertices of a parallelogram, with PQ parallel to RS. Find two pairs of coordinates of the point S.

3. A rhombus *ABCD* has *A* at (0, 2) and *B* at (5, 3). If the diagonals intersect at (0, 3), find the co-ordinates of *C* and *D*.

4. In ΔABC, $\overrightarrow{AB} = \binom{-4}{6}$ and $\overrightarrow{AC} = \binom{3}{-8}$. If P is the mid point of $\overrightarrow{AB}$, express $\overrightarrow{CP}$ as a column vector. Ans $\binom{-5}{11}$

Vector Equality

Two or more vectors are said to be equal if the *x* and *y* components are of the same value. That is if A = $\binom{x}{y}$ and B = $\binom{x}{y}$, then A and B are equal vectors written as A = B.

Worked Examples

1. Given that A = $\binom{x}{y}$, B = $\binom{3}{1}$ and C =$\binom{6}{5}$, find the values of *x* and *y* if *A* + *B* = *C*.

Solution

A + B = C

$\binom{x}{y} + \binom{3}{1} = \binom{6}{5}$

$\binom{x+3}{y+1} = \binom{6}{5}$

$\binom{x}{y} = \binom{6-3}{5-1}$

$\binom{x}{y} = \binom{3}{4}$

2. If M = $\binom{7}{2}$, N = $\binom{a}{b}$ and R= $\binom{10}{12}$, find the values of *a* and *b* if M – N = R.

Solution

M – N = R

$\binom{7}{2} - \binom{a}{b} = \binom{10}{12}$

$-\binom{a}{b} = \binom{10}{12} - \binom{7}{2}$

$-\binom{a}{b} = \binom{10-7}{12-2}$

$-\binom{a}{b} = \binom{3}{10}$

$\binom{a}{b} = -\binom{3}{10}$

$\binom{a}{b} = \binom{-3}{-10}$,

∴ *a* = -3 and *b* = -10

3. Given that $\binom{x+15}{y+17} = \binom{22}{18}$, find the values of *x* and *y*.

Solution

$\binom{x+15}{y+17} = \binom{22}{18}$

$\binom{x}{y} = \binom{22-15}{18-17}$

$\binom{x}{y} = \binom{7}{1}$

∴ *x* = 7 and *y* = 1

4. If $\binom{13}{4} + \binom{m}{10} = \binom{20}{n}$. Find *m* and *n*.

Solution

$\binom{13}{4} + \binom{m}{10} = \binom{20}{n}$

$\binom{13+m}{4+10} = \binom{20}{n}$

$\binom{13+m}{14} = \binom{20}{n}$

⇒13 + *m* = 20

m = 20 –13 and *n* = 14

m = 7 and *n* = 14

Solved Past Question

The vectors *a* = $\binom{-1}{4}$, *b* = $\binom{x}{y}$ and *c* = $\binom{3}{2}$are in the same plane. If 3*a* – 2*b* = *c*, find:

i. the vector b

ii. $|\mathbf{d}|$ and express your answer in the form $p\sqrt{q}$, where p and q are integers and $d = b - c$.

Solution

i. $a = \binom{-1}{4}$, $b = \binom{x}{y}$, $c = \binom{3}{2}$

$3\binom{-1}{4} - 2\binom{x}{y} = \binom{3}{2}$

$\binom{-3}{12} - \binom{2x}{2y} = \binom{3}{2}$

$\binom{-3}{12} - \binom{3}{2} = \binom{2x}{2y}$

$\binom{-6}{10} = \binom{2x}{2y}$

$\Rightarrow 2x = -6$ and $2y = 10$

$x = \frac{-6}{2}$ and $y = \frac{10}{2}$

$x = -3$ and $y = 5$

$b = \binom{x}{y} = \binom{-3}{5}$

ii. $d = b - c$

$d = \binom{-3}{5} - \binom{3}{2} = \binom{-6}{3}$

$|\text{d}| = \sqrt{(-6)^2 + 3^2} = \sqrt{36 + 9} = 3\sqrt{5}$ units

Exercises 9.16

A. 1. If $A = \binom{x\ +\ 3}{y\ -\ 5}$ and $B = \binom{10}{2}$, find the values of x and y, if A is equal to B.

2. If the vectors $\binom{V\ -\ 5}{6} = \binom{13}{U\ -\ 2}$. Find the values of V and U.

3. Find the values of a and b, if $A = \binom{6\ +\ a}{12\ +\ b}$ and $B = \binom{15}{10}$ are equal.

4. Find x and y from the vector equation $\binom{x}{2x} + \binom{2y}{-3y} = \binom{1}{9}$

5. Find the values of x and y in the equation, $\binom{x+3}{2} - \binom{y}{x+y} = \binom{2}{-1}$

6. If $p = \binom{4}{-3}$ and $q = \binom{-5}{7}$,

i. calculate $|q - p|$, correct to one d. p,

ii. find the vector r such that $p + r = 2q$.

The Values of Scalars in Equal Vectors

Given a vector equation with two different scalars, such as $m\binom{x}{y} + n\binom{p}{q} = \binom{a}{b}$, where m and n are scalars, the values of the scalars (m and n) are found as follows:

I. Expand the given equation

II. Equate the x component of the L.H.S of the equation to the x – component of R.H.S. and name it as eqn (1)

III. Similarly, equate the y component of the L.H.S of the equation to the y – component of R.H.S. and name it as eqn (2)

IV. Solve the pair of equations obtained simultaneously to obtain the values of the scalars

Worked Examples

If $a = \binom{1}{-1}$, $b = \binom{2}{3}$ and $c = \binom{-4}{-11}$, find the scalars m and n such that $c = ma + nb$, where m and n are scalars.

Solution

$a = \binom{1}{-1}$, $b = \binom{2}{3}$ and $c = \binom{-4}{-11}$

$c = m\,a + n\,b$

$\binom{-4}{-11} = m\binom{1}{-1} + n\binom{2}{3}$

$\binom{-4}{-11} = \binom{m}{-m} + \binom{2n}{3n}$

$\Rightarrow -4 = m + 2n$……………(1)

$-11 = -m + 3n$……………(2)

eqn (1) + eqn(2)

$-15 = 5n$
$n = -3$

Put $n = -3$ in eqn (1)
$-4 = m + 2(-3)$
$-4 = m - 6$
$-4 + 6 = m$
$2 = m$
$m = 2$
$(m, n) = (2, -3)$

Exercises 9.17

1.If $a = \binom{1}{-3}$, $b = \binom{2}{3}$ and $c = \binom{8}{3}$, find scalars p and q such that $c = pa + qb$,

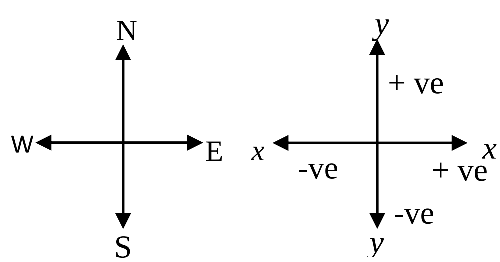
2. If $a = \binom{2}{3}$, $b = \binom{4}{7}$ and $c = \binom{8}{11}$, find:
i. m and n such that $c = ma + nb$, where m and n are scalars.
ii. $|\mathbf{d}|$ if $d = c - 2a$

3. The vectors $p = \binom{\mathbf{2}}{\mathbf{3}}$, $q = \binom{\mathbf{2}}{\mathbf{5}}$ and $r = \frac{1}{2}(q - p)$
i. find the vector r.
ii. if $mp + nq = \binom{\mathbf{4}}{\mathbf{3}}$, find m and n, where m and n are scalars.

4. If $p = \binom{2}{-1}$, $q=\binom{1}{3}$ and $r= \binom{5}{-6}$, find:
a. m and n, such that $r = mp + nq$, where m and n are scalars.
b. find $|g|$, if $g = 3q + r$

Cartesian Form of a Vector

Comparing the cardinal points to the OXY, plane,



Comparing the two planes, it can be seen that the North – pole represents y – axis positive (+ ve), the South Pole represents y – axis negative (–) whilst the West represents x – axis negative (–) and the East side represents x – axis positive (+). That is; (N = + y), (S = - y), (W = - x) and (E = + x). Therefore in the Cartesian form, the following generalizations can be made:

I. If $\overrightarrow{AB} = \binom{+x}{+y}$, then $\overrightarrow{AB}$ can be written in Cartesian form as $\overrightarrow{AB} = \binom{E}{N}$,

II. If $\overrightarrow{AB} = \binom{-x}{+y}$, then $\overrightarrow{AB}$ can be written in Cartesian form as $\overrightarrow{AB} = \binom{W}{N}$,

iii. If $\overrightarrow{AB} = \binom{-x}{-y}$, then $\overrightarrow{AB}$ can be written as Cartesian form $\overrightarrow{AB} = \binom{W}{S}$,

iv. If $\overrightarrow{AB} = \binom{+x}{-y}$, then $\overrightarrow{AB}$ can be written as

Cartesian form as $\overrightarrow{AB} = \binom{E}{S}$,

Note: Negative and positive signs are not used to represent vectors in the Cartesian form. For example, $\overrightarrow{AB} = \binom{-x}{-y} \neq \binom{-W}{-S}$

Worked Examples

Write the following vectors in Cartesian form.

1. $\overrightarrow{ST} = \binom{3}{-9}$ 2. $\overrightarrow{PQ} = \binom{-4}{-3}$

3. $\overrightarrow{MB} = \binom{-8}{7}$ 4. $\overrightarrow{AB} = \binom{5}{6}$

Solutions

1. $\overrightarrow{ST} = \binom{3}{-9}$ is represented in the $x - y$ plane as $\binom{+x}{-y}$, which is also equivalent to $\binom{E}{S}$ in the Cartesian form. Therefore:

1. $\overrightarrow{ST} = \binom{3}{-9} = \binom{3E}{9S}$ 2. $\overrightarrow{PQ} = \binom{-4}{-3} = \binom{4\,W}{3\,S}$

3. $\overrightarrow{MB} = \binom{-8}{7} = \binom{8\,W}{7\,N}$ 4. $\overrightarrow{AB} = \binom{5}{6} = \binom{5E}{6N}$

Exercises 9.18

A. Express the vectors in Cartesian form:

1. $\overrightarrow{PQ} = \binom{8}{10}$ 2. $\overrightarrow{AB} = \binom{5}{-2}$

3. $\overrightarrow{MN} = \binom{-6}{12}$ 4. $\overrightarrow{KL} = \binom{-9}{-3}$

B. Express the vectors in the form $\binom{a}{b}$ where *a* and *b* are rational numbers

1. $\binom{7m\ W}{4m\ N}$ 2. $\binom{2m\ E}{9m\ S}$ 3. $\binom{5m\ W}{5m\ S}$

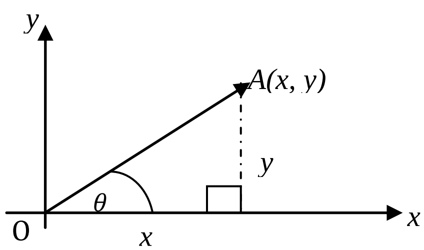
4. $\binom{3m\ E}{6\ m\ N}$ 5. $\binom{12km\ E}{19km\ S}$ 6. $\binom{0km\ W}{7km\ S}$

Angle of a Position Vector with the x - axis

To find the angle a position vector makes with the x – axis:

I. Write the vector in the form $\overrightarrow{OA} = a = \binom{x}{y}$

II. Make a sketch of the vector in the OXY plane as shown below:



III. Calculate the angle θ (the angle the vector makes with the x – axis, using the ratio:

a. $\tan\theta = \frac{Opposite}{Adjacent} = \frac{y}{x}$, b. $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

IV. Avoid negative signs in the calculations because they indicate the quadrant within which the vector falls

Worked Examples

Find the angle the following vectors make with the x – axis to the nearest degree;

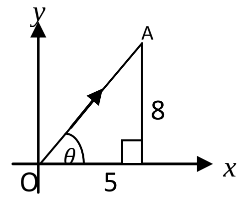
1. $a = \binom{5}{8}$ 2. $b = \binom{-7}{2}$ 3. $c = \binom{4}{-9}$ 4. $d = \binom{-3}{-11}$

Solution

Let θ be the angle the vector makes with the x – axis

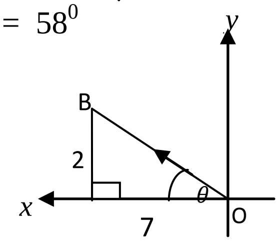
1. $a = \binom{5}{8}$

$\tan\theta = \frac{8}{5}$



$\theta = \tan^{-1}\left(\frac{8}{5}\right) = 58^0$

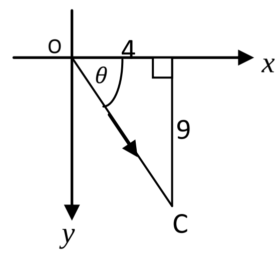
2. $b = \binom{-7}{2}$



Tan θ $= \frac{2}{7}$

$\theta = \tan^{-1}\left(\frac{2}{7}\right)$

$\theta = 16^0$



3. $c = \binom{4}{-9}$

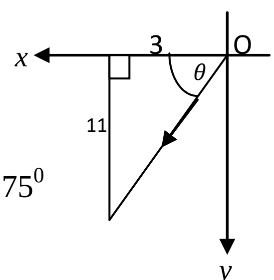
$\tan\theta = \frac{9}{4}$

$\tan^{-1}\left(\frac{9}{4}\right) = 66^0$

4. $d = \binom{-3}{-11}$

tan $= \frac{11}{3}$

$\tan^{-1}\left(\frac{11}{3}\right) = 75^0$



Exercises 9.19

Calculate the angle each of the following vectors makes with the x – axis:

1. $a = \binom{5}{11}$ 2. $b = \binom{7}{4}$ 3. $c = \binom{-6}{-10}$

4. $u = \binom{-9}{2}$ 5. $v = \binom{-3}{8}$ 6. $w = \binom{-12}{-7}$

Magnitude – Bearing Vectors as Column or Component Vectors [*a* units, b^0 to $\binom{x}{y}$]

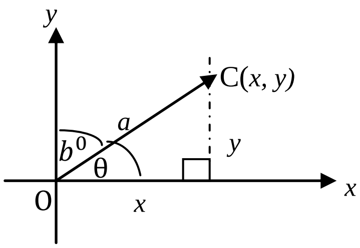
A vector written in the form ***(a units, b^0)*** is called ***a magnitude – bearing vector*** because ***a*** is the magnitude of the vector and b^0 is the bearing of the vector. Such vectors can be written in the form $\binom{\boldsymbol{x}}{\boldsymbol{y}}$, called ***column vector.***

Method 1

Reminder : SOH, CAH, TOA

Given $\overrightarrow{OC} = (a,\ b^0)$ to be written in the column or component form $\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}$:

I. Make a sketch of the vector in the OXY plane as shown below;



II. Let the angle the vector makes with the x – axis be θ. From the diagram,

$\cos\theta = \frac{x}{a}$

$x = a\cos\theta$ ………………..(1)

$\sin\theta = \frac{y}{a}$

$y = a\sin\theta$ ………………..(2)

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\ Cos\ \theta \\ a\ Sin\ \theta \end{pmatrix}$

Note:

1. If $0^0 < b \leq 90^0$, then $(a,\ b^0)$ falls in the first quadrant.

$\therefore$ The angle the vector makes with the x – axis. $\theta = 90^0 - b^0 \Rightarrow (a,\ b^0) = \begin{pmatrix} a\ Cos\ \theta \\ a\ Sin\ \theta \end{pmatrix}$

2. If $90^0 < b \leq 180^0$, then $(a,\ b^0)$ falls in the fourth quadrant.

$\therefore$ The angle the vector makes with the x – axis, $\theta = b^0 - 90^0 \Rightarrow (a,\ b^0) = \begin{pmatrix} a\ Cos\ \theta \\ -a\ Sin\ \theta \end{pmatrix}$

3. If $180^0 < b \leq 270^0$, then $(a,\ b^0)$ falls in the third quadrant.

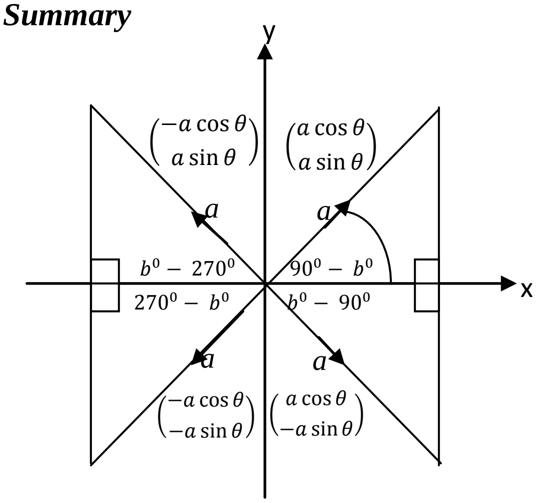
$\therefore$ The angle the vector makes with the x – axis, $\theta = 270^0 - b^0 \Rightarrow (a,\ b^0) = \begin{pmatrix} -a\ Cos\ \theta \\ -a\ Sin\ \theta \end{pmatrix}$

4. If $270^0 < b \leq 360^0$, then $(a,\ b^0)$ falls in the second quadrant.

$\therefore$ The angle the vector makes with the x – axis, $\theta = b^0 - 270^0 \Rightarrow (a,\ b^0) = \begin{pmatrix} -a\ Cos\ \theta \\ a\ Sin\ \theta \end{pmatrix}$

5. When the bearing is given in cardinal point such as N b^0E, N b^0W, S b^0W and S b^0 E, the angle between the points, b^0, is always formed with the vertical axis (y – axis). The angle formed with the x – axis, $\theta = 90^0 - b^0$

Summary



Worked Examples

Express the following as column vectors:

1. $\overrightarrow{OA}$ = (5km, 025^0)   2. $\overrightarrow{OB}$ = (7km, 126^0)

3. $\overrightarrow{OC}$ = 10km, 237^0)   4. $\overrightarrow{OD}$ = 3km, 310^0)

5. $\overrightarrow{OE} = (6km\ ,\ S23^0W)$

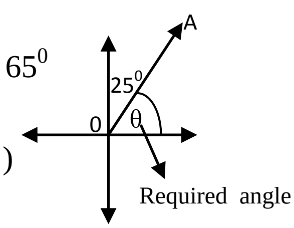
Solution

1. $\overrightarrow{OA} = (5km,\ 025^0)$ lies in the first quadrant.

Let a = 5km and $b = 025^0$

The angle the vector makes with the x – axis, $\theta = 90^0 - b^0$

$\theta = 90^0 - 25^0 = 65^0$



$\overrightarrow{OA}$ = (5km, 025^0)

$= \begin{pmatrix} a\ Cos\ \theta \\ a\ Sin\ \theta \end{pmatrix}$

$= \begin{pmatrix} 5\ Cos\ 65^0 \\ 5\ Sin\ 65^0 \end{pmatrix} = \begin{pmatrix} 2.1131 \\ 4.5354 \end{pmatrix}$

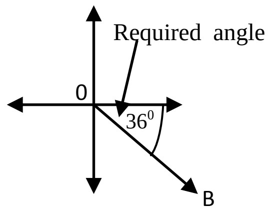
2. $\overrightarrow{OB} = (\ 7km,\ 126^0)$ lies in the fourth quadrant. Let a = 7km and $b = 75^0$

The angle the vector makes with the x – axis,

$= b^0 - 90^0$

$\theta = 126^0 - 90^0$

$\theta = 36^0$



$\overrightarrow{OB} = (\ 7km,\ 126^0)$

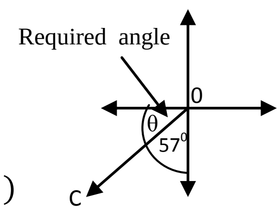
$= \begin{pmatrix} a\ Cos\ \theta \\ -a\ Sin\ \theta \end{pmatrix} = \begin{pmatrix} 7\ Cos\ 36^0 \\ -7\ Sin\ 36^0 \end{pmatrix} = \begin{pmatrix} 5.663 \\ -4.115 \end{pmatrix}$

3. $\overrightarrow{OC} = (\ 10km,\ 237^0)$ lies in the third quadrant. Let a = 10km and $b = 237^0$

The angle the vector makes with the x – axis, $\theta = 270^0 - b^0$

$\theta = 270^0 - 237^0$

$\theta = 33^0$



$\overrightarrow{OC} = (10km,\ 237^0)$

$= \begin{pmatrix} -a\ Cos\ \theta \\ -a\ Sin\ \theta \end{pmatrix} = \begin{pmatrix} -10\ Cos\ 33^0 \\ -10\ Sin\ 33^0 \end{pmatrix} = \begin{pmatrix} -8.387 \\ -5.446 \end{pmatrix}$

4. $\overrightarrow{OD} = (\ 3km,\ 310^0)$ lies in the second

quadrant. Let a = 3km and $b = 310^0$

The angle the vector makes with the x – axis,

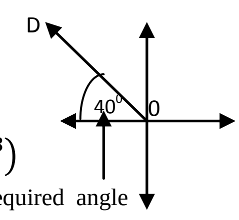
$\theta = b^0 - 270^0$

$\theta = 310^0 - 270^0 = 40^0$

$\overrightarrow{OD} = (3km, 310^0)$

$= \begin{pmatrix} -a\ Cos\ \theta \\ a\ Sin\ \theta \end{pmatrix}$

$= \begin{pmatrix} -3\ Cos\ 40^0 \\ 3\ Sin\ 40^0 \end{pmatrix} = \begin{pmatrix} -2.298 \\ 1.923 \end{pmatrix}$



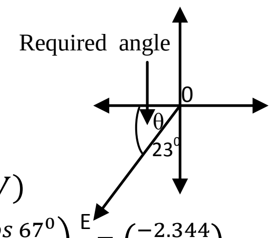
5. $\overrightarrow{OE} = (6km\ ,\ S23^0W)$ lies in the third quadrant. Let a = 6km and $b = 23^0$

The angle the vector makes with the x – axis,

$\theta = 90^0 - b^0$

$\theta = 90^0 - 23^0$

$\theta = 67^0$



$\overrightarrow{OE} = (6km, S23^0W)$

$= \begin{pmatrix} -a\ Cos\ \theta \\ -a\ Sin\ \theta \end{pmatrix} = \begin{pmatrix} -6\ Cos\ 67^0 \\ -6\ Sin\ 67 \end{pmatrix} = \begin{pmatrix} -2.344 \\ -5.523 \end{pmatrix}$

Method 2

This method also makes use of the fact that given $\overrightarrow{OA} = (\ a,\ b^0) = \begin{pmatrix} a\ Cos\ \theta \\ a\ Sin\ \theta \end{pmatrix}$, where a is the magnitude of the vector and θ is the angle the vector makes with the x – axis in the anti – clockwise direction. Simply put;

I. For all given bearings less than 90^0,

i.e $90^0 < b < 90^0$, $\theta = 90^0 - b^0$

II. For all given bearings greater than 90^0,

i.e $90^0 < b < 360^0$, $\theta = 450^0 - b^0$

With this method, the quadrant where the vector lies is not necessary. It shows up after the vector is expressed in the form: $\begin{pmatrix} x \\ y \end{pmatrix}$

Note:

When the bearing is given in the form of cardinal points such as:

a. N b^0 E, then b = $b^0 < 90^0$

$\Rightarrow \theta = 90^0 - b^0$

b. N b^0W, then $b = (90^0 - b^0) + 270^0 > 90^0$

$\Rightarrow \theta = 450^0 - (90^0 - b^0) + 270^0$

c. S b^0 W, then $b = 180 + b^0 > 90^0$

$\Rightarrow \theta = 450^0 - (180 + b^0)$

d. S b^0 E, then $b = (90^0 - b^0) + 90^0 > 90^0$

$\Rightarrow \theta = 450^0 - (90^0 - b^0) + 90^0$

Worked Examples

Express the following as column vectors:

1. $\overrightarrow{OA} = (\ 5km,\ 025^0)$

2. $\overrightarrow{OB} = (7km\ ,\ 126^0)$
3. $\overrightarrow{OC} = (\ 10km,\ 237^0)$
4. $\overrightarrow{OD} = (3km\ ,\ 310^0)$
5. $\overrightarrow{OE} = (6km\ ,\ S23^0W)$

Solution

1. $\overrightarrow{OA} = (\ 5km,\ 025^0)$
Let a = 5km and $b = 25^0 < 90^0$
$\Theta = 90^0 - b^0$
$\theta = 90^0 - 25^0 = 65^0$

$\overrightarrow{OA} = (\ 5km,\ 025^0)$
$= \begin{pmatrix} 5\ Cos\ 65^0 \\ 5\ Sin\ 65^0 \end{pmatrix} = \begin{pmatrix} 2.1131 \\ 4.0958 \end{pmatrix}$

2. $\overrightarrow{OB} = (\ 7km,\ 126^0)$
Let a = 7km and $b = 126^0 > 90^0$
$\theta = 450^0 - b^0$
$\theta = 450^0 - 126^0 = 324^0$

$\overrightarrow{OB} = (7km,\ 126^0)$
$= \begin{pmatrix} 7\ \ Cos\ 324^0 \\ 7\ Sin\ 324^0 \end{pmatrix} = \begin{pmatrix} 5.663 \\ -4.115 \end{pmatrix}$

3. $\overrightarrow{OC} = (10km,\ 237^0)$

Let a = 10km and $b = 237^0 > 90^0$
$\theta = 450^0 - b^0$
$\theta = 450^0 - 237^0 = 213^0$

$\overrightarrow{OC} = (10km, 237^0)$
$= \begin{pmatrix} 10\ Cos\ 213^0 \\ 10\ Sin\ 213^0 \end{pmatrix} = \begin{pmatrix} -8.387 \\ -5.446 \end{pmatrix}$

4. $\overrightarrow{OD} = (3km\ ,\ 310^0)$
Let a = 3km and $b = 310^0 > 90^0$
$\theta = 450^0 - b^0$
$\theta = 450^0 - 310^0 = 140^0$

$\overrightarrow{OD} = (3km, 310^0)$
$= \begin{pmatrix} 3\ Cos\ 140^0 \\ 3\ Sin\ 140^0 \end{pmatrix} = \begin{pmatrix} -2.298 \\ 1.928 \end{pmatrix}$

5. $\overrightarrow{OE} = (6km\ ,\ S23^0W)$
Given S b^0 W, $b = 180 + b^0 > 90^0$
$\theta = 450^0 - (180 + b^0)$
For S 23^0 W, $b = 180^0 + 23^0$
$= 180^0 > 90^0$
$\theta = 450^0 - 203^0 = 247^0$

$\overrightarrow{OE} = (6km, \text{S } 23^0 \text{ W})$
$= \begin{pmatrix} 6\ Cos\ \ 247^0 \\ 6\ Sin\ 247^0 \end{pmatrix} = \begin{pmatrix} -2.344 \\ -5.523 \end{pmatrix}$

Some Solved Past Questions

1. A(9, 5) and B(3, 11) are points in the OXY plane. If C is the midpoint of *AB*, find:
i. $\overrightarrow{OC}$,
ii. the value of the acute angle between OC and the x – axis, correct to the nearest degree.

Solution

i. A(9, 5) and B(3, 11)
$\overrightarrow{OC} = \frac{1}{2}\overrightarrow{AB}$
$\overrightarrow{OC} = \frac{1}{2}(a + b)$
$\overrightarrow{OC} = \frac{1}{2}\ [\begin{pmatrix} 9 \\ 5 \end{pmatrix} + \begin{pmatrix} 3 \\ 11 \end{pmatrix}]$
$\overrightarrow{OC} = \frac{1}{2}\begin{pmatrix} 12 \\ 16 \end{pmatrix} = \begin{pmatrix} 1/2 \times 12 \\ 1/2 \times 16 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$

ii. Let θ be the value of the acute angle between OC and the x – axis,
OC= $\begin{pmatrix} 6 \\ 8 \end{pmatrix}$
$\tan\theta = \frac{8}{6}$
$\theta = \tan^{-1}\left(\frac{8}{6}\right) = 53^0$

2. M(9, 7) and N (7, 23) are points in the OXY plane.
a. Find the coordinates of the point R such

that $\overrightarrow{OR} = \overrightarrow{OM} + \frac{1}{2}\overrightarrow{MN}$

b. Calculate ;

i. / $\overrightarrow{OR}$ /

ii. correct to the nearest degree, the angle that $\overrightarrow{OR}$ makes with the x - axis.

Solution

a. $\overrightarrow{OR} = \overrightarrow{OM} + \frac{1}{2}\overrightarrow{MN}$

$\overrightarrow{OR} = \overrightarrow{OM} + \frac{1}{2}(\overrightarrow{ON} - \overrightarrow{OM})$

$\overrightarrow{OR} = \binom{9}{7} + \frac{1}{2}[\binom{7}{23} - \binom{9}{7}]$

$\overrightarrow{OR} = \binom{9}{7} + \frac{1}{2}\binom{-2}{16}$

$\overrightarrow{OR} = \binom{9}{7} + \begin{pmatrix} -2 \times 1/2 \\ 16 \times 1/2 \end{pmatrix} = \binom{9}{7} + \binom{-1}{8} = \binom{8}{15}$

b. i. / $\overrightarrow{OR}$ / = $\sqrt{8^2 + 15^8}$ = $\sqrt{289}$ = 17 units

ii. Let θ be the angle $\overrightarrow{OR}$ makes with the x axis;

tan θ = $\frac{15}{8}$

θ = $tan^{-1}\left(\frac{15}{8}\right)$ = 62^0

Exercises 9.20

A. Express the following as column vectors:

1. $\overrightarrow{OQ}$ = (9km, 053^0) 2. $\overrightarrow{OR}$ = (10km, 106^0)

3. $\overrightarrow{OS}$ = (14km, 217^0) 4. $\overrightarrow{OT}$ = (6km, 332^0)

B. Express as vectors of the form $\binom{x}{y}$

1. $\overrightarrow{OP}$ = (5km, S44^0W)
2. $\overrightarrow{OV}$ = (12km, S60^0E)
3. $\overrightarrow{OW}$ = (8km, N 72^0 W)
4. $\overrightarrow{OM}$ = (10km, N 57^0 E)

Column Vectors as Magnitude –Bearing Vectors (a units, b^0)

Given the vector $\overrightarrow{OA} = \binom{x}{y}$, it can be written in the form **(a units, b^0)** called ***magnitude – bearing vecto r***by going through the following steps:

I. Find the magnitude of $\overrightarrow{OA}$ by using the formula; / $\overrightarrow{OA}$/ = $\sqrt{x^2 + y^2}$

II. Find the angle (acute), **θ**, the vector makes with the x – axis by using the formula;

tan **θ** = $\frac{y}{x}$, ⇒**θ** = $tan^{-1}\left(\frac{y}{x}\right)$

III. Identify the quadrant where the vector lies. Thus:

a. If $\binom{x}{y}$is in the first quadrant, the bearing $\boldsymbol{b^0 = 90^0 - \theta}$

b. If $\binom{-x}{y}$is in the second quadrant, the bearing $\boldsymbol{b^0 = 270^0 + \theta}$

c. If $\binom{-x}{-y}$is in the third quadrant, the bearing $\boldsymbol{b^0 = 270^0 - \theta}$

d. If $\binom{x}{-y}$is in the fourth quadrant, the bearing $\boldsymbol{b^0 = 90^0 + \theta}$

Worked Examples

Express in magnitude – bearing form;

1. $\overrightarrow{OA} = \binom{5}{3}$ 2. $\overrightarrow{OB} = \binom{-7}{4}$

3. $\overrightarrow{OC} = \binom{-6}{-8}$ 4. $\overrightarrow{OD} = \binom{10}{-8}$

Solution

1. $\overrightarrow{OA} = \binom{5}{3}$

/ $\overrightarrow{OA}$/ = $\sqrt{5^2 + 3^2}$ = $\sqrt{34}$ = 6 units

tan θ = $\frac{3}{5}$

θ = $tan^{-1}\left(\frac{3}{5}\right)$

θ = 31^0 (nearest degree)

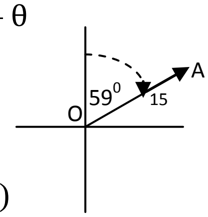
$\overrightarrow{OA} = \binom{5}{3}$lies in the first quadrant. Therefore the bearing, $b^0 = 90^0 - \theta$

But θ = 31^0,

⇒$b^0 = 90^0 - 31^0$

$b^0 = 59^0$

$\overrightarrow{OA}$= (15 units, 059^0)



2. $\overrightarrow{OB} = \binom{-7}{4}$

$/\overrightarrow{OB}/ = \sqrt{7^2 + 4^2} = \sqrt{65}$ = 8 units

$\tan\theta = \frac{4}{7}$

$\theta = \tan^{-1}\left(\frac{4}{7}\right) = 30^0$ (nearest degree)

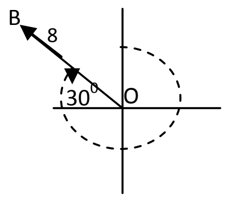
$\overrightarrow{OB} = \binom{-7}{4}$ lies in the second quadrant.

Therefore the bearing, $\boldsymbol{b^0 = 270^0 + \theta}$

But $\theta = 30^0$,

$\Rightarrow b^0 = 270^0 + 30^0 = 300^0$

$\overrightarrow{OB}$= (8 units, 300^0)



3. $\overrightarrow{OC} = \binom{-6}{-8}$

$/\overrightarrow{OC}/ = \sqrt{6^2 + 8^2} = \sqrt{100}$ = 10 units

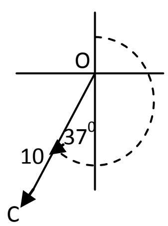
$\theta = tan^{-1}\left(\frac{8}{6}\right) = 53^0$ (nearest degree)

$\overrightarrow{OC} = \binom{-6}{-8}$ lies in the third quadrant. Therefore the bearing, $b^0 = 270^0 - \theta$

But $\theta = 53^0$,

$\Rightarrow b^0 = 270^0 - 53^0 = 217^0$

$\overrightarrow{OC}$= (10 units, 217^0)



4.$\overrightarrow{OD} = \binom{10}{-8}$

$/\overrightarrow{OD}/ = \sqrt{10^2 + 8^2} = \sqrt{164}$ = 13 units

$\theta = tan^{-1}\left(\frac{8}{10}\right) = 39^0$ (nearest degree)

$\overrightarrow{OD} = \binom{10}{-8}$ lies in the fourth quadrant.

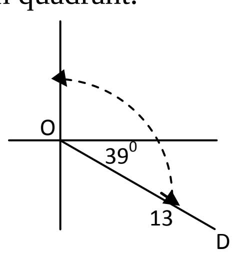
Therefore the bearing,

$\boldsymbol{b^0 = 90^0} + \theta$

But $\theta = 39^0$,

$\Rightarrow b^0 = 90^0 + 39^0 = 129^0$

$\overrightarrow{OD}$ = (13 units, 129^0)



Exercises 9.21

A. Write as magnitude – bearing vector:

1. $\overrightarrow{OP} = \binom{12}{7}$ 2. $\overrightarrow{OQ} = \binom{-3}{-9}$ 3. $\overrightarrow{OR} = \binom{-8}{6}$

4. $\overrightarrow{OS} = \binom{11}{-4}$ 5. $\overrightarrow{OT} = \binom{5}{8}$ 6. $\overrightarrow{OU} = \binom{-10}{-5}$

B. **Express in the form ($\boldsymbol{a}$ units, $\boldsymbol{b^0}$):**

1. $\overrightarrow{OM} = \binom{-2}{-10}$ 2. $\overrightarrow{ON} = \binom{11}{14}$ 3. $\overrightarrow{OB} = \binom{12}{-9}$

4. $\overrightarrow{OA} = \binom{20}{-10}$ 5. $\overrightarrow{OV} = \binom{7}{7}$ 6. $\overrightarrow{OM} = \binom{-17}{9}$

Application

UsingVector Approach to Solve Bearings

I. Make a sketch or accurate drawing of the diagram representing the problem.

II. Apply the triangular law of vectors which states that $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$, for a right triangle with vertices, A, B and C.

III. Express the bearing of A from B as a vector and the bearing of C from B as a vector and substitute in $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ to get the component vector $\overrightarrow{AC} = \binom{x}{y}$.

IV. To find the distance between A and C is to find the magnitude of $\overrightarrow{AC}$, such that:

$/\overrightarrow{AC}/ = \sqrt{x^2 + y^2}$

V. To find the bearing of A from C is to find the angle θ,$\overrightarrow{AC}$ makes with the *x* – axis, using the relation, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

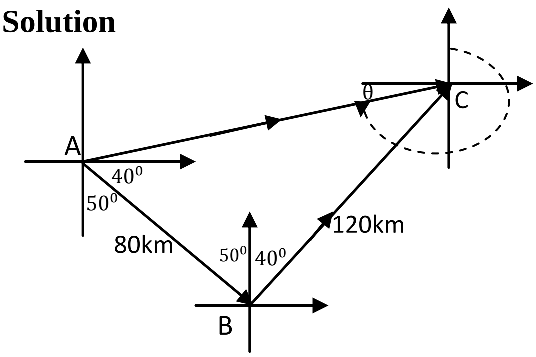
VI. To obtain the bearing of A from C, take measurement from the north pole of C to the line A (direction of A).

Worked Examples

1. An aircraft flew from port *A* on a bearing of 130^0 to another port *B* 80km apart. It then flew from port *B* on a bearing of 040^0 to port *C*, a distance of 120 km. Calculate:

i. the distance from port A to port C, to the nearest kilometer;
ii. the bearing of port A from port C, to the nearest degree.

Solution



From the diagram and the triangular law of vectors, $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

Resolving along the x – axis,

$\overrightarrow{AC} = \begin{pmatrix} 80\cos 40^0 \\ -80\sin 40^0 \end{pmatrix} + \begin{pmatrix} 120\cos 50^0 \\ 120\sin 50^0 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} 61.2836 \\ -51.4230 \end{pmatrix} + \begin{pmatrix} 77.1345 \\ 91.9253 \end{pmatrix} = \begin{pmatrix} 138.4181 \\ 40.5023 \end{pmatrix}$

$/ \overrightarrow{AC} / = \sqrt{(138.4181)^2 + (40.5023)^2}$

$|AC| = \sqrt{20{,}800.0067}$ = 144km (Nearest km)

ii. From the diagram, bearing of A from C
$= 90^0 + 90^0 + (90^0 - \theta^0)$

Angle $\overrightarrow{AC}$ makes with the x – axis,

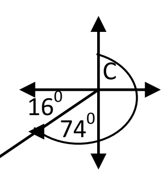
$\tan\theta = \frac{x}{y}$

$\tan\theta = \left(\frac{40.5023}{138.4181}\right)$

$\theta = \tan^{-1}\left(\frac{40.5023}{138.4181}\right)$

$\theta = 16^0$ (nearest degree) at the x – axis

Bearing of A from C;
$= 90^0 + 90^0 + (90^0 - \theta^0)$
$= 90^0 + 90^0 + (90^0 - 16^0)$
$= 254^0$

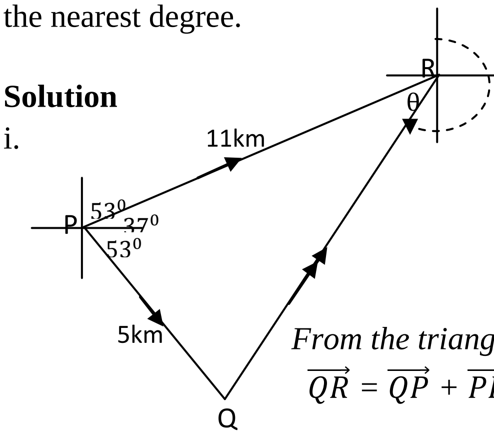


$\overrightarrow{CA}$ = (144km, 254^0) compare to solution at pg 274

2. Three schools, P, Q and R are situated as follows: Q is on a bearing of 5km, 143^0 from P and R is on the bearing 11 km, 053^0 from P.
i. Find the distance between school Q and School R, to the nearest kilometer,
ii. Find the bearing of school Q from school R to the nearest degree.

Solution

i.



From the triangular law, $\overrightarrow{QR} = \overrightarrow{QP} + \overrightarrow{PR}$

Resolving along the x – axis,

$\overrightarrow{QR} = -\begin{pmatrix} 5\cos 53^0 \\ -5\sin 53^0 \end{pmatrix} + \begin{pmatrix} 11\cos 37^0 \\ 11\sin 37^0 \end{pmatrix}$

$\overrightarrow{QR} = \begin{pmatrix} -3.0091 \\ 3.9932 \end{pmatrix} + \begin{pmatrix} 8.7850 \\ 6.6200 \end{pmatrix} = \begin{pmatrix} 5.7759 \\ 10.6132 \end{pmatrix}$

$/ \overrightarrow{QR} / = \sqrt{(5.7759)^2 + (10.6132)^2}$

$/ \overrightarrow{QR} / = \sqrt{146.001}$ = 12km (Nearest kilometer)

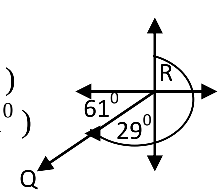
ii. From the diagram, bearing of Q from R
$= 90^0 + 90^0 + (90^0 - \theta^0)$

Angle $\overrightarrow{QR}$ makes with the x – axis,

$\tan\theta = \frac{x}{y} = \left(\frac{10.6132}{5.7759}\right)$

$\theta = \tan^{-1}\left(\frac{10.6132}{5.7759}\right) = 61^0$ at the x – axis

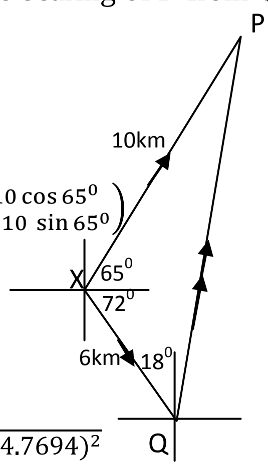
Bearing of Q from R;
$= 90^0 + 90^0 + (90^0 - \theta^0)$
$= 90^0 + 90^0 + (90^0 - 61^0)$
$= 209^0$



$\overrightarrow{RQ}$ = (12km, 209^0) compare to solution at pg 274/5

3. The bearing of P from X, 10km away is 025^0. Another point Q is 6km from X and on a bearing of 162^0. Calculate:

i. the distance PQ; ii. the bearing of P from Q.



Solution

i. From the diagram,

$\overrightarrow{QP} = \overrightarrow{QX} + \overrightarrow{XP}$

$\overrightarrow{QP} = -\begin{pmatrix} 6\cos 72^0 \\ -6\sin 72^0 \end{pmatrix} + \begin{pmatrix} 10\cos 65^0 \\ -10\sin 65^0 \end{pmatrix}$

$\overrightarrow{QP} = \begin{pmatrix} 2.3721 \\ 14.7694 \end{pmatrix}$

$\overrightarrow{PQ} = -\begin{pmatrix} 2.3721 \\ 14.7694 \end{pmatrix}$

/ $\overrightarrow{PQ}$ / = $\sqrt{(-2.3721)^2 + (-14.7694)^2}$

/ $\overrightarrow{PQ}$ / = $\sqrt{223.762}$ = 15km (Nearest kilometer)

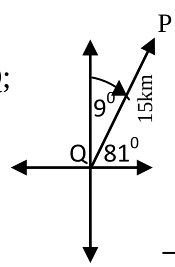
ii.From the diagram, bearing of P from Q

= $90^0 - \theta^0$

Angle $\overrightarrow{PQ}$ makes with the x – axis,

$\tan\theta = \left(\frac{14.7694}{2.3721}\right)$

$\theta = \tan^{-1}\left(\frac{14.7694}{2.3721}\right) = 81^0$ at the x – axis



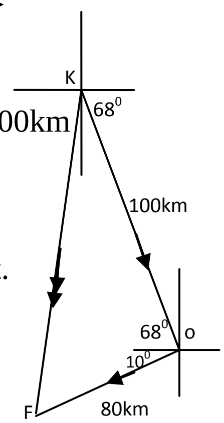
Bearing of P from Q;

= $90^0 - \theta^0$

= $90^0 - 81^0 = 009^0$

$\overrightarrow{QP}$ = (15km, 009^0)

5. From Kwadaso, I travelled 100km on a bearing of 158^0, and then 80km on a bearing of 260^0. Find the distance and bearing from my finishing point.



Solution

From the diagram,

$\overrightarrow{KF} = \overrightarrow{KO} + \overrightarrow{OF}$

$\overrightarrow{KF} = \begin{pmatrix} 100\cos 68^0 \\ -100\sin 68^0 \end{pmatrix} + \begin{pmatrix} -80\cos 10^0 \\ -80\sin 10^0 \end{pmatrix}$

$\overrightarrow{KF} = \begin{pmatrix} 2.3721 \\ 14.7694 \end{pmatrix} = \begin{pmatrix} -41.3240 \\ -106.6102 \end{pmatrix}$

/ $\overrightarrow{KF}$ / = $\sqrt{(-41.3240)^2 + (-106.6102)^2}$

/ $\overrightarrow{KF}$ / = $\sqrt{13073.4077}$ = 114.3390 km

The distance from the finishing point is 114km (to the nearest kilometer)

The angle $\overrightarrow{KF}$ makes with the x - axis

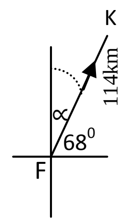
$\theta = \tan^{-1}\left(\frac{106.6102}{41.3240}\right)$

$\theta = 69^0$ (nearest degree) at the x – axis

Bearing of K from F;

= $90^0 - 69^0 = 21^0$

$\overrightarrow{FK}$ = (114km, 021^0)



Exercises 9.22

Use the vector method in each case.

1. Two boats A and B leave a port at the same time. A travels 15km on a bearing of 020^0 while B travels 14km on a bearing of 290^0. Calculate, correct to *two* decimal places, the

a. distance between A and B,

b. bearing of A from B.

2. From Bamako, Enugu is 1800km away on a bearing of 166^0 and Monrovia is 900 km away on a bearing of 206^0. Calculate :

i. the distance between Enugu and Monrovia,

ii. the bearing of Enugu from Monrovia.

3. The bearing of Q from P is 150^0 and the bearing of P from R is 015^0. If Q and R are 24km and 32km respectively from P:

i. represent this information on a diagram;

ii. calculate the distance between Q and R, correct to two decimal places;

iii. find the bearing of R from Q, correct to the nearest degree.

10 STATISTICS I

Idea of Statistics
It is a branch of Mathematics which deals with the collection and study of numbers in order to get facts or information.

Data: It is information collected from various sources through observation or from results of an experiment. These are raw data, array and grouped data.

Types of Data

Raw Data: It is a data which is in the form in which it was collected from source.

***Array*:** It refers to data which has been arranged in ascending or descending order of magnitude.

Grouped Data: It is a data which has been arranged into classes like 0 – 9, 10 – 19 etc.

Discrete data: It is data that can be counted, for example the number of people, cattle, sheep, table, chair etc. The count of discrete data is whole numbers, since we cannot get half a person, or $2\frac{1}{2}$ sheep on a farm. Similarly, the number which appears on a die when thrown can only be 1, 2, 3, 4, 5 or 6. Other examples of discrete data are vehicles passing a station, and the grade obtained by a student in a class.

Continuous data: It is a data which result from measurement, for example, the height of a person, the height of a tree or the, the time taken to complete an activity. The count of continuous data is not a whole number but rounded off. For instance, a tree of height 9.8m might actually be 9.812. Age is also another example of continuous data. Although the values are usuallytaken as whole numbers like 15, 18, 26 years, they can assume intermediate values like $15\frac{1}{2}$ or 18 years, 2 months…..

Exercises 10.1
Determine whether the following data is continuous or discrete.
1. The number of children in the various families in the community .
2. The monthly income of workers in a factory.
3. The marks obtained by a student in an examination.
4. The distance travelled by a day student to school each school day.
5. The amount of pocket money given to students.

Sources and Collection of Data
A data or information can be collected from many sources such as schools, hospitals and business.

In schools, records such as attendance, personal records, results of an examination and others are kept. At the hospitals, records of attendance, diseases treated number of patients treated etc are also collected and kept.

In order to collect data, a survey must be conducted. The result of a survey is called a ***raw data***. For example, 11 students in Asuofua D/A J.H.S. one were asked to mention the days of the week in which they were born and the following results were obtained; *Thursday, Saturday Thursday, Wednesday, Tuesday, Friday, Wednesday, Monday, Tuesday, Wednesday, Sunday,*

When data are collected from a source as above, it is called a ***Raw Data***.

Raw data is not in organized form. There is therefore the need to process raw data in an organized form called ***information*** to enhance understanding. From the above example, information can be obtained in a tabular form as follows:

Day	Tally	No. of pupils
Sunday	/	1
Monday	//	1
Tuesday	///	2
Wednesday	///	3
Thursday	////	2
Friday	///	1
Saturday	//	1

In this case, one can easily know the number of student born on each day of the week.

Measures of Central Tendencies
(Mode, Median and Mean)

The mode, median, and mean arethe three most common examples of measures of central tendencies. They are also called ***averages***. They refer to a single digit which enables one to assess the position in which a group is with respect to others.

Mode

The mode is the most commonly occurring number or the number that repeats itself most in a set of data.The mode is not a calculated value but a sight value which must be carefully identified by sight.

Worked Example

The ages of 10 K.G pupils are recorded as; 4, 4, 5, 5, 6, 5, 6, 6, 6, 6. Find the modal age.

Solution

The mode or the modal age is 6 because it's the number that appears most.

Median

The median of a set of numbers is the middle value or the arithmetic mean of the two middle values when they are ordered in ascending or descending order of magnitude.

Steps in finding the median of a raw data with "n" numbers;

Method I

I. Re – arrange the numbers in either ascending or descending order.

II. Count to locate the middle value as the median.

III. If two numbers, ***a*** and ***b***, appear as the middle values, find the sum of the two and divide by 2 to obtain the median. That is median = $\frac{a\ +\ b}{2}$.

Method II

I. Arrange the numbers in order of magnitude and count to ensure that they are up to *"n"* numbers.

II. If *n* is odd, then the median is the middle term.

That is; $\frac{1}{2}(n+1)^{th}$ ***term or position***

III. If *n* is even, then the median is the arithmetic mean of the two middle terms. That is:

$\frac{1}{2}(n)^{th}$ ***and*** $\frac{1}{2}(n)^{th} + 1$ ***terms or positions***

Worked Examples

1. Find the median of the following numbers:
47, 30, 56, 31, 55, 43 and 44

Solution

Method I

Re-arrange the numbers in ascending order to get:
30, 31, 43, 44, 47, 55, 56,
The middle number is 44. ⇒median = 44

Method II

Re-arrange the numbers in ascending order to get:
30, 31, 43, 44, 47, 55, 56,
Total number of observations is 7
Therefore, n = 7 and n is odd

Median = $\frac{1}{2}(n+1)^{th}$ *term or position*

= $\frac{1}{2}(7+1)^{th}$ *term or position*

= $\frac{1}{2}(8)^{th}$ = 4^{th} *position*

30, 31,43, 44, 47, 55, 56,

4^{th} Position number

Therefore, the median is 44.

2. Find the median of the following numbers; 4, 5, 4, 6, 7, 5, 7, 4, 6, 5

Solution

Method I

Re-arrange the numbers in ascending order to get:
4, 4, 4, 5, 5, 5, 6, 6, 7, 7
The middle numbers are 5 and 5. Therefore, median = $\frac{5\ +\ 5}{2}$ = 5

Method II

Re-arrange the numbers in ascending order to get:
4, 4, 4, 5, 5, 5, 6, 6, 7, 7

Total number of entries, n = 10 (even) Median
= $\frac{1}{2}(\mathrm{n})^{\mathrm{th}}$ and $\frac{1}{2}(\mathrm{n})^{\mathrm{th}}+1$ terms

$\frac{1}{2}(10)^{\mathrm{th}}$ = 5^{th} and $\frac{1}{2}(10)^{\mathrm{th}}$ + 1 = 6^{th}

The median positions are 5^{th} and 6^{th}

4, 4, 4, 5, 5, 5, 6, 6, 7, 7

6^{th} Position number

5^{th} Position number

Median = $\frac{5\ +\ 5}{2}$ = 5

Mean

The mean is the ratio of the sum of all the values in a raw data to the number of entries.

The mean is denoted by $\bar{\mathbf{x}}$.Thus, given the set of values: $x_1, x_2, x_3, x_4, ..x_n$, the mean,
$(\bar{x})$ = $\frac{x_1+x_2+x_3+x_4+x_n}{n}$ = $\frac{\sum x}{n}$
Where $\boldsymbol{n}$ is the position of the last number of the entries.

Worked Examples

1. Find the mode, median and mean of the following numbers: 4, 24, 10, 17, 19, 21, 10

Solution

Mode = 10 (Most frequently number)

Median: Ordering the numbers in ascending order, 4, 10, 10, 17, 19, 21, 24
The middle number is 17. ⇒median = 17

$\bar{x}$ = $\frac{4\ +\ 10\ +\ 10+\ 17\ +\ 19+\ \ 21+\ 24}{7}$ = $\frac{105}{7}$ = 15

2. The masses of 12 boxes measured to the nearest kg are: 26, 23, 27, 28, 29, 22, 23, 27, 20, 20, 24, and 25. Find the mode, median and mean

Solution
Mode = 20, 23 and 27

In ascending order: 20, 20, 22, 23, 23, 24, 25, 26, 27, 27, 28, 29
The middle numbers are 24 and 25.
Median = $\frac{24+25}{2} = \frac{49}{2}$ = 24.5

Mean ($\bar{x}$) = $\frac{x_1 + x_2 + x_3 + x_4 + x_n}{n}$
Where *n* = number of entries = 11
$\bar{x} = \frac{4+5+5+5+5+5+6+6+6+6+6}{11} = \frac{59}{11}$ = 5.36(2 d.p.)

3. Find the mean, median and modal heights of the following distribution: 170cm, 171cm, 173cm, 174cm, 177cm, 177, 184cm, 186cm,

Solution
i. Mean ($\bar{x}$) = $\frac{x_1 + x_2 + x_3 + x_4 + x_n}{n}$
Where *n* = number of entries = 8
$\bar{x} = \frac{170 + 171 + 173 + 174 + 177 + 184 + 186}{8}$ = 176.50 cm

ii. To find the median, arrange the numbers in ascending order to obtain: 170cm, 171cm, 173cm, 174cm, 177cm, 177, 184cm, 186cm.
The two middle values are 174 and 177.

Median = $\frac{174+177}{2}$ = 175. 50 cm

iii. The modal height = 177cm

Exercises 10.2
1. The following temperature in °C was recorded in 10 cities in Europe; - 4, 5, 2, 0, - 6, - 4, 3, - 6, - 4, -7. Find the modal temperature.

2. The marks scored by 8 pupils in a science test are 3, 7, 8, 8, 5, 8, 4, 8.What is the median mark?

3. The scores of 10 students in an examination are given as follows; 45, 12, 75, 81, 54, 51, 24, 67, 19 and 39. What is the median and mean score?

4. The price of 12 commodities in cedis was recorded as: 5, 2, 5, 3, 3, 5, 3, 2, 4, 3, 3, 2.
i. What the modal price?
ii. Find the median price.
iii. Determine the mean price.

5. The ages in years of 8 boys are: 14, 14, 15, 15, 12, 11 13, 10.What is the average age?

Calculations Involving Mean
I. Represent any unknown number by any preferred variable.
II. Identify the total number of entries.
III. Find the sum or total of the given data.
III. Identify the value of the mean, if given.
IV. Use the formula below and workout the value of the variable.
Mean = $\frac{Sum\ of\ entries}{number\ of\ entries}$

Worked Examples
1. The mean of the numbers 4, 3, 3, and *x* is 5, Find the value of *x.*

Solution
$\bar{x} = \frac{4+3+3+x}{4} = 5$
$\Rightarrow 4 + 3 + 3 + x = 4 \times 5$
$10 + x = 20$
$x = 20 - 10$
$x = 10$

2. The average of the numbers 4, 10, 24, *x* and 16 is 13. Find the value of *x.*

Solution

$\bar{x} = \frac{4+10+24+x+16}{5} = 13$
$\Rightarrow 4 + 10 + 24 + x + 16 = 13 \times 5$
$54 + x = 65$
$x = 65 - 54 = 11$

3. The average age of a class of 30 students is 18 years and the average age of another class of 45 students is 20 years. Find the average age of the students of the two classes.

Solution
If the average age of 30 students is 18 years
$\Rightarrow$the total age of 30 students = 30×18
= 540 years

If the average age of 45 students is 20 years
$\Rightarrow$the total age of 45 students = 45×20
= 900 years

Total number of students in the combined class
= 30 + 45 = 75 students

Total age of 75 students,
= (540 + 900) years = 1,440 years

Average age for the two classes;
$= \frac{1440}{75} = 19.2$ years

4. The average test mark for a class of 20 students is 65. Kofi's mark was recorded as 35 instead of 55. What is the correct average mark for the test?

Solution
Total marks when 35 was recorded.
$= 20 \times 65 = 1300$

The difference between the correct and incorrect mark = 55 – 35 = 20

Correct average age = $\frac{1300 + 20}{20} = 66$ years

5. The ages of 5 women are 48, y, 52, 50 and $(2y - 5)$. If their average age is 47 years, find the age of the oldest woman.

Solution
$\bar{x} = \frac{48 + y + 52 + 50 + 2y - 5}{5} = 47$
$\frac{145 + 3y}{5} = 47$
$145 + 3y = 5 \times 47$
$145 + 3y = 235$
$3y = 235 - 145$
$3y = 90$
$y = 30$
$\Rightarrow$The ages of the women are 48, 30, 52, 50 and 55
The oldest woman is 55 years old.

Exercises 10.3
1. The following are the ages in years of members of a group:8, 11, 8, 10, 6, 7, $3x$, 11, 11. If the mean age is 9 years, find the value of x.

2. The mean of 10 positive numbers is 16. When another number is added, the mean becomes 18. Find the number added.

3. The mean age of 4 men is 19 years, 11 month. When a fifth man joins them, the mean age of all the 5 men is 20 years 7 months. How old is the fifth man?

4. The mean of 5 numbers x, 2, 3, 5, and 9 is 3, find the value of x .

5. 40% of the employees of a factory are workers. All the remaining employees are executives. The annual income of each worker is Gh¢390.00 and

the annual income of each executive is Gh¢420.00. What is the average annual income of all the employees in the factory?

6. A year ago, the average annual income of Jack and Jill was Gh¢3,800.00, the average annual income of Jill and Jess was Gh¢4,800.00 and the average annual income of Jess and Jack was Gh¢5,800.00. What was the average annual income of the three people?

7. The average of all scores in a certain algebra test was 90. If the average of the 8 male students score was 87 and the average of the female students score was 92, how many females took the test?

Frequency and Frequency Diagrams

The number of times an event scores in a given data is called its ***frequency***.

Frequency Diagrams for Ungrouped Data

It is a table that shows the event and the number of times each event occurs. It is usually divided into four (4) sections as shown below:

Range(x)	Tally	f	fx
		$\sum f =$	$\sum fx =$

Column 1(x) **:** is made up of the events usually arranged downward in increasing order. It is also called the ***range.***

Column 2 (Tally)**:** consist of slight tilted strokes like /, that represents the number of times an event occurs in a data. That is / for 1, // for 2, /// for 3 and //// for 4. However, at every fifth successive occurrence, a horizontal line is drawn to across the first 4 strokes. Thus ~~////~~ for 5 and ~~////~~ ~~////~~ // for 12.

Column 3 (frequency): is made up of the digital representation of the tally.

Column 4 (fx): consist of the product of each event and its corresponding frequency.

$\sum f$ is the sum of all the frequencies.

$\sum fx$ is the sum of the product of frequencies and their corresponding events.

Worked Examples

1. The ages of 20 school children were recorded as follows;

13 9 15 17 13 17 15 11 9 9
9 11 9 11 15 11 15 11 11 11

Make a frequency table for the data.

Solution

Age (x)	Tally	Frequency (f)	fx
9	~~////~~	5	45
11	~~////~~ //	7	77
13	//	2	26
15	////	4	60
17	//	2	34
		$\sum f = 20$	$\sum fx = 242$

2. The table below gives family sizes recorded by 30 respondents. Construct a frequency table for the data.

5 4 3 4 4 5 4 4 4 3 3 3 5 4 3
6 3 2 4 5 3 3 4 3 3 6 5 2 5 5

Solution

Sizes (x)	Tally	Frequency (f)	fx
2	//	2	4
3	~~////~~ ~~////~~	10	30

4	~~////~~ ////	9	36
5	~~////~~ //	7	35
6	//	2	12
		$\sum f = 30$	$\sum fx = 117$

Exercises 10.4

1. The scores obtained by 35 students in a Mathematics test are given below;

17 16 16 16 17 18 16 18 18 17 18
17 19 18 19 19 18 18 17 20 19 20
20 19 17 17 17 20 16 20 19 17 17
16 19

Construct a frequency table for the data.

2. Dinah tossed two dice 40 times and recorded the total scores. The results are presented below:

8 4 8 4 10 5 10 6 3 10 6 8 6 3 6
5 3 7 4 3 7 4 11 7 4 11 9 3 11 5
5 10 3 3 11 3 7 3 4 3

Construct a frequency table for the data

3. A survey was carried out to determine the ages of 40 children, the results are as follows:

16 14 13 13 15 11 13 11 12 13
14 12 16 14 14 13 13 12 13 13
11 12 13 13 13 15 12 12 11 12
15 13 12 12 13 12 11 16 13 14

Construct a frequency diagram for the data.

Note*:the frequency table can be drawn without the column 4 or fx. The column 4 becomes necessary when the questions demand a calculation of the mean.*

Mode, Median and Mean on an Ungrouped Frequency Table

I. Mode

The mode in the frequency table is any number at column1 that has the highest frequency in column3 of the table. Note that the mode is not the highest frequency (in column 3) but its corresponding value in column 1.

From the table below;

Ages (x)	A	B	C	D	E
Frequency (f)	3	1	2	5	3

The mode is D because it has the highest frequency of 5, but the mode is not 5 because it is the highest frequency. The mode is not selected from column 3 but from column 1 of the frequency distribution table.

II. Mean

On the frequency distribution table, the mean ($\bar{x}$) is calculated by the formula;**($\bar{x}$) =** $\frac{\sum fx}{\sum f}$

The respective values are then substituted in the formula to obtain the mean.Care must therefore be taken to compute the summation of "f" and "fx" in order to obtain accurate values of $\sum fx$ and $\sum f$.

The value of the mean is usually corrected to 2 decimal places.

Worked Example

Copy and complete the table below and use it to find the mean to 2 decimal places.

x	Tally	Frequency(f)	fx
1	//		2
2	///	3	
3	~~////~~ /	6	
4			4
		$\sum f =$	$\sum fx=$

Solution

x	Tally	Frequency(f)	fx
1	//	2	2
2	///	3	6
3	~~////~~ /	6	18
4	/	1	4
		$\sum f = 12$	$\sum fx = 30$

Mean ($\bar{x}$) = $\frac{\sum fx}{\sum f} = \frac{30}{12} = 2.50$

2. The table below shows the frequency distribution of the sizes of shoes produced by a factory

Size	5	6	7	8	9	10	11
Frequency	4	6	9	15	14	7	5

Calculate, correct to the nearest whole number, the mean size of the shoes

Solution

Mean ($\bar{x}$) $= \frac{\sum fx}{\sum f}$

$\sum fx = (5 \times 4) + (6 \times 6) + (7 \times 9) + (8 \times 15) + (9 \times 14) + (10 \times 7) + (11 \times 5)$

$= 20 + 36 + 63 + 120 + 126 + 70 + 55 = 490$

$\sum f = 4 + 6 + 9 + 15 + 14 + 7 + 5 = 60$

$(\bar{\boldsymbol{x}}) = \frac{\sum fx}{\sum f} = \frac{490}{60} = 8$ (Nearest whole number)

iii. Median

To find the median from a frequency distribution table

I. Find the total frequency i.e. $\sum f$.

II. Identify whether $\sum f$ is an even number or an odd number.

Note that to locate the middle of anything is to divide by 2.

If $\sum f$ is odd, determine the median by adding 1 to $\sum f$ to make it even and the sum divided by 2 to give the position of the median. That is: ***Median position*** = $\left(\frac{\sum \mathbf{f+1}}{\mathbf{2}}\right)^{\mathbf{th}}$

By adding frequencies from the top to the bottom or the other way round or counting the strokes of the tally, the number which corresponds to the median position, $\left(\frac{\sum f+1}{2}\right)^{th}$ is the median.

If $\sum f$ is even, then two median positions which are $\left(\frac{\sum \mathbf{f}}{\mathbf{2}}\right)^{\mathbf{th}}$ and $\left(\frac{\sum \mathbf{f}}{\mathbf{2}}\right)^{\mathbf{th}} + \mathbf{1}$ are used to determine the two middle numbers, the sum of which is divided by 2 to get the median value.

Worked Examples

1. The height in centimeters of some school children were recorded as: 165, 165, 155, 174, 159, 169, 155, 150, 155, 154

i. Draw a frequency table for the distribution.

ii. Use your table to find the mode and the median of the distribution.

Solution

i.

Height (cm)	Tally	Frequency(f)
150	/	1
154	/	1
155	///	3
159	/	1
165	//	2
169	/	1
174	/	1
		$\sum f = 10$

Mode is the highest frequency = 155cm

Median: since $\sum f$ = 10 (even number)

So $\left(\frac{\sum f}{2}\right)$th and $\left(\frac{\sum f}{2}\right)$ th + 1

$= \left(\frac{10}{2}\right)^{th}$ and $\left(\frac{10}{2}\right)^{th} + 1$

= 5th and 6th positions

By adding frequencies from the top, the 5th position = 155 and the 6th position = 159

$\therefore$ Median = $\frac{155+159}{2} = \frac{314}{2}$ = 157cm.

2. The table below shows the marks scored out of 10 by some candidates in a test.

Marks	1	2	3	4	5	6	7	8
Frequency	2	3	5	7	8	13	7	5

From the table, find:
i. the modal mark.
ii. how many candidates took the test?
iii. calculate the mean and median mark for the test.

Solution

Marks(x)	Tally	Frequency(f)	fx
1	//	2	2
2	///	3	6
3	~~////~~	5	15
4	~~////~~ //	7	28
5	~~////~~ ///	8	40
6	~~////~~ ~~////~~ ///	13	78
7	~~////~~ //	7	49
8	~~////~~	5	40
		$\sum f$ = 50	$\sum fx$= 258

i. The modal mark is the mark that has the highest frequency = 6

ii. The total number of candidates who took the test = 2 + 3 + 5 + 7 + 8 + 13 + 7 + 5 = 50

iii. Mean ($\bar{x}$) = $\frac{\sum fx}{\sum f} = \frac{258}{50}$ = 5.6

Median :$\sum f$ = 50 (an even number)

$\therefore \left(\frac{\sum f}{2}\right)^{th}$ and $\left(\frac{\sum f}{2}\right)^{th} + 1$

$= \left(\frac{50}{2}\right)^{th}$ and $\left(\frac{50}{2}\right)^{th} + 1$

= 25th and 26th position

= 5 and 6

$\therefore$ Median = $\frac{5+6}{2} = \frac{11}{2}$ = 5.5

3. The table below shows the distribution of ages in years of children who were treated in a clinic.

Age (*yrs*)	1	2	3	4	5
Frequency(f)	8	5	2	4	6

Find the mean, mode and median.

Solution

Ages x	Tally	Frequency (f)	fx
1	~~////~~ ///	8	6
2	~~////~~	5	8
3	//	2	6
4	////	4	12
5	~~////~~ //	6	25
		$\sum f$ = 25	$\sum fx$ =57

i. Mean ($\bar{x}$) = $\frac{\sum fx}{\sum f} = \frac{57}{25}$ = 2.28

ii. The modal age is the age with the highest frequency = 1

iii. Median: $\sum f$ = 25 (odd number)

$\left(\frac{\sum f+1}{2}\right)^{th} = \left(\frac{25+1}{2}\right)^{th} = \left(\frac{26}{2}\right)^{th} = 13^{th}$ position.

But 13^{th} position = 2

∴The median = 2

4. The table below shows the distribution of marks scored by a group of students.

Marks	1	2	3	4	5	6	7	8	9	10
No. of student	2	2	3	3	*k*	2	2	0	1	1

If the mean of the distribution is 4.6, find the value of *k*.

Solution

Marks	Frequency *(f)*	*fx*
1	2	2
2	2	4
3	3	9
4	3	12
5	*k*	5*k*
6	2	12
7	2	14
8	0	0
9	1	9
10	1	10
	$\sum f$ = 16 + *k*	$\sum fx$ = 72 + 5*k*

Mean $(\bar{x}) = \frac{\sum fx}{\sum f} = 4.6$

$\Rightarrow \frac{72+5k}{16+k} = 4.6$

$72 + 5k = 4.6(16 + k)$

$72 + 5k = 73.6 + 4.6k$

$5k - 4.6k = 73.6 - 72$

$0.4k = 1.6$

$k = 4$

Exercises 10.5

1. The following is a record of scores obtained by 30 J.H.S. 2 pupils in a test marked out of 5:

5 3 2 4 5 2 4 3 1 3 3 4 2 3 4

5 3 4 3 2 4 3 1 2 3 3 2 4 2 1

a. Construct a frequency table for the data.

b. Find the median marks.

c. Calculate the mean of the distribution.

2. The table below gives the ages of the members of a juvenile club.

Ages (yrs)	8	9	10	11
Frequency	5	10	6	8

i. How many people are in the club?

ii. Determine the median and mean ages.

iii. What is the modal age?

3. The table below shows the distribution of 23 students visiting a clinic on a particular day.

Ages (*x*)	15	16	17	18	19	20
No. of stud	2	3	7	5	2	4

i. What is the modal age of the students?

ii. Find the mean and median age.

4. The score obtained by 35 students in a mathematics test are given below:

12 26 15 14 14 12 16 26 12 16

15 15 14 26 26 15 14 19 12 19

26 19 19 19 14 15 26 12 14 15

26 12 15 26 12

i. Construct a frequency table for the data.

ii. Use your table to calculate the mean and median.

iii. What is the mode of the distribution?

Relative Frequency

The relative frequency of an event is the probability of an event on a frequency table. To be precise, it is the frequency of an event compared to the total frequency.

The relative frequency of an event,

$\boldsymbol{R_f} = \frac{\boldsymbol{frequency\ of\ an\ event}}{\boldsymbol{Total\ frequency}}$

Worked Examples

1. A survey was conducted to determine the number of children per family at Pasro D/A J.H.S. 1. The results were recorded in the table below:

No. of Chn/family	No. of family
2	2
3	10
4	9
5	7
6	2

Find the relative frequency of;

a. 3 children per family,

b. 5 children per family.

Solution

a. From the table, the total number of families

= 2 + 10 + 9 + 7 + 2 = 30,

The number of families with 3 children per family = 10

$R_f = \frac{frequency\ of\ an\ event}{total\ frequency} = \frac{10}{30} = \frac{1}{3}$

b. Total number of families = 30

Number of families with 5 children per family = 7

$R_f = \frac{\text{Frequency of an event}}{\text{Total frequency}} = \frac{7}{30} = \frac{1}{5}$

2. A fair die was tossed 40 times. The outcomes are given in the table below;

SCORE ON DIE	FREQUENCY
1	5
2	6
3	11
4	8
5	6
6	4

Calculate the relative frequency of the event that the score is an even number.

Solution

Total frequency

= 5 + 6 + 11+ 8 + 6 + 4

= 40

Even numbers on the die are 2, 4, 6.

Frequency of 2 + frequency of 4 + frequency of 6

= 6 + 8 + 4 = 18

$R_f = \frac{\text{frequency of event}}{\text{total frequency}} = \frac{18}{40} = \frac{9}{20}$

Exercises 10.6

1. If a number is selected at random from the table below, what is the probability that the number is 5?

Number	1	3	5	7	9
Frequency	25	15	8	10	2

2. The frequency distribution table below shows the ages of some nurses:

Age (yrs)	24	29	34	39	44
Frequency	7	8	5	6	4

If a person is selected at random from the nurses, find the probability that the person chosen is;

a. 34 years b. more than 34 years

3. The frequency table below shows the age distribution of 20 pupils in a class.

Age (yrs)	9	10	11	12
Frequency	2	8	6	4

What is the probability that a pupil selected at random is:
a. 9 years old b. less than 12 yrs old.

Graphical Representation of Data
Data collected can be represented in various forms such as;
i. Pie chart ii. Bar chart
iii. Histogram iv. Cumulative frequency curve
v. Leaf and stem plot

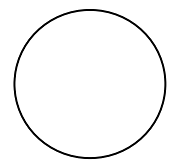
Pie – Chart
It is the diagrammatic representation of data or information by using degrees or sectors of a circle. Pie charts are also called *circle graph.*

Calculating the Angles of a Pie Chart
To get the angles of a pie-chart:
I. Remember the fact that the sum of angles in a circle is 360°
II. Find the sum of all the items under consideration.
III. Calculate the angle of each item by using the formula;

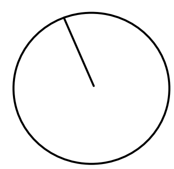
An angle $= \frac{\textbf{the item}}{\textbf{sum of items}} \times \mathbf{360°}$
For e.g. given the items *a*, *b*, and *c*
Angle $a = \frac{a}{a+b+c} \times 360°$
Angle$b = \frac{b}{a+b+c} \times 360°$
Angle$c = \frac{c}{a+b+c} \times 360°$

iv. Use the values of the angles calculated to draw the sectors in the circle. Make sure that all the angles calculated sum up to 360^0.

Drawing a Pie – Chart
1. Draw a sizeable circle using a compass and a well sharpened pencil.



2. Draw a line from the center of the circle to touch the circumference as a starting line.



3. Place the protractor on the radius (straight line) such that the base line and the center of the protractor coincide exactly with the radius and the center of the circle respectively.
Note:
All angles can be drawn in one direction, either clockwise or anticlockwise. Continue the process by using the second radius of the central angle drawn, to draw subsequent angles until all central angles are drawn.
Write the name and angle of each sector in that sector.

Worked Examples
1. The table below shows the distribution of pupils in a J.H.S. 1 who speak some of the Ghanaian languages.

Ghanaian language	No. of students who speak the Language
Nzema	5
Ga	20
Twi	30
Ewe	25
Fante	10

i. Draw a pie chart for the distribution.
ii. What is the modal language?

Solution
i. Total number of students who speak the language = 5 + 20 + 30 + 25 + 10 = 90.

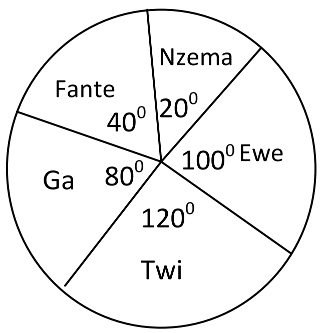
Nzema = $\frac{5}{90} \times 360° = 20°$

Ga = $\frac{20}{90} \times 360° = 80°$

Twi = $\frac{30}{90} \times 360° = 120°$

Ewe = $\frac{25}{90} \times 360° = 100°$

Fante = $\frac{10}{90} \times 360° = 40°$



Not drawn to scale

i. The modal language is the language with the highest frequency = Twi.

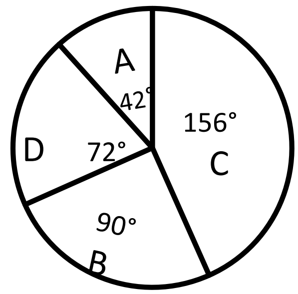
2. The following table shows the distribution of grades obtained by 120 students in an examination.

Grade	*A*	*B*	*C*	*D*
No. of students	14	30	52	24

i. Draw a pie chart for the distribution.

Solution
Total number of students
=14 + 30 + 52 + 24 =120

Item	Calculation	Angles
A	$\frac{14}{120} \times 360°$	42°
B	$\frac{30}{120} \times 360°$	90°
C	$\frac{52}{120} \times 360°$	156°
D	$\frac{24}{120} \times 360°$	72°



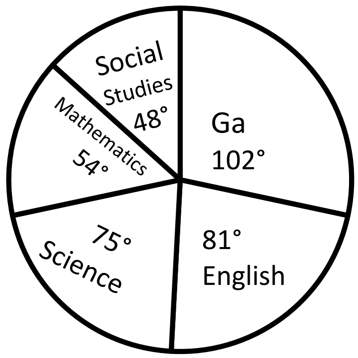
3. The table below shows the performance of Solomon in his final examination.

Subject	Score
English	54%
Mathematics	36%
Ga	68%
Science	50%
Social studies	32%

Draw a pie chart to represent this information,

Solution
Total score = 54 + 36 + 68 + 50 + 32 = 240

Item	Calculation	Angles
English	$\frac{54}{240} \times 360°$	81°
Maths	$\frac{36}{240} \times 360°$	54°
Ga	$\frac{50}{240} \times 360°$	102°
Science	$\frac{24}{240} \times 360°$	75°
Soc studies	$\frac{32}{240} \times 360°$	48°



Taxes	Gh¢16
Miscellaneous	Gh¢22

i. Draw a pie chart for this information.
ii. What item received the highest expenditure?
iii. What fraction of the amount is spent on clothing?

3. The table below shows the number of students who offered certain subjects in a school.

Subject	No. of students
Mathematics	45
Physics	39
Chemistry	28
Biology	14
Economics	36
History	18

Draw a pie chart for the above information

4. In an election, the number of votes won by five political parties A, B, C, D and E in a village were recorded as follows:

Party	A	B	C	D	E
No. of votes	14	11	19	52	24

i. Draw a pie chart to illustrate the information.
ii. What percentage of the total votes did the winner obtain?

Calculations Involving Pie - Charts

Sometimes a central angle of a pie chart could be represented by a variable. To determine the value of the variable, find the sum of all the central

Exercises 10.7

1. The Smith family has an annual income of Gh¢15,000.00. They planned to budget it in the following way:

Expenses	Amount(%)
Food	10%
Shelter	25%
Clothing	20%
Advancement	15%
Operating expenses	18%
Savings	12%

i. Make a pie chart to show the distribution of the money.
ii. What fraction of the money is spent on savings?

2. Mr. Green’s income is Gh¢180.00 per week. He spends his income as follows:

Rent	Gh¢36
Food	Gh¢66
Clothing	Gh¢10
Electricity	Gh¢18
Medicals	Gh¢12

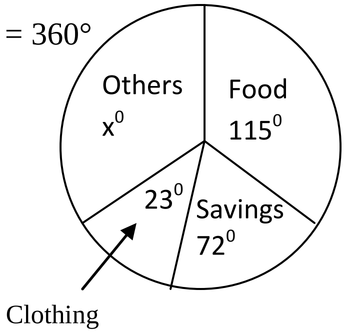
angles and equate it to 360°.For example, in the diagram below;

$x° + 115° + 72° + 23° = 360°$

$x° + 210° = 360°$

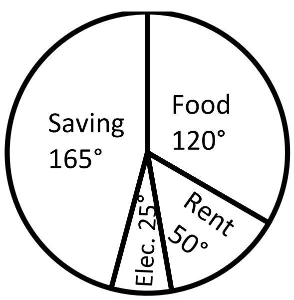
$x° = 360° - 210°$

$x = 150°$



Applying Method of Direct Proportion

Also the value of an item could be given to determine the value of other items on pie a chart. Similarly, the value of the total item can be given to find the value of other items. Such problems are solved by method of direct of proportion. For example, in the pie chart below;



If the amount spent on electricity is Gh¢10.00, the values of all the other items can be calculated as follows;

<u>Method 1</u>

Rent;

If 25° = Gh¢10, then 50° = $\frac{10 \times 50°}{25°}$ = Gh¢20.00

The amount spent on rent is Gh¢20.00

Food;

If 25° = Gh¢10, then 120° = $\frac{120 \times 10°}{25°}$ = Gh¢48.00

The amount spent on food is Gh¢48.00,

Savings;

If 25° = Gh¢10, then 165° = $\frac{165 \times 10°}{25°}$ = Gh¢66.00.

The amount spent on savings is Gh¢66.00,

Total amount;

If 25° = Gh¢10, then 360° = $\frac{360 \times 10°}{25°}$ = Gh¢144.00

The total amount spent is Gh¢144.00,

Alternatively;

Total amount

= 10 + 20 + 48 + 66

= Gh¢144.00

<u>Method 2</u>

Using the given value as a standard, express each item as a proportion of the angle of Electricity and the given value.

⇒Electricity : Rent

25° : 50° = 10 : x

$\frac{25}{50} = \frac{10}{x}$

$\frac{25x}{25} = \frac{50 \times 10}{25}$ = Gh¢20.00

Amount spent on Rent is Gh¢20.00

Electricity: Food

25° : 120° = 10 : x

$\frac{25}{120} = \frac{10}{x}$

$\frac{25x}{25} = \frac{120 \times 10}{25}$ = Gh¢48.00

The amount spent on food is Gh¢48.00,

Electricity: Savings

25° : 165° = 10 : x

$\frac{25}{165} = \frac{10}{x}$

$\frac{25x}{25} = \frac{165 \times 10}{25}$ = Gh¢66.00

The amount spent on savings is Gh¢66.00, Electricity: Total

$25° : 360° = 10 : x$

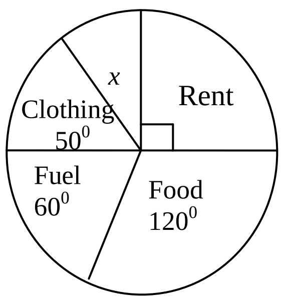
$\frac{25}{360} = \frac{10}{x}$

$\frac{25x}{25} = \frac{360 \times 10}{25}$

x = Gh¢144.00 (Total Amount)

Worked Examples

1. The pie chart below shows the monthly expenditure of Mr. Awuah whose monthly income is Gh¢180.00



i. What is the size of the angle representing Savings?

ii. How much does Mr. Awuah spend on Rent?

iii. What fraction of Mr. Awuah's income is spent on Food?

Solution

i. Let x represent angle of Savings

$= x + 50° + 60° + 90° + 120° = 360°$

$\Rightarrow x + 320° = 360°$

$x = 360° – 320° = 40°$

ii. Mr. Awuah's monthly income is Gh¢180.00, then amount spent on rent;

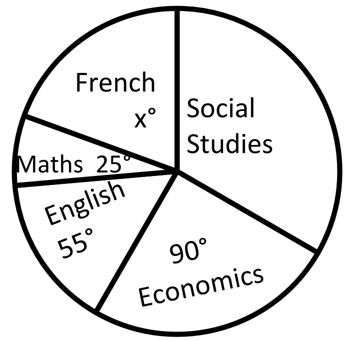
360° = Gh¢180

$90° = \frac{90 \times 180}{360°}$ = Gh¢45.00

iii. Fraction spent on Food;

$\frac{120°}{360°} = \frac{1}{3}$

2. The pie chart below shows the performance of Serwaa in her final examination.



i. What is the angle for French?

ii. If Serwaa scored 60% in Social Studies, what was her score in Economics?

Solution

Let x represent the angle for French;

$x + 55° + 90° + 120° + 25° = 360°$

$x + 290° = 360°$

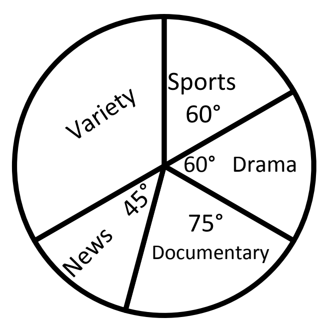
$x = 360° − 290° = 70°$

ii. If 120° = 60%

Then $90° = \frac{90}{120} \times 60\% = 45\%$.

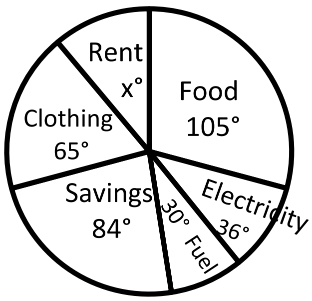
Exercises 10.8

1. The pie chart below shows the program analysis of a T.V station. The station telecast 6 hours each day.



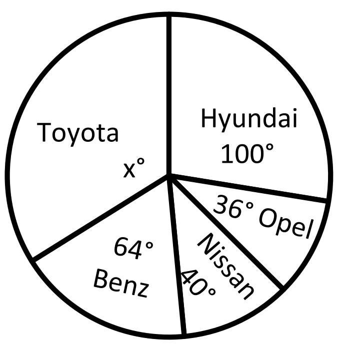
i. What is the size of the angle marked 'variety'?
ii. How many hours in a day do the station telecast news?

2. The pie chart below shows the monthly expenditure of Mr. Brown who receives Gh¢5,400.00 a month.



i. What is the angle for rent?
ii. Calculate the amount Mr. Brown put in his saving account.
iii. What fraction of his income was spent on rent?

3. The pie chart below represents the number of different cars that were parked at a car park.



If 9 Opel cars were parked,
i. Find the angle representing Toyota.
ii. How many Toyota cars were parked at the park?
iii. How many cars were parked at the park?
iv. What percentage of the cars parked is Nissan?

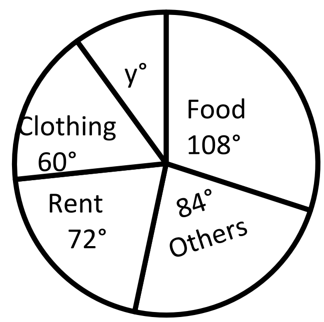
4. The following table gives a data from a newspaper survey in Kumasi. 400 people were asked to choose anyone of the newspaper they prefer to read.

Newspaper	No. of people
Graphic	150
Ghanaian Times	100
Mirror	75
Daily Guide	x
Free Press	25

i. Find the value of x.

ii. Draw a pie chart for the distribution.

5. The pie chart below shows how Mr. Brown spent his salary in the month of January, 2010. He spent Gh¢180.00 on rent.



i. Find angle y representing savings for the month.
ii. What's the man's salary for the month?
iii. By how much does expenditure on others exceed that on clothing.

Bar Chart / Graph
Any graph in which bars of various lengths are used to represent different quantities is called ***a Bar Graph*** or ***a Bar Chart.***

Drawing a Bar Chart

The following points are helpful in drawing the bar chart:

I. The bars must be of equal size.

II. The bars must be separated by blank spaces of the same size (equal spacing).

III. The *x*- axis and *y*- axis must be divided into equal segments and labeled to a convenient scale.

IV. The frequencies are always shown on the *y* – axis whilst the other component being marks, ages, scores, height or whatever are shown on the *x*- axis.

V. Present the values on the *x*-axis in such a way that it can use more than half of the graph sheet.

VI. The length of each bar must correspond to its frequency.

VII. The title of the chart must be written

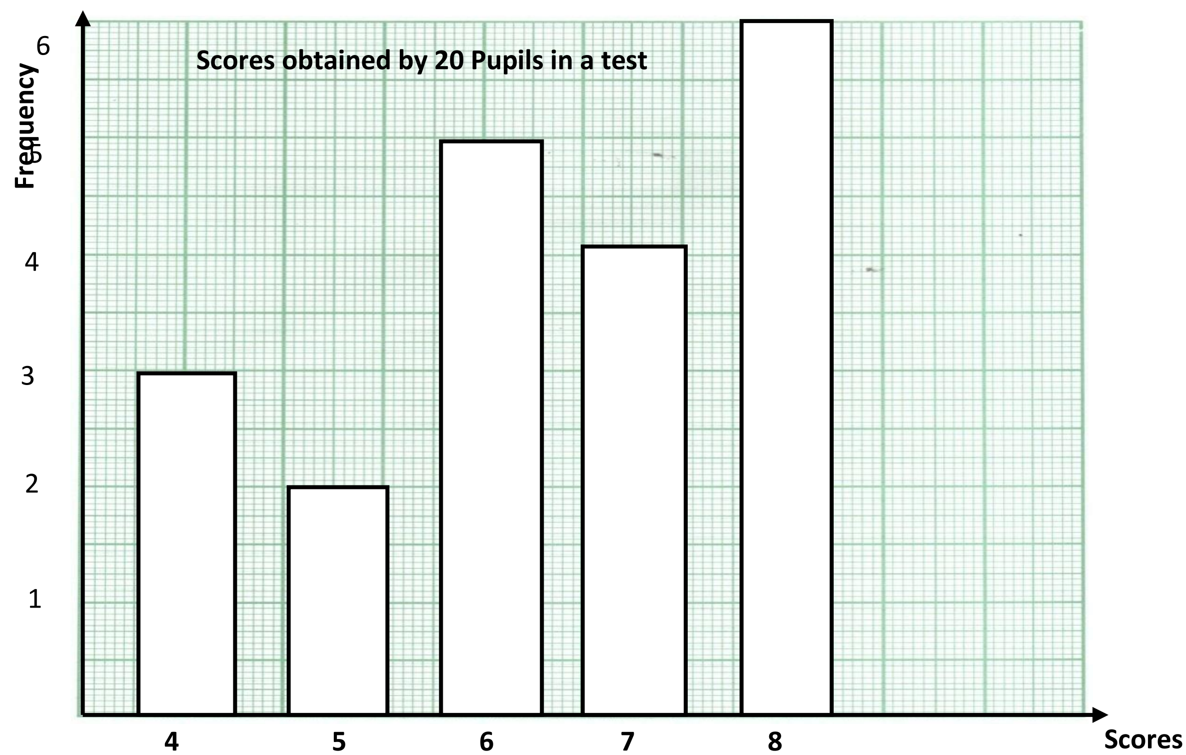
Worked Examples

1. The table below shows the scores obtained by 20 pupils in a test

Scores	4	5	6	7	8
Frequency	3	2	5	4	6

i. Draw a bar chart for the distribution

Solution



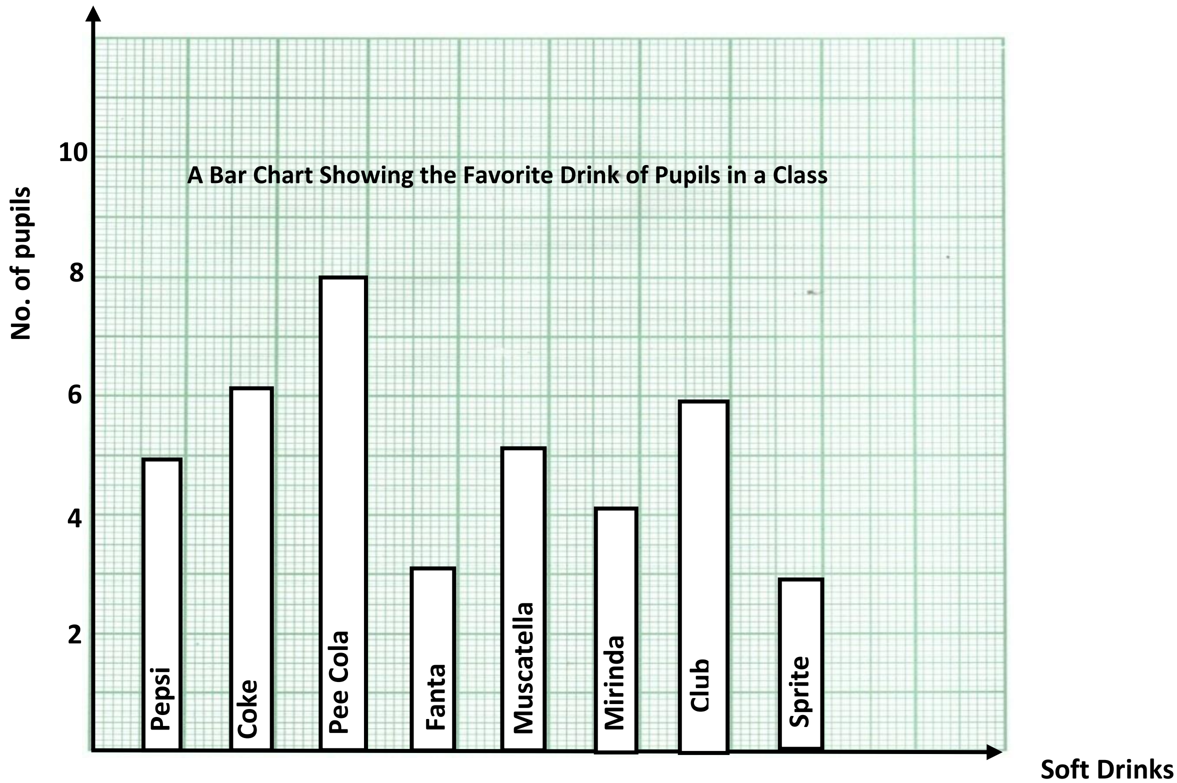
3. The following table is a result of a survey conducted in a class of a J.H.S to find the favorite soft drink of each pupil in the class.

Soft drink	No. of pupils
Pepsi	5
Coke	6
Pee cola	8
Fanta	3

Muscatella	5
Mirinda	4
Club	6
Sprite	3

Draw a bar chart showing this information, using a scale of 2cm to 1 unit on the vertical axis.

Solution



Exercises 10.9

1. The ages of 30 students in a class is shown below:

15 13 12 14 13 12 15 15 15 14 13 12 12 14 14 15 11 12 11 13 15 15 16 13 15 16 13 12 11 12.

i. Construct a frequency table for the data.

ii. Calculate from the table, the mode, median and mean.

iii. Draw abar chart for the data.

2. A group of 20 students took an examination in Twi and the following results were recorded (out of 10 marks).

6 5 4 6 7 5 7 4 6 5
10 8 8 9 9 9 9 6 6 9

i. Construct a frequency distribution table for this set of data.

ii. Use your table to determine the median and mean.

iii. Draw a bar chart for the information.

3. A die was tossed 50 times and the following results were recorded:
3 2 6 4 6 2 5 1 4 5 2 5 4 1 2 6
1 3 2 6 3 1 3 5 6 5 2 4 6 4 2 1
1 6 1 2 4 2 5 6 1 6 6 2 1 3 4 2
5 1

i. Construct a frequency table distribution table for the data.

ii. Determine the mode, median and mean from the table.

iii. Use the information to draw a bar chart.

4. Asibey performs an experiment with two dice, the numbers that showed up were added to obtain results shown in the table below:

Score	2	3	4	5	6	7	8
Frequency	2	2	5	1	1	3	4

Using this information, draw a bar chart.

5. The frequency table below is the mark distribution of a class of 35 pupils in a Mathematics examination.

marks	1	2	3	4	5	6	7	8	9	10
Freq	2	4	5	6	10	1	2	3	1	1

Use the distribution to draw a bar chart.

Extracting Information from a Bar Chart

In some situations, the bar chart may be drawn for students to answer some questions. It is quite necessary to note the following:

i. Consider the height of the tallest or the longest bar (Mode).

ii. Consider the height of each bar

iii. Add up the frequencies of all the bars

iv. Find the differences between the frequencies of all the bars

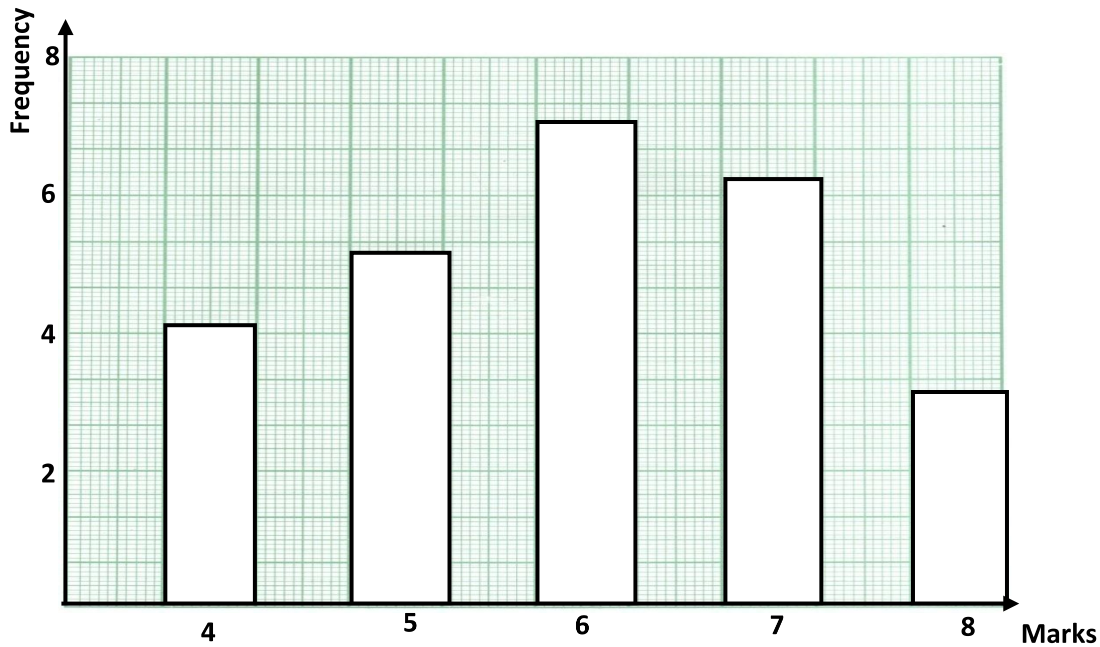
Worked Examples

1. The bar chart below shows the mark distribution in a test.

i. What is the modal mark?

ii. How many pupils took the test?

Solution



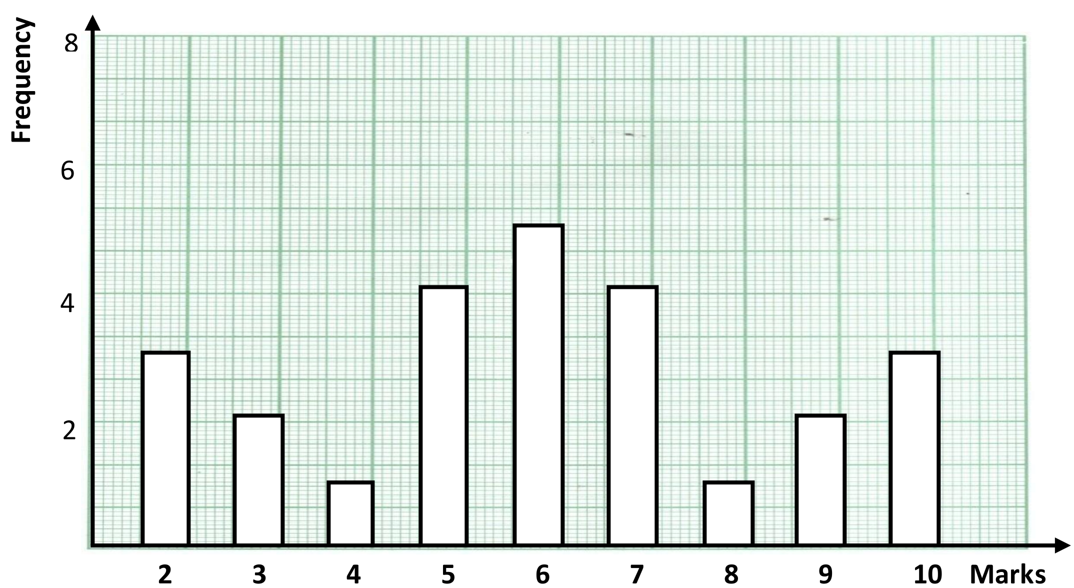
Solution

i. The modal mark is the mark with the longest (or highest) frequency (or bar) = 6

To find the total number of pupils who took the test, add up the frequencies of all the bars

= 4 + 5 + 7 + 6 + 3 = 25.

2. The bar chart below is for the distribution of marks in a class test.



i. Write down the frequency table for the distribution.
ii. Use the table to find the mean mark. If the pass mark is 4, how many pupils failed the test?

Solution

a. i

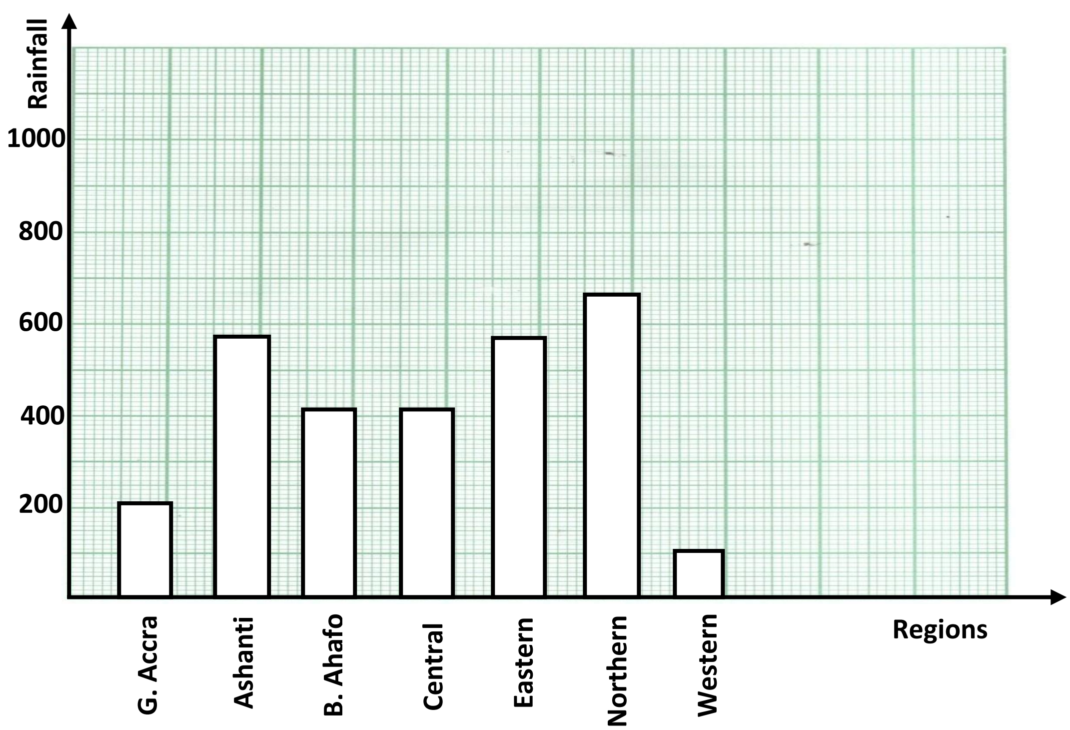
Marks(x)	Tally	Frequency(f)	fx
2	///	3	6
3	//	2	6
4	/	1	4
5	////	4	20
6	卌	5	30
7	////	4	28
9	//	2	18
10	///	3	30
		$\Sigma f = 25$	Σfx =150

ii. The mean ($\bar{x}$) = $\frac{\Sigma fx}{\Sigma f} = \frac{150}{25} = 6$

The number of people who failed the test if the pass mark is 4 = 3 + 2 = 5

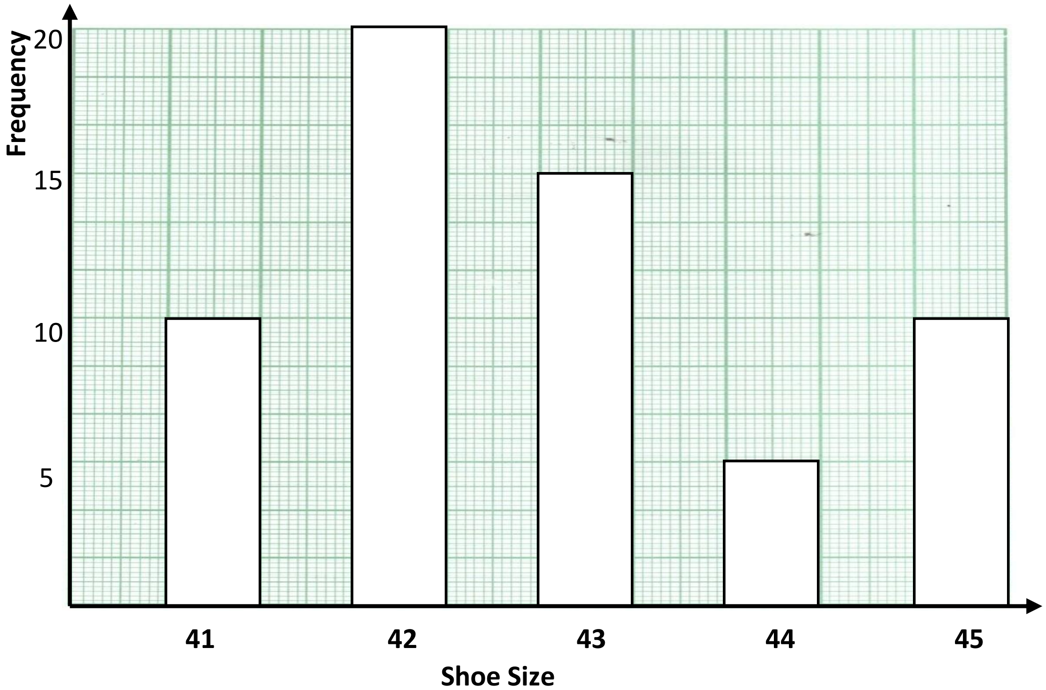
Exercise 10.10

1. The bar chart below shows the monthly rainfall distribution (in mm) for some selected regions in Ghana



Use the bar chart to answer the following questions:
i. Which region recorded the highest amount of rainfall
ii. What was the least amount of rainfall recorded?
iii. How much more rainfall was recorded in the Northern region than Ashanti.
iv. What was the total rainfall recorded by the entire regions in the chart?

2. The bar chart below shows the sizes of shoes sold in a shop



From the bar chart;
i. Which shoe size is the most common?
ii. Which shoe size is the least common?
iii. How many more size 43 shoes were sold than the size 41?
iv. How many shoes were sold in all?

Grouped Frequency Table
When the values of the entries are put into well organized groups and presented in a tabular form, a grouped frequency table is obtained. The groups into which the entries are put are referred to as ***classes***

The grouped frequency table is usually used for data's with many entries. Below is an example of a grouped frequency table

Class Marks (x)	Frequency (f)
25 – 29	3

30 – 34	4
35 – 39	7
40 – 44	6
45 – 49	2

On the above table, column 1 represents the group put into the classes: 25 – 29, 30 – 34, 35 – 39, 40 – 44, 45 – 49 and column 2 represents the respective frequencies. If the class 25 – 29 has a frequency of 3, it means that there are three different or same numbers within 25 and 29

Class Limits

Consider the table below;

Class Marks (x)	Frequency (f)
25 – 29	3
30 – 34	4
35 – 39	7
40 – 44	6
45 – 49	2

It is seen that the classes contain a range of values i. e. 25 – 29, 30 – 34, 35 – 39, 40 – 44, 45 – 49. This means that each class has a smallest value and a largest value. These two values are called ***Class limits*.** Thus, we have the lower class limit representing the smallest number and the upper class limit representing the largest number. For example in the class 25 – 29, the lower class limit is 25 and the upper class limit is 29

Lower class limit $\longrightarrow$ **25 – 29** $\longleftarrow$ ***upper class limit***

Class Boundaries

It is the point where one class separates itself from the other. It is also explained as the smallest and largest values an item in that class can take. For grouped frequency distribution, each class boundary is usually half way between the upper limit of one class and the lower limit of the next class. That is:

$= \frac{1}{2}$***(upper limit of one class + lower limit of the next class)***

For instance, to determine the class boundary of the class 25 – 29 and the next class 30 – 34,

I. Identify the upper class limit of 25 – 29 as 29 and the lower class limit of 30 – 34 as 34

II. Find the sum of the limits identified in I as 29 + 30 = 59

III. Find half of the sum obtained in II as:

$\frac{1}{2}(59) = 29.5$

IV. Recognize 29.5 as the upper class boundary of the class 24 – 29 and the lower class boundary of the next class 30 – 34

Steps in Finding the Class Boundary of a Grouped Data

To find the class boundary of a grouped data, follow the steps below;

I. Find the difference between the upper class limit of a particular class interval and the lower class limit of the next class and divide by 2. For example, the class boundary of 25 – 29, 30 – 34… is determined as $\frac{1}{2}(30 - 29) = 0.5$

II. Subtract the answer obtained from the lower class limit of each class and add the answer to the upper class limit of each class to obtain the class boundaries.

Worked Examples

Copy and complete the table below;

Marks (x)	Class Boundaries
5 – 9	
10 – 14	

15 – 19	
20 – 24	
25 – 29	

Solution

Method 1

$\frac{1}{2}(9+10)$= 9.5 $\quad\quad$ $\frac{1}{2}(14+15)$ = 14.5

$\frac{1}{2}(19+20)$ = 19.5 $\quad\quad$ $\frac{1}{2}(24+25)$ = 24.5

Marks (x)	Class Boundaries
5 – 9	4.5 – 9.5
10 – 14	9.5 – 14.5
15 – 19	14.5 – 19.5
20 – 24	19.5 – 24.5
25 – 29	24.5 – 29.5

Method 2

Consider 5 – 9 and 10 – 14

$\frac{1}{2}(10 - 9)$ = 0.5

Subtract 0.5 obtained from the lower class limit of each class and add 0.5 to the upper class limit of each class

Marks (x)	Class Boundaries
5 – 9	4.5 – 9.5
10 – 14	9.5 – 14.5
15 – 19	14.5 – 19.5
20 – 24	19.5 – 24.5
25 – 29	24.5 – 29.5

Class Mid – point/Class Mark

It is the average of the lower and upper limits of a class. This is achieved by adding the lower and upper class limits and dividing the sum by two to get the mid – point of the class. The midpoint is often called ***class mark***. For example, the class mark of the class 5 – 9 is determine as :$\frac{5+9}{2}$ = $\frac{14}{2}$ = 7

Worked Example

Copy and complete the table below:

Marks (x)	Class Mark
5 – 9	
10 – 14	
15 – 19	
20 – 24	
25 – 29	

Solution

(Calculations are not necessary)

Marks (x)	Class Mark
5 – 9	7
10 – 14	12
15 – 19	17
20 – 24	22
25 – 29	27

Class Size/Width

The class width or class size or class interval is the difference between the upper and lower class boundaries of aclass. Forinstance, the class width or class size of a class with boundaries 4.5 and 9.5 = 9.5 – 4.5 = 5

Format of a Grouped Frequency Table

Intervals	Class Boundary	Class Marks (x)	Frequency (f)	fx
			$\sum f =$	$\sum fx =$

Worked Examples

Below is a record of the marks (out of 50) obtained by 22 students in a test.

14 12 8 19 13 27 22 21 32 30 35 39 36 37 42 45 40 41 44 46 44 46

Prepare a group frequency table for the data using the intervals 1 – 10 , 11 – 20 …

Solution

Marks	Class Boundaries	Class Marks	Frequency (f)	fx
1 - 10	0.5 – 10.5	5.5	1	5.5
11 – 20	10.5 – 20.5	15.5	4	62
21 – 30	20.5 – 30.5	25.5	4	102
31 – 40	30.5 – 40.5	35.5	6	213
41 – 50	40.5 – 50.5	45.5	7	318.5
			$\sum f = 22$	$\sum fx = 701$

Exercises 10.11

1. The following is a record of scores out of 100 obtained by 40 candidates in a mathematics examination

50 70 66 52 56 68 64 52 68 56
56 50 50 54 64 50 58 58 60 56
74 50 72 51 59 54 52 60 66 58
52 56 53 58 68 52 64 62 52 56

Draw a grouped frequency table for the intervals 50 – 54, 55 – 59, 60 – 64 …

2. The following is a record of scores out of 100 obtained by 35 candidates in a mathematics examination

36 18 44 30 40 10 36 20 34 46
12 44 34 30 48 18 20 42 16 44
32 40 48 28 30 48 18 20 42 16
44 32 40 48 28

Draw a grouped frequency table for the intervals 10 – 14, 15 – 19, …

3. The marks scored by a group of students in a test are given below;

14 62 51 37 70 42 12 52 31 53 45 70 72
54 76 27 21 58 57 22 68 13 56 75 54 40
59 63 19 59 70 51 41 55 56 36 56 73 57
50 . Group the data in tens and construct a frequency table

Mean of a Grouped Data

To calculate the mean from a grouped frequency table, make sure the table is of the format shown below:

Class Interval	Class Mark(x)	f	fx
		$\sum f =$	$\sum fx =$

II. Carefully complete the table and accurately compute the values of $\sum f$ and $\sum fx$.

III. Substitute the values of $\sum f$ and $\sum fx$ into the formula below to obtain the mean value of the data : **Mean ($\bar{x}$) =** $\frac{\sum fx}{\sum f}$

Mode of a Grouped Data

For grouped data, the mid – point of the class with the highest frequency may be taken as the crude estimate of the mode. The class in which the mode occurs is called the ***modal class***

Median of a Grouped Data

For a grouped data, it is a bit complicated to determine the median but not the class within which the median value falls. These are few hints:

I. Find half of the total frequency and identify the value.

II. Count from the top of the table to obtain a sum of frequencies that equals the value obtained.

III. Identify the class within which the values falls as the ***median class.***

A good estimate of the median is best obtained from a cumulative frequency curve

Worked Examples

1.The table below shows the distribution of the masses of parcels in a local post office

Masses (Kg)	Frequency
20 – 24	2
25 – 29	3
30 – 34	7
35 – 39	26
40 – 44	29
45 – 49	25
50 – 54	6
55 – 59	2

Find from the table:

i. the mean ii. the mode

iii. the modal class iv. the median class

Solution

Masses (*kg*)	Class Marks(x)	f	fx
20 – 24	22	2	44
25 – 29	27	3	81
30 – 34	32	7	224
35 – 39	37	26	962
40 – 44	42	29	1218
45 – 49	47	25	1175
50 – 54	52	6	312
55 – 59	57	2	114
		$\sum f$= 100	$\sum fx$ = 4130

Mean ($\bar{x}$) = $\frac{\sum fx}{\sum f} = \frac{\mathbf{4130}}{\mathbf{100}}$ = 41.30 kg

ii. The mode is the midpoint of the highest frequency = 42

iii. From the table, the highest frequency is 29 and falls within the class 40 – 44. Therefore, the modal class is 40 – 44

iv. The median class

$= \frac{1}{2} \times \sum f = \frac{1}{2} \times 100 = 50$

Adding frequencies from the top, 50 can be located in the class 40 – 44. Therefore, the median class = 40 – 44

2. The marks obtained by 40 students in an examination are as follows;

63	76	87	61	78	85	77	87	74	77
80	77	74	88	72	78	79	89	85	90
77	70	81	69	75	78	73	86	83	91
69	96	65	88	84	74	84	81	83	75

Copy and complete the table below using the data above

Class Boundaries	(f)	Class Midpoint (x)	fx
59.5 – 64.5		62	
64.5 – 69.5		67	
69.5 – 74.5			
74.5 – 79.5			
79.5 – 84.5			
84.5 – 89.5			
89.5 – 94.5			
94.5 – 99.5		97	97
Total	$\sum f$ = 40		$\sum fx$=

b. i. Using the relation $\bar{x}=\frac{\sum fx}{\sum f}$, or otherwise, find the mean $\bar{x}$

ii. Calculate the probability that a student chosen at random obtained at least 75 marks

Solution

a.

Class	Class Boundaries	(f)	Class Midpt (x)	fx
60 - 64	59.5 – 64.5	2	62	124
65 – 69	64.5 – 69.5	3	67	201
70 – 74	69.5 – 74.5	6	72	432
75 – 79	74.5 – 79.5	11	77	847
80 – 84	79.5 – 84.5	7	82	574
85 – 89	84.5 – 89.5	8	87	696
90 – 94	89.5 – 94.5	2	92	184
95 – 99	94.5 – 99.5	1	97	97
		$\sum f$ = 40		$\sum fx$=3155

b. i. $\bar{x} = \frac{\sum fx}{\sum f} = \frac{3155}{40} = 78.88$

ii. P(at least 75) = $\frac{11+7+8+2+1}{40} = \frac{29}{40}$

Exercises 10.12

1. The table below shows the distribution of marks scored by a group of 30 students. Find: the mean. Ans = 27.4

Marks	10 – 14	15 – 19	20 – 24	25 – 29	30 – 34	35 – 39
No of Students	1	3	4	6	7	4

2. The marks obtained by candidates in a mathematics examination were first grouped 0 – 9, 10 – 19, 20 – 29, 30 – 39 and so on.
The midpoint of the mark of each group was taken as the mark representing the group. The following table gives the distribution of the marks;

Midpoint	4 .5	14.5	24.5	34.5	44.5
Freq.	18	19	x	12	9
Midpoint	54.5	64.5	74.5	84.5	
Freq.	5	2	2	1	

If the mean mark for the candidate was found to be 26.06;

i. determine the value of x

ii. draw a histogram for the distribution

iii. find the probability that a candidate chosen at random obtained 55 marks or more

The Assumed Mean

The assumed mean as the name suggests, is an assumption or a guess of the mean. The assumed mean is usually denoted by the letter "A". It doesn't need to be correct or even closer to the actual mean and the choice of the assumed mean is usually at one's own discretion (usually any of the class marks estimated to be closer to the mean) except where the question explicitly states a certain assumed mean value.

The assumed mean is used to calculate: the actual mean, the variance and the standard deviation.

Given the assumed mean, the actual mean is calculated by the formula: $\bar{x} = A + \frac{\sum fD}{\sum f}$

These values are obtained by completing the frequency table such as the one below:

class	***x***	***d* = *x* - A**	**Freq. (*f*)**	***fd***
			$\sum f$	$\sum fd$

Worked Examples

1. The student body of a certain school were polled to find out what their hobbies were. The number of hobbies of each student was recorded and the data obtained were grouped into classes as shown on the table below.

Number of hobbies	Frequency
0 – 4	45
5 – 9	58
10 – 14	27
15 – 19	30
20 – 24	19
25 – 29	11
30 – 34	8
35 – 39	2

Using an assumed mean of 17, find the mean of the number of hobbies of the students in the school.

Solution

Assumed mean, A = 17

Class	x	d = x – A	f	fd
0 – 4	2	-15	45	-675
5 – 9	7	-10	58	-580
10 – 14	12	-5	27	-135
15 – 19	17	0	30	0
20 – 24	22	5	19	95
25 – 29	27	10	11	110
30 – 34	32	15	8	120
35 – 39	37	20	2	40
			$\sum f =$ 200	$\sum fd =$ -1025

$\bar{x} = A + \frac{\sum fD}{\sum f}$

But A = 17, $\sum f$ = 200 and $\sum fd$ = - 1025

$\bar{x} = 17 + \frac{-1025}{200}$

$\bar{x} = 17 - 5.125 = 11.88$

2. The table below shows the marks obtained by students in an examination

Marks	0 – 9	10 – 19	20 – 29	30 – 39	40 – 49
Freq	2	3	15	10	10

Assuming the mean is 15, calculate the actual mean mark

Solution

A = 15

class	x	*d* = x – A	f	fd
0 – 9	4.5	-10.5	2	-21
10 – 19	14.5	-0.5	3	-1.5
20 – 29	24.5	9.5	15	142.5
30 – 39	34.5	19.5	10	195
40 – 49	44.5	29.5	10	295
			$\sum f$ = 40	$\sum fd =$ 610

$\bar{x} = A + \frac{\sum fD}{\sum f}$

But $A = 15$, $\sum f = 40$ and $\sum fd = 610$

$\bar{x} = 15 + \frac{610}{40} = 15 + 15.25 = 30.25$

Exercises 10.13

1. Consider the test results of 42 form three students in Mathematics

marks	40 – 49	50 – 59	60 – 69	70 – 79	80 – 89
Freq.	1	4	19	15	3

Taking the assumed mean as 64.5, find the actual mean of the distribution

2. The test results of 50 students were recorded as follows:

Marks	50 – 59	60 – 69	70 – 79	80 – 89	90 – 99
Freq.	6	11	19	9	5

Assuming the mean is 74.5, calculate the mean mark

3.The table below shows the marks obtained by students in an examination.

Marks	0 – 9	10 – 19	20 – 29	30 – 39	40 – 49
Freq	2	3	15	10	10

Assuming the mean is 15, calculate the actual mean mark.

4. Calculate the mean of the following data using the assumed mean method, given the assumed mean as 30.5;

Class	Frequency
10 – 19	11
20 – 29	5
30 – 39	13
40 – 49	4
50 – 59	3
60 – 69	4

The Histogram

A histogram is a graph showing the number of occurrences of items of data, and is drawn using vertical bars, similar to that of a bar chart.Group data is usually represented graphically by histograms.

The bars used for histograms are usually of standard size. The width of each bar is equivalent to the class interval representing it. Also, the area of the bars isproportional to the total frequency of the items in the class they represent

Steps in Drawing a Histogram

I. Choose a suitable scale for both vertical and horizontal axes, when no scale is given. The scale must be large enough to ensure that the histogram covers at least half of the graph sheet. Usually, scales using multiples of 5 or 10 are the best

II. Draw two axes, the vertical and horizontal axes. Label the vertical axes for the frequencies and the horizontal according to the given data. E.g. Ages (years), Marks Height (cm), not forgetting the units where applicable

III. The bars can be drawn in two ways explained as follows:

a. Using the class boundaries, draw bars using the lower and upper class boundaries

b. Using the class marks, place the class mark at the center of the bars and ensure that each bar has the same width corresponding to the class width.

IV. Use the symbol to indicate that the graph does not start from zero on the horizontal axis. The vertical axis should start from zero

V. Give the graph a title

Histogram for Ungrouped Data

Worked Examples

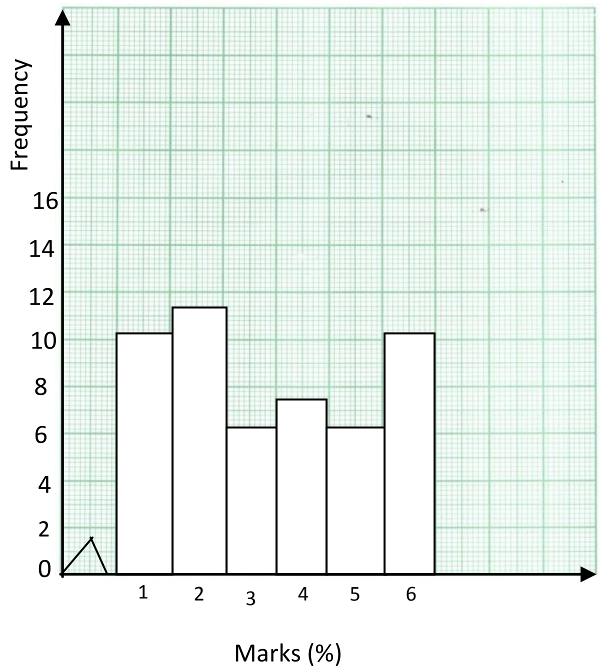
1. A die was tossed 50 times and the following were recorded

3 2 6 4 6 2 5 1 4 5 2 5 4 1
2 6 1 3 2 6 3 1 3 5 6 3 2 4
6 4 2 1 1 6 1 2 4 2 5 6 1 6
6 2 1 3 4 2 5 1

Use the data to draw a histogram

Solution

Score	Tally	Frequency
1	#####	10
2	##### /	11
3	### /	6
4	### //	7
5	### /	6
6	#####	10



Histogram for a Grouped Data

Worked Examples

1. Below is a record of the marks (out of 50) obtained by 22 students in a test.

14 12 8 19 13 27 22 21 32 30 35
39 36 37 42 45 40 41 44 46 44 46

i. Prepare a group frequency table for the data using the intervals 5 – 9 , 10 – 14 …

ii. From the table, determine the mean mark

iii. Draw a histogram to represent the data

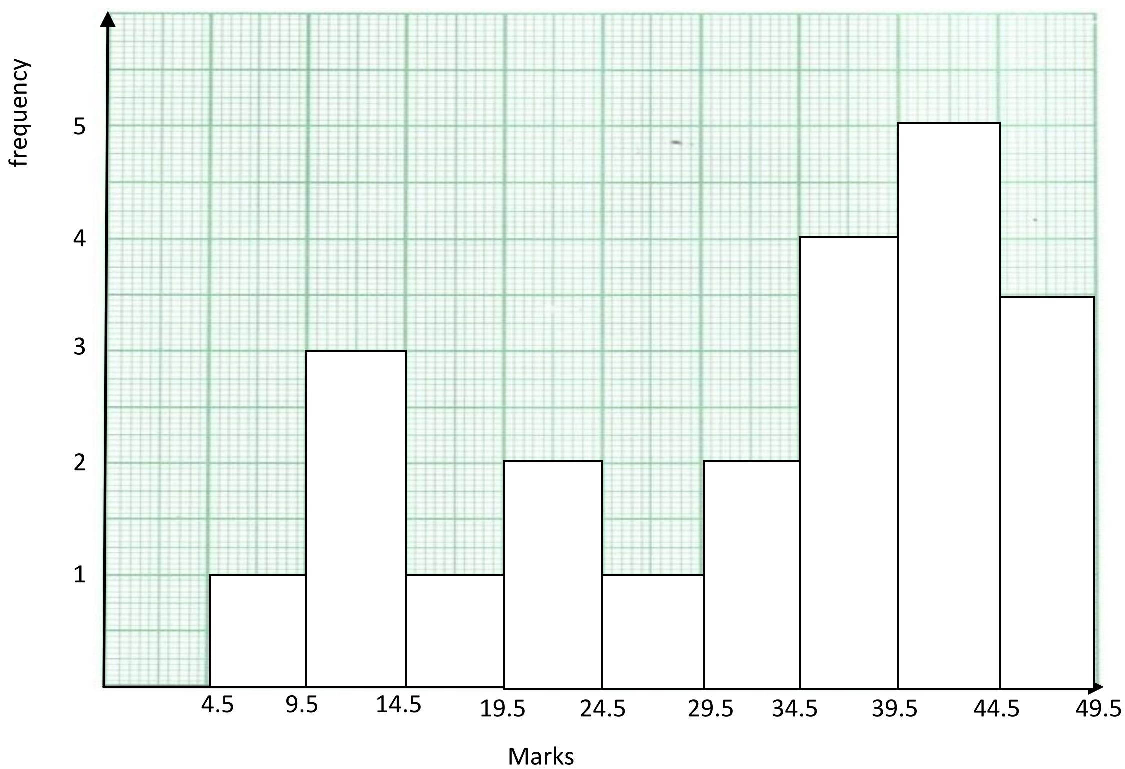
Solution

i.

Marks	Class Boundaries	Class Marks (x)	Frequency (f)	fx
5 – 9	4.5 – 9.5	7	1	7
10 – 14	9.5 – 14.5	12	3	36
15 – 19	14.5 – 19.5	17	1	17
20 – 24	19.5 – 24.5	22	2	44
25 – 29	24.5 – 29.5	27	1	27
30 – 34	29.5 – 34.5	32	2	64
35 – 39	34.5 – 39.5	37	4	148
40 – 44	39.5 – 44.5	42	5	210
45 – 49	44.5 – 49.5	47	3	141
			$\sum f$ = 22	$\sum fx$ = 694

ii. Mean ($\bar{x}$) = $\frac{\sum fx}{\sum f}$ = $\frac{694}{22}$ = 31.55

iii.



2. Copy and complete the table below and use it to draw a histogram using:
i. the class boundaries
ii. the class mark

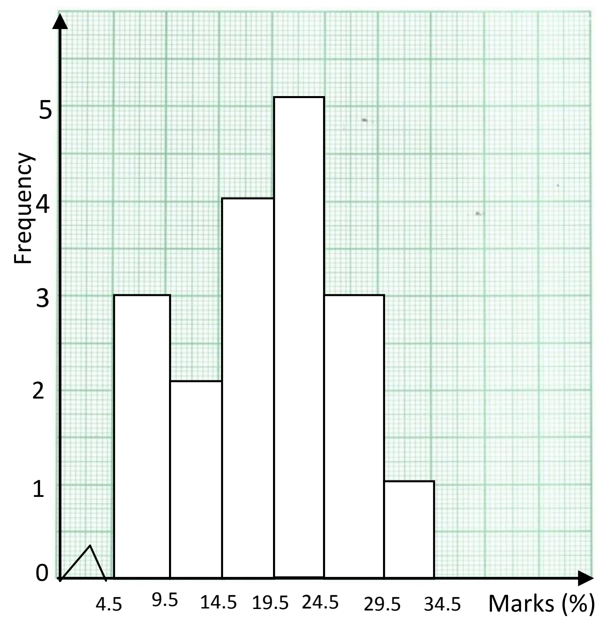
Ages (years)	Frequency
5 – 9	3
10 – 14	2

15 – 19	4
20 – 24	5
25 – 29	3
30 – 34	1

Solution

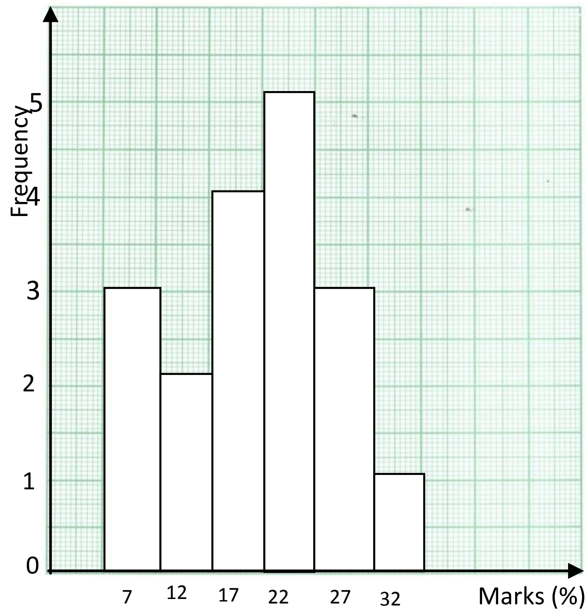
i. Using class boundaries

Ages (yrs)	Class boundaries	Frequency
5 – 9	4.5 – 9.5	3
10 – 14	9.5 – 14.5	2
15 – 19	14.5 – 19.5	4
20 – 24	19.5 – 24.5	5
25 – 29	24.5 – 29.5	3
30 – 34	29.5 – 34.5	1



ii. Using the class mark

Ages (yrs)	Class Mark	Frequency
5 – 9	7	3
10 – 14	12	2
15 – 19	17	4
20 – 24	22	5
25 – 29	27	3
30 – 34	32	1



__Estimating the Mode from a Histogram__

The class which has the highest frequency is called the ***modal class***. The actual mode is contained in the modal class.

The mode of can be estimated from the histogram by following the steps below:

I. Draw the histogram for the data.

II. Draw two diagonals from the top corners of the adjacent bars

a. Carefully draw a straight line from the top right corner of the tallest bar to the top right corner of the bar on the left

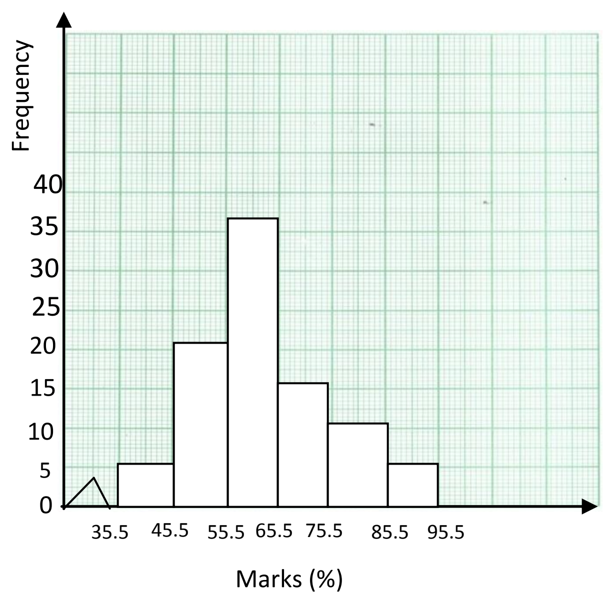
b. Draw another straight line from the top left corner of the tallest bar to the left corner of the bar to the right.

III. Draw a perpendicular from the point of intersection of the diagonals onto the horizontal axis.

IV. Read off the value of the point where the perpendicular touches the horizontal axis, as the mode.

Worked Examples

1. Determine the mode of the histogram below



Solution

The modal class has the tallest bar

= (55.5 – 65.5)

From the histogram, the modal mark = 55.9

Estimating the Median from a Histogram

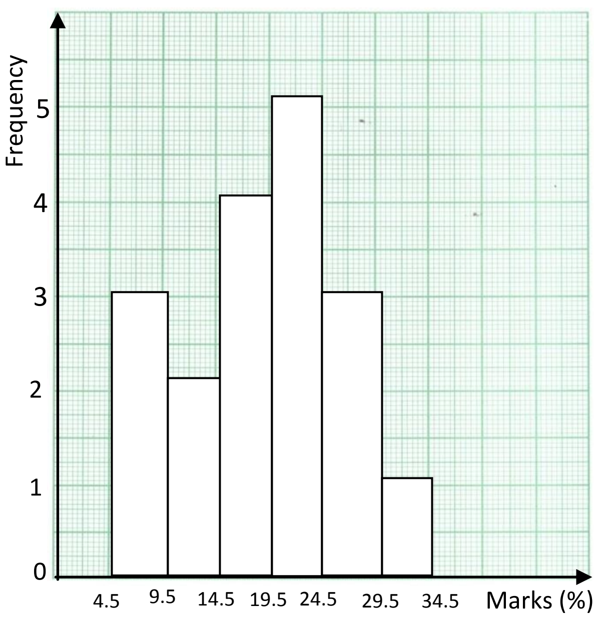
The Median can be estimated from the histogram by carefully following the steps below:

I. Draw the histogram for the distribution

II. Find the sum of the areas of all the rectangles

III. Calculate half of the sum of the areas of all the rectangles and locate it vertically on the histogram

IV. If the vertical line falls exactly on a boundary, then read off the boundary value as the median. However, if the vertical line does not fall on a boundary, then calculate the area of the additional strip and add it to the area up to the boundary value just on the left, to obtain the median.



Solution

Area of each bar = L × B,

B = (9.5 – 4.5) = 5

$A_1 = 3 \times 5 = 15$

$A_2 = 2 \times 5 = 10$

$A_3 = 4 \times 5 = 20$

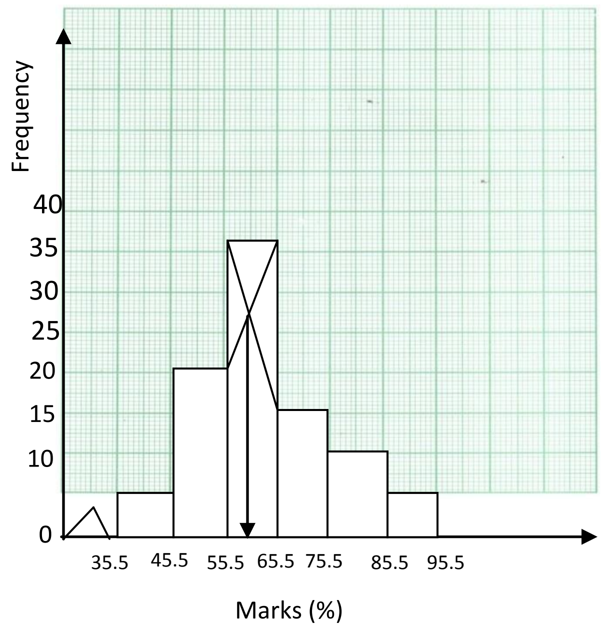
$A_4 = 5 \times 5 = 25$

$A_5 = 3 \times 5 = 15$

$A_6 = 1 \times 5 = 5$

Worked Examples

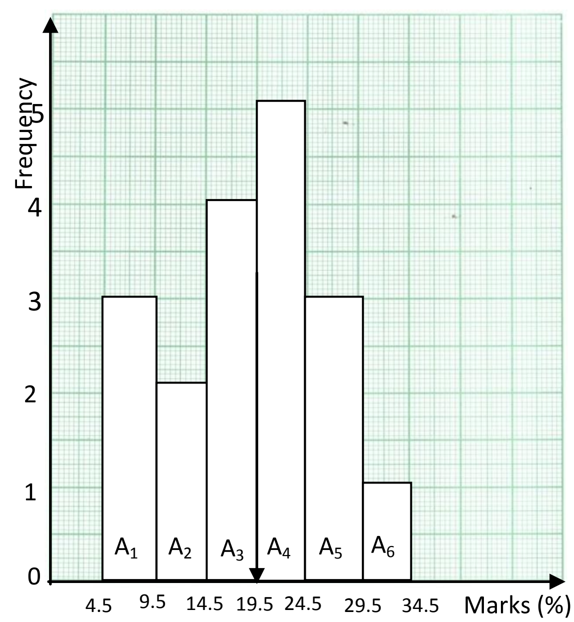
Determine the median of the histogram.



Total area = 15 + 10 + 20 + 25 +15 + 5 = 90

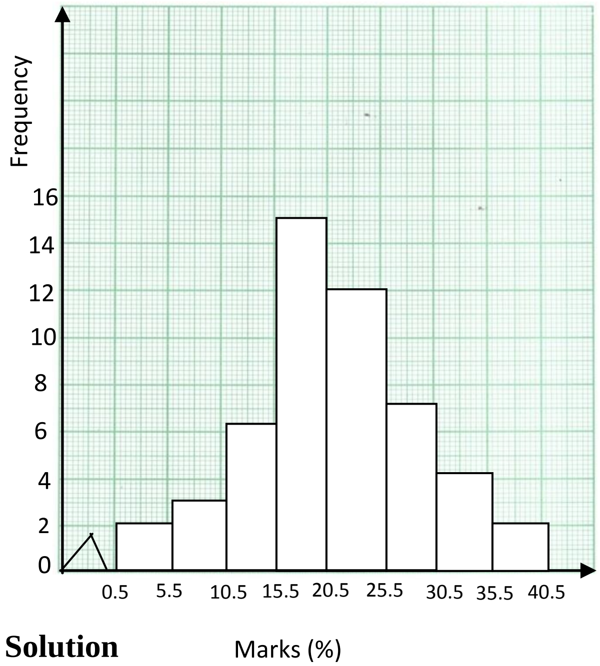
Half of total area = $\frac{90}{2}$ = 45

Median class = (14.5 – 19.5) and (19.5 – 24.5)



Median = 19.5 (shown on the graph)

2. What is the median of the histogram below?



Solution

Area of each bar = L × B,

B = (5.5 – 0.5) = 5

$A_1 = 2 \times 5 = 10$ $\quad$ $A_5 = 12 \times 5 = 60$

$A_2 = 3 \times 5 = 15$ $\quad$ $A_6 = 7 \times 5 = 35$

$A_3 = 6 \times 5 = 30$ $\quad$ $A_7 = 4 \times 5 = 20$

$A_4 = 15 \times 5 = 75$ $\quad$ $A_8 = 2 \times 5 = 10$

Total area = 10 + 15 + 30 + 75 + 60 + 35 + 20 + 10 = 255

Half of total area = $\frac{255}{2} = 127.5$

Median class = (15.5 – 20.5) meaning after 15.5 but not up to 20.5

From the histogram, area of the shaded region or rectangle = $15 \times x = 15x$

Half the area = $A_1 + A_2 + A_3$ + area of shaded region = 127.5

$$10 + 15 + 30 + 15x = 127.5$$

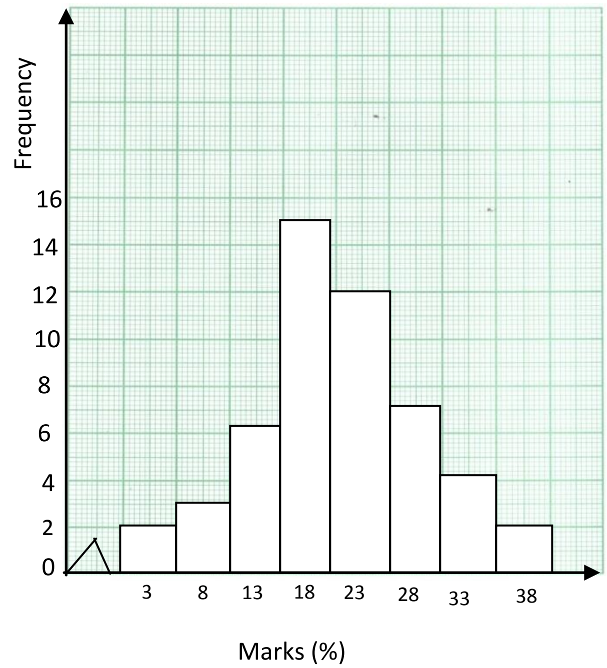
$$15x = 127.5 - 10 - 15 - 30$$

$$15x = 72.5$$

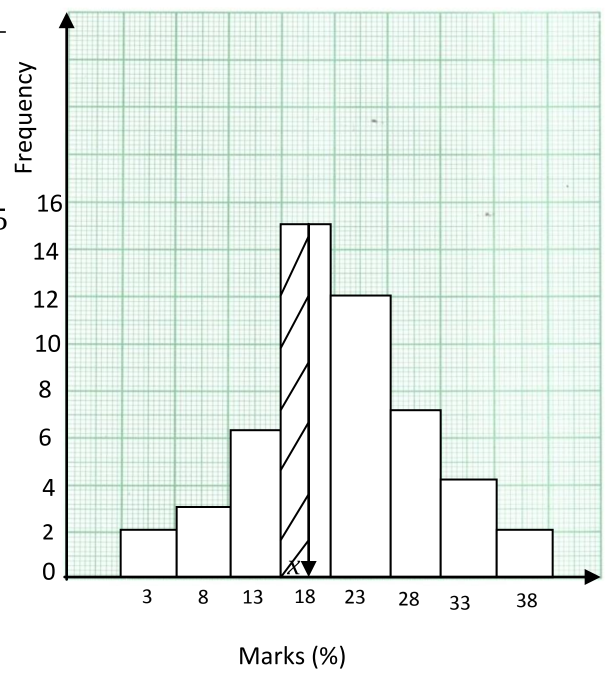
$$x = \frac{72.5}{15} = 4.$$

Estimated median = 15.5 + 4.8 = 20.3

2. What is the median of the histogram below?



Solution



Area of each bar = L × B,
B = (8 – 3) = 5

$A_1 = 2 \times 5 = 10$
$A_2 = 3 \times 5 = 15$
$A_3 = 6 \times 5 = 30$
$A_4 = 15 \times 5 = 75$
$A_5 = 12 \times 5 = 60$
$A_6 = 7 \times 5 = 35$
$A_7 = 4 \times 5 = 20$
$A_8 = 2 \times 5 = 10$

Total area
= 10 + 15 + 30 + 75 + 60 + 35 + 20 + 10 = 255

Half of total area = $\frac{255}{2}$ = 127.5

Median class = (18)

Class boundary = $\frac{18-13}{2}$ = 2.5

Lower class boundary of the median class;
= 13 + 2.5 = 15. 5

From the histogram, area of the shaded region or rectangle = 15 × x = 15x

Half the area = $A_1 + A_2 + A_3$ + area of shaded region = 127.5

10 + 15 + 30 + 15x = 127.5

15x = 127.5 – 10 – 15 – 30

15x = 72.5

$x = \frac{72.5}{15} = 4.8$

Estimated median = Lower class boundary + x
= 15.5 + 4.8 = 20.3

Exercises 10.15

1. The marks obtained by 30 pupils in an examination are as follows;

71 65 83 78 74 63 73 87 78 80 93
78 88 76 80 68 77 68 70 84 90 69
75 61 66 90 95 85 77 76

a. Arrange these marks in a frequency distribution table using class intervals of 61 – 65, 66 – 70, 71 – 75…

b. i. Draw a histogram of the distribution

ii. Use your histogram to estimate the mode

c. Calculate the mean of the distribution.

2. The table gives the heights measured to the nearest meter of 291 trees.

Height(m)	2	3	4	5	6	7	8	9
No of trees	14	12	42	83	118	12	7	3

a. Draw a histogram to illustrate the information.

b. Calculate the mean of the distribution correct to the nearest meter.

c. From the histogram, estimate the mode and median of the heights.

3. The frequency table below shows the height (to the nearest cm) of 26 girls in JHS 1.

Class interval (cm)	Frequency
140 – 143	1
144 – 147	2
148 – 151	7
152 – 155	5
156 – 159	8
160 – 163	3

a. Draw a histogram of the distribution.

b. Calculate the mean of the distribution correct to the nearest centimeter.

c. From the histogram, estimate the mode and median of the heights.

4.The marks obtained by some candidates in an examination are;

27 36 40 48 50 55 695 8 57 60 61 63
61 63 65 68 65 68 67 70 55 71 69
73 72 75 76 78 76 80 81 83 88 89 90

i. Form a group frequency table for this data, using the class intervals 20 – 29, 30 – 39…

ii. From the table, calculate the mean correct to 1 decimal place.

iii. Draw a histogram for the data.

5. The following are the marks scored by 50 students in a Mathematics test.

41	45	56	58	28	56	40	55	29	34
28	52	8	43	16	24	44	51	24	26
28	23	25	36	38	28	36	47	17	37
42	21	23	46	24	53	39	47	39	40
32	27	27	34	24	48	48	32	46	32

i. Prepare a frequency table for these data using equal class intervals of 1 – 10, 11 – 20 and so on.
ii. What is the modal class of the distribution?
iii. Illustrate the data by means of a suitable diagram.
iv. Using an assumed mean of 35.5 or otherwise, calculate from your table, the mean of the distribution.

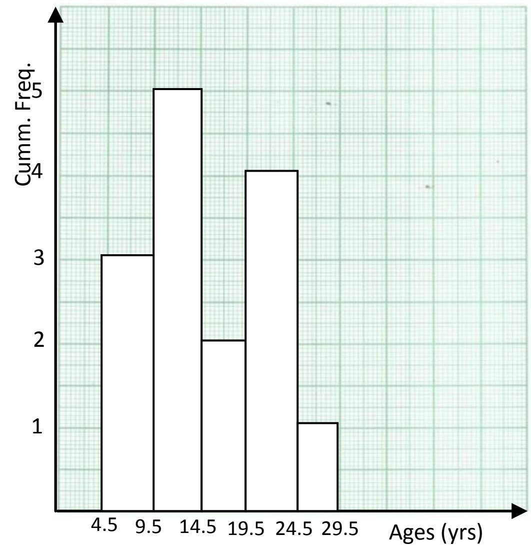
Preparing a Frequency Distribution Table for a Given Histogram

A. Class Boundary Histogram

I. On the vertical axis, identify the height of each bar as the frequency.
II. For each bar, identify the first and second numbers on the horizontal axis as the lower class bounadary and upper class boundary respectively, and by this technique, generate all the class boundaries.
III. For each class boundaries, add 0.5 to the lower class boundary and subtract 0.5 from the upper class boundaries to generate each class width or intervals.

Worked Examples

Prepare a frequency distribution table for the histogram below;



Solution

I. Frequencies = Height of bars = 3, 5, 2, 4, 1
II. Class boundaries = 4.5 – 9.5, 9.5 – 14.5, 14.5 – 19.5, 19.5 – 24.5, 24.5 – 29.5
III. Add 0.5 to lower class boundaries and subtract 0.5 from upper class boundaries of the same group to generate the class size.
$\Rightarrow$5 – 9, 10 – 14, 15 – 19, 20 – 24, 25 – 29

Class Size	Class Boundary	Frequencies
5 – 9	4.5 – 9.5	3
10 – 14	9.5 – 14.5	5
15 – 19	14.5 – 19.5	2
20 – 24	19.5 – 24.5	4
25 – 29	24.5 – 29.5	1

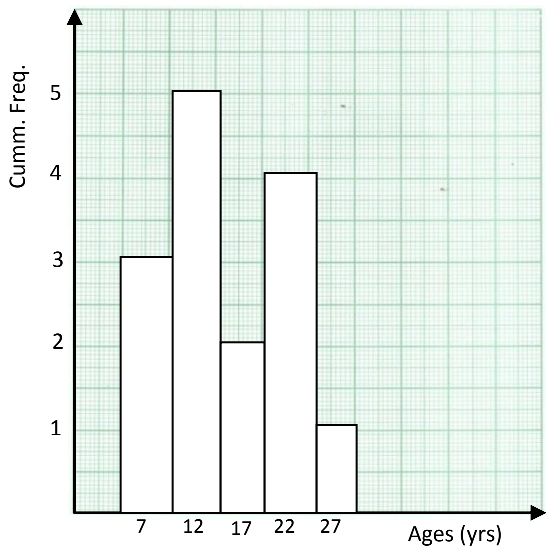
B. Class Mark/Mid Point Histogram

I. On the vertical axis, identify the height of each bar as the frequency
II. On the horizontal axis, find the common difference (d) between the class marks, and divide by two to obtain an answer, say ***A.***
$\Rightarrow \frac{d}{2} = \boldsymbol{A}$

III. Subtract and add the answer obtained in act II, that is ***A,*** from and to each class mark to obtain the lower class boundary and the upper class boundary respectively of that class

IV. For each class boundaries, add 0.5 to the lower class boundary and subtract 0.5 from the upper class boundaries to generate each class width or intervals.

Worked Example



Solution

I. Frequencies = Height of bars = 3, 5, 2, 4, 1

II. Difference between the first and second class marks = 12 – 7 = 5, the $\frac{5}{2} = 2.5$

III. Subtract and add 2.5 from and to each class mark to obtain the class boundaries

7 – 2.5 = 4.5 and 7 + 2.5 = 9.5
(4.5 – 9.5)

12 – 2.5 = 9.5 and 12 + 2.5 = 14.5
(9.5 – 14.5)

17 – 2.5 = 14.5 and 17 + 2.5 = 19.5
(14.5 – 19.5)
22 – 2.5 = 19.5 and 22 + 2.5 = 24.5
(19.5 – 24.5)
27 – 2.5 = 24.5 and 27 + 2.5 = 29.5
(24.5 – 29.5)

IV. Add 0.5 to the lower class boundary and subtract 0.5 from the upper class boundaries
(5 – 9) (10 – 14), (15 – 19), (20 – 24)
(25 – 29) to obtain class size/width

Class Size	Class Boundary	Frequencies
5 – 9	4.5 – 9.5	3
10 – 14	9.5 – 14.5	5
15 – 19	14.5 – 19.5	2
20 – 24	19.5 – 24.5	4
25 – 29	24.5 – 29.5	1

The Frequency Polygon

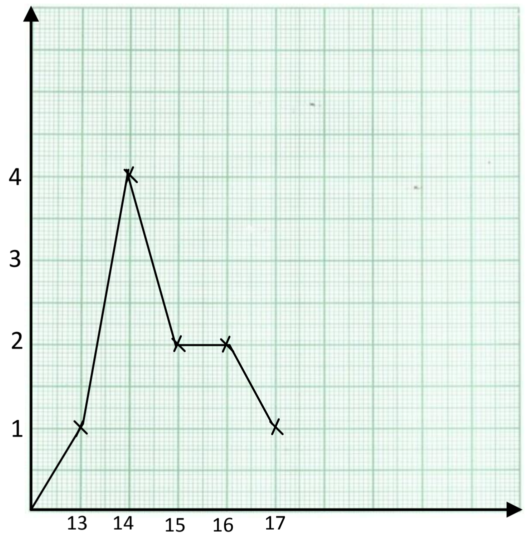
It is a line graph in which frequencies are plotted against corresponding class marks. To be precise, if you mark two additional intervals on each side of a histogram and then join the mid points of these intervals to the mid points of horizontal sides of the rectangle with straight lines, a frequency polygon is obtained

Worked Examples

1. Use the table below to draw a frequency polygon

Age (yrs)	13	14	15	16	17
Frequency	1	4	2	2	1

Solution

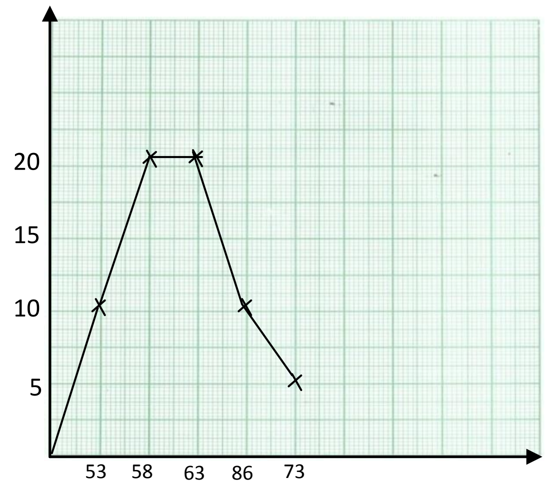


2. Given the following frequency distribution table of marks in a language test.

Score	Class mid – point	Frequency
51 – 55	53	10
56 – 60	58	20
61 – 65	63	20
66 – 70	68	10
71 – 75	73	5

Draw a frequency polygon for the distribution

Solution



Exercises 10.17

The marks scored by a group of students in a test are given below;

14 62 51 37 60 42 12 53 52 3153
45 20 72 54 26 31 27 60 11 21 58 57 22 68 8 56 37 54 40 59 63 19 59 20 51
41 55 38 56 36 56 33 57

i. Group the data in tens and construct a frequency table

ii. Draw a cumulative frequency polygon for the table

2. Fifty bags of sweets were opened. The number of sweets in each bag is recorded in the table below:

No of sweets	18 – 22	23 – 27	28 – 32	33 – 37	38 – 42
Frequency	6	14	16	11	3

Represent this information in the form of a frequency polygon

11 RIGID MOTION

Transformation

Transformation is the process of changing the shape, size and position of an object. The Methods used are;

1. Translation
2. Enlargement
3. Reflection
4. Rotation
5. Mapping

Translation

In translation, a translation vector written in the form $\binom{a}{b}$ is added to the given point(s) or object to produce the image. For e.g, if $A\binom{x}{y}$ is translated by the vector $\binom{a}{b}$, then the image of A, which is $\boldsymbol{A_1 = A}$ ***+ translation vector***.

That is: $\boldsymbol{A_1 = \binom{x}{y} + \binom{a}{b} = (x + a, y + b)}$

Properties of Translation

1. Every point moves the same distance in the same direction
2. The size of an angle remains the same.
3. The length of the line remains the same.
4. The shape of the object remains the same.
5. The size of the object does not change.

Worked Examples

1. If $A = (7, 5)$ is translated by the vector $\binom{3}{4}$, find the image, A_1

Solution

$A_1 = A +$ translation vector

$A_1 = \binom{7}{5} + \binom{3}{4} = \binom{7+3}{5+4} = \binom{10}{9} = (10, 9)$

2. Find the image of $P(2, 6)$ when translated by the vector $\binom{2}{3}$.

Solution

$P_1 = P +$ Translation vector

$P_1 = \binom{2}{6} + \binom{2}{3} = \binom{2+2}{6+3} = \binom{4}{9} = (4, 9)$

3. Find the image of Q (-3, 4) under a translation by the vector $\binom{-3}{4}$.

Solution

$Q_1 = Q +$ Translation vector.

$Q_1 = \binom{-3}{4} + \binom{-3}{4} = \binom{-3-3}{4+4} = \binom{-6}{-8} = (-6, -8)$

4. Given A (5, 8), B (-2, 4) and C (9, 7), find the images $A^1B^1C^1$ of ABC respectively under a translation by the vector $\binom{-5}{-3}$.

Solution

$A_1 =$ A + Translation vector

$A_1 = \binom{5}{8} + \binom{-5}{-3} = \binom{5-5}{8-3} = \binom{0}{5} = (0, 5)$

$B_1 = B +$ Translation vector

$B_1 = \binom{-2}{4} + \binom{-5}{-3} = \binom{-2-5}{4-3} = \binom{-7}{1} = (-7, 1)$

$C_1 = C +$ Translation vector

$C_1 = \binom{9}{7} + \binom{-5}{-3} = \binom{9-5}{7-3} = \binom{4}{4} = (4, 4)$

Exercises 11.1

1. The points $P(1, 7)$, $Q(-2, -9)$ and $R(6, 4)$ are translated by the vector $\binom{-4}{5}$, find the images $P_1Q_1R_1$ of PQR, such that $P \to P_1$, $Q \to Q_1$ and $R \to R_1$

2. Given $R(7, 5)$, $S(4, 6)$ and $T(-2, -3)$, find the co-ordinates of the images R_1, S_1, T_1 of R, S, T under a translation by the vector $\binom{-2}{3}$ such that :

$R \rightarrow R_1$, $S \rightarrow S_1$ and $T \rightarrow T_1$.

3. The point R(-3, 2) is translated by the vector $\binom{4}{-3}$. It is then given a further translation by the vector $\binom{-2}{5}$. Determine the final image of R

Finding the Object Given the Image A_1 and the Translation Vector (Tv),

Given the image A_1 and the translation vector (Tv), the coordinates of the object, A, is calculated by the relation:

A = ***Image (A_1) – Translation vector (Tv)***

$\boldsymbol{A\binom{x}{y} = \binom{x_1}{y_1} - \binom{a}{b} = A\ (x_1 - a, y_1 - b\ )}$

Worked Examples

1. The image of an object translated by the vector $\binom{1}{4}$ is M_1 (7, 5). What is the coordinates of the object, *M*.

Solution

M = Image (M_1) – Translation vector (Tv)

$M = \binom{7}{5} - \binom{1}{4} = (7 - 1, 5 - 4) = (6, 1)$

The coordinates of the object *M* is *M*(6, 1)

2. Find the coordinates of the object *P*, translated by the translation vector $\binom{-3}{-5}$ to produce the image P_1(4, 2).

Solution

P = *Image (P_1) – Translation vector (Tv)*

$P = \binom{4}{2} - \binom{-3}{-5} = (4 + 3, 2 + 5) = (7, 7)$

The coordinates of the object *P* is (7, 7)

3. The point P(*x*, *y*) is mapped onto P^1(5, 9) by the vector $\binom{-6}{7}$. Find the coordinates of P.

Solution

P = *Image (P_1) – Translation vector (Tv)*

$P = \binom{5}{9} - \binom{-6}{7} = (5 + 6, 9 - 7) = (11, 2)$

The coordinates of the object *P* is (11, 2)

Exercises 11.2

1. The respective images of *ABC* under translation by the translation vector $\binom{4}{-3}$ are A_1(0, 5), B_1(- 3, 8) and C_1(- 5, -1). Find the coordinates of *ABC*.

2. K^1(9, - 4) is the image of *K* under translation by the translation vector $\binom{5}{-2}$. Determine the coordinates of *K*.

3. If *U*(*x*, *y*) under translation by the translation vector $\binom{-4}{-1}$ produced the image U_1(-7, - 4). Find the values of *x* and *y*.

4. The image of a point under a translation by the vector $\binom{-3}{-5}$ is (2, -2). Find the coordinates of the point. Ans (5, 3)

Finding the Translation Vector Given the Image A_1 and the Object, A.

Given the image, A_1 and the object A, the translation vector (Tv) is calculated by the relation:

Tv = Image (A_1) – Object (A)

$\boldsymbol{Tv = A_1\binom{x_1}{y_1} - A\binom{x}{y} = \binom{x_1 - x}{y_1 - y}}$

Worked Examples

1. The image of P (4, 1) under translation by the vector, M, is P_1(7, 6). What is the coordinates of the translation vector, M?

Solution

$Tv = A_1\binom{x_1}{y_1} - A\binom{x}{y} = \binom{x_1 - x}{y_1 - y}$

$M = P_1\binom{7}{6} - P\binom{4}{1} = \binom{7-4}{6-1} = \binom{3}{5}$

The translation vector is $M\begin{pmatrix}3\\5\end{pmatrix}$

2. If $V_1(5, 2)$ is the image of $V(9, 0)$ under translation, find the translation vector.

Solution

$Tv = A_1\begin{pmatrix}x_1\\y_1\end{pmatrix} - A\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x_1 - x\\y_1 - y\end{pmatrix}$

$Tv = V_1\begin{pmatrix}5\\2\end{pmatrix} - V\begin{pmatrix}9\\0\end{pmatrix} = \begin{pmatrix}5-9\\2-0\end{pmatrix} = \begin{pmatrix}-4\\0\end{pmatrix}$

3. $P_1(8, -2)$ is the image of the point $P(5, 2)$ by the translation vector V. Find:

a. the vector, V.

b. the coordinates of the point Q which maps onto the points $Q_1(5, -2)$ under V.

c. $\overrightarrow{P_1Q_1}$

d. $|P_1Q_1|$

Solution

a. $Tv = A_1\begin{pmatrix}x_1\\y_1\end{pmatrix} - A\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x_1 - x\\y_1 - y\end{pmatrix}$

$Tv = P_1\begin{pmatrix}8\\-2\end{pmatrix} - P\begin{pmatrix}5\\2\end{pmatrix} = \begin{pmatrix}8-5\\-2-2\end{pmatrix} = \begin{pmatrix}3\\-4\end{pmatrix}$

b. Object + TV = Image

$Q + TV = Q_1$

$Q + \begin{pmatrix}3\\-4\end{pmatrix} = \begin{pmatrix}5\\-2\end{pmatrix} = \begin{pmatrix}5\\-2\end{pmatrix} - \begin{pmatrix}3\\-4\end{pmatrix} = \begin{pmatrix}5-3\\-2--4\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$

c. $P_1(8, -2)$ and $Q_1(5, -2)$

$\overrightarrow{P_1Q_1} = \overrightarrow{OQ_1} - \overrightarrow{OP_1}$

$= \begin{pmatrix}5\\-2\end{pmatrix} - \begin{pmatrix}8\\-2\end{pmatrix} = \begin{pmatrix}5-8\\-2--2\end{pmatrix} = \begin{pmatrix}-3\\0\end{pmatrix}$

d. $P_1(8, -2)$ and $Q_1(5, -2)$

$|P_1Q_1| = \sqrt{(5-8)^2 + (-2 - -2)^2}$

$= \sqrt{(-3)^2 + 0^2} = 3$

Exercises 11.3

1. If $B_1(-5, 4)$ is the image of $B(3, 7)$ under translation, find the translation vector.

2. Under a translation, a point $L(-8, 2)$ has an image $L_1(-2, 1)$. Find the translation vector.

3. Under a translation, $(4, 7) \rightarrow (1, -3)$. What is the translation vector?

4. $A(4, 3)$, $B(-3, -2)$, $C(0, -7)$ and $A_1(5, 7)$, $B_1(4, 7)$, $C_1(1, 2)$ are such that $A \rightarrow A_1$, $B \rightarrow B_1$ and $C \rightarrow C_1$ under translation by different translation vectors. Identify the translation vector of each.

Enlargement

An enlargement is a transformation which increases the size of shapes by a constant *k*. This constant (number) is called the ***scale factor of enlargement***.

The scale factor does not have to be a whole number all the time. It could even be a number less than one, causing the object to be smaller. Under such conditions, the transformation can be called either ***an enlargement or a reduction***.

Whenever an enlargement is done, one point remains fixed. This point is called the ***center of enlargement***. Thus, if $A(x, y)$, is enlarged by the scale factor, *k*, from the origin, then the image of *A*, is: A_1 = scale factor × A.

$\Rightarrow A_1 = k \times \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}k \times x\\k \times y\end{pmatrix} = (kx, ky)$

Properties of Enlargement

Generally, in an enlargement with scale factor, *k*,.

I. Each length is multiplied by the scale factor.

II. Each angle remains constant.

III. The shape of the figures is maintained but not necessarily the size.

IV. Corresponding lines are parallel.

Worked Examples

1. Find the image of A (5, 4), under enlargement from the origin with scale factor 2.

Solution

A_1 = scale factor × A

$A_1 = 2 \times \binom{5}{4} = \binom{2 \times 5}{2 \times 4} = \binom{10}{8}$ = (10,-8)

2. What is the coordinates of the image of Q (8, -12) under enlargement with scale factor, $\frac{1}{5}$ from the origin?

Solution

Q_1 = scale factor × Q

$Q_1 = \frac{1}{5} \times \binom{8}{-12} = \begin{pmatrix} \frac{1}{5} \times 8 \\ \frac{1}{5} \times -12 \end{pmatrix} = \binom{1.6}{-2.4}$ = (1.6, -2.4)

Exercises 11.4

1. Find the coordinates of the image of A(3, -7) under enlargement with scale factor -2

2. What is the coordinates of the image of Q (-7, -11) under enlargement with scale factor, $\frac{1}{4}$ from the origin?

3. The vertices of a quadrilateral ABCD are (-5, 4), (1, 3), (-2, 2) and (1, 2) respectively. ABCD is enlarged by a scale factor, 3 to produce $A_1B_1C_1D_1$. Find the coordinates of the new figure formed.

Enlargement with Negative Scale Factor

An enlargement with a negative scale factor turns a shape upside down as well changing its size. That is if ***A(x, y)*** is enlarged by – *k*, then $\boldsymbol{A_1 = -k \binom{x}{y} = A_1(-kx, -ky)}$

Worked Examples

1. Find the image of Q(3,7) under enlargement from the origin with scale factor -2.

Solution

Q_1 = scale factor × Q

$Q_1 = -2 \times \binom{3}{7} = \binom{-2 \times 3}{-2 \times 7} = \binom{-6}{-14}$ =(-6, -14)

2. What is the image of B (-6, -1) under enlargement from the origin with scale factor -3.

Solution

B_1 = Scale factor × B

$B_1 = -3 \times \binom{-6}{-1} = \binom{-3 \times -6}{-3 \times -1} = \binom{18}{3}$ = (18, 3)

3. Given P (5,-3), Q (-7, 2) and R(-6, -8). Find the image $P_1 Q_1R_1$ such that $P \rightarrow P_1$, $Q \rightarrow Q_1$ and $R \rightarrow R_1$ under enlargement with scale factor - 3 from the origin.

Solutions

P_1 = Scale factor × P

$P_1 = -3 \times \binom{5}{-3} = \binom{-3 \times 5}{-3 \times -3} = \binom{-15}{9}$ = (-15, 9)

Q_1 = Scale factor × Q

$Q_1 = -3 \times \binom{-7}{2} = \binom{-3 \times -7}{-3 \times 2} = \binom{21}{-6}$ = (21, -6)

R_1 = Scale factor × R

$R_1 = -3 \times \binom{-6}{-8} = \binom{-3 \times -6}{-3 \times -8} = \binom{18}{24}$ = (18, 24)

Exercises 11.5

1. If *A*(0, 5), *B*(-4, 7) and *C*(-2, -3) is enlarged by scale factor -2 from the origin, what is the co-ordinates of the image $A_1B_1 C_1$ such that $A \rightarrow A_1$, $B \rightarrow B_1$and $C \rightarrow C_1$?

2. Find the co-ordinates of the image of *R*,*S*, *T* under enlargement with scale factor $-\frac{1}{3}$, such that

$R \to R_1$, $S \to S_1$ and $T \to T_1$ from the origin given $R(5, -3)$, $S(7, 4)$, $T(-6, -1)$.

3. Find the coordinates of the image of P(-9, 12) under enlargement with scale factor, $-\frac{2}{3}$ from the origin

Enlargement from a Given Point/Centre

When the point $A(x, y)$ is enlarged from a point other than the origin, say (a, b), with scale factor, k,

I. Subtract the center of enlargement from the coordinate of the object. i.e: $\mathbf{A_1}\begin{pmatrix} x-a \\ y-b \end{pmatrix}$

II. Multiply the results by the scale factor, k. That is: $A_1\begin{pmatrix} x-a \\ y-b \end{pmatrix}k = A_1\begin{pmatrix} kx-ka \\ ky-kb \end{pmatrix}$

III. Add the results to the center of enlargement to obtain the coordinates of the image, A_1. That is: $A_1\begin{pmatrix} kx-ka \\ ky-kb \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$

The steps are easily remembered by the abbreviations **S.M.A,** meaning Subtract, Multiply and Add.

Worked Examples

1. Find the image of M(5, 3) under enlargement from the point (0, 2) with scale factor – 4.

Solution

M(5, 3), $k = -4$, point of enlargement (0, 2)

i. *Subtract the point of enlargement from the coordinates of the object*

$M_1\begin{pmatrix} 5-0 \\ 3-2 \end{pmatrix} = M_1\begin{pmatrix} 5 \\ 1 \end{pmatrix}$

ii. *Multiply $M_1\begin{pmatrix} 5 \\ -1 \end{pmatrix}$ by the scale factor, -4*

$M_1\begin{pmatrix} 5 \\ 1 \end{pmatrix} \times -4 = M_1\begin{pmatrix} 5\times -4 \\ 1\times -4 \end{pmatrix} = M_1\begin{pmatrix} -20 \\ -4 \end{pmatrix}$

iii. *Add the results to the center of enlargement*

$M_1\begin{pmatrix} -20 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} = M_1\begin{pmatrix} -20 \\ -2 \end{pmatrix} = M_1(-20, -2)$

2. The triangle ABC has vertices at A(-2, 1), B(2, 3), C (0, 5). This triangle is enlarged from the point (0,1) by a scale factor of $\frac{1}{2}$. Write down the new coordinates A_1, B_1, C_1 of ABC such that $A \to A_1$, $B \to B_1$, $C \to C_1$

Solution

A(-2, 1), $k = \frac{1}{2}$, center of enlargement (0, 1)

$A_1\begin{pmatrix} -2-0 \\ 1-1 \end{pmatrix} = A_1\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

$A_1\begin{pmatrix} -2 \\ 0 \end{pmatrix} \times \frac{1}{2} = A_1\begin{pmatrix} -2/2 \\ 0/2 \end{pmatrix} = A_1\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$A_1\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = A_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} = A_1(-1, 1)$

$B_1\begin{pmatrix} 2-0 \\ 3-1 \end{pmatrix} = B_1\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$B_1\begin{pmatrix} 2 \\ 2 \end{pmatrix} \times \frac{1}{2} = B_1\begin{pmatrix} 2/2 \\ 2/2 \end{pmatrix} = B_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$B_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = B_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} = B_1(1, 2)$

$C_1\begin{pmatrix} 0-0 \\ 5-1 \end{pmatrix} = C_1\begin{pmatrix} 0 \\ 4 \end{pmatrix}$

$C_1\begin{pmatrix} 0 \\ 4 \end{pmatrix} \times \frac{1}{2} = C_1\begin{pmatrix} 0/2 \\ 4/2 \end{pmatrix} = C_1\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$C_1\begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = C_1\begin{pmatrix} 0 \\ 3 \end{pmatrix} = C_1(0, 3)$

A_1(-1, 1), B_1(1, 2), C_1(0, 3)

3. A triangle has vertices $A(1, 1)$, $B(2, 4)$ and C (5, 8)

i. Calculate the new coordinates of these vertices if the triangle is translated by the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

ii. If instead, the triangle is enlarged using A as center of enlargement, with scale factor 2, find the coordinates of the images of B and C.

Solution

i. $A(1, 1)$, $B(2, 4)$ and C $(5, 8)$

Translation by the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

O + Tv = I

$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = A_1\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $A_1(2, 0)$

$B\begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = B_1\begin{pmatrix} 3 \\ 3 \end{pmatrix}$, $B_1(3, 2)$

C$\binom{5}{8}$+ $\binom{1}{-1}$ = C$_1\binom{6}{7}$, $C_1(6, 7)$

ii. Centre of enlargement A(1, 1)
Scale factor = 2
B$\binom{2}{4}$ – $\binom{1}{1}$ = $\binom{1}{3}$
$\binom{1}{3}\times 2 = \binom{2}{6}$
$\binom{2}{6}$+ $\binom{1}{1}$= B$_2\binom{3}{7}$ $B_2(3, 7)$

C$\binom{5}{8}$ – $\binom{1}{1}$ = $\binom{4}{7}$
$\binom{4}{7}\times 2 = \binom{8}{14}$
$\binom{8}{14}$ + $\binom{1}{1}$= C$_2\binom{9}{15}$ $C_2(9, 15)$

Exercises 11.6

A. 1. Triangle ABC has vertices A(1, 2), B(3, 2), C(5, 0). The triangle is enlarged by a scale factor of $2\frac{1}{2}$ about the point (1, 4). Write down the coordinates of the transformed triangle.

2. If the line segment AB with vertices A(1, 2) and B(3, 2), is reduced by a scale factor of $\frac{1}{2}$ about the point (1, 0), find the coordinates of the images A_1, B_1.

3. The rectangle PQRS has corners at P (1, 1), Q(4, 1), R(4, -5), S(1, -5). The rectangle is enlarged by the factor of $1\frac{1}{3}$ about the point (4, -2). Write down the coordinates of the vertices of the enlarged rectangle.

4. Write down the coordinates of the images of the point (3, 2) under enlargement with the following centers and scale factors;

i. (3, 2), -5   ii. (9, 8), $-\frac{1}{2}$

Finding the Object given the image A_1 and the Scale Factor, k

Given the image $A_1(x, y)$ and the scale factor, K, the object A is calculated by the rule;

$$\boldsymbol{A = A`_1\begin{pmatrix} x/k \\ y/k \end{pmatrix} = \left(\frac{x}{k},\ \frac{y}{k}\right)}$$

Worked Examples

1. The image produced by M under enlargement with scale factor 2 is $M_1(8, 12)$. What is the coordinates of M?

Solution

$$A = A_1\begin{pmatrix} x/k \\ y/k \end{pmatrix} = \left(\frac{x}{k},\ \frac{y}{k}\right)$$

Given $M_1(12, 8)$ and scale factor, 2
$x = 8$, $y = 12$ and $k = 2$

$$M = M_1\begin{pmatrix} 8/2 \\ 12/2 \end{pmatrix} = \left(\frac{8}{2},\ \frac{12}{2}\right) = (4, 8)$$

2. The image of R when it is enlarged by a scale factor – 3 is R_1 (3, -9). Find the coordinates of the R

Solution

$$A = A`_1\begin{pmatrix} x/k \\ y/k \end{pmatrix} = \left(\frac{x}{k},\ \frac{y}{k}\right)$$

Given $R_1(3, -9)$ and scale factor, -3
$x = 3$, $y = -9$ and $k = -3$

$$R = R_1\begin{pmatrix} 3/-3 \\ -9/-3 \end{pmatrix} = \left(\frac{3}{-3},\ \frac{-9}{-3}\right) = (-1, 3)$$

Exercises 11.7

1. Under enlargement with scale factor 3, $B \rightarrow B_1$. If the coordinate of B_1 is (15, 6), what is the coordinates of B?

2.The image of A when it is enlarged by a scale factor –2 is A_1 (-6, -2). Find the coordinates of R

3. The vertices of a triangle are PQR.. If P is enlarged by a scale factor, $\frac{1}{2}$, Q is enlarged by a scale factor 4, and R is enlarged by scale factor, –

$\frac{2}{3}$, such that P $\rightarrow$ P_1(-6, 2), Q $\rightarrow$ Q_1(0, 1) and *R* $\rightarrow$$R_1$(-9, -12), determine the coordinates of *PQR*

Reflection

Here an image is produced or formed by reflecting an object or point in either the *x* – axis or *y* – axis. Thus, we have -:

1. Reflection in the *x* – axis or line *y* = 0
2. Reflection in the *y*- axis or line *x* = 0
3. Reflection in a given line

Reflection in the x- axis (Line y = 0)

This is represented by the mapping:

$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x \\ -y \end{pmatrix}$

Object *Image*

Worked Examples

1. If *A*(3, 5) is reflected in the *x* – axis, find the image A_1

Solution

$A\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow A_1\begin{pmatrix} x \\ -y \end{pmatrix}$

$A\begin{pmatrix} 3 \\ 5 \end{pmatrix} \rightarrow A_1\begin{pmatrix} 3 \\ -5 \end{pmatrix}$, A_1(3, -5)

2. Find the image of *P*(-2, -4) when it is reflected in the *x*- axis.

Solution

In the *x*- axis, $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x \\ -y \end{pmatrix}$

$P\begin{pmatrix} -2 \\ -4 \end{pmatrix} \rightarrow P_1\begin{pmatrix} -2 \\ 4 \end{pmatrix}$, P_1(-2, 4)

3. What is the image of Q (7, 9) under reflection in the *x*- axis?

Solution

Reflection in the *x* – axis

$Q\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow Q_1\begin{pmatrix} x \\ -y \end{pmatrix}$

$Q\begin{pmatrix} 7 \\ 9 \end{pmatrix} \rightarrow Q_1\begin{pmatrix} 7 \\ -9 \end{pmatrix}$, Q_1(7, -9)

4. Find the image of M(-11, 5) under reflection in the line *y* = 0

Solution

$M\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow M_1\begin{pmatrix} x \\ -y \end{pmatrix}$

$M\begin{pmatrix} -11 \\ 5 \end{pmatrix} \rightarrow M_1\begin{pmatrix} -11 \\ -5 \end{pmatrix}$, M_1(-11, -5)

Solved Past Questions

1. a. Find the image of the position vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ under translation by the vector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

b. If A(2, 3) is reflected in the *x* – axis, find the image A^1 of A

Solution

a. $\begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

b. $A\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow A_1\begin{pmatrix} x \\ -y \end{pmatrix}$

$A\begin{pmatrix} 2 \\ 3 \end{pmatrix} \rightarrow A_1\begin{pmatrix} 2 \\ -3 \end{pmatrix}$, A_1(2, -3)

Reflection in the y- axis (Line x = 0)

This is represented by the mapping,

$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -x \\ y \end{pmatrix}$

Object *Image*

Worked Examples

1. If A (3, 8) is reflected in the *y* – axis, find its image, A_1

Solution

$A\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow A_1\begin{pmatrix} -x \\ y \end{pmatrix}$

$A\binom{3}{8} \rightarrow A_1\binom{-3}{8}$, $A_1(-3, 8)$

2. Given A (3, 2), B (7, 5) and C(-5, -4), find the image $A_1B_1C_1$ of ABC under reflection in the line $x = 0$, such that $A \rightarrow A_1$, $B \rightarrow B_1$ and $C \rightarrow C_1$.

Solution

For reflection in the line $x = 0$ or y- axis;

$\binom{x}{y} \rightarrow \binom{-x}{y}$

$A\binom{3}{2} \rightarrow A_1\binom{-3}{2}$, A_1 (-3, 2)

$B\binom{7}{5} \rightarrow B_1\binom{-7}{5}$, B_1 (-7, 5)

$C\binom{-5}{-4} \rightarrow C_1\binom{5}{-4}$, C_1 (5,-4)

Exercises 11.8

1. The image of P(3, 0) is P_1. If P is reflected inthe y – axis, find the coordinates of P_1.
2. Given that U (6, 1), V (3, 3) and W $(4, -7)$. Find the images $U_1V_1W_1$ of U, V, W under reflection in the line $x = 0$.

Reflection in a Given Line

1. Reflection in the line y = x or y – x = 0

This is represented by the mapping:

$\binom{x}{y} \rightarrow \binom{y}{x}$

Object   Image

Worked Example

1. P is the point (3, 7), write down the coordinate of the point obtained by reflecting P in the line $y = x$

Solution

For reflection in the line $y = x$ or $y – x = 0$

$\binom{x}{y} \rightarrow \binom{y}{x}$

$\binom{3}{7} \rightarrow \binom{7}{3}$,   (7, 3)

2. Reflection in the line y = -x or y + x = 0

This is represented by the mapping:

$\binom{x}{y} \rightarrow \binom{-y}{-x}$

Object   Image

Worked Example

1. Find the coordinates of the image of the point (5, 2) under reflection in the line $y = -x$

Solution

For reflection in the line $y = -x$;

$\binom{x}{y} \rightarrow \binom{-y}{-x}$

$\binom{5}{2} \rightarrow \binom{-2}{-5}$,   (-2, -5)

2. What is the coordinates of the image of A (-7, 5) under reflection in the line $y = – x$

Solution

For reflection in the line $y = – x$;

$\binom{x}{y} \rightarrow \binom{-y}{-x}$

$A\binom{-7}{5} \rightarrow A_2\binom{-5}{7}$, $A_2(-5, 7)$

3. Reflection in the line x = a or x – a = 0

This is represented by the mapping:

$\binom{x}{y} \rightarrow \binom{2(a)-x}{y}$

Object   Image

Worked Example

1. The point P(2, 1) is reflected in the line $x = 3$. Find the coordinates of its image.

Solution

For reflection in the line $x = a$

$\binom{x}{y} \rightarrow \binom{2(a)-x}{y}$

But $x = 2$, $y = 1$and line $a = 3$

Substitute in $\binom{x}{y} \rightarrow \binom{2(a)-x}{y}$

$\Rightarrow P\binom{2}{1} \rightarrow P_1\binom{2(3)-2}{1}$

$P\binom{2}{1} \rightarrow P_1\binom{4}{1}$ $P_1(4, 1)$

2. Under reflection in the line $x = 7$, the point A(2, 3) is mapped onto the point A_1:

i. Find the coordinates of A_1

ii. If O is the origin, find the area of ΔOAA_1

Solution

i. For reflection in the line $x = a$

$\binom{x}{y} \rightarrow \binom{2(a)-x}{y}$

But $x = 2$, $y = 3$ and line $a = 7$

Substitute in $\binom{x}{y} \rightarrow \binom{2(a)-x}{y}$

A $\binom{2}{3} \rightarrow A_1\binom{2(7)-2}{3}$

A $\binom{2}{3} \rightarrow A_1\binom{12}{3}$ $\quad$ $A_1(12, 3)$

ii. O(0, 0), A(2, 3) and $A_1(12, 3)$

$A = \frac{1}{2}bh$

But $b = /AA_1/ = \sqrt{(12-2)^2 + (3-3)^2}$

$= \sqrt{10^2 + 0^2} = \sqrt{10^2} = 10$

$h = y$ axis of /OA/

$h = \sqrt{(3-0)^2} = \sqrt{3^2} = 3$

$A = \frac{1}{2}bh = \frac{1}{2} \times 10 \times 3 = 15$sq. units

4. Reflection in the line y = a or y – a = 0

This is represented by the mapping:

$\binom{x}{y} \rightarrow \binom{x}{2(a)-y}$

Object $\quad$ *Image*

Worked Example

1. The point P(-3, 4) is reflected in the line $y = 2$. Find the image of P.

Solution

For reflection in the line $y = a$

$P_1\binom{x}{y} \rightarrow P_2\binom{x}{2(a)-y}$

But $x = -3$, $y = 4$ and $a = 2$

Substitute in $P_1\binom{x}{y} \rightarrow P_2\binom{x}{2(a)-y}$

$P_1\binom{-3}{4} \rightarrow P_2\binom{-3}{2(2)-4}$

$P_1\binom{-3}{4} \rightarrow P_2\binom{-3}{0}$ $\quad$ $P_1(-3, 0)$

2. Find the image of the point P(-12, 4) under a reflection in the line $y = -3$

Solution

For reflection in the line $y = a$

$P_1\binom{x}{y} \rightarrow P_2\binom{x}{2(a)-y}$

But $x = -12$, $y = 4$ and $a = -3$

Substitute in $P_1\binom{x}{y} \rightarrow P_2\binom{x}{2(a)-y}$

$P_1\binom{-12}{4} \rightarrow P_2\binom{-12}{2(-3)-4}$

$P_1\binom{-12}{4} \rightarrow P_2\binom{-12}{-10}$ $\quad$ $P_1(-12, -10)$

Exercises 11.9

1. The coordinates of the vertices of a quadrilateral are A(3, 1), B(-3, -4), C(0, -4) and D(5, -2). ABCD is reflected in the line $x = -4$, calculate the coordinates of the images of A, B, C and D.

2. A kite has vertices at the four points O(0,6), P(-3,-2), R(0, -6) and T(3, -2)

i. Find the coordinates of the images of the kite when it is reflected in the line $y = 3$

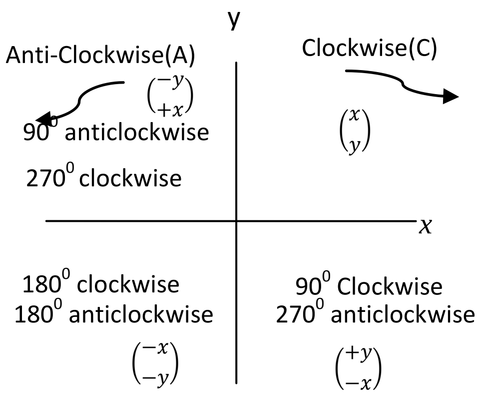
ii. If the image kite is reflected in the line $x = 2$, determine the coordinates of the of the vertices of the final kite obtained.

3. A parallelogram *ABCD* has vertices *A*(1, 3), *B*(4, 5) *C*(9, 5) and *D*(6, 3). It is given two transformations, a reflection in the x – axis, followed by a reflection in the line $x = -2$. Determine the coordinates of the vertices of the parallelogram in both positions.

4. The vertices of a triangle are A(2,0), B(-3, 0), C(-5, 3). The triangle is reflected in the line y = -4 and followed by a reflection in the line x = 4. Determine the coordinates of the of the vertices of the triangle in both positions.

Rotation

Rotation is the turning of an object about or around a fixed point or the origin. Rotation can be clockwise or anticlockwise through 90^0, 180^0, 270^0 and 360^0.



Clockwise and Anticlockwise Rotations about the Origin

1. Clockwise Rotation through 90^0 or Anti – clockwise Rotation through 270^0 about the origin.

This is described by the mapping;

$\binom{x}{y} \rightarrow \binom{y}{-x}$

Object Image

Worked Example

Find the image $A_1B_1C_1$ of A(5,4), B(-1,6) and C(-7, -2) under a clockwise rotation through 90^0 about the origin.

Solution

For clockwise rotation through 90^0 about the origin; $\binom{x}{y} \rightarrow \binom{y}{-x}$

$A\binom{5}{4} \rightarrow A_1\binom{4}{-5}$, A_1 (4, -5)

$B\binom{-1}{6} \rightarrow B_1\binom{6}{1}$, B_1 (6, 1)

$C\binom{-7}{-2} \rightarrow C\binom{-2}{7}$, C_1(-2, 7)

2. Clockwise Rotation through 180^0 or Anti – clockwise Rotation through 180^0 about the Origin.

This is described by the mapping;

$\binom{x}{y} \rightarrow \binom{-x}{-y}$

Object Image

Worked Example

Find the image of $P(7,4)$, Q(-5, -3) and $R(-10,8)$ under clockwise rotation through 180^0 about the origin.

Solution

For clockwise rotation through 180^0 about the origin:

$\binom{x}{y} \rightarrow \binom{-x}{-y}$

$P\binom{7}{4} \rightarrow P_1\binom{-7}{-4}$, P_1(-7, -4)

$Q\binom{-5}{-3} \rightarrow Q_1\binom{5}{3}$, Q_1(5, 3)

$R_1\binom{-10}{8} \rightarrow R_1\binom{10}{-8}$, R_1(10, -8)

3. Clockwise Rotation through 270^0 clockwise or Anti – clockwise Rotation Through 90^0 about the origin

This is represented by the mapping:

$\binom{x}{y} \rightarrow \binom{-y}{x}$

Object Image

Worked Examples

Given that $U(2,3)$, V(-6, -5) and $W(7,9)$. Find U_1, V_1 and W_1, the respective images of U, V, and W through a clockwise rotation of 270^0 about the origin.

Solution

For clockwise rotation of 270^0 about the origin*:*

$\binom{x}{y} \rightarrow \binom{-y}{x}$

$U\binom{2}{3} \rightarrow U_1\binom{-3}{2}$,

$V\binom{-6}{-5} \rightarrow V_1\binom{5}{-6}$

$W\binom{7}{9} \rightarrow W_1\binom{-9}{7}$

Solved Past Questions

The point P(-5, 1) is rotated through 180^0. Find the coordinates of its image

Solution

Rotation through 180^0 is represented by

$\binom{x}{y} \rightarrow \binom{-x}{-y}$

$P\binom{-5}{1} \rightarrow P_1\binom{5}{-1}$ $P_1(5, -1)$

2. The point P(3, -1) is rotated about the origin through an angle of 270^0 in the clockwise direction. Find the image of P

Solution

Rotation through 270^0clockwise is represented by: $\binom{x}{y} \rightarrow \binom{-y}{x}$

$P\binom{3}{-1} \rightarrow P_1\binom{-(-1)}{3}$ $P_1(1, 3)$

Exercises 11.10

Given A(3, 4), B(5, -2) and C(-9 , -3), find:

1. The image $A_1B_1C_1$ of ABC under clockwise rotation through 90^0 about the origin.
2. The image $A_2 B_2 C_2$ of ABC under a rotation through 180^0 clockwise about the origin.
3. The image $A_3 B_3 C_3$ of $A_1B_1C_1$ under a clockwise rotation through 270^0 about origin.
4. P(- 4, 3), Q(1, 2) and R (3,5) are the vertices of a triangle.

i. The triangle is rotated about the origin through 90^0 clockwise. Determine the coordinates of the vertices of P_1, Q_1 and R_1 of the image triangle $P_1 Q_1 R_1$

ii. Triangle $P_1Q_1R_1$ is also rotated through 90^0 clockwise to obtain triangle $P_2Q_2R_2$. Write down the coordinates of $P_2Q_2R_2$

Finding the Angle of Rotation Given the Object and its Image

Given the coordinates of the object and its corresponding image, the angle of rotation can be found by investigating the kind of rotation either clockwise or anti clockwise connecting them.

Mapping	Transformation
$\binom{x}{y} \rightarrow \binom{-y}{x}$	Rotation through 90^0 about the origin
$\binom{x}{y} \rightarrow \binom{y}{-x}$	Rotation through -90^0 about the origin
$\binom{x}{y} \rightarrow \binom{-y}{-x}$	Rotation through 180^0 about the origin

Note that a rotation of -90^0denotes a clockwise rotation of 90^0 and a rotation of 90^0 means an anti – clockwise rotation of 90^0.

Exercises 11.11

Each of the following pair is a point and its image under a rotation about the origin. State the angle of rotation in each case

1. (6, 3) → (-3, 6) 2. (5, -2) → (-2, -5)

3. (4, -1) → (-4, 1) 4. (-2, -3) → (3, -2)

Rotation About a Fixed Point

Clockwise and Anti – clockwise Rotations Through a given Angle about a Point (a, b)

When the point $A(x, y)$ is rotatedthrough 90^0about a pointsay (a, b), other than the origin, the image A_1 is found as shown below;

I. Identify the object as A(x, y), the angle of rotation and the center of rotation as (a, b)
II. Subtract the center of rotation from the given coordinate of the object. i.e. $A_1\binom{x-a}{y-b}$
III. Rotate the results (in II) through the given angle.
IV. Add the results (in III) to the center of rotation.
The processes are easily remembered by the abbreviations **S. R. A.** where **S** means **"Subtract", R** means **"Rotate"** and **A** means **"Add"**

Worked Examples

Find the image of the coordinates of the point (7, 8) under an anticlockwise rotation of 90^0 with centre (1, 5).

Solution

Object = (7, 8),
Angle of rotation = 90^0 anticlockwise
Centre of rotation = (1, 5)
Coordinate of the object – Centre of rotation
i.e. $A_1\binom{7-1}{8-5} = \binom{6}{3}$
$\binom{6}{3}$ under an anticlockwise rotation of 90^0
$\binom{x}{y} \rightarrow \binom{-y}{x}$
$\binom{6}{3} \rightarrow \binom{-3}{6}$
Add the results to the centre of rotation.
$\binom{-3}{6} + \binom{1}{5} = \binom{-2}{11}$, (-2, 11)

Exercises 11.12

1. Find the image of the coordinates of the point (-3, 8) under an anticlockwise rotation of 90^0 with centre (2, 3).
2. Find the image of the coordinates of the point (-5, -7) under a clockwise rotation of 90^0 with centre (-4, 8).
3. Find the image of the coordinates of the point (-7, 2) under a clockwise rotation of 270^0 with centre (-1 , - 5).

4. Find the image of the coordinates of the point (0, 9) under an anticlockwise rotation of 180^0 with centre (2, -2).

The Image of an Object undera Mapping

To find the image of an object under a given mapping:
I. Identify the x and y coordinates of the object and the given mapping
II. Substitute the values of the x and y coordinates in the rule of the mapping
III. Workout or simplify or evaluate to obtain the coordinates of the image of the object

Worked Examples

1. A point P(-6, 8) undergoes transformation under the mapping $\binom{x}{y} \rightarrow \binom{x-4}{-2+y}$
i. Find the coordinates of the image P^1
ii. Find $\overrightarrow{PP^1}$
iii. Calculate $|PP^1|$

Solution

i. Given P(-6, 8) ⇒x = -6, y = 8,
Substitute in $\binom{x}{y} \rightarrow \binom{x-4}{-2+y}$
P → P^1
$P\binom{-6}{8} \rightarrow P^1\binom{-6-4}{-2+8} \rightarrow P^1\binom{-10}{6}$ $P^1(-10, 6)$

ii. $\overrightarrow{PP^1} = OP^1 - OP = \binom{-10}{6} - \binom{-6}{8} = \binom{-4}{-2}$
iii. $|PP^1| = \sqrt{(-4)^2 + (-2)^2}$
$= \sqrt{16+4} = \sqrt{20} = 4.5$ *units*

2. Find the image of (3, -7) under the transformation $\binom{x}{y} \rightarrow \binom{2x+y}{y-3x}$.

Solution

Given (3, -7) ⇒x = 3, y = -7 substitute in

$\binom{x}{y} \rightarrow \binom{2x + y}{y\ - 3x}$

$\binom{3}{-7} \rightarrow \binom{2(3) - 7}{-7\ - 3(3)}$

$\binom{x}{y} \rightarrow \binom{6 - 7}{-\ 7 - 9}$

$\binom{x}{y} \rightarrow \binom{-1}{-16}$ (-1, - 16)

3. Find the image of (5, -8) under the mapping $\binom{x}{y} \rightarrow \binom{-\ y}{x + 2y}$.

Solution

Given (5, -8),

⇒x = 5, y = -8 substitute in

$\binom{x}{y} \rightarrow \binom{-\ y}{x + 2y}$

$\binom{5}{-8} \rightarrow \binom{-\ (-8)}{5 + 2(-8)}$

$\binom{5}{-8} \rightarrow \binom{8}{-11}$ (8, - 11)

4. What is the image of (-6, 4) under the mapping $\binom{x}{y} \rightarrow \binom{\frac{1}{2}y}{x + 2y}$?

Solution

Given (-6, 4),⇒x = -6, y = 4 substitute in

$\binom{x}{y} \rightarrow \binom{\frac{1}{2}y}{x + 2y}$

$\binom{-6}{4} \rightarrow \binom{\frac{1}{2}(4)}{-6 + 2\ (4)}$

$\binom{-6}{4} \rightarrow \binom{2}{2}$ (2, 2)

Exercises 11.13

1. Find the image of P(-4, -5) under the mapping $\binom{x}{y} \rightarrow \binom{-y - x}{x - y}$

2. Determine the image of A(-10, 5) under the mapping $\binom{x}{y} \rightarrow \binom{x + 5}{\frac{3x}{2} + 2y}$

Drawing the Graph of Transformation

I. Draw two perpendicular lines, Ox and Oy that divide the graph sheet into two equal

halves or unequal halves

II. Identify the origin or center as (0, 0)

III. Name the vertical axis as $y$and horizontal axis as x

IV. Mark and number both axes according
to the given scale and intervals provided.

V. Plot the given points (of the object).

VI. Workout the image whether under translation, enlargement, reflection, rotation or mapping and plot and form the image (s)on the graph sheet.

Worked Examples

1.Using a scale of 2cm to 2units on both axes, draw two perpendicular lines $0x$ and $0y$ on a graph sheet for the intervals -8 $\leq x \leq$ 8 and -8 $\leq y \leq 8$

b. Plot P (3, 1), Q (3, 4), R (-2, 2) and join the points with a rule to form triangle PQR

c. Draw the image P_1 Q_1 R_1 of triangle PQR under translation by the translation vector$\binom{4}{-3}$, where P → P_1, Q → Q_1 and R → R_1

Solution

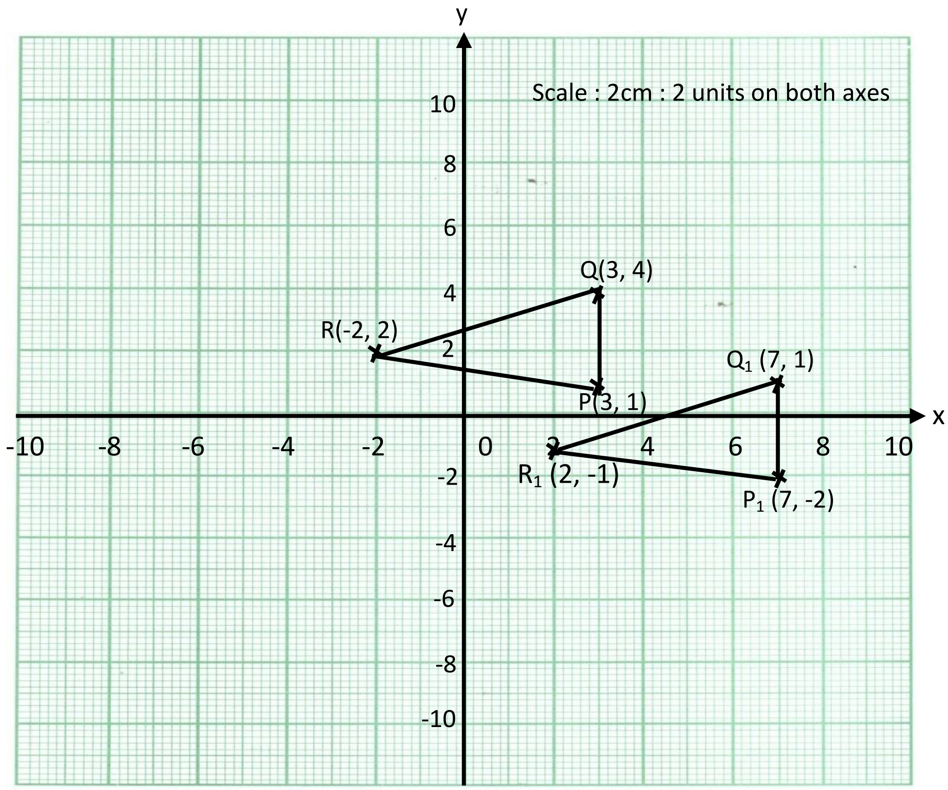
P (3, 1), Q (3, 4), R (-2, 2)

Object + Translation Vector = Image

P $\binom{3}{1}$ + $\binom{4}{-3}$ = $P_1\binom{7}{-2}$, P_1(7, -2)

Q $\binom{3}{4}$ + $\binom{4}{-3}$ = $Q_1\binom{7}{1}$, Q_1(7, 1)

R $\binom{-2}{2}$ + $\binom{4}{-3}$ = $R_1\binom{2}{-1}$, R_1(2, -1)



2. Using a scale of 2cm to 2units on both axes, draw two perpendicular lines Ox and Oy on a graph sheet for the intervals -8 $\leq x \leq$ 8 and -8 $\leq y \leq$ 8

b. Plot the points A (3, 5), B (-1, 2) and C (4, 2) and join them with a rule to form triangle ABC

c. Draw the image P_1 Q_1 R_1 of triangle PQR under reflection in the x - axis, such that A $\rightarrow$ A_1, B $\rightarrow$ B_1 and C $\rightarrow$ C_1. Clearly indicate the coordinates of the vertices of the triangles drawn

Solution

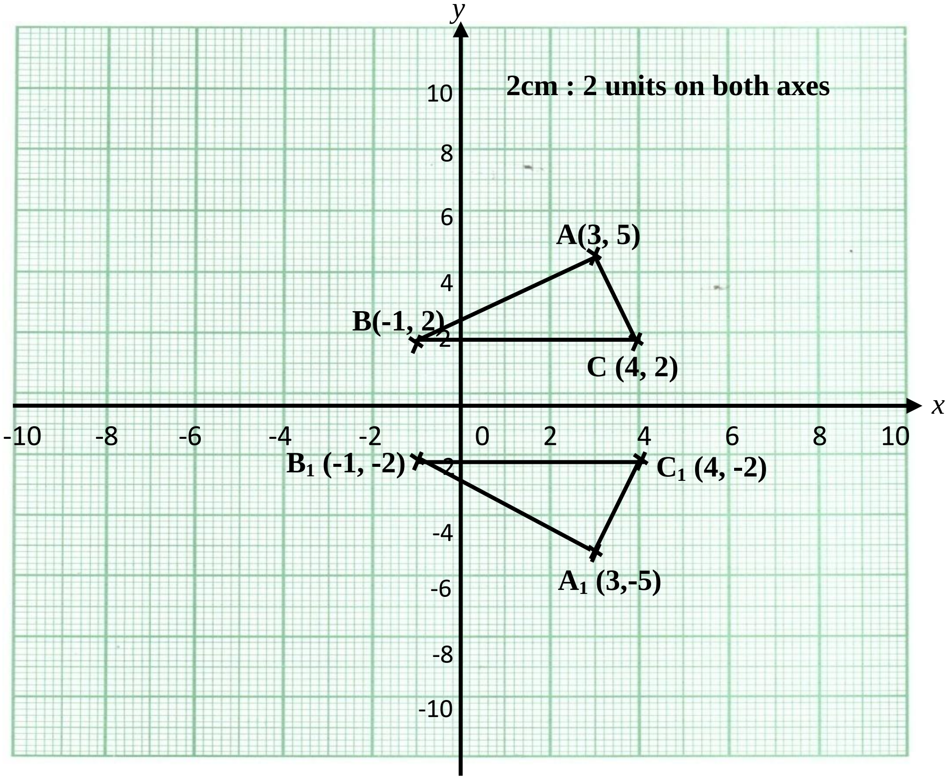
A (3, 5), B (-1, 2) , C (4, 2).

Reflection in the x – axis $\binom{x}{y} \rightarrow \binom{x}{-y}$

$A\binom{3}{5} \rightarrow A_1\binom{3}{-5}$, $A_1(3, -5)$

$B\binom{-1}{2} \rightarrow B_1\binom{-1}{-2}$, $B_1(-1, -2)$

$C\binom{4}{2} \rightarrow C_1\binom{4}{-2}$, $Q_1(4, -2)$



3. i. Using a scale of 2cm to 2 units on both axes, draw two perpendicular lines 0x and 0y on a graph sheet for the intervals -8 $\leq x \leq$ 8 and -8 $\leq y \leq$ 8

ii. Plot the points A (3, 5), B (-1, 2) and C (4, 2) and join them
with a rule to form triangle ABC

iii. Draw the image A_1 B_1 C_1 of triangle ABC under reflection in the y - axis, such that A → A_1, B → B_1 and C → C_1. Clearly indicate the coordinates of the vertices of the triangles drawn

Solution

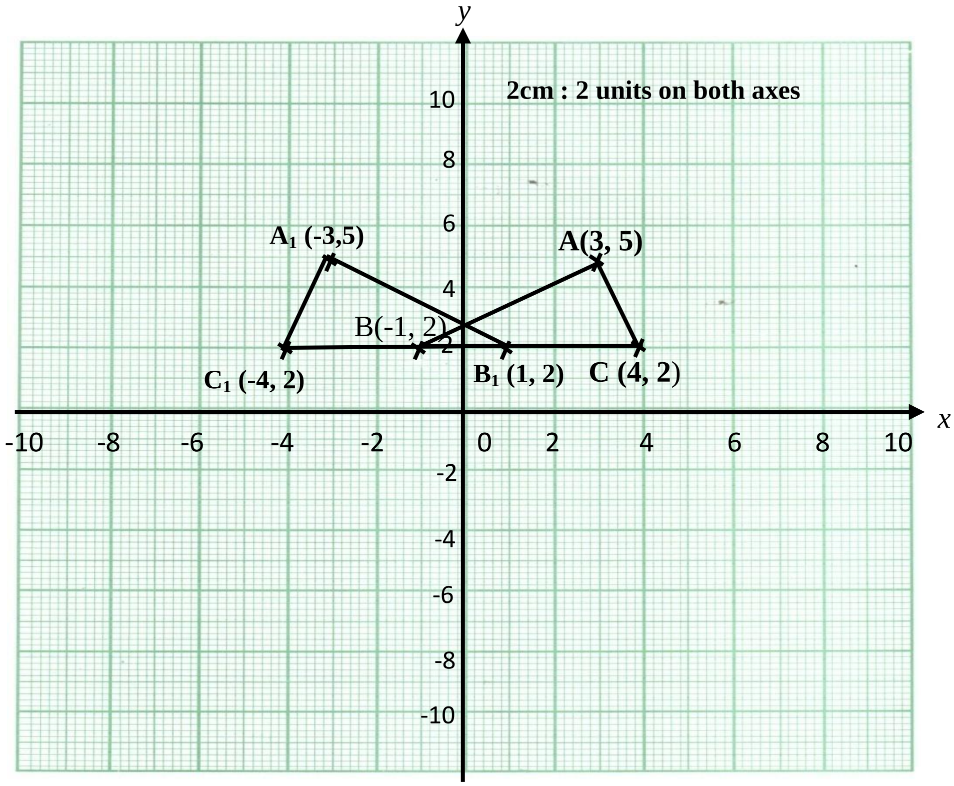
ii. A (3, 5), B (-1, 2), C (4, 2)

iii. Reflection in the y – axis $\binom{x}{y} \rightarrow \binom{-x}{y}$

$A\binom{3}{5} \rightarrow A_1\binom{-3}{5}$, A_1(-3, 5)

$B\binom{-1}{2} \rightarrow B_1\binom{1}{2}$, B_1(1, 2)

$C\binom{4}{2} \rightarrow C_1\binom{-4}{2}$, C_1(-4, 2)



4. a. Using a scale of 2cm to 1unit on both axes, draw two perpendicular lines 0x and 0y on a graph sheet for the intervals -6 $\leq x \leq$ 6 and -6 $\leq y \leq$ 6

b. Plot the points A (3, 1), B (-2, 3) and C (-5, -1) and join them with a rule to form $\Delta\ ABC$

c. Draw the image $A_1\ B_1\ C_1$ of triangle *ABC* under rotation through 90^0 anti – clockwise about the origin, such that A → A_1, B → B_1 and C → C_1. Clearly indicate the coordinates of the vertices of the triangles drawn

Solution

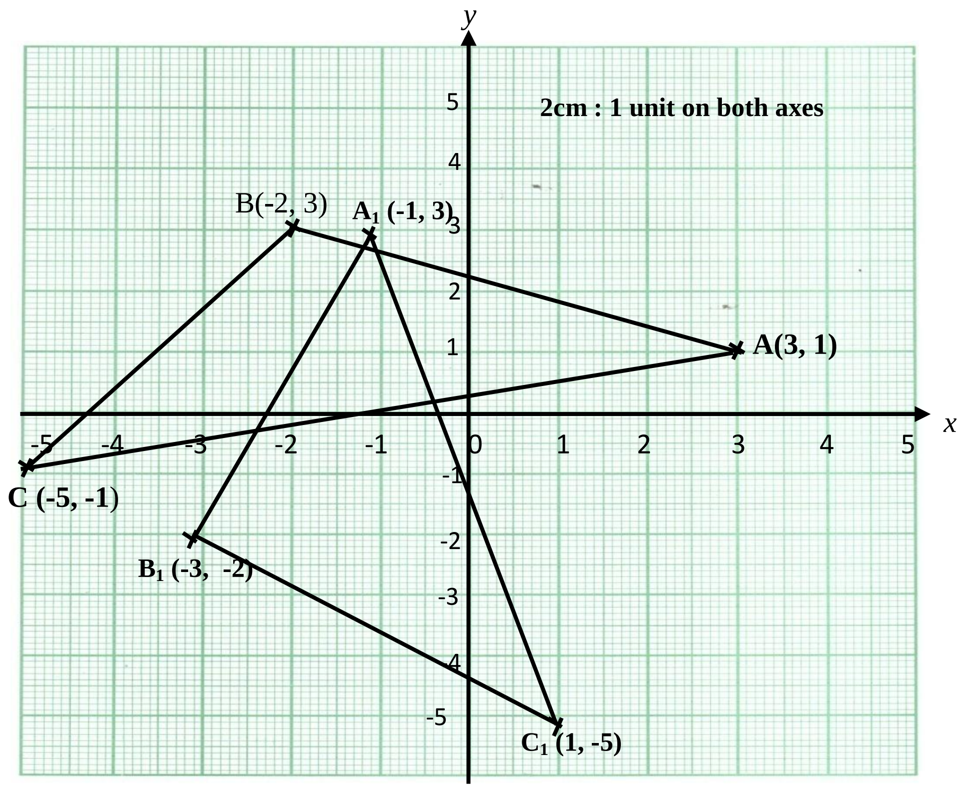
a. A (3, 1), B (-2, 3) and C (-5, -1)

c. Anti - clockwise rotation through 90^0 about the origin$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -y \\ x \end{pmatrix}$

A$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \rightarrow A_1\begin{pmatrix} -1 \\ 3 \end{pmatrix}$, A_1(-1, 3)

B$\begin{pmatrix} -2 \\ 3 \end{pmatrix} \rightarrow B_1\begin{pmatrix} -3 \\ -2 \end{pmatrix}$, B_1(-3, -2)

C$\begin{pmatrix} -5 \\ -1 \end{pmatrix} \rightarrow C_1\begin{pmatrix} 1 \\ -5 \end{pmatrix}$, C_1(1, -5)



Some Solved Past Questions

1.i. Using a scale of 2cm to 1 unit on both axes, draw the x and y axes for $-6 \leq x \leq 6$ and $-4 \leq y \leq 6$.

ii. Plot the points $\mathrm{A}(3, 1)$, $\mathrm{B}(1, 1)$ and $\mathrm{C}(3, 5)$ and describe the triangle ABC.

iii. Find the equation of AC

iv. Draw the triangle $A_1B_1C_1$ which is the reflection of triangle ABC in the $y - axis$, where, $A \to A_1$, $B \to B_1$ and $C \to C_1$. Indicate clearly the coordinates of A_1, B_1 and C_1.

v. Draw triangle $A_2B_2C_2$ which is the image of triangle ABC under the mapping $\binom{x}{y} \to \binom{2x}{1-y}$, where $A \to A_2$, $B \to B_2$ and $C \to C_2$, indicating clearly the coordinates of A_2, B_2 and C_2.

Solution

ii. $\mathrm{A}(3, 1)$, $\mathrm{B}(1, 1)$ and $\mathrm{C}(3, 5)$

Triangle ABC is a right triangle

ii. Equation of AC;

$\mathrm{A}(3, 1)$ and $\mathrm{C}(3, 5)$

Grad of AC, $m = \frac{5-1}{3-3} = \frac{4}{0}$ (undefined)

iv. Reflection of Δ ABC in the $y - axis$,

$\binom{x}{y} \to \binom{-x}{y}$

A $\binom{3}{1} \to$ A$_1\binom{-3}{1}$ A$_1$(-3, 1)

B $\binom{1}{1} \to$ B$_1\binom{-1}{1}$ B$_1$(-1, 1)

C$\binom{3}{5} \to$ C$_1\binom{-3}{5}$ C$_1$(-3, 5)

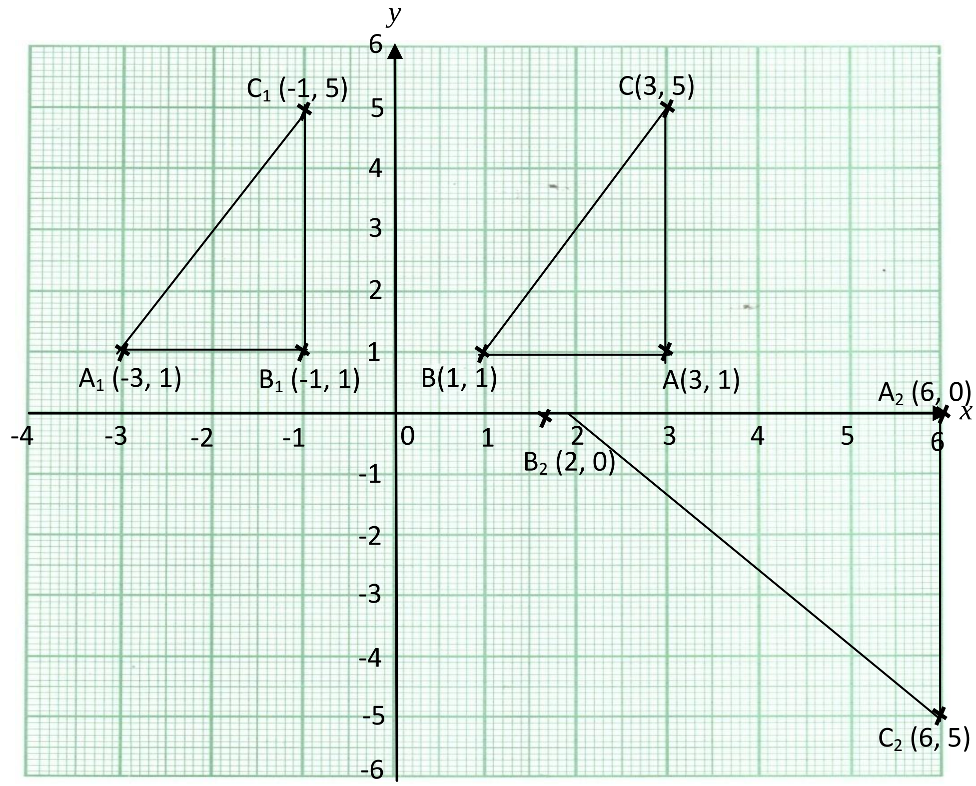
v. Coordinates of $A_2B_2C_2$ under the mapping;

$$\binom{x}{y} \to \binom{2x}{1-y}$$

A $\binom{3}{1} \to$ A$_2\binom{2(3)}{1-1}$ A$_2$(6, 0)

B $\binom{1}{1} \to$ B$_2\binom{2(1)}{1-1}$ B$_2$(2, 0)

C$\binom{3}{5} \to$ C$_2\binom{2(3)}{5}$ C$_2$(6, 5)



2. Using a scale of $2cm$ to 2 units on each axis, draw on a sheet of paper two perpendicular axesOx and Oy for the interval $-10 \leq x \leq 8$ and $-10 \leq y \leq 10$. Draw:

i. Triangle PQR with vertices$P(2, 8)$, $Q(5, 8)$ and $R(2, 4)$.

ii. The image triangle $P_1Q_1R_1$ of triangle PQR a reflection in the line$y$ = 2where P → P_1, Q → Q_1 and R → R_1

iii. The image triangle $P_2Q_2R_2$of triangle PQR under a half turn about the origin O, where P → P_2, Q → Q_2 and R → R_2.

iv. Describe precisely the transformation that will map triangle $P_2Q_2R_2$ onto triangle $P_1Q_1R_1$ where $P_2 \rightarrow P_1$, $Q_2 \rightarrow Q_1$ and $R_2 \rightarrow R_1$.

Solution

i. $P(2, 8)$, $Q(5, 8)$ and $R(2, 4)$.

ii. Reflection in the line $y = 2$

$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x \\ 2(2) - y \end{pmatrix}$

$P\begin{pmatrix} 2 \\ 8 \end{pmatrix} \rightarrow P_1\begin{pmatrix} 2 \\ 2(2) - 8 \end{pmatrix} \rightarrow P_1\begin{pmatrix} 2 \\ -4 \end{pmatrix}$, $P_1 = (2, -4)$

$Q\begin{pmatrix} 5 \\ 8 \end{pmatrix} \rightarrow Q_1\begin{pmatrix} 5 \\ 2(2) - 8 \end{pmatrix} \rightarrow Q_1\begin{pmatrix} 5 \\ -4 \end{pmatrix}$, Q_1 (5, -4)

$R\begin{pmatrix} 2 \\ 4 \end{pmatrix} \rightarrow R_1\begin{pmatrix} 2 \\ 2(2) - 4 \end{pmatrix} \rightarrow R_1\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, R_1 (2, 0)

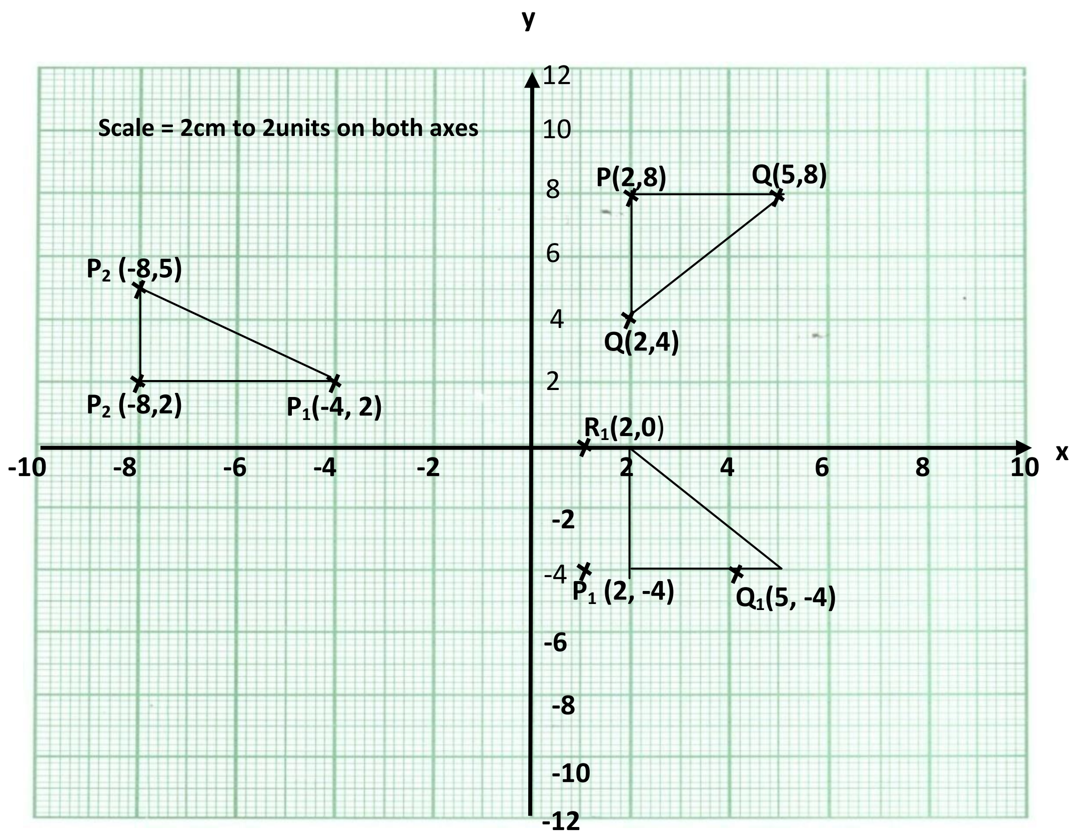
iii. Half turn about the origin, O

$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -y \\ x \end{pmatrix}$

$P\begin{pmatrix} 2 \\ 8 \end{pmatrix} \rightarrow P_2\begin{pmatrix} -8 \\ 2 \end{pmatrix}$ $P_2(-8, 2)$

$Q\begin{pmatrix} 5 \\ 8 \end{pmatrix} \rightarrow Q_2\begin{pmatrix} -8 \\ 5 \end{pmatrix}$ $Q_2(-8, 5)$

$R\begin{pmatrix} 2 \\ 4 \end{pmatrix} \rightarrow Q_2\begin{pmatrix} -4 \\ 2 \end{pmatrix}$ $R_2(-4, 2)$



Exercises 11.14

1. i. Draw on a sheet of graph paper two perpendicular axes for $-5 \leq x \leq 5$ and $-6 \leq y \leq 6$, using a scale of 2cm to 1 unit on both axes;

ii. Draw on the graph sheet a quadrilateral FGHJ with $(1, 1)$, $(3, 1)$, $(3, 3)$ and $(1, 3)$.

iii. Draw the image $F^1G^1H^1J^1$, where $F \rightarrow F^1, G \rightarrow G^1, H \rightarrow H^1$ and $J \rightarrow J^1$ under the mapping $\binom{x}{y} \rightarrow \binom{x-2y}{y-2x}$, showing clearly the coordinates of the vertices.

iv. Describe precisely the shape of the image $F^1G^1H^1J^1$

v. What is the ratio of the area of FGHJ to the area of $F^1G^1H^1J^1$?

vi. What two transformations will map F onto F^1?

2. i. Using a scale of 2cm to 2units on both axes draw two perpendicular lines Ox and Oy on a graph sheet.

ii. Mark the x-axis from -10 to 10 and the y-axis from -12 to 12.

iii. Plot P (2, 3), Q (0, 5), R (5, 3). Join the points with a rule to form triangle PQR. What type of triangle is PQR?

iv. Draw the image $P_1Q_1R_1$ of triangle PQR, under enlargement with scale factor 2 from the origin, such that $P \rightarrow P_1$, $Q \rightarrow Q_1$ and $R \rightarrow R_1$.

v. Draw the image P_2 Q_2 R_2 of triangle PQR, such that $P \rightarrow P_2$, $Q \rightarrow Q_2$ and $R \rightarrow R_2$ under reflection in the x- axis. Show all the linesof transformation.

vi. Draw $P_3Q_3R_3$ of PQR the image under rotation through 270^0 anticlockwise such that $P \rightarrow P_{3,}$ Q

$\rightarrow Q_3$ and R $\rightarrow R_3$ Clearly indicate all the co-ordinates of the triangles drawn.

3. i. Using a scale of 2cm to 2 units on both axes, draw two perpendicular lines O*x* and O*y*
on a graph sheet.
ii. Mark the *x*-axis from -10 to 10 and the *y*-axis from -12 to 12.
iii. Plot the points A (-9, -4), B (-7,-2), C (-5,-7) and D (-3,-5) Join all the points with a rule and name the figure formed.
iv. Draw the image $A_1B_1C_1D_1$ of figure *ABCD* under rotation through 270^0 clockwise about the origin, such that $A \rightarrow A_1$, $B \rightarrow B_1$, $C \rightarrow C_1$ and $D \rightarrow D_1$.
v. Draw the image $A_2B_2C_2D_2$ of figure ABCD under enlargement from the origin with scale factor -1,such that $A \rightarrow A_2$, $B \rightarrow B_2$, $C \rightarrow C_2$ and $D \rightarrow D_2$
vi. With line y = 0 as line of reflection, reflect ABCD to get A_3 B_3 C_3 D_3 such that $A \rightarrow A_3$,
$B \rightarrow B_3$, $C \rightarrow C_3$ and $D \rightarrow D_3$

4. a. Using a scale of 2cm to 1 unit on both axes, for $-4 \leq x \leq 5$ and $-5 \leq x \leq 5$, draw Δ PQR with vertices $P(-1, 2), Q(1, 4)$ and $R(2, 1)$
b. Draw the image $\Delta P_1Q_1R_1$ of triangle PQR under a reflection in the line $y = 0$, where P$\rightarrow P_1$, Q$\rightarrow Q_1$ and R$\rightarrow R_1$ the image $\Delta P_2Q_2R_2$ of Δ PQR under a translation by the vector $\binom{-2}{1}$, where P$\rightarrow P_2$, Q$\rightarrow Q_2$ and R$\rightarrow R_2$
c. Draw the image $\Delta P_3Q_3R_3$ of Δ PQR under an anticlockwise rotation of 90° about the origin where $P \rightarrow P_3, Q \rightarrow Q_3$ and $R \rightarrow R_3$
d. Name two coincident image points in your diagram.
e. Describe precisely a single transformation that maps triangle $P_1Q_1R_1$ unto $\Delta P_3Q_3R_3$, where $P_1 \rightarrow P_3, Q_1 \rightarrow Q_3$ and $R_1 \rightarrow R_3$

5. i. Using a scale of 2cm to 2 units on both axes, draw two perpendicular lines O*x* and O*y* on a graph sheet.
ii. Mark the *x*-axis from -10 to 10 and the *y*-axis, from -12 to 12.
iii. Plot the points *A* (1, 3), *B* (6, 5) and *C* (4, 0) . Join the points to form triangle *ABC*.
What type of triangle is formed?
iv. Draw triangle $A_1B_1C_1$, the image of triangle ABC under translation by the vector$\binom{4}{3}$, such that $A \rightarrow A_1$, $B \rightarrow B_1$, and $C \rightarrow C_1$
v. Draw the image $A_2B_2C_2$ of ΔABC under enlargement with scale factor -1, from the origin, such that $A \rightarrow A_2$, $B \rightarrow B_2$, $C \rightarrow C_2$
vi. With line x = 0 as mirror line, draw the image $A_3B_3C_3$ of triangle ABC, such that $A \rightarrow A_3$, $B \rightarrow B_3$ and $C \rightarrow C_3$ .Show all the lines transformation of triangle ABC and triangle $A_3B_3C_3$

6. i. Using a scale of 2cm to 1 unit on *x*- axis and 2cm to 2 units on *y*-axis axes, draw two perpendicular lines O*x* and O*y* on a graph sheet.
ii. Mark the *x* - axis from -5 to 5 and the y-axis from -12 to12.
iii. Plot the points,A($\frac{1}{2}$,-3), B($1\frac{1}{2}$,-2) and C (3,-5). Join A to B and B to C with a rule.
iv. Draw the image $A_1B_1C_1$ of ABC under enlargement with scale factor -1 from the origin, such that $A \rightarrow A_1$, $B \rightarrow B_1$ and $C \rightarrow C_1$
v. Draw the image A_2 B_2 C_2 of ABC under reflection in the line x = 0,such that $A \rightarrow A_2$, $B \rightarrow B_2$ and $C \rightarrow C_2$.
vi. Draw the image $A_3B_3C_3$ of ABC under rotation through 270^0 clockwise about the origin, such that $A \rightarrow A_3$ $B \rightarrow B_3$ and $C \rightarrow C_3$. Clearly indicate all the coordinates of the figure drawn.
vii. What single transformation maps $A_1B_1C_1$ to $A_2B_2C_2$?

7. i. Using a scale of 2cm to 1unit on both axes, draw two perpendicular lines O*x* and O*y* on a graph sheet.

ii. Mark the *x*-axis from -10 to 10 and the *y*-axis from -12 to 12.

iii. Plot the points P(1, 1), Q(3, 1), R(3, 3) and S(1, 4) and join them with a ruler to form figure PQRS. What type of quadrilateral is PQRS?

iv. Draw image $P_1Q_1R_1S_1$ of PQRS, under enlargement with scale factor (-1) from the origin, such that $P \rightarrow P_1$, $Q \rightarrow Q_1$, $R \rightarrow R_1$ and $S \rightarrow R_1$

v. Draw image $P_2Q_2R_2S_2$ of PQRS under clockwise rotation through 90^0about the origin such that $P \rightarrow P_2$, $Q \rightarrow Q_2$, $R\rightarrow R_2$ and $S \rightarrow S_2$

vi. Draw the image $P_3Q_3R_3S_3$ of P, Q, R, S under reflection in the line $x = 0$, such that $P \rightarrow P_3$, $Q \rightarrow Q_3$, $R \rightarrow R_3$ and $S \rightarrow S_3$.

8. i. Using a scale of 2cm to 2 units on both axes, draw on a sheet of graph paper two perpendicular axes O*x* and O*y* for the interval $-10 \leq x \leq 10$ and $-12 \leq y \leq 12$.

ii. Plot the following points: A (2, 4), B (2, 1) and C(-1, 1). Join the points to form triangle ABC.

iii. On the same graph sheet draw the image triangle $A_1B_1C_1$ under translation by vector $\binom{2}{3}$, such that $A \rightarrow A_1$, $B \rightarrow B_1$ and $C \rightarrow C_1$.

iv. Draw the image triangle $A_2B_2C_2$ of triangle ABC under reflection in the y – axis such that$A \rightarrow A_2$, $B \rightarrow B_2$ and $C \rightarrow C_2$.

v. Draw the image triangle $A_3B_3C_3$of triangle ABC under enlargement with scale factor 2, such that $A\rightarrow A_3$,$B \rightarrow B_3$ and $C \rightarrow C_3$.

vi. Draw the image triangle $A_4B_4C_4$ of triangle ABC under clockwise rotation through 270^0 about the origin, such that $A \rightarrow A_4$,$B \rightarrow B_4$ and $C \rightarrow C_4$. Clearly indicate the coordinates of all the triangles drawn.

9. Using a scale of 2cm to 2 units on each axes, draw on a graph sheet of paper two perpendicular line O*x* and O*y* for the intervals $-8\leq x \leq 12$ and $-12\leq y \leq 12$

i. Draw ΔABC with coordinates A(5, 7), B(3, 4) and C(7, 3)

ii. Draw the image $\Delta A_1 B_1 C_1$ of ΔABC under translation by the vector $\binom{4}{-2}$, such that $A \rightarrow A_1$, $B \rightarrow B_1$ and $C \rightarrow C_1$

iii. Draw the image $\Delta A_2B_2C_2$ of ΔABC under reflection in the line $y = -2$, such that $A \rightarrow A_2$, $B \rightarrow B_2$ and $C \rightarrow C_2$

iv. Draw $\Delta A_3B_3C_3$ of ΔABC the image under rotation through 90^0 anticlockwise about the origin, such that $A \rightarrow A_3$, $B \rightarrow B_3$ and $C \rightarrow C_3$

v. find the gradient of line B_2B_3

Scale Factor *(k)*

A geometric figure can be enlarged or reduced from a point called ***center of enlargement or reduction*** respectfully**.**

When a geometric figure is enlarged or reduced, a new figure is formed. The new figure formed is called the ***Image (I)***of the original figure, called the ***Object (O).***

The ratio of the length of the image to its corresponding object length is a constant called ***Scale factor*** of enlargement or reduction, denoted by *k :* Thus,

$$k = \frac{\text{Image length}}{\text{Object length}}$$

In case of comparative areas,

$$k^2 = \frac{\text{Image area}}{\text{Object area}}$$

In case of comparative volumes,

$$k^3 = \frac{\text{Image volume}}{\text{Object volume}}$$

Effects Of The Scale Factor

I. If k is positive, the object and its image are on the same side of the center of enlargement.

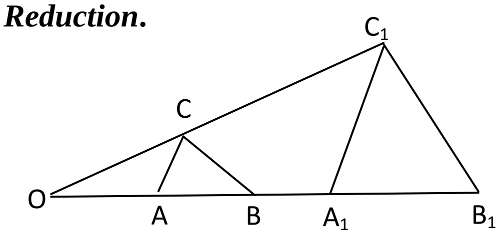
II. If k is negative, the object and its image are on opposite sides of the centre of enlargement.

III. If $k = 1$ or -1, it implies that the object and its image are of the same size.

II. If $k > 1$, the image size is bigger than the object size

IV. If $k < 1$, the object size is bigger than the image size.

On the other hand, if k is a proper fraction, the object size is reduced in a process called ***Reduction***.



In the diagram above,

I. O is the center of enlargement.

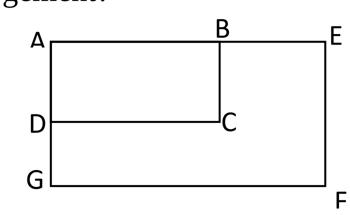
II. ABC is the object and $A_1B_1C_1$ is the image.

III. Scale factor, $k = \frac{A_1B_1}{AB} = \frac{A_1C_1}{AC} = \frac{B_1C_1}{BC}$

Worked Examples

1. In the diagram below, rectangle *AEFG* is an enlargement of rectangle *ABCD*. If */AB/ = 5cm* and */AE/ = 15cm*, what is the scale factor of enlargement?

1.



Solution

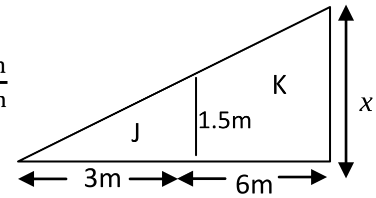
$k = \frac{image\ length}{object\ length} = \frac{/AE/}{/AB/} = \frac{15\text{cm}}{5\text{cm}} = 3\text{cm}$

2. In the diagram below, K is an enlargement of J.

i. Calculate the scale factor

ii. Find the value of x

Solution



i. $k = \frac{\text{image length}}{\text{object length}}$

$\text{k} = \frac{9\text{m}}{3\text{m}} = 3$

ii. $k = \frac{I}{O}$, but O = 1.5m, I = x and $k = 3$

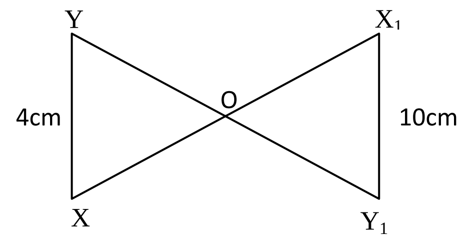
$3 = \frac{x}{1.5}$,

$x = k \times O = 3 \times 1.5 = 4.5\text{m}$ OR

$x = 3 \times 1.5 = \frac{30 \times 15}{100} = \frac{450}{100} = 4.5\text{m}$

Exercises 11.15

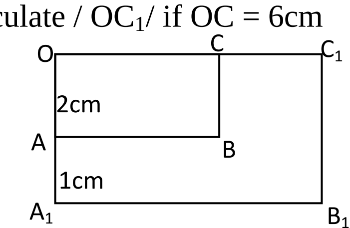
1. In the figure below, triangle OX_1Y_1 is an enlargement of OXY. Find the scale factor.



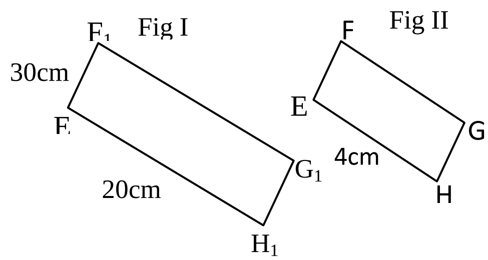
2. In the diagram below,

a. Find the scale factor of the enlargement.

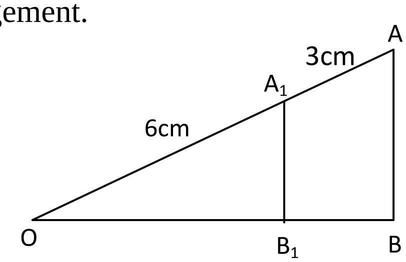
b. Calculate $/OC_1/$ if OC = 6cm



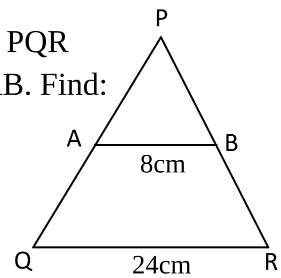
3. In the diagrams below, fig I is an enlargement of Fig II. Find the side EF of Fig II.



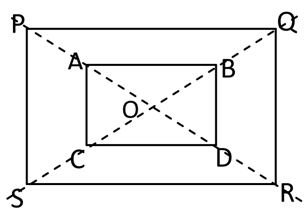
4. In the triangle below, OA_1B_1 is an enlargement of triangle OAB. Find the scale factor of enlargement.



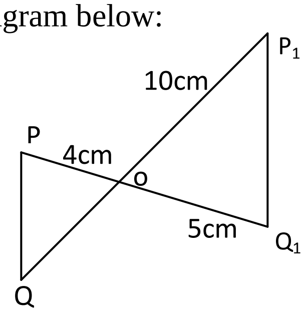
5. In the diagram below, PQR is an enlargement of PAB. Find:
i) the scale factor of enlargement;
ii) /PB/ if /PR/ = 15cm.



6. In the diagram below, the square PQRS is an enlargement of square ABCD from the center O. The area of square ABCD is $4cm^2$ and the area of square PQRS is $9cm^2$. Find the scale factor of enlargement.



7. Find the scale factor of enlargement and /OQ/ from the diagram below:



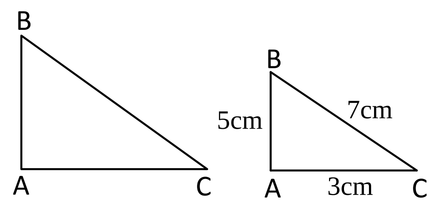
8. The radius of a lorry tyre is 54 cm and that of a motor bike is 18cm. If the tyre of the motor bike is the image of the tyre of the lorry, find the scale factor of enlargement.

Challenge problem

1. The area of a hockey pitch is $1408m^2$ and its image which is a volley pitch has an area of $88m^2$. If the length of the hockey pitch is 44m,
i. Determine the scale factor of enlargement ii. Determine the length and breadth of the volley pitch.

More Worked Examples

Given that triangle $A_1B_1\ C_1$ is an enlargement of triangle ABC with scale factor 3,



Find the length of the following; i) $A_1\ B_1$ ii) $A_1\ C_1$ iii) B_1C_1

Solution

i. $k = \frac{I}{O} = \frac{/A1B1/}{/AB/}$

$/A_1\ B_1/ = k\ /AB/$

But $k = 3$, and $/AB/ = 5$cm

$/A_1\ B_1\ / = 3 \times 5$ cm = 15cm

ii. $k = \frac{I}{O} = \frac{A1C1}{AC}$,

But $k = 3$ and $/AC/ = 3$cm

$/A_1\ C_1\ / = k\ /AC/ = 3 \times 3$cm = 9cm

iii) $k = \frac{I}{O} = \frac{/B1C1/}{/BC/}$,

But $k = 3$ and $/\ B\ C\ / = 7$ cm

$/\ B_1\ C_1\ / = k\ /\ BC\ / = 3 \times 7$ cm = 21cm

Exercises 11.16

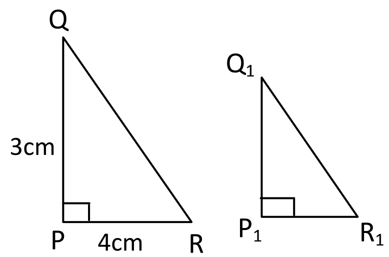
Triangle $P_1Q_1R_1$ is an enlargement of triangle

PQR with scale factor 2. Find the following lengths;

i. $P_1 Q_1$

ii. $P_1 R_1$

iii. $Q_1 R_1$

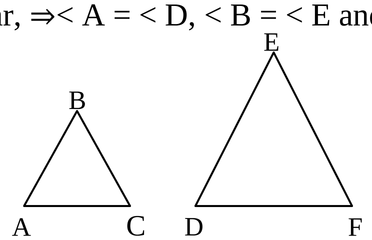


Similar Figures

If one figure is an enlargement of another, then they have the same shape, but not necessarily the same size. In this case they are said to be ***similar.***

If two figures are similar, then they have the same angles. There is also a fixed ratio between the sides of the figures called ***the scale factor, k, of enlargement.***

In particular, if triangles ABC and DEF are similar, ⇒< A = < D, < B = < E and < C = < F



Likewise, there is a fixed ratio between the sides of ΔABC and the sides of ΔDEF

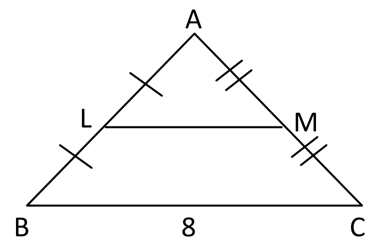
$k = \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

The ratio of sides is the same for both triangles: $\frac{AB}{BC} = \frac{DE}{EF} : \frac{AB}{DE} = \frac{AB}{AC} = \frac{DE}{DF} : \frac{AC}{BC} = \frac{BC}{EF} = \frac{DF}{EF}$.

For clarity and better understanding, there is the need to draw the figures separately but it is not required all the time.

Worked Examples

1. L and M are the mid points of the sides *AB* and *AC* of the triangle *ABC*.



i. Find a pair of similar triangles

ii. If *BC* = 8cm, Find *LM*

Solution

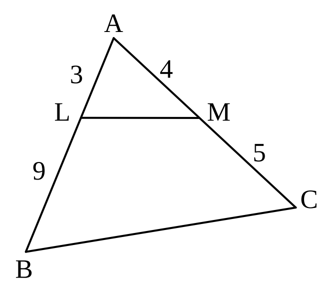
Δ *ALM* is a reduction of Δ *ABC*, with scale factor, $k = \frac{1}{2}$ (midpoint)

Hence Δ *ABC* is similar to Δ *ALM*.

It follows that *LM: BC* is equal to $\frac{1}{2}$

$LM = 8 \times \frac{1}{2} = 4$

2. L and M are on the sides *AB* and *AC* of the triangle *ABC*. *AM* = 4, *MC* = 5, *AL* = 3, *LB* = 9.



Find the ratio *LM : BC*

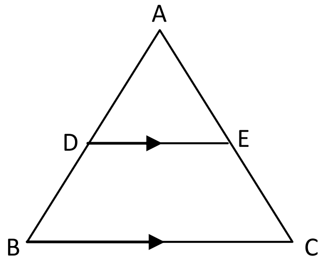
Solution

AB = 12, *AC* = 9. So the ratios *AL : AC* and AM : *AB* are both equal to 1 : 3

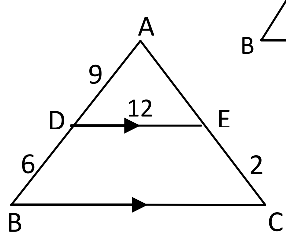
The angle *A* is the same in both triangles. So *ABC* and *AML* are similar. It follows that *LM : BC* = *AM : AB* = 1 : 3

3. The figure below shows that triangle ADE is an enlargement of triangle ABC. If /*DB*/ = 6cm, /*AD*/ = 9cm, /*CE*/= 2cm and /*DE*/ = 12cm,

calculate; i. /*BC*/ ii. /*AE*/



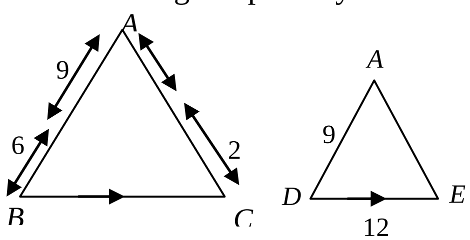
Solution



Object = Δ *ABC*

Image = Δ *ADE*

Redraw the triangle separately as shown below;



Δ *ABC* and Δ*ADE* are similar. Thus, /*AE*/ corresponds to side /*AC*/, /*AD*/ corresponds to /*AB*/ and / *DE*/ corresponds to /*BC*/

$\Rightarrow \frac{/AE/}{AC} = \frac{/AD/}{/AB/} = \frac{/DE/}{/BC/}$

Consider, $\frac{/AD/}{/AB/} = \frac{/DE/}{/BC/}$

$\Rightarrow \frac{9}{9+6} = \frac{12}{BC}$

$= \frac{9}{15} = \frac{12}{BC}$

9 × /*BC*/ = 12 × 15

/ *BC* / = $\frac{12 \times 15}{9}$ = 20 cm

ii. To find /*AE*/,

$\frac{/AE/}{/AC/} = \frac{/AD/}{/AB/}$,

$\frac{/AE/}{/AC/} = \frac{9}{15}$

15 × /*AE*/ = 9 × (/*AE*/ + 2)

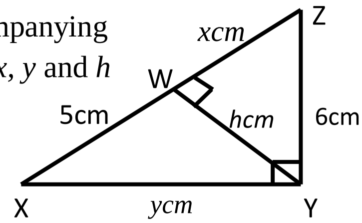
15/*AE*/ = 9/*AE*/ + 18

15/*AE*/ − /9*AE*/ = 18

6 /*AE*/ = 18

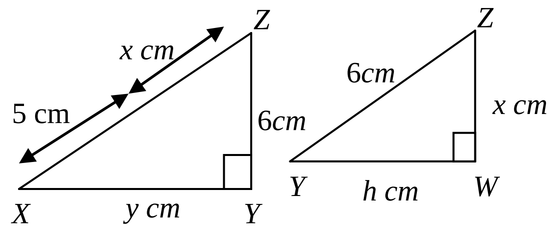
/*AE*/ = $\frac{18}{6}$ = 3 cm.

4. In the accompanying diagram, find *x*, *y* and *h*



Solution

Redraw the triangle separately as shown:



Δ *ZYW* is equivalent to Δ *ZXY;*

So, $\frac{/ZW/}{/ZY/} = \frac{/ZY/}{/XZ/}$

$\Rightarrow \frac{x}{6} = \frac{6}{x+5}$

x (*x* + 5) = 6 × 6 (cross products)

$x^2 + 5x = 36$

$x^2 + 5x - 36 = 0$

(*x* − 4) (*x* + 9) = 0 *(By factirisation)*

x − 4 = 0 or *x* + 9 = 0

x = 4 or *x* = − 9

So *x* = 4 units

Consider triangle YZW

$h^2 = 6^2 - x^2$, but *x* = 4

$h^2 = 36 - 16$

$h^2 = 20$

$h = \sqrt{20}$ cm

By Pythagoras theorem,

$y^2 = 5^2 + h^2$

$y^2 = 5^2 + \sqrt{20}^2$

$y^2 = 45$

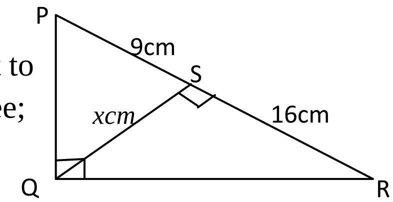
$y = \sqrt{45}$cm

5. In the diagram below, $< PQR = < PSQ = 90^0$. $/PS/ = 9$cm, $/SR/ = 16$cm and $/SQ/ = x$cm. Find:

a. the value of x;

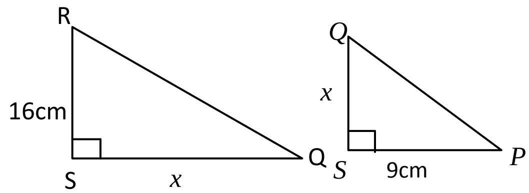
ii. $< QRS$ correct to the nearest degree;

iii. $/PQ/$.



Solution

a. Redraw the figure separately as below;



$\Delta\, QRS$ is similar to $\Delta\, PQS$

$\Rightarrow \frac{16}{x} = \frac{x}{9}$

$x^2 = 16 \times 9$

$x^2 = 144$

$x = \sqrt{144} = 12$cm

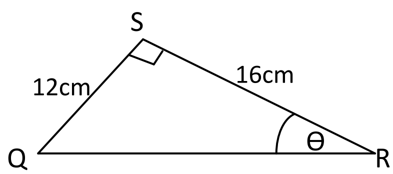
b. From $\Delta\, QRS$,

$\tan\theta = \frac{12}{16}$

$\theta = \tan^{-1}\left(\frac{12}{16}\right)$

$\theta = 36.6$

$\theta = 37^0$ (correct to the nearest degeree)

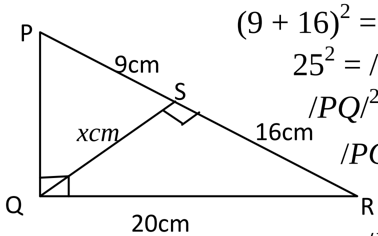


c. From $\Delta\, QRS$,

$/QR/^2 = 12^2 + 16^2$

$/QR/ = \sqrt{144 + 256} = \sqrt{400} = 20$cm

From ΔPQR,



$(9 + 16)^2 = /PQ/^2 + 20^2$

$25^2 = /PQ/^2 + 20^2$

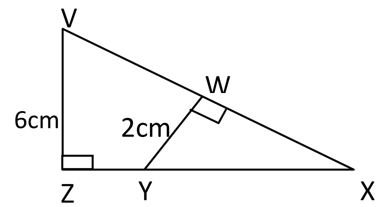
$/PQ/^2 = 25^2 - 20^2$

$/PQ/ = \sqrt{625 - 400}$

$/PQ/ = \sqrt{225}$

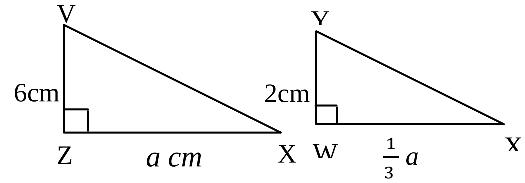
$/PQ/ = 15$cm

6. In the diagram, VXZ is a triangle, $/VZ/ = 6$cm, $/WY/ = 2$ cm and $<VZX = XWY = 90^0$. Find the ratio of the area of $\Delta\, VZX$ to the area of $\Delta\, WXY$



Solution

Redraw the triangle separately as shown:



$\Delta\, VZX$ is similar to $\Delta\, WXY$

$k = \frac{/YW/}{/VZ/} = \frac{2}{6} = \frac{1}{3}$

$\Rightarrow$ If $/ZX/ = a$, $/WX/ = \frac{1}{3}a$

Area of $\Delta\, VZX = \frac{1}{2}b\,h$

$= \frac{1}{2} \times a \times 6 = 3a$ cm^2

Area of $\Delta\, WXY = \frac{1}{2}\, b\, h = \frac{1}{2} \times \frac{1}{3}\, a \times 2 = \frac{1}{3}\, a$ cm^2

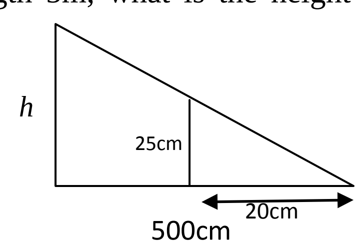
The ratio of the area of $\Delta\, VZX$ to the area of $\Delta\, WXY = 3a : \frac{1}{3}\, a = 3 : \frac{1}{3}$

$3 \times 3 : 3 \times \frac{1}{3} = 9 : 1$ *(Multiply through by 3)*

6. A pole and a stick stand vertically on a level ground. The stick is 25 cm long and cast a shadow of length 20 cm. If the pole casts a shadow of length 5m, what is the height of the pole?

Solution

Let the height of the pole be h



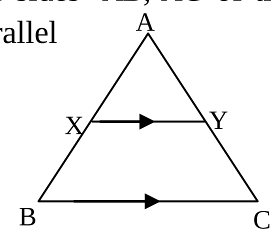
From the similar triangles,

$\frac{h}{25} = \frac{500}{20}$

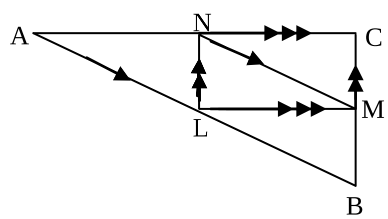
$\Rightarrow h = \frac{25 \times 500}{20}$ = 625 cm

Exercises 11.17

1. X and Y lie on the sides AB, AC of triangle ABC, so that XY is parallel to BC as shown in the figure below.
Write down a pair of similar triangles.

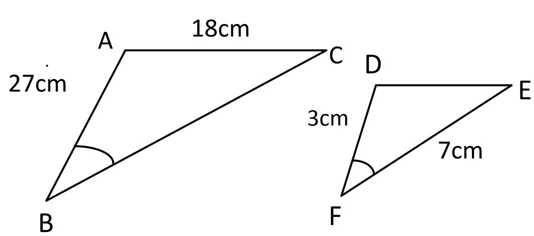


2. L, M and N lie on the sides AB, BC, CA of Δ ABC. LM, MN, NL are parallel to CA, AB, BC respectively as shown below,



Write down as many similar triangles as you can.

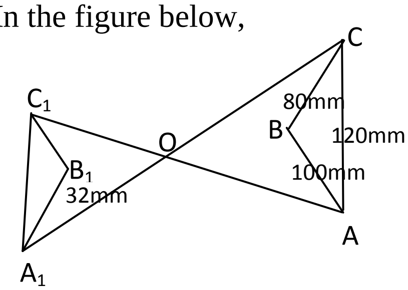
B. The two triangles below are similar with the marked angles equal. Use this to answer question 1 to 3.



1. Write a proportion involving the sides of the two triangles.
2. Calculate $/BC/$ Ans.= 63cm
3. Find $/DE/$ Ans.= 2cm

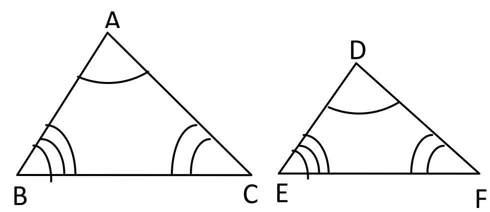
4. A vertical pole 6m high, cast a shadow 9m long at the same time that a tree cast a shadow 30m long. Find the height of the tree . A = 625 cm

5. In the figure below,



i. Write three equal ratios of the sizes of the two similar triangles.
ii. If $/A_1B_1/$ = 32mm, find $/A_1C_1/$ and $/B_1C_1/$.

D. The triangles ABC and DEF below are similar



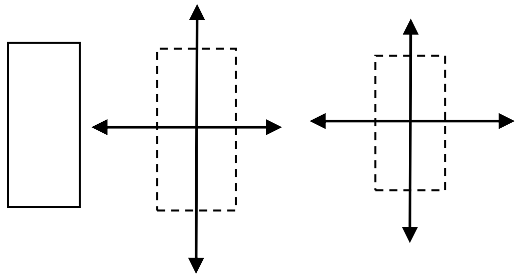
1. If $AB = 4$, $DE = 2$ and $AC = 6$, find DF
2. If $AB = 3$, $AC = 2$ and $DE = 6$, find DF
3. If $BC = 9$, $EF = 12$ and $DE = 8$, find AB
4. If $AB = \frac{3}{4}$, $DE = \frac{1}{2}$ and $BC = 1\frac{1}{2}$, find EF

Rotational Symmetry

Any figure that can be turned through an angle (say $90^{0,}$ 180^0, 270^0 and 360^0) about a fixed point such that the image fits onto the original shape of the figure is said to have rotational symmetry.

The number of times the figure fits onto the original shape when turned through is called the ***order of rotational symmetry***. The order of rotational symmetry of a rectangle is verified below;

Original shape



Since two rotations fit the original shape of the object, the order of rotation of a rectangle is 2.

Object	Order of Symmetry
square	4
Rectangle	2
Kite	1
Circle	Infinity
Regular pentagon	5
Rhombus	2

Exercises 11.18

1. Name 3 shapes that have both line of symmetry and rotational symmetry.

2. Determine the order of rotational symmetry of each of the following:

a. Isosceles triangle b. Equilateral triangle

c. Regular Pentagon d. Regular Hexagon.

12 RATIO AND RATES

Meaning of Ratio

Ratio is the comparison of two or more similar quantities. Thus, we compare pen to pen, pencil to pencil, money to money etc.

In general the ratio of two numbers x and y is written as $\boldsymbol{x : y}$ read as **“x is to y”.**

Ratio can also be expressed as a fraction. Thus, $x : y = \frac{x}{y}$ and $y : x = \frac{y}{x}$.

All ratios must be expressed in its simplest form with no unit. If the units are not the same, change one unit to the unit of the other before simplification

Worked Examples

1. There are 24 boys and 21 girls in a class, find the ratio of boys to girls.

Solution

Let B represent Boys and G represent Girls.

a. Ratio of boys to girls = B : G

$= 24 : 21 = \frac{24}{21} = \frac{8}{7} = 8:7$

2. Two sticks measure 25cm and 100cm respectively, find the ratio of their respective lengths

Solution

$25\text{cm} : 100\text{cm} = \frac{25}{100} = \frac{1}{4} = 1 : 4$

3. A mother is 60 years and the daughter is 28 years, what is the ratio of mother’s age to the daughter’s age.

Solution

Ratio of mother’s age to daughter’s age $= 60 : 28 = \frac{60}{28} = \frac{15}{7} = 15 : 7$

4. Express 9 days to 3 weeks as a ratio.

Solution

Express the ratiosin similar units.That is:

9 days: 3 weeks, but7 days = 1 Week

$\therefore$ 3 weeks = 3 × 7 days = 21 days

The new ratio is 9 days: 21 days

$\frac{9}{21} = \frac{3}{7}$ or $3 : 7$

5. Express the ratio 20cm to 15m in the form 1: n

Solution

Change 15m to cm

Thus 15m = 1500cm

New ratio is 20cm: 1500cm

$\frac{20}{1500} = \frac{1}{75} = 1 : 75$

6. Simplify the ratio $4 : 1\frac{3}{4}$

Solution

$4 : 1\frac{3}{4}$

$4 : \frac{7}{4}$ $\quad \begin{pmatrix} \textit{change mixed fraction} \\ \textit{to improper fraction} \end{pmatrix}$

$4 \times 4 : 4 \times \frac{7}{4}$ $\quad \begin{pmatrix} \textit{Multiply both} \\ \textit{sides by } 4 \end{pmatrix}$

$16 : 7$

Exercises 12.1

A. Express each ratio in its simplest form;

1. 85p : Gh¢5
2. 5cm : 20mm
3. 500g : 2kg
4. 3 km : 600m

B. 1. Tom and Jerry were given 124 oranges to share. If Tom received 24 oranges,
a. find the ratio of Jerry's oranges to that of Tom.
b. express Jerry's share as a ratio of the total oranges.

2. Romeo is 9 years older than Juliet. If Juliet is 18 years old; find the ratio of the age of Juliet to that of Romeo.

3. Express 6 days is to 3 weeks as a ratio in its simplest form.

4. Three friends are discussing pocket money. The first gets Gh¢3.60 a week, the second gets Gh¢4.80 and the third gets Gh¢6.00 a week. Write their pocket money as a ratio in its simplest form

Ratio of Two variables in an Equation
A. In a quadratic equation,
I. Solve it the quadratic way by finding the sum of roots and products of roots to obtain the factors.
II. Equate the roots obtained to zero.
III. Solve each of the equations to obtain the required ratio.

B. In a linear equation,
I. Group like terms at each side of the equation.
II. Solve to obtain the required ratio.

Worked Example
1. In $10x^2 - 9xy + 2y^2 = 0$ find the ratio $x : y$

Solution
$10x^2 - 9xy + 2y^2 = 0$
$10x^2 - 5xy - 4xy + 2y^2 = 0$
$(10x^2 - 5xy) - (4xy + 2y^2) = 0$
$5x(2x - y) - 2y(2x - y) = 0$
$(5x - 2y)(2x - y) = 0$
$5x - 2y = 0$
$5x = 2y \quad \begin{pmatrix} Read\ from\ left\ toright \\ and\ from\ right\ to\ left \end{pmatrix}$
$x = 2$ *and* $y = 5$
$\frac{x}{y} = \frac{2}{5}$
Therefore the ratio $x : y = 2 : 5$

$2x - y = 0$
$2x = y \quad \begin{pmatrix} Read\ from\ left\ toright \\ and\ from\ right\ to\ left \end{pmatrix}$
$x = 1$ and $y = 2$
$\frac{x}{y} = \frac{1}{2}$
Therefore, the ratio $x : y = 1 : 2$

2. Find the ratio $y : x$ in $4x^2 + 6xy - 4y^2 = 0$

Solution
$4x^2 + 6xy - 4y^2 = 0$
$4x^2 - 2xy + 8xy - 4y^2 = 0$
$(4x^2 - 2xy) + (8xy - 4y^2) = 0$
$2x(2x - y) + 4y(2x - y) = 0$
$(2x + 4y)(2x - y) = 0$
$2x + 4y = 0$ or $2x - y = 0$
$2x = -4y$ or $2x = y$

When $2x = -4y \quad \begin{pmatrix} Read\ from\ left\ toright \\ and\ from\ right\ to\ left \end{pmatrix}$
$x = -4$ and $y = 2$
$\frac{y}{x} = \frac{2}{-4} = \frac{1}{-2}$
Therefore, $y : x = 1 : -2$

When $2x = y \quad \begin{pmatrix} Read\ from\ left\ toright \\ and\ from\ right\ to\ left \end{pmatrix}$
$x = 1$ or $y = 2$
$\frac{y}{x} = \frac{2}{1}$
Therefore, $y : x = 2 : 1$

3. If $9\left(\frac{x - 2y}{x + 2y}\right) = 1$, find the ratio of $x : y$

Solution

$9\left(\frac{x-2y}{x+2y}\right) = 1$

$\frac{9x-18y}{x+2y} = 1$

$9x - 18y = x + 2y$

$9x - x = 2y + 18y$

$8x = 20y \qquad \left(\begin{matrix} Read\ from\ left\ toright \\ and\ from\ right\ to\ left \end{matrix}\right)$

$x = 20$ and $y = 8$

$\frac{x}{y} = \frac{20}{8}$

$\frac{x}{y} = \frac{5}{2}$

Therefore $x : y = 5 : 2$

4. $3x - 4 = 2(y - 2)$. Find $\frac{x}{y}$, where $y \neq 0$

Solution

$3x - 4 = 2(y - 2)$

$3x - 4 = 2y - 4$

$3x = 2y - 4 + 4$

$3x = 2y \left(\begin{matrix} Read\ from\ left\ toright \\ and\ from\ right\ to\ left \end{matrix}\right)$

$x = 2$ and $y = 3$

$\frac{x}{y} = \frac{2}{3}$

5. If $4(a - 2) = 3b - 8$, find $\frac{a}{b}$, $b \neq 0$

Solution

$4(a - 2) = 3b - 8$

$4a - 8 = 3b - 8$

$4a = 3b - 8 + 8$

$4a = 3b$

$a = 3$ and $b = 4$

$\frac{a}{b} = \frac{3}{4}$

Exercises 12.2

Find the ratio *x* : *y*in the following:

1. $x^2 + xy - 6y^2 = 0$ 5. $\frac{3(x-4y)}{x+y} = 2$
2. $x^2 + xy - 2y^2 = 0$ 6. $x(x - y) = y(x + 3y)$
3. $3x^2 - 7xy + 4y^2 = 0$
4. $\frac{x+2y}{x} = \frac{7}{5}$ 7. $\frac{x}{y} + \frac{20y}{x} = 9$

Proportion

A proportion is an equation which says that two or more ratios are equal. In other words, when two ratios are equated, they form a proportion.

A proportion is read as " *x* is to *y*" as "*a* is to *b*" expressed as $x : y = a : b$ or $\frac{x}{y} = \frac{a}{b}$

The products ***xb*** and ***ay*** are called ***cross products.*** Cross products are always equal.

That is, in $\frac{x}{y} = \frac{a}{b}$, $x \times b = a \times y$

Proving whether Two Ratios form a proportion

To prove whether two or more ratios form a proportion, check their cross products

I. If they are equal, the ratios form a proportion

II. If they are unequal, the ratios do not form a proportion

Worked Examples

1. Show whether 3 : 4 and 18 : 24 form a proportion.

Solution

$\frac{3}{4} = \frac{18}{24}$

$3 \times 24 = 4 \times 18$ *By cross multiplication*,

$72 = 72$

$\therefore$ 3 : 4 and 18 : 24 are proportional

2. Do 3 : 6 and 18 : 36 form a proportion ?

Solution

$\frac{3}{6} = \frac{18}{36}$

$3 \times 36 = 6 \times 18$

$108 = 108$

$\therefore$ 3 : 6 and 18 : 24 are proportional

3. What proportion of Gh¢150 is Gh¢5?

Solution
This means, express Gh¢5 as a fraction of Gh¢150
$\frac{5}{150} = \frac{1}{30}$

4. 10 men and 12 women work in a factory. What proportion of the work force are men and what proportion are women?

Solution
10 men + 12 men = 22 workers
Proportion of men = $\frac{10}{22} = \frac{5}{11}$
Proportion of women = $\frac{12}{22} = \frac{6}{11}$

Exercises 12.3
A. Determine whether the following pair of ratios form a proportion or not:
1. 2 : 16 and 5 : 40 2. 7 : 3 and 21 : 6
3. 12 : 4 and 9 : 3 4. 15 : 5 and 5 : 15

B. Answer the following:
1. What proportion of Gh¢25 is Gh¢5?
2. What proportion of 10 is 2.5?
3. What proportion of 5 kg is 600g?

C. 1. In a sports stadium, it is found that the ratio of boys to girls is 21:19. What proportions are boys and what proportions are girls?

2. Of the 200 people who took their driving test last week, 60 passed. What proportion failed the driving test?

3. Nancy throws a die 600 times and finds that a "6" turns up 80 times. What proportion of the throws were sixes?

Proportion with a Variable
If one side of a proportion contains a variable, the value of the variable is workout as follows;
I. Find the cross products of the ratios.
II. Divide both sides of the equation by the coefficient of the variable to obtain the value of the variable.

In summary, if $\frac{a}{b} = \frac{c}{d}$, then the cross product is $a \times d = b \times c$

Worked Examples
1. The ratio 9 : x is equivalent to 36 : 20, what is the value of x?

Solution
$9 : x = 36 : 20$
$\frac{9}{x} = \frac{36}{20}$
$9 \times 20 = 36 \times x$
$180 = 36x$
$\frac{180}{36} = \frac{36x}{36}$
$5 = x$ or $x = 5$

2. The ratio 8:12 is equivalent to y : 9. What is the value of y ?

Solution
$8 : 12 = y : 9$
$\frac{8}{12} = \frac{y}{9}$
$\Rightarrow 12 \times y = 8 \times 9$
$12y = 72$
$y = \frac{72}{12} = 6$

3. If 13 : 9 = 39 : 9k, find the value of k.

Solution
$13 : 9 = 39 : 9k$
$\frac{13}{9} = \frac{39}{9k}$

$13 \times 9k = 9 \times 39$

$117k = 351$

$k = \frac{351}{117} = 3$

4. Given that $(m + 2) : 1 = 6 : 2$, find m

Solution

$\frac{m+2}{1} = \frac{6}{2}$

$2(m + 2) = 1 \times 6$

$2m + 4 = 6$

$2m = 6 - 4$

$2m = 2,$

$x = 1$

5. Solve $3 : (n - 7) = 5 : 10$

Solution

$\frac{3}{n-7} = \frac{5}{10}$

$5(n - 7) = 10 \times 3$

$5n - 35 = 30$

$5n = 30 + 35$

$5n = 65$

$n = \frac{65}{5} = 13$

Solved past Questions

1. If $a : b = 4 : 5$, find $a + b : 2a - b$

Solution

If $a : b = 4 : 5$

$\frac{a}{b} = \frac{4}{5}$

$\Rightarrow a = 4$ and $b = 5$

$a + b = 4 + 5 = 9$

$2a - b = 2(4) - 5 = 8 - 5 = 3$

$a + b : 2a - b = 9 : 3 = 3 : 1$

2. If $3x^2 : 8 = 24 : 16$, find the value of x given that $x > 0$

Solution

$3x^2 : 8 = 24 : 16$

$\frac{3x^2}{8} = \frac{24}{16}$

$\frac{3x^2}{8} = \frac{3}{2}$

$2 \times 3x^2 = 3 \times 8$

$6x^2 = 24$

$x^2 = \frac{24}{6}$

$x^2 = 4$

$x = \sqrt{4} = 2$

Exercises 12.4

A. Find the value of the variables;

1. $y : 28 = 5 : 7$
2. $2 : x = 12 : 30$
3. $2 : 11 = a : 33$

B. Find value of x in the following:

1. $\frac{18}{14} = \frac{x}{7}$ 2. $\frac{4}{x} = \frac{x}{36}$ 3. $\frac{2}{3} = \frac{10}{5x}$

4. $\frac{6}{x} = \frac{1}{5}$ 5. $\frac{7}{3} = \frac{42}{3x}$ 6. $\frac{20}{8x} = \frac{5}{6}$

C. Find the values of the variables:

1. $5 : 4 = 30 : 3x$ 3. $32 : 14 = 2x : 7$

2. $4x : 48 = 1 : 3$ 4. $21 : 2x = 7 : 10,$

D. Find the value of the variable:

1. $(3 - 2y) : 4 = 5 : 4$ 4. $2 : (5 + y) = 4 : 2$

2. $5(2a - a) : 3 = 25 : 15$

3. $(3y - 3) : 6 = 6 : 4$ 5. $3 : 7 = 6 : (b - 2)$

Direct Proportion

Direct proportion involves comparison of two quantities such that an increase in one quantity automatically causes an increase in the other quantity. Likewise, a decrease in one quantity requires a decrease in the other. For instance, if the cost of 3 books is Gh¢30.00, any increase in the number of books will cost more than

Gh¢30.00 and any decrease in the number of books will cost less than Gh¢30.00.

To solve problems involving direct proportions,

I. Represent the required value by any preferred variable; examplex, y, m, n *etc*

II. Identify the ratios involved and form a simple proportion.

III. Solve for the value of the variable

Worked Examples

1. The cost of 3 crates of eggs is Gh¢21.00. How much will you pay for 6 crates of eggs?

Solution

Method 1

The two quantities involved are "eggs" and "money". Therefore, form a ratio of eggs and equate it to the ratio of money. That is:

3crates : 6 crates = 21 : x

$\frac{3}{6} = \frac{21}{x}$

$3 \times x = 21 \times 6$ *By cross multiplication,*

$3x = 126$

$x = \frac{126}{3} = 44$

$\therefore$ The 6 crates eggs will cost Gh¢42.00

Method 2

By unitary method,

First find the cost of 1 unit and multiply by the total number required.

i.e 3 eggs cost Gh¢21

$\therefore$ 1 crate will cost $\frac{21 \times 1}{3}$ = Gh¢7.00

If 1 crate costsGh¢7.00;

then 6 eggs will cost = 6 × Gh¢7 = Gh¢42.00

2. If the weight of 17 bags of rice is 680kg. What is the weight of 7 bags ?

Solution

Let y represents the weight of 7 bags of rice

By direct proportion:

17 : 7 = 680: y

$\frac{17}{7} = \frac{680}{y}$

$17 \times y = 7 \times 680$

$17y = 4{,}760$

$y = \frac{4{,}760}{17} = 280$

$\therefore$ The weight of 7 bags of the rice is 280kg.

3. If 120km trip requires 8 gallons of petrol, how many gallons of petrol is needed for a 600km trip?

Solution

Let x represent the liters of petrol required.

By direct proportion,

120 km : 600km = 8 : x

$\frac{120}{600} = \frac{8}{x},$

$120 \times x = 600 \times 8$

$120x = 4{,}800$

$x = \frac{4{,}800}{120} = 40$

$\therefore$ 40gallons is needed for the 600km trip.

4. At Mr. Opoku's shop, 3 wrist watches are sold for Gh¢100.00. How many watches can you purchase with Gh¢600.00?

Solution

Let m be the number of wrist watches.

100 : 600 = 3 : m

$\frac{1}{6} = \frac{3}{m}$

$1 \times m = 6 \times 3$

$m = 18$

$\therefore$ 18 wrist watches cost Gh¢ 600.00

Exercises 12.5

1. If 2 yards of a floor carpet cost Gh¢ 40.00, how

many yards can be bought for Gh¢ 2,400.00?

2. Mr. Jones purchased 6 crates of eggs for Gh¢180.00, find the cost of a crate of egg?

3. Mr. Green walks 16 minutes to his shop every day. Calculate in hours, the time he walks to his shop in 5 days.

4. A bus has a capacity of 63 passengers.If 8 buses are hired for a trip, how many passengers can make up the trip.

5. In an organized football competition, the duration of a match is 30minutes, how many matches can be played in a total of 420 minutes.

Indirect Proportion

Indirect proportion involves comparison of two quantities such that an increase in one quantity causes a decrease in the other quantity and vice versa. For instance, if 4 girls sweep a class in 20 minutes, provided they work at the same rate, then a decrease in the number of girls will cause an increase in the time (more than 20minutes) required to finish. Likewise, an increase in the number of girls will cause the work to be finished earlier, thus a decrease in the time (less than 20 minutes) required to finish the work.

To solve problems involving indirect proportion,

I. Identify the two ratios.

II. Form a proportion but interchange the positions of the terms of the second ratio. i.e if we have $\boldsymbol{a : b = c : d}$, then the indirect proportion is $\boldsymbol{a : b = d : c}$

III. Solve for the value of the unknown variable

Worked Examples

1. It takes 6 days for 4men to weed a plot of land. How long will it take 8men working at the same rate, to weed the same plot of land?

Solution

Ratio of men = 4: 8

Ratio of days = 6: x

As indirect proportion,

$4 : 8 = x : 6$

$\frac{4}{8} = \frac{x}{6}$

$8 \times x = 6 \times 4$

$8x = 24$

$x =$ 3days

$\therefore$ 8 men will use 3days to weed the same plot

2. It took 12 minutes for 8 girls to sweep a class. How many girls can sweep the same classroom at the same work rate in 4 minutes?

Solution

Ratio of minutes = 12: 4

Ratio of girls = 8 : x

As indirect proportion,

$12 : 4 = x : 8$

$\frac{12}{4} = \frac{x}{8}$

$4 \times x = 12 \times 8$

$4x = 96$

$x = \frac{96}{4}$ = 24 girls

$\therefore$ 24 girls can sweep the room 4 minutes.

3. If a car traveling at 80km/h takes 3 hours to cover a certain distance, how long will it take another car traveling at 60km/h to cover the same distance?

Solution

Ratio of distance = 80 : 60

Ratio of 1 hour =3 : x

As indirect proportion,

$80 : 60 = x : 3$

$\frac{80}{60} = \frac{x}{3}$

$60 \times x = 80 \times 3$

$60x = 240$

$x = \frac{240}{60} =$ 4 hours

4. A Lorry traveling at 120km/h took 3 hours to cover a certain distance. How long would it take another Lorry traveling at 90km/h to cover the same distance?

Solution

Ratio of distance = 120 : 90

Ratio of time = 3 : x

As indirect proportion,

120: 90 = x: 3

$\frac{120}{90} = \frac{x}{3}$

$90 \times x = 120 \times 3$

$90x = 360$

x = 4 hours.

Exercises 12.6

1. If 30 men can dig a pit in 21 days, how many days will 14 men take to dig the same pit, working at the same rate?

2. It took 6 students 1hour, 18 minutes to weed their compound. If 7 students are added, how long will it take them to weed the same plot of land at the same rate?

3. Eight boys weeded a piece of farmland in 6 hours. How long will it take 2 boys to weed the same piece of farmland, working at the same rate?

4. If 10 painters can take 8 days to paint a house, how long will it take 20 painters to paint the same house working at the same rate?

5. Mansah fetched 84 buckets of water to fill a poly tank in 300 minutes. If she fills the same poly tank with her other 5 friends,

i. how many buckets of water will each person fetch?

ii. in how many minutes will the poly tank be full, if they work at the same rate ?

Sharing According to a Given Ratio

It involves the situation whereby a given quantity is shared according to a given ratio between two or more people. The steps involve are;

I. Identify the sharers and their respective ratio or part.

II. Find the total ratio.

III. Identify the quantity to be shared.

IV. Express each ratio as a fraction of the total ratio and multiply by the total quantity.

For instance, to share K oranges in the ratio a : b, to let's say Jenifer and Rosemary respectively,

Jennifer's part = a, Rosemary's part = b

Total ratio = $a + b$

Jennifer's share = $\frac{a}{a+b} \times \mathbf{k}$

Rosemary's share = $\frac{b}{a+b} \times \mathbf{k}$

Worked Examples

Type 1

1. Jack and Jill shared Gh¢60,000.00 between them in the ratio 3 : 2, find:

i. the share of each person.

ii. how much did Jack receive than Jill?

Solution

i. Jack : Jill = 3 : 2

Total ratio = 3 + 2

Total amount = Gh¢60,000.00

Jack'share = $\frac{3}{5}$ × 60 = Gh¢36,000

Jill's share = $\frac{2}{5}$ × Gh¢60,000 = Gh¢24,000

ii. Gh¢36,000.00 – Gh¢24,000.00
 = Gh¢12,000.00
Jack received Gh¢12,000.00 more than Jill.

2. Pat, Tom and Ken share a profit of Gh¢150,000.00 in the ratio 1 : 4 : 3 respectively. How much did each person get?

Solution
Pat : Tom : Ken = 1 : 4 : 3
Total ratio = 1 + 4 + 3 = 8
Total amount = Gh¢150,000.00

Pat received = $\frac{1}{8}\times 150{,}000 = 18{,}700.50$
Therefore,Pat's share = Gh¢18,700.50

Tom = $\frac{4}{8} \times 150{,}000 = 75{,}000$
Therefore Tom received Gh¢75,000.00

Ken's share = $\frac{3}{8} \times 150{,}000 = 56{,}200.50$
Therefore, Ken received = Gh¢56,250.00

3. Three people shared Gh¢540,000.00 in the ratio 2: 3: 4. Find the least amount received.

Solution
From 2 : 3: 4, the least share is 2

Total ratio = 2 + 3 + 4 = 9
Amount shared = Gh¢540,000
Least share = $\frac{2}{9}$ × Gh¢540,000 = Gh¢120,000.00

4. Three men shared Gh¢480,000.00 in the ratio 7 : 8: 9. Find the difference between the least and the greatest shares.

Solution
Amount shared = Gh¢480,000
Ratio = 7 : 8 : 9.

Total ratio = 7 + 8 + 9 = 24
Least ratio = 7
Least share = $\frac{7}{24}$ × Gh¢480,000 = Gh¢140,000

Greatest ratio = 9
Greatest share = $\frac{9}{24}$ × Gh¢480,000 = Gh¢180,000

Difference = Gh¢180,000 – Gh¢140,000
 = Gh¢40,000.00

5. The sum of the ages of a man and his wife is 81 years. If the ratio of their ages is 5 : 4, find the age of the younger person.

Solution
Total ratio = 5 + 4 = 9
Total age = 81
Younger person's age = $\frac{4}{9} \times 81 = 36$ years

6. Three children Kwabena, Esi and Yaw were given 160 oranges to share. Kwabena got $\frac{1}{4}$ of the Oranges. Esi and Yaw shared the remainder in the ratio 3 : 2.
i. How many oranges did Esi receive?
ii. How many more oranges did Yaw receive than Kwabena?

Solution
Total number of oranges = 160
Kwabena's share = $\frac{1}{4}\times 160 = 40$ oranges
The remainder = 160 − 40 = 120 oranges
Esi : Yaw = 3 : 2
Total ratio = 3 + 2 = 5

i. Esi's share = $\frac{3}{5} \times 120 = 72$ oranges
Therefore, Esi received = 72 oranges

ii. Yaw's share = 120 −72 = 48 Oranges
Kwabena's share = 40 Oranges
(48 − 40) Oranges = 8 oranges
Therefore, Yaw received 8 oranges more than Kwabena

7. In sharing 95 oranges with Dede, Fofo kept 45 of them and share the rest equally with Dede. Calculate the share of each person.

Solution
Total oranges = 95
Fofo's first share = 45 oranges
Remaining = (95 − 45) = 50 oranges
50 oranges shared equally means a ratio 1:1

Fofo's second share = $\frac{1}{2} \times 50$
= 25 oranges
$\therefore$ Fofo's total share = 40 + 25 = 65oranges

Dede's share = $\frac{1}{2} \times 50 = 25$ oranges

8. In his will, a father left an estate worth Gh¢76,000.00. Out of this, Gh¢16,000.00 was reserved for various purposes. The rest of the amount was shared among his three children. The eldest son received 20% of the amount. The remaining amount was shared between the other son and the daughter in the ratio 3 : 2 respectively. Calculate:
i. the amount that the eldest son received,
ii. the amount that the daughter received,
iii. the difference between the amounts the two sons received.

Solution
i. Amount = Gh¢ (76,000 – 16,000) = Gh¢ 60,000

Eldest son's share,
= 20% of 60,000
= $\frac{20}{100} \times 60{,}000$ = Gh¢12,000.00

Remaining amount;
= Gh¢(60,000 – 12,000) = Gh¢48,000.00

ii. Total ratio = 3 + 2
Daughter's share = $\frac{2}{5} \times 48{,}000$ = Gh¢19,200

iii. Son's share = $\frac{3}{5} \times 48{,}000$ = Gh¢28,800.00
Difference = Gh¢(28,800 – 12,000) = Gh¢16,800

Type 2
Worked Examples
1. The ratio of the number of boys to the number of girls in a school of 432 pupils is 5 : 4. If the number of boys increased by 12, the new ratio of boys to girls is 7 : 6. Find the increase in the number of girls.

Solution
Total number of boys and girls = 432
Ratio of boys to girls = 5 : 4
Total ratio = 9
Number of boys = $\frac{5}{9} \times 432 = 240$
Number of girls = $\frac{4}{9} \times 432 = 192$

If the number of boys increased by 12, the number of girls increased by x. Therefore, ratio of boys to girls;
$240 + 12 : 192 + x = 7 : 6$
$\Rightarrow \frac{240+12}{192+x} = \frac{7}{6}$
$\frac{252}{192+x} = \frac{7}{6}$
$6 \times 252 = 7(192 + x)$ *Cross multiplication,*
$1512 = 1344 + 7x$
$7x = 1512 – 1344$
$7x = 168$
$x = 24$

2. There are 5 more girls than boys in a class. If 2 boys join the class, the ratio of boys to girls will be 5 : 4. Find the :
i. number of girls in the class;
ii. total number of pupils in the class.

Solution
i. Let the number of boys and girls be x respectively.

Ratio of girlsto boys = $x : x$
5 more girls than boys = $x + 5 : x$

If 2 more boys join the class, the ratio of boys to girls will be 5 : 4
$x + 5 : x + 2 = 5 : 4$
$\Rightarrow \frac{x+5}{x+2} = \frac{5}{4}$
$4(x + 5) = 5(x + 2)$
$4x + 20 = 5x + 10$
$20 – 10 = 5x – 4x$
$10 = x$
$x = 10$ girls

But number of girls = $x + 5 = 10 + 5 = 15$
Totaal number of pupils in the class
= number of boys + number of girls
= $x + x + 5$
= 10 + 10 + 5 = 25 pupils

Some solved Past Questions
1. An amount of Gh¢300,000.00 was shared among Ama, Kojo and Esi. Ama received Gh¢60,000.00. Kojo received $\frac{5}{12}$ of the remainder, while the rest went to Esi. In what ratio was the money shared?

Solution
Amount shared = Gh¢300,000
Ama's share = Gh¢60,000
Remainder = Gh¢(300,000 – 60,000)
= Gh¢240,000

Kojo's share = $\frac{5}{12}$ × Gh¢240,000 = Gh¢100,000

Esi's share
= Gh¢(300,000 – 60,000 – 100,000)
= Gh¢140,000

Ratio of Ama : Kojo : Esi
= 60,000 : 100,000 : 140,000 = 2 : 5 : 7
The money was shared in the ratio 2 : 5 : 7

2. A man gave out Gh¢24,000.00 to his three brothers X, Y and Z, to be shared among them. If X takes twice as much as Y and Y is given one – third of what Z takes, how much did each of them receive?

Solution
Let Z's share = z
Let Y's share =$\frac{1}{3}$ z
X's share = $2\left(\frac{1}{3}\right)$ Z
Total share = Gh¢24,000

⇒ (X + Y + Z) shares = 24,000
By substitution,
$z + \frac{1}{3}z + 2\left(\frac{1}{3}\right)Z = 24,000$
$z + \frac{z}{3} + \frac{2Z}{3} = 24,000$
$3z + z + 2z = 72,000$
$6z = 72,000$
$Z = \frac{72,000}{6} = 12,000$
Therefore the share of Z is Gh¢12,000.00

The share of Y = $\frac{1}{3}$ (z)
= $\frac{1}{3}$ (12,000) = Gh¢4,000.00

The share of X = $\frac{2Z}{3} = \frac{2(12,000)}{3}$ = Gh¢8,000

Exercises 12.7

1. Shamo, Bako and Aspa were given Gh¢40,000.00 to share. Shamo received $\frac{1}{5}$ of the amount and the rest was shared between Bako and Aspa in the ratio 5 : 3,
i. Calculate the share of Shamo, Bako and Aspa
ii. How much did Bako receive than Shamo?

2. If Bole and Bamboi are to share an amount of Gh¢80,000.00 in the ratio 5 : 3 respectively, what will be the difference between the share of the two persons?

3. A man gives his twochildren some spending money according to the ratio of their ages. If the children are aged 11 and 13 and he gives out a total of Gh¢40,800.00, how much does each child gets.

4. A sum of money is divided between three men X, Y and Z in the ratio 5 : 3 : 1. If Y has Gh¢3.50 more than Z, calculate how much X has.

5. An amount of Gh¢165,000.00 was shared among Okra, Harry and Regina as follows: Okra received $\frac{1}{3}$ of the amount and Gh¢1,500.00 out of the rest was given to Regina. Thereafter, the rest of the money was shared between Harry and Regina in the ratio 5 : 2.
i. Calculate the share of each person.
ii.How much did Okra receive than Regina?

6. Naa and Ayeley were given Gh¢70,000.00 to share. They agreed to give Gh¢ 6,000.00 to their mother and shared the rest in the ratio 3 : 5 respectively.
i. How much is the share of each person?
ii. How much did Ayeley receive than Naa?

7. Three brothers contribute money to a savings fund according to the ratio of their ages. The brothers are aged 16, 17 and 19.
i. If the eldest brother has to contribute Gh¢9,500.00 to the fund, how much money must each of the brothers contribute?
ii. If the younger brother has to contribute Gh¢5,600.00, how much money must each of the brothers contribute?

8. Mr. Asamoah shared Gh¢200,000.00 among his three sons Pak, Tom and Ben as follows; he gave $\frac{1}{4}$ of the money to Pak,$\frac{2}{5}$ of the money to Tom and Gh¢20,000.00 to Ben. He then shared the rest of the money between Tom and Ben in the ratio 2:3 respectively.
i. Calculate the total share of each person.
ii. Calculate the sum of money received by Ben and Pak.
iii. Which of the three sons receive the least share and by how much does it differ from the greatest share?

9. A brother and a sister share 2400 kola nuts in the ratio of 5 : 3. The brother then shares his kola nut with two friends in the ratio of 3 : 2 : 1 respectively. How many nuts does each friend receive?

Finding the Total Amount (Quantity) Shared Given the Ratio and the Share of One Person

Given the ratio and the share of one person, the total amount (quantity) of an item shared can be calculated. Any of the following methods can be used;

Method I

Take for instance, A and B shared a certain quantity, x, in the ratio $a : b$ respectively and A's share is m, then A's share can be expressed as shown below;

A's Share = $\frac{a}{a+b} \times x = m$

$m = \frac{ax}{(a+b)}$

Making x the subject,

$(a + b)\,m = a\,x$

$\boldsymbol{x = \frac{(a+b)m}{a}}$

Given the share of the first person, the quantity shared can be calculated by the formula;

$\boldsymbol{x = \frac{(a+b)m}{a}}$

Where $\boldsymbol{x}$ is the quantity shared, $\boldsymbol{(a + b)}$ is the total ratio, $\boldsymbol{a}$ is the ratio of the first person ***(A),*** and $\boldsymbol{m}$ is the share of A.

Similarly, if B's share is n, then B's share can be expressed as;

B's Share $= \frac{b}{(a+b)} \times x = n$

$n = \frac{bx}{(a+b)}$

Making x the subject,

$(a + b)n = bx$

$x = \frac{(a+b)n}{b}$

Given the share of the second person (B), the quantity shared can be calculated by the formula**;**

$\boldsymbol{x = \frac{(a+b)\,n}{b}}$, where $\boldsymbol{x}$ is the quantity shared, $\boldsymbol{(a + b)}$ is the total ratio, $\boldsymbol{b}$ is the ratio of the second person *(B*), and$\boldsymbol{n}$ is the share of the second person.

II. (Simple Proportion)

I. Represent the total quantity shared by any preferred variable (x)

II. Find the sum of the given ratio (a)

III. Identify the given share of one person, (y) and its corresponding ratio (b)

IV. Form a proportion of the corresponding ratio to the total ratio and the given share to the total quantity shared. That is:

$$\boxed{\boldsymbol{b : a = y : x \quad OR \quad \frac{b}{a} = \frac{y}{x}}}$$

Worked Examples

1. Adam and Eve shared a certain amount of money in the ratio 3:5. If Adam received Gh¢9,000.00, how much was shared between them?

Solution

Method I

Let the amount shared be x.

Adam's ratio (a) = 3

Eve's ratio (b) = 5

Total ratio = 3 + 5 = 8

Adam's share (m) = Gh¢9,000.00

Quantity shared, $x = \frac{(a+b)m}{a}$

But a = 3, b = 5 and m = 9000

$a + b = 3 + 5 = 8$

$x = \frac{(a+b)m}{a} = \frac{(3+5)900}{3} = \frac{8 \times 900}{3} = 24000$

The amount shared is Gh¢24,000.00

Method II

Let the amount shared be x

Given ratio = Adam and Eve = 3 : 5

Total ratio = 3 + 5 = 8 and its corresponding amount = x

Adam's ratio = 3 and its corresponding amount = Gh¢9000

3 : 8 = 9000 : x *By simple proportion,*

$\frac{3}{8} = \frac{900}{x}$

$3x = 8 \times 9000$

$x = \frac{8 \times 900}{3}$ = Gh¢24,000

The amount shared is Gh¢24,000.00

Method III

Adam's ratio = 3,

Adams share = Gh¢9,000

Eve's share = x

As proportion,

Adam and Eve = 3 : 5 = 900 : x

$\frac{3}{5} = \frac{\text{Gh¢}900}{x}$

3 x = 5 × Gh¢9000

$\frac{3x}{3} = \frac{5 \times \text{Gh¢ } 900}{3}$ = Gh¢15,000.00

Therefore, Eve's share is Gh¢15,000.00

But total amount shared

= Adam's share + Eve's share

= Gh¢ (9000.00 + 1,500.00) = Gh¢2,4000.00

2. Bole and Bamboi shared an amount of money in the ratio 2:5. If Bamboi received Gh¢4,500.00, how much money did they share?

Solution

Let the amount shared be x

Given ratio = Bole and Bamboi = 2 : 5

Total ratio = 2 + 5 = 7 and its corresponding amount = x

Bamboi's ratio = 5 and its corresponding amount = Gh¢4,500

5 : 7 = 4,500 : x *By simple proportion,*

$\frac{5}{7} = \frac{4,500}{x}$

5x = 7 × 4,500

$x = \frac{7 \times 4,500}{5}$ = Gh¢6,300

The amount shared is Gh¢6,300.00

3. Mr. Musa shared an amount to his three sons in the ratio 3 : 5 : 7. The one who had the least amount received Gh¢50,000.00, find the amount that was shared.

Solution

Method I

Let the amount shared be x.

The least ratio (a) = 3

The least share (n) = Gh¢50,000

Total ratio ($a + b + c$) = 3 + 5 + 7 = 15

The least share $= \frac{3}{15} \times x = 50,000$

$\Rightarrow \frac{3x}{15} = 50,000$

3x = 15 × 50,000

$x = \frac{15 \times 50,000}{3}$ = Gh¢250,000

The amount shared is Gh¢250,000.00

4. Three school children shared some oranges as follows; Akwasi got $\frac{1}{3}$ of the total and the remainder was shared between Abena and Jantua in the ratio 3 : 2. If Jantua got 24 oranges,

i. find the total number of oranges shared;

ii. how many oranges did Akwasi get?

Solution

i. Let the total number of oranges be x

Akwasi's share = $\frac{1}{3}x$

Remainder = $x - \frac{1}{3}x = \frac{3x - x}{3} = \frac{2x}{3}$

Total ratio = 3 : 2 = 5

Jantua's share = $\frac{2}{5} \times \frac{2}{3}x = 24$

$\frac{4}{15}x = 24$

4x = 24 × 15

4x = 360

$x = \frac{360}{4} = 90$

Total number of oranges shared was 90

ii.Akwasi's share = $\frac{1}{3}x$, but x= 90

= $\frac{1}{3} \times 90$ = 30 oranges

Solved Past Question

1. Three friends contributed to the capital of a company in the ratio 3 : 2 : 5. If the least contributor paid Gh¢30,000.00, how much was their total contribution?

Solution

Method I

Let the total contribution be x

Ratio = 3 : 2 : 5

Total ratio = 3 + 2 + 5 = 10

Least ratio = 2

Least contribution = $\frac{2}{10} \times x = 30,000$

$2x = 10 \times 30,000$

$2x = 300,000$

$x = \frac{300,000}{2} = 150,000.$

The total contribution is Gh¢150,000.00

Method II

By forming a proportion,

Least ratio : Total ratio = Least contribution : Total contribution

3 : 10 = 30,000 : x

$\frac{3}{10} = \frac{30,000}{x}$

$3x = 10 \times 30,000$

$x = \frac{10 \times 30,000}{3} = 150,000$

The total contribution is Gh¢150,000.00

3. Afia, Kwame and Abena shared the profit they earned in a month from a joint business in the ratio 2 : 4 : 3 respectively. If Abena and Kwame together got a total of Gh¢420,000.00, what was the total profit that the three shared?

Solution

Method I

Let the total profit shared be x

Afia, Kwame and Abena = 2 : 4 : 3

Total ratio = 2 + 4 + 3 = 9

Kwame and Abena's ratio = 4 + 3 = 7 and its corresponding amount = Gh¢420,000.00

By simple proportion,

7 : 9 = 420,000 : x

$\frac{7}{9} = \frac{420,000}{x}$

$7x = 9 \times 420,000$

$x = \frac{9 \times 420,000}{7} = 540,000$

The total profit shared is Gh¢540,000.00

Method II

Let the total profit shared be x

Afia, Kwame and Abena = 2 : 4 : 3

Total ratio = 2 + 4 + 3 = 9

Kwame and Abena's total ratio = 4 + 3 = 7

Kwame and Abena's share =

$\frac{7}{9} \times x = 420,000$

$7x = 9 \times 420,000$

$x = \frac{9 \times 420,000}{7} = 540,000$

The total profit shared is Gh¢540,000.00

2.Three partners Ali, Baba and Musa shared their profit in the ratio 7 : 13 : 5 respectively. At the end of a certain year Baba had Gh¢840,000.00 more than Ali. What was the total profit shared by the three partners?

Solution

Method I

Let the profit shared be$x$

Total ratio = 7 + 13 + 5 = 25

Ali's share =$\frac{7x}{25}$

Baba's share =$\frac{13x}{25}$

Ali's share =$\frac{5x}{25}$

Baba had Gh¢840,000.00 more than Ali

⇒Baba's share = y + 840,000

$\frac{13x}{25} = \frac{7x}{25}$ + 840,000
$\frac{13x}{25} - \frac{7x}{25}$ = 840,000
$\frac{6x}{25}$ = 840,000
6x = 25 × 840,000
$x = \frac{25 \times 840,000}{6}$ = 3,500,000.00
Total profit shared was Gh¢3,500,000.00

Method II
Let the total profit shared be x
Ali, Baba and Musa = 7 : 13 : 5
Total ratio = 7 + 13 + 5 = 25
Let Ali's share = y
⇒ Baba's share = y+ 840,000
7 : 25 = y : x
$\frac{7}{25} = \frac{y}{x}$
7x = 25y
$x = \frac{25y}{7}$ ………. (1)

13 : 25 = (y + 840,000) : x
$\frac{13}{25} = \frac{y + 840,000}{x}$
13x = 25(y + 840,000)

13x = 25y + 21,000,000
$x = \frac{25y + 21,000,000}{13}$ ……………. (2)

Equating eqn (1) and eqn (2)
$\frac{25y}{7} = \frac{25y + 21,000,000}{13}$
13 × 25y = 7(25y + 21,000,000)
325y = 175y + 147,000,000
325y – 175y = 147,000,000
150y = 147,000,000
$y = \frac{147,000,000}{150}$ = Gh¢980,000
Substitute y = 980,000 in eqn (1)
$x = \frac{25 \times 980,000}{7}$ = Gh¢3,500,000.00

3. Three candidates K, L and M were voted into office as school prefects. K secured 45% of the votes. L had 33% of the votes and M had the rest of the votes. If M secured 1,430 votes, calculate;
i. the total number of votes casted,
ii. how many more votes K received than L.

Solution
Total percentage = 100%
Percentage of M's votes
= (100 – 45 – 33)% = 22%
Let the total votes be x
M's votes = $\frac{22}{100} \times x$ = 1,430
⇒$\frac{22x}{100}$ = 1,430
22x = 1,430 × 100
$x = \frac{1,430 \times 100}{22}$ = 6,500

ii. K's vote = $\frac{45}{100}$ × 6,500 = 2,925votes
L's vote = $\frac{33}{100}$ × 6,500 = 2,145 votes

Votes K received than L;
= 2,925 – 2,145 = 980 votes

4. The development budget of a district council includes expenditure on feeder roads, schools and water supply. The expenditure on road, schools and water supply are in the ratio 7 : 15 : 2. If the expenditure on roads is Gh¢28 million, find the expenditure on;
i. Schools,
ii. Water supply,
iii. What is the total budget for these three projects?
iv. The cost of maintaining libraries is Gh¢900,000 and this is net from the expenditure on schools. What percentage, correct to three significant figures of the expenditure on schools is spent on maintaining libraries?

Solution

i. Road : schools : water supply = 7 : 15 : 2

Total ratio = 7 + 15 + 2 = 24

Let the expenditure on schools be = x

Ratio of road = 7

Expenditure on road = 28m

Proportion of road and schools

Ratio = Expenditure

7 : 15 = 28 : x

$\frac{7}{15} = \frac{28}{x}$

$7x = 28 \times 15$

$x = \frac{28 \times 15}{7}$ = 60 million

The expenditure on schools is Gh¢60 million

ii. Let the expenditure on water supply = y

Proportion of road and water supply

Ratio = Expenditure

7 : 2 = 28 : y

$\frac{7}{2} = \frac{28}{y}$

$7y = 2 \times 28$

$y = \frac{2 \times 28}{7}$ = 8 million

iii. Let the total expenditure be m

Proportion of Road and Total expenditure

7 : 24 = 28million : m

$\frac{7}{24} = \frac{28}{m}$

$7m = 24 \times 28$

$m = \frac{24 \times 28}{7}$ = 96 million

The total expenditure spent on the three projects is Gh¢96 million

iv.Expenditure on schools = 60 million

Cost of maintaining the library = Gh¢900, 000

Percentage of expenditure on schools spent on maintaining libraries

$= \frac{900{,}000}{60{,}000{,}000} \times 100\% = 1.5\%$

Exercises 12.8

1. Jacob and Esau were given some exercise books to share in a respective ratio of 4:3. If Jacob had 120 books, how many books did they share?

2. Sugri is 16 years old and Dabre is 14 years older than Sugri. When they shared a certain yards of clothing in the ratio of their ages, Dabre had 15 yards. How many yards of cloth did they share?

3. Sampson and Delilah shared some money in the ratio 3:5 respectively. Calculate the total amount shared if Sampson had Gh¢150,000.00.

4. A number of pens were shared among Shadrack, Mershack and Abednego in the ratio 3 : 1 : 4 respectively. If Merchack had 14 pens, find the total number of pens shared.

5. The ratio of the population of Secondi and Takoradi is 3 : 4 respectively. If the population of Takoradi is 17,540 people, what is the population of Secondi?

Challenge Problem

1. Baffour Ba shared some money among his three family members namely; Ananse, Aso and Ntekuma. Ananse received $\frac{1}{5}$ of the money and the remaining was shared between Aso and Ntekuma, where Ntekuma receive Gh¢5,200.00 constituting $\frac{1}{6}$ of the remaining amount.

i. Find the total amount of money shared by the three family members.

ii. Find the total share of each person.

iii. How much did Aso received than Ananse?

Rate

A rate is the ratio of two different quantities. This implies that rate is the comparison of two different quantities.

A rate has a unit expressed as the number of units of the first quantity to that of the second quantity. For example, if a household consumes 20units of electricity in 5days, then the rate as explains above requires a comparison of the two quantities , the first being " 20 units of electricity " and the second " 5days" ; so as to arrive at " a per unit" of the second quantity.

$\Rightarrow$ Rate = $\frac{1st quantity}{2nd quantity}$ = $\frac{20units}{5days}$

Rate = 4 units per day written as:"4 units/day"

Worked Examples

1. A car travels 120km in 2 hours. Find:

i. the rate at which the car travels in km/h.

ii. the rate at which the car travels in m/s

Solution

i. Rate = $\frac{1st\ quantity}{2nd\ quantity}$ = $\frac{120km}{2h}$ = 60km/h

ii. 120km = 120 × 1,000m = 120,000m

2hr = 2 × 3600s = 72,000s

Rate = $\frac{120,000m}{7,200s}$ = 16.6m/s

2. A man earns Gh¢400.00 for working 8 hours. How much does he earn per hour?

Solution

Rate = $\frac{1st\ quantity}{2nd\ quantity}$ = $\frac{Gh¢\ 400}{8h}$ = Gh¢50 per hour

Therefore, he earns Gh¢50.00 per hour.

3. A household consumes 2,800 liters of water per week. Find the daily rate of consumption.

Solution

Rate = $\frac{1st\ quantity}{2nd\ quantity}$ = $\frac{2,800\ litres}{7days}$ = 400 *liters per day*

4. Jojo bought 3kg of meat for Gh¢810.00. Find the cost per kilogram.

Solution

Cost per kilogram = $\frac{cost}{kilogram}$

But cost = Gh¢810.00 and kilogram = 3kg Cost per kilogram = $\frac{¢\ 810}{3kg}$= Gh¢270 per kg

5. A certain man pays Gh¢6,000.00 for staying in a hotel for 8 nights. What was the rate per night?

Solution

Rate = $\frac{Gh¢6,000}{8}$ = Gh¢750

The rate per night is Gh¢750.00

6. Musa works at a restaurant for 4 hours. He is paid Gh¢250.00 per hour. How much does he earn?

Solution

Gh¢250.00 is for 1 hour.

Therefore, 4 hours = 4 × 250 = Gh¢1,000.00

He earns Gh¢1,000.00

7. If 154 pages of a story book was read by Rachel in 2 weeks. How many pages did she read per day?

Solution

Rate = $\frac{1st\ quantity}{2nd\ quantity}$

But 1st quantity = 154 pages

2nd quantity = 2 weeks = 14 days

Rate = $\frac{154}{14}$ = 11*pages/day*.

Exercises 12.9

1. A factory worker earned Gh¢1,200.00 for 4 hours work. How much does he earn per hour?

2. A girl paid Gh¢1,300.00 for fetching 13 tanks of water. Find the rate per bucket.

3. A shopkeeper is marked Gh¢3.20 per hour. How much does he receive for working for:
i. 10 hours ii. 4 hours a day for 5 days?

4. A man receives an annual salary of Gh¢ 3,204.00. How much does he receive at the end of every Month?

5. Mr. Green filled his tank with 42 gallons of petrol at a cost of Gh¢2,940.00. Find the rate per gallon.

6. At Mammon Shoe Factory, each worker is paid Gh¢24.80 for 12 hours. Find the pay per hour of each worker.

7. A school boy wrote 374 words to cover a page of a sheet of paper. If the sheet contains 17 lines, how many words filled a line?

8. At a certain hospital, it was reordered that 138 H.I.V. patients die every 6 months. What is the monthly rate of death?

Challenge Problems

1. A typist charges Gh¢20.00 for the first 8 pages and 50*p* for each additional page typed. Find the cost of typing 93 pages.

2. At his communication centre, Mr. Green charges 150*p* for the first 15 minutes call and 130*p* for each additional minute. How much will you pay for a 50 minutes call?

Average Speed (S)

The average speed(s) is the rate of the total distance (d) to the total time (t). That is:

Average speed(s) = $\frac{\textbf{Total distance } (\boldsymbol{d})}{\textbf{Total time } (\boldsymbol{t})}$

Simply, $\boldsymbol{s = \frac{d}{t}}$

The average speed (s) can be in *m/s,km/h,km/minetc*

Worked Examples

1. A car travels 72 kilometers in an hour. Find its speed in meters per seconds

Solution

S = ? t = 1hr, d = 72km/h

Since the speed is required in meters per seconds, express the distance in meters and the time in seconds.

1 hour = 60 min, but 1 min = 60 seconds

$\Rightarrow$60 minutes =$\frac{60\, sec \times 60\, min}{1\, min}$ = 3,600s

t = 3,600s

If 1km = 1,000m

$\Rightarrow$, 72km = $\frac{72km \times 100km}{1km}$ = 72,000m

S = $\frac{d}{t} = \frac{72,000m}{3600s}$ = 20m/s

3. A van travels 154km in $1\frac{3}{4}$hour. Find its speed in km/h.

Solution

S = ?, d = 154km, $t = 1\frac{3}{4}h = \frac{7}{4}h$

S = $\frac{d}{t} = \frac{154}{\frac{7}{4}} = \frac{154km \times 4}{7hrs}$ = 88km/h

4. A cyclist covers 900m in 5minutes. What is his average speed in m/s?

Solution

S = ?, d = 900m and t = 5mins.

But 1 min = 60sec

$\therefore$ 5mins $= \frac{5 \times 60}{1} =$ 300s.

$S = \frac{d}{t} = \frac{900m}{300s}$ = 3m/s

5. An athlete runs 100 meters in 10 seconds. Find his speed in kilometers per hour.

Solution

$S = \frac{d}{t}$

But d = 100m = $\frac{100}{1,000}$ km = 0.1km

t = 10s = $\frac{10}{3600}$ h,

$S = \frac{0.1}{10/3600} = \frac{0.1 \times 3,600}{10}$ = 36km/h

6. A car travels 60km in 48minutes. Find the average speed of the car in km/h.

Solution

d = 60km,

t = 48min = $\frac{48}{60}h$= 0.8h

$S = \frac{d}{t} = \frac{60km}{0.8h}$ = 75km/h

Exercises 12.10

1. A cyclist covers 700km in 2hours. What is his average speed in: i. km/h ii. m/s

2. A 5,000 meter runner covered such a distance in 25 minutes. Calculate his rate of speed in:
i. m/s ii. km/min.

3. A runner covered $^1/_4$of a 4,000km marathon race in 2 hours and covered the rest in 5 hours. What is the average speed of the runner in km/h?

4. A van travels 344km in $2\frac{3}{4}$hours. Find its speed in:
i. km/h. ii. m/s

5. A car travels 72km in $1\frac{1}{2}$ hours and the travels at a constant speed of 28 km/h for a further $1\frac{1}{2}$ hours. Calculate:
i. the average speed for the first $1\frac{1}{2}$ hours;
ii. the average speed for the first $2\frac{1}{2}$ hours;
iii. the average speed for the whole journey.

Finding the Time Given the Speed and Distance

Given the average speed, *s*, and the total time, *t*, the distance, *d*, is calculated by the formula:

Total time (t) = $\frac{Total\ distance(d)}{Speed(s)}$

Simply, $t = \frac{d}{s}$

The total time can be measured in *seconds, minutes, hour etc.*

Note:

The phrase***"how long"***means, find the time.

Worked Examples

1. A train travels at a speed of 80km per hour. How long will it take to travel a distance of 320km?

Solution

S = 80km/h, d = 320km and t =?

$t = \frac{d}{s} = \frac{320km/h}{80km}$ = 4 hours.

2. What is the time in minutes taken to cover 81km at an average speed of 27km/h?

Solution

S = 27km /h, d = 81km and t =?

But, $t = \frac{d}{s} = \frac{81km}{27km/h} = 3h$

But 1 hour = 60mins

$\therefore$ 3 hour = $\frac{3h \times 60\ min}{1h}$ = 180mins

Exercises 12 .11

1. Jude traveled a distance of 12km on a bicycle at an average speed of 8km/h. How long did it take him to make the trip?

2. A boy cycles 75km to school at a speed of 25km/h. How long did it take him to reach the school?

3. Mr. Jimmy traveled a distance of 50km on a bicycle at an average speed of 10km/h. Find the time he used to make the trip in minutes?

4. A train traveling at 105km/h goes through a tunnel 630km long. Calculate the time in seconds, a passenger in the train spends inside the tunnel.

Unknown Distance Given the Speed and Time

Given the average speed, *s*, and the total time, *t*, the distance, *d*, is calculated by the formula:

Total distance = Speed × Time taken

Simply: $d = s \times t$

The total distance, *d*, can be measured in *cm, m, km etc.*

Note: The phrase ***"how far"*** means find the distance

Worked Examples

1. A car is traveling at 40 km/h. How far does it travel in $2\frac{1}{2}$ hours?

Solution

S = 40km/h, $t = 2\frac{1}{2}$hrs and d = ?

But $d = s \times t$

$= 40 \times 2\frac{1}{2}$

$= 40 \times \frac{5}{2} = 100$km

2. How far will an aero plane with speed 415km/h go in 3hrs?

Solution

S = 415km/h, t = 3hrs and d = ?

$d = s \times t = 415\text{km}/h \times 3h = 1,245$ km

3. The speed of an athlete is 12m/s. What distance in kilometers can he cover in a time of 9.5 seconds?

Solution

S = 12m/s, t = 9.5s and d = ?

$d = s \times t = 12\text{m/s} \times 9.5\text{ s} = 114\text{m}$

$d = \frac{114\ km}{1000} = 0.114$km

4. Kwame rode a bicycle for a distance of x km and walked for another $^1/_2$ an hour at a rate of 6km/hour. If Kwame covered a total distance of 10km, find the distance x he covered by bicycle.

Solution

Total distance = 10 km

Bicycle = $x$km

Walking = $\frac{1}{2}h \times 6$ km/h = 3 km.

Total distance = bicycle + walking

10km = $x$km + 3 km

x = (10 – 3) km = 7 km

The distance covered by the bicycle is 7 km

Exercises 12.12

1. Calculate the distance covered by a car in 9 hours at a speedof 80km/h.

2. A Boeing 747 aircraft has a speed of 225km/h. What distance can it cover in 12hours?

3. What distance will an athlete with speed 10 m/s cover in 8 minutes?

4. A Yutong bus travels from Kumasi to Tumu at a speed of 80km/h. If the bus takes off from Kumasi at 7 : 00 a.m and reaches its destination at 11 : 30 a.m, how far is Tumu from Kumasi?

5. A train traveled from Tarkwa to Obuasi at a speed of 120km/h in 4 hours. It then travelled from Obuasi to Ejisu at a speed of 90km/h in 5hours. Calculate the total distance covered by the train from Tarkwa to Ejisu.

Solved Past Question

1. An aero plane leaves Barcelona at 10 : 10 p.m and reaches Accra 4,415 km away at 5 : 50 a. m the next morning. Calculate, correct to the nearest whole number, the average speed of the aero plane in km/h.

Solution

$S = \frac{d}{t}$,

But d = 4,415km

t = 10 : 10 p.m – 5 : 50 a. m

t = 7h 40 min

40 min to hr = $\frac{40}{60}$ h = 0.67h

$\therefore t = 7h + 40$ min = (7 + 0.67) h = 7.67h

$\Rightarrow S = \frac{4{,}415\ km}{7.67\ h}$ = 576km/h

2. A bus travelled a distance of 56km at an average speed of 70 km per hour. It travels a further 60km at an average speed of 50 km per hour. Calculate for the whole journey:

i. the total time taken,   ii. the average speed.

Solution

i. $t = \frac{d}{s}$

d_1 = 56km, s_1 = 70km/h

$t_1 = \frac{56\ km}{70\ km/h}$ = 0.8h

d_2 = 60km, S_2= 50 km/h

$t_2 = \frac{60\ km}{50\ km/h}$ = 1.2h

Total time, $t = t_1 + t_1$

t = 0.8h + 1.2h = 2h

ii. Average speed = $\frac{Total\ distance}{Total\ time}$

Total distance = (56 + 60) km = 116km

Total time = 2hr

Average speed = $\frac{116\ km}{2h}$ = 58km/h

Scale Drawing *(Scale of a Map)*

The scale of a map is the ratio of the distance on a map to theactual corresponding distance on the ground. Simply put,

Scale = $\boldsymbol{\frac{map\ length}{ground\ length}}$

For similar areas and volumes, if the ratio of lengths is $a : b$, the ratio of the areas is $a^2 : b^2$, and the ratio of the volumes is $a^3 : b^3$

On a map, the ratio *map length : ground length* is known as ***representative fraction***

The scale is usually expressed as a ratio in the form 1 : n . For instance, if the scale of a map is given as 1 : 10,000, it means 1 unit on the map represents 10,000 of the same units on the ground.

Finding the Actual Size/Distance Given the Scale andSize/ Distance on the Map

To find the actual size or distance (x) of an object given its size (y) on the map and the scale:

I. Represent the actual size/distance by any preferred variable, say x

II. Identify the object's size on the map, say y

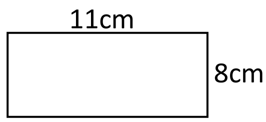
III. Identify the given scale, say *a : b*

IV. Equate the given scale to the ratio of thesize on the map to the actual size. That is :

$$a : b = y : x \text{ OR } \frac{a}{b} = \frac{y}{x}$$

V. Solve for the value of *x*, to obtain the actual size of the object

Consider the scale drawing of the football field below;



Given a scale of 1 : 1,000, it means that 1cm on the map represents 1,000cm on the ground. Therefore the actual size of the field length is:

1 : 1,000 = 11cm :*x*cm

$\frac{1}{1000} = \frac{11}{x}$,

$x = 11 \times 1000 = 11{,}000$cm

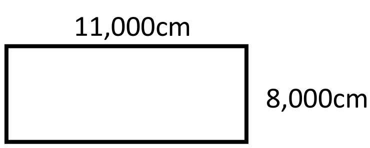
The actual size of the field width is:

1: 1,000 = 8cm : *y* cm

$\frac{1}{1000} = \frac{8}{y}$,

$y = 8 \times 1{,}000 = 8{,}000$cm

Therefore, the actual size of the field on the ground is shown below:



This implies that the actual area of the field

= 11, 000cm × 8, 000cm

= 88, 000, 000 cm^2

Worked Examples

The model of an aircraft is made on a scale of 1 : 20

i. If the aircraft is 25m long, how long is the model?

ii. If the area of the wing surface is on the model is $0.22m^2$, what is the area of the aircraft

iii. The air crafts carries 30 $m^2$of fuel. What is the capacity of the model's fuel tank?

Solution

i. Scale of lengths = 1 : 20 = *x* : 25

$\frac{1}{20} = \frac{x}{25}$

$20x = 25$

$x = 1.25$m

ii. Scale of areas = 1 : 20^2 = 1: 400

1 : 400 = 0.22 : *y*

$\frac{1}{400} = \frac{0.22}{y}$

$y = 400 \times 0.22 = 88m^2$

iii. Scale of volumes = 1 : 20^3

1 : 8,000 = *n* : 30

$\frac{1}{8000,} = \frac{n}{30}$

$n = \frac{30}{8{,}000} = 3.75 \times 10^{-3} m$

2. If the scale of a map is given as 1 : 900, what is the distance represented by 4cm on the map?

Solution

Method 1

Let the actual distance be *x*

If 1 : 900

$\Rightarrow$ 4 cm = $\frac{4cm \times 900}{1}$= 3,600cm

Method 2

Let the actual distance be *x*

1 : 900 = 4cm : *x*cm

$\frac{1}{900} = \frac{4cm}{xcm}$

$x = 900 \times 4cm = 3{,}600$cm

3. On a map, 1cm represents 120km. The distance between two towns on the map

is 5cm. Find the actual distance between the two towns.

Solution

Method 1

If 1cm represents 120km,

Then 5cm = $\frac{5cm \times 120km}{1cm}$ = 600km

$\therefore$ The distance between the two towns is 600km.

Method 2

Scale 1 : 120 = 5 : x

$\frac{1}{120} = \frac{5}{x}$

$x = 120 \times 5 = 600$ km

4. The scale of a map is 1cm to 100km. The distance between city P and city Q on the map is 23cm. What is the actual distance between city P and city Q.

Solution

Method 1

If 1cm represents 100km,

then 23cm = $\frac{23cm \times 100km}{1cm}$ = 2,300 km

$\therefore$ The actual distance between city P and Q is 2,300km.

5. The map of a large town is drawn to the scale 1 : 100,000. What is the distance in kilometer represented by a line segment 4cm long on the map?

Solution

If 1cm represents 100,000cm

Then 4cm = $\frac{4cm \times 100,000}{1cm}$ = 400,000cm

But 100,000cm = 1km

$\therefore$ 400,000cm = $\frac{400,000}{100,000} \times$ 1km = 4 km

6. On a map, two towns P and Q are 15.5cm apart. the scale of the map is 1 cm : 4 km. Calculate the actual distance between P and Q.

Solution

Method 1

1cm : 4km

15.5cm = $\frac{15.5cm \times 4km}{1cm}$ = 62km

$\therefore$ The distance between P and Q is 62km.

7. On a map, 1cm represents 4.5km. What is the actual distance between two towns which are 4cm apart on the map?

Solution

If 1cm represents 4.5km

Then 4cm = $\frac{4.5km \times 4cm}{1cm}$ = 18 km

$\therefore$ The distance between the two towns is 18km

Exercises 12.13

1. The scale of a map is 1 cm to 1,000 km. What is the distance between two towns which are:

i. 8cm apart on the map

ii. $3\frac{1}{2}$ cm apart on the map

iii. 103 mm apart on the map

iv. 2.8 cm apart on the map

2. a. The scale of a map is 1 cm to 5km. Write the scale as a ratio in the form 1 : n

b. What is the distance on the map between two towns which are:

i. 15km apart ii. 28 km apart,

iii. 47.5 km apart.

3. The scale of a map is 1 : 100,000. What is the distance in kilometers between two towns that are 12cm apart on the map?

4. The area of a football field is represented on a map as 8cm × 5cm. Find the actual size and the area of the field if the map is drawn to the scale 1 : 900.

5. The map of a region is drawn on a scale of 5cm to 1 km.

i. Express this scale as representative fraction of the map.

ii. Find the area of the region in km^2 which corresponds to 125 cm^2

6. The scale of a map is 1 : 100,000.

i. What distance in km, is represented by 2.2 cm on the map?

ii. If the length and breadth of a rectangular region on the map are measured correct to the nearest cm as 2.2 cm and 0.7 cm, what are the limits between which the exact length and breadth must lie?

7. A reservoir is to be constructed to hold 3.2×10^7 m^3 of water when full, an accurate model of it is built to a scale of 1 : 200.

i. When the model is full of water, the greatest depth is 18cm. What will be the greatest depth in the reservoir when full?

ii. If the surface area of the water in the model is 40 m^2, calculate the corresponding surfacearea of the water in the reservoir.

iii. Write down the volume of water in the reservoir when full.

Finding the Scale Given the Size of an Object on the Map and the Actual Size of the Object on the Ground

To find the scale to which an object is represented on a map;

I. Identify the size of the object on the map, (*a*)

II. Identify the actual size of the object on the ground (*b*)

III. Form a ratio of the size on the map to the size on the ground. That is :

Scale = $\frac{map\ length\ (a)}{ground\ length(b)} = a : b$

IV. Simplify the ratio in the form 1 : *n* to obtain the scale in the same unit

Worked Examples

1. The length of a pole 24,000cm is represented on a map as 8cm. What is the representative scale of the map?

Solution

$Scale = \frac{map\ length}{ground\ length}$

$= \frac{8cm}{24{,}000\ cm} = \frac{1}{3{,}000} = 1 : 3{,}000$

The scale of the map is 1 : 3,000

2. The distance between two towns on a map is 2cm. If the actual distance between the towns is 8km, find the scale of the map.

Solution

$Scale = \frac{map\ length}{ground\ length}$

$= \frac{2cm}{8km} = \frac{1\ cm}{4km} = 1cm : 4km$

But 4 km = 4 × 1,000 m = 4,000m

$= \frac{4{,}000}{1/100}$ cm $= \frac{4{,}000 \times 100\ cm}{1} = 400{,}000$ cm

The scale of the map is 1 : 400,000

Exercises 12.14

1. On a scale drawing, the length of a ship is 0.42cm. If the actual length of the ship is 84m, what is the scale?

2. The length of a field, 1.2km long is represented on map by line 40mm long. What is the scale of the mapping?

3. The actual length of a pole is 15,000 cm. If it is represented on the map as 3 mm, to which scale is the map drawn?

4. On a photograph, a man whose height is 170 cm appears as 2cm. Find the scale of the photograph.

5. Two towns are 100km apart. On a scale drawing, these towns are 2.5cm away from each other. Determine the scale of the drawing.

Foreign Exchange Conversion
In order to transact business with other people in other countries, there is the need to buy (by conversion), the currency of that country. This is done at the forex Bureau, Banks or "Black market" that set to exchange rates namely; ***'sell at'*** and ***'buy at'*** rates for every currency.

To change or convert one currency to another,
I. Identify the exchange rate
II. Identify the given currency
III. Represent the required currency by any preferred variable.
IV. Form a ratio of the exchange rate and equate to the ratio of the given currency to the required currency(form a proportion).
V. Work out for the value of the variable to obtain the required currency.

Worked Examples
1. If $1 = Gh¢2.50, calculate the dollar equivalent
i. Gh¢600.00 ii.Gh¢30.00
iii. Gh¢9,000.00 iv.Gh¢120.00

Solution
Using simple proportion,
i. If Gh¢1.50 = $ 1

$\Rightarrow$ Gh¢600 = $\frac{\text{Gh¢ } 600 \times \$1}{\text{Gh¢ } 1.50}$ = $400.00
ii. If Gh¢1.50 = $1
$\Rightarrow$ Gh¢30 = $\frac{\text{Gh¢ } 30 \times \$ 1}{\text{Gh¢ } 1.50}$ = $20.00
iii. If Gh¢1.50 = $1
$\Rightarrow$ Gh¢9,000 = $\frac{\text{Gh¢ } 9{,}000 \times \$1}{\text{Gh¢ } 1.50}$ = $ 6,000.00
iv. If Gh¢1.50 = $1
$\Rightarrow$ Gh¢120 = $\frac{\text{¢ } 120 \times \$ 1}{\text{¢ } 1.50}$ = $80.00

2. If $1.00 = Gh¢1.34, what is the cedi value of an article which cost $ 1.65

Solution
Let $x$be the cedis value.
By method of direct proportion,
Dollar = Cedi
1 : 1.65 = 1.34 : x
$\frac{1}{1.65} = \frac{1.34}{x}$
x = 1.65 × 1.34 = Gh¢2.21

3. If £ 1.00 = Gh¢ 2.20, how much cedis will you get for converting £ 40.00?

Solution
Let y be the cedi value.
By method of direct proportion,
£ = Gh ¢
1:40 = 2.20: y
$\frac{1}{40} = \frac{2.20}{y}$
y = 40 × 2.20 = Gh¢ 88.00

4. You require €250 (euro) to purchase a lap top computer. If the current conversion rate is €1.00 = Gh¢1.74. How much money in cedis will you need to buy the laptop?

Solution

Let x be the money require in cedis

By method of direct proportion,

€ = Gh¢

1 : 250 = 1.74 : x

$\frac{1}{250} = \frac{1.74}{x}$

$x = 1.74 \times 250$ = Gh¢435.00

Exercises 12.15

A. Change the following amount in cedis to pounds given that £1.00 = Gh¢6.70

1.Gh¢51,000.00	3. Gh¢17,000.00
2. Gh¢102,000.00	4.Gh¢680,000.00

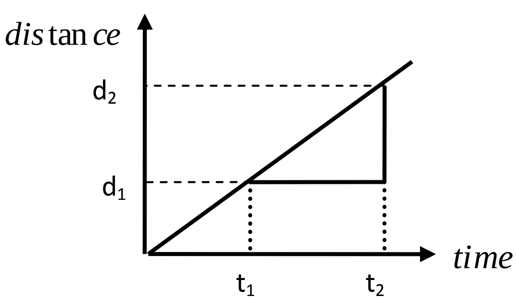
B. Change the following amount in cedis to dollars, if $1.00 = Gh¢4.75

1. Gh¢35,700.00	3.Gh¢15,980.00
2. Gh¢119,000.00	4.Gh¢79,900.00

Travel Graph

It is a graph that shows the relationship between distance and time, for a journey.

For most travel graphs, distance is usually represented on the vertical axis (y - axis) and time, on the horizontal axis (x – axis).



From the distance – time graph above,

Average speed (S) = $\frac{Change\ in\ distance}{Change\ in\ time} = \frac{d_1 - d_2}{t_1 - t_2}$

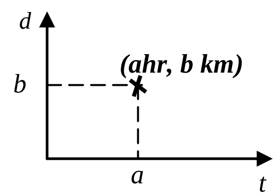
Drawing a Travel Graph

Observe the following when drawing a travel graph:

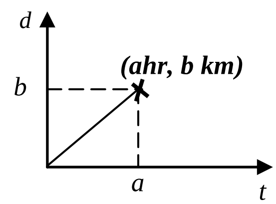
I. Name the x – axis as time and the y – axis as distance

II. Number the axes according to the given scale. If the scale is not given, use a convenient scale.

III. Identify the time movement and its corresponding distance movement and plot on the graph. i.e. (*time, distance*)

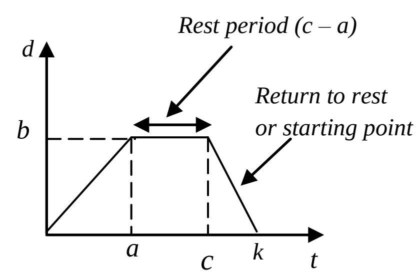


IV.Join the points of movements by a slanted line such as ⁄(usually beginning from the origin)



V. Join rest periods or periods of no movement by a horizontal line such as ─.

VI. A return to the starting point(finally at rest) is represented by a slanted line such as \ to touch the time axis.



Worked Examples

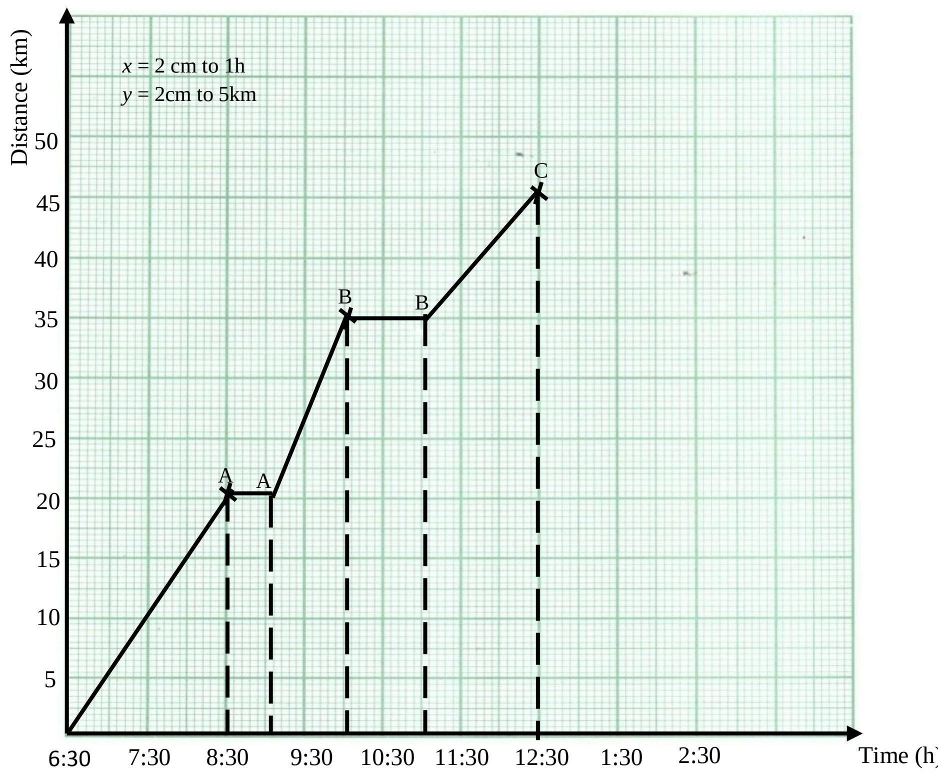
1. Mr. Brown travelled with his car at 6 : 30 a.m and for the first 2 hours covered a distance of 20 km to reach town A. He stopped for 30 minutes at town A and continued the journey travelling a distance of 15 km in an hour to town B, where he rested for an hour. He then travelled from town B to town C, 10km apart in a time of $1^1/_2$hours.

a. Using a scale of 2 cm to 1 hour on the time axis and 2 cm to 5 km on the distance axis, draw the graph of Mr. Brown's journey

b. From the graph, find:
i. the time he reached town *A*
ii. the time he left town *A*
iii. the time he reached town *B*
iv. the time he left town *B*
v. the time he reached town *C*

c. Find the speed of Mr. Brown in moving:
i. from his house to town *A*
ii. from town *A* to town *B*
iii. from town *B* to town *C*
iv. His average speed in travelling from his house to town

Solution
a. A = (8: 30, 20km), Rest = (8 : 30 to 9: 00),
B(10:00, 35km), Rest = (10:00 to 11: 30), C(1:30, 45km)



b. From the graph:
i. he reached town *A* at 8 : 30 a.m
ii. he left town *A* at 9 : 00 a.m
iii. he reached town *B* at 10 : 00 a.m
iv. he left town *B* at 11 : 30 a.m
v. he reached town *C* at 1 : 30 pm
c. i. Speed from his house to town *A*;

s = $\frac{d}{t} = \frac{20\ km}{2\ h}$ = 10 km/h

ii. Speed from town *A* to town *B;*
s = $\frac{d}{t} = \frac{15\ km}{1\ h}$ = 15 km/h

iii. Speed from town B to town C;

$s = \frac{d}{t} = \frac{10\ km}{2\ h}$ = 5 km/h

v. His average speed in traveling from his house to town C.

$= \frac{S_1 + S_2 + S_3}{3} = \frac{(10 + 15 + 5)}{3}$km/h = 10km/h

2. A boy left home at 8 : 00 a. m. and travelled on foot to visit his grandma at town Q, 14km away from his house. In a time of 30 minutes, he covered a distance of 6 km to town P. He rested and after 20 minutes continued his journey to his grandma's place in 40 minutes. If he spent 40 minutes at his grandma's place before walking 50 minutes back to his house,

a. draw the travel graph of the boy using a scale of 2cm to 20 minutes on the time axis and 2cm to 2 km on the distance axis

b. i. Find the speed of the boy before he reached town P

ii. At what time did he leave town P?

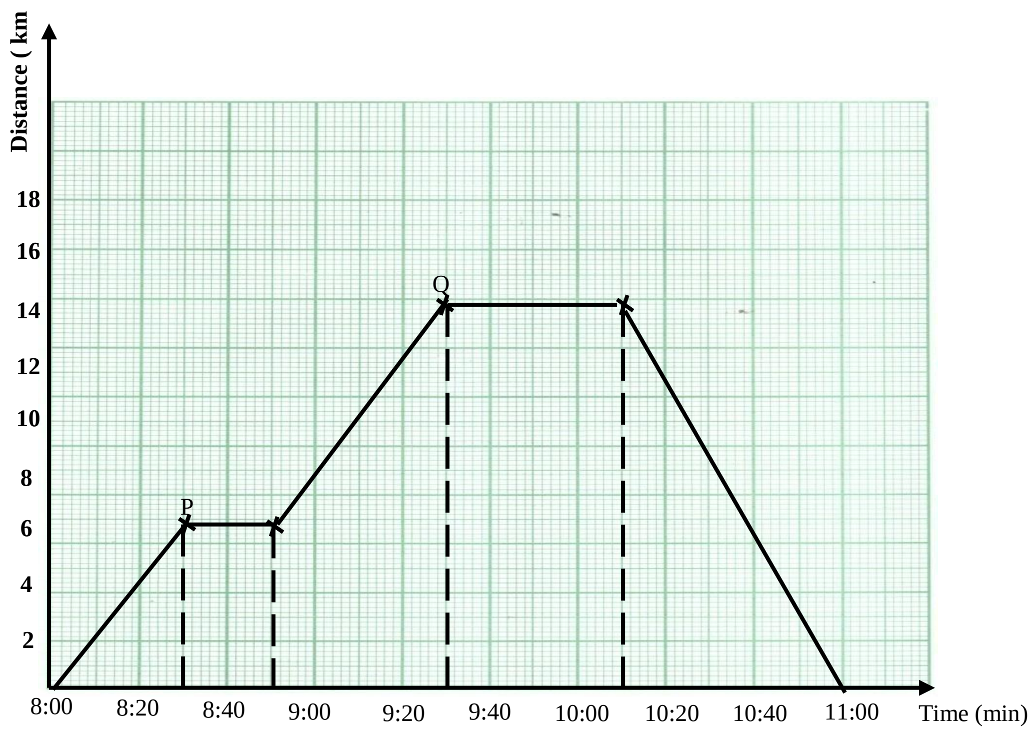
iii. Find the speed of the boy in moving from town P to town Q

iv. At what time did he leave his grandma's place?

v. Find the exact time the boy reached home

vi. Find the average speed of the boy in walking from his house to town.

Solution



b. i. The speed of the boy before he reached town P

$s = \frac{d}{t} = \frac{6\ km}{30\ min}$ = 0.2 km/min

ii. He left town P at 8 : 50 a.m

iii. The speed of the boy in moving from town P to town Q

$s = \frac{d}{t} = \frac{8\ km}{40\ min}$ = 0.2 km/min

iv. He left his grandma's place at 10: 10 a.m

v. The boy reached home at 11 : 00 a.m

vi. Find the average speed of the boy in walking from his house to town Q

S_1 = 0.2 km/min and S_2 = 0.2 km/min

Average speed = $\frac{(0.2+\ 0.2\ )}{2}$ km/min = 0.2km/min

Some Solved Past Questions

1. a. Kwame left his home in Takoradi at 7 : 00 a.m to walk to Apramdu, 14 km away. After walking 2.5 km during the first 30 minutes, he was given a ride in a car to Apramdu. The car travelled at an average speed of 40km/h. Draw Kwame's travel graph using a scale of 2cm to 10 minutes on the time axis and 2cm to 2km on the distance axis

b. Kojo also left Kwame's house at 7 a.m and cycled to Apramdu, arriving there at the same time as Kwame. Using the same axes and scale as in (a), draw Kojo's travel graph

c. Use your graph to determine the speed in km/h, at which Kojo cycled, correct to 2 s.f

Solution

Total distance = 14 km

Plot 30 min from 7 : 00 a.m. against 2.5 km from 0km = (30 min, 2.5km)

Remaining distance = (14 – 2.5)km = 11.5km

Let x represent the time in minutes used by the car to cover 11.5km

But the car travelled at an average speed of 40km /h

⇒the car traveled40km every 1 hour, 11.5km will be covered in x hours. That is:

40 km = 1h

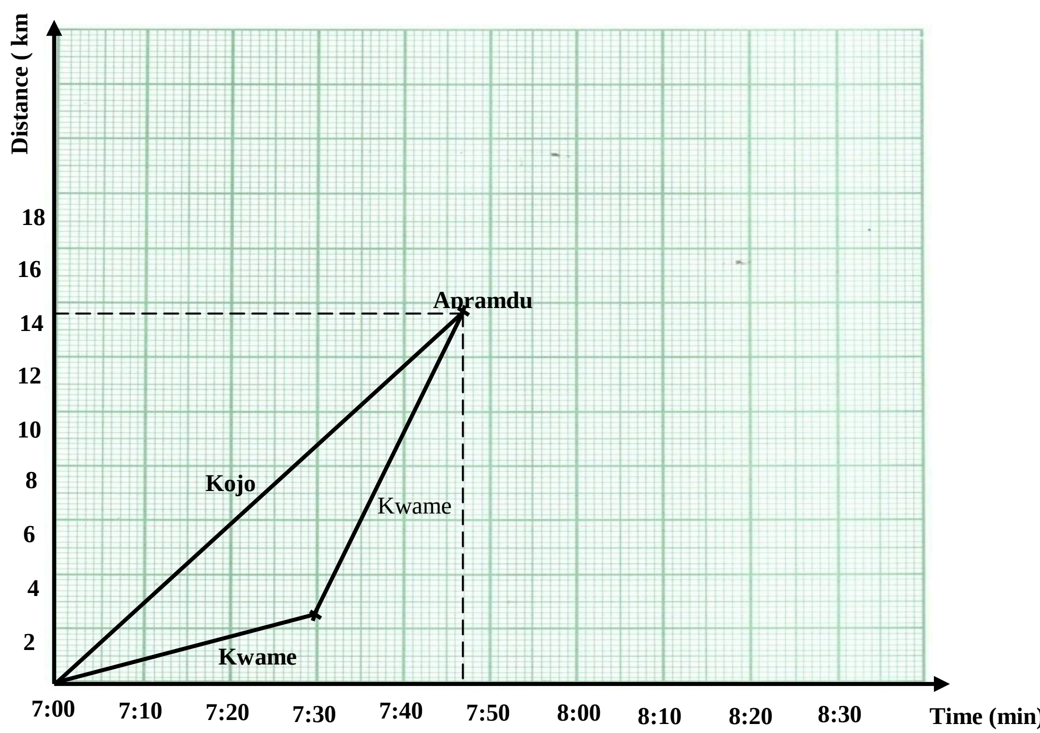
11.5km = x h

$\frac{40}{11.5} = \frac{1}{x}$

$x = \frac{11.5}{40}$ = 0.2875 h

0.2875 h = 0.2875 × 60 min = 17.25min

Plot 17.25 minutes from 7 : 30a.m against 11.5km from 2.5km = (7 : 47 a.m, 14km)



c. *Kojo's speed* = $\frac{Distance covered}{Time taken}$

Distance covered = (14 – 0)km = 14 km

Time taken = 7 : 47 – 7: 00 = 47 min

Time taken = $\frac{47}{60}$ h = 0.78h

Kojo's speed = $\frac{14km}{0.78h}$ = 18km/h

3. Two friends Kojo and yaw travelled the same route from town A to town B, a distance of 53km. Kojo's started at 6:30a.m. and for the first one and half hours, he moved at a constant speed and covered 28 km. He stopped for 30 minutes and continued his journey at 12.5km/h to reach town B. Yaw started from town A at 9:00 a.m and overtook kojo 30 minutes later. Yaw continued with the
same speed till he got to town B.

a. using a scale of 2cm to 30 minutes and 2cm to 5 km, draw the distance - time graph for these friends.

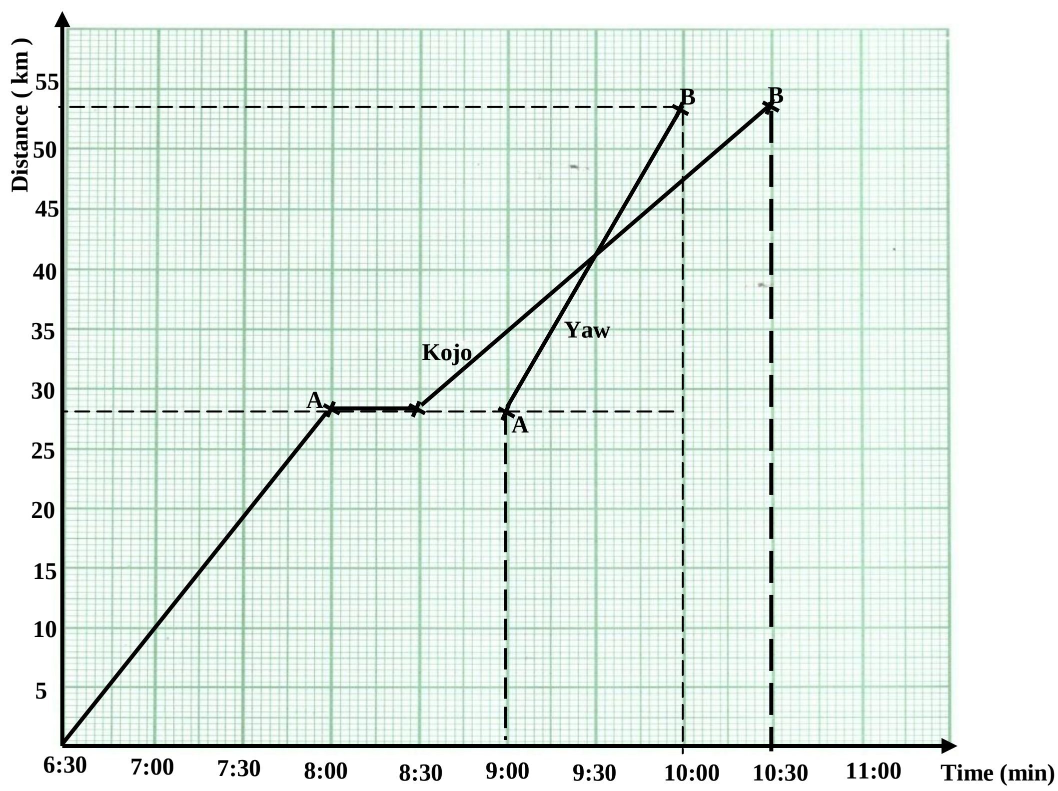
b. Use your graphs to find:

i. When Kojo reached town B

ii. When Yaw reached town B

iii. Yaw's speed

Solution



b. i. Kojo reached town B at 10 : 30

ii. Yaw reached town B 10 : 00

iii. Yaw's speed = $\frac{distance\ covered}{time\ taken}$

= $\frac{25\ km}{1h}$ = 25 km/h

4. Mr. Brakatu and Mr. Jojo walk from house A to B, 11.2km away. Mr. Jojo starts at 10.00 a.m. and

walks at a constant speed of 4.8km/h, but makes a call lasting 30 minutes at a point on the journey 6.4km away from house A. Mr. Brakatu starts at 11.00 a.m. and walks at a constant speed of 6.4km/h.

a. Taking 2cm to represent 20 minutes on the time axis and 2cm to represent 1km on the distance axis, draw the travel graphs of Mr. Brakatu and Mr. Jojo using the same axis

b. Use your graphs to find;

i. where Mr. Brakatu overtakes Jojo,

ii. the times Mr. Brakatu and Mr. Jojo arrived at their destination.

Solution

Total distance = 11.2km

Mr. Jojo's starting time = 10: 00 a.m

He travelled a constant speed 4.8km/h

⇒ 4.8km is covered every 1 hour

6.4km will be covered in the time

$= \frac{6.4}{4.8}h = \frac{6.4}{4.8} \times 60$ min = 80 min

Plot 80 min from 10 : 00 a. m. and 6.4km from 0km = (11:20 a.m, 6.4km)

Remaining distance = (11.2 – 6.4) km = 4.8km

Call time = 30 minutes from 11: 20 a. m. and ends at 11:50 a. m.

Remaining distance, 4.8 km was covered in the next 1 hour. Plot 1 hour from 11:50 a.m. and 4.8km from 6.4km = (12:50p.m, 11.2km)

Mr. Brakatu's starting time = 11: 00 p.m.

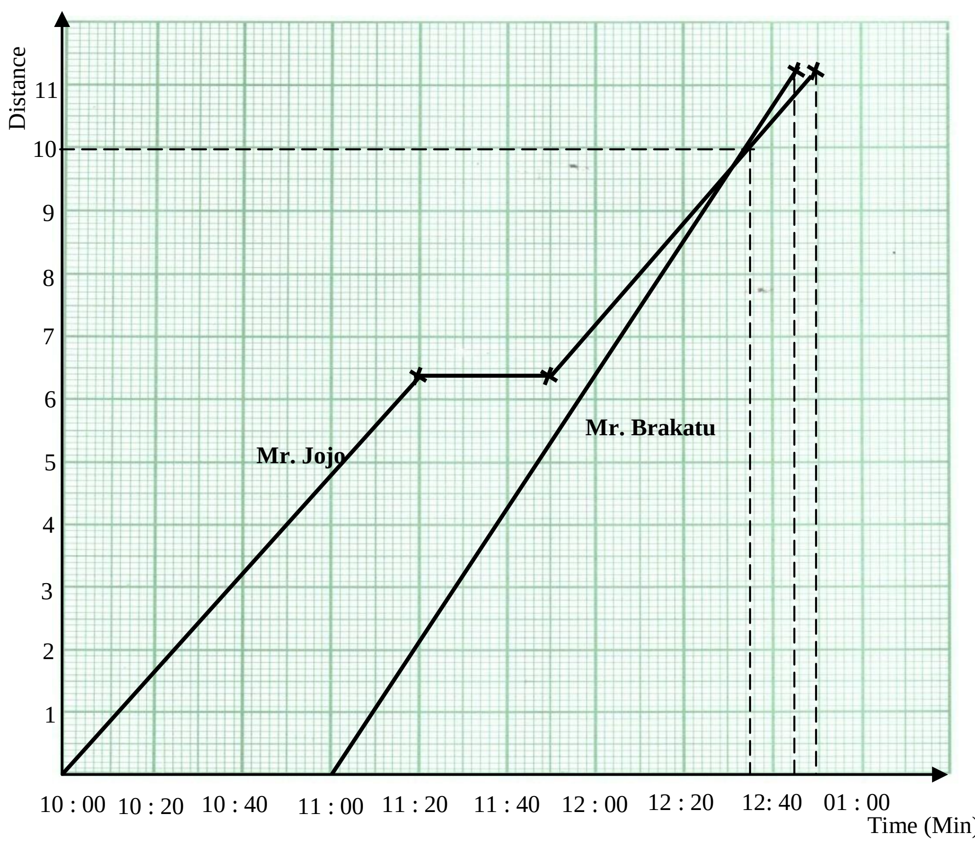
Walks at a constant speed of 6.4km/h

⇒ 6.4km covered in 1 hour

11.2km will be covered in $\frac{11.2}{6.4}h = \frac{11.2}{6.4}h \times 60$ min

= 105 minutes = 1hr 45min

Plot 1 hr 45min, from 11:00 p.m and 11.2km from 0km = (12:45 p.m, 11.2km)



b. i. At the time 12:35 p. m. and a distance of 10km

ii. Mr. Brakatu arrived at 12 : 45 p.m. and Mr. Jojo arrived at 12 : 50 p.m.

Exercises 12.16

1. A cyclist set out at 9 : 00 a. m. for a destination 40 km away. He cycled at a speed of 15km/h unto10:30, when he rested for half an hour. He then completed his journey at a speed of 20km/h. Using a suitable scale, draw a distance – time graph to represent the journey and use your graph toestimate the time at which the cyclist reached his destination.

2. A cyclist P starts from his house at 7 : 30 a.m and rides to his office at an average speed of 12km/h. A motorist Q starts from P's house at 8 : 00 a. m. and travels in the same direction as P at an average speed of 50km/h. After travelling 5km, the motorist meets a friend and stops for 9 minutes. He then continues his journey at the same average speed as before.

i. Using a scale of 2 cm to 15 minutes on the horizontal (time) axis and 2 cm to 5km on the vertical (distance) axis, draw thegraphs of the journeys of P and Q

ii. Use your graph to find;

a. When the motorist Q overtakes the cyclist P

b. How far P is from his house when Q overtakes him.

c. The distance between P and Q at 8:15 a.m.

3. At 8 : 00 a.m. a man set out on his bicycle from Oforikrom on the main road to Femesua. He rides at a constant speed of 12km/h for 45 minutes and then stops and talk to a friend. After he has been talking for 24 minutes, he suddenly realizes that he has forgotten something and returns immediately to Oforikrom at 18km/h. Draw the graph of his journey and find how far he is from Oforikrom at 9: 30 a. m. (Take as horizontal scale 2cm to represent 20 minutes, and as vertical scale 2cm to 1km

b. If the distance and walks at 7km/h towards Oforikrom.

4. A cyclist starts a 30 kmjourney at 9: 00 a.m. He maintains an average speed of 20km/h for the first three – quarters of an hour and then stops. Subsequently he continues his journey at an average speed of 30 km/h, arriving at his destination at 11: 00. Draw the distance - time graph and state, in minutes, the duration of his stop.

5. A boy rows upstream for $1\frac{1}{2}$ km at 2 km/h, waits half an hour and then rows back with the stream, taking 25 minutes. Another boy walks up the river bank at 3 km/h, starting from the same place as the first boy but 1 hour later.

i.Draw a graph, with time on the horizontal axis, showing the distances of the two boys from their start

ii. Find where and when the two boys meet.

iii. Find the speed of the boat downstream, in km/h

iv. What is the speed of the stream?

6. A train leaves Onu station at 6 : 00 a. m. for Krom station. It travels at an average speed of 48km/h throughout stopping for 6 minutes when it is 30 km from Onu. Another train leaves Onu at 6 : 25 a.m. by the same route and travels non-stop at an average speed of 80 km/h. If both trains arrive at Kron at the same time,

a. draw the graphs of the journeys of the two trains, using a scale of 2 cm to 10 minutes on the horizontal axis and 2 cm to 5 km on the vertical axis.

b. Use your graphs to find:

i. the distance between the two trains at 6 : 40 a.m.

ii. the time the two trains arrive at Kron station

iii. how far Kron station is from Onu station.

7.Manjang left Banjul on his motor bicycle at 9.00 a. m to travel to Lamin, 24 km away. He was expected to arrive at Lamin at 10: 12 a. m but had to stop 8km from Banjul for 20 minutes. Draw a graph for the journey

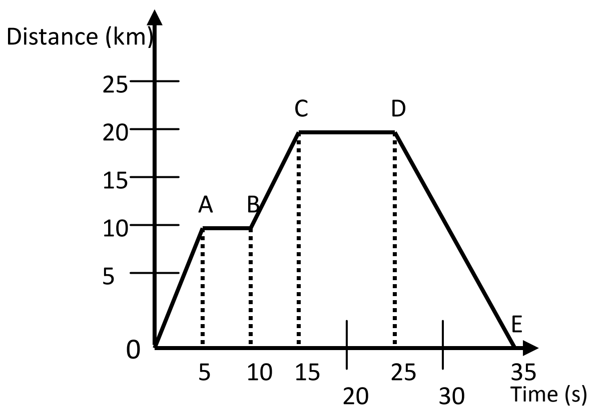
i. Use your graph to find:

a. his average speed for the last stage of the journey,

b. his average speed for the last stage of the journey if he was to arrive at Lamin at 10:12 a.m

Interpretation of Travel Graph

Consider the travel graph graph below;



I. A body starts from rest 0 and moved a distance of 10km in 5minutes to reach A.

II. At A, the body rested for 5 minutes (A− B) before moving from B.

III. From B, the body moved 10km to C where it rested for 10 minutes (C−D) before moving from D at a distance of 20km in 10minutes to rest at E.

Calculations

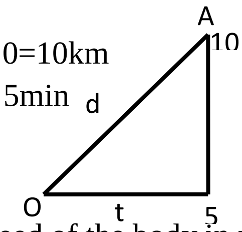
i. To calculate the speed(s) of body in moving from O to A, it follows that;

S $= \frac{d}{t}$

But $d = d_2 - d_1 = 10 - 0 = 10$km

$t = t_2 - t_1 = 5 - 0 = 5$min

$\Rightarrow S = \frac{10km}{5min} = 2km/min$



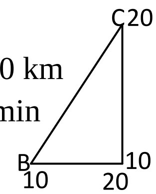
ii. To calculate the speed of the body in moving from B to C

S $= \frac{d}{t}$

$d = d_2 - d_1 = 20 - 10 = 10$ km

$t = t_2 - t_1 = 20 - 10 = 10$ min

$\Rightarrow S = \frac{10km}{10min} = 1km/min$



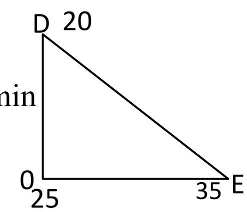
iii. To calculate the speed (s) of the body in moving from D to E.

S $= \frac{d}{t}$, but $d = d_2 - d_1$

d = 20 − 0 = 20km

$t = t_2 - t_1 = 35 - 25 = 10$min

$\Rightarrow S = \frac{20km}{10min} = 2km/min$

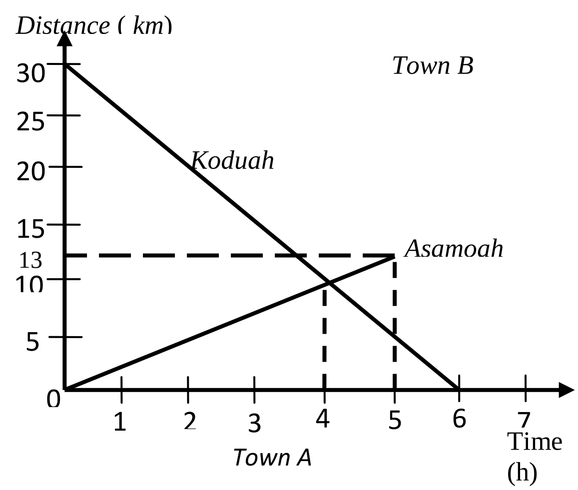


2. Consider the graph below which shows the journey made by two friends Asamoah and Koduah

a. At what time do they meet each other?

b. What is Asamoah's average speed?

c. What is Koduah's average speed?



Solution

a. From the graph, they meet each other at $4:00pm$

b. Asamoah's average speed, $S = \frac{d}{t}$

But $d = d_2 - d_1$

$d = (13 - 0)$ km $= 13$km

$t = t_2 - t_1 = (5 - 0) = 5$h

$\Rightarrow S = \frac{13\text{km}}{5\text{h}} = 2.6$km/h

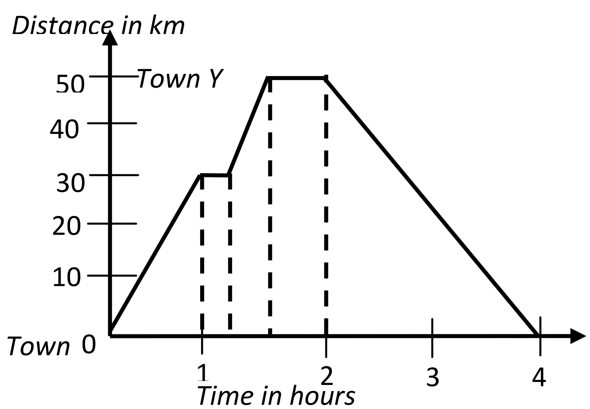
c. Koduah's average speed, $S = \frac{d}{t}$

But $d = d_2 - d_1 = (30 - 0)$km $= 30$km

$t = t_2 - t_1 = (6 - 0)\ h = 6h$

$\Rightarrow S = \frac{\text{d}}{\text{t}} = \frac{30\text{km}}{6\text{h}} = 5km/h$

Worked Examples



The above travel graph describes the journey of a cyclist from Town X to Y.

a. What is the average speed for the return journey from Town Y to Town X?

b. State the period within which the cyclist traveled to Town Y after his first rest.

c. How many minutes did the cyclist spend in town Y?

Solution

i. $S = \frac{d}{t}$

Distance from Town Y to Town X,

$d = d_2 - d_1 = (50 - 0)\ km = 50km.$

Time traveled from Town Y to Town X,

$t = t_2 - t_1 = (1\frac{1}{2} - 0)$ hr $= 1\frac{1}{2}$h

Now $S = \frac{d}{t} = \frac{50km}{1\frac{1}{2}h} = 33.33km/h$

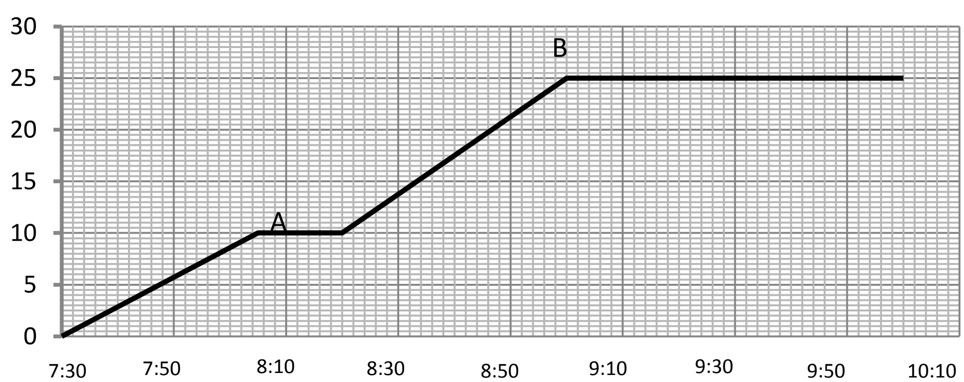
ii. Period of first rest = 1.00pm to 1:30pm

iii. Time spend in town Y,

$t = t_2 - t_1 = 2{:}00pm - 1{:}30pm = 30$mins

Exercises 12.17

1. The travel graph below shows the journey of Mr. Brown who travelled in his car to his mother's place at town B, 25km from his house.



Determine from the graph,
i. the time he spent in town A
ii. the time he left town A
iii. the time he spent in town B
iv. the time he reached town A
v. his speed in travelling from town A to B

Population Density

It is the degree to which an area is filled with people. Mathematically, it is the ratio of the number of people in a particular area to the size of the area.

Population density = $\frac{\boldsymbol{Number\ of\ people(p)}}{\boldsymbol{Size\ of\ area}}$

Worked Examples

1. A village has an area of 80 hectares and a population of 2,500 people. Calculate the population density of the town in people/ hector correct to the nearest whole person

Solution

Population density = $\frac{Nmber\ of\ people}{Size\ of\ area} = \frac{2{,}500\ people}{80\ hectares}$

Population density = 31people /hectare

2. The population density of a village is 520 people/km^2. If the area of the village is about 3.3 km^2, find its population to the nearest 100 people.

Solution

Population density = $\frac{Nmber\ of\ people}{Size\ of\ area}$

520 people/km^2 = $\frac{Population}{3.3\ km^2}$

⇒Population = 520 people/km^2 × 3.3km^2

Population = 1,716 people

Exercises 12.18

1.Oseikrom is a town that coversan area of 250 hectares. If it is populated with 5,000 people, calculate the density of the population correct to the nearest whole person.

2. A village has a population of 1,712people and an area of 214 hectares. Calculate the population density of the town in people/ hector correct to the nearest whole person.

3. The population density of a village is 172 people/km^2. If the area of the village is about 33km by 20 km, find its population to the nearest 100 people.

4. The population density of a village is 316 people/km^2. If the area of the village is about 345 km^2, find its population to the nearest 100 people.

PERCENTAGES 1

Definition

A percentage is a fraction with a denominator of 100. For example, $\frac{3}{100}$ = 3%. Therefore, to express anything as a percentage is to divide it into 100 equal parts. The 100 equal parts becomes the denominator and the portion taken out of the 100 is the numerator. If say **"*a*"** is the part taken out of the 100, then we have **"*a*"** percent. That is: $\frac{a}{100}$ = a%, read as: ***"a percent".***

For eg, $\frac{35}{100}$ = 35%, $\frac{87}{100}$ = 87% etc

Worked Examples

Write the following as percentages;

1. $\frac{15}{100}$ 2. $\frac{40}{100}$

Solution

1. $\frac{15}{100}$ = 15% 2. $\frac{40}{100}$ = 40%

Exercises 13.1

Write the following as percent;

1. $\frac{25}{100}$ 2. $\frac{3.2}{100}$ 3. $\frac{167}{100}$ 4. $\frac{53.4}{100}$

Percentages as Common Fractions

To convert percentages to common fractions and vice – versa, take note of the fact that, the percentage sign, % = $\frac{1}{100}$. This means that:

$a\% = a \times \frac{1}{100} = \frac{a}{100}$, expressed in the simplest form, to obtain the fraction.

Worked Examples

Express the following percentages as a fraction in its lowest term;

1. 28% 2.35%

Solution

1. $28\% = 28 \times \frac{1}{100} = \frac{28}{100} = \frac{7}{25}$
2. $35\% = 35 \times \frac{1}{35} = \frac{35}{100} = \frac{7}{20}$

Exercises 13.2

Convert the following percentages to fractions in its lowest term;

1. 72% 2.80% 3.0.5% 4.$112\frac{1}{2}$%

Fractions and Decimal as Percentages

To convert a fraction to a percentage, multiply the fraction by 100. That is:

$\frac{a}{b}$ as a percentage = $\boldsymbol{\frac{a}{b} \times 100 = \frac{100a}{b}}$

To write a decimal as a percentage,express the decimal as a number with a denominator of 100 and write the numerator as the percentage. For e.g. 0.21 as a percentage; $0.21 = \frac{21}{100}$ = 21%

Worked Examples

A. Change the following to percentages;

1. $\frac{2}{5}$ 2. $\frac{3}{4}$ 3. $\frac{7}{50}$

B. Write the following as percentages;

1) 0. 475 2) 0.3 3) 0.15

Solutions

A. *Multiply each fraction by 100*

1. $\frac{2}{5} \times 100 = \frac{200}{5} = 40\%$
2. $\frac{7}{50} \times 100 = \frac{700}{50} = 14\%$
3. $\frac{3}{4} \times 100 = \frac{300}{4} = 75\%$

B. *Express each decimal as a fraction with a denominator of 100 and pick the numerator as the percentage;*

1. $0.475 = \frac{47.5}{100} = 47.5\%$
2. $0.3 = \frac{30}{100} = 30\%$
3. $0.15 = \frac{15}{100} = 15\%$

Exercises 13.3

A. Express the fractions as percentages;

1. $\frac{27}{50}$ 2. $\frac{3}{8}$ 3. $2\frac{3}{5}$ 4. $\frac{11}{4}$ 5. $1\frac{3}{4}$

B. Write the decimals as percentages;

1. 0.625 2. 0.09 3. 0.72

Ordering a Combination of Fractions, Decimals and Percentages

To order a combination of common fractions, decimal fractions and percentages, it is advisable to:

<u>Method 1</u>

Multiplying each number by 100% and order them accordingly.

<u>Method 2</u>

Express all the numbers as a fraction with common denominator, usually 100 or a power of 10 if possible and order them according to the value of their numerators.

Worked Examples

1. Arrange in descending order; 54%,$\frac{5}{4}$, 0.52, $\frac{3}{5}$

Solution

<u>Method 1</u>

Multiply each number by 100%

$54\ \% = \frac{54}{100} \times 100\% = 54\%$

$\frac{5}{4} \times 100\% = 125\%$

$0.52 \times 100\% = 52\%$

$\frac{3}{5} \times 100\% = 60\%$

Comparing the products,

$\Rightarrow 125 > 60 > 54 > 52$

$\therefore$ Descending order: $\frac{4}{5}, \frac{3}{4}$, 54%, 0.52

<u>Method 2</u>

Express each number with a common denominator of 100

$54\% = \frac{54}{100}$

$\frac{4}{5} = \frac{4 \times 20}{5 \times 20} = \frac{80}{100}$

$0.52 = \frac{52}{100}$

$\frac{3}{4} = \frac{3 \times 25}{4 \times 25} = \frac{75}{100}$

Comparing numerators,

$\frac{80}{100} > \frac{75}{100} > \frac{54}{100} > \frac{52}{100}$

$\therefore$ In descending order ; $\frac{4}{5}, \frac{3}{4}$, 54%, 0.52%

2. Arrange the following in descendingorder: 53%, $\frac{3}{5}$, 0.52

Solution

Multiplyingeach number by 100%

$53\% = \frac{53}{100} \times 100\% = 53\%$

$\frac{3}{5} \times 100\% = \frac{300}{5}\% = 60\%$

$0.52 \times 100\% = 52\%$

Comparing the products,

60 > 53 > 52

$\therefore$ In descending order; $\frac{3}{5}$, 53%, 0.52

3. Arrange in ascending order; 0.72, 87%, $\frac{7}{10}$

Solution

Multiplying each number by 100%.

1. $0.72 \times 100\% = 72\%$

2. $87\ \% = \frac{87}{100} \times 100\% = 87\%$

3. $\frac{7}{10} \times 100\% = 70\%$

Comparing the products,

$87 > 72 > 70$

In ascending order we have $\frac{7}{10}$, 0.72, 87%

Exercises 13.4

A. Arrange in ascending order;

1. 0.6, $\frac{3}{2}$, 0.15, $\frac{5}{4}$
2. 40%, 0.37, $\frac{11}{20}$, $\frac{12}{25}$
4. 51%, 0.7, $\frac{9}{4}$
5. $\frac{11}{5}$, 0.22, $\frac{7}{2}$, 33%

B. Arrange in descending order;

1. 0.9, $\frac{7}{10}$, 85%
3. $\frac{3}{4}$, 0.31, 65%, $\frac{5}{2}$
2. 78%, $\frac{3}{4}$, 0.77, $\frac{5}{6}$
4. $\frac{1}{5}$, 0.55, 25%, $\frac{8}{10}$

Relative Error and Percentage Error

Relative error (often used in measurement) is a comparison or ratio of the absolute error to the actual / original value.

Mathematically,

Relative error $= \frac{Absolute\ \ error}{Original\ \ value}$;

where absolute error is the difference between the two values.

When the relative error is multiplied by 100%, it is called ***percentage error.***

Mathematically:

Percentage error $= \frac{Absolute\ \ error}{Original\ \ value} \times 100\%$

Percentage error = Relative error × 100%

Worked Examples

1. Find the percentage error in approximating 2.7 as 3.

Solution

Absolute error = 3 – 2.7= 0.3

Original value = 2.7

Percentage error $= \frac{\text{Absolute error}}{\text{Original value}} \times 100\%$

$= \frac{0.3}{2.7} \times 100\% = 11.1\ \%$

2. Find the percentage error in approximating $4\frac{1}{2}$ as 6.

Solution

Absolute error $= 6 - 4\frac{1}{2}$. But $4\frac{1}{2} = 4.5$

Absolute error = 6 – 4.5 = 1.5

Original value = 4.5

Percentage error $= \frac{\text{Absolute error}}{\text{Original value}} \times 100\%$

$= \frac{1.5}{4.5} \times 100\% = 33.3\ \%$

Exercises 13.44

Find the percentage error :

1) 9.8 as 10
2) 1.2 as 1.8
3) 15.2 as16.0
4) 5.4kg as 5.8kg
5). $3\frac{1}{2}$ as 4.2
6) $5\frac{1}{3}$ as $5\frac{2}{3}$

Percentage of a Given Quantity

To find the percentage of a given quantity (y),

express the percentage (x%) as a fraction and multiply by the given quantity.

$\Rightarrow$***x% of y*** $= \frac{x}{100} \times y$

Worked Examples

1. Find 35% of Gh¢400.00

Solution

$x\%$ of $y = \frac{x}{100} \times y$

35% of Gh¢400 = $\frac{35}{100} \times 400 = 140$

$\therefore$ 35% of Gh¢400.00 = Gh¢140.00

2. What is 25% of 80 books?

Solution

25% of 80 books

$\frac{25}{100} \times 80 = 20$

$\therefore$25% of 80 books = 20 books

3. Find $2\frac{1}{2}\%$ of Gh¢20.00.

Solution

Change $2\frac{1}{2}\% = \frac{5}{2}\%$

$\frac{2}{5}\%$ of Gh¢20.00 = $\frac{\frac{5}{2}}{100} \times 20 = \frac{5}{200} \times 20$

= Gh¢0.50 = 50p

4. Given that $7\frac{1}{2}\%$ of k is 33, find k.

Solution

$7\frac{1}{2}\%$ of k is 33

$\Rightarrow 7\frac{1}{2}\% \times k = 33$

$\frac{15/2}{100} \times k = 33$

$\frac{15}{2\times100} \times k = 33$

$\frac{15k}{200} = 33$

$15k = 200 \times 33$

$k = \frac{200 \times 33}{15} = 440$

Exercises 13.5

A. Find the following;

1. 26% of 5,000 pupils
2. 30% of 4,020 sheep
3. 90% of Gh¢20,000.00
4. Find $12\frac{1}{2}\%$ of Gh¢80,000.00

B. Find the following

1. $15\frac{1}{4}\%$ of Gh¢40,000.00
2. $14\frac{1}{3}\%$ of Gh¢33,400.00
3. $\frac{1}{2}\%$ of Gh¢16,000.00
4. $2\frac{1}{2}\%$ of Gh¢10,000.00

Expressing one Quantity as a Percentage of Another

To express a quantity, as a percentage of a similar quantity, express the quantity as a fraction of the total quantity and multiply by 100 percent. That is:

x as a percentage of $y = \frac{x}{y} \times 100\%$

Worked Examples

1. Express 25 as a percentage of 75.

Solution

$\frac{25}{75} \times 100\% = \frac{100\%}{3} = 33.3\%$

2. What percentage of 5 is 0.25?

Method 1

If 5 = 100%

$\Rightarrow 0.25 = \frac{0.25 \times 100\%}{5} = 5\%$

Method 2

$\frac{0.25}{5} \times 100\% = \frac{25}{5} = 5\%$

3. What percentage of Gh¢250.00 is Gh¢50.00?

Solution

$\frac{50}{250} \times 100\% = 20\%$

4. Express 190 girls as a percentage of 250 pupils.

Solution

$\frac{190}{250} \times 100\% = 76\%$

5. The population of Ghana was 5,000,000 in 1957. The population in 1998 was estimated to be 17,000,000. Find the percentage increase in population from 1957 to 1998.

Solution
Population in 1957 = 5,000,000
Population in 1998 = 17,000, 000

Increase in population
= 17,000,000 – 5,000,000 = 12,000,000
= $\frac{12{,}000{,}000}{5{,}000{,}000} \times 100\ \% = 240\ \%$

Exercises 13.6
1. What percentage of Gh¢1,250.00 is Gh¢50.00?

2. A tank contains 400 liters of water. If 100 liters is used, what percentage is left?

3. Mr. Green spent Gh¢3,000.00 out of his Gh¢40,000.00. What percentage of his money was spend?

4. If 300 candidates sit for an examination and 180 pass, what percentage of the candidates fail?

5. A candidate who gets 20% marks failed by 10 marks but another candidate who gets 42% marks gets 12 % more than the passing mark. Find the maximum marks.

Increasing a Quantity by a Given Percentage
<u>Method 1</u>
If ***y*** is increased by ***x***%, then:
New value = $\frac{(\mathbf{100}+\boldsymbol{x})}{\mathbf{100}} \times \boldsymbol{y}$

<u>Method 2</u>
If **y** is the given quantity increased by **x%,**
1. Then the increment, $\mathbf{I} = \frac{\boldsymbol{x}}{\mathbf{100}} \times \boldsymbol{y}$
2. ***New value*** = $y + \left(\frac{\boldsymbol{x}}{\mathbf{100}} \times \boldsymbol{y}\right)$

Worked Examples
1. Increase Gh¢500.00 by 30%.

Solution
<u>Method 1</u>
New value = $\frac{(100+x)}{100} \times y$,
But x = 30 and y = 500
New value= $\frac{(100+30)}{100} \times$ Gh¢¢500
New value = $\frac{130}{100} \times$ Gh¢500 = Gh¢650.00

<u>Method 2</u>
New value = $y + \left(\frac{x}{100} \times y\right)$
But x = 30 and y =Gh¢500
New value =$\left(\frac{30}{100} \times Gh¢500\right)$ +Gh¢500
New value = Gh¢(150 + 500) = Gh¢650.00

2. If an item which cost Gh¢250.00 is increased by 60%, what is the new price of the item?

Solution
New value = $\frac{(100+x)}{100} \times y$
But x = 60 and y = Gh¢250
New Value = $\frac{(100+60)}{100} \times$ Gh¢250
New value = $\frac{(160)}{100} \times$ Gh¢250 = Gh¢400.00

3. The population of Asuofua J.H.S was known to be 540 in 2010. Two years later, it increased by 20%. Find the student population in 2012.

Solution
I = $\frac{(100+x)}{100} \times y$

But $x = 20$ and $y = 540$

$I = \frac{(100 + 20)}{100} \times 540$ = 648 people

Decreasing a Quantity by a Given Percentage

Method 1

If y is decreased by x%, then;

New value = $\frac{(100 - x)}{100} \times y$

Method 2

If $\boldsymbol{y}$ is decreased by $\boldsymbol{x}$%,

1. then the decrement, $\boldsymbol{D} = \frac{x}{100} \times y$

2. ***New value*** $= \boldsymbol{y} - \left(\frac{x}{100} \times y\right)$

Worked Examples

1. Decrease Gh¢400.00 by 10%

Solution

Method 1

New value = $\frac{(100 - x)}{100} \times y$,

But $x = 10$ and y = Gh¢400

New value = $\frac{(100 - 10)}{100} \times$ Gh¢¢400

New value = $\frac{90}{100} \times 400$ = Gh¢360.00

Method 2

New value = $y - \left(\frac{x}{100} \times y\right)$

But $x = 10$ and y = Gh¢ 400

New value = Gh¢400 – $\left(\frac{10}{100} \times 400\right)$

New value = Gh¢(400 – 40) = Gh¢360.00

2. The number of patients treated at a hospital on a certain day was recorded as 240. If the number is decreased by 20% the next day, determine the number of patients that were treated the next day.

Solution

New value = $\frac{(100 - x)}{100}$ × y

But $x = 20$ and $y = 240$

New value = $\frac{(100 - 20)}{100} \times 240$

New value = $\frac{80}{100} \times 240$ = 192 patients

3. The cost of a rice cooker is Gh¢1,440.00. If the cost decreases by 24%, find the new price of the rice cooker.

Solution

New value = $\frac{(100 - x)}{100} \times y$,

But $x = 24$ and y = Gh¢1440

New value = $\frac{(100 - 24)}{100} \times 1440$

New value = $\frac{76}{100} \times 1440$ = Gh¢,1094.40

Exercise 13.7

1. A 10% service charge is added to a restaurant bill of Gh¢200.20. What is the total amount paid?

2. An article cost Gh¢720.80. If the cost is increased by 25%, find the new cost.

3. In a school of 800 boys, each boy plays either football or cricket or both. Given that 62% of the boys play football and 58% play cricket, calculate the number of boys who play both football and cricket.

4. i. The area of a square is 400 cm^2. Find the length of a side.
ii. If the length is now decreased by 20%, find the length of the new square

5. A student was given Gh¢2,300.00 by his father. If his mother gave him 28% of what his father gave him in addition, find the total money of the student.

6. A vendor sells 60% of apples he has and throws away 15% of the remainder. The next day, he sells 50% of the remainder and throws away the rest. What percentage of his apples does he throws away?

7. In an election contested by two parties *A* and *B*, party *A* secured 12% of the total votes more than party *B*. If party *B* got 132,000 votes and there are no invalid votes, by how many votes did it lose the election?

Discount

It is the reduction in the original price of an item. The discount or reduction is calculated
as a percentage of the original price.
To find the discount, you require;
a. Original price / Marked price
b. Discount rate
c. New price

Discount = Rate × Original price

New price = Original price – Discount

Worked Example

1. Mr. Brown wants to buy a calculator whose price is quoted as Gh¢5300.00 with 10% discount. Find:
i.the discount,
ii. how much he will pay for the item.

Solution

<u>Method 1</u>

Original price = Gh¢5,300.00

Rate = 10% = $\frac{10}{100}$

i. Discount = Rate × Original price

Discount = $\frac{10}{100}$ × 5300 = Gh¢530.00

ii. New price = Original price – Discount
New price = Gh¢ (5,300 – 530) = Gh¢4,770.00

<u>Method 2</u>

New price = original price × discount rate
Discount rate = 100% − 10% = 90%
Discount rate = $\frac{90}{100}$
∴ New price = $\frac{90}{100}$ × 5300 = Gh¢4,770.00
Discount = Gh¢(5300 – 4770) = Gh ¢530.00

2. Mr. Red paid Gh¢270.00 for T.V. set after he had been given a discount of 10%. Find the marked price of the T.V set.

Solution

Discount = 100% − 10% = 90%
⇒Mr. Red paid 90% = Gh¢270.00.
⇒the original price which is 100% will be *x*.

By proportion,
i.e. 90 : 100 = 270 : x
$\frac{90}{100} = \frac{270}{x}$
90 × x = 270 × 100
90 x = 27,000
x = $\frac{27,000}{90}$ = Gh¢300.00

3. Find the discount of 15% on Gh¢1,600.00 worth of goods. Find how much you will pay
for the goods.

Solution

Discount = Rate × Original price
Rate = 15% = $\frac{15}{100}$
Original price = Gh¢1,600.00
Discount = $\frac{15}{100}$ × 1600 = Gh¢240.00

New price = Original price –Discount

= Gh¢(1600 – 240) = Gh¢1,360.00

4. What is the discount of 25% on a computer that cost Gh¢5,600.00? Find the cost of the computer.

Solution
Discount = Rate × Original price,
where Rate = 25% $=\frac{25}{100}$
Original price = Gh¢5,600.00
Discount = $\frac{25}{100}$ × 5600 = Gh¢1,400.00

The cost of the computer ,
= Original price – Discount
= Gh¢(5,600 – 1,400) = Gh¢4,200.00

5. Calculate a discount of 12 $\frac{1}{2}$ % on an item that cost Gh¢24,000.00. Find the new price of the item.

Solution
Rate = 12 $\frac{1}{2}$ % = $\frac{25}{2}$ %
Original price = Gh¢24,000
Discount = $\frac{25/2}{100}$ × Gh¢24,000 = Gh¢3,000.00

The cost of the item;
= Original price – Discount
= Gh¢(24,000 – 3,000) = Gh¢21,000.00

Exercises 13.8
A. Find the following discounts:
1. 14% on Gh¢50,000.00
2. 17% on Gh¢10,000.00
3. 12% on Gh¢150,000.00
4. 19% on Gh¢110,000.00

B. Find the following:
1. 7 $\frac{1}{2}$% discount on on Gh¢22, 000.00
2. 12 $\frac{1}{3}$% discount on Gh¢210, 000.00
3. 15 $\frac{1}{4}$% discount on Gh¢120, 000.00
4. 10 $\frac{3}{4}$% discount on Gh¢36, 000.00

C.1. A shop is offering a discount of 15% on all goods. Find the sale price of goods normally priced at: i. Gh¢1650 ii. 9600 p

2. A water kettle and a rice cooker are sold at Gh¢410.00 and Gh¢528.00 respectively. If a discount of 20% is given on the cost of the kettle and a discount of 25 % is given on the cost of the rice cooker;
i. how much will you pay for 5 kettles?
ii. how much will you pay for 7 rice cookers?
iii. find the total cost of buying 5 kettle and 7 cookers at the given discount.

3. If a shop reduces all its prices by 5 $\frac{1}{3}$ %,
i. by how much is a generator that cost Gh¢72,000.00 reduced?
ii. how much will you pay for the generator?

Finding the Discount Rate Given the Marked Price and the Discount Value
Given the marked price and the discount value, the discount rate is calculated by the formula: ***Rate*** = $\boldsymbol{\frac{Discount\ value}{Marked\ price} \times 100\%}$

Worked Examples
1. A discount of Gh¢480.00 was given on an article marked Gh¢24,000.00. What was the percentage discount?

Solution
Rate = $\frac{Discount\ value}{Marked\ price}$× 100%

But discount value = Gh¢480.00 and Marked price = Gh¢24,000.00

Rate = $\frac{480}{24000} \times 100 = 2\%$

2. The original price of a calculator is Gh¢600.00. If it is reduced to Gh¢420.00, find the rate of discount.

Solution

$Rate = \frac{Discount\ value}{Marked\ price} \times 100\%$

But discount value = Gh¢(600 – 420)

= Gh¢180.00

Original price = Gh¢600.00

Rate = $\frac{180}{600} \times 100 = 30\%$

Exercises 13.9

1. If goods normally sold at Gh¢112.00 are offered at a sale at Gh¢98.00 express the discount as percentage of the normal price.

Finding the Original Price / Marked Price Given the Discount Rate and the New Price.

Given the discount rate and the new price, the original or marked price of an item is calculated by the formula:

$$\boldsymbol{Marked\ price = \frac{100 \times New\ price}{(100 - Discount\ rate)}}$$

Worked Examples

1. A store gives 10% discount for an article bought and paid for immediately. If a girl pays Gh¢18.00 cash for a dress, what is the marked price of the dress?

Solution

$Marked\ price = \frac{100 \times New\ price}{(100 - Discount\ rate)}$

But new price = Gh¢18.00 and rate = 10%

Marked price = $\frac{100 \times 18}{(100 - 10)} = \frac{100 \times 18}{90}$ = Gh¢20.00

2. Kojo paid Gh¢480.00 for T.V. set after he has been given a discount of 20%. Find the marked price.

Solution

$Marked\ price = \frac{100 \times New\ price}{(100 - Discount\ rate)}$

But new price = Gh¢480.00

Discount rate = 20%

Marked price = $\frac{100 \times 480}{(100 - 20)} = \frac{100 \times 480}{80}$ = Gh¢600.00

Exercises 13.10

1. A bookshop reduced the price of its books by 15%. Janet bought a book for Gh¢2,700.00 during the reduction sale. What was the original price of the book**?**

2. A company reduced the prices of its goods by 13% during promotional sales. A product was sold for Gh¢43,500.00 during the promotional sales. What was the price before the promotional sales?

Commission

It is the share or portion of a total amount of sales (money) usually given to the sales person. It is expressed as a percentage of the total sales.

Mathematically:

Commission = Rate × Total Amount.

$\Rightarrow Total\ sales = \frac{Commission}{Rate}$

Worked Examples

1. A shopkeeper receives a commission of 20% on her daily sales. If she makes a daily sales of Gh¢1,050.00, calculate her commission.

Solution

Rate = 20% =$\frac{20}{100}$, total sales = Gh¢1050.00

Commission = Rate × Total sales

Commission =$\frac{20}{100}$ × Gh¢,1050 = Gh¢210.00

2. A house agent's commission on the sales of a house is 7%. If he sell the house for Gh¢12,500.00, find his commission.

Solution

Commission = Rate × Total sales

Rate = 7% = $\frac{7}{100}$, amount = Gh¢12,500.00

∴ Commission = $\frac{7}{100}$ × Gh¢12,500 = Gh¢875.00

3. A sales girl is paid her basic monthly salary of Gh¢15,000.00. In addition, she earns a commission of 2% on all her sales. In a particular month, her sales amounted to Gh¢1.5 million. Calculate her gross income for that month.

Solution

Monthly salary = Gh¢15,000

Total sales = Gh¢1.5 million

Commission = 2% of Gh¢1.5 million

Commission = $\frac{2}{100}$ × 1,500,000 = Gh¢30,000.00

Gross income for that month;

= Monthly salary + Commission

= Gh¢(15,000 + 30,000) = Gh¢45,000.00

4. A bottle of soft drink cost Gh¢2.00. The commission paid on one bottle is 2% of the cost price. Find the commission on 24 bottles of the soft drink.

Solution

Commission on 1 bottle = Rate × total sales

But rate = 2% = $\frac{2}{100}$

Total sales of 1 bottle = Gh¢2.00

Commission on one bottle

= $\frac{2}{100}$ × Gh¢2 = Gh¢0.04 = 4p

Commission on 24 bottles = 24 × 4p = 96p

5. A crate of egg cost Gh¢2.50. If Naa sells 10 crates at a rate of 30%, find her commission.

Solution

Commission = Rate × Total amount

Rate = 30% = $\frac{30}{100}$

Total amount =10 crates × Gh¢2.50 = Gh¢25.00

Commission = $\frac{30}{100}$ × Gh¢25 = Gh¢7.50

Exercises 13.11

A. Calculate the following commissions;

1. 26% on Gh¢5,440.00
2. 13% on Gh¢705.00
3. 17 ¼% on Gh¢108.00
4. 14 $\frac{2}{3}$ % on Gh¢4,800.00
5. $1\frac{5}{6}$ % on Gh¢1,140.00

B. 1. A book seller was given 20% commission on his total sales of Gh¢62,500.00. Calculate his commission.

2. A car dealer receives 15% commission on the sales of each car. If he sells a car worth Gh¢7,226.00, how much commission will he receive?

3. A house agent is paid a commission of 9% on the sale of a semi – detached house. Calculate his commission received on five semi – detached building each at a cost of Gh¢2,200.00.

4. Mr. Kokotii is a car dealer and a house agent as well. If he sells a car on behalf of his client, he takes a commission of 16% on the total cost and if he sells a house on behalf of his client, he takes commission of 21% on the total cost. Calculate his total commission if he sells three cars each at Gh¢7,650.00 and two houses each at a cost of Gh¢2,550.00

5. Jonadab wanted Thammar as his girlfriend, but he could not approach her due to some fearfulness. He lured the services of his bosom friend, Ahitofel who agreed to take a commission of 8%on Jonadab's weekly wages of Gh¢20,820.00. Find the amount received by Ahitofel.

Finding Total Sales Given the Rate and Commission

Total Sales = $\frac{Commission}{Rate}$

Worked Examples

1. Aku received a commission of Gh¢9,000. 00 on some books sold. Find her total sales if she was given 12% commission.

Solution

Total sales = $\frac{Commission}{Rate}$

Rate = 12% = $\frac{12}{100}$

Commission = Gh¢9,000.00

Total sales = $\frac{9000}{\frac{12}{100}} = \frac{9000\times100}{12}$ =75,000.00

∴ Total sales = Gh¢75,000.00

2. A shopkeeper received Gh¢3,000.00 commission for selling some books If she had 10% commission find the total sales she made.

Solution

Commission = Gh¢3,000.00

Rate = 10% = $\frac{10}{100}$

Total sales =$\frac{Commission}{Rate}$

Total sales = $\frac{300}{\frac{10}{100}} = \frac{3,000\times100}{10}$ = Gh¢30,000.00

3. Find total amount if the commission on 5% of some goods sold is Gh¢110.00.

Solution

Rate = 5% = $\frac{5}{100}$

Commission = Gh¢110.00

Total sales = $\frac{Commission}{Rate}$

Total sales = $\frac{110}{\frac{5}{100}} = \frac{110\times100}{5}$ = Gh¢2,200.00

4. At what amount will 14% give a commission of Gh¢160.00.

Solution

Rate =14% = $\frac{14}{100}$, Commission = Gh¢160.00.

Total amount = $\frac{Commission}{Rate}$

Total amount = $\frac{160}{\frac{14}{100}} = \frac{160\times100}{14}$ = Gh¢1,142.86

Exercises 13.12

A.1. Kofi received Gh¢8,000.00 commission on the sales he made in a particular day. The commission is 20% of the sales he made. Calculate his total sales for the day.

2. A trader received a commission of $12\frac{1}{2}$% on the sales she made in a certain month. If the commission she received was Gh¢30,000.00, what is the total sales she made?

3. A trader received a commission of 10% on the sales made in a month. His commission was Gh¢35,000.00. Find the total sales made for the month.

4. A salesman received a commission of 5% on sales in a month. If his commission is Gh¢1,150.00, calculate his total sales in a month.

Finding the Rate, Given the Total Sales and Commission

Rate = $\frac{Commission \times 100\%}{Total\ Sales}$

Worked Examples

1. A shopkeeper receives Gh¢500.00 on Gh¢2,000.00 total sales of books from a publisher. What is the rate of commission?

Solution

Commission = Gh¢500.00,

Total sales = Gh¢2,000.00

Rate = $\frac{commission \times 100\%}{Total\ Sales} = \frac{500 \times 100\%}{2000} = 25\%$

2. At what rate will Gh¢720,000.00 sales yield a commission of Gh¢36,000.00?

Solution

Commission = Gh¢36,000.00

Total amount = Gh¢720,000.00

Rate = $\frac{Commission \times 100\%}{Total\ Sales} = \frac{36{,}000 \times 100\%}{72{,}0000} = 5\%$

3. A book seller received Gh¢500.00 commission on a Gh¢4,000.00 sale of books on behalf of a publisher. Calculate the rate of commission?

Solution

Rate = $\frac{Commission \times 100\%}{Total\ Sales} = \frac{500 \times 100\%}{4000} = 12.5\%$

Exercises 13.13

1. A shopkeeper received Gh¢50.00 commission from a publisher by selling 500 copies of a book each at GH¢5.00. At what rate was the commission calculated?

2. A shopkeeper received Gh¢3,000.00 commission from a publisher by selling 500 copies of a book at Gh¢20.00 each. What was his rate of commission?

3. A book seller received Gh¢2,500.00 on a Gh¢20,000.00 sale of books on behalf of a publisher. What was the rate of commission?

4. A house agent makes a commission of Gh¢103,500 when he sells a house for Gh¢690,000.00. Calculate the percentage of his commission.

ChallengeProblems

1. Every week, a sales man is paid a basicsalary of Gh¢22.00. In addition, he receives a commission at arate of $\frac{4}{5}\%$ on the first Gh¢1,000.00 of his weekly sales and at the rate of 2% on weekly sales over Gh¢1,000.00. Calculate:
i. his earning in a week in which his sales totaled Gh¢1,600.00
ii. his sales in a week when his total earnings amounted to Gh¢35.00

2. A salesman is paid a commission at a rate of $1\frac{1}{4}\%$ on all sales over Gh¢10,000.00, plus a weekly wage of Gh¢22.50. Find his average weekly earnings if his sales for the year amounted to Gh¢88,000.00.

Profit and Loss

Profit (P) -: It is how much or how far the price of an item exceeds the cost.

Loss (L) -: It is how much or how far the price of an item recedes the cost.

The price at which goods are bought is called ***cost price(C.P)*** and the price at which the goods are sold is called ***selling price (S.P).***

Profit occurs when the S.P. is greater than the C.P and loss occurs when the S.P is less than the C.P.

***Mathematically*:**

a. Profit = Selling price – Cost price

$$P = S.P - C.P$$

b. Loss = Cost price – Selling price

$$L = C.P - S.P$$

Worked Examples

1. Mansah bought some oranges at Gh¢240.00 and sold them for Gh¢300.00 Find her profit.

Solution

C.P = Gh¢240.00 and S.P = Gh¢300.00

P = S. P − C.P

P = Gh¢ (300 −240) = Gh¢60.00

2. A radio which cost Gh¢118.00 was sold for Gh¢96.00. Find the profit or loss.

Solution

C.P = Gh¢118.00 and S.P = Gh¢96.00

L = C.P – S.P

L = Gh¢ (118 – 96) = Gh¢22.00

3. A woman trader bought 458 oranges for Gh¢5,496.00. She kept 62 oranges for her family and sold the rest at Gh¢16.00 each. Calculate to one decimal place, her profit.

Solution

C. P. of 458 oranges = Gh¢5,496

C. P. of 1 orange = $\frac{Gh¢5{,}496}{458}$ = Gh¢12.00

Oranges sold = 458 – 62 = 396 oranges

C.P of 396 oranges at Gh¢12.00;

= 396 × Gh¢12 = Gh¢4,752.00

S.P of 396 oranges each at Gh¢16.00;

= 396 × Gh¢16 = Gh¢6,336.00

P = S. P – C. P

P = Gh¢(6,336 – 4,752) = Gh¢1,584.00

Exercises 13.14

1. Mr. Oteng bought 210 oranges for Gh¢10.50 and sold them at 3 for 20p. How much profit did he make?

2. Mr. Brown bought some goods for Gh¢4,261.00 and sold it for Gh¢5,972.00. Find his profit.

3. The cost price of a refrigerator is Gh¢60,303.00. If it is sold for Gh¢52,664.00, calculate the loss incurred.

4. A bag of cement cost Gh¢23.00. If Mr. White buys 115 bags and sells them at Gh¢25.00 each, calculate his profit.

Profit and Loss Percentage

The profit or loss percent can be expressed as the percentage of the cost price (C.P). Thus, the profit percent is calculated by,

$$P\% = \frac{Profit}{Cost\ price} \times 100\% = \frac{P}{C.P} \times 100\%$$

The loss percent is calculated by,

$$L\% = \frac{Loss}{Cost\ price} \times 100\ \% = \frac{L}{CP} \times 100\ \%$$

Worked Examples

1. A shoe which cost Gh¢16.00 was sold for Gh¢ 10.00. Find the loss percentage.

Solution

C. P = Gh¢16.00 and S. P = Gh¢10.00

L = C.P − S.P.

L = Gh¢(16 − 10) = Gh¢6.00

L% = $\frac{P}{CP} \times 100\%$ = $\frac{6}{16} \times 100\%$ = 37.5%

2. An item which cost Gh¢200.00 was sold for Gh¢250.00. Calculate the profit percent.

Solution

C.P = Gh¢200.00 and SP = Gh¢250.00

P = S.P – C.P

P = Gh¢ (250 – 200) = Gh¢50.00

P% = $\frac{P}{CP} \times 100\%$ = $\frac{50}{200} \times 100\%$ = 25%

3. The cost of a table is Gh¢30.00. If it is sold for Gh¢36.00, Find the percentage profit or loss.

Solution

C.P = Gh¢ 30.00 and S.P = Gh¢36.00.

P = S.P − C.P

P = Gh¢(36 – 30) = Gh¢6.00

P% = $\frac{P}{CP} \times 100\%$ = $\frac{6}{30} \times 100$ = 20%

4. A man buys eggs at Gh¢45.00 per crate of 30 eggs. He finds that 20% of the eggs are broken but sells the rest at Gh¢30.00 a dozen. Find his percentage profit to nearest whole number.

Solution

Total number of eggs = 30

Cost price = Gh¢45.00

20% of the eggs broken = $\frac{20}{100} \times 30$ = 6 eggs

The rest = 30 – 6 = 24 eggs

12 eggs = 1 dozen

24 eggs = $\frac{24}{12}$ = 2 dozens

But 1 dozen = Gh¢30.00

2 dozens = 2 × Gh¢30 = Gh¢60.00

C. P = Gh¢45 and S. P. = Gh¢60

P = S.P – C. P

P = Gh¢(60 – 45) = Gh¢15.00

P% = $\frac{P}{CP} \times 100\%$ = $\frac{15}{45} \times 100\%$ = 33%

Solved Past Question

A woman bought 130kg of tomatoes for Gh¢52,000.00. She sold half of them at a profit of 30%. The rest of the tomatoes began to go bad. She then reduced the selling price per kg by 12%. Calculate:

i. the new selling price per kg

ii. the percentage profit on the whole transaction if she threw away 5kg of bad tomatoes.

Solution

i. C.P. of 130 kg of tomatoes = Gh¢52,000.00

Cost of Half (65kg) = Gh¢26,000

Profit of 30% on Gh¢26,000

P = $\frac{30}{100}$ × Gh¢26,000 = Gh¢7,800

S.P = Gh¢(26,000 + 7,800) = Gh¢33,800

⇒S. P of 1 kg = $\frac{Gh¢33,800}{65}$ = Gh¢520.00

The new selling price per kg;

= 12% reduction in the S. P of 1 kg

= Gh¢520 – $\left(\frac{12}{100} \times \text{Gh¢}520\right)$

= Gh¢520 – Gh¢62.4 = Gh¢457.60

The new S.P per kg is Gh¢457.60 per kg

ii. S.P of 65kg = Gh¢33,800
Remaining = (65kg – 5kg of bad ones) = 60 kg

60 kg sold at Gh¢457.60 each
= 60 × Gh¢457.60 = Gh¢27,456.00

Total selling price (S.P);
= S.P of 65kg + S.P of 60 kg
= Gh¢(33,800 + 27,456) = Gh¢61,256.00

P = S.P – C.P.
P = Gh¢61,256 – Gh¢52,000 = Gh¢9,256.00

P% = $\frac{P}{C.P}$ × 100 % = $\frac{9,256}{52,000}$ × 100 % = 17.8%

2. A retailer bought 600 copies of a book at Gh¢4.50 each. He sold 500 copies at Gh¢6.00 each but gave a discount of 5p in the cedi. He sold the remainder of the books half the selling price but without discount. Calculate the retailer's percentage gain, correct to 3 significant figures.

Solution
C.P. of 600 copies of a book at Gh¢4.50 each
= 600 × Gh¢4.50 = Gh¢2,700.00

S.P. of 500 copies at Gh¢6.00 each but gave a discount of 5p in the cedi
⇒ A copy = Gh¢6.00 – 5p = Gh¢5.95

S.P of 500 copies at Gh¢5.95 each
= 500 × Gh¢5.95 = Gh¢2,975.00

Remaining copies = 600 – 500 = 100
100 copies at half of Gh¢6.00 (No discount)
100 × Gh¢3 = Gh¢300.00

Total selling price (S.P)
= S.P of 500 + S.P of 100
= Gh¢2,975 + Gh¢300 = Gh¢3,275.00

P = S.P – C. P
P = Gh¢3,275 – Gh¢2,700 = Gh¢575.00

P% = $\frac{575}{2,700}$ × 100 % = 21.3%

Exercises 13.15

1. Dansowaa bought a sewing machine for Gh¢50,000.00 and sold it for Gh¢92, 000.00. Find the percentage of her profit.

2. Mr. Green bought 300 crates of eggs each at a cost of Gh¢6.20. After selling all the eggs, he had Gh¢1,200.00. Find his profit or loss percent.

3. Mr. White buys an old car for Gh¢4,700.00 and spends Gh¢800.00 on its repairs. If he sells the car for Gh¢5,800.00, what is his gain percent?

4. Mr. Brown is a shopkeeper. He purchased 15 boxes of key soap each at a cost of Gh¢12,000.00 and 17 boxes of voltic mineral water each at a cost of Gh¢8,000.00. After selling all the goods, he had Gh¢237,000.00. Calculate his profit percent or loss percent.

5. The total cost of 544 oranges is Gh¢11,968.00. If 102 oranges are kept and the rest sold at Gh¢20.00 each, find the percentage of the profit or loss.

6. A woman bought 412 oranges for Gh¢7,416.00. She kept 80 of them and sold the rest at Gh¢23.00 each. Find her profit percent.

7. A television set cost a retailer Gh¢500.00. He offers to sell it either for a cash price of Gh¢650.00 or a down payment of 25% of the

cash price followed by 24 monthly installments of Gh¢20.50. Calculate his percentage profit calculated on his cost price and his selling price for each method of selling Ans: 30%, 23.1%

Challenge Problem

1. A merchant spent Gh¢2,250.00 in buying 1,000 articles. He fixed the selling price to allow himself a profit of 20% on his cost, and sold four – fifths of the articles at this price. He then reduced his selling price by one – third and sold the remainder of his stock at this new price. Calculate his profit as a percentage of his outlay. Ans: 12%

2. In a certain store, the profit is 20% of the cost. If the cost increased by 25% but the selling price remains constant, approximately what percentage of the selling price is the profit? Ans : 16.67%

3. Potatoes are bought at Gh¢20.00 per 50kg and are sold at Gh¢3.50 per kg. Find, correct to the nearest whole number, the profit as a percentage of:

a. the cost price Ans : (40%)

b. the selling price Ans : 28.6%

4. A shop marks an article so as to make a profit of 30% on the cost price. In a sale, a discount of 10% was allowed off the marked price. If the article was sold in the sale, state the actual percentage profit made by the shop.

Finding the Cost Price or Selling Price Given the Percentage Profit

A. Finding the Cost Price

Given the profit percentage and the selling price, the cost price is calculated as follows:

<u>Method 1</u>

I. Identify the given percentage profit (P%) and the selling price (S.P)

II. Substitute the values of P and S.P. in the formula: **C.P. =** $\boldsymbol{\frac{100}{100+P}}$**× S.P.**

III. Simplify and obtain the answer in the unit of the given currency.

<u>Method 2</u>

This is equivalent to increasing the cost price by the given percentage. That is :

C.P .+ (P% of C.P.) = S.P.

I. Identify the given percentage profit (P%) and the selling price(S.P)

II. Substitute the values of P and S.P in the formula: **C.P. +** $\boldsymbol{\left(\frac{P}{100} \times \text{C.P.}\right)}$ **= S.P.**

III. Solve for C.P and obtain the answer in the unit of the given currency.

Worked Examples

1. A set of furniture was sold for Gh¢300.00 at a profit of 20%. Findthe cost price.

Solution

<u>Method 1</u>

S.P. = Gh¢300.00,P % = 20%, C.P. =?

C.P. = $\frac{100}{100 + P}$ × S.P.

C.P = $\frac{100}{100 + 20}$ × Gh¢300 = Gh¢250.00

<u>Method 2</u>

S.P. = Gh¢300.00,P % = 20%, C.P. = x

C.P. + $\left(\frac{P}{100} \times \text{C.P}\right)$ = S.P

x + $\left(\frac{20}{100} \times x\right)$ = Gh¢300

$x + \frac{x}{5}$ = 300

$5x + x = 5 \times 300$

$6x = 5 \times 300$

$x = \frac{5 \times 300}{6}$ = Gh¢250

The cost price is Gh¢250.00

2. A kettle was sold for Gh¢575.00 at a profit of 15%. Find the cost price of the kettle.

Solution

S.P. = Gh¢575.00, Profit % =15%   C.P. =?

C.P. = $\frac{100}{100+P}$ × S.P.

C.P. = $\frac{100}{100+15}$ × Gh¢575 = Gh¢500.00

3. A wall clock is sold for Gh¢210.00 at a profit of 5%. What is the cost price of the wall clock?

Solution

S.P. = Gh¢210.00 and P% = 5% C.P.= ?

C.P. = $\frac{100}{100+P}$ × S.P

C.P. = $\frac{100}{100+5}$ × Gh¢210 = Gh¢200.00

B. Finding the Selling Price

Given the profit percentage and the cost price, the selling price is calculated as follows:

I. Identify the given percentage profit (P%) and the cost price (C.P)

II. Substitute the values of P and C.P. in the formula: **S.P =** $\mathbf{\frac{100+P}{100}}$ **× C.P.**

III. Simplify and obtain the answer in the unit of the given currency.

Method 2

The S.P is equal to increasing the C.P by the given percentage (P%). That is:

S.P. = C.P. + (P% of C.P.)

I. Identify the given percentage profit (P%) and the cost price (C.P)

II. Substitute the values of P and C.P in the formula: **C.P +**$\mathbf{\left(\frac{P}{100} \times C.P\right)}$ **= S.P.**

III. Simplify and obtain the answer in the unit of the given currency.

Worked Examples

1. A bag of rice which cost Gh¢50.00 was sold at a profit of 20%. Find the selling price.

Solution

Method 1

C.P. = Gh¢50.00, P% = 20%, S.P. =?

S.P. = $\frac{100+P}{100}$ × C.P.

S.P. = $\frac{100+20}{100}$ × Gh¢50 = Gh¢ 60.00

Method 2

C.P. = Gh¢ 50.00 and, P% = 20%, S.P. =?

S.P = C.P + $\left(\frac{P}{100} \times C.P\right)$

S.P = 50 + $\left(\frac{20}{100} \times 50\right)$ = Gh¢60

2. A bicycle was boughtat Gh¢300.00. If it was sold at a profit of 24%, find the selling price of the bicycle.

Solution

C.P. = Gh¢300.00 and P % = 24%, S.P.= ?

S.P. = $\frac{100+P}{100}$ × C.P.

S.P. = $\frac{100+24}{100}$ × Gh¢300 = Gh¢372.00

3. A lorry tyre which cost Gh¢525.00 was sold at a profit of 12%. What was the selling price of the tyre?

Solution

C.P. = Gh¢525.00 and P% = 12% S.P. = ?

S.P. = $\frac{100+P}{100}$ × C.P.

S.P. = $\frac{100+12}{100}$ × Gh¢525 = Gh¢588.00

Solved Past Question

A trader buys some goods whose mark price is Gh¢100,000.00 at a discount of 2.5%. After selling the goods, the trader is required to pay tax at a rate of 20% on the profit she makes in excess of Gh¢10,000.00. How much should she sell the goods so that after the tax she would make a net profit of Gh¢20,000.00?

Solution

Marked price = Gh¢100,000.00

Discount = $\frac{2.5}{100}$ × Gh¢100,000 = Gh¢2,500.00

C.P = Gh¢100,000 – Gh¢2,500

C.P = Gh¢97,500.00

Let the profit before tax be Gh¢P

20% of profit paid as tax leaving a net profit of Gh¢20,000

P – $\left(\frac{20}{100} \times \text{P}\right)$ = Gh¢20,000.00

P – $\frac{P}{5}$ = 20,000

5P – P = 5 × 20,000

4P = 100,000

P = $\frac{100{,}000}{4}$ = Gh¢25,000

P = S.P – C.P

S.P = P + C.P

S.P = Gh¢25,000 + Gh¢97,500 = Gh¢122,500.00

Finding the Cost Price or Selling Price Given the Percentage Loss

A. Finding the Cost Price

Given the loss percentage and the selling price, the cost price is calculated as follows:

<u>Method 1</u>

I. Identify the given loss percentage (L%) and the selling price (S.P)

II. Substitute the values of L and S.P in the formula: **C.P =** $\frac{\mathbf{100}}{\mathbf{100}-\boldsymbol{L}}$ **× S.P**

III. Simplify and obtain the answer in the unit of the given currency.

<u>Method 2</u>

This is equivalent to decreasing the cost price by the given percentage. That is :

C.P – (P% of C.P) = S.P

I. Identify the given percentage loss (L%) and the selling price(S.P)

II. Substitute the values of L and S.P in the formula: **C.P –** $\left(\frac{\boldsymbol{P}}{\mathbf{100}} \times \textbf{C.P}\right)$ **= S.P**

III. Solve for C.P and obtain the answer in the unit of the given currency.

Worked Examples

1. Mr. Owusu sold his used T.V. set for Gh¢255.00. If his loss was 15% , how much did he buy the T.V. set?

Solution

<u>Method 1</u>

S.P. = Gh¢255.00 and C.P = ? L % = 15%

C.P = $\frac{100}{100-L}$ × S.P

C.P = $\frac{100}{100-15\%}$ × Gh¢255

C.P = $\frac{100}{85}$ × Gh¢225

C.P = Gh¢300.00

<u>Method 2</u>

S.P. = Gh¢255.00 and C.P = x,L % = 15%

C.P – $\left(\frac{P}{100} \times \text{C.P}\right)$ = S.P

x – $\left(\frac{15}{100} \times x\right)$ = Gh¢255

$x - \frac{15x}{100}$ = 255

100x – 15x = 100 × 255

85x = 100 × 255

$x = \frac{100 \times 255}{85}$ = Gh¢300.00

2. A mobile phone is sold for Gh¢720.00 at a loss of 20%. What is its cost price?

Solution

S.P. = Gh¢ 720.00 and C.P = ? L % = 20%

C.P = $\frac{100}{100 - L}$ × S.P

C.P = $\frac{100}{100 - 20\%}$ × Gh¢720 = Gh¢900.00

B. Finding the Selling Price

Given the loss percentage (L%) and the cost price, the selling price is calculated as follows:

I. Identify the given percentage loss (L%) and the cost price (C.P)

II. Substitute the values of L and C.P in the formula: ***S.P =*** $\boldsymbol{\frac{100 - L\,\%}{100}}$ **×** ***C.P***

III. Simplify and obtain the answer in the unit of the given currency.

Method 2

The S.P is equal to decreasing the C.P by the given percentage (P%). That is:

S.P = C.P + (P% of C.P)

I. Identify the given percentage loss (P%) and the cost price (C.P.)

II. Substitute the values of L and C.P in the formula: **S.P. = C.P.** –$\left(\boldsymbol{\frac{P}{100}} \times \textbf{C. P.}\right)$

III. Simplify and obtain the answer in the unit of the given currency.

Worked Examples

1. At what price must an item bought for Gh¢350.00 be sold so as to incur a loss of 40%.?

Solution

Method 1

C.P = Gh¢350, L % = 40 and S.P = ?

S.P = $\frac{100 - \text{L}}{100}$ × C.P

S.P = $\frac{100 - 40}{100}$ × Gh¢350

S.P = $\frac{60}{100}$ × Gh¢350 = Gh¢210.00

Method 2

C.P = Gh¢350, L % = 40 and S.P = ?

S.P = C.P – $\left(\frac{P}{100} \times \text{C. P}\right)$

S.P = Gh¢350 – $\left(\frac{40}{100} \times 350\right)$ = Gh¢210.00

2. A man claims to have sold his car at a loss of 24%. How much did he sell the car if he actually bought it for Gh¢7,500.00?

Solution

C.P = Gh¢7,500, L % = 24% and S.P = ?

S.P = $\frac{100 - \text{L}}{100}$ × C.P

S.P = $\frac{100 - 24}{100}$ × Gh¢7,500

S.P = $\frac{76}{100}$ × Gh¢7500 = Gh¢5,700.00

Solved Past Questions

1. A trader sold 1,750 articles for Gh¢525, 000.00 and made a profit of 20%.

i. Calculate the cost price of each article.

ii. If he wanted 45% profit on the cost price, how much should he have sold each of the articles?

Solution

i. C.P = $\frac{100}{100 + P}$ × S.P

P = 20% and S.P = Gh¢525,000

C.P = $\frac{100}{100 + 20}$ × 525,000 = Gh¢437,500.00

Cost price of one item;

= $\frac{Gh¢\ 437{,}500}{1{,}750}$ = Gh¢250.00

ii. Increase the cost price by 45% ;

= 437,500 + $\left(\frac{45}{100} \times 437{,}500\right)$

= 437,500 + 196,875 = Gh¢634,375.00

Each article should be sold for ;

$= \frac{634,375.00}{1,750}$ = Gh¢362.50

2. A manufacturer makes a wireless set at a cost of Gh¢6,000.00 and sold it to a wholesaler at a profit of 10%. The wholesaler sold it to a retailer at a profit of 25%. Find:
i. the cost to the wholesaler;
ii. the selling price of the wholesaler;
iii. if a customer who paid cash had the price reduced to Gh¢6,200.00, find the percentage discount allowed on the marked price to the customer, to two significant figures.

Solution
<u>Method 1</u>
i. The cost to the wholesaler is the same as the selling price of the manufacturer;
S.P = $\frac{100 + P}{100} \times C.P$
P = 10% and C. P = 6,000
S.P = $\frac{100 + 10}{100}$ × 6,000 = Gh¢6,600.00

ii. The wholesaler bought the wireless set at Gh¢6,600 and sold to the retailer at 25%. Therefore, the selling price is:
S.P = $\frac{100 + \text{P\%}}{100}$ × C.P
S.P = $\frac{100 + 25}{100}$ × 6,600.00 = Gh¢8,250.00

iii. Discount = Marked price – New price
Marked price for the customer is Gh¢8,250.00 and the new price for the customer is Gh¢6,200.00
Discount = Gh¢(8,250 – 6,200) = Gh¢2,050.00
Rate = $\frac{Discount\ value}{Marked\ price}$ × 100%
Rate = $\frac{2050}{8,250}$ × 100% = 24.8%

<u>Method 2</u>
i. The manufacturer increased the cost, Gh¢6,000.00 by 10% and sold to the wholesaler. Therefore, the selling price or cost to the wholesaler;
$= 6,000 + \left(\frac{10}{100} \times 6,000\right)$
= 6,000 + 600 = Gh¢6,600.00

ii. The wholesaler increased the cost, Gh¢6,600 by 25% and sold it to the retailer.
The selling price of the wholesaler is;
$= 6,600 + \left(\frac{25}{100} \times 6,600\right)$
= 6,600 + 1,650 = Gh¢8,250.00

iii. Discount = Marked price – New price
Marked price for the customer is Gh¢8,250.00 and the new price for the customer is Gh¢6,200.00
Discount = Gh¢(8,250 – 6,200) = Gh¢2050.00

Rate = $\frac{Discount\ value}{Marked\ price}$ × 100%
Rate = $\frac{2050}{8,250}$ × 100% = 24.8%

4. A publisher prints 30,000 copies of an edition of a book. Each copy of the book costs the publisher Gh¢0.45 and it is sold to the public for Gh¢0.76. The publisher agrees to pay the author 10% of the selling price of the first 6,000 copies sold, and 12% of the selling price for all copies sold in excess of 6,000.00. Altogether, 25,000 copies of the books are sold. Calculate correct to the nearest cedis;
i. the total amount received by the author;
ii. the net profit the publisher makes after paying the author (Assume that the unsold copies are donated to various libraries);
iii. the authors total receipts as a percentage of the publishers net profit, correct to one decimal place.

Solution
i. S. P of a book = ¢0.76
S.P of first 6,000 books;
= Gh¢0.76 × 6,000 = Gh¢4,560.00

Authors share of the first 6,000 copies;
= 10% of the selling price
= $\frac{10}{100}$ × Gh¢4,560 = Gh¢456.00

Remaining copies of books;
= 25,000 – 6,000 = 19,000

S. P of a copy of the remaining copies;
= Gh¢0.76 per copy

S.P of the remaining 19,000 copies;
= Gh¢0.76 × 19,000 = Gh¢14,440.00

Authors share of the remaining 19,000 copies; = 12% of the selling price;
= $\frac{12}{100}$ × 14,440 = Gh¢1,732.80

Total amount received by the author;
= Gh¢(456.00 + 1,732.80) = Gh¢2,188.80

ii. Publishers share of the first 6,000 copies;
= Gh¢(4,560 – 456) = Gh¢4,104.00

Publishers share of the remaining 19,000 copies;
= Gh¢(14,440.00 – 1732.80) = Gh¢12,707.20

Publishers total share of the 25,000 copies;
= Gh¢(4,104.00 + 12,707.20) = Gh¢16,811.20

But cost of printing 30,000 copies at Gh¢0.45 each = Gh¢0.45 × 30,000 = Gh¢13,500.00

Net profit = Total share – Printing cost
= Gh¢(16,811.20 – 13,500.00)
= Gh¢3,311.20

iii. Authors total receipts as a percentage of the publisher's net profit;
= $\frac{Total\ receipts}{Net\ Profit}$ × 100%
= $\frac{2188.80}{3,311.20}$ × 100% = 66.1%

5. A trader buys 3,150 articles at a cost of Gh¢6.00 per article. He fixes the selling price so that, if only 3,000 articles are sold, he will make a profit of 40% on his total cost.
i. Calculate the selling price of one article
ii. If in fact, he sells 3,150 articles at this price, find the actual profit as a percentage of the total cost price
iii. If he had wanted 100% profit, how much should he have sold the 3,150 articles?

Solution
i. C.P. of the 3,150 articles at Gh¢6.00 each;
= Gh¢18,900

C.P. of the 3,000 articles at Gh¢6.00 each;
= 3,000 × Gh¢6.00 = Gh¢18,000.00

C.P. = Gh¢18,900.00, P% = 40%, S.P. = ?
S.P. = $\frac{100 + P}{100}$ × C.P.
⇒S.P. = $\frac{100 + 40}{100}$ × Gh¢18,900 = Gh¢26,460.00

Cost of 1 article = $\frac{Gh¢26,460}{3,000}$ = Gh¢8.82

ii. Selling price of 3,150 articles at Gh¢8.82;
= 3,150 × Gh¢8.82 = Gh¢27,783.00

P = S. P – C. P
P = Gh¢(27,783 – 18,900) = Gh¢8,883.00

The actual profit as a percentage of the total cost price = $\frac{8,883}{189,00}$ × 100 % = 47%

iii. C.P. = Gh¢18,900, P% = 100%, S.P = ?

S.P. = $\frac{100 + P}{100}$ × C.P

S.P. = $\frac{100 + 100}{100}$ × Gh¢18,900 = Gh¢37,800

6. A trader bought a radio for Gh¢68.00 and sold it at a profit of $7\frac{1}{2}$%.The following week, the cost price of the same type of radio increased by 8%. By what percentage, correct to two significant figures, must the trader increased his selling price in order to make the same profit.

Solution

C.P = Gh¢68.00

Profit =$7\frac{1}{2}$ % of Gh¢68.00

Profit = $\frac{7.5}{100}$ × Gh¢68 = Gh¢5.10

S.P. = P + C.P

S.P. = Gh¢5.10 + Gh¢68.00 .= Gh¢73.10

Increased in C.P(Gh¢68.00) by 8% ;

= 68 + $\left(\frac{8}{100} \times 68\right)$ = Gh¢73.44

S.P = P + C. P

S.P = Gh¢5.10 + Gh¢73.44 = Gh¢78.54

The new selling price = Gh¢78.54

Let *x* represent the percentage increase in the old S.P (Gh¢73.10) to the new S.P (Gh¢78.54)

73.10 + $\left(\frac{x}{100} \times 73.10\right)$ = 78.54

⇒ 73.10 + $\frac{73.10x}{100}$ = 78.54

⇒ 7310 + 73.10x = 7854

73.10x = 7854 – 7310

73.10x = 544

x = $\frac{544}{73.10}$ = 7.4% (2 s.f.)

She has to increase the selling price by 7.4%

Exercises 13.16

1. A bag of rice which cost Gh¢4,000.00 was sold at a profit of 20%. Find the selling price.

2. A trader bought 200 oranges at 8 for Gh¢10.00. At what price must she sell all of them to make a profit of 8%?

3. Find the selling price of goods bought at Gh¢2.80 and sold at :

a. a profit of 15% of the cost price,

b. a loss, of 15% of the cost price.

4. Maamele bought a car for Gh¢1,500.00. He later sold it for at a profit of 20%. What was the selling price?

5. A merchant buys two articles for Gh¢600.00. He sells one of them at a profit of 22% and the other at a loss of 8% and makes no profit or loss in the end. What is the selling price of the article that he sold at a loss?

B. 1. A radio set was sold for Gh¢ 250.00 at a profit of 25%, Find the cost price.

2. A car was sold for Gh¢ 44,080.00 at a loss of 12%. Find the cost price.

3. A man sold some articles for Gh¢ 18,000.00 and made a profit of 20%. Find the cost price of the articles.

4. A refridgerator was sold for Gh¢3,600.00 at a loss of $10\frac{1}{2}$%. Find the cost price.

Simple Interest

Definition of Terms

Simple Interest (I): When money is borrowed or deposited at the bank, for a certain period, it attracts an extra amount at a given rate. This extra money is called ***simple interest or interest (I)*** for short.

The actual money borrowed or deposited at the bank is called ***Principal (P).***

The period for which the money is borrowed or deposited at the bank is called ***time (T).*** The time can be in years, months, weeks or days.

The percentage at which the interest on the amount (money) borrowed or deposited is calculated is called the ***rate*** of interest.

Mathematically:

Interest = $\frac{Principal \times Time \times Rate}{100}$

Simply put, $\mathbf{I = \frac{PTR}{100}}$ ……….. (1)

From eqn(1),

I = $\frac{PTR}{100}$, *P*,*T* and *R* can be made the subject to get the following equations respectively**;**

$\mathbf{P = \frac{100I}{RT}}$, $\mathbf{T = \frac{100I}{PR}}$, $\mathbf{R = \frac{100I}{PT}}$

Amount (A)

Amount is the sum of principal and interest.

Amount = Principal + Interest

A = P + I

For repayment on instalment:

1. Monthly instalment = $\frac{Amount}{Number\ of\ months}$
2. Yearly instalment = $\frac{Amount}{Number\ of\ years}$

Worked Examples

1. Find the simple interest on Gh¢500.00 for 3 years at a rate of 5% per anum

Solution

P = Gh¢500.00, T = 3 years, R = 5 % I = ?

But, I = $\frac{PTR}{100} = \frac{500\times3\times5}{100}$ = Gh¢75. 00

2. A man deposited Gh¢1,520.00 at a bank for 6years at a rate of 10% p.a. Calculate the interestearned at the end of the period.

Solution

P = Gh¢1,520.00, T = 6yrs, R =10%, I =?

I = $\frac{PTR}{100} = \frac{1520\times6\times10}{100}$ = Gh¢ 912.00

3. Kusi borrowsGh¢24,000.00 from a bank. He pays after 2 years at a rate of 12% simple interest. Find the interest and the amount to be paid at the end of 2years.

Solution

P = Gh¢24,000, T = 2yrs, R = 12%, I =?

I = $\frac{PTR}{100} = \frac{24{,}000\times2\times12}{100}$ = Gh¢5,760.00

A = P + I,

But P = Gh¢24,000 and I = Gh¢5,760

A = Gh¢24,000 + Gh¢5,760 = Gh¢29,760.00

4. Kate borrowed Gh¢8,260.00 from a bank at a rate of 12% per anum simple interet.Calculate;

i.the interest at the end of 7 $\frac{1}{2}$ years.

ii. the amount repaid at the end of 7 $\frac{1}{2}$ years.

iii. the amount paid per month, if she agreed to repay the loan amount at equal monthly installment over the 7 $\frac{1}{2}$ year period.

Solution

P = Gh¢8,260, T = $7\frac{1}{2} = \frac{15}{2}$yrs, R = 12, I = ?

i. I = $\frac{PTR}{100} = \frac{8260\times15\times12}{100\times2}$= Gh¢7,434.00

ii. A = P + I

But P = Gh¢8,260 and I = Gh¢7,434

A = Gh₵8,260 + Gh₵7,434 = Gh₵15,694.00

iii. Monthly instalment = $\frac{Amount}{Number\ of\ month}$

Amount = Gh₵15,694.00

Number of months = $7\frac{1}{2}$ years = 90 months

Monthly Ins = $\frac{\text{Gh₵}15{,}694.00}{90\ months}$ = Gh₵174.38

5. Find the simple interest on Gh₵2,080.00 at $3\frac{1}{2}$ % per anum for 6months.

Solution

P = Gh₵2,080.00, R = $3\frac{1}{2}$ % = $\frac{7}{2}$ %

T = 6months = $\frac{1}{2}$ years and I = ?

I = $\frac{PTR}{100}$ = $\frac{2{,}080\times1\times100}{100\times2\times2}$ = Gh₵520.00

6. A trader was charged 2 pesewas per month for every Gh₵1.00 he borrowed from a bank.

i. At what percentage rate per anum was the interest charged?

ii. How much would the trader pay as interest on a loan of Gh₵5,000.00 for 6 months?

Solution

i. Per anum means by the year (12months)

⇒ P = 1.00 = 100p, I = 2p, T = 1 month

R = $\frac{100I}{PT}$ = $\frac{100\times2}{100\times1}$ = 2%

2% per month for 12 months,

= 2% × 12 = 24% per anum

ii. P = Gh₵5,000, I = ?, R = 24%,

T = 6 months = $\frac{1}{2}$ year

I = $\frac{PTR}{100}$ = $\frac{5{,}000\times{}^{1}/_{2}\times24}{100}$ = Gh₵600.00

Exercises 13.17

1. Find the simple interest on Gh₵500,000.00 for 5 years at a rate of 2% years at $5\frac{1}{2}$ % per anum.

2. Find the simple interest on Gh₵8,000.00 for $2\frac{1}{2}$ years at $5\frac{1}{2}$ %.

3. Find the interest on Gh₵480,000.00 for $5\frac{1}{2}$ years at $12\frac{1}{2}$ % per anum.

4. a. Find the total amount to be paid at the end of 4 years if Mrs. Diana borrows Gh₵31,520.00 at $22\frac{1}{2}$ % per anum with simple interest.

b. If she repays the loan at equal monthly instalment over the period, calculate the amount paid at the end of each month.

To Calculate for the Principal

To calculate for the rate,

I. Use the formula, **P =** $\frac{\mathbf{100I}}{\mathbf{TR}}$

II. Substitute the values of I, T and R, in the formula and simplify to obtain the value of P.

Worked Examples

1. The interest on a certain amount deposited at the bank for 4years at a rate of 5% p.a. is Gh₵1,600.00. How much money was deposited?

Solution

T = 4yrs, R = 5%, I = GH₵1,600.00, P = ?

P = $\frac{\mathbf{100I}}{\mathbf{TR}}$ = $\frac{1600\times100}{4\times5}$ = Gh₵8,000.00

2. What amount will yield an interest of Gh₵1,560.00 for 6 years at a rate of 20% p.a?

Solution

I = Gh₵1,560, T = 6yrs, R = 20%, P =?

P = $\frac{\mathbf{100I}}{\mathbf{TR}}$ = $\frac{1{,}560\times100}{20\times6}$ = Gh₵1,300.00

Exercises 13.18

1. What principal will yield an amount of Gh¢24,400.00 in 4 years at 20% per anum simple interest.

2. Find the principal that will earn a simple interest of Gh¢25,000.00 in 6 years at 17 $\frac{1}{2}$ % per anum.

3. A man deposited an amount of money in his savings account for 5 years. The rate of interest was 14% p.a. If the interest was Gh¢35.00, find the amount deposited.

4. Find the sum of money on which the simple interest for 8 years at 4 $\frac{1}{2}$ % per anum is Gh¢360.00.

To Calculate for Rate

To calculate for rate, use the formula:

R = $\boldsymbol{\frac{100I}{PT}}$

Note:

The rate is always expressed as a percentage.

Worked Examples

1. At what rate will Gh¢5,000.00 attract an interest of Gh¢300.00 for 2years?

Solution

P = Gh¢5,000, I = Gh¢300, T = 2 yrs and R = ?

R = $\frac{100I}{PT}$ = $\frac{300\times100}{5{,}000\times2}$ = 3%

2. The interest on Gh¢1,500.00 deposited at a bank for 3years was Gh¢90.00 Find the interest rate per anum.

Solution

T = 3years, I = Gh¢90, P = Gh¢1,500, R =?

R = $\frac{100\times I}{P\times T}$ = $\frac{90\times100}{1{,}500\times3}$ = 2%

3. The simple interest on Gh¢60,000.00 for 3 $\frac{3}{4}$ years is Gh¢5,625.00 Find the rate percent per anum.

Solution

P = Gh¢60,000.00, T = 3 $\frac{3}{4}$ years = $\frac{15}{4}$

I = Gh¢5,625.00, R = ?

R = $\frac{100\ I}{PT}$ = $\frac{100\times5{,}625}{60{,}000\times{}^{15}/_{4}}$ = $\frac{100\times5{,}625\times4}{60{,}000\times15}$ = 2.5%

Exercises 13.19

1. At what rate of interest will Gh¢8,000.00 yield a simple interest of Gh¢3,840.00 in 6 years.

2. A what rate of interest will Gh¢2,650.00 yield a simple interest of Gh¢4,770.00 in 3 years?

3. An amount of Gh¢2,500.00 invested for 5 years yielded a simple interest of Gh¢3,750.00.Find the rate of interest per anum.

To Calculate for Time

The time is calculated by the formula,

T = $\boldsymbol{\frac{100I}{PR}}$

Worked Examples

1. What time will Gh¢5,000.00 yield an interest of Gh¢1,000.00 at a rate of 5% p.a?

Solution

P = Gh¢5,000, I = Gh¢1,000,R = 5%, T = ?

T = $\frac{100I}{PR}$ = $\frac{100\times1{,}000}{5{,}000\times5}$ = 4 years

2. Calculate the time in which Gh¢6,000.00 will attract an interest of Gh¢1,200.00 at 20% per anum?

Solution

P = Gh¢6,000, I = Gh¢1,200, R = 20%, T=?

T = $\frac{100I}{PR} = \frac{100 \times 1,200}{6,000 \times 20}$ = 1 year

Exercises 13.20

1. How many years will Gh¢5,000.00 yield a simple interest of Gh¢1,000.00 at a rate of 5% per anum?

2. In how many years will Gh¢50,000.00 yield a simple interest of Gh¢100,000.00 at a rate of 5% per anum?

3. A man borrows Gh¢4,000.00 and at the end of each of the three following years he pays the lender Gh¢400.00; part of this is in payment of interest at 5% per anum on the amount of his debt during the year, and the rest reduces the debt. How much does he still owe at the end of the third year?

Solved Past Question

Atiamo borrows Gh¢900.00 from Oppong and the same amount from Asamoah to start a business. He agrees to pay Oppong 15% of the profit of the business each year. He also agrees to pay Asamoah 10% of the annual profits in addition to 3% interest on his loan each year. Find how much Oppong is paid in a year which Asamoah receives Gh¢48,000.00.

Solution

a. Let the profit for the year be x,
Amount borrowed from Oppong = Gh¢900.00
Amount borrowed from Asamoah = Gh¢900.00

Agreement with Oppong;
= 15% of the profit of the business each year

Agreement with Asamoah;
= 10% of the annual profits in addition to 3% interest on his loan each year

When Asamoah receives Gh¢48,000.00
⇒ 10% of x + 3% interest on Gh¢900
= Gh¢48,000
= $\frac{10}{100} \times x + I$ = Gh¢48,000
⇒$\frac{x}{10} + I$ = Gh¢48,000

But 3% interest on Gh¢900 for 1 year
P = 900, R = 3%, T = 1 year, I = ?
I = $\frac{PTR}{100} = \frac{900 \times 1 \times 3}{100}$ = Gh¢27.00

Put I = Gh¢27 in $\frac{x}{10}$ + I = Gh¢48,000
$\frac{x}{10}$ + Gh¢27 = Gh¢48,000
x + Gh¢270 = Gh¢480,000
x = Gh¢480,000 – Gh¢270 = Gh¢479,730.00

Oppong's payment = 15% of x

But x = Gh¢479,730

Oppong's payment = $\frac{15}{100}$ × Gh¢479,730
= Gh¢71,959.50

Ratio and Percentages Combined

Worked Examples

1. A school presents 200 candidates for examination. The ratio of boys to girls was 5 : 3 respectively.
i. How many boys were presented for the examination?
ii. If 24% of the girls and 28% of the boys passed with distinction, what percentage of the candidates obtained distinction?

Solution

i. Total number of candidates = 200
Ratio of boys to girls = 5 : 3
Total ratio = 5 + 3 = 8

Number of boys = $\frac{5}{8}$ × 200 = 125 boys

ii. Number of boys = 125

Number of girls = 200 – 125= 75girls

Girls who passed with distinction;
24% of the girls = $\frac{24}{100}$ × 75 = 18 girls

Boys who passed with distinction;
28% of the boys = $\frac{28}{100}$ × 125 = 35 boys

Students who passed with distinction;
= 18 + 35 = 53

Percentage of candidates who obtained distinction = $\frac{53}{200}$ × 100% = 26.5%

2. A soap factory finds that the cost of materials and labor to produce a certain brand of soap are in the ratio 3 : 5. The factory sells to the wholesaler at a profit of 27 $\frac{1}{2}$ % and the wholesaler sells to a retailer at a profit of 25%
a. If the retailer pays Gh¢1,530.00 for each box of soap, calculate how much it cost the factory in:
i. materials;
ii. labor to produce a box of soap;
b. the labor cost went up 20% (with no increase in the cost of materials), but the factory decided not to increases the price charged to the wholesaler. Calculate, correct to the nearest whole number, the new percentage profit the factory made.

Solution
a. The wholesaler's cost price;
S.P = Gh¢1,530, P% = 25% and C.P. = ?
C.P = $\frac{100}{100\,+\,p}$ × S.P

C.P = $\frac{100}{100\,+\,25}$ × Gh¢1,530 = Gh¢1,224.00

The factories cost price;
S.P = Gh¢1,224.00, P % = 27.5%, C.P. = ?
C.P = $\frac{100}{100\,+\,p}$ × S.P
C.P = $\frac{100}{100\,+\,27.5}$ × Gh¢1,224 = Gh¢960.00

Cost of materials : cost of labor = 3 : 5.
Total ratio = 3 + 5 = 8
i. Cost of materials = $\frac{3}{8}$ ×Gh¢960 = Gh¢360.00

ii. Cost of labor = $\frac{5}{8}$ × Gh¢960 = Gh¢600.00

b. 20% of labor cost (Gh¢600.00)
= $\frac{20}{100}$ × Gh¢600 = Gh¢120.00

The cost of labor;
= Gh¢600 + Gh¢120 = Gh¢720.00

New cost of materials and labor;
= Gh¢(360 + 720) = Gh¢1,080.00

New Profit for the factory;
S.P. = Gh¢1,224, C..P = Gh¢1,080
P = S.P – C.P
P = Gh¢1,224 – Gh¢1,080 = Gh¢144.00

P% = $\frac{144}{1,080}$ × 100 = 13.3%
The new percentage profit is13.3%

3. A manufacturer finds that the cost materials and labor to make a certain articles are in the ratio 3 : 5 respectively. The manufacturer sells to a retailer at a profit of 27 $\frac{1}{2}$ % and theretailer sells to a customer at a profit of 25%. The customer pays Gh¢6,375.00,

a. Calculate how much the article cost the manufacturer for materials and for labor.
b. If there is 20% rise in labor cost but no increases in the cost of materials and the manufacturer decides not to increase the price charged to the retailer, calculate the percentage profit which the manufacturer then makes.

Solution
a. The retailer's cost price;
S.P. = Gh¢6,375, P% = 25% and C.P. = ?
C.P. = $\frac{100}{100\ +\ p}$ × S.P.
C.P. = $\frac{100}{100\ +\ 25}$ × Gh¢6,375 .= Gh¢5,100.00

The manufacturer's cost price;
S.P. = Gh¢5,100.00, P % = 27.5%, C.P. = ?
C.P. = $\frac{100}{100\ +\ p}$ × S.P.
C.P. = $\frac{100}{100\ +\ 27.5}$ × Gh¢5,100 = Gh¢4,000.00

Cost of materials : cost of labor = 3 : 5.
Total ratio = 3 + 5 = 8
i. Cost of materials = $\frac{3}{8}$ × Gh¢4,000
= Gh¢1,500.00

ii. Cost of labor = $\frac{5}{8}$ × Gh¢4,000
= Gh¢2,500.00

b. 20% of labor cost (Gh¢2,500.00)
= $\frac{20}{100}$ × Gh¢2,500 = Gh¢500.00

The cost of labor;
= Gh¢2,500 + Gh¢500 = Gh¢3,000.00

New cost price of materials and labor;
= Gh¢(1,500 + 3,000) = Gh¢4,500.00

New Profit of the manufacturer;
S.P = Gh¢5,100, C.P = Gh¢4,500
P = S.P – C.P
P = Gh¢5,100 – Gh¢4,500 = Gh¢600.00

P% = $\frac{600}{4{,}500}$ × 100 = 13.3%
The factory makes a new percentage profit of 13.3%

4. The estimated cost of a house was Gh¢ 6,400.00. It was made up of the cost of labor, materials and the contractor's charge in the ratio 12 : 15 : 5 respectively. During construction, as a result of inflation, the cost of labor increased by *r* % and the cost of materials by 2*r*% while the contractors charge remains the same. If after the increase, the cost of labor was two – thirds of the cost of materials, find:
a. the value of *r*,
b. the new cost of the house.

Solution
a. Estimated cost = Gh¢6,400.00
Labor : materials : contractor' charge
12 : 15 : 5
Total ratio = 12 + 15 + 5 = 32
Labor charge = $\frac{12}{32}$ × Gh¢6,400
= Gh¢2,400.00
Materials charge = $\frac{15}{32}$ × Gh¢6,400
= Gh¢3,000.00
Contractors charge = $\frac{5}{32}$ × Gh¢6,400
= Gh¢1,000.00

Increase in labor by *r*%
= 2,400 + $\left(\frac{r}{100} \times 2{,}400\right)$
= 2,400 + 24*r*

Increase in materials by 2*r*% ;
= 3,000 + $\left(\frac{2r}{100} \times 3{,}000\right)$

= 3,000 + 60r

After the increase,

⇒ Cost of labor = $\frac{2}{3}$ (cost of materials)

2,400 + 24r = $\frac{2}{3}(3{,}000 + 60r)$

3(2,400 + 24r) = 2(3,000 + 60r)

7,200 + 72r = 6,000 + 120r

7,200 – 6,000 = 120r – 72r

1,200 = 48r

$r = \frac{1{,}200}{48}$ = 25%

b. The new cost of the house;

Labor charge = 2,400 + $\left(\frac{r}{100} \times 2400\right)$

= 2,400 + 24r

= 2,400 + 24(25)

= Gh¢3,000.00

Materials charge = 3,000 + $\left(\frac{2r}{100} \times 3{,}000\right)$

= 3,000 + 60r

= 3,000 + 60(25)

= Gh¢4,500.00

Contractors charge = Gh¢1,000.00

The new cost of the house ;

= Gh¢3,000 + Gh¢4,500 + Gh¢1,000

= Gh¢8,500.00

5. The price of an article is Gh¢6,000.00. It is made up as follows:

Cost of material = 20%,

Cost of manufacture = 50% and the rest is profit. If the cost of materials falls by 10% and the cost of manufacture increases by 10% but the price remains the same.

a. Find:

i. the new profit,

ii. the percentage change in profit.

b. find the new price of the article if the original profit is maintain.

Solution

a. Price of the article = 6,000.00

Cost of material = 20%,

Cost of manufacture = 50%

Profit = (100 – 20 – 50)% = 30%

Cost of materials = $\frac{20}{100}$ × Gh¢6,000

= Gh¢1,200.00

Cost of manufacture = $\frac{50}{100}$ × Gh¢6,000

= Gh¢3,000.00

Profit = $\frac{30}{100}$ × Gh¢6,000 = Gh¢1,800.00

i. 10% fall in the cost of materials;

= Gh¢1,200 –$\left(\frac{10}{100} \times \text{Gh¢}1{,}200\right)$

= Gh¢1,080.00

10% increase in the cost of manufacture;

= Gh¢3,000 +$\left(\frac{10}{100} \times \text{Gh¢}3{,}000\right)$

= Gh¢3,300.00

Material + Manufacture + Profit = Price

Gh¢1,080 + Gh¢3,300 + Profit = Gh¢6,000

Profit = Gh¢6,000 – Gh¢1,080 – Gh¢ 3,300

Profit = Gh¢1,620.00

Exercises 13.21

1. A manufacture sells an article to a retailer at a profit of 10%. The retailer in turn sells the article to a customer at a profit of 25%. If the customer pays Gh¢4,400.00, find the cost of manufacturing the article.

2. An article cost Gh¢x to produce and this is

divided between labor, materials and overheads in the ratio 5 : 4 : 1

a. Express in terms of x the cost of materials for making 1,000 articles.

Labor cost increased by 8% and materials by 20%, but overheads are reduced by 20% through increased production.

b. Write down an expression for the new cost per article.

c. Hence find the overall percentages increase in the cost of producing the articles.

Hire Purchase

It is the system by which the cost of goods or items is paid by installments, after initial deposit has been made. The deposit is usually a percentage of the total cost of the goods or items.

Note the following:

1. Deposits
= Percentage deposited × Cash price
$= \frac{x}{100} \times$ cash price

2. Balance = Cash price – Deposit

3. Total amount paid for goods
= Deposit + Total monthly installments

4. Interest
= Total monthly installments – Balance

5. When finding the rate of interest, average time is taken from the period where deposit is made and should be in years. This is because the first half of the average time is used to settle the principal/balance and the second half is used to settle the interest. The addition of one resultsfrom the fact that from the beginning of the first month, to the end of the last month, n is equal to$n + 1$. It is also explained as the month of deposit (1) + the months of instalment (n) = $n +$ 1. For example, average time for 6months equal instalments = $\frac{6+1}{2} = \frac{7}{2}$ months = 3.5 months
If 12 months = 1 year
⇒3.5 months = $\left(\frac{3.5}{12}\right)$years

6. Interest rate is calculated on the balance, using formula; $R = \frac{100I}{PT}$

Worked Examples

1. The cash price of a mobile phone is Gh¢1,800.00. If Mr. Brown pays 44% of the cash price as deposit and agrees to pay the rest in a monthly installment of Gh¢120.00 for 10 months, find;
i. the total amount he paid for the phone,
ii. the interest paid,
iii. the approximate interest rate.

Solution

Cash price = Gh¢1,800.00
Cash deposit = $\frac{44}{100}$ × Gh¢1800 = Gh¢792.00
Balance (P) = Cash price – cash deposited
= Gh¢1,800 – Gh¢792
= Gh¢1,008.00

Gh¢120 monthly installments for 10 months;
= Gh¢120 × 10 = Gh¢1,200.00

Total amount paid;
= Deposit + Total monthly installment
= Gh¢792 + Gh¢1,200 = Gh¢1,992.00

ii. Balance = Gh¢1,008.00
Total monthly installment = Gh¢1,200.00

Interest = Total monthly installments – Balance
Interest = Gh¢(1,200.00 – 1,008.00)
I = Gh¢192.00

Alternatively,

Interest = Total amount paid – cash price

Interest = Gh¢1,992.00 – Gh¢1,800.00

I = Gh¢192.00

iii. R = ?, I = Gh¢192, P = Gh¢1,008

Average time, T = $\frac{1+10}{2} = \frac{11}{2}$ months

If 12 months = 1 year

$\frac{11}{2}$ months = $\left(\frac{11}{2\times12}\right) = \left(\frac{11}{24}\right)$ years

From $I = \frac{PTR}{100}$,

$R = \frac{100I}{PT} = \frac{192 \times 100}{1008 \times {}^{11}/_{24}} = \frac{192 \times 100 \times 24}{1008 \times 11} = 41.5\%$

The approximate rate of interest is 41.5%

2. The cash price of a gas cooker was Gh¢60,000.00. A man paid 25% of the cash price as deposit. He then paid Gh¢8,165.00 a month for 6 months.

i. How much did he pay altogether for the cooker?

ii. Find the interest charged.

iii. Find the approximate rate of interest.

Solution

Cash price = Gh¢60,000.00

Cash deposit = 25% of Gh¢60,000

= $\frac{25}{100}$ × Gh¢60,000 = Gh¢15,000.00

Balance (P) = Cash price – cash deposit

= Gh¢(60,000 – 15,000)

= Gh¢45,000.00

Monthly installments of Gh¢8,165.00 for 6 months = 6 × Gh¢8,165 = Gh¢48,990.00

Total payment;

= Deposit + Total monthly installment

= Gh¢15,000 + Gh¢48,990 = Gh¢63,990.00

ii. Interest = Total payment – cash price

I = Gh¢63,990 – Gh¢60,000.00 = Gh¢3,990.00

iii. R = ?, I = Gh¢3,990, P = Gh¢45,000

Average time, T = $\frac{1+6}{2} = \frac{7}{2}$ months = $\left(\frac{3.5}{12}\right)$ years

$R = \frac{100I}{PT} = \frac{100 \times 3,990}{45,000 \times {}^{3.5}/_{12}} = 30.4\%$

The approximate rate of interest is 30.4%

3. A man bought a boy's bicycle costing Gh¢24,000 on hire purchase. He ended up paying 6 $\frac{1}{4}$% more than the cash price. If he made an initial deposit of 25% of the cash price and then paid the rest in six equal monthly installments, find;

i. the initial deposit,

ii. the approximate rate of interest,

iii. the amount of each installment.

Solution

i. Cash price = Gh¢24,000

Deposit = 25% of the cash price

= $\frac{25}{100}$ × Gh¢24,000 = Gh¢6,000.00

Balance (P) = Gh¢24,000 – Gh¢6,000

= Gh¢18,000.00

Total payment;

= 6 $\frac{1}{4}$ % more than the cash price

= Gh¢24,000 + $\left(\frac{25/4}{100} \times \text{Gh¢}24,000\right)$= Gh¢25,500.00

Interest = Total payment – Cash price

= Gh¢25,500 – Gh¢24,000 = Gh¢1,500.00

Balance (P) = Cash price – Cash deposit

= Gh¢24,000 – Gh¢6,000 = Gh¢18,000

P = Gh¢18,000, I = Gh¢1,500, R = ?

T = $\left(\frac{1+6}{2}\right)$ = 3.5 months = $\left(\frac{3.5}{12}\right)$ years

R = $\frac{100I}{PT} = \frac{100 \times 1{,}500}{18{,}000 \times {}^{3.5}/_{12}}$ = 28.6%

iii. Amount of each installment;

= $\frac{Amount}{Time\ allowed}$ = $\frac{Gh¢19{,}500}{6}$ = Gh¢3,250

Calculating the Monthly Installments at a Given Interest Rate per Anum

If after the initial deposit, the balance is paid at a given interest rate p. a. at a given monthly installments;

I. Identify the balance as the principal (P),identify the given rate (R%) andidentify the time (T) as the period of instalments in years.

II. Substitute the values of P, T (not the average) and R in $\boldsymbol{I = \frac{PTR}{100}}$ to determine the interest on the balance.

III. Find the sum of the principal and interest to obtain the amount paid during the period of installments. That is : A = P + I

IV. Divide the amount by the period of installment to obtain the monthly installments.

⇒Monthly installment = $\boldsymbol{\frac{(Principal + Interest)}{Time\ allowed}}$

V. Obtain the total payment by finding the sum of the initial depositand the total monthly installments. That is:

Total Payment = Deposit + Principal + Interest

VI. Determine the percentage cost of increase by the formula: ***P% =*** $\boldsymbol{\frac{I}{C.P}}$ ***× 100%***

Worked Examples

1. The conditions of hire purchase on the sale of a computer by Mr. Brown are as follows: initial payment of 25% of the cash price and the balance paid at 20% per anum in 12 months equal installment. If you wish to purchase this computer sold at a cash price of Gh¢1,480.00, calculate:

i. the amount you will pay at the end of every month.

ii.the total amount you will pay for the computer

iii. the percentage profit the owner will earn if he bought the computer for Gh¢1,500.00

Solution

Cash price of the computer = Gh¢1,480.00

Deposit = $\frac{25}{100}$ × Gh¢1480 = Gh¢370.00

Balance (P) = Cash price – Deposit

= Gh¢1,480 – Gh¢370 = Gh¢1,110.00

20% interest (I) on balance at 12months (1year), I = ?

P = Gh¢1,110, T = 12months = 1yr, R= 20%

I = $\frac{PTR}{100} = \frac{1110 \times 1 \times 20}{100}$ = Gh¢222.00

Amount to be paid in 12 months (total monthly installments)

A = P + I

A= Gh¢1,110 + Gh¢222 = Gh¢1,332.00

Amount to be paid every month (Monthly installments)

= $\frac{Amount}{Time\ allowed}$ = $\frac{Gh¢1{,}332}{12}$ = Gh¢111.00

ii. Total amount to be paid (Total payment)

= Deposit + Total monthly installment

= Gh¢370 + Gh¢1,332 = Gh¢1,702.00

iii. C.P = Gh¢1,500

S.P under hire purchase = Gh¢1,702.00

Profit (P) = S.P – C. P

P = Gh¢1,702 – Gh¢1,500 = Gh¢202.00

P% = $\frac{P}{C.P}$ × 100% = $\frac{202}{1{,}500}$ × 100% = 13.5%

2. Jones bought a car for Gh¢6, 800.00. He later put it for sale at Gh¢8,800.00. He agreed to sell it to Ruby under the following hire purchase terms: an initial payment of 20% of the price and the balance paid at 15% per simple interest per anum in twelve monthly equal installments. Calculate;
a. the amount paid at the end of every month.
b. the total amount Ruby paid for the car.
c. the percentage profit Jones made on the cost price of the car.

Solution
a. Cash price = Gh¢8,800.00
Deposit = 20% of Gh¢8,800
$= \frac{20}{100} \times 8,800 =$ Gh¢1,760.00

Balance = Cash Price – Deposit
= Gh¢8,800, – Gh¢1,760 = Gh¢7,040.00

15% interest on Balance at 12 months equal installments ;
P = Gh¢7,040, T = 1, R = 15%, I = ?
$I = \frac{PTR}{100} = \frac{7,040 \times 1 \times 15}{100} =$ Gh¢1,056.00

Amount paid in 12 months,
(Total monthly installments), A = P + I
A = Gh¢7,040.00 + Gh¢1,056.00 = Gh¢8,096.00

Amount paid at the end of every month
(Monthly Installments)
$= \frac{Amount}{Time\ allowed} = \frac{Gh¢8,096}{12} =$ Gh¢674.67

b. Total amount paid(Total payment)
= Deposit + Total monthly installment
= Gh¢(1,760.00 + 8,096.000) = Gh¢9,856.00

c. C.P = Gh¢6,800.00
S.P under hire purchase = Gh¢9,856.00
Profit (P) = S.P – C. P
P = Gh¢9,856 – Gh¢6,800 = Gh¢3,056.00

$P\% = \frac{P}{C.P} \times 100\% = \frac{3056}{6,800} \times 100\% = 44.9\%$

3. Mr. Amos bought a car for Gh¢2.5 million cedis. He paid 40% of the cost and paid the rest in equal monthly instalments. He took 8 years to make full payment for the car. Interest was charged at 18% simple interest. Calculate;
i. the monthly installments,
ii. the total amount he paid for the car,
iii. the percentage increase in the cost of the car.

Solution
Cash price of the car = Gh¢2,500,000.00
Deposit = 40% of Gh¢2,500,000.00
$= \frac{40}{100} \times$ Gh¢2,500,000 = Gh¢1,000,000.00

Balance (P) = Cash price – Deposit
= Gh¢2,500.000 – Gh¢1,000,000
= Gh¢1,500,000.00

18% interest (I) on balance for 8 years;
P = Gh¢1,500,000,T = 8years,R = 18%
$I = \frac{PTR}{100} = \frac{1,500,000 \times 8 \times 18}{100} =$ Gh¢2,160,000.00

Total monthly installments;
A = P + I
A= Gh¢1,500,000 + Gh¢2,160,000
A = Gh¢3,660,000.00

8 years = 96 months
Monthly installments for 8 years;
$\frac{Gh¢3,660,000}{96} =$ Gh¢38,125.00

ii. Total amount paid(Total payment);
= Deposit + Total monthly installment
= Gh¢1,000,000 + Gh¢3,660,000
= Gh¢4,660,000.00

iii. C.P. = Gh¢2,500,000.00

S.P under hire purchase = Gh¢4,660,000

Profit (P) = S.P – C. P

P = Gh¢4,660,000 – Gh¢2,500,000

P = Gh¢2,160,000.00

P% = $\frac{P}{C.P}$ × 100% = $\frac{2,160,000}{2,500,000}$ × 100% = 86.4%

4. Mr. White bought a plot of land whose cash price was Gh¢45,500.00. He made an initial deposit of 18% of the cash price and paid the rest at 30% rate of interest per annum in 4 months equal installment. Calculate:

i. the monthly installments,

ii. the total amount he paid for theland,

iii. the percentage increase in the cost of the land.

Solution

Cash price of the land = Gh¢45,500.00

Deposit = 18% of Gh¢45,500.00

= $\frac{18}{100}$ × Gh¢45,500 = Gh¢8190.00

Balance (P) = Cash price – Deposit

= Gh¢45,500 – Gh¢8,190 = Gh¢37,310.00

30% rate of interest (I) charged on balance for 4 months

P = Gh¢37,310, T = 4 months = $\left(\frac{4}{12}\right)$ yrs R = 30%

I = $\frac{PTR}{100}$ = $\frac{37,310 \times 4 \times 30}{100 \times 12}$= Gh¢3,731.00

Total monthly instalments;

A = P + I

A = Gh¢37,310 + Gh¢3,731= Gh¢41,041.00

Monthly installments for 4 months;

$\frac{\text{Gh¢}41,041}{4}$ = Gh¢10,260.25

ii. Total amount paid (Total payment);

= Deposit + Total monthly installment

= Gh¢8,190 + Gh¢41,041.00 = Gh¢49,231.00

iii. C.P = Gh¢45,500.00

S.P under hire purchase = Gh¢49,231.00

Profit (P) = S.P – C. P

P = Gh¢45,500.00 – Gh¢49,231.00 = Gh¢3,731

P% = $\frac{P}{C.P}$ × 100% = $\frac{3,731}{45,500}$ × 100% = 8.2%

Exercises 13.22

1. The cash price of a DSTV decoder is Gh¢20,000.00. If a lady pays 25% of the cash price as deposit and agrees to pay Gh¢3,000.00 a month a for 6 months, find;

i. the total amount she paid for the decoder,

ii. the interest paid.

iii. the approximate interest rate.

2. The proprietor of a certain boarding school purchased some student mattresses for which the total cash price was Gh¢90,000.00. He made an initial deposit of 33% of the cash price and agreed to pay the rest in monthly installments of Gh¢6,200.00 for 10 months. Find the approximate rate of interest.

3. The cash price of a VW saloon car was Gh¢14,000.00. If Mr. Brown paid 50% of the cash price as deposit and then agreed to pay Gh¢1,700.00 every month for 6 months.

i. How much did he pay altogether?

ii. Find the interest charged.

iii. Find the approximate rate of interest.

4. How much interest is paid on Gh¢4,500.00 borrowed from a financial institution and repaid in 30 months installment of Gh¢220.00 per month?

5. Mr. Okraku bought a car at Gh¢8,800.00 and sold it later to Mr. Kwantabisa, for a cash price of Gh¢10,200.00. If Mr. Kwantabisa paid $33\frac{1}{2}$ % of the price and paid the balance at 20% simple interest per anum in 6 monthly equal installments, calculate;

i. the amount paid every month,

ii. the total amount paid by Mr. Kwantabisa,

iii. the percentage profit Mr. Okraku made on the cost price of the car.

6. Mr. Dadzie agreed to buy a car that cost Gh¢25,000.00. He paid 40% of the cost price of the car and agreed to pay the rest at an interest of 18% per anum for 12 months. Calculate;

i. the monthly installment,

ii. the total amount he paid for the car,

iii. the percentage increase in the cost of the car.

7. A woman buys furniture for which the cash price is Gh¢180,000.00. She pays 33 $\frac{1}{3}$ % cash deposit and agrees to pay Gh¢13,500.00 a month for 10 months. Find the approximate rate of interest.

8. A company was granted a loan of Gh¢750,000.00. The loan was to be repaid in four quarterly monthly installments of Gh¢220,000.00 starting three months after the loan has been granted. Find the approximate rate of interest.

9. The conditions of hire purchase on the sale of a house are as follows: initial payment of 45% of the cash price and the remaining paid at 25% per anum in 6 months equal installment. If the cash price of the land is Gh¢78,480.00, calculate:

i. the charge per month,

ii. the selling price of the house,

iii. the percentage profit charged on the cost of the house.

10. A man buys a house at Gh¢20,000.00. He pays 30% of the cost out of his own resources and takes a loan for the remainder at 2 $\frac{1}{2}$ % simple interest per annum. Calculate;

i. the total amount the man pays for the house if he takes 8 years to settle the loan ans: Gh¢22,800.00

ii. the percentage increase in the cost of the house to the man as a result of the loan.

iii. his gain percent if, after settling the loan he renovates the house at a cost of Gh¢2,200.00 and then sells it for Gh¢33,000.00

11. Mr. Ababio bought a refrigerator costing Gh¢160,000.00 from a consumer credit union on hire purchase. He made an initial deposit of Gh¢100,000.00 and agreed to pay the rest in 15 monthly installments of Gh¢5,000.00 each. What is the approximate rate of interest charged?

14 MODULO ARITHMETIC

Defining Modulo

Modular arithmetic, also known as ***remainder arithmetic*** is a type of arithmetic in which an integer is represented by its remainder when divided by another integer. In other words, the modulo of any integer is found by dividing the integer by the given modulo and the remainder identified as the answer.

Modulo is written as ***"mod"*** for short. For instance, 23 (mod 7) means the remainder when 23 is divided by 7. That is:

1. $23 \div 7 = 3 \times 7 + 2 = 2$
2. $23 \div 7 = 3$ R $2 = 2$ OR
3. $23 - 7 - 7 - 7 = 2$.

Similarly, 35 (mod 8) could fit any of the following:

1. $35 \div 8 = 4 \times 8 + 3 = 3$
2. $35 \div 8 = 4\frac{3}{8} = 3$
3. $35 - 8 - 8 - 8 - 8 = 3$

Note that a particular interest is attached to the value of the remainder and not the quotient. Thus, 23(mod 7) = 2 and 35 (mod 8) = 3

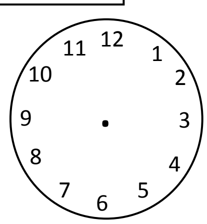
Linear and Cyclic Variables

1. The Analogy of the Clock

The numbers displayed on the analog clock representing the hours are:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

These aredisplayed on the face of the clock below:



Assuming the time is 8 O' clock now. In the next 7hours,which means moving in the clockwise direction 7 hours from 8 (i.e. 8hr + 7hr), the time will be 3 O' clock instead of 15 O' clock.This isexplained as: 8hr + 7hr = 15hr – 12hr = 3hrs. Since the numbers maximize at 12hours and begin the cycle, any sum greater than 12 produces an answer which is the difference between the sum and 12, conditioned to be less than 12, otherwise continue tosubtract 12, till a number less than 12 is obtained

Similarly, if it is 8 O' clock now, 9 hours ago, the time was 11 O'clock, explained asmoving in the anti – clockwise direction 9 hours from 8 (8hrs – 9 hrs.), the answer conditioned to be a positive number less than 12, otherwise, add 12 successfully until the first positive number is obtained. i.e. 8hrs – 9 hrs= -1 + 12 = 11 hrs

Thus, on theanalogue clock,the hands of the clock pass over the numbers, wrap up at 12, and thereafter, begin the cycle again. The clock therefore operates under modulo 12. The difference is that, 12 is used in place of 0.

The fact that one can move back on the face of the clock to determine a previous time and also move forward to pre – determine the time after a given hours, the numbers are said to be ***linear variables.*** The fact that the numbers cycle around themselves andespecially around a maximum point (12), makes them ***cyclic variables***

Worked Examples

1. If it is 9 O' clock now, what will be the time after the following hours?

i. 5hours ii.11hours iii. 23 hours

Solution

i. 9hrs + 5hrs = 14hrs – 12hrs = 2 O'clock

ii. 9hrs + 11hrs = 20hrs – 12hrs = 8 O'clock

iii. 9hrs + 23hrs = 32hrs – 12hrs – 12hrs
= 8 O'clock

2. If it is 4 O' clock now, what was the time the following hours before 4 O'clock?

i. 13 hours ii. 27 hours iii. 35 hours

Solution

i. 4hrs – 13 hrs = - 9hrs + 12 = 3 O' clock

ii. 4hrs – 27hrs = -23hrs + 12 + 12 = 1 O' clock

iii. 4hrs – 35hrs = -31 + 12 + 12 + 12 = 5 O'clock

Exercises 14.1

A. Assuming the time is 6 O' clock now, what will be the time in the following hours

1.7 hours 2.9 hours 3.18 hours

4.26 hour 5.37 hours 6. 44 hours

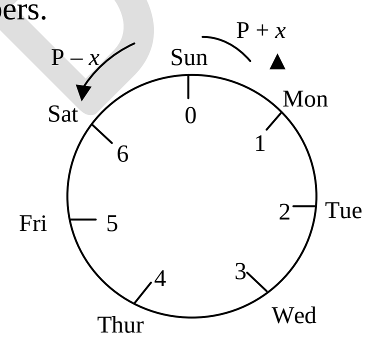
B. If it is 10 O' clock now, what was the time the following hours before 10O'clock?

1. 14 hours 2.25 hours 3. 52 hours

2. Analyzing the Days of the Week

Let the days of the week namely: Sunday, Monday, Tuesday, Wednesday, Thursday, Friday and Saturday be represented by the numbers 0, 1, 2, 3, 4, 5, 6 respectively.

Below is a circular representation of the days and their numbers.



If P = {0, 1, 2, 3, 4, 5, 6}, then:

I. x number of days after P is determined as P + x. For e.g. 5 days after Wednesday is determined as 3(Wed) + 5 = 8 – 7 = 1 represented by Monday

II. x number of days before P is determined as P – x. For e.g. 4 days before Wednesdayis determined as 3(Wed) – 4 = -1 + 7 = 6 representing Saturday

III. Because P can move in both directions, it is called a ***linear variable***and because P moves in a definite cycle, it is called a ***cyclic variable***

Worked Examples

1. What day of the week is it 6 days after Saturday?

Solution

6 days after Saturday;

= 6 (sat) + 6 = 12 – 7 = 5(Fri)

6 days after Saturday is Friday

2. What day is 28 days after Wednesday?

Solution

28 days after Wednesday;

= 3 (Wed) + 28 = 31 – 7 – 7 – 7 – 7 = 3(Wed)

28 days after Wednesday is Wednesday

3. What day was 17 days before Friday?

Solution

17 days before Friday;

= 5 (Fri) – 17 = -12 + 7 + 7 = 2 (Tue)

17 days before Friday was Tues

Exercises 14.2

A. Find the modulus in each of these;

1.The system of the days of the week

2. The system of the month of the year

B. If today is Tuesday, what day will it be in the next:

1. 16 days 2.39 days 3. 73days

C. If today is Friday, what was the day, the following days before Friday;

1. 17 days 2. 33 days 3. 71 days

3. ***Analizing the Months of the Year***

Below is the table of the list of months of the year and the number of days;

Month	Days	Month	Days
Jan	31	July	31
Feb	28/29 (leap yr)	Aug	31
Mar	31	Sept	30
Apr	30	Oct	31
May	31	Nov	30
June	30	Dec	31

Assign numbers to the months as shown below:

Jan (1)	Feb (2)	Mar (3)	Apr (4)
May (5)	Jun (6)	Jul (7)	Aug (8)
Sept (9)	Oct (10)	Nov (11)	Dec (0)

Similar to the number of days, we move forward (clockwise direction) to determine a month ahead (after a given month) and backward (anticlockwise direction) to determine past months (before a given month). For example,

1. 9 months after September is determined as:

Sept (9) + 9

= 18 (mod 12) = 18 – 12 = 6 (representing June)

2. 25 months before May is determine as :

May (5) – 25

= -20 (mod 12) = - 20 + 12 + 12 = 4 (representing April)

Worked Examples

1. What month was it 88 months, before October?

Solution

Oct (10) + 88 = 98 (mod 12)

= 98- 12 – 12 -12 – 12 – 12 – 12 – 12 – 12

= 2 (Feb)

Alternatively;

Oct (10) + 88 = 98 (mod 12)

$98 \div 12 = 8\frac{2}{12}$

R = 2 (Feb)

2. What month is it 94 months after February?

Solution

Feb (2) + 94 = 96 (mod 12) = 0 (December)

3. In a particular year, August 11 was a Tuesday. What day was 3rd October the same year?

Solution

Tue (2) + 20 days of Aug + 30 days of Sept + 3 days of Oct

= 55 mod 7 = 6 (Representing saturday)

4. In a particular leap year, 19th March, was a Friday. What day was it December 21st the previous year.

Solution

Fri (5) – 19 days of Mar – 29 days of Feb – 31 days of Jan – 10 days of Dec

= - 84 (mod 12)

= - 84 + 12 + 12 + 12 + 12 + 12 + 12 + 12

= 0 (Representing Sunday)

5. The days of a non leap year are numbered in the scale of seven: i.e. 5th Jan is day 5, 8th Jan is day 11, 28th Jan is day 40.

i. What number is given to:

a. 20th Feb   b. 31st Dec?

ii. If 1st January is a Tuesday, what day of the week is:

a. day 52   b. day 142?

Solution

i. a. 5th Jan is day 5, $\Rightarrow$5(base 7) = 0 R 5 = 5

8th Jan is day 11, $\Rightarrow$8 (base 7) = 1 R1 = 11

28th Jan is day 40, $\Rightarrow$28 (base 7) 4 R 0 = 40

$\Rightarrow$ 20th Feb

= 28 days (Jan) + 3 days(Jan) + 20 days (Feb)

= 51 days

= 51 (base 7) = 7 R 2 = 72

But the scale is 7 $\Rightarrow$ 7 = 10

Therefore, 72 in scale 7 = 102.

The number given to 20th Feb is 102

b.31stDec

= 28 days (Jan) + 3 days (Jan) + 28 days (Feb) + 31 (Mar) + 30 (Apr) + 31 (may) + 30 (Jun) + 31 (Jul) + 31(Aug) + 30(Sept) + 31(Oct) + 30 (Nov) + 31 (Dec)

= 365days

= 365 (base 7)

= 52 (base 7) R 1

= 7 R 3 R 1

= 731

But the scale is 7 $\Rightarrow$ 7 = 10

So 731 = 1031

The number given to 31st Dec is 1031

ii. a. If 1st Jan is a Tues,

$\Rightarrow$ 1(mod 7) = 0 R 1 = 1

Tuesday = 1

Day 52

= 1(Tue) + (5 × 7) + 2 = 38(mod 7) = 3 (Wed)

b. Day 142

= 1 (Tue) + (1 × 7 × 7) + (4 × 7) + 2

= 1 (Tue) + 49 + 28 + 2 = 80 (mod 7) = 3 (wed)

Exercises 14.2B

1. The month is now May. What month will it be in;   a. 33 months time?   b. 119 months time?

2. What was the month, the following months before the given months?

a. 41 months before November?

b. 103 months before June?

3. In a certain leap year, April 3 was a Monday. What day was the following days;

a. 15th March of the same year;

b. January 24th , the same year;

c. November 1, the previous year;

d. June 14, the same year.

4. Jenifer was born on Friday 15th February, 2012 which happened to be a leap year.

a. If Martha was born on 29th July 2013, on which day of the week was she born?

b. If Joe was born 73 days after Martha, on which day was he born?

c. If Linda was born on 1st October, 2011, find the day on which she was born.

d. If someone is born on 10th May 2020, find the exact day in which the birthday falls

5. The days of a non leap year are numbered in the scale of seven: i.e. 5th Jan is day 5, 8th Jan is day 11, 28th Jan is day 40.

i. What number is given to:

a. 3rd March   b. 16th Oct?

ii. If 1st January is a Friday, what day of the week is: a. day 60 b. day 100?

Other Applications

Modulo is applicable to situations where an activity occurs at regular intervals. For example, organization of market days and the days at which a group of people take turns to engage in a daily or timely activities.

I. Identify the particular activity, whether market days or duty days.

II. Represent the days of the week as follows: Sun = 0, Mon= 1, Tue = 2, Wed = 3, Thur = 4, Fri = 5, Sat = 6

III. Make a "sequence of days" on which the activity repeats or happens.

IV. Represent the days of the week on the face of the clock and match according to the sequence.

V. Use the "sequence of days" to answer all questions.

Worked Examples

1. Four security officers (Nii, Noi, Ayittey and Odame) take turns towatch a factory. Nii is on duty on a Monday this week.

i. When will he be on duty the fourth time?

ii. When will his next four duties fall?

iii. How many weeks will it be before he is next on duty on a Monday?

iv. How many days will he be on duty on Saturday after Monday?

Solution

i. Let Sun = 0, Mon = 1, Tue = 2, Wed = 3, Thu = 4, Fri = 5, Sat = 6

Sequences of days on duty from Monday

1, 5, 2, 6, 3, 0, 4, 1

Alternatively,

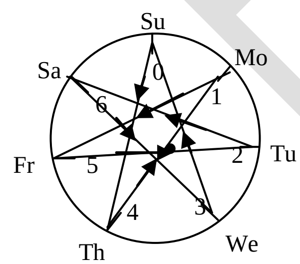
The sequence of days can also be generated as follows:

$1 + 4 (\text{mod } 7) = 5 \rightarrow 5 + 4 (\text{mod } 7) = 2 \rightarrow$
$2 + 4 (\text{mod } 7) = 6 \rightarrow 6 + 4 (\text{mod } 7) = 3 \rightarrow$
$3 + 4 (\text{mod } 7) = 0 \rightarrow 0 + 4 (\text{mod } 7) = 4 \rightarrow$
$4 + 4 (\text{mod } 7) = 1$

Sequences of days on duty from Monday

1, 5, 2, 6, 3, 0, 4, 1

Diagram of the sequence



From the sequence, his fourth duty will be on Saturday (6).

ii. Counting four times after Monday (1) his next four duties will fall on Wednesday(3)

iii. From the sequence,
Difference between the days = 4 days
Monday (1) to Monday (1) = 7 duties
Number of days he will be on duty on Monday
= 4 days × 7 = 28 days

Number of Weeks;
If 7 days = 1 week,
28 days = 4 weeks

iv. Difference between the days = 4 days
Monday (1) to Saturday (6) = 3 duties
Number of days he will be on duty on Saturday after Monday = 4days × 3 = 12 days

2. At a certain village, market days are organized every ten days. If a market day falls on Tuesday;

i. on which day will the fourth market day fall?
ii. how many days will it take a market day to fall on Tuesday again?

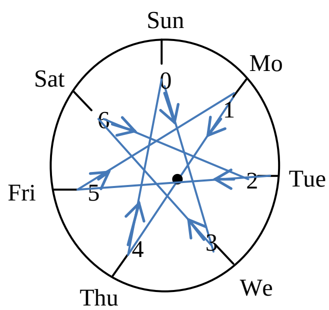
Solution

i. Let Sun = 0, Mon = 1, Tue = 2, Wed = 3, Thu = 4, Fri = 5, Sat = 6

Sequence of his duty day from Tuesday

2, 5, 1, 4, 0, 3, 6, 2

Diagram of the sequence



i. From the sequence, the fourth market day will fall on Thursday (4)

ii. From the sequence,
Difference between the days = 10 days
Tuesday (2) to Tuesday (2) = 7 market days
Number of days a market day will be on Tuesday
= 10 days × 7 = 70 days

3. At a transport station, a bus leaves every 2 hours. If the first bus leaves at 5: 00 am, when will the 10^{th} bus leave?

Solution

5 + (2 × 9) mod 12
= 5 + 18 (mod 12) = 23 (mod 12) = 11: 00 pm

Exercises 14.3

1. A train regularly leaves a station at every 3 hours. If the first train leaves at4 : 00 pm, at what time will the eighth train leave?

2. A town has a market day everynine days. If a market day falls on Tuesday this week, find how many days it will be before a market day falls on;
i. Thursday ii. Wednesday iii. Monday

3. There are five traffic officers at a station. One traffic officer takes turn to be on duty every day of the week. Mawusi, who is one of the traffic officers, is on duty on Friday this week;
i. what will be his next day on duty?
ii. how many weeks will it be before Mawusi, is on duty on a Friday again?

4. In a village, market days are held every four days. This week, a market day falls on a Monday.
i. List the sequence of market days.
ii. When will the next market day fall after tuesday?
iii.When will the next fivemarket days fall after monday?
iv. When will the tenth market day fall after monday?
v. How many days will it take a market day to fall on Monday again?

5. In a particular year, August 16 was a Wednesday. On which day of the week was October 11, the same year?

6. In Boateng's nursery, some seeds were sown on Saturday. The seedlings were transplanted 23 days later. If the plants were pruned after 86 days, on which day of the week were they pruned?

The Modulo Equation

Take $\frac{20}{3} = 6\,\frac{2}{3}$ for instance. This is expressed in modulo arithmetic as 20 (mod 3) = 2 (mod 3). But the ***mod*** is supposed to be at only one side of the equation. Therefore, the statement: 20 (mod

3) = 2 (mod 3) can be simplified as shown below, any of which is acceptable:

I. 20 (mod 3) = 2: keeping the mod at the left side of the equation

II. 20 = 2 (mod 3): keeping the mod at the right side of the equation. These are called ***equivalent modulo***

Integers for a Given Modulo

Take for instance, the division of any integer by 4. It is seen that the possible remainders, when any integer is divided by 4 are 0, 1, 2, 3. Thus, the set of integers (remainders), representing division of integers by 4 is reduced to {0, 1, 2, 3}. The divisor, (4) in this case, is called the ***modulus*** and the arithmetic is said to be in ***modulo 4.*** Modulo 4 can therefore take one of the values {0, 1, 2, 3}. Likewise, modulo 6 can take one of the values {0, 1, 2, 3, 4, 5}

In general, modulo *n* can take the values **0** to (***n* – 1**)

Exercises 14.4

Write out the set of integers for each of the following modular system:

1. mod 7 2. mod 8 3. mod 9 4. mod 11

Hints for Finding the Modulo

I. If the dividendis less than the divisor (modulo number), the dividend remains the answer. For e.g. 5(mod 7) = 5, 3(mod 8) = 3 *etc*

II. Modulo 5 of all numbers ending in 0, 1, 2, 3, 4 is the last or ending digit. e.g. 12(mod 5) = 2, 170(mod 5) = 0, 1094(mod 5) = 4 etc

III. Modulo 5 of all numbers ending in5, 6, 7, 8, 9 is found by subtracting 5 from the last or ending digit. e.g. 19(mod 5) = 9 – 5 = 4. Therefore, 19(mod 5) = 4

The Modulo of a Negative Number

For all negative numbers, the modulo is determined by addingthe modulo number successfully until a positive number is obtained as answer.

Worked Examples

Simplify each of the following in the modulo given.

1.-8(mod 3) = 3. - 61 (mod 9) =
2. -38 (mod12) = 4. -117 (mod 36) =

Solution

1. – 8(mod 3) = -8 + 3 + 3+ 3 = 1
-8(mod 3) = 1

2. -38 (mod12)
= -38 + 12 + 12 + 12 + 12 = 10.
-38 (mod12) = 10

3. -61 (mod 9)
= -61 + 9+ 9 + 9+ 9+ 9 + 9 + 9+ 9 = 2
-61 (mod 9) = 2

4. -117 (mod 36)
= -117 + 36 + 36 + 36 + 36 = 27
-117 (mod 36) = 27

Exercises 14.5

A. Simplify:

1. -68 (mod 7) =.	5. -57(mod 13) =
2. -17 (mod 4) =	6. -88(mod 15) =
3. -33 (mod 11) =	7. -221(mod36) =
4. -105 (mod 44) =	8. -175(mod 30) =

Equivalent or Congruent Moduli

1. In modulo arithmetic, two or more integers are said to be equivalent if they give the same remainder under the same mod. For e.g,

23(mod 7) = 2 and 16(mod 7) = 2 are said to be equivalent written as:
23(mod 7) ≡ 16(mod 7) = 2

2. Similarly, in 8(mod 5) = **3**, the equivalent is written as8 ≡ 3(mod 5)

Worked Examples
1. Simplify the following:
i. 14 (mod 4)   ii. 20 (mod 8)

Solution
i. 14 (mod 4) = 14÷4 = $3\frac{2}{4}$ = 2
  14 ≡ 2 (mod 4)

ii. 20 (mod 8) = 20 ÷ 8 = $2\frac{4}{7}$ = 4
  20 ≡ 4 (mod 8)

2. Determine whether the following pairs of numbers are equivalent in the moduli given
i. 29 and 77(mod 6)
ii. 19 and 30 (mod 7)

Solution
i. 29(mod 6) = 29 ÷ 6 = $4\frac{5}{6}$ = 5
77(mod 6) = 77÷ 6 = $12\frac{5}{6}$ = 5
29(mod 6) = 77(mod 6) = 5

Therefore 29 and 77(mod 6) are equivalent

ii. 19(mod 7) = 19 ÷ 7 = $2\frac{5}{7}$ = 5
30(mod 7) = 30 ÷ 7 = $4\frac{6}{7}$ = 6
19(mod 7) ≠ 30 (mod 7)
Therefore 19(mod 7) ≠ 30 (mod 7)
are not equivalent

Exercises 14.6

A. Which of the following pairs of numbers are equivalent in the modulo specified
1. 17 and 31 (mod 7)   2. 19 and 25 (mod 6)
3. 67 and 87 (mod 9)   4. 42 and 63 (mod 8)
5. 44 and 80 (mod 9)   6. 57 and 96 (mod 13)

Modulo Operations (Arithmetic)
(*Addition, Subtraction, Multiplication and Division)*
The act of performing the operations: Addition, Subtraction, Multiplication and Division involving modulo are called ***Simplification of modulo arithmetic***.

To simplify modulo arithmetic:
I. Perform the operation (+, - , ×, ÷)
II. Convert the answer to the given modulo

Worked Examples
Simplify the following:
1. 14 + 4 (mod 4) =   5. 13 × 33(mod 5)   =
2. 13 + 33(mod 8) =   6. 300 ÷ 12 (mod 11) =
3. 26 – 9 (mod 6) =   7. 78 ÷2(mod 4)   =
4. 12 – 34 (mod 8) =

Solution
1. 14 + 4 (mod 4) = 18(mod 4) = 2 or
= 2 (mod 4)
2. 13 + 33(mod 8) = 46 (mod 8) = 6 or
= 6 (mod 8)
3. 26 – 9 (mod 6) = 17 (mod 6) = 5 or
= 5(mod 6)
4. 12 – 34 (mod 8) = -22(mod 8)
= -22 + 8 + 8 + 8
= 2 or = 2 (mod 8)
5. 23 × 2 (mod 9) = 46(mod 9) = 1 or
= 1 (mod 9)
6. 13 × 33(mod 5) = 429 (mod 5) = 4 or
= 4 (mod 5)
7. 300 ÷ 12 (mod 11) = 25(mod 11) = 3 or

$= 3(\text{mod } 11)$

8. $78 \div 2(\text{mod } 4) = 39(\text{mod } 4) = 3$ or

$= 3\ (\text{mod } 4)$

Exercises 14.7

A. Simplify the following:

1. 57(mod 7) 2. 94(mod 8) 3. 66(mod 14)
4. 32(mod 12) 5. 98(mod 15) 6. 72(mod 17)

B. Simplify in the modulo given;

1. -8 + 11 (mod 13) =
2. -50 + 99 (mod 5) =
3. 120 + 77 (mod 23) =
4. 331 + 111 (mod 35) =
5. 168 + 133 (mod 16) =
6. 14 + 63 + 28(mod 8) =

C. Find the following products:

1. 13 × 11 (mod 7) = 4. 89 × 37 (mod 55) =
2. 64 × 17 (mod 28) = 5. 27 × 19 (mod 26) =
3. 37 × 32 (mod 42) = 6. 29 × 39 (mod 22) =

The Modulo Number in a Given Equation

When the modulo number is represented by a variable as in $2 = 12\ (\text{mod } x)$, the value of the variable is found by going through the following process:

I. Find the difference between the numbers.
II. Find all the factors of the difference identified.
III. Substitute each factor of the difference in the modulostatement provided and identify the one that makes the statement true as the value of the variable.

Note:

The answer obtained from the difference satisfies the statement and therefore is a truth set of the variable under no given conditions

Worked examples

1. If $2 = 12\ (\text{mod } x)$, find the value of x

Solution

$2 = 12\ (\text{mod } x)$,
I. Difference between the numbers
$12 - 2 = 10$
II. Factors of 10 = {1, 2, 5, 10}
III. *By substitution*,
$0 = 12(\text{mod } 1)$, $x \neq 1$
$0 = 12(\text{mod } 2)$, $x \neq 2$
$2 = 12(\text{mod } 5)$, $x = 5$
$2 = 12(\text{mod } 10)$, $x = 10$
By comparison,
$12(\text{mod } x) = 12\ (\text{mod } 5) = 12\ (\text{mod } 10)$.
$\therefore x = 5$ or $x = 10$

2. Find the value of x in $24(\text{mod } x) = 3$

Solution

Difference between the numbers
$24 - 3 = 21$
Factors of 21 = {1, 3, 7, 21}
By substituting the factors,
$24(\text{mod } 1) = 0$, $x \neq 1$
$24(\text{mod } 3) = 0$, $x \neq 3$
$24(\text{mod } 7) = 3$, $x = 7$
$24(\text{mod } 21) = 3$, $x = 21$
$24(\text{mod } x) = 24(\text{mod } 7) = 24\ (\text{mod } 21) = 3$.
$\therefore x = 7$ or $x = 21$

3. Determine the modulo in which $6 + 8 = 4$ was performed.

Solution

Let the mod in which the operation was performed be x
$6 + 8\ (\text{mod } x) = 4$
$14(\text{mod } x) = 4$
x is a factor of $14 - 4 = 10$
Factors of 10 = {1, 2, 5, 10}

By substitution,

14 (mod 1) = 0, $x \neq 1$

14(mod 2) = 0, $x \neq 2$

14(mod 5) = 4, $x = 5$

14 (mod 10) = 4, $x = 10$

$\Rightarrow x = 5$ or $x = 10$

Therefore, the operation was performed in modulo 5 or modulo 10.

4. Perform $50 \div 2 = 7(\text{mod } n)$, where $1 \leq n \leq 18$

Solution

Under the condition $1 \leq n \leq 18$, $n = 18$

$50 \div 2 = 7(\text{mod } n)$

$25 = 7(\text{mod } n) \equiv 25(\text{mod } n) = 7$

$25 - 7 = 18$

Factors of 18 = 1, 2, 3, 6, 9, 18

When $n = 1$, 25(mod 1) = 0, $n \neq 1$

When $n = 2$, 25(mod 2) = 3, $n \neq 2$

When $n = 3$, 25(mod 3) = 4, $n \neq 3$

When $n = 6$, 25(mod 6) = 1, $n \neq 6$

When $n = 9$, 25(mod 9) = 7, $n = 9$

Under the given condition, $x : x = 9$

Exercises 14.8

Determine the modulo:

1. $13 = 3 \text{ (mod } x)$ 2. $25 = 4(\text{mod } x)$
3. $54 = 10(\text{mod } x)$ 4. $42 = 12(\text{mod } x)$
5. $38 = 5(\text{mod } x)$ 6. $60 = 15(\text{mod } x)$

Finding the Number under Modular Operation

The number whose modulo is being operated may be represented by a variable as in $5x = 2(\text{mod } 4)$

Take note of the fact that modular n uses the numbers 0 to $(n - 1)$. For instance modulo 6 uses the numbers 0, 1, 2,3,4,5.

By substitution, the values of 0 to $(n - 1)$ that makes the statement true is the value of the variable.

Again, in the statement $5x = 2$ (mod 4) for instance, considering the R.H.S. of the equation, the modulo number, 4 is greater than 2, and this gives the impression that the divisor is 4 as usual, the remainder is 2, and the dividend is $5x$ at the L. H. S. The statement $5x = 2$ (mod 4) can therefore be written as $5x$ (mod 4) = 2.

Worked Examples

1. If $7x = 3$ (mod 6), find x

Solution

Method 1

In mod 6, x can take one of the values of 0, 1, 2, 3, 4, 5. By substitution, investigate to get the value of x that satisfies $7x = 3$mod 6

When $x = 0$, 7(0) = 0 (mod 6)

When $x = 1$, 7(1) = 7(mod 6) = 1

When $x = 2$ 7(2) = 14(mod 6) = 2

When $x = 3$ **7(3) = 21(mod 6) = 3**

When $x = 4$ 7(4) = 28(mod 6) = 4

When $x = 5$ 7(5) = 35(mod 6) = 5

$\therefore x = 3$, satisfies $7x = 3$(mod 6)

Method 2

The statement $7x = 3$ (mod 6) can also be written as $7x$ (mod 6) = 3

Mod 6 = {0, 1, 2, 3, 4, 5}

when x = 0

7(0) mod 6 = 0 (mod 6) = 0

when x = 1,

7(1) mod 6 = 7(mod 6) = 1

when x = 2,

7(2) mod 6 = 14(mod 6) = 2

When x = 3,

7(3) mod 6 = 21(mod 6) = 3

when x = 4,

7(4) mod 6 = 28(mod 6) = 4

when x = 5,

7(5) mod 6 = 35(mod 6) = 5

$\therefore x = 3$, satisfies $7x \pmod 6 = 3$

2. Solve$x + 9 = 3 \pmod 5$

Solution
$x + 9 = 3 \pmod 5$ can be written as
$x + 9 \pmod 5 = 3$
mod 5 = {0, 1, 2, 3, 4},
Substitute the values {0, 1, 2, 3, 4}, in
$x + 9 \pmod 5 = 3$

When $x = 0$,
$0 + 9 \pmod 5 = 9 \pmod 5 = 4$
When $x = 1$,
$1 + 9 \pmod 5 = 10 \pmod 5 = 0$
When $x = 2$,
$2 + 9 \pmod 5 = 11 \pmod 5 = 1$
When $x = 3$,
$3 + 9 \pmod 5 = 12 \pmod 5 = 2$

When $x = 4$,
$4 + 9 \pmod 5 = 13 \pmod 5 = 3$
$\therefore x = 4$, satisfies $x + 9 \pmod 5 = 3$

3. Find the value of x in $5^2 + x^2 = 2 \pmod 4$

Solution
$5^2 + x^2 = 2 \pmod 4$ can also be written as
$5^2 + x^2 \pmod 4 = 2$
mod 4 = { 0, 1, 2, 3}
When $x = 0$,
$5^2 + (0)^2 \pmod 4 = 25 \pmod 4 = 1$
When $x = 1$,
$5^2 + (1)^2 \pmod 4 = 26 \pmod 4 = 2$
When $x = 2$,
$5^2 + (2)^2 \pmod 4 = 29 \pmod 4 = 1$
When $x = 3$,
$5^2 + (3)^2 \pmod 4 = 34 \pmod 4 = 2$
Therefore $x = 1$ and $x = 3$ satisfies the statement $5^2 + x^2 \text{mod } 4 = 2$

5. Find the truth set of $34 = x \pmod 6$

Solution
The statement $34 = x \pmod 6$ can be written as $34 \pmod 6 = x$.
But $34 \pmod 6 = 4$
$\therefore x = 3$

Exercises 14.9
A. Find the solution set in each case;

1. $2x = 2 \pmod 4$
2. $2x + 1 = 1 \pmod 3$
3. $x^2 = 4 \pmod 6$
4. $x^2 + 1 = 0 \pmod 5$
5. $3x = 1 \pmod 4$
6. $3x + 4 = 5 \pmod 7$
7. $x^3 = 3 \pmod 5$
8. $x^2 - 5 = 4 \pmod 8$

B. Find the values of the variables

1. $7 \times 3 = y \pmod 8$
2. $6 \times 5 = y \pmod 7$
3. $3 \times 3 = y \pmod 5$
4. $5 \times 5 = y \pmod 6$
5. $4 \times 3 = y \pmod 5$

C. Find the solution set of the following;

1. $6 + x = 2 \pmod 8$
2. $8 + x = 3 \pmod{12}$
3. $4 + x = 3 \pmod{12}$
4. $4 + x = 2 \pmod 6$
5. $x + 5 = 3 \pmod 7$

Modulo ArithmeticTables
Modular arithmetic tables are usually constructed under addition $\oplus$ and multiplication $\otimes$ for a given modulo.

Given modulo$n$, the table is constructed using the numbers 0 to $(n - 1)$. Thus, for modulo 5, the set of values are {0,1, 2, 3, 4} and for modulo 6, the range of values are 0 , 1, 2, 3, 4, 5. For large modulo, a set of values may be given for operation.

On the table, the range of values for the mod or the 0 to $(n - 1)$ values occupy the first column and the first row, whilst the operator, whether +

or × is circled to look like ⊕ or ⊗ respectively, and placed at the left top corner of the table as shown below:

⊕	0		
0			

⊗	0		
0			

Thereafter;

I. Perform the operation,

II. Divide the answer by the mod

III. Record only the remainder in the boxes or cells to complete the table

Worked Examples

1. Write out the replacement set or elements of the following modulo

a) Modulo 2 b) modulo 5 c) modulo 8

Solution

1. a. Mod 2 = {0, 1}

b. Mod 5 = {0, 1, 2, 3, 4}

c. Mod 8 = {0, 1, 2, 3, 4, 5, 6, 7}

2. Draw an addition table for arithmetic modulo 6. Use your table to find;

i. (2⊕5) ⊕ (5⊕2) ii. 4 ⊕4

ii. If 4 ⊕k = 0, find the value of k

Solution

⊕	**0**	**1**	**2**	**3**	**4**	**5**
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

i. From the table:

(2⊕5) = 1 and (5⊕2) = 1

(2⊕5) ⊕(5⊕ 2) = 1 ⊕ 1 = 2

ii. 4 ⊕4 = 2

iii. 4 ⊕ k = 0

From the table,

When k = 0, 4 ⊕ 0 = 4

When k = 1, 4 ⊕ 1 = 5

When k = 2, 4 ⊕ 2 = 0

When k = 3, 4 ⊕ 3= 1

When k = 4, 4 ⊕ 4 = 2

When k = 5, 4 ⊕ 5 = 3

4 ⊕ k = 4 ⊕ 2 = 0

⇒k = 2

The truth set is { k : k = 2}

2. Construct a table for multiplication ⊗ modulo 6 for the set S = {1, 2, 3, 4, 5.}

Use the table to find:

i. 4 ⊗ 2, ii. (3 ⊗1) ⊗(5⊗ 2),

iii. (1 ⊗4) ⊗ (5⊗5) ⊗(2⊗4),

iv. the truth set of m⊗m = m.

Solution

⊗	**1**	**2**	**3**	**4**	**5**
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

From the table:

i. 4 ⊗2 = 2

ii. (3 ⊗1) = 3 and (5⊗2) = 4

Therefore (3 ⊗ 1) ⊗(5⊗2) = 3 ⊗4 = 0

iii. $(1 \otimes 4) = 4$, $(5\otimes 5) = 1$ and $(2\otimes 4) = 2$

Therefore $(1\otimes 4) \otimes (5\otimes 5) \otimes (2\otimes 4)$

$= (4 \otimes 1 \otimes 2)$

$= 4\otimes 1 = 4$

$= 4\otimes 2 = 2$

iv. $m\otimes m = m$

From the table,

When $m = 1$, $1 \otimes 1 = 1$

When $m = 2$, $2 \otimes 2 = 4$

When $m = 3$, $3 \otimes 3 = 3$

When $m = 4$, $4 \otimes 4 = 4$

When $m = 5$,

$5 \otimes 5 = 1$, $1 \otimes 1 = 1$, $3 \otimes 3 = 3$ and $4\otimes 4 = 4$

$\Rightarrow m = 3$ or $m = 4$

Truth set = $\{m : m = 1, 3 \text{ or } m = 4\}$

3. Draw a multiplication table in modulo 6 and use your table to find the truth set following equations;

i. $x\otimes x$, ii. $y\otimes (y + 3) = y + 2$,

Solution

⊗	**0**	**1**	**2**	**3**	**4**	**5**
0	**0**	**0**	**0**	**0**	**0**	**0**
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

ii. $y\otimes (y + 3) = y + 2$

From the table,

When $y = 0$

$0 \otimes (0 + 3) = 0 + 2$

$0 \otimes 3 \neq 2$

When $y = 1$

$1 \otimes (1 + 3) = 1 + 2$

$1 \otimes 4 \neq 3$

When $y = 2$

$2 \otimes (2 + 3) = 2 + 2$

$2 \otimes 5 = 4$

When $y > 2$, the statement is undefined.

Therefore $y = 2$, satisfies the statement.

4. a. Draw the multiplication $\otimes$ table for arithmetic modulo 7.

b. Using the table,

i. state with reasons, whether or not the operation is commutative,

ii. evaluate $(4 \otimes 6) \otimes (5\otimes 4)$,

iii. find the truth set of $n\otimes n = n$.

Solution

⊗	**0**	**1**	**2**	**3**	**4**	**5**	**6**
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

i. ***Method 1***

⊗	**0**	**1**	**2**	**3**	**4**	**5**	**6**
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

The table is symmetrical about the leading diagonal. Therefore, the operation $\otimes$ is commutative.

Method 2

i. From the table,

$2 \otimes 3 = 6$ and $3 \otimes 2 = 6$

$2 \otimes 3 = 3 \otimes 2 = 6$

Therefore the operation $\otimes$ is commutative .

ii. From the table,

$(4 \otimes 6) \otimes (5 \otimes 4)$

$= 3 \otimes 4 = 5$

iii. $n \otimes n = n$

From the table,

$0 \otimes 0 = 0$, $1 \otimes 1 = 1$, $2 \otimes 2 = 4$,

$3 \otimes 3 = 2$, $4 \otimes 4 = 2$, $5 \otimes 5 = 4$

$6 \otimes 6 = 1$,

Therefore, $n = 0$ or $n = 1$

Truth set = $\{ n : n = 0 \text{ or } n = 1\}$

5. a. Draw an addition and multiplication tables for the set K = {3, 6, 9, 12} in arithmetic modulo 15.

b. From your table:

i. evaluate $(3 \otimes 3) \oplus (6 \otimes 9)$,

ii. find the truth set of $n \otimes (n \oplus 3) = 3$.

Solution

a.

$\oplus$	**3**	**6**	**9**	**12**
3	6	9	12	0
6	9	12	0	3
9	12	0	3	6
12	0	3	6	9

$\otimes$	**3**	**6**	**9**	**12**
3	9	3	12	6
6	3	6	9	12
9	12	9	6	3
12	6	12	3	9

b. i. $(3 \otimes 3) \oplus (6 \otimes 9) = 9 \oplus 9 = 3$

ii. $n \otimes (n \oplus 3) = 3$

When $n = 3$

$3 \otimes (3 \oplus 3) = 3 \otimes 6 = 3$

When $n = 6$

$6 \otimes (6 \oplus 3) = 6 \otimes 9 = 9$

When $n = 9$

$9 \otimes (9 \oplus 3) = 9 \otimes 12 = 3$

When $n = 12$

$12 \otimes (12 \oplus 3) = 12 \otimes 0 =$ Not defined

$\Rightarrow n \otimes (n \oplus 3) = 9 \otimes (9 \oplus 3) = 3$

Therefore $n = 3$ or $n = 4$

Exercises 14.10

1. i. Construct a table for multiplication $\otimes$modulo 9 on the set P = {2, 3, 4, 5,7}

ii. Find the truth set of $x \otimes x = 4$, from the table, where $x \in$ P

2. i. Make an addition and multiplication table for 'clock arithmetic' with a clock showing the numbers 0, 1, 2, 3, 4; e.g. $2 + 3 = 0$, $4 \times 3 = 2$

ii. If x is a variable on {0, 1, 2, 3, 4}, with the table, solve;

a. $x + 3 = 0$ b. $x + 2 = 1$

c. $3x + 2 = 4$ d. $4x + 3 = 1$

3. a. Draw a table for multiplication, $\otimes$, modulo 7 on the set M = {2, 3, 4, 5, 6}

b. Use your table to find on the set M, the truth set of $r \otimes (r \otimes 6) = 3$

4. a. Draw a table for multiplication $\otimes$ in modulo 12 on the set {1, 5, 7, 11}

b. Using the table;

i. state with reasons whether or not the operation $\otimes$ is commutative,

ii. evaluate $5 \otimes (7 \otimes 11)$,

iii. find the truth set of $n \otimes n = 1$.

5. a. Draw addition $\oplus$ and multiplication $\otimes$ tables on the set R = {2, 5, 7, 11} in arithmetic modulo 12

b) From your table:

i) evaluate $(5 \otimes 7) \oplus (7 \otimes 11)$,

ii) find r if $r \otimes (r \oplus 2) = 11$.

6. i. Copy and complete the following table for multiplication modulo 7 on the set $\{1, 2, 3, 6\}$

$\otimes$	1	2	3	6
1				
2				
3				
6				

ii. Use the table to find the truth set of $n \otimes n \otimes n = n$

7. i. Copy and complete the table below for multiplication modulo 5 on the set {1, 2, 3, 4}

$\otimes$	1	2	3	4
1				
2				
3			4	
4				

ii. Use the table to find the truth set of $3 \otimes n = 1$

8. Copy and complete the following table for addition$\oplus$ and multiplication$\otimes$ mod 5

$\otimes$	1	2	3	4
1				
2				
3				
4				

$\oplus$	1	2	3	4
1				
2				
3				
4				

b) From the tables, find:

i) $(2 \otimes 4) \oplus 4$ ii) $(3 \oplus 4) \otimes 2$

9. a. Copy and complete the addition $\oplus$ and multiplication$\otimes$ tables for arithmetic modulo 7

$\oplus$	1	2	3	4	5	6
1	2	3	4	5	6	0
2	3		5	6	0	
3	4		6	0		2
4	5		0		2	
5	6		1	2	3	4
6		1		3		5

$\otimes$	1	2	3	4	5	6
1	1			4	5	6
2	2		6	1	3	
3	3	6	2	5	1	4
4		1			6	
5	5	3	1		4	
6		5		3	2	1

b. Use your tables to solve for x;

i. $x \oplus 3 = 6$ ii. $3 \otimes x = 1$ iii. $(6 \oplus x) \otimes 4 = 2$

Modulo and Binary Combined

To solve problems involving a combination of binary operation and modulo arithmetic:

I. Identify the binary operation and the given modulo.

II. Perform the binary operation and convert the answer to the given modulo.

III. Complete the table of values if any, with the answers obtained from the conversion.

Worked Examples

The operation * is defined by $a * b = ab + 2$ in arithmetic modulo 5

a. Draw a table for * on the set {1, 2, 3, 4}

b. Use the table;

i. evaluate $(2 * 1) + (2 * 3)$

ii. find the truth set of $n * (n * 3) = 3$

Solution

a.

*	1	2	3	4
1	3	4	0	1
2	4	1	3	0
3	0	3	1	4
4	1	0	4	3

b. From the table

i. (2 * 1) = 4 and (2 * 3) = 3

Therefore (2 * 1) + (2 * 3) = 4 + 3 = 7

But 7 = 2(mod 5)

ii. n * (n * 3) = 3

From the table,

When n = 1,

1 * (1 * 3) = 1 * 0 (Not defined on the set)

When n = 2,

2 * (2 * 3) = 2 * 3 = 3

When n = 3,

3 * (3 * 3) = 3 * 1 = 0

When n = 4,

4 * (4 * 3) = 4 * 4 = 3

$\Rightarrow n * (n * 3) = 2 * (2 * 3) = 4 * (4 * 3) = 3$

Therefore n = 2 or n = 4

2. An operation * is defined by $m * n = mn + 2$, in arithmetic modulo 7.

i. Copy and complete the table for the operation * on the set {1, 3, 5, 6}

*	1	3	5	6
1	3	5	0	1
3	5	4		
5	0			
6		6	4	3

ii. From the table in (i), find the truth set of:

a. 3 * n = 3 b. m * m = 4

Solution

i. $m * n = mn + 2$, in arithmetic modulo 7

3 * 5 = (3)(5) + 2 = 17(mod 7) = 3

3 * 6 = (3)(6) + 2 = 20 (mod 7) = 6

5 * 3 = (5)(3) + 2 = 17(mod 7) = 3

5* 5 = (5)(5) + 2 = 27(mod 7) = 6

5 * 6 = (5)(6) + 2 = 32(mod 7) = 4

6 * 1 = (6)(1) + 2 = 8(mod 7) = 1

*	1	3	5	6
1	3	5	0	1
3	5	4	3	6
5	0	3	6	4
6	1	6	4	3

ii. From the table,

a. 3 * n = 3 b. m * m = 4

3 * 5 = 3 3 * 3 = 4

$\Rightarrow n = 5$ $\Rightarrow m = 3$

Solved Past Question

1. The operation* is defined on the set {2, 4, 6} by $m * n$ = the unit digit in the product mn.

a. Copy and complete the table.

*	2	4	6
2	4	8	2
4		6	
6			

b. Use your table to solve the equations:

i) x * 4 = 8 ii) $e * e = e$ iii) (4 * f) * 4 = f

Solution

i. $m * n$ = the unit digit in the product mn.

4* 2 = 8, 4* 6 = 4, 6* 2 = 2,

6* 4 = 4, 6* 6 = 6

*	2	4	6
2	4	8	2
4	8	6	4
6	2	4	6

b. i. x * 4 =8

$2 * 4 = 8$
$\Rightarrow x = 2$

ii. $e * e = e$
$6 * 6 = 6$
$\Rightarrow e = 6$

iii. $(4 * f) * 4 = f$
when $f = 4$, $(4 * 4) * 4$
$= 6 * 4 = 4$

When $f = 6$, $(4 * 6) * 4$
$= 4 * 4 = 6$
$\{ f : f = 4 \text{ or } f = 6\}$

Exercises 14.11

1. The binary operation * is defined by the statement $a * b = 2ab + 1$ in arithmetic modulo 6

a. Use this to copy and complete the table below for *on the set {1, 2, 3, 4, 5}

*	1	2	3	4	5
1	3	5	1	3	5
2					
3	1	1		1	1
4			5		
5	5		1	5	3

b. Use the table to evaluate;

i. 3 * (3 * 5)

ii. (2 * 4) * (2 * 5)

iii. the truth set of $(1 * n) * n = 3$

iv. why will you say the operation * is commutative?

2. The operation * is defined by $a * b = a(b + 2)$ in arithmetic modulo 8.

a. Prepare a table for * on the set P = {1, 3, 5, 7}

b. From the table;

i. evaluate (3 * 5) * 7

ii. find the truth set of $n^2 = n$

3. The table below defines the operation ◙ on the set X = {*a*, *b*, *c*, *d*}

◙	*a*	*b*	*c*	*d*
a	*b*	*d*	*a*	*c*
b	*d*	*c*	*b*	*a*
c	*a*	*b*	*c*	*d*
d	*c*	*a*	*d*	*b*

a. Find giving reason(s) whether the following are correct:

i. X is closed with respect to ◙,

ii. there is an identity element.

b. Find the where possible, the inverse of the elements *a*, *b*, *c* and *d*

c. Find *p* and *q* if:

i. (*d* ◙ *a*) ◙ (*c* ◙ *b*) = *p*,

ii. (*q* ◙ *a*) ◙ (*c* ◙ *d*) = *b*.

15 INDICES AND LOGARITHMS

Positive Integral Indices

The task of multiplying a number by itself, a certain number of times gives rise to the idea of powers of numbers. For example, $2 \times 2 \times 2 \times 2$ can be explained as 2 multiplied by itself 4 times. This can be written in short form as: $2 \times 2 \times 2 \times 2 = 2^4$

In the expression, 2^4, 2 is called the ***base***and 4 is called the ***index or exponent*** and it is read as ***"two exponent four"***.

Generally, in an expression of the form a^m,*a* is called the ***base*** and ***m*** is called the ***index or exponent or power*** and it is read as ***"a exponent m"*** or ***"a to the power of m"***.

Laws of indices

1.	$a^m \times a^n = a^{m+n}$	5.	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
2.	$\frac{a^m}{a^n} = a^{m-n}$	6.	$a^0 = 1$
3.	$(a^m)^n = a^{mn}$	7.	$a^{-m} = \frac{1}{a^m}$
4.	$(ab)^m = a^m b^m$	8.	$a^{m/n} = \left(a^{1/n}\right)^m$

Application of the Laws of Indices

1. To simplify an exponential expresson means, leave the answer in exponent form
2. To evaluate an exponential expression means to find the value of the expression.

In dealing with complex exponential expressions, apply the laws of indices one after the other. Also:

I. Arrange the numbers (base) in ascending order.
II. Align the base when the problem involves division.

Worked Examples

1. Simplify $\frac{5^7 \times 3^5}{3^2 \times 5^2}$

Solution

Re-arrange the terms to have

$\frac{3^5 \times 5^7}{3^2 \times 5^2} = 3^{5-2} \times 5^{7-2} = 3^3 \times 5^5$

2. Evaluate $\frac{5^7 \times 3^5}{3^2 \times 5^2}$

Solution

$\frac{3^5 \times 5^7}{3^2 \times 5^2} = 3^{5-2} \times 5^{7-2} = 3^3 \times 5^5$

$= 27 \times 3125 = 84{,}375$

3. Simplify, $\frac{9^{10} \times 2^8}{2^0 \times 9^9}$

Solution

$\frac{9^{10} \times 2^8}{2^0 \times 9^9} = \frac{2^8 \times 9^{10}}{2^0 \times 9^9} = 2^{8-0} \times 9^{10-9} = 2^8 \times 9$

4. Evaluate $\frac{9^{10} \times 2^8}{2^0 \times 9^9}$

Solution

$\frac{9^{10} \times 2^8}{2^0 \times 9^9} = \frac{2^8 \times 9^{10}}{2^0 \times 9^9} = 2^{8-0} \times 9^{10-9}$

$= 2^8 \times 9 = 256 \times 9 = 2{,}304$

5. Simplify $\frac{7^7 \times 4^{12}}{4^8 \times 7^5}$

Solution

$\frac{7^7 \times 4^{12}}{4^8 \times 7^5} = \frac{7^7 \times 4^{12}}{7^5 \times 4^8} = 7^{7-5} \times 4^{12-8} = 7^2 \times 4^4$

6. $\frac{5^{11}\times 8^2\times 5^6}{8^9\times 5^5}$

Solution

$\frac{5^{11}\times 8^2\times 5^6}{8^9\times 5^5} = \frac{5^{11}\times 5^6\times 8^2}{5^5\times 8^9} = \frac{5^{17}\times 8^2}{5^5\times 8^9} = 5^{12}\times 8^{-7}$

Exercises 15.1

A. Simplify the following;

1. $\frac{5^7\times 3^2\times 5^2}{3^7\times 3^{-6}\times 5^8}$ 2. $\frac{m^2 n}{3m}$ 3. $\frac{27a^7}{3a^3}$

4. $\frac{2^{10}\times 3^2}{3^6\times 2^8}$ 5. $\frac{8x^{12}y^6}{4x^4y^3}$ 6. $\frac{(2d^2)(5ad^2)}{4d^3}$

B. Evaluate the following;

1. $\frac{6^2}{2^2\times 3}$ 2. $\frac{2^3\times 3^4\times 3^3}{2^2\times 2\times 3^5}$

3. $\frac{2^7\times 3^4\times 5^3}{2^3\times 3^2\times 5^2}$,   4. $(3^6\div 3^4)\times 27$

5. $(2^6\times 3^4)\div(2^4\times 3^2)$

Zero Power of a Natural Number

Study the pattern below carefully with reference to the division rule:

$\boldsymbol{a^m\div a^n = a^{m-n}}$

$2^4 = 2\times 2\times 2\times 2$

$\frac{2^4}{2} = \frac{2\times2\times2\times2}{2} = 2^3 = 2\times 2\times 2$

$\frac{2^3}{2} = \frac{1\times2\times2\times2}{2} = 2^2 = 2\times 2$

$\frac{2^2}{2} = \frac{2\times2}{2} = 2^1 = 2$

$\frac{2}{2} = \frac{2}{2} = 2^0 = 1$

Since the value of $2^1\div 2^1 = 2^0 = 1$, it follows that $a^n\div a^n = a^0$, so $a^0 = 1$ ($a\neq 0$)

Worked Examples

Simplify the following;

1. $\frac{12^7\times 12^2}{12^9}$

Solution

$\frac{12^7\times 12^2}{12^9} = \frac{12^9}{12^9} = 12^{9-9} = 12^0 = 1$

2. 4×69^0

Solution

$4\times 69^0 = 4\times 1 = 4$

Exercises 15.2

Evaluate each of the following:

1. $\frac{2^3\times 2^5}{2^7}$ 2. $\frac{5^7\times 3^8}{3^4\times 5^4\times 5^3\times 3^8}$

3. $\frac{7^5\times 7^7\times 7^2}{7^8\times 7^6}$ 4. $(8^{11}\times 8^0\times 8)\div 8^8$

5. Show that; $(13^6\times 13^4)\div(13^5\times 13^5) = 1$

Rational Indices

By the law of indices,

$4^{1/2}\times 4^{1/2} = 4$…………(1)

$2\times 2 = 4$…………………(2)

Comparing eqn (1) and eqn (2)

$4^{1/2} = 2$

But $\sqrt{4} = 2$. Therefore, we define $4^{1/2}$ as the square root of 4 written as $\sqrt{4} = 4^{1/2}$

In general, $\boldsymbol{\sqrt{a} = a^{1/2}}$

For e.g. $\sqrt{9} = 9^{1/2} = (3^2)^{1/2} = 3$

Similarly, $a^{1/3}\times a^{1/3}\times a^{1/3} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a$

We define $a^{1/3}$ as the cube root of a written as; $\boldsymbol{\sqrt[3]{a} = a^{1/3}}$

In general, $\boldsymbol{\sqrt[n]{a} = a^{1/n}}$

For e.g. $\sqrt[3]{27} = 27^{1/3} = (3^3)^{1/3} = 3$

Generally, if $\boldsymbol{a^{m/n}}$ is taken to be the *nth* root of $\boldsymbol{a^m}$, then $\boldsymbol{a^{m/n} = \sqrt[n]{a^m}}$

For e.g. $\sqrt{25}^3 = 25^{3/2} = (5^2)^{3/2} = 5^3$

Worked Examples

1. Evaluate $9^{3/2}$

Solution

$9^{3/2} = (3^2)^{3/2} = 3^3 = 3 \times 3 \times 3 = 27$

2. Evaluate $\left(\frac{1}{4}\right)^{3/2}$

Solution

$(1/4)^{3/2} = \frac{(1)^{3/2}}{(4)^{3/2}} = \frac{(1)^{3/2}}{(2^2)^{3/2}} = \frac{1}{2^3} = \frac{1}{8}$

3. Simplify $(4 \times 49)^{\frac{1}{2}}$

Solution

$(4 \times 49)^{\frac{1}{2}} = 4^{1/2} \times 49^{1/2}$

$= (2^2)^{1/2} \times (7^2)^{1/2} = 2 \times 7$

4. Evaluate $(27 \times 125)^{\frac{1}{3}}$

Solution

$(27 \times 125)^{\frac{1}{3}} = 27^{1/3} \times 125^{1/3}$

$= (3^3)^{1/3} \times (5^3)^{1/3} = 3 \times 5 = 15$

5. Evaluate $\frac{16^{1/3} \times 4^{1/3}}{8}$

Solution

Method 1

$\frac{16^{1/3} \times 4^{1/3}}{8} = \frac{(16 \times 4)^{1/3}}{8} = \frac{64^{1/3}}{8} = \frac{(4^3)^{1/3}}{8} = \frac{4}{8} = \frac{1}{2}$

Method II

Express each as an exponent with base 2 and apply the laws of indices.

Exercises 15.3

A. i. Express with signs of the form $\sqrt[n]{a^m}$

1. $a^{\frac{3}{5}}$ 2. $x^{\frac{1}{2}}$ 3. $b^{\frac{5}{2}}$

ii. Write the following in exponent form;

1. $\sqrt[4]{x^4}$ 2. $\sqrt[3]{x^5}$ 3. $\sqrt[3]{x^6}$

B. Find the values of the following:

1. $81^{\frac{3}{4}}$	4. $(-8)^{1/3}$	7. $49^{\frac{3}{2}}$
2. $27^{\frac{2}{3}}$	5. $8^{\frac{4}{3}}$	8. $9^{\frac{5}{2}}$
3. $4^{\frac{3}{2}}$	6. $64^{1/3}$	9. $1296^{1/2}$

C. Evaluate the following;

1. $\left(\frac{98}{2}\right)^{1/2}$	2. $\left(\frac{8}{27}\right)^{2/3}$	3. $\left(\frac{27}{216}\right)^{1/3}$
4. $\left(\frac{27}{8}\right)^{1/3}$	5. $\left(\frac{16}{81}\right)^{1/4}$	6. $\left(\frac{4}{9}\right)^{1/2}$
7. $64^{\frac{1}{3}} \times 216^{\frac{1}{3}}$		8. $(8 \times 729)^{\frac{1}{3}}$

Find *x* if

1. $(x + 1)^5 = 243$ 2. $(x - 1)^5 = \frac{32}{243}$

Negative Indices

From the law of indices, $2^0 \div 2^1 = 2^{-1}$, but $2^0 = 1$ and $2^1 = 2$

By substitution, $1 \div 2 = \frac{1}{2} = 2^{-1}$ and therefore, 2^{-1} is taken to be equal to $\frac{1}{2}$

Similarly, $2^0 \div 2^3 = 2^{-3} = \frac{1}{2^3}$, where 2^{-3} is the reciprocal of 2^3

In general,

1. $\boldsymbol{a^{-n} = \frac{1}{a^n}}$, **($a \neq 0$)** and $\boldsymbol{a^{-n}}$ is the reciprocal of $\boldsymbol{a^n}$

2. $\boldsymbol{a^{-mn} = \frac{1}{a^{mn}}}$ **($a \neq 0$)** and $\boldsymbol{a^{-mn}}$ is the reciprocal of $\boldsymbol{a^{mn}}$

Worked Examples

1. Express $\frac{1}{4^2}$ as a negative index.

Solution

1. $\frac{1}{4^2} = 4^{-2}$

2. Write $\frac{1}{32}$ in the form 2^{-a}

Solution

$\frac{1}{32} = \frac{1}{2\times2\times2\times2\times2} = \frac{1}{2^5} = 2^{-5}$

3. Simplify $\frac{2^2 \times 2^3}{2^8}$

Solution

$\frac{2^2 \times 2^3}{2^8} = \frac{2^5}{2^8} = 2^{5-8} = 2^{-3} = \frac{1}{2^3}$

4. Find the value of $(27/8)^{-2/3}$

Solution

$(27/8)^{-2/3}$

$= (8/27)^{2/3} = \frac{8^{2/3}}{27^{2/3}} = \frac{(2^3)^{2/3}}{(3^3)^{2/3}} = \frac{2^2}{3^2} = \frac{4}{9}$

5. Find the value of $\left(\frac{1}{81}\right)^{-1/4}$

Solution

$\left(\frac{1}{81}\right)^{-1/4} = (81)^{1/4} = (3^4)^{1/4} = 3$

6. Find the value of $\left(\frac{\mathbf{1}}{\mathbf{16}}\right)^{\mathbf{-3/2}}$

Solution

$\left(\frac{1}{16}\right)^{-3/2} = (16)^{3/2} = (4^2)^{3/2} = 4^3 = 64$

Method II

$8^{-1/3} = \left(\frac{1}{8}\right)^{1/3} = \frac{1^{1/3}}{8^{1/3}} = \frac{1}{(2^3)^{1/3}} = \frac{1}{2}$

Exercises 15.4

A. Express as negative exponents;

1. $\frac{1}{25y}$ 2. $\frac{1}{2^x}$ 3. $\frac{x^{-2}}{x^{-1}}$
4. $\frac{27}{729} = 3^x$ 5. $\frac{1}{125}$ 6. $\frac{32}{128}$

B. Find the values of the following:

1. $\left(\frac{1}{2}\right)^{-1}$ 4. $\left(\frac{4}{9}\right)^{-1/2}$ 7. $\left(\frac{1}{3}\right)^{-2}$
2. $\left(\frac{1}{4}\right)^{-1/2}$ 5. $\left(\frac{2}{3}\right)^{-2}$ 8. $\left(\frac{16}{81}\right)^{-1/4}$
3. $\left(-\frac{1}{2}\right)^{-2}$ 6. $\left(\frac{27}{8}\right)^{-4/3}$ 9. $\left(\frac{1}{4}\right)^{-1/2}$

C. Find the values of the following;

1. $64^{-2/3}$ 2. $81^{-3/4}$ 3. $4^{-3/2}$
4. $16^{-3/4}$ 5. $125^{-1/3}$ 6. $\frac{4^{-3/2}}{8^{-2/3}}$

D. Simplify the following;

1. $27^{1/2} \times 243^{1/2}$ 5. $32^{1/6} \times 16^{1/12}$
2. $32^{1/6} \div 8^{1/6}$ 6. $4^{1/3} \div 16^{1/12}$
3. $\frac{27^{1/2} \times 243^{1/2}}{243^{4/5}}$ 7. $\frac{6^{1/2} \times 96^{1/4}}{216^{1/5}}$
4. $\frac{12^{1/3} \times 6^{1/3}}{81^{1/6}}$ 8. $\frac{8^{1/6} \times 4^{1/3}}{32^{1/6} \times 16^{1/12}}$

Exponential Equations

An equation involving powers of numbers, is called an ***exponential equation.*** For example, $2^3 = 8$, $3^{2x+3} = 9$ are exponential equations

The steps in solving exponential equations are;
1. Express all the numbers as exponents of the same base (usually a prime number).
2. Express both sides as a power of the same number, using the laws of indices.
3. Equate the exponents and solve the equation.

Worked Examples

1. Find the value of x in $2^x = 8$

Solution

$2^x = 8$

$2^x = 2 \times 2 \times 2$

$2^x = 2^3$

$x = 3$

2. In $2^{x-4} = 8$, find the value of x

Solution

$2^{x-4} = 8,$

$2^{x-4} = 2 \times 2 \times 2$
$2^{x-4} = 2^3$
$x - 4 = 3$
$x - 4 + 4 = 3 + 4$
$x = 7$

3. Solve for x given that $7^{x+3} = 49$

Solution
$7^{x+3} = 49$
$7^{x+3} = 7 \times 7$
$7^{x+3} = 7^2$
$x + 3 = 2$
$x = 2 - 3 = -1$

4. If $\frac{1}{32^x} = 256$, determine the value of x

Solution
$\frac{1}{32^x} = 256$
$\frac{1}{2^{5x}} = 2^8$
$2^{-5x} = 2^8$
$-5x = 8$
$x = -\frac{8}{5}$

5. If $2^{x+2} \times 8^x = 1$, find the value of x

Solution
$2^{x+2} \times 2^{3x} = 2^0$
$x + 2 + 3x = 0$
$4x + 2 = 0$
$4x = -2$
$x = -\frac{2}{4} = -\frac{1}{2}$

6. Solve $\left(\frac{1}{125}\right)^x - \left(\frac{1}{25}\right)^{3/4} = 0$

Solution
$\left(\frac{1}{125}\right)^x - \left(\frac{1}{25}\right)^{3/4} = 0$
$125^{-x} - 25^{-3/4} = 0$
$5^{-3x} - 5^{-2\left(\frac{3}{4}\right)} = 0$
$5^{-3x} - 5^{-\left(\frac{3}{2}\right)} = 0$
$5^{-3x} = 5^{-\left(\frac{3}{2}\right)}$ (*Equating exponents)*
$-3x = -\frac{3}{2}$
$-3x \times 2 = -\frac{3}{2} \times 2$ (*Multiply through by 2)*
$-6x = -3$
$x = \frac{-3}{-6} = \frac{1}{2}$

7. Given that $16^{3/4} = 2^y$, find the value of y

Solution
$16^{3/4} = (2^4)^{3/4} = 2^3$
But $16^{3/4} = 2^y$
$2^y = 2^3$, therefore $y = 3$

Exercises 15.5

A. Solve the following;

1. $49^x = 7$
2. $5^{-x} = 125$
3. $729^x = 9$
4. $9^x = 27$
5. $10^{3n} = 10000$
6. $9^x = 729$

B. Solve the following equations:

1. $3^{2x-5} = 27$
2. $7^{x+1} = 343^x$
3. $3^{x+4} = 9^4$
4. $10^{4x-5} = 1000$
5. $3^{2x-1} = 81$
6. $4^{x+1} = 128$
7. $144^{3x} = 12^{x-5}$
8. $2^{x-4} = 8^{x-5}$
9. $5^{3x-1} = \sqrt{5}$
10. $25^{-2x} = 125^{x+7}$

C. Find the value of the variables:

1. $\left(\frac{3}{5}\right)^x = \frac{9}{25}$
2. $3^x = \frac{1}{3}$
3. $2^n = \frac{1}{16}$
4. $\left(\frac{1}{5}\right)^x = 125$
5. $\frac{1}{5^x} = \frac{1}{125}$
6. $5^x = \frac{1}{125}$
7. $\left(\frac{1}{6}\right)^{-3x-2} = 36^{x+1}$
8. $\left(\frac{1}{4}\right)^x = 64$

Logarithms

From the calculator, it is seen that the log of 100 to the base 10 is 2. This is stated mathematically as $\log_{10} 100 = 2$. This means that $100 = 10^2$, where 10 is the base and 2 is the index (logarithm). Note that $\log_{10}$ is written simply as log.

Relating Indices and Logarithms

In general, if $\boldsymbol{a = b^c}$ then the logarithm of $\boldsymbol{a}$ is expressed as: $\boldsymbol{Log_b a = c}$ read as *"the logarithm of **a** to the base **b** is **c**"*

Thus:

$$\boxed{\textit{If } a = b^c, \textit{ then } \log_b a = c}$$

Any statement in index form has an equivalent log form, for example,

$8 = 2^3$ *(index form)*

$\Rightarrow \log_2 8 = 3$ *(log form)*

Worked Examples

1. Express $2^4 = 16$ in logarithmic notation

Solution

In $a = b^c$, $\log_b a = c$

If $2^4 = 16$, then $\log_2 16 = 4$

2. Write $8^{-2/3} = \frac{1}{4}$ in logarithmic form

Solution

If $8^{-2/3} = \frac{1}{4}$, then $\log_8 \frac{1}{4} = {-2}/{3}$

3. Express in index notation $\log_2 32 = 5$

Solution

If $\log_2 32 = 5$, then $32 = 2^5$

Exercises 15.6

A. Write the following in logarithm form:

1. $125 = 5^3$ 4. $0.01 = 10^{-2}$
2. $343^{1/3} = 7$ 5. $25^{1/2} = 5$
3. $256^{3/4} = 64$ 6. $2^{-3} = \frac{1}{8}$

B. Express in exponential form:

1. $\log_9 1 = 0$
2. $\log_4 2 = \frac{1}{2}$
3. $\log_5 625 = 4$
4. $-2 = \log_3 \frac{1}{9}$
5. $\log_{27} 3 = \frac{1}{3}$
6. $\log (0.1 \times 10^8) = 7$

Laws of Logarithms

Logarithms and their operations are govern by some basic properties and laws as shown below for a, x and y belonging to the set of real numbers;

1. $\boldsymbol{\log_a (xy) = \log_a x + \log_a y}$

e.g. $\log_2 (4 \times 2) = \log_2 4 + \log_2 2$

$= \log_2 2^2 + \log_2 2$

$= 2 + 1 = 3$

2. $\boldsymbol{\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y}$

e.g. $\log_2\left(\frac{4}{2}\right) = \log_2 4 - \log_2 2$

$\log_2\left(\frac{4}{2}\right) = \log_2 2^2 - \log_2 2 = 2 - 1 = 1$

3. $\boldsymbol{\log_a a = 1}$

e.g. $\log_3 3 = 1$

4. $\boldsymbol{\log_a x^y = y \log_a x}$

e.g. $\log_2 4 = \log_2 2^2 = 2 \log_2 2 = 2 \times 1 = 2$

5. $\boldsymbol{\log_a 1 = 0}$

e.g. $\log_{10} 1 = 0$

6. $\boldsymbol{\log_a\left(\frac{xy}{z}\right) = \log_a x + \log_a y - \log_a z}$

e.g. $\log_a 2 + \log_a 6 - \log_a 4$

$= \log_a\left(\frac{2 \times 6}{4}\right) = \log_a\left(\frac{12}{4}\right) = \log_a 3$

7. $\boldsymbol{\log_a x = \frac{1}{\log_x a}}$

E.g. $\log_{1000} 10 = \frac{1}{\log_{10} 1000}$

$= \frac{1}{\log_{10} 10^3} = \frac{1}{3\log_{10} 10} = \frac{1}{3 \times 1} = \frac{1}{3}$

8. $\log_a b = \frac{\log_x b}{\log_x a}$ (change of base)

e.g. $\log_4 64 = \frac{\log_2 64}{\log_2 4} = \frac{\log_2 2^6}{\log_2 2^2} = \frac{6}{2} = 3$

9. $\log_a x = y$**, then** $x = a^y$

e.g. If $\log_2 8 = 3$, then $8 = 2^3$

Exercises 15.7

A. Express as a single logarithm:

1. log 2 + log 5
2. log 2 + log 12 – log 6
3. log 3 + log 16 – log 4 – log 6
4. $(\log_6 4 + \log_6 9)^2$
5. $(\log_5 25 + \log_5 15 - \log_5 75)^4$

Common Logarithmic Expressions

(Applying the Laws of Logarithms)

A logarithmic expression is a logarithm statement that is not equated or does not contain an equal sign.

To evaluate a logarithmic expression is to find the value of the expression by applying the laws of indices. Hence, answers should **not** contain the word ***"log".***

To simplify a logarithmic expression is to apply the laws to the extend that no law is applicable. Hence, a simplified answer should contain the word***"log"***

Simplifying Logarithms Expressions

Type I: Form log $_a$ ***b***

I. Express b as a product of two factors to obtain $\log_a(c \times d)$

II. Apply the laws to obtain $\log_a c + \log_a d$

III. Express c and d as exponents, if possible and apply the exponential law of logarithm

IV. Apply other laws if necessary

Worked Examples

1. Simplify $\log_2 72$

Solution

$\log_2 72 = \log_2(8 \times 9)$

$= \log_2 8 + \log_2 9$

$= \log_2 2^3 + \log_2 3^2$

$= 3 + 2\log_2 3$

2. Simplify$\log_5 40$

Solution

$\log_5 40 = \log_5(5 \times 8)$

$= \log_5 5 + \log_5 8$

$= 1 + \log_5 2^3$

$= 1 + 3\log_5 2$

Exercises 15.8

Simplify;

1. $\log_3 72$ 2. $\log_3 63$ 3. $\log_5 150$

4. $\log_8 120$ 5. $\log_2\sqrt{98}$ 6. $\log \sqrt[3]{180}$

Type II: Involving Addition and Subtraction

1. Ensure that all the logs are of the same base
2. Express the given expression as a single logarithm of the form $\log_a b$.
3. Go throught the processes of type I to complete simplification, only if a is a factor of b.
4. If a is a not a factor of b, then leave your answer in the form: $\log_a b$

Worked Examples

1. Simplify 3 log 3 – log 27

Solution

3 log 3 – log 27

$= \log 3^3 + \log 27$

$= \log 27 - \log 27 = \log\left(\frac{27}{27}\right) = \log 1$

2. Simplify $\log_4 9 + \log_4 21 - \log_4 7$

Solution

$\log_4 9 + \log_4 21 - \log_4 7$

$= \log_4\left(\frac{9 \times 21}{7}\right) = \log_4 27 = \log_4 3^3 = 3\log_4 3$

3. Simplify $\log_2 3 - \log_2 15 + \log_2 50$

Solution

$\log_2 3 - \log_2 15 + \log_2 50$

$= \log_2\left(\frac{3}{15} \times 50\right)$

$= \log_2 10$

$= \log_2(2 \times 5)$

$= \log_2 2 + \log_2 5$

$= 1 + \log_2 5$

Exercises 15.9

A. Simplify:

1. $\log_5 12 + \log_5 10$
2. $2\log_7 9 - \log_7 81$
3. $\log 45 - \log\frac{9}{2}$
4. $\log_3 24 + \log_3 15 - \log_3 10$
5. $\log_7 98 + \log_7 30 - \log_7 15$

Evaluating Logarithmic Expressions

Type I: Form $\log_a b$

Given a logarithmic expression of the form $\boldsymbol{\log_a b}$ to determine its value:

I. Equate the expression to any preferred variable, say x. i.e. $\boldsymbol{\log_a b = x}$

II. Write the logarithmic expression as an exponential equation. That is: $\boldsymbol{b = a^x}$

III. Express the exponential equation as equation with a common base, if possible.

IV. If not possible, take the log on both sides of the equation. $\boldsymbol{\log b = \log a^x}$

V. Simplify the variable side of the equation. i.e. $\boldsymbol{\log b = x \log a}$

VI. Make the variable the subject and work out to obtain the value of the variable. i.e. $x = \frac{\log b}{\log a}$

Worked Examples

1. Use tables or calculator to find an approximate value of $\log_2 7$

Solution

Let $\log_2 7 = x$

$\Rightarrow 7 = 2^x$

Taking log on both sides

$\log 7 = \log 2^x$

$\log 7 = x\log 2$

$x = \frac{\log 7}{\log 2} = 2.807$ (3 d. p.)

2. Evaluate $\log_2 64$

Solution

Let $\log_2 64 = x$

$64 = 2^x$

$2^6 = 2^x$ (*Express* 64 *as an index to the base* 2)

$x = 6$ *(Equating exponents)*

3. Evaluate $\frac{1}{2}\log_3 81$

Solution

<u>Method 1</u>

$\frac{1}{2}\log_3 81 = \frac{1}{2}\log_3 3^4$

$= 4 \times \frac{1}{2} \times \log_3 3,$

$= 4 \times \frac{1}{2} \times 1 = 2$

<u>Method 2</u>

$\frac{1}{2}\log_3 81 = \log_3 81^{1/2} = \log_3 (9^2)^{1/2} = \log_3 9$

Let $\log_3 9 = x$

$9 = 3^x$

$\Rightarrow 3^2 = 3^x \quad x = 2$

4. Evaluate $\log_{0.1} 10$

Solution

Let $\log_{0.1} 10 = x$

$10 = 0.1^x$

$10 = \left(\frac{1}{10}\right)^x$

$10 = (10^{-1})^x$

$10 = 10^{-x}$

$1 = -x$

$x = -1$ *(Try other methods)*

5. Evaluate $\log_3 27\sqrt{3}$

Solution

Let $\log_3 27\sqrt{3} = x$

$27\sqrt{3} = 3^x$

$3^3 \times 3^{\frac{1}{2}} = 3^x$

$3^{3+\frac{1}{2}} = 3^x$

$3^{\frac{7}{2}} = 3^x$

$x = \frac{7}{2}$

8. Find the value of $\log_5 \frac{1}{25}$

Solution

$\log_5 \frac{1}{25}$

$= \log_5 25^{-1}$

$= \log_5 5^{2(-1)}$

$= \log_5 5^{-2}$

$= -2 \log_5 5$

$= -2 \times 1 = -2$ **(Try an alternative method)**

Exercises 15.10

A. Evaluate:

1. $\log_{10} 10^7$ 4. $\log_3 243$ 7. $\log_8 2$

2. $\log_4 1$ 5. $\log_{27} 3$ 8. $\log_{\frac{2}{3}} \frac{4}{9}$

3. $\log_5 125$ 6. $\log_{0.01} 10$ 9. $\log_2 32$

B. **Find the values of the following:**

1. $\frac{1}{2}\log_3 81$ 4. $\frac{2}{3}\log_2 64$ 7. $-\log_2 \frac{1}{2}$

2. $\frac{1}{3}\log_2 8$ 5. $\frac{1}{2}\log_{49} 7$ 8. $\frac{-1}{2}\log_2 4$

3. $\log_{27}\left(\frac{1}{3}\right)$ 6. $\log_7 \frac{1}{49}$ 9. $\log_{16} 8$

C. Evaluate to 2 significant figures:

1. $\log \sqrt{22}$ 3. $\log \sqrt[3]{34}$

2. $\log \sqrt{50}$ 4. $\log_2 2\sqrt{2}$

Type II: Involving Addition and Subtraction

1. Ensure that all the logs are of the same base
2. Express the given expression as a single logarithm of the form: $\boldsymbol{\log_a b}$.
3. Equate the single log expression to any preferred variable to obtain a logarithmic equation.
4. Write the logarithmic equation as an exponential equation.
5. Solve for the value of the variable in the exponential equation as in type I.

Worked Examples

Evaluate to four decimal places;

1. Evaluate $\log_5 12 + \log_5 10$ to four decimal places

Solution

$\log_5 12 + \log_5 10$

$= \log_5 (12 \times 10) = \log_5 120$

Let $\log_5 120 = y$

$120 = 5^y$

$\log 120 = \log 5^y$

$\log 120 = y \log 5$

$y = \frac{\log 120}{\log 5} = 2.9746$ (4 d.p)

2. Evalutae $(\log_6 4 + \log_6 9)^2$.

Solution

Method 1

$\log_6 (4 \times 9)^2 = \log_6 (36)^2 = \log_6 1296$

Let $\log_6 1296 = y$

$1296 = 6^y$

$\log 1296 = \log 6^y$

$\log 1296 = y \log 6$

$y = \frac{\log 1296}{\log 6} = 4$

Method 2

$2 \log_6 (4 \times 9)$

$= 2 \log_6 36$

$= 2 \log_6 6^2 = 2 \times 2 \log_6 6 = 2 \times 2 \times 1 = 4$

3.Evaluate$\log_4 9 + \log_4 21 - \log_4 7$

Solution

$\log_4 9 + \log_4 21 - \log_4 7$

$= \log_4 \left(\frac{9 \times 21}{7}\right) = \log_4 27$

Let $\log_4 27 = x$

$\Rightarrow 27 = 4^x$

$\log 27 = \log 4^x$

$\log 27 = x \log 4$

$x = \frac{\log 27}{\log 4} = 2.3775$

4. Evaluate without using tables or calculators:

$\log_{10}\left(\frac{75}{10}\right) - 2 \log_{10} \left(\frac{5}{9}\right) + \log_{10} \left(\frac{100}{243}\right)$

Solution

$\log_{10}\left(\frac{75}{10}\right) - 2 \log_{10}\left(\frac{5}{9}\right) + \log_{10}\left(\frac{100}{243}\right)$

$= \log_{10}\left(\frac{75}{10}\right) - \log_{10}\left(\frac{5}{9}\right)^2 + \log_{10}\left(\frac{100}{243}\right)$

$= \log_{10}\left(\frac{75}{10}\right) - \log_{10}\left(\frac{25}{81}\right)^2 + \log_{10}\left(\frac{100}{243}\right)$

$= \log_{10}\left(\frac{75}{10} \div \frac{25}{81} \times \frac{100}{243}\right)$

$= \log_{10}\left(\frac{75}{10} \times \frac{81}{25} \times \frac{100}{243}\right)$

$= \log_{10}\left(\frac{3}{1} \times \frac{1}{1} \times \frac{10}{3}\right) = \log_{10} 10 = 1$

5. Without using Mathematical tables or calculators , simplify;

$\frac{1}{2} \log_{10}\left(\frac{25}{4}\right) - 2 \log_{10}\left(\frac{4}{5}\right) + \log_{10}\left(\frac{320}{125}\right)$

Solution

$\frac{1}{2} \log_{10}\left(\frac{25}{4}\right) - 2 \log_{10}\left(\frac{4}{5}\right) + \log_{10}\left(\frac{320}{125}\right)$

$\log_{10}\left(\frac{25}{4}\right)^{1/2} - \log_{10}\left(\frac{4}{5}\right)^2 + \log_{10}\left(\frac{320}{125}\right)$

$\log_{10}\left(\frac{5}{2}\right) - \log_{10}\left(\frac{16}{25}\right) + \log_{10}\left(\frac{64}{25}\right)$

$\lg\left(\frac{5}{2} \div \frac{16}{25} \times \frac{64}{25}\right)$

$\lg\left(\frac{5}{2} \times \frac{25}{16} \times \frac{64}{25}\right)$

$\lg\left(\frac{5}{1} \times \frac{1}{1} \times \frac{2}{1}\right) = \lg(5 \times 2) = \lg 10$

Exercises 15.11

Find the values of the following:

1. $\log\sqrt{35} + \log\sqrt{2} - \log\sqrt{7}$
2. $\log_6 24 + \log_6 15 - \log_6 10$
3. $\log_7 98 + \log_7 30 - \log_7 15$
4. $\log 3 - \log 15 + \log 50$
5. $\log 5 + \log 16 - \log 4 - \log 10$

Division of Logarithms

To simplify expressions involving division of logarithms

I. Express each number as an exponent with a common base

II. Apply the law: $\log_a x^y = y\log_a x$

III. Simplify by crossing out common factors

Worked Examples

Simplify : $\frac{\log 125}{\log 25}$

Solution

$\frac{\log\ 125}{\log\ 25}$

Express 125 *and* 25 *as a power of* 5

$\frac{\log\ 125}{\log\ 25} = \frac{\log\ 5^3}{\log\ 5^2} = \frac{3\ \log\ 5}{2\log\ 5} = \frac{3}{2}$

2. Simplify $\frac{\log\ 27}{\log\ 6 - \log\ 2}$

Solution

$\frac{\log\ 27}{\log\ 6 - \log\ 2}$

Make log 6 – log 2 a single log

$\Rightarrow \log\left(\frac{6}{2}\right)$

= log 3

Substitute in place of log 6 – log 2

$\Rightarrow \frac{\log\ 27}{\log\ 3}$

$\frac{\log\ 27}{\log\ 3} = \frac{\log\ 3^3}{\log\ 3^1} = \frac{3\log\ 3}{\log\ 3} = 3$

3. Simplify $\frac{\log\ a^3 - \log\ a}{\log\ a}$

Solution

$\frac{\log\ a^3 - \log\ a}{\log\ a} = \frac{\log\ {a^3}/{a}}{\log\ a} = \frac{\log\ a^2}{\log\ a} = \frac{2\ \log\ a}{\log\ a} = 2$

4. Given that log $p = x$ and log $q = y$, express log $\frac{pq}{100}$ in terms of x and y

Solution

log$p = x$ and log $q = y$

$p = 10^x$ and $q = 10^y$

$pq = 10^x \times 10^y$

$= 10^{x+y}$

$\Rightarrow \log\frac{pq}{100}$

$= \log\frac{10^{x+y}}{100}$

$= \log\frac{10^{x+y}}{10^2}$

$= \log_{10} 10^{x+y} - \log_{10} 10^2$

$= x + y\log_{10} 10 - 2\log_{10} 10$

$= x + y - 2$

Exercises 15.12

Simplify:

1. $\frac{\log\ 81}{\log\ 9}$ 2. $\frac{\log\ 50 + \log\ 2}{\log 8 - \log 80}$ 3. $\frac{\log\ 49}{\log\ 343}$

4. $\frac{\log_{49} 7}{\log_{16} 4}$ 5. $\frac{\log_2 8 + \log_2 2\sqrt{2}}{\log_2 \frac{1}{\sqrt{2}} - \log_2 \sqrt{2}}$ 6. $\frac{\log\ 8}{\log\ 2}$

Solving Exponential Equations

In an exponential equation, when the exponent of a number at a side of the equation is a variable,

I. Take the log on both sides of the equation.

II. Simplify the variable part of the equation.

III. Make the variable the subject and work out (simplify) to obtain the value of the variable.

Worked Examples

1. Solve the equation $2^x = 5$, using tables or calculators to 5 significant figures

Solution

In $2^x = 5$,

Taking log on both sides,

$\log 2^x = \log 5$

$x \log 2 = \log 5$

$x = \frac{\log\ 5}{\log\ 2} = 2.3219$ (5 s.f)

2. Solve the exponential equation $4^{3x} - 1 = 8$

Solution

$4^{3x} - 1 = 8$

$4^{3x} = 8 + 1$

$4^{3x} = 9$

$\log\ 4^{3x} = \log 9$

$3x \log 4 = \log 9$

$x = \frac{\log\ 9}{3\ \log\ 4} = 0.5283$ (4 d.p)

Exercises 15.13

A. Solve the equations:

1. $3^x = 5$ 2. $3^x = 2$
3. $3^{4x} = 4$ 4. $2^x \times 2^{x+1} = 10$
5. $(1/2)^x = 6$ 6. $(2/3)^x = 1/6$

B. **Solve for *n*, to 4 significant figures;**

1. $4^n = 20$ 2. $3^n = 2500$
3. $5^n = 680$ 4. $2^n = 31$
5. $4^{n+2} = 3460$ 6. $3^{2n-1} = 4800$
7. $5^{n-1} = 2^n$ 8. $5^{2n-1} = 4^{n+1}$

Substitution

Sometimes the estimated value of the log of a number may be given to simplify or evaluate other logarithmic expressions. In such cases:

I. Express the given log or simplified as an exponent or a product of the number whose estimated value has been given.

II. Apply the law: $\mathbf{\log_a x^y = \log_a x}$, if necessary.

III. Substitute the estimated value into the expression and simplify

Worked Examples

Given that log 2 = 0.3010, find without using tables or calculators the values of;

i. log 8 ii. log 80

Solution

i. $\log 8 = \log 2^3 = 3 \log 2$,

Substitute log 2 = 0.3010

$= 3 \times 0.3010 = 0.9030$

ii. $\log 80 = \log (8 \times 10)$

$= \log 8 + \log 10$

$= \log 2^3 + \log 10$

$= 3 \log 2 + \log 10$

$= 3 \times 0.3010 + 1$

$= 1.9030$

2. Given that $\log_3 5 = 1.465$, evaluate without using tables $\log_3 25 + \log_3 15$

Solution

$\log_3 25 + \log_3 15$

$= \log_3 5^2 + \log_3 (3 \times 5)$

$= 2 \log_3 5 + \log_3 3 + \log_3 5$

Substitute $\log_3 5 = 1.465$

$\Rightarrow 2 \times 1.465 + 1 + 1.465$

$= 5.395$

3. If log 5 = 0.6990, find $\log 25^{1/3}$

Solution

$\log 25^{1/3} = \log(5^2)^{\frac{1}{3}}$

$= \log(5)^{\frac{2}{3}} = \frac{2}{3} \log 5$,

Substitute log 5 = 0.6990

$\Rightarrow \frac{2}{3} \times 0.6990 = 0.4660$

Some Solved Past Questions

1. If $\log_x 3 = 0.5283$ and $\log_x 7 = 0.9358$, find logarithm to base *x* of ;

i. 27 ii. $\sqrt{7}$

Solution

$\log_x 3 = 0.5283$ and $\log_x 7 = 0.9358$,

i. $\log_x 27$

$= 3 \log_x 3^3$

$= 3 \log_x 3$

But $\log_x 3 = 0.5283$

$\Rightarrow 3 \times 0.5283 = 1.5849$

ii. $\log_x \sqrt{7}$

$= \log_x 7^{1/2}$

$= \frac{1}{2} \log_x 7$

But $\log_x 7 = 0.9358$

$\Rightarrow \frac{1}{2} \times 0.9358 = 0.4679$

2. If log x = 0.5 and log y= 1.5, find $x + y$

Solution

Log $_{10}x$= 0.5 and log y_{10}= 1.5

$x = 10^{0.5}$ and $y = 10^{1.5}$

$x + y = 10^{0.5} + 10^{1.5} = 34.8$

3. Given that log 6 = 0.778, find without using tables or calculators the value of log 600.

Solution

log 600 = log (6 × 100)

= log 6 + log 100

= log 6 + log 10^2

= log 6 + 2 log 10

= log 6 + 2(1)

Substitute log 6 = 0.778

⇒ 0.778 + 2 = 2.778

4. Given that $\log_{10} 2 = 0.3010$ and $\log_{10} 5 = 0.6990$, evaluate $\log_{10} 50 - \log_{10} 40$

Solution

$\log_{10} 50 - \log_{10} 40$

$= \log_{10} \left(\frac{50}{40}\right) = \log_{10} \left(\frac{5}{4}\right) = \log_{10} 5 - \log_{10} 4$

$= \log_{10} 5 - \log_{10} 2^2$

$= \log_{10} 5 - 2 \log_{10} 2$

$= 0.6990 - 2(0.3010) = 0.097$

<u>Method 2</u>

$\log_{10} 50 - \log_{10} 40$

$= \log_{10} (5 \times 10) - \log_{10} (5 \times 8)$

$= \log_{10} (5 \times 10) - \log_{10} (5 \times 2^3)$

$= \log 5 + \log 10 - \log 5 - \log 2^3$

$= \log 5 + \log 10 - \log 5 - 3 \log 2$

$= 1 - 3 \log_{10} 2$

$= 1 - 3(0.3010) = 0.097$

5. log 9 = 0.9542, find the value of log 0.009

Solution

log 0.009

$= \log (9 \times 10^{-3})$

$= \log 9 + \log_{10} 10^{-3}$

$= \log 9 + (-3) \log_{10} 10$

$= 0.9542 - 3\ (1) = -2.0458$

6. Given that $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$, evaluate$\log_{10} 0.24$

Solution

$\log_{10} 0.24$

$= \log_{10} \left(\frac{24}{100}\right)$

$= \log_{10} 24 - \log_{10} 100$

$= \log_{10} (2^3 \times 3) - \log_{10} (10^2)$

$= \log_{10} 2^3 + \log_{10} 3 - \log_{10} 10^2$

$= 3 \log_{10} 2 + \log_{10} 3 - 2 \log_{10} 10$

$= 3\ (0.3010) + (0.4771) - 2\ (1)$

$= -\ 0.6199$

Exercises 15.14

A. Given that log 2 = 0.3010 and log3 = 0.4771, find the values (correct to 4 s.f)

1. log 12 2. $\log \sqrt[3]{60}$ 3. log 18
4. log 1.5 5. 2 log 21 – log 98

6. Given that log 5 = 0.6989, find correct to four places of decimals, the value of log 40

7. Given $\log_a 3 \approx 0.48$ and $\log_a 5 \approx 1.72$, evaluate $\log_a 45$ without the use of a calculator

8. If $\log_5 2 = 0.431$ and $\log_5 3 = 0.682$, find the value of $\log_5 \left(\frac{3}{8}\right) + 2 \log_5 \left(\frac{4}{5}\right) - \log_5 \left(\frac{2}{5}\right)$

C. If $\log_5 2$ = 0.431 and $\log_5 3$ = 0.682, find the values of the following;

1. $\log_5 6$ 2. $\log_5 27$ 3. $\log_5 8$
4. $\log_5 12$ 5. $\log_5 \sqrt{3}$ 6. $\log_5\left(\frac{2}{3}\right)$
7. $\log_5\left(\frac{1}{4}\right)$ 8. $\log_5 100$ 9. $\log_5 1.5$

Involving Change of Base

Given $\log_a b$ to simplify using change of base:

I. Find the common factor of the base (a) and the number (b) to obtain c

II. Express the logarithm of b to base c to the logarithm of a to base c as a common ratio to complete the process. i.e.$\log_a b = \frac{\log_c b}{\log_c a}$

III. Simplify where possible

Proof

For $y = \log_b M$,

$\Rightarrow b^y = M$ in exponential form .

(Take base – a logarithm of both sides)

It follows that $\log_a (b^y) = \log_a M$

$y \log_a b = \log_a M$, *(Apply power property of logarithms)*

$y = \frac{\log_a M}{\log_a b}$ (*Divide by* $\log_a b$)

But $\log_b M = y$

$\log_b M = \frac{\log_a M}{\log_a b}$

Worked Examples

1. Evaluate $\log_9 27$

Solution

$\log_9 27 = \frac{\log_3 27}{\log_3 9} = \frac{\log_3 3^3}{\log_3 3^2} = \frac{3}{2}$

2. Simplify $\log_8 92$

Solution

$\log_8 92 = \frac{\log_2 92}{\log_2 2^3} = \frac{\log_2 92}{3\log_2 2} = \frac{\log_2 92}{3}$

Exercises 15.15

A. If $f(x) = 2\log_5 x$, evaluate:

1) $f(5)$ 2) $f(25)$ 3) $f\left(\frac{1}{5}\right)$ 4) $f(\sqrt{5})$

B. Use the change of base method to simplify:

1. $\log_6 45$ 2. $\log_2 30$ 3. $\log_4 128$
4. $\log_{12} 72$ 5. $\log_8 2$ 6. $\log_{30} 100$

Logarithm Equations

A logarithmic equation is a logarithm statement that is equated or contains an equal sign. In other words, it is an equation involving logarithms.

To solve logarithmic equation is to find the value or values of the variable that makes the statement or equation true.

There are two methods for solving equations involving logs:

Method 1

1. Get a single log on both sides of the equation,

II. Equate the expressions and solve for the value of the variable i.e. write the equation in the form: $\log_b x = \log_b y \Rightarrow x = y$ and solve

Method 2

I. Get a single log in the equation,

II. Change from logarithmic form to exponential form. i.e. write the equation in the form: $\log_b a = c$ **as** $a = b^c$

Note:

1. Make sure that all logs have the same base. If necessary, use the change of base law to write logs to the same base

2. Logs are defined for only positive numbers. Therefore reject any solution that give rise to log (negative number)in the original equation

Worked Examples

Solve $\log_2 x = 3 - \log_2 (x - 2)$

Solution

Method 1

$\log_2 x = 3 - \log_2 (x - 2)$

$\log_2 x + \log_2 (x - 2) = 3$

$\log_2 x (x - 2) = 3$

$\log_2 (x^2 - 2x) = 3$

$x^2 - 2x = 2^3$

$x^2 - 2x = 8$

$x^2 - 2x - 8 = 0$

$(x - 4)(x + 2) = 0$

$x - 4 = 0$ or $x + 2 = 0$

$x = 4$ or $x = -2$

Reject $x = -2$. Therefore, $x = 4$

Method 2

$\log_2 x = 3 - \log_2 (x - 2)$

$\log_2 x + \log_2 (x - 2) = 3$

$\log_2 x(x - 2) = \log_2 8$

$\log_2 (x^2 - 2x) = \log_2 8$

$x^2 - 2x = 8$

$x^2 - 2x - 8 = 0$

$(x - 4)(x + 2) = 0$

$x - 4 = 0$ or $x + 2 = 0$

$x = 4$ or $x = -2$

Reject $x = -2$. Therefore, $x = 4$

2. Solve $\log_{10} x^2 + \log_{10} 2 = \log_{10} 50$

Solution

$\log_{10} x^2 + \log_{10} 2 = \log_{10} 50$

$= \log_{10}(2x^2) = \log_{10} 50$

$\Rightarrow 2x^2 = 50$

$x^2 = 25$

$x^2 = 5^2$

$x = 5$

Some Solved Past Questions

1. Solve for x in $3 \log x + \log 3 = \log 81$

Solution

$3 \log x + \log 3 = \log 81$

$\log x^3 + \log 3 = \log 81$

$\log 3x^3 = \log 81$

$\Rightarrow 3x^3 = 81$

$x^3 = 27$

$x^3 = 3^3$

$x = 3$

2. Solve $\log (8x + 1) - \log (2x + 1) = \log (x + 2)$

Solution

$\log (8x + 1) - \log (2x + 1) = \log (x + 2)$

$\log_{10}\left(\frac{8x+1}{2x+1}\right) = \log_{10} (x + 2)$

$\frac{8x+1}{2x+1} = x + 2$

$8x + 1 = (x + 2)(2x + 1)$

$8x + 1 = x(2x + 1) + 2(2x + 1)$

$8x + 1 = 2x^2 + x + 4x + 2$

$8x + 1 = 2x^2 + 5x + 2$

$0 = 2x^2 + 5x - 8x + 2 - 1$

$2x^2 - 3x + 1 = 0$

$a = 2$, $b = -3$ and $c = 1$

Substitute in $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(1)}}{2(2)}$

$x = \frac{3 \pm \sqrt{9 - 8}}{4}$

$x = \frac{3 \pm \sqrt{1}}{4}$

$x = \frac{3 + \sqrt{1}}{4}$ or $x = \frac{3 - \sqrt{1}}{4}$

$x = 1$ or $x = \frac{1}{2}$

3. Find the value of x if $\log_4 16 = \log_x 36$

Solution

$\log_4 16 = \log_x 36$

$\log_4 4^2 = \log_x 36$

$2 \log_4 4 = \log_x 36$
$2 = \log_x 36$
$\Rightarrow x^2 = 36$
$x = 36^{1/2} = (6^2)^{1/2} = 6$

Exercises 15.16
A. Solve the following equations:
1. $\log (5x + 2) = \log 2$
2. $\log (2x - 3) = \log 3$
3. $\log_4 (x + 2) - \log_4 3 = \log_4 (x - 1)$
4. $\log_4 (x + 6) - \log_3 x = \log_3 5$
5. $\log_2 3 + \log_2 x = \log_2 12$
6. $\log_3 x - \log_3 4 = \log_3 2$
7. $\log_5 3(x + 5) + \log_5 (x - 1) = \log_5 8$
8. $\log_a (x - 6) + \log_a (x - 4) = \log_a x$
9. $\log_3 (x^2 - 10) - \log_3 x = 2 \log_3 3$

B. Solve the following equations;
1. $\log_2 (x + 2) - \log_2 x = 3$
2. $\log (2x - 1) - \log 5 = 1$
3. $\log (x - 7) - \log 3 = 2$
4. $\log_2 9 + \log_2 (x + 3) = 3$
5. $\log_5 x + \log_5 (x + 4)$

C. Find the value of *x*
1. $\log x - 1 = \log (x - 9)$
2. $\log (2x + 1) = 1 - \log x$
3. $\log_2 (x + 2) + \log_2 (x - 2) = 5$
4. $\log_9 x = \frac{1}{2} + \log_9 (5x + 18)$
5. $\log_2 (2x - 5) + \log_2 (x - 2) = 4$

Involving Simultaneous Equations
Two or more logarithmic equations involving two different variables are solved simultaneously either by method of substitution or elimination.

Worked Examples
1. Solve the simultaneous equations:
$\log (5x - y) = \log 9$ and $\log (3x + 2y) = \log 8$

Solution
$\log (5x - y) = \log 9$………………..(1)
$\log (3x + 2y) = \log 8$ ………………(2)

Divide both sides by log
$5x - y = 9$……………(1)
$3x + 2y = 8$…………(2)

eqn (1) × 2
$10x - 2y = 18$………(3)

eqn (2) + eqn (3)
$(3x + 10x) + (2y - 2y) = 8 + 18$
$13x = 26$
$x = 2$

Put $x = 2$ in eqn (1)
$5(2) - y = 9$
$10 - y = 9$
$y = 10 - 9 = 1$
Therefore, $x = 2$, $y = 1$

Exercises 15.17
Solve the pair of equations;
1. $\log_2 (2x + y) = 3$and $\log_2 (3x - 4y) = 0$
2. $\log_3 (3x - y) = \log_3 (y + 1)$ and $\log_3 2 + \log_3 (x + y) = 2$
3. $\log x^2 = \log y$ and $\log (2x + y) - \log 3 = 0$
4. $\log_2 2 + \log_2 (x + y) = \log_2 4$ and $\log_2 x + \log_2 y = 2$, $x > 0$, $y > 0$

The Characteristic and the Mantissa
The logarithm of any number consists of two parts namely: the ***characteristic*** which is the whole number and the ***mantissa*** which is the decimal part. For e.g., in $\log_{10} 300 = 2.4771$, the whole number at the left of the decimal point, 2 is the characteristic of $\log_{10} 300$ and the decimal part which consist of 0.4771 is the mantissa of $\log_{10} 300$

Finding the Characteristic

To find the characteristic of any number;

Method 1

I. Express the number in standard form/

II. Identify the power of ten as the characteristic of the number.

Method 2

The following guidelines are used to identify the characteristic;

I. If the number is equal to or greater than one, subtract one from the total number of digits to get the characteristic of that number. For e.g. 300 have three digits. Therefore, its characteristic is 3 – 1 = 2. Similarly, the characteristic of 71 is 2 – 1 = 1 and the characteristic of 6 is 1 – 1 = 0

II. If the number is a decimal number greater than or equal to one, subtract one from the total number of digits before (or at the right of) the decimal point . For e.g. in 300.12, there are three digits at the left of the decimal point. Therefore, the characteristic is 3 – 1 = 2. Similarly, the characteristic of 7.1 is 1 – 1 = 0

III. If the number is a decimal number less than one,

a. Identify the number of zeros immediately at the right of the decimal point and add one to it

b. Negate the sum in (a) to get the characteristic of the number.

Activities *a* and *b* are simplified as;

– (Number of immediate zeros after the decimal point plus one) For e.g. 0.000304 have three zeroes immediately after the decimal point. Therefore, the characteristic is calculated by negating the sum of three and one to get negative four. That is: – (3 + 1) = – 4

Worked Examples

Find the characteristic of the following:

1. 0.00574 2. 40322 3.36.19

Solution

Method 1

1) $0.00574 = 5.74 \times 10^{-3}$

The power of 10 = -3

Characteristics = -3

Method 2

Number of zeros after the decimal point

= 2 + 1 = 3

Negate the sum to get -3

Characteristics = -3

2. 40322

$40322 = 4.0322 \times 10^{4}$

Characteristics = 4

3. 36.19

$= 3.619 \times 10^{1}$

Characteristic = 1

Exercises 15.18

Find the characteristic of the following:

1. 4120 2. 0.000203 3. 6.12

4. 211.0045. 78×10^{-3}6. 0.344

Finding the Mantissa

The table of logarithm also known as *Four Figure Table* is used to find the mantissa (or the decimal part less than one). The four figure table has three columns as shown below;

34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

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I. The *x*- column contains the first two significant figures of a given number.

II. The second column, headed 0 – 9 gives the logarithm of the first three significant figures.

III. The third column, also known as "*difference column*" headed 1 – 9 is the difference to be added to the value of the log of the first three significant figures in act II.

Finding the Log of a Number

a. Identify the numbe.r

b. Find the characteristic of the number.

c. Put a decimal point at the right of the characteristic obtained.

d. Use the four figure table to determine the mantissa (decimal part).

Worked Examples

Use log table to find the following:

1. log 2.5 2. log 2.54 3. log 2.548

Solution

log 2.5 (equal to 2.50)

I. At column *x* of the log table, look for 25

II. At the second column, find the value of 25, under the column headed by 0

III. Thus, log 2.5 = 0.3979

2. log 2.54

I. At column *x* of the log table, look for 25

II. At the second column, find the value of 25, under the column headed by 4

III. Thus, log 2.54 = 0.4048

3. log 2.548

I. At column *x* of the log table, look for 25

II. At the second column, find the value of 25, under the column headed by 4 to obtain 0. 4048

III. On the same row, under the difference column, determine the number under 8, which is 14.

IV. Interpret 14 as 0.0014 and add it to 0.4048

Thus log 2.548 = 0.4048 + 0.0014

= 0.4072

3. Use tables to find the log of 2548

Solution

log 2548

2.548×10^3

Characteristic = 3

25 in the first column, under 4, difference 8

= 0.4048 + 0.0014

= 0.4062

log 2548 = 3 + 0.4022 = 3.4022

4. Use tables to find log 0.0004069

Solution

Method 1

log 0.0004069

4.069×10^{-4}

Characteristic = – 4

From the log table, find 40,under 6, difference 9

= 0.6085 + 0.0010 = 0.6095

log 0.0004069 = – 4 + 0.6095

= $\bar{4}$ .6095

Method 2

log 0.0004069 = log $\left(\frac{4.069}{10000}\right)$

= log 4.069 – log 10000

= log 4.069 – log 10^4

= 0.6095 – 4

= – 4 + 0.6095

= $\bar{4}$ .6095

Exercises 15.19

Use tables to find the following:

1. log 0.2468
2. log 24.68
3. log 2468
4. log 0.0004
5. log 3.456
6. log 7.24

The Anti – logarithm of a Number

Anti - logarithm is the reverse of logarithm. It implies that if ***log (a) = b, then a = anti – log (b)***

The ***anti – logarithm table*** is used to find the anti – log of a number. The anti – logarithm table is similar to that of the logarithm so the process involved in finding the logarithm of a number is likewise, similar to that of finding the anti-logarithm of a number. A portion of the four figure table is shown below:

TABLE 2 *(contd.)*: ANTILOGARITHMS OF NUMBERS

x	0	1	2	3	4	5	6	7	8	9	Differences								
											1	2	3	4	5	6	7	8	9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	6	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	6	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	5	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	8	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11

The rule for finding the antilog is as follows:
I. Find the antilog of the decimal part of the log of the given number
II. If the integral part (characteristic) is n, shift the decimal point n places to the right and if it is – n, shift the decimal point $n$places to the left

Worked Examples
1. Find the anti – log of 2.5127

Solution
To find the anti-log of 2.5127,
I. Identify the characteristic as 2 (meaning 10^2)
II. Consider the decimal part, 0.5127
III. From the anti–log table, read 0.51 under 2, difference 7.That is 0.51 under 2 = 3251
Interpreted as = 3.251
Difference 7 = 5
Interpreted as = 0.005
IV. Add the two values to obtain 3.251 + 0.005 = 3.256
V. Multiply 3.256 by 10^2
= 3.256 × 10^2 = 325.6
Anti – log of 2.5127 = 325.6

2. Find the anti – log of -3.2856

Solution
In -3.2856,
Characteristic = -3 (meaning 10^{-3})
0. 28 under 5, difference 6 = 1. 928 + 0.003
= 1.931
1.931 × 10^{-3} = 0.001931
Anti – log of -3.2856 = 0.001931

3. Find the anti – log of 0.0749

Solution
To find the anti – log of 0.0749,
Characteristic = 0 (meaning 10^0)
Read 0.07 under 4 difference 9
= 1.186 + 0.002
= 1.188
⇒ 1.188 ×10^0
= 1.188 × 1 = 1.188
Anti – log of 0.0749 = 1.188

Exercises 15.20
Find the antilogarithm:

1. 0.6914	2. 0.0975	3. 3.5007
4. – 4.2031	5. – 2.4348	6. – 1.8977

Finding the Value of a Variable Given the Value of the Logarithm of the Variable
Given the value of the logarithm of avariable, the value of the variable is easily found by the use of ***anti – logarithm table***.

Worked Examples
1. If log x = 0.0615, find the value of x

Solution
log x = 0.0615
x = anti – log (0.0615)
From the anti –log table,
I. In the first column, look for 0.06 under 1 in the second column to obtain = 1151
II. On the same row as 1151, and under difference column headed by 5 which is 1
III. Interpret 1151 as 1.151 and 1 as 0.001
IV. Add to get 1.151 + 0.001 = 1.152
Thus x = 1.152

2. If log x = 1.0607, find the value of x

Solution
log x = 1.0607
= 1 + 0.0607
= log 10 + log 1.5
= log (10 × 1.15)
= log 11.5

log x = log 11.5

⇒x = 11.5

3. If log x = $\bar{2}$.1319, find the value of x

Solution

log x = $\bar{2}$.1319

log x = $\bar{2}$. + 0.1319

= - log 100 + log 1.355

= log 1.355 – log 100

= log $\left(\frac{1.355}{100}\right)$

= log 0.01355

log x = log 0.01355

⇒x = 0.01355

Exercises 15.21

Use tables to find the value of *x*

1. log x = 0.521 2. log x = 0.0521
3. log x = $\bar{3}$.451

Using Tables to Evaluate Exponential Values, Values in Roots and Others

Worked Examples

Use tables to find the following:

1. 1.234^2 2. 14.02^3 3. $\sqrt{1.234}$ 4. $\sqrt[3]{0.4276}$

Solutions

1. Let $x = 1.234^2$

log x = log $(1.234)^2$

= 2 log 1.234

= 2 × 0.0913 = 0.1826

Anti – log 0.1826 = 1.523

x = 1.523

2. Let $x = 14.02^3$

log x = log $(14.02)^3$

= 3 log 14.02

= 3 × (1.1467)

= 3.4401

Anti – log 3.4401 = 2.755×10^3

x = 2755

3. Let $x = \sqrt{1.234}$

log x = log $(1.234)^{1/2}$

= $\frac{1}{2}$ log 1.234

= $\frac{1}{2}$ × 0.0913

= 0.04565

Anti – log 0.0456 = 1.111

x = 1.111

4. Let $x = \sqrt[3]{0.4276}$

log x = log $(0.4276)^{1/3}$

= $\frac{1}{3}$ × log 0.4276

= $\frac{1}{3}$ log (4.276×10^{-1})

= $\frac{1}{3}$ lg 4.276 + lg10^{-1}

= $\frac{1}{3}$ (0.6311 – 1)

= - 0.123

Anti – log - 0.123 = 0.7534

x = 0.7534

Exercises 15.22

A. Use tables to find the value of :

1. 2.423^2 2. 3.456^3 3. 0.0123^2
4. 12.12^3 5. 24^2 6. 0.9^3

B. Use tables to find the value of :

1. $\sqrt{1.231}$ 2. $\sqrt{0.0112}$ 3. $\sqrt{12.45}$
4. $\sqrt[3]{24.24}$ 5. $\sqrt[3]{148.5}$ 6. $\sqrt[3]{123.4}$

Computations with Logarithms

With the use of the log – tables, computations (like multiplication and division) can be done. The following examples are illustrations.

Worked Examples

Use tables to perform the following:

1. 1.345 ×0.0164 2. 0.2352 ÷ 3.342

3. $\frac{1.345 \times 0.0164}{3.342}$

4. $\frac{3.342 \times 1.345}{0.0164 \times 0.2352}$

5. $\frac{0.0164 \ \times 13.45^3}{\sqrt{3.342}}$

6. $\frac{\sqrt[3]{1.345} \ \times 0.2352^2}{0.0164 \ \times 1.241^2}$

Solutions

Method 1

Let x = 1.345 ×0.0164

log x = log (1.345 ×0.0164)

= log 1.345+ log 0.0164

= 0.1287 + $\bar{2}$.2148

= 0.1287 + (-2) + 0.2148

= -2 + 0.1287 + 0.2148

= - 2 + 0.3435 = $\bar{2}$. 3435

Anti – log $\bar{2}$. 3435 = 0.02206

x = 0.02206

Method 2

No	Log
1.345 0.0164	0.1287 + $\bar{2}$.2148
0.02206	$\bar{2}$.3435

2. 0.2352 ÷ 3.342

Method 1

Let x = 0.2352 ÷ 3.342

log x = log$\left(\frac{0.2352}{3.342}\right)$

= log 0.2352 –log 3.342

= $\bar{1}$.3715 – 0.5240

= -1 + 0. 3715 – 0.5240

= -2 + 1 + 0.3715 – 0.5240

= -2 + 1.3715 – 0.5240

= -2 + 0.8475

= $\bar{2}$.8475

Anti –log $\bar{2}$.8475 = 0.07039

x = 0.07039

Method 2

No	log
0.2352 3.342	$\bar{1}$ .3715 – 0.5240
0.07039	$\bar{2}$. 8475

3. $\frac{1.345 \times 0.0164}{3.342}$

No	Log
1.345 0.0164	0.1287 + $\bar{2}$ .2148
 3.342	$\bar{2}$.3435 - 0.5240
0.006600	$\bar{3}$.8195

4. $\frac{3.342 \times 1.345}{0.0164 \times 0.2352}$

No	Log
3.342 1.345	0.5240 + 1.287
 0.0164	0.6527 – $\bar{2}$.2148
 0.2352	2.4379 – $\bar{1}$.3715
1165	3.0664

5. $\frac{0.0164 \ \times 13.45^3}{\sqrt{3.342}}$

No	Log
0.0164 13.45^3	$\bar{2}$.2148 3 × 1.1287 = + 3.3861
 $\sqrt{3.342}$	= 1.6009 $\frac{1}{2}$ × 0.524 = – 0.2620
21.83	= 1.3389

6. $\frac{\sqrt[3]{1.345} \ \times 0.2352^2}{0.0164 \ \times 1.241^2}$

No	Log
$\sqrt[3]{1.345}$ 0.2352^2	$\frac{1}{3}$ × 0.1287 = 0.0429 2 × $\bar{1}$.3715 = + $\bar{2}$.7430
 0.0164	= $\bar{2}$.7859 = – $\bar{2}$.2148
	= 0.5711

1.245^4	$4 \times 0.0938 = -0.3752$
1.571	$= 0.1959$

Exercises 15.22

With the help of four – figure tables, compute the following:

A. 1. 3.142×0.123 2. 12.12×0.1134

3. 5.23×4.86 4. 698×62

5. $3.103 \times 4.3 \times 0.021$ 6. $0.002 \times 4.34 \times 1.23$

B. 1. $869 \div 43$2. $6.524 \div 2.122$

3. $3.361 \div 5.124$ 4. $0.0899 \div 0.33$

5. $4.68 \div 4.1 \div 32$

C. 1. $\frac{1.249 \times 3.624}{3.128}$ 2. $\frac{234.1 \times 36.1}{4.31}$

3. $\frac{1.23 \times 741}{21 \times 4.86}$ 4. $\frac{3.178 \times 4.123}{1.142 \times 0.0123}$

5. $\frac{\sqrt{3.142} \times 2.139^2}{\sqrt{41.12} \times 3.123^3}$ 6. $\frac{45.67 \times \sqrt[3]{9.921}}{6.627^2 \times \sqrt{11.234}}$

16 SIMULTANEOUS EQUATIONS

Introduction

It involves solving two or more equations at the same time. Each of the involving equations consists of two variables of the same identity.

Study this case carefully; "when three times a number is added to another number, the result is 16 and the sum of the same numbers is 4." Equations written for cases of this nature are called ***simultaneous equations*.**

Two separate equations are written and solved at the same time. The set of ordered pairs that satisfy or make both statements true, is called the ***truth set or solution set of the equations*.**

Solving Simultaneous Equations

There are 3 general ways of solving simultaneous equations namely;

1. Elimination method.
2. Substitution method
3. Graphical method

Any of the methods can be used to solve simultaneous equation unless specified.

Elimination Method

In this method, one of the variables, having the same co – efficient in both equations is eliminated (by addition or subtraction).

<u>Type 1: (Equal co-efficients)</u>

This involves the situation where a variable has the same coefficient in both equations. Here:

i. elimination is done by addition or subtraction.

ii. then divide both sides of the equation by the coefficient of the existing variable to obtain its value.

iii. Substitute the value of the variable obtained in either eqn (1) or eqn (2) and solve to obtain the value of the other.

vi. Write the answer as ordered pairs (x, y).

Worked Examples

1. Find the values of x and y in the pair of equations; $3x + 2y = 16$ and $5x + 2y = 20$.

Solution

$3x + 2y = 16$ ……………….(1)

$5x + 2y = 20$ ………………..(2)

y has the same co – efficient in both equations, eliminate y by subtraction.

$\therefore$ eqn (2) – eqn (1)

$(5x - 3x) + (2y - 2y) = (20 - 16)$

$2x = 4$,

$x = \frac{4}{2} = 2$.

Put $x = 2$ in eqn (1)

$3(2) + 2y = 16$

$6 + 2y = 16$

$2y = 16 - 6$

$2y = 10$,

$y = \frac{10}{2} = 5$

$\therefore$ The truth set = $\{(x, y): x = 2, y = 5\}$

2. Solve the equations $5x - 6y = 21$ and $3x - 6y = 3$ simultaneously.

Solution

$5x - 6y = 21$ ……………(1)

$3x - 6y = 3$………………(2)

Eliminate y by subtraction;

∴ eqn (2) – eqn (1);

$(3x - 5x) + [- 6y - (- 6y)] = (3 - 21)$

$-2x = -18,$

$x = 9.$

Put $x = 9$ in eqn (1)

$5(9) - 6y = 21$

$45 - 6y = 21$

$- 6y = 21 - 45$

$- 6y = - 24$

$y = \frac{-24}{-4} = 6$

∴ The truth set = $\{(x, y): x = 4, y = -\frac{4}{9}\}$

3. Solve the system;

$x - y = 12$ and $2x + y = 13$

Solution

$x - y = 12$ ……..………(1)

$2x + y = 13$……………(2)

Eliminate y by addition;

∴ eqn (2) + eqn (1)

$(2x + x) + [y + (-y)] = (13 + 12)$

$3x = 25,$

$x = \frac{25}{3}$

Put $x = \frac{25}{3}$ in equation (1)

$\frac{25}{3} - y = 12$

$- y = 12 - \frac{25}{3}$

$- y = \frac{11}{3}$

$y = -\frac{11}{3}$

∴ The truth set = $\{(x, y): x = \frac{25}{3}, y = -\frac{11}{3}\}$

4. Find the values of x and y which satisfy the system $- 4x + 5y = - 8$ and $- 4x + y = 12$.

Solution

$- 4x + 5y = - 8$ ………………..(1)

$- 4x + y = 12$……………………(2)

Eliminate x by subtraction;

∴ eqn (2) – eqn (1)

$(- 4x - (- 4x) + (y - 5y) = [12 - (-8)]$

$- 4y = 20,$

$y = - 5.$

Put $y = 5$ in eqn (1)

$- 4x + 5(-5) = -8$

$- 4x - 25 = - 8$

$- 4x = - 8 + 25$

$- 4x = 17$

$x = -\frac{17}{4}$

∴ $(x, y) = (-\frac{17}{4}, - 5)$

5. Solve the equations; $3x - 7y = - 12$ and $3x - 5y = - 6$ simultaneously.

Solution

$3x - 7y = - 12$ …………………(1)

$3x - 5y = - 6$ …………………..(2).

Eliminate x by subtraction;

eqn (2) – eqn (1);

$(3x - 3x) + [(-5y - (-7y)] = [- 6 - (-12)]$

$2y = 6,$

$y = 3.$

Put $y = 3$ in eqn (1)

$3x - 7(3) = - 12$

$3x - 21 = -12$

$3x = -12 + 21$

$3x = 9$

$x = 3$

∴ $(x, y) = (3, 3)$

6. What values of x and y satisfy the system
$- 7x – 2y = 10$ and $7x + 4y = 22$?

Solution
$- 7x – 2y = 10$ …………….(1)
$7x + 4y = 22$………………(2)

Eliminate x by addition;
eqn (2) + eqn (1)
$[(7x + (-7x)] + [4y + (-2y)] = (22 + 10)$
$2y = 32$,
$y = 16$.

Put $y = 16$ in eqn (1)
$- 7x - 2(16) = 10$
$- 7x – 32 = 10$
$- 7x = 10 + 32$
$- 7x = 42$
$x = - 6$
$\therefore (x, y) = (- 6, 16)$

7. Solve $4x + y = 10$ and $- 4x – 3y = 2$ simultaneously.

Solution
$4x + y = 10$ ……………………(1)
$- 4x – 3y = 2$ …………………..(2)

Eliminate x by addition;
eqn (2) + eqn (2)
$[(- 4x + 4x)] + (-3y + y) = (2 + 10)$
$- 2y = 12$,
$y = - 6$.

Put $y = - 6$ in eqn (1);
$4x – 6 = 10$
$4x = 10 + 6$
$4x = 16$
$x = 4$
$\therefore (x, y) = (4, - 6)$

Type 2: (Multiplication by a constant)

If one of the coefficients of the variables in the equations is a factor of the other of the same kind in the other equation, multiply by a constant number to obtain the same coefficient and name it equation (3). Then carry on with the elimination process.

Worked Examples
1. Find a and b for the pair of equations:
$4a + 4b = 36$ and $6a – 2b = 22$

Solution
$4a + 4b = 36$…………….(1)
$6a – 2b = 22$…………….(2)

$2 \times$ eqn (2);
$12a – 4b = 44$…………….(3)

Now, consider eqn (1) and eqn (3)
$4a + 4b = 36$……………..(1)
$12a – 4b = 44$…………….(3)

Eliminate b by addition,
eqn (1) + eqn (3);
$16a = 80$
$a = 5$

Put $a = 5$ in eqn (1);
$4(5) + 4b = 36$
$20 + 4b = 36$
$4b = 36 – 20$
$4b = 16$
$b = 4$
$(a, b) = (5, 4)$

2. Solve the pair:
$6a + 2b = 18$ and $5a + 6b = 28$

Solution

$6a + 2b = 18$……………...(1)
$5a + 6b = 28$ ……………..(2)

$3 \times$ eqn (1);
$18a + 6b = 54$ ……..…….(3)

eqn (3) – eqn (2);
$13a = 26$
$a = 13$

Put $a = 13$ in eqn (1);
$6(13) + 2b = 18$
$78 + 2b = 18$
$2b = -60$
$b = -30$
$(a, b) = 13, -30)$

Type 3: Uncommon coefficient

If the co - efficients are unequal, obtain equal ones by;

I. Multiplying the co – efficient of x in eqn (1) by all the terms of equation (2) and likewise, multiply the co – efficient of x in eqn (2) by all the terms of equation (1). This gives rise to two new equivalent equations (3) and eqn (4) respectively with common coefficient in one variable. The coefficient of y can also be used in a similar fashion to obtain eqns (3) and (4).

II. Taking eqn (3) and eqn (4) into consideration, this bring us back to ***type 1***. Therefore, eliminate the variable with the same co – efficient either by adding or subtracting one equation from the other respectively.

Worked Examples

1. Solve the following pair of equations simultaneously; $3x + y = 10$ and $2x + 2y = 4$

Solution

Name the equations as
$3x + y = 10$ …………………..(1)
$2x + 2y = 4$ ………………….. (2)

Obtain equal co efficient of x in both equations;
$2 \times$ eqn (1);
$6x + 2y = 20$………………….. (3)

$3 \times$ eqn (2);
$6x + 6y = 12$ …………………..(4)

eqn (3) – eqn (4);
$(6x - 6x) + (2y - 6y) = (20 - 12)$
$-4y = 8$,
$y = \ = -2$

Put $y = -2$ in eqn (1);
$3x + (-2) = 10$
$3x - 2 = 10$
$3x = 10 + 2$
$3x = 12$
$x = 4$
$\therefore (x, y) = (4, -2)$

2. Solve $4x + 3y = 6$ and $3x + 2y = 3$ simultaneously;

Solution

$4x + 3y = 6$ ---- (1)
$3x + 2y = 3$ ---- (2)

$3 \times eqn$ (1);
$12x + 9y = 18$ …………………(3)
$4 \times$ eqn (2);
$12x + 8y = 12$ …………………(4)

eqn (3) – eqn (4);
$(12x - 12x) + (8y - 6y) = 18 - 6$
$2y = 12$,
$y = \frac{12}{2} = 6.$

Put $y = 6$ in eqn (1);
$4x + 3(6) = 6$
$4x + 18 = 6$
$4x = 6 – 18$
$4x = −12$,
$x = -\frac{12}{4} = −3$
∴ The truth set = $\{(x, y): x = −3, y = 6\}$

3. If $3x – 7y = 2$ and $x – 2y = 4$, what are the values of x and y.

Solution
$3x – 7y = 2$ ………………..(1)
$x – 2y = 4$………………….(2)

1 × eqn (1);
$3x – 7y = 2$ ……………….. (3)

3 × eqn (2);
$3x – 6y = 12$ ……………… (4)

eqn (4) – eqn (3);
$(3x - 3x) – 6y – (−7y) = 12 – 2$
$− 6y + 7y = 10$
$y = 10$

Put $y = 10$ into eqn (2);
$x – 2(10) = 4$
$x – 20 = 4$
$x = 4 + 20 = 24$
∴ The truth set = $\{(x, y) : x = 24, y = 10\}$

4. Solve these equations simultaneously;
$3x + 5y = - 6$ and $2x – 3y = 15$

Solution
$3x + 5y = - 6$……………(1)
$2x – 3y = 15$……………(2)

2 × eqn (1);
$6x + 10y = -12$…………(3)

3 × eqn (2);
$6x – 9y = 45$ …………..(4)

eqn (3) – eqn (4);
$19y = -57$
$y = -3$

Put $y = -3$ in eqn (2);
$2x – 3(-3) = 15$
$2x + 9 = 15$
$2x = 15 – 9$
$2x = 6$
$x = 3$
∴ $(x, y) = (3, -3)$

Exercises 16.1
A. Solve by elimination method;
1. $3x – 4y = 2$ and $7x + 2y = 16$
2. $3x – 5y = 19$ and $4x – 7y = 26$
3. $3a – 2b = 5$ and $a + 4b = 4$
4. $3x + 4y = 5$ and $5x + 7y = 3$
5. $3x – 5y = 8$ and $x + 2y = 10$

B. Solve the following by elimination;
1. $5x + 2y = 11$ and $3x + 2y = 9$
2. $3x + 4y = 23$ and $2x + 4y = 18$
3. $6x + 2y = – 6$ and $4x + 3y = 1$
4. $– x + 3y = 3$ and $2x + 3y = –11$
5. $x + 5y = – 12$ and $2x – y = 9$
6. $7x – 4y = 40$ and $4x + 5y = 43$

C. Solve the system of equations:
1. $5x + 2y = 14$ and $6x + 2y = 16$
2. $7x + 3y = 27$ and $6x + 3y = 24$
3. $10x – 2y = 30$ and $3x – 2y = 2$
4. $9x – 6y = 42$ and $6x – 6y = 18$
5. $5x + 3y = 69$ and $7x – 3y = 75$

Substitution Method

Method 1

I. Make one variable the subject of any of the equations and substitute the derived equation into the other equation. This means that when a variable is made the subject of equation (1), the derived equation is substituted in equation (2) and vice – versa.

II. At this stage, one variable is eliminated. The result is then simplified to get the value of the existing variable.

III. Find the value of the eliminated variable by substituting the value of the known variable in any of the equations to obtain the ordered pairs that satisfy both equations.

Method 2

I. Choose a preferred variable.

II. Make that variable the subject of eqn (1) and name the derived equation as eqn (3)

III. Make that same variable the subject of equation (2) and name the derived equation as eqn (4).

IV. At this point, equate equation (3) and equation (4) and solve to obtain the value of theinvolving variable.

V. Put the value of the variabke obtained in either equation (1) or (2) to obtain the value of the other variable.

VI. Express the answer as ordered pairs (x, y).

Worked Examples

1. Solve: $3x + y = 10$ and $2x + 2y = 4$ by method of substitution.

Solution

Name the equations as;

$3x + y = 10$ ……….(1)

$2x + 2y = 4$………..(2)

Make y the subject of eqn (1).

$\Rightarrow y = 10 - 3x$

Put/substitute $y = 10 - 3x$ in eqn (2);

$2x + 2(10 - 3x) = 4$ (y is eliminated).

$2x + 2(10 - 3x) = 4$

$2x + 20 - 6x = 4$

$2x - 6x = 4 - 20$

$-4x = -16$

$x = \frac{-16}{-4} = 4$

Put x = 4 in eqn (1) to obtain the value of y

$3(4) + y = 10$

$12 + y = 10$

$y = 10 - 12 = -2$

Truth set = {(x, y) : $(4, -2)$}

Method 2

$3x + y = 10$ ………………...(1)

$2x + 2y = 4$………………...(2)

Make y the subject of eqn (1);

$\Rightarrow y = 10 - 3x$ …………….(3)

Make y the subject of eqn (2);

$2y = 4 - 2x$

$y = \frac{4-2x}{2}$……………………(4)

eqn (3) = eqn (4);

$10 - 3x = \frac{4-2x}{2}$

$2(10 - 3x) = 4 - 2x$

$20 - 6x = 4 - 2x$

$-6x + 2x = 4 - 20$

$-4x = -16$

$x = 4$

Put $x = 4$ in eqn (1);

$3(4) + y = 10$

$12 + y = 10$

$y = 10 - 12 = -2$

$(x, y) = (4, -2)$

2. What are the values of $x$and y in the following equations?

$3x + 2y = 3$ and $4x + 3y = 6$

Solution

$3x + 2y = 3$……... (1)

$4x + 3y = 6$………(2)

Make y the subject of eqn (1);

$2y = 3 - 3x$

$\frac{2y}{2} = \frac{3-3x}{2}$

$y = \frac{3-3x}{2}$

Put $y = \frac{3-3x}{2}$ *in eqn (2);*

$4x + 3\left(\frac{3-3x}{2}\right) = 6$

$2 \times 4x + 2 \times 3\left(\frac{3-3x}{2}\right) = 2 \times 6$

$8x + 3(3 - 3x) = 12$

$8x + 9 - 9x = 12$,

$8x - 9x = 12 - 9$ (*Grouping like terms)*

$-x = 3$

$x = -3$

Put $x = -3$ *into eqn (1);*

$3(-3) + 2y = 3$

$-9 + 2y = 3$

$2y = 3 + 9$

$2y = 12$

$y = \frac{12}{2} = 6$

Truth set = $\{(x, y) : (-3, 6)\}$

3. What are the values of x and y if $y + 2x = 3$ and $3y - 4x = 4$. Use substitution method.

Solution

$y + 2x = 3$ ……….. …(1)

$3y - 4x = 4$………….. (2)

Make y the subject of eqn (1)

$y = 3 - 2x$

Put $y = 3 - 2x$ *into eqn (2);*

$3(3 - 2x) - 4x = 4$

$9 - 6x - 4x = 4$

$-6x - 4x = 4 - 9$

$-10x = -5$

$x = \frac{1}{2}$

Put $x = \frac{1}{2}$ *into eqn (1);*

$y + 2\left(\frac{1}{2}\right) = 3$

$y + 1 = 3$

$y = 3 - 1 = 2$

Truth set = $\{(x, y) : (\frac{1}{2}, 2)\}$

Exercises 16.2

A. Solve the following by substitution;

1. $7x - 3y = 40$ and $4x + 5y = 43$
2. $3x - 2y = -8$ and $4x + 4y = -14$
3. $x - y = 12$ and $2x - y = 8$
4. $8x - 2y = -14$ and $4x - 3y = 3$
5. $2x - 2y = 2$ and $5x - 6y = 1$

B. Solve by any preferred method;

1. $4x + 5y = -8$ and $3x + y = 5$
2. $2x + 7y = 4$ and $x + 3y = 1$
3. $6x + 11y = 16$ and $2x - 3y = -8$
4. $9x + 7y = 8$ and $-4x + 3y = 27$
5. $2x - 9y = 22$ and $8x - 11y = 48$

Solved Past Questions

1. If $3x - y = 5$ and $2x + y = 15$, evaluate; $x^2 + 2y$

Solution

$3x - y = 5$ ……….(1)

$2x + y = 15$………(2)

eqn (1) + eqn (2);

$5x = 20$

$x = 4$

Put $x = 4$ in eqn (1);

$3(4) – y = 5$

$12 – y = 5$

$12 – 5 = y$

$y = 7$

$(x, y) = (4, 7)$

$x^2 + 2y = (4)^2 + 2(7) = 30$

Involving Fractional Equations

1. When the equations involve one fraction, multiply each term of equation by the denorminator of the fraction to obtain equivalent equation
2. When the equations involve one or more fractions, find the L.C.M. of the fractions and multiply the L.C.M. by each term of the equation to obtain equivalent equation(s).

Solve the equivalent equations by any preferred method unless stated.

Worked Examples

1. Solve the system of equations;

$\frac{1}{2}x + \frac{3}{4}y = 2$ and $x + 3y = 7$

Solution

1. $\frac{1}{2}x + \frac{3}{4}y = 2$……………(i)

$x + 3y = 7$ ………………(ii)

Multiply through eqn (1) by L. C. M = 4

$(4)\frac{1}{2}x + (4)\frac{3}{4}y = (4)\ 2$

$2x + 3y = 8$……………………………(1)

$x + 3y = 7$ ……………………………..…(2)

eqn (1) – eqn (2);

$(2x – x) + (3y – 3y) = (8 – 7)$

$x = 1$

Put $x = 1$ in eqn (1);

$2(1) + 3y = 8$

$2 + 3y = 8$

$3y = 8 – 2$

$3y = 6$

$y = 2$

$(x, y) = (1. 2)$

2. Find the values of a and b that satisfy the equations; $\frac{6}{5}a + \frac{3}{2}b = -4$ and $\frac{1}{3}a – \frac{5}{4}b = 1$

Solution

$\frac{6}{5}a + \frac{3}{2}b = -4$……………(i)

$\frac{1}{3}a – \frac{5}{4}b = 1$…………….(ii)

Multiply through eqn (1) by L.C.M = 10

$(10)\frac{6}{5}a + (10)\frac{3}{2}b = -4\ (10)$

$2(6a) + 5(3b) = -40$

$12a + 15b = -40$ …………….(1)

Multiply through eqn (2) by L.C.M = 12

$(12)\frac{1}{3}a – (12)\frac{5}{4}b = (12)$

$4a - 3(5b) = 12$

$4a – 15b = 12$ …………….(2)

Solving eqn (1) and eqn (2) simultaneously;

$12a + 15b = -40$ …………….(1)

$4a – 15b = 12$ ………………..(2)

eqn (1) + eqn (2);

$16a = -28$

$a = -\frac{7}{4}$

Put $a = -\frac{7}{4}$ in eqn (1)

$12\left(-\frac{7}{4}\right) + 15b = -40$
$12(-7) + 4(15b) = -40(4)$
$-84 + 60b = -160$
$60b = -160 + 84$
$60b = -76$
$b = -\frac{19}{15}$
$(a, b) = \left(-\frac{7}{4}, -\frac{19}{15}\right)$

3. Solve the simultaneous equations
$3x – 2y + 2 = 0$ and $\frac{x-1}{3} = \frac{y-2}{5}$

Solution
$3x – 2y + 2 = 0$…………..(i)
Re- arrange to obtain;
$3x – 2y = – 2$ ……………(1)
$\frac{x-1}{3} = \frac{y-2}{5}$……………….(ii)

Multiply through by L.C.M of 3 and 5 = 15
$15 \times \frac{x-1}{3} = 15 \times \frac{y-2}{5}$
$5(x – 1) = 3(y – 2)$
$5x – 5 = 3y – 6$
$5x – 3y = -6 + 5$
$5x – 3y = -1$……………….(2)

eqn (1) × 5
$15x – 10y = – 10$…………..(3)

eqn (2) × 3
$15x – 9y = -3$……………..(4)

eqn (3) – eqn (4);
$-y = -7$
$y = 7$

Put $y = 7$ in eqn (1)
$3x – 2(7) = – 2$
$3x – 14 = – 2$
$3x = – 2 + 14$
$3x = 12$
$x = 4$
$(x, y) = (4, 7)$

4. Solve the simultaneous equations;
$2x – 3y = 5$ and $\frac{x}{2} + \frac{y}{3} = – 2$

Solution
$2x – 3y = 5$……………(1)

$\frac{x}{2} + \frac{y}{3} = – 2$
$(6)\frac{x}{2} + (6)\frac{y}{3} = – 2\ (6)$
$3x + 2y = – 12$ …………(2)

eqn (1) × 3
$6x – 9y = 15$……………..(3)

eqn (2) × 2
$6x + 4y = – 24$……………(4)

eqn (3) – eqn (4)
$-13y = 39$
$y = -\frac{39}{13} = – 3$

Put $y = – 3$ in eqn (1)
$2x – 3(-3) = 5$
$2x + 9 = 5$
$2x = 5 – 9$
$2x = -4$
$x = -\frac{4}{2} = – 2$
$(x, y) = (-2, -3)$

Solved Past Question
1. If $\frac{1}{2}p + q = 1$ and $p – \frac{1}{2}q = 7$, find $(p + q)$

Solution
$\frac{1}{2}p + q = 1$ ……………..(1)
$p – \frac{1}{2}q = 7$………………(2)

2 × eqn (1)

$p + 2q = 2$……………..(3)

eqn (2) – eqn (3)

$-\frac{5}{2}q = 5$

$-5q = 2 \times 5$

$-5q = 10$

$q = -2$

Put $q = -2$ in eqn (1)

$\frac{1}{2}p + (-2) = 1$

$\frac{1}{2}p = 1 + 2$

$\frac{1}{2}p = 3$

$p = 2 \times 3 = 6$

$(p, q) = (6, -2)$

$(p + q) = 6 + (-2) = 4$

Exercises 16.3

Find the truth set;

1. $\frac{2}{5}x + \frac{1}{3}y = 1$ and $\frac{3}{5}x - \frac{1}{9}y = 7$
2. $\frac{1}{2}x + \frac{1}{3}y = 2$ and $4x - y = 5$
3. $\frac{1}{6}(x - 1) + y = 6$ and $\frac{1}{4}(y - 1) + x = 8$
4. $y = \frac{1}{4}(x - 2)$ and $\frac{1}{3}x + y = 10$

Challenge Problems

Solve the pair of equations simultaneously;

1. $2x + y = 3(y - x) + 7$ and $\frac{x}{3} = 2 - \frac{y}{4}$
2. $\frac{x+1}{2} - \frac{y+3}{3} = 4$ and $x + \frac{y-3}{2} = \frac{1}{2}$

Involving Quadratics

If one of the involving equations is operated by a multiplication sign, or one is a quadratic, the equations can only be solved by substitution method, which in turn becomes a quadratic equation, solved by factorization or method of completing squares.

Worked Examples

1. Solve the simultaneous equations

$xy = 12$ and $x + y = 7$

Solution

$xy = 12$……….(1)

$x + y = 7$ …… (2)

From eqn (1)

$x = \frac{12}{y}$

Put $x = \frac{12}{y}$ in eqn (2)

$\frac{12}{y} + y = 7$ *(Multiply through by y)*

$12 + y^2 = 7y$

$y^2 - 7y + 12 = 0$

$(y^2 - 4y) - (3y + 12) = 0$

$y(y - 4) - 3(y - 4) = 0$

$(y - 3)(y - 4) = 0$

$y - 3 = 0$ or $y - 4 = 0$

$y = 3$ or $y = 4$

From eqn (2), when $y = 3$

$x + 3 = 7$

$x = 7 - 3 = 4$

When $y = 4$,

$x + 4 = 7$

$x = 7 - 4 = 3$

Truth set = $\{(x, y) : (4, 3)$ and $(4, 3)\}$

2. Solve the equations:

$x - y = -2$ and $xy = 24$

Solution

$x - y = -2$ ……………(1)

$xy = 24$……………….(2)

From eqn (1);

$x = -2 + y$

Put $x = -2 + y$ *in eqn (2)*
$(-2 + y)y = 24$
$-2y + y^2 = 24$
$y^2 - 2y - 24 = 0$
$(y - 6)(y + 4) = 0$
$y = 6$ or $y = -4$

Put $y = 6$ *in eqn (2)*
$6x = 24$
$x = 4$

Put $y = -4$ in eqn (2)
$-4x = 24$
$x = -6$
Truth set = {(x, y) : (4, 6) and (- 6, - 4) }

3. Find the truth set of the equations
$x - y = 5$ and $x^2 + 2y = -2$

Solution
$x - y = 5$ ……….(1)
$x^2 + 2y = -2$ …….(2)

From eqn (1);
$-y = 5 - x$
$y = -(5 - x)$
$y = -5 + x$

Put $y = -5 + x$ in eqn (2);
$x^2 + 2(-5 + x) = -2$
$x^2 - 10 + 2x = -2$
$x^2 + 2x - 10 + 2 = 0$

$x^2 + 2x - 8 = 0$
$(x - 2)(x + 4) = 0$ (by factorization)
$\Rightarrow x - 2 = 0$ or $x + 4 = 0$
$x = 2$ or $x = -4$

Put $x = 2$ in eqn (1);
$2 - y = 5$
$-y = 5 - 2$
$y = -3$

Put $x = -4$ in eqn (1);
$-4 - y = 5$
$-y = 5 + 4$
$y = -9$
Truth set = {(x, y) : (2, -3) and (- 4, - 9) }

Exercises 16.4

A. Solve the following systems of equations:

1. $y^2 = 4x$ and $y = x$
2. $xy = 64$ and $4x - y = 60$
3. $y^2 = 4x + 1$ and $y = x + 1$

B. Solve the simultaneous equations

1. $y + x = 11$ and $x^2 + y^2 = 61$
2. $y = 2x + 5$ and $x^2 + y^2 = 5$
3. $y = 2x - 6$ and $x^2 + y^2 = 72$
4. $y = x + 2$ and $x^2 + y^2 = 34$
5. $x^2 + y^2 = 40$ and $y = x - 4$

Challenge Problems

Solve the pair of equations;

1. $\frac{1}{x} + \frac{1}{y} = 3$ and $\frac{2}{x} + \frac{3}{y} = 7$
2. $\frac{3}{x} + \frac{4}{y} = 2$ and $\frac{4}{x} - \frac{1}{y} = 3$
3. $\frac{x+1}{y+1} = 2$ and $\frac{2x+1}{2y+1} = \frac{1}{3}$
4. $\frac{c + d + 2}{7} = \frac{c}{3} = \frac{d}{2}$
5. $\frac{x + 2y + 1}{4} = \frac{3x + y + 1}{8} = \frac{2x + 3y + 2}{9}$

Involving Exponential Equations

When given an exponential equation, apply the law of indices and reduce it to a linear equation or a quadratic equation. Then solve by any preferred method unless stated

Worked Examples

1. Solve the simultaneous equations;
$3^{2x-y} = 27$ and $4^{(x+y)} = \frac{1}{2^6}$

Solution

$3^{2x-y} = 27$

$3^{2x-y} = 3^3$

$2x - y = 3$……………(1)

$4^{(x+y)} = \frac{1}{2^6}$

$2^{2(x+y)} = \frac{1}{2^6}$

$2^{2(x+y)} = 2^{-6}$

$2(x + y) = -6$

$2x + 2y = -6$ ……….(2)

eqn (1) – eqn (2);

$-3y = 9$

$y = \frac{9}{-3} = -3$

Put y = -3 in eqn (1)

$2x - (-3) = 3$

$2x = 0$

$x = 0$

Truth set = {(x, y) : (0, -3) }

2. Solve simultaneously, the equations:

$5x - 4y = 6$ and $3^{3(y-x)} = \frac{1}{27}$

Solution

$5x - 4y = 6$…………..(1)

$3^{3(y-x)} = \frac{1}{27}$

$3^{3(y-x)} = 27^{-1}$

$3^{3(y-x)} = 3^{3(-1)}$

$3(y - x) = 3(-1)$

$3y - 3x = -3$

$-3x + 3y = -3$…………(2)

$3 \times$ eqn (1);

$15x - 12y = 18$……….(3)

$4 \times$ eqn (2);

$-12x + 12y = -12$………(4)

eqn (3) + eqn (4);

$3x = 6$

$x = 2$

Put $x = 2$ in eqn(1)

$5(2) - 4y = 6$

$10 - 4y = 6$

$-4y = 6 - 10$

$-4y = -4$

$y = 1$

$(x, y) = (2, 1)$

Exercises 16.5

A. Solve the following pair of equations

1. $5^{x+y} = 25$ and $49 = \frac{1}{7^y}$
2. $2a + 3b = 3$ and $2^{\frac{a}{b}} = \frac{1}{32}$
3. $y = x^2 - 4$ and $y = 2x - 1$
4. $x + 6y = 3$ and $2^{x+y} = \frac{1}{4}$
5. $3^a \times 3^b = 243$ and $3^a \div 3^{2b} = 3^5$

Graphical Method

An equation of the form $ax + by + c = 0$ is a linear equation. Its graph can be drawn by finding the points where it cuts the x and$y$ – axes. The coordinates of the points of intersection (x,y) of the graph of two linear relations is the truth set of the two relations. This is easily found by going through the following steps:

I. Find where the straight line cut's the x and y – axes. To find the intercept on the y – axis, put x = 0, and solve for y in the given relations. To find intercept on x– axis, put y = 0 and solve for x in both relations.

II. Prepare two separate tables of values foreach relation using x and y values obtained.

III. Plot the points on a graph sheet and join them with a ruler, to obtain two straight lines that intersects each other.

IV. Locate the point of intersection of the two lines. The co-ordinates of this point is the solution set of the equations.

Worked Examples

1. Find the truth set of $x + y = 6$ and $x - y = 2$ by graphical method.

Solution

For $x + y = 6$

Intercept on the x – axis ,

When $y = 0$, , $x + 0 = 6$, $x = 6$

$(x, y) = (6, 0)$

Intercept on the y – axis,

When $x = 0$, $0 + y = 6$, $y = 6$

$(x, y) = (0, 6)$

Table of values for $x + y = 6$

x	6	0
y	0	6

For $x - y = 2$

Intercept on x – axis, $y = 0$

When $y = 0$, $x - 0 = 2$, $x = 2$

$(x, y) = (2, 0)$

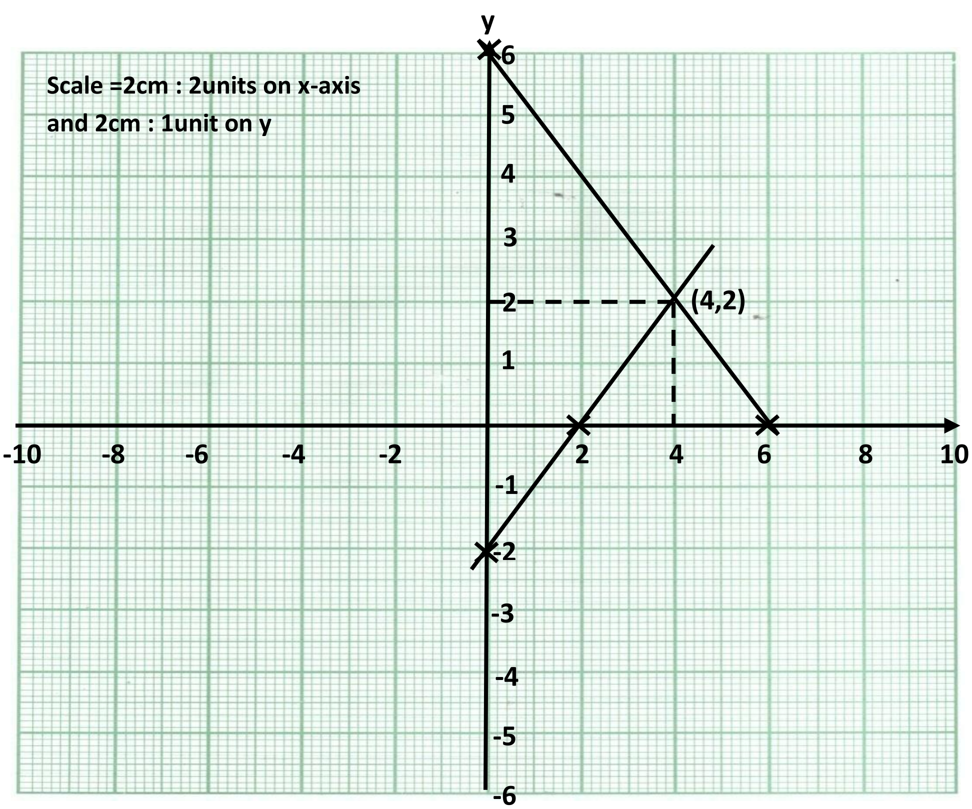
Intercept on y – axis, $x = 0$

When $x = 0$, $0 - y = 2$, $y = -2$;

$(x, y) = (0, -2)$

Table of values for $x - y = 2$

x	0	2
y	-2	0



From the graph, the two lines meet at the point where $x = 4$ and $y = 2$

∴ The truth set is: $x = 4, y = 2$

Exercises16.6

A. Solve by graphical method.

1. $y = 5 - x$ and $y = 2x - 1$
2. $y = 3x - 2$ and $y = \frac{1}{3}x + 2$
3. $y = x - 6$ and $y = 4 - 2x$
4. $y = x + 4$ and $y = 5 - 3x$
5. $y - \frac{1}{2}x = 3$ and $y + \frac{2}{3}x = 5$

Challenge problems

Show that the straight lines with equations; $3x + 4y = 2$, $5x - 2y = 12$ and $2x + 3y = 1$, all intersect at the same point. Illustrate in a sketch.

Word Problems

Equations involving two variables may be stated in words. Students are advised to write mathematical equations to represent the statements and solve them by any preferred method unless specified. It is also advisable to use x and y for the involving variables.

Worked Examples

1. The sum of two numbers is 13 and their difference is 3. Find the two numbers.

Solution

Method 1: (Substitution)

Let x and y be the two numbers. The involving equations are;

$x + y = 13 \ldots\ldots\ldots.(1)$

$x - y = 3 \ldots\ldots\ldots\ldots(2)$

Make y the subject of eqn (1)

$y = 13 - x$

Put $y = 13 - x$ *in eqn (2;)*

$x - (13 - x) = 3$

$x - 13 + x = 3$

$x + x = 3 + 13$

$2x = 16$

$x = \frac{16}{2} = 8$

Put $x = 8$ *in eqn (2);*

$8 - y = 3$

$-y = 3 - 8$

$-y = -5$

$y = 5$

Truth set = { (x, y) : (8 , 5)}

Method 2 (elimination)

$x + y = 13 \ldots\ldots\ldots.\ (1)$

$x - y = 3 \ldots\ldots\ldots..\ (2)$

eqn (1) + eqn (2);

$(x + y) + (x - y) = 13 + 3$

$x + x + y - y = 16$

$2x = 16$

$x = 8$

Put $x = 8$ in eqn (2);

$8 + y = 13$

$y = 13 - 8 = 5$

Truth set = {(x, y) : (8, 5)

2. The sum of two numbers is -19 and their difference is 11. Find the numbers.

Solution

Let x andy be the two numbers.

$x + y = -19$ ……….. (1)

$x - y = 11 \ldots\ldots\ldots\ldots$ (2)

By elimination,

eqn (1) + eqn (2)

$(x + y) + (x - y) = -19 + 11$

$x + y + x - y = -8$
$2x = -8$
$x = -4$

Put $x = -4$ in eqn (1)
$-4 + y = -19$
$y = -19 + 4 = -15$
Truth set = $\{(x, y) : (-4, -15)\}$

3. Mr. Brown bought 3 rulers and 6 pencils all for 90p. When he purchased 4 rulers and 2 pencils, he paid the same amount. Determine the cost of 1 ruler and 1 pencil.

Solution
Let r represent ruler and p represent pencil
$3r + 6p = 90$……………(1)
$4r + 2p = 90$……………(2)

eqn (2) × 3
$12r + 6p = 270$…………(3)

eqn (3) – eqn (1);
$9r = 180$
$r = 20$

Put r = 20 in eqn (1);
$3(20) + 6p = 90$
$6p = 30$
$p = 5$
Therefore the cost of one ruler is 20p and the cost of one pencil is 5p.

4. A pile of 40 coins consist of 5p coins and 50p coins. If the total sum of money is Gh¢12.80, find the number of each kind of coin in the pile.

Solution
Let x represent the 5p coin and y represent the 50p coin.
$x + y = 40$………………(1)
$5x + 50y = 1{,}280$ (changed to pesewas)

Divide through by 5
$x + 10y = 256$…………..(2)
eqn (2) – eqn (1);
$9y = 216$
$y = \frac{216}{9} = 24$

Put $y = 24$ in eqn (1);
$x + 24 = 40$
$x = 40 - 24 = 16$
Therefore, there are 24, 50p coins and 16, 5p coins.

5. A family of three adults and two children paid Gh¢8.00 for a journey. Another family of four adults and three children paid Gh¢11.00 as the fare for the same journey. Calculate the fare for:
i. an adult,
ii. a child,
iii. a family of four adults and five children.

Solution
i. Let a represent adult and c represent children
$3a + 2c = 8$……….(1)
$4a + 3c = 11$………(2)

4 × eqn (1);
$12a + 8c = 32$……….(3)
3 × eqn (2)
$12a + 9c = 33$………(4)

eqn (4) – eqn (3);
$c = 1$

ii. Put $c = 1$ in eqn (1);
$3a + 2(1) = 8$……….(1)
$3a + 2 = 8$
$3a = 8 - 2$

$3a = 6$
$a = 2$
Therefore, the fare for an adult is Gh¢2.00 and the fare for a child is Gh¢1.00

iii. $4a + 5c$
$= 4(2) + 5(1)$
$= 8 + 5$
$= 13$
The fare for a family of four adults and five children is Gh¢13.00

6. If the price of a book is reduced by Gh¢5.00, a person can buy 5 more books for Gh¢300.00. Find the original price of the book.

Solution
Let the price of a book be x
Number of copies of the book be y
$xy = 300$……………………..(1)
$(x – 5)(y + 5) = 300$…………(2)

From eqn (1);
$y = \frac{300}{x}$

Put $y = \frac{300}{x}$ in eqn (2);
$(x - 5)\left(\frac{300}{x} + 5\right) = 300$
$x\left(\frac{300}{x} + 5\right) - 5\left(\frac{300}{x} + 5\right) = 300$
$300 + 5x - \frac{1500}{x} - 25 = 300$
$5x - \frac{1500}{x} = 25$
$5x^2 - 1500 = 25x$
$5x^2 - 25x - 1500 = 0$
$x^2 - 5x - 300 = 0$
$(x + 15)(x - 20) = 0$ (by factorization)
$x = -15$ or $x = 20$
The original price of a book is Gh¢20.00

7. The product of two positive numbers is 20. The sum of squares is 41. Find the numbers.

Solution
Let the numbers be x and y;
$xy = 20$……………………..(1)
$x^2 + y^2 = 41$………………...(2)

From eqn (1);
$y = \frac{20}{x}$………………(3)

Put $y = \frac{20}{x}$ in eqn (2);
$x^2 + \left(\frac{20}{x}\right)^2 = 41$
$x^2 + \frac{400}{x^2} = 41$
$(x^2)^2 + 400 = 41x^2$
$(x^2)^2 - 41x^2 + 400 = 0$

Let $x^2 = k$
$k^2 - 41k + 400 = 0$
$(k - 16)(k - 25) = 0$
$k = 16$ or $k = 25$

Now;
$x^2 = 16$ or $x^2 = 25$
$x^2 - 16 = 0$ or $x^2 - 25 = 0$
$x^2 - 4^2 = 0$ or $x^2 - 5^2 = 0$
Applying difference of two squares;
$x = 4$, $x = -4$ or $x = 5$, $x = -5$
$x = 4$ or $x = 5$ (positive numbers)

Put $x = 4$ or $x = 5$ in eqn (3)
When $x = 4$, $y = \frac{20}{4} = 5$;
When $x = 5$, $y = \frac{20}{5} = 4$
The numbers are 4 and 5.

Some Solved Past Questions
1. The cost of a packet of sugar is x cedis and the

cost of a tin of milk is y cedis. If 3 packets of sugar and 4 tins of milk cost Gh¢635.00 and 4 packets of sugar and 3 tins of milk cost Gh¢695.00, write two equations connecting x and y. Hence, find x and y.

Solution

$3x + 4y = 635$…………...(1)
$4x + 3y = 695$…………...(2)
4 × eqn (1);
$12x + 16y = 2540$……….(3)

3 × eqn (2)
$12x + 9y = 2085$………..(4)

eqn (3) – eqn (4);
$7y = 455$
$y = \frac{455}{7} = 65$

Put $y = 65$ in eqn (1);
$3x + 4(65) = 635$……….(1)
$3x + 260 = 635$
$3x = 635 – 260$
$3x = 375$
$x = \frac{375}{3} = 125$
x = Gh¢125.00 and y = Gh¢65.00

2. A train fare for a school child is half the fare of a teacher. The total fare for 120 children and 15 teachers for an excursion is Gh¢180.00.
i. Find the fare of a child
ii. How many children will go on the excursion with 20 teachers for a total fare of Gh¢240.00?

Solution

Let a child be represented by c and a teacher by t
$c = \frac{1}{2}t$…………………………(1)
$120c + 15t = 180$ …………….(2)

Put eqn (1) into eqn (2);
$120\left(\frac{1}{2}\right)t + 15t = 180$
$60t + 15t = 180$
$75t = 180$
$t = \frac{180}{75} = 2.4$

Put $t = 2.4$ eqn (1);
$c = \frac{1}{2}(2.4) = 1.2$
Therefore each teacher paid Gh¢2.40 and each child paid Gh¢1.20

ii. Let A represent the number of students required to go on the excursion
$A(c) + 20t = 240$
But $c = 1.2$ and $t = 2.40$
$1.2A + 20(2.40) = 240$
$1.2A + 48 = 240$
$1.2A = 240 – 48$
$1.2A = 192$
$A = \frac{192}{1.2} = 160$
Therefore, 160 students will go on the excursion with 20 teachers

Exercise 16.7

1. The sum of two numbers is 26 and their difference is 4. Find the two numbers.

2. Two numbers sum up to 30. If the difference between the numbers is 8,find the numbers.

3. There are two numbers x and y. Six times the larger number exceeds twice the smaller number by 156. If the difference of the two numbers is 18, find the numerical strength of x and y.

4. A certain number decreased by half of another number is -3. When twice the second number is

subtracted from twice the first number, the result is -2. Identify the two numbers.

5. I bought 12 pencils and 10 rulers for Gh¢2.10. At the same place, I bought 20 pencils and 4 rulers for Gh¢1.60.What is the price of one pencil and that of one ruler?

6. There were 200 people at a concert. Some paid 30*p* each and the rest paid 20*p* each and the total takings were Gh¢50.80. How many paid 20*p*?

7. Mr. Ben saves money by putting every 50*p* and every 20*p* coin he receives in a box. After a while, he finds out that he has 54 coins, amounting to Gh¢17.10. How many 50*p* coins does he have?

8. The price of admission to an Obra show was Gh¢100.00 for an adult and Gh¢50.00 for children. The total amount raised from the sale of 600 tickets was Gh¢50,000. Find:
i. the number of adults admitted;
ii. the number of children admitted.

9. At a concert 500 tickets were sold: the cheaper ones cost Gh¢5.00 and the more expensive one Gh¢9.00. The total receipts were Gh¢3,220. Let x and y be the numbers of cheap and expensive tickets respectively. Form two equations in x and y, and hence find how many cheap tickets were sold.

10. The value of a fraction expressed as $\frac{p}{q}$ is $\frac{2}{3}$. If 3 is subtracted from the numerator and added to the denominator, its value becomes $\frac{3}{7}$. Find the values of p and q.

Challenge Problems

1. In a pub, brandy is 10p more expensive than whisky. A round of three brandies and 5 whiskies cost Gh¢7.10. Let x be the cost of a brandy and y the cost of a whisky. Find the cost of a whisky.

2. There are five more boys than there are girls in a class. If there were one more girl in the class, the ratio of boys to girls would be 5 to 4. How many boys and girlsare in the class?

4. The expression $ax - by$ has the value 6 when $x = 4$ and $y = 3$. It also has the value 6 when $x = 2\frac{1}{2}$ and $y = \frac{3}{4}$. Find the values of a and b.

5. Four years ago, a man was six times as old as his son, but 5 years' time, he will be only three times as old as his son. What are their ages now?

6. If the price of a book is reduced by Gh¢4.00, Rita can buy 8 more books for Gh¢192.00. Find the original price of the book.

7. If 2 is added to the numerator and denominator of a certain fraction, it becomes $\frac{9}{10}$ and if 3 is subtracted from the numerator and denominator of the same fraction, it becomes $\frac{4}{5}$. Find the fractions. Ans $\frac{7}{8}$

8. If twice the age of a son is added to the age of a father, the sum is 56. But if twice the age of the father is added to the age of son, the sum is 82. Find the ages of the father and son.

9. The produt of two positive integers is 45 and the sum of their square is 106. Find the numbers.

Applications to a Number and its Reverse

Take the two – digit number, 42 from instance. The place value of 4 is tens and the place value of 2 is unit. Thus, 42 can be expanded as:

$42 = 40 + 2$

$= 4(10) + 2(1)$

If 42 is reversed, we obtain 24, where 2 is the tens digit and 4 is the unit digit also expanded as:

$24 = 20 + 4$

$= 2(10) + 4(1)$

Similarly, given the two digits number, xy, it can be expanded to obtain:

$xy = x(10) + y\ (1)$

$= 10x + y$

If xy is reversed, yx is obtained, also expanded as;

$yx = y\ (10) + x\ (1)$

$= 10y + x$

Therefore, to solve problem involving an unknown two digit number and its reverse, express them in the expanded form as shown above and form the appropriate equation(s)

Worked Examples

1. A number with two digits is equal to four times the sum of its digits. The number formed by reversing the order of the digit is27 greater than the given number. Find the number.

Solution

Let the two digit number be xy.

$\Rightarrow xy = 10x + y$

Four times the sum of the digits $= 4(x + y)$

$\Rightarrow 10x + y = 4(x + y)$

$10x + y = 4x + 4y$

$10x – 4x = 4y – y$

$6x = 3y$

$x = \frac{3y}{6}$………………(1)

Let the reverse of the digit be yx.

$= 10y + x$ is 27 greater than the given number;

$\Rightarrow 10y + x = 27 + 10x + y$

$10y – y + x – 10x = 27$

$9y – 9x = 27$ (Divide through by 9)

$y – x = 3$…………(2)

Put $x = \frac{3y}{6}$ in eqn (2);

$y – \frac{3y}{6} = 3$

$6y – 3y = 3 \times 6$

$3y = 18$

$y = \frac{18}{3} = 6$

Put $y = 6$ *in eqn (1)*

$x = \frac{3(6)}{6} = 3$

Therefore, the number $xy = 36$

2. The sum of the digits of a two digit number is 9. The number is 27 more than the original number with its digit reversed. Find the number.

Solution

Let the two digit number be xy

$x + y = 9$…………..(1)

$10x + y = 27 + 10y + x$

$10x – x + y – 10y = 27$

$9x – 9y = 27$

$x – y = 3$………….(2)

eqn (1) – eqn (2);

$2y = 6$

$y = 3$

Put $y = 3$ *in eqn (1);*

$x + 3 = 9$

$x = 9 – 3 = 6$

Therefore, the number $xy = 63$

3. A two digit number is 6 more than 4 times the sum of its digits. The digits from the left to the right are consecutive even integers. Find the number.

Solution

Let the two digit number be xy

$xy = 6 + 4(x + y)$

$10x + y = 6 + 4(x + y)$………..(1)

For consecutive even integers, if one is x, the other (y) is $x + 2$

Substitute $y = x + 2$ into eqn (1)

$10x + (x + 2) = 6 + 4(x + x + 2)$

$10x + x + 2 = 6 + 4(x + x + 2)$

$11x + 2 = 6 + 4(2x + 2)$

$11x + 2 = 6 + 8x + 8$

$11x – 8x = 6 + 8 – 2$

$3x = 12$

$x = \frac{12}{3} = 4$

But $y = x + 2 = 4 + 2 = 6$

Therefore, the number $xy = 46$

4. A two digit number is 5 times the sum of its digits. The digits from left to right form consecutive integers. Find the digits.

Solution

Let the two digits number be xy

$\Rightarrow xy = 5(x + y)$

$10x + y = 5(x + y)$ ……….(1)

From left to right, the digits form consecutive integers

$\Rightarrow$If one is x, the other (y) is $x + 1$

Substitute $y = x + 1$ in eqn (1)

$10x + y = 5(x + y)$

$10x + (x + 1) = 5(x + x + 1)$

$10x + x + 1 = 5(2x + 1)$

$11x + 1 = 10x + 5$

$11x – 10x = 5 – 1$

$x = 4$,

But $y = x + 1 = 4 + 1 = 5$

The number is $xy = 45$

Exercises 16.8

1. Find a number of two digits which exceeds four times the sum of its digits by 3 and which is increased by 18 when its digits are interchanged

2. If a certain number with two digits is divided by the sum of the digits, the quotient is 6 and the remainder is 5. The difference between the given number and the number formed by reversing the digit is 18. Find the given number

3. The unit digit of a two digit number is two less than the tens digit. The number is two more than six times the sum of the digits. Find the number.

4. A two digit number is five times the sum of its digits. When 9 is added to the number, the results is the original number with its digits reversed. Find the number.

17 PERCENTAGES II

Compound Interest

It is a kind of interest that accrues at the end of a given period and serves as an element of expanding or strengthening the principal at the beginning of another or a subsequent period of transaction. In this situation, an initially earned interest is added to the initially deposited principal, thereby creating a higher principal which then earns higher interest at the end of the stipulated period.

The Compound Interest Formula

A. Payment without Instalment

Here, compound interest is calculated or payable at the end of each year. Compound Interest is equal to the sum of interest at the end of each year over the given number of years.

Hence, if a principal, *p*, is deposited or borrowed at an interest rate, *r*, and compounded yearly for a period of *t* years, the accumulated value, A, is:

A = $\boldsymbol{p(1+r)^t}$

The compound interest, I, is calculated by the the formula : ***I = A – P***

If an accumulated value, A, is desired after *t* years and the money is deposited at an interest rate, *r*, and compounded yearly, the present value is

P = $\boldsymbol{\frac{A}{(1+r)^t}}$

Worked Examples

1. Find the compound interest on Gh¢300.00 for 3 years at 20% per anum

Solution

<u>Method 1</u>

P = Gh¢300, T = 3 years R = 20% I = ?

Interest at the end of 1st year

P_1= 300, T = 1, R = 20%

$I_1 = \frac{P_1TR}{100} = \frac{300 \times 20 \times 1}{100}$ = Gh¢60.00

Amount at the end of 1st year

$A_1 = P_1 + I_1$

A_1 = Gh¢300 + Gh¢60 = Gh¢360.00

Interest at the end of 2nd year

P_2= 360 T = 1 R = 20%

$I_2 = \frac{P_2TR}{100} = \frac{360 \times 20 \times 1}{100}$ = Gh¢72.00

Amount at the end of 2nd year

$A_2 = P_2 + I_2$

A_2= Gh¢360 + Gh¢72 = Gh¢432.00

Interest at the end of 3rd (final) year

$I_3 = \frac{P_3TR}{100} = \frac{432 \times 20 \times 1}{100}$ = Gh¢86.40

Amount at the end of 3rd (final) year

$A_3 = P_3 + I_3$

A_3 = Gh¢432 + Gh¢86.40 = Gh¢518.40

Compound Interest over the 3 – year period

= 3rd year amount – First (initial) Principal

= $A_3 - P_1$

= Gh¢(518.40 – 300.00) = Gh¢218.40

<u>Method 2</u>

P = Gh¢300, T = 3years, R = 20% , I = ?

Interest at the end of 1st year

P_1= Gh¢300, T = 1 and R = 20%

$I_1 = \frac{P_1TR}{100} = \frac{300 \times 20 \times 1}{100}$ = Gh¢60.00

Amount at the end of 1st year

$A_1 = P_1 + I_1$

A_1 = Gh¢300 + Gh¢60 = Gh¢360.00

Interest at the end of 2nd year

P_2= Gh¢360.00 T = 1 and R = 20%

$I_2 = \frac{P_2TR}{100} = \frac{360 \times 20 \times 1}{100}$ = Gh¢72.00

Amount at the end of 2nd year

$A_2 = P_2 + I_2$

A_2 = Gh¢360 + Gh¢72 = Gh¢432.00

Interest at the end of 3rd (final) year

$I_3 = \frac{P_3TR}{100} = \frac{432 \times 20 \times 1}{100}$ = Gh¢86.40

Amount at the end of 3rd (final) year

$A_3 = P_3 + I_3$

A_3 = Gh¢(432.00 + ¢86.40) = Gh¢518.40

Compound Interest is equal to the sum of interest at the end of each year over the 3 – year period. That is:

Compound Interest = 1st year Interest + 2nd year interest + 3rd (final) year interest

$= I_1 + I_1 + I_3$

= Gh¢(60 + 72 + 86.40) = Gh¢218.40

Method 3(using the formula)

P = Gh¢300, t = 3 years, r = 20%, A = ?

A = $p(1+r)^t$

A = $300\left(1+\frac{20}{100}\right)^3$ (*By substitution*)

A = $300\,(1.2)^3$ = 300 × 1.728 = 518.40

Compound interest, I = A – P

= Gh¢(518.40 – 300.00) = Gh¢218.40

2. Find the compound interest on Gh¢800.00 for 2 years at $3\frac{1}{2}$% per anum.

Solution

P = Gh¢800, T = 2, R = $3\frac{1}{2}$% = 3.5% and I = ?

Interest at the end of first year

$I_1 = \frac{P_1TR}{100} = \frac{800 \times 1 \times 3.5}{100}$ = Gh¢28.00

Amount at the end of first year

$A_1 = P_1 + I_1$

A_1 = Gh¢800 + Gh¢28 = Gh¢828.00

Interest at the end of second (final) year

$I_2 = \frac{PTR}{100} = \frac{828 \times 1 \times 3.5}{100}$ = Gh¢28.98

Amount at the end of second (final) year

$A_2 = P_2 + I_2$

A_2 = Gh¢828 + Gh¢28.98 = Gh¢856.98

Compound interest

= 2nd (final) year amount – 1st (Initial) principal

$= A_2 - I_1$

= Gh¢856.98 – Gh¢800.00= Gh¢56.98

Alternatively

Compound Interest

= 1st year Interest + 2nd (final) year interest

$= I_1 + I_2$

= Gh¢28 + Gh¢28.98 = Gh¢56.98

Exercises 17.1

A. Find the compound interest on:

1. Gh¢3,200.00 for 3 years 12% per anum.
2. Gh¢2,400.00 for 4 years at 17.5% .
3. Gh¢50,200.00 for 3 years at 15% .

B. 1. A man deposited Gh¢80,000.00 at a bank at 12% compound interest per anum. Find his total amount at the end of the third year.

2. Mr. Brown borrowed Gh¢2,500.00 from Bayport financial services for 4 years at 28% compound interest per anum. Calculate the

interest and amount to be paid at the end of the period.

3. Mrs. Kaya deposited an amount of Gh¢1,320.00 at a bank at 17 $\frac{1}{2}$% per anum. Find her compound interest at the end of three years.

4. A district had 75,000 inhabitants at the beginning of the year 2007. The population grew at a constant rate of $2\frac{1}{2}$ % per anum. What was the population at the beginning of 2010?

Calculating the Time

Identify the given principal(p) , amount (A), and the rate (r).

II. Substitute in the formula : A = $p(1+r)^t$

III. Take logarithm on both sides of the equation and make t the subject. However, other alternatives can also be used.

Worked Example

A man invests Gh¢12,000.00 at a rate of 10% per anum. At what time will his investment amount to Gh¢15,972.00?

Solution

Method 1

A = 15,972, p = 12,000,r = 10% = 0.1 t = ?

Substitute in A = $p(1+r)^t$

15,972 = 12,000$(1+0.1)^t$

$\frac{\mathbf{15{,}972}}{\mathbf{12{,}000}} = (1+0.1)^t$

1.331 = $(1.1)^t$

lg 1.331 = lg$(1.1)^t$

lg 1.331 = t lg 1.1

$t = \frac{lg\ 1.331}{lg\ 1.1}$ = 3 years

Method 2

10% of Gh¢12,000 = Gh¢1,200

Amount at the end of first year;

Gh¢12,000 + Gh¢1,200 = Gh¢13,200

10% of Gh¢13,200 = Gh¢1,320

Amount at the end of second year;

Gh¢13,200 + Gh¢1,320 = Gh¢14,520

10% of Gh¢14,520 = Gh¢1,452

Amount at the end of third year;

Gh¢14,520 + Gh¢1,452 = Gh¢15,972

Exercise 17.1B

Mr. Brown borrowed Gh¢2,500.00 from a bank at 28% compound interest per anum. How many years did it takehis interest to amount to Gh¢4,210.89

B. Compound Interest for Instalments

For instalments, payments or compound interest within a year are payable as follows:

1. Monthly, meaning 12 times a year.
2. Quarterly, meaning 4 times a year.
3. Half – yearly, meaning 2 times a year.

Method 1

I. Divide the rate, r, by the number of payments/instalment within a year.

II. Use this rate to calculate the interest, I, and the amount, at the end of each instalment.

III. Find the sum of interest for the total number of instalments made to obtain the compound interest.

Method 2

The Compound Interest Formula

If a principal, p, is deposited or borrowed at an interest rate, r, and compounded a certain number of times, n, within a year for a period of t years, the accumulated value is: $\mathbf{A} = \boldsymbol{p}\left(\mathbf{1}+\frac{\boldsymbol{r}}{\boldsymbol{n}}\right)^{\boldsymbol{nt}}$

Where A is the amount at the end of the given period, p is the principal, r is the rate , n is the number of times per year interest is compounded and t is the time in years.

The compound interest, **I**, is calculated by the the formula : ***I = A – P***

Worked Examples

Find the compound interest on Gh¢2,000.00 for 1 year at 32% per anum payable every quarter.

Solution

Method 1

P_1= 2,000, R = 32%,

T = 1yr = $\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)$

If rate per 1 year = 32%, then rate per $\frac{1}{4}$(quarter) a year = $\frac{1}{4}$ × 32% = 8%. Therefore rate to be used quarterly (3months) = 8% for 4 quarters

Interest at the end of 1st quarter (3 months)

P_1 = 2000, R = 32% and T = 1

$I_1 = \frac{P_1TR}{100} = \frac{2000 \times 1 \times 8}{100}$ = Gh¢160.00

Amount at the end of first quarter

$A_1 = P_1 + I_1$

A_1 = Gh¢2,000 + Gh¢160 = Gh¢2,160.00

Interest at the end of 2nd quarter (3months)

P_2 = 2160, T = 1 and R = 8%

$I_2 = \frac{p_2TR}{100} = \frac{2160 \times 1 \times 8}{100}$ = Gh¢172.80

Amount at the end of second quarter

$A_2 = P_2 + I_2$

A_2 = Gh¢2,160 + Gh¢ 172.80 = Gh¢2.332.80

Interest at the end of 3rd quarter (3 months)

P_3 = Gh¢2,160 T = 1 and R = 8%

$I_3 = \frac{P_3TR}{100} = \frac{2332.80 \times 1 \times 8}{100}$ = Gh¢186.62

Amount at the end of third quarter

$A_3 = P_3 + I_3$

A_3 = Gh¢2,332.80 + Gh¢186.62 = Gh¢2519.42

Interest at the end of 4th quarter (3 months)

P_4 = 2519.42, T = 1 and R = 8%

$I_4 = \frac{P_4TR}{100} = \frac{2519.42 \times 1 \times 8}{100}$ = Gh¢201.55

Amount at the end of third quarter

$A_4 = P_4 + I_4$

A_4 = Gh¢2,519.42 + Gh¢201.55 = Gh¢2,720.97

Compound interest

= $I_1 + I_2 + I_3 + I_4$

= Gh¢(160 + 172.80 + 186.62 + 201.55)

= Gh¢720.97

Method 2(using the formula)

P = 2,000, r = 32% = 0.32, n = 4 t = 1

Substitute in A = $p\left(1+\frac{r}{n}\right)^{nt}$

A = $2{,}000\left(1+\frac{0.32}{4}\right)^{4\times 1}$

A = $2{,}000(1+0.08)^4$ = Gh¢2,720.97

I = A – P

I = Gh¢2,720.97 – Gh¢2,000.00 = Gh¢720.97

2. Find the compound interest on Gh¢400.00 for 2 years at 10% per anum, if interest is added half a year

Solution

Method 1

Interest per 1 year = 10%

Interest per $\frac{1}{2}$a year = 5%

If 1 year is 2 – half years
Then 2 years is 4 – half years

Interest at the end of first half year
P_1 = 400, R = 5% and T = 1
$I_1 = \frac{P_1TR}{100} = \frac{400 \times 1 \times 5}{100}$ = Gh¢20.00

Amount at the end of first half year
$A_1 = P_1 + I_1$
A_1 = Gh¢400 + Gh¢20 = Gh¢420.00

Interest at the end of second half year
P_2 = 420, T = 1 and R = 5%
$I_2 = \frac{p_2TR}{100} = \frac{420 \times 1 \times 5}{100}$ = Gh¢21.00

Amount at the end of second half year;
$A_2 = P_2 + I_2$
A_2 = Gh¢420 + Gh¢21 = Gh¢441.00

Interest at the end of third half year;
P_3 = 441, T = 1 and R = 5%
$I_3 = \frac{P_3TR}{100} = \frac{441 \times 1 \times 5}{100}$ = Gh¢22.05

Amount at the end of third half year;
$A_3 = P_3 + I_3$
A_3 = Gh¢441 + Gh¢22.05 = Gh¢463.05

Interest at the end of fourth half year;
P_4 = 463.05, T = 1 and R = 5%
$I_4 = \frac{P_4TR}{100} = \frac{463.05 \times 1 \times 5}{100}$ = Gh¢23.15

Amount at the end of third half year;
$A_4 = P_4 + I_4$
A_4 = Gh¢463.05 + Gh¢23.15 = Gh¢486.20

Compound interest = $I_1 + I_2 + I_3 + I_4$
= Gh¢(20.00 + 21.00 + 22.05 + 23.15)
= Gh¢86.20

Method 2(using the formula)
P = 400, r = 10% = 0.1, n = 2, t = 2
Substitute in $A = p\left(1 + \frac{r}{n}\right)^{nt}$
$A = 400\left(1 + \frac{0.1}{2}\right)^{2\times2}$
$A = 400(1 + 0.05)^4$ = Gh¢486.20
I = A – P
I = Gh¢486.20 – Gh¢400.00 = Gh¢86.20

Exercises 17.2

A. Find the compound interest:

1. Gh¢2,060.00 for 1 year at 20% per anum, payable every quarter
2. Gh¢1,540.00 for 2 years at 24% per anum, paid half a year
3. Gh¢72,000.00 for a year, payable half a year at a rate of 24% per anum

B. Find the amount at compound interest:

1. Gh¢28,000.00 for 1 year at 20% p.a payable half yearly.
2. Gh¢150,000.00 for 1 $\frac{1}{2}$ years at 30% p.a payable half yearly.
3. Gh¢120,000.00 for $1\frac{1}{2}$ years at 22% p.a payable half yearly.
4. Gh¢200,000.00 for 2 years at 28% p.a payable quarterly.
5. Gh¢250,000.00 for 3 years at 30% p.a payable quarterly.

Finding the Outstanding Balance or Refund
I. Calculate the interest and amount at the end of each period of payment.
II. Deduct the amount of repayment to obtain a reducing balance.

III. Calculate the interest and amount using on the reducing balance and deduct repayment until the last repayment is made.

IV. Note that a ***negative outstanding balance*** indicates a ***refund***.

Worked Examples

1. A man borrowed Gh¢150,000.00 at 25% per anum compound interest paid half yearly. The man agreed to repay Gh¢50,000.00 at the end of each half year. Find the amount of loan outstanding at the beginning of the fourth half year

Solution

	GH¢
Principal for first half year	150,000.00
Interest at 12.5% for 1st half year =	18,750.00
Amount at the end of 1st half year =	168,750.00
Deduct 1st repayment	50,000.00
Principal for 2nd half year	118,750.00
Interst at 12.5% for 2nd half year	14,843.75
Amount at the end of 2nd half year	133,593.75
Deduct 2nd repayment	50,000.00
Principal for 3rd half year	83,593.75
Interst at 12.5% for 3rd half year =	10,449.22
Amount at the end of 3rd half year	94,042.97
Deduct 3rd repayment	50,000.00
Outstanding Loan	44,042.97

2. Mr. Okra borrowed Gh¢20,000.00 at 12% per anum compound interest paid half yearly. He agreed to repay Gh¢9,000.00 at the end of each half year. Find the amount of outstanding refund at the end of the third half year.

Solution

	GH¢
Principal for first half year	20,000.00
Interest at 6% for 1st half year	1,200.00
Amount at the end of 1st half year	21,200.00
Deduct 1st repayment	9,000.00
Principal for 2nd half year	12,200.00
Interst at 6% for 2nd half year	732.00
Amount at the end of 2nd half year	12,932.00
Deduct 2nd repayment	9,000.00
Principal for 3rd half year	3,932.00
Interst at 6% for 3rd half year	235.92
Amount at the end of 3rd half year	4,167.92
Deduct 3rd repayment	9,000.00
Outstanding Loan	- 4,832.08

$\Rightarrow$ Refund = Gh¢4,832.08

Exercises 17.3

1. A woman borrows Gh¢50,000.00 at 15% per anum compound interest payable half – yearly. She agrees to repay Gh¢30,000.00 at the end of each year. Find the amount of loan outstanding at the beginning of the fourth half – year.

2. At the end of the year, Tony's bank increased the rate of interest from 15% to 18% per anum payable half – yearly. How much more interest would Tony receive on his savings of Gh¢20,000.00 at the end of the year.

3. A man borrowed Gh¢25,000.00 at 24% per anum compound interest payable monthly for 5 months. He paid back Gh¢5,000.00 at the end of each of the first 4 months. Calculate how much he paid to clear the debt at the end of the fifth month.

4. Mr. Adanko borrowed Gh¢2,500.00 from a certain financial institution at a rate of 17% per anum.

i. Find the compound interest at the end of 3 years.

ii. If for every month, Mr. Adanko pays Gh¢155.00 to the financial institution, find his refund at the end of the 3 year period.

5. If you take a loan of Gh¢7,000.00 at 21% compound interest and as part of the loan agreement, you pay a fixed deposit of Gh¢2,400.00 every year; find the outstanding loan at the end of three years.

Finding the Rate given the Amount at the End of a Certain Year

I. Identify the principal (P) and the amount (A)

II. Find the interest (I) by the formula:

$I = A - P$

III. Identify the values of P, I and T (time) and Substitute in $R = \frac{100I}{PT}$ to obtain the value of R.

Worked Example

1. Mr. Brown invested Gh¢180.00 at Ghana Commercial Bank for 3 years at compound interest. If at the end of the first year, his money amounted to Gh¢207.00, calculate:

i. the rate per cent per anum

ii. the total interest earned by Mr. Brown at the end of the three – year period

Solution

Method 1

i. P = Gh¢180, T = 1year, R = ?

I = Gh¢207 – Gh¢180 = Gh¢27.00

From $I = \frac{PTR}{100}$,

$R = \frac{100I}{PT} = \frac{100 \times 27}{180 \times 1} = 15\%$

ii. Interest at the end of 1st year

P_1 = Gh¢180, T = 1 and R = 15% I = ?

$I_1 = \frac{P_1TR}{100} = \frac{180 \times 15 \times 1}{100}$ = Gh¢27.00

Amount at the end of 1st year

$A_1 = P_1 + I_1$

A_1 = Gh¢180 + Gh¢27 = Gh¢207.00

Interest at the end of 2nd year;

P_2 = Gh¢207, T = 1 and R = 15%

$I_2 = \frac{P_2TR}{100} = \frac{207 \times 15 \times 1}{100}$ = Gh¢31.05

Amount at the end of 2nd year;

$A_2 = P_2 + I_2$

A_2 = Gh¢207 + Gh¢31.05 = Gh¢238.05

Interest at the end of 3rd (final) year;

P_3 = Gh¢238.05 T = 1, and R = 15%

$I_3 = \frac{P_3TR}{100} = \frac{238.05 \times 15 \times 1}{100}$ = Gh¢35.71

Amount at the end of 3rd (final) year;

$A_3 = P_3 + I_3$

A_3 = Gh¢238.05 + Gh¢35.71 = Gh¢237.76

Compound Interest over the 3 – year period;

= 3rd (final) year amount – Initial Principal

$= A_3 - P_1$

= Gh¢237.76 – Gh¢180.00 = Gh¢93.76

Total interest earned is Gh¢93.76

Method 2

Compound Interest = 1St year Interest + 2nd year interest + 3rd (final) year interest

$= I_1 + I_2 + I_3$

= Gh¢(27 + 31.05 + 35.71) = Gh¢93.76

Total interest earned at the end of the 3 – year period is Gh¢93.76

Depreciation

It is the loss in the value of goods or items. When items are used for some period of time, its value decreases (depreciates) as a result of old age. The value at which an item depreciates is calculated as a percentage of the cost price. Therefore, if the rate of depreciation is ***r%*** then;

1. Depreciation = $\frac{\boldsymbol{Reduction\ in\ value}}{\boldsymbol{cost\ of\ value} \times \mathbf{100}}$

2. Depreciation = $\frac{\boldsymbol{r}}{\mathbf{100}}$ × ***Cost price***

3. New price = $\frac{(\mathbf{100}-\boldsymbol{r})}{\mathbf{100}}$ × ***Cost price.***

4. Cost/original price = $\frac{\mathbf{100}}{(\mathbf{100}-\boldsymbol{r})}$× ***New price***

Depreciation value or new value,at a constant rate, at a given period of time is best calculated by the formula:$\boldsymbol{V} = \boldsymbol{p}\ (\mathbf{1} - \boldsymbol{r}\%)^{\boldsymbol{n}}$
Where V is the depreciated or new value,
p is the initial value,
r% is the depreciation rate
n is the number of calculations

Depreciation (D) = Initial value – New value
D = P – V

Worked Examples

1. The value of a sewing machine depreciated by 10% of its value in one year. The cost price of the machine was Gh¢1,500.00. Find its new price (value) when it is 1 year old.

Solution

Method 1

New price = $\frac{(100-r)}{100}$ × C.P
But, *r* = 10, and C.P = Gh¢1,500.
New price = $\frac{(100-10)}{100}$ × 1,500 = Gh¢1,350.00

Method 2

Depreciation = $\frac{r}{100}$ × C.P,
But *r* = 10 and C.P = Gh¢1,500
Depreciation = $\frac{10}{100}$ × 1,500 = Gh¢150.00
New Price = Cost price – Depreciation
New price = Gh¢1,500 – Gh¢150 = Gh¢1,350.00

Method 3:(using the formula)

P = 1,500, *r*% = 10%= 0.1, *n* = 1
$V = p\ (1 - r\%)^n$
V = 1,500$(1 - 0.1)^1$ = Gh¢1,350.00

2. The value of a T. V. set depreciated by15% of it cost price of Gh¢5,400.00. Find the value of the T.V after 2 years.

Solution

Method 1

At the end of the first year,
New value = $\frac{(100-r)}{100}$ × C.P
But *r* = 15, C.P = Gh¢5,400
New value = $\frac{(100-15)}{100}$ × 5,400
New value = $\frac{85}{100}$ × 5,400 = Gh¢4,590.00

⇒ New value at the end first year
= Gh¢4,590.00

At the end of the second year;
r = 15, and C. P = Gh¢4,590.00
New value = $\frac{(100-15)}{100}$ × 4,590 = Gh¢3,901.50

Method 2

At the end of the first year;
Depreciation = $\frac{r}{100}$ × C.P,
But *r* = 15 and C.P = Gh¢5,400
Depreciation = $\frac{15}{100}$ × 5,400 = Gh¢810.00

New value at the end of first year
= Cost price – Depreciation
= Gh¢5,400.00 – Gh¢810.00 = Gh¢4,590.00

At the end of the second year;
Depreciation = $\frac{r}{100}$ × C.P,
But *r* = 15 and C.P = Gh¢4,590

Depreciation = $\frac{15}{100} \times 4{,}590$ = Gh¢688.50

New value at the end of firt year
= Cost price – Depreciation
= Gh¢4,590.00 – Gh¢688.00 = Gh¢3901.50

Method 3:(using the formula)
P = 54,00, r% = 15%= 0.15, n =2
$V = p\ (1 - r\%)^n$
V = $5{,}400(1 - 0.15)^2$ = Gh¢3,901.50

3. The cost of an item depreciated according to the table below:

First year	Nil
Second year	25%
Third year	10%

If the cost of the item is Gh¢6,000.00, find its cost at the end of three years

Solution
Method 1
Cost price = Gh¢6,000.00
First year = Nil, C.P = Gh¢6,000.00

Second year = 25%
New price = $\frac{(100 - r)}{100} \times$ C.P
= $\frac{(100-25)}{100} \times 6{,}000$ = Gh¢4,500

Third year = 10%
New value = $\frac{(100 - r)}{100} \times$ C.P
= $\frac{(100-10)}{100} \times 4{,}500$ = Gh¢4,050.00
The cost of the item at the end of three years is Gh¢4,050.00

Method 2
First year = Nil
C. P at the end of first year = Gh¢6,000.00

Second year = 25%
Depreciation = $\frac{r}{100} \times$ C.P,
But r = 25 and C.P = Gh¢6,000
Depreciation = $\frac{25}{100} \times 6{,}000$ = Gh¢1,500.00

New Price = Cost price – Depreciation
New price = Gh¢6,000.00 – Gh¢1,500.00
= Gh¢4,500.00

C.P at the end of second year = Gh¢4,500.00

Third year = 10%
Depreciation = $\frac{r}{100} \times$ C.P,
But r = 10 and C.P = Gh¢4,500.00
Depreciation = $\frac{10}{100} \times 4{,}500$ = Gh¢450.00

New Price = Cost price – Depreciation
New price = Gh¢4,500.00 – Gh¢450.00
= Gh¢4,050.00

4. A business man purchased a copier for Gh¢8,500.00 and anticipates it will depreciate in value by Gh¢1,250.00 per year.
i. What is the copier's value after 4 years?
ii. How many years will it take the copier's value to decrease to Gh¢2,250.00?

Solution
Cost of machine Gh¢ 8,500
Depreciation per year Gh¢ 1,250
Depreciation for 4 years = 4 × Gh¢1,250
= Gh¢5,000
Copier's value after 4 years;
= Gh¢8,500 – Gh¢5,000 = Gh¢3,500.00

ii. Value of copier after 5 years;
= Gh¢3,500 – Gh¢1,250 = Gh¢2,250.00
Therefore, it will take 5 years for the copier's value to decrease to Gh¢2,250.00?

Exercises 17.4

1. The value of a refrigerator depreciated by 15% of its value each year. If the value of the refrigerator was Gh¢2,040.00, calculate its new price after 1 year.

2. The value of a new machine depreciated each year by 10% of its value at the beginning of that year. The value of the machine when new was Gh¢200,000.00. Find its value when it was 3 years old

3. The value of a printer depreciates by 12% of its value every year. Calculate the new price of the calculator after three years, if its value when new is Gh¢1,250.00.

4. The cost price of a Chinese mobile phone depreciated 5% after one year and 10% after two years and 20% after three years. If the value of the mobile phone is Gh¢800.00, find its value after three years.

5. A car which was bought for Gh¢9,000.00 when new was valued at Gh¢7,500.00 at the end of the first year. It then depreciated each year by $12\frac{1}{2}$% of its value at the beginning of that year. Calculate:

i. the depreciation at the end of the first year;

ii. the value of the car at the end of the third year.

6. In a certain country, the official valuation for cars up to 4 years old is calculated according to the following table:

Year after manufacture	Depreciation on original purchase value
First year	Nil
Second year	10%
Third year	18%
Fourth year	30%

The original purchase value of a car is Gh¢9,000.00. What will be its official valuation during each of the first 4 years?

7. The cost of an H.P. computer and a B.M.W. vehicle depreciates with respect to the table below;

Year	HP Computer	B.M.W
First	14%	15%
Second	25%	24%
Third	20%	20%

If the cost of the computer is Gh¢1,850.00 and the B.M.W is Gh¢24,400.00

i. Determine the cost of the B. M. W. after 3 years

ii. Find the cost of the computer after 3 years

iii. What is the total cost of the B.M.W. and the H.P computer after 3 years?

Partnership in Business

It is the situation whereby two or more people contribute money to set up business with the aim of maximizing profit. The profit accrued from partnership isusually shared annually according to the ratio of the contributions, responsibilities and other agreed terms and conditions.

Forms of Business Partnership

1. Sole Proprietorship

A Sole Proprietorship is one individual or married couple in business alone. Sole proprietorships are the most common form of business structure. This type of business is simple to form and operate, and may enjoy greater flexibility of management, fewer legal controls, and fewer taxes. However, the business owner is personally liable for all debts incurred by the business.

2. General Partnership

A General Partnership is composed of 2 or more persons (usually not a married couple) who agree to contribute money, labor, or skill to a business. Each partner shares the profits, losses, and management of the business and each partner is personally and equally liable for debts of the partnership. Formal terms of the partnership are usually contained in a written partnership agreement.

3. Limited Partnership

A Limited Partnership is composed of one or more general partners and one or more limited partners. The general partners manage the business and share fully in its profits and losses. Limited partners share in the profits of the business, but their losses are limited to the extent of their investment. Limited partners are usually not involved in the day – to-day operations of the business. Filing with the state is required.

4. Limited Liability Partnership (L.L.P.)

A Limited Liability Partnership (L.L.P.) is similar to a General Partnership except that normally a partner doesn't have personal liability for the negligence of another partner. This business structure is used most by professionals, such as accountants and lawyers. Filing with the state is required.

Sharing Profit according to Ratio of Contribution

Worked Examples

1. Four business partners Amos, Ben, Cudjoe and Dadzie share a profit of Gh¢11,000,000.00. Amos takes 30% of the profit and the remainder is shared among Ben, Cudjoe and Dadzie in the ratio 4 : 4 : 3 respectively. Find the share of each person

Solution

Total profit = Gh¢11,000,000.00

Amos share = 30% of the profit

$= \frac{30}{100} \times 11{,}000{,}000 =$ Gh¢3,300,000.00

The rest = Gh¢(11,000,000 - 3,300,000)

= Gh¢7,700,000.00

Total ratio = 4 + 4 + 3 = 11

Ben's share = $\frac{4}{11} \times 7{,}700{,}000$

= Gh¢2,800,000.00

Cudjoe's share = $\frac{4}{11} \times 7{,}700{,}000$

= Gh¢2,800,000.00

Dadzie's share = $\frac{3}{11} \times 7{,}700{,}000$

= Gh¢2,100,000.00

2. Three friends Ato, Oko and Edem entered into a business partnership. They contributed Gh¢3.0 million, Gh¢2.4 million and Gh¢3.6 million respectively. It was agreed that profits will be shared in proportions to their contributions. After one year of operation, the profit made was 2.7 million cedis

i. Find the amount received by each partner as his share of the profit

ii. Express Edem's share of the profit as a percentage of his investment

Solution

Total profit = Gh¢2.7 million

= Gh¢2,700,000

Ratio of contribution;

30 : 24 : 36 = 5 : 4 : 6

Total ratio = 5 + 4 + 6 = 15

Ato's share = $\frac{5}{15} \times$ Gh¢2,700,000

= Gh¢900,000.00

OKo's share = $\frac{4}{15}$ × Gh¢2,700,000
= Gh¢720,000.00

Edem's share = $\frac{6}{15}$ × Gh¢2,700,000
= Gh¢1,080,000.00

ii. Edem's share of the profit as a percentage of his investment $= \frac{profit}{Contribution} \times 100\ \%$

$= \frac{1,080,000}{3,600,000} \times 100\% = 30\%$

Sharing Profit according to a Given Ratio, Responsibilities and other conditions

Worked Examples

1. Abubakar and Babatunde entered into a business partnership. The capital from the business is made up of Gh¢25,000.00 from Abubakar and Gh¢7,000.00 from Babatunde. They agreed to share the yearly profit in the following manner; Babatunde as managing director is paid Gh¢1,000.00 and additional 7.5% of the total profit. Each partner is paid a sum equal to 3% of the capital he invested. The remainder of the profit is shared between the partners in the ratio of their contributions to the capital. If the profit at the end of a certain year was Gh¢6,400.00, calculate the total amount each partner received from the profit.

Solution

Abubakar's contribution = Gh¢25,000
Babatunde's contribution = Gh¢7,000
Profit shared = Gh¢6,400
Babatunde's first share = Gh¢1,000.00

Babatunde's second share
= 7.5% of the total profit
$= \frac{7.5}{100} \times 6,400 =$ Gh¢480.00

Babatunde's third share
= 3% of the capital he invested
$= \frac{3}{100} \times 7,000 =$ Gh¢210.00

Abubakar's first share
= 3% of the capital he invested
$= \frac{3}{100} \times 25,000 =$ Gh¢750.00

Remaining profit
= Gh¢6,400 – Gh¢(1,000 + 480 + 210 + 750)
= Gh¢ (6,400 – 2,440) = Gh¢3,960

The remaining Gh¢3,960 shared in the ratio of their capital contributions
⇒Abubakar : Babatunde = 25,000 : 7,000 = 25 : 7
Total ratio = 25 + 7 = 32
Babatunde's fourth share
$= \frac{7}{32} \times 3,960 =$ Gh¢866.25

Babatunde's total share
= Gh¢ (1,000 + 480 + 210 + 866.25)
= Gh¢ 2,556.25

Abubakar's second share
$= \frac{25}{32} \times 3,960 =$ Gh¢3,093.75

Abubakar's total share
= Gh¢ (750 + 3,093.75) = Gh¢ 3,843.75

2. Jonas, George and Green are partners in a business and their contribution to the capital are respectively Gh¢15,000.00, Gh¢25, 000.00 and Gh¢30,000.00. They agreed to share 40% of any net profit in the ratio of their contribution to the capital. In 2011, their profit before tax was Gh¢16,800.00 and 45% was paid to the government as tax.

i. Calculate the share of the profit received by each partner.

ii. Green invested his share of the profit in 2011 at 12 % per anum simple interest. Express Green's share of the profit in 2011 together with interest earned on it in 8 years
as a percentage of his initial contribution to the capital, giving your answer to two decimal places.

Solution

Total profit = Gh¢16,800.00

Tax = 45% of total profit

$= \frac{45}{100} \times$ Gh¢16,800 = Gh¢7560.00

The Rest (Net profit) ;

= Gh¢16,800.00 – Gh¢7,560.00 = Gh¢9,240.00

40% of net profit;

$= \frac{40}{100} \times$ Gh¢9,240 = Gh¢3,696.00

Amount to Shared = Gh¢3,696.00

Ratio of contribution = 15 : 25 : 30 = 3 : 5 : 6

Total ratio = 3 + 5 + 6 = 14

Jonas' share $= \frac{3}{14} \times$ Gh¢3,696 = Gh¢792.00

Gorge's share $= \frac{5}{14} \times$ Gh¢3,696 = Gh¢1320.00

Green' share $= \frac{6}{14} \times$ Gh¢3,696 = Gh¢1,584.00

ii. P = Gh¢1,584.00 T = 8years, R = 12%

$I = \frac{PTR}{100} = \frac{1,584 \times 8 \times 12}{100}$ = Gh¢1,520.64

Green's share + Interest

= Gh¢1,584.00 + Gh¢1,520.64 = Gh¢3,104.64

Green's amount as a percentage of his total contribution $= \frac{Amount}{Initial\ contribiution} \times 100\%$

$= \frac{3,104.64}{30,000} \times 100\% = 10.35\%$ (2 d.p)

A Partner Joining the Business Later

When a partner does not start the business but joins later, his share of the profit is calculated from the time he joins, but not from the beginning of the year.

Note:

1. That, we have a calendar year of 12 months so profits are easily shared according to the number of months of joining the business and the capital contributed.
2. That, the net profit is shared in the ratio of the product of the capital contributions and time since each partner joins the company.

Worked Examples

1. Mr. Blain and Mr. Thomas entered in a partnership with capitals of Gh¢12,600.00 and Gh¢1,920.00 respectively. After three months they were joined by Mr. Ceesay with a capital of Gh¢16,200.00. It was agreed that the profit should be shared in proportion to their capitals. During the first three month of the year, the business made a profit of 24% of the working capital and during the remaining nine months, the profit was 32% of the working capital.

i. Find the amount received by each partner as his share of the profit for the year.

ii. Express Mr. Blain's share of the profit as a percentage of his investment.

Solution

Working Capital of Mr. Blain and Mr. Thomas;

= Gh¢12,600.00 + Gh¢1,920.00 = Gh¢14,520.00

First three month profit;

= 24% of the working capital

$= \frac{24}{100} \times 14,520$ = Gh¢3,484.00

Ratio of contribution
12,600 : 1,920 = 105 : 16

Total ratio = 105 + 16 = 121
Mr. Blain's share = $\frac{105}{121}$ × Gh¢3,484
= Gh¢3,023.31
Mr. Thomas' share = $\frac{16}{121}$ × Gh¢3,484
= Gh¢460.69

After three months capital contribution by Mr. Blain, Mr. Thomas and Mr. Ceesay;
= Gh¢(12,600.00 + 1,920.00 + 16,200.00)
= Gh¢30,720.00

Profit of remaining 9 months;
= 32% of the working capital
= $\frac{32}{100}$ × Gh¢30,720 = Gh¢9,830.40

Share of profit according to ratio of contribution;
= 12,600 :1,920 :16,200 = 105 : 16 : 135
Total ratio = 105 + 16 + 135 = 256

Mr. Blain's share = $\frac{105}{256}$ × Gh¢9,830.40
= Gh¢4,032.00
Mr. Thomas' share = $\frac{16}{256}$ × Gh¢9,830.40
= Gh¢614.40

Mr. Ceesay's share = $\frac{135}{256}$ × Gh¢9,830.40
= Gh¢5,184.00

Mr. Blains total share;
= Gh¢3,023.31 + Gh¢4,032.00 = Gh¢7,055.31

Mr. Thomas' total share;
= Gh¢460.69 + Gh¢614.40 = Gh¢1,075.09

Mr. Ceesay's total share = Gh¢5,184.00

ii. Mr. Blain's share of the profit as a percentage of his investment;
= $\frac{share\ of\ profit}{investment}$ × 100%
= $\frac{7{,}055.31}{12{,}600}$ × 100% = 56%

2. Yaw started a business with Gh¢2,400, 000.00. After 6 months he was joined by Esi who contributed Gh¢3,000.00. Two months later, Yaw and Esi were joined by Kwasi who contributed Gh¢3,300.00. They agreed to share the profit as follows: 20% to Yaw as manager of the company, 4% to Kwasi as assistant manager. The rest of the profit was then shared in the ratio of the product of their capital in the company and the time since each of them joined the company. If the profit at the end of the year after Yaw had started the company was Gh¢1,005,000.00, calculate the total amount received by each of the three partners of the company.

Solution
Profit = Gh¢1,005,000.00,

Yaw's first share of 20% of profit as manager,
= $\frac{20}{100}$ × Gh¢1,005,000.00 = Gh¢201,000.00

Kwasi's first share;
= 4% of profit as assistant manager
= $\frac{4}{100}$ × Gh¢1,005,000.00 = Gh¢40,200.00

The rest of the profit;
= Gh¢(1,005,000 – 201,000 – 40,200)
= Gh¢763,800.00

The rest of the profit shared in the ratio of the product of their capital and time since each of them joined the company

Yaw's product of capital and time of joining;
= Gh¢2,400,000 × 12 = Gh¢28,800,000

Esi's product of capital and time of joining;
= Gh¢3,000 × 6 = Gh¢18,000

Kwasi's product of capital and time of joining
= Gh¢3,300 × 4 = Gh¢13,200

Ratio of product of capital and time of joining
Yaw : Esi : Kwasi
28,800,000 : 18,000 : 13,200
288,000 : 180 : 132 = 72,000 : 45: 33
Total ratio = 72,000 + 45 + 33 = 72,078
Total amount = Gh¢763, 800.00

Yaw's second share = $\frac{72,000}{72078}$ × Gh¢763,800
= Gh¢762,973.44
Kwasi's second share = $\frac{33}{72078}$ × Gh¢763,800
= Gh¢349.70

Esi's share = $\frac{45}{7,208}$ × Gh¢763, 800 = Gh¢476.86

Total amount received by Yaw;
= Gh¢201,000.00 + Gh¢762,973.44
= Gh¢763,973.44

Total amount received by Kwasi ;
= Gh¢40,200.00 + Gh¢349.70 = Gh¢40,549
Total amount received by Esi = Gh¢476.86

3.Three partners shared the profit in a business in the ratio 5 : 7 : 8. They had partnered for 14 months, 8 months and 7 months respectively. What was the ratio of their investments?

Solution
Let their investments be Gh¢ x for 14 months, Gh¢y for 8 months and Gh¢ z for 7 months respectively.
Then, $14x : 8y : 7z = 5 : 7 : 8$.
$\Rightarrow \frac{14x}{8y} = \frac{5}{7}$
$7 \times 14x = 5 \times 8y$
$98x = 40y$
$y = \frac{98x}{40} = \frac{49x}{20}$
$\Rightarrow \frac{14x}{7z} = \frac{5}{8}$
$8 \times 14x = 5 \times 7z$
$112x = 35z$
$z = \frac{112x}{35} = \frac{16x}{5}$
$\therefore x : y : z = x : \frac{49x}{20} : \frac{16x}{5}$
$x : y : z = 20 : 49 : 64$

Exercises 17.5
1. David and Goliath contributed Gh¢21,000.00 and Gh¢7,000.00 towards a business. At the end of one year, the business yielded a profit of Gh¢42,000.00 which they shared according to the ratio of their contributions.
i. How much was received by each partner?
ii. Express the share of each person as a percentage of the total profit.

2. A joint business was set up by Summer, Butler and Winter with respective contributions of Gh¢5million, Gh¢3.5 million and Gh¢1.5million. The profit was agreed to be shared in proportion to their contributions.
i. Calculate the share of each person if the profit made after one year was Gh¢30million. ii. What percentage of their contributions was received as profit by the partners?

3. The capital for a joint business is made up of Gh¢7,500.00 and Gh¢12,000.00 contributions by Samson and Delilah respectively. They agreed to share the profit as follows; as the manager,

Samson will be paid Gh¢500.00 in addition to 10% of the total profit. Each person will be paid 5% of the contributions made and the rest of the profit will be shared according to the ratio of their contributions. Calculate the share of each partner if they made a net profit of Gh¢8,000.00 at the end of the year.

4. Candy, Sandy and Wendy jointly thought of engaging themselves in a business venture. It was agreed that Candy would invest Gh¢6,500.00 for 6 months, Sandy, Gh¢8,400.00 for 5 months and Wendy, Gh¢10,000.00 for 3 months. Candy wants to be the working member for which, he was to receive 5% of the profits. The profit earned was Gh¢7,400.00. Calculate the share of Sandy in the profit.

5. Esi and Mansah entered into a business partnership. Esi contributed 35% of the capital while Mansah contributed the rest. At the end of the year, they made a profit of Gh¢5,600.00. 15 % of the profit was paid into a reserve fund while 25% of the remaining profit was paid as income tax. They then shared the remaining profit in the ratio of their contributions. If Mansah contributed Gh¢10,400. 00, find;

i. the total contributions of Esi and Mansah;

ii. the total amount paid as income tax,

iii. correct to one decimal place, Esi's profit as a percentage of her contribution.

6. Joel, Martin and Bruce subscribe Gh¢50,000.00 for a business. Joel subscribes Gh¢4,000.00 more than Martin and MartinGh¢5,000.00 more than Bruce. Out of a total profit of Gh¢35,000.00, how much will Joel receive if it is shared according to the ratio of their subscription.

7. Tonton and Sansan entered into partnership with capitals in the ratio 4 : 5. After 3 months, Tonton withdrew $\frac{1}{4}$ of his capital and Sansan withdrew $\frac{1}{5}$ of his capital. The gain at the end of 10 months was Gh¢7,600.00. Calculate Tonton's share of the profit .

8. *A*, *B*, *C* rent a pasture. *A* puts 10 oxen for 7 months, *B* puts 12 oxen for 5 months and *C* puts 15 oxen for 3 months for grazing. If the rent of the pasture is Gh¢175.00, how much must *C* pay as his share of rent?

9. A began a business with Gh¢85,000. He was joined afterwards by B with Gh¢42,500.00. For how much period does B join, if the profits at the end of the year are divided in the ratio of 3 : 1?

10. Cain and Abel started a business in partnership investing Gh¢20,000.00 and Gh¢15,000.00 respectively. After six months, Joseph joined them with Gh¢20,000.00. What will be Abel's share in total profit of Gh¢2,500.00 earned at the end of 2 years from the starting of the business?

Calculating the Profit Given the Ratios, Terms and Conditions

Sometimes, students are requested to find the total profit shared given the ratio of contributions, the share of the profit of one person and other terms and conditions. In such situations

I. Calculate the capital contributions of each person.

II. Represent the total profit by any preferred variable.

III. Calculate the share or shares of each person out of the total profit

IV. Find the total of the shares (total amount used).

V. Determine the remaining profit using the relation:

Remaining profit = Profit - total amount used

VI. Determine the one whose share has been given and find the share of that person out of the remaining fraction

VII. Find the total share(s) of the person in activity VI and equate it to the given total profit of that person

VIII. Solve for the value of the variable, to get the total profit shared

Worked Examples

1. Keren and Kelvin contributed a capital of Gh¢3,200.00 to purchase a vehicle for *"trotro"* business. The capital was contributed in the ratio 2 : 3. The profit for a year was shared under the following agreed conditions and terms; Keren the driver received 15% of the total profit and Kelvin,
the conductor (mate) received 10% of the profit. Each partner is paid additional 5% of the capital contributed and the rest of the profit was shared between them in the ratio of their contributions to the capital. If Keren's share of the total profit was Gh¢2,340.00, calculate

i. the total profit for the year

ii. Kelvin's share of the profit as a percentage of his contribution to the capital.

Solution

Capital = Gh¢3,200.00

Keren : Kelvin = 2 : 3

Total ratio = 2 + 3 = 5

Capital contributed by each person;

Keren = $\frac{2}{5} \times 3{,}200$ = Gh¢1,280.00

Kelvin = $\frac{3}{5} \times 3{,}200$ = Gh¢1,920.00

Let the total profit be x

Keren's first share = $\frac{15x}{100} = \frac{3x}{20}$

Keren's second share = $\frac{5}{100} \times 1{,}280$ = Gh¢64.00

Kelvin's first share = $\frac{10x}{100} = \frac{x}{10}$

Kelvin's second share = $\frac{5}{100} \times 1{,}920$ = Gh¢96.00

Total amount used

$= \frac{3x}{20} + 64 + \frac{x}{10} + 96$

$= \frac{3x}{20} + \frac{x}{10} + 64 + 96$

$= \frac{x}{4} + 160$

$= \frac{x + 640}{4}$

Remaining profit;

= Profit – Total amount used

$= x - \left(\frac{x + 640}{4}\right)$

$= \frac{4x - (x + 640)}{4} = \frac{4x - x - 640}{4} = \frac{3x - 640}{4}$

Ratio for sharing the remaining profit;

Keren : Kelvin = 2 : 3

Total ratio = 2 + 3 = 5

Keren's share of the remaining profit;

$= \frac{2}{5} \times \left(\frac{3x - 640}{4}\right)$

$\therefore$ Keren's share of the total profit

= His first share + his second share + his share of the remaining profit

$= \frac{3x}{20} + 64 + \frac{2}{5}\left(\frac{3x - 640}{4}\right)$

But Keren's total profit = Gh¢2,340.00

$\Rightarrow \frac{3x}{20} + 64 + \frac{2}{5}\left(\frac{3x - 640}{4}\right)$ = Gh¢2,340

$\frac{3x}{20} + 64 + \frac{2}{20}(3x - 640) = 2{,}340$

$(20)\frac{3x}{20} + (20)\,64 + (20)\frac{2}{20}(3x - 640) = (20)\,2340$

$3x + 1280 + 2(3x - 640) = 46,800$

$3x + 1280 + 6x - 1280 = 46,800$

$3x + 6x = 46,800$

$9x = 46,800$

$x = 5,200$

The total profit for the year was Gh¢5,200.00

ii.Kelvin's first share;

$= \frac{x}{10} = \frac{Gh¢5,200}{10}$ = Gh¢520.00

Kelvin's second share = Gh¢96.00

Kelvins total share of the profit;

= Gh¢520 + Gh¢96.00 = Gh¢616.00

Kelvin's profit as a percentage of his contribution

$= \frac{kelvins\ profit}{kelvins\ contribution} \times 100\%$

$= \frac{616}{1920} \times 100\% = 32\%$

2. Alex and Jimmy entered into a business partnership in January 2013. The total capital was Gh¢27,000.00 which they agreed to contribute in the ratio 2: 1 respectively. The annual profit from 2013 was shared as follows:Alex was paid $5\frac{1}{2}\%$ of the total profit for his service as a manager. The remainder of the profit was shared between them in the ratio of their contributions to the capital. If Alex received a sum of Gh¢6,850.00out of the profit. Calculate:

i. the total profit for the year;

ii. Jimmy's share of the profit as a percentage of his initial contribution to the capital;

iii. If Alex had to pay tax at 30% on the amount he received, how much did he pay?

Solution

i. Capital = Gh¢27,000.00

Alex : Jimmy = 2 : 1

Total ratio = 2 + 1 = 3

Capital contributed by each person;

Alex = $\frac{2}{3} \times 27,000$ = Gh¢18,000.00

Jimmy = $\frac{1}{3} \times 27,000$ = Gh¢9,000.00

Let the total profit be x

Alex's first share = $\frac{5.5x}{100}$

Remaining Profit = $x - \frac{5.5x}{100} = \frac{100x - 5.5x}{100} = \frac{94.5x}{100}$

Alex's second share = $\frac{2}{3}\left(\frac{94.5x}{100}\right) = \frac{189x}{300}$

Total share of Alex = $\frac{5.5x}{100} + \frac{189x}{300}$

But Alex's total share = Gh¢6,850

$\Rightarrow \frac{5.5x}{100} + \frac{189x}{300} = 6,850$

$300 \times \frac{5.5x}{100} + 300 \times \frac{189x}{300} = 300 \times 6,850$

$3(5.5x) + 189x = 2,055,000$

$16.5x + 189x = 2,055,000$

$205.5x = 2,055,000$

$x = \frac{2,055,000}{205.5} = 10,000$

The total profit shared is Gh¢10,000.00

ii. Jimmy's share = Profit – Alex's share

= Gh¢ 10,000 – Gh¢ 6,850 = Gh¢3,150.00

Jimmy's share of the profit as a percentage of his initial contribution to the capital;

$= \frac{3,150}{9,000} \times 100\% = 35\%$

iii. 30% tax paid by Alex on his share;

$= \frac{30}{100} \times$ Gh¢6,850

= Gh¢2,055.00

Exercises 17.6

1. Martha and Mercy invest in a business in the ratio 3 : 2. If 5% of the total profit goes to charity and Martha's share is Gh¢855.00. Find the total profit shared:

2. Kofi and Yaw entered into business partnership with a total capital of Gh¢81 million. They agreed to contribute the capital in the ratio 2 : 1 respectively. The profit was shared as follows; Kofi was paid 5% of the total profit for his services as the manager. Each partner was paid 3% of the capital he invested. The remainder of the profit was then shared between them in the ratio of their contributions to the capital. If Kofi's share of the total profit was Gh¢7.5 million, calculate

a. the total profit for the year to the nearest thousand cedi

b. Yaw' share of the profit as a percentage of his contribution to the capital

Banking

It is the act of engaging in the business of keeping money for savings or exchange. Institutions responsible for these activities are called ***Banks.***

In Ghana, there is the central bank called Bank of Ghana, established to have oversight responsibilities on all banking activities. Some of the banks in Ghana are Ghana Commercial Bank, Agricultural Development Bank, SG – SSB, National Investment Bank, Eco – Bank, Barclays Bank, Standard Charted Bank etc. Added to the list are Rural and Community banks (Micro-banks) who work under the auspices of the ARB Apex Bank.

The customer of a bank is identified by a unique number called ***account number***, required for all transactions.

Bank Transactions and Services

Some common transactions and services provided by the banks are explained below:

Savings Account

The savings account allows the customer to deposit money at the bank and earns interest at a given rate on the deposited amount for a given period of time (usually one year). With savings account, banks insist on minimum balance that customers have to leave in their accounts and also insist on customers withdrawing money in person.

Current Accounts

The current accounts allow customers to deposit and withdraw money from their accounts at any time within banking hours. With current accounts, the customer withdraws money by the use of a cheque and withdrawal can be done through a third person. The customer is charged for all services provided by the bank.

Fixed Deposit accounts

Fixed deposit accounts allows a customer to deposit money with a bank for a given (fixed) period of time at a given (fixed) rate of interest payable at the due or fixed date. With fixed deposit, if the customer makes a partial or full withdrawal of the deposited amount before the fixed period, it constitutes breach of contract and as such he or she forfeits the interest. It is also called ***time deposit***

Inward Money Transfer

This service allows money to be transferred from one person to another from within and outside the country through the banks for a fee. Eg. Money gram, western union, bankers draft etc . In the case of transfer of foreign currency, the bank pays the customer the cedi equivalent.

Overdraft

This service allows a customer to withdraw more money than he or she has in his accounts at an interest. This facility is usually granted for a short period of time (one or two months).

Loans

This service allows customers to take money from the bank at a given rate of interest for a given period of time. With the loan facility, banks usually request from the customer, documents of buildings or any valuable property as collateral evidence. Others also ask for a third person called ***guarantor*** to offer to pay the loan with its interest to the bank in case the customer defaults repayment.

Bank Charges

For operating an account with a bank or requesting some services or transactions from a bank, the bank charges the customer some fees. Some of the bank charges or fees are listed below:

1. Cost of turnover, COT (applicable to current accounts
2. Withdrawal charges
3. Overdraft charges
4. Bankers Drat Charges
5. Telegraphic Transfer charges
6. Periodic charges
7. Money Transfer charges
8. Cheque deposit charges
9. Card user charges

Completing a Pay-in-slip and Payment Cheques:

The pay in slip and payment cheques has the following empty spaces to be completed

1. Name of customer /depositor
2. Accounts number
3. Branch where account is held
4. Date
5. Amount deposited (in figures and words)
6. Signature and others

Value Added Tax (V. A. T)

It is a tax or extra money charged on goods and services. This charge is calculated as the percentage of the basic cost of the good or services.

The cost of an item without V.A.T is called the ***basic cost*** or ***VAT exclusive cost*** (always 100%) and the cost of item with VAT is called ***VAT inclusive cost*** orsimply, ***total cost or selling price***.The V.A.T inclusive cost is equal to the basic cost plus the VAT charge.
i.e. (100% + r %)

V.A.T is usually calculated as a percentage.

Mathematically,

1. *V.A.T = Rate × Basic cost*

2. *VAT inclusive cost /selling price*

= Basic cost + (Rate × Basic cost)

= Basic cost + VAT

In Ghana, the VAT rate is 12.5%, subject to change

Note

1. Given the cost price of an item, it simply implies the basic cost or VAT exclusive cost
2. Given the selling price of an item, it simply implies VAT inclusive cost

Worked Examples

1. The cost of a bracelet is Gh¢1,000.00.

i. How much is the VAT charged at a rate of $12\frac{1}{2}$%?

ii. What is the VAT inclusive cost of the bracelet?

Solution

i. V. A .T = Rate × Basic cost

Rate = $12\frac{1}{2}$ % = 12.5%

Basic cost = Gh¢1,000.00

V. A. T = Rate × Basic cost.

V. A. T = $\frac{12.5}{100}$× 1,000 = Gh¢125.00

ii. VAT inclusive cost = Basic cost + VAT

= Gh¢1,000.00 + Gh¢125.00 = Gh¢1,125.00

2. The VAT exclusive charge of a monthly electricity bill is Gh¢200.00.Find the V.A.T of 12.5% on the bill and hence, find the total monthly bill.

Solution

Rate = 12.5%, Basic cost = Gh¢200.00

VAT = Rate × Basic cost

VAT = 12.5% × Gh¢200

= $\frac{12.5}{100}$ × Gh¢200 = Gh¢25.00

Monthly bill = Basic charge + VAT

= Gh¢(200.00 + 25.00) = Gh¢225.00

3. The VAT rate of a certain country is 20%.
If the cost of a refrigerator is Gh¢620.00, calculate:

i. the VAT charged on the refrigerator.

ii. the total cost of the refrigerator

Solution

i.VAT = Rate × Basic cost

But rate = 20% , basic cost = Gh¢620.00

VAT = 20% × Gh¢620

= $\frac{20}{100}$ × Gh¢620 = Gh¢124.00

ii. Selling price = Basic cost + VAT charge

= Gh¢620.00 + Gh¢124.00

= Gh¢744.00

Exercises 17.7A

1. Rashid wants to buy a car costing Gh¢8,000.00. If VAT is due at a rate of $12\frac{1}{2}$ % of the cost of the car, how much will he pay for the car?

2. The cost of a colour television is Gh¢2,200.00.

i. Find the VAT at $17\frac{1}{2}$ %.

ii. How much will you pay for the television?

3. The VAT rate of a certain country is 20%. If the cost of a Toyota Lancruiser vehicle in that country is Gh¢36,600.00, calculate:

i. the VAT on the vehicleii. the selling price of the car

4. At a certain store, the VAT exclusive cost of a set of football jersey is Gh¢375.00, if VAT is charged at a rate of 12%, find:

i. the VAT on five sets of jerseys

ii. the selling price of the five jerseys

Finding the Basic Cost or the VAT Charge Given the VAT Inclusive Cost

From the relation,

VAT inclusive cost /selling price/ total cost

= Basic cost + (Rate × Basic cost)

Let S represent the VAT inclusive cost, *b* represent the basic cost and *r* the VAT rate

⇒ S = *b* + *br*%

1. $\boldsymbol{b} = \frac{\boldsymbol{S}}{(\mathbf{1}+\boldsymbol{r}\%)}$

2. VAT charge = Inclusive cost – Basic cost

Worked Examples

1. An electricity bill including VAT of 12 $\frac{1}{2}$ % is Gh¢450.00.

i. How much is the bill exclusive of VAT?

ii. How much VAT is charged on the bill?

Solution

i. VAT inclusive charge, S = Gh¢450.00, VAT rate, *r* = 12.5% and basic cost (same as VAT exclusive cost), *b* = ?

$\boldsymbol{b} = \frac{\boldsymbol{S}}{(\mathbf{1}+\boldsymbol{r}\%)} = \frac{\mathbf{450}}{(\mathbf{1}+\mathbf{12.5}\%)}$ = Gh¢400.00

The bill exclusive of VAT is Gh¢400.00

ii. VAT charge = Inclusive cost – Basic cost

= Gh¢450 .00 – Gh¢400.00 = Gh¢50.00

2. A school bag is sold at Gh¢594.00. How much is the VAT charged on the bag if the VAT rate is 12.5%?

Solution

VAT inclusive cost, S = Gh¢594.00, VAT rate, *r* = 12.5% and basic cost (same as VAT exclusive cost), *b* = ?

$\boldsymbol{b} = \frac{\boldsymbol{S}}{(\mathbf{1}+\boldsymbol{r}\%)} = \frac{\mathbf{594}}{(\mathbf{1}+\mathbf{12.5}\%)}$ = Gh¢528.00

The basic cost of the bag is Gh 528.00

ii. VAT charge = Inclusive cost – Basic cost

= Gh¢594 .00 – Gh¢528.00 = Gh¢66.00

Exercises 17.7B

1. The selling price of a digital camera is Gh¢400.00. If VAT is charge at a rate of 12.5%, calculate:

i. the basic cost of the camera.

ii. the amount charged as VAT.

2. A calculator sells at Gh¢800.00 at a VAT rate of 12.5%. Find:

i. the basic cost of the calculator.

ii. the VAT charge on the calculator.

3. A telephone bill including VAT of 12.5 % is Gh¢490.00. How much was the bill before VAT was added.

4. A electricity bill including VAT of 12.5% is Gh¢4,374.00. How much was the bill before VAT was added.

5. The VAT paid on an article amounts to Gh¢900.00 at a rate of 10%. What is the total price of the article including VAT?

National Health Insurance Levy (NHIL)

It is a charge in percentage, calculated on the cost of goods and services to provide health assistance to people. The rate on NHIL in Ghana is 2 $\frac{1}{2}$ %, subject to changes.

1. NHIL = Rate × Basic cost

2. ***NHIL inclusive cost / selling price***

= Basic cost + (Rate × Basic cost)

= Basic cost + NHIL charge

Note

1. Given the cost price of an item, it simply implies the basic cost or NHIL exclusive cost

2. Given the selling price of an item, it simply implies NHIL inclusive cost

Worked Examples

1. Find the NHIL on a television that cost Gh¢ 612.00

Solution

NHIL = Rate × Basic cost

where Rate = $2\frac{1}{2}\% = 2.5\%$

NHIL = $\frac{2.5}{100}$ × Gh¢ 612 = Gh¢15.30

2. Calculate the NHIL on a refrigerator that cost Gh¢240.00

Solution

NHIL = Rate × Basic cost

where Rate = $2\frac{1}{2}\% = 2.5\%$

Basic cost = Gh¢240.00

NHIL = $\frac{2.5}{100}$ × Gh¢240.00 = Gh¢6.00

3. A yard of G.T.P. cost Gh¢3.80. Find the NHIL on 30 yards of G.T.P.

Solution

NHIL = Rate × Basic cost

where rate = $2\frac{1}{2}\% = 2.5\%$

Basic cost = Gh¢3.80 × 30 = Gh¢114.00

NHIL = $\frac{2.5}{100}$ × 114 = Gh¢2.85

Exercises 17.8

A. Calculate the NHIL of $2\frac{1}{2}$ % on the following, if NHIL is exclusive.

1. A car that cost Gh¢8,000.00.
2. An electricity bill that cost Gh¢4,600.00
3. A refrigerator that cost Gh¢1,800.00
4. A bag that cost Gh¢1,000.00
5. A bill that cost Gh¢4,700.00

B. Calculate the NHIL of $5\frac{1}{2}$ % on the following and determine the NHIL inclusive cost;

1. A car that cost Gh¢400.00
2. An electricity bill of Gh¢2,200.00
3. A telephone bill of Gh¢1,120.00.
4. Twenty bags of rice each at a cost of Gh¢120.00

Finding the Basic Cost or the NHIL Cost Given the NHIL Inclusive Cost

From the relation, *NHIL inclusive cost = Basic cost + (Rate × Basic cost)*

Let C represent NHIL inclusive cost, B represent basic cost and r represent rate of NHIL

⇒ C = B + (Br)

But r = 2.5% (unless stated otherwise)

C = B + (2.5% B)

C = B (1 + r%)……….(1)

From eqn (1), given the NHIL inclusive cost, C, the basic cost, B, is calculated as:

1. B = $\frac{C}{1+r\,\%}$

2. NHIL cost = Inclusive Cost – Basic cost

Worked Examples

1. A man purchased a set of furniture at an NHIL inclusive cost of Gh¢6,314.00. If NHIL is charged at a rate of 2.5%, find:

i. the basic cost of the set of furniture

ii. the amount paid as NHIL

Solution

i. NHIL inclusive cost, C = Gh¢6,314.00, Basic cost, B =? and r = 2.5%

B = $\frac{C}{1+2.5\%} = \frac{6314}{1+2.5\%}$ = Gh¢6,160.00

ii. NHIL cost = Inclusive Cost – Basic cost

NHIL cost = Gh¢6,314 – Gh¢6,160 = Gh¢154.00

2. What is the NHIL on a car sold for Gh¢20,541.00 at rate of $2\frac{1}{2}$ %? Hence, find the NHIL on the car.

Solution

i. NHIL inclusive cost, C = Gh¢20,541.00, Basic cost, B =? and r = 2.5%

Substitute in B = $\frac{C}{1+2.5\%} = \frac{20541}{(1+2.5\%)}$

= Gh¢20,040.00

ii. NHIL cost = Inclusive Cost – Basic cost

NHIL cost = Gh¢20,541 – Gh¢20,040

= Gh¢501.00

Exercises 17.9

1. A Plasma television is sold at Gh¢4,220.00. If NHIL is charged at a rate of 2.5%, calculate;
i. the basic cost of the plasma television.
ii. the NHIL cost.

2. The cost of a scientific calculator including NHIL is Gh¢738.00. What is the NHIL on the calculator, if NHIL is charged at a rate of 2.5%?

3. The NHIL inclusive cost of a car is Gh¢25,584.00. If you purchase this car at the said cost, what will be your contribution to NHIL, if the rate of NHIL is 5.5%?

4. At a certain boutique, the NHIL inclusive cost of a necklace is Gh¢940.00 and that of a bracelet is Gh¢730.00. If the rate of NHIL is 5.5%;
i. find the total basic cost of 15 necklaces and 20 bracelets
ii. what is the total health insurance contribution on the 15 necklaces and 20bracelets?

Insurance

In order to take care of future eventualities such as death, illness, accident, fire outbreaks, injuries and damages to properties, people take insurance policy.

The person who takes the insurance policy is called the ***policy holder*** and the company that offers the insurance is called the ***insurance company***.

The policy holder pays some amount of money at regular intervals to the insurance company. The money paid by the policy holder is intended to compensate him in case of occurrence of any of the eventualities mentioned above. The amount paid regularly to the insurance company is called ***premium*** and the compensation or benefit paid to the policy holder is called ***insuredvalue*** .The premium is usually a percentage of the insured value.

Mathematically;

Premium = Rate × Insured value….. (1)

Insured value = $\frac{\boldsymbol{Premium}}{\boldsymbol{Rate}}$***………….. (2)***

Rate = $\frac{\boldsymbol{premium}}{\boldsymbol{insured\ value}}$ ***× 100………… (3)***

Worked Examples

1. Mr. Oppong insured his car against accident. The value of the car was estimated as Gh¢9,500.00. If the rate of insurance is 15%, calculate the annual premium paid to the company.

Solution

Premium = Rate × Insured value

But rate = 15% = $\frac{15}{100}$

Insured value = Gh¢9,500

Premium = 15% × 9,500

= $\frac{15}{100}$ × 9,500 = Gh¢1,425

The premium paid annually is Gh¢1,425.00

2. Mrs. Shirley pays a premium of Gh¢117.00 at a rate of 13% star insurance company. Calculate the insured value if she contributes towards insuring her car against accident.

Solution

$Insured\ value = \frac{Premium}{Rate}$

But premium = 117 and rate = 13% = $\frac{13}{100}$

Insured value = $\frac{117 \times 100}{13}$ = Gh¢900.00

The insured value of the car is Gh¢900.00

3. Kennedy insured his house against fire. If the insured value of the house is Gh¢2,351.00 at a rate of 9%, find the yearly premium he pays to the insurance company.

Solution

Premium = Rate × Insured value.

But rate = 9% = $\frac{9}{100}$,

Insured value = Gh¢2,351

Premium = 9% × 2,351

= $\frac{9}{100}$ × 2,351= Gh¢211.59

4. Calculate the insured value of Mr. Peter's car if he contributes a premium of Gh¢144.00 a year at a rate of 6%

Solution

$Insured\ value = \frac{Premium}{Rate}$

Premium = Gh¢144, Rate = 6% = $\frac{6}{100}$

Insured value = $\frac{44}{\frac{6}{100}} = \frac{144 \times 100}{6}$ = Gh¢2,400

The insured value is Gh¢2,400.00

5. Find the rate of insurance given the insured value as Gh¢2400, 00 and the premium as Gh¢144.00.

Solution

$Rate = \frac{premium}{insured\ value} \times 100\%$

Premium = Gh¢144 and Insured value = Gh¢2,400

Rate = $\frac{144}{2400} \times 100 = 6\%$

Exercises 17.10

1. The value of a car is Gh¢10,200.00, If it is insured against accident at rate of 9%. Calculate the premium paid annually to the insurance company.

2. Mr. Brown insured his hotel against fire. The insured value of the hotel is Gh¢15,220.00. Calculate the yearly premium paidto the insurance companyby Mr. Brown at a rate of 15%

3. King George is a policy holder of a certain insurance company. He insured his vehicle value at Gh¢120.00 per month. What is his yearly premium?

4. Mrs. Pomaa contributes a yearly premium of Gh¢189.00 to state insurance company as commitment to his insured yutong bus against accident at a rate of 20%. Find the insured value of the Yutong bus.

5. Calculate the insured value of a property which attracts a yearly premium of Gh¢204.00 at a rate of 17%.

6. Maame Duku pays Gh¢156.00 yearly premium as a property insurance to Enterprise insurance company at a rate of 12%. Calculate the insured value of her property.

7. The premium paid on an insured car valued at

Gh¢5,000.00 is Gh¢250.00, find the rate per anum.

8. The value of a ship is estimated at Gh¢3,600.00. If the owner contributes a premium of Gh¢324.00 a year; calculate the insurance rate.

Income Tax

It is the portion of a person's salary or wage paid to the government. Income tax levies are used by the government for developmental projects such as schools, roads, hospitals, transport and others. Income tax is deducted from workers salary before it is paid to them.

In Ghana the agency responsible for collecting income tax is Internal Revenue Service (I.R.S).

Note the following terms:

Gross Income: The total income before tax is deducted.

Tax Allowance: The part of the income that is not taxed

Taxable Income: It is the part of the income that is taxed

Net Income:It is the part of the salary left after all taxes has been taken. That is:

Net salary = Gross salary – Total tax

a. Tax free income

= sum of all tax allowances

b. Taxable income

= Annual Income – tax allowances

c. Annual tax paid

= Tax rate × Taxable amount

d. Monthly tax paid = $\frac{\boldsymbol{annualtaxpaid}}{\mathbf{12}}$

e. Net annual income

= Annual gross income – Annual tax paid

f. Net monthly income = $\frac{net\ annual\ income}{12}$

Note that words like "NIL" "NO" FREE ALLOWANCE" against any amount of money represent tax free (allowance) income.

Worked Examples

1. Suppose income tax regulations are as follows:

Allowances:

Personal allowance:	Gh¢45.00
Marriage allowance:	Gh¢30.00
Child allowance:	Gh¢25.00
(For each child under 18years)	
Disability Allowance:	= Gh¢20.00

Tax on Taxable income:

5% for first Gh¢60.00

10% for second Gh¢60.00

20% for the remainder

a. Calculate the tax paid by Mr. Peter, a physically challenged person, whose salary is Gh¢3,800.00 per annum, if he has a wife and three of his children under 18years.

b. Calculate Mr. Peter's net salary.

Solution

Income		Gh¢3,800.00
Allowances	Gh¢	
Personal	45.00	
Marriage	30.00	
Disability	20.00	
Child	25 × 3 = 75.00	
Total allowance	170.00	<u>Gh¢170.00</u>
Taxable income		<u>Gh¢3,630.00</u>

Taxes

5% tax on first Gh¢60.00

$= \frac{5}{100} \times 60$ = Gh¢3.00

10% tax on second Gh¢60.00

$= \frac{10}{100} \times 60 =$ Gh¢6.00

Remainder = Gh¢3,630 – (60 + 60)
= Gh¢(3,630 – 120) = Gh¢3,510.00

20% tax on the remainder,
$= \frac{20}{100} \times$ Gh¢3,510 = Gh¢702.00

Total Tax = Gh¢(3 + 6 + 702) = Gh¢711.00

Net Salary = Gross salary – Total tax
Net salary = Gh¢3,800 – Gh¢711
= Gh¢3,089.00

2. Mr. Green earns a salary of Gh¢5,800.00 per annum. He has three children whose ages are 19 years, 21 years and 23 years. If Mr. Green is visually impaired but has devoiced his wife, calculate the tax paid by him and his net salary using the following regulations;

Allowances:
Personal allowance =Gh¢40.00
Marriage allowance =Gh¢ 30.00
Child allowance = Gh¢20.00, but for each child under 18,
Disability allowance = Gh¢25.00

Tax on taxable income:

10% on first Gh¢50.00

15% on next Gh¢50.00

20% on the remainder

Solution
Gross income Gh¢ 5,800.00
Allowances: *Gh¢*
Personal 40.00
Disability 25.00
Total allowance 65.00 Gh¢65.00
Taxable income Gh¢5,735.00

Taxes
10% tax on first Gh¢50.00 = $\frac{10}{100} \times 50 =$ Gh¢5.00
15% tax on next Gh¢50.00 = $\frac{15}{100} \times 50 =$ Gh¢7.50

Remainder = Gh¢5,735 – Gh¢(50 + 50)
= Gh¢5,735 − Gh¢ 100 = Gh¢5,635.00

20% tax on the remainder;
$= \frac{20}{100} \times 5{,}635 =$ Gh¢1,127.00

Total tax = Gh¢(5 + 7.50 + 1,127)
= Gh¢1,139.50
Mr. Green paid a total tax of Gh¢1,139.50

Net salary = Gross income – Total tax
= Gh¢5,800.00 − Gh¢1,139.50
= Gh¢4,660.50

3. Suppose the income tax regulation of a country is as follows.

Allowances	Amount (Gh¢)
Personal	100
Marriage	400
Disability	500
Car maintenance	600
Child (under 18)	250
Dependant relative	200

	Taxable income	Tax Rate
	Gh¢	Gh¢
First	800.00	Nil
Next	1,200.00	5.5%
Next	1,200.00	10.5%
Next	600.00	15%
Remainder		5%

Mr. Green is a widower with 2 children aged 20 and 21 years. Recently, he lost one of his legs through an accident, which did not involve his own car. If his salary per anum is Gh¢4,850.00, calculate:
i. his annual taxable income and how much he pays as income tax.
ii. how much he receives as net annual income.

Solution

Income (Gh¢)		**4,850**
Personal	100	
Disability	500	
Car maintenance	600	
Tax free	800	2,000
Taxable income		**2,850**

Taxes
5.5% on Gh¢1,200
$\frac{5.5}{100} \times 1,200 = 66$
Remaining = 2,850 – 1,200 = 1650

10.5% on Gh¢1,200
$\frac{10.5}{100} \times 1,200 = 126$
Remaining = 1,650 – 1,200 = 450

15% on Gh¢450 (not Gh600)
$\frac{15}{100} \times 450 = 67.5$

Total taxes
= 66 + 126 + 67.5 = 259.5
Total tax paid is Gh¢259.5

ii. Net Salary = Gross – taxes
= 4,850 – 259.5 = 4,590.50
Net annual income = Gh¢4,590.50

3. Darling earns a monthly salary of Gh¢6,500.00. He is allowed a tax free of Gh¢500.00. If he pays 17% tax in his taxable income, calculate;
i.his taxable income,
ii.his total tax paid,
iii.his net salary.

Solution
i. Gross monthly salary = Gh¢6,500.00
Tax allowance = Gh¢500.00
Taxable income = ?

Taxable income
= Gross monthly salary – Tax allowance
= Gh¢6,500.00 – Gh¢500.00 = Gh¢6,000.00

ii. Tax paid is 17% of taxable income
$= \frac{17}{100} \times 6,000$ = Gh¢1,020.00

iii. His net salary = Gross salary – Total tax
Net salary = Gh¢6,500 − Gh¢1,020
Net salary = Gh¢5,480.00

4. Mrs. Akushika's monthly taxable income is Gh¢9,850.00. Use the table below to calculate:
i.how much she pays as income tax
ii.her net monthly income.

	Taxable income	Tax Rate
	Gh¢	Gh¢
First	600.00	Nil
Next	600.00	50%
Next	600.00	10%
Next	1,200.00	15%
Next	1,200.00	25%
Next	5,400.00	35%
Remainder		55%

Solution

i. Gross income Gh¢9,850.00

Allowances	*Gh¢*	
First	600.00	
Total allowance	600.00	<u>Gh¢600.00</u>
Taxable income		<u>Gh¢9,250.00</u>

Taxes

50% of Gh¢600.00 = $\frac{50}{100} \times 600$ = Gh¢300.00

10% of Gh¢600.00 = $\frac{10}{100} \times 600$ = Gh¢60.00

15% of Gh¢1,200.00 = $\frac{15}{100} \times 1{,}200$ = Gh¢180.00

25% of Gh¢1,200.00 = $\frac{25}{100} \times 1{,}200$ = Gh¢300.00

35% of Gh¢5,400.00 = $\frac{35}{100} \times 5{,}400$ = Gh¢1,890.00

Remainder

= Gh¢9,250 – Gh¢(600 + 600 + 1,200 + 1,200 + 5,400)

= Gh¢9,250 − Gh¢9,000 = Gh¢ 250.00

55% tax on remainder = $\frac{55}{100} \times 250$ = Gh¢137.50

Total tax

= Gh¢(300 + 60 + 180 + 300 + 1,890 + 137.50)

= Gh¢2,867.50

ii. *Net income = Gross income – Total tax.*

Net income = Gh¢ 9,850.00 − Gh¢2,867.50

Net income = Gh¢6,982.50

Therefore, her net income is Gh¢6,982.50.

Some Solved Past Questions

1. In Ghana, the annual income tax payable by an individual in a certain year was assessed at the following rate:

Rate of Tax	
For every cedi of the first 300.00	Nil
For every cedi of the next 240.00	5p
For every cedi of the next 480.00	$7\frac{1}{2}$p
For every cedi of the next 480.00	10p
For every cedi of the next 960.00	$12\frac{1}{2}$p
For every cedi of the next 1,140.00	15p

a. Calculate the income tax payable by Agbedefu who earned Gh¢3,000.00 per anum

b. What percentage of Agbedefu's annual income was payable as income tax?

Solution

Gross income Gh¢3,000

Allowances	*Gh¢*	
First 300		
Total allowance	300	<u>Gh¢300</u>
Taxable income		<u>Gh¢2,700.00</u>

Taxes

5% of Gh¢240.00 = $\frac{5}{100} \times 240$ = Gh¢12.00

Remaining = Gh¢2,700 – Gh¢240 = Gh¢2,460.00

7.5% of Gh¢480.00 = $\frac{7.5}{100} \times 480$ = Gh¢36.00

Remaining = Gh¢2,460 – Gh¢480 = Gh¢1,980.00

10% of Gh¢480.00 = $\frac{10}{100} \times 480$ = Gh¢48.00

Remaining = Gh¢1,980 – Gh¢480 = Gh¢1,500.00

12.5% of Gh¢960.00 = $\frac{12.5}{100} \times 960$ = Gh¢120.00

Remaining = Gh¢1500 – Gh¢960 = Gh¢540.00

15% of Gh¢540 (Not Gh¢1,140)

= $\frac{15}{100} \times 540$ = Gh¢81.00

Total tax paid (Income tax);

= Gh¢(12 + 36 + 48.00 + 120 + 81)

= Gh¢297.00

b. $\frac{Income\ tax}{Annual\ income} \times 100\%$
$= \frac{297}{3000} \times 100\% = 9.9\%$

Exercises 17.11

A. In each of the following cases, calculate the income tax.

Annual Salary (Gh¢)	Allowance (Gh¢)	Tax Rate (%)
800.00	45.00	10%
1500.00	500.00	15%
11,070.00	3,550.00	20%
25,000.00	5,200.00	30%

B.1. Madam Fausty receives Gh¢5,000.00 a year. She is allowed a tax free of Gh¢300.00. If she contributes 12% to social security and 2% p. a to health insurance, calculate:
i. her taxable income,
ii. her total tax,
iii. her net salary.

2. Mama Jane's monthly salary is Gh¢4,500.00. She is allowed a tax free of 12% of her monthly salary. She contributes 17 $\frac{1}{2}$ % to social security and 2 $\frac{1}{2}$% per month to health insurance. Determine:
i. her tax allowance,
ii. her total tax payment per month.
iii. her net monthly salary.

3. A man earns Gh¢38,400.00 per anum and pays an income tax of Gh¢480.00 per month. What percentage of his salary does he pay as income tax?

4. The Chief Executive Officer of Down – Below Football Club is married with two children. He is paid a yearly salary of Gh¢18,000.00. His tax free allowances are as follows:
Personal allowance = Gh¢1,200.00
Wife allowance = Gh¢600.00
Children allowance = Gh¢250.00 per child.

Tax on taxable income:
20% on the first Gh¢2,000.00
15% in the next Gh¢4,000.00
Calculate the officer's:
i. taxable income,
ii. total tax paid a month,
iii. net monthly salary.

5. A company's director married with 6 children is on an annual salary of Gh¢12,000.00. His tax free allowances are as follows;
Personal allowance Gh¢1,200.00 plus 10 % of excess of his salary over Gh¢10,000.00
Wife allowance Gh¢600.00
Children allowance Gh¢250.00 per child for the first four children
Dependent relatives Gh¢550.00

On the taxable income, the rates of tax are as follows:
10p in the Gh¢ on the first Gh¢2,000.00
15p in the Gh¢ on the next Gh¢4,000.00
22 $\frac{1}{2}$ p in the Gh¢ on the next Gh¢5,000.00
30p in the Gh¢ on the next Gh¢10,000.00
37 $\frac{1}{2}$ p in the Gh¢ on the rest. Calculate:
a. the director's taxable incomeb. the average monthly tax he pays to the nearest pesewa,
b. the percentage of his monthly salary he pays as tax, correct to one decimal place.

6. A man's annual salary is Gh¢36,900.00. He is entitled to $\frac{2}{9}$ of this amount free of tax and a further tax free allowance of Gh¢700.00. On his remaining salary after these allowances, he pays 10p in the cedi on the first Gh¢6,000.00, 40p in

the cedi on the next Gh¢4,500.00 and 17p in the cedi on the balance if any.

a. How much tax does he pay annually?

b. Calculate his annual net salary

Challenge Problem

1. Mr. White receives a legacy of Gh¢100,000.00. He considers two ways of investing it, as shown below: Calculate his net
income in each case.

i. In a savings bank, he can invest Gh¢5,000.00 in the ordinary branch, bearing 4% p. a. interest and the rest in the special branch, bearing 7% p. a. interest. Interest received in the ordinary branch is tax free: interest received in the special branch is subject to tax at a standard rate of Gh¢300.00

ii. He can buy oil shares costing Gh¢20.50 each which pay a dividend annually of Gh¢1.20 per share, all of which is liable to income tax at the standard rate.

2. Suppose the following allowances were granted free of income tax from an employer's annual salary.

Employees personal allowance: Gh¢30,000.00

Employees wife allowance: Gh¢24,000.00

Allowance for each child under 18: Gh¢12, 000.00. Find :

a. the annual taxable income,

b. the net income of the following employees;

i. Mr. Benson whose income is Gh¢195,000.00 p. a and who is married but has no child,

ii. Mr. Okoro who also earns Gh¢195,000.00 and married with two children under the age of 18,

iii. Mr. Keji whose salary is Gh¢195,000.00 per anum and who has 4 children, two of whom are above the age of 18, but a widower.

Custom Duties

Taxes are charged on goods or items imported into the country as well as articles that are produced locally. Taxes paid on articles brought from other countries are import duties and sale tax. These are collectively called ***custom duties,*** for e.g., cars, clothing, shoes, and other goods brought from abroad.

Taxes are also paid on goods produced, sold or used within a country. This is called ***excise duties***. For e.g., soft drinks, cement, textiles and other made in Ghana goods.

The tax or duty amount depends on the value of the article. Thus, the higher the value of the article, the bigger the tax amount.

Mathematically, ***Duty = Rate × Item value.***

Worked Examples

1. An import duty of 25% is charged on some goods. How much duty must be paid on a computer valued at Gh¢12,500.00?

Solution

Duty on computer = Rate × Item value

But rate = 25% = $\frac{25}{100}$, value = Gh¢12,500

Duty = $\frac{25}{100} \times 12{,}500$ = Gh¢3,125.00

2. The import duty of 15% is charge on a calculator. If the calculator is valued at Gh¢300.00 Calculate the duty that must be paid on the calculator

Solution

Duty = *Rate* × *value*

Rate = 15% = $\frac{15}{100}$, value = Gh¢300.00

Duty = $\frac{25}{100} \times 300$ = Gh¢45.00

3. The exercise duty on a bag of cement is 5%. How much will a factory pay for producing 270 bags of cement each costing Gh$\cancel{c}$300.00 per bag?

Solution

Method 1

Duty on 1 bag of cement
= Rate × value of bag
But rate = 5% = $\frac{5}{100}$ and value = Gh¢300.00
Duty = $\frac{5}{100} \times 300$ = Gh¢15.00

Duty on 1 bag is Gh¢15.00.
∴ Duty on 270 bags = 270× Gh¢15.00
= Gh¢4,050.00

Method 2

1 bag of cement cost Gh¢300.00
270 bags will cost $270 \times Gh¢300.00$
= Gh¢81,000.00

5% duty on Gh¢81,000.00
= $\frac{5}{100} \times\ 81{,}000$ = Gh¢4,050.00

Exercises 17.12

1. An import duty of 25% is charged on a Toyota car valued at Gh¢55,000.00. How much duty will you pay if you own the car?

2. If the import duty on a bag of flour is 5%. How much duty is paid on 200 bags, if each bag cost Gh¢400.00

3. An import duty of Gh¢15.00 is charged on a barrel of crude oil. How much duty will be paid on 600 barrels of crude oil?

4. The exercise duty on a bag of cement is 12%. How much will a man pay for importing 240 bags of cement each at a cost of Gh¢350.00

Household Bills (Tariffs)

Tariff is defined the system of prices which companies charge for the services they provide.

In Ghana, some of the common household tariffs are electricity, water and telephone bills. These services are provided by the Electricity Company of Ghana, Ghana Water Company and Ghana Telecommunications respectively. Each service is charged monthly according to a given schedule or pattern in accordance with the amount of units consumed.

Worked Examples

1. In a household, the meter reading for water at the end of October 2009 was 7848 thousand litres. The meter reading at the end of November, 2009 was 7908 thousand litres. The household was charged for the consumption at the following rates:
The first 10 thousand litres at Gh¢500.00 per thousand litres
The next 30 thousand litres at Gh¢1,300.00 per thousand litres
The next 40 thousand litres at Gh¢1,820.00 per thousand litres. Calculate:
a. the consumption at the end of November
b. the total charge for the consumption.

Solution

a.The consumption at the end of November
= End of November - end of October
= 7908 thousand litres – 7848 thousand
= 60 thousand liters = 60,000 liters

b. Units consumed = 60,000 units
First 10 thousand litres at Gh¢500.00 per thousand litres = 10 × Gh¢500 = Gh¢5,000

The next 30 thousand litres at Gh¢1,300.00 per thousand litres = 30 × Gh¢1,300
= Gh¢39,000,000
Remaining liters= 60,000 – 10,000 – 30,000
= 20,000 liters

20 thousand litres at Gh¢1,820.00 per thousand litres = 20 × Gh¢1,820 = Gh¢36,400.00
Total amount;
= Gh¢(5,000 + 39,000 + 36,400) = Gh¢80,400.00

Method 2

a. Units consumed = 7908 thousand litres – 7848 thousand
= 60 thousand liters
= 60,000 liters

b.

Charges (Gh¢)	Total Charge (Gh¢)	Amount (Gh¢)	Remaining Liters
10,000*l* *at* 500/1000*l*	10 × 500	5,000	60,000 –10,000 = 50,000*l*
30,000*l* *at* 1300 /1000*l*	30 × 1300	39,000	50,000 –30,000 = **20,000*l***
40,000*l* ***at*** **1820** /1000*l*	**40 × 1820**	**72,800**	**20,000 – 40,000 = - 20,000*l***
20,000*l* *at* 1820/1000*l*	20 × 1820	36400	0

Total charge = Gh¢(50,000 + 39,000 + 36400)= Gh¢ 80,400,000.00

Note:

The negative sign (on the table) indicates that there is no remaining liters. Therefore, all figures on that row are not used in the calculation.

2. In a certain year, Kwaku paid for electricity consumed each month as follows; the cost of the first 30 units was Gh¢1.50 per unit; the cost of the next 30 units was Gh¢1.70 per unit; the cost of each additional unit is 50p

i. If Kwaku used 420 units of electricity in July that year, calculate the amount he paid.

ii. If kwaku paid Gh¢194.00 in August last year, calculate the number of units of electricity he used.

iii. Find correct to 2 decimal places, the percentage change in Kwaku's consumption of electricity from July to August that year

Solution

i. Total units consumed = 420
30 units was 1.50 per unit
= 30 × Gh¢1.50= Gh¢45.00

The next 30 units was 1.70 per unit
= 30 × Gh¢1.70 = Gh¢51.00

Remaining units = 360 units
360 units at 50p per unit
= 360 × 50p = Gh¢180.00

Total amount
= Gh¢45 + Gh¢51 + Gh¢180 = Gh¢276.00

ii. Amount paid in August last year = Gh¢194.00
Less cost of first 30 units = Gh¢45.00
Less cost of next 30 units = Gh¢51.00
Remaining = Gh¢98.00
Total cost of each additional units;
= Gh¢98.00

But cost of each additional units;
= 50p = Gh¢ 0.5
Units consumed for Gh¢98.00 at 50p per unit
= $\frac{98}{0.5}$ = 196 units

Total units consumed last year august
= 30 units + 30 units + 196 units = 256 units

Method 2

Units consumed = 420

Charges (Gh¢)	Total Charge (Gh¢)	Amount (Gh¢)	Remaining Units
30 at 1.5 each	30 × 1.5	45	420 – 30 = 390
30 at 1.7 each	30 × 1.7	51	390 – 30 = 360
Additional at 0.5 each	360 × 0.5	180	

Total amount = Gh¢(45 + 51 + 180) = Gh¢276.00

b. Amount Charged = Gh¢194.00

Charges (Gh¢)	Amount (Gh¢)	Less Amount (Gh¢)	Units
30 at 1.5 each	30 × 1.5 = 45	194 – 45 = 149	30
30 at 1.5 each	30 × 1.7 = 51	149 – 51 = 98	30
Remaining at 50p each	$\frac{98}{0.5}$		196

Total units consumed = 30 + 30 + 196 = 256 units

3. In a chargeable period, a household consumed 700 units of electricity. Use the table below to find the cost.

Electricity Company of Ghana
Power Tariff Schedule
Domestic 50 units or less @ Gh270.00
Next 150 units @ Gh4.35 each
Next 400 units @ Gh6.86 each
Additional units @ Gh14.72 each

Commercial Each unit supplied @ Gh17.00
Service charge Gh115.00

Solution
First 50 units = Gh¢270.00
Next 150 units = Gh¢(150 × 4.35) = Gh¢652.50
Next 400 units = Gh¢(400 × 6.86) = Gh¢2,744.00
Additional 100 units = Gh¢(100 × 14.72)
= Gh¢1,472.00

Total units = 700
Total amount;
= Gh¢(270 + 652.50 + 2,744 + 1,472)
= Gh¢5,138.50

4. The monthly electricity charges in a country are calculated as follows;
First 50 units = Gh¢4,000.00
Next 100 units = Gh¢120.00 per unit
Next 150 units = Gh¢150.00 per unit
Next 300 units = Gh¢220.00 per unit
Remaining unit = Gh¢350 per unit
a. How much did Mr. Owusu pay for using 720 units in a month?
b. A man paid Gh¢73,260.00 for electricity consumed in a month. How many units of electricity did he consume?

Solution
Total units consumed = 720 units
Remaining units
= 720 – 50 – 100 – 150 – 300
= 120 units

First 50 units = Gh¢4,000
Next 100 units = 100 × Gh¢120.00
= Gh¢12,000
Next 150 units = 150 × Gh¢150.00
= Gh¢22,500
Next 300 units = 300 × Gh¢220.00
= Gh¢66,000.00
Remaining unit = 120 × Gh¢350 = Gh¢42,000
Total amount;
= Gh¢4,000 + Gh¢12,000 + Gh¢22,500 + Gh¢ 66,000 + Gh¢42,000
= Gh¢146,500.00

ii. Amount paid = Gh¢73,260.00
Less first 50 units = Gh¢4,000
Less next 100 units = 100 × Gh¢120.00
= Gh¢12,000
Less next 150 units = 150 × Gh¢150.00
= Gh¢22,500
Remaining = 34,760 at 220 per unit

Units consumed for Gh¢34,760.00 at Gh¢220 per unit = $\frac{\text{Gh¢}34{,}760}{\text{Gh¢}220}$ = 158 units

Total units consumed by the man;
= 50 + 100 + 150 + 158
= 458 units

Exercises 17.13
1. The monthly electricity bill for a household is calculated by adding a fixed charge Gh¢15,500.00 to the cost of the number of units of electricity used. If the cost per unit is Gh¢500.00, what is the bill for a household that uses 111 units in a month?

2. Consider the table below and use it to answer the questions that follow:

Electricity Company of Ghana	
Power Tariff Schedule	
From July	
Domestic	50 units or less @ Gh270.00
	Next 150 units @ Gh4.35 each
	Next 400 units @ Gh6.86 each
	Additional units @ Gh14.72 each
Commercial	Each unit supplied @ Gh17.00
	Service charge Gh115.00

i. How much does a man pay for using 900 units of electricity at home?
ii. A clerk made use of 35 units in his office, how much did he pay just for the units he used?
iii. If the bill for a commercial office is Gh¢15,415.00. Find the number of units consumed.

3. In a certain year, Mr. Harry paid for domestic electricity consumed each month as follows:

The cost of the first 30 units used is 100p per unit; the cost of the next 30 units used is 70p per unit and the cost of each additional unit used is 50p per unit,

i. Mr. Harry used 430 units of electricity in august that year, calculate the amount he paid

ii. if he paid Gh¢301.00 the next month, find the units of electricity he consumed.

4. The Ghana Water Company charges the following rates for water consumed each month:

First 20,000 liters at Gh¢400.00 per 1,000 liters,

Next 30,000 litres at Gh¢520 .00 per 1,000 litres,

Remaining liters at Gh¢900 per 1,000 liters

i. If Mr Boadi used 62,000 liters in a particular month, calculate his total bill for the month,

ii. If he paid Gh¢28,000.00 in a particular month, calculate the liters of water he consumed to the nearest liter.

5. The Ghana water company tariff for a certain year is given below:

First 10 liters	Gh¢ 0.50 per liter
Next 10 liters	Gh¢ 0.75 per liter
Next 10 liters	Gh¢ 1.00 per liter
Next 10 liters	Gh¢ 1.25 per liter
Remaining liters	Gh¢ 1.50 per liter

If Mr Brown used 200 liters of water in April of that particular year, how much did he pay?

Challenge Problem

Below is the tarrif schedule for electricity consumption in a particular country;

First 50 units or less = Gh¢0.27 per unit

Next 50 units = Gh¢0.31 per unit

Next 50 units = Gh¢0.68 per unit

Next 30 units = Gh¢0.80 per unit

Additional units = Gh¢0.95 per unit

Service Charge = Gh¢2.00

Street light = 5% of additional units

If a household initial and final meter readings for a particular month is 4122kw and 4473kw, calculate the total bill for that month.

18 VARIATIONS

Meaning of Variations

In mathematical sense, variation is concerned with ways and means by which one variable relates or depends on other variable(s). The type of variations includes;

1. Direct variation
2. Inverse variation
3. Joint or Combined variation
4. Partial variations or variation in parts

Direct Variation

It is a kind of relation that exists between two quantities such that an increase in one quantity causes an increase in the other quantity and vice – versa. For e.g. If 4 pens (x)cost Gh¢2.00 (y), then an increase in the number of pens means the cost will be more than Gh¢2.00 and a decrease in the number of pens means the cost will be less than Gh¢2.00. In this instance, the cost of pens (y) is said to be in a direct proportion to the number of pens(x). Simply put ***"y is (directly) proportional to x".***This kind of relationship is called a ***direct variation***.

In variations, the appropriate phrase to use instead and their mathematical expressions are explained below:

1. ***"y varies directly as x"***,is expressed mathematically as:$y \propto x$,

$y = kx$…….. (1), where k is called the constant of variation

From (1), $k = \frac{y}{x} = \frac{y_1}{x_1}$, where $x_1$and $y_1$are corresponding values of x and y

2. *"y varies as the square of x"*is expressed mathematically as:$y \propto x^2$,

$y = kx^2$, where k is the constant

From $y = kx^2$,

$k = \frac{y}{x^2} = \frac{y_1}{{x_1}^2}$, where $x_1$and $y_1$are corresponding values of x and y

3. *"y varies as the square root of x"*, is expressed mathematically as:$y \propto \sqrt{x}$

$y = k\sqrt{x}$, where k is the constant

From $y = k\sqrt{x}$,

$k = \frac{y}{\sqrt{x}} = \frac{y_1}{\sqrt{x_1}}$, where $\sqrt{x_1}$ and y_1 are corresponding values of $\sqrt{x}$ and y

4. ***"y varies as the cube of x"***, is expressed mathematically as: $y \propto x^3$

$y = kx^3$, where k is the constant of variation

From $y = kx^3$,

$k = \frac{y}{x^3} = \frac{y_1}{{x_1}^3}$, where $y_1$and ${x^3}_1$are the corresponding values of y and x^3

5. ***"y varies as the cube root of x"***,is expressed mathematically as: $y \propto \sqrt[3]{x}$

$\Rightarrow y = k\sqrt[3]{x}$, where k is the constant

From $y = k\sqrt[3]{x}$,

$k = \frac{y}{\sqrt[3]{x}} = \frac{y_1}{\sqrt[3]{x_1}}$, where $\sqrt[3]{x_1}$ and $y_1$are corresponding values of $\sqrt[3]{x}$ and y

Worked Examples

1. Given that p varies directly as the square of q and $p = 4$ when $q = 2$. Find q when $p = {}^1/_4$

Solution

p varies directly as the square of q

$p \propto q^2$

$\Rightarrow p = kq^2$…………….(1)

From eqn (1)

$k = \frac{p}{q^2}$

When $p = 4$ and $q = 2$

$k = \frac{p}{q^2} = \frac{4}{2^2} = \frac{4}{4} = 1$

Put $k = 1$ in equation (1) to obtain the variation equation as $p = q^2$

When $p = {}^1/_4$, q = ?and $k = 1$

From the variation equation;

$q^2 = p$

$q = \sqrt{p} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

2. In the table below, y varies directly as x changes. Find the values of a and b

y	1	2	a
x	3	b	15

Solution

y varies directly as x

$y \propto x$

$\Rightarrow y = kx$ ……….(1)

From eqn (1)

$k = \frac{y}{x}$

From the table when $y = 1$, $x = 3$

$k = \frac{y}{x} = \frac{1}{3}$

Put $k = \frac{1}{3}$in eqn (1) to obtain the variation equation as$y = \frac{1}{3}x$

From the table, when $y = 2$, $x = b$ and $k = \frac{1}{3}$

From the variation equation,

$x = 3y$

$x = 3\times 2 = 6$

From the table, when $y = a$, $x = 15$ and $k = \frac{1}{3}$

From the variation equation

$y = \frac{1}{3}x$

$y = \frac{1}{3} \times 15 = 5$

Therefore, $a = 5$, and $b = 6$

3. P varies as the square of $(Q + 1)$ and P is 2 when Q is 3.

i. Write an equation connecting P and Q.

ii. Find the possible value of Q, when $P = 8$.

Solution

i. P varies as the square of $(Q + 1)$

$P \propto (Q + 1)^2$

$\Rightarrow P = k(Q + 1)^2$ ………..(1)

From Eqn (1)

$k = \frac{P}{(Q+1)^2}$

When P = 2 and Q = 3,

$k = \frac{P}{(Q+1)^2} = \frac{2}{(3+1)^2} = \frac{2}{16} = \frac{1}{8}$

Put $k = \frac{1}{8}$ in eqn (1)

$P = \frac{1}{8}(Q + 1)^2$

The equation connecting P and Q *is:*

$P = \frac{1}{8}(Q + 1)^2$

ii. When $p = 8$, Q = ?and $k = \frac{1}{8}$

$8 = \frac{1}{8}(Q + 1)^2$

Multiply both sides by 8

$64 = (Q + 1)^2$

$\sqrt{64} = \sqrt{(Q+1)^2}$

$8 = Q + 1$

$Q = 8 - 1 = 7$

$\Rightarrow$when $p = 8$, Q = 7

4. Given that y varies directly as x^2 and that $y = 20$ when $x = 2$, find the value of y when $x = 3$

Solution

y varies directly as x^2

$y \propto x^2$

$\Rightarrow y = kx^2$……….(1)

From eqn (1)

$k = \frac{y}{x^2}$

When $y = 20$, $x = 2$ and $k = ?$

$k = \frac{20}{2^2} = \frac{20}{4} = 5$

Put $k = 5$ in eqn (1) to obtain the variation equation as $y = 5x^2$

When $x = 3$, $y = ?$ and $k = 5$

From the variation equation

$y = 5x^2 = 5 \times 3^2 = 5 \times 9 = 45$

Therefore, when $x = 3$, $y = 45$

5. The resistance to an aeroplane varies directly as the squareof its speed. If the resistance is 1260 ohms when the air craft is travelling 45 m/s, what is the resistance at 65m/s?

Solution

Let R represent the resistance and s represent the speed

$\Rightarrow R \propto s^2$

$R = ks^2$

When R = 1,260, s = 45

$1260 = k(45)^2$

$k = \frac{1260}{45^2}$

Hence, $R = \frac{1260}{45^2}s^2$

When s = 65, R = ? and $k = \frac{1260}{45^2}$

$R = \frac{1260}{45^2}(65)^2 = \frac{1260 \times 65^2}{45^2} = 2{,}630$

6. If A varies directly as the squere of r and V varies directly as the cube of r, find the percentage increase in A and V if r is increased by 20 percent.

Solution

$A \propto r^2$

$A = kr^2$

When r is increased by 20%,

$= r + \left(\frac{20}{100} + r\right)$

$= r + \frac{r}{5} = \frac{5r+r}{5} = \frac{6r}{5}$

$A = k\left(\frac{6r}{5}\right)^2 = \frac{36kr^2}{25}$

Increase in area;

$= \frac{36kr^2}{25} - kr^2$

$= \frac{36kr^2 - 25kr^2}{25}$

$= \frac{11kr^2}{25}$

Percentage increses in A;

$= \frac{\frac{11kr^2}{25}}{kr^2} \times 100\%$

$= \left(\frac{11kr^2}{25} \div kr^2\right) \times 100\%$

$= \left(\frac{11kr^2}{25} \times \frac{1}{kr^2}\right) \times 100\%$

$= \frac{11}{25} \times 100\ \% = 44\%$

Workout the % increase in V   Ans : (72%)

Exercises 18.1

A. Express the following as an equation:

1. u varies as the square root of v
2. The cube of a varies as the square of b
3. If $y \propto x^2$, state the factor by which y is multiplied when x is multiplied by -3

B. 1. v varies as w and $w = 5$ when $v = 40$. Find: i. v when $w = 8$ ii. w when v = 4

2. If y varies asx^3 and $y = 54$ when $x = 3$, find y when $x = 1.5$and obtain an expression for y in terms of x

3.u varies directly as the square of v. If $u = 14$ when$v = 2$, find the value of $u$when $v = 3$

4. It is known that y varies as x and $y = 9$ when $x = 3$. Find y in terms of $x$and the value
of y when $x = 4$

5. The area, A of a sector, containing a specified angle, varies as the square of its
radius, r. When r is 2cm, A is 3.6cm^2
a. Find A when r is 7cm
b. Find r when A is 8.1cm^2

6. Find the values of x and y in the table below:

T	1	2	3	x	5
S	y	4	6	8	10

7. y varies as the square root of x, and $y = 3$ when $x = 1.44$. Find:
i. y when $x = 0.36$ ii. x when $y = 10$

8. In the table below, find the missing entry assuming that V $\propto h$

h	125	275
V	25	

9. Given that the mass of a sphere varies directly as the cube of its radius, complete the table.

Mass	54	
Radius	3	4

10. P varies as the square of the sum of x and y when P is 4, x is 3 and $y = 5$. Find P when $x = 6$ and $y = 8$

11. Find, $\sqrt{x+y}$ from the table below leaving your answer in surd form if p varies directly as q^2

P	2	x	32
q	1	2	y

12. For the entries in the table below, verify that $S^2 \propto$ W and hence find the equation connecting W and s^2

S	1	2	5
W	2	8	50

13. The breaking weight, W tonnes of a rope of diameter d mm is proportional to the square of its diameter. Given that W = 6 when $d = 40$, find W when $d = 24$/

14. The amount, in litres of petrol consumed varies directly as the number of kilometres travelled by a car. It is known that a certain saloon car used up 4.5 litres of petrol when it travelled a distance of 53km. Find how much petrol is needed for the car to cover 275km.

15. The distance, d meters travelled by a falling object in t seconds is given by the formula $d = kt^2$. When $t = 4$, $d = 80$
i. Find d when $t = 8$, and t when $d = 1125$
ii. Change the subject of the formula to t, and copy and complete the statement: 'The time in seconds an object takes to fall varies ……….'

16. The area of a circular sector containing a given anglevaries as the square root of the radius of the circle. If the area of the sector is 2cm^2

when the radius is 1.6cm, find the area of the sector containing the same angle when the radius of the circle is 2.7cm.

Inverse Variations

It is a kind of relationship that exists between two quantities such that an increase in one quantity causes a decrease in the other quantity and vice versa at standard conditions. For e.g. If six girls(x) can sweep a classroom in 10 minutes(y), then an increase in the number of girls implies that the work can be completed in less than 10 minutes and a decrease in the number of girls implies that the work can be completed in more than 10 minutes. In this instance, the time is said to be in an inverse proportion to the number of girls. Simply put, ***"y is inversely proportional to x".*** This kind of relationship is called ***inverse variation***,

In variations, the appropriate phrase to use instead and their mathematical expressions are explained below:

1. "***y varies inversely as x"*** is expressed mathematically as $y \propto \frac{1}{x}$,, therefore $y = \frac{k}{x}$, where k is the constant of variation**.**

From $y = \frac{k}{x}$,$k = yx = y_1x_1$, wherey_1x_1, are corresponding values of x and y respectively

2. "***y varies inversely as the square of x"*** is expressed mathematically as $y \propto \frac{1}{x^2}$,, therefore $y = \frac{k}{x^2}$, where k is the constant of variation**.** From $y = \frac{k}{x^2}$, $k = yx^2 = y_1{x_1}^2$, where, $y_1{x_1}^2$, are corresponding values of x and y respectively

3. "***y varies inversely as the cube of x"***is expressed mathematically as $y \propto \frac{1}{x^3}$,therefore $y = \frac{k}{x^3}$, where k is the constant of variation**.**

From $y = \frac{k}{x^3}$, $k = yx^3 = y_1{x^3}_1$, where,$y_1{x^3}_1$ are corresponding values of x and y respectively

4. ***"y varies inversely as the cube root of x"*** is expressed mathematically as $y \propto \frac{1}{\sqrt[3]{x}}$,, therefore $y = \frac{k}{\sqrt[3]{x}}$, where k is the constant of variation**.** From$y = \frac{k}{\sqrt[3]{x}}$, $k = y\sqrt[3]{x} = y_1\sqrt[3]{x_1}$, where,$y_1\sqrt[3]{x_1}$are corresponding values of x and y respectively.

Worked Examples

1. Find the variation constant and an equation of variation if y varies inversely as x, and $y = 32$ when $x = 0.2$.

Solution

"y varies inversely as the *x*"

$y \propto \frac{1}{x}$

$y = \frac{k}{x}$ ……..(1)

From eqn (1)

$k = xy$

When $x = 0.2$, $y = 32$ and $k = ?$

$k = 0.2 \times 32 = 6.4$

The variation constant $k = 6.4$

Hence, the equation of variation is$y = \frac{6.4}{x}$

2. y varies inversely as the square of x. When $x = 3$, $y = 100$. Find:

a. the equation connecting x and y

b. the value of x when $y = 25$

c. the value of y when $x = 15$

Solution

a. y varies inversely as the square of *x*

$y \propto \frac{1}{x^2}$

$y = \frac{k}{x^2}$ ……….(1)

$\Rightarrow k = x^2y$

$k = 3^2 \times 100 = 900$

Substitute k =100 in eqn (1)

$y = \frac{100}{x^2}$

Hence,the equation connecting x and $y$is $y = \frac{900}{x^2}$

b. When y = 25, x = ?and$k$ = 900

From eqn (1)

$x^2 = \frac{k}{y}$

$x = \sqrt{\frac{k}{y}} = \sqrt{\frac{900}{25}} = 6$

c. When x = 15, y = ?and k = 900

From eqn (1)

$y = \frac{k}{x^2} = \frac{900}{15^2} = \frac{900}{225} = 4$

3. R varies inversely as the cube of S. If R = 9 when S = 3, find S when R = $\frac{243}{64}$

Solution

R varies inversely as the cube of S

$R \propto \frac{1}{S^3}$

$R = \frac{k}{S^3}$ …..(1)

$k = S^3y = 3^3 \times 9 = 243$

Hence, the variation equation is $R = \frac{243}{S^3}$

When R = $\frac{243}{64}$, S = ? and$k$ = 243

From eqn (1)

$S^3 = \frac{k}{R}$

$S = \sqrt[3]{\frac{k}{R}} = \sqrt[3]{\frac{243}{243/64}} = \sqrt[3]{\frac{243 \times 64}{243}} = \sqrt[3]{64} = 4$

4. If R varies inversely as the square of (3q – 2) and R = 4 when q = 2, find:

i. R when q =1 ii. q when R = 16

Solution

i. R varies inversely as the square of (3q – 2)

$\Rightarrow R \propto \frac{1}{(3q-2)^2}$

$R = \frac{k}{(3q-2)^2}$………(1)

$k = R(3q-2)^2$

When R = 4, q = 2 and k= ?

$k = 4[3(2)-2]^2 = 4(6-2)^2 = 64$

Hence, the variation equation is $R = \frac{64}{(3q-2)^2}$

When q = 1, R = ? and k = 64

From eqn (1)

$R = \frac{k}{(3q-2)^2} = \frac{64}{(3\,(1)-2)^2} = 64$

ii. When R = 16, q = ? and$k$ = 64

From eqn (1)

$R(3q-2)^2 = k$

$(3q-2)^2 = \frac{k}{R}$

$3q - 2 = \sqrt{\frac{k}{R}}$

$3q = \sqrt{\frac{k}{R}} + 2$

$q = \frac{\sqrt{\frac{64}{16}} + 2}{3} = \frac{(\sqrt{4} + 2)}{3} = \frac{4}{3}$

5. If y is inversely proportional to x + 2 and y = 48 when x = 10, find x when y = 30

Solution

"y is inversely proportional to (x + 2)"

$y \propto \frac{1}{x+2}$

$y = \frac{k}{x+2}$………(1)

$k = y(x+2)$

When$y$ = 48, x = 10

$k = 48\,(10+2) = 48(12) = 576$

Hence, the variation equation is $y = \frac{576}{x+2}$

When x = ?, y = 30

$30 = \frac{576}{x + 2}$

$30(x + 2) = 576$

$30x + 60 = 576$

$30x = 576 – 60$

$30x = 516$

$x = \frac{516}{30} = 17.2$

Some Solved Past Questions

1. P varies inversely as the square of (Q + 1) and P is 2 when Q is 3

a. Write an equation connecting P and Q

b. Find the possible values of Q when P = 8

Solution

P varies inversely as the square of (Q + 1)

$P \propto \frac{1}{(Q+1)^2}$

$P = \frac{k}{(Q+1)^2}$……….(1)

$k = P(Q + 1)^2$

When P = 2, Q = 3 and k= ?

$k = 2(3 + 1)^2$

$k = 2(4)^2 = 2 \times 16 = 32$

Substitute k = 32 in $P = \frac{k}{(Q+1)^2}$ to obtain

$P = \frac{32}{(Q+1)^2}$

The equation connecting P and Q is

$P = \frac{32}{(Q+1)^2}$

b. When P = 8, Q = ?and$k$ = 32

From eqn (1)

$P(Q + 1)^2 = k$

$(Q + 1)^2 = \frac{k}{P}$

$Q + 1 = \sqrt{\frac{k}{P}}$

$Q = \sqrt{\frac{k}{P}} - 1 = \sqrt{\frac{32}{8}} - 1 = \sqrt{4} - 1 = 2 - 1 = 1$

2. A garrison of 400 men had food for 40 days. After 10 days, 200 more men joinedthem. How long will the food last now? Assume that the amount of food taken by each man is almost the same.

Solution

400 men ate the food for 40 days. Hence, they can now eat the remaining food for 40 – 10 = 30 days.

Let the remaining food last for x days. When 200 men joined them, then the number of men becomes 400 + 200 = 600

No of men	400	600
No of days	30	x

From the table above,

$400 \times 30 = 600x$

$\Rightarrow x = 20$ days

3.The following table shows the safe exposure time for people with less sensitive skin.

UV Index	Safe Exposure Time (in mins)
2	150
4	75
6	50
8	37.5
10	30

a. Determine whether the table indicates direct variation or inverse variation.

b. Find an equation of variation that approximates the data. Use the data point (6, 50)

c. Use the equation to predict the safe exposure time for a person with less sensitive skin when the UV rating is 3

Solution

a. The table indicates an inverse variation

b. Index, I = 6 and time, T = 50

From the table, I $\propto \frac{1}{T}$

I = $\frac{k}{T}$ ……….(1)

From eqn (1)

k = I × T = 6 × 50 = 300

Substitute k= 300 in eqn (1)

I = $\frac{300}{T}$

c. When I = 3, K = 300 and T = ?

From eqn(1)

T = $\frac{K}{I} = \frac{300}{3}$ = 100

Exercises

A. Express the following as an equation:

1. p varies inversely as q^4
2. p varies inversely as v
3. H varies inversely as the square of d
4. T varies inversely as the square root of g

B. 1. y varies inversely as x and y = 6 when x = 6. Find: i. y when x = 24 ii.x when y = 32

2. A quantity y varies inversely as another quantity x. If $y = \frac{1}{3}$ when x = 9, express y in terms of x

3. Given that y is inversely proportional to x^2 and that x = 10, when y = 5, find the value of y when x =15

4. Given that y is inversely proportional to x and that y = 24 when $x= \frac{1}{2}$, find the value of y when x = 16

5. Given that y is inversely proportional to the square of x and that y = 24 when $x= \frac{1}{2}$,

i. Find the equation connecting x and y

ii. Find the value of y when x = 3

6. Given that y is inverselyproportional to x and that y = 4 when $x= \frac{1}{2}$, find the value of y when x = $2\frac{1}{2}$

7. Given that N is inversely proportional to r^2 and that N = 3, when r = 5, find the value of N when r = 10

8. If s varies inversely as t^3 and s = 4 when t = 2, find s in terms of t and the value of s when t = 7

9. Given that y is inversely proportional to (x – 2) and that y = 6 when x = 5, calculate the value of y when x = 7

10. It is given that the force between two particles is inversely proportional to the square of the distance between them. If the force is F when the distance between them is r and cF. When the distance is 5r, write down the value of c. Ans 0.04

Joint or Combined Variation

It is a kind of relationship that exists between three (or more) quantities such that one quantity relates to the others in a direct proportion. In other words, joint variation is a type of variation where a quantity varies directly as the product of two or more other quantities. When one variable varies directly as two or more others, the word

jointly is sometimes used. For example, in V = πr^2h, V varies jointly as h and r^2

The joint variation is of three types. Its mathematical expressions are explained below:

I. The variation is direct for all the variables

E.g. The variable z varies directly as x and directly as y. That is z = kxy, where k is the constant of proportionality

II. The variation is inverse for all the variables.

E.g. The variable z varies inversely as x and inversely as y. That is z = $\frac{k}{xy}$, where k is the constant of proportionality

III. The variation is a combination of direct and inverse variations

E. g. z varies directly as x and inversely as y are expressed as z $\propto x$ and z $\propto \frac{1}{y}$ respectively. By combining the two, z $\propto \frac{x}{y}$

$\Rightarrow$ z = $\frac{Kx}{y}$

Worked Examples

1. If s varies directly as q and directly as r and s = 40 when q = 2 and r = 4,

i. Write an equation connecting s, q and r

ii. Find the value of S, when q = 6 and r = 8

iii. Find the value of q when S = 60 and r = 0.5

Solution

"S varies directly as q and directly as r"

S$\propto q$ *and* S $\propto r$

S $\propto qr$

S = kqr……………….(1)

$k = \frac{s}{qr}$

When s = 40, q = 2 and r = 4,

$k = \frac{40}{2 \times 4}$= 5

Put k = 5 in eqn (1)

S = 5qr

The equation connecting s, q and r is

S = 5qr

ii. S= ?q = 6, r = 8 and k = 5

S = 5 × 6 × 8 = 240

iii. q= ? S = 60, r = 0.5 and k = 5

s = 5qr

60 = 5 × q × 0.5

$q = \frac{60}{5 \times 0.5}$= 24

2. z varies directly as x and inversely as the square of y. Find z in terms of x and y , given that z = 10 when x = 8 and y = 2. Find;

i. the value of z when x = 12, y = 4

ii. the value of x when y = 5, z = 4

Solution

"z varies directly as x and inversely as y^2"

z $\propto x$ and z$\propto \frac{1}{y^2}$

z$\propto \frac{x}{y^2}$

z= $\frac{kx}{y^2}$……………(1)

Whenz = 10, x = 8, y = 2 and k= ?

$10 = \frac{8k}{2^2}$

10 × 2^2 = 8k

10 × 4 = 8k

$k = \frac{10 \times 4}{8}$ = 5

Put k = 5 in eqn (1)

Hence the variation equation is z= $\frac{5x}{y^2}$

i. When z = ?x = 12, y = 4 and k = 5

z$= \frac{5(12)}{4^2} = \frac{60}{16}$ = 3.75

ii. When$x$= ?y = 5, z = 4 and k = 5

$4 = \frac{5x}{5^2}$

$4 \times 5^2 = 5x$

$4 \times 25 = 5x$

$x = \frac{4 \times 25}{5} = 20$

3. A quantity f varies jointly as the square of v and the inverse of r. If f = 12 when v = 2 and r = 3, find:

a. the constant of variation

b. the value of v when r = 4 and f = 23

Solution

a. "f varies jointly as the square of v and the inverse of *r"*

$f \propto v^2$ and $f \propto \frac{1}{r}$

$f \propto \frac{v^2}{r}$

$f = \frac{kv^2}{r}$……………(1)

If f = 12 when v = 2 and r = 3,

$\Rightarrow 12 = \frac{k(2)^2}{3}$

$12 \times 3 = 4k$

$k = \frac{12 \times 3}{4} = 9$

Hence, the constant of variation, k = 9

Put k = 9 in eqn (1)

$f = \frac{9v^2}{r}$

Hence, the variation equation is $f = \frac{9v^2}{r}$

b.v = ? , r = 4, f = 23 and k = 9

$23 = \frac{9v^2}{4}$

$4 \times 23 = 9v^2$

$\frac{4 \times 23}{9} = v^2$

$\Rightarrow v = \sqrt{\frac{92}{9}} = \frac{\sqrt{92}}{3} = 3.2$

4. x varies jointly as the square of m and the cube of n. When x = 9, $m = {}^3/_4$ and $n = {}^1/_2$

a. determine the relationship between X, m and n

b. calculate, correct to three significant figures the values of the following:

i. x when $m = \frac{2}{3}$ and $n = \frac{1}{5}$

ii.m when x = 5 and $n = \frac{1}{8}$

Solution

a. "x varies jointly as the square of m and the cube of *n"*

$x \propto m^2$ and $x \propto n^3$

$x \propto m^2n^3$

$x = km^2n^3$ ……….(1)

When$x$ = 9, $m = {}^3/_4$and $n = {}^1/_2$, k = ?

$k = \frac{x}{m^2n^3} = \frac{9}{\left(\frac{3}{4}\right)^2\left(\frac{1}{2}\right)^3} = \frac{9}{\left(\frac{9}{16}\right)\left(\frac{1}{8}\right)} = 128$

Put k = 128 in eqn (1)

$x = 128m^2n^3$ (variation equation)

The relationship between x, m and n *is:*

$x = 128m^2n^3$

b. i. When$x$= ?, $m = \frac{2}{3}$, $n = \frac{1}{5}$ and k = 128

But $x = 128m^2n^3$ (variation equation)

$x = 128\left(\frac{2}{3}\right)^2\left(\frac{1}{5}\right)^3$

$x = 128\left(\frac{4}{9}\right)\left(\frac{1}{125}\right) = 128\left(\frac{4}{1,125}\right)$ = 0.455 (3 s. f)

ii. When m = ?, x = 5 , $n = \frac{1}{8}$

From $x = 128m^2n^3$ (variation equation)

$5 = 128m^2\left(\frac{1}{8}\right)^3$

$5 = 128m^2\left(\frac{1}{512}\right)$

$5 \times 512 = 128m^2$

$m^2 = \frac{5 \times 512}{128}$

$\Rightarrow m = \sqrt{\frac{5 \times 512}{128}} = 4.47$

5. Find the equation of variation if y varies jointly as x and z and inversely as the square of w, and $y = 105$ when $x = 3$, $z = 20$ and $w = 2$

Solution

"*y* varies jointly as *x* and *z* and inversely as the square of *w*",

$\Rightarrow y \propto x\,z$ and $y \propto \frac{1}{w^2}$

$y \propto \frac{\mathrm{xz}}{w^2}$

$y = \frac{\mathrm{kxz}}{w^2}$..............(1)

When $y = 105$, $x = 3$, $z = 20$ and $w = 2$

$105 = \frac{k(3)(20)}{2^2}$

$105 = \frac{60k}{4}$

$105 \times 4 = 60k$

$k = \frac{105 \times 4}{60} = 7$

Put $k = 7$ in eqn (1)

$y = \frac{7\mathrm{xz}}{\mathrm{w}^2}$ (variation equation)

6. t varies inversely as v and inversely as w. If $v = 7$, $w = 3$ and $t = 9$. Find an equation connecting t, v and w

Solution

"*t* varies inversely as *v* and inversely as *w*"

$t \propto \frac{1}{v}$ and $t \propto \frac{1}{w}$

$t \propto \frac{1}{vw}$

$t = \frac{k}{vw}$..............(1)

When $v = 7$, w = 3 and $t = 9$

$9 = \frac{k}{(7)(3)}$

$9 = \frac{k}{21}$

$9 \times 21 = k$

$k = 189$

Put $k = 189$in eqn (1)

$t = \frac{189}{vw}$

The equation connecting t, v and w *is* $t = \frac{189}{vw}$

7. y varies directly as $(x - 2)$ and varies inversely as x^2. When $x = 12$, $y = 8$.

i. Find the relation between x and y

ii. Find the value of y when $x = 4$

Solution

"*y* varies directly as (*x* – 2) and varies inversely as x^2 ,

$\Rightarrow y \propto (x - 2)$ and $y \propto \frac{1}{x^2}$

$y \propto \frac{(x-2)}{x^2}$

$y = \frac{k(x-2)}{x^2}$..............(1)

When $x = 12$, $y = 8$

$8 = \frac{k(12-2)}{12^2}$

$8 = \frac{10k}{144}$

$8 \times 144 = 10k$

$k = \frac{8 \times 144}{10} = 35.2$

Put $k = 35.2$ in eqn (1)

$y = \frac{35.2(x-2)}{x^2}$

When $y = ?$, $x = 4$

$y = \frac{35.2(4-2)}{4^2} = \frac{35.2 \times 2}{4^2} = 4.4$

8. x varies directly as y^2 and y varies inversely as z. If $y = 3$ and z = 2 when $x = 12$, find:

i. an expression for x in terms of z alone.
ii. the values of x and y when $z = 4$

Solution

"x varies directly as y^2 and y varies inversely as z"

$x \propto y^2$ and $y \propto \frac{1}{z}$

$x = cy^2$……………(1)

$y = \frac{k}{z}$ ………………(2)

Where c and k are variation constants

From eqn (1),

When $x = 12$, $y = 3$ substitute in $x = cy^2$

$12 = c(3)^2$

$12 = 9c$

$c = \frac{12}{9} = \frac{4}{3}$

Put $c = \frac{4}{3}$ in eqn (1)

$x = \frac{4}{3}y^2$

Hence, variation equation is $x = \frac{4}{3}y^2$…(i)

From eqn (2),

When $z = 2$, $y = 3$ substitute in $y = \frac{k}{z}$

$3 = \frac{k}{2}$

$2 \times 3 = k$

$k = 6$

Hence, variation equation is $y = \frac{6}{z}$…(ii)

Now put $y = \frac{6}{z}$ into $x = \frac{4}{3}y^2$

$x = \frac{4}{3} \times \left(\frac{6}{z}\right)^2 = \frac{4}{3} \times \frac{36}{z^2} = \frac{144}{3z^2} = \frac{48}{z^2}$

The expression for x in terms of z is $x = \frac{48}{z^2}$

b. From eqn (ii), $y = \frac{6}{z}$

when $z = 4$, $y = \frac{6}{4} = \frac{3}{2} = 1.5$

From the expression, $x = \frac{48}{z^2}$

When z = 4, $x = \frac{48}{(4)^2} = \frac{48}{16} = 3$

Hence, when z = 4, $x = 3$ and $y = 1.5$

9. Given that p varies directly as q whiles q varies inversely as r, how does p varies with r

Solution

"p varies directly as q whiles q varies inversely as r"

$p \propto q$

$\Rightarrow p = k_1 q$……….(1)

Where k_1 is the constant of variation

Also, $q \propto \frac{1}{r}$

$\Rightarrow q = \frac{k_2}{r}$…….(2)

Where k_2 is the constant of variation

Substitute eqn (2) into eqn (1)

$P = \frac{k_1 k_2}{r}$ where $k_1 \times k_2 = k$

$P = \frac{k}{r}$

$\Rightarrow P \propto \frac{1}{r}$, P varies inversely as r

Some Solved Past Questions

1. x varies directly as the square root of t and inversely as S. When $x = 4$, $t = 9$ and $s = 18$
i. Express x in terms of s and t.
ii. Find x when $t = 81$ and $s = 27$.

Solution

i. *"x varies directly as the square root of t and inversely as S"*

$\Rightarrow x \propto \sqrt{t}$ and $x \propto \frac{1}{S}$

$x \propto \frac{\sqrt{t}}{S}$

$x = \frac{k\sqrt{t}}{S}$ …………..(1)

When $x = 4$, $t = 9$ and $s = 18$

$4 = \frac{k\sqrt{9}}{18}$

$4 \times 18 = k\sqrt{9}$

$k = \frac{4 \times 18}{\sqrt{9}} = \frac{4 \times 18}{3} = 24$

Put k = 24 in eqn (1)

Hence, the variation equation is $x = \frac{24\sqrt{t}}{s}$

ii. When x=? t = 81 and s = 27

$x = \frac{24\sqrt{81}}{27} = \frac{24 \times 9}{27} = 8$

2. Three quantities P, Q and R are connected so that P varies directly as R and inversely as the square root of Q. If P = 6 when R = 12 and Q = 25.

a. Find an expression for P in terms of Q and R

b. Find the value of Q when P = 30 and R = 9

Solution

a. "P varies directly as the as R and inversely as the square root of R".

$P \propto R$ and $P \propto \frac{1}{\sqrt{Q}}$

$P \propto \frac{R}{\sqrt{Q}}$

$P = \frac{kR}{\sqrt{Q}}$..............(1)

If P = 6, R = 12 and Q = 25

$6 = \frac{12k}{\sqrt{25}}$

$6 = \frac{12k}{5}$

$6 \times 5 = 12k$

$30 = 12k$

$k = \frac{5}{2}$

Put $k = \frac{5}{2}$ in eqn (1)

Hence, the variation equation is $P = \frac{5R}{2\sqrt{Q}}$

When Q = ?, P = 30 and R = 9 and $k = \frac{5}{2}$

$30 = \frac{5 \times 9}{2\sqrt{Q}}$

$30 \times 2\sqrt{Q} = 5 \times 9$

$60\sqrt{Q} = 45$

$\sqrt{Q} = \frac{45}{60} = 0.75$

$Q = (0.75)^2 = 0.5625$

4. P varies directly as the square of S and inversely as R. When S = 6 and R = 3, P = 46. Find the value of P when S = 10 and R = 5.

Solution

"P varies directly as the square of S and inversely as R"

$P \propto S^2$ and $P \propto \frac{1}{R}$

$P \propto \frac{S^2}{R}$

$P = \frac{kS^2}{R}$..............(1)

When S = 6 and R = 3, P = 46, k = ?

$46 = \frac{k(6)^2}{3}$

$46 \times 3 = 36k$

$k = \frac{46 \times 3}{36} = \frac{23}{6}$

Put $k = \frac{23}{6}$ in eqn (1)

$P = \frac{23S^2}{6R}$

Hence, the equation of variation is $P = \frac{23S^2}{6R}$

When P = ?, S = 10, R = 5 and $k = \frac{23}{6}$

$P = \frac{23(10)^2}{6(5)} = \frac{23 \times 100}{30} = 76.67$

3. x, y and z are such that x varies directly as z and inversely as the cube root of y. If x = 8, y = 27 and z = 4. Find:

a. an expression for x in terms of y and z

b. the value of y when x= 12 and z = 10

Solution

a. "x varies directly as$z$ and inversely as the cube root of y"

$x \propto z$ and $x \propto \frac{1}{\sqrt[3]{y}}$

$x \propto \frac{z}{\sqrt[3]{y}}$

$x = \frac{kz}{\sqrt[3]{y}}$..............(1)

When$x$ = 8, y = 27 and z = 4

$8 = \frac{K(4)}{\sqrt[3]{27}}$

$8 \times \sqrt[3]{27} = 4k$

$8 \times 3 = 4k$

$k = \frac{8 \times 3}{4} = 6$

Put k = 6 in eqn (1)

Hence, the variation equation is $x = \frac{6z}{\sqrt[3]{y}}$

b. When y = ?, x = 12, z = 10 and k = 6

$x = \frac{6Z}{\sqrt[3]{Y}}$

$12 = \frac{6(10)}{\sqrt[3]{Y}}$

$\sqrt[3]{y} = \frac{6(10)}{12} = \frac{60}{12} = 5$

$y = 5^3 = 125$

4. P varies directly as Q and inversely as the square root of R. If P = 3 when Q = 4 and R = 25

a. Write an equation connecting P, Q and R

b. Find R when P = 6 and Q = 7 (correct to the nearest whole number)

Solution

a. "P varies directly as Q and inversely as the square root of R"

$P \propto Q$and $P \propto \frac{1}{\sqrt{R}}$

$P \propto \frac{Q}{\sqrt{R}}$

$P = \frac{kQ}{\sqrt{R}}$.................(1)

When P = 3, Q = 4, R = 25 and k= ?

$P = \frac{kQ}{\sqrt{R}}$

$3 = \frac{k(4)}{\sqrt{25}}$

$3 \times \sqrt{25} = 4k$

$3 \times 5 = 4k$

$k = \frac{3 \times 5}{4} = \frac{15}{4}$

Put $k = \frac{15}{4}$ in eqn (1)

$P = \frac{15Q}{4\sqrt{R}}$

The equation connecting P, Q and R is $P = \frac{15Q}{4\sqrt{R}}$

b. When R = ? P = 6, Q = 7 and $k = \frac{15}{4}$

$P = \frac{15Q}{4\sqrt{R}}$

$6 = \frac{15(7)}{4\sqrt{R}}$

$6 \times 4\sqrt{R} = 15 \times 7$

$\sqrt{R} = \frac{15 \times 7}{6 \times 4} = 4.375$

R = $\sqrt{4.375} = 2$

5. The electrical resistant R ohms of a wire varies directly as the length, Lcm and inversely as the square of the diameter $d$cm

a. Express d in terms of L, R and the constant of variation,k

b. Find the value of d, correct to 2 decimal places, when L = 15cm, R = 0.23ohms and $k = 1.25 \times 10^{-3}$

Solution

a. "R varies directly as Lcm and inversely as the square of d"

$R \propto L$and $R \propto \frac{1}{d^2}$

$R \propto \frac{L}{d^2}$

$R = \frac{KL}{d^2}$

$d^2 = \frac{KL}{R}$

$d = \sqrt{\frac{kL}{R}}$

b. When d= ?, L = 15cm, R = 0.23 and $k = 1.25 \times 10^{-3}$

$d = \sqrt{\frac{kL}{R}} = \sqrt{\frac{(1.25\times 10^{-3}) \times 15}{0.23}}$ = 0.08cm

6. The force of attraction, F, between two bodies, varies directly as the product of their masses M and m and inversely as the square of the distance d, between them. Given that F =20N when M = 25kg, m = 10kg and d = 5 meters. Find:

a. an expression for F in terms of M, m and d,

b. the distance d, when F = 30N, M = 7.5kg and m = 4kg.

Solution

a. "F varies directly as the product of M and m and inversely as the square of d".

$F \propto Mm$ and $F \propto \frac{1}{d^2}$

$F \propto \frac{Mm}{d^2}$

$F = \frac{kMm}{d^2}$ ……………(1)

When F = 20, M = 25, m = 10, d = 5, k= ?

$F = \frac{kMm}{d^2}$

$20 = \frac{k(25)(10)}{5^2}$

$20 \times 5^2 = 250k$

$k = \frac{20 \times 25}{250} = 2$

Put k = 2 in eqn (1)

$F = \frac{2Mm}{d^2}$

Hence, the variation equation is $F = \frac{2Mm}{d^2}$

b. when d = ?, F = 30, M = 7.5 and m = 4

$F = \frac{2Mm}{d^2}$

$30 = \frac{2(7.5)(4)}{d^2}$

$d^2 = \frac{60}{30} = 2$

$d = \sqrt{2}$

Exercises 18.3

A. 1.Express the following as an equation, where *k* is a variation constant

i. Z varies jointly as x^2 and y

ii. V varies directly as h and r^2 jointly

iii. R varies directly as the square root of x and varies inversely as the square of y

B. 1. z varies directly as the square of x, and inversely as $\sqrt{y}$. Find z in terms of x and y, given that z = 8 when x = 6 and y = 9. Find the value of z when x = 2 and y = 4.

2. q varies as x and inversely as z^2, and q = 10 when x = 30 and z = 14. Find:

i. the formula connecting q, x and z,

ii. q when x = 36 and z = 42.

3. The volume V of a given mass of gas varies directly as the absolute temperature T and inversely as the pressure P, When V = 490, T = 350 and P = 750. Calculate V when T = 400 and P = 560.

4. If y varies jointly as $x$and $(x + 2)$, when x = 4, y = 72. Find the value of y if x = 7.

5. If c varies directly as the cube of m and inversely as n, and c = 8 when m = 4 and n = 12, find c in terms of m and n and find the value of k when m = 2 and n = 16.

6. If y varies jointly as x^2 and z^3, and $y = 12$ when $x = 2$ and $z = 3$, find y in terms of $x$and z and the value of y when $x = 3$ and $z = 4$

7. Given that C varies as the square of x and inversely as the cube of y, and that C = 18 when $x = 3$ and $y = 2$. Find the value of:
i. C when $x = 2$ and $y = 4$
ii. x when C = 48 and $y = 2$

8. In the table below, $W \propto \frac{Q}{R^2}$, where W, R and Q positive integers. Solve for W_2 and r_3

W	R	Q
3	4	4
W_2	1	2
8	r_3	6

Challenge Problems
1. It is given that x varies directly as y, and that y varies inversely as the square root of z.
If $x = 3$ and $y = 12$ when $z = 4$
i. Calculate the values of x and y when $z = 9$
ii. Express x in terms of z

2. The force between two magnetic poles varies inversely as the square of their distance apart. If the distance apart varies as the square of the time,
a. Find how the force varies with the time, t
b. If the force is 0.25 N, when $t = 3$
i. Find the force when $t = 4$
ii. Find the time when the force is 0.001 N

Partial Variation or Variation in Parts
It is a type of variation which consist of two parts namely a fixed (constant) part and a variable part. For instance, when a sound system and chairs are hired for an activity, the sound system is hired at fixed cost whilst the cost of the chairs varies according to the number required. Some of its mathematical expressions are explained below:
I. y is partly fixed (constant) and partly varies as x is expressed as$y$ = C and$y \propto x$,
y = C and $y = kx$,
By putting them together, $y = C + (kx)$, where C and k are variation constants

II. y is partly fixed (constant) and partly varies as the inverse of x is expressed as
y = C and $y \propto \frac{1}{x}$,
y = C and $y = \frac{k}{x}$,
By putting them together, $y = C + \left(\frac{k}{x}\right)$, where C and k are variation constants

III. ypartly varies as xand partly varies as z expressed as$y \propto x$ and$y \propto z$,
$y = Cx$ and$y = kz$
By puttingthem together,$y = Cx + (kz)$, where C and k are variation constants

Solving Partial Variation Problems
Depending on the given variation phrase,
I. Form two equations by substituting the set of values of the given quantities
II. Solve the equations simultaneously to obtain the values of the constants, C and k
III. Substitute the values of C and $k$obtained in the variation equation to establish the actual relationship between the given quantities

Worked Examples
1. n is partly constant and partly varies as m.
When $m = 7$, $n = 11$ and when $m = 10$, $n = 23$. Find:
i. the equation connecting n and m
ii. the value of n when $m = 25$
iii. the value of m when $n = 43$

Solution

n is partly constant and partly varies as m.

$n = c$ and $n \propto m$

$n = c$ and $n = km$

$\Rightarrow n = c + km$

When $m = 7$, $n = 11$,

$11 = c + k(7)$

$11 = c + 7k$……………(1)

When $m = 10$, $n = 22$,

$23= c + k(10)$

$23 = c + 10k$……………(2)

eqn (2) – eqn (1)

$12 = 3k$

$k = \frac{12}{3} = 4$

Put $k = 4$ in eqn (1)

$11 = c + 7(4)$

$11 = c + 28$

$c = 11 – 28 = -17$

Put $k = 4$ and $c = -17$ in $n = c + km$ to obtain the equation of variation as $n = -17 + 4m$

ii. When $n = ?$ $m = 25$, Substitute in $n = -17 + 4m$

$n = -17 + 4(25) = 83$

iii. When $m = ?n = 43$,

Substitute in $n = -17 + 4m$

$43 = -17 + 4(m)$

$43 + 17 = 4m$

$60 = 4m$

$m = \frac{60}{4} = 15$

2. A variable y is partly constant and partly varies as the square of x. When $x = 2$, $y = 6$ and when $x = 6$, $y = 10$

i. Find the equation connecting x and y

ii. What is the value of y when $x = 24$

iii. Find the value of x when $y = 12$

Solution

"*y* is partly constant and partly varies as the square of *x*"

$y = c$ and $y \propto x^2$

$y = c$ and $y = kx^2$

$\Rightarrow y = c + kx^2$

When $x = 2$, $y = 6$ substitute in $y = c + kx^2$

$6 = c + k(2)^2$

$6 = c + 4k$ ……………(1)

When $x = 6$, $y = 10$ substitute in$y = c + kx^2$

$10 = c + k(6)^2$

$10 = c + 36k$ ……………(2)

Solving eqn (1) *and eqn* (2) *simultaneously*

$6 = c + 4k$ ………………(1)

$10 = c + 36k$ ……………(2)

eqn (2) – eqn (1)

$4 = 32k$

$k = \frac{1}{8}$

Put $k = \frac{1}{8}$ in eqn (1)

$6 = c + 4\left(\frac{1}{8}\right)$

$6 = c + \frac{4}{8}$

$6 \times 8 = 8 \times c + 8 \times\frac{4}{8}h$

$48 = 8c + 4$

$48 – 4 = 8c$

$44 = 8c$

$c = \frac{11}{2}$

Substitute $c = \frac{11}{2}$ and $k = \frac{1}{8}$ in $y = c + kx^2$

$y = \frac{11}{2} + \frac{1}{8}x^2$

The equation connecting x and y is $y = \frac{11}{2} + \frac{1}{8}x^2$

ii. When y = ? , $x = 24$

$y = \frac{11}{2} + \frac{1}{8}x^2$

$y = \frac{11}{2} + \frac{1}{8}(24)^2$

$y = \frac{11}{2} + \frac{1}{8}(576) = \frac{11}{2} + 72 = 77.5$

iii. When $y = 12$, x = ?

$12 = \frac{11}{2} + \frac{1}{8}x^2$

$8 \times 12 = 8 \times \frac{11}{2} + 8 \times \frac{1}{8}x^2$

$96 = 44 + x^2$

$96 - 44 = x^2$

$52 = x^2$

$x = \sqrt{52} = 7$ (nearest whole number)

3. Q is partly constant and partly varies as the square of R. If Q is 40, R is 1 and if Q is13, R is 2. Find the value of Q when R is 3.

Solution

Q is partly constant and partly varies as the square of R

$Q = c$ and $Q \propto R^2$

$Q = c$ and $Q = kR^2$

$\Rightarrow Q = c + kR^2$

When $Q = 40$, $R = 1$ substitute in $Q = c + kR^2$

$40 = c + k(1)^2$

$40 = c + k$ ……………(1)

When $Q = 13$, $R = 2$, put in $Q = c + kR^2$

$13 = c + k(2)^2$

$13 = c + 4k$ ……………(2)

Solving eqn (1) and eqn (2) simultaneously

$40 = c + k$ ……………(1)

$13 = c + 4k$ …………..(2)

eqn (2) – eqn (1)

$-27 = 3k$

$k = -9$

Put $k = -9$ in eqn (1)

$40 = c + (-9)$

$c = 40 + 9 = 49$

Put $c = 49$ and $k = -9$ in $Q = c + kR^2$

$Q = 49 + (-9)R^2$

The equation connecting x and y is $Q = 49 - 9R^2$

When Q = ?, R = 3, put in $Q = 49 - 9R^2$

$Q = 49 - 9(3)^2 = 49 - 81 = -32$

4. The cost of producing a radio component is partly constant and partly varies inversely with the number made per day. If 100 components are made per day, the cost is Gh¢2.00 per article; If 200 components are made per day the cost is reduced to Gh¢1.50. What would be cost of a component if 500 were made per day?

Solution

Let c represent the cost of producing a radio component and n represent the number made per day

$\Rightarrow c$ is partly constant and partly varies inversely with n

$c = k_1$ and $c \propto \frac{1}{n}$

$c = k_1$ and $c = \frac{k_2}{n}$

$c = k_1 + \frac{k_2}{n}$

When $c = 2$, $n = 100$

$2 = k_1 + \frac{k_2}{100}$

$100 \times 2 = 100k_1 + 100 \times \frac{k_2}{100}$

$200 = 100k_1 + k_2$ ……….(1)

When $c = 1.5$, $n = 200$

$1.5 = k_1 + \frac{k_2}{200}$

$200 \times 1.5 = 200k_1 + 200 \times \frac{k_2}{200}$

$300 = 200k_1 + k_2$ ……….(2)

Solving eqn (1) and eqn (2) simultaneously

eqn (2) – eqn (1)

$100 = 100k_1$

$k_1 = 1$

Put $k_1 = 1$ in eqn (1)

$200 = 100(1) + k_2$

$200 = 100 + k_2$

$200 - 100 = k_2$

$k_2 = 100$

Put $k_1 = 1$ and $k_2 = 100$ in $c = k_1 + \frac{k_2}{n}$

$\Rightarrow c = 1 + \frac{100}{n}$

When $c = ?$, $n = 500$, put in $c = 1 + \frac{100}{n}$

$c = 1 + \frac{100}{500}$

$c = 1 + \frac{1}{5}$

$5 \times c = 5(1) + 5\left(\frac{1}{5}\right)$

$5c = 5 + 1$

$5c = 6$

$c = \frac{6}{5} = 1.2$

The cost of a component would be Gh¢1.20

5. The cost, c, of maintaining a motor car in a factory is partly constant and partly varies inversely as the number, n, of cars produced per day. The cost of producing 4 cars per day is Gh¢1,600.00 and that of producing 5 cars per day is Gh¢1,420.00 Find the relation between c and n.

Solution

c is partly constant and partly varies inversely as n

$c = k_1$ and $c \propto \frac{1}{n}$

$c = k_1$ and $c = \frac{k_2}{n}$

$c = k_1 + \frac{k_2}{n}$

When $c = 1600$, $n = 4$

$1600 = k_1 + \frac{k_2}{4}$

$1600 \times 4 = 4 \times k_1 + 4 \times \frac{k_2}{4}$

$6400 = 4k_1 + k_2$ ……….(1)

When $c = 1420$, $n = 5$

$1420 = k_1 + \frac{k_2}{5}$

$1420 \times 5 = 5 \times k_1 + 5 \times \frac{k_2}{5}$

$7100 = 5k_1 + k_2$ ……….(2)

Solving eqn (1) and eqn (2) simultaneously

eqn (2) – eqn (1)

$700 = k_1$

$k_1 = 700$

Put $k_1 = 700$ in eqn (1)

$6{,}400 = 4(700) + k_2$ ……….(1)

$6{,}400 = 2{,}800 + k_2$

$k_2 = 6{,}400 - 2{,}800 = 3{,}600$

Put $k_1 = 700$ and $k_2 = 3{,}600$ in $c = k_1 + \frac{k_2}{n}$

$\Rightarrow c = 700 + \frac{3600}{n}$

The relation between c and n is $c = 700 + \frac{3600}{n}$

6. The cost of maintaining a school is partly constant and partly varies as the number of students. With 50 students, the cost is Gh¢15,705.00 and with 40 students, it is Gh¢13,305.00. Find the cost when there are 44 students.

Solution

Let C = Cost of maintaining a school

n = number of students

C = $a + bn$, where a, and b are constants

When c = 15,705 and n = 50

15,705 = a + 50b……….(1)

When c = 13,305 and n = 40

13,305 = a + 40b……….(2)

eqn (2) – eqn (1);

2400 = 10b

b = 240

Put b = 240 in eqn (1)

15,705 = a + 50(240)

15,705= a + 12,000

a = 15,705 – 12,000 = 3,705

⇒The variation equation is C = 3,705 + 240n

Now, put a = 3705, b = 240 and n = 44 in

$C = a + bn$

C = 3,705 + (240) (44)

C = 3,705 + 10,560 = Gh¢14,265.00

7. The cost, C, of weeding a rectangular plot of land is partly constant and partly varies jointly as the length, L, and the breadth B, of the plot. For a plot of length 50m and breadth 20m, the cost of weeding is Gh¢85,000.00 and for a plot of length 40m and breadth 30m, the cost of weeding is Gh¢100, 000.00

i. Find the relationship between C, L and B.

ii. Calculate the cost of weeding a plot of length 50 m and breadth 40m.

Solution

i. C is partly constant and partly varies jointly as L and B

C= $k_1$and c ∝ LB

$C = k_1$ and $c = k_2$LB

$C= k_1 + k_2$ LB

When C = 85,000, L = 50 and B = 20 put in

$C = k_1 + k_2$ LB

85,000 = $k_1 + k_2$ (50) (20)

85,000 = $k_1 + 1000k_2$ ……….(1)

When c = 100,000, L = 40 and B = 30 put in $C= k_1 + k_2$ LB

100,000 = $k_1 + k_2$ (40) (30)

100,000 = $k_1 + 1200k_2$ ……….(2)

Solving eqn (1) and eqn (2) simultaneously;

eqn (2) – eqn (1)

15,000 = 200k_2

k_2 = 75

Put k_2 = 75 in eqn (1)

85,000 = $k_1 + 1000(75)$ ……….(1)

85,000 = $k_1 + 75{,}000$

85,000 – 75,000 = k_1

k_1 = 10,000

Put k_1 = 10,000 and k_2 = 75 in

C = 10,000 + 75 LB

The relationship between C L and B is

C= 10,000 + 75 LB

ii. When C = ? L = 50 and B = 40

C = 10,000 + 75 LB

C = 10,000 + 75 (50) (40)

C = 10,000 + 150,000 = 160,000

8.The cost (c) of producing n bricks is the sum of a fixed amount (h) and a variable (y)

a. y varies directly as n, write (in symbols) the relation between;

i. y and n

ii. c, h and n

b. If it cost Gh¢520.00 to produce 200 bricks and Gh¢1,030.00 to produce 400 bricks, calculate the cost of producing 300 bricks.

Solution

a. i. $C = h + y$

$y = kn$

ii. $C = h + kn$

b. When C = Gh¢520.00, and $n = 200$ substitute in $C = h + Kn$

$520 = h + k(200)$……….(1)

When C = Gh¢1030.00, and $n = 400$ substitute in $C = h + Kn$

$1030 = h + k(400)$……….(2)

eqn (2) – eqn (1);

$510 = 200k$

$k = \frac{510}{200} = 2.55$

Put $k = 2.55$ in eqn (1);

$520 = h + (2.55)(200)$

$520 = h + 510$

$h = 520 - 510 = 10$

Put $k = 2.55$, $h = 10$ and c = 300 in

$C = h + kn$

C = 10 + (2.55) (300) = 10 + 765 = 775

The cost of producing 300 bricks is Gh¢775. 00

Exercises 18.4

A. Write the partial variations equations:

i.y is partly constant and partly varies as x^2

ii. y varies partly as x and partly as x^3

iii. The variable y varies partly as the square of x and partly as the constant

iv. The variable V varies partly as k and partly as the inverse of the cube of r

v. The variable z varies partly as the square root of x and partly as the constant

B. 1. T varies directly as r and partly constant. If T = 35, $r = 2$ and if T = 55, $r = 4$

i. Write down an equation connecting T and r

ii. Find the value of:

a. r when T = 75b. T when $r = 1.5$

2. y is partly constant and partly varies as x. If $y = 2$ when $x = 4$ and $y = 6$ when $x = 6$, find y when $x = 9$

3. x and y are connected by the relation of the form $y = ax + {}^{b}/_{x}$. When $x = 1$, $y = 2$ and when $x = 4$, $y = 15\frac{1}{2}$. Find the relation and also the value of y when $x = 4$

4. y varies partly as x and is partly constant . When $x = 0$, $y = 1$ and when $x = 1$, $y = 0$.

a. Find the constant of variation and the relation between x and y

b. calculate the value of y when $x = 2$

5.The annual cost C, of running a certain car is made up of two parts, one of which is fixed and the other varies as the distance, d, run by the car in the year. In one year, it ran 6,000 km at a cost of Gh¢9,000.00; in the next year, it ran 7,200km at a total cost of Gh¢9,500.00. Find:

i. the relationship between C and d,

ii. how much it cost to run the car in a year during which it ran 1200km.

6. The cost (C) of paying electricity bills per month is partly constant and varies directly as the

square of the number of appliances (*n*) used in the household. If Mr Brown uses 5 appliances, he pays Gh¢82,000.00 and if he uses 7 appliances he pays Gh¢250,000.00 How much will Mr. White pay, if he uses 13 appliances?

7. A super market pays its sales personnel on weekly basis. At the end of each week, each sales person receives a basic wage plus a
bonus which varies directly as the number of complete weeks the particular person has worked in the shop. At the end of her fourth week, a sales girl received a pay package containing Gh¢2,060.00. Six weeks later her pay had jumped to Gh¢2,150.00. Find the exact relation for determining how much the shop sales personnel are paid every week.

8. The cost per mile, *p*, of running mycar in Kumasi is partly constant and partly inversely proportional to *n*, the number of miles travelled per month. When $n = 500$, $c = 10$, and when $n = 1{,}000$, $c = 8$.
i. Find the formula giving c in terms of *n.*
ii. What would be the cost per mile at a travelling distance of 1500 miles per month?

9. The daily cost of running a school is assumed to be partly constant and partly proportional to the number of pupils. If a school of 300 pupils cost Gh¢125.00 per day to run, and a school of 400 costs Gh¢160.00 per day, find the formula giving the expected cost in terms of the number of pupils, and hence find the expected daily cost of running a school with 500 pupils.

10. The cost, C, of running a training course is partly constant and partly varies as the number of candidates (*n*) and the number of weeks (w) that the course lasts. When 110 candidates attended the course for 10 weeks, the running cost was Gh¢120,000.00 and when 150 candidates attended the course for 6 weeks, the running cost was Gh¢100,000.00. Find:
i. the equation connecting *c*, *n* and *w*
ii. the cost of running the course for 100 candidates for 12 weeks

11. The cost (c) of producing *n* bricks is the sum of a fixed amount, *h*, and a variable amount *y*, where *y* varies directly as *n*. If it costs Gh¢950.00 to produce 600 bricks and Gh¢1,030 to produce 1000 bricks,
i. Find the relationship between *c, h* and *n.*
ii. Calculate the cost of producing 500 bricks

12. V is the sum of two parts. One part varies directly as *t*, one part varies directly as t^2. When $t = 1$, V = 14 and when $t = 2$, V = 7
i. Find the relationship between *t* and V.
ii. When $t = 3$, find the value of V.

Measures of Positions

Measures of positions are techniques that divide a set of data into equal groups. The different measures of positions are ***quartiles, deciles and percentiles***

To determine measurement of position, the data must be sorted from the lowest to the highest.

Quartiles

The quartiles are the three values of the variable that divide an ordered data set into four equal parts.Q_1(lower quartile), Q_2(middle quartile) and Q_3(upper quartile) determine the values for 25%, 50% and 75% of the data. Q_2 coincides with the median.

Determining Quartiles

I. Order or arrange the data from smallest to largest

II. Find the place that occupies every quartile, using the expression, $\frac{kN}{4}$, where k = 1, 2, 3, for Q_1, Q_2, and Q_3 respectively

Quartiles of Odd Number of Data

Quartiles of odd number of data are determined by the formula,$\frac{k(n+1)^{th}}{4}$, where k = 1, 2, 3 for Q_1, Q_2, and Q_3 respectively

$Q_1 = \frac{(n+1)^{th}}{4}$, $Q_2 = \frac{2(n+1)^{th}}{4}$, $Q_3 = \frac{3(n+1)^{th}}{4}$,

Worked Example

Find the lower, middle and upper quartiles of the data 2, 5, 9, 3, 2, 6, 4

Solution

Let the lower quartile be Q_1, the median be Q_2 and the upper quartile be Q_3

Ordering the data: 2, 2, 3, 4, 5, 6, 9

***n* = 7**

$Q_1 = \frac{(7+1)^{th}}{4} = \frac{8^{th}}{4} = 2^{nd}$ position

2, 2, 3, 4, 5, 6, 9

↓

$Q_1 = 2$

$Q_2 = \frac{2(7+1)^{th}}{4} = \frac{16^{th}}{4} = 4^{th}$ position

2, 2, 3, 4, 5, 6, 9

↓

$Q_2 = 4$

$Q_3 = \frac{3(7+1)^{th}}{4} = \frac{24^{th}}{4} = 6^{th}$ position

2, 2, 3, 4, 5, 6, 9

↓

$Q_3 = 6$

Quartiles of Even Number of Data

Quartiles of even number of data, for eample *a, b, c, d, e, f, g, h,* are determined by the following procedure:

I. Order the data and divide the data into two equal parts as shown below:

a, b, c, d, e, | *f, g, h, k, l*

II. Identify the average of ***e*** and ***f*** as the median or Q_2

a, b, c, d, *e* | *f* *g, h, k, l*

$$\mathbf{Q_2 = \frac{e+f}{2}}$$

III. If the data from ***a*** to ***e*** is an odd number, identify the middle number of ***a*** to ***e*** as the first quartile, Q_1

$a, b, c, d, e \mid f, g, h, k, l$

Q_1

IV. If the data from f to l is an odd number, identify the middle number of f to l as the third quartile, Q_3

$a, b, c, d, e, \mid f, \; g, h, \; k, \; l$

Q_1 $\qquad$ Q_3

V. If the data from a to e and f to l are even numbers, identify the average of the two middle numbers from ***a*** to ***e*** as Q_1 and the average of the two middle numbers from ***f*** to ***l*** as Q_3.

$a, \; b, c, \; d \; e \; f, g, h, k, \; l$

$Q_1 = \frac{b+c}{2}$ $\qquad$ $Q_2 = \frac{e+f}{2}$ $\qquad$ $Q_1 = \frac{h+k}{2}$

Using the Formula

For even number of data, two positions are required for each quartile. Thus:

$Q_1 = \frac{1}{4}(n)^{th}$ and $\frac{1}{4}(n)^{th} + 1$

$Q_2 = \frac{2}{4}(n)^{th}$ and $\frac{2}{4}(n)^{th} + 1$

$Q_3 = \frac{3}{4}(n)^{th}$ and $\frac{3}{4}(n)^{th} + 1$

Use the positions to identify the two numbers and divide the sum by two to obtain the value of the quartile.

Worked Examples

1. Determine the quartiles of the data; 2, 2, 5, 6, 7, 7, 9, 3, 3, 4, 8, 9

Solution

Method 1

Ascending order:

2 2 3 3 4 5 6 7 7 8 9 9

Q_1 $\qquad$ Q_2 $\qquad$ Q_3

$Q_1 = \frac{3+3}{2} = 3$ $\quad$ $Q_2 = \frac{5+6}{2} = 5.5$ $\quad$ $Q_3 = \frac{7+8}{2} = 7.5$

Method 2

Ascending order :

2 2 3 3 4 5 6 7 7 8 9 9

n = 12

$Q_1 = \frac{1}{4}(12)^{th}$ and $\frac{1}{4}(12)^{th} + 1$

$= 3^{rd}$ and 4^{th} positions

$Q_1 = \frac{3+3}{2} = 3$

$Q_2 = \frac{2}{4}(12)^{th}$ and $\frac{2}{4}(12)^{th} + 1$

$= 6^{th}$ and 7^{th} positions

$Q_2 = \frac{5+6}{2} = 5.5$

$Q_3 = \frac{3}{4}(12)^{th}$ and $\frac{3}{4}(12)^{th} + 1$

$= 9^{th}$ and 10^{th} positions

$Q_3 = \frac{7+8}{2} = 7.5$

2. Find Q_1, Q_2 and Q_3 of the data; 2, 7, 6, 14, 7, 11, 9, 3, 4, 4, 8

Solution

2, 3, 4, 4, 6, 7, 8, 9, 11, 14

$n = 10$

Q_1 position $= \frac{1}{4}(10)^{th}$ and $\frac{1}{4}(10)^{th} + 1$

Q_1 position $= 2.5^{th}$ and 3.5^{th} positions

Q_1 position $= \left(\frac{2.5+3.5}{2}\right)^{th} = 3^{rd}$ position

$Q_1 = 4$

Q_2 position = $\frac{2}{4}(10)^{th}$ and $\frac{2}{4}(10)^{th} + 1$

= 5^{th} and 6^{th} positions

= 6 and 7

$Q_2 = \frac{6+7}{2} = 6.5$

Q_3 position = $\frac{3}{4}(10)^{th}$ and $\frac{3}{4}(10)^{th} + 1$

Q_3 position = 7.5^{th} and 8.5^{th}

Q_3 position = $\left(\frac{7.5+8.5}{2}\right)^{th}$

Q_3 position = 8^{th} position

$Q_3 = 8$

Deciles

The deciles (D) are the nine values of the variable that divide an ordered set of data into ten equal parts. The deciles determine the values for 10%, 20%, 30%...90%. The median corresponds with D_5.

Calculating Deciles of Raw Data

I. Re- arrange the data in ascending order

II. Identify the number of entries as N

III. Find the place or position that occupies every decile, using the relation, $D = \frac{k(N+1)^{th}}{10}$, where k = 1, 2, 3…9 for D_1, D_2, D_3...D_9 respectively.

Worked Examples

1. Find D_4, D_5 and D_7 of the data : 6, 4, 7, 6, 11, 8, 5, 3, 2, 9, 10, 12, 3

Solution

2, 3, 3, 4, 5, 6, 6, 7, 8, 9, 10, 11, 12

N = 13

$D_4 = \frac{4(13+1)^{th}}{10} = \frac{56}{10} = 5.6^{th}$ position

It implies that D_4 lies between the 5^{th} and 6^{th} positions. But 5^{th} position = 5 and 6^{th} position = 6

$\Rightarrow D_4 = \frac{5+6}{2} = 5.5$

$D_5 = \frac{5(13+1)^{th}}{10} = \frac{70}{10} = 7^{th}$ position

$\Rightarrow D_5 = 6$

$D_7 = \frac{7(13+1)^{th}}{10} = \frac{98}{10} = 9.8^{th}$ position

It implies that D_7 lies between the 9^{th} and 10^{th} positions. But 9^{th} position = 8 and 10^{th} position = 9

$\Rightarrow D_7 = \frac{8+9}{2} = 8.5$

2. From the data; 4, 1, 3, 12, 3, 2, 11, 6, 11, 10, 9, 10, 8, 7, determine the values of D_3, D_5 and D_8

Solution

1, 2, 3, 3, 4, 6, 7, 8, 9, 10, 10, 11, 11, 12

N = 14

$D_3 = \frac{3(14+1)^{th}}{10} = \frac{45}{10} = 4.5^{th}$ position

$\Rightarrow D_3$ lies between the 4^{th} and 5^{th} positions

$D_3 = \frac{3+4}{2} = 3.5$

$D_5 = \frac{5(14+1)^{th}}{10} = \frac{75}{10} = 7.5^{th}$ position

$\Rightarrow D_5$ lies between the 7^{th} and 8^{th} positions

$D_5 = \frac{7+8}{2} = 7.5$

$D_8 = \frac{8(14+1)^{th}}{10} = \frac{120}{10} = 12^{th}$ position

$\Rightarrow D_8 = 11$

Percentiles

Percentiles are the 99 values of the variable that divide an ordered data set into 100 equal parts.

The percentiles determine the values for 1%, 2%, 3%...99% of the data. $D_1 = P_{10}$, $D_2 = P_{20}$… The median corresponds to P_{50}

Determining Percentiles

I. Re- arrange the data in ascending order.

II. Identify the number of entries as ***n.***

III. Find the place or position that occupies every percentile, using the relation, P = $\frac{k(N+1)^{th}}{100}$, where k =1, 2, 3…99 for P_1, P_2, P_3 … P_{99} respectively.

Worked Examples

1. Find P_{15} and $P_{84}$of the data : 6, 4, 7, 6, 11, 8, 5, 3, 2, 9, 10, 12, 3

Solution

2, 3, 3, 4, 5, 6, 6, 7, 8, 9, 10, 11, 12

N = 13

$P_{15} = \frac{15(13+1)^{th}}{100}$ = 2.1th position

$\Rightarrow P_{15}$ lies between the second and third positions.

But second position = 3 and third position = 3

$P_{15} = \frac{3+3}{2} = 3$

$P_{84} = \frac{84(13+1)^{th}}{100}$ = 11.76 th position

$\Rightarrow P_{84}$ lies on the 11^{th} and 12^{th} positions

$P_{84} = \frac{10+11}{2} = 10.5$

Exercises 19.1

A. Find Q_1, Q_2 and Q_3 from the data:

1. 2, 5, 8, 8, 3, 4, 5, 1.
2. 8, 9, 1, 2, 7, 3, 2, 4, 7, 8.
3. 7, 11, 9, 2, 4, 2, 10, 4, 6.
4. 10, 11, 14, 11, 18, 19, 11, 12, 13, 15, 16.

B. Find D_3, D_5 and D_8 of the following data:

1. 2, 5, 8, 8, 3, 4, 5, 1
2. 7, 11, 9, 2, 4, 2, 10, 4, 6
3. 10, 11, 14, 11, 18, 19, 11, 12, 13, 15, 16

C. Find P_{35}, P_{50}, and P_{70}

1. 2, 5, 8, 8, 3, 4, 5, 1.
2. 8, 9, 1, 2, 7, 3, 2, 4, 7, 8.
3. 7, 11, 9, 2, 4, 2, 10, 4, 6.
4. 10, 11, 14, 11, 18, 19, 11, 12, 13, 15, 16.

Cumulative Frequency Table

Consider the table below:

Marks (x)	Frequency (f)
11 – 20	7
21 – 30	3
31 – 40	4
41 – 50	6
51 – 60	5

To draw up a cumulative frequency table,identify the upper class boundaries of the class intervals. Thus, for the class interval 11 – 20, with frequency 7, it means that 7 students had a mark of 20.5(upper class boundary) or less. For the class interval 21 – 30, we consider students who had marks less than 30.5, and this is found by adding the frequency of the previous class to that of itself. i.e. 7 + 3 = 10

The cumulative frequency table for the above information is drawn as shown below:

Marks less than	Frequency (f)	Cumulative frequency (f)
20.5	7	7
30.5	3	7 + 3 = 10
40.5	4	10 + 4 = 14
50.5	6	14 + 6 = 20
60.5	5	20 + 5 = 25

Worked Examples

1. The heights of 100 girls were measured to the nearest centimetre. The results were as follows

Height (Cm)	Frequency
130 – 134	4
135 – 139	7
140 – 144	14

145 – 149	20
150 – 154	24
155 – 159	16
160 – 164	9
165 – 169	6

a. Draw upa cumulative frequency table

b. How many girls were 149.5cm tall or more?

Solution

Height less than	Frequency	Cumulative Frequency
134.5	4	4
139.5	7	11
144.5	14	25
149.5	20	45
154.5	24	69
159.5	16	85
164.5	9	94
169.5	6	100

b. Number of girls who were 149.5cm tall or more

= Total girls – Number of girls who were not 149.5cm tall or more = 100 – 25 = 75

2. The following is the records of marks(%) obtained by some students in a test

59 59 64 50 74 79 33 57 57 53 67 49

80 57 32 57 76 48 74 24 56 50 56

50 60 39 52 57 30 61 73 40

a. Using the class intervals of 20 – 29, 30 – 39, 40 – 49 …construct a cumulative frequency table for the data

b. How many students obtained 49.5% or less?

c. How many students obtained 69.5% or more?

Solution

a.

Marks less than	Frequency	Cumulative Frequency
29.5	1	1
39.5	3	4
49.5	3	7
59.5	15	22
69.5	4	26
79.5	5	31
89.5	1	32

b.Students who obtained 49.5% or less **= 7**

c. Students who obtained 69.5% or more?

32 – 26 = 6 students

Exercises 19.2

1. The followingtable gives the masses of 100 clients of a company to the nearest kilogram.

Mass (Nearest kg)	Freq	Mass (Nearest kg)	Freq
55 – 59	3	85 – 89	11
60 – 64	5	90 – 94	7
65 – 69	10	95 – 99	4
70 – 74	18	100 – 104	2
75 – 79	20	105 – 109	2
80 – 84	17	110 – 114	1

a. Draw up a cumulative frequency table.

b. Using the frequency table, find how many clients had mass more than 79.5kg.

2. The following table gives the masses of 160 babies born in a maternity home to the nearest tenth of a kilometre.

Mass	Frequency
1.5 – 1.9	5
2.0 – 2.4	15
2.5 – 2.9	60
3.0 – 3.4	54

3.5 – 3.9	25
4.0 – 4.4	1

a. Draw up a cumulative frequency table

b. Using the cumulative frequency table, find how many babies had mass of less than 29.5kg at birth.

c. What proportion of the total number of births is this?

Cumulative Frequency Curve (Ogive)

Steps to follow when drawing the cumulative frequency curve:

I. Prepare a cumulative frequency table.

II. Draw two perpendicular axes on a graph sheet.

III. Label the two axes, (horizontal axis for upper class boundaries and vertical axis for cumulative frequencies)

IV. Choose a suitable/ given scale and mark of the values.

V. Plot the points from the cumulative frequency table drawn up in step I

VI. Join the points with a smooth curve, but not a ruler.

Worked Examples

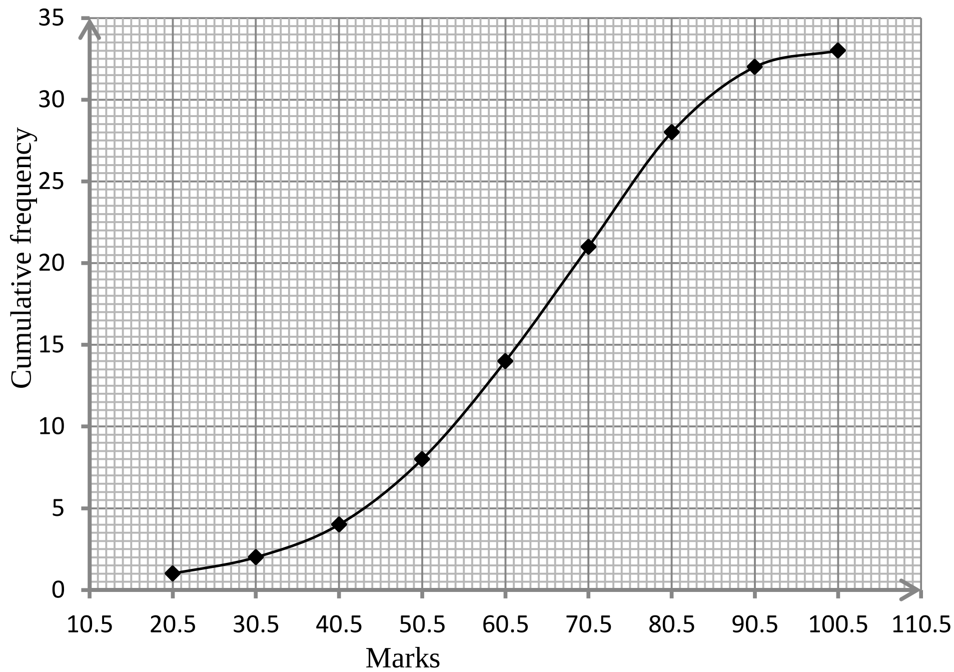
1.The table below gives the distribution of marks obtained by 33 candidates in an examination.

Marks	Frequency
11 – 20	1
21 - 30	1
31 – 40	2
41 – 50	4
51 – 60	6
61 – 70	7
71 – 80	7
81 – 90	4
91 – 100	1

Draw a cumulative frequency curve for the data.

Solution

Marks	Marks Less than	Frequency	Cumulative Frequency
11 – 20	20.5	1	1
21 - 30	30.5	1	2
31 – 40	40.5	2	4
41 – 50	50.5	4	8
51 – 60	60.5	6	14
61 – 70	70.5	7	21
71 – 80	80.5	7	28
81 – 90	90.5	4	32
91 – 100	100.5	1	33



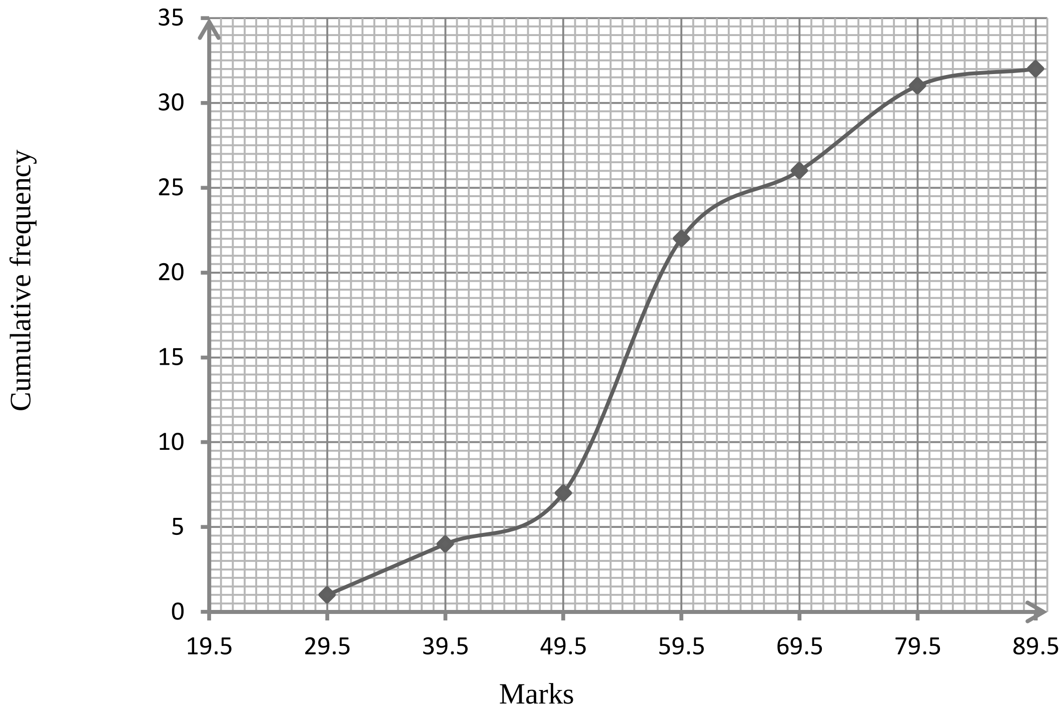
2. A class of students obtained the following marks (%) in a test

59 59 64 50 74 79 33 57
57 53 67 49 80 57 52 57
76 48 74 24 56 50 56 50
60 39 52 57 30 61 73 40

Use this information to draw a cumulative frequency table and a cumulative frequency curve, using the intervals 20 – 29, 30 – 39…

Solution

Marks	Marks less than	Frequency	Cumulative frequency
20 – 29	29.5	1	1
30 – 39	39.5	3	4
40 – 49	49.5	3	7
50 – 59	59.5	15	22
60 – 69	69.5	4	26
70 – 79	79.5	5	31
80 – 89	89.5	1	32



Exercises 19.3

1. The marks scored by 30 students in a mathematic test are given below:

38 31 50 18 51 63 10 34 42 89 73 11
33 31 41 25 76 13 26 23 29 30 51 91
37 64 19 86 09 20

a. Using class intervals 1 – 20, 21 – 40 …Prepare for the distribution, a cumulative frequency table

b. Use your table to draw a cumulative frequency curve

2. The height of 40 citrus plants in a school farm are recorded, to the nearest centimetre as follows:

103 116 127 101 118 125 119 127 114 117
120 117 114 128 112 118 119 129 125 130
117 110 121 109 115 118 113 126 123 131
109 117 105 122 124 114 124 121 123 115

a. Form a group frequency table using the intervals 100 – 104, 105 – 109 …

b. Calculate the mean height of the plant

c. Form a cumulative frequency table and draw the ogive

3. The marks obtained by 40 students in an examination are as follows:

51 46 38 68 21 51 58 64 72 33 86 48
67 93 71 63 44 50 22 91 78 66 52 81
43 64 53 82 45 58 57 72 62 77 61 74
88 35 43 56

a. Using class intervals of 20 – 29 , 30 – 39 , 40 – 49 … construct a frequency distribution table for the data.

b. Draw a cumulative frequency curve for the distribution.

Estimating the Quartiles from a Cumulative Frequency Curve

The positions of quartiles on a cumulative frequency curve involving N observations are determined as follows:

$Q_1 = \frac{N}{4}$, $Q_2 = \frac{N}{2}$ and $Q_3 = \frac{3N}{4}$

The median Q_2 is estimated on the cumulative frequency curve as follows:

I. Find $\left(\frac{n}{2}\right)^{th}$

II. Look for this value on the vertical axis (the cumulative frequency axis)

III. Draw a line perpendicular to the vertical axis at $\frac{n}{2}$ mark and extend it till it meets the cumulative frequency curve

IV. Draw a line from the point of intersection with the curve perpendicular to the horizontal axis

V. Read off the corresponding value on the horizontal axis as Q_2

This same procedure is used to estimate the values of Q_1 and Q_3

Note that Q_1 corresponds to the 25th percentile and Q_3 corresponds to the 75th percentile

Worked Examples

1. The following are the marks obtained by some students in an achievement test.

90	25	31	35	52	50	48	15
19	40	60	83	23	38	40	70
55	65	43	68	58	57	62	83
46	33	09	75	59	46	71	05
63	19	50	47	42	47	28	21

a. Construct a cumulative frequency table for the distribution using the class intervals 1 – 10, 11 – 20, 21 – 30 …

b. Draw a cumulative frequency curve for the distribution

c. Use your graph to find:

i. the median;

ii. the lower quartile;

ii. the upper quartile;

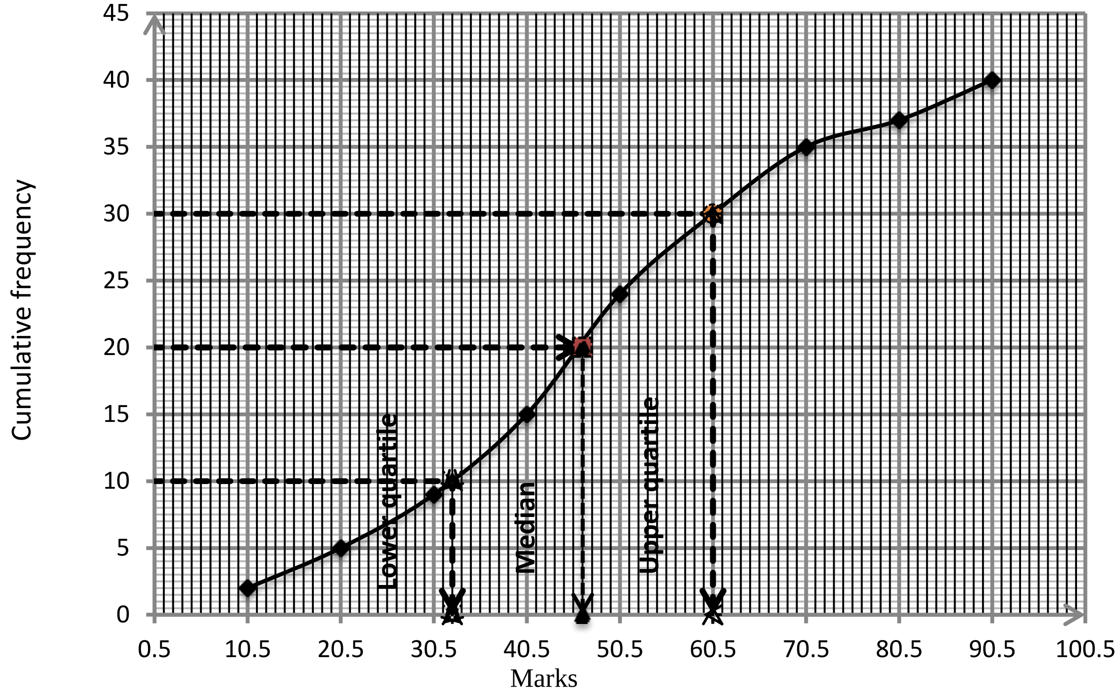
iv. the marks obtained by a student who was 15th in the class.

Solution

a.

Marks	Marks less than	Frequency	Cumulative frequency
1 – 10	10.5	2	2
11 – 20	20.5	3	5
21 – 30	30.5	4	9
31 – 40	40.5	6	15
41 – 50	50.5	9	24
51 – 60	60.5	6	30
61 – 70	70.5	5	35
71 – 80	80.5	2	37
81 – 90	90.5	3	40

b.



c. From the cumulative frequency curve,

i. Median (Q_2) $= \left(\frac{n}{2}\right)^{th}$

But $n = 40$

$Q_1 = \left(\frac{40}{2}\right)^{th}$ = 20th position = 46.5

ii. Lower quartile (Q_1) = $\left(\frac{n}{4}\right)^{th}$

$Q_1 = \left(\frac{40}{4}\right)^{th}$ = 10th position = 32.5

iii. Upper quartile (Q_3) = $\left(\frac{3n}{4}\right)^{th}$

$Q_1 = \left(\frac{3(40)}{4}\right)^{th}$ = 30th position = 60.5

iv. From the graph, the marks obtained by a student who was 15th in the class is 52.5

Exercises 19.4

1. The marks scored in Mathematics by 50 students in an examination are recorded below

93 88 85 62 68 84 73 81 90 68
75 59 53 71 73 79 57 73 60 93
72 78 71 95 61 65 97 67 74 62
60 74 83 68 66 78 61 89 94 77
71 79 68 60 96 78 76 82 75 95

Using class intervals 50 – 54, 55 – 59 … construct a cumulative frequency table for the distribution

b. Draw the cumulative frequency curve.

c. Use your graph to estimate;

i. the median mark,

ii. the lower quartile,

iii. the probability that a student selected at random scored a mark between 80 and 89.

2. The marks obtained by 40 students in an examination are as follows:

51 46 38 68 21 51 58 64 72 33 86 48 67 93
71 63 44 50 22 91 78 66 52 81 43 64 53 82
45 58 57 72 62 77 61 74 88 35 43 56

a. Using class intervals of 20 – 29, 30 – 39 , 40 – 49 … construct a frequency distribution table for the data

b. Draw a cumulative frequency curve for the distribution
c. Use your curve to estimate;
i. the upper quartile
ii. the pass mark if 31 students passed

3. The following are the marks obtained by a number of students in an examination;

79	80	49	68	70	51	41	10	18	28
19	29	30	36	33	50	43	49	41	45
47	55	55	60	50	40	31	32	10	20
44	11	21	29	30	25	26	35	33	48
42	70	51	20	26	34	35	34	67	71

a. Construct a cumulative frequency table for the distribution using the intervals 1 – 10, 11 – 20…
b. Draw a cumulative frequency curve for the distribution.
c. Using your cumulative frequency curve:
i. determine the pass mark if one – fourth of the class passed.
ii. Find the probability that a student selected at random had distinction if the minimum mark for distinction is 75%

4. The table below shows the distribution of marks of 800 candidates in an examination.

Marks (%)	0 – 9	10 – 19	20 – 29	30 – 39	40 – 49
Freq.	10	40	80	140	170
Marks (%)	50 – 59	60 – 69	70 – 99	80 – 89	90 – 99
Freq	130	100	70	40	20

a. i. Construct a cumulative frequency table.
ii. Draw the ogive.
iii. Use your ogive to determine the 50th percentile

5. The table below shows the distribution of marks of candidates in an examination,

Marks	Marks less than	Freq.	Cumulative frequency
0 – 9	9.5	4	
10 – 19	19.5	7	
20 – 29	29.5	5	
30 – 39		10	
40 – 49		13	
50 – 59		20	
60 – 69		15	
70 – 79		13	
80 – 89		3	
90 – 99		1	

a. Copy and complete the table.
b. Draw a cumulative frequency curve for the distribution.

6. The table below gives the distribution of the ages of all people (the nearest thousands) in a certain village

Age (yrs)	0 – 4	5 – 9	10 – 14	15 – 19	20 – 24
Freq.	2	3	6	15	12
Marks (%)	25 – 29	30 – 34	35 – 39		
Freq	7	4	1		

Draw a cumulative frequency table and hence, draw a cumulative frequency curve for the distribution

Mean Deviation of a Raw Data

The mean deviation or average deviation is the arithmetic mean of the absolute deviations and it is denoted by $D_{\bar{x}}$. Thus:

$$D_{\bar{x}} = \frac{/x_1 - \bar{x}/ + /x_2 - \bar{x}/ + \cdots /x_n - \bar{x}/}{N}$$

The steps involved in finding the mean deviation about the mean are as follows:

I. Find the mean of the given data

II. Subtract the mean from each of the observations and record your results

III. Find absolute values of the deviations obtained.

IV. Add the absolute values obtained.

V. Divide the results of the last step by the number of observations to obtain the mean deviation of the data.

Class	$\boldsymbol{x}$	$\boldsymbol{f}$	***fx***	***d*** $= \lvert \boldsymbol{x} - \bar{\boldsymbol{x}} \rvert$	***fd***
		$\sum \boldsymbol{f}$ =	$\sum \boldsymbol{fx}$ =		$\sum \boldsymbol{fd}$ =

Worked Examples

Find the mean deviation about the mean for the following set of data: 6, 7, 10, 12, 13, 4, 8, 12

Solution

Data = 6, 7, 10, 12, 13, 4, 8, 12

Mean = $\frac{6+7+10+12+13+4+8+12}{8} = \frac{72}{8} = 9$

Deviations from the mean;

= 6 – 9, 7 – 9 , 10 – 9 , 12 – 9 , 13 – 9 , 4 – 9 , 8 – 9 , 12 – 9

= -3, -2, 1, 3, 4 -5, -1, 3

Absolute deviations;

= /-3/, /-2/, /1/, /3/, /4/, /-5/, /-1/, / 3/

= 3, 2, 1, 3, 4, 5, 1, 3

Sum of absolute deviations;

= 3 + 2 + 1 + 3 + 4 + 5 + 1 + 3 = 22

Mean deviation about the mean = $\frac{22}{8}$ = 2.75

Exercises 19.5

1. Calculate the mean deviation of the following distribution: 9, 3, 8, 8, 9, 8, 9, 18

2. Find the mean deviation of the following set of numbers: 3, 4, 4, 4, 5, 8, 8, 9

3. For the set of numbers 11, 14, 14, 13, 15, 20, 17, determine the mean deviation

Mean Deviation of a Grouped Data

To find the mean deviation ($D_{\bar{x}}$) of a grouped data,

I. Prepare and complete a tablesuch as the one below:

II. From the table, determine the mean by the formula: **Mean ($\bar{\boldsymbol{x}}$) =** $\frac{\sum \boldsymbol{fx}}{\sum \boldsymbol{f}}$

III. Calculate the mean deviation by the formula: $D_{\bar{x}} = \frac{\sum fd}{\sum f}$

IV. Leave the answer to two decimal places unless stated.

Worked Example

The marks obtained by 20 students in a test (marked out of 20) were recorded as follows;

12 9 14 13 2 9 14 5 12 7

18 13 6 10 8 10 15 3 12 16

i. Construct a group frequency table for the data, using the class intervals 1 – 5, 6 – 10, 11 – 15 …

ii. Use your table to calculate the mean deviation of the distribution.

Solution

Class	x	f	fx	$d = \|x - \bar{x}\|$	fd
1 – 5	3	3	9	7.25	21.75
6 – 10	8	7	56	2.25	15.75
11 – 15	13	8	104	2.75	22
16 – 20	18	2	36	7.75	15.5
		$\sum f = 20$	$\sum fx = 205$		$\sum fd = 75$

$$(\bar{x}) = \frac{\sum fx}{\sum f} = \frac{205}{20} = 10.25$$

$$\text{Mean deviation, } D_{\bar{x}} = \frac{\sum fd}{\sum f} = \frac{75}{20} = 3.75(2.\text{d.p})$$

Exercises 19.6

1. The scores obtained by 42 pupils in a test were:

25 20 18 34 26 10 17 15 15 35 30 10 16
19 20 11 10 10 05 07 25 10 23 08 05 15
05 13 20 10 05 13 07 22 06 05 10 09 12
15 35 26

i. Group the data as 5 – 9, 10 – 14 … and prepare a frequency table for the data

ii. From the table, determine the mean deviation.

2. The scores out of 100 in a science test were:

95 81 78 67 78 84 60 72 66 60 33 36 60
72 27 63 42 18 42 36 27 66 54 27 42 63
27 18 33 30 24 39 45 24 39 30 33 30 30
33 45 9

a. Draw a frequency table for the intervals: 0 – 9, 10 – 19, 20 – 29 …

b. From the table, determine:

i. the mean deviation.

ii. the modal class and the median class.

3. The heights of 25 boys are measured to the nearest cm and are then grouped as follows:

Height(cm)	Frequency
151 – 155	4
156 – 160	8
161 – 165	7
166 – 170	5
171 – 175	1

Calculate the mean deviation

4. The following is the records of the marks of 40 candidates in an examination

65 84 91 58 43 86 73 33 76 80
57 33 53 29 40 27 72 19 52 67
37 14 18 92 13 45 61 39 23 22
22 41 27 51 63 47 19 35 39 76

a. Using class intervals of 11 – 20, 21 – 30 etc, prepare for the distribution a frequency table and use your table to find the mean deviation of the distribution.

b. Find the modal class and the median class.

Measures of Dispersion

Measures of dispersion indicate the degree of 'spread' of the data. The most common statistics used as measures of dispersion are the range, the interquartile range, variance and standard deviation

The Range

The range of the numbers in a group of data is the difference between the greatest number in the data and the least number in the data. For example, given the list 11, 10, 5, 13, 21, the range of the numbers is 21 – 5 = 16

The Interquartile Range

The interquartile range is defined as the difference between the third quartile and the first quartile. i. e. $Q_3 - Q_1$. Thus, the interquartile range measures the spread of the middle half of the data

Semi - Interquartile Range

The semi – interquartile range is another measure of dispersion. To find the semi –interquartile range, divide the inter quartile range by 2.

i.e. $\frac{Q_3 - Q_1}{2}$

Worked Examples

The height of 15 boys to the nearest cm is given as: 138, 136, 143, 143, 147, 158, 146, 145, 137, 135, 143, 140, 129, 156, 149

From these data, find:

i. the range

ii. the quartiles,

iii. the inter quartile range,

iv. the semi – interquartile range.

Solution

i. Arranging the data in order of size,

129, 135, 136, 137, 138, 140, 143, 143, 143, 145, 146, 147, 149, 156, 158

The range 158 – 129 = 29

ii. n = 15

$Q_1 = \frac{(15+1)^{\text{th}}}{4} = \frac{16}{4} = 4^{\text{th}}$ position = 137cm

$Q_2 = \frac{2(15+1)^{\text{th}}}{4} = \frac{32}{4} = 8^{\text{th}}$ position = 143cm

$Q_3 = \frac{3(15+1)^{\text{th}}}{4} = \frac{48}{4} = 12^{\text{th}}$ position = 147cm

iii. Interquartile range

$= Q_3 - Q_1 = 147 - 137 = 10$

iv. Semi – interquartile range

$= \frac{Q_3 - Q_1}{2} = \frac{147-137}{2} = \frac{10}{2} = 5$

Exercises 19.7

1. For the data: 11, 11, 13, 15, 17, 19, 23, 31, 45, 47, 49, find the range and the inter quartile range

2. Find the range, the quartiles and the interquartile range of the data 30, 64, 49, 45, 30, 55, 47, 49

4. Find the interquartile range and the semi – interquartile range of the following data: 43, 34, 45, 31, 40, 22, 30, 17, 25, 18, 27

Variance

The variance is defined as a measure of how the data distributes itself about the mean. It is computed as the average of the squared differences from the mean.

To calculate the variance, follow the steps below:

I. Work out the mean.

II. Then for each number subtract the mean and square the results(the squared difference)

III. Then work out the averages of those squared differences.

Note:

When you have n data values that are:

1. The population: divide by $\boldsymbol{n}$ when calculating the variance. That is $\boldsymbol{\sigma^2} = \frac{\sum(X - \bar{x})^2}{n}$, where $\bar{x}$ is the mean and $\boldsymbol{n}$ is the number of scores.

2. A sample (a selection taken from a bigger population): divide by $\boldsymbol{n} - \mathbf{1}$ when calculating the variance. That is: $\sigma^2 = \frac{\sum(X - \bar{x})^2}{n - 1}$

Worked Examples

Find the variance of the following scores in a test: 17, 12, 14, 13, 19

Solution

17, 12, 14, 13, 19, $n = 5$

Mean $(\bar{x}) = \frac{17+12+14+13+19}{5} = \frac{75}{5} = 15$

Variance

$(\sigma^2) = \frac{(17-15)^2 + (12-15)^2 + (14-15)^2 + (13-15)^2 + (19-15)^2}{5}$

$\sigma^2 = \frac{(2)^2 + (-3)^2 + (-1)^2 + (-2)^2 + (4)^2}{5}$

$\sigma^2 = \frac{4 + 9 + 1 + 4 + 16}{5} = \frac{34}{5} = 6.8$

Properties of Variance

1. The variance is always positive or in the event that the values are equal, the variance is zero.
2. If all the values of the variables are added by the same number, the variance does not change.
3. If all the values of the variables are multiplied by the same number, the variance is multiplied by the square of that number.

Exercises 19.8

Determine the variance:

1. 30, 64, 49, 45, 30, 2. 22, 30, 17, 25, 18, 27

Standard Deviation

Standard deviation is the measure of the extent to which the given values or data are spread or dispersed around the mean. It is the square root of the variance.

The steps to follow in computing the standard deviation are as follows:

I. Find the mean $(\bar{x})$ of the distribution.

II. For each given value, find the deviation from the mean. i.e. $x - \bar{x}$.

III. Find the square of each deviation.

i. e. $(x - \bar{x})^2$

IV. Find the sum of all the squared deviations.

i. e. $\sum(x - \bar{x})^2$

V. Divide this sum by the number of values in the data. i. e. $\frac{\sum(x - \bar{x})^2}{n}$

VI. Find the positive square root of the answer obtained. i. e. $\sqrt{\frac{\sum(x - \bar{x})^2}{n}}$

Generally, the standard deviation, S, is calculated by the formula: $S = \sqrt{\frac{\sum(x - \bar{x})^2}{n}}$

Standard Deviation of a Raw Data

I. Find the mean $(\bar{x})$ of the distribution by the formula: $\bar{x} = \frac{x_1 + x_2 + x_3 + x_4 + x_n}{n} = \frac{\sum x}{n}$

II. Prepare and complete a table of values as shown below:

x	$x - \bar{x}$	$(x - \bar{x})^2$
		$\sum(x - \bar{x})^2 =$

II. Carefully identify the values of n and $\sum(x - \bar{x})^2$ and substitute in the formula:

$S = \sqrt{\frac{\sum(x - \bar{x})^2}{n}}$ to obtain the standard deviation of the data, ususally to two decimal places.

Worked Examples

1. Calculate the standard deviation for the data: 2, 4, 4, 7, 8

Solution

Mean ($\bar{x}$) = $\frac{2+4+4+7+8}{5}$ = $\frac{25}{5}$ = 5

x	x - $\bar{x}$	$(x - \bar{x})^2$
2	2 – 5 = -3	$(-3)^2$ = 9
4	4 – 5 = -1	$(-1)^2$ = 1
4	4 – 5 = -1	$(-1)^2$ =1
7	7 – 5 = 2	$(2)^2$ = 4
8	8 – 5 = 3	$(3)^2$ = 9
		$\sum(x - \bar{x})^2$ = 24

But S = $\sqrt{\frac{\sum(x-\bar{x})^2}{n}}$, where $\sum(x - \bar{x})^2$ = 24, n = 5

S = $\sqrt{\frac{24}{5}}$ = 2.1909 = 2.19 (2 d. p)

2. Find the standard deviation for the data:
3, 4, 5, 6, 8, 10

Solution

Mean $\bar{x}$ = $\frac{3+4+5+6+8+10}{6}$ = $\frac{36}{6}$ = 6

x	x - $\bar{x}$	$(x - \bar{x})^2$
3	3 – 6 = -3	$(-3)^2$ = 9
4	4 – 6 = -2	$(-2)^2$ = 4
5	5 – 6 = -1	$(-1)^2$ =1
6	6 – 6 = 0	$(0)^2$ = 0
8	8 – 6 = 2	$(2)^2$ = 4
10	10 – 6 = 4	$(4)^2$ = 16
		$\sum(x - \bar{x})^2$ = 34

S = $\sqrt{\frac{\sum(x-\bar{x})^2}{n}}$, where $\sum(x - \bar{x})^2$ = 34 and n = 6

S = $\sqrt{\frac{34}{6}}$ = 2.3805 = 2.38(2 d. p)

3. The scores obtained by students in a test are: 21, 25, 27, 25, 27, 21, 24, 23, 23 and 24. Calculate the standard deviation.

Solution

$\bar{x}$ = $\frac{21+25+27+25+27+21+24+23+23+24}{10}$ = 24

x	x - $\bar{x}$	$(x - \bar{x})^2$
21	-3	9
21	-3	9
23	-1	1
23	-1	1
24	0	0
24	0	0
25	1	1
25	1	1
27	3	9
27	3	9
		$\sum(x - \bar{x})^2$ = 40

But S = $\sqrt{\frac{\sum(x-\bar{x})^2}{n}}$, where $\sum(x - \bar{x})^2$ = 40 and n = 10

S = $\sqrt{\frac{40}{10}}$ = 2.00 (2 d. p)

Exercises 19.9

1. The marks scored by 8 pupils in a science test are 3, 7, 8, 8, 5, 8, 4, 8.What is the standard deviation?

2. The scores of 10 students in an examination are given as follows; 45, 12, 75, 81, 54, 51, 24, 67, 19 and 39. What is the standard deviation of the score?

3. The price of 12 commodities in cedi was recorded as: 5, 2, 5, 3, 3, 5, 3, 2, 4, 3, 3, 2
Calculate the standard deviation.

4. The ages in years of 8 boys are: 14, 14, 15, 15, 12, 11 13, 10. Calculate the standard deviation.

Standard Deviation of Ungrouped Data

To find the standard deviation of an ungrouped data, prepare and complete a table as shown below:

x	f	fx	fx^2
	$\sum f =$	$\sum fx =$	$\sum fx^2 =$

II. From the table, calculate the mean by the formula: $\bar{x} = \frac{\sum fx}{\sum f}$

III. Carefully substitute the values of $\sum f$, $\sum fx^2$ and $\bar{x}$ in $S = \sqrt{\frac{\sum fx^2}{\sum f} - (\bar{x})^2}$ and simplify to obtain the value of the standard deviation.

Worked Examples

1.The marks scored by 15 pupils in a test are as follows; 14, 14, 11, 13, 17, 14, 11, 13, 20, 19, 17, 11, 20, 14, 17. Find:

a. the mean mark,

b. the standard deviation, correct to 2 decimal places.

Solution

x	f	fx	fx^2
11	3	33	363
13	2	26	338
14	4	56	784
17	3	51	867
19	1	19	361
20	2	40	800
	$\sum f$ =15	$\sum fx =$ 225	$\sum fx^2 =$ 3513

From the table,

$\bar{x} = \frac{\sum fx}{\sum f} = \frac{225}{15} = 15$

Substitute the values of $\bar{x}$, $\sum f$ and $\sum fx^2$ in

$S = \sqrt{\frac{\sum fx^2}{\sum f} - (\bar{x})^2} = \sqrt{\frac{3513}{15} - (15)^2} = 3.03$(2d.p)

2. The table below gives the distribution of the number of letters per word in the first 50 words of an easy.

No. of letters	1	2	3	4	5	6	7	8	9
No. of words	1	3	x	10	12	y	6	4	1

a. If the average number of letters per word is 5, find the values of x and y

b. Calculate correct to two decimal places, the standard deviation of the distribution

Solution

x	f	fx	fx^2
1	1	1	1
2	3	6	12
3	x	$3x$	$9x$
4	10	40	160
5	12	60	300
6	y	$6y$	$36y$
7	6	42	294
8	4	32	256
9	1	9	81
	$\sum f$ = $(37 + x + y)$	$\sum f x$ = $(190 + 3x + 6y)$	$\sum fx^2 =$ $(1104 + 9x + 36y)$

a. From the table,

$\sum f = (37 + x + y) = 50$

$37 + x + y = 50$

$x + y = 50 – 37$

$x + y = 13$……….(1)

$\bar{x} = \frac{\sum fx}{\sum f} = \frac{190 + 3x + 6y}{50}$

But $\bar{x} = 5$

$\Rightarrow \frac{190 + 3x + 6y}{50} = 5$

$190 + 3x + 6y = 50 \times 5$

$190 + 3x + 6y = 250$

$3x + 6y = 250 - 190$

$3x + 6y = 60$

$x + 2y = 20$……….(2)

eqn (2) – eqn(1)

$(x - x) + (2y - y) = (20 - 13)$

$y = 7$

Put $y = 7$ in eqn (1)

$x + 7 = 13$……….(1)

$x = 13 - 7 = 6$

$(x, y) = (6, 7)$

ii. $\sum f\,x = (190 + 3x + 6y)$

$= 190 + 3(6) + 6(7)$

$= 190 + 18 + 42 = 250$

$\sum fx^2 = (1104 + 9x + 36y)$

$= 1104 + 9(6) + 36(7)$

$= 1104 + 54 + 252 = 1410$

Substitute the values of $\bar{x}$, $\sum f$ and $\sum fx^2$ in in S

$= \sqrt{\frac{\sum fx^2}{\sum f} - (\bar{x})^2} = \sqrt{\frac{1410}{50} - (5)^2} = 1.79$ (2 d. p)

3. The table below gives the frequency distribution of marks scored by 400 students in a test.

Marks(x)	Frequency (f)	fx
1	14	14
2	30	60
3	32	96
4	40	160
5	52	260
6	80	480
7	59	413
8	56	448
9	21	189
10	16	160

Find correct to one decimal place;

a. the mean mark of the distribution

b. the standard deviation of the distribution

Solution

Marks (x)	Frequency (f)	fx	fx^2
1	14	14	14
2	30	60	120
3	32	96	288
4	40	160	640
5	52	260	1300
6	80	480	2880
7	59	413	2891
8	56	448	3584
9	21	189	1701
10	16	160	1600
	$\sum f = 400$	$\sum fx = 1880$	$\sum fx^2 = 15018$

a. $\bar{x} = \frac{\sum fx}{\sum f} = \frac{1880}{400} = 4.7$

b. Substitute the values of $\bar{x}$, $\sum f$ and $\sum fx^2$ in S

$= \sqrt{\frac{\sum fx^2}{\sum f} - (\bar{x})^2} = \sqrt{\frac{15018}{400} - (4.7)^2} = 15.5$ (2 d. p)

Exercises 19.10

1. Calculate the standard deviation of the frequency distribution below:

Marks	1	2	3	4	5	6
Freq	4	2	1	2	1	2

2. The following is a record of scores obtained by 30 J.H.S. 2 pupils in a test marked out of 5:

5 3 2 4 5 2 4 3 1 3 3 4 2 3 4
5 3 4 3 2 4 3 1 2 3 3 2 4 2 1

Construct a frequency table for the data and calculate the standard deviation.

3. The marks obtained by 16 pupils in a test were recorded as follows: 4, 3, 8, 7, 7, 6, 6, 4, 5, 1, 4, 7, 8, 4, 3, 2.

i. Construct a frequency distribution table for this data.
ii. Calculate the mean mark.
iii. Calculate the standard deviation.

4. The table below shows the marks obtained by some students in a test.

Marks	5	6	7	8	9	10
No. of student	5	4	2	3	7	7

i. How many students took part in the test?
ii. Find from the table, the median score.
iii. Calculate the mean score.
iv. What is the standard deviation of the distribution?

5. The table gives the frequency distribution of marks obtained by a number of students in a test.

Marks	3	4	5	6	7	8
Frequency	5	m	$m+1$	9	4	1

If the mean mark is 5, calculate the;
i. value of m
ii. mode and median

Standard Deviation of Grouped data

For a grouped data, the standard deviation is calculated as follows:

I. Prepare a table frequency as shown below:

class	*Mid-point x*	*f*	*fx*	*fx*^{2}
$a-c$				
$d-f$				
		$\sum f$ =	$\sum fx$ =	$\sum fx^2$ =

II. From the table, calculate the mean by the formula: $\boldsymbol{\bar{x}} = \frac{\sum fx}{\sum f}$

III. Carefully substitute the values of $\sum f$, $\sum fx^2$ and $\bar{x}$ in $\mathbf{S} = \sqrt{\frac{\sum fx^2}{\sum f} - (\bar{x})^2}$ and simplify to obtain the value of the standard deviation to two decimal places

Note: x is the class mid-point of each class.

Worked Examples

1. Calculate the standard deviation of the frequency distribution below:

Age	1 – 3	4 – 6	7 – 9	10 –12	13 – 15
Freq	4	10	20	11	5

Solution

class	*x*	*f*	*fx*	*fx*^{2}
1 – 3	2	4	8	16
4 – 6	5	10	50	250
7 – 9	8	20	160	1280
10 – 12	11	11	121	1331
13 – 15	14	5	70	980
		$\sum f$ = 50	$\sum fx$ = 409	$\sum fx^2$ = 3857

From the table, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{409}{50} = 8.18$

Put $\bar{x} = 8.18$, $\sum fx^2 = 3857$ and $\sum fx = 409$ in the formula;

$$S = \sqrt{\frac{\sum fx^2}{\sum f} - (\bar{x})^2} = \sqrt{\frac{3857}{50} - (8.18)^2} = 3.20$$

Exercises 19.11

1. Find the standard deviation for the data below: Ans $\bar{x}$= 8.24 S = 3.27

Age	1 – 3	4 – 6	7 – 9	10 – 12	13 – 15
Freq	2	5	10	5	3

2. Calculate the standard deviation of the frequency distribution below: S = 6.87

Age	1 – 5	6 – 10	11 – 15	16 – 20	21 – 25
Freq	2	5	10	5	3

3. The following is the marks scored by 50 students in a test:

21 35 52 70 55 42 09 48 57 36
46 15 35 12 29 61 48 22 43 58
25 42 01 60 44 38 54 47 69 30
47 18 35 32 21 50 11 24 41 50
53 30 54 47 34 48 60 45 16 33

i. Make a frequency table for these data using equal class intervals of 1 – 10, 11 – 20, …

ii. Calculate the standard deviation from the table

iii. Draw a histogram for the data

4. The following table gives the distribution of height (in cm) of 60 boys

Height (Cm)	Frequency
144 – 153	4
154 – 163	15
164 – 173	16
174 – 183	11
184 – 193	14

Calculate the mean and the standard deviation of the distribution of heights.

Using the Assumed Mean to Calculate the Standard Deviation

Using an assumed mean, A, to calculate the standard deviation, prepare and complete a frequency table similar to the one below:

class	***x***	***d = x – A***	***f***	***fd***	fd^2
			$\sum f$	$\sum fd$	$\sum fd^2$

The standard deviation is then calculated by the formula:

$$S = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2}$$

Worked Example

1. The table below shows the marks obtained by students in an examination

Marks	0 – 9	10 – 19	20 – 29	30 – 39	40 – 49
Freq	2	3	15	10	10

Assuming the mean is 15, calculate the standard deviation.

Solution

$A = 15$

class	x	$d = x - A$	f	fd	fd^2
0 – 9	4.5	-10.5	2	-21	220.5
10 – 19	14.5	-0.5	3	-1.5	0.75
20 – 29	24.5	9.5	15	142.5	1353.75
30 – 39	34.5	19.5	10	195	3802.5
40 – 49	44.5	29.5	10	295	8702.5
			$\sum f = 40$	$\sum fd = 610$	$\sum fd^2 = 14,080$

$$S = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2} = \sqrt{\frac{14080}{40} - \left(\frac{610}{40}\right)^2} = 10.93 \text{ (2 .d.p)}$$

Exercises 19.12

1. Consider the test results of a form three class in Mathematics

marks	40 – 49	50 – 59	60 – 69	70 – 79	80 – 89
Freq.	1	4	19	15	3

Taking the assumed mean as 64.5, find the standard deviation of the distribution

2. The test results of 50 students were recorded as follows:

Marks	50 – 59	60 – 69	70 – 79	80 – 89	90 – 99
Freq.	6	11	19	9	5

Choose a suitable working mean and use it to calculate the standard deviation

3. Calculate the standard deviation of the following data using the assumed mean method.

class	Frequency
10 – 20	4
21 – 30	5
31 – 40	4
41 – 50	4
51 – 60	3
61 – 70	4

The Range and Interquartile Range from Frequency Distribution

The Range

The range can be determined from the frequency distribution table in two ways

1. Finding the difference between the mid points of the highest class and the lowest class.
2. Finding the difference between the highest value and the lowest value

Worked Example

From the frequency distribution table below, find the range.

Marks	1 – 10	11 – 20	21 – 30	31 – 40	41 – 50	51 – 60
Freq.	2	6	7	14	20	35

Solution

<u>Method 1</u>

Midpoint of the highest class

$= \frac{51 + 60}{2} = 55.5$

Midpoint of the lowest class

$= \frac{1 + 10}{2} = 5.5$

Range = 55.5 – 5.5 = 50

Method 2

From the table, the highest value = 60 and the lowest value = 1

Range = 60 – 1 = 59

Exercise 19.13

Determine the range of the frequency distribution table below:

Marks	11 – 20	21 – 30	31 – 40	41 – 50	51 – 60	61 – 70
Frequency	5	21	15	10	7	3

The Interquartile Range

The inter quartile range for data in a frequency distribution is still the difference between the third quartile (Q_3) and the first quartile (Q_1). i. e. $Q_3 - Q_1$

But the first quartile is determined by the formula: $\mathbf{Q_1 = L + \frac{\frac{n}{4} - CF}{f} i}$, where:

L is the lower boundary of the class containing the first quartile,

n is the number of frequencies,

CF is the cumulative frequency of all the classes preceding the class in which the first quartile lies,

f is the frequency in which the first quartile falls.

The Third quartile $\mathbf{Q_3 = L + \frac{\frac{3n}{4} - CF}{f} i}$ where all variable carry the same meaning except L, CF, *f* and *i* which refer to the values needed from the third quartile

Worked Example

The following table shows the distribution of the masses of parcels in a local post Office

Marks (kg)	20 – 24	25 – 29	30 – 34	35 – 39	40 – 44	45 – 49	50 – 54	55 – 59
Frequency	2	3	7	26	29	25	6	2

Find the interquartile range for the data.

Solution

Classes	Class boundaries	Frequency	Cumulative frequency
20 – 24	19 .5 – 24.5	2	2
25 – 29	24.5 – 29.5	3	5
30 – 34	29.5 – 34.5	7	12
35 – 39	34.5 – 39.5	26	38
40 – 44	39.5 – 44.5	29	67
45 – 49	44.5 – 49 .5	25	92
50 – 54	49.5 – 54.5	6	98
55 – 59	54.5 – 59 .5	2	100

Determine the first quartile as follows:

$Q_1 = L + \frac{\frac{n}{4} - CF}{f} i$

But $\frac{n}{4} = \frac{100}{4} = 25$

$25 > 12$ but $25 < 38 \Rightarrow Q_1$ lies in fourth class. Therefore L = 34.5

Cumulative frequencies of all the classes preceding the class containing Q_1 is 12, so CF =12

The frequency of the class containing Q_1 is 26. Therefore $f = 26$

The class interval of the class containing Q_1 is 39.5 – 34.5 = 5. Therefore $i = 5$

In all, there are 100 observations, so $n = 100$

By substitution,

$$Q_1 = 34.5 + \frac{\frac{100}{4} - 12}{26} \times 5$$

$$= 34.5 + \frac{25 - 12}{26} \times 5 = 37.0$$

Determine the third quartile as follows:

$$Q_3 = \mathbf{L} + \frac{\frac{3n}{4} - CF}{f}i$$

But $\frac{3n}{4} = \frac{3(100)}{4} = 75$

$75 > 67$ but $75 < 92 \Rightarrow Q_3$ lies in sixth class. Therefore L = 44.5

Cumulative frequencies of all the classes preceding the class containing Q_3 is 67, so CF = 67

The frequency of the class containing Q_3 is 25. Therefore $f = 25$

The class interval of the class containing Q_1 is 49.5 – 44.5 = 5. Therefore $i = 5$

In all, there are 100 observations, so $n = 100$

By substitution,

$$Q_3 = \mathbf{L} + \frac{\frac{3n}{4} - CF}{f}i$$

$$Q_3 = 44.5 + \frac{\frac{3(100)}{4} - 67}{25} \times 5$$

$$= 44.5 + \frac{75 - 67}{25} \times 5 = 46.1$$

The interquartile range;

$= Q_3 - Q_1 = 46.1 - 37.0 = 9.1$

Exercises 19.14

Find the range and interquartile range of each of the data below;

1.

Weight	116 – 118	119 – 121	122 – 124	125 – 127	128 – 130
Frequency	7	19	28	16	5

2.

Marks	40 – 44	45 – 49	50 – 54	55 – 59	60 – 64	65 – 69	70 – 74
Frequency	4	11	20	31	19	11	4

20 PROBABILITY

Definition

Probability is defined as the ratio of the number of successful outcomes of a random experiment to the total number of possible outcomes.

Mathematically,

Probability = $\frac{\textbf{\textit{number of succesful outcomes}}}{\textbf{\textit{total number of possible outcomes}}}$

It may also be defined as the ratio of event to sample space. In this case,

Probability(P) = $\frac{\textbf{\textit{number of event}}}{\textbf{\textit{numberofsample space}}} = \frac{\boldsymbol{n(E)}}{\boldsymbol{n(S)}}$

Measures of probability are fractions between 0 and 1 i.e $0 \leq P(E) \leq 1$. The probability of zero (0) means an event will not occur and a probability of one means the event will always occur.

Random Experiment

It is an experiment whose results depend on chance; meaning the result of the experiment cannot be predetermined. For instance, the toss of a coin is a random experiment because it cannot be determine with surety and certainty, the face that will show up.

Similarly, the game of football is a random experiment because; one cannot predetermine the outcome (whether win, draw or lose) with certainty. Again, in boxing bout, the winner cannot be predetermined so it is a random experiment.

Outcome of an Experiment

It is the result we get from an experiment. In tossing the 20p coin for instance, either the coat of arms or the cocoa pod will show up. Therefore, there are only two outcomes when a 20p coin is tossed namely,

1. Coat of arms 2. Cocoa pod

Likewise, in the game of football, only three outcomes are expected namely,

1. Win 2. Draw 3. Lose

The outcome of an experiment deals with numbers.

Sample Space of an Experiment

It is the set of all possible outcomes of an experiment. It is denoted by S; For instance, if one side of a coin is represented by H and the other side by T, then the sample space, for a coin is S = $\{H, T\}$. Similarly, the sample space for a Ludo die is S =$\{1, 2, 3, 4, 5, 6\}$.

Event

An event is the occurrence with which a particular interest is attached. It is always the subset of the sample space. For example, in a die experiment, the sample space, S = {1, 2, 3, 4, 5, 6}. If one is only interested in the occurrence of an even number, then the event, E = {2, 4, 6}

Also, in the same experiment, if one is interested in the occurrence of a number more than 4, then the event, E = {5, 6}

The first letter of an event is usually used to represent the probability of that event. Thus, the probability of an even number can be written as P(E), the probability of an odd number can be as P(O) and that of a number less than 5 as P(<5).

Exercises 20.1

List all the possible outcomes of the following;

1. Days of the week on which a baby is born.

2. Months of the year in which workers are paid.
3. Time of the day in which greetings are offered.
4. Observing the sex of a baby.
5. Playing the game of football.
6. Taking a penalty kick.
7. Tossing a die?

Worked Examples

1. A ludo die is thrown once, what is the probability of scoring;
i. an even number
ii. an odd number
iii. a number less than 5

Solution

i. Sample space S = $\{1, 2, 3, 4, 5, 6\}$

n(S) = 6, E =$\{2, 4, 6\}$, n(E) = 3

But P = $\frac{n(E)}{n(S)}$, P(E) = $\frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}$

ii. Sample space S =$\{1, 2, 3, 4, 5, 6\}$,

n(S) = 6 E =$\{1, 3, 5\}$, n(E) = 3

But P = $\frac{n(E)}{n(S)}$, P(O) = $\frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}$

iii. Sample space S =$\{1, 2, 3, 4, 5, 6\}$,

n(S) = 6 , E =$\{1, 2, 3, 4\}$,n(E) = 4

But P = $\frac{n(E)}{n(S)}$,

P(<5) = $\frac{n(E)}{n(S)} = \frac{4}{6} = \frac{2}{3}$

2. A fair die is thrown once.
i. Write down the set of all possible outcomes
ii. Find the probability of obtaining a multiple of 2.
iii. What is the probability of obtaining a prime number?

Solution

i. The set of all possible outcomes
S = $\{1, 2, 3, 4, 5, 6\}$

ii. S = $\{1, 2, 3, 4, 5, 6\}$, E= $\{2, 4, 6\}$
n(S) = 6, n(E) = 3
P(M_2) =$\frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}$

iii. S = $\{1, 2, 3, 4, 5, 6\}$, E = $\{2, 3, 5\}$
n(S) = 6, n(E) = 3
P(P) =$\frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}$

3. In a class of 24 students, 8 of them are girls and 16 of them are boys. Find the probability of picking at random,
i. a boy ii. a girl

Solution

Let G represent girls and B represent boys
n(S) = 24, n(B) = 16, n(G) = 8
i. P(B) =$\frac{n(E)}{n(S)} = \frac{16}{24} = \frac{2}{3}$

ii. P(G) = $\frac{n(E)}{n(S)} = \frac{8}{24} = \frac{1}{3}$

4. A bag contains 12 red pens and 16 blue pens. If a pen is randomly selected, what is probability that it is of: i. Blue color ii. Red color

Solution

Let R represent red pens and B represent blue pens
n(S) = n(B) + n(R) = 16 +12 =28
n(S) = 28, n(R) = 12**,** n(B) = 16
i. P (B) =$\frac{n(E)}{n(S)} = \frac{16}{28} = \frac{4}{7}$

ii. P (R) =$\frac{n(E)}{n(S)} = \frac{12}{28} = \frac{3}{7}$

5. A bag contains 12 mangoes of which 4 are not ripe. What is the chance of picking at random a ripe mango from the bag?

Solution
n(S) = 12**,** n(not ripe) = 4**,**
n(ripe) = n(S) – n(not ripe)
= 12 – 4 = 8
P(ripe) =$\frac{n(E)}{n(S)} = \frac{8}{12} = \frac{2}{3}$

Exercises 20.2
A. 1. What is the probability of an event that which is certain to happen?

2. What is the probability of an event that is impossible?

3. Given that the probability that an event will happen is $a(0 \leq a \leq 1)$, what is the probability that the event will not happen?

B. 1. A number is selected at random from the set of whole number from 1 to 9 inclusive. What is the probability that the number is even?

2. A number is chosen at random from the set W = {11, 12, 13,…20}. Find the probability that the number is at least 15.

3. A bag contains 12 good and 9 bad oranges. If an orange is picked at random from the bag, what is the probability that it is a good orange?

4. A box contains 40 pens of equal size. 10 of them are green and 18 red. If a pen is chosen at random from the box, what is the probability that it is neither green nor red?

5. There are 100 cars in park. 28 of them are blue and red 34. If a car is selected at random, what the probability that it neither blue nor red?

6. Find the probability that a number chosen at random from the natural numbers 2 to 12 inclusive is:
i. either a prime or a multiple of 3
ii. not a prime

The Probability of a Letter in a Word
Sometimes, a word may be given to determine the probability of selecting a letter from it. Learners must therefore take note of the following:
1. that the set of English alphabets, A = {*a*, *b*, *c*…*z*} and the number of alphabets, n(A) = 26
2. that the set of English vowels, V = {*a*, *e*, *i*, *o*, *u*} and the number of vowels, n(V) = 5
3. the rest of the alphabets excluding the vowels are consonants and therefore, the number of consonants, n(C) = 21

For instance, in the word "HIPPOPOTAMUS," the chance of selecting a vowel at random, if the letters are written on a card and placed in a box, is determine as follows;
S ={H, I, P, P, O, P, O, T, A, M, U, S}, n(S) = 12,
V = {I, O, O, A, U}, $n(V) = 5$
P(V) = $\frac{n(E)}{n(S)} = \frac{5}{12}$

Worked Examples
1. A letter is selected from the letters of the English alphabets. What is the probability that the letter is selected from the word MATHEMATICS?

Solution
Alphabets, S = {*a*, *b*, *c*…z} n(S) = 26
Word, W = {MATHEICS}, n(W) = 8
P(W) = $\frac{n(W)}{n(S)} = \frac{8}{26} = \frac{4}{13}$

2. The letters of the word "*examination*" are written on a card and placed in a box. If Mr. Greenpicks a letter at random from the box, what is the probability that he picks:

i. a vowel? ii. the letter "I"?
iii. the letter "O"? iv. the letter "M"?

Solution

i. Let S represent sample space and
V represents vowels
S = {e, x, a, m, i, n, a, t, i, o, n}, n(S) = 11,
V = $\{e, i, a, a, i, o\}$, n(V) = 6
P(V) = $\frac{n(E)}{n(S)} = \frac{6}{11}$

ii. n(S) = 11, n(I) = 2
P(I) = $\frac{n(E)}{n(S)} = \frac{2}{11}$

iii. n(S) = 11, n(O) = 1
P(O) = $\frac{n(E)}{n(S)} = \frac{1}{11}$

iv. n(S) = 11, n(M) = 1
P(M) = $\frac{n(E)}{n(S)} = \frac{1}{11}$

3. Determine the probability of picking the following at random when the letters of the word "**Length"** is place in a bag.
i. a vowel, ii. a consonant, iii. the letter "g."

Solution

S = {l, e, n, g, t, h}, V = {e}
n(S) = 6, n(V) = 1
i. P(V) = $\frac{n(E)}{n(S)} = \frac{1}{6}$

ii. S = {l, e, n, g, t, h}, C = {l, n, g, t, h}
n(S) = 6, n(C) = 5
P(C) = $\frac{n(E)}{n(S)} = \frac{5}{6}$

iii. n(S) = 6, n(g) = 1
P(g) = $\frac{n(E)}{n(S)} = \frac{1}{6}$

4. A letter is chosen at random from the letters of the word "STATISTICS". Find the probability that the letter is:
i. S ii. not a vowel.

Solution

i. S = {S, T, A, T, I, S, T, I, C, S}, E = {S, S, S}
n(S) = 10, n(E) = 3
P(S) = $\frac{n(\mathrm{E})}{n(\mathrm{S})} = \frac{3}{10}$

ii.S = {STATISTICS}, $\bar{V}$ = {STTSTCS}
n(S) = 10, $n(\bar{V})$ = 7
P($\bar{V}$) = $\frac{n(E)}{n(S)} = \frac{7}{10}$ = 0.7

Exercises 20.3

1. In the word *"encyclopedia",* what is the probability of randomly selecting; ?
i) a vowel ii) the letter e iii) a consonant

2. The letters of the word ***"Photosynthesis"*** is written on a card and placed in a box. Whatis the probability of selecting at random,
i. a letter that is not a vowel from the box?
ii. a letter that is S
3. A letter is selected from the letters of the English alphabets. What is the probability that the letter is selected from the word OPPOSITION?

Equally Likely Events

Equally likely events are events of the same experiment that have equal chance of occurring. In other words, they are two or more events of the same experiment that have the same probability. For e.g, if a die is tossed once:
S = {1, 2, 3, 4, 5, 6} n(s) = 6,
Even numbers, E ={2, 4, 6 }, n(E) = 3

Odd numbers, O = $\{1, 3, 5\}, n(O) = 3$.

The probability of scoring an even number,
P(E) = $\frac{n(\text{E})}{n(\text{S})} = \frac{3}{6} = \frac{1}{2}$
Likewise, the probability of scoring an odd number, P(O) = $\frac{\text{n(E)}}{\text{n(S)}} = \frac{3}{6} = \frac{1}{2}$

These two events, P(E) and P(O), are said to have equally likely outcomes because they have the same chance of occurrence or probability which is $\frac{1}{2}$. Two or more of such events are called ***equally likely events*** and they are said to have ***equally likely outcomes.***

Exercises 20.4

1. A number is chosen from the set S = {1, 2, 3, 4, 5, 6} show by working, four events whose outcome is equally likely.

2. In the word ***"existentialism,"*** identify any four events of the letters that have equal outcomes.
3. Show by working, any three events of the letters that are likely to have equal outcomes in the word ***"preposition"***

Finding the Number of an Event, *n*(E), Given the Probability P, and Number of Sample Space, *n*(S)

To find the n(E), given P and n(S), make n(E), the subject of the formula, P(E) = $\frac{n(\text{E})}{n(\text{S})}$ to obtain,
n(E) = P × n(S)   where,
n(E) = number of event,
n(S) = number of sample space
P = probability of obtaining that event.

Worked Examples

1. There are 20 beads in a bag. The probability that a red beads is taken at random from the bag is $\frac{1}{4}$. Find the number of red beads in the bags.

Solution

n(S) = 20, n(E) = ? P(Red) = $\frac{1}{4}$
But n(E) = P(R) ×n(S).
n(E) = $\frac{1}{4} \times 20 = 5$
∴The number of red beads in the bag is 5

2. A box contains 25 balls some of which are blue and others green. If the probability of picking a green ball randomly from the box is $\frac{2}{5}$.
i. Determine the number of green balls in the box.
ii. How many blue balls are in the box?

Solution

i. n(S) = 25, n(E) = ?,   P(Green) = $\frac{2}{5}$
But n(E) = P(Green) ×n(S).
n(E) = $\frac{2}{5} \times 25 = 10$
∴There are 10 green balls in the box

ii. n(Green) + n(Blue) = n(S)
But n(G) = 10, n(S) = 25 and n(B) =?
n(B) = n(S) – n(G)
n(B) = 25 – 10 = 15
Therefore, there are 15 blue balls in the box

3. The probability of selecting a girl at random from a class of 27 students is $\frac{1}{3}$.
i. How many girls are in the class?
ii. Find the number of boys in the class.

Solution

i. n(S) = 27, n(E) = ? P(Girls) = $\frac{1}{3}$
But n(E) = P(Girls) ×n(S)
n(E) = $\frac{1}{3} \times 27 = 9$
∴There are 9 girls in the class

ii. n(Girls) + n(Boys) = n(S)
But n(G) = 9, n(S) = 27 and n(B) =?

n(B) = n(S) – n(G)
= 27 – 9 = 18
∴ There are 18 boys in the class

4. A school bus has a capacity of 40 people, some of which are boys and others girls. The probability of picking a girl at random is $\frac{3}{4}$. How many girls are in the bus?

Solution
n(S) = 40, n(E) = ?, P(Girls) = $\frac{3}{4}$
But n(E) = P(R) × n(S)
n(E) = $\frac{3}{4}$ × 40 = 30 girls

5. There are 16 white, 20 blue and a number of green identical tennis balls in a box. The probability of picking a green tennis ball from the box is $\frac{7}{25}$. Find the probability of picking a blue tennis ball from the same box.

Solution
Let W represents white tennis ball,
B represents blue tennis ball,
G represents green tennis ball,
x represents total number of balls in the bag.
n(W) = 16, n(B) = 20, n (x) = ?
P(G) = $\frac{7}{25}$, P(W) = $\frac{16}{x}$, P(B) = $\frac{20}{x}$
But P(W) + P(B) + P(G) = 1
$\Rightarrow \frac{16}{x} + \frac{20}{x} + \frac{7}{25} = 1$
$\frac{16}{x} + \frac{20}{x} = 1 - \frac{7}{25}$
$\frac{16}{x} + \frac{20}{x} = \frac{25}{25} - \frac{7}{25}$
$\frac{36}{x} = \frac{18}{25}$
$18x = 36 \times 25$
$x = \frac{36 \times 25}{18} = 50$

⇒The total number of balls in the bag is 50

n(S) = 50, n(B) = 20
P(B) = $\frac{n(B)}{n(S)} = \frac{20}{50} = \frac{2}{5}$

Some Solved Past Questions
1. The probability of picking a footballer from a group of 120 sportsmen is $\frac{5}{8}$. How many sportsmen were not footballers?

Solution
<u>Method 1</u>
Let the number of footballers be n(F) and the number of sportsmen who were not footballers be $n(\bar{F})$
n(F) = $\frac{5}{8}$ × 120 = 75
$n(\bar{F})$ = 120 – 75 = 45

<u>Method 2</u>
P(F) + $P(\bar{F})$ = 1
$P(\bar{F})$ = 1 - P(F)
But P(F) = $\frac{5}{8}$
$P(\bar{F}) = 1 - \frac{5}{8}$
P $(\bar{F}) = \frac{8}{8} - \frac{5}{8} = \frac{8-5}{8} = \frac{3}{8}$
n(F) = $\frac{3}{8}$ × 120 = 45

2. A bag contains some balls of which $\frac{1}{4}$ are red. Forty more balls of which five are red are added. If $\frac{1}{5}$ of all the balls are red, how many balls were there originally?

Solution
Let the number of balls be x
n (R) = $\frac{1}{4}x$

If forty more balls are added, total number of ball $= x + 40$

Additional 5 red balls;

n (R) $= \frac{1}{4}x + 5$

$\frac{1}{5}$ of all the balls are red;

$\frac{1}{5}(x + 40) = \frac{1}{4}x + 5$

$20 \times \frac{1}{5}(x + 40) = 20 \times \frac{1}{4}x + (20)5$

$4(x + 40) = 5x + 100$

$4x + 160 = 5x + 100$

$60 = 5x - 4x$

$60 = x$

Therefore, 60 balls were there originally

3. Three blue balls, five green balls and a number of red balls are put together ina sack. One ball is picked at random from the sack. If the probability of picking a blue ball is $\frac{1}{6}$, find:

i. the number of red balls in the sack,

ii. the probability of picking a green ball.

Solution

i.n(B) = 3, n(G) = 5 and n(R) = x

$n(S) = 3 + 5 + x$

$= 8 + x$

P(B) $= \frac{n(E)}{n(S)} = \frac{3}{8 + x}$, but P(B) $= \frac{1}{6}$

$\frac{3}{8 + x} = \frac{1}{6}$

$3 \times 6 = 8 + x$

$18 = 8 + x$

$x = 18 - 8 = 10$

But n(R) = x

n(R) = 10

ii. P(G) $= \frac{n(G)}{n(S)} = \frac{5}{8 + x}$, but $x = 10$

P(G) $= \frac{5}{8 + 10} = \frac{5}{18}$

Exercises 20.5

1. Nissan and Benz cars numbered 48 are parked for sale. If the likelihood of Mr. White buying a Nissan car is rated as $\frac{3}{8}$,

i. how many Nissan cars are on the park?

ii. find the number of cars that are Benz.

2. A box contains 50 pens. The probability of picking a blue pen at random from the box is $\frac{3}{5}$.

i. How many blue pens are in the box?

ii. If the rest of the pens are red, how many are they in the box?

3. There are 54 identical balls of white and green colors in a bag. The probability of picking a green ball from the bag at random is $\frac{2}{3}$. Find the number of green balls in the bag.

4. A bag contains a number of balls, 30 of which are red and the remainder blue. If a ball is chosen at random, the probability that it is red is 0.60. Find the number of blue balls in the bag. Ans = 20

5. A box contains30 white, 25 blue and a number of green identical balls. The probability of picking a green ball from the box is $\frac{4}{15}$. Find:

i. the total number of balls in the bag

ii. the number of green balls in the bag

iii. the probability of picking a blue ball from the box.

6. A bag contains 60 balls, some of which are red and some blue. The probability of a ball drawn at random, being red is $\frac{1}{15}$. Find:

i. the number of blue balls in the bag,

ii. the number of red balls that should be added to the bag to change the probability to $\frac{1}{2}$.

Rules of Probability

1. If the probability that an event E, will occur, denoted by P(E) = a, then the probability that the event will not occur denoted by:

$P(\bar{E})$ = 1 – P(E) = 1 – *a*.

For e.g. If the probability that George will go to school today is $\frac{5}{7}$, then the probability that he will not go to school is :$1 - \frac{5}{7} = \frac{7}{7} - \frac{5}{7} = \frac{2}{7}$

2. For any two events A and B,

P (A) + P (B) = 1…… (1)

From eqn (1)

P (A) = 1 – P (B)…. .. (2)

P (B) = 1 – P (A)… …(3)

For e.g if the probability of obtaining a "head" when a coin is tossed is $\frac{1}{3}$, the probability of obtaining a "tail" is:

$1 - \frac{1}{3} = \frac{3}{3} - \frac{1}{3} = \frac{3-1}{3} = \frac{2}{3}$

Worked examples

1. There are 30 males and females in a class. The probability of selecting a female at random is $\frac{1}{6}$,

i. what is the probability of selecting a male?

ii. how many males are in the class?

Solution

i. P(M) + P(F) = 1, But P(F) = $\frac{1}{6}$, and P(M) = ?

$P(M) = 1 - P(F) = 1 - \frac{1}{6} = \frac{6}{6} - \frac{1}{6} = \frac{6-1}{6} = \frac{5}{6}$

ii. n(S) = 30, P(M) = $\frac{5}{6}$$n$(M) = ?

But P(M) = $\frac{n(M)}{n(S)}$

$\Rightarrow \frac{5}{6} = \frac{n(M)}{30}$

$n(M) = \frac{5 \times 30}{6} = 25$

There are 25 males in the class

2. A box contains a number of green and yellow beads. If the probability of choosing a yellow bead is $\frac{17}{25}$, find the probability of choosing a green bead.

Solution

P(G) + P(Y) = 1

But P(Y) = $\frac{17}{25}$, P(G) = ?

P(G) = 1 – P(Y)

$P(G) = 1 - \frac{17}{25} = \frac{25}{25} - \frac{17}{25} = \frac{6-17}{25} = \frac{8}{25}$

3. A class consists of some boys and girls. The probability of picking a girl for a quiz is $\frac{1}{6}$. Find the probability of picking a boy.

Solution

P(B) + P(G) = 1

P(G) = $\frac{7}{13}$, P(B) = ?

P(B) = 1 – P(G)

$P(B) = 1 - \frac{7}{13} = \frac{13}{13} - \frac{7}{13} = \frac{13-7}{13} = \frac{6}{13}$

4. The probability that it will rain on a particular day is $\frac{1}{5.}$. What is the probability that it will not rain on that day?

Solution

Let the probability that it will rain be P(R) = $\frac{1}{5}$, the probability that it will not rain ,

$P(\bar{R})$= 1 – P(R)

$= 1 - \frac{1}{5} = \frac{5}{5} - \frac{1}{5} = \frac{5-1}{5} = \frac{4}{5}$

5. A bag contains a number of red, blue and green pens. The probability of picking at random a red pen is $\frac{2}{7}$ and the probability of picking at random a green pen is $\frac{1}{6}$. What is the probability of picking a blue pen at random from the bag?

Solution

P(R) = $\frac{2}{7}$, P(G) = $\frac{1}{6}$ and P(B) = ?

P(R) + P(B) + P(G) = 1

$\frac{2}{7}$ + $\frac{1}{6}$ + P(G) = 1

$\frac{19}{42}$ + P(G) = 1

P(G) = 1 – $\frac{19}{42}$ = $\frac{23}{42}$

Exercises 20.6

1. A carton consists of Pepsi and Fanta drinks. The probability of taking a Fanta at random is $\frac{11}{15}$, what is the probability of taking Pepsi?

2. The probability of picking a yellow T' shirt at random from a bag containing yellow and red T'shirts is $\frac{13}{33}$. What is the probability of picking a red T'shirt at random from the bag?

3. A bag contains a collection of Nokia and Samsung mobile phones. If the probability of picking a Nokia phone at random from the bag is $\frac{5}{18}$, what is the probability of not picking a Nokia mobile phone from the bag?

4. A bag contains a number of red, black and white tenisballs. The probability of picking at random a black tenis ball from the bag is $\frac{4}{9}$ and the probability of picking at random a white tenis ball from the bag is $\frac{2}{7}$. What is the probability of picking a red tenis ball at random from the bag?

Challenge Problems

1. All possible two – digit numbers are formed from the digits 1, 2, 3, 4, 5, 6. Find the probability that one of these numbers chosen at random will be divisible;

i. by 5 ii. by 6

Compound Events

Events can be combined by the words "*or*" and "*and*". Events which are combined as such are called ***combined events.*** The word ***"or"*** corresponds to union (U) and the word ***"and"*** corresponds to Intersection(∩). These are further explained below:

I. AUB is the event that occurs if A or B (or both) occur. In probability, **"*or*"** means addition. i.e

P(A or B) = P(A) + P(B)

⇒P(A ∪ B) = P(A) + P(B)

II. A∩B is the event that occurs if A and B occur together.In probability, **"*and*"** means multiplication. That is:

P(*A* and *B*) = P(*A*) × P(*B*)

⇒P(*A* ∩ *B*) = P(*A*) × P(*B*)

Worked Examples

1. A fair die is thrown once, what is the probability of obtaininga factor of 4 or number greater than 4?

Solution

i. S = {1, 2, 3, 4, 5, 6}

n(S) = 6,

Fcators of 4 = {1, 2, 4} *n*(F4) = 3

Numbers of greater than 4 = {5, 6} *n*(G4) = 2

P(F4) = $\frac{n(F4)}{n(s)}$ = $\frac{3}{6}$

P(G4) = $\frac{n(G4)}{n(s)}$ = $\frac{2}{6}$

P(F4) or P(G4) = P(F4) + P(G4)

⇒P(F4∪G4) = $\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$

2. A number is chosen at random from the set X = {1, 2, 3…10). What is the probability that the number chosen is either a factor of 24 or multiple of 5?

Solution

Let the probability of choosing a factor of 24 be P(F24) and the probability of choosing a multiple of 5 be P(M5)

X = {1, 2, 3…10}, n(S) = 10

Factors of 24 = {1, 2, 3, 4, 6, 8}, n(F24) = 6

Multiples of 5 = {5, 10}, n(M5) = 2

P(F24) = $\frac{n(F24)}{n(s)} = \frac{6}{10} = \frac{3}{5}$

P(M5) = $\frac{n(M5)}{n(s)} = \frac{2}{10} = \frac{1}{5}$

⇒P(F24 or M5) = P(A) + P(B)

$= \frac{3}{5} + \frac{1}{5} = \frac{3+1}{5} = \frac{4}{5}$

3. The probabilities that Manful, John and Ernest will pass an examination are $\frac{2}{3}$, $\frac{5}{8}$ and $\frac{3}{4}$ respectively. Find the probability that all three will pass the examination.

Solution

P(M and J and E),

= P(M) × P(J) × P(E),

$= \frac{2}{3} \times \frac{5}{8} \times \frac{3}{4} = \frac{5}{16}$

4. The probabilities that Sammy, Eddy and Jimmy will hit a target are $\frac{2}{3}$, $\frac{3}{4}$ and $\frac{4}{5}$ respectively. Find the probability that;

i. only Sammy will hit the target,

ii. only Sammy and Eddy will not hit the target.

Solution

i. P(S) = $\frac{2}{3}$, P(E) = $\frac{3}{4}$, P(J) = $\frac{4}{5}$

Probability that only Sammy will hit the target means Eddy and Jimmy will not hit the target;

⇒P(S)= $\frac{2}{3}$ and P($\bar{E}$) = $\frac{1}{4}$ and P($\bar{J}$) = $\frac{1}{5}$

= P(S and $\bar{E}$ and $\bar{J}$)

= P(S∩ $\bar{E}$ ∩ $\bar{J}$)

= P(S) × P($\bar{E}$) × P($\bar{J}$)

$= \frac{2}{3} \times \frac{1}{4} \times \frac{1}{5} = \frac{1}{30}$

ii. Probability that only Sammy and Eddy will not hit the target;

⇒P($\bar{S}$ and $\bar{E}$and J)

= P($\bar{S}$) × P($\bar{E}$) × P(J)

$= \frac{1}{3} \times \frac{1}{4} \times \frac{4}{5} = \frac{1}{15}$

5. The probabilities that Randy, Sandy and Wendy will win the U. S. lottery are $\frac{3}{4}$, $\frac{3}{5}$ and $\frac{5}{6}$ respectively. Find the probability that;

i. Randy or Sandy or Wendy will not win,

ii. Only Randy and Sandy or only Randy and Wendy will win.

Solution

i. P(R) = $\frac{3}{4}$, P(S) = $\frac{3}{5}$, P(W) = $\frac{5}{6}$

The probability that Randy or Sandy or Wendy will not win;

⇒P($\bar{R}$ or $\bar{S}$ or $\bar{W}$)

P($\bar{R}$) + P($\bar{S}$) + P($\bar{W}$)

$= \frac{1}{4} + \frac{2}{5} + \frac{1}{6} = \frac{49}{60}$

ii. Probability that only Randy and Sandy or only Randy and Wendy will win,

⇒ P(R and S and $\bar{W}$) or P(R and W and $\bar{S}$)

P(R) × P(S) × P($\bar{W}$) or P(R) × P(W) × P($\bar{S}$)

$= \left(\frac{3}{4} \times \frac{3}{5} \times \frac{1}{6}\right)$ or $\left(\frac{3}{4} \times \frac{5}{6} \times \frac{2}{5}\right)$

$= \left(\frac{3}{40}\right)$ or $\left(\frac{1}{4}\right) = \frac{3}{40} + \frac{1}{4} = \frac{3+10}{40} = \frac{13}{40}$

Exercises 20.7

1. The probability of picking a red pen from a bag is $\frac{3}{11}$ and that of picking a green pen from a box is $\frac{5}{9}$. Find the probability of picking;
i. both pens,
ii. only one color,
iii. none of the pens.

2. The probabilities that Sam, Dan and Jake will go to school on monday are $\frac{2}{5}$, $\frac{1}{4}$ and $\frac{4}{7}$ respectively. Find the probability that:
i. all of them will go to school on Monday.
ii. all of them will not go to school on monday

3. The probabilities that Mr. Brown will wear a white shirt to school is $\frac{3}{7}$, the probability that he will wear a black trousers to school is $\frac{5}{6}$ and the probability that he will wear a yellow shoe is $\frac{4}{9}$. What is the probability that:
i. he will wear only a white shirt and a black trousers to school?
ii. he will wear neither a white shirt nor a black trousers nor a yellow shoe to school?
iii. he will wear all the three to school?

4. The probabilities that *A*, *B*, *C* will occur are $\frac{2}{3}$, $\frac{5}{8}$ and $\frac{3}{4}$ respectively. What is the probability that;
i. *A* or *B* or *C* will not occur?
ii. Only *A* and *B* or only *A* and *C* will occur?

5. The probabilities of three teams *P*, *Q*, *R* winning a football competition are $\frac{1}{4}$, $\frac{1}{8}$ and $\frac{1}{10}$ respectively. Calculate the probability that;
a. either *P* or *Q* will win,
b. either *P* or *Q* or *R* will win,
c. None of these teams will win.

The Coin and Die Experiments

A. Tossing a Coin a Number of Times

1. If a fair coin is tossed once, the possible outcomes are, S = {H,T}, and the number of possible outcomes, *n*(S) = 2

2. If a fair coin is tossed twice or two different coins are tossed once, the possible outcomes are S = {HT, TH, TT, HH}, the number of possible outcomes, *n*(S) = 4. This is illustarated as :

	H	T
H	**HH**	**HT**
T	**TH**	**TT**

3. If a fair coin is tossed three times, the possible outcomes, S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}, the number of possible outcomes, *n*(S) = 8. This is illustrated as :

	HH	HT	TH	TT
H	**HHH**	**HHT**	**HTH**	**HTT**
T	**THH**	**THT**	**TTH**	**TTT**

The number of possible outcomes of the above experiments is summarized in the table below;

No. of times a coin is tossed	No. of possible outcomes
1	$2 = 2^1$
2	$4 = 2^2$
3	$8 = 2^3$
n	2^n

From the table if a coin is tossed *n* times, the number of possible outcomes = 2^n

Worked Examples

A fair coin is tossed twice.

i. Write down the set of possible outcomes.

ii. What is the probability of obtaining?

a. exactly two heads, c. at least one head,

b. no head, d. a head and a tail,

Solution

i. S = {HH, HT, TH, TT}, n(S) = 4

ii. a. S = {HH, HT, TH, TT}, n(S) = 4

Let A denote the event of obtaining two heads

A = {HH}, n(A) = 1

P(A) = $\frac{n(A)}{n(s)} = \frac{1}{4}$

The probability of obtaining two heads = $\frac{1}{4}$

b. Let B be the event of obtaining no head;

B = {TT}, n(B) = 1

P(B) = $\frac{n(B)}{n(s)} = \frac{1}{4}$

The probability of obtaining no head = $\frac{1}{4}$

c. Let C be the event of obtaining at least one head;

C = {HH, HT, TH}, n(C) = 3

P(C) = $\frac{n(C)}{n(s)} = \frac{3}{4}$

The probability of obtaining at least one head = $\frac{3}{4}$

d. Let D denote the event of obtaining a head and a tail

D = {HT, TH}, n(D) = 2

P(B) = $\frac{n(D)}{n(s)} = \frac{2}{4} = \frac{1}{2}$

The probability of obtaining a head and a tail = $\frac{1}{2}$

2. Three fair coins are tossed once.

a. Write down the set of all possible outcomes

b. What is the probability of obtaining?

i. Exactly two heads?

ii. No tail?

iii. Not more than one head?

Solution

a. The set of possible outcomes,

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT},

b. i. Let A represent the event of obtaining exactly two heads;

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT},n(S) = 8,

A = {HHT, HTH, THH}, n(A) = 3

P(A) = $\frac{n(A)}{n(s)} = \frac{3}{8}$

ii. Let B represent the event of obtaining no tail;

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}, n(S) = 8,

B = {HHH}, n(B) = 1

P(B) = = $\frac{n(B)}{n(s)} = \frac{1}{8}$

iii. Let C represent the event of obtaining not more than one head;

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT},n(S) = 8,

C = {HTT, THT, TTH, TTT}, n(C) = 4

P(C) = $\frac{n(C)}{n(s)} = \frac{4}{8} = \frac{1}{2}$

B. Throwing a Ludo Die

1. If a fair die is thrown once, the sample space, S = {1, 2, 3, 4, 5, 6} and the number of sample space, n(S) = 6

2. If a fair die is thrown twice or two dice are thrown together, the sample space, S, is obtained in a tabular form shown below:

Die 1

Die 2	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

Number of possible outcomes, $n(S) = 36$

Note:

*1. The pair of numbers through which the diagonal drawn from **n** to **n** passes sum up to**n***

*2. The pair of numbers at the left (up) of a diagonal drawn from **n** to **n**are less than n*

*3. The pair of numbers at the right (down) of a diagonal drawn from **n** to **n** are greater or more than **n***

Worked Examples

1. Two fair dice, A and B, each with faces numbered 1 to 6are thrown together.

i. Construct a table showing all the equally likely outcomes

ii. From your table, list the pair of numbers on the two dice for which the sum is

a. 5 b. 10 c. more than 10 d. at least 10

iii. Find the probability that the two dice show:

a. Different scores b. the same scores

iv. Find the probability that the sum of the numbers on the two dice is:

a. 5 b. 10 c. more than 10 d. at least 10

Solution

i.

Die 1

Die 2	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

From the table, the total number of possible outcomes, $n(S) = 36$

ii. a. The pair of numbers for which the sum is 5 = (1, 4), (2, 3), (3, 2), (4, 1)

b. The pair of numbers for which the sum is 10 = (4, 6), (5, 5), (6, 4)

c. The pair of numbers for which the sum is more than 10 = (5, 6), (6, 5) and (6, 6)

d. The pair of numbers for which the sum is at least 10 = (4, 6), (5, 5), (6, 4), (5, 6), (6, 5) and (6, 6)

iii.a. Let A represent the pair of numbers that show different scores ;

$n(A) = 30$ and $n(S) = 36$

$P(A) = \frac{n(A)}{n(s)} = \frac{30}{36} = \frac{5}{6}$

b. Let B represent the pair of numbers that show the same number;

= (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) and (6, 6)

$n(B) = 6$ and $n(S) = 36$

$P(B) = \frac{n(B)}{n(s)} = \frac{6}{36} = \frac{1}{6}$

iv.a. Let C represent the pair of numbers that sum up to 5 = (1, 4), (2, 3), (3, 2), (4, 1)

$n(C) = 4$ and $n(S) = 36$

$P(C) = \frac{n(C)}{n(s)} = \frac{4}{36} = \frac{1}{9}$

b. Let D represent the pair of numbers that sum up to 10 = (4, 6), (5, 5), (6, 4)

$n(D) = 3$ and $n(S) = 36$

$P(D) = \frac{n(D)}{n(s)} = \frac{3}{36} = \frac{1}{12}$

c. Let E represent the pair of numbers that sum up more than 10 = (5, 6), (6, 5), (6, 6)

n(E) = 6 and n(S) = 36

P(E) = $\frac{n(E)}{n(s)} = \frac{6}{36} = \frac{1}{6}$

d. Let F represent the pair of numbers that sum up to at least 10
= (4, 6), (5, 5), (6, 4), (5, 6), (6, 5) and (6, 6)

n(F) = 6 and n(S) = 36

P(F) = $\frac{n(F)}{n(s)} = \frac{6}{36} = \frac{1}{6}$

2. Two fair dice are thrown at the same time.

a. Draw the sample space for the possible outcomes.

b. Find the probability of obtaining:

i. a total score of 6 or 8,

ii. the same number on the two dice,

iii. a total not less than 5.

Solution

a.

Die 1 (columns), Die 2 (rows)

	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

b. E (6) = [(5, 1) (4, 2), (3, 3), (2, 4), (1, 5)]

n(E6) = 5

E (8) = [(5, 3) (4, 4), (3, 5), (2, 6)]

n(E8) = 4

From the table, the total number of possible outcomes, n(S) = 36

i. The probability of a total score of 6 or 8

P(6 or 8) = $\frac{n(\text{E }6)}{n(\text{S})} + \frac{n(\text{E }8)}{n(\text{S})} = \frac{5}{36} + \frac{5}{36} = \frac{5}{18}$

ii. The probability of the same number on the two dice

E = [(1, 1) (2, 2) (3, 3) (4, 4) (5, 5) (6, 6)]

n (E) = 6 n(s) = 36

P(same number) = $\frac{6}{36} = \frac{1}{6}$

iii. A total of not less than 5

n (E) = 30 n(s) = 36

P(not less than 5) = $\frac{30}{36} = \frac{5}{6}$

3.In the throw of two fair dice, what is the probability of throwing?

i. a pair of even numbers;

ii. a pair of prime numbers;

iii. a total score that is at most 4.

Solution

i.

Die 1 (columns), Die 2 (rows)

	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

n(S) = 36

A pair of even numbers, E;

(2, 2), (2, 4), (2, 6), (4,2), (4, 4), (4, 6), (6, 2), (6, 4), (6, 6) n(E) = 9

P(E) = $\frac{9}{36} = \frac{1}{4}$

ii. a pair of prime numbers, P
= (2, 3), (2, 5), (3, 2), (3, 3), (3, 5), (5, 2), (5, 3), (5, 5) n(P) = 8

P(P) = $\frac{8}{36} = \frac{2}{9}$

iii. A total score that is atmost 4,

E = (1,1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)

n(E) = 6

P(E) = $\frac{6}{36} = \frac{1}{36}$

4. Two dice are thrown together and the scores are added. Copy and complete the following table of total scores;

	1	2	3	4	5	6
1	2					
2						
3				7		
4						
5						
6						12

i. What is the probability of scoring exactly 9? ii. What is the probability of scoring an even?
iii. What is the probability of scoring either 7 or 11?

Solution

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$n(S) = 36$

i. Probability of scoring exactly 9 = $\frac{1}{9}$

ii. Probability of scoring an even number = $\frac{1}{2}$

iii. Probability of scoring either 7 or 11 $= \frac{2}{9}$

C. Throwing a Coin and a Die Together

If a coin and a die are thrown together or one after the other, the outcomes are obtained as shown in the table below;

	Dice						
Coin		1	2	3	4	5	6
	H						
	T						

Worked Examples

1. In a game, a player throws a die and tosses a coin. If the coin lands heads, he scores twice the number on the die. If the coin lands tails, he scores three times the number on the die. What is the probability of scoring more than 6?

Solution

	Dice						
Coin		1	2	3	4	5	6
	H	2	4	6	8	10	12
	T	3	6	9	12	15	18

$n(S) = 12$,

More than 12 = { 8, 9, 10, 12, 12, 15, 18}

n(More than 12) = 7

P(More than 12) = $\frac{7}{12}$

Other Related Experiments

In some related instances, two setsof elements may be provided for an experiment. For example, given A = {1, 2, 3} and B = {t, h, e}, if A and B occur at the same time, then the set of possible outcomes or sample space is represented as shown below:

	1	***2***	***3***
t	1t	2t	3t
h	1h	2h	3h
e	1e	2e	3e

Number of sample space, $n(S) = 9$

Worked Examples

1. A number is selected from each of the sets {2, 3, 4} and {1, 3, 5}. Find the probability that:
i. the sum of the two numbers is less than 7 and greater than 3;
ii. the sum of the numbers is a prime number;

Solution

	2	3	4
1	(1, 2)	(1, 3)	(1, 4)
3	(3, 2)	(3, 3)	(3, 4)
5	(5, 2)	(5, 3)	(5, 4)

n(S) = 9

Sum of the two numbers less than 7 and greater than 3 = (1, 3), (1, 4), (3, 2), (3, 3)

n(E) = 4

P = $\frac{n(E)}{n(s)} = \frac{4}{9}$

ii.sum of the numbers that is a prime number

E = (1, 2), (1, 4), (3, 2), (3, 4), (5, 2), n(E) = 5

P(P) = $\frac{5}{9}$

2. If two letters are selected from the sets {G, O, L, D} and {C, O, A, L}, what is the probability that:

i. both letters are vowels,

ii. the letters contain at least one vowel.

Solution

	G	**O**	**L**	**D**
C	CG	CO	CL	CD
O	OG	OO	OL	OD
A	AG	AO	AL	AD
L	LG	LO	LL	LD

n(S) = 16

V = {OO, AO}, n(V) = 2

P(V) = $\frac{n(V)}{n(S)} = \frac{2}{16} = \frac{1}{8}$

ii. Let X represent the set that contains at least one vowel

⇒ X = { CO, OG, OO, OL, OD, AG, AO, AL, AD, LO }, n(X) = 10

P(X) = $\frac{n(X)}{n(S)} = \frac{10}{16} = \frac{5}{8}$

3.A two digit numeral (base ten) is formed by choosing both digits at random from the set {6, 7, 8, 9}. The same digit may be chosen twice. Find the probability that the number is;

a. even, b. divisible by 4, c. prime.

Solution

i.

	6	7	8	9
6	66	67	68	69
7	76	77	78	79
8	86	87	88	89
9	96	97	98	99

n(S) = 16

Even numbers, E = {66, 68, 76, 78, 86, 88, 96, 98}

n(E) = 8

P(E) = $\frac{8}{16} = \frac{1}{2}$

ii. Numbers divisible by 4,

= {68, 76, 88, 96}

n(numbers divisible by 4) = 4

P(numbers divisible by 4) = $\frac{4}{16} = \frac{1}{4}$

iii. Prime numbers = {67, 79, 89, 97}

n(prime) = 4

P(Prime) = $\frac{4}{16} = \frac{1}{4}$

Exercises 20.8

1.The following is an incomplete table of possible outcomes when a die is thrown twice

Die 1

Die 2		1	2	3	4	5	6
	1	1,1	1,2	1,3			
	2						
	3						
	4						
	5						
	6						

i. Copy and complete the table

ii. Use your table to find the probability of throwing:

a. no six,   b. at least one five,   c. two sixes.

2.a. A fair coin is tossed three times,write down the set of possible outcomes.

b. What is the probability of obtaining?

i. no tail,

ii. exactly two heads,

iii. at most two heads,

iv. at least one head.

3. A pair of fair dice each numbered 1 to 6 are tossed together. Find the probability of scoring:

i. a sum of at least 9

ii. a sum of 7

iii. a sum that is atmost 10

4. Show as an array of ordered pairs all possible outcomes in throwing a pair of dice. Hence find the probability of getting:

i. two sixes,

ii. at least one six,

iii. a total score of six,

iv. a total score of not six,

v. a total score of eleven,

vi. a total score which is a multiple of three.

5. I throw two dice, one after the other. Make an array of possible outcomes, and find the probability that I score:

i. a sum that is exactly 8,

ii. a sum that is at least 8,

6. Draw a possibility space to show all the outcomes when thrown together a normal die and a die with its sides numbered 1, 1, 1, 2, 2, 3.Use this probability space to find the probability that:

i. the total score is more than 4,

ii. a double is thrown,

iii. there is a multiple of three on each die,

iv. the total score is a prime number.

7. A boy tosses a coin and throws a die. If the coin lands heads, he scores the number shown on the die. If the coin lands tail, he scores double the die number. What is the probability that he will score an odd number?

8. A number is selected from each of the sets {1, 2, 3, 4} and {3, 5, 7}. Find the probability that:

i. the sum of the two numbers is greater than 5 and less than 8,

ii. the sum of the numbers is an even number,

iii. the sum ofthenumbers is at most 8 or atleast 11.

9. If two letters are selected from the sets M = {C, A, T} and N = {D, U, C, K}, what is the probability that:

i. both letters are vowels,

ii. the letters contain at least one vowel,

iii. none is a vowel.

10. Two dice are thrown together and the scores multiplied. Tabulate all the possible results, and find the probability that the resulting score is:

a. odd   b. less than 20   c. a factor of 36

11. A two digit number in base ten is formed by choosing the tens digit from the set {4, 5, 6, 7, 8} and the units digits from the set {1, 2, 3, 9). Find the probability that the resulting number is:

a. even,

b. greater than 50,

c. both even and greater than 50,

d. either even or greater than 50.

12. A two digit numeral (base ten) is formed by choosing both digits at random from the set {6, 7,

8, 9}. The same digit may be chosen twice. Find the probability that the number is;

a. odd, b. prime .

13. A = {1, 2, 3}; B = {4, 5, 6, 7}; C = {(a, b) : $a\in$ A, $b\in$ B}.
P = {8, 9}; B = {10, 11, 12, 13, 14}; R = {(p, q) : $p\in$ P, $q\in$ Q}. Find:
a. n (C)
b. the probability that a member of C ∪ R, chosen at random , is a member of C

Mutually Exclusive Events

Any two events that cannot occur at the same time are said to be ***mutually exclusive***. For example, in throwing the ludo die once, the sample spaces, S = {1, 2, 3, 4, 5, 6,}. Suppose one is interested in playing the number 2 or 5. It is clear that 2 occurs once on the die and 5 also occurs once on the die; hence these two numbers cannot show up at the same time in a single throw. The event of playing 2 and 5 are said to be ***mutually exclusive events*** in this instance.

In general, if A and B are mutually exclusive events, then the probability of A or B occurring in a single draw is given by:

$P(A \text{ or } B) = \frac{n(A)}{n(s)} + \frac{n(B)}{n(s)}$

$\boldsymbol{P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)}$

This is called the ***addition rule of probability.***

For mutually exclusive events**,**

$\boldsymbol{P(A \text{ and } B) = P(A \cap B) = 0}$

Worked Examples

1. i. Write down the sample space for the set of first ten positive integers.

ii. What is the probability of selecting either 1 or 10?

Solution

i. S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

ii. S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} n(S) = 10

Let P(1) denote the probability of selecting 1 P(10) denote the probability of selecting 10

$n(1) = 1$ and $n(10) = 1$

$P(1) = \frac{n(1)}{n(S)} = \frac{1}{10}$

$P(10) = \frac{n(10)}{n(S)} = \frac{1}{10}$

$\Rightarrow P(1 \text{ or } 10) = P(1 \cup 10)$

$= P(1) + P(10)$

$= \frac{1}{10} + \frac{1}{10} = \frac{2}{10}$

2. In a football match, the probability of Kotoko scoring a goal is 0.25 and that of Hearts scoring is 0.41. What is the probability of Kotoko or Hearts Scoring?

Solution

Let P(K) represent the probability of Kotoko scoring and P(H) represent the probability of Hearts scoring

P(K) = 0.25 and P(H) = 0.41

P(K or H) = P(KUH) = P(K) + P(H)

= 0.25 + 0.41 = 0.66

3. Given that X and Y are mutually exclusive events such that P(X) = 0.5, P(Y) = n and P(X or Y) = 0.8,

i. find the value of n;

ii. what is the probability of X or Y occurring in terms of X and Y?

Solution

P(X or Y) = P(X) + P(Y)

But P(X or Y) = 0.8, P(X) = 0.5, P(Y) = n

By substitution,

P(X or Y) = P(X) + P(Y)

$0.8 = 0.5 + n$

$0.8 – 0.5 = n$

$0.3 = n$

ii. P(X or Y) = P(X) + P(Y)

4. Two mutually exclusive events A and B are such that P(A) = $\frac{2}{5}$ and P(B) = $\frac{1}{4}$. What is the probability of either A or B occurring?

Solution

P(A or B) = P(A) + P(B)

But P(A) = $\frac{2}{5}$and P(B) = $\frac{1}{4}$

$\Rightarrow$P(A∪B) = $\mathbf{\frac{2}{5} + \frac{1}{4} = \frac{13}{20}}$

The probability of A or B occurring is $\mathbf{\frac{13}{20}}$

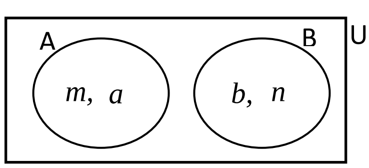
Exercises 20.9

1. If A and B are mutually exclusive events and P(A) = $\frac{1}{3}$ and P(B) = $\frac{2}{5}$, find the probability of A or B occurring.

2. In a certain study, pupils were classified as being underweight or overweight or having normal weight. The probability that a pupil is underweight is 0.3 and the probability that a pupil has a normal weight is 0.5 and the probability that a pupil is overweight is 0.2. Find the probability that a pupil selected at random is either underweight or over weight.

Dependent and Independent Events

An event , A is said to be independent on an event B, if the occurrence or non– occurrence of event B does not affect the occurrence of event A. The diagram below further explains independent events.



A is independent of ***B***. Likewise ***B*** is independent of ***A***

1. AUB = A +B = {*a*, *b*, *m*, *n*}

In terms of probability,

P(A∪B) = P(A) + P(B)

This is the ***addition rule of probability*** for independent events. That is P$\left(\frac{A}{B}\right)$ = P(A)

2. If ***A*** and ***B*** are two independent events then the probability that both occur is the product of their separate probabilities.

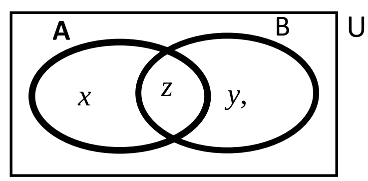
i.e. ***P(A and B) = P(A∩ B) = P(A) × P(B)***

This relation is called the ***multiplication rule of probability***.

Similarly, for three independent events ***A***, ***B*** and ***C***,

P(A∩B∩C) = P(A) × P(B) × P(C)

Two events that are not independent are said to be dependent. In other words, an event A is said to be dependent on event B if the occurrence of event A affects the occurrence of event B and vice - versa



AUB = A + B = {*x*, *z*} + {*z*, *y*}

AUB = {*x*, *y*, *z*, *z*} – {z}, but A∩B = z

AUB = A + B – (A∩B)

In terms of probability,

P(AUB) = P(A) + P(B) – P(A∩B)

Worked Examples

A. 1. Use the given probabilities to determine whether or not the events A and B are independent events:

P(A) = $\frac{1}{4}$, P(B) = $\frac{2}{5}$, P(A∩B) = $\frac{1}{10}$

Solution

If two events A and B are independent events, then P(A∩B) = P(A) × P(B)

P(A∩B) = $\frac{1}{10}$

P(A) × P(B) = $\frac{1}{4} \times \frac{2}{5} = \frac{2}{20} = \frac{1}{10}$

P(A∩B) = P(A) × P(B) = $\frac{1}{10}$

Events A and B are independent.

2. Use the probabilities given to determine whether or not, events X and Y are independent: P(X) = 0.3, P(Y) = 0.7 and P(X∩Y) = 0.14

Solution

If two events X and Y are independent events, then P(X∩Y) = P(X) × P(Y)

P(X∩Y) = 0.14

P(X) × P(Y) = 0.3 × 0.7 = 0.21

P(X∩Y) ≠ P(X) × P(Y)

Therefore events X and Y are not independent

3. Two independent events A and B are such that P(A) = $\frac{5}{12}$ and P(A∩B) = $\frac{1}{8}$ Find:

i. P(B) ii. P(A∪B)

Solution

i. If A and B are two independent events, then P(A∩B) = P(A) × P(B)

But P(A) = $\frac{5}{12}$ and P(A∩B) = $\frac{1}{8}$

By substitution,

$\frac{1}{8} = \frac{5}{12} \times$P(B)

P(B) = $\frac{1}{8} \div \frac{5}{12} = \frac{1}{8} \times \frac{12}{5} = \frac{12}{40} = \frac{3}{10}$

ii. *P(AUB) = P(A) + P(B)*

$= \frac{5}{12} + \frac{3}{10} = \frac{50 + 36}{120} = \frac{86}{120} = \frac{43}{60}$

4. i. X and Y are two independent events such that P(X) = $\frac{3}{7}$ and P(X∩Y) = $\frac{2}{5}$, find P(Y) .

ii. Find P(XUY) if the events X and Y are not independent.

Solution

If X and Y are two independent events, then P(X∩Y) = P(X) × P(Y)

P(X) = $\frac{3}{7}$ and P(X∩Y) = $\frac{2}{5}$

By substitution,

$\frac{2}{5} = \frac{3}{7} \times$ P(Y)

P(Y) = $\frac{2}{5} \div \frac{3}{7} = \frac{2}{5} \times \frac{7}{3} = \frac{14}{15}$

ii. *P(XUY) = P(X) + P(Y)* – P(X∩Y)

$= \frac{3}{7} + \frac{14}{15} - \frac{2}{5} = \frac{101}{105}$

B. 1. A number is chosen at random from the set of factors of 30. What is the probability that the number chosen is a prime number or a multiple of 3?

Solution

S = {1, 2, 3, 5, 6, 10, 15,30}, n(S) = 8

P = {2, 3, 5}, n(P) = 3

M = {3, 6, 15, 30}, n(M) = 4

P ∩M = {3}, n(P ∩O} = 1

P(P) = $\frac{n(P)}{n(S)} = \frac{3}{8}$ P(M) = $\frac{n(M)}{n(S)} = \frac{4}{8}$

P(P ∩M} = $\frac{1}{8}$

P(P or O) = P(P) + P(O) – P(P∩O)

P(P ∪ O) = P(P) + P(O) – P(P∩O)

$= \frac{3}{8} + \frac{4}{8} - \frac{1}{8} = \frac{6}{8} = \frac{3}{4}$

2. A number is chosen at random from the integers 1 to 12 inclusive. Find the probability that the number chosen is :

i. a multiple of 2 or a multiple of 3,
ii. a factor of 18 or a greater than 9.

Solution
i. S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12},
n(S) = 12
Let M_2 represent the set of multiples of 2 and M_3 represent the set of multiples of 3

M_2 = {2, 4, 6, 8, 10, 12}, $n(M_2) = 6$
M_3= {3, 6, 9, 12}$n(M_3) = 4$
$M_2 \cap M_3$ = {6, 12) $n(M_2 \cap M_3) = 2$
$\text{P}(M_2 \; or \; M_3) = \text{P}(M_2) + \text{P}(M_3) - \text{P}(M_2 \cap M_3)$
$\text{P}(M_2 \; U \; M_3) = \text{P}(M_2) + \text{P}(M_3) - \text{P}(M_2 \cap M_3)$
But$\text{P}(M_2) = \frac{n(M_2)}{n(S)} = \frac{6}{12} = \frac{1}{2}$
$\text{P}(M_3) = \frac{n(M_3)}{n(S)} = \frac{4}{12} = \frac{1}{3}$
$\text{P}(M_2 \cap M_3) = \frac{2}{12} = \frac{1}{6}$
$\text{P}(M_2 \; U \; M_3) = \text{P}(M_2) + \text{P}(M_3) - \text{P}(M_2 \cap M_3)$
$= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}$

ii. Factors of 18, F = {1, 2, 3, 9}, n(F) = 4
Numbers greater than 9, G = {10, 11, 12}
n(G) = 3
P(F∩G) = { }⇒ (Independent event)
⇒P(FUG) = P(F) + P(G)
$= \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$

3. The probability that Romeo passes his end of term examination is $\frac{1}{5}$ and the probability that Juliet passes the same examination is$\frac{2}{3}$. Find the probability that:
i. both Romeo and Juliet pass the examination
ii. both fail the examination,
iii. at least one of them passes the examination.

Solution
i. Let $\text{P}(R)$ represent the probability that Romeo passes,
⇒ the probability that Romeo fails is $\text{P}(\bar{R})$,
Let P(J) represent the probability that Juliet passes the exams.
⇒$\text{P}(\bar{J})$ represent the probability that she fails
The probability that both Romeo and Juliet pass the examination,
P(R∩J) = P(R) × P(J)
But P(R) = $\frac{1}{5}$ and P(J) = $\frac{2}{3}$
By substitution,
P(R∩J) = P(R) × P(J)
P(R∩J) = $\frac{1}{5} \times \frac{2}{3} = \frac{2}{15}$
ii. The probability that both Romeo and Juliet fail the examination is given by:
$\text{P}(\bar{R} \cap \bar{J}) = \text{P}(\bar{R}) \times \text{P}(\bar{J})$
$\text{P}(R) + \text{P}(\bar{R}) = 1$, but $\text{P}(R) = \frac{1}{5}$
$\frac{1}{5} + \text{P}(\bar{R}) = 1$
$\text{P}(\bar{R}) = 1 - \frac{1}{5} = \frac{5}{5} - \frac{1}{5} = \frac{4}{5}$
$\text{P}(J) + \text{P}(\bar{J}) = 1$, but $\text{P}(J) = \frac{2}{3}$
$\Rightarrow \frac{2}{3} + \text{P}(\bar{J}) = 1$
$\text{P}(\bar{J}) = 1 - \frac{2}{3} = \frac{3}{3} - \frac{2}{3} = \frac{1}{3}$

$\text{P}(\bar{R} \cap \bar{J}) = \text{P}(\bar{R}) \times \text{P}(\bar{J}) = \frac{4}{5} \times \frac{1}{3} = \frac{4}{15}$

iii. The probability that at least one of them passes the examination is given by:

<u>Method I</u>
P(RUJ) = P(R) + P(J) – P(R∩Y)
By substitution,
P(RUJ) = $\left(\frac{1}{5} + \frac{2}{3}\right) - \frac{2}{15}$
$= \left(\frac{3+10}{15}\right) - \frac{2}{15} = \frac{13}{15} - \frac{2}{15} = \frac{11}{15}$

<u>Method II</u>
P(R) $\text{P}(\bar{J})$ + *P(J)* $\text{P}(\bar{R})$ + *P(R) P(J)*

$= \frac{1}{5} \times \frac{1}{3} + \frac{2}{3} \times \frac{4}{5} + \frac{1}{5} \times \frac{2}{3}$

$= \frac{1}{15} + \frac{8}{15} + \frac{2}{15} = \frac{11}{15}$

4. The probability that Maame goes to school on a market day is $\frac{1}{5}$ and the probability that her brother, Frank also attends school on the same day is $\frac{3}{4}$. What is the probability that:

i. both will be absent?

ii. at least one will be present?

iii. exactly one of them will be present?

Solution

Let $\mathrm{P}(M)$ represent the probability of Maame going to school;

$\mathrm{P}(\bar{M})$ represent the probability ofMaame not going to school.

$\mathrm{P}(F)$ represent the probability of Frank going to school;

$\mathrm{P}(\bar{F})$ represent the probability of Frank not going to school.

$\mathrm{P}(M) + \mathrm{P}(\bar{M}) = 1$, but $\mathrm{P}(M) = \frac{1}{5}$

$\frac{1}{5} + \mathrm{P}(\bar{M}) = 1$

$\mathrm{P}(\bar{M}) = 1 - \frac{1}{5} = \frac{5}{5} - \frac{1}{5} = \frac{4}{5}$

$\mathrm{P}(F) + \mathrm{P}(\bar{F}) = 1$, but $\mathrm{P}(F) = \frac{3}{4}$

$\mathrm{P}(\bar{F}) = 1 - \frac{3}{4} = \frac{4}{4} - \frac{3}{4} = \frac{1}{4}$

i. The probability that both were absent

$\mathrm{P}(\bar{M} \cap \bar{F}) = \mathrm{P}(\bar{M}) \times \mathrm{P}(\bar{F})$

$= \frac{4}{5} \times \frac{1}{4} = \frac{4}{20} = \frac{1}{5}$

ii. The probability that at least one was present;

Method I

$P(M)P(\bar{F})$ or $P(F)P(\bar{M})$ or $P(M)P(F)$

$= \frac{1}{5} \times \frac{1}{4} + \frac{3}{4} \times \frac{4}{5} + \frac{1}{5} \times \frac{3}{4}$

$= \frac{1}{20} + \frac{12}{20} + \frac{3}{20} = \frac{16}{20} = \frac{4}{5}$

Method 2

$P(M \cup F) = P(M) + P(F) - P(M \cap F)$

But $P(M \cap F) = P(M) \times P(F)$

$= \frac{1}{5} \times \frac{3}{4} = \frac{3}{20}$

$P(M \cup F) = \left(\frac{1}{5} + \frac{3}{4}\right) - \frac{3}{20} = \frac{4}{5}$

Method 3

$P(M \cup F) = 1 - P(\bar{M} \cap \bar{F})$

$= 1 - \frac{1}{5} = \frac{5}{5} - \frac{1}{5} = \frac{4}{5}$

iii. The probability that exactly one of them is present:

$P(M) \times \mathrm{P}(\bar{F}) + P(F) \times \mathrm{P}(\bar{M})$

$= \frac{1}{5} \times \frac{1}{4} + \frac{3}{4} \times \frac{4}{5} = \frac{1}{20} + \frac{12}{20} = \frac{13}{20}$

5. Bag X contains 10 balls, ofwhich 3 are red and 7 are blue. Bag Y contains 10 balls of which 4 are red and 6 blue. A ball is drawn at random from each bag. Find the probability that:

i. both are red, ii. at least one is blue.

Solution

i. The probability that both are red;

⇒Bag X is red and Bag Y is red

$n(\mathrm{X}) = 10$, $n(\mathrm{R}) = 3$, $\mathrm{P}(\mathrm{R_x}) = \frac{3}{10}$

$n(\mathrm{Y}) = 10$, $n(\mathrm{R}) = 4$, $\mathrm{P}(\mathrm{R_y}) = \frac{4}{10}$

$\mathrm{P}(\mathrm{R_x} \text{ and } \mathrm{R_y}) = \mathrm{P}(\mathrm{R_x} \cap \mathrm{R_y})$

$= \mathrm{P}(\mathrm{R_x}) \times \mathrm{P}(\mathrm{R_y}) = \frac{3}{10} \times \frac{4}{10} = \frac{12}{100} = \frac{3}{25}$

ii. The probability that at least one is blue;

$= \mathrm{P}(\mathrm{B_x} \cap \mathrm{B_y})$ or $\mathrm{P}(\mathrm{B_x} \cap \mathrm{R_y})$ or $\mathrm{P}(\mathrm{R_x} \cap \mathrm{B_y})$

$= \mathrm{P}(\mathrm{B_x} \cap \mathrm{B_y}) + \mathrm{P}(\mathrm{B_x} \cap \mathrm{R_y}) + \mathrm{P}(\mathrm{R_x} \cap \mathrm{B_y})$

$= \left(\frac{7}{10}\right)\left(\frac{6}{10}\right) + \left(\frac{7}{10}\right)\left(\frac{4}{10}\right) + \left(\frac{6}{10}\right)\left(\frac{6}{10}\right)$

$= \frac{42}{100} + \frac{28}{100} + \frac{18}{100} = \frac{88}{100} = \frac{22}{25}$

Exercises 20.10

1. If X and Y are two independent events such that P(X) = $\frac{3}{8}$ and P(X∩Y) = $\frac{3}{20}$, find:
i. P(Y) ii. P(XUY)

2. From the given probabilities, determine whether or not events A and B are independent events:
i. P(A) = $\frac{1}{4}$, P(B) = $\frac{1}{5}$ and P(A∩B) = $\frac{1}{9}$
ii. P(A) = $\frac{1}{2}$, P(B) = $\frac{3}{5}$ and P(A∩B) = $\frac{3}{10}$

3. The probability that Mugu will go to church on sunday is $\frac{1}{4}$ and that of his brother, Yaaro is $\frac{5}{6}$. What is the probability that:
i. both will go to church?
ii. at least one of them will go to church?
iii. exactly one of them will go to church?

4. Mrs. Brown is expecting a baby. The probability that the baby will be a boy is $\frac{1}{2}$ and the probability that the baby will be fair in complexion is $\frac{1}{4}$. What is the probability that the baby will be fair complexioned boy?

5. A number is selected at random from the set S = {1, 2, 3…15}. What is the probability that the number is less than 10 or even?

6. A number is selected from the set of factors of 30. What is the probability that the number is either a prime number or a multiple of 3?

7. The probability that an athlete A breaks a record of a race is $\frac{1}{2}$; the probability that an athlete B breaks the record in the same race is $\frac{1}{3}$. Find the probability that:
i. both will break the record;
ii. neither breaks the record;
iii. exactly one of them breaks the record.

8. Two proof readers, M and N, read the proofs of a book, and working independently, correct misprints. The probability of M detecting a misprint is P(M) = 0.8 and N is P(N) = 0.75, find:
i. the probability that a misprint is detected by both M and N,
ii. the probability that a misprint is detected by neither M nor N,
iii. the probable number of undetected misprints if M finds 212.

9. Statistics show that the probability of a baby being a boy is 0.55. Find the probability that the first three children born in the family include at least two boys.

10. Two men, Sugri and Dabre play the game of 'oware'. Sugri usually wins two games to every one won by Dabre. If they play three games, what is the probability that:
i. Sugri wins all of them?
ii. Dabre wins twice and Sugri once?

Selection with Replacement

"***Selection with replacement***" means selecting an item from a lot and putting it back from where it was taken. The addition and multiplication rules of probability stated above are used to determine the probability of *A* or *B* (two or more items of different or the same kind) from a lot with replacement.

Generally, if a selection is done with replacement;

1. The number of items in the various groups does not change.
2. The total number of items in the lot (Sample space) does not change.

Worked Examples

1. A bag contains 4 red balls and 6 black balls. A ball is selected from the bag after which it is replaced and a second ball selected and replaced. What is the probability that both balls are red?

Solution

Both balls are red means, the first ball is red and the second ball is also red;
S = 4red balls and 6black balls,
n(S) = 10 balls
n(R) = 4 and *n*(B) = 6
Let P(R) represents the probability of selecting a red ball;
P(R∩R) = P(R) × P(R)
$= \frac{4}{10} \times \frac{4}{10} = \frac{16}{100} = \frac{4}{25}$

2. A box contains 5 white balls, 3 black balls and 2 red balls of the same size. A ball is selected at random from the bag and replaced. Find the probability of obtaining:
a. two red balls,
b. two white balls,
c. two white balls or two black balls.

Solution

a. Let the probability that the first ball is red be $P(R_1)$ and the probability that the second ball is red be $P(R_2)$
$P(R_1 \cap R_2) = P(R_1) \times P(R_2)$
But $P(R_1) = \frac{2}{10}$, and $P(R_2) = \frac{2}{10}$
$P(R_1 \cap R_2) = P(R_1) \times P(R_2)$
$= \frac{2}{10} \times \frac{2}{10} = \frac{4}{100} = \frac{1}{25}$

b.Let the probability that the first ball is white be $P(W_1)$ and the probability that the second ball is white be $P(W_2)$
$P(W_1 \cap W_2) = P(W_1) \times P(W_2)$
But $P(W_1) = \frac{5}{10}$, and $P(w_2) = \frac{5}{10}$
$P(W_1 \cap W_2) = P(W_1) \times P(W_2)$
$= \frac{5}{10} \times \frac{5}{10} = \frac{25}{100} = \frac{1}{4}$

c. Let the probability that the first ball is black be $P(B_1)$ and the probability that the second ball is black be $P(B_2)$
$P(B_1 \cap B_2) = P(B_1) \times P(B_2)$
But $P(B_1) = \frac{3}{10}$, and $P(B_2) = \frac{3}{10}$
$P(B_1 \cap B_2) = P(B_1) \times P(B_2)$
$= \frac{3}{10} \times \frac{3}{10} = \frac{9}{100}$

$P(W_1 \cap W_2)$ or $P(B_1 \cap B_2)$
$= P(W_1) \times P(W_2) + P(B_1) \times P(B_2)$
$= \left(\frac{5}{10} \times \frac{5}{10}\right) + \left(\frac{3}{10} \times \frac{3}{10}\right)$
$= \frac{25}{100} + \frac{9}{100} = \frac{34}{100}$

Exercises 20.11

1. A bag contains 3 red pens and 7 blue pens. What is the probability that the first pen drawn is red and the second is blue, if the selection is done with replacement?

2. A box contains 4 black balls and 8 white balls of the same size. What is the probability that the first ball drawn is white and the second ball is black if the selection is with replacement?

3. Assming that a child is equally likely to be a boy or a girl. A family contains 3 childred. State the probability that :
i. they are all boys,
ii. they are all girls,

iii. the family contains both boys and girls.

4. If two numbers are selected at randon, one after the other, with replacement from the set A = {5, 6, 7, 8, 9}, find the probability of selecting at least one prime number.

Solution

	5	6	7	8	9
5	5, 5	5, 6	5, 7	5, 8	5, 9
6	6, 5	6, 6	6, 7	6, 8	6, 9
7	7, 5	7, 6	7, 7	7, 8	7, 9
8	8, 5	8, 6	8, 7	8, 8	8, 9
9	9, 5	9, 6	9, 7	9, 8	9, 9

E = (5, 5), (5, 6), (5, 7), (5, 8), (5, 9), (6, 5), (6, 7) (7, 5), (7, 6), (7, 7), (7, 8), (7, 9) (8, 5), (8, 7), (9, 5), (9, 7)

n(E) = 16 n(S) = 25

P(at least one prime number) = $\frac{16}{25}$

Selection without Replacement

"***Selection without replacement***" means selecting an item from a lot without putting it back from where it was taken.

Generally, if a selection is done without replacement,

1. The number of balls or items in the various groups reduces
2. The total number of items in the lot (Sample space) reduces
3. The probability of subsequent draws depends on the previous selection

Worked examples

1. A box contains 10 red bulbs and 15 green balls of the same type and size. A bulb is selected at random and is not replaced, a second ball is then selected at random, find the probability that:

i. both bulbs are red,

ii. the first ball is green and the second is red.

Solution

i.*"Both balls are red"* means, the first ball is red and the second is also red;

Let P(R_1)be the probability of the first red ball and P(R_2) be the probability of the second red ball;

P$(R_1$and$R_2)$= P$(R_1 \cap R_2)$= P$(R_1) \times P(\,R_2)$

But P(R_1) = $\frac{10}{25} = \frac{2}{5}$ and P(R_2) = $\frac{9}{24} = \frac{3}{8}$

By substitution,

P$(R_1 \cap R_2)$ = P$(R_1) \times P(\,R_2)$

$= \frac{2}{5} \times \frac{3}{8} = \frac{6}{40} = \frac{3}{20}$

ii.Let P(G_1) bethe probability of the first green ball and P(R_2) bethe probability of the second red ball;

P$(G_1$and $R_2)$ = P$(G_1 \cap R_2)$

= P$(G_1) \times P(\,R_2)$

But P(G_1) = $\frac{15}{25} = \frac{3}{5}$ and P(R_2) = $\frac{10}{24} = \frac{5}{12}$

By substitution,

P$(G_1 \cap R_2)$ = P$(G_1) \times P(\,R_2)$

$= \frac{3}{5} \times \frac{5}{12} = \frac{15}{60} = \frac{1}{4}$

2. A box contains 10 marbles, 7 of which are black and the rest red. Two marbles are drawn, one after the other without replacement. Find the probability of getting:

i. a red and a black marble,

ii. two black marbles,

iii. twomarbles of the same color.

Solution

i. n(S) = 10

Number of black marbles = 7

Number of red marbles = 10 – 7 = 3

Let P(R_1) bethe probability of the first red marble and P(B_2) bethe probability of the second black marble;

P$(R_1 \cap B_2)$= P$(R_1) \times P(\,B_2)$

But P(R_1) = $\frac{3}{10}$ and P(B_2) = $\frac{6}{9} = \frac{2}{3}$

By substitution,

P$(R_1 \cap B_2)$ = P$(R_1) \times P(\,B_2)$

$= \frac{3}{10} \times \frac{7}{9} = \frac{21}{90} = \frac{7}{30}$

ii.Let P(B_1) be the probability of the first black marble and P(B_2) be the probability of the second black marble;

P$(B_1 \cap B_2)$= P$(B_1) \times P(\,B_2)$

But P(B_1) = $\frac{7}{10}$ and P(B_2) = $\frac{6}{9} = \frac{2}{3}$

By substitution,

P$(B_1 \cap B_2)$= P$(B_1) \times P(\,B_2)$

$= \frac{7}{10} \times \frac{2}{3} = \frac{14}{30} = \frac{7}{15}$

iii. Two marbles of the same color;

= P$(B_1 \cap B_2)$ or P$(R_1 \cap R_2)$

= [P$(B_1) \times P(\,B_2)$] + [P$(R_1) \times P(\,R_2)$]

$= \left(\frac{7}{10} \times \frac{2}{3}\right) + \left(\frac{3}{10} \times \frac{2}{9}\right)$

$= \frac{14}{30} + \frac{6}{90} = \frac{8}{15}$

Some Solved Past Questions

1. A bag contains 6 red, 8 black and 10 yellow identical beads. Two beads are picked at random, one after the other without replacement. Find the probability that:

a. both are red,

b. one is black and the other yellow.

Solution

a. n (R) = 6 , *n*(B) = 8 and *n*(Y) 10

n(S) = 6 + 8 + 10 = 24

P(R$_1$) = $\frac{6}{24}$

Without replacement,

n(R) = 5 and *n* (S) = 23

P(R$_2$) = $\frac{5}{23}$

P(R$_1$ and R$_2$) =P(R$_1$∩ R$_2$)

= P(R$_1$) × P(R$_2$)

$= \frac{6}{24} \times \mathbf{\frac{5}{23}} = \mathbf{\frac{30}{552}} = \mathbf{\frac{5}{92}}$

b. P(R$_1$ and Y$_2$) = P(R$_1$∩Y$_2$)

= P(R$_1$) × P(Y$_2$)

$= \frac{6}{24} \times \frac{10}{23} = \frac{60}{552} = \frac{5}{46}$

2. Three bags *P*, *Q* and *R* contains red, blue and white balls respectively of equal sizes. The raio of the ball in the bag are *P : Q* = 2 : 3, and *Q* : *R* = 4 : 5. All the balls are removed into a big bag and properly mixed together.

a. Find the probability of picking a:

i. red ball ii. blue ball iii. white ball

b. If two balls are picked at random one after the other with replacement, find the probability of picking:

i. a white ball and a blue ball

ii. blue ball first, then a red ball.

Solution

a. $\frac{P}{Q} = \frac{2}{3}$ and $\frac{Q}{R} = \frac{4}{5}$

$\frac{P}{Q} = \frac{2}{3} \times \frac{4}{4} = \frac{8}{12}$

$\frac{Q}{R} = \frac{4}{5} \times \frac{3}{3} = \frac{12}{15}$

⇒ P = 8, Q = 12 and R = 15

n(R) = 8, *n*(B) = 12 and *n*(*W*) = 15

n(S) = 8 + 12 + 15 = 35

i. P(*R*) = $\frac{8}{35}$, P (*B*) = $\frac{12}{35}$ and P (*W*) = $\frac{15}{35}$

b. i. P(white and blue) ;
= P(W and B)
$= \frac{15}{35} \times \frac{12}{35} = \frac{180}{1225} = \frac{36}{245}$

ii. P(Blue and Red) ;
P(B and R) = $\frac{12}{35} \times \frac{8}{35} = \frac{96}{1225}$

Exercises 20.12

1. A bag contains 3 red pens and 7 bluepens. What is the probability that the first pen drawn is red and the second pen is blue if selection is done without replacement?

2. A box contains 5 red, 4 white and 3 blue balls. If three balls are drawn from the box one after the other without replacing them, find the probability that the first ball is red, the second is white and the third is blue.

3. A man has 9 identical balls in a bag. Out of these, 3 are black, 2 are blue and the remaining is red.
i. If a ball is drawn at random, what is the probability that it is:
a. not blue,b. not red.
ii. If 2 balls are drawn at random one after the other, what is the probability that both of them will be:
a. black, if there is no replacement?
b. blue if there is replacement?

4. A bag of mixed toffees contains 20 cream, 15 milo and 10 nuts.
i. If I select one toffee at random, what is the probability that it is a nut?
ii. If the first toffee is not a nut and I eat it, what is the probability that the next one I pick
will be a nut?

5. There are 5 bulbs in a box, 2 of which are defective. Two bulbs are selected at random, one after the other without replacement. Find the probability that:
i. both bulbs are defective,
ii. the first is defective and the second is not.

6. A bag of sweets contains 7 toffees, 3 chocolate and 5 chewing gums, all wrapped identically. Sweets are drawn out one at a time and not replaced. Find the probability that the first:
i. drawn is a toffee,
ii. drawn is a toffee and the second a chocolate,
iii. and second drawn are both toffees.

7. Calculate the probability of an event of drawing two red balls from a set of 5 white and 5 red balls in a bag when the ball drawn first:
i. is replaced before the second draw,
ii. is not replaced before the second draw.

8. Calculate the probability of drawing three prime numbers in succession from a set of 10 cards numbered 1 – 10, where the drawn cards are not replaced before the next draw.

9. A bag contains three green, 2 white and 4 black balls. If I draw two balls in succession, without replacingthe first, what is the probability that;
I. I have a green and a black ball.
ii. I draw two balls of the same color.

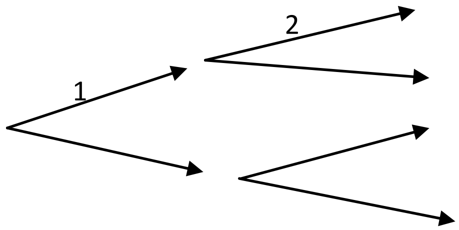
Probability and Tree Diagrams

A tree diagram called probability tree diagram, can be used to solve problems involving combined or compound events

Steps:

I. For each action in the problem, show the possible outcomes (or events) at the end of a branch of the tree

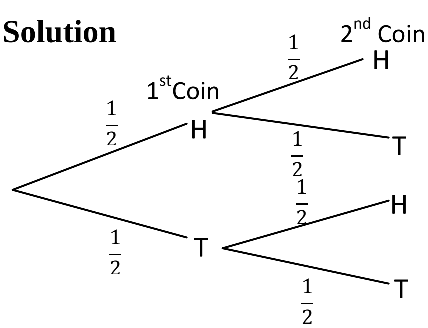
II. Each action in the problem is shown by a stage in the diagram. For example, for two actions, there will be two actions as shown below:



III. At the end of each route along the branches of the tree, the final outcome is written.
IV. Work out the probability of each outcome shown by a branch on the diagram. Write each probability on it branch of the tree, not forgetting the fact that probabilities on adjacent branches add up to 1.
V. Determine the probability of the final outcome by finding the route that leads to the final outcome. Multiply together the probabilities from any branch as you go along.

Worked Examples
1. A fair coin is tossed twice. List all the possible outcomes using a tree diagram.

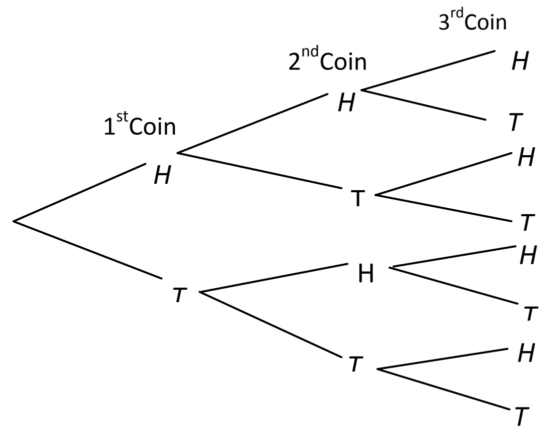
Solution



Possible outcomes = {HH, HT, TH, TT}

2. With the aid of a tree diagram, list all the possible outcomes when a coin is tossed three times

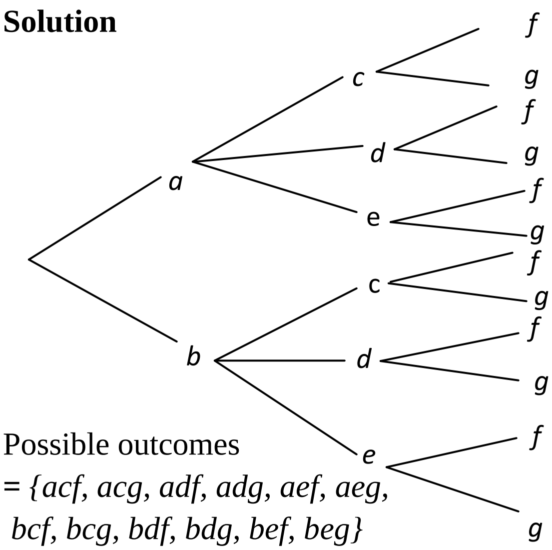
Solution



Possible outcomes = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

3. One element is drawn at random from each of the three sets, A = {*a*, *b*}, B = {*c*, *d*, *e*}, C = {*f*, *g*}. Use tree diagram to list all possible outcomes

Solution



Possible outcomes
= *{acf, acg, adf, adg, aef, aeg, bcf, bcg, bdf, bdg, bef, beg}*

4. There are 5 red pens and 2 red pens in a bag. A pen is taken at random from the bag with replacement. A second pen is then taken from the bag. What is the probability that:
i. both pens are red?
ii. both pens are of the same color?

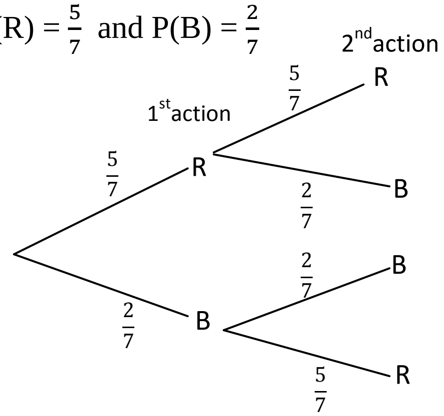
Solution

i. Number of red pens (R) = 5

Number of blue pens (B) = 2

Total number of pens = 5 + 2 = 7

P(R) = $\frac{5}{7}$ and P(B) = $\frac{2}{7}$



The probability that both pens are red;

P(R∩R) = P(R) × P(R)

$= \frac{5}{7} \times \frac{5}{7} = \frac{25}{49}$

ii. The probability that both pens are of the same color;

P(R∩R) or P(B∩B)

= P(R) × P(R) + P(B) × P(B)

$= \left(\frac{5}{7} \times \frac{5}{7}\right) + \left(\frac{2}{7} \times \frac{2}{7}\right)$

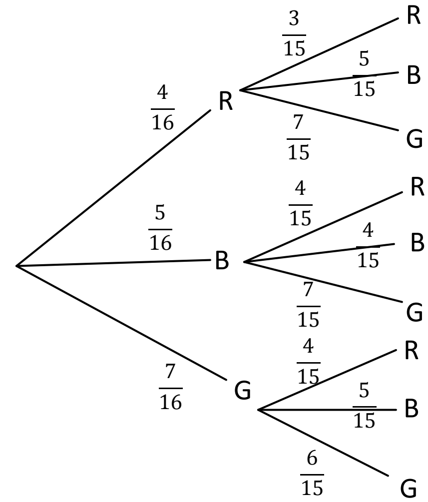
$= \frac{25}{49} + \frac{4}{49} = \frac{29}{49}$

5. A bag contains 4 red, 5 blue and 7 green marbles, which are all identical except for color. A marble is selected at random, the color is noted and it is not replaced in the bag. A second marble is selected at random from those remaining in the bag. Find the probability that the marbles selected are:

i. both red, ii. of the same color ?

Solution

i.



i. Probability that both marbles are red;

P(R∩R)

$= \frac{4}{16} \times \frac{3}{15} = \frac{12}{240} = \frac{1}{20}$

ii. Probability that both marbles are of the same color;

P(R∩R) or P(B∩B) or P(G∩G)

$= \left(\frac{4}{16} \times \frac{3}{15}\right) + \left(\frac{5}{16} \times \frac{4}{15}\right) + \left(\frac{7}{16} \times \frac{6}{15}\right)$

$= \frac{12}{240} + \frac{20}{240} + \frac{42}{240} = \frac{74}{240} = \frac{37}{120}$

Exercises 20.13

1. Three coins are tossed once. Use the tree diagram to list all the possible outcomes.

2. In two independent events, neither of the two has an influence on the other. A game is played by rolling a die and then tossing a coin. You win if the die shows 3 or a 4 and the coins shows heads.

i. Use tree diagram to find the sample space.

ii. What is your probability of winning?

3. Two balls are drawn without replacement from a bag containing 12 similar balls, 4 red, 2 yellow and 6 blue.
i. Represent the possible outcomes on a tree diagram.
ii. From the tree diagram, find the probability of selecting two blue balls.
iii. From the tree diagram, find the probability of selecting one red and one yellow ball.

4. There are 5 red pens and 3 blue pens in a box. A pen is chosen at random from the box. Use tree diagram to find the probability that:
i. both pens are blue,
ii. both pens are of the same color.

5. A coin is tossed three times in succession. Draw the probability tree diagram for the various outcomes and hence write down the probability of obtaining ;
i. two heads only,
ii. one head only,
iii. no heads.

6. A bag contains 5 white balls and 3 black balls. One ball is drawn from the bag and a second is then drawn. Find the probability of drawing one ball of each color.

Describing a Quadratic Equation

Any equation which can be put in the form $ax^2 + bx + c = 0$, where $a \neq 0$ is called a ***quadratic equation***. In $\boldsymbol{ax^2 + bx + c = 0}$,
a is called the co-efficient of x^2,
b is called the co-efficient of x and
c is called the constant term.

When $a = 1$, the quadratic equation becomes $x^2 + bx + c = 0$.
For all quadratic equations, the forms; $ax^2 + bx + c = 0$ and $y = ax^2 + bx + c$ are equivalent equations and as such $y = 0$

Factors of Quadratic Equations

Make use of the knowledge of factorization of quadratic expressions to find the factors of a quadratic equation. Make sure the quadratic equation is equated to zero to assume its standard form.
i.e. $x^2 + b\,x + c = 0$. By illustration;
x^2 - (sum of roots)x + (product of roots)= 0
Thus, the factors of $x^2 + b\,x + c = 0$ is found as follows;

$$\boldsymbol{x^2 + \underset{\substack{\downarrow \\ m+n}}{b}\,x + \underset{\substack{\downarrow \\ m\times n}}{c} = 0 \ldots \quad (1)}$$

By substitution,
$x^2 + mx + nx + mn = 0$
$(x^2 + mx) + (nx + mn) = 0$
$x(x + m) + n(x + m) = 0$
$(x + n)\ (x + m) = 0.$
Thus $(x + n)\ (x + m)$ are the factors of
$x^2 + b\,x + c = 0$

Worked Examples

Factorize the following equations;
1. $x^2 + 6x + 8 = 0$ 2. $x^2 - 13x + 36 = 0$
3. $x^2 - 2x - 15 = 0$ 4. $x^2 + 4x - 12 = 0$

Solutions

1.$x^2 + 6x + 8 = 0$
$(x^2 + 4x) + (2x + 8) = 0$
$x(x + 4) + 2(x + 4) = 0$
$(x + 2)\ (x + 4) = 0$

2.$x^2 - 13x + 36 = 0$
$(x^2 - 4x) - (9x + 36) = 0$
$x(x - 4) - 9(x - 4) = 0$
$(x - 4)\ (x - 9) = 0$

3. $x^2 - 2x - 15 = 0$
$(x^2 - 5x) + (3x + 15) = 0$
$x(x - 5) + 3(x - 5) = 0$
$(x + 3)\ (x - 5) = 0$

4. $x^2 - 4x - 12 = 0$
$(x^2 - 6x) + (2x - 12) = 0$
$x(x - 6) + 2(x - 6) = 0$
$(x + 2)\ (x - 6) = 0$

Exercises 21.1`
Find the factors of the following;
1. $x^2 + 11x + 18 = 0$ 2. $x^2 + 2x - 3 = 0$
3. $x^2 + x - 30 = 0$ 4. $y^2 + 2y - 8 = 0$
5. $3x^2 - 13x - 10 = 0$ 6. $7x^2 + 11x - 6 = 0$

Truth set of a Quadratic Equation

The truth set of a quadratic equation is the set of values that satisfy the equation or makes the equation true. It is also called the ***solution set or the roots*** of the equation.

To solve a quadratic equation is to find the values of x the make the equation true (or equal to zero). This is done by applying any of the following methods:

1. Factorization
2. Completing the square
3. Quadratic formula
4. The graphical method

Method of Factorization

To solve quadratic equations of the form:

$x^2 + bx + c = 0$,

I. Find two factors of the constant term, c, that sum up to the coefficient of x, as illustrated below

$$\boldsymbol{x^2 + \underset{\substack{\downarrow \\ m+n}}{b}x + \underset{\substack{\downarrow \\ m\times n}}{c} = 0 \ldots \ (1)}$$

II. Substitute $mx + nx = bx$ in eqn (1) to obtain four terms as:

$x^2 + mx + nx + mn = 0$

III. Group the terms and factorize completely as shown below;

$(x^2 + mx) + (nx + mn) = 0$

$x(x + m) + n(x + m) = 0$

$(x + n) \ (x + m) = 0.$

IV. At this point, the implication is that either $x + n = 0$ or $x + m = 0$

$x = -n$ or $x = -m$

Therefore, the truth set or solution set of the equation is $\{x : x = -n \text{ or } x = -m\}$

Worked Examples

1. Find the truth set of $x^2 - x - 6 = 0$

Solution

$x^2 - x - 6 = 0,$

$x^2 + 2x - 3x - 6 = 0$ *(By factorization)*

$(x^2 + 2x) - (3x - 6) = 0$

$x(x + 2) - 3(x + 2) = 0$

$(x + 2)(x - 3) = 0$

$x + 2 = 0$ or $x - 3 = 0$

$x = -2$ or $x = 3$

Truth set = $\{x : x = -2 \text{ or } x = 3\}$

2. Solve $x^2 - 13x + 36 = 0$

Solution

$x^2 - 9x - 4x + 36 = 0$

$(x^2 - 9x) - (4x + 36) = 0$

$x(x - 9) - 4(x - 9) = 0$

$(x - 4)(x - 9) = 0$

$(x - 4) = 0$ or $(x - 9) = 0$

$x = 4$ or $x = 9$

The truth set = $\{x : x = 4 \text{ or } x = 9\}$

3. Solve $2x^2 + 5x - 3 = 0$

Solution

$2x^2 + 5x - 3 = 0$

$2x^2 - x + 6x - 3 = 0$

$(2x^2 - x) + (6x - 3) = 0$

$x(2x - 1) + 3(2x - 1) = 0$

$(2x - 1) = 0$ or $x = -3$

$2x = 1$ or $x = -3$

$x = \frac{1}{2}$ or $x = -3$

The truth set = $\left\{x : x = \frac{1}{2} \text{ or } x = -3\right\}$

Solving Related Problems

Worked Examples

1. Write the equation $x - 11 + \frac{24}{x} = 0$ in the form $ax^2 + bx + c = 0$, and hence find the truth set

Solution

$x - 11 + \frac{24}{x} = 0$

Multiply through by x

$x^2 - 11x + 24 = 0$

$(x - 3)(x - 8) = 0$

$x - 3 = 0$ or $x - 8 = 0$

$x = 3$ or $x = 8$

2. If (p, q) is the truth set of the equation $x^2 + 10x = 96$, evaluate $(p + q)$

Solution

$x^2 + 10x = 96$
$x^2 + 10x - 96 = 0$
$x^2 + 16x - 6x - 96 = 0$
$(x^2 + 16x) - (6x - 96) = 0$
$x(x + 16) - 6(x + 16) = 0$
$(x - 6)(x + 16) = 0$
$x - 6 = 0$ or $x + 16 = 0$
$x = 6$ or $x = -16$
Truth set $(6, -16) = (p, q)$
$\Rightarrow (p + q) = 6 + (-16) = -10$

Exercises 21.2

A. Solve the following equations;

1. $x^2 - 3x - 4 = -6$
2. $x^2 - 6x - 16 = 0$
3. $x^2 + 13x + 25 = -5$
4. $14 - 5x - x^2 = 0$
5. $x^2 + x - 12 = 0$
6. $x^2 + 16x + 63 = 0$
7. $24 + 11x + x^2 = 0$
8. $44 + 15x + x^2 = 0$

B. Solve the following equations;

1. $x^2 + 7x - 5x - 35 = 0$
2. $2x^2 - 13x + 20 = 5$
3. $17 - 15x - 2x^2 = 0$
4. $3x^2 + 2x - 8 = 0$

C. Use factors to solve these equations

1. $x^2 - 6x + 5 = 0$
2. $x^2 + 9x + 20 = 0$
3. $4x^2 + 4x + 1 = 0$
3. $x(x - 8) = 33$

Method of Completing Squares

The method of completing the squares depend on the identities:

1. $(a + b)^2 = a^2 + 2ab + b^2$
2. $(a - b)^2 = a^2 - 2ab + b^2$

Worked Examples

1. Complete the squares in the following;

$(\square x + \square)^2 = 25x^2 + 70x + \square$

Solution

Comparing the statements

$(\square x + \square)^2 = 25x^2 + 70x + \square$ and
$(a + b)^2 = a^2 + 2ab + b^2$, it is seen that:
i. $a^2 = 25x^2$
$a = \sqrt{25x^2} = 5x$

ii. $2ab = 70x$
But $a = 5x$
$2(5x)b = 70x$
$10bx = 70x$
$b = 7$ so $b^2 = 7^2 = 49$
Therefore the complete statement is
$(5x + 7)^2 = 25x^2 + 70x + 49$

2. Copy and complete the statement
$(\frac{1}{2}x - \square)^2 = x^2 - x + \square$

Solution

Comparing the statements:

$(\frac{1}{2}x - \square)^2 = \square x^2 - x + \square$ and
$(a - b)^2 = a^2 - 2ab + b^2$ it is seen that
i. $(\frac{1}{2}x - \square)^2 = (a - b)^2$
Therefore $a = \frac{1}{2}x$ and $a^2 = \frac{1}{4}x^2$

ii $-2ab = -x$, but $a = \frac{1}{2}x$
$-2(\frac{1}{2})xb = -x$
$-b = -1$
$b = 1$, so $b^2 = 1$
Therefore the complete statement is:
$(\frac{1}{2}x - 1)^2 = \frac{1}{4}x^2 - x + 1$

3. If $m, n \in z$, find m and n such that
$x^2 + 12x + m = (x + n)^2$

Solution

Comparing the statements:

$x^2 + 12x + m = (x + n)^2$ and

$a^2 + 2ab + b^2 = (a + b)^2$

i.$x^2 + 12x + m = a^2 + 2ab + b^2$

Therefore $a = x$

ii. $2ab = 12x$, but $a = x$

$2bx = 12x$

$b = \frac{12}{2} = 6$

But $b = n$ and $b^2 = m$

$6 = n$ and $6^2 = m$

By substitution,

$x^2 + 12x + m = (x + n)^2$

$x^2 + 12x + 36 = (x + 6)^2$

Therefore $m = 36$ and $n = 6$

Exercises 21.3

1. The truth set of the equation $ax^2 + bx = 4$ is {-3, 2}. Find the numerical values of the constants a and b

Quadratic Equations of the Form:

$x^2 + bx + c = 0$

To complete the squares of quadratic equations of the form**:** $x^2 + bx + c = 0$

I. Transpose the constant, c,to the right side of the equation to assume the opposite sign i.e. $x^2 + bx = -c$

II. Divide the coefficient of x (number attached to x) by 2. i.e.$\frac{b}{2}$

III. Square $\frac{b}{2}$ *to get* $\left(\frac{b}{2}\right)^2$ *and add* $\left(\frac{b}{2}\right)^2$ *to both sides of the equation*

i.e. $x^2 + bx + \left(\frac{b}{2}\right)^2 = -c + \left(\frac{b}{2}\right)^2$

IV. Complete the squares at the left side of the equation to get

$\left(x + \frac{b}{2}\right)^2 = - c + \left(\frac{b}{2}\right)^2$

V. Introduce a square root sign on both sides of the equation

i.e$\left(x + \frac{b}{2}\right) = \pm\sqrt{-c + \left(\frac{b}{2}\right)^2}$

6. *Make x the subject of the equation*

$x = \frac{b}{2} \pm \sqrt{-c + \left(\frac{b}{2}\right)^2}$

Find the value or values of x that satisfies the equation

Worked Examples

1. Find the truth set of $x^2 + 8x + 15 = 0$

Solution

$x^2 + 8x + 15 = 0$

$x^2 + 8x = -15$

$x^2 + 8x + \left(\frac{8}{2}\right)^2 = -15 + \left(\frac{8}{2}\right)^2$

$x^2 + 8x + 4^2 = -15 + 4^2$

$(x + 4)^2 = -15 + 16$

$(x + 4)^2 = 1$

$x + 4 = \pm\sqrt{1}$

$x = -4 + \sqrt{1}$ or $x = -4 - \sqrt{1}$

$x = -4 + 1$ or $x = -4 - 1$

$x = -3$ or $x = -5$

Truth set = $\{x : x = -3 \text{ or } x = -5\}$

2. Solve $x^2 + 3x - 28 = 0$

Solution

$x^2 + 3x - 28 = 0$

$x^2 + 3x = 28$

$x^2 + 3x + \left(\frac{3}{2}\right)^2 = 28 + \left(\frac{3}{2}\right)^2$

$\left(x + \frac{3}{2}\right)^2 = 28 + \frac{9}{4}$

$\left(x + \frac{3}{2}\right)^2 = \frac{121}{4}$

$x + \frac{3}{2} = \pm\sqrt{\frac{121}{4}}$

$x + \frac{3}{2} = \pm \frac{11}{2}$

$x = -\frac{3}{2} + \frac{11}{2}$ or $x = -\frac{3}{2} - \frac{11}{2}$

$x = \frac{8}{2}$ or $x = -\frac{14}{2}$

$x = 4$ or $x = -7$

Truth set = $\{x : x = -4 \text{ or } x = -7\}$

3. Find the truth set of $x^2 - 5x - 14 = 0$

Solution

$x^2 - 5x - 14 = 0$

$x^2 - 5x = 14$

$x^2 - 5x + \left(\frac{-5}{2}\right)^2 = 14 + \left(\frac{-5}{2}\right)^2$

$\left(x - \frac{5}{2}\right)^2 = 14 + \frac{25}{4}$

$\left(x - \frac{5}{2}\right)^2 = \frac{81}{4}$

$x - \frac{5}{2} = \pm\sqrt{\frac{81}{4}}$

$x - \frac{5}{2} = \pm\frac{9}{2}$

$x = \frac{5}{2} \pm \frac{9}{2}$

$x = \frac{5}{2} + \frac{9}{2}$ or $\frac{5}{2} - \frac{9}{2}$

$x = \frac{14}{2}$ or $x = -\frac{4}{2}$

$x = 7$ or $x = -2$

Truth set = $\{x : x = 7 \text{ or } x = -2\}$

Quadratic Equations of the Form:

$ax^2 + bx + c = 0, a > 1$

To complete the squares of quadratic equations of the form**:** $ax^2 + bx + c = 0$,

I. Transpose the constant, c, to the right side of the equation to assume the opposite sign i.e. $ax^2 + bx = -c$

II. Divide through the equation by a, which is the coefficient of x^2, i.e. $\frac{ax^2}{a} + \frac{b}{a}x = -\frac{c}{a}$

$= x^2 + \frac{b}{a}x = -\frac{c}{a}$

III. Find $\frac{1}{2}$ of $\frac{b}{a}$ i.e. $\frac{1}{2} \times \frac{b}{a} = \frac{b}{2a}$

IV. Square $\frac{b}{2a}$ to get $\left(\frac{b}{2a}\right)^2$ and add $\left(\frac{b}{2a}\right)^2$ to both sides of the equation

i.e. $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -c + \left(\frac{b}{2a}\right)^2$

V. Complete the squares at the left side of the equation to get

$\left(x + \frac{b}{2a}\right)^2 = -c + \left(\frac{b}{2a}\right)^2$

VI.Introduce the square root sign on both sides of the equation

i.e. $\left(x + \frac{b}{2a}\right) = \pm\sqrt{-\text{c} + \left(\frac{b}{2a}\right)^2}$,

VII. Make x the subject to get the value or values of x that satisfies the equation;

$x = -\frac{b}{2a} \pm \sqrt{-\text{c} + \left(\frac{b}{2a}\right)^2}$

Worked Examples

1. Find the truth set of $3x^2 + 8x + 5 = 0$

Solution

$3x^2 + 8x + 5 = 0$

$3x^2 + 8x = -5$

$x^2 + \frac{8}{3}x = -\frac{5}{3}$

$x^2 + \frac{8}{3}x + \left(\frac{8}{2\times3}\right)^2 = -\frac{5}{3} + \left(\frac{8}{2\times3}\right)^2$

$x^2 + \frac{8}{3}x + \left(\frac{8}{6}\right)^2 = -\frac{5}{3} + \left(\frac{8}{6}\right)^2$

$x^2 + \frac{8}{3}x + \left(\frac{4}{3}\right)^2 = -\frac{5}{3} + \left(\frac{4}{3}\right)^2$

$\left(x + \frac{4}{3}\right)^2 = -\frac{5}{3} + \frac{16}{9}$

$\left(x + \frac{4}{3}\right)^2 = \frac{1}{9}$

$x + \frac{4}{3} = \pm\sqrt{\frac{1}{9}}$

$x + \frac{4}{3} = \pm\frac{1}{3}$

$x = -\frac{4}{3} \pm \frac{1}{3}$

$x = -\frac{4}{3} + \frac{1}{3}$ or $x = -\frac{4}{3} - \frac{1}{3}$

$x = \frac{-4+1}{3}$ or $x = \frac{-4-1}{3}$

$x = -\frac{3}{3}$ or $x = -\frac{5}{3}$

$x = -1$ or $x = -\frac{5}{3}$

Truth set = $\left\{x : x = -1 \text{ or } x = -\frac{5}{3}\right\}$

2. Find the truth set of $3x^2 + 2x - 5 = 0$

Solution

$3x^2 + 2x - 5 = 0$

$3x^2 + 2x = 5$

$x^2 + \frac{2}{3}x = \frac{5}{3}$

$x^2 + \frac{2}{3}x + \left(\frac{2}{2\times3}\right)^2 = \frac{5}{3} + \left(\frac{2}{2\times3}\right)^2$

$x^2 + \frac{2}{3}x + \left(\frac{2}{6}\right)^2 = \frac{5}{3} + \left(\frac{2}{6}\right)^2$

$x^2 + \frac{2}{3}x + \left(\frac{1}{3}\right)^2 = \frac{5}{3} + \left(\frac{1}{3}\right)^2$

$\left(x + \frac{1}{3}\right)^2 = \frac{5}{3} + \frac{1}{9}$

$\left(x + \frac{1}{3}\right)^2 = \frac{16}{9}$

$x + \frac{1}{3} = \pm\sqrt{\frac{16}{9}}$

$x + \frac{1}{3} = \pm\frac{4}{3}$

$x = -\frac{1}{3} \pm \frac{4}{3}$

$x = -\frac{1}{3} + \frac{4}{3}$ or $x = -\frac{1}{3} - \frac{4}{3}$

$x = \frac{-1+4}{3}$ or $x = \frac{-1-4}{3}$

$x = \frac{3}{3}$ or $x = -\frac{5}{3}$

$x = 1$ or $x = -\frac{5}{3}$

Truth set = $\left\{x : x = 1 \; or \; x = -\frac{5}{3}\right\}$

3. Find the truth set of $3x^2 - 2x - 5 = 0$

Solution

$3x^2 - 2x - 5 = 0$

$3x^2 - 2x = 5$

$x^2 - \frac{2}{3}x = \frac{5}{3}$

$x^2 - \frac{2}{3}x + \left(-\frac{2}{2\times3}\right)^2 = \frac{5}{3} + \left(-\frac{2}{2\times3}\right)^2$

$x^2 - \frac{2}{3}x + \left(-\frac{2}{6}\right)^2 = \frac{5}{3} + \left(-\frac{2}{6}\right)^2$

$x^2 - \frac{2}{3}x + \left(-\frac{1}{3}\right)^2 = \frac{5}{3} + \left(-\frac{1}{3}\right)^2$

$\left(x - \frac{1}{3}\right)^2 = \frac{5}{3} + \frac{1}{9}$

$\left(x - \frac{1}{3}\right)^2 = \frac{16}{9}$

$x - \frac{1}{3} = \pm\sqrt{\frac{16}{9}}$

$x - \frac{1}{3} = \pm\frac{4}{3}$

$x = \frac{1}{3} \pm \frac{4}{3}$

$x = \frac{1}{3} + \frac{4}{3}$ or $x = \frac{1}{3} - \frac{4}{3}$

$x = \frac{1+4}{3}$ or $x = \frac{1-4}{3}$

$x = \frac{5}{3}$ or $x = -\frac{3}{3}$

$x = \frac{5}{3}$ or $x = -1$

Truth set = $\left\{x : x = \frac{5}{3} \text{ or } x = -1\right\}$

Exercises 21.4

A. Complete the squares in the following;

1. $(x + 3)^2 = x^2 + \square\, x + 9$
2. $(x - 5)^2 = x^2 - \square x + \square$
3. $(x - \square)^2 = x^2 - 14x + \square$
4. $(x - \square)^2 = x^2 - 10x + \square$
5. $(3x - 2)^2 = 9x^2 - \square x + 4$
6. $(x + \frac{1}{2})^2 = x^2 - \square x + \square$

B. Solve by completing the square;

1. $x^2 + 7x - 3 = 0$
2. $10 + 3x - 2x^2 = 0$
3. $4x^2 - 6x - 1 = 0$
4. $x^2 + 9x + 20 = 0$
5. $x^2 + 4x - 21 = 0$
6. $3x^2 + 12x + 6 = 0$

The Quadratic Formula

The quadratic formula is derived from the quadratic equation $\boldsymbol{ax^2 + bx + c = 0}$ as follows:

In $\boldsymbol{ax^2 + bx + c = 0}$

I. Subtract $\boldsymbol{c}$ *from both sides*

$ax^2 + bx = -c$

II. Divide through by $\boldsymbol{a}$

$x^2 + \left(\frac{b}{a}\right)x = -\left(\frac{c}{d}\right)$

III. Add $\left(\frac{b}{2a}\right)^2$ *to both sides of the equation*

$x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \left(\frac{c}{d}\right)$

$\boldsymbol{x^2} + \left(\frac{\boldsymbol{b}}{\boldsymbol{a}}\right)x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{\boldsymbol{c}}{\boldsymbol{d}}$

IV. Factorize the left – hand side

$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

Take the square root of both sides

$x + \frac{b}{2a} = \pm\sqrt{\left(\frac{b^2 - 4ac}{4a^2}\right)}$

$x + \frac{b}{2a} = \frac{\pm\sqrt{(b^2 - 4ac)}}{2a}$

Subtract $\frac{\boldsymbol{b}}{\boldsymbol{2a}}$ *from both sides*

$x = -\frac{b}{2a} \pm \frac{\sqrt{(b^2 - 4ac)}}{2a}$

So if $\boldsymbol{ax^2 + bx + c = 0}$, then

$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

This formula is used to solve quadratic equations which cannot readily be solved by factorization. In other words, to solve the quadratic equation $ax^2 + bx + c = 0$

I. Attempt to factorize the quadratic expression on the left - hand side

II. If factorization is not possible, use the formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Worked Examples

1. Find the truth set of $3x^2 + 8x + 5 = 0$

Solution

$3x^2 + 8x + 5 = 0$ and $ax^2 + bx + c = 0$ compared, $a = 3$, $b = 8$ and $c = 5$

Substitute in $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-8 \pm \sqrt{8^2 - 4(3)(5)}}{2(3)}$

$x = \frac{-8 \pm \sqrt{64 - 60}}{6}$

$x = \frac{-8 \pm \sqrt{4}}{6}$

$x = \frac{-8 \pm 2}{6}$

$x = \frac{-8 + 2}{6}$ or $x = \frac{-8 - 2}{6}$

$x = \frac{-6}{6}$ or $x = \frac{-10}{6}$

$x = -1$ or $x = -\frac{5}{3}$

Truth set = $\left\{x : x = -1 \text{ or } x = -\frac{5}{3}\right\}$

2. Find the truth set of $3x^2 + 2x - 5 = 0$

Solution

$3x^2 + 2x - 5 = 0$ and $ax^2 + bx + c = 0$ *compared,* $a = 3, b = 2$ *and* $c = -5$

Substitute in $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-2 \pm \sqrt{2^2 - 4(3)(-5)}}{2(3)}$

$x = \frac{-2 \pm \sqrt{4 + 60}}{6}$

$x = \frac{-2 \pm \sqrt{64}}{6}$

$x = \frac{-2 \pm 8}{6}$

$x = \frac{-2 + 8}{6}$ or $x = \frac{-2 - 8}{6}$

$x = \frac{6}{6}$ or $x = \frac{-10}{6}$

$x = 1$ or $x = -\frac{5}{3}$

Truth set = $\left\{x : x = 1 \text{ or } x = -\frac{5}{3}\right\}$

3. Find the truth set of $3x^2 - 2x - 5 = 0$

Solution

$3x^2 - 2x - 5 = 0$ and $ax^2 + bx + c = 0$

compared, $a = 3$, $b = -2$ *and* $c = -5$

Substitute in $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = \frac{-(-2) \pm \sqrt{(-2)^2-4(3)(-5)}}{2(3)}$

$x = \frac{2 \pm \sqrt{4+60}}{6}$

$x = \frac{2 \pm \sqrt{64}}{6}$

$x = \frac{2 \pm 8}{6}$

$x = \frac{2+8}{6}$ or $x = \frac{2-8}{6}$

$x = \frac{5}{3}$ or $x = -1$

Truth set = $\left\{x : x = \frac{5}{3} \text{ or } x = -1\right\}$

3. Find the roots of $\frac{x}{x+1} - \frac{x+1}{x} = 1$ to one decimal place

Solution

$\frac{x}{x+1} - \frac{x+1}{x} = 1$

$\frac{x^2 - (x+1)(x+1)}{(x+1)(x)} = 1$

$\frac{x^2 - x^2 - 2x - 1}{x^2 + x} = 1$

$\frac{-2x - 1}{x^2 + x} = 1$

$-2x - 1 = x^2 + x$

$x^2 + x + 2x + 1 = 0$

$x^2 + 3x + 1 = 0$

Let $a = 1$, $b = 3$ and $c = 1$ and substitute in the quadratic formula: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(1)}}{2(1)}$

$x = \frac{-3 \pm \sqrt{5}}{2}$

$x = \frac{-3+\sqrt{5}}{2}$ or $x = \frac{-3-\sqrt{5}}{2}$

$x = -0.4$ or $x = -2.6$

The roots are $x = -0.4$ or $x = -2.6$

Exercises 21.5

A. Solve by using the formula:

1. $3x^2 - 7x - 1 = 0$
2. $4 - 2x - 2x^2 = 0$
3. $x^2 - 6x + 5 = 0$
4. $5x^2 - 3x - 7 = 0$
5. $3x^2 = 7x + 2$
6. $x^2 - 3x = 0$

B. Solve by any suitable method;

1. $2x^2 + 5x - 12 = 0$
2. $6x^2 - x = 2$
3. $3x^2 + 14x + 8 = 0$
4. $8x^2 = 9 - 6x$

C. Find the truth set of the equations:

1. $\frac{x-7}{x+3} = \frac{x+2}{x-1}$
2. $\frac{x}{3} = \frac{x+7}{x-1}$
3. $\frac{x}{5} = \frac{2x-9}{x-4}$
4. $\frac{5}{x-4} = \frac{x}{2x-9}$
5. $x - 1 = 1 + \frac{8}{x}$
6. $\frac{1}{2x} + \frac{9}{x+7} = 1$

Challenge Problems

Solve the equations, where $x \in$ R

1. $(x+5)^2 = 81$
2. $\left(x + \frac{1}{4}\right)^2 = 9$
3. $(2x-1)^2 = \frac{1}{4}$
4. $(x-4)^5 = 64$

Factors of Equations of the Form:

$a^2 - b^2 = 0$

To find the factors of equations of the form; $a^2 - b^2 = 0$, use the method of difference of two squares as shown below;

$a^2 - b^2 = (a+b)(a-b) = 0$. This is the factors of $a^2 - b^2 = 0$

Worked Example

Factorize $x^2 - 9 = 0$

Solution

$x^2 - 9 = 0$

$x^2 - 3^2 = 0$

$(x-3)(x+3) = 0$

Solving Equations of the Form: $a^2 - b^2 = 0$

To solve equation of the form $a^2 - b^2 = 0$ is to find the values of ***a*** and ***b*** that make the statement true. This is easily done by using the method of difference of two squares which is illustrated by the statement:

$\boldsymbol{a^2 - b^2 = (a + b)(a - b) = 0}$

$\boldsymbol{= (a + b) = 0 \text{ or } (a - b) = 0}$

$\boldsymbol{\Rightarrow a = - b \text{ or } a = b.}$

The truth set is $a = - b$ or $a = b$.

Worked Examples

1. Solve $x^2 - 9 = 0$

Solution

$x^2 - 9 = 0$

$x^2 - 3^2 = 0$

$x^2 - 3^2 = (x + 3)(x - 3) = 0$

$(x + 3) = 0$ or $(x - 3) = 0$

$x = -3$ or $x = 3$

2. Solve $4x^2 - 25 = 0$

Solution

$4x^2 - 25 = 0$

$= (2x)^2 - (5)^2$

$= (2x + 5)(2x - 5) = 0$

$= (2x + 5) = 0$ or $(2x - 5) = 0$

$2x = -5$ or $2x = 5$

$x = \frac{-5}{2}$ or $x = \frac{5}{2}$

Truth set is $x = \frac{-5}{2}$ or $x = \frac{5}{2}$

3) Solve $2x^2 = 3$

Solution

$2x^2 = 3$

$2x^2 - 3 = 0$

$(\sqrt{2}x)^2 - (\sqrt{3})^2 = 0$

$(\sqrt{2}x - \sqrt{3})(\sqrt{2}x + \sqrt{3}) = 0$

$(\sqrt{2}x - \sqrt{3}) = 0$ or $\sqrt{2}x + \sqrt{3} = 0$

$\sqrt{2}x = \sqrt{3}$ or $\sqrt{2}x = - \sqrt{3}$

$x = \frac{\sqrt{3}}{\sqrt{2}}$ or $x = \frac{-\sqrt{3}}{\sqrt{2}}$

$x = \frac{-\sqrt{3}}{\sqrt{2}}$ or $x = \frac{\sqrt{3}}{\sqrt{2}}$

4. If $a^2 - b^2 = (a + b)(a - b)$, evaluate $9.32^2 - 0.68^2$

Solution

If $a^2 - b^2 = (a + b)(a - b)$, then

$(9.32)^2 - (0.68)^2$

$= (9.32 + 0.68)(9.32 - 0.68)$

$= (10)(8.64)$

$= 86.4$

Exercises 21.6

A. Factorize the following;

1. $x^2 - 16 = 0$	2. $9x^2 = 144$
3. $x^2 = 81$	4. $9x^2 - 25 = 0$
5. $64x^2 - 169 = 0$	6. $4x^2 - 25 = 0$

B. Find the truth set :

1. $x^2 - 16 = 0$	2. $9x^2 = 144$
3. $x^2 = 81$	4. $9x^2 - 25 = 0$
5. $64x^2 - 169 = 0$	6. $4x^2 - 25 = 0$

Challenge problems

1. Solve $2x^2 - 1 = 0$

2. Find the truth set of the equation;
$16(x + 1)^2 - 2\sqrt{5} = 0$

3. Given $x^2 - y^2 = 77$ and $x + y = 11$, find the values of: i. $x - y$ (Ans7) ii. x and y (Ans9, 2)

4. Factorize the right – hand side of the formula $A = \pi(x^2 - y^2)$. Then calculate A when 3.14 is used as an approximation for π and

i. $x = 5.2, y = 4.8$ ii. $x = 65, y = 35$

Word Problems

In solving word problems involving quadratic equations, write the mathematical equation for the problem and solve it, taking note of the fact that the problem does not end with solving the quadratic equation. You should therefore go back to the word problem and answer the question it demands.

Worked Examples

1. The sum of two numbers is 18. The sum of the squares of the numbers is 194. Find the numbers.

Solution

Let x be the number

Then the other number is $(18 - x)$.

Sum of squares = $x^2 + (18 - x)^2$.

But this is given as 194

$x^2 + (18 - x)^2 = 194$

$x^2 + x^2 - 36x + 324 = 194$

$2x^2 - 36x + 130 = 0$

$x^2 - 18x + 65 = 0$

$x^2 - 5x - 13x + 5 = 0$

$(x^2 - 5x) - (13x + 65) = 0$

$x(x - 5) - 13(x - 5) = 0$

$(x - 13)(x - 5) = 0$

$x - 13 = 0$ or $x - 5 = 0$

$x = 13$ or $x = 5$

Therefore, the numbers are 5 and 13.

2. A certain rectangle has perimeter of 48cm and area of 128cm^2. Find the length and breadth of the rectangle.

Solution

Let the length be L cm and breadth be B

P = 2(L + B) ‘

But P = 48

$\Rightarrow 2(L + B) = 48$

$L + B = \frac{48}{2}$

L + B = 24cm

$\Rightarrow$B = 24 – L……….(1)

But area of rectangle = L × B = 128cm^2

L × B = 128cm^2…….(2)

Put eqn (1) into eqn (2)

$\Rightarrow L(24 - L) = 128\text{cm}^2$

$24L - L^2 = 128$

$L^2 - 24L + 128 = 0$

$L^2 - 8L - 16L + 128 = 0$

$(L^2 - 8L) - (16L + 128) = 0$

$L(L - 8) - 16(L - 8) = 0$

$(L - 16)(L - 8) = 0$

$L - 16 = 0$ or $L - 8 = 0$

$L = 16$ or $L = 8$

When L =16, B = 24 – 16 = 8

When L = 8, B = 24 – 8 = 16

Therefore, the length and breadth of the rectangle are 16cm and 8cm respectively.

3. The breadth of a rectangle is 1cm less than its length. If the area is 42cm^2, find the breadth.

Solution

Let the breadth be $x$cm. It means thatthe length is $(x + 1)$cm

But area = length × Breadth

$42\text{cm}^2 = x(x + 1)$

$42\text{cm}^2 = x(x + 1)$

$41 = x^2 + x$

$x^2 + x - 42 = 0$

$x^2 + 7x - 6x - 42 = 0$

$(x^2 + 7x) - (6x - 42) = 0$

$x^2(x + 7) - 6(x + 7) = 0$

$(x - 6)(x + 7) = 0$

$x - 6 = 0$ or $x + 7 = 0$
$x = 6$ or $x = -7$

Since the breadth of a rectangle cannot be negative, the breadth of the rectangle is 6cm.

4. The present ages of a man and his son are 44 and 13 years. How many years ago was the product of their ages 140?

Solution
Let the number of years ago be x
$(44 - x)(13 - x) = 140$
$44(13 - x) - x(13 - x) = 140$
$572 - 44x - 13x + x^2 = 140$
$x^2 - 57x + 572 = 140$
$x^2 - 57x + 572 - 140 = 0$
$x^2 - 57x + 432 = 0$
$a = 1$, $b = -57$ and $c = 432$
Substitute in $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Substitute in $x = \frac{-(-57) \pm \sqrt{(-57)^2 - 4(1)(432)}}{2(1)}$
$x = \frac{57 \pm \sqrt{3249 - 1728}}{2}$
$x = \frac{57 \pm \sqrt{3249 - 1728}}{2}$
$x = \frac{57 \pm \sqrt{1521}}{2}$
$x = \frac{57 \pm 39}{2}$
$x = \frac{57 + 39}{2}$ or $x = \frac{57 - 39}{2}$
$x = 48$ or $x = 9$
Since the ages must be positive number of years, $x = 9$ years ago.

5. It is known that a number y is either 0, 1 or 4. Form an open sentence or equation in which y is the variable.

Solution
$y = 0$ or $y = 1$ or $y = 4$
$y = 0$, $y - 1 = 0$ or $y - 4 = 0$
$\Leftrightarrow y(y - 1)(y - 4) = 0$
$\Leftrightarrow y(y^2 - 5y + 4) = 0$
$\Leftrightarrow y^3 - 5y^2 + 4z = 0$

6. A man cycles 20km from C to B. If he increases his speed by 3km/h, he saves 30 minutes on the journey. Find his original speed in km/h.

Solution
Let x km/h be the original speed
The time taken for the journey at this speed is $\frac{20}{x}$ hours
Time taken for the journey at this increased speed would be $= \left(\frac{20}{x+3}\right)$ hours
Thus we have $\frac{20}{x} - \frac{20}{x+3} = \frac{1}{2}$
$20(x + 3) - 20(x) = \frac{1}{2}x(x + 3)$ *Multiply through by x(x + 3)*
$20x + 60 - 20x = \frac{x^2 + 3x}{2}$
$x^2 + 3x - 120 = 0$
$\Rightarrow a = 1$, $b = 3$ and $c = -120$
Substitute in $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-120)}}{2}$
$x = \frac{-3 \pm \sqrt{4 + 60}}{6}$
$x = \frac{-3 \pm \sqrt{489}}{2}$
$x = \frac{-3}{2} \pm \frac{22.11}{2}$
$x = -1.5 \pm 11.06$
$x = -1.5 + 11.06$ or $x = -1.5 - 11.06$
$x = 9.56$ or $x = -12.56$
Therefore the original speed is 9.56km/h

Exercises 21.7
1. The sum of a number and its square is 6. Find the number.
2. The square of a number is 17 times that number, what is the number?

3. The area of a rectangle is $84cm^2$. If the length is 5cm greater than the width, find the length of the rectangle.

4. A farmer encloses a rectangular piece of land of area $2,500cm^2$ with fencing of total length 250m. Find the breadth of the rectangular land.

5. Asaase is 7 years younger than Atikopo. If the product of their ages is 78, find Asaase's age.

6. 18 added to the square of a number is equal to 11 eleven timesthe number. Find the number.

7. My brother's age is 7 less than mine. If I square his age and subtract one, I get the same results as when I multiply my age by 5. How old am I?

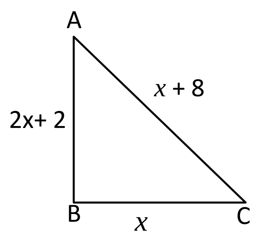
8. Squaring a number gives the same result as multiplying it by 10 and adding 24 to the result. Find the possible values for the number. (Ans 12, -2)

Challenge Problems

1. A number on base ten has two digits, the second one being two more than the first. The number itself is 22 more than the product of it's digits. Find the number.

2. Due to a fault, the speed of a train was reduced by 10km/h over a journey of 100km. The journey took 20 minutes longer. Find the usual speed of the train.

3. What value must be chosenfor x so that < B is a rightangle?



Word Problems Type 2

1. A man bought some shirts for Gh¢720.00. If each shirt was Gh¢2.00 cheaper, he would have received four more shirts. Calcualte the number of shirts bought.

Solution

Let x be the cost of a shirt and y be the number of shirts bought.

$xy = 720$…………………..(1)

If each shirt was Gh¢2.00 cheaper he would have received four more shirts.

$(x - 2)\,(y + 4) = 720$……..(2)

From eqn (1);

$x = \frac{720}{y}$

Put $x = \frac{720}{y}$ in eqn (2)

$\left(\frac{720}{y} - 2\right)(y + 4) = 720$

$\frac{720y}{y} + \frac{4(720)}{y} - 2y - 8 = 720$

$720 + \frac{2880}{y} - 2y - 8 = 720$

$\frac{2880}{y} - 2y - 8 = 720 - 720$

$\frac{2880}{y} - 2y - 8 = 0$ (multiply through by y)

$2,880 - 2y^2 - 8y = 0$

$2y^2 + 8y - 2880 = 0$

$(y - 36)\,(y + 40) = 0$

$y = 36$ or $y = -40$ (ignore $y = -40$)

The number of shirts bought is 36

2. Each term a house master was given Gh¢21.00 to share equally among his students as pocket money. One term, an extra five students were assigned to his house, but the total pockets money was unchanged. As a result, each student recieved 10p less that term. Find the original number of students in his house.

Solution

Let x be the number of students and y be the amount received by each student.

$xy = 2100$

$y = \frac{2100}{x}$…………………..(1)

Extra 5 student assigned to the house and each student received 10p less that term

$y - 10 = \frac{2100}{x+5}$……………(2)

Put eqn (1) in eqn (2);

$\left(\frac{2100}{x}\right) - 10 = \frac{2100}{x+5}$ (Multiply through by x)

$2100 - 10x = \frac{2100\,x}{x+5}$ (Multiply through by $x + 5$)

$2100(x + 5) - 10x(x + 5) = 2100x$

$2100x + 10500 - 10x^2 - 50x = 2100x$

$10500 - 10x^2 - 50x = 2100x - 2100x$

$10500 - 10x^2 - 50x = 0$

$10x^2 + 50x - 10500 = 0$

$(x - 30)(x + 35) = 0$

$x = 30$ or $x = -35$ (ignore $x = -35$)

The number of students in the house is 30

Exercises

1. A shop keeper buys some books for Gh ¢80.00 If he had bought four more books for the same amount, each would have cost Gh¢1.00 less. Find the total number of books he bought .

2. A shop keeper buys some books for Gh¢ 1,200.00. If he had bought ten more books for the same amount, each book would have cost him Gh¢20.00 less. How many books did he buy? Ans 20

3. Alice buys a number of books for Gh¢40.00. If he had bougth two more books for the same amount, each book would have cost Gh¢1.00 less. How many books did he buy?

4. If the price of a radio is reduced by Gh¢4.00 a a man can buy 8 more radio sets for Gh¢192.00. Find the original price of the radio.

5. A firm employed a number of workers who were paid minimum daily wage. The total wages bill for these labourers was Gh¢36.00. When the miminum dialy wage was increased by 10p, the firm reduced the number of labourers by 5 and the total wage bill was still Gh¢36.00. Find the new minimum daily wage.

6. A shop sells only one type of radio. Every week, there is a total income of about Gh¢450.00 from sales of these radio. When the shop had reduction sales, the owner calculated that if he reduced the price of the radio by Gh¢5.00 and sold an extra 5 radio per week, he would obtain an extra Gh¢50.00 per week. What was the original piece of each radio?

7. A piece of copper wire cost Gh¢240.00. If it was 4 meter longer and price of each meter of copper wire was Gh¢3.00 less, the total cost of the piece would remain unchanged. Find the length of the copper wire.

8. A group of student went to a restaurant for a meal. When the bill of Gh¢175.00 was brought by a waiter, two of the cheeky ones from the group just sneaked off before the bill was paid, which resulted in the payment of extra Gh¢10 by each remaining student. How many students were in the group as well? Ans 7

Graphical Solution of Quadratic Equations

The graph of $ax^2 + bx + c = 0$, $a > 0$ is a parabola of U – shape and the graph of $ax^2 + bx + c$, $a < 0$ is a parabola of ∩ – shape. The points at which the parabola cuts the x – axis is the roots of the equation or the truth set or the solution set or the zeros of the equation.

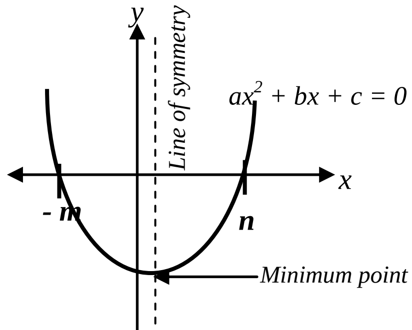


Fig. I

From the diagram above (Fig. I), the intercept on the x – axis is -m and n. Therefore the truth set is $x = - m$ or $x = n$

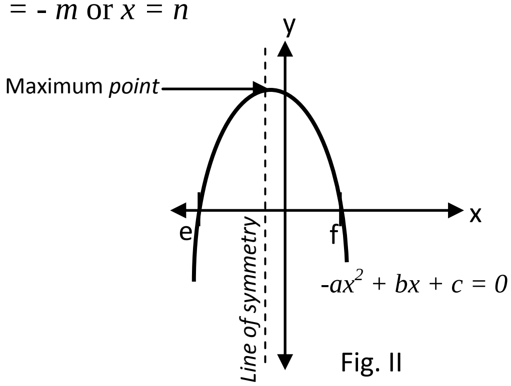


Fig. II

From the diagram above (Fig. II), the intercept on the x – axis is -e and f. Therefore the truth set is $x = e$ or $x = f$

Drawing the Graph of $ax^2 + bx + c = 0$,

To draw the graph of $ax^2 + bx + c = 0$, $a \neq 0$ I. Prepare a table of values for a given range of values of x.

II. Plot the points (x, y) on a graph sheet, using a given or a convenience scale;

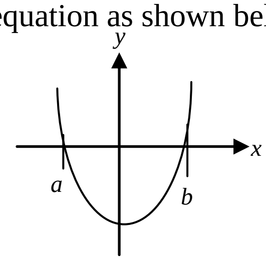
III. Join the points to make a free hand sketch of the required parabola/curve;

The Roots or Zeros of QuadraticEquation

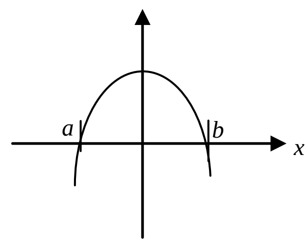
The roots of a quadratic equation is the value(s) of x for which the equation, $y = 0$. The roots of the quadratic equation are also called the ***zeros of the equation***.

From the graph, the roots of a function is determine with cognizance to the nature of the parabola in relation to the x – axis ;

I. Whether U or ∩ – shaped, the points at which the parabola cuts the x – axis is the roots or zeros of the equation as shown below.

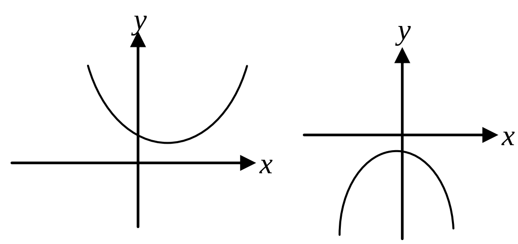


The zeros or truth set of the function is x = a or x = b



The zeros or truth set of the function is $x = a$ or $x = b$

II. Whether U or ∩ – shaped, if the parabola does not cut the x – axis, the equation is said to have no roots or no zeros as shown below



In both diagrams, the functions have no roots or no zeros.

Equation of Axes or Line of Symmetry

Parabolas can be described as being symmetrical, meaning that a line can be drawn through a parabola, to divide it into two equal parts, creating mirror images of each other. The straight line bisecting the parabola is called ***a line of symmetry***.

For all quadratic equations of the form, $y = ax^2 + bx + c$, $a \neq 0$, the line of symmetry has the equation, $\boldsymbol{x = -\frac{b}{2a}}$

Likewise, on the graph, if the curve cuts the x – axis at say x_1 and x_2, the equation of the axis of symmetry is calculated as, $\boldsymbol{x = \frac{x_1 + x_2}{2}}$

The Minimum and Maximum Points

The vertex of a parabola is the point of intersection of the line of symmetry and the parabola itself. The vertex is the turning point: either maximum (highest) or minimum (lowest) point of the parabola.

When the parabola is U – shaped, it is said to have ***a minimum or least turning point*** and when it is ∩ – shaped, it is said to have ***a maximum or greatest turning point.***

Values of x and y at the Turning Point

For all equations of the form:

$y = ax^2 + bx + c$, $a \neq 0$, at the turning point,

$x = -\frac{b}{2a}$

To get the value of y at the turning point, substitute $x = -\frac{b}{2a}$ *in* $y = ax^2 + bx + c$

The value of y obtained is called the ***maximum or minimum value***, depending on the nature of the parabola

Worked Examples

1. What is the value of x and y at the turning point of $y = 2x^2 - 8x + 3$

Solution

In $y = 2x^2 - 8x + 3$, $a = 2$ and $b = -8$

At the turning point, $x = -\frac{b}{2a} = -\frac{(-8)}{2 \times 2} = 4$

Put $x = 4$ in $y = 2x^2 - 8x + 3$

Put $x = 4$ in $y = 2x^2 - 8x + 3$

$y = 2(4)^2 - 8(4) + 3 = 3$

$\therefore$ At the turning point of $2x^2 - 8x + 3 = 0$,

$x = 4$ and $y = 3$

Worked Examples

1. Draw the graph of $y = x^2 - x - 6$, for the values $-3 \leq x \leq 4$, for the scales 2cm to 2 units on both axes. Find from the graph:

i. the equation of the line of symmetry,

ii. the minimum point of the parabola,

iii. the minimum value of the curve,

iv. the truth set of $x^2 - x - 6 = 0$

Solution

<u>Method 1</u>

$x^2 - x - 6 = 0$

Let $y = x^2 - x - 6$

When $x = -3$, $y = (-3)^2 - (-3) - 6 = 6$

When $x = -2$, $y = (-2)^2 - (-2) - 6 = 0$

When $x = -1$, $y = (-1)^2 - (-1) - 6 = -4$

When $x = 0$, $y = (0)^2 - (0) - 6 = -6$

When $x = 1$, $y = (1)^2 - (1) - 6 = -6$

When $x = 2$, $y = (2)^2 - (2) - 6 = -4$

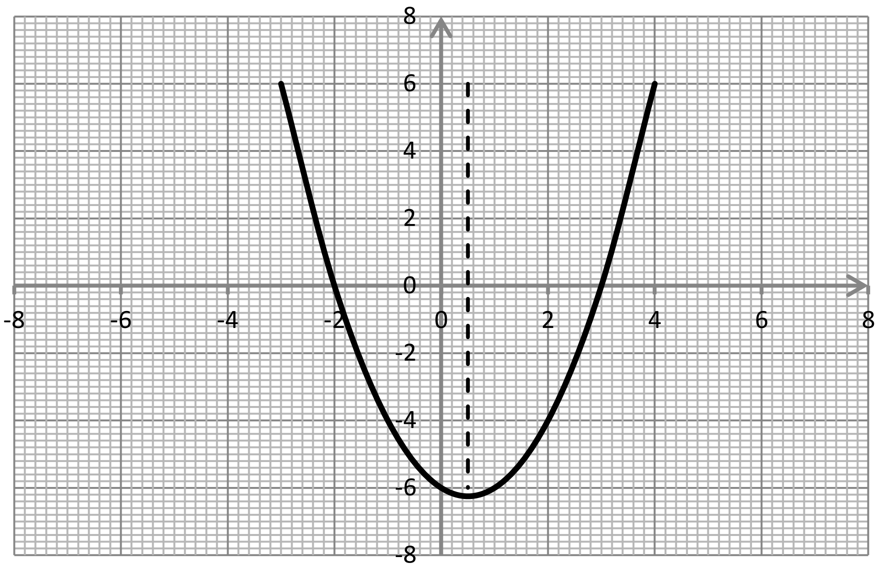
When $x = 3$, $y = (3)^2 - (3) - 6 = 0$

Table of values

x	-3	-2	-1	0	1	2	3	4
y	6	0	-4	-6	-6	-4	0	6

Method 2

x	-3	-2	-1	0	1	2	3	4
x^2	9	4	1	0	1	4	9	16
$-x$	- 3	- 2	- 1	- 0	- 1	- 2	- 3	-4
- 6	- 6	- 6	- 6	- 6	- 6	- 6	- 6	-6
y	6	0	- 4	- 6	- 6	- 4	0	6



i.Equation of the axis of symmetry,

$x = -\frac{b}{2a}$.

But from$y = x^2 - x - 6$, $a = 1$, $b = -1$

$\Rightarrow x = -\frac{-1}{2(1)} = \frac{1}{2} = 0.5$

The equation of the axis of equation is $x = 0.5$ (Shown on the graph)

ii. To find the minimum point, substitute $x = 0.5$ in $x^2 - x - 6 = 0$

$\Rightarrow$when $x = 0.5$, $y = (0.5)^2 - 0.5 - 6 = - 6.25$

$(x, y) = (0.5, -6.25)$

The minimum point $(x, y) = (0.5, - 6.25)$

iii. The minimum value of the curve is the y value of the minimum point. Hence, $y = - 6.25$

iv. The truth set of $x^2 - x - 6 = 0$, is the point where the curve cuts the axis.

From the graph, $x = -2$ or $x = 3$.

2. The following is an incomplete table for, $y = 4x^2 - 8x - 21$ for $-2.0 \leq x \leq 4.0$

x	-2.0	-1.5	-1.0	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	11		-9	-21				-21		-9	0	

a. Copy and complete the table above.

b. Draw the graph of $y = 4x^2 - 8x - 21$, using a scale of 2cm to 1 unit on x axis and 2 cm to 5 units on the y - axis.

c. Find from the graph:

i. the equation of the line of symmetry,

ii. the minimumpoint of the parabola,

iii. the minimum value of the curve,

iv. the truth set of $y = 4x^2 - 8x - 21$.

Solution

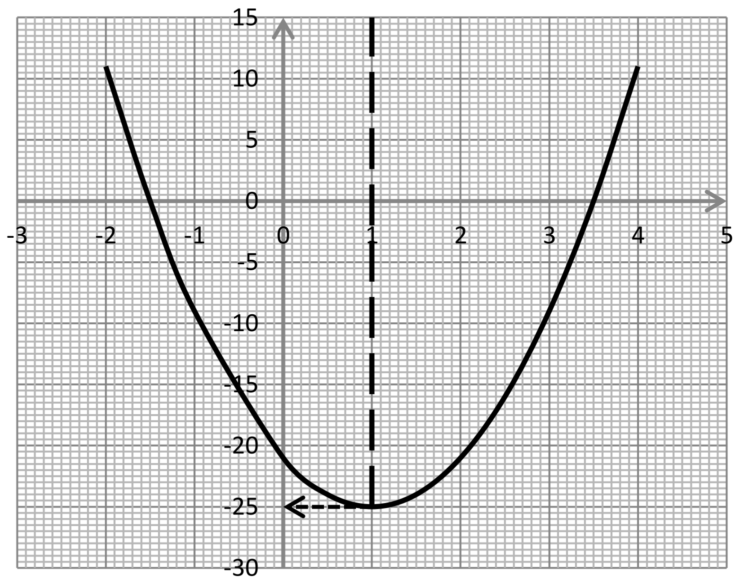
a. In $y = 4x^2 - 8x - 21$

When $x = -1.5$, $y = 4(-1.5)^2 - 8(-1.5) - 21 = 0$
When $x = 0.5$, $y = 4(0.5)^2 - 8(0.5) - 21 = -24$
When $x = 1$, $y = 4(1)^2 - 8(1) - 21 = -25$
When $x = 1.5$, $y = 4(1.5)^2 - 8(1.5) - 21 = -24$
When $x = 2.5$, $y = 4(2.5)^2 - 8(2.5) - 21 = -16$
When $x = 4$, $y = 4(4)^2 - 8(4) - 21 = 11$

Completed table of values

x	-2.0	-1.5	-1.0	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	11	0	-9	-21	-24	-25	-24	-21	-16	-9	0	11

b.



c. i. Equation of the axis of symmetry,

$\boldsymbol{x = -\frac{b}{2a}}$

But from $4x^2 - 8x - 21$, $a = 4$ and $b = -8$

$\boldsymbol{x = -\frac{-8}{2(4)} = \frac{8}{8} = \frac{2}{2}} = 1$

$x = 1$ (shown on the graph)

ii. The minimum point of the parabola;

Substitute $x = 1$ in $y = 4x^2 - 8x - 21$

When $x = 1$, $y = 4(1)^2 - 8(1) - 21 = -25$

$(x, y) = (1, -25)$

The minimum point $(x, y) = (1, -25)$OR

Read directly from the graph to obtain;

$(x, y) = (1, -25)$

iii. The minimum value is the value of y at the minimum point. Hence, $y = -25$

iv. The truth set of $y = 4x^2 - 8x - 21$
Since the curve cuts the x – axis at the points -1.5 and 3.5, the truth set is $x = -1.5$ or $x = 3.5$

3. Copy and complete the table for the relation $y = 3 - 2x - x^2$, for the interval $-5 \leq x \leq 3$

x	-5	-4	-3	-2	-1	0	1	2	3
y	-12		0		4				

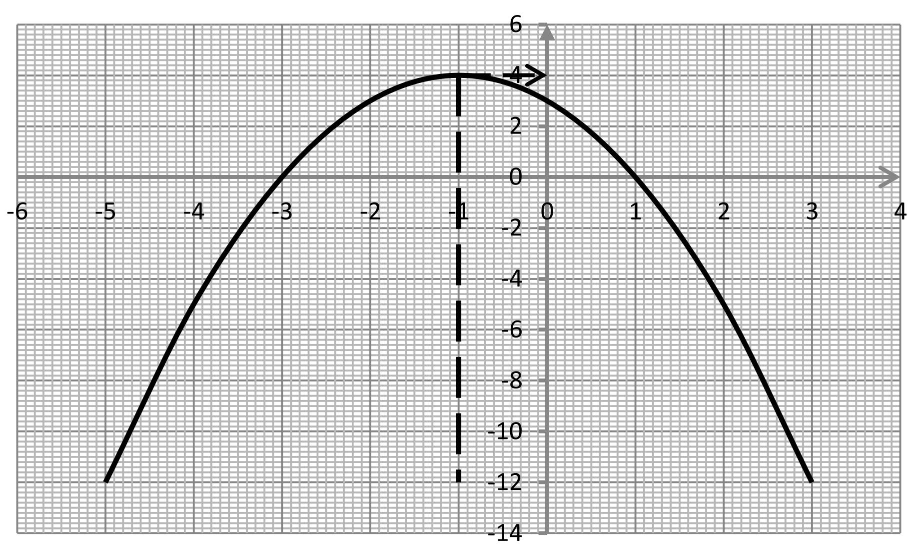
b. Using a scale of 2 cm to 1 unit on$x$ – axis and 2 cm to 2 units on y – axis, draw the graph of the relation$y = 3 - 2x - x^2$
c. Use your graph to find:
i. the equation of the axis of symmetry
ii. the truth set of $y = 3 - 2x - x^2$
iii. themaximum point of the curve
iv. the maximum value of the curve

Solution
a. In $y = 3 - 2x - x^2$
When $x = -4$, $y = 3 - 2(-4) - (-4)^2 = -5$
When $x = -2$, $y = 3 - 2(-2) - (-2)^2 = 3$
When $x = 0$, $y = 3 - 2(0) - (0)^2 = 3$
When $x = 1$, $y = 3 - 2(1) - (1)^2 = 0$
When $x = 2$, $y = 3 - 2(2) - (2)^2 = -5$
When $x = 3$, $y = 3 - 2(3) - (3)^2 = -12$

Completed table of values

x	-5	-4	-3	-2	-1	0	1	2	3
y	-12	-5	0	3	4	3	0	-5	-12

b.



c. i. Equation of the axis of symmetry,

$\boldsymbol{x = -\frac{b}{2a}}$

But from $y = 3 - 2x - x^2$, $a = -1$ and $b = -2$

$\boldsymbol{x = -\frac{-2}{2(-1)}} = -\frac{2}{2} = -1$

⇒The equation of axis of symmetry is $x = -1$ (Shown on the graph)

ii. To find the maximum point of the parabola, substitute $x = -1$ in $y = 3 - 2x - x^2$

When $x = -1$, $y = 3 - 2(-1) - (-1)^2 = 4$

$(x, y) = (-1, 4)$

The maximum point $(x, y) = (-1, 4)$

OR read directly from the graph to obtain

$(x, y) = (-1, 4)$

iii. The maximum value is the value of y at the maximum point. Hence, $y = 4$

iv. The truth set of $y = 3 - 2x - x^2$

Since the curve cuts the x – axis at the points; -3 and 1, the truth set is $\{x : x = -3 \text{ or } x = 1\}$

4. Copy and complete the following table of values for the relation $y = 4 + 5x - 2x^2$, for the interval $-3 \leq x \leq 5$

x	-3	-2	-1	0	1	2	3	4	5
y		-14	-3			6			-21

b. Using 2 cm to 1 unit on the x – axis and 2 cm to 5 units on the y – axis, draw the graph of $y = 4 + 5x - 2x^2$

c. From your graph, find:

i. the value of x for which y is maximum,

ii. the value of x for which $1 + 5x - 2x^2 = 0$

Solution

a. In $y = 4 + 5x - 2x^2$

When $x = -3$, $y = 4 + 5(-3) - 2(-3)^2 = -29$

When $x = 0$, $y = 4 + 5(0) - 2(0)^2 = 4$

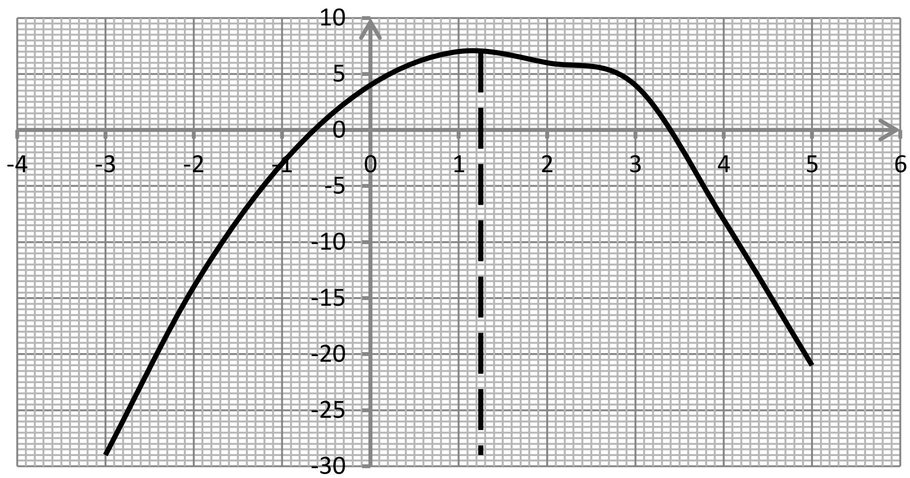
When $x = 1$, $y = 4 + 5(1) - 2(1)^2 = 7$

When $x = 3$, $y = 4 + 5(3) - 2(3)^2 = 1$

When $x = 4$, $y = 4 + 5(4) - 2(4)^2 = -8$

Completed table of values

x	-3	-2	-1	0	1	2	3	4	5
y	-29	-14	-3	4	7	6	4	-8	-21



c. i. Equation of the axis of symmetry,

$\boldsymbol{x = -\frac{b}{2a}}$

But from $y = 4 + 5x - 2x^2$, $a = -2$ and $b = 5$

$x = -\frac{5}{2(-2)} = 1.3$

The equation of axis of symmetry is $x = 1.3$ (Shown on the graph)

ii. The maximum point of the parabola;
Substitute $x = 1.25$ in $y = 4 + 5x - 2x^2$
When $x = 1.25$, $y = 4 + 5(1.25) - 2(1.25)^2 = 7$
The maximum point $(x, y) = (1.25, 7)$

iii. The maximum value is the value of y at the maximum point. Hence, $y = 7$

iv. The truth set of $y = 4 + 5x - 2x^2$;
Since the curve cuts the x – axis at the points -0.6 and 3.4, the truth set is $x = -0.6$ or $x = 3.4$

Exercises 21.8

1. a. Copy and complete the table below for the relation $y = x^2 - 4x + 3$ for $-2 \leq x \leq 6$

x	-2	-1	0	1	2	3	4	5	6
y	15						3		

b. i using a scale of 2cm to 1 unit on x – axes and 2cm to 2 units on y – axis, draw the graph of $y = x^2 - 4x + 3$

c. Use your graph to find;
i. the equation of the axis of symmetry of the curve,
ii. the zeros of $y = x^2 - 4x + 3$,
iii. the least point of the curve,
iv. the least value of the curve.

2. a. Copy and complete the following table for $y = x^2 - x - 2$ for $-3 \leq x \leq 4$

x	-3	-2	-1	0	1	2	3	4
y	10				-2			

b. i. using a scale of 2cm to 2units on both axes, draw the graph of $y = x^2 - x - 2$

c. Use your graph to find:
i. the zeros of $y = x^2 - x - 2$,
ii. the equation of the axis of symmetry,
iii. the minimum point of the curve,
iv. the minimum value of the curve.

3. Copy and complete the following table for $y = 2x^2 - 3x - 9$ for $-2 \leq x \leq 3$

x	-2	-1	0	1	2	3
y	5		-9			

b. Use the table to draw the graph of $y = 2x^2 – 3x – 9$, for the scales 2cm : 2 unit on x – axis, and 2 cm : 2 units on x – axis.
c. Use your graph to find;
i. the equation of the axis of symmetry,
ii. the zeros of $y = 2x^2 – 3x – 9$,
iii. the least point of the curve,
iv. the least value of the curve.

4. a. Using x values of -2, -1, 0, 0.5, 1, 2, 3, draw the graph of $y = x^2 – x$ using a scale of 2 cm to1 unit on both axis.
b. i. What is the equation of the line about which the graph is symmetrical?
ii. What is the minimum value of the curve?

5. a. Copy and complete the table below for the relation $y = 2 – x – x^2$ in the interval $– 5 \leq x \leq 4$

x	-5	-4	-3	-2	-1	0	1	2	3	4
y				0			0			

b. Use your table to draw the graph of $y = 2 – x – x^2$ for the scales 2cm to 1 unit on x – axis and 2 cm to 5units on y – axis
c. From the graph, find:
i. the values of x for which $y = 2 – x – x^2 = 0$,
ii. the equation of the axis of symmetry,
iii. the maximum point of the curve,
iv. themaximum value of the curve.

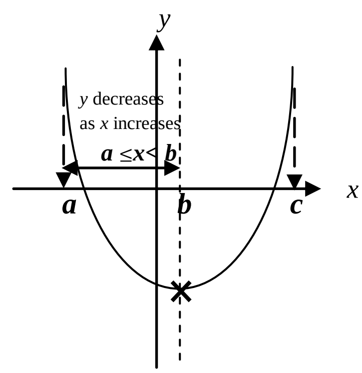
Range of Values of *x* for which *y* Decreases as *x* Increases

A. When the curve is ∪– shaped

I. Identify the x value of the turning point of the curve as ***b***

II. Identify the x value of the curve at the extreme left as ***a***

III. Substitute the values of ***a*** and ***b*** obtained from the graph in $a \leq x < b$ to get the range of values of x for which y decreases as x increases.This shown in the diagram below:



Explanation

Each point of the curve is represented by a pair of points *(x, y).* From the extreme left, as the curve moves downward, the *y*values descend representing a decrease in*y*values. At the same time, its corresponding x values move to the right or ascend representing an increase in the x values. This occurs until the curve reaches its turning point. Within this range (from the extreme left to the turning point), we say the ***y values decreases as the x values increases***

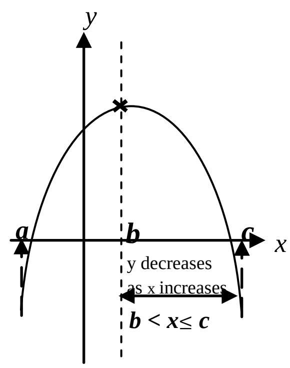
B. When the curve is ∩ - shaped

I. Identify the x value of the turning point of the curve as ***b;***

II. On the x – axis, identify the extreme value of the curve at the right as ***c;***

III. Substitute the values of ***b*** and ***c*** obtained from the graph in $b < x \leq c$ to get the range of values of x for which y decreases as x increases.

This is shown in the diagram below:



Explanation

From the turning point, as the curve moves down the right, the y values descend representing a decrease in y values. At the same time, its corresponding x values move to the right or ascend representing an increase in the x values. This occurs till theend of the curve at the extreme right. Within this range (from the turning point to the extreme right of the curve), we say the ***y values decrease as the x values increase.***

Range of Values of *x* for which *y* Increases as *x* Increases

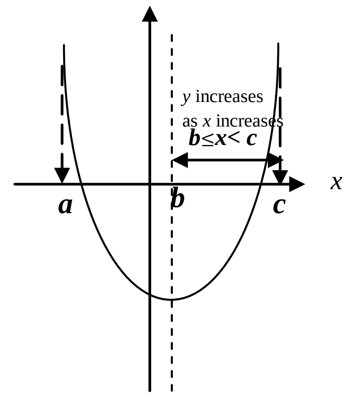
A. For U – shaped

I. Identify the x value of the turning point of the curve as ***b***

II. On the x – axis, identify the extreme value of the curve at the right as ***c***

III. Substitute the values of ***b*** and ***c*** obtained from the graph in $\boldsymbol{b \leq x < c}$ to get the range of values of x for which y increases as x increases.

This is shown in the diagram below:



Explanation

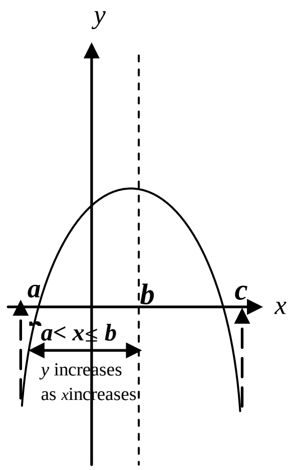
From the turning point, as the curve moves up the right, the y values ascend representing an increase in y values. At the same time, its corresponding x values move to the right or ascend representing an increase in the x values. This occurs till the end of the curve at the extreme right. Within this range (from the turning point to the extreme right of the curve), we say the ***y values increase as the x values increase***

B. ***For ∩ – shaped***

I. Identify the x value of the turning point of the curve as ***b***

II. On the x – axis, identify the extreme value of the curve at the right as ***a***

III. Substitute the values of ***a*** and ***b*** obtained from the graph in $\boldsymbol{a < x \leq b}$ to get the range of values of x for which y increases as x increases.This shown in the diagram below:



Explanation

From the extreme left, as the curve moves up, the y values ascend representing anincrease in y values. At the same time, its corresponding x values move to the right or ascend representing an increase in the $x$values. This occurs till the curve reaches its turning point. Within this range (from the extreme left to the turning point), we say the ***y values increase as the x values increase.***

Worked Examples

1**.** a. Copy and complete the following table for $y = x^2 - 5x - 2$ for $-1 \leq x \leq 6$

x	-1	0	1	2	3	4	5	6
y	4			-8		-6		

b. Using a scale of 2 cm to 1 unit on the x – axis

and 2 cm to 2 units on the y – axis, draw the graph of the relation

c. From your graph, find:

i. the equation of axis of symmetry of y,

ii. the turning point of y,

iii. the minimum value of y,

iv. the truth set of $x^2 - 5x - 2 = 0$,

v. the range of values of x for which y decreases as x increases,

vi. the range of values of x for which y increases as x increases.

Solution

In $y = x^2 - 5x - 2$,

When $x = 0$, $y = (0)^2 - 5(0) - 2 = -2$

When $x = 1$, $y = (1)^2 - 5(1) - 2 = -6$

When $x = 3$, $y = (3)^2 - 5(3) - 2 = -8$

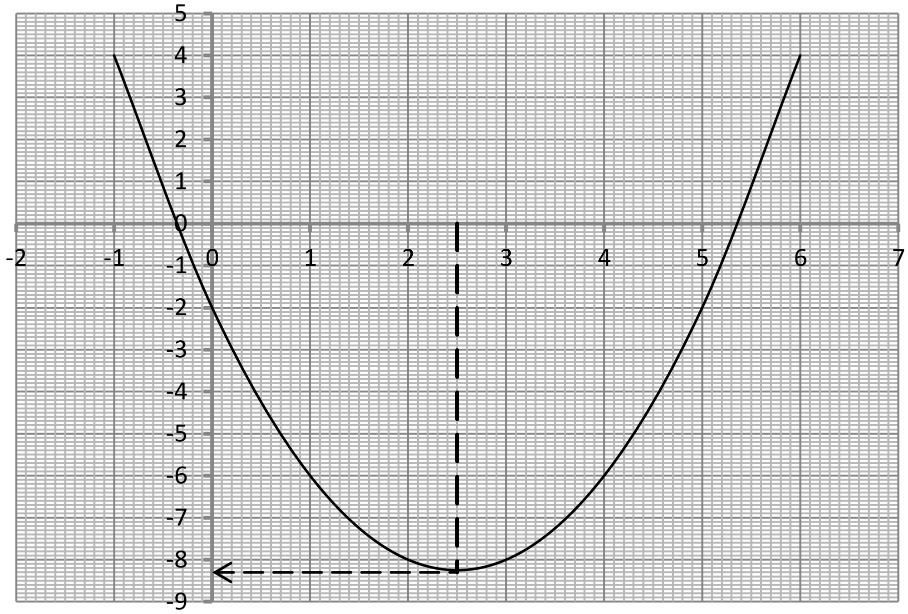
When $x = 5$, $y = (5)^2 - 5(5) - 2 = -2$

When $x = 1$, $y = (6)^2 - 5(6) - 2 = 4$

Completed table of values

x	-1	0	1	2	3	4	5	6
y	4	-2	-6	-8	-8	-6	-2	4

b.



c. i. Equation of the axis of symmetry, $\boldsymbol{x = -\frac{b}{2a}}$.

But from $y = x^2 - 5x - 2$, $a = 1$, $b = -5$

$\Rightarrow x = -\frac{-5}{2\,(1)} = \frac{5}{2} = 2.5$

$x = 2.5$ (show the line $x = 2.5$ on the graph)

ii. To find the minimum point of the parabola, substitute $x = 2.5$ in $y = x^2 - 5x - 2$

When $x = 2.5$, $y = (2.5)^2 - 5(2.5) - 2 = -8.3$

$(x, y) = (2.5, -8.3)$

The minimum point $(x, y) = (2.5, -8.3)$ OR read directly from the graph to obtain $(x, y) = (2.5, -8.3)$

iii. The minimum value is the value of y at the minimum point. Hence, the minimum value is $y = -8.3$

iv. The truth set of $y = x^2 - 5x - 2$;

Since the curve cuts the x – axis at the points –0.2 and 5.4, the truth set is $\{x : x = -0.2 \text{ or } x = 5.2\}$

v. The range of values of x for which y decreases as x increases = $\boldsymbol{a \leq x < b}$.

From the graph, $a = -1$ and $b = 2.5$substitute in $a \leq x < b = -1 \leq x < 2.5$

vi. The range of values of x for which y increases as x increases = $\boldsymbol{b < x \leq c}$.

From the graph, $b = 2.5$ and $c = 6$, substitute in $b < x \leq c = 2.5 < x \leq 6$

2. a. Copy and complete the following table of values for the relation $y = 10 + 6x - 3x^2$ for $-3 \leq x \leq 5$

x	-3	-2	-1	0	1	2	3	4	5
y				10	13		1	-14	

b. Using a scale of 2 cm to 1 unit on the x – axis and 2 cm to 5units on the y – axis, draw the graph of the relation.

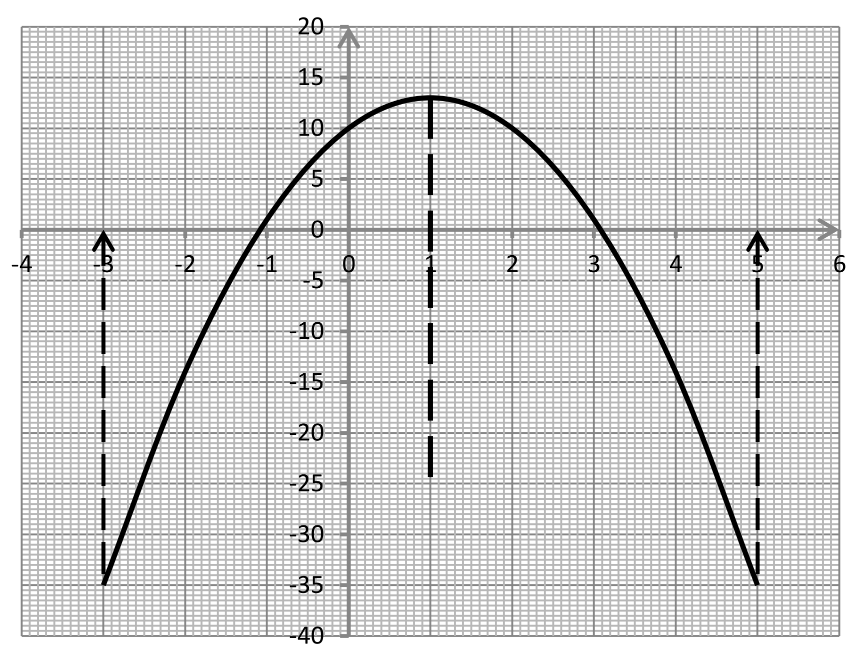
c. Find the following from the graph:
i. the truth set of $10 + 6x - 3x^2 = 0$,
ii. the equation of the axis of symmetry,
iii.the turning point of $y = 10 + 6x - 3x^2$.
iv. the maximum value of $y = 10 + 6x - 3x^2$
v. the range of values of x for which y decreases as x increases;
vi. the range of values of x for which y increases as x increases.

Solution

a. $y = 10 + 6x - 3x^2$
When $x = -3$, $y = 10 + 6(-3) - 3(-3)^2 = -35$
When $x = -2$, $y = 10 + 6(-2) - 3(-2)^2 = -14$
When $x = -1$, $y = 10 + 6(-1) - 3(-1)^2 = 1$
When $x = 2$, $y = 10 + 6(2) - 3(2)^2 = 10$
When $x = 5$, $y = 10 + 6(5) - 3(5)^2 = -35$

Completed table of values

x	-3	-2	-1	0	1	2	3	4	5
y	-35	-14	1	10	13	10	1	-14	-35



c. i. The curve cuts the x – axis at $x = -1.1$ and $x = 3.1$. Therefore, the truth set is $\{x : x = -1.1 \text{ or } x = -3.1\}$

ii. From $y = 10 + 6x - 3x^2$, equation of the axis of symmetry, $\boldsymbol{x = -\frac{b}{2a}} = -\frac{\mathbf{6}}{\mathbf{2(-3)}} = \frac{6}{6} = 1$

iii. Put $x = 1$ in $y = 10 + 6x - 3x^2$
When $x = 1$, $y = 10 + 6(1) - 3(1)^2 = 13$.
The maximum point, $(x, y) = (1, 13)$

iv. The maximum value is the value of y at the maximum point. Therefore, $y = 13$.

v. The range of values of x for which y decreases as x increases $= 1 \leq x < 5$.

vi. The range of values of x for which y increases as x increases $= -3 < x \leq 1$.

3. The following is an incomplete table for the relation $y = 2x(4x - 7) - 9$, where $-2 \leq x \leq 3$

x	-2	-1	0	0.5	1	1.5	2	3	4
y	51		-9	-14					63

a. Copy and complete the table

b. Using a scale of 2cm to 1 unit on the x – axis and 1 cm to 5 units on the y – axis, draw the graph of the relation

c. Estimate, correct to one decimal place;

i. the x – coordinates of the point where y starts increasing with respect to x

ii. the values of x for which $2x(4x - 7) + 4 = 0$

Solution

a. $y = 2x(4x - 7) - 9$

$y = 8x^2 - 14x - 9$

When $x = -1$, $y = 8(-1)^2 - 14(-1) - 9 = 13$

When $x = 1$, $y = 8(1)^2 - 14(1) - 9 = -15$

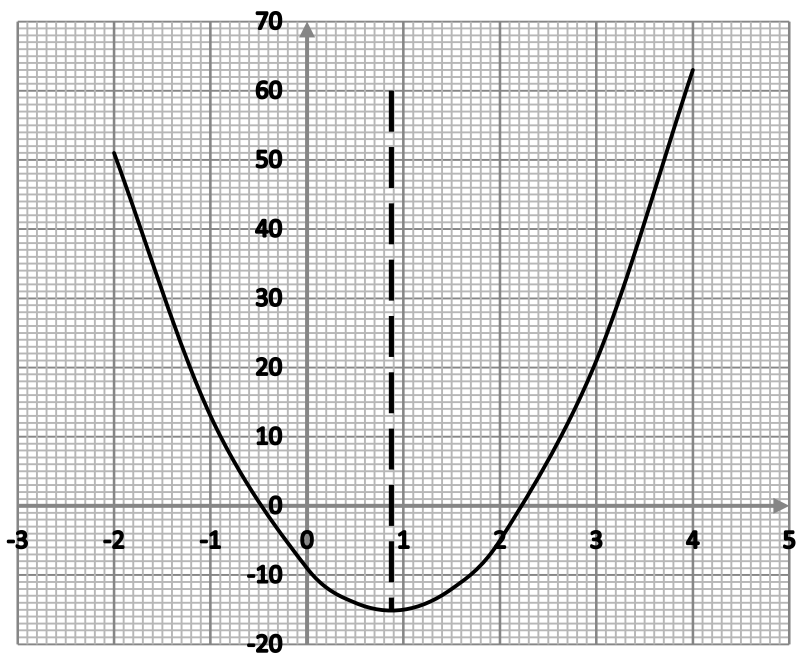
When $x = 1.5$, $y = 8(1.5)^2 - 14(1.5) - 9 = -12$

When $x = 2$, $y = 8(2)^2 - 14(2) - 9 = -5$

When $x = 3$, $y = 8(3)^2 - 14(3) - 9 = 21$

x	-2	-1	0	0.5	1	1.5	2	3	4
y	51	13	-9	-14	-15	-12	-5	21	63

b.



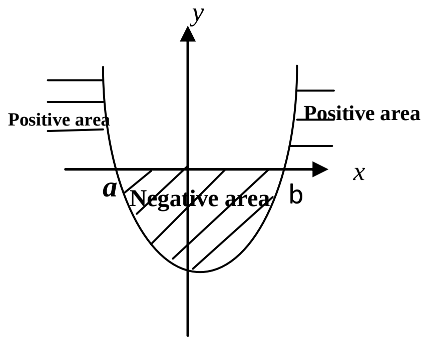
C.i. The x – coordinates of the point where y starts increasing with respect to $x = 0.9 \leq x < 4$

ii. Values of x for which $2x(4x - 7) + 4 = 0$

The curve cuts the x – axis at $x = -0.5$ and $x = 2.2$. Therefore, $\{x : x = 0.5 \text{ or } x = 2.2\}$

Range of Values of x for which y is Negative or $y < 0$

a. When the curve is U – shaped

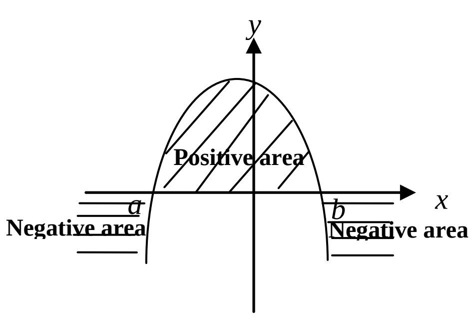


Explanation

From the diagram above, when $y = 0$, $x = a$ or $x = b$. This means that $y < 0$ or 'y is negative' is the region below the x – axis.

The values of ***a*** to ***b*** and ***b*** to ***a*** on the x axis produce negative values of y respectively. Therefore, the values of x for which $y < 0$ or y is negative for a ∪ – shaped parabola is $x > a$ and $x < b$ and the range is ***a* < *x* < *b* or *b* > *x* > *a***

b. When the curve is ∩ - shaped



Explanation

From the diagram above, when $y = 0$, $x = a$ or$x = b$. This means that $y < 0$ or y is negative is the region below the x – axis,

The values of x at the left of ***a*** and at the right ***b***produce negative values of y respectively. Therefore, the values of x for which $y < 0$ or y is negative for an ∩ – shaped parabola is $x < a$ and $x > b$ and the range is ***a* > *x* > *b*** or ***b* < *x* < *a***

Worked Examples

1. Copy and complete the following table for the relation $y = \frac{1}{2}(2x - 1)(x + 2)$

x	-3.5	-3	-2	-1	-0.5	0	1	2
y	6	3.5	0					6

b. Using a scale of 2 cm to 1 unit on each axis, draw the graph of the relation $y = \frac{1}{2}(2x - 1)(x + 2)$ for $-3.5 \leq x \leq 2$

c. Use your graph to find;

i. the value of x for which y is least,

ii. the truth set of $\frac{1}{2}(2x - 1)(x + 2) = 4$,

iii. the values of x for which y is negative.

Solution

In $y = \frac{1}{2}(2x - 1)(x + 2)$

When $x = -1$,

$y = \frac{1}{2}[2(-1) - 1][(-1) + 2] = -1.5$

When $x = -0.5$,

$y = \frac{1}{2}[2(-0.5) - 1][(-0.5) + 2] = -1.5$

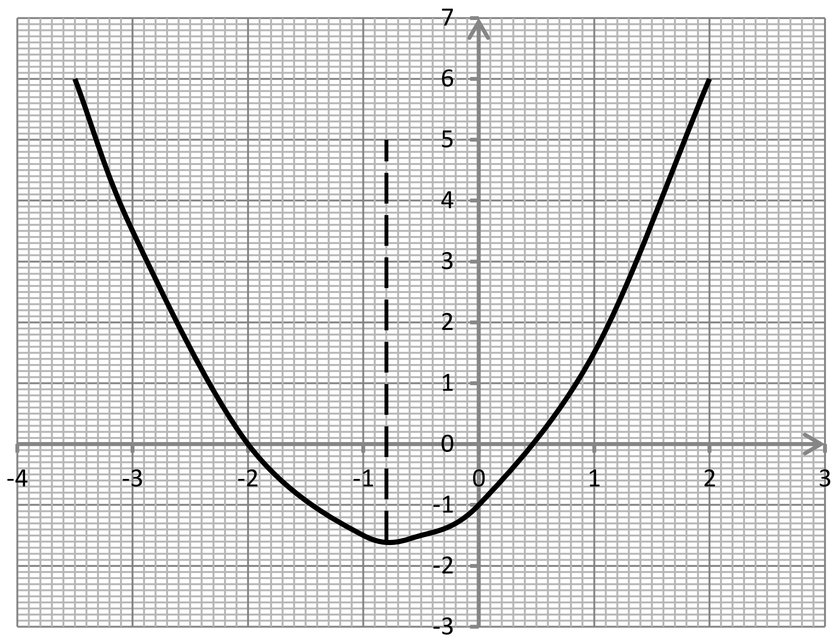
When $x = 0$,

$y = \frac{1}{2}[2(0) - 1][(0) + 2] = -1$

When $x = 1$,

$y = \frac{1}{2}[2(1) - 1][(1) + 2] = 1.5$

x	-3.5	-3	-2	-1	-0.5	0	1	2
y	6	3.5	0	-1.5	-1.5	-1	1.5	6



c. From the graph

i. the value of x for which y is least = -0.8

ii. the truth set of $\frac{1}{2}(2x - 1)(x + 2) = 0$
= {x :x = -2 or x = 0.5}

iii. the values of x for which y is negative is
$x > -2$ and $x < 0.5$
Range = $-2 < x < 0.5$

2. Copy and complete the table of values for the relation $y = 3 - 2x - x^2$ for the interval $-5 \leq x \leq 3$

x	-5	-4	-3	-2	-1	0	1	2	3
y	-12		0			3			-12

b. Using a scale of 2cm to 1 unit on the x – axis and 2 cm to 2 units on the y – axis, draw the graph of the relation

c. From your graph, find:

i. the equation of the line of symmetry of the curve

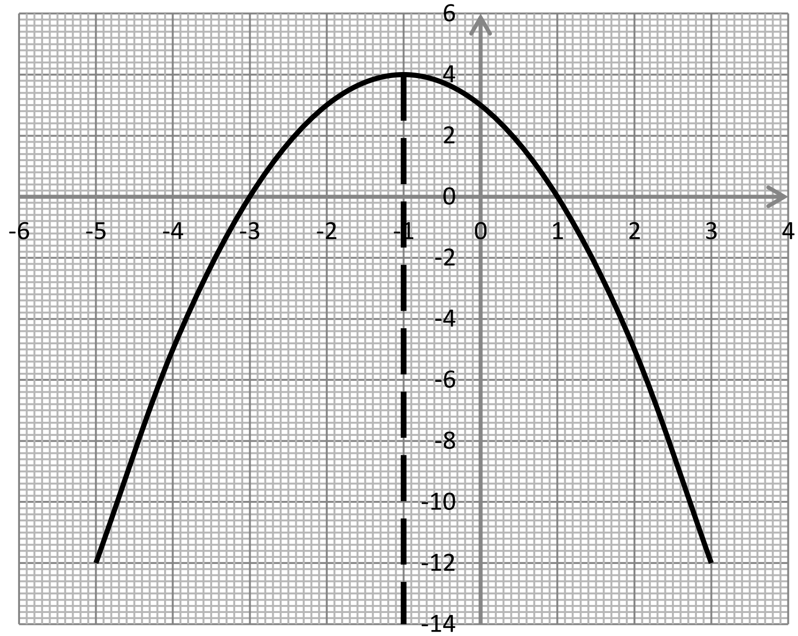
ii. the values of x for which the relation:
$y = 3 - 2x - x^2$ is less than zero

Solution

a. From $y = 3 - 2x - x^2$
When $x = -4$, $y = 3 - 2(-4) - (-4)^2 = -5$
When $x = -2$, $y = 3 - 2(-2) - (-2)^2 = 3$
When $x = -1$, $y = 3 - 2(-1) - (-1)^2 = 4$
When $x = 1$, $y = 3 - 2(1) - (1)^2 = 0$
When $x = 2$, $y = 3 - 2(2) - (2)^2 = -5$

x	-5	-4	-3	-2	-1	0	1	2	3
y	-12	-5	0	3	4	3	0	-5	-12

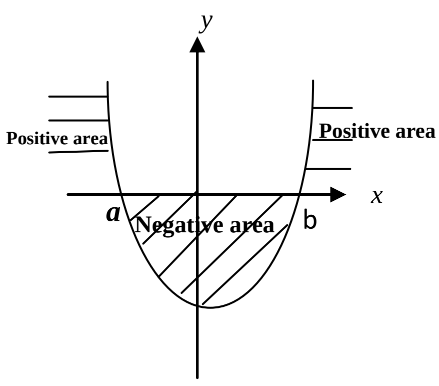
b.



ii. The values of x for which the relation,
$3 - 2x - x^2$ is less than zero is $x < -3$ and $x > 1$
Range = $-3 > x > 1$

Range of Values of x for which y is Positive or $y > 0$

a. When the curve is U – shaped

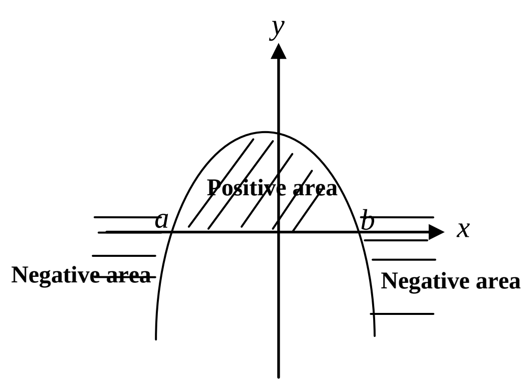


Explanation

From the diagram above, when $y = 0$, $x = a$ or $x = b$. This means that $y > 0$ or y is positive is the region above the x – axis,

The values of x at the left of ***a*** and at the right ***b*** produce positive values of y respectively. Therefore, the values of x for which $y > 0$ or y is positive for an ∩ – shaped parabola is $x < a$ and $x > b$ and the range is $b < x < a$.

b. When the curve is ∩ - shaped



Explanation

From the diagram above, when y = 0, x = a or x = b. This means that y > 0 or 'y is positive' is the region above the x – axis.

The values of $\boldsymbol{a}$ to $\boldsymbol{b}$ and $\boldsymbol{b}$ to $\boldsymbol{a}$ on the x axis produce negative values of $y$respectively. Therefore, the values of x for which y > 0 or y is positive for an ∩ – shaped parabola is $x > a$ and $x < b$ and the range is $a < x < b$

Worked Examples

1. Copy and complete the following table for the relation $y = \frac{1}{2}(2x-1)(x+2)$

x	-3.5	-3	-2	-1	-0.5	0	1	2
y	6	3.5	0					6

b. Using a scale of 2 cm to 1 unit on each axis, draw the graph of the relation:

$y = \frac{1}{2}(2x-1)(x+2)$for $-3.5 \le x \le 2$

c. Use your graph to find:

i. the value of x for which y is least,

ii. the truth set of $\frac{1}{2}(2x-1)(x+2) = 4$,

iii. the values of x for which y is positive.

Solution

In $y = \frac{1}{2}(2x-1)(x+2)$

When x = -1,

$y = \frac{1}{2}[2(-1)-1][(-1)+2]$= -1.5

When x = -0.5,

$y = \frac{1}{2}[2(-0.5)-1][(-0.5)+2]$= -1.5

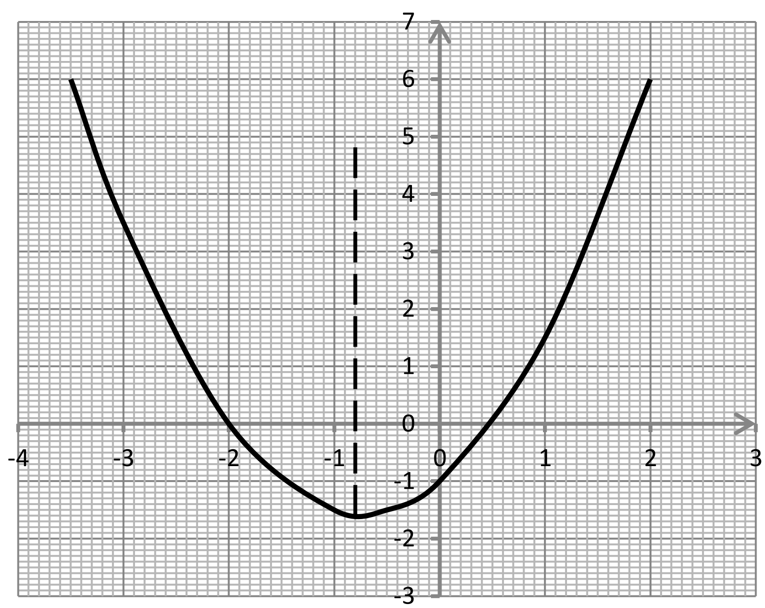
When x = 0,

$y = \frac{1}{2}[2(0)-1][(0)+2]$= -1

When x = 1,

$y = \frac{1}{2}[2(1)-1][(1)+2]$= 1.5

x	-3.5	-3	-2	-1	-0.5	0	1	2
y	6	3.5	0	-1.5	-1.5	-1	1.5	6



c. From the graph:

i. the value of x for which y is least = -0.8

ii. the truth set of $\frac{1}{2}(2x-1)(x+2) = 0$
= {x :x = -2 or x = 0.5}

iii. the values of x for which y is positive is
x< -2 and x > 0.5
Range = -2 > x > 0.5 OR 0.5 < x < -2

2. Copy and complete the table of values for the relation $y = 3 - 2x - x^2$ for the interval
$-5 \leq x \leq 3$.

x	-5	-4	-3	-2	-1	0	1	2	3
y	-12		0			3			-12

b. Using a scale of 2cm to 1 unit on the x – axis and 2 cm to 2 units on the y – axis, draw the graph of the relation.

c. From your graph, find:

i. the equation of the line of symmetry of the curve,

ii. the values of x for which the relation,
$3 - 2x - x^2$ is greater than zero.

Solution

a. From $y = 3 - 2x - x^2$

When x = - 4, $y = 3 - 2(-4) - (-4)^2 = -5$

When x = - 2, $y = 3 - 2(-2) - (-2)^2 = 3$

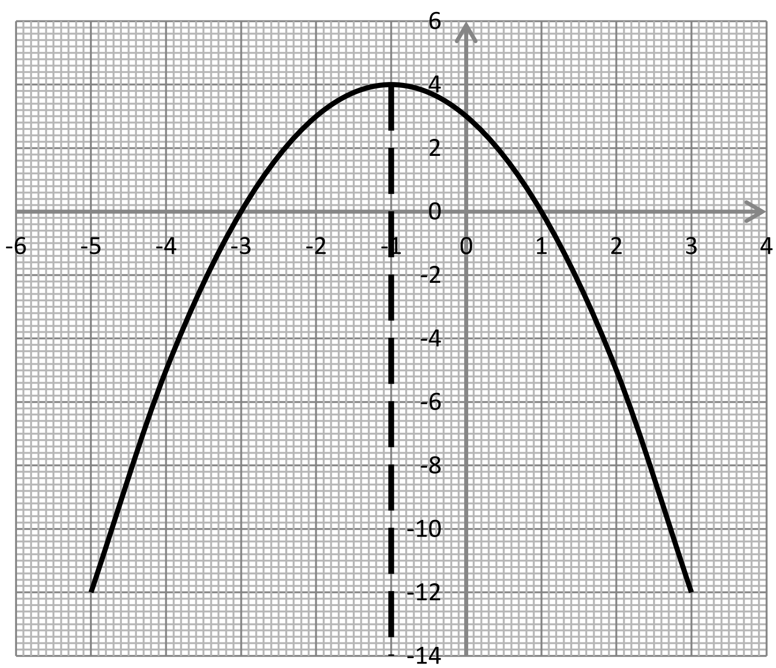
When x = - 1, $y = 3 - 2(-1) - (-1)^2 = 4$

When x = 1, $y = 3 - 2(1) - (1)^2 = 0$

When x = 2, $y = 3 - 2(2) - (2)^2 = -5$

x	-5	-4	-3	-2	-1	0	1	2	3
y	-12	-5	0	3	4	3	0	-5	-12

b.



ii. The values of x for which the relation $y = 3 - 2x - x^2$ is greater than zero is $x > -3$ and $x < 1$

Range = $-3 < x < 1$

Exercises 21.9

1. Copy and complete the table of values for the relation $y = x^2 - 2x - 3$ for the interval $-2 \leq x \leq 4$

x	-2	-1.5	-1	0	1	2	2.5	3	3.5	4
y	-5		0	-3				0		

b. Using a scale of 2cm to 1 unit on both axes, draw the graph of the relation

c. Use your graph to find:

i. the equation of the line of symmetry of the curve,

ii. the solution set of $x^2 - 2x - 3 = 0$,

iii. the range of values of x for which y is negative.

2. Construct a table of values and draw the graph of $y = x^2 + 2x + 3$ for the interval $-5 \leq x \leq 3$, using a scale of 2cm to 1 unit on x – axis and 2 cm to 2 units on y axis

b. From the graph, find:

i. the axis of symmetry,

ii. the least value of the curve,

iii. the range of values of x for which y decreases as x increases,

iv. the range of values of x for which y increases as x increases.

3. a. Draw the graph of $y = x^2 + 2x - 3$ for the values of x from -5 to 3 for the scales 2cm :1 unit on x – axis and 2cm : 2 units on y – axis

b. From the graph, find:

i. the equation of the axis of symmetry,

ii. the minimum point of the curve,

iii. the range of values of x for which y decreases as x increases,

iv. the range of values of x for which y increases as x increases.

4. a. Construct a table of values for $y = 3 + 5x - 2x^2$ in the interval $-4 \leq x \leq 5$

b. i. Draw the graph of $y = 3 + 5x - 2x^2$ using the scale 2cm to 1 unit on x – axis and 2cm to 5 units on y – axis.

c. Find:

i. the range of values of x for which y decreases as x increases.

ii. the range of values of x for which y increases as x increases.

iii. the values of x for which y is positive

iv. the values of x for which y is negative

5. The following is an incomplete table for the relation $y = (x + 1)(3 - x)$ for $-3.0 \leq x \leq 5.0$

x	-2	-1	0	1	1.5	2	2.5	3	4	5
y		0	3				1.75			

a. Copy and complete the table.

b. Taking 2cm as 1 unit on x – axis and 2cm to 2 units on y – axis , draw the graph of the relation for the given interval.

c. Use the graph to find;

i. the truth set of $(x + 1)(3 - x) = 0$,

ii. the greatest value of y,

iii. the value of x for which y is the greatest

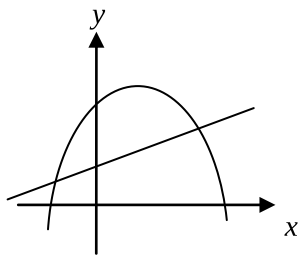
iv. in the given range, the value of x for which y decreases as x increases

Solving Quadratic and a Linear Equation Simultaneously on the Same Graph

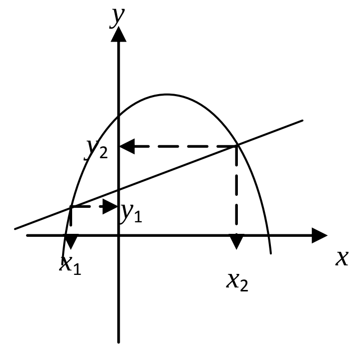
To solve a quadratic and a linear equation on the same graph:

I. Prepare or copy and complete the table of values for each of the equations.

II. Draw the graph of the curve (quadratic equation) and the straight line (linear equation) on the same graph sheet.



III. Identify the point of intersection of the curve and the straight line as the solution set of the two equations as shown in the diagrams below:



Worked Examples

1. a. Copy and complete the following table for $y = 3 + 2x - x^2$ for the interval $-2 \leq x \leq 4$

x	-2	-1	0	1	1.5	2	2.5	3	3.5	4
y		0	3				1.75			-5

b. Taking 2cm as 1 unit on both axes draw the graph of the relation for the given interval

c. Draw on the same axes, the graph of $x - y = 0$

d. Using your graphs solve $3 + 2x - x^2 = x$

Solution

a. In $y = 3 + 2x - x^2$

When $x = -2$, $y = 3 + 2(-2) - (-2)^2 = -5$

When $x = 1$, $y = 3 + 2(1) - (1)^2 = 4$

When $x = 1.5$, $y = 3 + 2(1.5) - (1.5)^2 = 3.75$

When $x = 2$, $y = 3 + 2(2) - (2)^2 = 3$

When $x = 3$, $y = 3 + 2(3) - (3)^2 = 0$

When $x = 3.5$, $y = 3 + 2(3.5) - (3.5)^2 = -2.25$

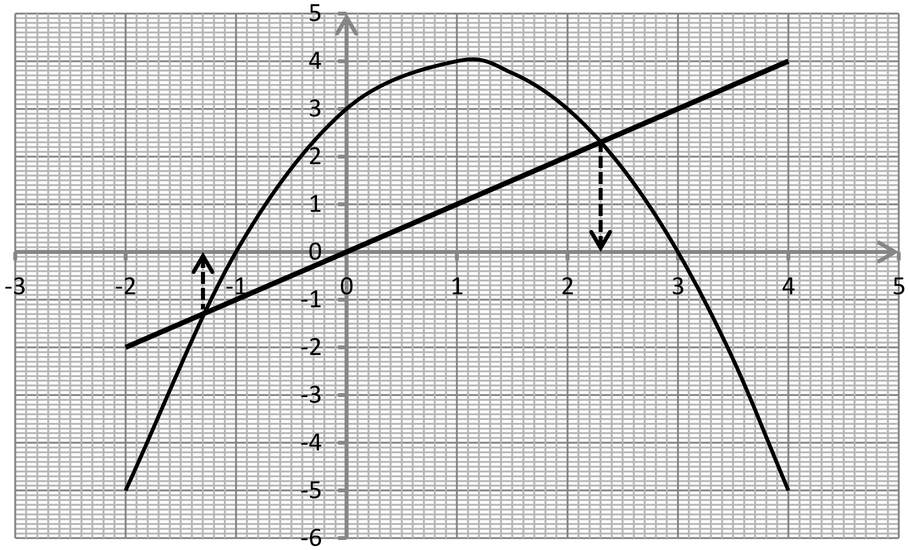
Completed table of values for $y = 3 + 2x - x^2$

x	-2	-1	0	1	1.5	2	2.5	3	3.5	4
y	-5	0	3	4	3.75	3	1.75	0	-2.25	-5

From the relation $x - y = 0$, $x = y$

Completed table of values for $x - y = 0$

x	-2	-1	0	1	1.5	2	2.5	3	3.5	4
y	-2	-1	0	1	1.5	2	2.5	3	3.5	4



d. From the graph the truth set of the equation $3 + 2x - x^2 = x$ is $\{x : x = 2.3 \text{ or } x = -1.3\}$

2. a. Using a scale of 2cm to represent 1 unit on the x – axis and 2cm to 2 units on the y – axis, draw the graph of the following relations for the interval $0 \leq x \leq 6$

i. $y = 1 + 6x - x^2$

ii. $y = 11 - x$

b. Use your graph to find;

i. the greatest value of $1 + 6x - x^2 = 0$

ii. the truth set of the simultaneous equation $y = 1 + 6x - x^2$ and $x + y = 11$

Solution

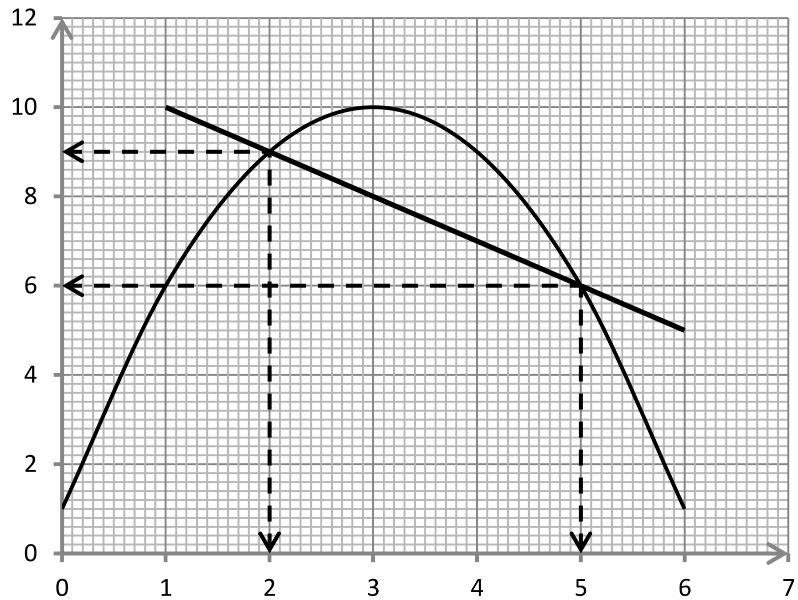
a. i. Table of values for $y = 1 + 6x - x^2$

x	0	1	2	3	4	5	6
1	1	1	1	1	1	1	1
$6x$	0	6	12	18	24	30	36
$-x^2$	0	-1	-4	-9	-16	-25	-36
y	1	6	9	10	9	6	1

ii. Table of values for $x + y = 11$

$\Rightarrow y = 11 - x$

x	0	1	2	3	4	5	6
11	11	11	11	11	11	11	11
$-x$	-0	-1	-2	-3	-4	-5	-6
y	11	10	9	8	7	6	5



b. i. From the graph, the greatest value of y is 10.
ii. The truth set of the simultaneous equation;
$y = 1 + 6x - x^2$ and $x + y = 11$ is $x = 2$ or $x = 5$.

Exercises 21.10

A. 1. On the same set of axes and taking values of x from – 4 to 4, draw the graphs of $y = x^2 - 2$ and $y = 4 - x$, using a scale of 2cm to 1 unit on x – axis and 2 cm to 2 units on y axis . Write down the coordinates of the two positions where the graphs cross.

2. 1. a. Copy and complete the following table for $y = 3 + 2x - x^2$ for the interval $-2 \leq x \leq 4$

x	-2	-1	0	1	1.5	2	2.5	3	3.5	4
		0				3		0		

b. Taking 2cm as 1 unit on x – axis and 2 cm to 2 units on y - axis, draw the graph of the relation for the given interval.
c. Draw on the same axes, the graph of $x - y = 0$
d. Using your graphs to;
i. solve the equation $3 + 2x - x^2 = x$,
ii. find the values of x for which $3 + 2x - x^2 = 2$.

3.a. Draw a table of values for the relation:
$(x - 1)(x - 1) = 6$ for $-4 \leq x \leq 7$
b. Using a scale of 2 cm to represent 2 units on x – axis and 2cm to 5 units on $y$axis, draw the graphs;
i. $(x - 1)(x - 1) = 6$,
ii. $x + y = 8$ within the same interval.

c. Use your graphs to find, correct to two significant figures, the truth set of the simultaneous equations $(x - 1)(x - 1) = 6$ and $x + y = 8$

B. Using a suitable scale, solve the following pair of equations graphically, for the range $-4 \leq x \leq 4$.

1. $y = x^2 - 5x$
   $y = x + 4$
2. $y = 6x - x^2$
   $y = x - 2$
3. $y = x^2 + 4x - 5x$
   $y = x + 6$
4. $y = x^2 + x - 1$
   $y = 2x$

5. $y = x^2 - 8x - 7$
$2y = x - 17$

6. $x^2 - y - 5 = 0$
$4x + 3y + 6 = 0$

Using the Graph of One equation to find the Truth Set of Related Equations

I. Name the equation of the curve as eqn (1)
II. Name the given or related equation as equation (2)
III. Ensure the two given equations have the same coefficients of x and x^2 and each is equated to zero.
IV. Subtract equation (2) from equation (1) to obtain the value of the line y.
V. Draw line y to touch the curve.
VI. Identify the value(s) of x at which line y intersects the curve as the truth set of the related equation or truth set of eqn (2)

Worked Examples

1. a. Copy and complete the table of values below for the relation $y = 10 + 6x - 3x^2$ for the range $-3 \leq x \leq 5$

x	-3	-2	-1	0	1	2	3	4	5
y				10	13		1	-14	

b. Using a scale of 2cm to1 unit on the x – axis and 2 cm to 5 units on the y – axis, draw the graph of the relation for the given intervals.
c. From the graph, find the truth set of the equations;
i. $10 + 6x - 3x^2 = 0$
ii. $5 + 2x - x^2 = 0$

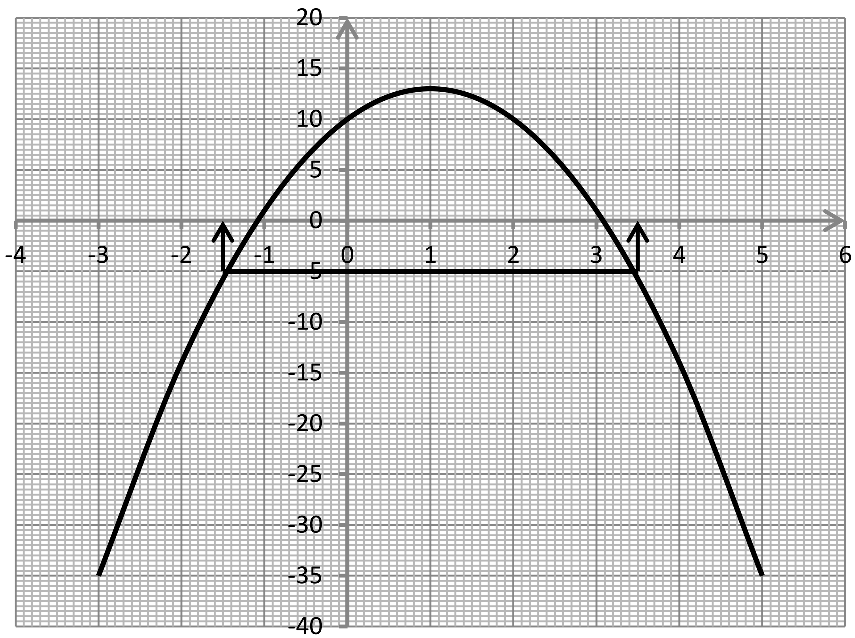
Solution

a. $y = 10 + 6x - 3x^2$
When $x = -3$, $y = 10+ 6(-3) - 3(-3)^2 = -35$
When $x = -2$, $y = 10+ 6(-2) - 3(-2)^2 = -14$
When $x = -1$, $y = 10+ 6(-1) - 3(-1)^2 = 1$
When $x = 2$, $y = 10+ 6(2) - 3(2)^2 =10$
When $x = 5$, $y = 10+ 6(5) - 3(5)^2 = -35$

Completed table of values

x	-3	-2	-1	0	1	2	3	4	5
y	-35	-14	1	10	13	10	1	-14	-35

b.



c. i. The curve cuts the x – axis at x = -1.1 and x = 3.1. Therefore, the truth set is {x :x = -1.1 or x = -3.1}

ii. From $10 + 6x - 3x^2 = 0$
$y = 10 + 6x - 3x^2$ ……………(1)

From $5 + 2x - x^2 = 0$
$y = 5 + 2x - x^2$ ………………(2)
3 × eqn (2)
$y = 15 + 6x - 3x^2$…………......(3)

eqn (1) – eqn (3)
y = -5 (shown on the graph)
From the graph, when y = -5, x = -1.5 or x= 3.5. Therefore the truth set of $y = 5 + 2x - x^2$ is
{x : x = -1.5 or x = 3.5}

2. Copy and complete the following table of values for the relation $y = 4 + 5x - 2x^2$ for $-3 \leq x \leq 5$

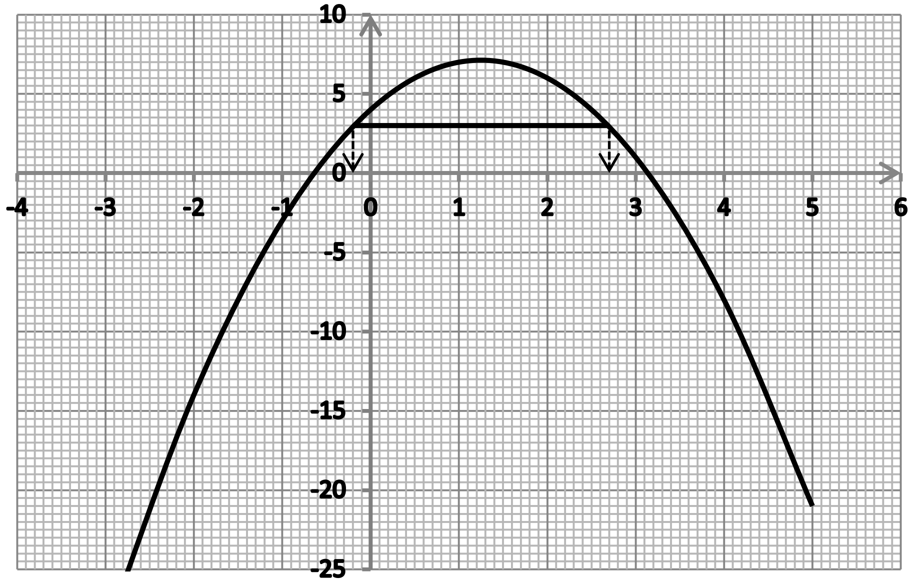
x	-3	-2	-1	0	1	2	3	4	5
y	-29		-3		7			-8	

b. Using 2 cm to 1 unit on the x – axis and 2 cm to 5 units on the y – axis, draw the graph of $y = 4 + 5x - 2x^2$
c. From your graph, find the values of x for which $1 + 5x - 2x^2 = 0$

Solution
In $y = 4 + 5x - 2x^2$,
When$x$ = -2, $y = 4 + 5(-2) - 2(-2)^2 = -14$
When$x$ = 0, $y = 4 + 5(0) - 2(0)^2 = 4$
When$x$ = 2, $y = 4 + 5(2) - 2(2)^2 = 6$
When$x$ = 3, $y = 4 + 5(3) - 2(3)^2 = 1$
When$x$ = 5, $y = 4 + 5(5) - 2(5)^2 = -21$

x	-3	-2	-1	0	1	2	3	4	5
y	-29	-14	-3	4	7	6	1	-8	-21

b.



c. From $4 + 5x - 2x^2 = 0$

$y = 4 + 5x - 2x^2$.........eqn (1)

From $1 + 5x - 2x^2 = 0$

$y = 1 + 5x - 2x^2$..........eqn (2)

eqn (1) – eqn (2)

$y = (4 - 1) + (5x - 5x) - 2x^2 - (-2x^2)$

$y = 3$

From the graph, when $y = 3$, $x = -0.2$ or $x = 2.7$.

Therefore, the values of x for which $1 + 5x - 2x^2 = 0$ is $x = -0.2$ or $x = 2.7$

Exercises 21.11

1. a. Copy and complete the table of values below for the relation $y = x^2 - 4x - 21$

x	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
y	11	0			-21	-24			-21	-16	-9	0	11

b. Using a scale of 2 cm to 2 units on the x – axis and 2 cm to 5 units on the y – axis, draw the graph of the relation

c. Use your graph to find:

i. the truth set of $x^2 - 4x - 16 = 0$

ii. the range of values of x for which $y > 0$

2. a. Copy and complete the following table for $y = 7 + 4x - 3x^2$, for the interval $-3 \leq x \leq 4$

x	-3	-2	-1	0	0.5	1	1.5	2	2.5	3	3.5	4
y	-32			7	8.25			3		-8	-15.75	

b. Taking 2cm as 1 unit on x – axis and 2 cm to 5 units on the y – axis, draw the graph of $y = 7 + 4x - 3x^2$, for the given interval.

c. Draw on the same axes, the graph of the relation $y + 2x + 2 = 0$

d. Using your graphs:

i. solve the equation $9 + 6x - 3x^2 = 0$

ii. find the values of x for which $7 + 4x - 3x^2 = 0$

iii. the range of values of x for which $y = 7 + 4x - 3x^2$ is negative.

3.The following is an incomplete table for the relation $y = 2x(4x - 7) - 9$, where $-2 \leq x \leq 4$

x	-2	-1	0	0.5	1	1.5	2	3	4
y	51		-9						63

a. Copy and complete the table.

b. Using a scale of 2cm to 1 unit on the x – axis and 2 cm : 10cm on the y – axis, draw the graph of the relation.

c. Estimate, correct to one decimal place;

i. the x – coordinates of the point starts increasing with respect to y,

ii. the truth set of $2x(4x - 7) + 4 = 0$.

4. Copy and complete the following table for $y = 4 + 3x - x^2$, for the interval $-2 \leq x \leq 5$

x	-2	-1	0	1	2	3	4	5
y	-6		4				0	

b. Using a scale of 2 cm to 1 unit on the x – axis and 2 cm to 2 units on the y – axis, draw on the same graph sheet, the graphs of the relations:

i. $y = 4 + 3x - x^2$,

ii. $2y + 3x = 6$.

c. Use your graph to find:

i. $4 + 3x - x^2 = 0$,

ii. $4 + 3x - x^2 = 3 - \frac{3}{2}x$,

iii. the maximum value of $y = 4 + 3x - x^2$.

5. a. Copy and complete the following table of values for the relation $y = x^2 - 2x + 5$ for the interval $-2 \leq x \leq 5$

x	-2	-1	0	1	2	3	4	5
y		8				8		2

b. Using a scale of 2cm to 1 unit on the x – axes and 2 cm to 2 units on the y – axis, draw the graph of the relation for the interval $-2 \leq x \leq 5$

c. Use your graph to determine:

i. the truth set of $x^2 - 2x - 2 = 0$

ii. the coordinates of the lowest point on the graph.

iii. the range of values of x for which y increases as x increases.

6. Draw the graph of $y = x^2 - 2x + 2$ for the interval $-2 \leq x \leq 4$, using a suitable scale. Use your graph to find the solution of the equation: $x^2 - 6x + 8 = 0$

7. Draw the graph of $y = -1 + 3x - x^2$ for the interval $-4 \leq x \leq 3$, using a suitable scale. Use your graph to find the solution of the equation $-3 + 4x - x^2 = 0$

Other Applications

Worked Examples

1. Below is an incomplete table of values for the relation $y = x^2 + mx + n$, where m and n are constants for the interval $-1.5 \leq x \leq 4.5$

x	-1.5	-1.0	0	1	2	3	4	4.5
y				-6				2.8

Use the table to find the values of m and n and use it to complete the table

b.Using a scale of 2 cm to 1 unit on both axes, draw the graph of the relation for the given intervals.

Solution

From the table, when $x = 1$, $y = -6$

Substitute in $y = x^2 + mx + n$

$\Rightarrow -6 = (1)^2 + m(1) + n$

$-6 = 1 + m + n$……………(1)

When $x = 4.5$, $y = 2.8$

Substitute in $y = x^2 + mx + n$

$\Rightarrow 2.8 = (4.5)^2 + m(4.5) + n$

$2.8 = 20.25 + 4.5m + n$……………(2)

eqn (2) – eqn (1)

$2.8 - (-6) = (20.25 - 1) + (4.5m - m) + (n - n)$

$8.8 = 19.25 + 3.5m$

$8.8 - 19.25 = 3.5m$

$-10.45 = 3.5m$

$m = \frac{-10.45}{3.5} = -3$

Put $m = -3$ into eqn (1)

$-6 = 1 - 3 + n$

$-6 - 1 + 3 = n$

$-4 = n$

$\Rightarrow n = -4$

Put $m = -3$ and $n = -4$ in $y = x^2 + mx + n$

$\Rightarrow y = x^2 + (-3)x + (-4)$

$y = x^2 - 3x - 4$

The required equation is $y = x^2 - 3x - 4$

When $x = -1.5$, $y = (-1.5)^2 - 3(-1.5) - 4 = 2.75$

When $x = -1$, $y = (-1)^2 - 3(-1) - 4 = 0$

When $x = 0$, $y = (0)^2 - 3(0) - 4 = -4$

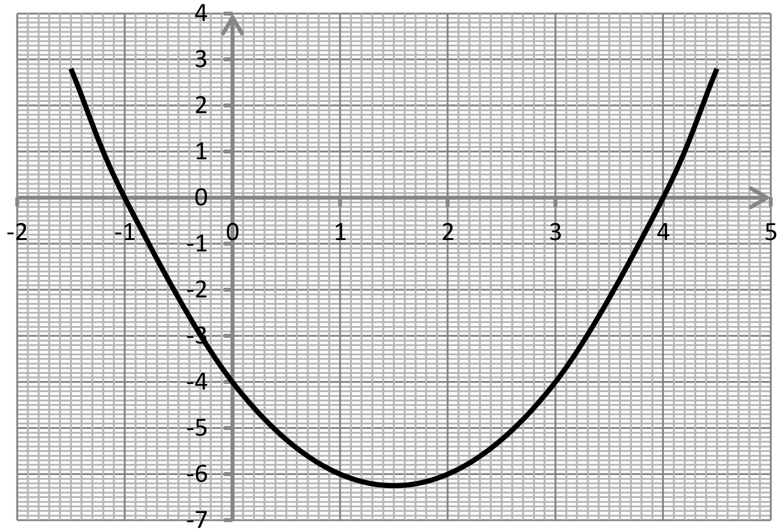
When $x = 2$, $y = (2)^2 - 3(2) - 4 = -6$

When $x = 3$, $y = (3)^2 - 3(3) - 4 = -4$

When $x = 4$, $y = (4)^2 - 3(4) - 4 = 0$

Completed table of values

x	-1.5	-1.0	0	1	2	3	4	4.5
y	2.8	-1	0	- 6	- 6	- 4	0	2.8



2. The roots of a quadratic equation $mx^2 + nx + r = 0$ are $\frac{1}{2}$ and -3

a. Find the values of the constants m, n and r

b. Using a scale of 2 cm to 1 unit on x – axes and 2 cm to 5 units on y - axis, draw the graph of the relation $y = mx^2 + nx + r$ for the intervals $-5 \leq \boldsymbol{x} \leq 4$

c. Use your graph to find the truth set of $2x^2 + 5x - 18 = 0$

Solution

a. $x^2 -$ *(sum of roots)*x + *(product of roots)* = 0

$x = \frac{1}{2}$ *or* $x = -3$

$x^2 - \left(\frac{1}{2} + -3\right)x + \left(\frac{1}{2} \times -3\right) = 0$

$x^2 - \left(-\frac{5}{2}\right)x + \left(-\frac{3}{2}\right) = 0$

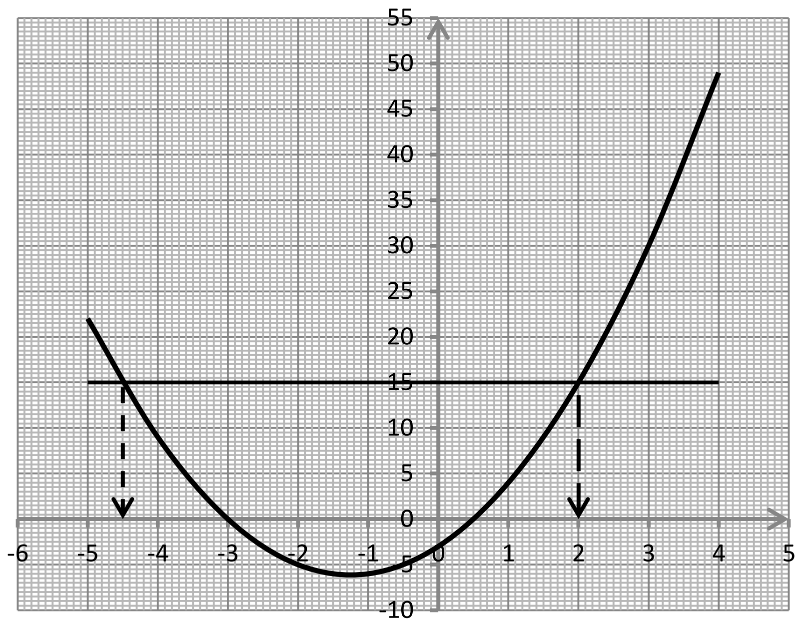
$x^2 + \frac{5}{2}x - \frac{3}{2} = 0$

$2x^2 + 5x - 3 = 0$ compared to $mx^2 + nx + r = 0$

$\Rightarrow m = 2$, $n = 5$ *and* $r = -3$

b. In $2x^2 + 5x - 3 = 0$

x	-5	-4	-3	-2	-1	0	1	2	3	4
$2x^2$	50	32	18	8	2	0	2	8	18	32
$+5x$	-25	-20	-15	-10	-5	0	5	10	15	20
-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3
y	22	9	0	-5	-6	-3	4	15	30	49



c. $2x^2 + 5x - 3 = y$ …………..(1)
$2x^2 + 5x - 18 = y$………….(2)

eqn (1) – eqn (2)
$y = 15$ **(Shown on the graph)**

The line $y = 15$ cuts the curve at $x = -4.5$ and $x = 2$. The truth set of $2x^2 + 5x - 18 = 0$ is $\{x : x = -4.5 \text{ or } x = 2\}$

Exercises 21.12

1.Copy and complete the following table for $y = 7 + 3x - x^2$, for the range $-3 \le x \le 5$

x	-3	-2	-1	0	1	2	3	4	5
y	-11			7					

b. Using a scale of 2 cm to 1 unit on the x – axis and 2 cm to 2 units on the y – axis, draw the graph of the relation $y = 7 + 3x - x^2$ for the given interval
c. Use your graph to find:
i. the solution set of $7 + 3x - x^2 = 0$,
ii. the solution set of $5 + 3x - x^2 = 0$,
iii. the maximum value of y.

2. The table is for the relation $y = px^2 - 5x + q$

x	-3	-2	-1	0	1	2	3	4	5
y	21	6		-12				0	13

a. i. Use the table to find dthe values of p and q
ii. Copy and complete the table.
b. Using scales of 2 cm to 1 unit on x – axis and 2 cm to 5 units on the y – axis draw the graph of the relation for $-3 \le x \le 5$.

2. Copy and complete the table for the relation $y = 2 + 2x - x^2$, for the interval $-1 \le x \le 3$

x	-1	-0.5	0	0.5	1	1.5	2	2.5	3
y	-1	0.75	2						

b. Using a scale of 4 cm to 1 unit on both axes, draw the graph of the relation
c. Draw on the axes, the graph of $x + y = 3$
d. Using your graphs;

i. solve the equation $2 + 2x - x^2 = 3 - 1$,
ii. find the truth set of $2 + 2x - x^2 = 1$.

The Gradient of a Curve at a Given Point

All straight lines have a fixed slope or gradient, known as "*m*" in the equation $y = mx + c$. Quadratic graphs on the other hand, have no such fixed or constant gradient. As the graph is curved, the value of the gradient changes all the time based on its position on the graph.

The gradient of a curve at a given point is determined as follows:
I. Identify the given point on the curve
II. Draw a tangent to the curve at that point
III. Use the tangent to construct a right angled triangle
IV. From the triangle, calculate the gradient by the formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$

Drawing the Tangent

A tangent is a straight line which touches the curve at one point. The tangent is drawn by placing a ruler so that it just touches the graph at the required point, and then adjust the ruler so that it just touches the graph at the given point.

Worked Examples

1. a. Copy and complete the table below for $y = x^2 - 4x + 3$ for -2 to -6

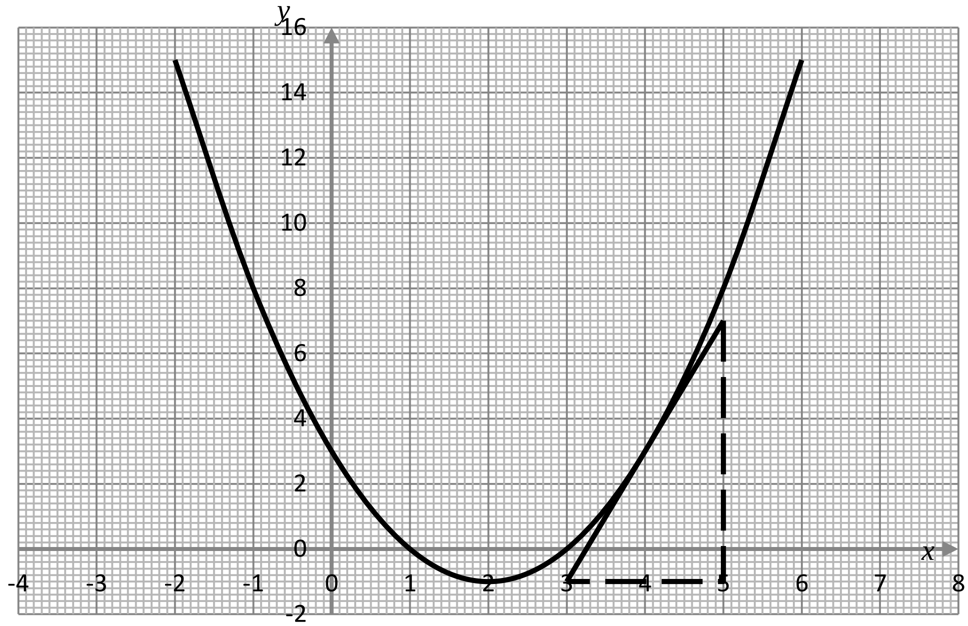
x	-2	-1	0	1	2	3	4	5	6
y	15		3			0			15

b. Using scale of 2cm to 1 unit on x – axis and 2cm to 2 units on y – axis, draw the graph of $y = x^2 - 4x + 3$
c. From the graph, determine the gradient of the curve the points $x = 4$

Solution

a. In$y = x^2 - 4x + 3$
When $x = -1$, $y = (-1)^2 - 4(-1) + 3 = 8$
When $x = 1$, $y = (1)^2 - 4(1) + 3 = 0$
When $x = 2$, $y = (2)^2 - 4(2) + 3 = -2$
When $x = 4$, $y = (4)^2 - 4(4) + 3 = 3$
When $x = 5$, $y = (5)^2 - 4(5) + 3 = 8$

x	-2	-1	0	1	2	3	4	5	6
y	15	8	3	0	-1	0	3	8	15



c. From the graph (triangle):
$x_1 = 3$, $x_2 = 5$, $y_1 = -1$, $y_2 = 7$,

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - -1}{5 - 3} = \frac{8}{2} = 4$

2. a. Copy and complete the following table of values for the relation $y = 4 + 5x - 2x^2$ for the intervals $-3 \leq x \leq 5$

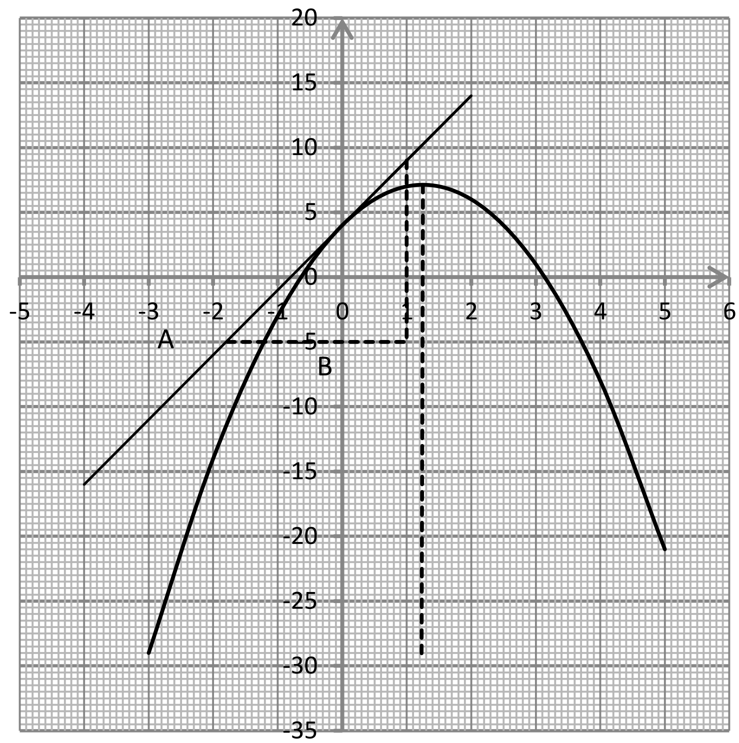
x	-3	-2	-1	0	1	2	3	4	5
y		-14		4	7				-21

b. Using 2cm to 1 unit and on the x – axis and 2cm to 5 units on the y – axis, draw the graph of $y = 4 + 5x - 2x^2$ for $-3 \leq x \leq 5$

c. From your graph , find;
i. the value of x for which y is maximum,
ii. the gradient at $x = 0$.

a. **Solution**
In $y = 4 + 5x - 2x^2$,
When $x = -2$, $y = 4 + 5(-2) - 2(-2)^2 = -14$
When $x = 0$, $y = 4 + 5(0) - 2(0)^2 = 4$
When $x = 2$, $y = 4 + 5(2) - 2(2)^2 = 6$
When $x = 3$, $y = 4 + 5(3) - 2(3)^2 = 1$
When $x = 5$, $y = 4 + 5(5) - 2(5)^2 = -21$

x	-3	-2	-1	0	1	2	3	4	5
y	-29	-14	-3	4	7	-14	1	-8	-21



c. From the graph:

i. the value of x for which y is maximum is
$x = 1.3$

ii. Gradient at $x = 0$

From the graph (triangle):

$x_1 = -1.8$, $x_2 = 1$, $y_1 = -5$, $y_2 = 9$,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - -5}{1 - -1.8} = \frac{14}{2.8} = \frac{1}{2}$$

Exercises 21.13

1. Draw the graph of $y = x^2$, using a scale of 2cm to1 unit on x – axis and 2cm to 2 units on y – axis . Determine gradient of the curve at the point (9,3).

2. Using a scale of 2cm to1 unit on x – axis and 2cm to 10 units on y – axis, draw the graph of $x^2 - 3x + 2 = 0$ for the interval, -2 to 4

ii. Find the gradient of the curve at the point (3, 2)

3. i. Draw the graph of $y = x(6 - x)$ from $x = 0$ to $x = 6$, taking 2cm to 1 unit on the x – axis, and 1 cm to 1 unit on the y – axis, and plotting points for which x = 1, 2, 2.5, 3, 3.5, 4, 5, 6

ii. Find the gradient of the curve at the point (4, 8) Ans: -2

iii. state the coordinates of the points at which the gradient of the curve is zero Ans : (3, 9)

4. i. Draw the graph of $y = 8x - x^2$ from $x = 0$ to $x = 8$, taking 2 cm to 1 unit on the x – axis, and 1 cm to 5 units on the y – axis.

ii. Find the gradient of the curve at the points where $x = 3$ Ans: 2

iii. Use the answers in ii to give the gradients where x = 5, 6 and 7 . Ans : - 2, - 4, - 6

5. Copy and complete the table below for the curve $y = 2x^2 - x - 15$

x	-4	-3	-2	-1	0	1	2	3	4
y	-21				-15				13

i. Draw the graph of $y = 2x^2 - x - 15$, to a scale of 2 cm to 1 unit on x axis and 2 cm to 5 units on y – axis.
ii. Find the gradient of the curve at the point where$x = -1$ **Ans: - 4.83**

6. Taking a scale of 2 cm to 1 unit on each axis, plot the points given by the following table on a graph sheet, and draw a smooth curve through them.

x	0	1	2	3	4	4.6	4.9	5
y	0	0.1	0.4	1	2	3	4	5

i. Draw the tangents to the curve at the points where $x = 2$ and $x = 4$
ii. Find the gradient of the curve at each of these point to 1 decimal place Ans: 0.3, 1.4

7. a. Copy and complete the following table of values for the relation $y = -x^2 + x + 2$ for the interval$-3 \leq x \leq 3$

x	-3	-2	-1	0	1	2	3
y		-4					-4

b. using scales of 2cm to 1 units on the x – axis and 2cm to 2 units on the y – axis draw a graph of the relation $y = -x^2 + x + 2$
c. From the graph, find the ;
i. the maximum value of y
ii. roots of the equation
iii. gradient of the curve at $x = -0.5$

Quadratic Equation and its Roots

The roots of a quadratic equation of the form $ax^2 + bx + c = 0$ is the values of x that satisfy the equation. It is also called the *truth set or solution set.*

Quadratic equations of the form:$ax^2 + bx + c = 0$ is related to its roots as shown below;
x^2 – (sum of roots)x + (product of roots)= 0
⇒ I. ***sum of roots = b***
II. ***product of roots = c***

Forming an Equation from its Roots

Given the roots of the equation as m and n,
I. Find the sum of roots as$b = m + n$ and the product of roots as$c = m \times n$.
II. Substitute $b = m + n$and$c = m \times n$ in the equation:
x^2– (sum of roots)x + (product of roots) = 0
to obtain $\boldsymbol{x^2 - (m + n)x + (m \times n) = 0}$
III. Simplify to obtain an equation in the form: $\boldsymbol{ax^2 + bx + c = 0}$

Worked Examples

1. Find the quadratic equation whose roots are 5 and 3

Solution

Method 1

Roots 5 and 3
Sum of roots, $b = 5 + 3 = 8$
Product of roots, $c = 5 \times 3 = 15$
By substitution,
x^2 – (sum of roots)x + (product of roots)= 0
$x^2 - 8x + 15 = 0$
The quadratic equation is $x^2 - 8x + 15 = 0$

Method 2

Roots of the equation are 5 and 3
Let the roots of the equation be x

$x = 5$ or $x = 3$
$x - 5 = 0$ or $x - 3 = 0$
$(x - 5)(x - 3) = 0$
$x(x - 3) - 5(x - 3) = 0$
$x^2 - 3x - 5x + 15 = 0$
$x^2 - 8x + 15 = 0$

2. Find the quadratic equation whose roots are – 4 and 7

Solution
Roots of the equation are – 4 and 7
Sum of roots, $b = -4 + 7 = 3$
Product of roots, $c = -4 \times 7 = -28$
By substitution,
$x^2 -$ *(sum of roots)x + (product of roots) = 0*
$x^2 - 3x - 28 = 0$
The quadratic equation is $x^2 - 3x - 28 = 0$

3. What quadratic equation has the roots -3 and -4?

Solution
Roots of the equation = – 3 and – 4
Sum of roots $= -3 + -4 = -7$
Product of the roots $= -3 \times -4 = 12$
By substitution,
$x^2 -$ *(sum of roots)x + (product of roots)= 0*
$x^2 - (-7)x + 12 = 0$
The quadratic equation is $x^2 + 7x + 12 = 0$

4. If the roots of a quadratic equation are $\frac{2}{5}$ and $\frac{1}{4}$, find its equation

Solution
Roots of the equation are $\frac{2}{5}$ and $\frac{1}{4}$
Sum of roots, $b = \frac{2}{5} + \frac{1}{4} = \frac{8+5}{20} = \frac{13}{20}$
Product of roots, $c = \frac{2}{5} \times \frac{1}{4} = \frac{2}{20}$
By substitution,
$x^2 -$ *(sum of roots)x + (product of roots)= 0*
$x^2 - \frac{13}{20}x + \frac{2}{20} = 0$
Multiply through by 20
$20x^2 - 13x + 2 = 0$
The quadratic equation is $20x^2 - 13x + 2 = 0$

6. A quadratic equation has roots – 3 and $\frac{1}{2}$, find the equation.

Solution
Roots of the equation are 3 and $-\frac{1}{2}$
Sum of roots, $b = 3 + \left(-\frac{1}{2}\right) = \frac{6-1}{2} = \frac{5}{2}$
Product of roots, $c = 3 \times -\frac{1}{2} = -\frac{3}{2}$
By substitution,
$x^2 -$ *(sum of roots)x + (product of roots)= 0*
$x^2 - \frac{5}{2}x - \frac{3}{2} = 0$
Multiply through by 2
$2x^2 - 5x - 3 = 0$
The quadratic equation is $2x^2 - 5x - 3 = 0$

Exercises 21.14
Find equation with the following roots:

1. 2 and 7	2. – 3 and 8
3. – 4 and – 2	4. – 7 and 2
5. – 5 and 6	6. 4 and – 7
7. – 2 and $\frac{3}{7}$	8. 4 and $-\frac{1}{3}$

22 MENSURATION I

The Idea of a Circle

A circle is a set of points in a plane which are at the same distance from a fixed point. The fixed point is called the ***centre of the circle*** and the set of points form the ***circumference*** of the circle.

It can also be defined as the path of s point which moves so that it is always equidistance from a fixed point called thecenter. It is also said to be the path around a circular region

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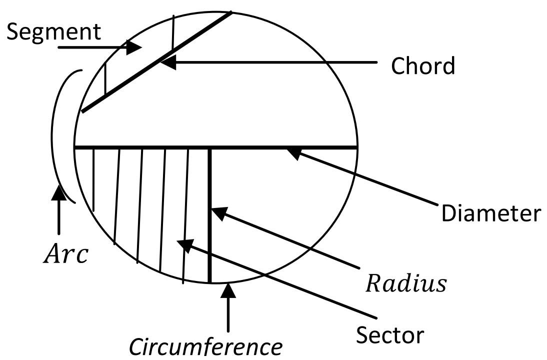
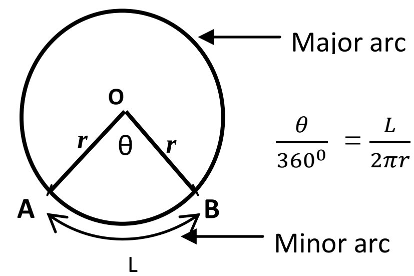
Parts of a Circle



Circumference: It is the distance Round a circular region. It is also the length or perimeter of a circle
Diameter: It is a straight line divides a circle into two equal halves
Semi - circle: It is half of a circle
Chord: It is a straight line that connects any two points on a circle
Arc: It is a any portion of the circle
Segment: It is the area bounded by an arc and a chord
Radius: It is a line drawn from the center of a circle to touch any part of the circumference. The plural is ***radii***
Sector: It is the area bounded by two radii and an arc

Length of an Arc, Area of a Sector and Perimeter of a Sector



If the central angle subtended by the minor arc AOB of a circle of radius r is θ, then by proportion $\frac{\theta}{360^0} = \frac{L}{2\pi r}$
The length of the arc is given by:
$$\mathbf{L} = \frac{\theta}{360^0} \times 2\pi r$$

If the central angle subtended by the major arc AOB of a circle of radius r is $360^0 - \theta$, then the length of the arc is given by:
$$\mathbf{L} = \frac{360^0 - \theta}{360^0} \times 2\pi r$$

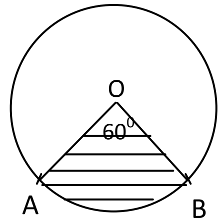
Similarly, if the central angle of a sector of a circle of radius r is θ, then by proportion:
$$\frac{\theta}{360^0} = \frac{A}{\pi r^2}$$
The area of the sector is given by: $\mathbf{A} = \frac{\theta}{360^0} \times \pi r^2$

The perimeter of the sector *AOB*, formed by the radii and the minor arc *AB* is given as:
P = r + r + length of arc
P = 2 r + L

Worked Examples

1. The area of a circle with centre O is 120 cm^2, angle *AOB* is 60^o. Find :
i. the area of the sector *AOB*.
ii. the length of the minor arc *AB*

iii. the perimeter of the monir sector



Solution

i. Area of sector *AOB*

$A = \frac{\theta}{360^0} \times \boldsymbol{\pi r^2}$

But $\theta = 60^o$, $\pi r^{\,2} = 120\text{cm}^2$

$A = \frac{60^0}{360^0} \times 120\text{cm} = 20\text{cm}^2$

ii. Length of the minor arc *AOB* =

$L = \frac{\theta}{360^0} \times 2\pi r$

But $\pi r^{\,2} = 120\text{cm}^2$

$r = \sqrt{\frac{120}{22/7}} = 6\text{cm}$

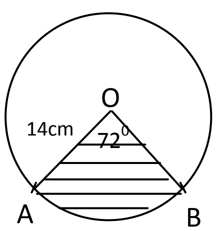
$L = \frac{60^0}{360^0} \times 2 \times \frac{22}{7} \times (6) = 6.29$ cm

iii. The perimeter of the sector

P = 2r + L

P = 2(6) + 6.29 = 18.29 cm

2.



The diagram above shows a circle with centre O and radius 14cm. The shaded region *AOB* is a sector with angle *AOB* = 72º. ($\pi = \frac{22}{7}$). Find:

i. the length of the minor arc *AB*

ii. the area of the shaded sector *AOB*.

iii. the perimeter of the shaded region

Solution

Length of arc *AB*,

$L = \frac{\theta}{360^0} \times 2\pi r$

But $\theta = 72^o$, $r = 14$cm and $\pi = \frac{22}{7}$

$L = \frac{72^0}{360^0} \times \frac{22}{7} \times 14 = 17.6$cm.

ii. Area of the shaded sector *AOB*,

$A = \frac{\theta}{\mathbf{360^0}} \times \boldsymbol{\pi r^2}$

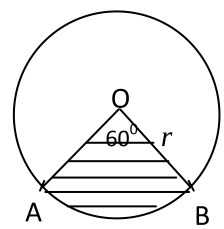
$\boldsymbol{A} = \frac{72^0}{360^0} \times \frac{22}{7} \times (14)^2 = 123.2\text{cm}^2$

iii. P = 2*r* + L.

But *r* = 14cm and L = 17.6cm

P = 2(14cm) + 17.6cm = 45.6cm

3. In the diagram below, O is the centre of the circle and *r* is its radius. Calculate the area of the non-shaded region.

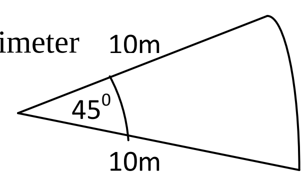


Solution

Area of the non-shaded region

$= \frac{360^0 - 60^0}{360^0} \times \pi r^2 = \frac{300^0}{360^0} \times \pi r^2 = \frac{5}{6}\pi r^2$

4. Calculate the perimeter of the figure below.



Solution

$\theta = 45^o$, $r = 10$m

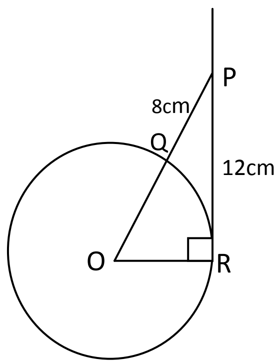
P = 2 *r* + L

$L = \frac{\theta \pi r}{180^0} = \frac{22 \times 10 \times 45}{7 \times 180^0} = 7.86$ (2 d. p)

P = 2 *r* + arc length

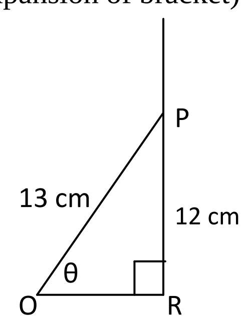
P = 2 × (10) + 7.86 = 27.86m.

5. In the diagram below, O is the center of the circle and PR is a tangent to the circle at R. If $|PR|$ = 12cm and $|PQ|$ = 8cm, calculate:
i. the radius of the circle;
ii. length of the minor arc QR, correct to two decimal places ($\pi = \frac{22}{7}$)



Solution
Let the radius of the circle be x,
OR= OQ = x.

By Pythagoras theorem,
$12^2 + x^2 = (8 + x)^2$
$144 + x^2 = 64 + 16x + x^2$ (expansion of bracket)
$x^2 - x^2 - 16x = 64 - 144$
$-16x = -80$
$x = \frac{80}{16}$
$x = 5$ cm



Length of minor arc QR,
$L = \frac{\theta}{360^0} \times 2\pi r = \frac{\theta \pi r}{180^0}$
But angle, θ , subtended by QR
$\sin\theta = \frac{opp}{hyo} = \frac{12}{13}$
$\theta = sin^{-1}\left(\frac{12}{13}\right) = 67^0$

Length of the minor arc QR,
$L = \frac{67^0}{360^0} \times 2 \times \left(\frac{22}{7}\right) \times 5 = 5.85$cm

6. The area of a sector of a circle of radius 12 cm, is 130cm^2. If the sector is folded such that is straight edges coincides to form a cone, find the radius of the base of the cone. $\pi = \frac{22}{7}$

Solution
2. a. $130 = \frac{\theta}{360} \times \frac{22}{7} \times 12 \times 12$
$327600 = 3168\theta$
$\theta = 103^0$

Length of arc of sector,
$L = \frac{103^0}{360^0} \times \frac{22}{7} \times 2 \times 12 = 21.7$cm

Base of the cone = Length of arc of sector
$2\pi r = 21.7$
$r = \frac{21.7}{2\pi} = 3.45$ cm

Exercises 22.1
1. The arc PQ subtends an angle of 40^0 at the center of a circle of radius 3cm. What is:
i. the length of the arc PQ?
ii. the area of the sector the angle subtends?
iii. the perimeter of the sector the angle subtends?

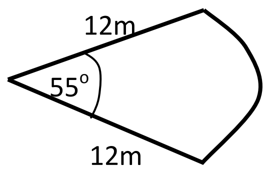
2. The angle of a sector of a circle of diameter 8cm is 135^o. Find;
i. the area of the sector,
ii. the length of the minor arc,
iii. the perimeter of the sector.

3. A circle of radius 6m subtends an angle of 105^0 at the center of the circle. Calculate:
i. the perimeter of the sector,
ii. the area of the sector (л $=\frac{22}{7}$)

4. A sector of a circle subtends an angle of 135^o at the center of a circle. If the radius of the circle is 14cm, calculate;
i. the length of the minor arc,
ii. the area of the sector,

iii the perimeter of the sector.

5. A garden plot is in the form of a sector of a circle as shown below. The boundary is an arc of a circle of radius and two radii each of length 10m, the central angle is 45^o. What is the perimeter of the plot?



Challenge Problems

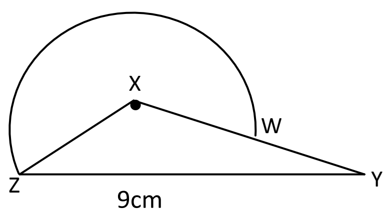
1. In the diagram below, triangle *XYZ* has *XY* = 6cm, *YZ* = 9 cm *XZ* = 4 cm. The point W lies on the line *XY*. The circular arc *ZW* is the major arc of the circle with centre *X* and radius 4cm.

i. find < *ZXY;*

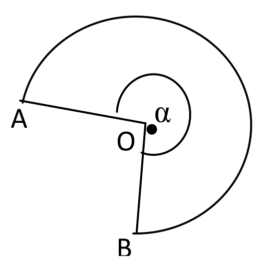
ii. the area of the major sector *XZWX;*

iii. the area of the whole figure;

iv. the perimeter *XWYZ.*

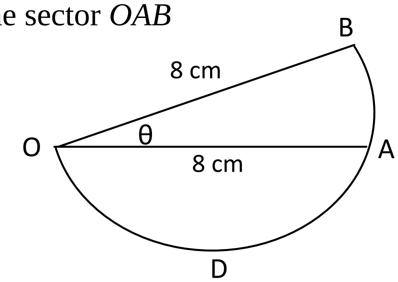


2. A sector *AB* is to be cut from a circle of cardboard with radius 25cm, and then folded so that radii *OA* and *OB* are joined to form a cone, with a slant height 25cm.



If the vertical height of the cone is to be 20 cm, what must the angle α be?

3. In the diagram below, *OAB* is a sector of circle with center O and radius 8cm. < *BOA* is in degrees. *OAC* is a semi-circle with diameter *OA*. The area of the semicircle *OAC* is twice the area of the sector *OAB*



i. Find the value of θ

ii. Find the perimeter of the complete figure in terms of π.

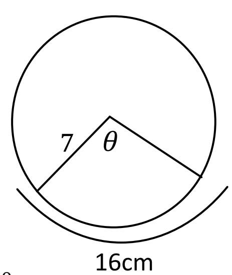
Finding the Central Angle

Given the radius, *r*, and the arc length,*l*, the central angle subtended by the arc, θ, is calculated by the relation: $\boldsymbol{\theta = \frac{360^0\, l}{2\pi\, r}}$

Given the radius of a circle, *r*, and the area of a sector, A, formed by the radii, the central angle is calculated by the relation: $\boldsymbol{\theta = \frac{360^0\, A}{\pi r^2}}$

Worked Examples

1. The diameter of a circle is 14cm. Find the angle which an arc of length 16cm subtends at the center of the circle



Solution

$\theta = \frac{360^0\, l}{2\pi\, r}$

$= \frac{360^0 \times 16}{2 \times 7 \times 22/7}$

$= \frac{360^0 \times 16\ \times 7}{2 \times 7 \times\ 22} = 131^0$

2. The area of the sector of a circle is $66cm^2$. If the radius of the the circle is 9cm, calculate the angle of the sector ($\pi = 3.14$)

Solution

Area of sector , A = 66, r = 9cm, and θ = ?

Substitute in $\theta = \frac{360^0\ A}{\pi r^2}$

$\theta = \frac{360^0 \times 66}{(3.14)\times 9^2} = \frac{23760}{254.34} = 93^0$

Exercises 22.2

1. The area of the sector of a circle with radius 6cm is 35.4cm^2. Calculate the angle of the sector (π = 3.14)

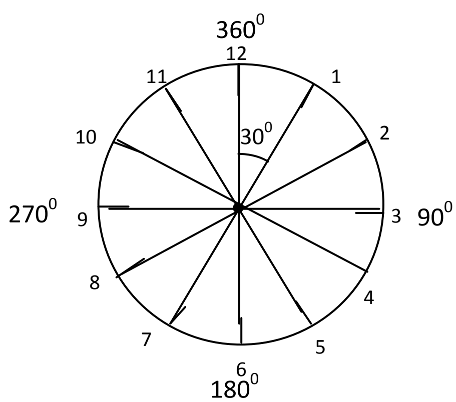
2. The area of the sector of a circle is 52.7cm^2. If the radius of the the circle is 8cm, calculate the angle of the sector (π = 3.14)

3. The length of an arc of a circle is 33 cm. Calculate the angle it substends at the center of the circle if its radius is 24cm. (π = 3.14)

Application to the Clock

The hands (minute and hour hands) of the analog clock engage in a circular motion, in the clockwise direction.

Consider that face of the clock below.



It implies that the angle, θ, turned by moving from one unit (hour) to another unit (hour) is 30^0.

Hence, the following conclusions can be made:

If the minute hand moves from say A to B in x minutes, the angle, θ, formed at the center between the points of movement is calculated as:

$\boldsymbol{\theta = \frac{x}{60} \times 360^0}$

Note that the minute arm forms the radius of the circular face of the clock. Hence, given the length of the minute arm, the area, A, covered by moving from one point to another is calculated by the formula: $\boldsymbol{A = \frac{\theta}{360^0} \times \pi r^2}$

The distance covered by the minute arm in moving from say A to B is equivalent to the length, L, of the arc AB and is calculated by the formula:

$\boldsymbol{L = \frac{\theta}{360^0} \times 2\pi r}$

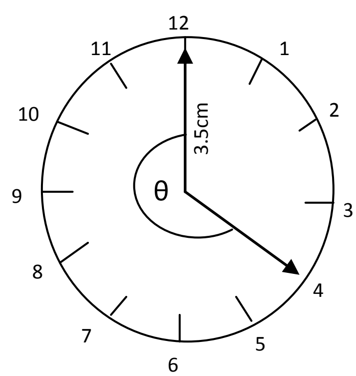
Worked Examples

1.If the minute hand of a clock with length $3^1/_2$cm moves from 9:20 a.m to 10:00 a.m, find: i. the area covered by the minute hand;

ii. the distance covered by the minute hand ($\pi = \frac{22}{7}$)

Solution

i.



Angle in a complete revolution = 360^0

Minutes in a complete revolution = 60

60 minutes = 360^0

Minutes between 9 : 20 and 10 : 00 = 40 min

Angle, θ, between 9 : 30 and 10 : 00

$\theta = \frac{40}{60} \times 360^0 = 240^0$

Area covered by the minute hand,

$A = \frac{\theta}{360^0} \times \pi r^2$

$A = \frac{240^0}{360^0} \times \frac{22}{7} \times (3.5)^2 = 25.6\text{cm}^2$

ii. Distance covered by the minute hand,

$L = \frac{\theta}{360^0} \times 2\pi r$

$L = \frac{240^0}{360^0} \times 2 \times \frac{22}{7} \times 3.5 = 14.67\text{cm}$

Angle between the Minute and Hour Hands

For one complete revolution of the hour hand;

12 hours = 360^0

1 hour = 30^0

For one complete revolution of the minute hand;

60 min = 360^0

1 min = 6^0

To find the angle between the minute and hour hand;

1. Convert the time to hours and multiply by 30^0 to obtain the angle covered by the hour hand.
2. Multiply the minutes by 6^0 to obtain the angle covered by the minute hand.
3. Find the difference between the two values.

Worked Examples

Find the angle between the hour and minute hand when it is 4 : 40 pm?

Solution

Change 4:40 hours;

$4\frac{40}{60} = 4\frac{2}{3} = \frac{14}{3}$ hours

Angle covered by hour hand;

$\frac{14}{3} \times 30^0 = 140^0$

Angle coverd by minute hand;

$40 \times 6^0 = 240^0$

Angle between the hour and minute hand;

$= 240^0 - 140^0 = 100^0$

Exercises 22.3

1. Complete the following table (give the smaller angle in each case)

Time	1:00	2:30	7:00	8:45	10:30	11:20
Angle	30^0	75^0				

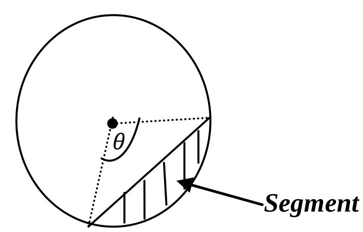
2. Determine the area and distance covered by the minute hand of length 5cm as it moves from 6 : 00 to 7 : 15.

B. Find the angle formed by the hour and minute hands at the following time.

1. 7 : 15 pm 2. 4 : 20 pm 3. 3 : 30 am
4. 2 : 20 am 3. 11 : 50 pm 4. 10 : 55 am

Area and Perimeter of Segments

The segment of a circle is the region bounded by a chord and the arc subtended by the chord



To find the area of a segment,

I. Find the area of the sector by the formula:

$A = \frac{\theta}{360^0} \times \pi r^2$

II. Find the area of the triangle by the formula: $A = \frac{1}{2}b\,h$ or $A = \frac{1}{2}ab\sin\theta$

III. Subtract the area of the triangle from the area of the sector to obtain the area of the segment.

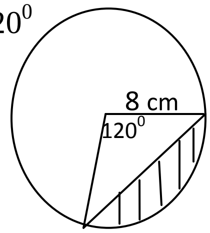
$$\boxed{A_{segment} = A_{sector} - A_{triangle}}$$

The perimeter of a segment is calculated as:
= length of the arc + length of chord

Worked Examples

1. Find the area of the segment of a circle with central angle of 120^0 and a radius of 8cm, to the nearest whole number

Solution



Area of sector subtended by 120^0

$A = \frac{\theta}{360^0} \times \pi r^2$

$A = \frac{120^0}{360^0} \times \frac{22}{7} \times 8^2 = 67.05\text{ cm}^2$

Area of triangle, A = $\frac{1}{2}ab\sin\theta$

$A = \frac{1}{2}(8)(8)\ \sin 120^0$

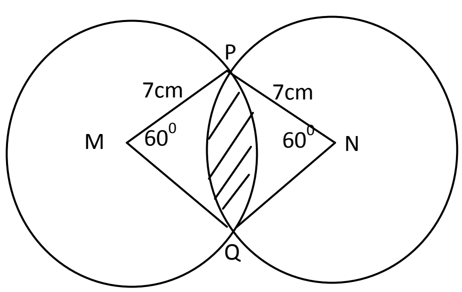
$A = 27.71\text{cm}^2$

$A_{segment} = A_{sector} - A_{triangle}$

$A_{segment} = 67.05 - 27.71$

$A_{segment} = 39\text{cm}^2$ (Nearest whole number)

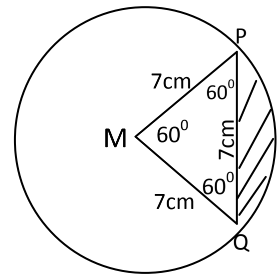
2. In the diagram, M and N are the centres of two circles of equal radii 7cm. The circles intercept at P and Q. If $< PMQ = <PNQ = 60^0$. Calculate correct to the nearest whole number, the area of the shaded portion (л = $\frac{22}{7}$)



Solution

r = 7cm and л = $\frac{22}{7}$

Area of sector subtended by 60^0;



$A = \frac{\theta}{360^0} \times \pi r^2$

$A = \frac{60^0}{360^0} \times \frac{22}{7} \times 7^2$

$A = 25.66\text{ cm}^2$

Area of triangle, $A = \frac{1}{2}ab\sin\theta$;

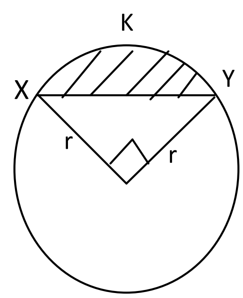
$A = \frac{1}{2}(7)(7)\ \sin 60^0 = 21.23\text{cm}^2$

$A_{segment} = A_{sector} - A_{triangle}$

$A_{segment} = 25.66 - 21.23 = 4.44\text{cm}^2$

For the two equal circles, the total area of segment = $2(4.44\text{cm}^2) = 9\text{cm}^0$
(Nearest whole number)

3. In the diagram below, O is the center of the circle radius r cm and $< XOY = 90^0$. If the area of the shaded part is 504cm^2, calculate the value of r



Solution

Area of sector subtended by 90^0

$A = \frac{\theta}{360^0} \times \pi r^2 = \frac{90^0}{360^0} \times \frac{22}{7} \times r^2 = \frac{22}{28}r^2\text{cm}^2$

Area of triangle,

$A = \frac{1}{2}ab\sin\theta$

$A = \frac{1}{2}r^2\sin 90^0 = \frac{1}{2}r^2\text{cm}^2$

$A_{segment} = A_{sector} - A_{triangle}$

$504\text{cm}^2 = \frac{22}{28}r^2 - \frac{1}{2}r^2$

$14{,}112 = 22r^2 - 14r^2$

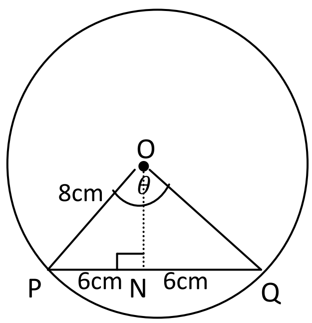
$14{,}112 = 8r^2$

$r^2 = \frac{14{,}112}{8} = 1{,}764$

$r = \sqrt{1764} = 42$ cm

4. PQ is a chord of a circle, center O. The radius of the circle is 8cm. $/PQ/ = 12$ cm. Find:
i. the angle subtended at the center by the minor arc PQ.
ii. the perimeter of the minor sector OPQ
iii. the perimeter of the minor segment. ($\pi = \frac{22}{7}$)

Solution



i. Δ PQR isisosceles

$\overline{ON}$ bisects $\overline{PQ}$

$< POQ = \theta$

$\text{Sin}\,\frac{1}{2}\theta = \frac{6}{8} = 0.75$

$\frac{1}{2}\theta = sin^{-1}(0.75)$

$\frac{1}{2}\theta = 48.59^0$

$\theta = 97.18^0$

The angle substended at the centre is 97.18^0

ii. the perimeter of the minor sector OPQ;

$P = 2r + l$

But $r = 8$ and $L = \frac{\theta}{360^0} \times 2\pi r$

$L = \frac{97.18^0}{360^0} \times 2\left(\frac{22}{7}\right) \times 8 = 13.57$cm

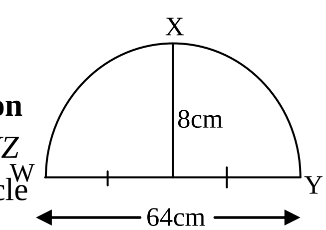
$P = 2(8) + 13.57 = 16 + 13.57 = 29.57$

iii. the perimeter of the minor segment;

= length of minor arc PQ + length of chord PQ,

$= 13.57 + 12 = 25.57$

Solved Past Question

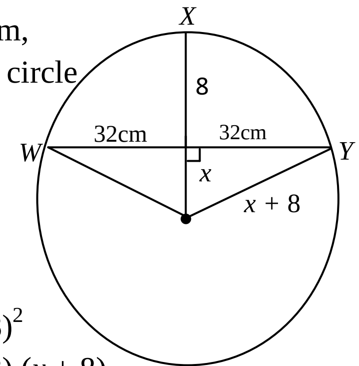


In the diagram, $WXYZ$ isa segment of a circle such that XZ is the perpendicular bisector of WY.

If $/XZ/ = 8$ cm and $/WY/ = 64$cm, calculate the radius of the circle

Solution

Draw a complete circle as shown below;



From the diagram, the radius of the circle

$r = x + 8$ cm

By Pythagoras theorem,

$x^2 + 32^2 = (x + 8)^2$

$x^2 + 32^2 = (x + 8)(x + 8)$

$x^2 + 1024 = x^2 + 16x + 64$

$1024 - 64 = 16x$

$16x = 960$

$x = \frac{960}{16} = 60$

$\Rightarrow r = 60\text{cm} + 8\text{ cm} = 68\text{cm}$

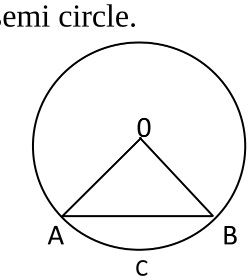
Exercises 22.4

1. Calculate the area of segment of a circle whose angle is 20^0 and its radius is 3 cm.

2. Calculate the area of a segment of a circle whose angle is 125^0 and its radius is 18 cm.

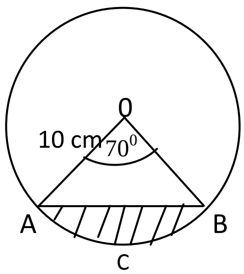
3. ABC is a semi circle, center O, such that $<BOC = 60^0$ and arc BC = 15cm. Find :
i. the radius of the circle,
ii. the length of the arc AB,
iii. the perimeter of the semi circle.

4. In the figure below, the diameter of the circle is 14cm. The length of the

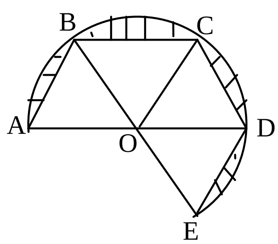


chord $\overline{AB}$ is 7cm. Calculate;
i. the length of the minor arc of a sector *OAB*,
ii. the perimeter of the sector *OAB*,
iii. the perimeter of the segment *ABC*.

5. Determine the area of the shaded region in the figure below:



6. In the diagram below, *ABCDEO* is two – thirds of a circle O. The radius *AO* is 7cm and */AB/* = */BC/* = */CD/* = */DE*. Calculate correct to the nearest whole number , the area of the shaded portion



Perimeters

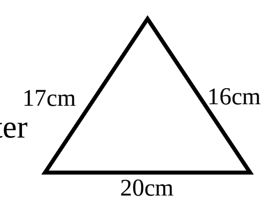
The distance around a plane figure is called its ***perimeter.*** All polygons have perimeters.

Perimeter of a Triangle

To calculate the perimeter of a triangle in which the length of the sides are given, add the lengths of the three (3) sides.

Worked Examples

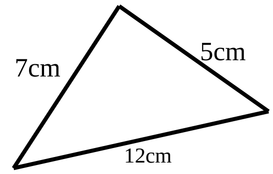
1. Calculate the perimeter of the figure below;



Solution

P = 20 cm + 16 cm + 17 cm = 53cm

2. Find the perimeter of the figure below.

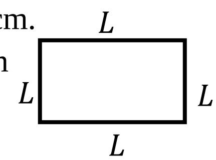


Solution

P = 5cm + 7 cm + 12 cm = 24 cm

Perimeter (P) of a Square

The perimeter ofa square is calculated by adding all the lengths of the sides of the square. Consider the square below with side, Lcm.



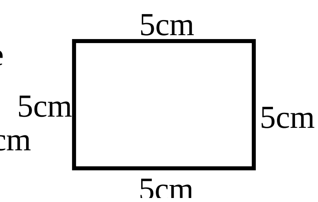
P = Lcm + Lcm + Lcm + Lcm
P = 4 Lcm

Worked Examples

1. Find the perimeter of a square with side 5cm.

Solution

Method 1



Perimeter of square
= 4L , but L = 5cm
P = 4 × 5cm = 20 cm

Method 2

P = 5cm + 5cm + 5cm + 5cm = 20cm

2. Calculate the perimeter of a square of side 13cm.

Solution

Perimeter = 4L, where L = 13cm
Perimeter = 4 ×13cm = 52 cm

3. A square has a side of 24cm. Calculate its perimeter.

Solution
$P = 4L = 4 \times 24$ cm = 96 cm

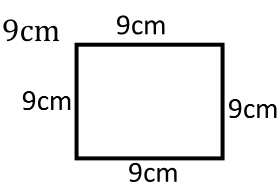
4. What is the perimeter of a square with side 11cm?

Solution
Method 1
$P = L + L + L + L$
$P = 11cm + 11cm + 11cm + 11cm = 44cm$

Method 2
$P = 4L = 4 \times 11cm = 44cm$

5. The length of a side of a square is 9cm. Calculate the perimeter of the square.

Solution
Method 1
$P = 4L$, but $L = 9cm$
$P = 4 \times 9cm$
P = 36cm



Method 2
$P = 9cm + 9cm + 9cm + 9cm$= 36cm

Length of a Square Given the Perimeter
To find the length of a square, given the perimeter, remember that P = 4L. Making L the subject, **L =** $\frac{P}{4}$, where **L** is the length of the square and **P** is the perimeter of the square.

Worked Examples
1. The perimeter of a square is 12cm. Find the length of a side of the square.

Solution
L= $\frac{P}{4}$, butP = 12cm and L = ?
By substituting the value of P,
$L = \frac{P}{4} = \frac{12}{4} = 3cm$

2. A square has a perimeter of 28cm.Calculate the length of a side of the square.

Solution
$L = \frac{P}{4}$, but P = 28cm and L = ?
$L = \frac{P}{4} = \frac{28}{4} = 7cm$

3. Find the length of the side of a square whose perimeter is 132cm.

Solution
$L = \frac{P}{4}$, but L =? and P = 132cm
$L = \frac{P}{4} = \frac{132cm}{4} = 33cm$

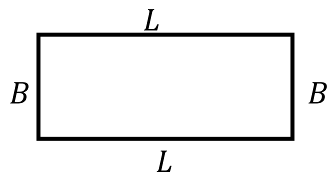
Exercises 22.5
A. Calculate the perimeter of a square with the following sides.
1) 27cm 2) 16cm 3) 19m
4) 140cm 5) 21cm 6) 40m

B. Calculate the length of a side of a square with the following perimeters.
1. 104cm 2. 168cm 3. 72cm
4. 148cm 5. 284cm 6. 100cm

C. Calculate the area of a square with the following perimeters.
1. 40cm 2. 28cm 3. 88cm
4. 140m 5. 36cm 6. 44cm

Perimeter of a Rectangle
The perimeter of a rectangle is calculate by adding up all the sides.

Consider the rectangle below;



P = L + B + L + B
P = 2L + 2B
P = 2(L + B)

$$\boxed{\boldsymbol{P = 2\,(L + B)}}$$

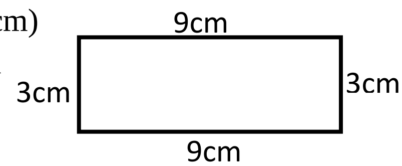
Worked Examples

1. Calculate the perimeter of a rectangle whose length and breadth are 9cm and 3cm respectively.

Solution

Method 1

Perimeter of a rectangle, P = 2(L + B)
But L = 9cm and B = 3cm
P = 2(9cm + 3cm)
P = 2 (12cm)
P = 24 cm



Method 2

P = 9cm + 3cm + 9cm + 3cm = 24 cm

2. A rectangle has a length of 13cm and breadth 7cm. Calculate its perimeter.

Solution

P = 2(L + B), but L = 13cm and B = 7cm
P = 2(13cm + 7cm) = 40 cm

3. The area a of a rectangle is $20cm^2$. If its length is 5cm, find its perimeter.

Solution

Area of a rectangle, A = L × B
But A = $20cm^2$, L = 5cm and B = ?
A = L × B
$20cm^2$ = 5cm × B
$\frac{20cm^2}{5cm}$ = B
B = 4cm

P = 2(L + B),where L = 5cm and B = 4cm
P = 2(5 cm + 4cm) = 18cm

4. What is the perimeter of a rectangle with area of $150cm^2$ and breadth 10cm.

Solution

Area of a rectangle, A = L × B, where
B = 10cm, L = ? and A =150 cm^2
A = L × B
$150cm^2$ = L × 10cm
L = $\frac{150\text{cm}^2}{10\text{cm}}$ = 15cm

P = 2(L + B),where L = 15cm and B = 10cm
P = 2(15 cm + 10cm) = 50cm

5. The perimeter of a rectangle is 26cm.If its length is 9cm. Calculate:
a. the breadth of the rectangle,
b. the area of the rectangle.

Solution

P= 2(L + B) but L = 9cm, B =? and
P = 26cm
26cm = 2 (9cm + B)
$\frac{26\text{cm}}{2} = \frac{2(9\text{cm}+\text{B})}{2}$
13cm = 9cm + B
13cm – 9cm = B
4cm = B or B = 4cm

Area of a rectangle, A = L × B
where L = 9cm and B = 4cm
A = 9cm × 4cm = $36cm^2$

Exercises

1. The length of a rectangle is 5cm more than its width and the area is 50 cm^2. Find the length, width and the perimeter. Ans: 5, 10, 30

2. The length of a rectangle is greater than its breadth by 3cm. If its area is 10cm^2, find the perimeter. Ans 14cm

3. The perimeter of a rectangle is 116 cm. If the length is 22 cm more than the breadth, form an equation and use it to find the length and breadth of the rectangle.

4. A table is three times as long as it is wide. If it was 3m shorter and 3m wider, it should be a square; how long and how wide is it?

5. The sides of a rectangle with area 135cm^2 are $3x$ cm and $(x + 4)$ cm long. Form an equation in x, and solve it. Write down the length and breadth of the rectangle.

Perimeter of Other Polygons

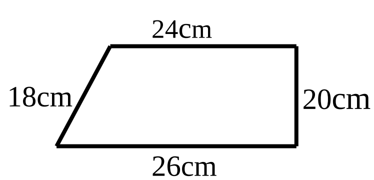
To find the perimeter of other polygons'

I. Identify all the dimensions of the polygon II. Add up all the dimensions to obtain the perimeter of the polygon

Worked Examples

Calculate the perimeter of the figures;

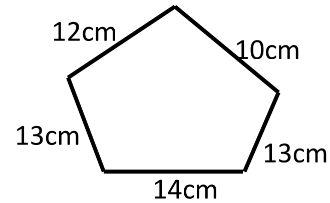
1.



Solution

P = 18cm + 26 cm + 20cm + 24cm = 88 cm

2.



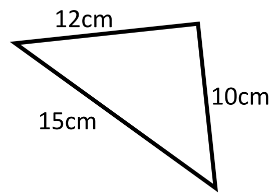
Solution

P = 12 + 13 + 14 + 13 = 62cm

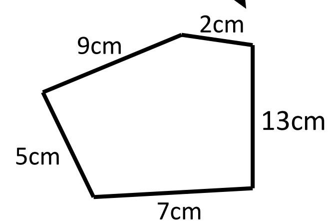
Exercises 22.6

Calculate the perimeter of the figures:

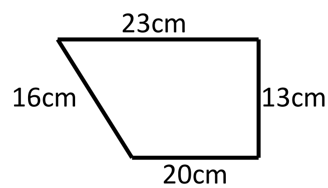
1.



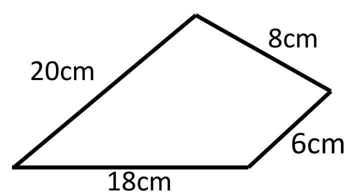
2.



3.



4.

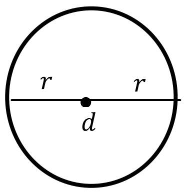


Perimeter of a Circle (Circumference)

The perimeter of a circle is called its ***Circumference***. The circumference, C, of a circle is calculated by the formula.

$$\boxed{C = \pi d} \quad \boxed{1}$$

Where d is the diameter of the circle.



But $d = r + r = 2r$

Substitute $d = 2r$ in eqn (1)

$$\boxed{C = 2\pi r} \quad \boxed{2}$$

Where $\boldsymbol{r}$ is the radius of the circle. Thus, any of the formulas: $\mathbf{C = \pi d}$ and $\mathbf{C = 2\pi r}$ can be used to calculate the circumference of a circle. The implication is simple:

I. when the diameter is given, use $\mathbf{C = \pi d}$

II. when the radius is given, use $\mathbf{C = 2\pi r}$

Worked Examples

1. Calculate the circumference of a circle with diameter 14cm ($\pi = \frac{22}{7}$)

Solution

Method 1

Given the diameter, d=14cm, we use $C = \pi d$

$C = \frac{22}{7} \times 14\text{cm} = 44\text{cm}$

Method 2

$C = 2\pi r$, but $d = 2r$

$r = \frac{d}{2} = \frac{14}{2} = 7\text{cm}$

$C = 2 \times \frac{22}{7} \times 7 \text{ cm} = 44 \text{ cm}$

2. What is the circumference of a circle with radius 21cm? ($\pi = \frac{22}{7}$)

Solution

Given the radius, we use the formula;

$C = 2\pi r$

$C = 2 \times \frac{22}{7} \times 21 \text{ cm} = 132 \text{ cm}$

3. A circle has a radius of 28cm. Find its perimeter ($\pi = \frac{22}{7}$)

Solution

$\mathbf{C = 2\pi r}$, but $r = 28\text{cm}$

$C = 2 \times \frac{22}{7} \times 28 = 176 \text{ cm}$

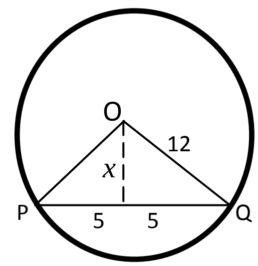
Solved Past Question

1. A circle with centre O, has radius 12 cm. A

chord PQ of the circle is 10cm long. Calculate:

i. the distance of the chord from the centre O.

ii. angle POQ



Solution

i. $\sin\theta = \frac{5}{12}$

$\theta = \sin^{-1}\left(\frac{5}{12}\right)$

$\theta = 25^0$

ii. $< POQ = 2\theta = 2 \times 25^0 = 50^0$

Exercises 22.7

Calculate the circumference of the circles

1. $r = 56\text{cm}$	2. $d = 14\text{cm}$	3. $d = 56\text{cm}$
4. $r = 42\text{cm}$	5. $d = 21\text{cm}$	6. $r = 10$ cm

Area of a Triangle

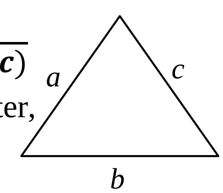
The area of a triangle is calculated with reference to the nature (dimensions) of the triangle.

1. Given three sides, a, b and c of the triangle, the area is given by ;

$\boldsymbol{A} = \sqrt{\boldsymbol{p(p-a)(p-b)(p-c)}}$

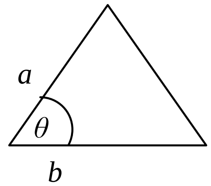
Where $\boldsymbol{p}$ is half of the perimeter,

Or $\boldsymbol{p} = \frac{\boldsymbol{a+b+c}}{\boldsymbol{2}}$

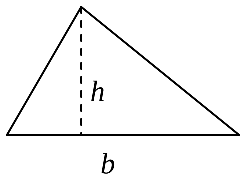


2. Given two sides a. and b, and an included angle, θ, the area of the triangle is calculated by the formula:

$\boldsymbol{A = \frac{1}{2} ab \sin \theta}$



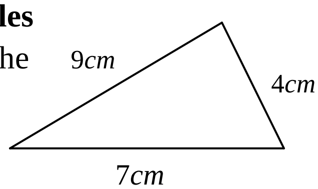
3. Given the length of the base and the height of the triangle, the area is calculated by the formula; $\boldsymbol{A = \frac{1}{2}bh}$.



Here, when given two sides, you can find the height by the use of Pythagoras theorem.

Worked Examples

Find the area of the triangle below;



Solution

Let a = 9cm, b = 4cm and c = 7cm

$p = \frac{9\ cm + 4\ cm + 7\ cm}{2} = \frac{20}{2}\ cm = 10\ cm$

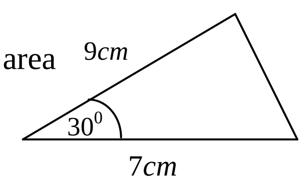
a = 9cm, b = 4cm , c = 7cm and p = 10cm

$A = \sqrt{p(p-a)(p-b)(p-c)}$

$A = \sqrt{10(10-9)(10-4)(10-7)}$

$A = \sqrt{10(1)(6)(3)} = \sqrt{180}$ = 13cm^2

2. Calculate the area of the triangle.



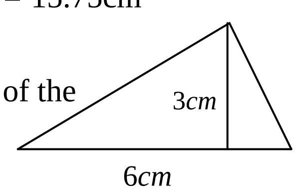
Solution

a = 9cm, b = 7cm, $\theta = 30^0$, substitute in

$A = \frac{1}{2} ab \sin \theta$

$A = \frac{1}{2}(9)(7) \sin 30^0 = 15.75$cm^2

3. What is the area of the triangle?



Solution

b = 6cm and h = 3cm, substitute in

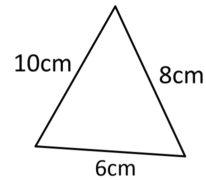
$A = \frac{1}{2}bh$

$A = \frac{1}{2} \times 6cm \times 3cm = 18cm^2$

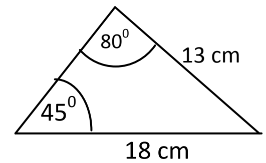
Exercises 22.7B

Find the area of each triangle.

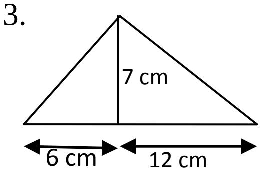
1.



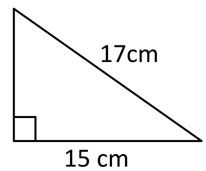
2.



3.



4.



Challenge Problems

1. Two chords and a diameter form a triangle inside a circle. The radius is 5cm and one chord is 2cm longer than the other. Find the perimeter and area of the triangle. Ans 24, 24cm^2

2. The length and width of a rectangular garden are 150m and 120m. A foot path of regular width is added to the boundary of the garden and the total area of the garden becomes 2,800m^2 more than its original area. Find the width of the foot path. Ans 5

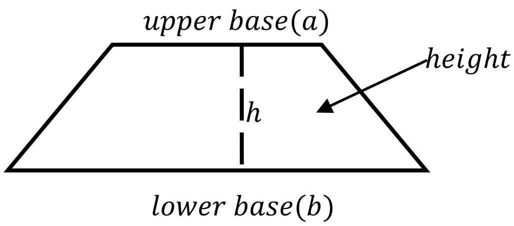
3. An isosceles triangle is inscribed in a circle in such a way that its longest side, which goes through the centre is $\sqrt{50}$ cm. Find the area of the triangle. Ans 12.5cm

Area of Quadrilaterals

1.Area of a Trapezium

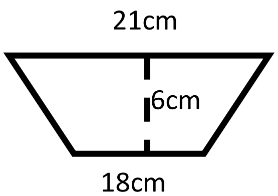
The formula for calculating the area of a trapezium is; $\boldsymbol{A = \frac{1}{2}(a+b)h}$ **;**

where A is the area of the trapezium,
a is the length of the upper base,
b is the length od the lower base,
h is the height.



Worked Examples

1. Calculate the area of the trapezium below;



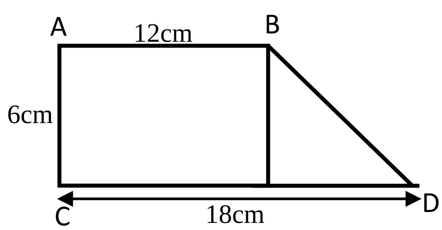
Solution

$A = \frac{1}{2}(a + b)h$

But $a = 21\text{cm}, b = 18\text{cm}$ and $h = 6\text{cm}$

$A = \frac{1}{2}$(12 cm + 18 cm) × 6cm $= 117\text{cm}^2$

2. Calculate the area of the figure ABCD.



Not drawn to scale

Solution

Area = $\frac{1}{2}$ $(a + b)\ h$

a = 12cm, b = 18cm, h = 6cm.

A = $\frac{1}{2}$ (12 cm + 18 cm) × 6 cm = 90cm^2

3. The lower base of a trapezium is 16cm and its height is 11cm. If it has an area of 268cm^2, find its upper base.

Solution

$A = \frac{1}{2}(a + b)h$

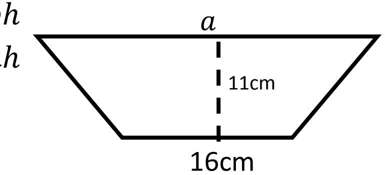
But A= 268cm^2, b =16, h =11 and a= ?

Make$a$ the subject of $A = \frac{1}{2}(a + b)h$

$2A = ah + bh$

$2A - bh = ah$

$a = \frac{2A - bh}{h}$



Put in b = 16cm, h = 11cm and A = 268cm^2

$a = \frac{2\times268 - 16\times11}{11}$ = 32.73cm (2dp)

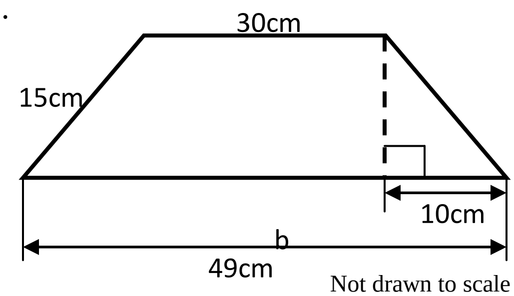
Note: In some cases, it is first necessary to calculate the height, h, by the use of Pythagoras theorem; and then substitute in the trapezium formula ; $\mathbf{A} = \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{a} + \boldsymbol{b})\boldsymbol{h}$,

where a and b are the opposite sides of the trapezium

More Worked Examples

Determine the area of the following figures;

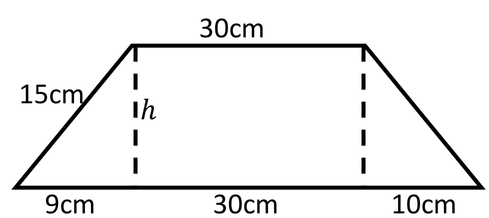
1.



Not drawn to scale

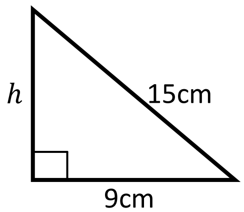
Solution

Considering the right – angled triangle at the extreme left,



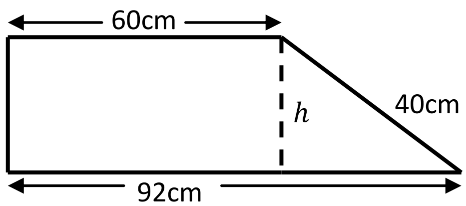
By Pythagoras theorem

$15^2 = h^2 + 9^2$
$15^2 - 9^2 = h^2$
$225 - 81 = h^2$
$144 = h^2$
$\sqrt{144} = h^2$
$h = 12\text{cm}$



Substitute $h = 12\text{cm}$, in $A = \frac{1}{2}(a+b)h$
Where a = 30cm, b = 49cm and h = 12cm
$\therefore\ A = \frac{1}{2}(30\text{cm} + 49\text{cm}) \times 12\text{cm}$
$A = \frac{1}{2} \times 79\text{cm} \times 12\text{cm} = 474\ \text{cm}^2$

2.



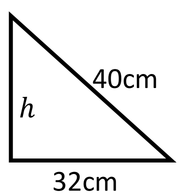
Not drawn to scale

Solution

Method 1

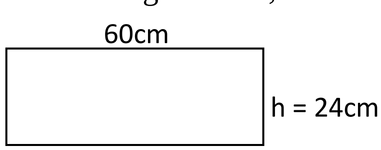
Consider the triangle below;.

$40^2 = h^2 + 32^2$
$40^2 - 32^2 = h^2$
$1600 - 1024 = h^2$
$h^2 = 576$
$h = \sqrt{576} = 24\text{cm}$



Area of the triangle;
$A = \frac{1}{2} b\, h$
Substitute b = 32cm and h = 24 cm
Area of $\Delta = \frac{1}{2} \times 32\text{cm} \times 24\ \text{cm} = 384\text{cm}^2$

Consider the rectangle below;



Area of the rectangle = L × B
But L = 60cm and B = h = 24cm
Area = 60cm × 24cm
A = 1,440cm^2

Total Area of the figure;
= Area of Δ + Area of the rectangle,
= 384cm^2 + 1440cm^2
= 1,824cm^2

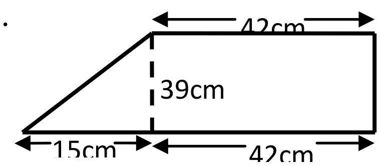
Method 2

By Pythagoras theorem, h = 24cm.
Substitute h = 24cm in the formula for the area of a trapezium = $\frac{1}{2}(a + b)h$
But a = 60cm, b = 92cm and h = 24cm
$A = \frac{1}{2}(60 + 92) \times 24$
A = 1,824cm^2

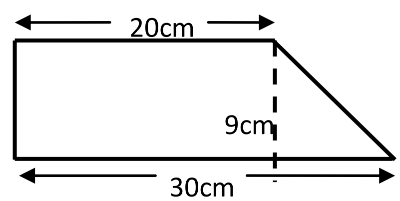
Exercises 22.8

A. Calculate the area of trapeziums;

1.



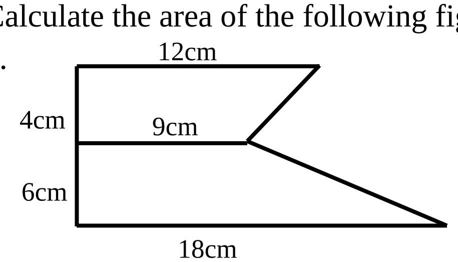
2.



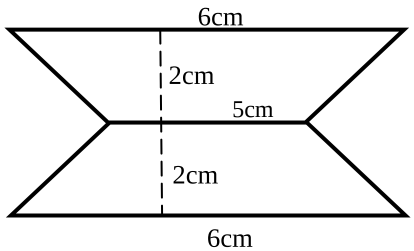
Challenge Problems

Calculate the area of the following figures;

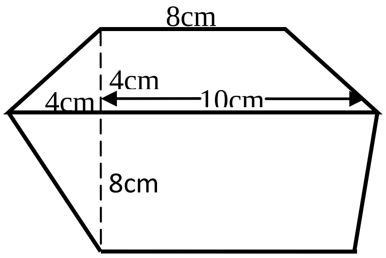
1.



2.

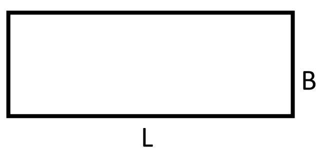


3.



Area of a Rectangle

The area of a rectangle is calculated by multiplying the length by the breadth.



Area = Length × Breadth

A = L× B or A = LB

Worked Examples

1. The length and breadth of a rectangle is 15cm and 10cm respectively. Find its area.

Solution

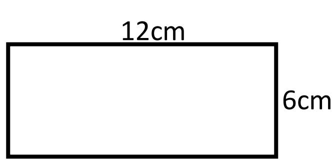
A = L × B

But $L = 15\text{cm}$ and $B = 10\text{cm}$.

$\therefore A = 15\text{cm} \times 10\text{cm} = 150\text{cm}^2$

2. A rectangle has a length of 12cm and breadth of 6cm. Find its area.

Solution



$A = L \times B$

$A = 12\text{cm} \times 6\text{cm}$

$A = 72\text{cm}^2$

3. The area of a rectangle is 15cm^2. If the length of the rectangle is 5cm. Calculate :

i. the breadth of the rectangle.

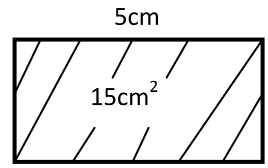
ii. the perimeter of the rectangle

Solution

i. $A = L \times B$

But $A = 15\text{cm}^2$

$L = 5\text{cm}$ and $B = ?$



$15\text{cm}^2 = 5\text{cm} \times B$

$\therefore B = \frac{15\text{cm}}{5\text{cm}} = 3\text{cm}$

Some Solved Past Questions

1. The area of a rectangle is 18cm^2. One of its sides is 9cm long. Find its perimeter.

Solution

$A = L \times B$, but $A = 18\text{cm}^2$, $L = 9\text{cm}$

$18\text{cm}^2 = 9\text{cm} \times B$

$B = \frac{18\text{cm}^2}{9\text{cm}} = 2\text{cm}$

$P = 2(L + B) = 2(9\text{cm} + 2\text{cm}) = 22\text{cm}$

2. An equilateral triangle has side 16cm. A square has the same perimeter as the equilateral triangle. What is the area of the square?

Solution

Perimeter of the triangle;

$P = 16\text{cm} + 16\text{cm} + 16\text{cm} = 48\text{cm}$.

Perimeter of the square = L + L + L + L = 4L.

But perimeter of square = perimeter of Δ.

$\Rightarrow 4L = 48\text{cm}$

$L = \frac{48\text{cm}}{4} = 12\text{cm}$

But Area of a square;

$A = L^2 = L \times L$

$A = 12\text{cm} \times 12\text{cm} = 144\text{cm}^2$

3. A square of side 6cm has the same area as rectangle of length 9cm. Find the breadth of the rectangle.

Solution

Let, B, be the breadth of the rectangle.

Area of the square = Area of the rectangle

$\Rightarrow L^2 = L \times B$

But length of the square = 6cm and the length of the rectangle = 9cm

By Substitution,

$6^2 = 9 \times B$

6 cm× 6cm = 9 cm × B

$B = \frac{36cm^2}{9cm} = 4cm$

4. The perimeter of a rectangle is 24cm. If the breadth of the rectangle is 4cm. Find the area of the rectangle

Solution

P = 2 (L + B). But P = 24cm, B = 4cm

24 cm = 2 (L + 4cm)

24 = 2L + 8cm

24cm – 8cm = 2L

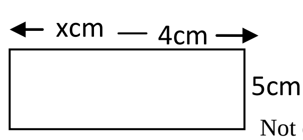
$L = \frac{16cm}{2} = 8cm$

But Area of a rectangle A = L × B.

L = 8cm and B = 4cm

A = 8cm × 4cm = $32cm^2$

5. The area of the figure below is $60cm^2$. Find the value of *x*



Not drawn to scale

Solution

A = L × B

A = (*x* + 4) × 5 (But A = $60cm^2$)

$\Rightarrow 60cm^2 = 5\,(x \text{ cm} + 4cm)$

60 = 5*x* + 20

60 − 20 = 5*x* + 20

40 = 5*x*,

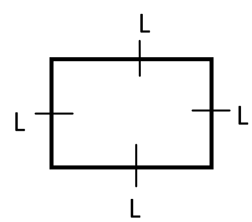
$x = \frac{40}{5} = 8cm$

Exercises 22.9

1. The length of a rectangle is 20 cm and the breadth 7cm, calculate the area.
2. The area of a rectangle $117cm^2$. If its length is 13 cm, what is the breadth?
3. Find the length of a rectangle whose area is $135cm^2$ and breadth 9 cm.
4. Find the breadth of a rectangle in which the area is 810 cm^2 and the length 45cm.
5. A rectangle has an area of 36 cm^2 and width of 3cm. Find its perimeter .

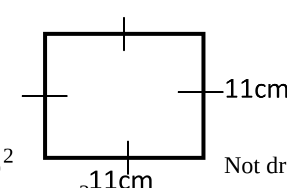
Area of a Square

A square is a rectangle with all sides equal.The area of a square is found by multiplying the length by itself. Since a square has all sides equal, it implies that if one side is L, then all other sides is L. That is; $A = L \times L = L^2$



Worked Examples

1. A square has a length of 11cm. What is its area?



Not drawn to scale

Solution

$A = L^2$,

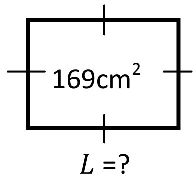
$\Rightarrow A = (11cm)^2$

A = 11 × 11 = 121 cm^2

2. A square has an area of $169cm^2$. Find its length.

Solution

$A = L^2$,



But $A = 169\text{cm}^2$, L = ?

$\Rightarrow 169\text{cm}^2 = L^2$

$\sqrt{169}\text{cm}^2 = \sqrt{L^2}$,

L = 13 cm

3. The area of a square is 49cm^2. Find the perimeter of the square.

Solution

$A = L^2 = L \times L$

But $A = 49\text{cm}^2$

$49 = L^2$

$\sqrt{49}\ \text{cm}^2 = \sqrt{L^2}$,

L = 7cm.

Perimeter of a square,

P = 4L

But L = 7cm

$\Rightarrow P = 4 \times 7 = 28$cm.

Exercises 22.10

A. Calculate the area of a square with the following lengths;

1) 5cm	2) 12cm	3) 16cm
4) 25cm	5) 23cm	6) 41cm

B. Find the length of a square with the following areas;

1) 81cm^2	2) 49cm^2	3) 225cm^2
4)169cm^2	5) 400cm^2	6) 121cm^2

Area of a Parallelogram

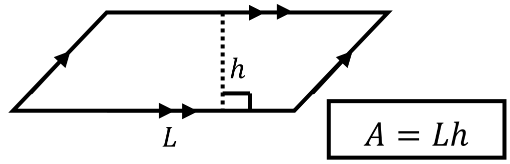
The area of a parallelogram is calculated by the formula: $\mathbf{A = LH}$, where;

A is the area;

L is the length of the base;

H is the height od the parallelogram;

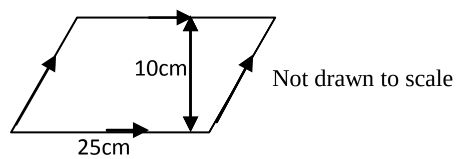
Consider the parallelogram below:



The height 'h' is the perpendicular distance between any two parallel sides and the base is one of these two parallel sides connected by the perpendicular.

Worked Examples

1. Calculate the area of the parallelogram



Solution

$A = \text{Length}(l) \times \text{Height(h)}$

where l = 25cm and h = 10cm

$A = 25\text{cm} \times 10\text{ cm} = 250\text{ cm}^2$

2. The length and height of a parallelogram are 27cm and 30cm respectively. What is its area?

Solution

$A = l \times h$

But $l = 27$cm and h = 30cm

$A = 27\text{cm} \times 30\text{cm} = 810\text{ cm}^2$

Exercises 22.11

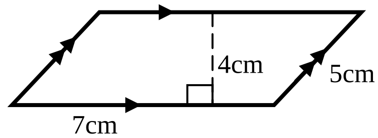
A. 1. A parallelogram has a base of length 10cm and a perpendicular height of 4cm. Find its area.

2. Calculate the area of a parallelogram whose length is 23cm and perpendicular height of 17cm.

3. What is the perpendicular height of parallelogram of area 810m^2 which has length of 30m?

B. Find the areas of the figures:

1.



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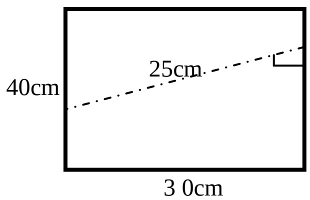

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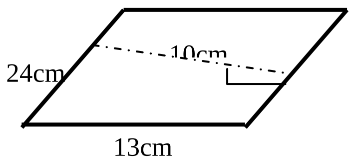
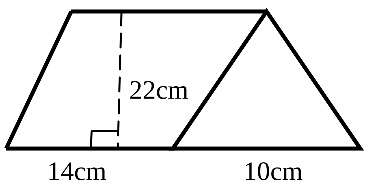


4.



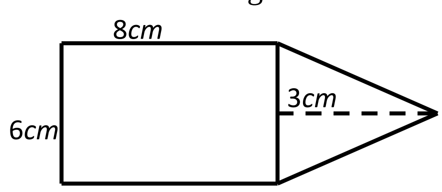
Area of Complex/Combined Shapes

Complex shapes are formed when two or more shapes are combined or put together to form a single shape. The areas of complex shapes are calculated by finding the sum of the areas of the separate solids or finding the difference of the areas of the separate solids.

Worked Examples

Calculate the area of the figures below.

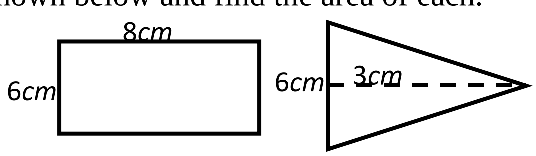
1.



Not drawn to scale

Solution

Split the shape into a rectangle and triangle as shown below and find the area of each.



Area of rectangle, $A_1 = L \times B$

$A_1 = 6\text{cm} \times 8\text{ cm} = 48\text{cm}^2$

Area of Δ $A_2 = \frac{1}{2}bh$

$A_2 = \frac{1}{2} \times 6 \times 3 = 9\text{cm}^2$

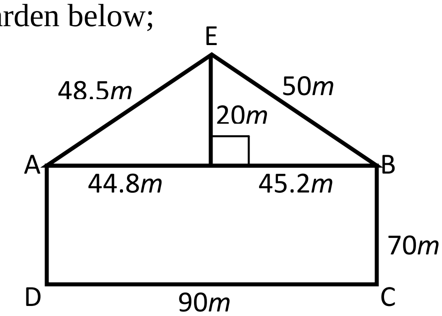
Total area of the shape, A

A = Area of rectangle (A_1) + Area of $\Delta(A_2)$

$A = A_1 + A_2$

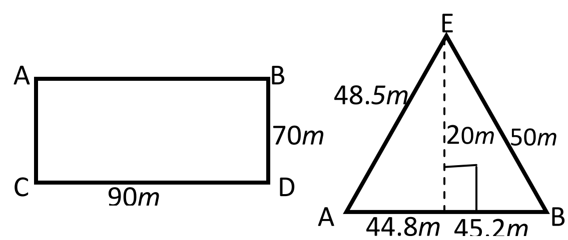
$A = 48\text{cm}^2 + 9\text{cm}^2 = 57\text{cm}^2$.

2. Find the total area and the perimeter of the garden below;



Solution

Split the figure to obtain the following shapes.



Area of rectangle *ABCD*,

$A_1 = L \times B$

$A_1 = 90\text{m} \times 70\text{m} = 6300\ m^2$

Area of triangle *ABE* (A_2);

$A_2 = \frac{1}{2}bh$

$A_2 = \frac{1}{2}(44.8 + 45.2) \times 20 = 900\text{ m}^2$

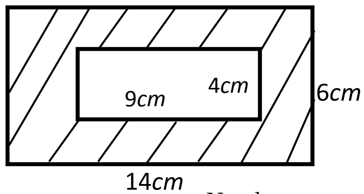
Total area = Area of *ABCD* + Area of *ABE*

$= A_1 + A_2$

$= 6300\text{m}^2 + 900\text{m}^2 = 7200\text{m}^2$

Perimeter = */AD/ + /DC/ + /CB/+ /BE/ + /EA/*
= 90 + 70 + 50 + 48.5 + 70 = 328.5 cm

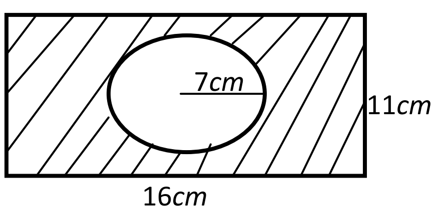
3. Find the area of the shaded region in the figure below:



Not drawn to scale

Solution
Let the area of the large rectangle be A_1 and the area of the small rectangle be A_2
$A_1 = L \times B = 14cm \times 6cm = 84cm^2$
$A_2 = L \times B = 9cm \times 4cm = 36cm^2$
Area of shaded region
$A = A_1 - A_2$
$A = (84 - 36)\ cm^2 = 48cm^2$

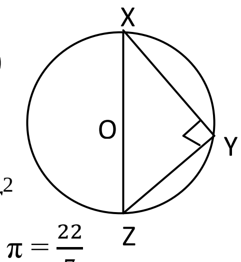
4. In the figure below, find the area of the shaded region ($\pi = \frac{22}{7}$)



Solution
Let the area of the rectangle be A_1 and the area of the circle be A_2.
$A_1 = L \times B = 11cm \times 16\ cm = 176cm^2$
$A_2 = \pi r^2 = \frac{22}{7} \times 7 \times 7 = 154\ cm^2$

Area of the shaded region;
A = Area of the rectangle – Area of the circle
$A = A_1 - A_2$
$A = 176\ cm^2 - 154\ cm^2 = 22cm^2$

5. In the diagram below, XYZ is cut off from the circle, center O. If /XZ/ = 35 cm and /YZ/ = 28cm, find the area of the remaining part of the circle ($\pi = \frac{22}{7}$)



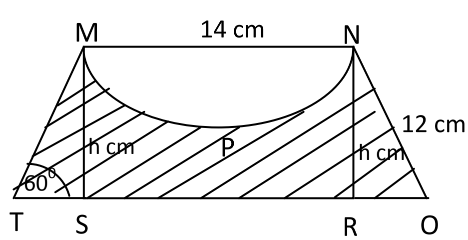
Solution
Area of the circle, $A = \pi r^2$
But $r = \frac{d}{2} = \frac{35}{2} = 17.5$, and $\pi = \frac{22}{7}$
$A = \frac{22}{7} \times 17.5 \times 17.5 = 962.5\ cm^2$

For ΔXYZ,
/XZ/ = 35 cm /YZ/ = 28 cm, /XY / = ?
By Pythagoras theorem,
$/XZ/^2 = /YZ/^2 + /XY/^2$
$35^2 = 28^2 + /XY/^2$
$/XY/^2 = 35^2 - 28^2$
$/XY/ = \sqrt{441} = 21$ cm

Area of Δ XYZ ;
$= \frac{1}{2}$ /XY/ /YZ/ sin θ
$= \frac{1}{2} \times 21 \times 28 \sin 90^0 = 294\ cm^2$

Area of the remaining part;
= Area of circle – Area of triangle
$= 962.5\ cm^2 - 294\ cm^2 = 668.5\ cm^2$

6. The diagram below shows a trapezium MNOT, in which MN // TO, /MN/= 14 cm, < MTO = 60^0 and /MT/ = /NO/ = 12 cm. If the semi – circle MPN is removed from the trapezium, calculate correct to the nearest cm^2, the area of the remaining portion ($\pi = \frac{22}{7}$).



Solution

Area of semi – circle = $\frac{1}{2}\pi r^2$

But $r = \frac{d}{2} = \frac{14}{2} = 7$ cm

Area of semi – circle = $\frac{1}{2} \times \frac{22}{7} \times 7^2 = 77\text{cm}^2$

Area of trapezium = $\frac{1}{2}(a + b)h$

$= \frac{1}{2}(/MN/ + /TO/)\,h$

Consider ΔMST,

/MT/ = /NO/ = 12 cm, h = ? /TS/ = ?

$\sin 60^0 = \frac{h}{12}$

$\Rightarrow h = 12 \sin 60^0 = 10.3923$ cm

By Pythagoras theorem,

$/TS/^2 + h^2 = MT^2$

$/TS/^2 + (10.3923)^2 = 12^2$

$/TS/^2 = 12^2 - (10.3923)^2$

$/TS/^2 = 36$

$/TS/ = \sqrt{36} = 6$cm

$\Rightarrow$But TS = RO = 6cm (ΔMST $\equiv$ ΔNRO)

$/TO/ = TS + SR + RO$

$/TO/ = 6 + 14 + 6 = 26$ cm

Area of trapezium;

$= \frac{1}{2}(/MN/ + /TO/)\,h$

$= \frac{1}{2} \times (14 + 26) \times 10.3923$

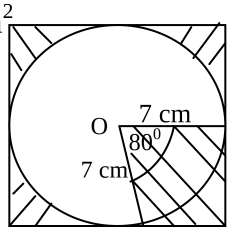
$= \frac{1}{2} \times 30 \times 10.3923\ 155.8845\ \text{cm}^2$

Area of shaded region

= Area of trapezium – Area of semi – circle

$= 155.8845\ \text{cm}^2 - 77\text{cm}^2$

$= 78.8845\ \text{cm}^2 = 79\ \text{cm}^2$



7. The figure shows a circle of radius 7cm inscribed in a square. Ifa portion of the circle is shaded with some portions of the square, calculate the area of the shaded portions.($\pi = \frac{22}{7}$)

Solution

Area of total shaded portion;

=Area of square – Area of unshaded sector

$= L^2 - \frac{\theta}{360} \times \pi r^2$

But r = 7cm , L = 2r =2(7cm) = 14cm, and

$\theta = 360^0 - 80^0 = 280^0$, substitute in

$L^2 - \frac{\theta}{360} \times \pi r^2$

$\Rightarrow 14^2 - \frac{280^0}{360^0} \times \frac{22}{7} \times 7^2$

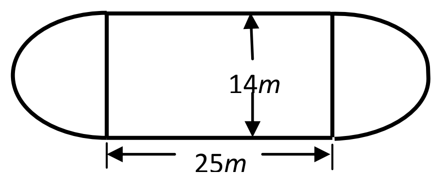
$= 196 - 119.7778 = 76.22\text{cm}^3$ (2 d.p)

Exercises 22.12

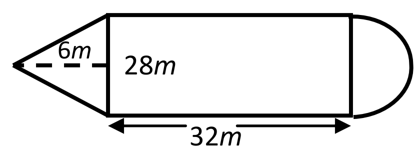
A. Determine the area of the following;

All diagrams are not drawn to scale

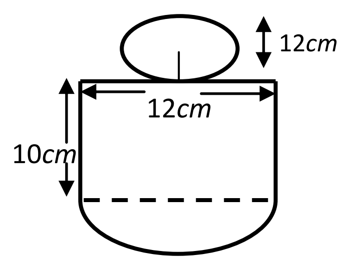
1.



2.

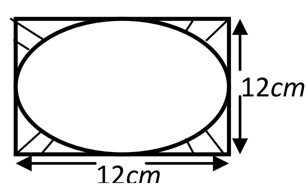


3.

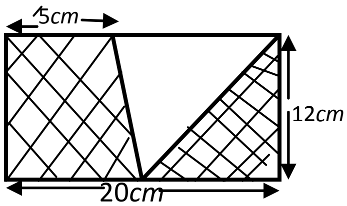


B. Determine the area of the shadedregions in following diagrams;

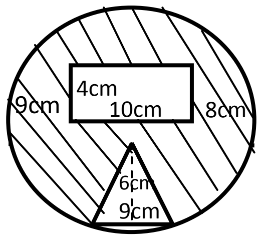
1.



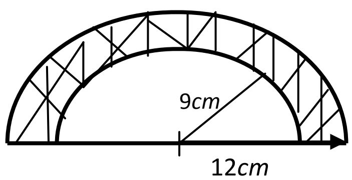
2.



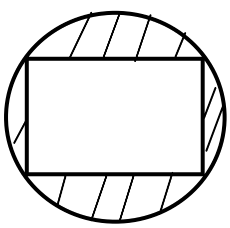
3.



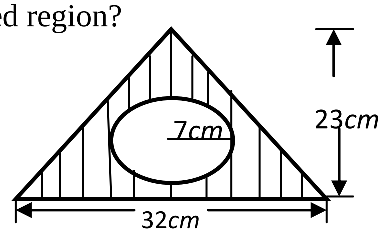
4.



C. 1. If the radius of the circle is 21cm and the area of the rectangle is 800cm^2, find the area of the shaded region in the diagram below;

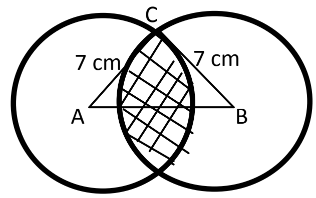


2. In the diagram below, what is the area of the shaded region?



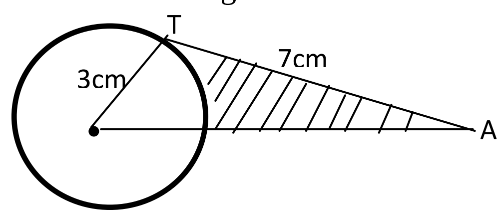
Challenge Problems

1. The diagram below shows two circles of radius 7cm with centres A and B. The distance AB is 12cm and the point C lies on both circles. The common region to both circles is shaded.



i. Calculate the angle CAB correct to the nearest degree.
ii. Find the perimeter of the shaded region.
iii. Find the area of the shaded region.

8. In the diagram below, O is the centre of the circle and AT is a tangent.



Calculate the area of the shaded region

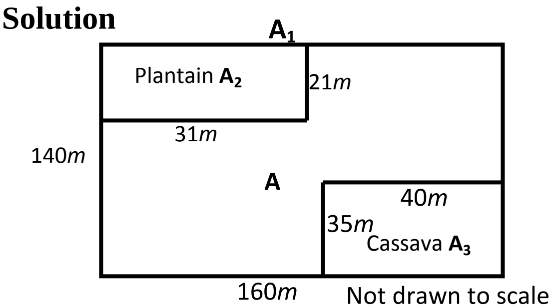
Word Problems

When problems involving area of complex shapes are stated in words, it is advisable to make a sketch representing the problem before attempting to solve.

Worked Examples

1. Mr. Martin had a plot of land of size $160\text{m} \times 140\text{m}$. On this piece of land, he cultivated plantain that covered an area of $31\text{m} \times 21\text{m}$, and cassava that covered an area of $35\text{m} \times 40\text{m}$. Find the area of the land that was not cultivated.

Solution



Let the area of the piece of land be A_1, the area of land cultivated with plantain be A_2,the area of the land planted with cassavabe $A_3$and the area of land that was not cultivated be A

$A_1 = L \times B = 140m \times 160m = 22{,}400m^2$

$A_2 = L \times B = 31m \times 21m = 651m^2$

$A_3 = L \times B = 35m \times 40m = 1{,}400m^2$

Total Area of land cultivated;

= Area of land planted with plantain + Area of land planted with cassava.

Total Area = $A_2 + A_3$

$= 651m^2 + 1{,}400m^2 = 2{,}051\ m^2$

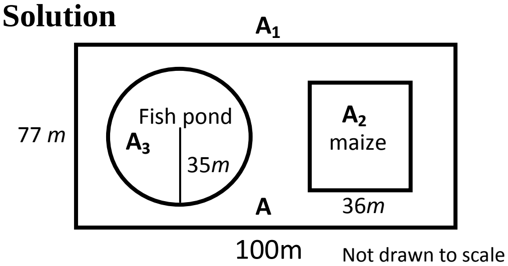
Area of land that is not cultivated (A);

$A = A_1 - (A_2 + A_3)$

$A = 22{,}400m^2 - 2{,}051m^2 = 20{,}349\ m^2$

2. Mrs. Doe has a plot of land that measures$100m \times 77m$. If she cultivates maize that covers an area of $36m \times 36m$and also construct a circular fish pond of radius 35m,calculate the area of her land that is not cultivated. ($\pi = \frac{22}{7}$)

Solution



Let A_1 be the area of the piece of land, A_2 be the area of the land planted with maize, A_3 be the area of land used for fish pond and A be the area of land that is not cultivated.

$A_1 = L \times B = 77\ m \times 100\ m = 7{,}700m^2$

$A_2 = L \times B = 36\ m \times 36\ m = 1{,}296m^2$

$A_3 = \pi r^2$

$A = \frac{22}{7} \times 35m \times 35m = 3{,}850m^2$

Total area of land cultivated;

$= A_2 + A_3 = 1{,}296m^2 + 3{,}850m^2 = 5{,}546m^2$

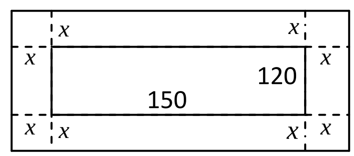
Area of land that is not cultivated;

$A = A_1 - (A_2 + A_3)$

$A = 7{,}700\ m^2 - (\ 1{,}296m^2 + 5{,}146m^2) = 2{,}554\ m^2$

3. The length and width of a rectangular garden are 150m and 120m. A foot path of regular width is added to the boundary of the garden and the total area of the garden becomes 2800m^2 more than its original area. Find the width of the foot path. **Ans : 5**

Solution



Area of first rectangle

$= 150 \times 120 = 18{,}000\ m^2$

Let the width of the foot path be x

Area of the second rectangle;

$(2x + 150)\ (2x + 120) = 18000 + 2800$

$4x^2 + 240x + 300x + 18000 = 20800$

$4x^2 + 540x + 18000 - 20800 = 0$

$4x^2 + 540x - 2800 = 0$

$x^2 + 135x - 700 = 0$

$(x - 5)\ (\ x + 140) = 0$

$x = 5$ or $x = -140$

$x = 5$

The width of the foot path is 5m

Exercises 22.13

1. The area of a rectangular plot is 70m × 90m. If it is cultivated with groundnut that covers an area of 25 m × 40m, find the size of the plot of land thatis not cultivated.

2. Mr. Brown rented a room of size 250 cm × 300 cm. His bed occupied a space of 130 cm × 145 cm and his refrigerator occupied 80cm × 60cm space. Calculate the area of the empty space in the room.

3. Mr. Henry purchased a plot of land that measured 120m × 120m. He put up a storey building that occupied a space of 40m × 60m. He then constructed a circular swimming pool of radius 21m on the same piece of land. Calculate the area of land that was left bare.

4. A circular plot has a radius of 63m. On this piece of land, shadrack wants to cultivate pineapple that will cover a space of 43m × 32m, Meshach needs 50m × 50m for groundnutproduction and Abednego requires a circular piece of land of radius 14m for tomato production.Calculate the area of land left for Daniel's cocoa production.

5. A square plot of land 96m wide is used to cultivate 625m^2 of cassava and 40m × 30m beans production. Calculate the area of plot that is not cultivated.

Challenge Problems

1. A piece of rectangular paper measures 8x by 12x. If a circle of radius 3x is cut out of the middle of the paper, what will be the area of the remaining piece of paper in terms of π.

B. A floor measures 15m by 8m is to be laid with tiles measuring 50cm by 25cm.
i. Find the number of tiles required.
ii. If the carpet is laid on the floor so that a space of 1m exist between its edges of the floor. What fraction of the room is not covered with carpet.

2. The area of a square exceeds twice that of another by 56cm^2. If the difference of the perimeter between the two is 24cm, find the area of the smaller square. **Ans 100 or 4**

3. A rectangular floor measures 10m by 6m. A piece of rectangular carpet measuring 6m by 4m is placed in the middle of it. Calculate the area of the uncovered floor.

4. A rectangular room holds a rectangular carpet in the centre measuring 8m by 16m. The width from the edge of the carpet to each wall is the same. The area not covered by the carpet is 112m. What is the width of the uncovered area around the carpet?

5. A carpet is laid around a rectangular dance floor measuring 10m by 8m. If the width of the carpet is 2m, find its area.

6. A rectangular piece of carpet is placed on the floor of a rectangular room leaving a margin of 1m around it. If the room measures 8m by 6m, find the area of thr room nit covered by the carpet.

7. Carpet cost Gh¢16.00 a square foot. A rectangular floor is 16 m long by 14m wide. How much will it cost to carpet the floor?

8. A rectangular bathroom measures 9.5m by 12m. The floor is covered by rectangular tiles measuring 1.5cm by 2cm. How many tiles are there on the bathroom floor?

23 PLANE GEOMETRY II

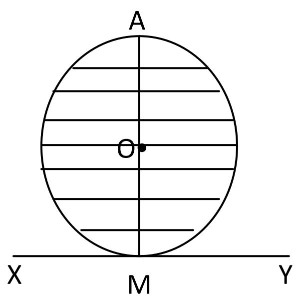
Tangent to a Circle

A tangent to a circle is:

i. A straight line which meets the circumference at one point only.

ii. Perpendicular or 90^0 to the diameter or radius drawn to the point of contact.

Consider the figure below;

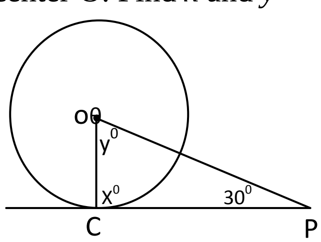


AM is the diameter which is the axis of symmetry, and O is the center of the circle. XY is the line parallel to the chords, at the end of the diameter.

Since all the chords are perpendicular to AM, XY is also perpendicular to AM and is called ***a tangent*** to the circle. M is called ***the point of contact***

Worked Examples

1. In the figure below, CP is a tangent to the circle, center O. Find x and y



Solution

CP is perpendicular to *OC*,

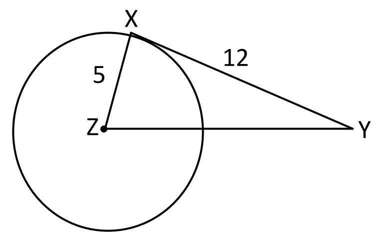
Therefore, <OCP = $x = 90^0$

$x + y + 30^0 = 180^0$, but $x = 90^0$

$y + 90^0 + 30^0 = 180^0$

$y = 180^0 - 90^0 - 30^0 = 60^0$

2. The figure below shows a circle, center Z with radius 5cm long and a tangent XY 12cm long. How far is Y from the center of the circle?



Solution

< ZXY = 90^0. Therefore triangle XYZ is a right – angled triangle.

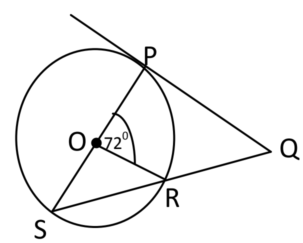
By Pythagoras theorem,

$5^2 + 12^2 = /YZ/^2$

YZ = $\sqrt{5^2 + 12^2}$

= $\sqrt{25 + 144} = \sqrt{169}$ = 13cm

3. In the diagram, $\overline{PS}$ is the diameter of the circle O.



If $\overline{PQ}$ is a tangent to the circle at P and $< POR = 72^0$, calculate the value of $< PQR$.

Solution

$< POR + < SOR = 180^0$

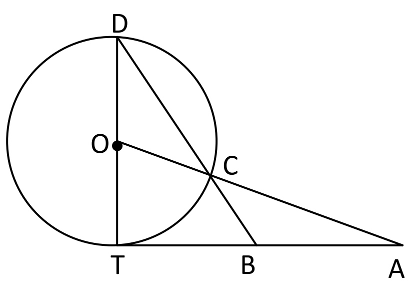
$72^0 + < SOR = 180^0$

$< SOR = 180^0 - 72^0 = 108^0$

∆SOR is an isosceles;
⇒< OSR = < ORS = x
$2x + 108^0 = 180^0$
$2x = 180^0 – 108^0$
$2x = 72^0$
$x = 36^0$

< PQR + < QSP + < SPQ = 180^0
< PQR + 36^0 + 90^0 = 180^0
< PQR = $180^0 – 36^0 – 90^0 = 54^0$

4. In the figure below, O is the center of the circle, TA is a tangent and < OAT = 23^0. Calculate < DBT



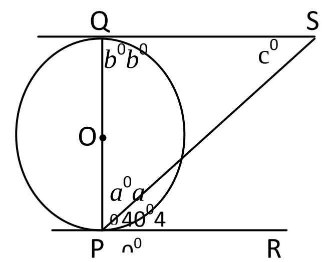
Solution
< OTA + < OAT + < TOA = 180^0
90^0 + 23^0 + < TOA = 180^0
< TOA = $180^0 – 90^0 – 23^0 = 67^0$

< TOA + < COD = 180^0
67^0 + < COD = 180^0
< COD = $180^0 – 67^0$ = < 113^0

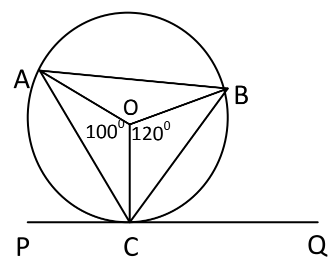
But ∆ OCD is an isosceles triangle
⇒< ODC = < OCD
< COD + < ODC + < OCD = 180^0
113^0 + 2 < ODC = 180^0
2 < ODC = $180^0 – 113^0$
2 < ODC = 67^0
< ODC = $\frac{67^0}{2} = 33.5^0$

< BTD + < ODC + < DBT = 180^0
⇒90^0 + 33.50 + < DBT = 180^0
< DBT = $180^0 – 90^0 – 33.5^0 = 56.5^0$

Exercises 23.1
1. In the figure below, PR and QS are tangents at opposite ends of the diameter PQ. Find *a*, *b* and *c*.

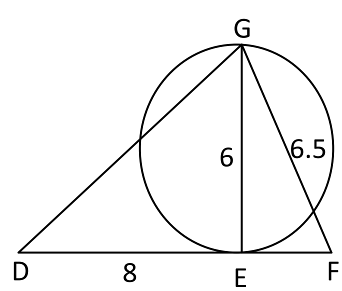


2. In the figure below, OA, OB and OC are radii, and PCQ is a tangent to the circle.



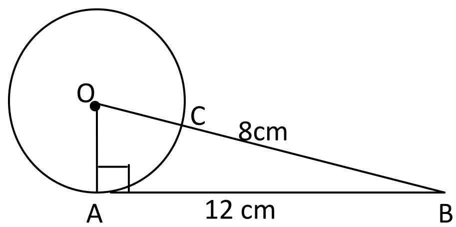
Copy the diagram, mark two right angles and fill in the sizes of all the other angles.

3. In the figure below, GE is a diameter, and DEF is a tangent to the circle. DE = 8cm, GE = 6cm and GF = 6.5cm. Calculate the lengths of DG and EF.



4. In the diagram below, AB is a straightline touching the circle center O at A. If /AB/ = 12cm,

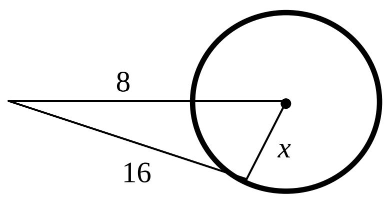
/BC/ = 8cm and angle OAB = 90^0, calculate the radius of the circle.



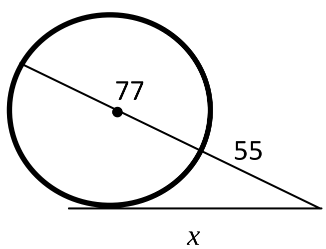
5. Two chords and a diameter form a triangle inside a circle. The radius is 5cm and one chord is 2cm longer than the other. Find the perimeter and area of the triangle. Ans 24, $24cm^2$

6. Determine the value of *x*

i.

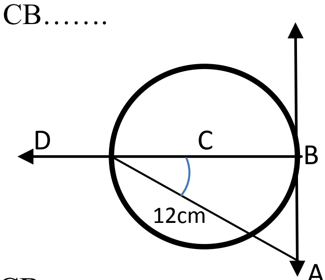


ii.

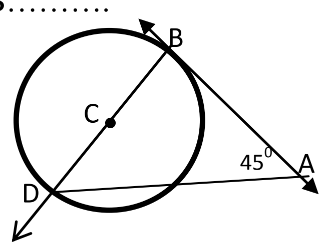


B. Solve for the missing information

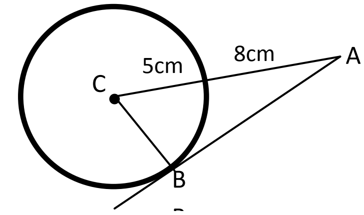
1. CB…….



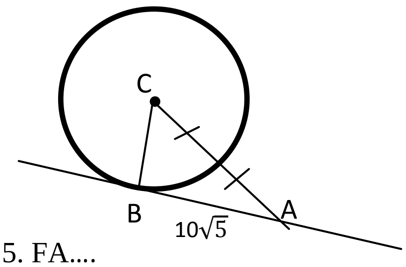
2. CB……….



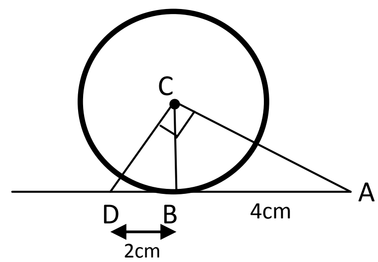
3. AB……..

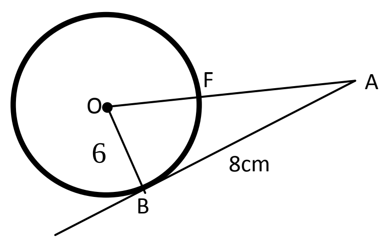


4. CB

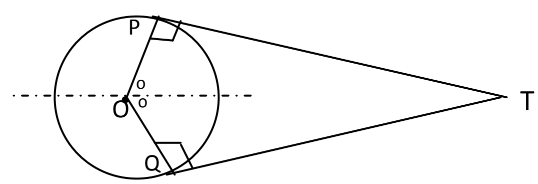


5. FA….





The Tangent – Kite

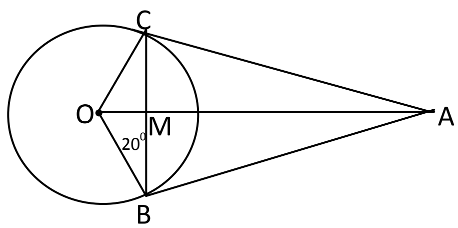


In the figure above, tangents from T touch the circle, center O, at P and Q.

OPTQ is called ***a tangent – kite***, and has axis of symmetry OT. Two tangents to a circle from a point outside the circle are equal in length. The line PT = QT. Thus, < OPT = < OQT

Worked Examples

1. AB and AC are tangents to the circle with center O in the figure below. If < OBM = 20^0, fill in the sizes of all the angles in the figure.



Solution

OCM = 20^0

< *OCA* = 90^0 (tangent to a circle)

⇒ < *OCM* + < *MCA* = < *OCA*

20^0 + < MCA = 90^0

< MCA = $90^0 – 20^0$

<MCA= 70^0

< AMC + < MCA + CAM = 180^0

But, AMC = 90^0 and < MCA = 70^0

⇒ $90^0 + 70^0$ + < CAM = 180^0

< CAM = $180^0 – 90^0 – 70^0 = 20^0$

< OCM + < OMC + < COM = 180^0

But OCM = 20^0 and OMC = 90^0

⇒ $20^0 + 90^0$ + < COM = 180^0

< COM = $180^0 – 20^0 – 90^0 = 70^0$

OA is the axis of symmetry. It implies that ΔOCA = ΔOBA

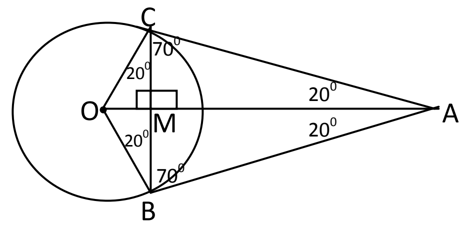
⇒< OCM = < OBM = 20^0

< COM = < BOM = 70^0

< OMC = < OMB = 90^0

< MCA = < MBA = 70^0

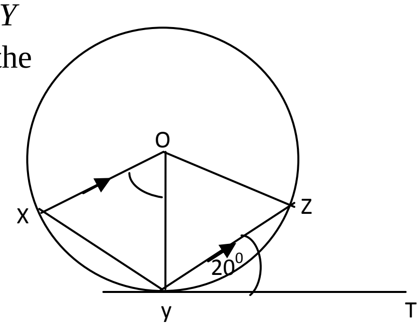
< CAM = < BAM = 20^0



2. In the diagram below, O is the center of the circle, *YT* is a tangent, < *TYZ* = 20^0, and *XO// YZ*

i. Find, < *XOY*

ii. Calculate the value of the reflex angle *XOZ*



Solution

i. /OY/ is perpendicular to /YT/

Therefore, < OYT = 90^0 and ZYT = 20^0

< OYZ + < ZYT = < OYT

< OYZ + 20^0 = 90^0

< OYZ = $90^0 – 20^0 = 70^0$

< XOY and < OYZ are alternate angles

But < OYZ = 70^0

Therefore, < XOY = 70^0

ii. < OYZ = < YOZ = 70^0 (Base angles of an isosceles triangle)

< XOZ + < XOY + < YOZ = 360^0

But < XOY = 70^0 and < YOZ = 70^0

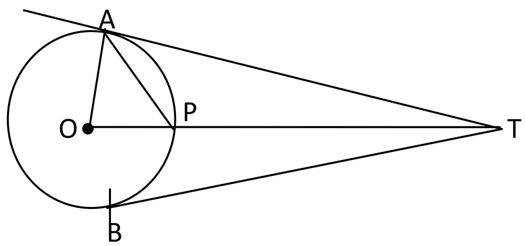
< XOZ + $70^0 + 70^0 = 360^0$

< XOZ = $360^0 – 70^0 – 70^0$

< XOZ = 220^0

3. In the figure below, TA and TB are tangents to the circle O. Given that < ATB = 20^0, find:

i. < AOT ii. < PAT



Solution

i. < ATB = 20^0

< OTA = < OTB = 10^0

< OAT + < OTA + < AOT = 180^0

$90^0 + 10^0$ + < AOT = 180^0

< AOT = $180^0 – 90^0 – 10^0 = 80^0$

ii. ΔOAP is Isosceles

< AOP = < AOT = 80^0

< OAP = < OPA

< OAP + < OPA + < AOT = 180^0

2 < OAP + 80^0 = 180^0

2 < OAP = $180^0 – 80^0$

2 < OAP = 100^0

< OAP = $\frac{100^0}{2}$ = 50^0

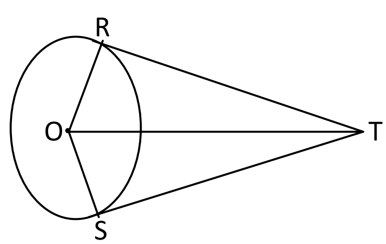
But < OAP + < PAT = 90^0

50^0 + < PAT = 900

< PAT = $90^0 – 50^0 = 40^0$

Exercises 23.2

1. In the figure below, ORTS is a tangent – kite.



Name:

i. one pair of congruent triangles;

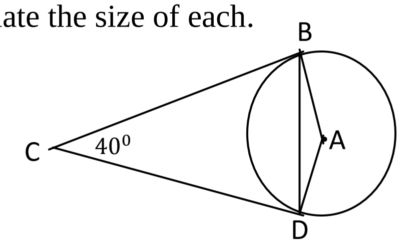
ii. two pair of equal lines;

iii. three pair of equal angles;

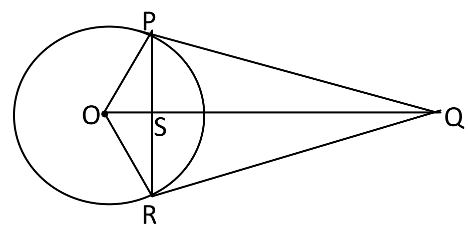
iv. a pair of right – angles;

v. if OR = 3cm, and OT = 8cm, calculate the length of the tangents RT and ST to 1 decimal place.

2. Figure *ABCD* below is a tangent- kite and < *BCD* = 40^0. Name two isosceles triangles and calculate the size of each.



3. In the figure below, *OPQR* is a tangent – kite.



i. Name the axis of symmetry and explain why *PR* is perpendicular to *OQ*

ii. If < *SPQ* = 70^0, calculate the size of each angle in the diagram

iii. If *OQ* = 7.2cm and *OP* = 2.7cm, calculate the length of the tangents to two significant figures

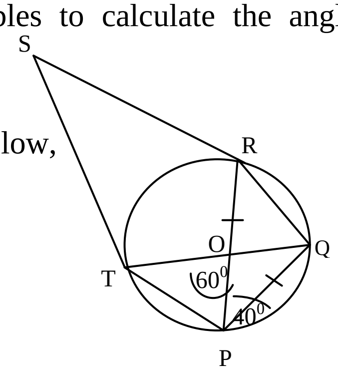
4. *AB* is a tangent touching a circle, center O, at *B*. *AO* = 17cm and *OB* = 8cm

i. Calculate the length of *AB.*

ii. If AB is produced 12cm to *C*, calculate how far *C* is from the center O to 1 d.p.

iii. Use your tables to calculate the angles of triangle *AOB.*

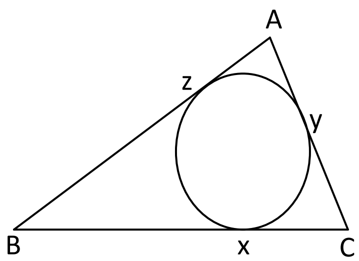
5. In the figure below, *PQRT* is a circle. /*PQ*/ = /*PR*/, < *PQR* = 40^0 and < *POT* = 60^0,



Find: i. < *TOR*   iii. < *PSQ*

Challenge Problems

1. In the figure below, tangents BC, CA and AB to the circle touch it at X, Y and Z.



i. Name three pair of lines of equal length in the figure.

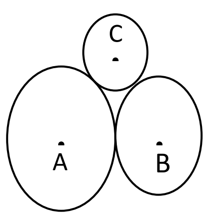
ii. If *BZ* = 4cm, *XC* = 5cm and *AY* = 3cm, Calculate the lengths of *AB*, *BC* and *CA*.

iii. If the perimeter of Δ *ABC* is 42cm, *BX* = 6cm and *CX* = 7cm, calculate the lengths of *AB* and *AC*

iv. If the perimeter of Δ *ABC* is 50cm, *AZ* = 10cm and *CX* = 12cm, calculate the length of *BC*.

v. If the perimeter of Δ *ABC* = 48cm and *AZ* = 6cm, calculate the length of *BC*.

2. Three circular discs, with centres *A*, *B* and *C* and radii 5cm, 4cm and 3cm respectively, are in contact as shown below;



i. If the first two circles touch at *X*, why are *A*, *X* and *B* collinear? (Hint: Think of a common tangent at *X*)

ii. Calculate the lengths of *BC*, *CA* and *AB*, and hence make an accurate drawing of the arrangement

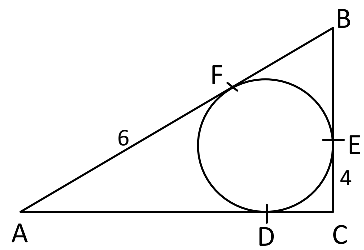
iii. If Δ *ABC* has *BC* = 6cm and *AB* = *AC* = 8cm, Calculate the radii of the three touching circles

3. In the figure below, a circle touches the sides of Δ *ABC* at *D*, *E* and *F*

i. Write down three pair of equal lines in the figure.

ii. If *BF* = 6cm, and *CE* = 4cm, calculate the length of *BC*.

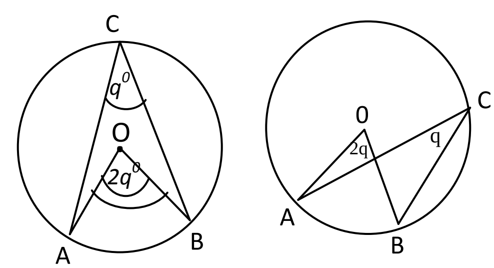
iii. If the perimeter of the triangle is 30cm, calculate the length of *AF*.



4. A point P is 22cm from the centre O of a circle. The angle between the tangents from P is 66^0. Calculate the radius of the circle.

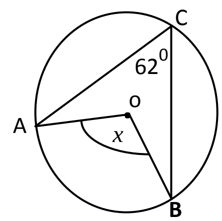
Properties of Angles in a Circle

I. Angles at the centre of a circle is double any angle at the circumference subtended by(or standing on) the same arc.



Worked Examples

1. In the figure below, find the value of the angle marked *x*.

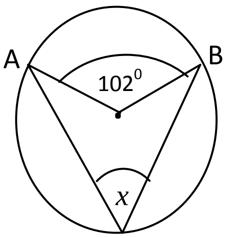


Solution

Let $q = 62^0$, then $x = 2q$

$x = 2 \times 62^0 = 124^0$

2. Find the value of x in the diagram below:



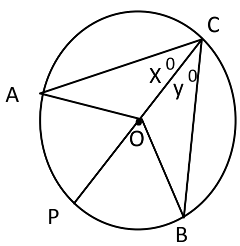
Solution

Let $x = q$, then $2q = 102^0$

$\Rightarrow 2x = 102^0$

$x = \frac{102^0}{2} = 51^0$

3. In the figure below, calculate the sizes of all the angles in terms of x and y. Show that < AOB = 2 < ACB

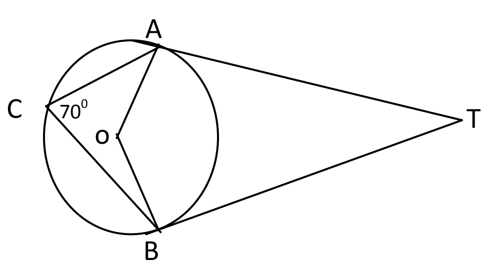


Solution

$< AOB = < AOP + < POB$

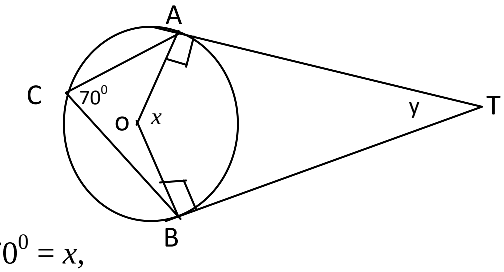
$= 2x^0 + 2y^0 \quad = 2(x + y)^0 \quad = 2 < ACB$

4. TA and TB are tangents to a circle centre O. If $< ACB = 70^0$, calculate angle ATB



Solution

Let < AOB = x and < ATB = y



$2 \times 70^0 = x$,

$x = 140^0$

$y + x + 90^0 + 90^0 = 360^0$ *(angles in a quadrilateral)*,

But $x = 140^0$

$y + 140^0 + 90^0 + 90^0 = 360^0$

$y = 360^0 - 140^0 + 90^0 + 90^0 = 40^0$

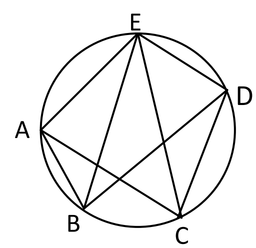
Exercises 23.3

1. In the figure below, name two angles at the circumference which stand on (or are subtended by):

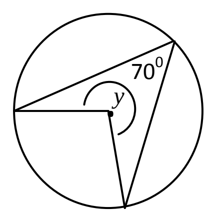
i. the minor arc BC,

ii. the minor arc DE,

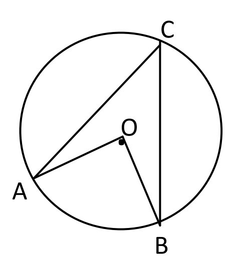
iii. the arc AED.



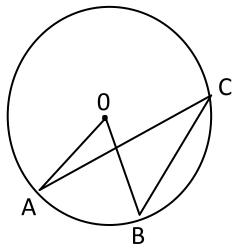
2. In each of the figures below, find the sizes of the angle marked with letter y:



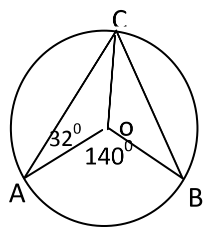
3. In the figure below, calculate $< AOB$, if $< ACB = 38^0$



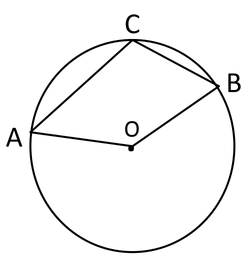
4. Calculate < *ACB* in the figure below, if
< *AOB* = 52^0



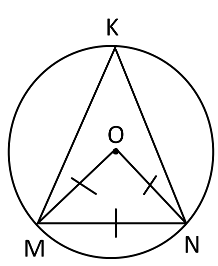
5. In the diagram below, O is the center of the circle. The points *A, B, C* are points on the circumference of the circle. Angles *CAO* and *AOB* are 32^0 and 140^0 respectively.
Calculate:
i. angle *OBC*
ii. angle *COB*



6. In the figure below, calculate the reflex
< *AOB*, if < *ACB* = 115^0

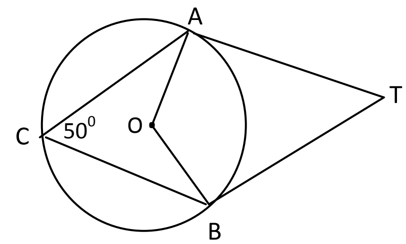


7. In the figure below, O is the center of the circle and *OM* = *MN* = *ON*.



Calculate the sizes of the angles at the center and the circumference subtended by:
i. the minor arc *MN*
ii. the major arc *MKN*

8. In the figure below, *TA* and *TB* are tangents, and O is the center of the circle.



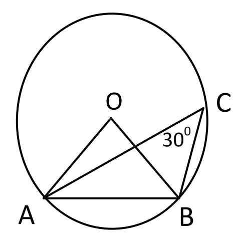
i. If < *ACB* = 50^0, calculate the size of <*ATB*
ii. If, *ACB* = x^0, calculate the size of < *ATB* (in terms of *x*)

Challenge Problems

1. Δ*XYZ* is inscribed in a circle with center O, and OY and OZ are joined. If <*YXZ* = 60^0, calculate the sizes of the angles of Δ *YOZ*

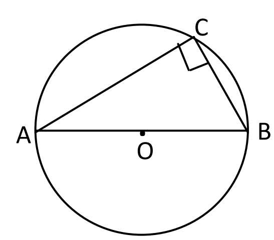
2. Δ*PQR* is inscribed in a circle with center O, and *OP*, *OQ* and *OR* are joined. If < *PQR* = 110^0 and < *QOR* = 130^0, calculate the sizes of the angles of Δ*PQR*

3. Below is a figure with center O and < *ACB* = 30^0



Show that *AB* is equal in length to a radius of the circle

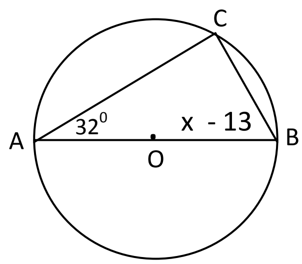
II. Every angle in a semi – circle is a right angle



i. < *AOB* = 180^0
ii. < *ACB* = 90^0

Worked Examples

1. In the figure below, find the value of the angle marked *x.*



Solution

<ACB = 90^0 (angle subtended at the circumference by a diameter).

$x - 13 + < ACB + 32^0 = 180^0$

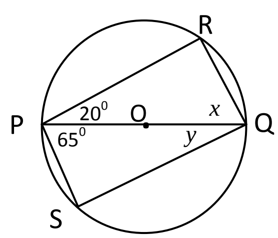
$x - 13^0 + 90^0 + 32^0 = 180^0$

$x = 180^0 - 90^0 - 32^0 + 13^0 = 61^0$

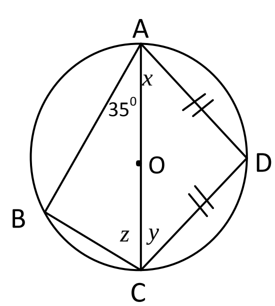
Exercises 23.4

A. In the figures below, O is the center of the circle. Mark in the sizes of all the angles in the circle.

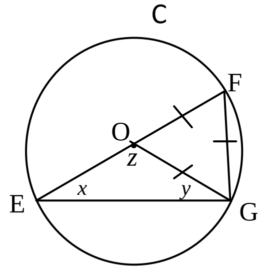
1.



2.



3.

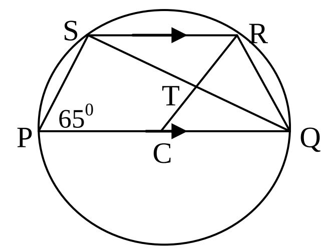


B. 1. *AB* is a diameter of a circle and *C* is a point on the circumference. < *BAC* = 45^0.

i. What is the value of < *ACB*?

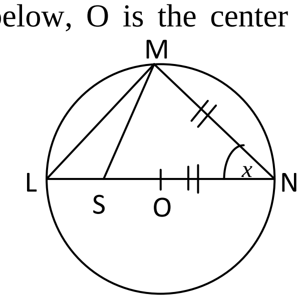
ii. Calculate < *ABC*.

2. In the diagram below, C is the center of the circle. *PQRS* is a parallelogram inscribed in the circle. *STQ* and *PCQ* are straight lines and RS is parallel to *QP*. Angle *SPC* = 65^0.



Show that triangles *CQT* and *RST* are similar and hence, find : < *CRQ*

3. In the diagram below, O is the center of the circle. */SN/ = /NM/* and < *LMS* = 44^0, Find the value of *x*



Challenge Problem

AB is a diameter of a circle of radius 10cm. *BC* is a chord of this circle of length 16cm.

i. Calculate the length of *AC*.

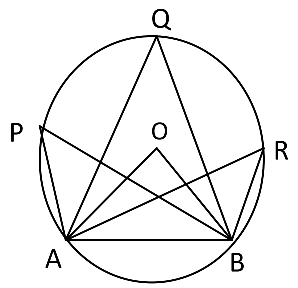
ii. If *D* is the image of *B* under reflection in *AC*, calculate the area of Δ *ABD*.

iii. If *E* is the image of *C* under reflection in *AB*, calculate the area of quadrilateral *ACBE*.

III. Angles in the same segment of a circle are equal

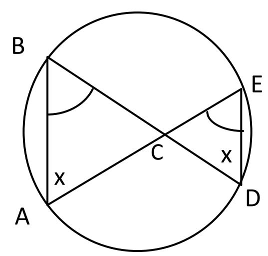
A segment of a circle is a region of the circle bounded by a chord and an arc. Every chord divides a circle into two segments.

Angles at the circumference of a circle standing on the same arc are often described as ***angles in the same segment*** and they are equal.



In the figure above, $< APB = < AQB = < ARB$

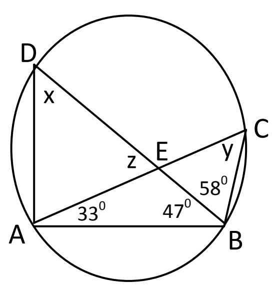
Similarly, in the figure below,



$< ABC = < CED$, $< BAC = < CDE$ and
$< ACB = < ECD$
ΔABC is similar to ΔDCE

Worked Examples

1. In the figure below, find the values of angles x, y and z. Which two triangles are similar?



Solution

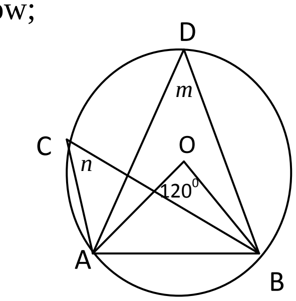
$33^0 + 47^0 + 58^0 + y^0 = 180^0$
$y = 180^0 - 33^0 - 47^0 - 58^0 = 42^0$

$x = y$ (angles in the same segment of a circle), but $y = 42^0$. Therefore, $x = 42^0$
$< BEC + y + 58^0 = 180^0$, but $y = 42^0$
$< BEC + 42^0 + 58^0 = 180^0$
$< BEC = 180^0 - 42^0 - 58 = 80^0$

$< BEC = z$ *(vertically opp. angles)*
But $< BEC = 80^0$
Therefore $z = 80^0$

ΔADE is similar to ΔBEC

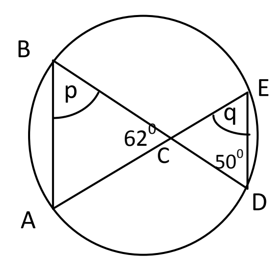
2. Find the values of angles m and n in the figure below;



Solution

$2m = 120^0$ (property I)
$m = \frac{120^0}{2} = 60^0$
$m = n$ (angles in the same segment of a circle), but $m = 60^0$. Therefore, $n = 60^0$

3. Determine the value of the angles marked p and q in the figure below:



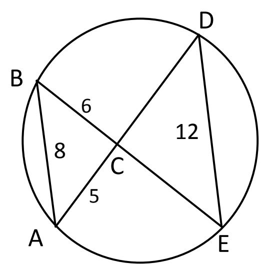
Solution

$62^0 = < DCE$ (Vertically opp. angles)
$< DCE + 50^0 + q = 180^0$, (But $< DCE = 62^0$)
$62^0 + 50^0 + q = 180^0$

$q = 180^0 - 62^0 - 50^0 = 68^0$

But $p = q$
$\Rightarrow p = 68^0$

4.i. Prove that triangles *ABC* and *EDC* in the figure below are similar.
ii. If *AB* = 8cm, *BC* = 6cm, *AC* = 5cm and *DE* = 12cm, calculate the lengths of *CD* and *CE*.



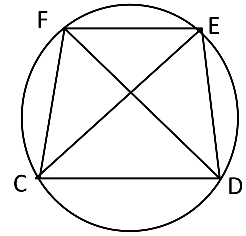
Solution
i. $< CDE = < CBA$ *(angles subtended at the circumference by the same arc AE)*
$< CAB = < CED$ *(angles subtended at the circumference by the same arc BD)*
$< BCA = < DCE$ *(vertically opposite angles)*

ii. $/DE/ \propto /BA/$
$/DE/ = k\,/BA/$
$\frac{/DE/}{/BA/} = k$ *(Scale factor)*
$k = \frac{12}{8} = 1.5$

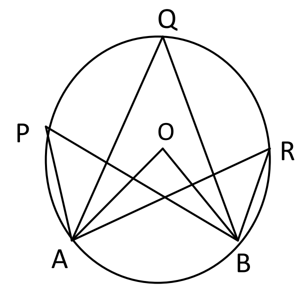
$/CD/ \propto /AC/$
$/CD/ = k\,/AC/$, but $k = 1.5$ and $/AC/ = 5$cm
$/CD/ = 1.5 \times 5\text{cm} = 7.5\text{cm}$

$/BC/ \propto /CE/$
$/BC/ = k\,/CE/$, but $k = 1.5$ and $/CE/ = 6$cm
$/BC/ = 1.5 \times 6\text{cm} = 9\text{cm}$

Exercises 23.5
1. In the figure below, mark four pairs of equal angles at the circumference

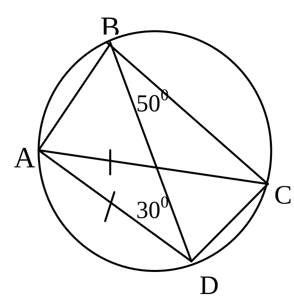


2. In the figure below, O is the center of the circle



i. If $< AOB = 80^0$, what are the sizes of angles *APB*, *AQB* and *ARB*
ii. If $< AOB = 72^0$, what are the sizes of angles *APB*, *AQB* and *ARB*
iii. If $< AOB = 2x$, what are the sizes of angles *APB*, *AQB* and *ARB*
iv. What can you say about the sizes of all angles like *APB*, *AQB* and *ARB*, which are subtended by the arc *AB*?

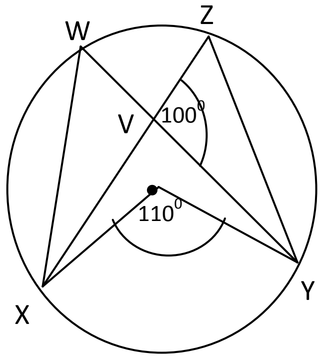
3.



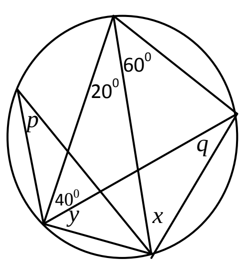
In thediagram above, *A*, *B*, *C* and *D* are points on a circle. $/AB/ = /AC/$, $< DBC = 50^0$ and $< ADB = 30^0$. Calculate the value of $< CAD$

4. In the diagram below, *WXYZ* is a circle with center O. *XZ* and *WY* intersect at V. $<XOY = 110^0$ and angle $< YVZ = 100^0$. Calculate:

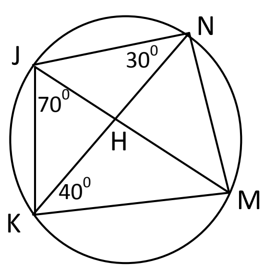
i. angle XZY ii. $< WXZ$



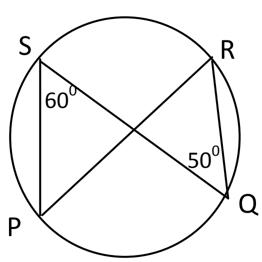
5. In the figure below, find the values of x, y, p and q.



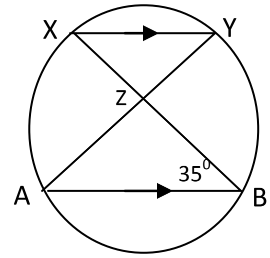
6. In the figure below, fill in the sizes of all the angles



7. Fill in the sizes of all the angles in the figurebelow. Why are the triangles similar?

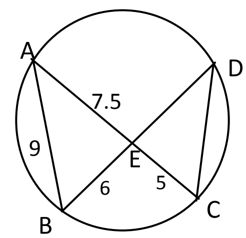


8. In the figure below, XY is parallel to AB. Calculate the sizes of as many angles as possible

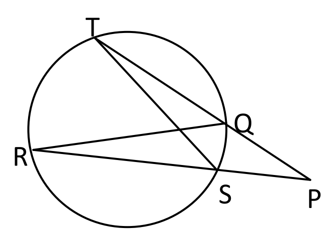


Challenge Problems

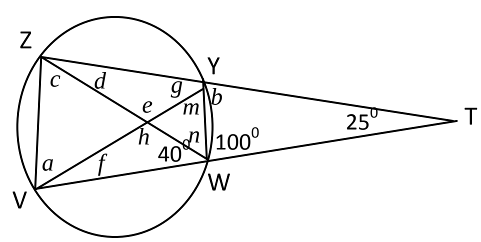
1. i. Explain why the triangles AEB and DEC in the figure below are similar.
ii. If AB = 9cm, AE = 7.5cm, BE = 6cm and CE = 4cm, calculate the lengths of DE and DC



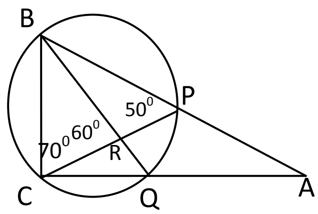
2. i. Prove that triangles PQR and PST in the figure below are similar.
ii. If TQ = 6cm, QP = 4cm, and PR = 8cm, calculate the lengths of PS



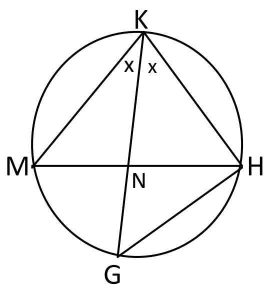
3. Find the values of the angles named with letters in the figure below;



4. In the figure below, $< BPC = 50^0$, $< BRC = 60^0$ and $< BCR = 70^0$. Calculate the angles of ΔABC



5. i. In the figure below, < *MKN* = < *GKH*. Prove that triangles *MKN* and *GKH* are similar.
ii. Name another pair of similar triangles in the figure, and prove that they are similar.

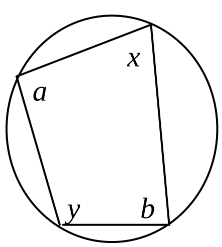


Cyclic Quadrilaterals

A quadrilateral whose vertices lie on the circumference of a circle is called ***a cyclic quadrilateral.*** The vertices are said to be ***concyclic points.***

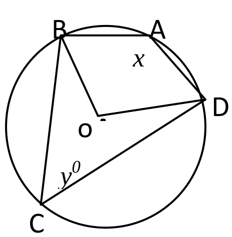
<u>Properties</u>

I. The opposite angles of a cyclic quadrilateral are supplementary (sum up to 180^0)



$\boldsymbol{x + y = 180^0}$
$\boldsymbol{a + b = 180^0}$

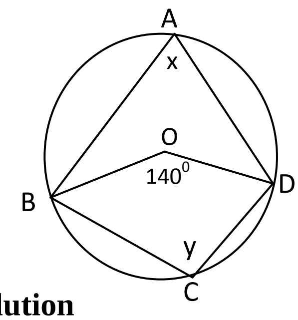
Similarly, in the figure below,



i. $\boldsymbol{x^0 + y^0 = 180^0}$
ii. <*CBA* + <*CDA* = 180^0

Worked Examples

1. In the figure below, O is the center of the circle. Calculate:
i. Reflex < *BOD* ii. *x* iii. *y* iv. $x + y$



Solution

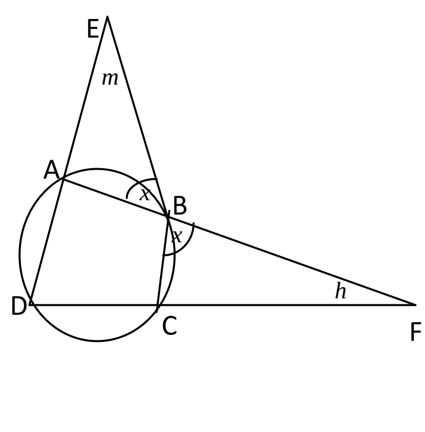
i.< *BOD* + $140^0 = 360^0$*(sum of angles in a circle)*
< *BOD* = $360^0 - 140^0 = 220^0$

ii. $2x = 140^0$ (*Property I*)
$x = \frac{140^0}{2} = 70^0$

iii. $x + y = 180^0$(*opposite angles of a cyclic quadrilateral*),
But $x = 70^0$
$70 + y = 180$
$y = 180^0 - 70^0 = 110^0$

iv. $x = 70^0$ and $y = 110^0$
$x + y = 70^0 + 110^0 = 180^0$ *(supplementary)*

2. In the diagram below, *ABCD* is a cyclic quadrilateral. *DAE*, *CBE* and *DCF* are straight lines. If $h + m = 96^0$, find the value of *x*.



Solution

From ΔABE,

$< BAD = x + m$ (exterior angle theorem)

From ΔCFB,

$< DCB = h + x$ (exterior angle theorem)

$< BAD + < DCB = 180$

$(x + m) + (h + x) = 180^0$

$2x + m + h = 180^0$ (But $m + h = 96^0$)

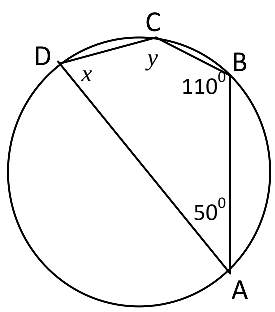
$2x + 96^0 = 180^0$

$2x = 180^0 - 96^0$

$2x = 84^0$

$x = \frac{84^0}{2} = 42^0$

3. Find the values of angles x and y in the figure below;



Solution

$x + 110^0 = 180^0$

$x = 180^0 - 110^0$

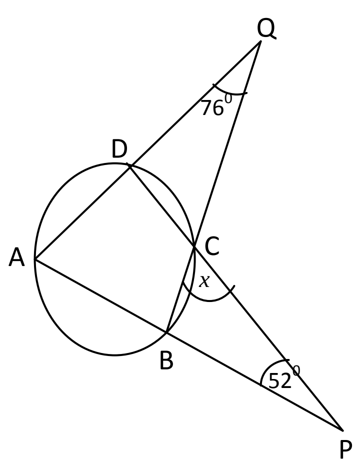
$x = 70^0$

$y + 50^0 = 180^0$

$y = 180^0 - 50^0 = 130^0$

Solved Past Question

In the diagram below, $ABCD$ is a cyclic quadrilateral. AB and DC are produced to meet at P, AD and BC are produced to meet at Q.



If $< DQC = 76^0$, $< BPC = 52^0$ and $< BCP = x$, calculate the value of x

Solution

From ΔDCQ,

$< ADC = x + 76^0$ (exterior angle theorem)

From ΔBCP,

$< ABC = x + 52^0$(exterior angle theorem)

$< ADC + < ABC = 180^0 \left(\begin{matrix}\text{opp.} < \text{ s of a} \\ \text{cyclic quadrilateral}\end{matrix}\right)$

$(x + 76^0) + (x + 52^0) = 180^0$

$x + 76^0 + x + 52^0 = 180^0$

$2x + 128^0 = 180^0$

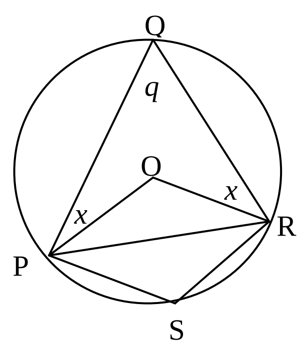
$2x = 180^0 - 128^0$

$2x = 52^0$

$x = \frac{52^0}{2} = 26^0$

2. The diagram below shows a circle $PQRS$ with center O, quadrilateral $OPSR$ is a rhombus.

$< QPO = < ORQ = x$ and $< PQR = q$.



Find: i. q

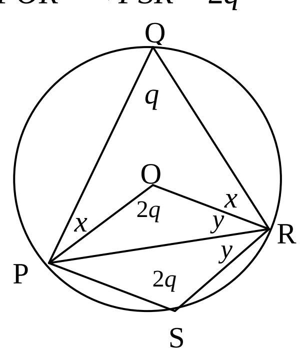
ii. x

iii. $< QRS$

Solution

i. $PORS$ is a rhombus, opposite angles are equal

$< POR = < PSR = 2q$



$q + 2q = 180^0$ (Opp <s of a cyclic quad)
$3q = 180^0$
$q = 60^0$

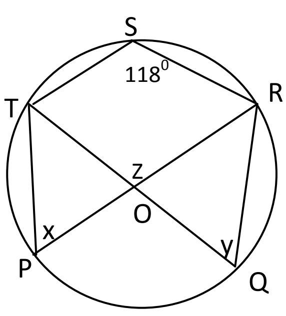
$2q = 2 \times 60^0 = 120^0$

Consider *OPQR*,
$< POR = 360^0 - 120^0 = 240^0$
$x + < POR + x + q = 360^0 \left(\begin{matrix}Sum\ of\ angles \\ in\ a\ quad\end{matrix}\right)$
$x + 240^0 + x + 60^0 = 360^0$
$2x + 300^0 = 360$
$2x = 360^0 - 300^0$
$2x = 60^0$
$x = 30^0$

iii. $< QRS = x + y + y$
Consider ΔPOR
$y + y + 2q = 180^0$
$2y + 120^0 = 180^0$
$2y = 180^0 - 120^0$
$2y = 60^0$
$y = 30^0$

$< QRS = x + y + y$
$< QRS = 30^0 + 30^0 + 30^0$
$< QRS = 90^0$

3. In the diagram below, O is the center of a circle, /*TS*/ = /*SR*/, angle *TPR* = *x*, < *TQR* = *y*, <*TOR* = z and < *TSR* = 118^0.



i. Find the relationship between *x, y, z*
ii. Find angle *STP*.

Solution
i. Consider □*PRST*;
$x + 118^0 = 180^0$
$x = 180^0 - 118^0 = 62^0$

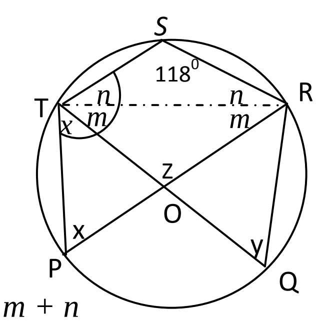
Consider □*QRST;*
$y + 118^0 = 180^0$
$y = 180^0 - 118^0 = 62^0$

Consider Δ *OQR* (isosceles);
$< ROQ + y + y = 180^0$
$< ROQ + 62^0 + 62^0 = 180^0$
$< ROQ = 180^0 - 62^0 - 62^0$
$< ROQ = 56^0$

$z + < ROQ = 180^0$ (< s on a staright line)
$z + 56^0 = 180^0$
$z = 180^0 - 56^0 = 124^0$

Now $x = 62^0$, $y = 62^0$ and $z = 124^0$
$62^0 + 62^0 = 124^0$
$x + y = z$

ii.



$< STP = x + m + n$

From Δ *TSR*;
$2n + 118^0 = 180^0$
$2n = 180^0 - 118^0$
$2n = 62^0$
$n = 31^0$

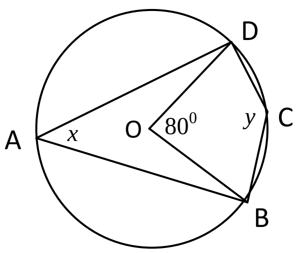
From Δ *TOR*;
$2m + 124^0 = 180^0$
$2m = 180^0 - 124^0$

$2m = 56^0$
$m = 28^0$

$< STP = 62^0 + 28^0 + 31^0$
$< STP = 121^0$

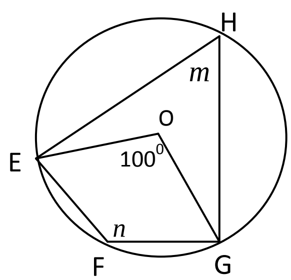
Exercises 23.6

1. In the figure below, O is the center of the circle.

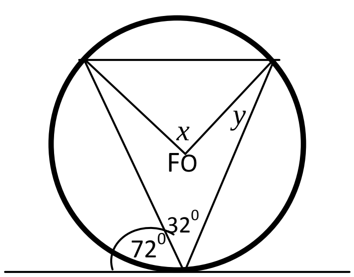


Calculate:
i. Reflex $< BOD$
ii. x
iii. y
iv. $x + y$

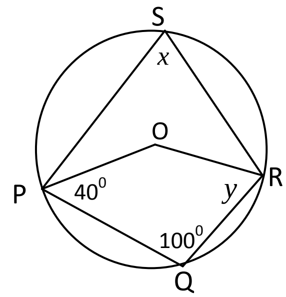
2. In the figure below, O is the center of the circle. Find the sizes of the angles marked m and n.



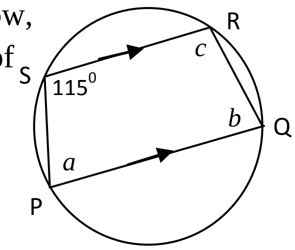
3. Find the size of angle y in the diagram below



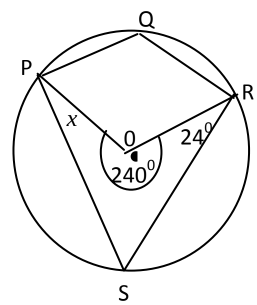
4. P, Q, R, S are four points on a circle center O below; $< PQR = 100^0$ and $< OPQ = 40^0$, find the value of y



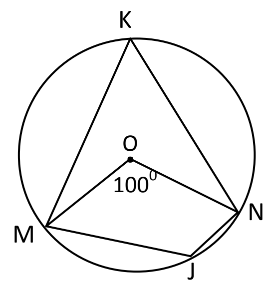
5. In the figure below, calculate the sizes of the angles marked a, and b



6. The diagram shows a circle $PQRS$ with center O. The reflex angle at O is 204^0, angle $QRS = 54^0$ and angle $OPS = x$, find the value of x.



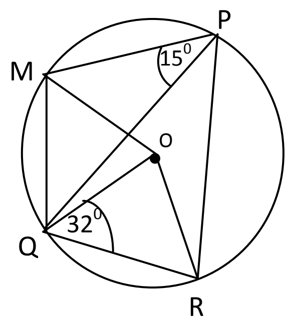
7. In the figure below, O is the center of the circle.



Calculate the sizes of:
i. $< MKN$ (standing on the minor arc MJN)
ii. $< MJN$ (standing on the major arc MKN

Challenge Problems

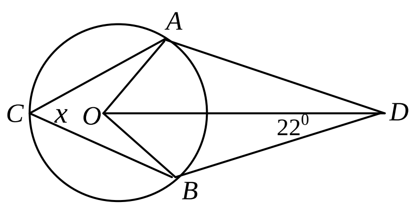
1. In the diagram below, O is the centre of the circle, $< OQR = 32^0 < MPQ = 15^0$. Calculate:



i. < *QPR*

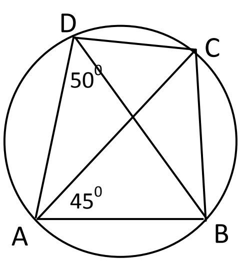
ii. < *MQO*

2. The points *A*, *B* and *C* are all on the circumference of a circle, andn O is the center.

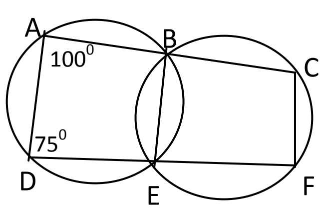


Work out the size of angle *x*

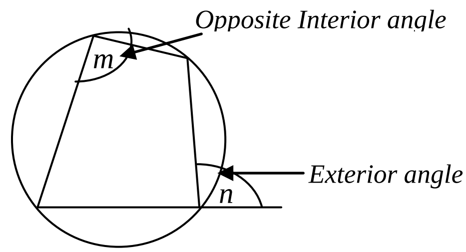
3. In the figure below, calculate the size of angle < *ABC* in two different ways.



4. Calculate the sizes of all the angles in the figure below in which *ABC* and *DEF* are double chords



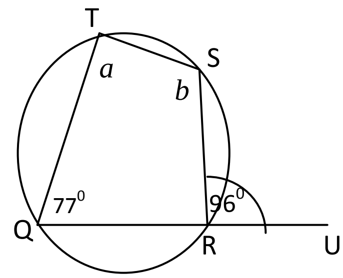
II. An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle



From the diagram above, $m = n$

Worked Examples

In the figure below, calculate the sizes of the angles named with letters.



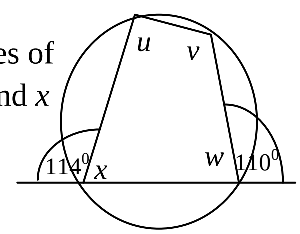
Solution

$a = 96^0$ $\left(\begin{array}{c} ext < and\ int.\ opp\ < \\ of\ a\ cyclic\ quad \end{array}\right)$

$b + 77^0 = 180^0$ $\left(\begin{array}{c} Opp < s\ of \\ a\ cyclic\ quad \end{array}\right)$

$b = 180^0 - 77^0 = 103^0$

2. Find the values of angles *u*, *v*, *w* and *x* in the figures below;



Solution

$v = 114^0$ and $u = 110^0$ $\left(\begin{array}{c} ext < of\ a\ cyclic\ quad \\ and\ its\ opp\ int, < \end{array}\right)$

$\Rightarrow u + w = 180^0$ $\left(\begin{array}{c} Opp < s\ of\ a\ cyclic \\ quad\ add\ up\ to\ 180^0 \end{array}\right)$

But $u = 110^0$

$110^0 + w = 180$

$w = 180^0 - 110^0 = 70^0$

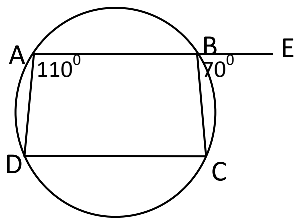
$x + v = 180^0$, but $v = 114^0$

$x + 114^0 = 180^0$

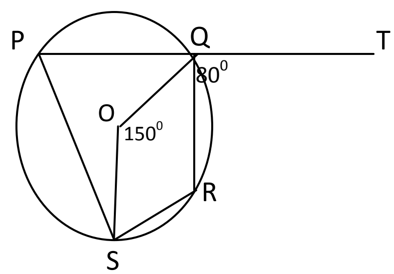
$x = 180^0 - 114^0 = 66^0$

Exercises 23.7

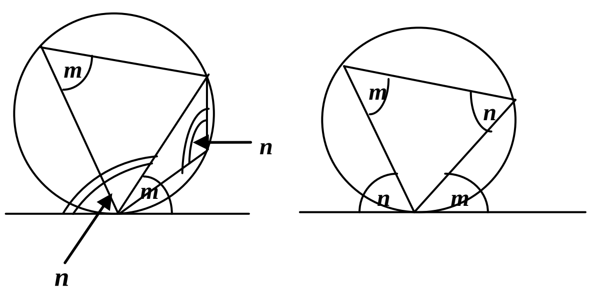
1. In the figure below, calculate < *BCD*.



2. In the figure below, O is the center of the circle. Calculate the angles of quadrilateral *PQRS.*

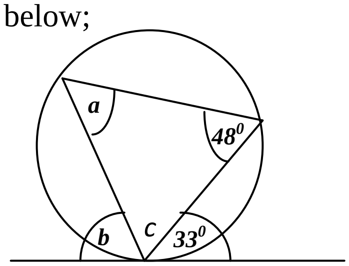


III. An angle between a tangent and a chord through the point of contact is equal to any angle in the opposite segment .



Worked Examples

Find the values of the angles *a*, *b* and *c* in the figure below;



Solution

From the diagram,

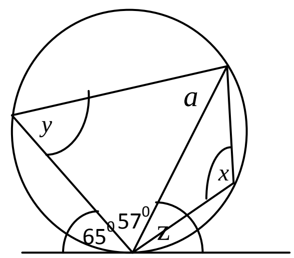
$b = 48^0$ and $a = 33^0$

$a + c + 48^0 = 180^0$,

$33^0 + c + 48^0 = 180^0$ (But $a = 33^0$)

$c = 180^0 - 33^0 - 48^0 = 99^0$

2. In the figure below, calculate the values of angles *x*, *y*, *z* and *a*.



Solution

$65^0 + 57^0 + z = 180^0$ (< s on a straight line)

$z = 180^0 - 65^0 - 57^0 = 58^0$

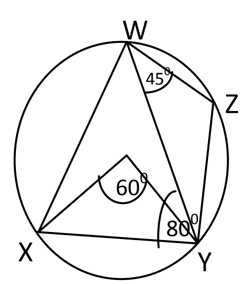
An angle between a tangent and a chord through the point of contact is equal to any angle in the opposite segment. Therefore; $z = y$, but $z = 58^0$

$\Rightarrow y = 58^0$

Also, $a = 65^0$

$x = 65^0 + 57^0 = 122^0$

3. In the diagram, *WXYZ* are points on the circumference of a circle center O. $<XOY = 60^0$, $<YWZ = 45^0$ and $< XYW = 80^0$. Calculate < *ZYW*.



Solution

$2 \times < XWY = 60^0$

$< XWY = \frac{60^0}{2} = 30^0$

XWZY is a cyclic quadrilateral and that opposite angles are supplementary (180^0)

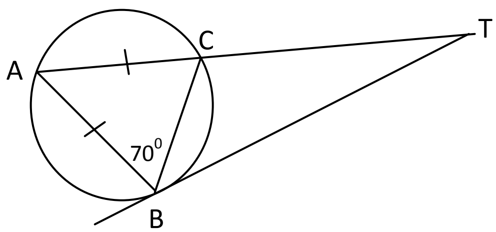
$< XWY + < YWZ + < XYW + < ZYW = 180^0$

But $< XWY = 30^0$, $< YWZ = 45^0$, $< XYW = 80^0$ and $< ZYW = ?$

$30^0 + 45^0 + 80^0 + < ZYW = 180^0$

$< ZYW = 180^0 - 30^0 - 45^0 - 80^0 = 25^0$

4. In the figure below, *AB* = *AC* and the tangent *BC* touches the circumcircle of triangle *ABC* at B. Given that $A\hat{B}C = 70^0$, find: i. $B\hat{C}T$ ii. $A\hat{T}B$



Solution

i. $< ABC = < BCA = 70^0$ (base < s of an Isosceles Δ)

$< ABC + < BCA + < CAB = 180^0$

$70^0 + 70^0 + < CAB = 180^0$

$< CAB = 180^0 - 70^0 - 70^0 = 40^0$

$< CBT = < CAB = 40^0$

$< BCT = < TAB + < ABT$

$< BCT = 40^0 + 70^0 = 110^0$

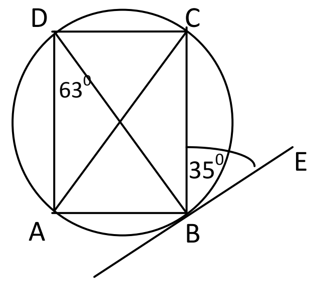
ii. $< ABT + < TAB + < ATB = 180^0$

$70^0 + 40^0 + 40^0 + < ATB = 180^0$

$< ATB = 180^0 - 70^0 - 40^0 - 40^0 = 30^0$

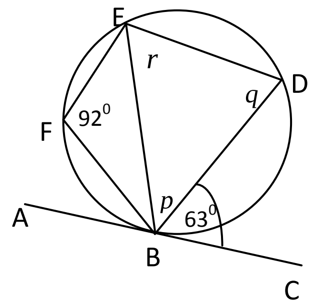
Exercises 23.8

1. In the figure below, *BE* is a tangent to the circle and *BD* is a line through the center of the circle. If angle *CBE* = 35^0 and angle *ADB* = 63^0, calculate the following;

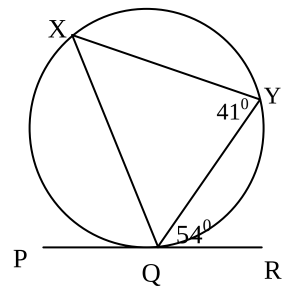


i. *ACB*
ii. *ACD*
iii. *ABC*
iv. *BAD*

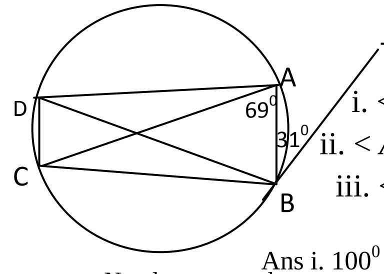
2. In the figures below, find the values of the angles marked with letters



3. In the diagram below, *PQR* is a tangent to the circle at *Q*. $< YQR = 54^0$ and $< XYQ = 41^0$. Find the size of $< XQY$.



4. In the figure below, *TB* touches the circle at *B* and *BD* is the diameter. Angle *TBA* = 31^0 and angle *BAC* = 69^0. Calculate:



i. $< ADC$
ii. $< ABC$
iii. $< CAD$

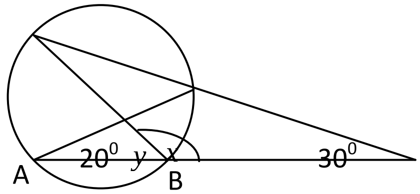
Ans i. 100^0 ii. 80^0 iii. 21^0

Not drawn to scale

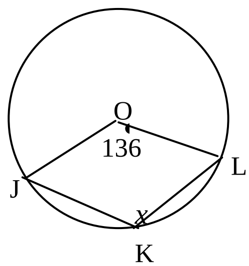
Review Exercises 23.9

In each of the following diagrams, find the value of the angles represented by letters.

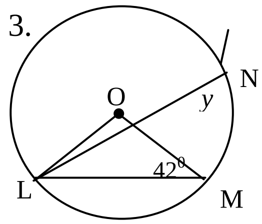
1.



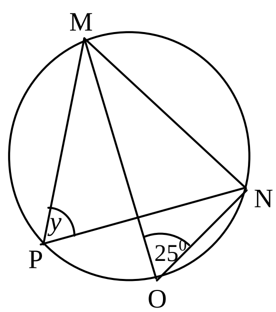
2.



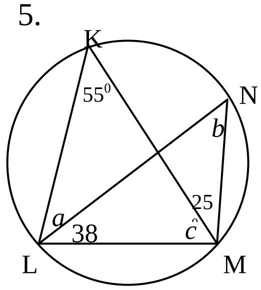
3.



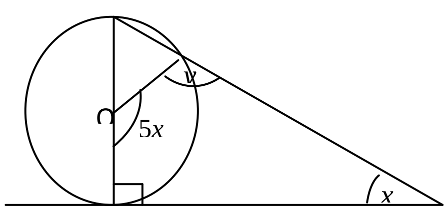
4.



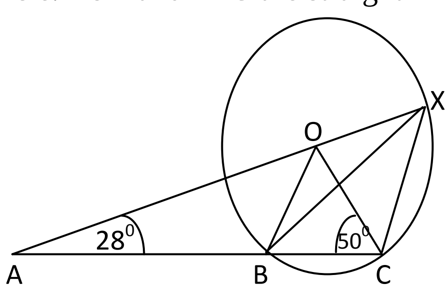
5.



6. In the figure below, O is the center of the circle. Find x and y

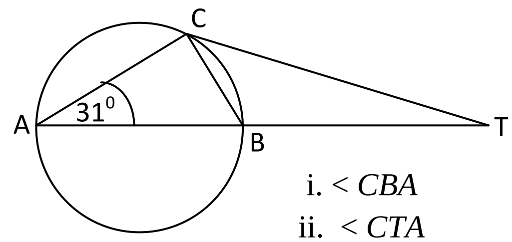


7. In the diagram below, O is the center of the circle. AOX and ABC are straight lines.



Given that $< OCB = 50^0$ and $< OAB = 28^0$, calculate $< BXC$ and $< XBC$.

8. In the diagram below, AB is a diameter of the circle and the tangent at C meets AB produced at T. Given that $< CAB = 31^0$, calculate;

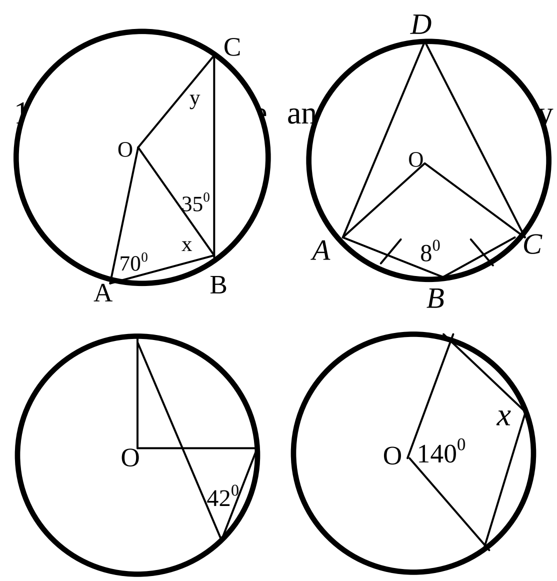


i. $< CBA$

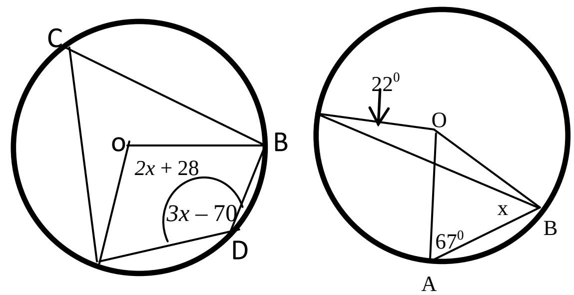
ii. $< CTA$

9. Calaculate the angles of x and y in the diagrams below

1 an y in the



11. Calaculate the angles of x and y in the diagrams below



Trigonometry

It is a branch of a mathematics that deals with measurement of lengths and angles of right-angled triangles.

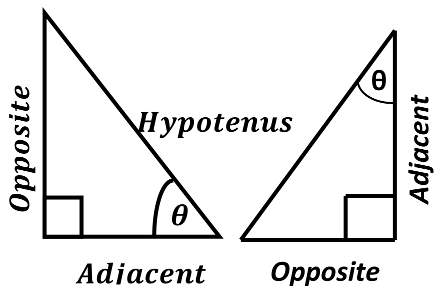
Labeling a Right - angled Triangle

A right- angled triangle is labelled as follows:

I. The side that faces the right angle, also known as the longest side is called the ***Hypotenuse (H)***,

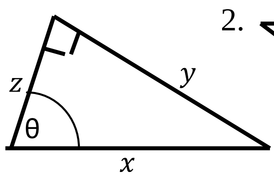
II. The side that faces the acute angle, θ, is called the ***Opposite (O)***

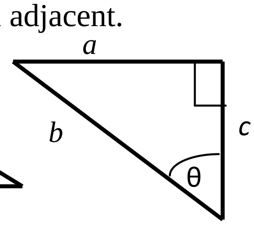
III. The third side is called the ***Adjacent (A)***



Exercises 24.1

A. For each of the following right – angled triangle, indicate the side that represents the hypotenuse, opposite and adjacent.

1. 

2. 

B. In triangle XYZ, x is a right angle

1. Which side is the hypotenuse?
2. Which side is opposite to angle y?
3. Which side is adjacent to angle y?
4. Which side is opposite angle Z?
5. Which side is adjacent to angle Z?

Sine, Cosine and Tangent of Angles

The three basic trigonometric ratios are:

1. Sine or sin
2. Cosine or cos
3. Tangent or tan

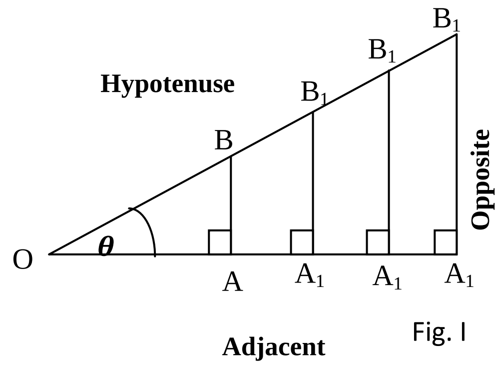


Fig. I

From fig. I, the ratios are defined as,

$$\sin\theta = \frac{opposite}{hypothenus} = \frac{O}{H}$$

$$= \frac{AB}{OB} = \frac{A_1B_1}{OB_1} = \frac{A_2B_2}{OB_2} = \frac{A_3B_3}{OB_3}$$

$$\cos\theta = \frac{adjacent}{hypothenus} = \frac{A}{H}$$

$$= \frac{OA}{OB} = \frac{OA_1}{OB_1} = \frac{OA_2}{OB_2} = \frac{OA_3}{OB_3}$$

$$\tan\theta = \frac{opposite}{adjacent} = \frac{O}{A}$$

$$= \frac{AB}{OA} = \frac{A_1B_1}{OA_1} = \frac{A_2B_2}{OA_2} = \frac{A_3B_3}{OA_3}$$

These ratios can easily be remembered by the acronyms: ***SOH, CAH, TOA***; where

SOH = sin θ = $\frac{O}{H}$,

CAH = cos θ = $\frac{A}{H}$

TOA = can θ = $\frac{O}{A}$

Trigonometric Ratios of 30^0 and 60^0

The trigonometric ratio of 30^0 and 60^0 are derived from an equilateral triangle of dimensions 2 – units as shown below;

By Pythagoras theorem,

$2^2 = |AH|^2 + 1^2$

$|AH|^2 = 2^2 - 1^2$

$|AH|^2 = 3$

$|AH| = \sqrt{3}$

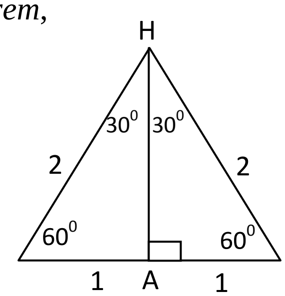


Fig. II

The dimensions of fig. II can be shown below:

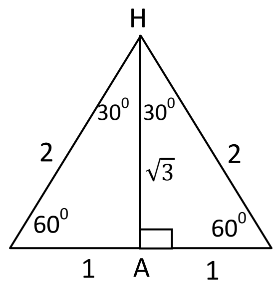


Fig. II

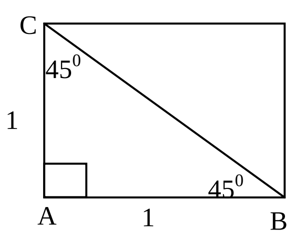
From this figure, the trigonometric ratios of 30^0 and 60^0 are obtained as follows:

$\sin 30^0 = \frac{1}{2}$, $\cos 60^0 = \frac{1}{2}$

$\sin 60^0 = \frac{\sqrt{3}}{2}$, $\cos 30^0 = \frac{\sqrt{3}}{2}$

$\tan 30^0 = \frac{1}{\sqrt{3}}$, $\tan 60^0 = \sqrt{3}$

Trigonometric Ratios of 45^0 and 90^0



By Pythagoras theorem,

$|BC|^2 = 1^2 + 1^2 = 2$

$|BC| = \sqrt{2}$

The dimension of the figure is shown below**;**

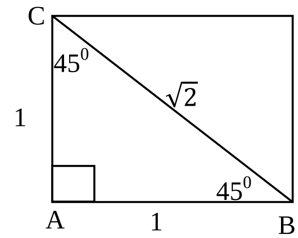


Fig. III

From this fig. III, the trigonometric ratios of 45^0 are obtained as follows;

$\cos 45^0 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\sin 45^0 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\tan 45^0 = 1$

The trigonometric ratios of 90^0 are obtained as follows

$\cos 90^0 = \frac{0}{\sqrt{2}} = 0$,

$\sin 90^0 = \frac{\sqrt{2}}{\sqrt{2}} = 1$,

$\tan 90^0 = \frac{\sqrt{2}}{0}$ = undefined

These ratios can be used instead of a calculator, especially when answers are required in the form of surds.

Summary Table:

Ratios of 30^0, 45^0, 60^0 and 90^0

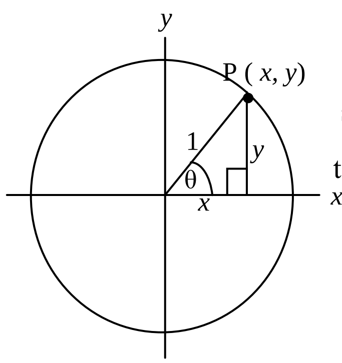
	30^0	45^0	60^0	90^0
sin	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	$\frac{\sqrt{3}}{2}$	1	$\sqrt{3}$	-

Trigonometry Ratios for Any Angle

The Unit Circle

The unit circle has its center at the origin(0, 0) and the length of the radius is 1.

Take any point $p(x, y)$, on the circle, making an angle of Θ, from the center.

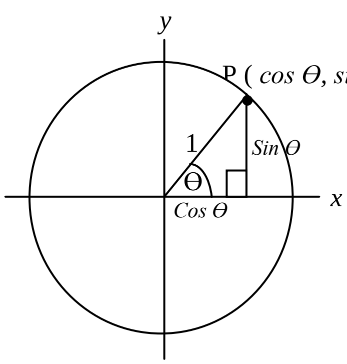


$\text{Cos }\theta = \frac{x}{1} = x$,

$\sin\theta = \frac{y}{1} = y$,

$\tan\theta = \frac{y}{x}$, $\tan\theta = \frac{Sin\ \theta}{Cos\ \theta}$

This very important results indicates that the coordinates of any point on the unit circle can be represented by $p(\cos\theta,\ \sin\ \theta)$, where θ is any angle



As the point p rotates, θ changes. These definitions of $\cos\theta$ and $\sin\ \theta$ in terms of the coordinates of a point rotating around the unit circle apply for all values of the angle θ^0,

Memory Aid: (Christian name, Surname)

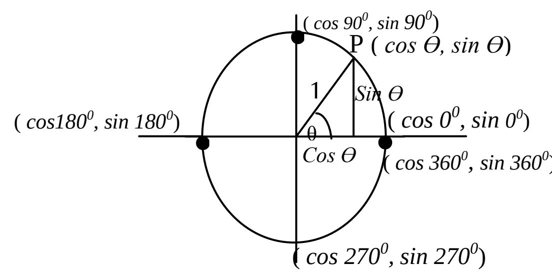
= (*cosθ* , *sin θ*) = (*x*, *y*)

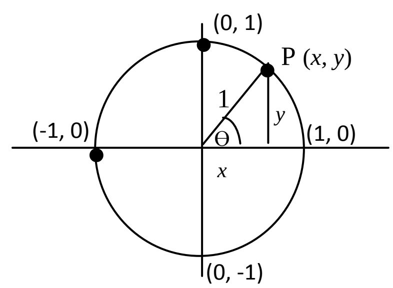
Note: Using Pythagoras theorem:

$\cos^2\theta + \sin^2\theta = 1$

Values of Sin, Cos and Tan for 0^0, 90^0, 180^0, 270^0 and 360^0

Both diagrams below represent the unit circle but using two different notations to describe any point p on the circle.

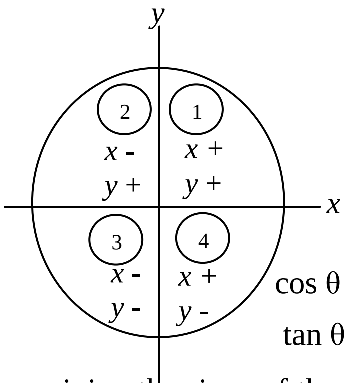




By comparing corresponding points on both unit circles, the values of sin, cos, and tan for 0^0, 90^0, $180^{0,}$ 270^0 and 360^0, can be read directly without the use of a calculator.

$(\text{Cos } 0^0, \sin 0^0) = (\cos 360^0, \sin 360^0) = (1, 0)$ $\text{Cos } 0^0 = \text{Cos } 360^0 = 1$ $\sin 0^0 = \sin 360^0 = 0$ $\tan 0^0 = \tan 360^0 = \frac{0}{1} = 0$	$(\text{Cos } 90^0, \sin 90^0) = (0, 1)$ $\text{Cos } 90^0 = 0$ $\sin 90^0 = 1$ $\tan 90^0 = \frac{1}{0}$(undefined)
$(\text{Cos } 180^0, \sin 180^0) = (-1, 0)$ $\text{Cos } 180^0 = \text{Cos } 360^0 = -1$ $\sin 180^0 = 0$ $\tan 180^0 = \frac{0}{-1} = 0$	$(\text{Cos } 270^0, \sin 270^0) = (0, -1)$ $\text{Cos } 270^0 = 0$ $\sin 270^0 = -1$ $\tan 270^0 = \tan 360^0 = \frac{-1}{0}$ (undefined)

The x and y axes divide the plane into four quadrants. Consider the unit circle below,

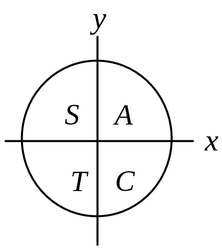


$\cos\theta = x$ and $\sin\theta = y$

$\tan\theta = \frac{\cos\theta}{\sin\theta} = \frac{x}{y}$

By examining the signs of the four quadrants, the sign of sin θ, cos θ, tan θ for any value of θ can be found

Summary of Signs

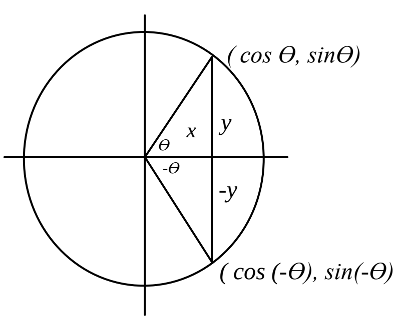


1st quadrant: sin cos and tan are ALL positive
2nd quadrant: sin is positive, cos and tan are negative.
3rd quadrant: tan is positive, sin and cos are negative.
4th quadrant: cos is positive, sin and tan are negative.

A very useful memory aid, **CAST**, in the diagram above shows the ratios that are positive for the angles between $0^0$and 360^0.

Negative Angles

Consider the unit circle showing angles θ and -θ



$\cos\theta = x$, $\sin\theta = y$ and $\tan\theta = \frac{y}{x}$

$\cos(-\theta) = x$, $\sin(-\theta) = -y$, $\tan(-\theta) = -\frac{y}{x}$

Thus;

***cos* (-θ) = *cos* θ**

***sin* (-θ) = - *sin* θ**

***tan* (-θ) = - *tan* θ**

Worked Examples

Find without using tables or calculators

1. $\sin(-60^0)$ 2. $\cos(-30^0)$ 3. $\tan(-225^0)$

Solutions

1. $\sin(-60^0) = -\sin 60^0 = -\frac{\sqrt{3}}{2}$

2. $\cos(-30^0) = \cos 30^0 = \frac{\sqrt{3}}{2}$

3. $\tan(-225^0) = -\tan 225^0 = -\tan 45^0 = -1$

Exercises 24.2

1. $\tan(-150^0)$ 2. $\tan(-210^0)$ 3. $\cos(-120^0)$
4. $\cos(-135^0)$ 5. $\sin(-270^0)$ 6. $\sin(-135^0)$

Methods for Finding the Trigonometric Ratio for Any Angle between 0^0 and 360^0

<u>Method 1</u>

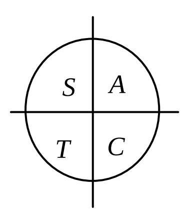
I. Draw a rough diagram for the angle in the $x - y$ plane, taking measurement in the anticlockwise direction from the positive x - axis

II. Determine in which quadrant the angle lies

III. Find its related acute angle to the nearest horizontal line (x - axis)

IV. Use the trigonometric ratios of the related angle

V. Use the ***CAST*** diagram below to find the signs of the ratios



Note:***sin***^{2} ***A*** = $(\boldsymbol{sin\ A})^2$ ***etc***

Method 2

If θ is the general angle, then

1. In the first quadrant, ($0^0 < \theta < 90^0$), all the trigonometry ratios are positive

2. In the second quadrant, ($90^0 < \theta < 180^0$), only sin is positive. That is:

a. sin (180 – θ) = sin θ

eg. sin 120^0 = sin ($180^0 – 120^0$)

= sin 60^0

b. cos (180^0 – θ) = – cos θ

eg.cos 120^0 = cos ($180^0 – 120^0$)

= – cos 60^0

c. tan (180^0 – θ) = – tan θ

eg.tan 120^0 = tan ($180^0 – 120^0$)

= – tan 60^0 = $\sqrt{3}$

3. In the third quadrant, ($180^0 < \theta < 270^0$), only tan is positive. That is

a. sin (θ– 180^0) = – sin θ

e.g. sin 210^0 = sin ($210^0 – 180^0$)

= – sin 30^0 = $-\frac{1}{2}$

b. cos (θ – 180^0) = – cos θ

e.g.cos 210^0 = cos ($210^0 – 180^0$)

= – cos 30^0 = $-\frac{\sqrt{3}}{2}$

c. tan (θ–180^0) = tan θ

e.g.tan 210^0 = tan ($210^0 – 180^0$)

= – tan 30^0 = $-\frac{1}{\sqrt{3}}$

4. In the fourth quadrant, ($270^0 < \theta < 270^0$), subtract the given angle Θ, from 360^0 to obtain the related angle. That is, Related angle = (360^0 – θ). Only cos is positive. That is explained below:

a. sin (360^0 - θ) = - sin θ

e.g. Sin 300^0 = sin ($360^0 – 300^0$)

= – sin 60^0 = $-\frac{\sqrt{3}}{2}$

b. cos (360^0 - θ) = cos θ

E.g. cos 300^0 = cos ($360^0 – 300^0$)

= cos 60^0 = $\frac{1}{2}$

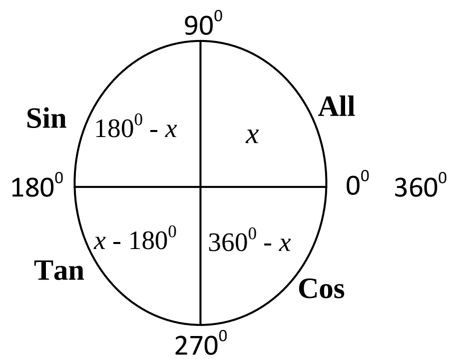
c. tan (360^0– θ) = – tan θ

E.g. tan 300^0 = tan ($360^0 – 300^0$)

= – tan 60^0

= $-\sqrt{3}$

Summary



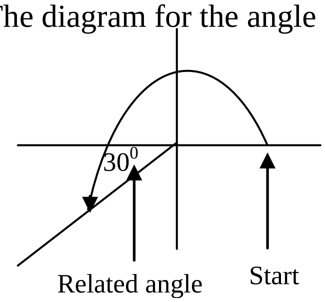
Worked Examples

1. Find cos 210^0, leaving your answer in surd form:

Solution

Method 1

I. The diagram for the angle is shown below;



II. 210^0 is in the 3rd quadrant

cos is negative in the 3rd quadrant

III. Related angle is 30^0

IV. ∴ cos 210^0 = - cos 30^0 = $-\frac{\sqrt{3}}{2}$

Method 2

Let $\theta = 210^0$

210^0 is in the 3rd quadrant where only tan is positive

Related angle = $(\theta - 180^0)$

$= (210^0 - 180^0) = 30^0$

$\Rightarrow 210^0$ is related to 30^0

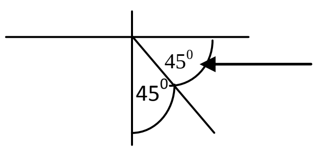
Sin $210^0 = -\cos 30^0 = -\frac{\sqrt{3}}{2}$

2. Find sin 315^0, leaving your answer in surd form:

Solution

Method 1

315^0 lies in the 3^{rd} quadrant as shown in the diagram below;



sin is negative in the fourth quadrant

Sin $315^0 = -\sin 45^0 = -\frac{1}{\sqrt{2}}$

Method 2

Let $\theta = 315^0$

$315^0$is in the 4th quadrant, where only cos is positive

Related angle = $(360^0 - \theta)$

$= (360^0 - 315^0) = 45^0$

$\Rightarrow 315^0$ is related to 45^0

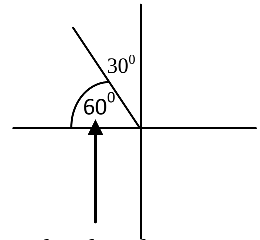
sin315^0 = - sin $45^0 = -\frac{1}{\sqrt{2}}$

3. Find without using tables, tan 120^0,

Solution

Method 1

120^0 lies in the 2nd quadrant



tan is negative in the 2nd quadrant

tan 120^0 = - tan 60 = - $\sqrt{3}$

Method 2

Let $\theta = 120^0$

$120^0$is in the 2nd quadrant, where only sin is positive

Related angle = $(180^0 - \theta)$

$= (180^0 - 120^0) = 60^0$

$\Rightarrow 120^0$is related to 60^0

sin120^0 = sin $60^0 = \frac{\sqrt{3}}{2}$,

Simplifying and Evaluating Trigonometric Expressions

Replace each angle with it's ratio and perform the included operation. If answer is required as a surd, do so by rationalizing the denorminator

Worked Examples

1. Evaluate sin 30^0 + 2 tan 45^0 without using calculators or tables

Solution

Sin 30^0 + 2 tan 45^0

$= \frac{1}{2} + 2\,(1) = \frac{1}{2} + 2 = 2\frac{1}{2} = \frac{5}{2}$

2. without using tables or calculator evaluate cos 45^0 + sin 30^0.

Solution

cos 45^0 + sin 30^0

$\frac{\sqrt{2}}{2} + \frac{1}{2} = \frac{1+\sqrt{2}}{2}$

3. Without using tables or calculators, simplify sin 135^0 – sin 315^0

Solution

sin 135^0 – sin 315^0

$\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$

4. $\cos^2 45^0 + \sin 30^0$

$= \left(\frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2} = \frac{2}{4} + \frac{1}{2} = 1$

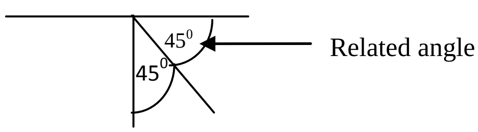
Solved Past Question

1. Without using mathematical table or calculator, evaluate 3 cos 315^0 – 2 sin 210^0, leaving your answer in surd form.

Solution

Method I

315^0 lies in the 3rd quadrant as shown in the diagram below;

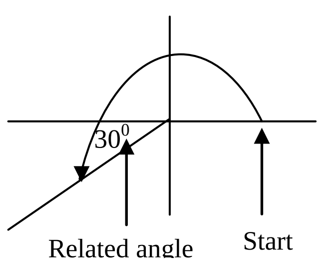


cos is positive in the fourth quadrant

cos 315^0 = cos 45^0 = $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\Rightarrow 3 \cos 315^0 = 3\left(\frac{1}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}}$

The diagram for angle 210^0 is shown below;



210^0 is in the 3rd quadrant and the related angle is 30^0, but sin is negative in the 3rd quadrant

$\therefore \sin 210^0 = -\sin 30^0 = -\frac{1}{2}$

$\Rightarrow 2 \sin 210^0 = 2(-\sin 30^0)$

$= 2\left(-\frac{1}{2}\right) = -1$

$\Rightarrow 3 \cos 315^0 - 2 \sin 210^0$

$\frac{3}{\sqrt{2}} - (-1) = \frac{3}{\sqrt{2}} + 1 = \frac{2+3\sqrt{2}}{2}$

Method 2

$3 \cos 315^0 - 2 \sin 210^0$

$= 3\left(\frac{\sqrt{2}}{2}\right) - 2\left(-\frac{1}{2}\right) = \frac{3\sqrt{2}}{2} + 1 = \frac{2+3\sqrt{2}}{2}$

2. Simplify: $\frac{\tan 60^0 + \tan 45^0}{\tan 60^0 - \tan 45^0}$, leaving your answer in surd form.

Solution

$\frac{\tan 60^0 + \tan 45^0}{\tan 60^0 - \tan 45^0}$

$= \frac{\sqrt{3}+1}{\sqrt{3}-1}$

$= \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$ ***Rationalise the denorminator***

$= \frac{(\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$

$= \frac{4+2\sqrt{3}}{2} = \frac{4}{2} + \frac{2\sqrt{3}}{2} = 2 + \sqrt{3}$

3. Evaluate $\frac{\tan 60^0 - \sin 45^0}{\cos 150^0}$, leaving your answer in surd form.

Solution

$\frac{\tan 60^0 - \sin 45^0}{\cos 150^0}$,

$= \frac{\sqrt{3} - \frac{\sqrt{2}}{2}}{-\frac{\sqrt{3}}{2}}$ *(solving the numerator)*

$= \frac{\frac{2\sqrt{3}-\sqrt{2}}{2}}{-\frac{\sqrt{3}}{2}}$

$= \frac{2\sqrt{3}-\sqrt{2}}{2} \div -\frac{\sqrt{3}}{2}$

$= \frac{2\sqrt{3}-\sqrt{2}}{2} \times -\frac{2}{\sqrt{3}}$

$= -\frac{2(2\sqrt{3}-\sqrt{2})}{2\sqrt{3}}$

$= -\frac{(2\sqrt{3} - \sqrt{2})}{\sqrt{3}}$

Rationalise the denorminator

$= -\frac{(2\sqrt{3} - \sqrt{2})}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= -\frac{\sqrt{3}(2\sqrt{3} - \sqrt{2})}{(\sqrt{3})(\sqrt{3})}$

$= -\frac{2 \times 3 - \sqrt{2 \times 3}}{3}$

$= -\frac{6 - \sqrt{6}}{3}$

4. If $\frac{2}{1 - cos 30^0} = a + b\sqrt{3}$, find the values of a and b.

Solution

$\frac{2}{1 - cos 30^0}$

$= \frac{2}{1 - \sqrt{3}/_2} = \frac{2}{\frac{2 - \sqrt{3}}{2}} = 2 \div \frac{2 - \sqrt{3}}{2} = 2 \times \frac{2}{2 - \sqrt{3}} = \frac{4}{2 - \sqrt{3}}$

$= \frac{4}{2 - \sqrt{3}} \times \frac{2 + \sqrt{3}}{2 + \sqrt{3}}$ ***Rationalise the denorminator***

$= \frac{4(2 + \sqrt{3})}{(2 - \sqrt{3})(2 + \sqrt{3})}$

$= \frac{8 + 4\sqrt{3}}{1}$

$= 8 + 4\sqrt{3}$

$\Rightarrow 8 + 4\sqrt{3} = a + b\sqrt{3}$

Therefore $a = 8$ and $b = 4$

Exercises 24.3

A. Evaluate, leaving your answer in surd form where necessary

1. $\cos 120^0 + \cos 225^0$ 2. $\tan 240^0 - \tan 330^0$

3. $\sin^2 60^0 + \cos^2 45^0$ 4. $\tan^2 30^0 - \sin^2 60^0$

B. 1. Without using tables or calculators, find the value of $\sin 60^0 \cos 45^0 + \cos 60^0 \sin 45^0$

2. Without using tables or calculators, find the value $1 + \cos^2 30$.

3. Express $\frac{1}{1 - sin\ 60^0}$ as a surd.

4. Express $\frac{5}{1 + cos\ 30^0}$ as a surd.

5. Without using table, evaluate $\frac{tan\ 45^0 - tan\ 30^0}{1 + tan\ 45^0\ tan\ 30^0}$.

6. Express $\frac{1 + tan\ 30^0}{1 - tan\ 30^0}$ in surd form.

C. 1. Find $\frac{x^2}{9} + \frac{y^2}{4}$, if $x = 3 \cos 60^0$ and $y = 2 \sin 45^0$.

2. Given that $x = \sin 45^0$ and $y = \cos 45^0$, find the value of $\frac{1 + x^2}{1 - y^2}$, without using tables or calculators.

3. If $x = \cos 60^0$ and $y = \sin 60^0$, evaluate $\frac{x^2 + y^2}{y^2 - x^2}$, without the use of tables or calculators

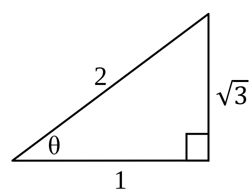
4. If $p = \sin 30^0$ and $q = \cos 30^0$, evaluate $\frac{p + q}{q^2}$, without tables or calculators.

5. If $x = \tan 30^0$ and $y = \tan 45^0$, find the value of $\frac{x + y}{xy}$.

Inverse of Trigonometric Ratios

It is the reverse process of determining the angle given the value of one of the trigonometric ratios. The inverse of a trigonometric ratio is written as ($^{-1}$). Thus, $\sin^{-1}$is pronounced ***"inverse sine x"*** or ***"sine inverse x"*** or ***"arc sine x"***

Consider the triangle below;



Ratio	*Angle*
$\sin \theta = \frac{\sqrt{3}}{2}$	$\theta = \sin^{-1} \frac{\sqrt{3}}{2}$
$\cos \theta = \frac{1}{2}$	$\theta = \cos^{-1} \frac{1}{2}$
$\tan \theta = \sqrt{3}$	$\theta = \tan^{-1} \sqrt{3}$

However, a problem arises with the $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$ notation.

Consider the following equations;

If $\sin\theta = \frac{1}{2}$, then $\theta = \sin^{-1}\frac{1}{2} = \ldots, -330^0, -210^0, 30^0, 150^0, 390^0, 510^0, \ldots$

If $\tan\theta = 1$, then $\theta = \tan^{-1}1 = \ldots, -315^0, -135^0, 45^0, 315^0, 405^0, 675^0, \ldots$

Hence, if there is no restrictions on the value of θ, then the equation $\sin\theta = \frac{1}{2}$ and $\tan\theta = 1$ have an infinite number of solutions.

This problem is overcomed by restricting the value of θ to the range $-90^0 \leq \theta \leq 90^0$.

The angles within this range are often called ***"principal values".***

Using these restrictions for θ, we often obtain a single value for our answer. Thus, $\sin^{-1}\frac{\sqrt{3}}{2} = 60^0$, not also 120^0, as 120^0 is not in the range -90^0 to 90^0

Worked Examples

1. Find the principal values of each of the following;

i. $\sin^{-1}\frac{1}{2}$ ii. $\cos^{-1}\frac{1}{\sqrt{2}}$ iii. $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Solution

i. $\sin^{-1}\frac{1}{2} = \sin^{-1}(0.5) = 30^0$

ii. $\cos^{-1}\frac{1}{\sqrt{2}} = \cos^{-1}(0.7071) = 45^0$

iii. $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -30^0$

2. Given that $\sin y = -\frac{\sqrt{3}}{2}$ and $\tan y = \sqrt{3}$. Find y.

Solution

The only quadrant in which sin is – ve and tan is + ve is the 3rd quadrant.

$\tan y = \sqrt{3} = 1.732$

$\Rightarrow y = \tan^{-1}(1.732) = 60^0, 240^0$

Exercises 24.4

Find the values of θ, where $0^0 \leq \theta \leq 90^0$

1. $\cos\theta = -\frac{1}{2}$	2. $\cos\theta = 0.6$
3. $\sin\theta = -0.7660$	4. $\cos\theta = 0.8$

Trigonometric Equations

Given a trigonometric equation, the intention is to find the value of the variable (in degrees) that satisfies the equation. The satisfactory value is called the ***truth set*** or ***solution set*** of the equation.

I. Identify the left hand side (L.H.S.) and the right hand side (R. H. S.)of the given equation.

II. Identify the location of the trigonometric ratio and divide both sides of the equation by its co-efficient, if any.

III. If the trigonometric ratio is at the L.H.S, find its inverse at the R. H. S and vice – versa to obtain an angle.

IV. Expand brackets if any and group like terms to work out the value of the variable with restrictions to the given range.

Worked Examples

1. Given that $\tan(x + 25^0) = 5.145$, where $0^0 \leq x \leq 90^0$, find to one decimal place the value of x.

Solution

$\tan(x + 25^0) = 5.145$

$x + 25^0 = \tan^{-1}(5.145)$

$x + 25^0 = 79^0$

$x = 79^0 - 25^0 = 54^0$

2. Find θ if $\cos(\theta + 60^0) = 0.0872$, where $0^0 \leq x \leq 90^0$

Solution

cos (θ + 60^0) = 0.0872

θ + 60^0 = $\cos^{-1}$ (0.0872)

θ + 60^0 = 85^0

θ = 85^0 – 60^0 = 25^0

3. Find the truth set of 1.414 sin p = 1, where $0^0 < p < 90^0$

Solution

1.414 sin p = 1

sin$p$ = $\frac{1}{1.414}$

$p = \sin^{-1}\left(\frac{1}{1.414}\right)$ = 45^0

4. If 1.5 cos x = 0.75, find x if $0^0 < x < 90^0$

Solution

1.5 cos x = 0.75

cos$x$ = $\frac{0.75}{1.5}$

$x = \cos^{-1}\left(\frac{0.75}{1.5}\right) = 60^0$

5. If x cos 60^0 = 1.5 and y sin 30^0 = 2, evaluate; $\sqrt{x^2 + y^2}$

Solution

x cos 60^0 = 1.5

$x = \frac{1.5}{\cos 60^0} = 3$

y sin 30^0 = 2

$y = \frac{2}{\sin\ 30^0} = 4$

Substitute x = 3 and y = 4 in $\sqrt{x^2 + y^2}$

$\Rightarrow\sqrt{3^2 + 4^2} = \sqrt{25} = 5$

Some Solved Past Questions

1. Given that 5 sin x = 4.33, where $0^0 \leq x \leq 90^0$, find x correct to the nearest degree.

Solution

5 sin$x$ = 4.33

sin x = $\frac{4.33}{5}$

$x = \sin^{-1}\left(\frac{4.33}{5}\right) = 60^0$

2. If 8 sin x + 2 = 5, find x correct to the nearest degree.

Solution

8 sin x + 2 = 5

8 sin x = 5 – 2

8 sin x = 3

sin x = $\frac{3}{8}$

$x = \sin^{-1}\left(\frac{3}{8}\right) = 22^0$

Exercises 24.5

A. Solve the following for $0^0 \leq \theta \leq 90^0$

1. sin $(2\theta + 30^0)$ = 0.8
2. tan $(3\theta - 45^0) = \frac{1}{2}$
3. 8 cos θ = 3
4. 4 tan θ = 3
5. sin $(2\theta - 12) = \frac{1}{2}$
6. cos θ = $\frac{3}{4}$
7. sin $(\theta - 30^0) = \frac{\sqrt{3}}{2}$
8. 3 cos (θ – 10^0) = 1

B. 1. If 10 sin $(x – 45)^0$ = 5, where $0^0 < x < 90^0$, find the value of x.

2. Given that 2 cos $x = \sqrt{3}$, where $0^0 < x < 90^0$, find the value of x.

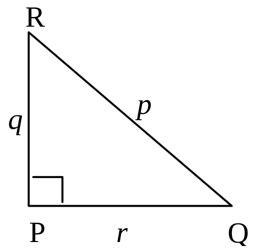
3. Solve the equation 2 $\sin^2$ θ = sin θ, for the values of θ from 0^0 to 180^0.

4. Solve the equation 2 $\cos^2$ θ = cos θ, for the values of θ from 0^0 to 180^0.

Equality of sine and Cosine of Complementary Angles

Complementary angles are two angles that sum up to 90^0

Consider the triangle below;



sin R = $\frac{r}{p}$ and cos Q = $\frac{r}{p}$

Hence sin R = cos Q = cos $(90^0 – R)$

Similarly, cos R = sin $(90^0 – R)$. Therefore, the sine of an angle is equal to the cosine of its complementary angle and vice – versa. For example:

$\sin 75^0 = \cos (90^0 – 75^0)$

$\sin 75^0 = \cos 15^0$

$\cos 30^0 = \sin (90^0 – 30^0)$

$\cos 30^0 = \sin 60^0$

Worked Examples

1. If $\sin (x + 30^0) = \cos 40^0$, find x where $0^0 < x < 90^0$

Solution

Method 1

Expressing cos 40^0 as a sin

$\sin (x + 30^0) = \cos 40^0$

$\sin (x + 30^0) = \sin (90^0 – 40^0)$

$\sin (x + 30^0) = \sin 50^0$

$x + 30^0 = 50^0$

$x = 50^0 – 30^0 = 20^0$

Method 2

Expressing $\sin (x + 30^0)$ *as a cos*

$\sin (x + 30^0) = \cos 40^0$

$\cos [90^0 – (x + 30^0)] = \cos 40^0$

$\cos (90^0 – x – 30^0) = \cos 40^0$

$\cos (60 – x)^0 = \cos 40^0$

$60^0 – x = 40^0$

$x = 60^0 – 40^0$

$x = 20^0$

2. Given that $\cos (2x – 23)^0 = \sin 47^0$. Find the value of x where $0^0 < x < 90^0$

Solution

$\cos (2x – 23)^0 = \sin 47^0$

$\cos (2x – 23)^0 = \cos (90 – 47^0)$

$\Rightarrow 2x – 23^0 = 43^0$

$2x = 23^0 + 43^0$

$2x = 66^0$

$x = 33^0$

3. Given that $\sin (5x – 28)^0 = \cos (3x – 50)^0$, find the value of x where $0^0 < x < 90^0$

Solution

$\sin (5x – 28)^0 = \cos (3x – 50)^0$

But $\sin (5x – 28)^0 = \cos 90^0 – (5x – 28)^0$

$= \cos 90 – 5x + 28^0$

$= \cos (90^0 + 28^o – 5x)$

$= \cos (118 – 5x)$

$\Rightarrow \cos (118 – 5x) = \cos (3x – 50)^0$

$118^0 – 5x = 3x – 50^0$

$118^0 + 50 = 3x + 5x$

$168^0 = 8x$

$x = 21^0$

3. Without, using tables or calculators, find the value of $\frac{2 \sin 40^0}{\cos 50^0} - \frac{\cos 22^0}{\sin 68^0}$.

Solution

$\frac{2 \sin 40^0}{\cos 50^0} - \frac{\cos 22^0}{\sin 68^0}$

But $\cos 50^0 = \sin (90^0 – 50^0) = \sin 40^0$

$\cos 22^0 = \sin (90^0 – 22^0) = \sin 68^0$

$\Rightarrow \frac{2 \sin 40^0}{\sin 40^0} - \frac{\sin 68^0}{\sin 68^0} = 2 – 1 = 1$

Exercises 24.6

1. If $\sin 5x = \cos 20^0$, find the value of x.

2. If sin 25^0 = cos$(y + 50)^0$, find the value of y

3. If cos $3y$ = sin $2y$ and $0^0 \leq y \leq 90^0$, find the value of y.

4. If sin$(x - 10)^0$ = cos$(x + 10)^0$ and $0^0 \leq x \leq 90^0$, calculate the value of x.

Expressing a Given Ratio as a Sum or Difference of Two Trigonometric Ratios

A given angle θ, can be expressed as a sum of two special angles, α and β. $\Rightarrow \theta = (\alpha + \beta)$.
Likewise, θ, can be expressed as a difference of two special angles, α and β. $\Rightarrow \theta = (\alpha - \beta)$. Hence, note the following identities:

1. $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
2. $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
3. $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
4. $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
5. $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$
6. $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

Worked Examples

1. Express 135^0 as a sum of two trigonometric ratios and find its value if sin $(a + b)$ = sin a cos b + cos a sin b.

Solution

$\sin 135^0 = \sin(90 + 45)^0$
But $\sin(a + b) = \sin a\cos b + \cos a\sin b$
$\Rightarrow \sin(90 + 45)^0$
$= \sin 90^0\cos 45^0 + \cos 90^0\sin 45^0$
$= (1)\left(\frac{\sqrt{2}}{2}\right) + (0)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$

2. Express cos 150^0 as a sum of two trigonometric ratios and hence find the value of cos 150^0, if cos $(x + y)^0$ = cos$x$cos$y$ – sin x sin y.

Solution

$\cos 150^0 = \cos(90 + 60)^0$
But $\cos(x + y)^0 = \cos x\cos y - \sin x\sin y$
$\Rightarrow \cos(90 + 60)^0$
$= \cos 90^0\cos 60^0 - \sin 90^0\sin 60^0$
$= (0)\left(\frac{1}{2}\right) - (1)\left(\frac{\sqrt{3}}{2}\right) = 0 - \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$

3. Express cos 15^0as adifference of two trigonometric ratios and hence find the value of cos 15^0, if cos (α – β) = cosα cosβ + sin α sin β.

Solution

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
$\cos 15^0 = \cos(45 - 30)^0$
$\cos(45 - 30)^0 = \cos 45^0\cos 30^0 + \sin 45^0\sin 30^0$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$
$$= \frac{\sqrt{2} + \sqrt{2}}{4}$$

Exercises 24.7

Express the following as a sum or difference of two trigonometric ratios and simplify leaving the answer in surd form:

1. cos 75^0 2. sin 165^0 3. sin 30^0
4. cos 165^0 5. cos 15^0 6. sin 105^0

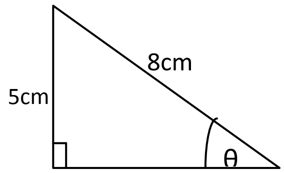
Applications to Right – angled Triangles

Finding the Value of the Interior Angles of a Right Triangle from Two Given Sides

I. Name an angle as θ, if not given in the diagram
II. Apply the appropriate trigonometric ratio that links the two given sides.
III. Find the inverse of the ratio applied to determine the value of the angle named θ.
IV. Make use of the fact that the sum of interior angles of a triangle is 180^0.

Worked Examples

1. Find sin θ and the value of θ in the right – angled triangle below;



Solution

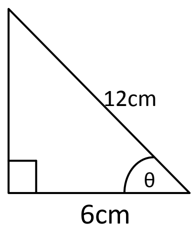
From the diagram,

O = 5cm and H = 8cm

sin θ = $\frac{O}{H} = \frac{5}{8}$

θ = $\sin^{-1}\left(\frac{5}{8}\right) = 39^0$ (to the nearest degree)

2. Find cos θ and the value of θ in the figure below;



Solution

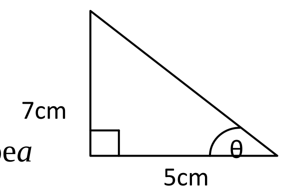
From the diagram,

A = 6cm and H = 12cm

cos θ = $\frac{A}{H} = \frac{6}{12}$

θ = $\cos^{-1}\left(\frac{6}{12}\right) = 60^0$

3. In the diagram below, find sin θ and the value of θ



Solution

Let the unknown side be a

$a^2 = 7^2 + 5^2$

$a^2 = 49 + 25$

$a^2 = 74$

$a = \sqrt{74} = 8.60$cm (2 d.p)

⇒O = 7cm and H = 8.60cm

sin θ = $\frac{O}{H} = \frac{7}{8.60}$

θ = $\sin^{-1}\left(\frac{7}{8.60}\right) = 54^0$(to the nearest degree)

Word Problems

1. A ladder 10 m long rests against a vertical wall so that the distance between the foot of the ladder and the wall is 4.5m.

i. Find the angle the ladder makes with the wall.

ii. Find the height above the ground at which the upper end of the ladder touches the wall.

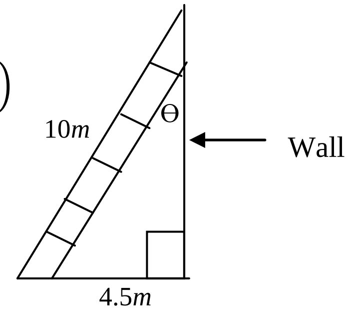
Solution

i. Let the angle the ladder makes with the wall be θ

sin θ = $\frac{O}{H} = \frac{4.5}{10}$

θ = $sin^{-1}\left(\frac{4.5}{10}\right)$

θ = 27^0



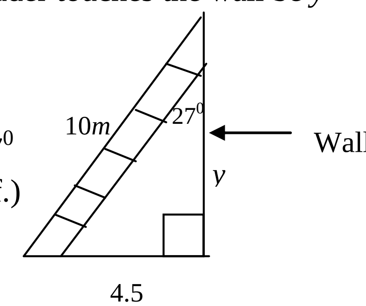
ii.The height above the ground at which the upper end of the ladder touches the wall be y

$\cos 27^0 = \frac{A}{H}$

$\cos 27^0 = \frac{y}{10}$

$y = 10 \cos 27^0$

$y = 8.9$ (2 s. f.)



2. A ladder is leaning against a vertical wall. The ladder is 6m long and the foot of the ladder is 2m from the base of the wall. Find the angle the ladder makes with the wall.

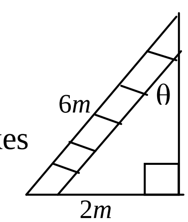
Solution

Let the angle the ladder makes with the wallbe θ

sin $\theta = \frac{2}{6}$

θ = $\sin^{-1}\left(\frac{2}{6}\right)$

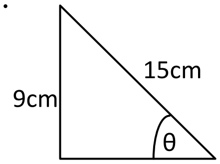
θ = 19^0



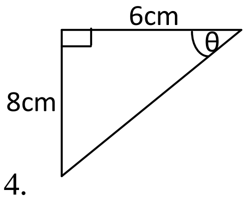
Exercises 24.8

In the diagrams below, find the ratio for angle θ and the value of angle θ

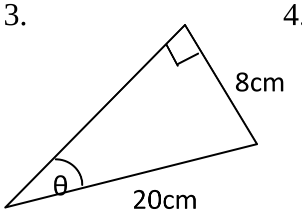
1.



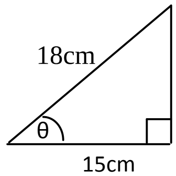
2.



3.



4.



B. 1. A ladder which is 8m long is placed against a vertical wall. If the ladder reaches 6.3m up the wall, calculate the angle the ladder makes with the wall.

2. A ladder is12m tall. If the ladder is placed against a vertical wall 7.5m high, calculate the angle the ladder makes with the wall.

3. A ladder is placed against a vertical wall of height 13m. Calculate the angle the ladder makes with the horizontal, if the ladder is 25m high.

4. Linda is flying a kite. She pins the end of the kite to the ground so that the string is pulled straight as the wind pushes on the kite. If the length of the kite string is 7.8m and the kite has a vertical height of 5.9 above the ground, what angle does the string of the kite makes with the horizontal ground?

The Length of the Sides of a Right – angled Triangle Given the Value one Acute Angle and a side

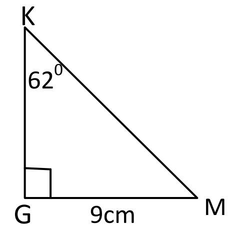
I. Identify the given side, the unknown side and the given acute angle.

II. Find the ratio of the angle that relates the other two sides (unknown side included).

III. Make the unknown side the subject of the equation and workout for its value.

Worked Examples

1. In the diagram below, KGM is a right triangleand angle GKM = 62^0. Find the length of KM and KG.



Solution

$\sin 62^0 = \frac{9}{|KM|}$

$|KM| = \frac{9}{\sin 62^0}$ = 10.2cm (3 s. f)

By Pythagoras theorem,

$|KM|^2 = |KG|^2 + 9^2$

But $|KM| = 10.2$

$10.2^2 = |KG|^2 + 9^2$

$10.2^2 - 9^2 = |KG|^2$

$|KG|^2 = 23.04$

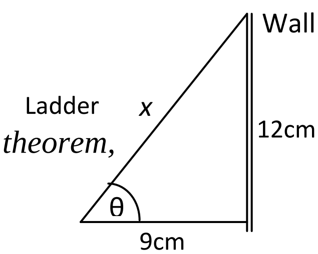
$|KG| = \sqrt{23.04}$ = 4.8cm

(sin, tan and cos can be used to find the side KG)

2. A ladder leans against a wall. The end of the ladder touches the wall 12m from the ground. The foot of the ladder is 9m away from the foot of the wall.

i. What is the length of the ladder?

ii. Calculate the angle the ladder makes with the ground.



Solution

i. *By Pythagoras theorem,*

$x^2 = 9^2 + 12^2$

$x = \sqrt{81 + 144}$

$x = \sqrt{225}$ = 15m

ii. Let θ be the angle the ladder makes with the ground

O = 12cm and A = 9cm, *TOA is applicable*

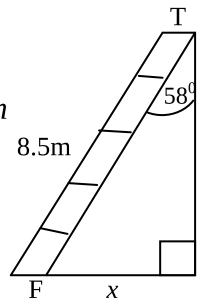
tan θ = $\frac{12}{9}$

θ = $\tan^{-1}\left(\frac{12}{9}\right)$ = 53^0 (Nearest degree)

3. A ladder 8.5meters long leans against a vertical wall. The top (T) of the ladder makes an angle of 58^0 with the wall. How far, correct to one decimal place, is the foot (F) of the ladder from the wall?

Solution

Let the distance between the ladder and the foot of the wall be *x;*



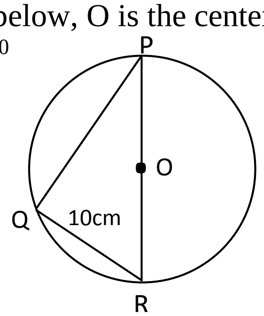
From the diagram

sin 58^0 = $\frac{x}{8.5}$

x = 8.5 sin 58^0

x = 7.2m

4. In the diagram below, O is the center of the circle, <QPR = 50^0 and /QR/ = 10cm Calculate to one decimal place, the radius of the circle.



Solution

<PQR = 90^0 (angle in a semi – circle)

Let PR = x

sin 50^0 = $\frac{10}{x}$

x = $\frac{10}{\sin 50^0}$ = 13.05cm

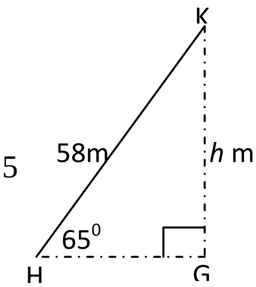
But the radius, $r = \frac{1}{2}x$

= $\frac{1}{2}$ × 13.05 = 6.5cm

5. A kite in the air has its string tied to the ground. If the length of the string is 58m, find the height of the kite above the ground when the string is taut and its inclination to the horizontal is 65^0.

Solution

Let k represent the kite,KG represents the height, *h*, of the kite above the ground and HG be the ground level;



sin 65^0 = $\frac{O}{H}$

sin 65^0 = $\frac{h}{58}$

h = 58 sin 65^0 = 52.5

Hence the kite is 53m high (2 s. f)

6. In the figure below ABCD is a rectangle in which AD = 43.7cm DE = 100cm, and E$\hat{D}$C = 27^0. Calculate:

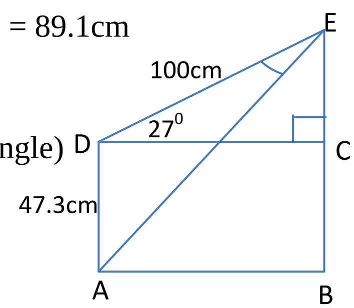
i. AB ii. EC iii. E$\hat{A}$B

Solution

i. From ΔCDE;

cos 27^0 = $\frac{/DC/}{100}$

DC = 100 cos 27^0 = 89.1cm



But DC = AB

(length of a rectangle)

AB = 89.1cm

ii. From Δ*CDE*;

sin 27^0= $\frac{/EC/}{100}$

/EC/ = 100 sin 27^0 = 45.4cm

iii. From ΔABE,

Let < EAB = θ

tan θ = $\frac{BE}{AB}$

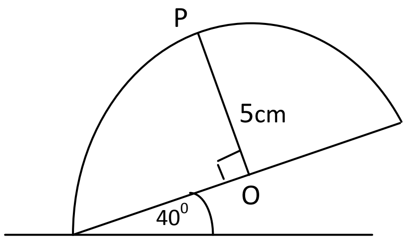
But BE = EC + CB = 45.4 + 47.3 = 92.7cm

AB = 89.1cm

$\tan\theta = \frac{92.7}{89.1}$

$\theta = \tan^{-1}\frac{92.7}{89.1} = 46^0$

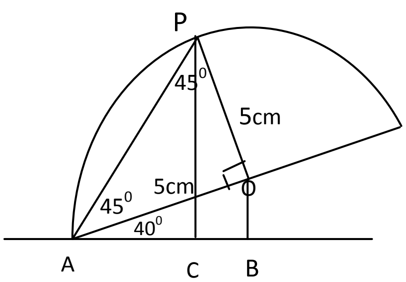
6. A protractor , in the shape of a semi circle, center O and radius 5cm, is held in vertical plane with one corner on a table as shown below;



The straight edge makes an angle of 40^0 with the horizontal. Calculate:
i. the height of O above the table,
ii. the height of P above the table.

Solution

i. Rename the figure as shown below;



/*OP*/ = /*OA*/ = 5cm

Δ*OAP* is isosceles, ⇒ base angles are equal

<*A*= 45^0 and <*P* = 45^0

Let the height of O above the table be /OB/

From ΔOAB;

$\sin 40^0 = \frac{OB}{5}$

/OB/ = 5 sin 45^0 = 3.5cm

ii. From ΔOAB,

$/AP/^2 = 5^2 + 5^2$

$/AP/^2 = 25 + 25$

$AP = \sqrt{50} = 7.1$cm

Let the height of P above the table be CP

From ΔOAB

$\sin 85^0 = \frac{CP}{AP}$

But AP = 7.1 cm

$\sin 85^0 = \frac{CP}{7.1}$

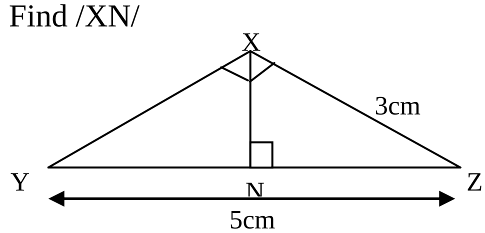
/CP/ = 7.1 sin 85^0 =7.1cm

Note:

Always watch the degree of accuracy permitted in answers to problems involving measured quantities. In the above example,
(no 5), the given measures 58 and 65 have only two significant digits. This as a matter of fact limits our answer to two significant digits

Some Solved Past Questions

1. In the diagram below, XYZ is a triangle, /XZ/ = 3cm, /YZ/ = 5 cm and <YXZ = < XNZ = 90^0. Find /XN/



Solution

Applying Pythagoras theorem on Δ XYZ,

$|YZ|^2 = |XY|^2 + |XZ|^2$

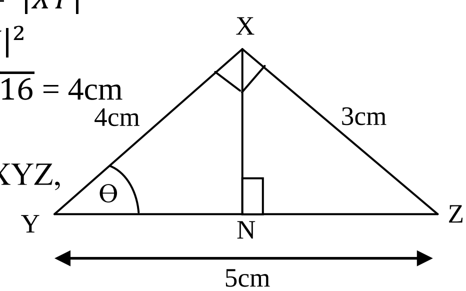
But $|YZ|$ = 5cm, $|XZ|$ = 3cm,

$5^2 = |XY|^2 + 3^2$

$5^2 - 3^2 = |XY|^2$

$16 = |XY|^2$

$|XY| = \sqrt{16} = 4$cm



From Δ XYZ,

$\sin\theta = \frac{3}{5}$

$\theta = \sin^{-1}\left(\frac{3}{5}\right)$

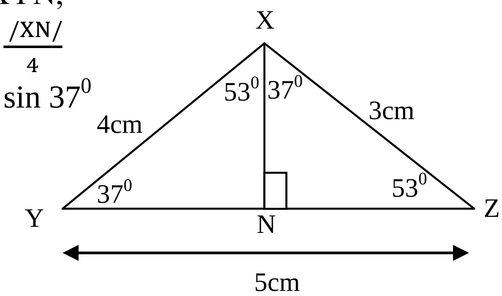
$\theta = 37^0$ (2. s. f)

From Δ XYN,

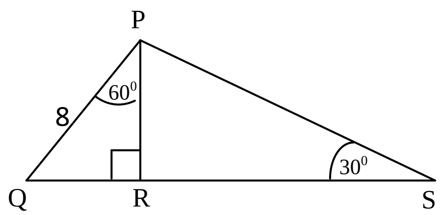
$\sin 37^0 = \frac{/XN/}{4}$

$/XN/ = 4 \sin 37^0$

$= 2.4$ cm



2. In the following diagram, PR is perpendicular to QS, angle QPR = 60^0, angle PSR = 30^0 and / PQ / = 8cm, find the length of QS



Solution

Let /QR/ = *x*and /RS/ = *y*

$\Rightarrow /QS/ = x + y$

$\sin 60^0 = \frac{O}{H}$

$\sin 60^0 = \frac{x}{8}$

$x = 8 \sin 60^0 \;= 7$ cm

$/PR/^2 + /RQ/^2 = /QP/^{\,2}$

$/PR/^2 = /QP/^{\,2} - /RQ/^{\,2}$

$/PR/^2 = 8^2 - 7^2$

$/PR/^2 = 15$

$/PR/ = 4$

$\tan 30^0 \;= \frac{4}{y}$

$y = \frac{4}{tan\ 30^0} \;= 7$ cm

But /QS/ = $x + y$ = 7 cm + 7cm = 14 cm

Exercises 24.9

A. 1. In a right – angled triangle *DEF*, the hypotenuse /*DE*/ = 35cm long, and angle D is 40^0. Find /*EF*/ and /*DF*/.

2. A ladder, 8m long leans against a wall and makes an angle of 60^0 with the ground;

a. how high up the wall does the ladder reach?

b. how far from the wall is the foot of the ladder?

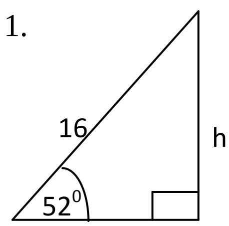
3. A ladder 12m long, leans against a wall and makes an angle of 53^0 with the ground;

i. how high up the wall does the ladder reach?

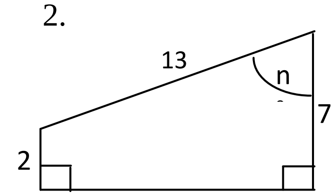
ii. how far from the wall is the foot of the ladder?

B. Find the angles and sides indicated by letters in the following:

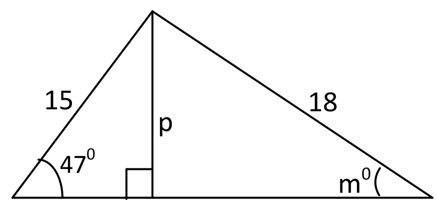
1.



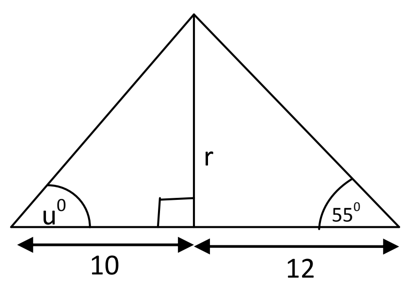
2.



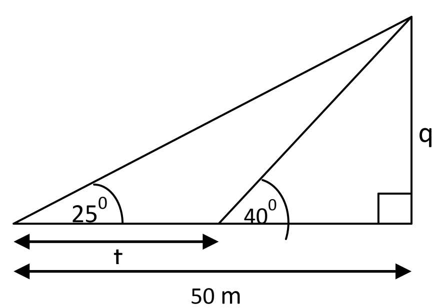
3.



4.

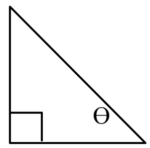


5.



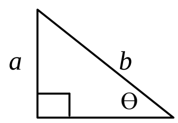
Determing Other Ratios from a Given One

Draw a right – angled triangle and name one acute angle as θ.

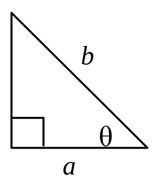


Then label the dimensions of the triangle by observing the following;

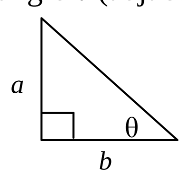
I. Given **sin θ=** $\boldsymbol{\frac{a}{b}}$,***a*** is the side that faces θ (*opposite*) and ***b*** is the side that faces the right angle (*hypotenuse*)



II. Given **cosθ =** $\boldsymbol{\frac{a}{b}}$, ***a*** is the side containing the right angle and angle θ(adjacent) and ***b*** is the side that faces the right angle (*hypotenuse*)



IV. Given **tan** θ = $\boldsymbol{\frac{a}{b}}$, ***a*** is the side that faces θ (opposite) and ***b*** is the side containing the right angle and angle θ (adjacent).



V. Apply Pythagoras theorem to get the length of the third side.

VI. Find the required trigonometric ratio.

Worked Examples

1. In a right – angled triangle, sin θ = $\frac{3}{5}$. Find:

i. cos θ and tan θ

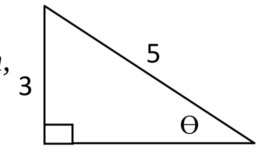
ii. the perimeter of the triangle

iii. the interior angles of the triangle

Solution

i. sin θ = $\frac{3}{5} = \frac{Opposite(O)}{Hypotenus(H)}$.

O = 3 and H = 5, Adjacent (A) = ?



By Pythagoras theorem,

$O^2 + A^2 = H^2$

$3^2 + A^2 = 5^2$

$A^2 = 5^2 - 3^2 = 16$

A = $\sqrt{16}$ = 4cm

Now *O* = 3cm and *H* = 5cm, *A* = 4cm

cos θ = $\frac{Adjacent\ (A)}{Hypotenus\ (H)} = \frac{4}{5}$

tan θ = $\frac{Opposite\ (O)}{Adjacent\ (A)} = \frac{3}{4}$

ii. Perimeter of the triangle;

P = O + A + H = (3 + 4 + 5)*cm* = 12*cm*

iii. The interior angles of the triangle are the

a. right – angle = 90^0

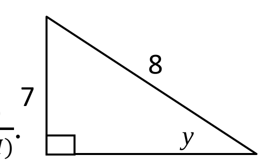
b. sin θ = $\frac{3}{5}$

$\sin^{-1}\left(\frac{3}{5}\right) = 37^0$ (to the nearest degree)

c. $180^0 - (90^0 + 37^0) = 53^0$

∴ The interior angles are 90^0, 37^0 and 53^0

2. Given that sin *y* = $\frac{7}{8}$, where $0^0 \leq y \leq 90^0$, find tan *y*. Hence, find the truth set of cos *x* = 1 – tan *y*, giving your answer correct to the nearestdegree.



Solution

sin *y* = $\frac{7}{8} = \frac{Opposite(O)}{Hypotenus(H)}$.

O = 7 and H = 8, A = ?

By Pythagoras theorem,

$O^2 + A^2 = H^2$

$7^2 + A^2 = 8^2$

$A^2 = 8^2 - 7^2 = 15$

A = $\sqrt{15}$ =4cm (to the nearest cm)

Now O = 7cm and H = 8cm, A = 4cm

tan y = $\frac{Opposite\ (O)}{Adjacent\ (A)} = \frac{7}{4}$

cos$x$ = 1 – tan y,

cos$x$ = 1 – $\frac{7}{4}$

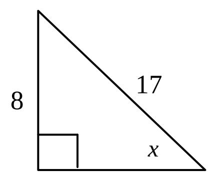
cos$x$ = − 0.75

x = - $cos^{-1}(0.75)$ = − 139^0 (Nearest degree)

3. If sin x = $\frac{8}{17}$, find the value of $\frac{tan\ x}{1 + 2\ tan\ x}$

Solution

Method 1



sin$x$ = $\frac{8}{17}$

Let the third side of thetriangle be a

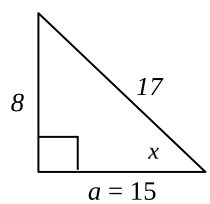
By Pythagoras theorem,

$17^2 = a^2 + 8^2$

$a^2 = 17^2 - 8^2$

$a^2 = 225$

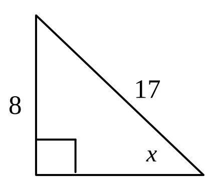
$a = \sqrt{225} = 15$



tan$x$ = $\frac{8}{15}$ and 2tan x = 2 × $\left(\frac{8}{15}\right) = \frac{16}{15}$

Substitute the values of tan x and 2 tan x in $\frac{tan\ x}{1+2\ tan\ x} = \frac{8/15}{1+16/8} = \frac{8/15}{3} = \frac{8}{15\times3} = \frac{8}{45}$

Method 2



sin$x$ = $\frac{8}{17}$

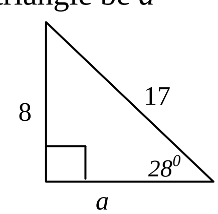
$x = \sin^{-1}\left(\frac{8}{17}\right) = 28^0$

Let the third side of the triangle be a

tan 28^0 = $\frac{8}{a}$

$a = \frac{8}{\tan 28^0} = 15$

⇒tan 28^0= $\frac{8}{15}$



$\frac{tan\ x}{1+2\ tan\ x} = \frac{\tan 28^0}{1+\ 2\tan 28^0} = \frac{8/15}{1+\ 2\,(8/15)}$

$= \frac{8/15}{1+16/8} = \frac{8/15}{3} = \frac{8}{15\times3} = \frac{8}{45}$

4. If 4 tan x = 3,where $0^0 \leq x \leq 90^0$,

i. find the value of x

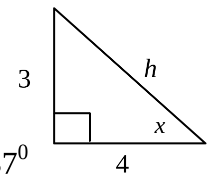
ii. evaluate $\frac{1 + \cos\ x}{2 - \cos\ x}$

Solution

4 tan x = 3

tan$x$ = $\frac{3}{4}$

$x = \tan^{-1}\left(\frac{3}{4}\right) = 37^0$

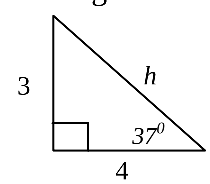


ii. ***Method 1***

Let the hypotenuse of the triangle be$h$ and $x = 37^0$

cos 37^0 = $\frac{4}{h}$

$h = \frac{4}{\cos 37^0}$ = 5cm



cos37^0 = $\frac{4}{5}$, but $x = 37^0$

⇒cos$x$ = $\frac{4}{5}$, substitute in $\frac{1 + \cos x}{2 - \cos x}$

$\frac{1 + \cos x}{2 - \cos x} = \frac{1+4/5}{2-4/5} = \frac{9/5}{6/5} = \frac{9}{6} = \frac{3}{2}$

ii.***Method 2***

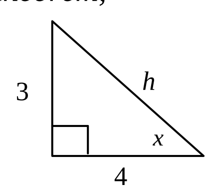
Let the hypotenuse of the triangle be h

By Pythagoras theorem,

$h^2 = 4^2 + 3^2$

$h^2 = 25$

$h = \sqrt{25}$ = 5cm



cos x = $\frac{4}{5}$ substitute in $\frac{1 + \cos x}{2 - \cos x}$

$\frac{1 + \cos x}{2 - \cos x} = \frac{1+4/5}{2-4/5} = \frac{9/5}{6/5} = \frac{9}{6} = \frac{3}{2}$

6. Given that $\sin x = \frac{5}{8}$, $0^0 < x < 90^0$ find correct to 3 decimal places $\frac{\cos x - \sin x}{2 \tan x}$

Solution

$\sin x = \frac{5}{8}$, but $\sin x = \frac{O}{H}$

$\Rightarrow$O = 5 and H = 8

By Pythagoras theorem,

$O^2 + A^2 = H^2$

$5^2 + a^2 = 8^2$

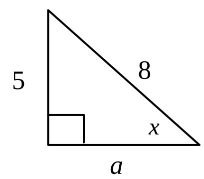
$a^2 = 8^2 - 5^2$

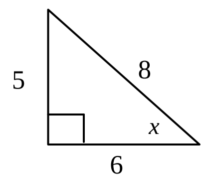
$a^2 = 64 - 25$

$a^2 = 39$

$a = \sqrt{39}$

$a = 6$





$\cos x = \frac{A}{H} \Rightarrow \cos x = \frac{6}{8}$

$\tan x = \frac{O}{A} \Rightarrow \tan x = \frac{5}{6}$

$\sin x = \frac{O}{H} \Rightarrow \sin x = \frac{5}{8}$

$\frac{\cos x - \sin x}{2 \tan x} = \frac{6/8 - 5/8}{2\ (5/6)} = \frac{1/8}{10/6} = \frac{1}{8} \times \frac{6}{10} = \frac{3}{40} = 0.075$

Exercises 24.10

A.1. Find without using tables, the values of $\cos \theta$ and $\tan \theta$, if $\sin \theta = \frac{-24}{25}$

2. Find without using tables, the values of $\cos \theta$ and $\tan \theta$, if $\sin \theta = \frac{5}{13}$

3. In a right – angled triangle, $\tan \theta = \frac{13}{8}$,

i. Find $\sin \theta$ and $\cos \theta$

ii. What is the perimeter of the triangle?

iii. Find the interior angles of the triangle

B.1.If $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$, find the value of $\sin A \cos B - \cos A \sin B$,

2. If $\cos p = \frac{3}{5}$, where $0^0 < p < 90^0$, find the value of $\frac{\tan p}{\cos p}$

3. Given that $\tan x = \frac{12}{5}$, where $0^0 < x < 90^0$, what is the value of $\sin x + \cos x$.

4. $\sin\theta = \frac{8}{17}$ and θ is obtuse, find the value of $\frac{1}{\cos\theta + 1}$.

5. If $\sin x = \frac{12}{13}$, where $0^0 \leq x \leq 90^0$, find the value of :

i. $1 - \cos^2 x$, ii. $1 + \cos^2 x$,

iii. $(1 + \cos x)^2$, iv. $(\tan x + 1)^2$

6. If $\sin \theta = 0.65$, find the following for $0^0 \leq x \leq 90^0$;

i. θ ii. $\tan \theta$ iii. $\frac{\cos \theta}{\tan \theta}$ iv. $\left(\frac{\cos \theta \theta}{\tan \theta \theta}\right)^2$

7. If $\cos x = 0.8$, find the value of the following for $0^0 \leq x \leq 90^0$:

i. $\cos^2 x - \sin^2 x$ ii. $\frac{2 \tan x}{\cos x - \sin x}$

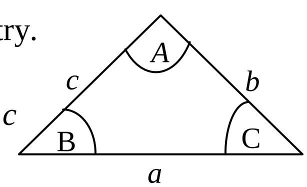
Application to Other Triangles

Notation

The diagram below shows the usual notation for a triangle in trigonometry.

Angles: *A*, *B*, *C*

Length of sides: *a*, *b*, *c*

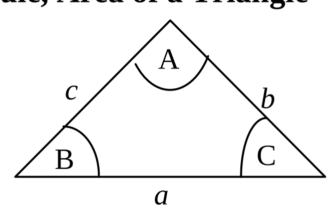


The length of the sides are donated by a lower case letters, and named after the angle they are opposite. i. e. *a* is opposite to angle *A*, *b* is opposite to angle *B* and *c* is opposite to angle *C*

Sine and Cosine Rule, Area of a Triangle

Sine Rule:

$\frac{a}{Sin\ A} = \frac{b}{\sin\ B} = \frac{c}{\sin\ C}$



Alternatively,

$\frac{Sin\ A}{a} = \frac{Sin\ B}{b} = \frac{Sin\ C}{c}$

Cosine Rule (for length of sides):

$a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = a^2 + c^2 - 2ac\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$

Alternatively (for size of angles);

$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

Area of △ abc

$A = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A$

When to Use the Sine Rule

Use the sine rule if you know:

1. Two angles and one side.
2. Two sides and an angle opposite one of these sides.

When to Use the Cosine Rule

Use the cosine rule if you know:

1. Two sides and an included angle.
2. The lengths of the three sides.

***Note*:**

1.As a general rule, if you cannot use the sine rule then use the cosine rule.

2. If two angles are given, workout the third angle straight away, as the three angles in a triangle add up to $180^{0.}$

3. The sine and cosine rules and the area of the triangle formulas also apply to a right – angled triangle, but with right – angled triangles the basic trigonometric ratios are used.

4. The largest angle of a triangle is opposite the largest side and the smallest angle is opposite the shortest side. There can be only one obtuse angle in a triangle.

Tackling Problems in Trigonometry

I. If not given, draw a diagram, and put in as much information as possible.

II. If two, or more, triangles are linked redraw the triangles separately.

III. Watch for common sides which link the triangles (i.e. carry common values from one triangle to another triangle).

IV. Use the sine or cosine rule as needed.

Worked Examples

1. In △ *abc*, /*ab*/ = 3, /*ac*/ = 5 and /*bc*/ = 7. Calculate:

i. the measure of the greatest angle of the triangle.

ii. the area of △ *abc*, leaving your answer in the form $\frac{a\sqrt{b}}{c}$, where *b* is a prime.

Solution

i. The largest angle is opposite the largest side

Using the cosine rule,

$a^2 = b^2 + c^2 - 2bc\cos A$

$7^2 = 5^2 + 3^2 - 2(5)(3)\cos A$

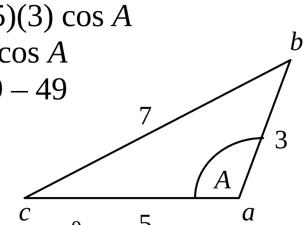
$49 = 25 + 9 - 30\cos A$

$30\cos A = 25 + 9 - 49$

$30\cos A = -15$

$\cos A = \frac{-15}{30} = -\frac{1}{2}$

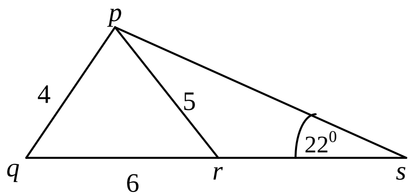
$A = cos^{-1}\left(-\frac{1}{2}\right) = 120^0$



ii. Area of △ *abc* $= \frac{1}{2}bc\sin A$

$= \frac{1}{2}(5)(3)\sin 120^0 = \frac{1}{2}(5)(3)\left(\frac{\sqrt{3}}{2}\right) = \frac{15\sqrt{3}}{4}$

2. In the diagram below, /*pq*/ = 4cm, /*pr*/ = 5cm, /*qr*/ = 6cm and / <*psr* / = 22^0



Find /*ps*/, correct to two places of decimals

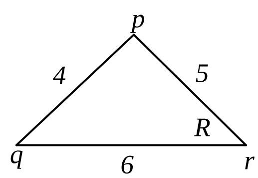
Solution

Work on the two angles triangles separately

I. Consider Δ *pqr,*

Given three sides,

use the cosine rule;



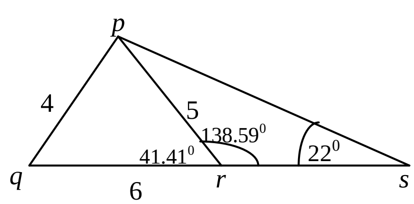
$\cos R = \frac{p^2 + q^2 - r^2}{2pq}$

$\cos R = \frac{6^2 + 5^2 - 4^2}{2(6)(5)} = \frac{45}{60} = \frac{3}{4}$

$R = cos^{-1}\left(\frac{3}{4}\right) = 41.41^0$ (2 d. p)

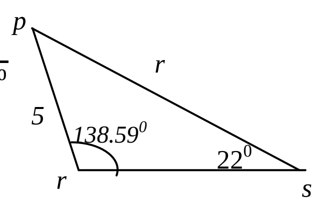
$\therefore$ / <*prs* / = $180^0 - 41.41^0$

= 138.59^0



II. Consider Δ *prs,*

Having two angles and one side, use the sine rule to find /*ps*/,



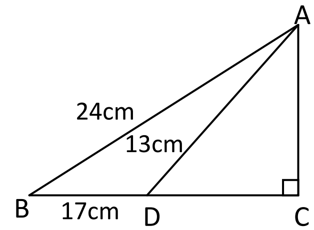
$\frac{r}{Sin\ 138.59^0} = \frac{5}{\sin 22^0}$

$r = \frac{5\ sin\ 138.59^0}{sin\ 22^0}$

r = 8.82849883

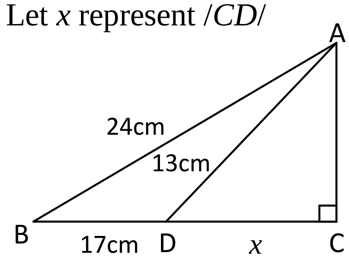
Thus /*ps*/ = 8.83cm (2 d. p)

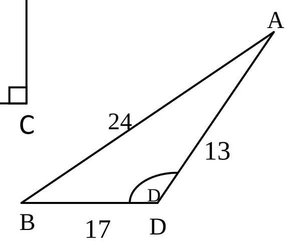
3. In the figure below, /*AB*/ = 24cm, /*BD*/ = 17cm, *AD* = 13cm and < *ACD* = 90^0. Find /*CD*/



Solution

Let x represent /*CD*/





Consider Δ *ABD,*

Using the cosine rule,

$24^2 = 17^2 + 13^2 - 2(17)(13) \cos D$

576 = 289 + 169 – 442 cos D

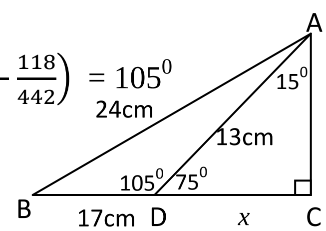
576 = 458 – 442 cos D

442 cos D = 458 – 576

442cosD = -118

$\cos D = \frac{-118}{442}$

$D = cos^{-1}\left(-\frac{118}{442}\right) = 105^0$



$\tan 15^0 = \frac{O}{H}$

$\tan 15^0 = \frac{x}{13}$

$x = 13 \tan 15^0 = 3.5$

Hence /*CD*/ = 3.5cm

Exercises 24.11

Unless otherwise stated, where necessary, give the lengths of sides and areas correct to 2 decimal places and give angles correct to one place of decimals

A. 1. In Δ *pqr*, p = 7 cm, $P = 30^0$, $Q = 84^0$. Find R, q and r

2. In Δ *abc*,b = 8 *cm*, c = 10*cm*, and $A = 60^0$. Find a

3. In an acute – angled triangle *ABC*, the lengths of *AB*, *AC* are 10.3 cm , 15.7 cm respectively, and the angle C is 40^0, calculate to the nearest tenth of a degree, the angle *A*

4. In ΔXYZ, $XY = 17.2$ cm, $YZ = 21.3$ cm, $ZX = 16.0$ cm. Find the three angles and the area of the triangle

5. In ΔABC, $a = 9.5$ cm, $b = 8.4$ cm, $C = 73^0$. Calculate;

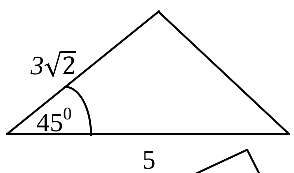
i. the area the triangle,

ii. the side *c*,

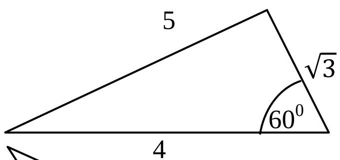
iii. angles *A* and *B*.

B. Without using calculator, find the area;

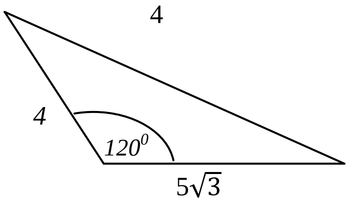
1.



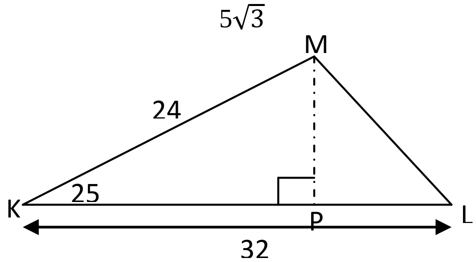
2.



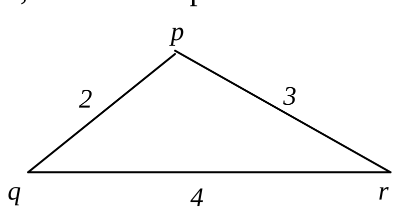
3.



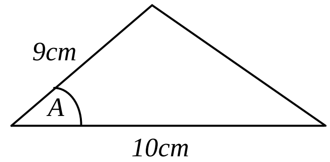
4.



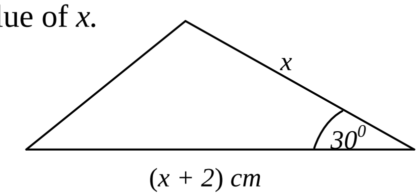
C. 1. Calculate the three angles in triangle *pqr* below, correct to 2 places of decimals



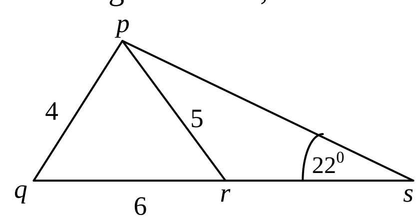
2. In the triangle below, $\cos A = \frac{4}{5}$. Without using tables or calculator, find the area of the triangle.



3. The area of the triangle below is 6cm^2. Find the value of *x*.



4. In the diagram below,

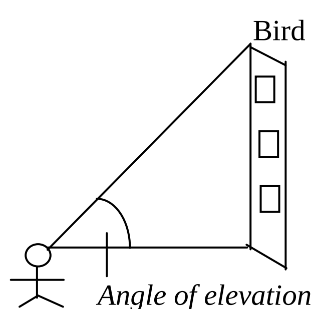


$/pq/ = 4$cm, $/pr/ = 5$cm, $/qr/ = 6$cm and $/ <psr / = 22^0$. Find $/ps/$, correct to two places of decimals

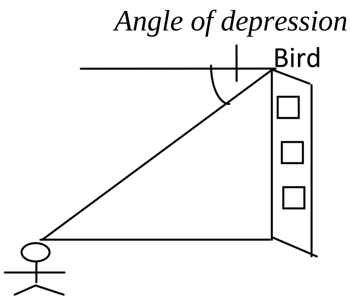
5. In Δ *ABC*, $A = 51$, $B = 68^0$, and $c = 4.5$m. Calculate; i. side *b*, ii. the area of ΔABC.

Angles of Elevation and Depression

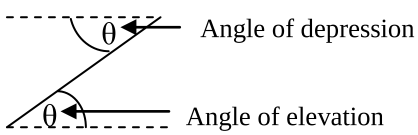
Suppose you are on the ground and you decide to see a bird on top of a tower. You have to turn your head through a certain angle before you see the bird. If you are looking in the horizontal direction before you turn your head up to see the bird, then the angle through which you turn your head is called the***angle of elevation***.



On the other hand, the bird will also have to turn its head downwards to see you. If it is also looking in the horizontal direction before it turns to see you from the top of the tower, then the angle through which the bird turn its head is called the ***angle of depression.***



In general, the angle of elevation is equal to the angle of depression.



To solve problems involving elevations and depressions;

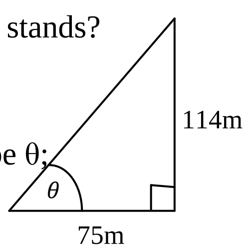
I. Make a sketch of the diagram to represent the problem.

II. Indicate the dimensions and angles if given

III. Represent the unknown sides by any preferred variable and an unknown angle by θ

Worked Examples

1. A surveyor checking his instruments is standing 75m away from the bottom of a tower block of flats which he knows is 114m high. What is the correct angle of elevation of the top of the block from where he stands?



Solution

Let the angle of elevation be θ;

From the diagram,

TOA is applicable

$\tan\theta = \frac{114}{75}$

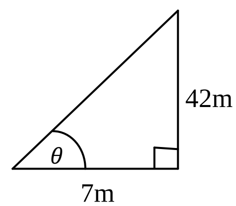
$\theta = \tan^{-1}\left(\frac{114}{75}\right) = 57^0$

2. A bird stands on top of a story building which is 42m tall. A girl standing 7m away from the story building observes the bird. Calculate the angle of elevation of the bird from where the girl stands

Solution

Let the angle of elevation be θ

From the diagram,



TOA is applicable

$\tan\theta = \frac{42}{7}$

$\theta = \tan^{-1}\left(\frac{42}{7}\right)$

$\theta = 81^0$

3. An airplane is flying at a height of 4 miles above the ground. The distance along the ground from the airplane to the airport is 9 miles. What is the angle of depression from the airplane to the airport?

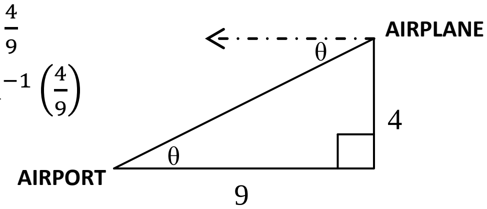
Solution

Let the angle of depression be θ

$\tan\theta = \frac{4}{9}$

$\theta = \tan^{-1}\left(\frac{4}{9}\right)$

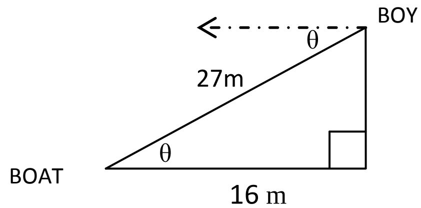
$\theta = 24^0$



4. A boy on top of a tree views a sea boat which is 27m away from where the boy is. If the ground distance between the boat and the tree is 16m, what is the angle of depression of the boy to the boat?

Solution

Let the angle of depression be θ



From the digram, CAH is applicable

$\cos\theta = \frac{16}{27}$

$\theta = \cos^{-1}\left(\frac{16}{27}\right) = 54^0$

Calculating the Height / Distance Given the Angle of Elevation or Angle of Depression

Given the angle of depression or the angle of elevation and the length of one side,

I. Represent the unknown side by any preferred variable

II. Find a trigonometric ratio (SOH, CAH, TOA) that links or relates the given angle, the known side and the unknown side

III. Make the unknown side the subject to obtain its value

<u>Type 1</u>

Here, elevation or depression is taken from the ground

Worked Examples

1. The top of a cliff 5 km away has an angle of elevation 14^0. Calculate the height of the cliff in meters.

Solution

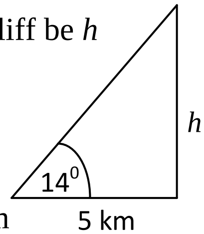
Let the height of the cliff be h

From the diagram,

TOA is applicable

$\tan 14^0 = \frac{h}{5}$

$\Rightarrow h = 5\tan 14^0 = 1.2\text{m}$



2. A bird sits on top of a lamppost. The angle of depression from the bird to the feet of an observer standing away from the lamppost is 34^0. The distance from the bird to the observer is 24m. How tall is the lamppost?

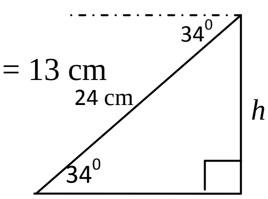
Solution

Let the height of the lamppost be h

From the diagram, SOH is applicable

$\sin 34^0 = \frac{h}{24}$

$h = 24\sin 34^0 = 13$ cm



3. From the top of a tower, the angle of depression from a point on the ground 10m away

from the base of the tower is 60^0.

i. How tall is the tower?

ii. What is the angle of elevation of a point M, half – way up the ladder?

Solution

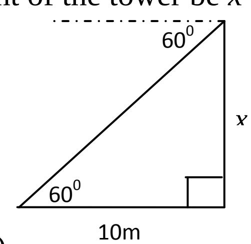
i. Let the height of the tower be x

$\tan 60^0 = \frac{x}{10}$

$x = 10\tan 60^0$

$x = 17.3$

$x = 17\text{m}$(2 s. f)



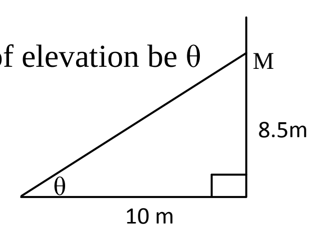
ii. Let the angle of elevation be θ

$\tan\theta = \frac{8.5}{10}$

$\theta = tan^{-1}\left(\frac{8.5}{10}\right)$

$\theta = 40.3^0$

Hence, the angle of elevation is about 40^0



4. A building is 45m high. At a distance away from the building, an observer notices that the angle of elevation to the top of the building is 51^0. How far is the observer from the base of the building?

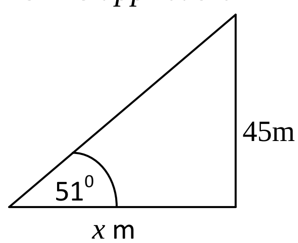
Solution

Let the distance between the observer and the base of the building be x

From the diagram, TOA is applicable

$\tan 51^0 = \frac{45}{x}$

$x = \frac{45}{\tan 51^0} = 36\text{m}$



Type 2

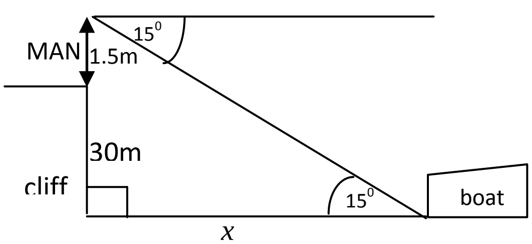
Here, elevation or depression is taken from an additional height (in the case of a person, the eyes)

Worked Examples

1. A man 1.5m tall stands on top of a vertical cliff which is 30m high and sees a fish boat on the sea at an angle of depression of 15^0. How far is the boat from the cliff?

Solution

Let the distance between the cliff and the boat be x



Opposite (O) = 31.5m, Adjacent (A) = x

From the diagram, TOA is applicable

$\tan 15^0 = \frac{31.5}{x}$

$x = \frac{31.5}{\tan 15^0} = 118\text{m}$

Therefore, the cliff is 118m high

3. A boy whose eye is 2m above the ground is standing at a distance of 24m from a tall building on a level ground. Find the height of the building from the ground, if the angle of elevation from the top of the building is 20^o

Solution

Letthe height of the building from the ground $= x + 2$

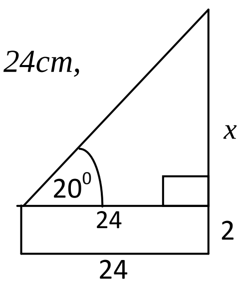
Given $O = x$ and $A = 24$cm,

TOA is applicable;

$\tan 20^0 = \frac{x}{24}$,

$x = 24 \tan 20^0$

$x = 8.7\text{m}$



But the height of the bulding;

$= x + 2$

$= 8.7m + 2\ m = 10.7m$

4. A pole 10m stands on the same horizontal level with a tree. From the top of the pole, the angle of elevation of a bird on top of the tree is 36^0. If the pole and the tree are 220m apart, calculate the height of the tree.

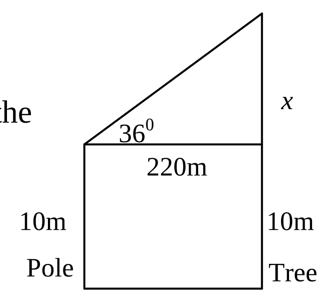
Solution

Let the height of the tree be $(x + 10)$m

$36^0 = \frac{x}{220}$

$x = 220 \tan 36^0$

$x = 160\text{m}$ (3 s.f)



But the height of the tree;

$= (x + 10)\text{m} = (160 + 10)\text{m} = 170\text{m}$

Exercises 24.12

1. The angle of elevation of the highest point of a tree viewed by a boy of height 1.5m standing

18m from the foot of the tree is 37^0. Find the height of the tree from the ground.

2. Find the angle of elevation of the sun when a vertical pole 2m high cast shadow 1.5m long on the horizontal ground.

3. The angle of elevation of the top of a church tower is viewed by a boy of height 1.65m standing 20m from the foot of the tower is 40^0. Find the height of the tower from the ground.

4. The angle of depression of a ship measured from the top of a light house 15m above the sea level is 12^0. Find the horizontal distance of the ship from the light house.

5. The angle of depression of a roundabout from the 8th floor of a block of a flat is 18^0. Each floor is 3m high and the observation is made by a girl of height 1.5m. Find the distance of the roundabout from the flats.

6. The angle of depression of a roundabout from the 10th floor of a block of flat is 15^0. Each floor is 350cm high and an observation is made by a man of height 125cm. Find the distance of the roundabout from the block of flats.

Double Elevations and Depressions

I. Make a sketch of the diagram showing the two angles of elevations or depression or both and the given dimensions.

II. Represent unknown sides by any preferred variable.

III. Form two trigonometric equations involving the angles of elevation or depression and the given dimensions.

IV. Solve the equations simultaneously

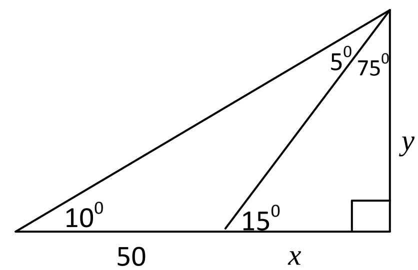
Worked Examples

Type 1

1. A boat is sailing directly towards a cliff. The angle of elevation of a point on the top of the cliff and straight ahead of the boat increases from 10^0 to 15^0 as the boat sails a distance of 50m. What is the height of the cliff?

Solution

Let y be the height of the cliff



$\tan 10^0 = \frac{y}{50+x}$……………….(1)

$\tan 15^0 = \frac{y}{x}$…………………..(2)

From eqn (1)

$y = (50 + x) \tan 10^0$…………(3)

From eqn (2)

$y = x \tan 15^0$ ……………….(4)

equating eqn(3) and eqn (4)

$(50 + x) \tan 100 = x \tan 150$

$50 \tan 100 + x \tan 100 = x \tan 150$

$50 \tan 10^0 = x \tan 15^0 – x \tan 10^0$

$50 \tan 10^0 = x (\tan 15^0 – \tan 10^0)$

$x = \frac{50^0 \tan 10^0}{\tan 15^0 - \tan 10^0} = 96m$

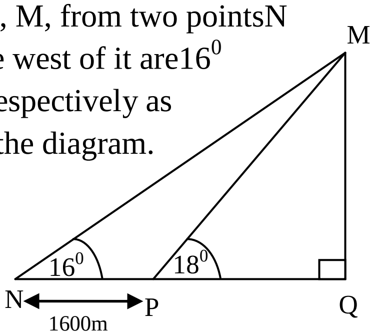
Put $x = 96$ in eqn (4)

$y = 96 \tan 15^0 = 26m$

Hence, the height of the cliff is 26m

2. A surveyor at sea level observed that the angle of elevation of the top of a

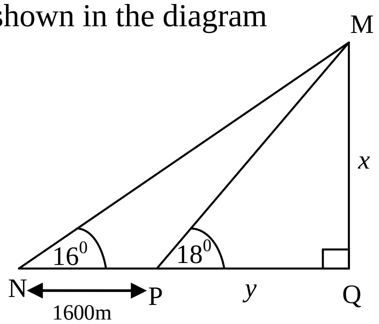
mountain, M, from two points N and P due west of it are 16^0 and 18^0 respectively as shown in the diagram.



If /NP/ = 1600m, and the base of the mountain, Q, is vertically below M, calculate the height of the mountain.

Solution

Let the height of the mountain be x and /PQ/ = y as shown in the diagram



$\tan 18^0 = \frac{x}{y}$……………………...(1)

$\tan 16^0 = \frac{x}{(1600 + y)}$ …………..…(2)

From eqn (1)
$y \tan 18^0 = x$…………………(3)

From eqn (2)
$(1600 + y) \tan 16^0 = x$………....(4)

eqn (3) = eqn (4)
$y \tan 18^0 = (1,600 + y) \tan 16^0$
$y \tan 18^0 = 1,600 \tan 16^0 + y \tan 16^0$
$y \tan 18^0 - y \tan 16^0 = 1,600 \tan 16^0$
$y(\tan 18^0 - \tan 16^0) = 1,600 \tan 16^0$
$y = \frac{1600 \tan 16^0}{\tan 18^0 - \tan 16^0} = 12,018m$

Put $y = 12,018m$ in eqn (3)
$12,018 \tan 18^0 = x$
$x = 3,905m$
$\therefore$ The height of the mountain is $3,905m$

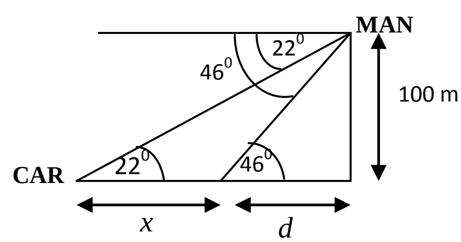
Method 2

The sine rule can also be used by working separately on each triangle

3. From the top of a 100m storey building. A man observes a car moving towards the building. If the angle of depression of the car changes from 22^0 to 46^0, during the period of observation, how far does the car travel?

Solution

Let the distance the car moved be $x + d$



$\tan 46^0 = \frac{100}{d}$ ……….(1)
$\Rightarrow d = \frac{100}{\tan 46^0} = 97\text{m}$

$\tan 22^0 = \frac{100}{x + d}$ ……….(2)

Put $d = 97$m in eqn (2)
$\tan 22^0 = \frac{100}{x + 97}$ ……….(2)
$\Rightarrow x + 97 = \frac{100}{\tan 22^0}$
$x = \left(\frac{100}{\tan 22^0}\right) - 97 = 151\text{m}$

$\Rightarrow x + d = 151m + 97\ m = 248\ m$

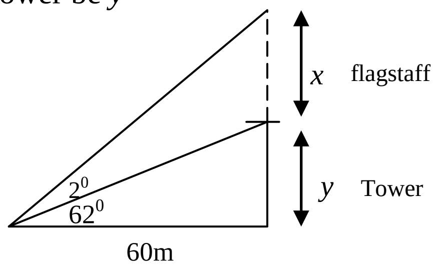
Type 2

1. A man standing 60m away from a tower notices that the angles of elevation of the top and

bottom of the flagstaff on top of the tower are 64^0 and 62^0 respectively. Find the height of the flagstaff, correct to one decimal place.

Solution

Let the height of the flagstaff be x and the height of the tower be y



$\tan 62^0 = \frac{y}{60}$ ………………..(1)

$y = 60 \tan 62^0$

$\tan 64^0 = \frac{x + y}{60}$………………(2)

$60 \tan 64^0 = x + y$

Put $y = 60 \tan 62^0$ in $60 \tan 64^0 = x + y$

$60 \tan 64^0 = x + 60 \tan 62^0$

$x = 60\tan 64^0 - 60 \tan 62^0 = 10.2\text{m}$ (1 d. p)

2. A tank 2m tall stands on top of a concrete pillar. From a point (P) on the same horizontal ground as the foot of the pillar, the angle of elevation of the top (T) and bottom (B) of the tank are 49^0 and 42^0 respectively.

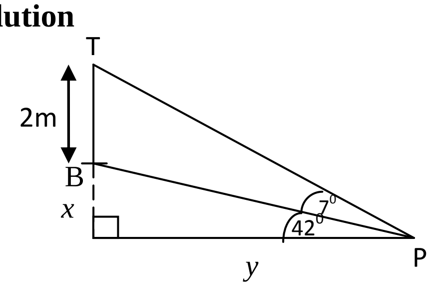
i. Draw a diagram to represent this information.

ii. Calculate, correct to one decimal place, the height of the pillar.

iii. Calculate, correct to one decimal place, /PB/

Solution

i



ii. Let the height of the pillar be x

$\tan 42^0 = \frac{x}{y}$………………..(1)

$\tan 49^0 = \frac{x + 2}{y}$ …………….(2)

From eqn (1)

$y \tan 42^0 = x$………………(3)

From eqn(2)

$y \tan 49^0 = x + 2$…………..(4)

Put $x = y \tan 42^0$ in eqn (4)

$y \tan 49^0 = y \tan 42 + 2$

$y \tan 49^0 - y \tan 42^0 = 2$

$y(\tan 49^0 - \tan 42^0) = 2$

$y = \frac{2}{\tan 49^0 - \tan 42^0} = 8m$

Put $y = 8$ in eqn (1)

$x = 8 \tan 42^0 = 7.20\text{m}$

Therefore, the height of the pillar is 7.2m

iii. *By Pythagoras theorem*

$|PB|^2 = x^2 + y^2$

But $x = 7.2$ and $y = 8$

$|PB|^2 = (7.2)^2 + (8)^2$

$|PB|^2 = 115.84$

$|PB| = \sqrt{115.84} = 10.7m$

<u>Type 3</u>

Worked Examples

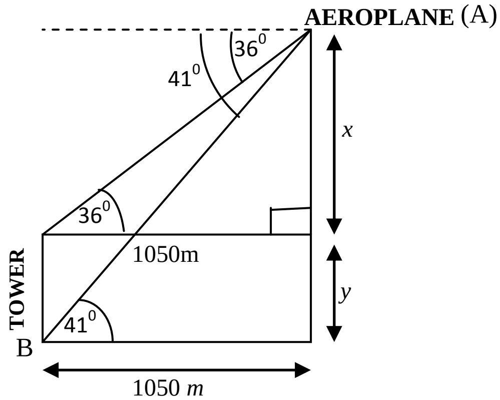
1. From an airplane in the air and at a horizontal distance of 1050m, the angles of depression of the top and base of a control tower at an instant are 36^0 and 41^0 respectively. Calculate correct to the nearest whole number, the;

i. height of the control tower,

ii. shortest distance between the aeroplane and the base of the control tower.

Solution

i. Let the height of the tower be y and the distance between the aeroplane and the base of the control tower be AB



$\tan 36^0 = \frac{x}{1050}$ ……….(1)

$\Rightarrow x = 1050 \tan 36^0 = 763\ m$

$\tan 41^0 = \frac{x + y}{1050}$ ……….(2)

$1050 \tan 41^0 = x + y$

But $x = 763$ m

$\Rightarrow 1050 \tan 41^0 = 763 + y$

$y = 1050 \tan 41^0 - 763 = 150m$

The height of the control tower is 150 m

ii. The shortest distance between the aeroplane and the base of the control tower = AB

$\cos 41^0 = \frac{1050}{/AB/}$

$/AB/ = \frac{1050}{\cos 41^0} = 1391$ m

The shortest distance between the aeroplane and the base of the control tower is 1391m

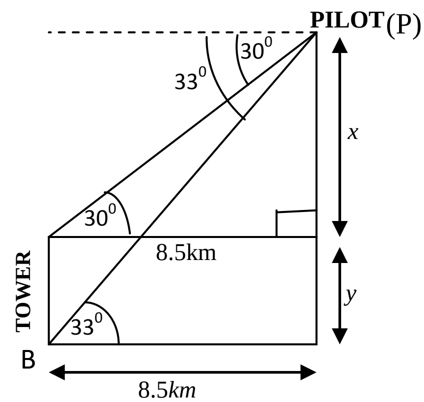
2. From a horizontal distance of 8.5km, a pilot observes that the angles of depression of the top and base of a control tower are 30^o and 33^o respectively. Calculate, correct to three significant figures:

i. the shortest distance between the pilot and the base of the control tower;

ii. the height of the control tower.

Solution

i. Let the shortest distance between the pilot and the base of the control tower PB



$\cos 33^0 = \frac{8.5}{/PB/}$

$/PB/ = \frac{8.5}{\cos 33^0} = 10.1$ m

The shortest distance between the aeroplane and the base of the control tower is 10.1m

ii. Let the height of the control tower be y

$\tan 30^0 = \frac{x}{8.5}$ ……….(1)

$\Rightarrow x = 8.5 \tan 30^0 = 4.90\ km$

$\tan 33^0 = \frac{x + y}{8.5}$ ……….(2)

$8.5 \tan 33^0 = x + y$

$y = 8.5 \tan 33^0 - x$

$y = 8.5 \tan 33^0 - 4.90 = 0.620km$

The height of the tower is 0.620m

Type 4

Worked Examples

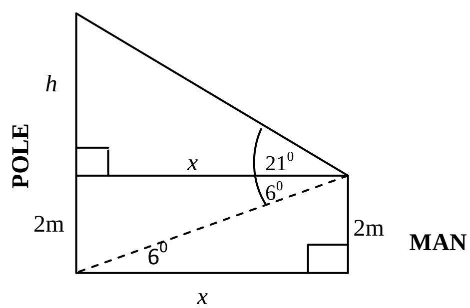
1. A man 2m tall stands on the same level ground as a vertical pole. He observes that the angle of elevation of the top of the pole is 21^0 and the angle of depression of the foot of the pole is 6^0. Find, correct to the nearest meter;

i. how far the pole is from the man,

ii. the height of the pole.

Solution

i. Let the distance between the pole and the man be x and the height of the pole be h+ 2



$\tan 6^0 = \frac{2}{x}$ ……….(1)

$\Rightarrow x = \frac{2}{\tan 6^0}$ = 19m

ii. From the diagram, *TOA* is applicable,

$\tan 21^0 = \frac{h}{x}$ ……….(2)

But x = 19m, put in eqn (2)

$\tan 21^0 = \frac{h}{19}$

$h = 19 \tan 21^0 = 7\ m$

$\Rightarrow h + 2 = 7m + 2m = 9\ m$

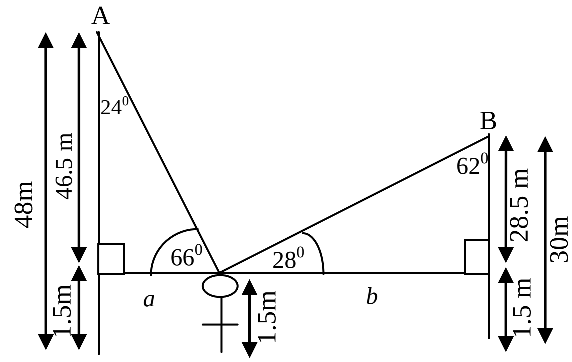
The height of the pole = 9 m

Some Solved Past Questions

1. Two towers A and B are 48m and 30m high respectively. Tower A lie to the west and B to the east of a man 1.5m tall. From the man's eye level, the angles of elevation of the top of A and B are 66^0 and 28^0 respectively. Calculate to three significant figures, the distance between A and B

Solution

Let /AB/ = $a + b$



From the diagram,

$\tan 24^0 = \frac{a}{46.5}$

$a = 46.5 \tan 24^0 = 20.7$m

$\tan 62^0 = \frac{b}{28.5}$

$b = 28.5 \tan 62^0 = 53.6$m

$/AB/ = a + b$

$/AB/$ = 20.7m + 53.6m = 74.3m

2. Two points A and C, on opposite sides of a vertical pole, are on the same level ground as the foot of the pole, B. The angles of elevation of the top of the pole D from A and C are 30^0 and 48^0 respectively. If the distance between A and C is 50m, find $/BD/$, the height of the pole.

Solution

Let $/BD/ = h$, $/CD/ = y$ and $/AD/ = x$ as shown in the diagram;

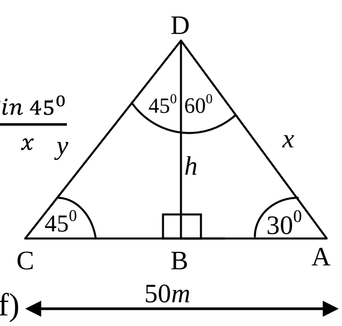
Using sine rule,

$\frac{Sin\ (45^0 + 60^0)}{50} = \frac{Sin\ 45^0}{x}$

$\frac{Sin\ 105^0}{50} = \frac{Sin\ 45^0}{x}$

$x = \frac{50 \sin 45^0}{\sin 105^0}$

$x = 36.60m$(4 s. f)



$\sin 30^0 = \frac{h}{x}$, but $x = 36.6m$

$\sin 30^0 = \frac{h}{36.6}$,

$h = 36.6 \sin 30^0 = 18.3m$

But $/BD/ = h = 18.3\ m$

Therefore, the height of the pole is $18.3m$

Alternatively,

Using sine rule,

$\frac{Sin\ (45^0\ +\ 60^0\ )}{50} = \frac{Sin\ 30^0}{y}$

$\frac{Sin\ 105^0}{50} = \frac{Sin\ 30^0}{y}$

$y = \frac{50 \sin 30^0}{\sin 105^0}$ = 25.88m(4 s. f)

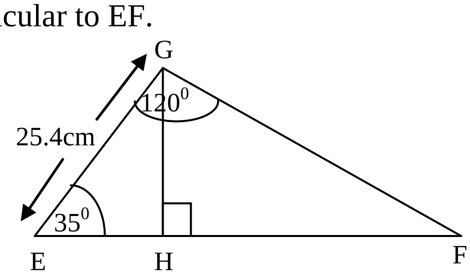
$\sin 45^0 = \frac{h}{y}$, but y = 25.88m

$\sin 45^0 = \frac{h}{25.88}$,

$h = 25.88 \sin 45^0 = 18.30m$

But $/BD/ = h = 18.3\ m$

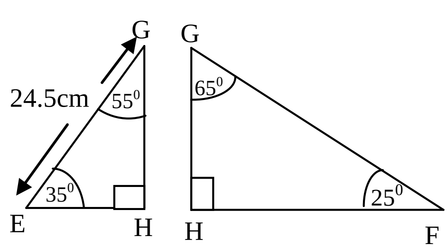
3. In the figure below, angle GEH = 35^0, angle EGF = 120^0, /EG/ = 25.4cm and GH is perpendicular to EF.



Calculate /*EF*/, correct to three significant figures

Solution

Draw the triangles separately as shown below;



From Δ *EHG*

$\sin 55^0 = \frac{/EH/}{25.4}$

$/EH/ = 25.4 \sin 55^0$ = 20.8cm

$\sin 35^0 = \frac{/GH/}{25.4}$

$/GH/ = 25.4 \sin 35^0$ = 14.6cm

From Δ *HFG*

$\sin 25^0 = \frac{/GH/}{/H\ F/}$, but /GH/ = 14.6

$\sin 25^0 = \frac{14.6}{/H\ F/}$

$/HF/ = \frac{14.6}{\sin 25^0}$ = 34.5cm

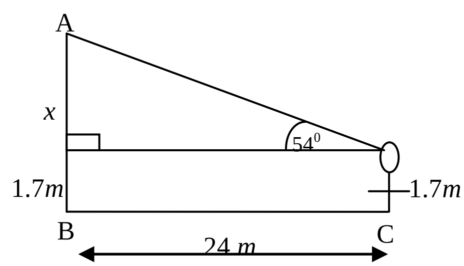
EF = EH + HF

EF = 20.8 + 34.5 = 55.3cm

4. A vertical pole *AB* is erected on a level ground. A man 1.7mtall stands at *C*, 24 m away from the foot *B*of the pole. The angle of elevation of the top *A* from the man is 54^0. Calculate, correct to one decimal place the height of the pole.

Solution

Let the heightof the man to the end of the vertical pole be x



From the diagram;

The height of the pole = (x + 1.7)m

But $x = 24 \tan 54^0 = 33.0m$ (1 d.p)

Substitute x = 33.0 in (x + 1.7)m

The height of the pole = (33.0 + 1.7)m = 34.7m

Exercises 24.13

1. Two ships, S and T, 40km apart, simultaneously observe by a radar of an aeroplane A at an angle of elevation of 10^0 and 6^0 respectively.Given that the aeroplane is directly above the line joining the ships, calculate;

i. the distance AS,

ii. the height of the areoplane above sea – level.

2. A surveyor wishes to measure the height of a round tower. Measuring the angle of elevation, he finds that the angle increases from 22^0 to 36^0 after walking 25 m towards the base of the tower. Calculate the height of the tower, correct to the nearest meter.

3. The length of the shadow of a pole on a level ground increased by 60 meters when the angle of elevation of the sun changes from 54^0 to 32^0. Calculate the height of the pole, correct to three significant figures.

4. From a Swiss mountain rescue station, 897m up a mountain, a manobserves that the angles of depression of two villages on the valley floor are 53^0 and 28^0 respectively. How far apart are the villages?

5. Two vertical electricity poles, each 18m high stands at the points A and B along a straight horizontal road. A straight footpath meets theroads at B from a point C on the foot path, 43 meters from the road. The angles of elevation of the top of the poles at A and B are respectively 14.6^0 and 16.4^0. Calculate, correct to one decimal place;

a. the distance $\overline{AC}$ and $\overline{BC}$,

b. the distance between the poles.

6. The feet *A* and *C*, of two vertical poles *AP* and *CR* are on the same horizontal plane as the foot *B* of a vertical flag pole, *QB* = 5m high. *A* is 10m east and *C* is 10m west of *B*. The heights of the poles are /*AP*/ = 6m and /*CR*/ = 5m. Calculate;

i. /*AR*/,

ii. the angle *CP* makes with the horizontal, correct to the nearest degreeiii. the angle the plane *ACQ* makes with the horizontal, correct to the nearest degree.

Trigonometry and Bearings

Revision: Refer to page 264 to 269 *(Other application of bearings).*

I. Draw a diagram to represent the problem.

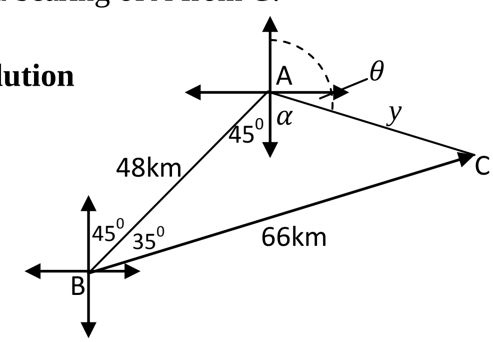
II. If the triangle formed is **not** a right angled triangle, use cosine rule or cosine rule to determine the value of an unknown side and the values of unknown angles.

III. To find the bearing of say *A* from *B*, is to find the total angle turned through in the clockwise direction from the north pole of *B* to the direction of *A*.

Worked Examples

1. *B* is 48km south – west of *A*, and *C* is 66km from *B* on a bearing of 080^0. Find, the distance and bearing of *A* from *C*.

Solution



Let the distance of *A* to *C* be *y*

From the diagram, ΔABC is not a right – angled triangle. Therefore, use the cosine rule to find the value of the unknown side and unknown the angles.

$\Rightarrow y^2 = 66^2 + 48^2 - 2\,(66)(48)\cos 35^0$

$y^{\,2} = 1469.85$

$y = \sqrt{1469.85} = 38$ km

From the diagram, < C

$48^2 = 66^2 + 38^2 - 2\,(66)(38)\cos C$

$48^2 - 66^2 + 38^2 = -\,2\,(66)(38)\cos C$

$-3496 = -\,5016\cos C$

$\cos C = \frac{3496}{5016}$

$C = \cos^{-1}\left(\frac{3496}{5016}\right) = 46^0$(nearest degree)

$35^0 + 45^0 + C + \alpha = 180^0$

$35^0 + 45^0 + 46^0 + \alpha = 180^0$

$\alpha = 180^0 - 126^0 = 54^0$

But $\alpha + \theta = 90^0$

$54 + \theta = 90$

$\theta = 90 - 54 = 36^0$

Bearing of C from A ;

$= 90 + \theta = 90 + 36^0 = 126^0$

2. A ship sails on a bearing of 138^0 from a port *P* for a distance of 15km to a port *Q* and then sails on a bearing of 012^0 to a port *R*. If the bearing of *R* from *P* is 053^0, find:

i. the distance from *Q* to *R;*

ii. the ships final distance from *P*.

Solution

i. Let the distance from Q to R be $\overrightarrow{\boldsymbol{/QR/}}$

From the diagram,

$(42 + 12)^0 + (37 + 48)^0 + \theta = 180^0$

$\theta = 180^0 - 54^0 - 85^0 = 41^0$

Using sine rule,

$\frac{\overrightarrow{/QR/}}{\sin\,(48+37)^0} = \frac{15}{\sin\theta}$

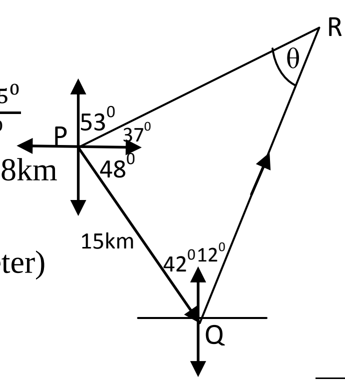
$\frac{\overrightarrow{/QR/}}{\sin 85^0} = \frac{15}{\sin 41^0}$

$\overrightarrow{/QR/} = \frac{15\sin 85^0}{\sin 41^0}$

$\overrightarrow{/QR/} = 22.7768$km

$\overrightarrow{/QR/} = 23$km

(Nearest kilometer)



ii. Let the ships final distance from P be $\overrightarrow{/RP/}$

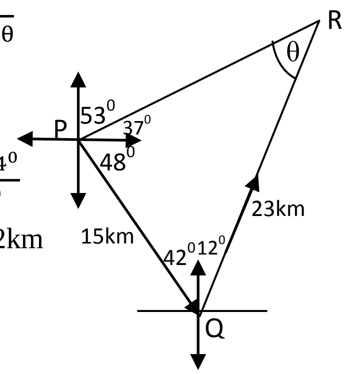
$\frac{\overrightarrow{/RP/}}{\sin\,(42+12)^0} = \frac{15}{\sin\theta}$

$\frac{\overrightarrow{/RP/}}{\sin 54^0} = \frac{15}{\sin 41^0}$

$\overrightarrow{/RP/} = \frac{15\sin 54^0}{\sin 41^0}$

$\overrightarrow{/RP/} = 18.4972$km

$\overrightarrow{/RP/} = 18$km



Alternatively;

Using the cosine rule;

$\overrightarrow{/RP/}^2 = 15^2 + 22.7768)^2 - 2(15)(22.7768)\cos 54^0$

$\overrightarrow{/RP/}^2 = 743.7826 - 401.6360$

$\overrightarrow{/RP/}^2 = 342.1466$

$\overrightarrow{/RP/} = \sqrt{342.1466} = 18.4972 = 18$km

Revision Exercises

1. An aircraft flies 100 km from *A* to *B* on a course 075^0, then 200 km from *B* to *C* on a course 343^0. How far north and how far east are *B* and *C* from *A*.

2. An object *Q* is 6km from *P* on a bearing N20^0E, and object R is 7.5 km from *P* on a bearing N75^0E. Calculate the distance and bearing of *Q* from *R*.

3. A ship sails due east at a steady speed of 24km/h while a fishing boat is towed north – east

at a steady speed of 10km/h from the same point. They start at the same time;

a. find the bearing of the ship from an observer on the boat at the end of 2 hours;

b. find the distance between the ship and the boat at that time (2 hours after the starting time.

4. Town Q is 20km due north of P and the bearing of town R from Q is 140^0. If R is 8km from Q, calculate;

a. the bearing of R from P, to the nearest degree,

b. how far north of P, R is, correct to *two* significant figures.

5. Kaya rides a bicycle to school everyday. From home, he rides for 3.5km on a bearing of 037^0 and then 1.4km on a bearing of 335^0 before arriving at school.

a. What is the total distance Kaya rides from home to school?

b. How far (i)north (ii) east

is Kaya's school from home?

c. find the bearing and distance of Kaya's school from his home.

A Slipping Ladder or Inclined Plane

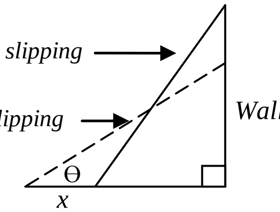
A ladder or any straight object (stake) leaning against a vertical wall may either slips backward or upward as shown below;

"Backwards"

When a ladder or an object slips backward:

I. A new angle of elevation, θ, is formed

II. There is an upward adjustment (reduction) in the length of the wall causing an increase in the distance between the ladder and the wall. This increment is represented by x in the diagram.

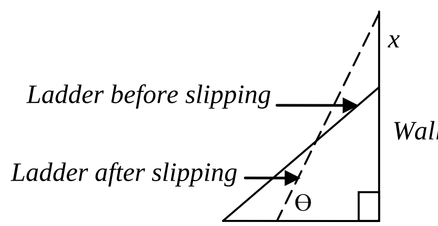


"Upwards"

When a ladder or a stake slips backward:

I. A new angle of elevation, θ, is formed.

II. There is ahorizontal adjustment (reduction) in the distance between the foot of the wall and the ladder causing an increase in the height of the wall. This increment is represented by x in the diagram.



To tackle involving problems:

I. Make a sketch of the figure.

II. Draw the figures separately as "before the ladder slipped" and "after the ladder slipped."

III. Work for the interior angles and the length of the sides of each triangle.

IV. Put the separate figures together with its dimensions.

Worked Example

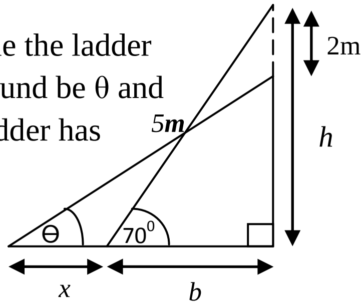
1. A ladder 5m long leans against a vertical wall at an angle of 70^0 to the ground. The ladder slips down the wall 2m. Find correct to two significant figures.

i. the new angle which the ladder makes with the ground,

ii. the distance the ladder has slipped back on the ground from its original position.

Solution

i. Let the new angle the ladder makeswith the ground be θ and the distance the ladder has slipped back be x



Draw the figures separately as "before the ladder slipped" and "after the ladder slipped".

"Before"

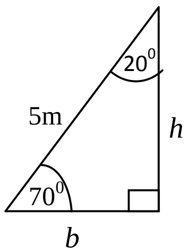
Let the height of the wall be $h$and the distance between the foot of the wall and the ladder be b

$\sin 70^0 = \frac{h}{5}$

$h = 5\sin 70^0 = 4.7$m

$\sin 20^0 = \frac{b}{5}$

$b = 5\sin 20^0 = 1.7$m



Alternatively

By Pythagoras theorem,

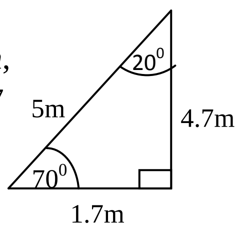
$5^2 = b^2 + h^2$, but $h = 4.7$

$5^2 = b^2 + (4.7)^2$

$5^2 - (4.7)^2 = b^2$

$b^2 = 2.91$

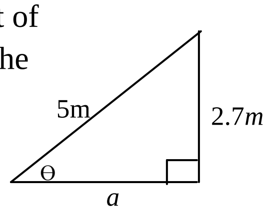
$b = \sqrt{2.91} = 1.7$m



"After"

Height of wall = $(4.7 - 2)m = 2.7m$

Let the distance between the ladder and the foot of the wall be a and the new angle the ladder makes with the ground be θ



$\sin\theta = \frac{2.7}{5}$

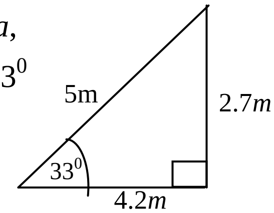
$\theta = \sin^{-1}\left(\frac{2.7}{5}\right) = 33^0$

ii. To calculate for a,

$\tan\theta = \frac{2.7}{a}$, but$\theta = 33^0$

$\tan 33^0 = \frac{2.7}{a}$

$a = \frac{2.7}{\tan 33^0} = 4.2m$



Alternatively

By Pythagoras theorem,

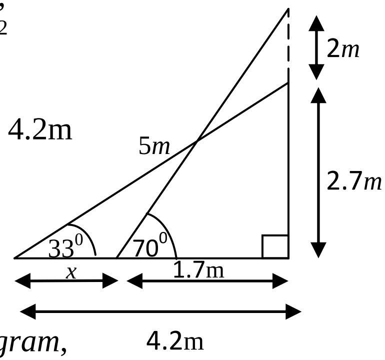
$5^2 = a^2 + 2.7^2$,

$5^2 - (2.7)^2 = a^2$

$a^2 = 17.71$

$a = \sqrt{17.71} = 4.2$m

Below is the dimension of the figure;



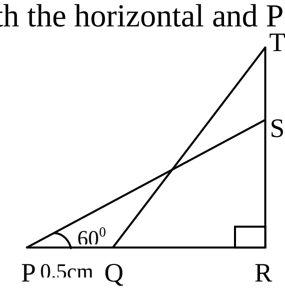
From the diagram,

$x + 1.7 = 4.2$

$x = 4.2 - 1.7 = 2.5$

Therefore, the ladder slipped back $2.5m$

3. In the diagram below, PS and QT are two ladders 10m and 12m long respectively, placed against a vertical wall TR. PS makes an angle of 60^0 with the horizontal and PQ = 0.5m.



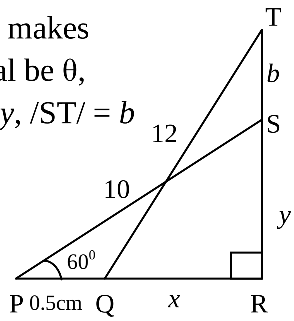
Calculate to two significant figures;

i. the angle which QT makes with the horizontal,

ii. the height of point T above the horizontal.

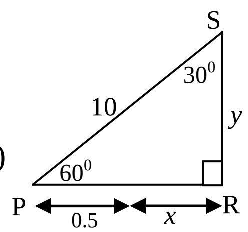
Solution

Let the angle QT makes with thehorizontal be θ, /QR/ = x, /SR/ = y, /ST/ = b



Draw the triangles separately as shown below and calculate all the angles and dimensions of each;

From Δ PRS,



$\sin 60^0 = \frac{y}{10}$

$y = 10 \sin 60^0$

$y = 8.7m$ (2 s. f.)

$\cos 60^0 = \frac{0.5 + x}{10}$

$10 \cos 60^0 = 0.5 + x$

$10 \cos 60^0 – 0.5 = x$

$x = 4.5m$

Alternatively,

By Pythagoras theorem,

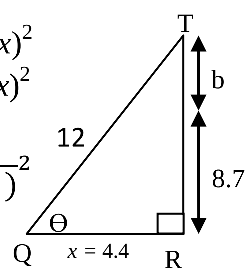
$10^2 = y^2 + (0.5 + x)^2$,

But $y = 8.7$

$10^2 = (8.7)^2 + (0.5 + x)^2$

$10^2 – (8.7)^2 = (0.5 + x)^2$

$24.31 = (0.5 + x)^2$



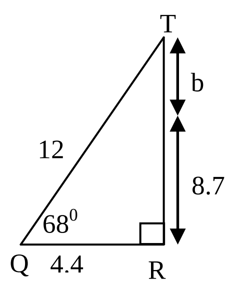
$\sqrt{24.31} = \sqrt{(0.5 + x)}^2$

$\sqrt{24.31} = 0.5 + x$

$x = \sqrt{24.31} – 0.5$

$x = 4.4m$

From ΔQRT,



$\cos \theta = \frac{4.4}{12}$

$\theta = \cos^{-1}\left(\frac{4.4}{12}\right) = 68^0$

(2. S. f)

$\sin \theta = \frac{b + 8.7}{12}$

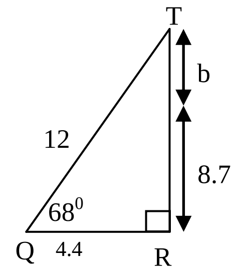
But $\theta = 68^0$

$\Rightarrow \sin 68^0 = \frac{b + 8.7}{12}$

$12 \sin 68^0 = b + 8.7$

$12 \sin 68^0 – 8.7 = b$

$b = 2.4m$



Alternatively

By Pythagoras theorem,

$12^2 = (4.4)^2 + (8.7 + b)^2$,

$12^2 – (4.4)^2 = (8.7 + b)^2$,

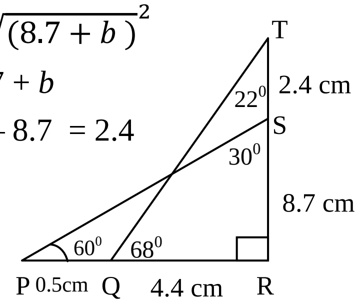
$124.64 = (8.7 + b)^2$,

$\sqrt{124.64} = \sqrt{(8.7 + b)}^2$

$\sqrt{24.31} = 8.7 + b$

$b = \sqrt{24.31} – 8.7 = 2.4$

Dimensions of the figure;



i. The angle QT makes with the horizontal is 68^0

ii. The height of point T above the horizontal

= RS + ST

= (8.7 + 2.4)cm = 11cm

Exercises 24.14

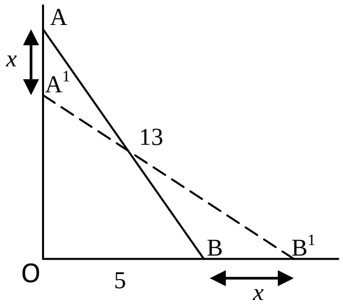
1. A pole 25m long is placed against a vertical wall such that its lower end is 7m from the foot of the wall on the same horizontal ground. If the upper end of the pole is pushed down by 2 m, calculate correct to two significant figures:

a. how much further away from the wall the lower end will move,

b. the angle the pole now makes with the horizontal.

2. A ladder $\overline{AB}$, of length 13m, rests against a vertical wall with its foot on a horizontal floor at

a distance of 5m from the wall. When the top slips down a distance x meters the foot moves out x meters. Find x



3. A ladder 22m long leans against a vertical wall at an angle of 38^0 to the ground. The ladder slips down the wall 3.2m. Find correct to two significant figures;

i. the new angle which the ladder makes with the ground,

ii. the distance the ladder has slipped back on the ground from its original position.

4. A ladder of length 20m, rests against a vertical wall with its foot on a horizontal floor at a distance of 12m from the wall. When the top slips down a distance x meters the foot moves out twice as much as x meters.

i. Find the value of x.

ii. How far does the foot moves out of the wall?

Three Dimensional Problems

<u>Type 1</u>

When solving problems in three dimensions:

I. Redraw each triangle separately.

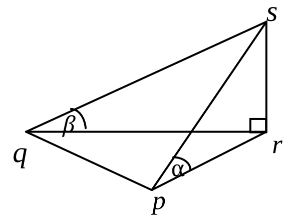
II. Apply the sine or cosine rule to these triangles.

III. Watch for common sides that links the triangles.

IV. Carry common values from one triangle to another.

Worked Examples

p, q and r are points on level ground. [s,r] is a vertical tower of height h. The angles of elevation of the top of the top of the tower from p and q are α and β, respectively



i. If /α/ = 60^0 and /β/ = 30^0, express /pr/ and /qr/ in terms of h

ii. Find /qp/ in terms of h, if tan <qrp = $\sqrt{8}$

Solution

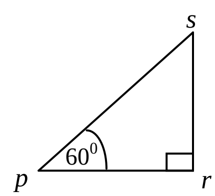
Redraw right – angled triangles prs and qrsseparately

i. tan 60^0 = $\frac{h}{/pr/}$

$\sqrt{3}$ = $\frac{h}{/pr/}$

$\sqrt{3}$/ pr / = h

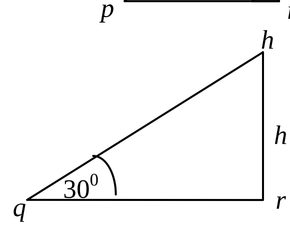
/ pr / = $\frac{h}{\sqrt{3}}$



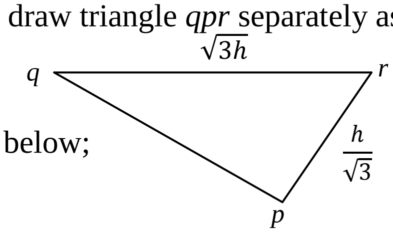
tan30^0 = $\frac{h}{/qr/}$

$\frac{1}{\sqrt{3}}$ = $\frac{h}{/qr/}$

/ pr / = $\sqrt{3}h$



ii. Re - draw triangle qpr separately as shown below;



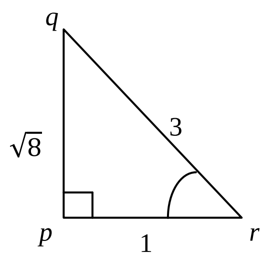
Using the cosine rule on Δ qpr

$|qp|^2$ = $|qr|^2$ + $|pr|^2$- 2 $|qr|.|pr|$cos < qrp

But cos < qrp *is obtained as shown below*

Given tan < qrp = $\sqrt{8}$ = $\frac{\sqrt{8}}{1}$, its triangle is shown below;

By Pythagoras theorem,
$|qr| = 3$
$\therefore \cos < qrp = \frac{1}{3}$



$|qp|^2 = \left|\sqrt{3}h\right|^2 + \left|\frac{h}{\sqrt{3}}\right|^2 - 2\left|\sqrt{3}h\right|.\left(\frac{h}{\sqrt{3}}\right)\left(\frac{1}{3}\right)$

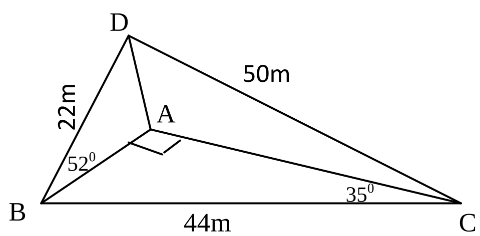
$|qp|^2 = 3h^2 + \frac{h^2}{3} - \frac{2}{3}h$

$|qp|^2 = \frac{9h^2 + h^2 - 2h^2}{3}$

$|qp|^2 = \frac{8h^2}{3}$

$|qp| = \sqrt{\frac{8h^2}{3}} = \sqrt{\frac{8}{3}}h$

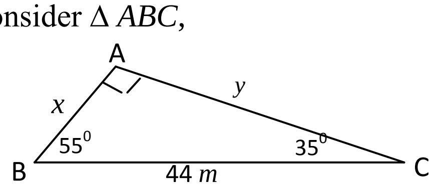
2. In the figure below, ΔABC is a right – angled triangle on a horizontal ground. AD is
a vertical tower.



$< BAC = 90^0$, $< ACB = 35^0$, $< ABD = 52^0$ /*BD*/ = 22m, /*DC*/ = 50m and / *BC*/ = 24m. Find:

i. the height of the tower,
ii. the angle of elevation of the top of the tower from *C*.

Solution
Let /AB/ = x, /*AC*/ = $y$and /*AD*/ = h
Consider Δ *ABC*,



Using the sin rule,

$\frac{\sin 90^0}{44} = \frac{\sin 35^0}{x}$

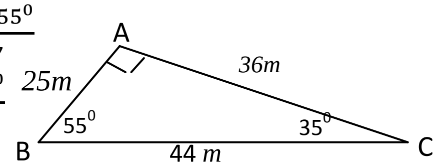
$x = \frac{44 \sin 35^0}{\sin 90^0} = 25m$

Similarly,

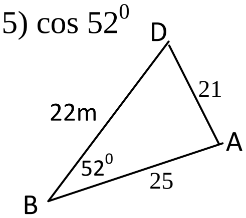
$\frac{\sin 90^0}{44} = \frac{\sin 55^0}{y}$

$y = \frac{44 \sin 55^0}{\sin 90^0}$

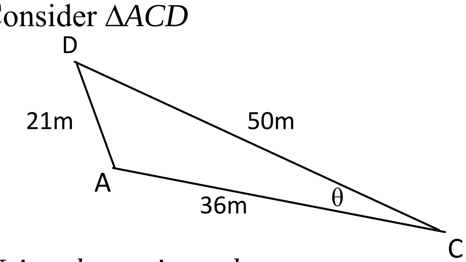
$y = 36m$



Consider Δ ABD,
Using the cosine rule,
$h^2 = 22^2 + 25^2 - 2(22)(25) \cos 52^0$
$h^2 = 484 + 625 - 677$
$h^2 = 432$ m
$h = \sqrt{432}$ m $= 21m$

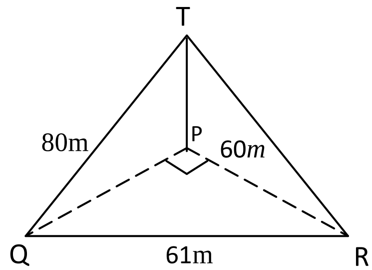


Consider Δ*ACD*

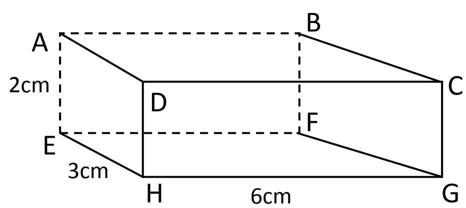


Using the cosine rule,
$21^2 = 36^2 + 50^2 - 2(36)(50) \cos \theta$
$2(36)(50) \cos \theta = 36^2 + 50^2 - 21^2$
$3600 \cos \theta = 3355$
$\cos \theta = \frac{3355}{3600}$
$\theta = cos^{-1}\left(\frac{3355}{3600}\right) = 21^0$

Exercises 24.15
1. In the figure below, *PQR* is a right – angled triangle on a horizontal ground. *PT* is a vertical tower. $< QPR = 90^0$, /*QR*/ = 61m, /*PR*/ = 60m and /*QT*/ = 80m.
Find: a. the height of the tower b. /*TR*/



2. The figure below represents a rectangular box. Given that AB = 6cm, AD = 3cm and AE = 2cm. Calculate the length of the diagonal AG.



3. ABC is an isosceles triangle in which $AB = AC$ = 10 cm and BC = 4cm. The mid points of AB and BC are X and Y respectively, and the line through X perpendicular to AB meets AY at O. Calculate angle BAC and the lengths of AO and OY

4. A right pyramid has a base ABC which is an equilateral triangle of side 5 cm. The height of the vertex V above the center O of
the base is 4m. Calculate:
i. the length /OA/, correct to two decimal places.
ii. the angle of inclination of the edge $\overline{AV}$ to the base ABC, correct to the nearest degree.
iii. the volume of the pyramid, to three significant figures.

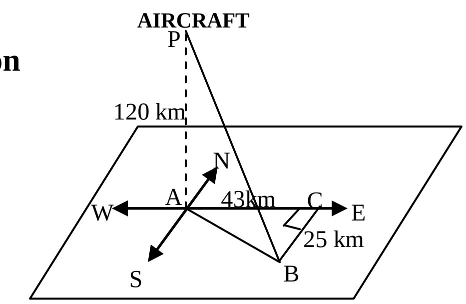
5. X, Y and Z are three points on a level ground. The bearing of Y from X is 030^0 and the bearing of Z from Y is 120^0. A vertical tower XT stands at X and the angle of elevation of T from Y is 68.2^0. Given that XY = 40 m and YZ = 30m, calculate:
i. XZ,
ii. the height of the tower.

Type 2

Worked Examples

1. An aircraft is flying at a height of 120km over a plain. It is immediately above town A. Town B is on a vector $\binom{43km}{-25km}$ from A. What is the distance of the aircraft from town B to the nearest kilometer?

Solution



From the diagram, the 'distance of the aircraft from town B' is the distance PB.

Since PA, AC and CB are perpendicular to each other;

$PB^2 = PA^2 + AC^2 + CB^2$

$PB^2 = 120^2 + 43^2 + 25^2$

PB^2 = 16,874 km

PB = 130 km (nearest kilometer)

25 SEQUENCES AND SERIES

Sequence

A sequence is an ordered list. It is a collection of numbers arranged in a definite order, connected by a simple rule. For example, perfect squares are listed as 1, 4, 9, 16, 25, 36…The dots indicates that it is impossible to list all of them since there are indefinite number of them.

However, this collection could be represented in several ways, one of which is to write 1, 4, 9, …n^2, $n \in N$. A list of numbers such as this is generally called a ***sequence***. Other examples of sequences are even numbers: 2, 4, 6,.. odd numbers: 1, 3, 5… However, the prime numbers: 2, 3, 5, 7…do not form a sequence because the numbers or the terms do not follow any definite order.

Worked Examples

Write down the next three terms in each of the following sequences:

1. 1, 3, 5, 7,… 2.4, 2, 0, -2…

3.$\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}$… 4. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$…

Solution

1. 1, 3, 5, 7, 9, 11, 13 2.4, 2, 0, -2, -4, -6, -8

3. $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}, \frac{1}{96}, \frac{1}{192}$ 4.$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}$

Exercises 25.1

Write down the next three terms in each of the following sequences:

1. 1, 2, 4, 8… 2. 1, 2, 6, 24, 120…

3. 1 , $\frac{2}{3}, \frac{3}{9}, \frac{4}{27}$… 4. - $\frac{1}{2}$, - $\frac{1}{4}$, - $\frac{1}{8}$…

5. 12, 8, 4,….. 6. -82, -88, -94…

Series

When the numbers or terms of a sequence are added or considered as a sum, it is called a ***series***. Thus we have a series of even number as 2 + 4 + 6 + 8… and the series of odd numbers as 1 + 3 + 5 + 7 …

A series may end after a definite number of terms. Such a series is called a ***finite series***. e.g. 2 + 4 + 6 + 8

On the other hand, a series may not end and it is then called an ***infinite series.***

e.g. 2 + 4 + 6 + 8…

Types of Sequences

There are two main types of sequence namely:

1. Arithmetic Progression (A. P) or Linear sequence
2. Geometric Progression (G.P) or Exponential Sequence

The *n*th Term of an Arithmetic Progression (A. P) or Linear Sequence

An arithmetic progression is a series in which any term is obtained from the previous term by adding a certain number called ***common difference***.

Consider the sequence: 1, 3, 5, 7, 9

1. The sequence is said to have five terms, denoted by n. Thus n = 5
2. Each term is represented by U. Therefore, 1, 3 , 5 , 7 , 9= U_1, U_2 ,U_3, U_4, U_n

In this case:

U_1 is called the first term of the sequence,
U_2 is called the second term of the sequence,
U_3 is called the third term of the sequence,
U_4 is called the fourth term of the sequence,

U_n is called the*nth* term of thesequence and of course the last term of the sequence.

3. The digit representing the first term is denoted by *a*. That is$U_1 = a$

4.$U_2 - U_1$ is called the common difference denoted by *d*. $U_2 - U_1 = d$. The common difference *d* can be positive or negative.

5. The sum of the first *n* terms of a linear sequence is denoted by S_n.

Thus, for $U_1, U_2, U_3, U_4, \ldots U_n$,

$S_n = U_1 + U_2 + U_3 + U_4 + \ldots U_n$

If any term is obtained from the previous term by adding the common difference, *d*, then it implies that if:

$U_1 = a$

$U_2 = a + (2 - 1)\, d = a + d$

$U_3 = a + d + d = a + (3 - 1)\, d = a + 2d$

$U_4 = a + d + d + d = a + (4 - 1)\, d = a + 3d$

$U_5 = a + d + d + d + d = a + 4d$

$U_n = a + (n - 1)\, d$ or $U_n = U_1 + (n - 1)\, d$

Therefore, the *nth* term, also called the general term of a linear sequence, denoted by U_n is given by; $\boldsymbol{U_n = a + (n - 1)\, d}$

Worked Examples

1. Find the twelfth term of the sequence:
19, 14, 9, 4…

Solution

In the sequence, 19, 14, 9, 4…

$a = 19$, $d = U_2 - U_1 = 14 - 19 = -5$, $n = 12$

Substitute in $U_n = a + (n - 1)\, d$

$U_{12} = 19 + (12 - 1)\ -5$

$U_{12} = 19 + 11(-5) = 19 - 55 = -36$

2. Find the 20th term of the sequence 7, $6\frac{1}{4}$, $5\frac{1}{2}$…

Solution

Let $a = 7$, d = $6\frac{1}{4} - 7 = -\frac{3}{4}$ and $n = 20$

$U_n = a + (n - 1)\, d$

$U_{20} = 7 + (20 - 1)\,(-\frac{3}{4})$

$= 7 + 19 \times (-\frac{3}{4})$

$= 7 + -\frac{57}{4} = \frac{28-57}{4} = -\frac{29}{4}$

Therefore, the 20th term is $-\frac{29}{4}$

Finding the General Term of an A.P.

The general term of an AP is the formula representing a given A.P. The formula is found as follows:

I. Identify the given sequence.

II. Identify the values of *a* and *d*.

III. Substitute the values of *a* and *d* in $U_n = a + (n - 1)\, d$ and simplify if possible.

Worked Examples

1. Find the general term of the sequence:
2, -1, -4…

Solution

From the sequence: 2, -1, -4…

$a = 2$, $d = -1 - 2 = -3$

Substitute in $U_n = a + (n - 1)\, d$

$U_n = 2 + (n - 1)\,(-3)$

$= 2 - 3n + 3$

$= 5 - 3n$

Therefore, the general term is $U_n = 5 - 3n$

2. The first term of a linear sequence is 13 and the common difference is 8. Find :
i. the general term, ii. the tenth term.

Solution

i. If $a = 13$ and $d = 8$

$\Rightarrow U_n = 13 + (n - 1)\, 8$

$U_n = 13 - 8 + 8n$

$U_n = 8n + 5$

Therefore, the general term is $U_n = 8n + 5$

ii. To find the tenth term,

$a = 13$, $d = 8$ and $n = 10$

Substitute in $U_n = 8n + 5$

$U_{10} = 8(10) + 5 = 85$

3. Write down the 12th and *n*th terms of the A.P, $\frac{1}{4} + \frac{7}{8} + \dots$

Solution

From the sequence: $\frac{1}{4} + \frac{7}{8} + \dots$

$a = \frac{1}{4}$, $d = \frac{7}{8} - \frac{1}{4} = \frac{7-2}{8} = \frac{5}{8}$, $n = 12$

Substitute in $U_n = a + (n-1)\,d$

$U_n = \frac{1}{4} + (n-1)\frac{5}{8}$

$U_{12} = \frac{1}{4} + (12-1)\frac{5}{8}$

$U_{12} = \frac{1}{4} + (11)\frac{5}{8} = \frac{1}{4} + \frac{55}{8} = \frac{2+55}{8} = \frac{57}{8}$

ii. The *n*th term, $U_n = a + (n-1)\,d$

But $a = \frac{1}{4}$, $d = \frac{5}{8}$

By substitution,

$U_n = \frac{1}{4} + (n-1)\frac{5}{8}$

$U_n = \frac{1}{4} + \left(\frac{5}{8}n - \frac{5}{8}\right)$

$U_n = \frac{1}{4} - \frac{5}{8} + \frac{5}{8}n$

$U_n = -\frac{3}{8} + \frac{5}{8}n$

Exercises 25.2

1. Write down the first five terms of each of the following linear sequence;

i. first term 5, common difference 3

ii. first term 3, common difference $\frac{1}{2}$

2. Write down a formula for the *n*th term of each of the following linear sequences:

i. 3, 6, 9, 12… ii. 0,5, 10, 15…

ii. 12, 9, 6, 3… iii. 2, 7, 12, 17….

In each case , write down the 20^{th} term.

Finding the First Term, a, and the Common Difference, d of an A.P.

Given the value of a term other than the first term and the common difference, d, the first term is calculated as follows:

I. Write an expression for the given term and the common difference

II. Equate the expression to the given value of the term

III. Solve for the value of *a* in the equation to obtain the first term of the A.P

Worked Examples

The fourth term of an A.P is 18, and the common difference is -5. Find the first term.

Solution

$U_4 = 18$, $d = -5$, $n = 4$

Substitute in $U_n = a + (n-1)\,d$

$U_4 = a + (4-1)(-5)$

$U_4 = a + 3(-5)$

$U_4 = a - 15$

But $U_4 = 18$

$\Rightarrow a - 15 = 18$

$a = 18 + 15 = 33$

Therefore, the first term of the A.P is 33

Exercises 25.3

1. In an A.P, the third term is 12, and the common difference is 7. Find the first term.

2. Find the first term of an A.P if the tenth term of the A.P is 77, and common difference -12

3. The fourth term of an A.P.is 20, and the common difference is 6. Find the first term.

4. What is the first term of an A.P. whose sixth term is 42 and common difference, 4.

Finite Arithmetic Sequence

It is an A.P. that comes to an end. In other words, it an A.P., that has a given last term (L).

Eg. 2, 4, 6, ..50

For a finite A.P;

$u_n = $ L

$\Rightarrow a + (n - 1)\, d = $ L

By changing the subject to *n*, the number of terms of an *A.P*, given the last term can be calculated.

Worked Examples

Determine the number of terms in the sequence; 5, 11, 17….77 and hence find the 30 the term.

Solution

5, 11, 17….77

$a = 5$, d = 6 $u_n = $ L = 77

$u_n = 5 + 6\,(n - 1)$

$u_n = 5 + 6\,n - 6$

$u_n = 6\,n - 1$

$\Rightarrow 6\,n - 1 = 77$

6n = 77 + 1

$6n = 78$

$n = 13$ terms

$u_{30} = 6\,(30) - 1 = 179$

2. The first term of an A.P. is 2 and the common difference is 2. Find the term that is equal to 46.

Solution

$a = 2$, $d = 2$, L = 46

Substitute in $U_n = a + (n - 1)\, d$

$U_n = 2 + (n - 1)\, 2$

$U_n = 2 + 2n - 2$

$U_n = 2n$

But U_n = L = 46

$\Rightarrow 2n = 46$

$n = 23$

Therefore 46 is the 23rd term

2. Find the number of terms of the A.P: 407 + 401 + … –133

Solution

From the A.P = 407 + 401 + … –133

$a = 407$, $d = 401 - 407 = -6$, L = -133, n =?

Substitute in $U_n = a + (n - 1)\, d$

$U_n = 407 + (n - 1)\, -6$

$U_n = 407 - 6n + 6$

$U_n = 407 + 6 - 6n$

$U_n = 413 - 6n$

But U_n = L = -133

$\Rightarrow 413 - 6n = -133$

$6n = 413 + 133$

$6n = 546$

$n = 91$

Therefore the A.P. has 91 terms

3. Find the number of terms of the linear sequence 3, 7, 11…31

Solution

From the linear sequence 3, 7, 11…31

$a = 3$, d = 7 – 3 = 4, n = ? and $l = 31$

$U_n = 3 + (n - 1)\, 4$

$U_n = 3 + 4n - 4$

$U_n = -1 + 4n$

But U_n = L = 31

$\Rightarrow 31 = -1 + 4n$

$31 + 1 = 4n$

$32 = 4n$

$n = 8$

Therefore, the number of terms is 8

4. The nth term of the sequence; 5, 8, 11…is 383. Find n

Solution

From the sequence; 5, 8, 11…

$a = 5$, $d = 8 - 5 = 3$, $U_n = 383$

Substitute in $U_n = a + (n - 1) d$

$U_n = 5 + (n - 1) 3$

$U_n = 5 + 3 (n - 1)$

$U_n = 5 + 3n - 3$

$U_n = 5 - 3 + 3n$

$U_n = 2 + 3n$

But $U_n = 383$

$\Rightarrow 2 + 3n = 383$

$3n = 383 - 2$

$3n = 381$

$n = 127$

Exercises 25.4

A. Find the number of terms of each of the following linear sequence:

1. 2, -9, -20, …-141
2. 2.7 + 3.2 + … + 17.7
3. 50 + 47 + 44 + … + 14
4. 2 – 9 – … – 130
5. 8, 10, 12,…32
6. 39, 33,…, -63

First Term, a, and Common Difference, d, of an A.P from Two or more given Terms

Given the values of two or more terms of an A. P. the first term, a, and the common difference, d, can be calculated as follows:

I. Identify the values of the given terms.

II. Write two (or more) equations, each for the given terms.

III. Solve the equations simultaneously to determine the respective values of a and d.

Worked Examples

1. The second term of an A.P is 15 and the fifth term is 21. Find the common difference and the first term.

Solution

$U_n = a + (n - 1) d$

$U_2 = a + (2 - 1) d = 15$

$\Rightarrow a + d = 15$……….(1)

$U_5 = a + (5 - 1) d = 21$

$\Rightarrow a + 4d = 21$…………….(2)

eqn (2) – eqn (1)

$(a - a) + (4d - d) = (21 - 15)$

$3d = 6$

$d = 2$

Put $d = 2$ in eqn (1)

$a + 2 = 15$

$a = 15 - 2 = 13$

First term is 13 ; the common difference is 2

2. Find the linear sequence whose eighth term is 12 and 12th term is -8.

Solution

$U_n = a + (n - 1) d$

$U_8 = a + (8 - 1) d = 12$

$\Rightarrow a + 7d = 12$……….(1)

$U_{12} = a + (12 - 1) d = -8$

$\Rightarrow a + 11d = -8$………(2)

eqn (2) – eqn (1)

$(a - a) + (11d - 7d) = (-8 - 12)$

$4d = -20$

$d = - 5$

Put $d = - 5$ in eqn (1)

$a + 7(- 5) = 12$

$a - 35 = 12$

$a = 12 + 35 = 47$

The first term is 47, common difference is -5

The sequence is 47, 42, 37, 32, 27, 22, 17, 12, 7, 2, -3, -8

3. The 8th and the 22nd terms of a linear are 38 and 108 respectively. Find:
i. the first term and the common difference,
ii. the general term of the sequence.

Solution
$U_n = a + (n - 1)\, d$
$U_8 = a + (8 - 1)\, d = 38$
$\Rightarrow a + 7d = 38$……….(1)
$U_{22} = a + (22 - 1)\, d = 108$
$\Rightarrow a + 21d = 108$………(2)

eqn (2) – eqn (1)
$(a - a) + (21d - 7d) = (108 - 38)$
$14d = 70$
$d = 5$

Put $d = 5$ in eqn (1)
$a + 7(5) = 38$
$a + 35 = 38$
$a = 38 - 35 = 3$
First term is 3 and the common difference is 5.

b. Substitute $a = 3$ and $d = 5$ in $U_n = a + (n - 1)\, d$
$U_n = 3 + (n - 1)\, 5$
$U_n = 3 + 5n - 5$
$U_n = 5n - 5 + 3$
$U_n = 5n - 2$
Therefore, the general term is $U_n = 5n - 2$

Some Solved Past Questions
1. The 8th term of an A.P. is five times the third term whilst the 7th term is 9 greater than the fourth term. Write the first five terms of the A.P.

Solution
$U_n = a + (n - 1)d$
$U_8 = a + 7d$
$U_3 = a + 2d$

8th term is five times the third term
$\Rightarrow a + 7d = 5(a + 2d)$
$a + 7d = 5a + 10d$
$a - 5a = 10d - 7d$
$- 4a = 3d$……………(1)

$U_7 = a + 6d$
$U_4 = a + 3d$

The 7th term is 9 greater than the fourth term
$\Rightarrow a + 6d + 9 = a + 3d$
$a - a + 6d - 3d = -9$
$3d = -9$
$d = -3$

Put $d = -3$ in eqn (1)
$- 4a = 3(-3)$
$- 4a = -9$
$a = 2.25$
$\Rightarrow$The first five terms of the sequence is:
2.25, - 0.75, - 3.75, - 6.75, -9.75

2. If 3, x, y, 18 are in arithmetic progression, find the values of x and y.

Solution
From 3, x, y, 18
$a = 3$, $U_4 = 18$, $d = ?$
$U_n = a + (n - 1)\, d$
$U_4 = 3 + (4 - 1)\, d$
$U_4 = 3 + 3d$

But $U_4 = 18$
$\Rightarrow\ 3 + 3d = 18$
$3d = 18 - 3$
$3d = 15$
$d = 5$

$\Rightarrow x = 3 + 5 = 8$

$y = 3 + 5 + 5 = 13$
The sequence is 3, 8, 13, 18

3. $2k + 1$, $3k$, $5k – 5$… are the first three terms of an arithmetic sequence.
i. Calculate the value of k
ii. Find the next two terms of the sequence

Solution
i. $3k – (2k + 1) = (5k – 5) – 3k$
$3k – 2k – 1 = 5k – 5 – 3k$
$3k – 2k – 5k + 3k = 1 – 5$
$-k = -4$
$k = 4$

ii. $2(4) + 1, 3(4), 5(4) – 5$
9, 12, 15…
Common difference = 12 – 9 = 3
9, 12,15, **18, 21**

Exercises 25.5
1. Find the first term, the common difference and an expression for the nth term for each of the A.P. below:
i. second term, 3 and third term 5,
ii. second term, 4 and fifth term 16.

2. The 2nd and 5th terms of a linear sequence are 26 and 62 respectively. Find:
i. the first term and the common difference,
ii. the general term of the sequence.

3. Find the 28th term of a linear sequence whose fourth term is -18 and ninth term is 12.

4. Find the 10th term of the linear sequence whose 2nd term is 28 and 17th term is -2.

Sum of a Linear Sequence or A.P.
Suppose we want to find the sum:
2 + 4 + 6 + 8… + 50 + 52
If the sum is represented by S, we have,
S = 2 + 4 + 6 + 8… + 50+ 52

Reversing the order of the terms:
S = 52 + 50 + 48 + 46 + 44… + 2

Adding the two sums,
S= 2 + 4 + 6 + 8…+ 50 + 52
<u>52 + 50 + 48 + 46 + 4…+ 2</u>
54 + 54 + 54………. 54 + 54

In this sum, we have 26 terms
Therefore 2S = 54 × 26
$S = \frac{54 \times 26}{2} = 702$

Now let U_1 be the first term, d the common difference and S_n the sum of n terms.
$\Rightarrow S_n = U_1 + (U_1 + d) + (U_1 + 2d) + [(U_1 + (n-1)d)]$
Reversing the order of the terms, we have
$S_n = [(U_1 + (n-1)d)] + (U_1 + 2d) + (U_1 + d) + U_1$
Add the two sums,
$2S_n = [2U_1 + (n-1)d] + [2U_1 + (n-1)d] + [2U_1 + (n-1)d]$

For n terms,
$2S_n = n\,[2U_1 + (n-1)d]$
$S_n = \frac{n\,[2U_1 + (n-1)d]}{2}$
$S_n = \frac{n}{2}[2U_1 + (n-1)d]$
But $U_1 = a$
$\Rightarrow S_n = \frac{n}{2}[2a + (n-1)d]$
$S_n = \frac{n}{2}[a + a + (n-1)d]$
But $a + (n-1)d = l$
$\boldsymbol{\Rightarrow S_n = \frac{n}{2}(a + l)}$

Given the last term U_n, the sum of the linear sequence is given by the relation,

$S_n = \frac{n}{2}[a + U_n]$ OR $S_n = \frac{n}{2}[2a + (n-1)d]$

Worked Examples

1. Find the sum of the first 10 terms of the A.P: 4, 2, 0, -2…

Solution

A.P = 4, 2, 0, -2…

From the sequence, $U_1 = a = 4$, d = 2 – 4 = -2 and $n = 10$

Substitute in $S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{10} = \frac{10}{2}[2(4) + (10-1)(-2)]$

$S_{10} = 5[8 + (9)(-2)]$

$S_{10} = 5(8 - 18)$

$S_{10} = 5(-10)$

$S_{10} = -50$

2. Find the sum of the first 20 terms of the arithmetic sequence whose first term is 2 and whose common difference is 4.

Solution

$n = 20$, $a = 2$, and $d = 4$

Substitute in $S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{20} = \frac{20}{2}[2(2) + (20-1)4]$

$S_{20} = 10[4 + (19)4]$

$S_{20} = 10(80)$

$S_{20} = 800$

3. The sum of the first ten terms of an A.P. is 351 and the tenth term is 51. Find the first term and the common difference.

Solution

$S_{10} = 351$, $a = ?$, $a_{10} = 51$, $n = 10$, $d = ?$

$S_n = \frac{n}{2}[a + U_n]$

$351 = \frac{10}{2}[a + 51]$

$351 = 5(a + 51)$

$351 = 5a + 255$

$351 - 255 = 5a$

$96 = 5a$

$A = 19.2$

To find the value of d, use the formula;

$a_n = a + (n-1)d$

$a_{10} = 19.2 + (10-1)d$

$a_{10} = 19.2 + 9d$

$51 = 19.2 + 9d$ **But $a_{10} = 51$**

$51 - 19.2 = 9d$

$31.8 = 9d$

$d = 3.5$

4. Find the sum of the first 20 terms of the linear sequence 3, 5, 7, 9…

Solution

From the linear sequence: 3, 5, 7, 9…

$a = 3$, $d = 5 - 3 = 2$ and $n = 20$

Substitute in $S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{20} = \frac{20}{2}[2(3) + (20-1)2]$

$S_{20} = 10[6 + (19)2]$

$S_{20} = 10(44)$

$S_{20} = 440$

Exercises 25.6 A

1. Find the sum of the first 20 terms of the linear sequence 4, 2, 0, -2…

2. Find the sum of the A.P. 3, 6, 9 … 300.

3. How many terms of the A.P;. 15 + 13 + 11 +… are required to make a total of -36.

4. Given the arithmetic sequence:
$y - 3$, $2y - 4$ and $23 - y$
i. Determine the value of y.

ii. Find the sum of the first 26 terms of the sequence.

5. The first three terms of an arithmetic sequence are $2k - 7$, $k + 8$ and $2k - 1$.
i. Calculate the value of the 15^{th} term of the sequence.
ii. Calculate the sum of the first 30 even terms of the sequence

Exercises 25.6B

1. The sixth and eleventh terms of a linear sequence are respectively are 23 and 48. Calculate the sum of the first 20 terms of the sequence.
2. The fifth term of an A.P. is 17 and the third term is 11, Find the sum of the first seven terms.

3. The first and thirtieth terms of an A.P. are 71 and -16 respectively. Find the sum of the first 50 terms of the sequence.

4. The sum of the first three terms of an A.P. is 3 and the sum of the first five terms is 20. Find the first five terms of the progression.

5. The fourth term of an A.P. is 15 and the sum of the first five terms is 55. Find the first term and the common difference, and write down the first five terms.

6. The sum of a number of consecutive terms of an A.P. is $-\frac{39}{2}$, the first term is $\frac{17}{2}$ and the common difference is -3. Find the number of terms.

The n^{th} Term of a Geometric Progression (G.P) or Exponential sequence

It is the type of sequence in which each term after the first is obtained by a multiplying the preceding term by a fixed/constant number or common factor called ***common ratio, r.***

Consider the sequence: 1, 3, 9, 27, 81…,
It is seen that each term is obtained by a multiplication of the preceding digit by 3;
1. The sequence is said to have five terms, denoted by n. Thus $n = 5$

2. Each term is represented by U. Therefore, 1, 3, 9 , 27, …n = U_1 , U_2 ,U_3 , U_4,… U_n
In this case:
U_1 is called the first term of the sequence,
U_2 is called the second term of the sequence
U_3 is called the third term of the sequence U_4 is called the fourth term of the sequence
U_n is called the *nth* term of the sequence and of course the last term of the sequence

3. The digit representing the first term is denoted by a. That is: $U_1 = a$

4.$\frac{U_2}{U_1} = \frac{U_3}{U_2}$is called the common ratio denoted by r.
i.e. $\frac{U_2}{U_1} = \frac{U_2}{U_1} = \frac{U_3}{U_2} = r$. The common ratio, r, can be positive or negative

5. The sum of the first n terms of a geometric sequence is denoted by S_n
Thus, for U_1, U_2, U_3, U_4, … U_n ,
$S_n = U_1 + U_2 + U_3 + U_4 + \ldots U_n$
(Note U_n = L, last term)

If any term is obtained from the previous term by multiplying the common ratio, r, then it implies that if:
$U_1 = a$
$U_2 = U_1r = U_1\, r^{2-1} = ar$
$U_3 = U_2r = U_1\, r^{3-1} = ar^2$
$U_4 = U_3r = U_1\, r^{4-1} = ar^3$

$U_5 = U_4r = U_1\, r^{5-1} = ar^4$

$\mathbf{U}_{\boldsymbol{n}} = \boldsymbol{U_1\, r^{n-1} = a\, r^{n-1}}$

$\mathbf{U}_{\boldsymbol{n}} = \boldsymbol{L = a\, r^{n-1}}$

Therefore, the *nth* term of a geometric sequence denoted by U_n is given by; $\mathbf{U}_{\boldsymbol{n}}\boldsymbol{= a\, r^{n-1}}$

Worked Examples

1. Find the 8th term of the geometric sequence; 12, 24, 48…

Solution

From the sequence:12, 24, 48…

$n = 8$, $a = 12$ and $r = \frac{24}{12} = 2$

Substitute in $U_n = ar^{n-1}$

$U_8 = 12\ (2)^{8-1} = 12 \times 2^7 = 1{,}536$

2. Find the ninth term of the G.P: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8} \dots$

Solution

From the sequence:$\frac{1}{2}, \frac{1}{4}, \frac{1}{8} \dots$

$n = 9$, $a = \frac{1}{2}$, and $r = \frac{1}{4} \div \frac{1}{2} = \frac{1}{4} \times \frac{2}{1} = \frac{1}{2}$

Substitute in $U_n = ar^{n-1}$

$U_9 = \frac{1}{2}\left(\frac{1}{2}\right)^{9-1} = \frac{1}{2}\left(\frac{1}{2}\right)^{8} = \left(\frac{1}{2}\right)\left(\frac{1}{256}\right) = \frac{1}{512}$

3. Find the 8th term of the geometric sequence:

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$

Solution

From the geometric sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$

$a = 1$, $r = \frac{1}{4} \div \frac{1}{2} = \frac{1}{4} \times \frac{2}{1} = \frac{1}{2}$ and $n = 8$

Substitute in $U_n = ar^{n-1}$

$U_8 = 1\left(\frac{1}{2}\right)^{8-1} = \left(\frac{1}{2}\right)^{7} = \frac{1}{128}$

4. The first term of a G.P. is 20 and the constant ratio is $\frac{5}{4}$. Find the sequence.

Solution

Generally, a G.P. is of the form a, ar, ar^2, ar^3…where a is the first term and r is the common ratio

$\Rightarrow a = 20$

$a_2 = 20 \times \frac{5}{4} = 25$

$a_3 = 20 \times \left(\frac{5}{4}\right)^2 = \frac{125}{4}$

$a_4 = 20 \times \left(\frac{5}{4}\right)^3 = \frac{625}{16}$

Therefore, the sequence is; 20, 25, $\frac{125}{4}, \frac{625}{16}$

Exercises 25.7

A. For each geometric sequence, find the constant ratio and the indicated term

1.4, 12, 36…(U_{12})

2. $\frac{1}{2}, \frac{3}{4}, \frac{9}{8}$ ……(U_{20})

3. 16, 64,256…(U_8)

4. 3, $-\frac{3}{2}$, $\frac{3}{4}$….(U_{10})

B. 1. Find a formula for the *n*th term of the geometric sequence with $r = 2$ and $a_1 = 3$

2. Find the *n*th term of the geometric sequence, $a = \frac{1}{4}$, $n = 10$, $r = 2$

3. Find a formula for the *n*th term of the geometric sequence: 192, – 48, 12, – 3

4. For the geometric sequence: 5, -20, -80, find U_9

5. In the following G.P, find the value of y : 3, 12, 48, $5y + 7$

A Finite Geometric Progression

A finite geometric progression is the one that has a given last term. For all finite G.P:

$\boldsymbol{U_n = ar^{n-1} = L}$

Worked Examples

1. Find the number of terms of the G.P. 2 + 4 + 8…+ 512

Solution

From the G.P., 2 + 4 + 8…+ 512,

$a = 2$, $r = \frac{4}{2} = 2$, L = 512 = U_n

$U_n = ar^{n-1}$

$512 = 2(2)^{n-1}$

$\frac{512}{2} = (2)^{n-1}$

$256 = (2)^{n-1}$

$2^8 = 2^{n-1}$

$8 = n - 1$

$n = 8 + 1 = 9$

The number of terms is 9

2. Find the number of terms of the G.P.

$-\frac{1}{4}, -\frac{1}{5}, -\frac{4}{25}.....-\frac{256}{3125}$

Solution

G.P.= $-\frac{1}{4}, -\frac{1}{5}, -\frac{4}{25}.....-\frac{256}{3125}$

$a = -\frac{1}{4}$, $r = -\frac{1}{5} \div -\frac{1}{4} = \frac{4}{5}$, L = $-\frac{256}{3125}$, n = ?

Substitute in ar^{n-1} = L

$-\frac{1}{4}\left(\frac{4}{5}\right)^{n-1} = -\frac{256}{3125}$

$-4 \times -\frac{1}{4}\left(\frac{4}{5}\right)^{n-1} = -\frac{256}{3125} \times -4$

$\left(\frac{4}{5}\right)^{n-1} = \frac{1024}{3125}$

$\left(\frac{4}{5}\right)^{n-1} = \left(\frac{4}{5}\right)^{5}$

$n - 1 = 5$

$n = 5 + 1 = 6$

Exercises 25.7B

1. Find the number of terms of the geometric seqence: 0.03, 0.06, 0.12 … 1.92.

2. In the geometric sequence; $-\frac{1}{3}, -\frac{1}{4} - \frac{3}{16}$ …. $-\frac{729}{16384}$, determine the number of terms.

3. Determine the number of terms of the geometric sequence, - 4 , 2, -1…$\frac{1}{2048}$.

4. The first term of a geometric sequenece is 2 , and the common ratio is 3, find the fifth term.

5. The first term of a geometric sequenece is 16 , and the common ratio is $-\frac{1}{3}$, find the fouth term.

Unknown First Term of a G.P, Given a Term and the Common Ratio,

Given the value of a term other than the first term and the common ratio, r,

I. Write an expression for the given term and the common ratio.

II. Equate the expression to the given value of the term.

III. Solve for the value of a in the equation to obtain the first term of the G.P.

Worked Examples

1. The third term of a G.P is 5 and the common ratio is $\frac{1}{2}$. Find the first term of the G. P.

Solution

$U_n = ar^{n-1}$

$n = 3$, $r = \frac{1}{2}$ and $U_3 = 5$

$U_3 = a\left(\frac{1}{2}\right)^{3-1} = a\left(\frac{1}{2}\right)^2 = a\left(\frac{1}{4}\right)$

$\Rightarrow a\left(\frac{1}{4}\right) = 5$ (But $U_3 = 5$)

$\frac{a}{4} = 5$

$a = 4 \times 5 = 20$

Exercises 25.8

1.Find the first term of the G.P whose fourth term is 9 and common ratio, $\frac{1}{3}$.

2. The third term of a G.P is -8 and the common ratio is 4. Find the first term.

3. A geometric sequence has a fifth term of 6 and a common ratio $-\frac{2}{3}$. What is the first term of the sequence?

4. Determine the value of the first term of a G.P. whose sixth term is -10 and constant ratio, -3.

5. A geometric sequence has a common ratio $-\frac{3}{5}$. If the fourth term of the sequence is $\frac{5}{6}$, find the first term of the sequence.

Finding the First Term, a, and the Common Ratio, r, of a G.P., Given the Values of Two or More Terms

Given the values of two or more terms of a G. P. the first term, *a*, and the common ratio, *r*, can be calculated as follows:

I. Identify the values of the given terms

II. Write two (or more) equations, each for the given terms

III. Solve the equations simultaneously to determine the respective values of *a* and *r*

Worked Examples

1. The fifth and ninth terms of an exponential sequence are 16 and 256 respectively. Find:

i. the first term and the common ratio ,

ii. the seventh term.

Solution

i.$U_5 = a\,r^{5-1}$

$U_5 = a\,r^4$

But $U_5 = 16$

$\Rightarrow 16 = ar^4$……………….(1)

$U_9 = ar^{9-1}$

$U_9 = ar^8$

But $U_9 = 256$

$\Rightarrow 256 = ar^8$……………….(2)

eqn (2) ÷ eqn (1)

$\frac{256}{16} = \frac{ar^8}{ar^4}$

$16 = r^4$

$r = \sqrt[4]{16}$

$r = 2$

Put $r = 2$ in eqn (1)

$16 = a(2)^4$

$16 = 16a$

$a = 1$

The first term is 1 and the common ratio is 2

b. The seventh term,

$U_7 = a\,r^{7-1}$

$U_7 = ar^6$

But $a = 1$ and $r = 2$

$U_7 = (1)\,(2)^6 = 2^6 = 64$

2. The second and fourth terms of an exponential sequence of positive terms are 9 and 4 respectively. Find:

i. the common ratio,

ii. the first term.

Solution

i. $U_n = ar^{n-1}$

$U_2 = ar^{2-1}$

$U_2 = ar$

But$U_2 = 9$

$\Rightarrow 9 = ar$ ……….(1)
$U_4 = ar^{4-1}$
$U_4 = ar^3$

But$U_4 = 4$
$\Rightarrow 4 = ar^3$ ……….(2)
eqn (1) ÷ eqn (2)
$\frac{4}{9} = \frac{\boldsymbol{ar^3}}{\boldsymbol{ar}}$
$\frac{4}{9} = r^2$
$r = \sqrt{\frac{4}{9}} = \frac{2}{3}$

ii. Put $r = \frac{2}{3}$ into eqn (1)
$9 = \frac{2}{3}\ (a)$
$9 \times 3 = 2a$
$a = \frac{9 \times 3}{2} = 13.5$
Therefore the first term is 13.5

3. The third and sixth terms of an exponential sequence are $\frac{1}{4}$ and $\frac{1}{32}$ respectively. Find:
i. the first term and the common ratio,
ii. the 8th term of the sequence.

Solution
$U_n = ar^{n-1}$
$U_3 = ar^{3-1}$
$U_3 = ar^2$
But$U_3 = \frac{1}{4}$
$\Rightarrow \frac{1}{4} = ar^2$……….(1)

$U_6 = ar^{6-1}$
$U_6 = ar^5$
But$U_6 = \frac{1}{32}$
$\Rightarrow \frac{1}{32} = ar^5$……….(2)

eqn (2) ÷ eqn (1)

$\frac{1}{32} \div \frac{1}{4} = \frac{\boldsymbol{ar^5}}{\boldsymbol{ar^2}}$
$\frac{1}{8} = r^3$
$\left(\boldsymbol{\frac{1}{2}}\right)^{\boldsymbol{3}} = r^3$
$\Rightarrow r = \frac{1}{2}$

Put $r = \frac{1}{2}$ into eqn (1)
$\frac{1}{4} = a\left(\boldsymbol{\frac{1}{2}}\right)^{\boldsymbol{2}}$
$\frac{1}{4} = \frac{1}{4}a$
$4a = 4$
$a = 1$
The first term is 1 and the common ratio is $\frac{1}{2}$

ii. The eighth term $U_8 = ar^{8-1}$
But $a = 1$ and $r = \frac{1}{2}$
$U_8 = 1\left(\frac{1}{2}\right)^{8-1} = \left(\frac{1}{2}\right)^7 = \frac{1}{128}$

Exercises 25.9
1. The fifth and ninth terms of an exponential sequence are 16 and 256 respectively. Find:
i. the first term and the common ratio,
ii. the sixth term.

2. The first and sixth terms of an exponential sequence are $13\frac{1}{2}$ and $\frac{1}{18}$. Find the commom ratio.

3. The sum of the first two terms of a G.P.is 3, and the sum of the second and third terms is – 6. Find the first term and the commom ratio.

Sum of a Geometric Progression or G.P
Consider the geometric sequence with first term, a and common ratio, r :
$a,\ ar, ar^2,\ ar^3,\ ar^{n-1},$
Suppose we want to find the sum of the GP:
$S_n = a + ar + ar^2 + ar^3 \ldots. + ar^{n-1}$…(1)
Multiply both sides of the equation by r

$rS_n = ar + ar^2 + ar^3 \ldots + ar^n \ldots\ldots(2)$

eqn (1) – eqn (2)

$S_n - rS_n = a - ar^n$

Factorize both sides of the equation

$S_n(1 - r) = a(1 - r^n)$

$S_n = \frac{a(1 - r^n)}{1 - r}$

On the other hand, eqn (2) – eqn (1)

$rS_n - S_n = ar^n - a$

Factorize both sides of the eqnto obtain

$S_n(r - 1) = a(r^n - 1)$

Make S_n the subject to obtain

$S_n = \frac{a(r^n - 1)}{r - 1}$

Therefore, the sum of the first n terms of a G.P. with first term, a, and common ratio, r is given by:

1. $S_n = \frac{a(1 - r^n)}{1 - r}$, for $r < 1$

2. $S_n = \frac{a(r^n - 1)}{r - 1}$, for $r > 1$

Worked Examples

1. Find the sum of the first ten terms of the G.P whose first term is $\frac{1}{2}$ and common ratio, 2

Solution

$a = \frac{1}{2}$, $r = 2$, $n = 10$

$S_n = \frac{a(r^n - 1)}{r - 1}$, for $r > 1$

$S_{10} = \frac{\frac{1}{2}(2^{10} - 1)}{2 - 1} = \frac{\frac{1}{2}(1024 - 1)}{1} = \frac{1}{2}(1023) = 511.5$

2. An exponential sequence is given by $\frac{1}{8}$, $\frac{1}{2}$, 2. Find:

i. the common ratio,

ii. the fifth term,

iii. the sum of the five term.

Solution

i. G. P = $\frac{1}{8}, \frac{1}{2}, 2$

$U_1 = \frac{1}{8}$, $U_2 = \frac{1}{2}$, $U_3 = 2$

$r = \frac{U_2}{U_1} = \frac{1/2}{1/8} = 4$

ii. $a = \frac{1}{8}$, $r = 4$

$U_5 = ar^{n-1}$

$U_5 = \frac{1}{8} \times 4^{5-1} = \frac{1}{8} \times 4^4 = \frac{1}{8} \times 256 = 32$

iii. $a = \frac{1}{8}$, $r = 4$, and $n = 5$

$S_n = \frac{a(r^n - 1)}{r - 1}$, for $r > 1$

$S_5 = \frac{\frac{1}{8}(4^5 - 1)}{2 - 1} = \frac{\frac{1}{8}(1024 - 1)}{1} = 127.88$

3. Find the sum of the first eight terms of the G.P. : 5 + 15 + …

Solution

$a = 5$, $r = \frac{15}{5} = 3$, $n = 8$

$S_n = \frac{a(r^n - 1)}{r - 1}$, for $r > 1$

$S_8 = \frac{5(3^8 - 1)}{3 - 1} = \frac{5(6561 - 1)}{2} = \frac{5(6560)}{2} = 16{,}400$

Some Solved Past Questions

The sum of the first and third terms of a G.P is 40 while the fourth and the sixth terms are in the ratio 1: 4. Find the :

i. common ratio, ii. the fifth term.

Solution

i. First term, $U_1 = a$

Third term, $U_3 = ar^2$

Sum of first and third terms,

$a + ar^2 = 40$………………..(1)

Fourth term, $U_4 = ar^3$
Six term, $U_6 = ar^5$

Fourth term: six term
$ar^3 : ar^5 = 1 : 4$
$\frac{ar^3}{ar^5} = \frac{1}{4}$
$\frac{1}{r^2} = \frac{1}{4}$
$r^2 = 4$
$r = \sqrt{4} = 2$ or - 2

ii. $a + ar^2 = 40$………………..(1)
$a(1 + r^2) = 40$
But $r^2 = 4$
$a(1 + 4) = 40$
$5a = 40$
$a = 8$
Therefore, the first term is 8

Exercises 25.10

1. Find the sum of the given geometric series $n = 20$, $r = -2$ and $a_1 = 1$

2. Calculate the sum of the six terms of the geometric sequence: 2 + 12 + … + 972

3. Find the sum of the first ten terms of the geometric series: 18 + 6 + 2 + ….

4. The third term of a G.P. is 10 and the sixth term is 80. Find the common ratio, the first term and the sum of the first six terms.

Word Problems

For word problems involving series and sequence,
I. Carefully read the problem.
II. Identify whether the problem is a series or a sequence or both.
III. Form a mathematical statement for the problem using series and sequence notations.
IV. Workout the problem and provide the required answer.

Worked Examples

1. A worker started saving Gh¢2,500.00 the first year, Gh¢5,000.00 the second year, Gh¢10,000.00 the third year and continued doubling his savings each year. How much will be saved in 11th year?

Solution

Gh¢2,500, Gh¢5,000, Gh¢10,000…..
A a = 2,500, $r = \frac{5,000}{2,000} = 2$, $n = 11$
$U_n = ar^{n-1}$
$U_{11} = 2,500(2)^{11-1}$
$U_{11} = 2,500(2)^{10}$
$U_{11} = 2,500(2)^{10}$
$U_{11} = 2500 \times 1024$
$U_{11} = 2,560,000$

2. Mr. Jimmy starts a job with an annual salary of Gh¢2,500.00 which increases by Gh¢500.00 every year. After working for 6 years, Mr. Jimmy is promoted to a new position associated with a new annual salary of Gh¢4,200.00 which increases by Gh¢220.00 every year. Calculate;
i. Mr. Jimmy's annual salary in the tenth year of service.
ii. his total earnings at the end of the tenth year of service.

Solution

i. $a = 2,500$, $d = 500$, $n = 6$
$U_n = a + (n – 1) d$
$U_6 = 2,500 + (6 – 1) 500$

$U_6 = 2{,}500 + 500(5)$

$U_6 = 5{,}000$

Annual salary at the end of the sixth year is Gh¢5,000.00

For the next four years

$a = 4{,}200$, $d = 220$, $n = 4$

$U_4 = a + (n - 1)\,d$

$U_4 = 4{,}200 + (4 - 1)\,220$

$U_4 = 4{,}200 + (\,3)\,220$

$U_4 = 4{,}860$

Mr. Jimmy's annual salary in the tenth year of service is Gh¢4,860.00

ii. His total earnings at the end of the tenth year of service $= S_6 + S_4$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_6 = \frac{6}{2}[2(2{,}500) + (6-1)500]$

$S_6 = 3 \times 7500 = 22{,}500$

$S_4 = \frac{4}{2}[2(4{,}200) + (4-1)220]$

$S_4 = 2 \times 9060 = 18{,}120$

But total earnings $= S_6 + S_4$

$= 22{,}500 + 18{,}120$

= Gh¢40,620.00

Exercises 25.11

1. The sum of the interior angles of a triangle is 180º, of a quadrilateral is 360º and of a pentagon is 540º. Assuming this pattern continues, find the sum of the interior angles of a dodecagon (12 sides).

2. A mine worker discovers an ore sample containing 500 mg of radioactive material. It is discovered that the radioactive material has a half-life of 1 day. Find the amount of radioactive material in the sample at the beginning of the 7th day.

3. A culture of bacteria doubles every 2 hours. If there are 500 bacteria at the beginning, how many bacteria will there be after 24 hours?

4. Tim earned Gh¢105.00 on the first day. If he earned two times the amount of money earned the day before each day, how much did he earn in the first 7 days?

5. A man starts savings on first April. He saves Gh¢1.00 the first day, Gh¢2.00 the second day, Gh¢4.00 the third day, and so on doubling the amount every day. If he managed to keep on saving under this system until the 15th of that month, how much would he have saved?

6. Doris was left a legacy of Gh¢600,000.00 by his father. She decided to let her children have a share of it but they have to earn it by catching flies. For the first fly they caught, she gave them Gh¢1.00, for the second fly, she gave them Gh¢2.00, for the third fly, Gh¢4.00, for the fourth Gh¢8.00 and so on. By the end of the day, her children had caught20 flies. How much of the Gh¢600,000.00 was left for herself?

7. If a lab technician has a salary of Gh¢22,000.00 her first year and is due to get a Gh¢500.00 raise each year;

i. what will her salary be in the seventh year?

ii. what is the total salary for seven years of work of the technician?

8. On the first day of October, a teacher suggests to his students that they read five pages of a novel and everyday thereafter, increases their daily reading by two pages. How many pages will they

read during October if the students follow the instruction?

9. If an air conditioning system is not completed by the agreed upon date, the contractor pays a penalty of Gh¢500.00 for the first day that it is over due,Gh¢600.00 for the second day, Gh¢700.00 for the third day and so on. If the system is completed 10 days late, then what is the total amount of penalties that the contractor must pay?

10. Mr. Jimmy starts a job with an annual salary of Gh¢3,000.00 which increases by Gh¢400.00 every year. After working for 10 years, Mr. Jimmy is promoted to a new position associated

with a new annual salary of Gh¢7,500.00 which increases by Gh¢360.00 every year. Calculate;
i. Mr. Jimmy's annual salary in the tenth year of service
ii. His total earnings at the end of the tenth year of service

Challenge Problem
Consider yourself, your parents, your grand parents, your great – grand parents, your great – great grand parents, and so on, back to your grand parents with the word "great" used in front 40 times. What is the total number of people you are considering?

26 GEOMETRICAL CONSTRUCTION

Introduction

A mathematical set which contains the required instruments is needed for geometrical constructions. Most at times students are expected to use a pair of compass and a ruler only.

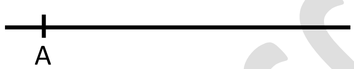
Construction of a Line Segment

A line segment is a straight line that has two endpoints. That is to say, it starts from one point, say ($\boldsymbol{A}$) and ends at another point say ($\boldsymbol{B}$).

To construct a line segment $\boldsymbol{AB}$ of length 4 cm.

i. With a ruler and a pencil, draw a straight line which is presumably more than 4cm

ii. With a ruler and a pencil, cut the line towards the left end and label that point A



iii. Measure a distance of 4cm on the ruler and with the compass point at A draw an arc to cut the line at the right end and label that point B.

Thus $|AB|$ = 4cm above is the required line segment

Note: *Do not rub the extensions.*

Bisecting a Line

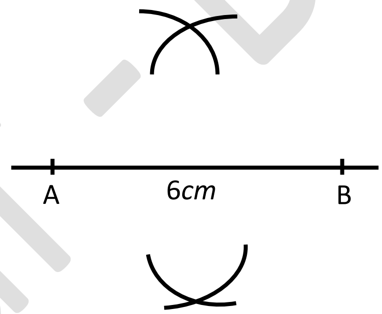
To bisect a line means to divide the line into two equal halves.

To bisect a line $|AB|$ = 6cm.

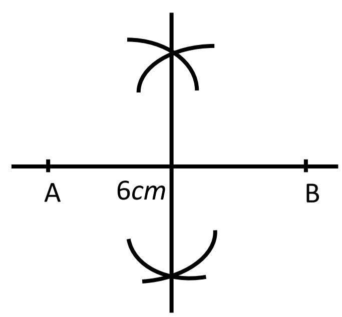
1. Draw $|AB|$ = 6cm

2. with the pin – point of your compass at A and a distance more than half of the line AB draw an arc above the line and another arc below the line.

3. Maintaining the same distance, set the compass point at B and construct an arc
above and below the line to cut the first pair of arcs drawn.



4. Join the points of intersection of the arcs with a straight line to get the bisector of line AB.



Exercises 26.2

Bisect the following line segments.

1. $|BC|$ = 8.5cm
2. $|MN|$ = 8cm
4. $|CD|$ = 7.5cm
5. $|GH|$ = 7cm

Construction of a Perpendicular at a Given Point on a Straight Line (AB)

I. Draw the required straight line AB.

II. Mark the point says M on the line.

III. With the compass point at M and at any convenient radius, construct two arcs to cut the line at the left (c) and right (d).

IV. Using any suitable radius, step on the left arc (c) and right arc (d) respectively and draw, two arcs above the line to intersect (meet) with each other at E.

V. Draw a line from E to the point M to get the required perpendicular line.

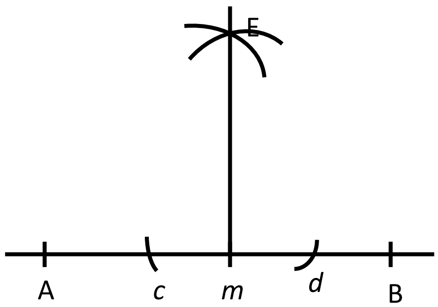


Fig. 1

Exercises 26.3

Construct a perpendicular at the given points on the following lines.

1. $|AB| = 7cm$, $m = 2cm$ from B.
2. $|PQ| = 8cm$, $n = 3cm$ from P.
3. $|PQ| = 8cm$, $n = 3cm$ from Q.
4. $|MN| = 6.5cm$, $r = 2.5cm$ from M.

Construction of a Perpendicular at the End of a Given Line (AB)

1. Draw the given line AB

2. To draw the perpendicular at the end A, extend the line to the left of A.

3. With the compass point at A and at any convenient radius, construct a semi- circle below the line to cut the line at C and D respectively.

4. Using any suitable radius step at C and D respectively and construct two arcs above the line CD to meet at E.

5. Join the arcs and the point A with a straight line to get the required perpendicular at the end point A.

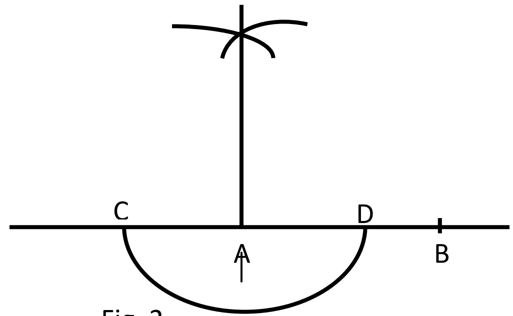


Fig. 2

Exercises 26.4

Construct a perpendicular at the end of the following lines.

1.$|AB|$= 6cm at A 2.$|AB|$= 6cm at B.

3. $|PQ|$ = 7.5cm at Q 4. $|PQ|$ = 7.5cm at P

5.$|MN|$ = 7cm at M 6.$|MN|$ = 7cm at N

Construction of a Perpendicular from a Given Point (K) to a Given Line AB

1. Choose any point say, K above the line AB.
2. With the compass pin at K, draw an arc to cut line AB at C and D.
3. With C as the centre and any convenient radius draw an arc below line AB, with the same radius and D as the centre , construct another arc below line AB to meet the first arc at point E.
4. Join K and the arcs at E with a straight line to obtain the perpendicular from K to AB.

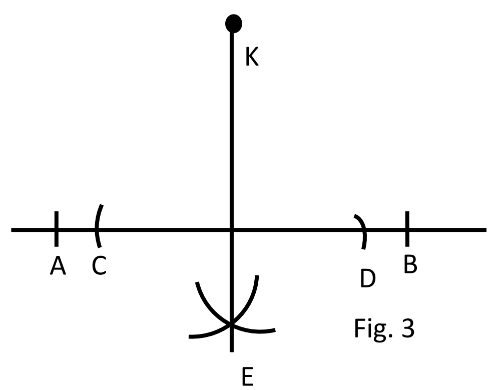


Fig. 3

Constructing a Line Parallel to a Given Line AB

1. Construct the line AB and mark a point K outside (on top of) AB.

2. Construct a perpendicular from K to line AB as in figure 3.

3. Extend the line KE through K and step at K with a convenient radius , mark two points above (L) and below (M) the line KM.

4. Bisect the line segment LM and identify the meeting points of the arcs as C and D.

5. Draw a line through C and D. Line CD is therefore the line parallel to AB.

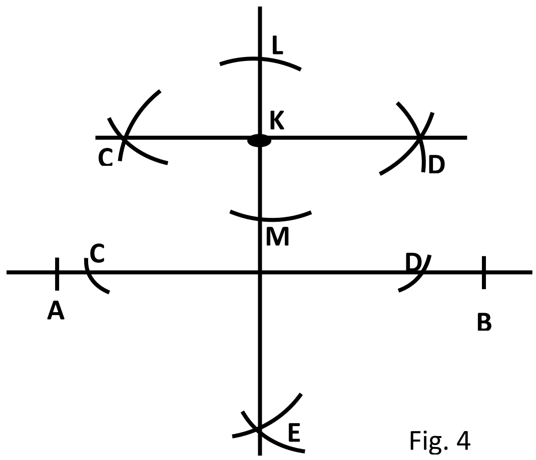


Fig. 4

Exercises 26.5

Construct a line parallel to the following lines.

1. $\lvert AB\rvert$ = 7.5cm	4. $\lvert PQ\rvert$ = 9cm
2. $\lvert MN\rvert$ = 8.5cm	5. $\lvert ST\rvert$ = 6.5cm
3. $\lvert GH\rvert$ = 8cm	6. $\lvert KL\rvert$ = 7cm

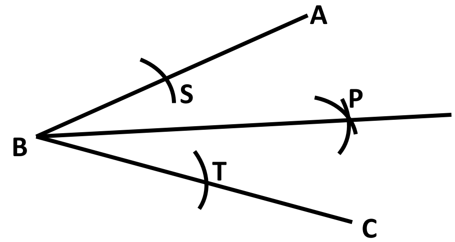
Angles

Bisecting an Angle

To bisect an angel means to divide the angel into two equal halves.

Steps:

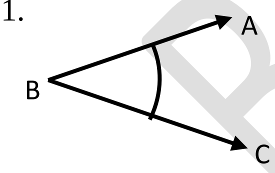
1. Draw an angle ABC.
2. With the compass point at B draw an arc to cross AB and BC at S and T respectively.
3. With the compass point at S and a suitable radius construct an arc inside the angel ABC.
4. With the same radius, place the point at T and construct another arc to meet the first one at P.
5. Draw a straight line from B through P. BP is the angel bisector of ABC.



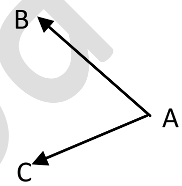
Exercises 25.6

Bisect the following angles;

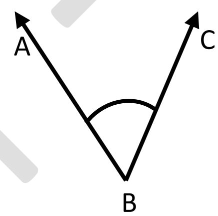
1.



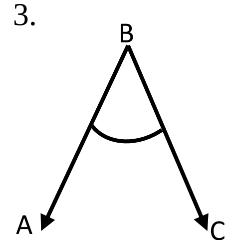
2.



2.



3.



Construction of Angles (90^0, 45^0, 120^0, 60^0, 30^0, 15^0, 75^0, 105^0) at a Given Point Using a Ruler and a Pair of Compasses Only

1. ***Construction of 90^0 and 45^0 at the point A on line AB***

i. Draw the line segment AB with extensions.

ii. With A as centre and at any convenient radius, draw a semi–circle on top of the horizontal line to cut it at C and D.

iii. With the compass point at C and the same radius, draw an arc to cut the semi – circle at E.

iv. With the compass point at D and the same radius (as in ii & iii) draw an arc to cut the semi – circle at F.

v. With the pin at E and F respectively and at any convenient radius, construct two arcs above the semi – circle using the same radius, so that the arcs intersect with each other at.

vi. Draw a straight line through the arcs at G to the point A to get an angel of 90^0at right.

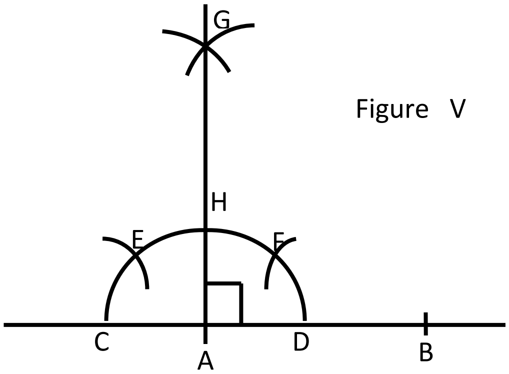
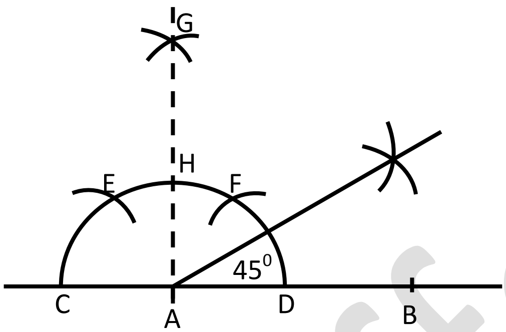


Figure V

To construct angle 45^0, construct 90^0 and bisect the 90^0 by placing the compass point at D and H



Exercises 26.7

Given that $|AB| = 5cm$, **construct** the following, all on separate diagrams

1. 90^0 at B
2. 90^0 at A
3. 45^0 at B
4. 45^0 at A,

2. Construction of $120^0$and 60^0 at a Point on Line AB

a. Draw the line AB with extensions.

b. With A as centre and at any convenient radius draw a semi – circle on top of the horizontal line to cut it at C and D.

c. With the compass point at C and the same radius, construct an arc to cut the semi- circle at E.

d. Maintaining the same radius, step at D and draw an arc to cut the semi – circle at F.

e. Draw a straight line from A to pass through E to obtain 120^0 at right.

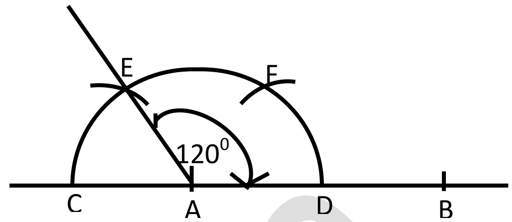


Figure VII.

f. To get 60^0, draw a line from A to pass through F.

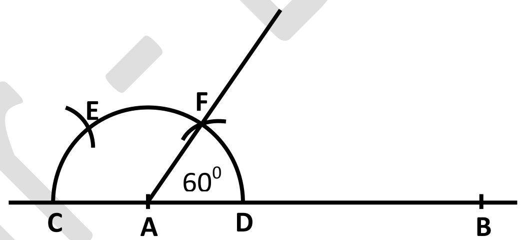


Figure VIII

3. Construction of 30^0 and 15^0 at a point A on line AB

i. Draw line AB and construct 60^0 at A.

ii. Construct a bisector of angle 60^0 by placing the compass point at D and F.

iii. A line from A through the arcs gives an angle of 30^0.

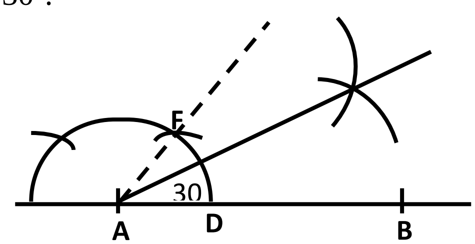


Figure IX

To construct 15^0, go through the process of constructing 30^0 and bisect the 30^0 to get 15^0 as shown below.

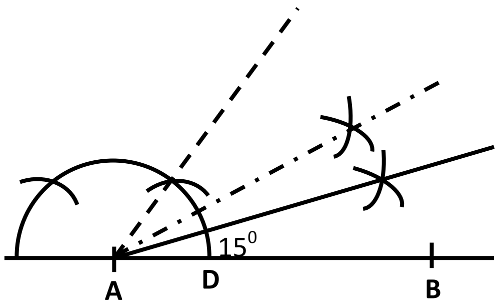


Figure X

Exercises 26.8

Construct the angles at the given point.

1.$|AB| = 5cm$,120^0 at A and B.

2. $|PQ| = 5.5cm$, 60^0 at P and Q..

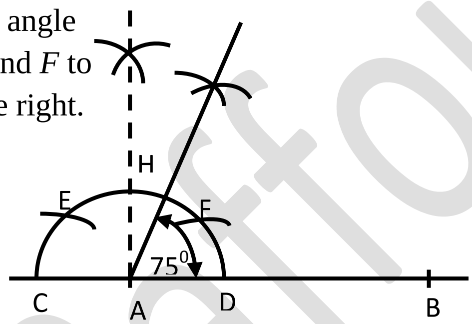
3. $|AB| = 5cm$, 30^0 at A and B.

4.$|PQ| = 7.5cm$, 15^0at P and Q.

4. Construction of 75^0 at a point A on line AB

a. Draw the line AB and construct 90^0 at A as shown below.

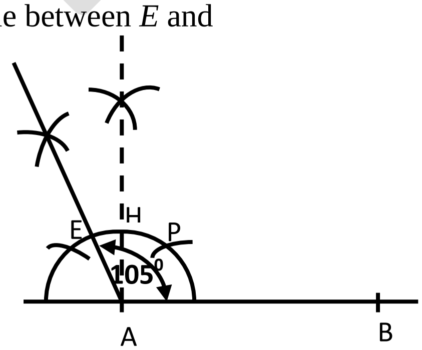
b. Bisect the angle between H and F to get 75^0 at the right.



5. Construction of 105^0 at a point A on a given line AB

i. Draw the line AB and construct 90^0 at A as shown in figure below.

ii. Bisect the angle between E and H to get 105^0 at the right.



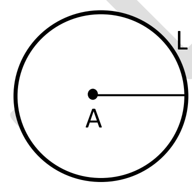
Exercises 26.9

Using a ruler and a compass only, construct the following angles at the given points on separate diagrams.

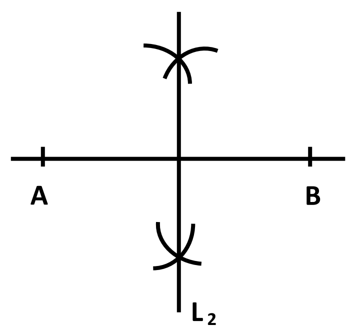
1. $|AB| = 6cm$, 75^0 at A and B
2. $|MN| = 6cm$, 75^0 at M and N
3. $|PQ| = 6cm$, 105^0at P and Q

The Idea of Locus

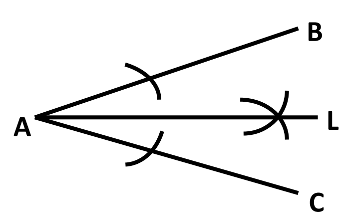
1. The locus of a point which is equidistance from one point means, draw a circle. Thus the locus of point, L_1 which is 6cm from A, means draw a circle with radius of 6cm from A, with A as centre.



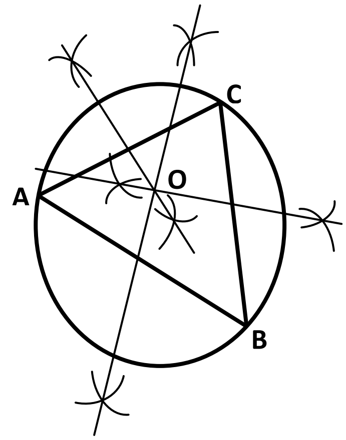
2. To construct locus of points L_2 which is equidistance from two points A and B means draw the bisector or the mediatorof AB as shown below.



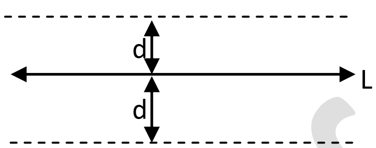
3. To construct the locus of points L_3 which is equidistance from lines AB and AC means, place the compass point at the common point A and bisect the angle between lines AB and AC.



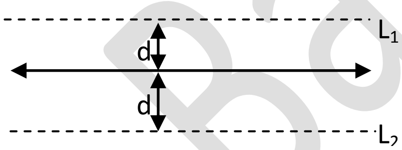
4. The locus of a point L_4 equidistance from three points *A*, *B* and *C* means bisect *AB*, *BC* and *AC* and use the point of intersection of the bisectorsto draw a circle through the three points *A*, *B* and *C*.



5. The locus of points at a fixed distance *d*, from a line line , L is a pair of parallel lines, *d* distance from L and on either side of L.



6. The locus of points equidistance from two parallel lines , L_1 and L_2 is a line parallel to both L_1 and L_2 and mid way between them.



Constructing Triangles

1. Equilateral triangles

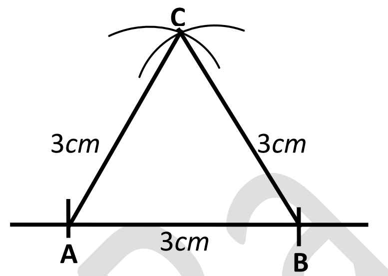
Set your compass to the length of the side of the required equilateral triangle. For example to draw an equilateral triangle of sides 3*cm*

I. Draw a line and mark a point *A*. Measure 3*cm*.

II. Place the point of the compass at *A* and draw a generous arc, 3*cm* from *A*, which cuts the line at *B*.

III. Maintaining the same radius, replace the point of the compass at *B* and draw another arc to cut the first at *C* .

IV. Join *A*, *B* and *C* to form the required triangle.



2. Isosceles triangles

In this case only two sides are equal. For example to draw a triangle *ABC* with sides 6*cm*, 6*cm* and 4*cm*.

I. Draw line *AB*.

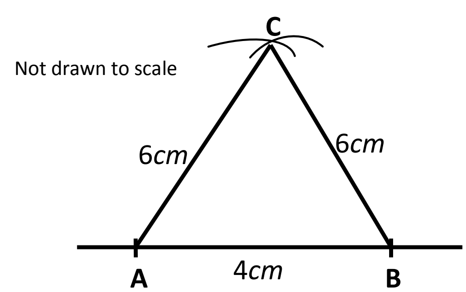
II. Open the compass to 4*cm* and mark the points *A* and *B* on the line so that /*AB*/ = 4cm

III. Set the compass to 6*cm*.

IV. Place the compass at *A* and draw one arc.

V. Maintaining the same distance, place the compass at *B* and draw another arc which cuts or meets the first at *C*.

VI. Draw lines *AC* and *BC*.



3. Scalene triangle

A scalene triangle has three unequal sides. For example to construct triangle *ABC* with sides 6cm, 5*cm* and 4*cm*.

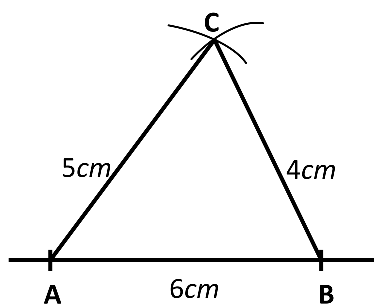
I. Draw a line and mark the point *A* near to one end of it.

II. With your compass set at 6*cm*, mark the point *B* on the line exactly 6*cm* from *A*.

III. Adjust your compass to 5*cm* and place the compass point at *A* and draw an arc above the line.

IV. Adjust your compass to 4*cm*, place the compass point at *B* and draw another arc to cut or meet the first one at *C*.

V. Join C to *A* and *B* to obtain the diagram shown below.



4. Right – Angled Triangles

It has one angle as 90^0. For example to construct a right-angled triangle of $|AB| = 6cm$, $|AC| = 8cm$ and the right angle at *C*.

I. Draw a straight line *AB*

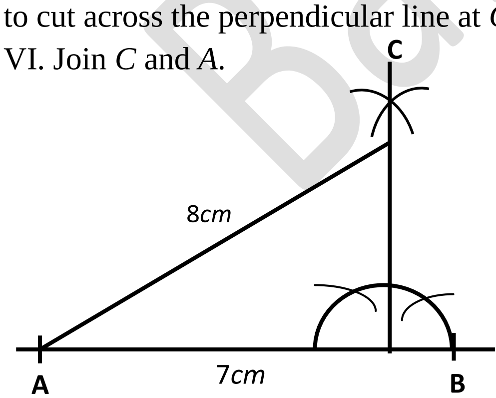
II. Using your compass mark point *B*, 8*cm* from *A* .

III. Construct a right angle at *B*.

IV. Open the compass to measure 8*cm*

V. Place the compass point at *A* and draw an arc to cut across the perpendicular line at *C*.

VI. Join *C* and *A*.



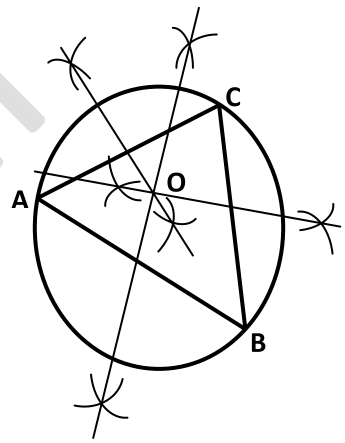
Exercises 26.10

Using a pair of compass and ruler only, construct the triangles with the following sides and name the type of triangle drawn in each case.

1. Triangle PQR with each side 7*cm*.
2. /AB/ = 8cm, /BC/= 7cm and /AC/ = 7cm
3./AB/ = 7cm, /BC/= 8cm and /AC/ = 8cm
4./AB/ = 9cm, /BC/= 7cm and /AC/ = 8cm

The Circum-circle

If the perpendicular bisectors of all the three sides of a triangle are constructed, they meet at a point O. With O as centre, a circle can be drawn to touch the three vertices of the triangle. The circle is said to be circum-scribed and it is called a ***"circum-circle".*** It'scentre is the circum-centre.



Worked examples

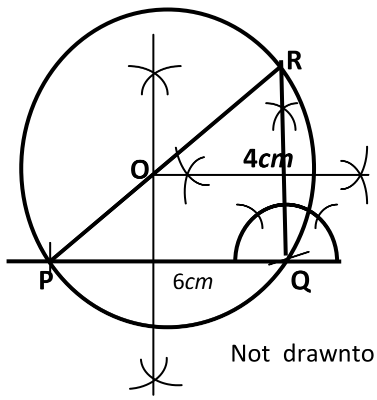
1. Using a ruler and a pair of compasses only, construct triangle *PQR* such that $|\boldsymbol{PQ}| = 6cm$, $|\boldsymbol{QR}| = 4cm$ and angle PQR = 90^0

ii. Construct the perpendicular bisectors of *PQ* and *QR* and name the intersection O.

iii. Draw a circle, with O as center and *OQ* is radius.

iv. Measure: (a)$|\boldsymbol{PR}|$(b) angle QPR

Solution



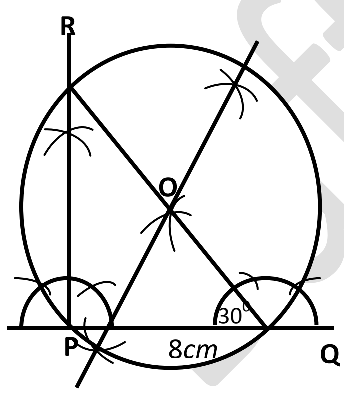
$|PR| = 7.2cm$, Angle QPR = 34^0

2. a. i. Using a ruler and a pair of compasses only, construct PQR such that $|PQ| = 8cm$ angle RPQ = 90^0 and angle PQR = 30^0, measure $|RQ|$.

ii. Construct the perpendicular bisector (mediator) of RQ. Let it meet RQ at O

b. With O as centre and radius OP, draw a circle. Measure $|OP|$

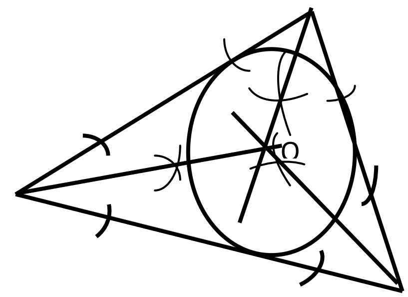
Solution



/*RQ*/= 9.4cm, and /*OP*/ = 4.4cm

The Inscribed Circle

When the angle bisectors of a triangle are constructed, the three bisectors meet at a point O. With O as centre, a circle can be drawn to touch all the three sides of the triangle. This is called the ***"inscribed circle".***



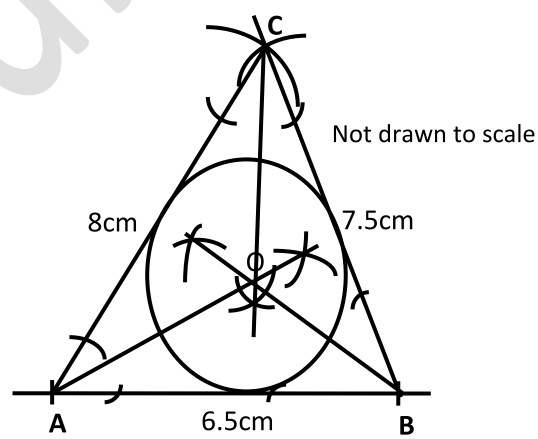
Worked examples

1. a. Using a ruler and a pair of compasses only, construct triangle *ABC* in which /*AB*/ = 6.5cm, /*BC*/ = 7.5cm and /*AC*/ = 8cm

b. Construct angle bisectors of angles *ABC*, *BCA* and *CAB* to meet at O

c. With O as centre, draw a circle to touch the sides of the triangle.

Solution



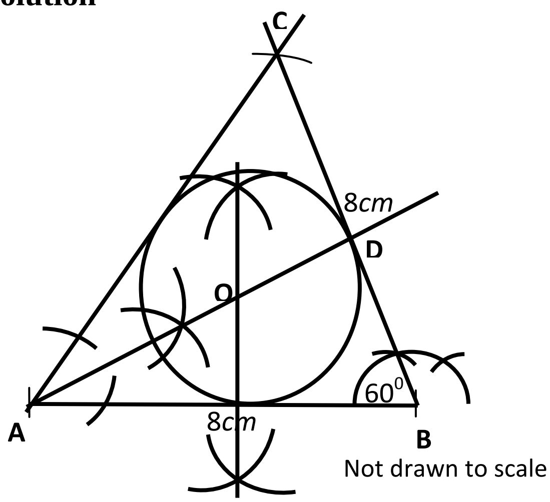
2. a. Using a ruler and a pair of compasses only, construct triangle *ABC* such that $|AB| = 8cm$, $|BC| = 8cm$ and angle *ABC* = 60^0. What type of triangle is triangle *ABC*?

b. Construct the bisector of angle *BAC* to meet *BC* at *D*. Measure$|AD|$

c. Construct the perpendicular bisector of *BA* to meet *AD* at O.

d. Using O as centre and radius OD, draw a circle to touch the three sides of the triangle.

Solution



a. Triangle ABC is an equilateral triangle and $|AD| = 7cm$.

Drawing Quadrilaterals

1. A square and a Rectangle

Set your compass to the length of the side of the required square. For example, to construct asquare $ABCD$ of sides 4.5cm

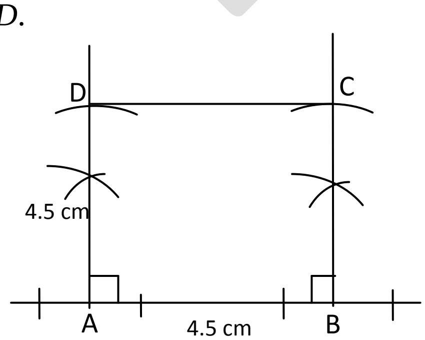
I. Construct a line segment AB = 4.5 cm

II. Construct perpendiculars at A and B

III. With A as centre and radius 4.5 cm, draw an arc to cut the perpendicular at A at the point D

IV. With B as centre and radius 4.5 cm, draw an arc to cut the perpendicular at B at the point C

V. Join C to D with a ruler to obtain square $ABCD$.



To construct a rectangle $ABCD$ of length 8 cm and breadth 6 cm;

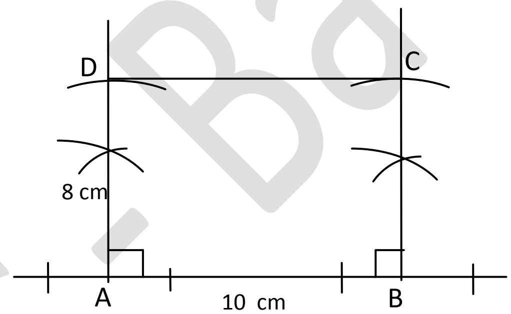
I. Draw a line segment AB = 8 cm

II. Construct perpendiculars at A and B

III. With A as centre and radius 8 cm, draw an arc to cut the perpendicular at A at the point D

IV. With B as centre and radius 8 cm, draw an arc to cut the perpendicular at B at the point C

V. Join C to D with a ruler to obtain square $ABCD$.



Exercises 26.11

1. Costruct a square ABCD of side 6.5 cm. Measure the length of the diagonal

2. Construc a rectangle with length 10 cm and breadth 7.5 cm. Measure the length of its diagonal

2. Constructing Parallelograms

a. Two Sides and the Length of the Diagonal.

For example, to draw parallelogram ABCD in which AB = 6 cm, BC = 4 cm and diagonal 7 cm

I. Draw AB = 6cm

II. With A as center and radius 7 cm, draw an arc

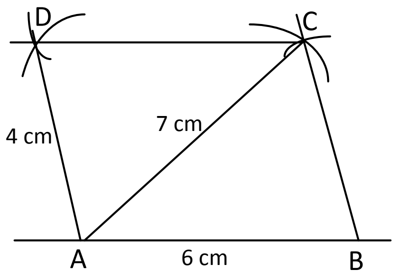
III. With B as center and radius 4 cm, draw another arc cutting the previus arc at C

IV. Join BC and AC

V. With A as center and radius 4 cm, draw an arc

VI. With Cas center and radius 6 cm, draw another arc cutting the previously drawn arc at D

VII. Join *DA* and *DC* to form parallelogram *ABCD*



b. One side and the length of Two diagonals

Note that the diagonals bisect each other. For example to construct parallelogram *ABCD* whose side is 5 cm and diagonals 6 cm and 6.6 cm

i. Draw *AB* = 5 cm

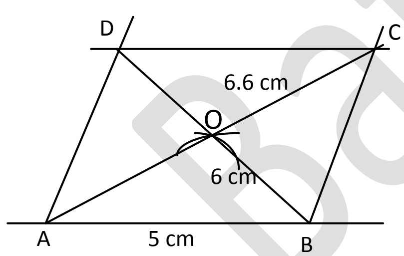
ii. With *A* as centre and radius 3.3 cm, draw an arc

iii. With *B* as center and radius 3 cm, draw another arc cutting the previous arc at O

iv. Join *OA* and *OB*

v. Produce *AO* to *C* such that *OC* = *AO* and produce *BO* to *D* such *OD* = *OB*

vi. Join *AD*, *BC* and *CD* to form parallelogram *ABCD*



c. Two sides and One Angle

Make use of the fact that consecutive angles are supplementary. Therefore, draw the given angle as well as its supplementary. For example to draw parallelogram *ABCD* with *AB* = 10 cm, *BC* = 7 cm and < *ABC*105^0

i. Construc *AB* = 10 cm

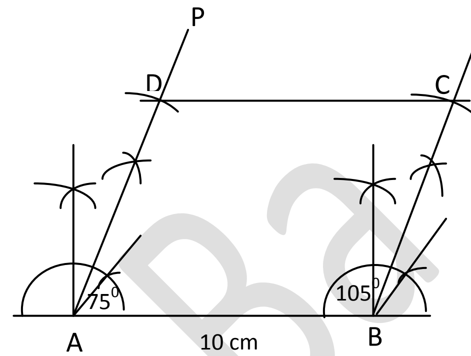
ii. Construct < 105^0 at *B*

iii. With center B and radius 7, cut BQ at C

iv. Construct < 75^0 (supplementary angle of < 105^0) at A

v. With A as center and radius 7 cm, cut AP at D

vi. Join C to D to obtain parallelogram ABCD



Exercises 26.12

1. Using a ruler and a pair od compasses only, construt a parallelogram PQRS with PQ = 9 cm, QR = 6 cm and < PQR = 120^0. Measure the length of QR

2. Using a ruler and a pair of compasses only, construct parallelogram *ABCD* in which *AB* = 7 cm, *BC* = 5 cm and diagonal 8.5 cm

3. Using a ruler and a pair of compasses only, to construct parallelogram PQRSin whichPQ = 7.5 cm and diagonals 11 cm and 8.4 cm

Constructing a Kite

Worked Examples

a. Using a ruler and a pair of compasses only, construct triangle *ABC* with $|AB|$ = 10*cm*, angle *ABC* = 30^0 and angle *CAB* = 60^0.

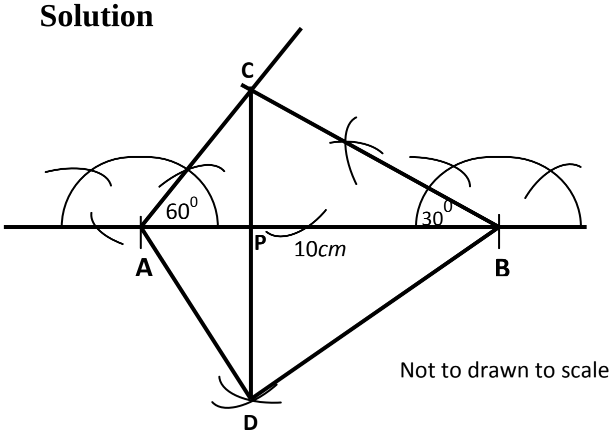
b. Construct a perpendicular from the point C to meet the line *AB* at P.

c. i. Extend the line *CP* to meet point *D* such that $|BC|$= $|BD|$.

ii. Join A to D and B to D.

d. What type of quadrilateral is *ADBC*?

Solution



d. Quadrilateral *ADBC* is a kite.

Constructing a Rhombus

Worked Examples

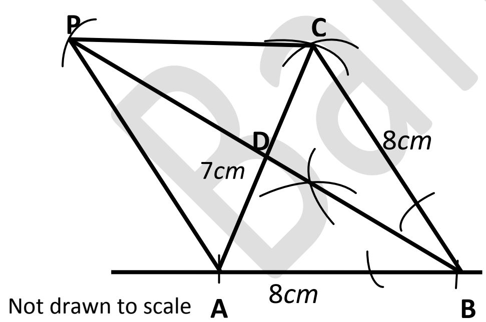
1. a. Using a ruler and a pair of compasses only, construct triangle ABC with $|AB|$ = 8cm, $|BC|$= 8cm and $|AC|$ = 7cm.

b. Bisect angle ABC and let the bisector meet AC at D. Produce$|BD|$ to P such that$|BD| = |DP|$. Join AP and CP

c. Measure i. angle ADB ii.$|AP|$

d. What kind of quadrilateral is ABCP?

Solution



i. Angle *ADB* = 90^0

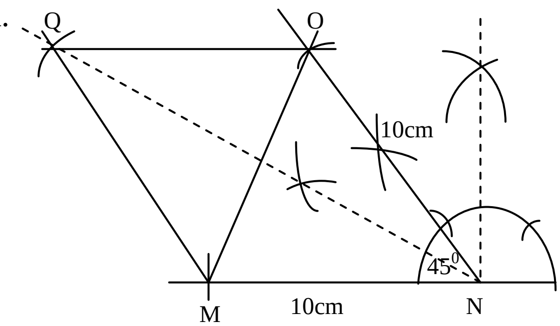
ii. Quadrilateral *ABCP* is a Rhombus.

2. a. Using a ruler and a pair of compasses only, construct a rhombus *MNOQ* of length 10cm and < *MNO* = 45^0

b. Measure; i. /MO/ ii. /NQ/

Solution

a.



b. i. /*MO*/ = 7.3cm ii. /*NQ*/ = 18cm

Drawing Other Figures

Sometimes students are required to construct some figures other than a circum-circle and an inscribed circle. Follow the question carefully and do exactly what it requires from you to obtain such figures.

Worked examples

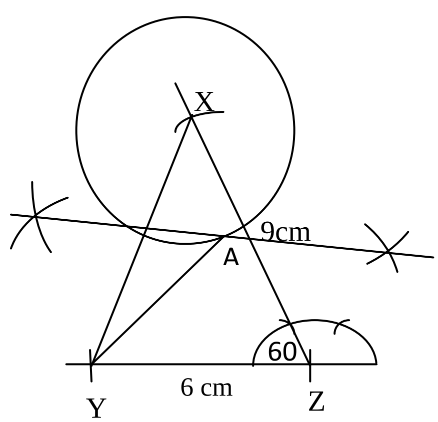
1. a. Using a ruler and a pair of compasses only, construct triangle *XYZ* in which$|\text{YZ}| = 6cm$, angle *XYZ* = $60^0$and $|\text{XZ}| = 9cm$. Measure |XY|

b. Construct the mediator of *YZ*.

c. Draw a circle with centre *X* and radius 5*cm*.

d. Measure |YA|, where *A* is the point of intersection of the mediator and the circle in the triangle region *XYZ*.

Solution



i. $|XY| = 9.9cm$ ii. $|YA| = 4.4cm$

2. a. using a ruler and a pair of compasses only, draw $|PQ| = 9cm,$

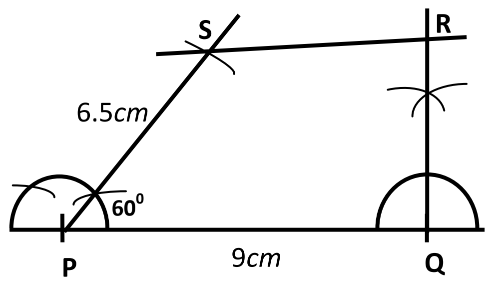
b. Construct a perpendicular to PQ at Q.

c. Construct angle QPS = 60^0 at the point P onPQ such that$|PS| = 6.5cm.$

d. Construct a line parallel to PQ through Q and the parallel line through S to meet at R.

e. Measure|PR|.

Solution



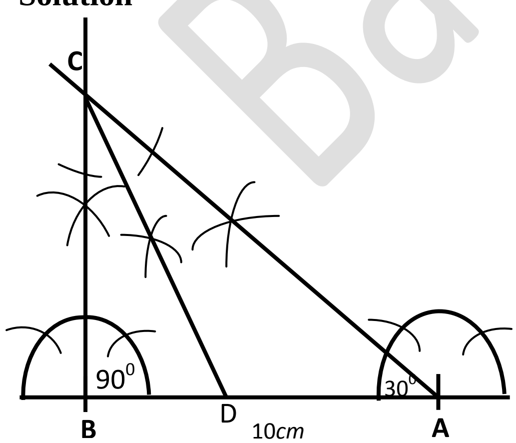
$|PR| = 10.6cm$

3. a. Using ruler and a pair of compasses only, construct triangle ABC such that $|BA| = 10cm,$ angle ABC = 90^0 and angle BAC = 30^0. Measure the length of BC.

b. Bisect the angle ACB to meet BA at D.

c. What type of triangle is CDA?

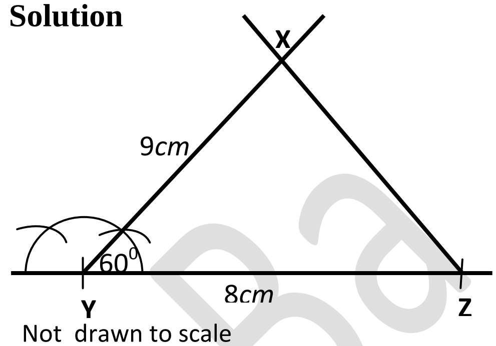
Solution



Triangle CDA is a scalene triangle.

4. Using a ruler and a pair of compasses only, construct triangle XYZ with $|ZY| = 8cm$, XYZ = $60^0$and $|XY| = 9cm.$
Measure: i. angle YZX ii. $|XZ|$

Solution



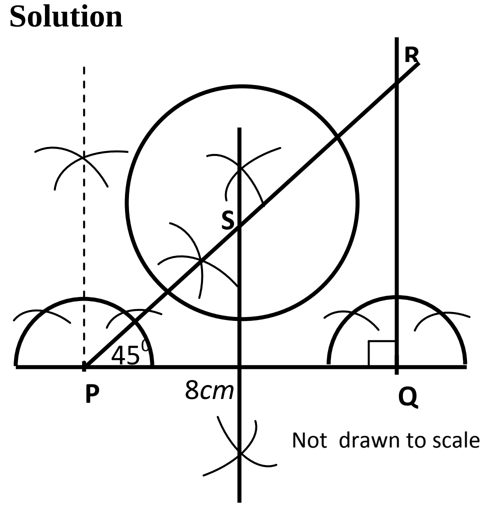
Angle YZX = 65^0 and $|XZ| = 8.7cm$

4. a. Using a ruler and a pair of compasses only, construct triangle PQR in which $|PQ| = 8cm$, angle QPR = 45^0 and angle PQR = 90^0. Measure |QR|

b. Construct the mediator of PQ to meet PR at the point S.

c. With S as the centre and radius $3cm$, construct a circle.

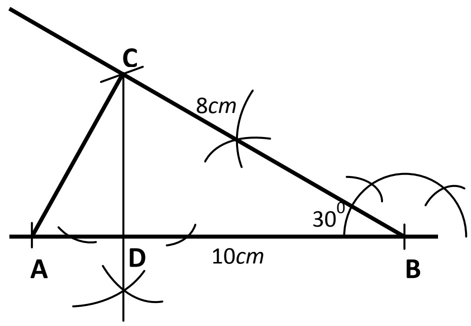
Solution



5. i. Using a pair of compasses and a ruler only, construct triangle ABC such that $|AB| = 10cm,$ and angle ABC = 30^0 and $|BC| = 8cm$. Measure angle ACB.

ii. Construct a perpendicular from C to meet line AB at D.

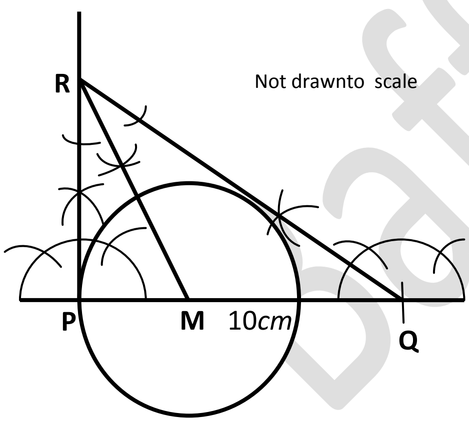
Solution



Angle ACB = 96^0 and $|CD| = 4cm$.

6. a. Using a ruler and a pair of compasses only, construct triangle PQR such that the length of PQ = 10cm, angle QPR = 90^0 and angle PQR = 30^0.
b. Bisect the angle QRP to meet PQ at M c. With M as centre, and radius MP, draw a circle.
d. Measure the radius of the circle.

Solution



Radius, MP = 3.3cm.

Exercises 26.13

1. a. Using a ruler and a pair of compassesonly, construct triangle ABC such that $|AB|$=7.5cm, BC = 8.5cm and $|AC|$ =10cm.
b. Construct perpendicular bisectors of AB and BC to meet at O.
c. With O as centre and radius OA, draw a circle to touch the vertices of the triangle.
d. Measure angles ABC and ACB.

2. a. Using a ruler and a pair of compasses only, construct triangle XYZ such that $|XY| = 8cm$, /ZY/ = 8.5cm and ZX = 7.5cm.
b. Construct the mediator of line YZ and the mediator of line XZ.
c. Locate O, the point of intersection of the mediators of lines YZ and XZ.
d. With O as centre and radius OY, draw a circle.
Ans: 1d. ABC = 78^0, ACB = 47^0
e. Measure the radius of the circle drawn in (d) and calculate the circumference of the circle ($\pi = 3.14$) .

3. Using a ruler and a pair of compasses only,
i. Construct Δ ABC where / AB/ = 7cm, /AC/ = 8cm and < A = 105^0.
ii.the locus of points 6cm, from C;
iii. Y, the locus of points equidistant from $\overrightarrow{AB}$ and $\overrightarrow{BC}$to cut X in P and R.
b. Measure i. $|\overrightarrow{BC}|$ ii. $|\overrightarrow{PR}|$

4. a. Using a ruler and a pair of compasses only, construct triangle ABC with sides /AB/ = 7cm, /BC/ = 8cm and /AC/ = 9cm
b. Construct the locus $L_{1,}$of the point equidistance from A and B, and the locus L_2, of the point equidistance from B and C to meet at O.
c. With centre O and radius OA, draw a circle to pass through the vertices of the triangle.
d. Measure and write down the radius of the circle you have drawn.

5. Using a ruler and a pair of compasses only,
i. Construct a Δ ABC, such that$|AB| = 6cm$, /AC/ = 8cm and angle BAC = 30^0.

ii. Construct the bisector of the angles ACB to meet line AB at D.
iii. Measure $|AD|$and $|BD|$.
iv.Write down the ratio$|AD|:|BD|$. Ans:2:1
Ans.(iii) /AD/ = 4cm, /BD/ = 2cm

6. a. Using a ruler and a pair of compasses only, construct triangle ABC such that $|AB| = 7cm$, /AC/ = 7cm and angle CAB = 60^0
b. Construct a perpendicular from C to meet AB
c. Construct /CD/ = 7cm, parallel to AB through C.
d. Join B to D with a ruler.
e. Name the figure formed. (Parallelogram)

7.a. Using a ruler and a pair of compasses only, construct triangle RST in which $|RS| = 6.5cm$, angle TRS = $45^0$and angle RST= 75^0. Measure$|RT|$. Ans: /RT/ 7.7cm
b. Construct the locus, $L_{1,}$ which is equidistance from the points R and S and the locus, $L_{2,}$ which is equidistance from the points S and T. Locate the meeting point of L_1 and L_2 as O.
c. With O as centre, draw a circle to touch the vertices of the triangle.

8. i. Using a ruler and a pair of compasses only, construct triangle ABC with /AB/ = 10cm, angle ABC = 30^0 and angle CAB = 60^0.
ii. Construct the locus of the point which is equidistance from line AC and CB to meet AB at P.
iii. Construct the locus of the point which is $2.5cm$ from P.

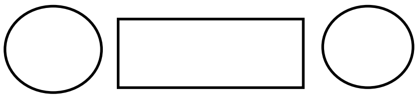
9. Using a ruler and a pair of compasses only;
i. Construct triangle XYZ where /XY/ = 8.5cm, angle ZXY = 45^0 and angle XYZ = 45^0.
ii. Construct a perpendicular from Z to pass through XY to T such that /ZT/ = 9cm.
iii. Join X to T and Y to T to form figure XTYZ.
iv. Name figure XTYZ. Ans: Rhombus

10. i. Using a ruler and a pair of compasses only, construct triangle ABC in which $|AB|= 7.5cm$, angle $BAC = 75^0$ and angle $ABC = 45^0$
ii. Bisect angles ACB, CBA and BAC. Let the bisectors meet at O.
iii. With centre O, draw a circle to touch the sides of the triangle.

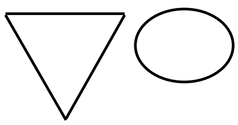
27 MENSURATION II

Net of Common Solids

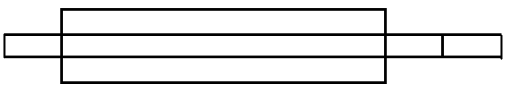
1. Net of a cylinder

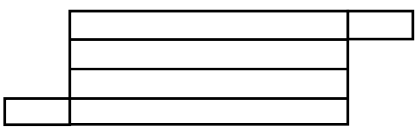


2. Net of a cone

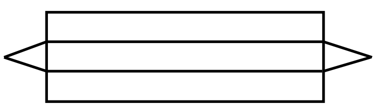


3. Net of a cuboids

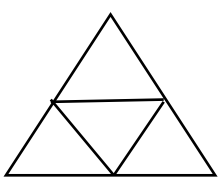




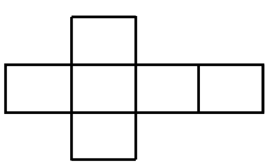
4. Net of a prism



5. Net of a pyramid



6. Net of a cube

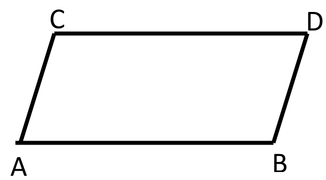


Surface and Surface Area

By surface, we mean an object that has only two dimensions – length and breadth but no thickness.

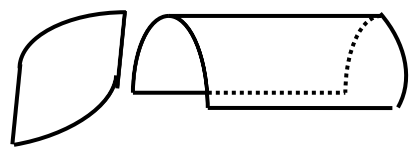
Study the examples below;

1.



This parallelogram has a flat surface so it can be called a **plane**.

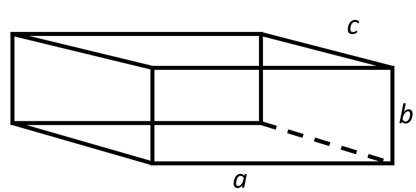
2.



In these examples, the surfaces are not flat but curved and are therefore **not planes**. All solids are bounded by a surface or surfaces. Surface area means ***the measure of surface.***

Surface Area of a Rectangular Solid

All rectangular solids have six surfaces. The surface area is found by calculating the area of all surfaces and then adding them up.



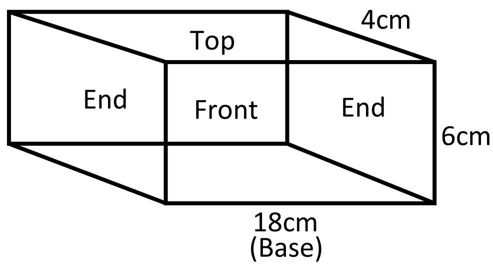
Total surface area of a rectangular solid,

$\mathbf{A = 2(}ab + ac + bc\mathbf{)}$

Worked Examples

1. Calculate the total surface area of a rectangular solid, which has a length, width and height of 18cm , 4cm and 6cm.

Solution



Let $a = 18$, $b = 6$ and $c = 4$

Area of front (ab)$= 18cm \times 6cm = 108\text{cm}^2$

Area of top (ac) $= 18cm \times 4cm = 72\text{cm}^2$

Area od end (bc) $= 6\text{cm} \times 4\text{cm} = 24\text{cm}^2$

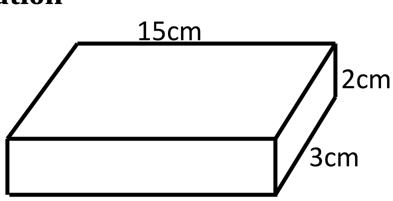
$\therefore$ Total surface area, A

$A = 2(108 + 72 + 24)\text{cm}^2$

$A = 2 \times 204\text{cm}^2 = 408cm^2$

2. How many square centimeters are in a total surface area of a rectangular solid of length 15cm, width 3cm and height 2cm.

Solution



Area of face $= 15\text{cm} \times 2\text{cm} = 30\text{cm}^2$

Area of top $= 15\text{cm} \times 3\text{cm} = 45\text{cm}^2$

Area of ends $= 2\text{cm} \times 3\text{cm} = 6\text{cm}^2$

$\therefore$ Total surface area,

$A = 2(30 + 45 + 6)\text{cm}^2 = 2 \times 81 = 162\text{cm}^2$

3. A rectangular box has length 6cm, breadth 7cm and height 8cm. Calculate the total surface area of the box.

Solution

Area of face $= 6\text{cm} \times 7\text{cm} = 42\text{cm}^2$

Area of top $= 6\text{cm} \times 8\text{cm} = 48\text{cm}^2$

Area of ends $= 8\text{cm} \times 7\text{cm} = 56\text{cm}^2$

$\therefore$ Total surface area,

$A = 2(42 + 48 + 56)\text{cm}^2$

$A = 2 \times 146 = 292\text{cm}^2$

Exercises 27.1

A.1. Calculate the total surface area of a rectangular box that is 18cm long, 12cm wide and 9cm high.

2. Calculate the total surface area of a rectangular solid, which has length, width andheight of 13cm, 9cm and 7cm respectively.

3. How many square centimeters are in a total surface area of a rectangular solid of length 12cm, width 7cm and height 5cm.

4. A rectangular box has length 18cm, breadth 21cm and height 24cm. Calculate the total surface area.

B. Calculate the total surface are of the rectangular solid with the dimensions

Length	Width	Height

1. 8m	3m	2m
2. 12cm	2cm	3cm
3. 5 cm	1 cm	2 cm
4.36 cm	5cm	3cm

Surface Area of a Cube

Since the cube is a rectangular solid, the method for calculating the surface area of a rectangular solid can be used to calculate the surface area of a cube. In a cube, all the edges are equal.

To find the total surface area of a cube, find the area of one face and multiply it (answer) by 6 (because it has 6 faces).

Mathematically,

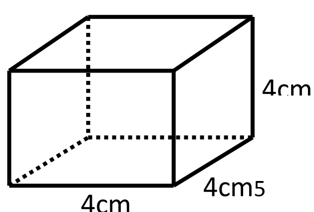
Area (A)= $\boldsymbol{6\,l^2}$, where

A is the total surface area of the cube

$\boldsymbol{l}$ is the length of an edge of the cube.

Worked Examples

1. Calculate the total surface area of a cubewith sides 4cm long.



Solution

Area $= 6\,l^2$,

but $l = 4\text{cm}$

Area $= 6 \times (4\text{cm})^2 = 96\text{ cm}^2$

2. A box is made of equal dimension of 25cm.Calculate the surface area of one of the face and the total surface area of the box.

Solution

i. Area of one face $= l^2$

But $l = 25$ cm

A= 25 cm × 25 cm = 625 cm^2

ii. Total surface area of the box , A = $6l^2$,

But $l = 25$ cm

Total surface area = $6 \times (25cm)^2 = 3{,}750\text{ cm}^2$

3. The volume of a cube is 125cm^3. Find the area of one of its faces.

Solution

$V = L^3$

Substitute$V = 125\text{cm}^3$

By substitution,

$125\text{cm}^3 = L^3$

$L = \sqrt[3]{125} = 5\text{cm}$ OR

$125\text{cm}^3 = L^3$

$5^3 = L^3$

Equating exponents and bases, $L = 5\text{cm}$

Area, $A = L^2$.

Substitute $L = 5\text{cm}$

$A = (5\text{cm})^2 = 5\text{ cm} \times 5\text{ cm} = 25\text{ cm}^2$

Exercises 27.2

Calculate the total surface area of a cube with the following sides;

1) 13cm 2) 17cm 3) 66m 4) 10cm

Surface Area of a Cone

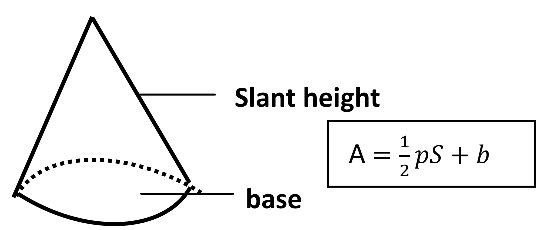
The total surface area of a cone is calculated by the formula; $\mathbf{A} = \frac{1}{2}\boldsymbol{pS} + \boldsymbol{b}$, where

A represent the total surface area

p *is the perimeter of the base*

S *is the slant height and*

b *is the area of the base*



But the perimeter of the base p = the circumference of the circle.Thus, $\boldsymbol{p = C = 2\pi r}$

The area of the base, b = the area of the circle.Thus, $b = \pi r^2$

By substitution, total surface area of the cone:

$$\mathbf{A} = \frac{1}{2} \times 2\pi r S + \pi r^2$$

$$\mathbf{A} = \pi r S + \pi r^2$$

Curved Surface Area of a Cone

The curved surface area of circular cone of base radius, r, and slant height, s, is calculated by the formula, $\mathbf{A} = \pi\, r\, S$

Work Examples

1. Calculate the total surface area of a solid with a slant height of 15cm and a base radius of 7cm.

Solution

$A = \pi rh + \pi r^2$

Substitute $\pi = \frac{22}{7}$, r = 7cm , h = 15 cm,

A = $\frac{22}{7} \times 7 \times 15 + \frac{22}{7} \times 7 \times 7$

A = 330 + 154 = 484cm^2

2. Find the total surface area of a cone of slant height 5cm and base radius 7cm.

Solution

$A = \pi rh + \pi r^2$

Substitute $\pi = \frac{22}{7}$, r = 7cm , h = 5 cm,

A = $\frac{22}{7} \times 7 \times 5 + \frac{22}{7} \times 7 \times 7$

A = 110 + 154

A = 264cm^2

Exercises 27.3

A. Calculate the total surface area of a cone with the following dimensions

Slant Height (h)	Radius of the base, r	Diameter of the base, d
1. 5 cm	3.5cm	
2. 25 cm		21cm
3. 16 cm		35cm
4. 15 cm	14 cm	

B. Find the curved surface area of a cone with the following measurements;

1. Slant height = 3cm, diameter = 14cm.
2. Slant height = 24cm, diameter = 14cm.
3. Slant height = 12cm, diameter = 9cm.
4. Find the radius of a cone with a curved surface area 44m^2 and a slant height 4m
5. What is the height of a cone with a curved surface area of 28πm^2 and a radius of 4m

Surface Area of a Pyramid

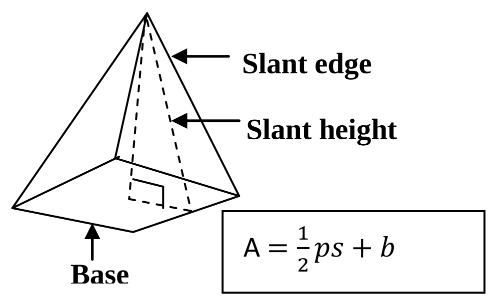
The formula for calculating the total surface area of a pyramid is; $\mathbf{A} = \frac{1}{2} ps + b$, where

A *is the total surface area*

p *is the perimeter of the base.*

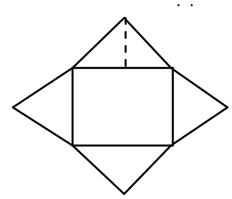
s *is the slant height and*

b *is the area of the base.*



Method 2

I. Draw the net of the pyramid (rectangular) to obtain four triangles surrounding the base.



II. Find the area of each triangle using the formula , $A = \frac{1}{2}\, bh$, where h is the slant height of the pyramid.

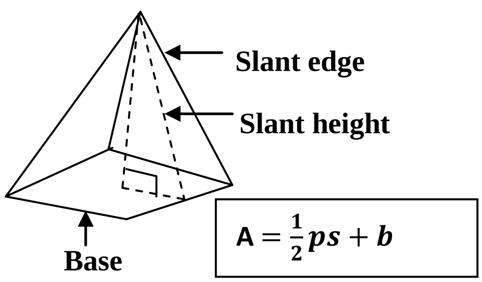
III. Find the area of the base using the formula,
$A = L \times B$

IV. Total surface area = Area of the base + area of each triangle.

Surface Area of a Pyramid

The formula for calculating the total surface area of a pyramid is; $\mathbf{A = \frac{1}{2} ps + b}$, where

A *is the total surface area*
p *is the perimeter of the base.*
s *is the slant height and*
b *is the area of the base.*



Method 2

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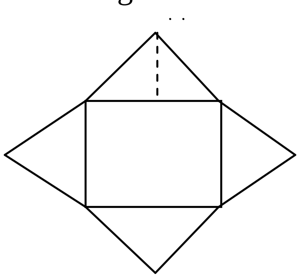


II. Find the area of each triangle using the formula , $A = \frac{1}{2} bh$, where h is the slant height of the pyramid.

III. Find the area of the base using the formula,
A = L × B

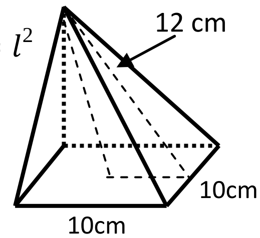
IV. Total surface area = Area of the base + area of each triangle.

Worked Examples

1. Calculate the total surface area of a solid pyramid with a square base of sides 10cm, and a slant height of 12cm.

Solution

Area of the base $(b) = l^2$
$b = 10 \text{ cm} \times 10 \text{ cm}$
$b = 100 \text{ cm}^2$

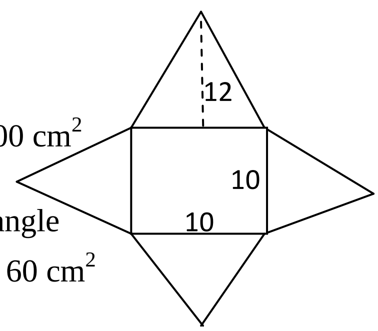


Perimeter of the base
P = 10 cm + 10cm + 10 cm + 10 cm = 40

Substitute p = 40 cm, h = 12 cm and b = 100
$A = \frac{1}{2} p\, s + b$
$A = \frac{1}{2} \times 40 \times 12 + 100 = 340\text{cm}^2$

Method 2

Area of the base
$A = 10 \times 10 = 100 \text{ cm}^2$



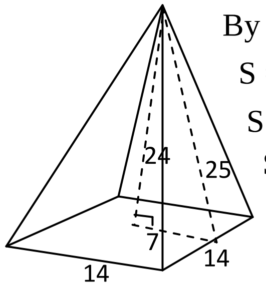
Area of each triangle
$A = \frac{1}{2}(10)(12) = 60 \text{ cm}^2$

Area of 4 triangles
= 4 (60)
= 240cm^2

Total surface area;
= Area of the base + area of each triangle.
= $100 \text{ cm}^2 + 240 \text{ cm}^2$
= 340 cm^2

2. The height of a pyramid is 24cm. If it has a square base of side 14cm, find its total surface area.

Solution



By pythagoras theorem,

$S = \sqrt{24^2 + 7^2}$

$S = \sqrt{625}$

$S = 25$ cm

Perimeter of base;

P = 14 + 14 + 14 + 14 = 56cm

Area of base;

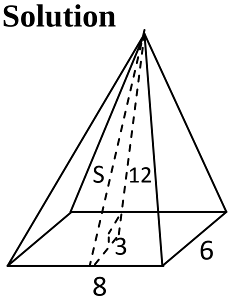
$b = 14 \times 14 = 196$ cm^2

Total surface area;

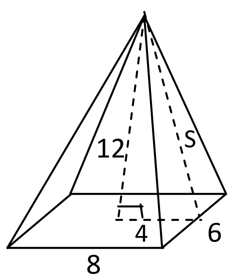
$A = \frac{1}{2} p\, s + b = \frac{1}{2}(56)(25) + 196 = 896$ cm^2

3. The height of a pyramid is 12cm. If it has a rectangular base of length 8cm and breadth 6cm, find the total surface area of the pyramid.

Solution



Pyramid 1

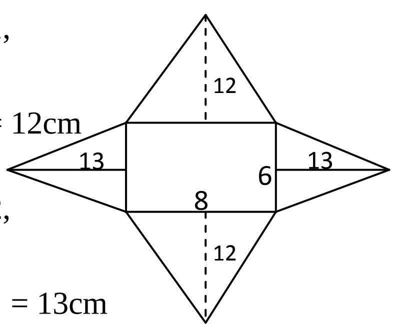


Pyramid 2

From pyramid 1,

$S = \sqrt{3^2 + 12^2}$

$S = \sqrt{9 + 144} = 12$cm



From pyramid 2,

$S = \sqrt{4^2 + 12^2}$

$S = \sqrt{16 + 144} = 13$cm

Area of trianlges of the net;

$= 2 \times \frac{1}{2}(8)(12) + \; 2 \times \frac{1}{2}(6)(13)$

$= 96$ cm^2 + 78cm^2 = 174 cm^2

Area of the base of the pyramid;

$A = 8 \times 6 = 48$ cm^2

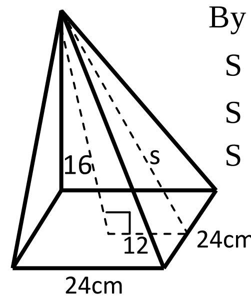
Total surface area;

= Area of the base + area of each triangle.

= 48 cm^2 + 174cm^2 = 222 cm^2

4. The base of a pyramid, 16cm high, is a square whose each side is of length 24cm. Find the slant height and the length of the slant edge of the pyramid.

Solution

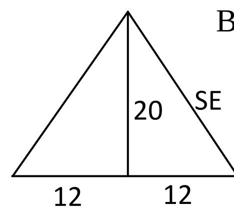


By pythagoras theorem,

$S = \sqrt{16^2 + 12^2}$

$S = \sqrt{400}$

$S = 20$ cm



By pythagoras theorem,

$SE = \sqrt{20^2 + 12^2}$

$SE = \sqrt{544}$

$SE = 23$ cm

Exercises 27.4

Calculate the total surface area of a pyramid with the following dimensions;

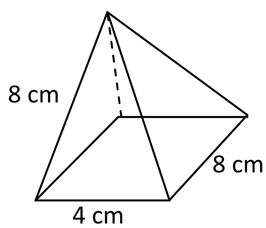
Slant height (h)	Length of side of Square base (b)
15cm	8cm
29cm	25cm
30cm	15cm
150cm	60cm

B. 1. The base of a rectangular pyramid is 10 cm by 18 cm. If the height of the pyramid is 15cm, find its total surface area.

2. The height of a pyramid is 12 cm. Find the total surface area of the pyramid if it has a square base of sides 10cm

3. A pyramid has a height 5cm. It has a rectangular base that measures 6cm by 8cm. Calculate its total surface area.

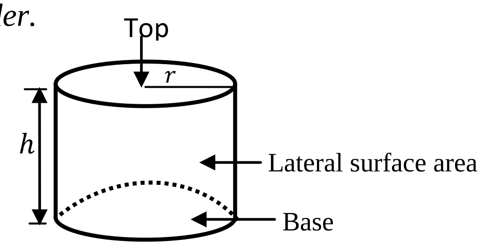
4. The pyramid below has arectangular base and faces that are isoscleles triangles. Finf the total surface area to the nearest whole number.



Surface Area of a Cylinder

The formula for calculating the total surface area of a solid cylinder is: $\boldsymbol{A = 2\pi r^2 + 2\pi r h,}$

where π is a constant value taken as $\frac{22}{7}$, *r is the radius of the cylinder and h is the height of the cylinder.*



NB: the top and base of the cylinder are circles.

***Total surface area*, *A*,**

= Area of top and base + lateral surface area

$$\boxed{\boldsymbol{A = 2\pi r^2 + 2\pi r h}}$$

For an opened cylinder, the total surface area, $\boldsymbol{A = 2\pi r h}$ ***(lateral surface area)***

For a cylinder closed or opened at one end, the total surface area , $\boldsymbol{A = \pi r^2 + 2\pi r h}$

Worked Examples

1. Calculate the total surface area of a solid cylinder with a height 10cm and radius 7cm.

Solution

$A = 2\pi r^2 + 2\pi r h$

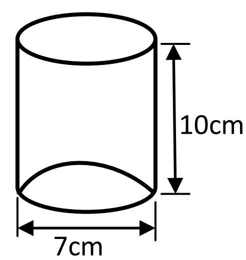
Substitute $r = \frac{22}{7}$, r = 7cm and h = 10 cm

$A = 2 \times \frac{22}{7} \times 7 \times 7 + 2 \times \frac{22}{7} \times 7 \times 10$

A = 308 + 440 = 616cm^2

2. What is the total surface area of a cylinder of height 10cm and diameter 7cm.

Solution



$A = 2\pi r^2 + 2\pi r h$

Substitute $\pi = \frac{22}{7}$,

$r = \frac{d}{2} = \frac{7}{2}$cm and h = 10 cm

$A = 2 \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} + 2 \times \frac{22}{7} \times \frac{7}{2} \times 10$

A = 77 + 220 = 297cm^2

3. The base radius of a closed cylinder is 4m. The height of the cylinder is 7m. Calculate its total surface area ($\pi = \frac{22}{7}$).

Solution

Total surface area of a closed cylinder,

A= $2\pi r (r + h)$,

Substitute r = 4m and h =7m

$A = 2 \times \frac{22}{7} \times 4\ (4 + 7)$ = $\underline{276.57\text{m}^2}$

4. A cylinder closed at one end has radius 7cm and height 20cm. ($\pi = \frac{22}{7}$).

i. Find the total surface area.

ii. If the cylinder is filled with water to a depth of 5cm, calculate the volume of water in it.

Solution

i. Total surface of a cylinder,

$A = \pi r^2 + 2\pi rh$

Substitute r = 7cm and h = 20cm

$A = \frac{22}{7} \times 7 \times 7 + 2 \times \frac{22}{7} \times 7 \times 20$

$A = 154 + 880 = 1{,}034\ cm^2$

ii. Volume of cylinder, $v = \pi r^2 h$

But r = 7cm and h = 5cm

$V = \frac{22}{7} \times (7)^2 \times 5 = 770cm^3$

5. A rectangular sheet of metal has length 44cm and breadth 10cm. If it is folded to form a cylinder with the breadth becoming the height, calculate:

i. the radius of the cylinder formed;

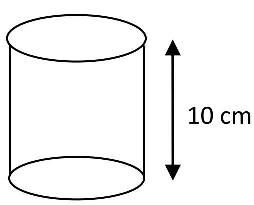
ii. the volume of the cylinder.

Solution

Length of metal sheet = 44cm,

Breadth = 10cm and $\pi = \frac{22}{7}$

C = 44cm = $2\pi r$



i. $2\pi r = 44cm$

$\frac{2\pi r}{2\pi} = \frac{44}{2\pi}$

$r = \frac{44}{2 \times \frac{22}{7}} = \frac{44 \times 7}{2 \times 22} = 7cm$

ii. $V = \pi r^2 h$

$V = \frac{22}{7} \times (7)^2 \times 10 = 1{,}540cm^3$

Exercises 27.5

A. Calculate the total surface area of a solid cylinder with the given dimensions;

1. r = 10.5 cm, h = 22cm 3. r = 35 cm, h = 26cm
2. r = 21 cm , h = 23cm 4. r = 7 cm , h = 17cm

B. Find the curved surface area of a cylinder with the following measurements

1. Height = 14cm, radius = 6cm
2. Diameter = 70cm, height = 9cm
3. Length = 12mm, radius = 2mm.

C. Find the radius of a cylinder with the following measurements;

1. Curved surface area = $44cm^2$ and height = 3.5cm
2. Curved surface area = $220m^2$, length = 7m
3. A sheet of paper, 20cm by 15cm, is rolled into a cylinder, so that its longer edges meet. Find the diameter of the cylinder so formed.

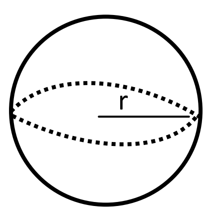
Surface Area of a Sphere

There formula for calculating the total surface area of a sphere is; $\boldsymbol{A = 4\pi r^2}$ where;

A represents the total surface area,

$\boldsymbol{\pi}$ is a constant $= \frac{22}{7}$

$\boldsymbol{r}$ is the radius of the sphere.



$$\boxed{\boldsymbol{A = 4\pi r^2}}$$

Worked Examples

1. Calculate the total surface area of a sphere of radius 14cm.

Solution

$A = 4\pi r^2$,

Substitute r = 14cm and $\pi = \frac{22}{7}$

$A = 4 \times \frac{22}{7} \times 14 \times 14 = 2{,}464\text{cm}^2$

2. The diameter of a sphere is 21cm.Calculate its total surface area.

Solution

$A = 4\pi r^2$.

But $r = \frac{d}{2} = \frac{21}{2}$cm and $\pi = \frac{22}{7}$

$A = 4 \times \frac{22}{7} \times \frac{21}{2} \times \frac{21}{2} = 1{,}386\text{cm}^2$

Exercises 27.6

Calculate the total surface area of spheres with the following radii and diameters;

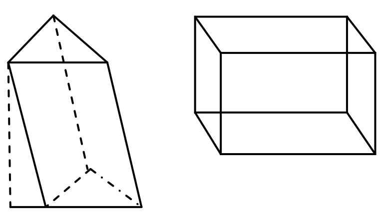
1. r = 7cm 2. r = 5cm 3 . r = 9cm

4. d = 35cm 5. d = 21cm 6. d = 28cm

Surface Area of a Prism

The lateral surface area of a prism is the sum of the areas of its lateral faces.

The total surface area of a prism is the sum of the areas of its lateral faces and its two bases.



Usually, if right or oblique is not mentioned, it is assumed that it is a right prism.

1. The general formula for the lateral surface area is $A = ph$, where p represents the perimeter of the base and h represents the height of the prism.

2. The general formula for the total surface area of a right prism is $A = ph + 2B$, where p represents the perimeter of the base, h represents the height of the prism and B the area of the base.

3. There is no easy way to calculate the surface area of an oblique prism in general. The best way is to find the areas of the bases and the lateral faces separately and add them.

Worked Examples

1. Find the lateral surface area of triangular prism with base edges 3cm, 4cm and 5 cm and altutide 8cm.

Solution

$p = 3 + 4 + 5 = 12$cm

Lateral surface area;

$A = ph$

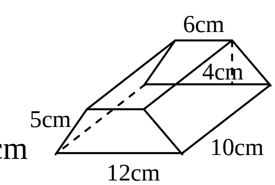
$A = 12(8) = 96\text{cm}^2$

2. Find the total surface area of a trapezoidal prism with parallel edges of base 6cm, and 12cm, the legs of the base 5cm each, the altitude of the base 4cm and a heihght of the prism 10cm.

Solution

Perimeter of the base is the sum of the lengths of the sides of the base.

P = 6 + 5 + 12 + 5 = 28cm



The base is an isosceles trapezoid. The area of base;

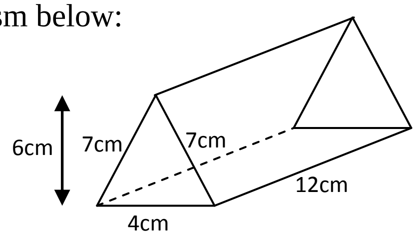
$B = \frac{1}{2}(a + b)h$

$B = \frac{1}{2}(6 + 12)(4) = 36\text{cm}^2$

Total surface area:

$A = ph + 2B$
$A = 28(10) + 2(36) = 280 + 72 = 352\text{cm}^2$

3. Calculate the surface area of the triangular prism below:



Solution
Perimeter of the base;
$p = 7 + 7 + 4 = 18\text{cm}$

Area of the base;
$B = \frac{1}{2}(4)(6) = 12\text{cm}^2$

Total surface area:
$A = ph + 2B$
$A = 18(12) + 2(12) = 216 + 24 = 240\text{cm}^2$

Exercises
1. A triangular prism has a triangular end with a base of 5cm and a height of 4cm. The length of each side is 8cm and the width of each side is 6cm. What is the surface area of the prism? A = 156cm^2

2. A triangular prism has a triangular end with a base of 8cm and a height of 6cm. The length of each side is 10cm and the width of each side is 6cm. What is the surface area of the prism? A = 428cm^2

3. A triangular prism has a triangular end with a base of 12cm and a heightof 9cm. The length of each side is 14cm and the width of each side is 9cm. What is the surface area of the prism? A = 486cm^2

Practical Problems
For all practical problems,
I. Separate the complex shape into individual shapes.
II. Find the appropriate areas.
III. Find the total surface area.
IV. Given the cost per m^2, multiply the cost by the total surface area.

Worked Examples
1. Calculate the cost of painting the four walls and ceiling of a room 30m long, 22m wide and10m high at Gh¢15.00 per 9m^2.

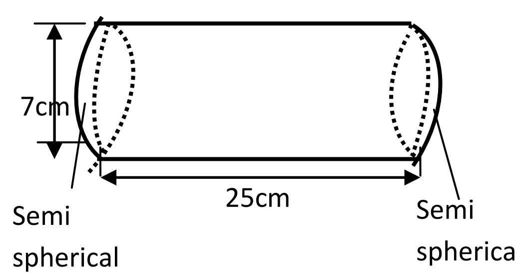
Solution
Area of 2 long walls $= 2(30 \times 10) = 600\text{m}^2$
Area of 2 short walls $= 2(22 \times 10) = 440\text{m}^2$
Area of ceiling $= 30 \times 22 = 660\text{m}^2$

Total surface area;
$= 600 + 440 + 660$
$= 1{,}700\ \text{m}^2$

Number of square meter $= \frac{1700}{9}$

Cost of painting of Gh¢15.00 per m^2
$= \frac{1700}{9} \times 15$
= Gh¢28.33

2.



Area of two semi- spherical ends $= 4\pi r^2$
$A = 4 \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} = 154\text{cm}^2$

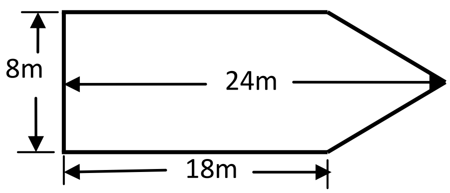
Lateral surface area of a cylinder $= 2\pi rh$,

$= 2 \times \frac{22}{7} \times \frac{7}{2} \times 25 = 550\text{cm}^2$

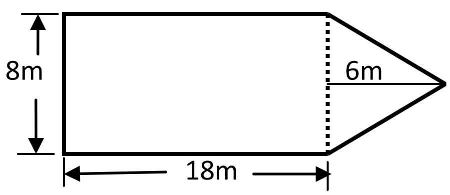
Total surface area,

$= 154 + 550 = 704\text{cm}^2$

3. Find the cost of covering with cocoa seedlings, a piece of land with dimensions as shown below if the cost of a seedling is Gh¢4.00 per m^2



Solution



Area of rectangle (A_1) = L × B.

Where $l = 18$m and $b = 8$m

$A_1 = 18 \times 8 = 144\text{m}^2$

Area of triangle (A_2) $= \frac{1}{2}bh$;

Where $b = 8$m and $h = 6$m

$A_2 = \frac{1}{2} \times 8\text{m} \times 6\text{m} = 24\text{m}^2$

Area of the total plot of the land;

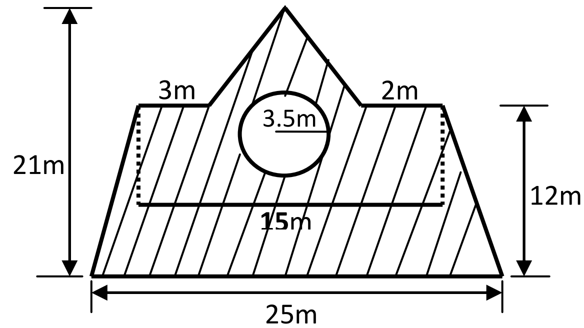
$= A_1 + A_2 = 144\text{m} + 24\text{ m}^2 = 168\text{ m}^2$

Cost of seedling;

= 168 × Gh¢4 = Gh¢672.00

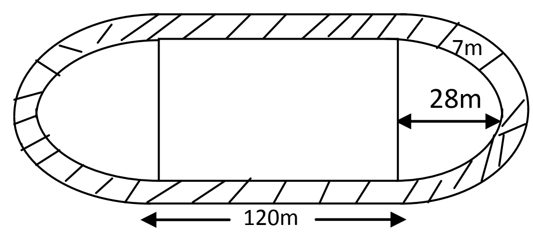
Exercises 27.7

1. What would be the cost of covering the surface area represented by theshaded portions of the diagram belowwith floor tile each at a cost of Gh¢2.00 per square meter.

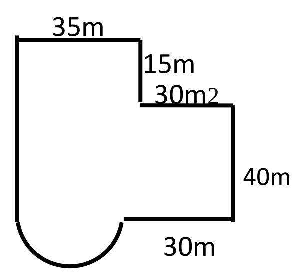


Ans = Gh¢493.00

2. The dimensions of a field track is shown in the diagram below. Calculate the cost of covering the track with tartan surface, if the cost of covering 1 square meter is Gh¢12.00.



3. What would be the cost of carpeting the floor of a house with the dimensions shown below, if the cost of is Gh¢14.00.

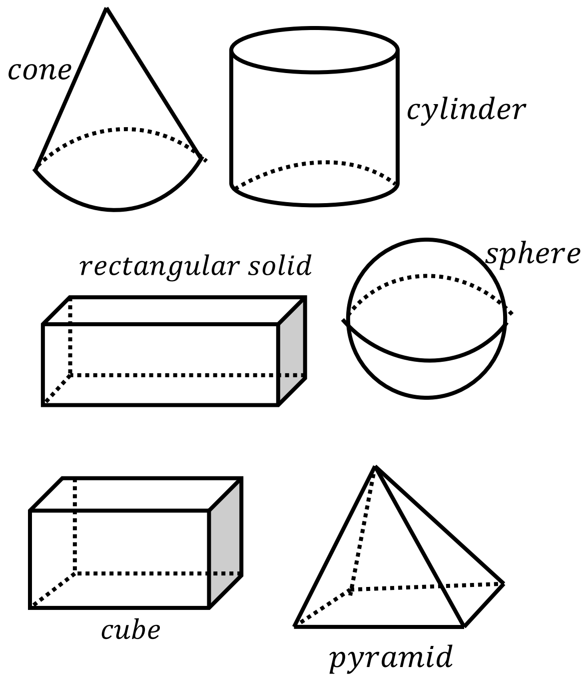


Meaning of Volume

When we want to know how much a container will hold, we need a measurement of space. Space involves three dimensions – length, breadth and height. This measurement of space is called ***volume.***

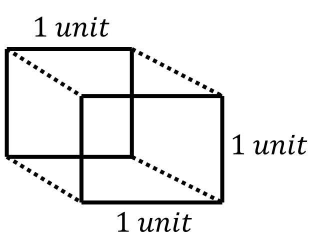
The measurement of volume is based on unit cubes.

Study the figures below;



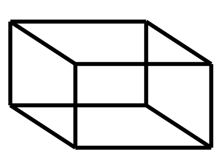
All the above are 3- dimensional solids because they have length, width and Height.

All solids contain volume measured in ***cubic meters (m^3) or cubic centimetres (cm^3).***
Consider the solid below:

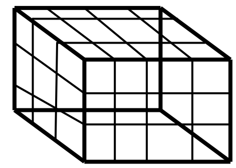


The solid is 1unit long, 1unit wide and 1unit high. It therefore has a volume of 1cubic unit.

In the diagram below:

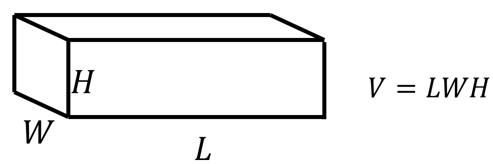


The area is found by finding out how many cubic units are contained in the solid as shown below;



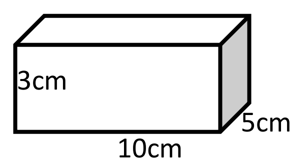
The volume is found by counting the entire unit cubes that completely fill the solid.

Volume of a Cuboid /Rectangular Solid
The volume of a rectangular solid is calculated by the formula; V = LWH, where**V** is the volume, **L** is the length, **W** is the width and **H** is the height of the solid



Worked Examples
1. Calculate the volume of the rectangular solid with length 10cm, width 5cm and height 3cm

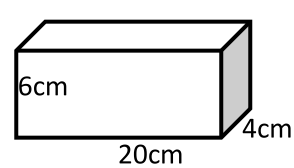
Solution



V = LWH
But L = 10cm, W = 5 and H = 3
$V = 10cm \times 5cm \times 3cm = 150cm^3$

2. A rectangular box has length 20cm, width 6cm and height 4cm. Find the volume of the box.

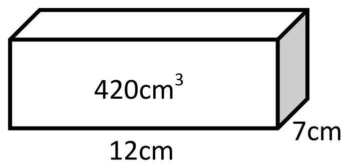
Solution



V = L W H
But $L = 20cm, W = 4cm$ and $H = 6cm$
$V = 20cm \times 4cm \times 6cm$

$V = 480cm^3$

3. The volume of a rectangular pond is $420cm^3$. If the pond is 12cm long and 7cm wide, calculate its height.



Solution

V = LWH

But V = 420 cm^3, L= 12cm, W = 7cm and H = ?

420 cm^3 = 12 cm × 7 cm × H

420 cm^3 = 84 cm^2 × H

H = $\frac{420\ cm^3}{84\ cm^2}$ = 5 cm

4. A rectangular notebook has a volume of $96cm^3$.If it is 4cm wide and 3cm high, how long is the book?

Solution

V = LWH

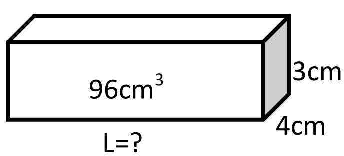
But V = 96 cm^3, L = ? W = 4cm and H = 3cm

96 cm^3 = L × 4cm × 3cm

420 cm^3 = L × 12 cm^2

L = $\frac{96\ cm^3}{12\ cm^2}$

L = 8 cm



5. The volume of water in cuboids is $9m^3$, the length of the cuboids is 3m and breadth is 2m. Calculate the depth of the water in the cuboids.

Solution

V = L B H,

But V = 9 m^3,L = 3m , B = 2 m and H = ?.

9 = 3 × 2 × H

9 = 6 H

H = $\frac{9}{6}$ = 1.5m

6. A rectangular box has a length 20cm width 6cm and height 4cm. Find how many cubes of size 2cm that will fit into the box.

Solution

V = LBH = 20 × 6 × 4 = 450cm^3

Volume of 1 cube = 2 × 2 × 2 = 8cm^3

Number of cubes = $\frac{450}{8}$ = 60 cm^3

Exercises 27.8

A. 1. Calculate the volume of a rectangular solid of which is 18cm long, 2cm wide and 3cm high.

2. The length, width and height of a rectangular tank is 16cm, 9cm and 11cm respectively. Calculate the volume of the tank.

3. A pit is 25m deep, 17m wide and 8m long. What is the volume of the pit?

4. Find the volume of a rectangular box that is 10cm long, 8cm wide and 2cm high.

5. The dimensions of a water tank in the form of a cuboid are 60cm by 15cm by 20cm. Find the capacity of the tank

B. 1. The volume of a rectangular tank is $4096cm^3$. If its width and height is 16cm and 8cm respectively, calculate its length.

2. A rectangular tower has a volume of $968cm^3$. If it has a length of 16cm and width 11cm, calculate its height.

3. A door has a length of 30cm and height 90cm. Calculate the width of the door if its volume is $8,100cm^3$.

4. A storey building is 15m high and 5m wide. If the storey- building occupies a space of $1,425m^3$, calculate its length.

.

Volume of a Cube

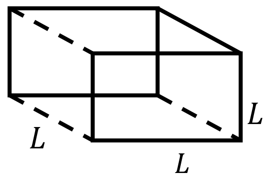
A cube is a rectangular solid in which the length, width and height are the same.

Therefore the formula $\mathbf{V = LWH}$ can be written as $\mathbf{V = L \times L \times L}$ where $\mathbf{L}$ is the length of the side of the cube.

In the diagram below;

$V = L \times L \times L$

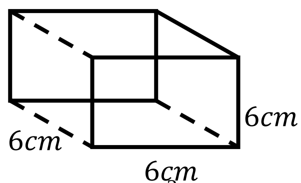
$V = L^3$



Worked Examples

1. Calculate the volume of a cube of side 6cm.

Solution



$V = L \times L \times L$

$V = L^3$

But L= 6cm

$V = 6\text{cm} \times 6\text{cm} \times 6\text{cm} = 216\text{cm}^3$

2. Find the length of the side of a cube whose volume is 1728cm^3.

Solution

Method 1

$V = L^3$,

But $V = 1{,}728\text{ cm}^3 = L^3$

$12 \times 12 \times 12 = L \times L \times L$

$\Rightarrow L = 12\text{ cm}$

Method 2

$L^3 = 1728\text{cm}^3$

$L = \sqrt[3]{1728\text{cm}^3} = 12\text{cm}$

3. The volume of a cube is 27cm^3. Find the area of one of its faces.

Solution

Let x be the side of the cube.

$27\text{ cm}^3 = x^3$

$3^3 = x^3$

$3\text{ cm} = x$ or $x = 3\text{cm}$

Area of one face $= x \times x = x^2 = 3^2$

$A = 3\text{cm} \times 3\text{cm} = 9\text{cm}^2$

4. The volume of water in a rectangular tank is 30cm^3. The length of the tank is 5cm and its breadth is 2cm. Calculate the depth of water in the tank.

Solution

$V = L \times B \times H$

But $V = 30\text{cm}^3$, $L = 5\text{cm}$, $B = 2\text{cm}$

$30\text{cm}^3 = 5\text{cm} \times 2\text{ cm} \times H$

$30\text{cm}^3 = 10\text{cm}^2 \times H$

$H = \frac{30\text{cm}^3}{10\text{cm}^2} = 3\text{cm}.$

Exercises 27.9

A. Calculate the volume of a cube with the following side;

1) 11.5cm 2) 12cm 3) 20cm

B. Find the length of the side of a cube with the following volumes:

1) 343cm^3 2) $2{,}744\text{cm}^3$ 3) $1{,}331\text{cm}^3$

C.1. The volume of a cube is 64cm^3. Find the length of one of its sides.

2. The volume of a cube is 27cm^3. Find the area of one face.

3. Find the volume of a cube whose length is 14cm

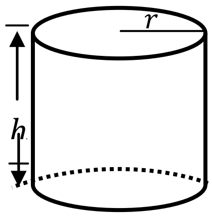
Volume of a Cylinder

The formula for finding the volume of a cylinder is $\mathbf{V = \pi r^2 h}$, where;

V is the volume of the cylinder,

π (pi) is a constant with value $\frac{22}{7}$ or 3.142,

R is the radius of the cylinder ,
h is the height of the cylinder.

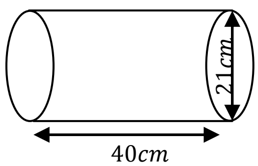


$V = \pi r^2 h$

Worked Examples

1. If a cylindrical tank is 21cm in diameter and 40cm in height, what is its volume?

Solution



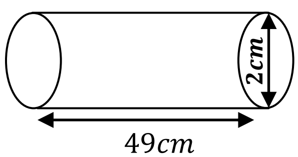
$V = \pi r^2 h,$

Where $\pi = \frac{22}{7}$

$r = \frac{21}{2}$ and $h = 40\text{cm}$

$V = \frac{22}{7} \times \frac{21\text{cm}}{2} \times \frac{21\text{cm}}{2} \times 40\text{cm} = 1{,}3860\text{cm}^3$

2. Calculate the volume of the figure below $(\pi = \frac{22}{7})$



Solution

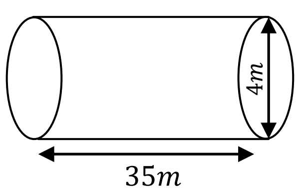
$V = \pi r^2 h$

But $r = 2\text{cm}, h = 49\text{cm}$ and $\pi = \frac{22}{7}$

$V = \frac{22}{7} \times (2\text{cm})^2 \times 49\text{cm} = 616\ \text{cm}^3$

3. A timber log of height 35m has a base radius of 4m. Calculate its volume.

Solution



$V = \pi r^2 h$

But $r = 4cm$, $h = 35cm$ *and* $\pi = \frac{22}{7}$

$V = \frac{22}{7} \times (4\text{cm})^2 \times 35\text{cm} = 1{,}760\text{cm}^3$

4. The volume of a cylinder is 220cm^3 the radius of the cross-section is 2.5cm. Find the height of the cylinder ($\pi = \frac{22}{7}$)

Solution

Volume of a cylinder, $V = \pi r^2 h$

But $V = 220\text{cm}^3$, $r = 2.5\text{cm}$, $\pi = \frac{22}{7}$, h =?

$220 = \frac{22}{7} \times (2.5)^2 \times h$

$7 \times 220 = 7 \times \frac{22}{7} \times (2.5)^2 \times h$

$1{,}540 = 137.5h$

$h = 11.2\text{cm}$

5. A cylinder has diameter 4cm and height of 14cm. ($\pi = \frac{22}{7}$). Find:

i. the circumference of the base

ii. the area of the base

iii. the volume of the cylinder.

Solution

i. $c = 2\pi r$,

$d = 4$ and $\frac{d}{2} = r = \frac{4}{2} = 2\text{cm}$

$C = 2 \times \frac{22}{7} \times 2 = 12.57\text{cm}$

ii. Area of the base, $A = \pi r^2$

$A = \frac{22}{7} \times (2)^2 = 12.57\text{cm}^2$

iii. $V = \pi r^2 h$

$V = \frac{22}{7} \times (2)^2 \times 14 = 176\text{cm}^3$

6. A rectangular sheet of metal has length 44cm and breadth of 10cm. If it is folded to form a cylinder with the breadth becoming the height, calculate:

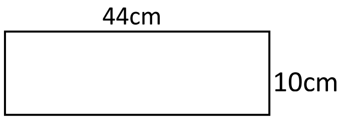
i. the radius of the cylinder formed,

ii. the volume of the cylinder ($\pi = \frac{22}{7}$).

Solution

i. Length of metal sheet = 44cm

Breadth = 10cm, $\pi = \frac{22}{7}$

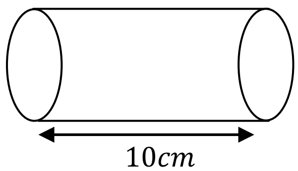


C = $2\pi r$ = 44cm

$2\,\pi\,r$ = 44cm

$\frac{2\pi r}{2\pi} = \frac{44}{2\pi}$,

$r = \frac{44 \times 7}{2 \times 22}$ = 7cm



ii. $V = \pi r^2 h$

V = $\frac{22}{7} \times (7)^2 \times 10$ = 1,540cm^3

7. A rectangular tank of length 22cm, width 9cm filled with water. The water is poured into a cylindrical container of radius 6cm, calculate the:

i. volume of the rectangular tank

ii. the depth of water in the cylindrical container

Solution

1. Volume of water in the rectangular tank,

V = L B H = 22cm × 9cm × 16cm = 3,168cm^3

ii. Volume of cylindrical container = $\pi\, r^2\, h$.

But volume of water in the rectangle tank

= volume of water in the cylindrical container

⇒3,168cm^3 = $\pi\, r^2\, h$,

But r = 6.

3168 = $\frac{22}{7} \times (6)^2 \times h$

7 × 3168 = $\frac{22}{7} \times 6 \times 6 \times h$

$h = \frac{22176}{792}$ = 28cm

8. A water tank in the form of cuboids of height 22cm and a rectangular base of length 7cm and width 5cm is filled with water. The water is then poured into a cylindrical container of diameter 14cm. Calculate the height of the water in the cylindrical container

Solution

Volume of water in the cuboids = L B H

V = 7 × 5 × 22 = 770cm^3

Volume of water in the container = Volume of water in the cylindrical container

⇒770 cm^3 = $\pi r^2 h$

But $r = \frac{d}{2} = \frac{14\text{cm}}{2}$ = 7cm

$770 = \frac{22}{7} \times 7 \times 7 \times h$

7 × 770 = 154 h

$h = \frac{7 \times 770}{154}$ = 5cm.

Exercises 27.10

For each question, take $\pi = \frac{22}{7}$

1. Find the volume of a cylindrical tank of radius is 42cm and height 45cm.

2. The volume of water in a cylindrical container is 308cm^3. If the height of the wateris 8cm. Find the radius of the base of the cylinder.

3. The volume of a cylindrical is 1,100cm^3. If the diameter of its base is 10cm, find its height.

Volume of a Pyramid

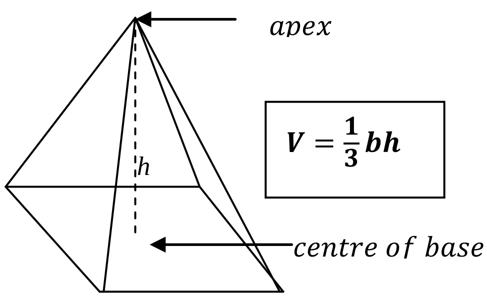
The formula for the volume of a pyramid is

$\boldsymbol{V = \frac{1}{3} bh}$, where;

V is the volume of the pyramid,

b is the area of the base of the pyramid,

h is the height of the pyramid.



The height of the pyramid is the distance from the centre of the base to the apex.

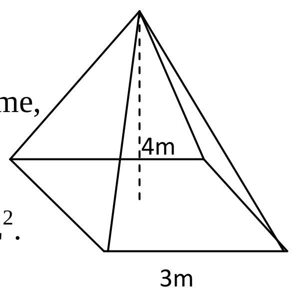
Worked Examples

1. Calculate the volume of a pyramid with a square base of sides 3m and height 4m.

Solution

$V = \frac{1}{3}bh$

To calculate the volume, first find b (the area of the base)



Area of base (b) $= L^2$.

But $L = 3$m

Area of base (b) $= 3\text{m} \times 3\text{m} = 9\text{m}^2$

Volume of pyramid, V $= \frac{1}{3}bh$

$V = \frac{1}{3} \times 9\text{m}^2 \times 4\text{m} = 12\text{m}^3$

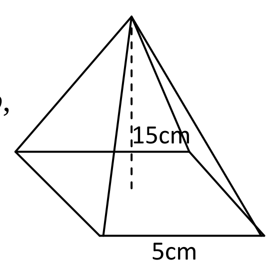
2. The height of a pyramid is 15cm. The side of the square base is 5cm. Find the volume.

Solution

$V = \frac{1}{3}bh$

But area of square base, b,

$b = 5\text{cm} \times 5\text{ cm} = 25\text{ cm}^2$



$V = \frac{1}{3} \times 25\text{cm}^2 \times 15\text{cm}$

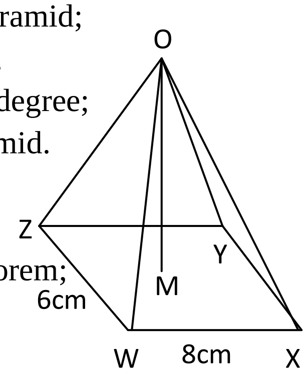
$V = 125\text{ cm}^3$

3. The diagram below shows a right pyramid with a rectangular base WXYZ and vertex O. If /WX/ = 8cm, /ZW/= 6 cm and /OX/ =13 cm, Calculate:

a. the height of the pyramid;

b. the value of <OXZ, correct to the nearest degree;

c. volume of the pyramid.



Solution

a. By Pythagoras theorem;

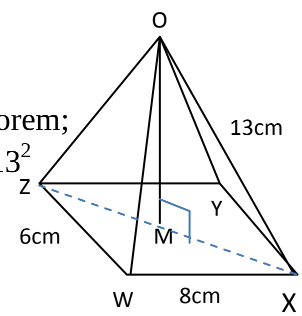
$/xz/^2 = 6^2 + 8^2$

$/xz/^2 = 100$

$/xz/ = \sqrt{100} = 10$cm

But $/MX/ = \frac{1}{2}/XZ/$

$= \frac{1}{2}(10) = 5$cm



By Pythagoras theorem;

$/OM/^2 + /MX/^2 = 13^2$

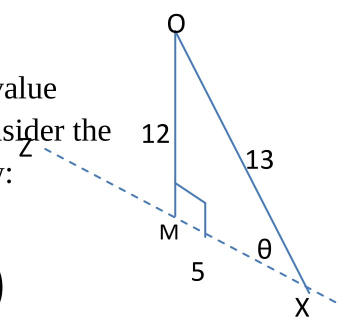
$/OM/^2 = 13^2 - 5^2$

$/OM/^2 = 144$

$/OM/ = \sqrt{144}$

OM = 12cm

b. To find the value of < OXZ, consider the diagram below:



$\cos\theta = \frac{5}{13}$

$\theta = cos^{-1}\left(\frac{5}{13}\right)$

$\theta = 67^0$

Volume of pyramid, V $= \frac{1}{3}bh$

But area of base, b = L B

$A = 6\text{ cm} \times 8\text{cm} = 48\text{cm}^2$

$V = \frac{1}{3}bh = \frac{1}{3} \times 48 \times 12 = 192\text{cm}^3$

Exercises 27.11

1. The length of the side of a square base of a pyramid is 5cm. If the height of the pyramid is 15cm. Calculate its volume.

2. A pyramid has a square base of side 8cm long. Its height is 10cm. Find its volume.

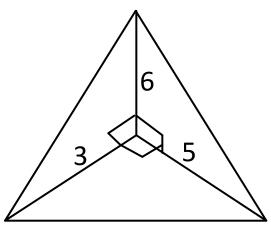
3. A pyramid has a rectangular base 15cm by 20cm and its height is 9cm. Find:

a. the area of the base

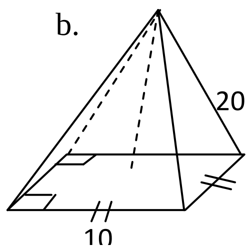
b. the volume of the pyramid

4. Calculate the volume of the figures below:

a.



b.



The Volume of a Cone

The formula for the volume of a cone is :

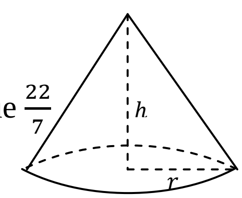
$\boldsymbol{V = \frac{1}{3}\pi r^2 h}$, where;

V is the volume of the cone

r is the radius of the cone

π (pi) is a constant with value $\frac{22}{7}$

h is the height of the cone.

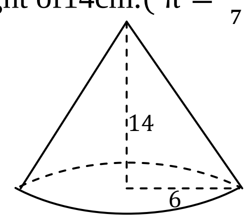


$$\boxed{V = \frac{1}{3}\pi r^2 h}$$

Worked Examples

1. Find the volume of a cone which has a base radius of 6cm and a height of14cm.($\pi = \frac{22}{7}$)

Solution



V = $\frac{1}{3}\pi r^2 h$

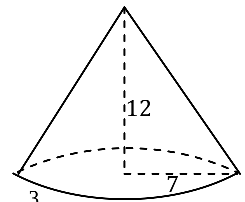
But r = 6 cm, h = 14 cm

V = $\frac{1}{3} \times \frac{22}{7} \times 6^2 \times 14$

V= 528cm^3

2. Calculate the volume of a cone which has a base radius of 7cm and a height of 12cm. $\pi = \frac{22}{7}$

Solution



V = $\frac{1}{3}\pi r^2 h$

But r = 7 cm, h = 12 cm

V = $\frac{1}{3} \times \frac{22}{7} \times 7^2 \times 12 = 616\text{cm}^3$

3. If the volume of a cone is $1{,}056\text{cm}^3$ and its height is 63cm, find its radius?($\pi = \frac{22}{7}$)

Solution

V = $\frac{1}{3}\pi r^2 h$,

But h = 63cm, r = ?, V = $1{,}056\text{cm}^3$ and $\pi = \frac{22}{7}$

$1{,}056\text{cm}^3 = \frac{1}{3} \times \frac{22}{7} \times r^2 \times 63\text{cm}$

$1{,}056\text{cm}^3 = 66\text{cm} \times r^2$

$r^2 = \frac{1056cm^3}{66cm}$

$r^2 = 16cm^2$,

$r = \sqrt{16cm^2} = 4\text{cm}$

4. The volume of a cone of height 7cm is 66cm^3. Find the radius of the cone.

Solution

$V = \frac{1}{3}\pi r^2 h$,

where $h = 7\text{cm}$, $r = ?$ $V = 66\text{cm}^3$

$66\text{cm}^3 = \frac{1}{3} \times \frac{22}{7} \times r^2 \times 7\text{cm}$

$198\text{cm}^3 = 22 \times r^2$

$r^2 = \frac{198cm^3}{22cm} = 9$

$r = \sqrt{9cm^2}$

r = 3 cm

5. The volume of a cone with base 5cm is 1,650cm^3. Calculate the height of the cone. $(\pi = \frac{22}{7})$

Solution

$V = \frac{1}{3}\pi r^2 h,$

But $r = 5\text{cm}, h = ?, V = 1{,}650\text{cm}^3$ and $\pi = \frac{22}{7}$

$1{,}650\text{cm}^3 = \frac{1}{3} \times \frac{22}{7} \times (5\text{cm})^2 \times h$

$1{,}650\text{cm}^3 = \frac{550\text{cm} \times \text{h}}{21}$

$1{,}650\text{cm}^3 \times 21 = 550\text{cm}^2 \times h$

$h = \frac{1650\ cm^3 \times 21}{550 cm^2} = 63\text{cm}$

6. A solid cone has base radius of 7cm. If the volume of the cone is 1,078cm^3, find the vertical height of the cone $(\pi = \frac{22}{7})$

Solution

$V = \frac{1}{3}\pi r^2 h$,

where $r = 7\text{cm}$, $\text{h} = ?$, $\text{V} = 1078\text{cm}^3$

$1078 = \frac{1}{3} \times \frac{22}{7} \times 7^2 \times h,$

$1078 = \frac{154}{3}h$

$3 \times 1{,}078 = 154h$

$h = \frac{3 \times 1078}{154} = 21\text{cm}$

Exercises 27.12

1. Calculate the height of a cone with volume 1,848cm^3 and radius 7cm.
2. What is the height of a cone whose volume is 88cm^3 and radius 22cm?
3. A cone has a radius of 14cm and height 28cm. Find its volume.
4. A cone has a height of 5.6cm and a radius of 5cm. Find the volume of the cone correct to 1decimal place **(Ans 146.7cm^3)**

Recasting a Solid

Recasting is the act of giving a solid or metal object a different form by melting it down and reshaping it. The amount of substance contained in the new solid is equal to the amount of substance in the old solid formed from it. Thus, though the shape of the object is changed, the volume remains the same.

Worked Examples

1. A solid metalcube of side 6cm is recast into asolid sphere. Find the radius of the sphere.

Solution

Volume of cube = Volume of Sphere

$L^3 = \frac{4}{3}\pi r^3$

$6^3 = \frac{4}{3}\pi r^3$

$216 = \frac{4}{3}\pi r^3$

$3 \times 216 = \frac{4}{3}\pi r^3 \times 3$

$648 = 4\pi r^3$

$\frac{648}{4} = \frac{4\pi r^3}{4}$

$162 = \pi r^3$

$r^3 = \frac{162}{\pi}$

$r^3 = \frac{162}{3.14}$

$r = \sqrt{\frac{162}{3.14}} = 3.72$ cm

2. A cone is 8cm high and radius 2 cm. The cone is melt and recast into a sphere. Find the diameter of the sphere.

Solution

Volume of cone = Volume of sphere

$\frac{1}{3}\pi r^2 h = \frac{4}{3}\pi r^3$

$\frac{1}{3}\pi (2)^2(8) = \frac{4}{3}\pi r^3$

$\frac{1}{3} \times 32\pi = \frac{4}{3}\pi r^3$

$32 = 4r^3$

$r^3 = \frac{32}{4}$

$r = \sqrt[3]{8} \quad = 2$ cm

But diameter of the sphere = $2r = 2 \times 2 = 4$cm

3. A solid metal cylinder of radius 6cm and height 18 cm is melted down and recast as a cone of radius 6cm. What is the height of the cone?

Solution

Volume of cylinder = $\pi r^2 h$

Volume of cone = $\frac{\pi r^2 h}{3}$

Volume of cylinder = Volume of cone

But radius of cylinder = 6 cm,

height of cylinder = 18 cm,

radius of cone = 6 cm,

height of cone = h

$\pi r^2 h = \frac{\pi r^2 h}{3}$

$\pi (6)^2 h = \frac{\pi\, 6^2 \times 18}{3}$

$\frac{h}{3} = 18$

$h = 3 \times 18$cm = 54cm

4. A solid metal sphere 12 cm in diameter is melted and recast into small solid metal spheres 4cm in diameter. How many of these smaller solid metal spheres will there be?

Solution

Volume of sphere = $\frac{4}{3}\pi r^3$

Large spheres

$r = \frac{d}{2} = \frac{12\ cm}{2} = 6$cm

$V = \frac{4}{3} \times \pi(6)^3 = 288\pi$

Small spheres

$r = \frac{d}{2} = \frac{4\ cm}{2} = 2$cm

$V = \frac{4}{3} \times \pi(2)^3 = \frac{32}{3}\pi$

Number of smaller spheres made from larger sphere

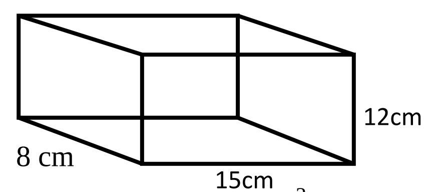
$= \frac{Volume\ of\ larger\ sphere}{Volume\ of\ smaller\ sphere} = \frac{288\pi}{32/_{3\pi}} = 27$ spheres

4. A solid rectangular block with dimension 15cm, 12cm and 8cm respectively is melted down completely and recast into a solid right circular cone of base radius 12cm. Calculate:

i. correct to two significant figures, the height of the cone ($\pi = 3.142$) .

ii. Use your results in (i) to calculate the slant height and hence the curved surface area of the cone, correct to two significant figures.

Solution



$V = LBW$

V= 15cm × 12cm × 8cm = 1,440cm^3

Volume of rectangular solid = Volume of cone

$\Rightarrow LBW = \frac{1}{3}\pi r^2 h$

$1,440 = \frac{1}{3}(3.142)(12)^2 h$

$h = \frac{3 \times 1440}{(3.142)(12)^2} = 9.5$ (2s.f.)

ii. Let the slant height of the cone be h

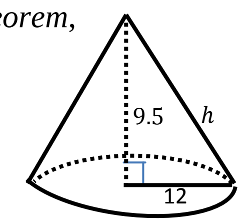
By Pythagoras theorem,

$h^2 = 9.5^2 + 12^2$

$h^2 = 234.25$

$h = \sqrt{234.25}$

$h = 15$cm (2 s.f)



The curved surface area of the cone**,**

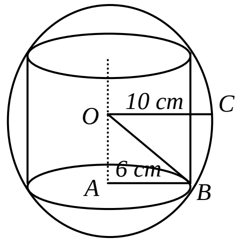
$A = \pi rl$

$A = (3.142) \times 12\text{cm} \times 15.4\ cm = 580\text{cm}^2$

5. From a sphere of radius 10cm, a right circular cylinder whose base diameter is 12 cm is carved

out. Calculate the volume of the right circular cylinder correct to two decimal places.

Solution



From the figure,

OB = OC = 10 cm (radius of sphere)

In ΔABC,

$OB^2 = OA^2 + AB^2$

$OA = \sqrt{10^2 - 6^2} = \sqrt{100 - 36} = \sqrt{64} = 8$ cm

Height of cylinder = 2 × OA = 2 × 8cm = 16 cm

Volume of right circular cylinder = $\pi r^2 h$

$= \frac{22}{7} \times 6^2 \times 16$cm

$= \frac{22}{7} \times 6$cm × 6 cm × 16cm

$= 1{,}810.29\text{cm}^3$

Exercises 27.12B

1. A solid metal ball of radius 2cm is melted and the metal obtained recast to form a right circular cone of radius 5 cm. Find the height of the cone.

2. Twenty – seven ball bearings of diameter 1 cm are melted down and recast to form a single sphere. Find its diameter.

3. A circular cone of height 12 cm and radius 9 cm is recast into a solid sphere. Calculate the surface area of the sphere.

4. A cone is 8.4cm high and the radius of its base is 2.1cm. It is melted and recast into a sphere. Find the radius of the cone.

5. A metallic sphere of radius 10.5cm is melted and recast into small cones, each of radius 3.5cm and height 3 cm. Find the number of cones thus obtained.

6. A metallic box of length 16cm, breadth 8cm and width 4 cm is recast into a solid cylinder of height 8cm. Calculate the surface area of the cylinder, to two decimal places.

Challenge Problem

The surface area of a solid metallic sphere is 1,256 cm^2. It is melted and recast into a solid right circular cone of radius 2.5 cm and height 8cm. Calculate;

i. the radius of the solid sphere

ii. the number of cones recast($\pi = 3.14$)

2. A 4cm × 4cm × 2 cm slab of cholcholate is melted then made into twenty thin cylindrical sticks of diameter 4mm. If 10% of the chocolate is wasted during the process, what is the length of one chocolate stick?

Frustum of a Right Circular Cone

A frustum of a cone or truncated coneis the results of cutting a cone by a plane parallel to the base and removing the part parallel to the apex as shown in the figure below:

r = radius of upper base

R = radius of lower base

s = slant height

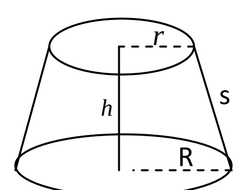
1. $\boldsymbol{s} = \sqrt{(R - r)^2 + h^2}$

2. Volume $= \frac{1}{3}\pi h(r^2 + rR + R^2)$

3. Lateral area/ curved surface area;

$= \pi\,(R + r)s$

$= \pi(R + r)\sqrt{(R - r)^2 + h^2}$



4. Total surface area, A

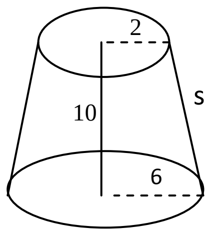
$A = \pi\,[s(R + r) + R^2 + r^2]$

Worked Examples

1. Calculate the lateral area, surface area and volume of a truncated cone of radii 2 cm and 6 cm and a height of 10cm.

Solution

R = 6 cm, r = 2 cmand h = 10 cm and $\pi = \frac{22}{7}$



Lateral area = $\pi (R + r)s$

But S = $\sqrt{(R-r)^2 + h^2}$

S = $\sqrt{(6-2)^2 + 10^2}$

S = 10.77 cm

Substitute s = 10.77,

R = 6 cm, r = 2 cm , $\pi = \frac{22}{7}$ in

Lateral area = $\pi (R + r)s$

$\Rightarrow \frac{22}{7}(2 + 6) \times 10.77 = 270.79 \text{ cm}^2$

Total surface area, A

$= \pi\,[s(R+r) + R^2 + r^2]$

$= \frac{22}{7}[10.77(6+2) + 6^2 + 2^2]$

$= \frac{22}{7}[86.16 + 36 + 4] = 396.50 \text{ cm}^2$

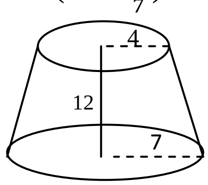
V = $\frac{1}{3}\pi h(r^2 + rR + R^2)$

V = $\frac{1}{3} \times \frac{22}{7} \times 10(2^2 + (2 \times 6) + 6^2)$

V = $\frac{1}{3} \times \frac{22}{7} \times 10(4 + 12 + 36)$

V = $\frac{1}{3} \times \frac{22}{7} \times 520 = 544.76 \text{ cm}^3$

2. In the figure below, find the volume of the frustum, if the radii of the bases are 4 cm and 7 cm and the height is 12 cm($\pi = \frac{22}{7}$)



Solution

Method 1

R = 7, r = 4, h = 12 and $\pi = \frac{22}{7}$ substitute in

V = $\frac{1}{3}\pi h(r^2 + rR + R^2)$

V = $\frac{1}{3} \times \frac{22}{7} \times 12(4^2 + (4 \times 7) + 7^2)$

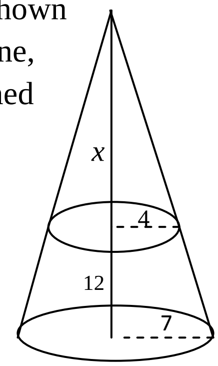
V = $\frac{1}{3} \times \frac{22}{7} \times 12(16 + 28 + 49)$

V = $\frac{1}{3} \times \frac{22}{7} \times 1{,}116 = 1{,}169 \text{ cm}^3$

Method 2

Complete the cone as shown

From the completed cone, similar figures are formed



$\Rightarrow \frac{x}{x+12} = \frac{4}{7}$

$4(x + 12) = 7x$

$4x + 48 = 7x$

$48 = 7x - 4x$

$48 = 3x$

$x = 16$ cm

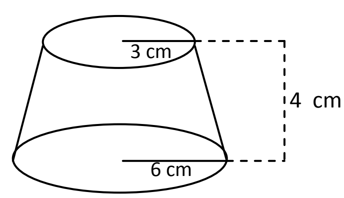
Volume of frustum;

= Vol of the complete cone – vol of the smaller cone

$= \frac{1}{3} \times \frac{22}{7} \times 7^2 \times (16 + 12) - \frac{1}{3} \times \frac{22}{7} \times 4^2 \times 16$

= 1, 437.33 – 268.19 = 1,169 cm^3

3. The figure below shows a cone whose upper part has been cut off. The base radius is 6 cm and the upper radius is 3 cm.



If the height of the remaining portion is 4 cm, calculate correct to the nearest whole number, the volume of the original cone.

Solution

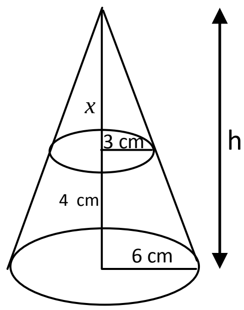
From the complete cone: similar right triangles are formed

$\Rightarrow \frac{x}{4} = \frac{3}{6}$

$6x = 4 \times 3$

$6x = 12$

$x = 2$



But $h = 4 + x$

$\Rightarrow h = 4 + 2 = 6$ cm

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times 6^2 \times 6 = 226 \text{ cm}^3$

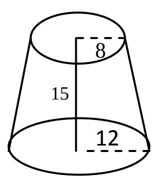
Exercises 27.13

1. Find the volume of the frustum of a cone with bases of radii 5 cm and 9 cm, given the height of the frustum as 6cm.

2. Calculate the lateral area, surface area and volume of a truncated cone of radii 10 cm and 12 cm and a slant height of 15 cm.

3. A frustum of height 5 cm is formed from a right circular cone whose height is 10 cm and whose base radius is 4 cm. Calculate the volume of the frustum.

4. In the figure below, calculate:
i. the lateral area of the figure,
ii. the surface area of the figure,
iii. the volume of the figure.



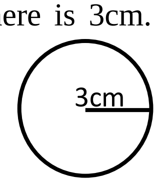
Volume of a Sphere

The formula for finding the volume of a sphere is: $\boldsymbol{V = \frac{4}{3}\pi r^3}$ where;

v is the volume, **π** is a constant and ***r*** is the radius.

Worked Examples

1. The radius of a sphere is 3cm. Calculate its volume.



Solution

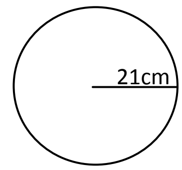
$V = \frac{4}{3}\pi r^3$

But But r = 3cm and $\pi = \frac{22}{7}$

$V = \frac{4}{3} \times \frac{22}{7} \times 3^3 = 113.14\text{cm}^3$

2. Calculate volume of a gold ring of radius 21cm.

Solution



$V = \frac{4}{3}\pi r^3$

But r = 21cm and $\pi = \frac{22}{7}$

$V = \frac{4}{3} \times \frac{22}{7} \times (21\text{cm})^3 = 38{,}808\text{cm}^3$

3. A circle has a radius of 14cm. Find
i. the surface area
ii. the volume of the circle.

Solution

i. Surface area, $A = 4\pi r^2$

$A = 4 \times \frac{22}{7} \times (14\text{cm})^2 = 2{,}464\text{cm}^2$

ii. $V = \frac{4}{3}\pi r^3$ where r = 14cm and $\pi = \frac{22}{7}$

$V = \frac{4}{3} \times \frac{22}{7} \times (14 \text{ cm})^3$

$V = 11{,}498.67\text{cm}^3$

Exercises 27.14

A. Calculate volume of a sphere with the following radii;

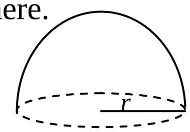
1) 4cm 2) 5cm 3) 6cm
4) 7cm 5) 8cm 6) 10cm

B. Calculate the radius of a sphere with the following volumes.

1) 36cm^3 2) 105cm^3 3) 400cm^3
4) 88cm^3 5) 2150cm^3 6) 616cm^3

The Hemisphere

A hemisphere is half of a full sphere.



1. Volume = $\frac{2}{3}\pi r^3$

2. Curved Surface Area; A = $2\pi r^2$

3. Total surface area, A = $3\pi r^2$

Worked Examples

A hemisphere has a radius of 3 cm. Calculate its:

i. volume,

ii. curved surface area,

iii. total surface area ($\pi = \frac{22}{7}$).

Solution

r = 3cm and $\pi = \frac{22}{7}$

i. V = $\frac{2}{3}\pi r^3$

V = $\frac{2}{3} \times \frac{22}{7} \times 3^3$ = 56.57 cm^3

ii. Curved Surface Area,

A = $2\pi r^2$

A = $2 \times \frac{22}{7} \times 3^2$ = 56.57 cm^2

iii. Total surface area,

A = $3\pi r^2$

A = $3 \times \frac{22}{7} \times 3^2$ = 84.86 cm^2

2. A water reservoir in the form of a cone mounted on a hemisphere is built such that the plane face of the hemisphere fits exactly to the base of the cone and the heiht of the cone is 6 times the radius of the its base.

a. Illustrate this information in a diagram.

b. If the volume of the reservoir is $333\frac{1}{3}\pi$ m^3, calculate, correct to the nearest whole number, the

i. volume of the hemisphere,

ii. total surface area of the reservoir ($\pi = \frac{22}{7}$).

Solution

b. Volume of the reservoir;

= Vol of the cone + Vol of the Hemisphere

$= \frac{1}{3}\pi r^2 h + \frac{2}{3}\pi r^3$

$= \frac{1}{3}\pi x^2 6x + \frac{2}{3}\pi x^3$

But Volume of the reservoir = $333\frac{1}{3}\pi$

$\Rightarrow \frac{1}{3}\pi x^2 6x + \frac{2}{3}\pi x^3 = 333\frac{1}{3}\pi$

$\frac{1}{3}\pi\, 6x^3 + \frac{2}{3}\pi x^3 = \frac{1000}{3}\pi$

$\pi\, 6x^3 + 2\pi x^3 = 1000\pi$

$6x^3 + 2x^3 = 1000$

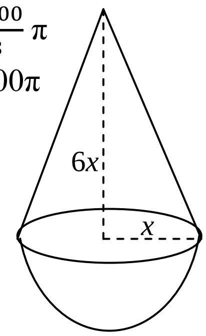
$\Rightarrow 8x^3 = 1000$

$x^3 = \frac{1000}{8}$

$x^3 = 125$

$x = \sqrt{125}$

$x = 5$



Radius of the cone/hemisphere, r = 5m

Height of the cone, $h = 6x = 6 \times 5$ = 30 m

i. Volume of the hemisphere , V = $\frac{2}{3}\pi r^3$

V = $\frac{2}{3} \times \frac{22}{7} \times 5^3$

V = 262 m^3 (nearest whole number)

ii. Total surface area of the reservoir;

= Curved surface area of the cone + curved surface area of the hemisphere

$= \pi rl + 2\pi r^2$

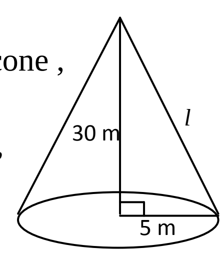
But slant height of the cone , l = ?

By Pythagoras theorem,

$l^2 = 30^2 + 5^2$

$l^2 = 925$

l = 30.4138 m



Substitute $r = 5$, $\pi = \frac{22}{7}$ and $l = 30.4138$ in

$A = \pi rl + 2\pi r^2$(Total surface area of reservoir)

$A = \frac{22}{7} \times 5 \times 30.4138 + 2 \times \frac{22}{7} \times 5^2$

$A = 477.9311 + 157.1429$

$A = 635$ m^2 (nearest whole number)

Exercises 27.15

1. A toy in the form of a cone mounted on a hemisphere of same radius 7cm. If the total height of the toy is 31 cm, find the total surface area.

2. A solid wooden toy in the shape of a rght circular cone mounted on a hemisphere. If the radius of the hemisphere is 4.2 cm and the total height of the toy is 10.2 cm, find the volume of the wooden toy?

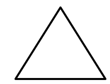
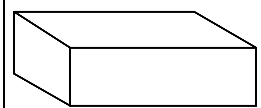
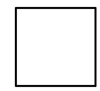
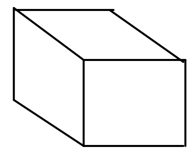
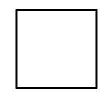
3. A vessel is in the form of a hollow hemisphere mounted by a hollow cylinder. The diameter of the hemisphere is 14 cm and the total height of the vessel is 13 cm. Find the inner surface area of the vessel.

4. A medicine capsule in the shape of a cylinder with two hemispheres stuck to each of its ends. The length of the entire capsule is 14 cm and the diameter of the capsule is 5 cm. Find its surface area.

Volume of a Prism

A prism is a solid object with identical ends, flat sides and the same cross section all along its length.

A cross section is the shape made by cutting straight across an object. A prism is said to be a polyhedron, meaning all sides are flat. Hence, a cylinder is not a prism because it has curved sides. The sides of a prism are parallelograms.

Prism	Shape	Cross section
Triangular		
Square		
Cubical		

The volume of a prism is the product of the area of one end and the length of the prism. This is expressed as :

V= Area of cross section × length

Worked Examples

A prism has a uniform cross section in the shape of a right – angle triangle whose two shorter sides are of lengths 2cm and 1.5cm. The length of the prism is 6cm. Write down:

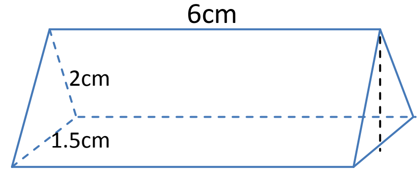
i. the area of the cross section;

ii. the volume of the prism.

Solution

i. Area of the cross section;

$A = \frac{1}{2} b\,h = \frac{1}{2} \times 1.5 \times 2 = 1.5\text{cm}^2$



Alternatively

$A = \frac{1}{2}\, b\, h \sin 90^0$

$A = \frac{1}{2}\,(1.5)\,(2) \sin 90^0 = 1.5\text{cm}^2$

ii. The volume of the prism, $V = Al$

But V = 1.5and l = 6

$V = 1.5\text{cm}^2 \times 6\text{cm} = 9\text{cm}^3$

Exercises 27.16

1. The figure below shows a prism of length 30 cm whose cross section is an equilateral triangle of side 4cm.

Calculate:

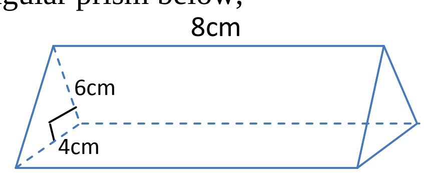
i. the area of the triangular end;

ii. the total surface area of the prism;

iii.the volume of the prism.

2. What is the volume of a prism whose ends have an area of $25m^2$ and which is $12m$ long

3. In the triangular prism below,



a. What is the area of the cross section of the prism?

b. What is the volume of the prism?

4. A rectangular prism is 13 cm long. If the volume of the prism is 936cm^3, find the area of one end of the prism

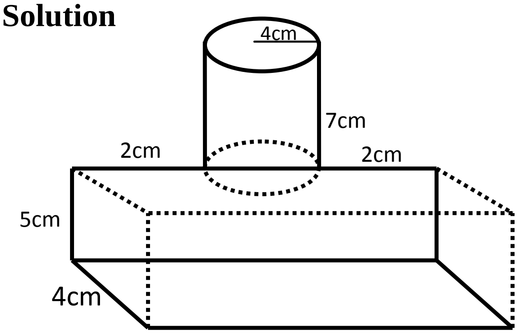
Volume of Combined Solids

Two or more solids may be combined to form a complex solid. In such solids, the volume is calculated by finding the sum of the volumes of the separate solids.

Worked Examples

1. Calculate the volume of the solid below.

Solution



Total volume of solid;

= Volume of cylinder + volume of cuboid

Volume of cylinder $= \pi r^2$

But $r = 4\text{cm}$ and $h = 7\text{cm}$

$V = \frac{22}{7} \times 4\text{cm}^2 \times 7\text{cm} = 352\text{cm}^3$

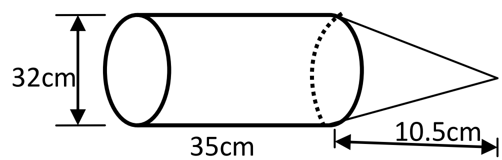
Volume of cuboids = L × W × H

But L = 12cm, w = 4cm and H = 5cm

$V = 12\text{cm} \times 4\text{cm} \times 5\text{cm} = 240\text{cm}^3$

Total volume $= 352\text{cm}^3 + 240\text{cm}^3 = 592\text{cm}^3$

2. In the figure below, calculate the total volume



Solution

Total volume;

= Volume of cylinder +Volume of cone

Volume of cylinder, $V = \boldsymbol{\pi r^2 h}$

But $r = \frac{d}{2} = \frac{32}{2} = 16\text{cm}, h = 35\text{cm}$

Volume of cylinder;

$= \frac{22}{7} \times (16\text{cm})^2 \times 35\text{cm} = 28{,}160\text{cm}^3$

Volume of cone, $V = \frac{1}{3}\pi r^2 h$,

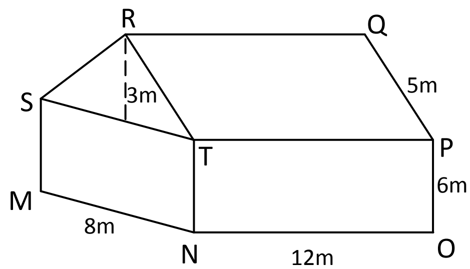
But h = 10.5cm

$V = \frac{1}{3} \times \frac{22}{7} \times 16^2 \times 10.5\text{cm} = 2{,}816\text{cm}^3$

Total Volume;

$= 28160\text{cm}^3 + 2816\text{cm}^3 = 30{,}976\ \text{cm}^3$

3. The diagram shows a hut used in storing grains which is in the shape of a triangular prism mounted on a cuboid. If the dimensions are as shown in the diagram,



Calculate;

a. the volume of the hut.

b. total external surface area of the hut.

Solution

Volume of Prism,

V = Area of cross – section × length

Area of cross – section, A,

= Area of triangle + area of rectangle

A = $\frac{1}{2}bh$ + L × B

A = $\frac{1}{2} \times 8 \times 3 + 8 \times 6 = 12 + 48 = 60\text{m}^2$

But length, L = 12m

V = A × L

V = $60\text{m}^2 \times 12\text{m} = 720\text{m}^3$

ii. Total surface area = area of the net

2(SRT) + 2(NMST) + 2(NOPT) + 2(QRTP)

2($\frac{1}{2}$ BH) + 2(LB) + 2(LB) + 2(LB)

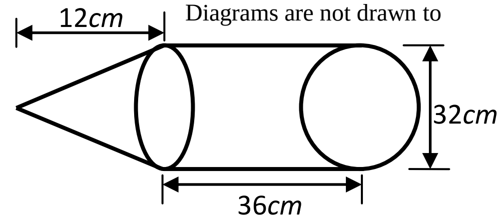
= 2($\frac{1}{2} \times 3 \times 8$) + 2(8× 6) + 2(12 × 6) + 2(5 × 12)

= 24 + 96 + 144 + 120 = 384 cm^2

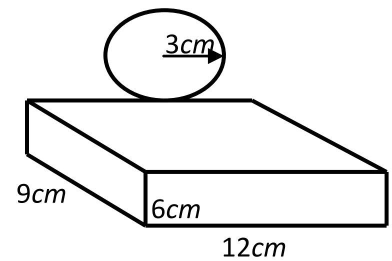
Exercises 27.17

Calculate the total volume of the solids below $\pi = \frac{22}{7}$

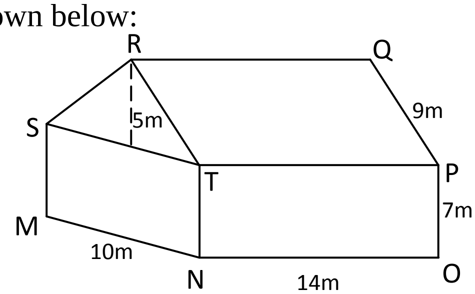
1. Diagrams are not drawn to



2.

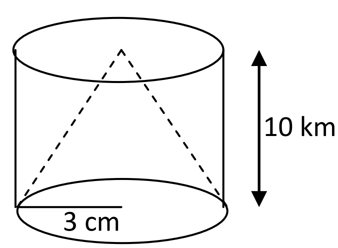


3. A model house is made by sticking a triangular prism on top of a rectangular block as shown below:



Calculate the volume of the model house.

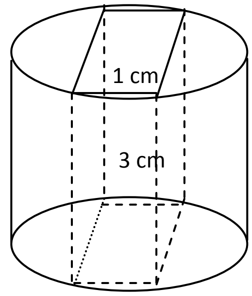
4. A cone is contained in a cylinder so that their bases and heights are the same as shown below:



Calculate the volume of the space lying in – between the cylinder and the cone ($\pi = \frac{22}{7}$)

6. A tent in the shape of a cylinder surmounted by a conical top. If the height and the diameter of the cylindrical part are 2.1 m and 4 m respectively, and the slant height of the top is 2.8 m, find:
i.the area of the canvas used for making the tent.
ii. the cost of the canvas of the tent at a rate of Gh¢500.00 per m^2

5. The figure below shows a cylinder circumscribing about a prism with a square base.



If the side of the square base of the prism is 1 cm and its height is 3 cm, find the volume and total surface area of the cylinder

28 LOGIC AND REASONING

Statements

A statement is a sentence which is either true or false, but not both.

Consider the following sentences:

1. Please sit down.
2. How old are you?
3. Mr. Osei is a hard working teacher.
4. Kumasi is the capital town of Ghana
5. A triangle has four sides.
6. The baby is either a boy or a girl.

Sentences 1 and 2 are not statements because they are neither true nor false.

Sentences 3, 4 and 5 are statements because they can either be true or false, depending on the context. The sixth sentence, is obviously a true statement.

Exercises 27.1

A. Which of the sentences are statements?

1. Ghana is in Africa.
2. All primes are odd numbers.
3. Will you go to school tomorrow?
4. 5 = 3 mod 3.
5. 3 is an even number.
6. 10 is a multiple of 5.

B. Which of the following sentences are statements? If they are statements, state whether they are true or false

1. Do your mathematics exercises.
2. There are 52 weeks in a year.
3. Drive with care.
4. $\sin 30^0 = \sin 60^0$.
5. All prime numbers can only be divided by 1 and itself.
6. 22 is a composite number.

Notation of Statements

In mathematics, statements are denoted by letters such as P, Q and R. For example, P: 5 + 6 = 13, simply means P is the statement 5 + 6 = 13. P is obviously a false statement because $5 + 6 \neq 13$.

Negation of a Statement

Consider the statement P: Linda is a girl. The statement "Linda is not a girl" is called the negation of P and it is denoted by ∼P, where the symbol ∼ means "not" and ∼P is read as "not P". Hence ∼P:"Linda is not a girl". Obviously, when P is true, the negation of P is false and vice – versa.

Worked Examples

Write down the negation of the statements

1. P : $5 \in$ {primes} 2. $7 \notin$ {even integers}.
3. P : $x < 2$ 4. P : $3 < x < 4$

Solution

1. ∼P : $5 \notin$ {primes} 2. ∼P : $7 \in$ {even integers}
3. ∼P : $x \nless 2$, or ∼P: $x \ngeq 2$ 4. ∼P : $x \leq 3$, or $x \geq 4$

Exercises 27.2

Write down the negation of each of the following statements and state whether the negation is true or false;

1. P : $3 \in$ {primes} 2. P : $4 \in$ {factors of 10}
3. P : $5 < 3$ 4. P : $x > 3$
5. P : $4 = 2$ 6. P : $-4 \leq x \leq 5$

Statements and Venn Diagrams

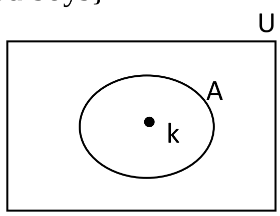
Similar to sets, statements can be represented in Venn diagrams. Identify the following from the given statements:

1. Universal set.

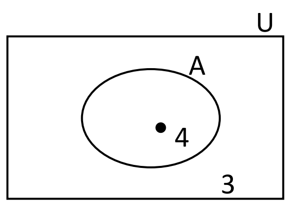
2. Subsets.
3. Complement.

Worked Examples
Represent each of the following statements on a Venn diagram.
i. P : Kofi is a good boy.
ii. Q : 4 is a factor of 100 but 3 is not.

Solution
i. U = {boys} A = {Good boys} k = Kofi
$k \in$ {Good boys}



ii. U = {Integers} A = {factors of 100}
4 $\in$ {factors of 100} A^1 = {3}



Exercises 27.3
Represent each of the following statements on a Venn diagram.
1. P : A square is a quadrilateral.
2. Q : Rich people are hardworking people.
3.R : Every science student studies mathematics but not history.
4. S : All my friends are intelligent people.
5. T : Every multiple of 10 is also a multiple of 5.

B. Illustrate on a Venn diagram.
1. P : All football players play volley ball.
Q : A volley ball player does not play hockey Blay plays volleyball but not football.

2. P : All my friends are intelligent.
Q : Intelligent people are quite people.
R : Dufie is a quite person but not intelligent.

3. P : A vegetarian does not eat meat.
Q : All my friends are vegetarians.
R : Jonathan is not a vegetarian but does not eat meat.
S : Rosemond eats meat.
T : Brown is a vegetarian but he is not my friend.

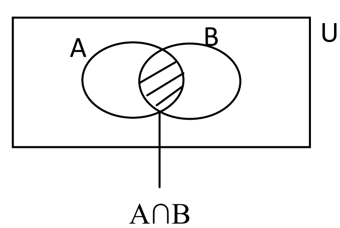
4. P : Rich men live in big houses.
Q : Mr. Owusu is a rich man.
R : Mr. Owusu lives in a big house.

5. P : All boys play football.
Q : Danny plays football.
R : Danny is a boy.

Compound Statements
A compound statement is a combination of two or more statements in a single statement. They are usually formed by the conjunction "and" and "or". For example, P : $x > -4$, Q : $x < 3$. The compound statement P and Q is the statement "$x > -4$" and "$x < 3$" written as P and Q : $-4 < x < 3$

In general, P and Q is denoted by $P \wedge Q$, where "$\wedge$" means "and"

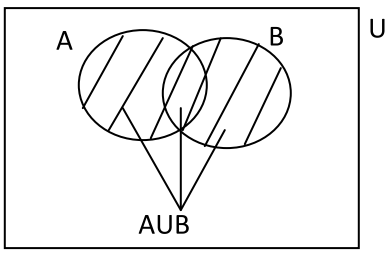
If P is a statement about a subset A and Q is a statement about subset B, then $P \wedge Q$, is a statement about the subset $A \cap B$ represented in the diagram below:



The compound statement "P and Q" is true only if both P and Q are true and false only if either P or Q is false.

Similarly, the compound statement P or Q is the statement "$x > -4$ " or "$x < 3$" written as:
P and Q : $-4 < x < 3$

In general, P or Q is denoted by $P \vee Q$, where " $\vee$ " means "or"
If P is a statement about a subset A and Q is a statement about subset B, then $P \vee Q$, is a statement about the set $A \cup B$.



The compound statements "P or Q" is true only if either P or Q is true and false if P is false and Q is also false.

Worked Examples
A. From the following pair of statements, form a compound statement using **"and"** and determine whether the compound statement is true or false.
1. P : A square has four sides, Q : A triangle has three sides
2. $P : 6 + 7 = 13$, $Q : 5 + 4 > 8$
3. $P : 4 \in$ {primes}, $Q : 4 \in$ {even numbers }

B. From the following pair of statements, form a compound statement using "or" and determine whether the compound statement is true or false.
1. $P : 4 \in$ {primes}, $Q : 4 \in$ {even numbers}
2. $P : 3 \in$ {factors of 27}, $Q : 5 \in$ {multiples of 10}
3. $P : 2 \in$ {odd numbers}, $Q : 2 \in$ {factors of 15}

Solution
A. 1. $P \wedge Q$: A square has four sides, and a triangle has three sides. $P \wedge Q$ is true, because P and Q are both true.

2. $P \wedge Q$: $6 + 7 = 13$ and $5 + 4 > 8$. $P \wedge Q$ is true, because both P and Q are true.

3. $P \wedge Q$: $4 \in$ {primes}$\cap$ {even numbers }
$P \wedge Q$ is false since P is false.

B. 1. $P \vee Q$: $4 \in$ {primes}$\cup${even numbers}
$\Rightarrow$ 4 is either a prime or an even number
$P \vee Q$ is true, since P is true

2. $P \vee Q$: 3 is a factors of 27 or 5 is a multiples of 10.
$P \vee Q$ is true since Q is true

3. $P \vee Q$: $2 \in$ {odd numbers} $\cup$ {factors of 15}
$\Rightarrow$ 2 is either an odd number or 2 is a factor of 15
$P \vee Q$ is false since P and Q are false.

Exercises 27.4
A. From the following pair of statements, form a compound statement using "and" and determine whether the compound statement is true or false.
1. P: $2 \in$ {primes}, Q : $2 \in$ { even number}
2. P: $3 \in$ {factors of 6}, Q :$3 \in$ { factors of 9}
3. P : $2 > -4 + 5$, Q : $-12 + 16 > 3$
4. P : $6 \notin$ {primes}, Q : $6 \in$ { composite}
5. P : $10 \in$ {even}, Q : $5 \in$ { primes}

B. From the following pair of statements, form a compound statement using "or" and

determine whether the compound statement is true or false

1. P : $\frac{1}{2} \in$ {integers}, Q : 2 $\in$ { prime numbers}
2. P : 5 $\in$ {factors of 14}, Q : 5 $\in$ {multiples of 10}.
3. P : 3< 2, Q : 2<3
4. P : 15 $\in$ {primes}, Q : 5$\in$ { prime}
5. P : 7 $\in$ {perfect squares}, Q : 8$\in$ { perfect cubes}

Implication

Consider the following statements;

1. If it is a dog, then it is an animal with four legs.
2. If two triangles are congruent then the two triangles are similar.
3. If $a = b$, then $a^2 = b^2$.
4. If he has passed the examination, then he is promoted.

Each of these sentences consist of two clauses – the "if clause" and the "then clause". Each of the two clauses is a statement in itself.

Let us, consider the "if clause" as statement P and the "then clause" as statement Q. It is observed that in each case, if statement P is true, then it implies that statement Q is also true. This is expressed as P implies Q, symbolically P $\Rightarrow$ Q. Sentences of the type P $\Rightarrow$ Q are referred to as ***implication.***

Statements 1, 2 and 3 may appear as follows:

1. It is a dog $\Rightarrow$ It is an animal with 4 legs
2. Two triangles are congruent $\Rightarrow$ Two triangles are similar
3. $a = b, \Rightarrow a^2 = b^2$

Note that the arrow "$\Rightarrow$" always points at the"then clause"

Worked Examples

Which implications are true or false;

1. $x^2 > 4$, then $x > 2$
2. If a number is a perfect square, then it is positive

Solution

1. False, $(-3)^2 > 4$, (But $-3 < 2$)
2. True

Converse of an Implication

For all implications, the arrow "$\Rightarrow$" always points at the "then clause" as shown in the following implications:

1. It is a dog$\Rightarrow$It is an animal with 4 legs.
2. Two triangles are congruent $\Rightarrow$ Two triangles are similar.
3. $a = b, \Rightarrow a^2 = b^2$

If the arrow is reversed like this "$\Leftarrow$", the implications appear as follows:

a. It is a dog $\Leftarrow$ It is an animal with four legs.
b. Two triangles are congruent $\Leftarrow$ Two triangles are similar.
c. $a = b \Leftarrow a^2 = b^2$

Implications *a*, *b* and *c* are the respective converses of the implications 1, 2 and 3

For converse (*a*);

a. It is a dog $\Leftarrow$ Ithas four legs

This means that "if it has four legs, then it is a dog"

The implication is false, because not only dogs have four legs. Since this implication is not true, it is written as:

It is a dog $\nLeftarrow$ It has four legs

Equivalent Statements

Consider the statement below:

‘If he passed the examination, then he is promoted’

1. The implication is:

‘He has passed the examination$\Rightarrow$ He is promoted’. This is a true statement.

2. The converse is ;

He has passed the examination$\Leftarrow$ He is promoted. This is also a true statement.

This implies that the implication as well as its converse is true. The implication and its converse can be simplified by the use of the arrow “$\Leftrightarrow$”. Thus, the implication appears as:

‘He has passed the examination$\Leftrightarrow$ He is promoted’

Generally, when an implication P $\Rightarrow$ Q and its converse Q $\Rightarrow$ P are both true, the statements P and Q are said to be ***equivalent,*** expressed as P$\Leftrightarrow$Q

Note:

P $\Rightarrow$ Q means, P implies Q.

P $\Leftarrow$ Q means, P is implied by Q.

P $\Leftrightarrow$ Q means, P implies and is implied by B.

Worked Examples

1. Two statements P and Q are defined by;

P : an angle is 90^0

Q : an angle is a right angle

Write down the following implication in full and determine whether P and Q are equivalent

a. P $\Rightarrow$Q   b. Q $\Rightarrow$ P   c. P $\Leftrightarrow$ Q

Solution

a. P $\Rightarrow$Q : If an angle is 90^0, then it is a right angle

b. Q $\Rightarrow$P : If an angle is a right angle, then it is 90^0

c. P $\Leftrightarrow$Q : If an angle is 90^0, then it is a right angle and vice – versa.

Since P $\Rightarrow$ Q and Q $\Rightarrow$P are true, P $\Leftrightarrow$Q is also true and hence P and Q are equivalent.

2. Two statements P and Q are defined by

P : $x \in$\{even numbers\}

Q : $4x \in$ \{even numbers \}

Determine whether P and Q are equivalent

Solution

P $\Rightarrow$Q : If x is even , then $4x$ is even

Q $\Rightarrow$P : If $4x$ is even , then x is even

P $\Rightarrow$Q is true, but Q $\Rightarrow$ P is false since 4×1 $\in$\{even numbers\} but $1 \notin$ \{even numbers\}

Hence P $\Leftrightarrow$ Q is false. Therefore P and Q are not equivalent

Exercises 27.5

A. For the following pair of statements P and Q, determine whether P and Q are equivalent

1. P : The sides of a triangle are equal
   Q : The sum of two angles of a triangle is 120^0
2. P : $\Delta\ ABC$ is isosceles triangle
   Q : $\Delta\ ABC$ is an equilateral triangle
3. P : A number is divisible by 10
   Q : The factors of a number include 2 and 5
4. P : $x^2 - 3x + 2 = 0$, Q : $x = 1$ or $x = 2$
5. P : $x + 2 > 3$, Q : $x + 1 > 3$

B. State whether the following implications are true or false

1. $x = 4 \Rightarrow x^2 = 16$ 2. $x^2 = 16 \Rightarrow x = 4$
3. $x < 0 \Rightarrow x^2 > 0$ 4. $x > 0 \Rightarrow x^2 > 0$
5. $x^2 > 0 \Rightarrow x > 0$
6. A polygon has n sides $\Leftrightarrow$It has n angles
7. Each angle of a triangle is $60^0$$\Leftrightarrow$It is an equilateral triangle
8. $\overline{PD}$ is the mediator of $\overline{AB}$$\Leftarrow$$\overline{PD}$ is perpendicular to $\overline{AB}$

B. Fill in the blank space with $\Rightarrow$, $\Leftarrow$ or $\Leftrightarrow$ where appropriate, if none is applicable, use $\nLeftrightarrow$

$n(A) = n(B)$		A and B are equivalent sets
$(a + b)^2 = 16$		$(ab)^2 = 16$
$x + y > 3$		$x > 3$ or $y > 3$
Area of a square is 25cm^2		The perimeter of the square is 20cm
$A = \{x : 1 < x < 6$, x is an integer$\}$		$A = \{2, 3, 4, 5\}$
It is a quadrilateral with 4 right angles		It is a square

C. Fill in the blanks of the following:

1. If he is a taxi driver, then he drives well
Paa is a taxi driver, therefore …..

2. If a triangle has two equal angles, then it is an isosceles triangle
In ΔABC, $< A = < B$. Therefore …

3. If a quadrilateral is a parallelogram, then its diagonal bisects each other. A rhombus is a parallelogram. Therefore ….

The Chain Rule of Implication

Consider the statements that P, Q and R are defined by :

P : $x > 7$, Q : $x > 5$, R : $x > 3$ and the implications

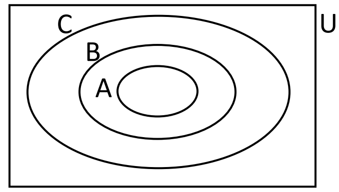
P $\Rightarrow$Q : $x > 7 \Rightarrow x > 5$

Q $\Rightarrow$R : $x > 5 \Rightarrow x > 3$

This can be represented in a Venn diagram as shown below:

$\cup$ = {numbers} , A = {numbers > 7}

B = {numbers > 5}, C = {numbers > 3}



It is observed that $A \subset B \subset C \Rightarrow A \subset C$

Thus, if a number is greater 7, then it is also greater than 3.

Generally, if $P \Rightarrow Q$ and $Q \Rightarrow R$, the conclusively, $P \Rightarrow R$. This is called the chain rule of implication, written as $P \Rightarrow Q \Rightarrow R$.

The chain rule can be extended to more than three statements .

Worked Examples

The following implications are true in a certain college;

* If a student studies science subjects, he studies mathematics

* If a student studies mathematics, he does not study history.

i. Represent the implications symbolically and use the chain rule of implication to deduce a valid implication.

ii. Illustrate in a diagram.

Solution

i. P : a student studies science subject

Q : a student studies mathematics

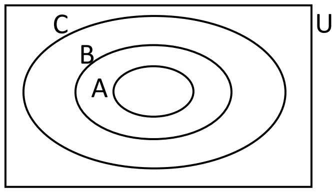
R : a student does not study history

The implications are:

$P \Rightarrow Q$; $\quad Q \Rightarrow R$

The implication P $\Rightarrow$ R is valid, because P $\Rightarrow$R : If a student studies science subjects, he does not study history

ii. U = {students in the college}
A = {Science students}
B = {mathematics students}
C = {non – history students}



Exercises 27.6

1. Consider the following statements:
P : x ∈{natural numbers }
Q : x ∈{rational numbers }
R : x ∈{ integers }
i. Form valid implications from the statements P, Q and R
ii. Form a chain of implication.

2. The following statements are true of inhabitants of Toase;
S_1 : only persons over 21 years pay basic rate.
S_2 only persons who pay basic rate can register as voters.
S_3 : only persons who register as voters can vote in an election.
i. Express the statements S_1, S_2 and S_3 as implications.
ii. Construct a chain of implications.
iii. Which of the following implications are valid deductions from S_1, $S_2$and S_3?
a. P_1 : every voter in an election is over 21 years old.
b. P_2 : every person who pays basic rate can vote in an election.
c. P_3 : every person who registers as a voter pays basic rate.
d. P_4 : every person over 21 years of age can register as a voter .

Truth Tables

A truth table is a table which gives all the truth values of a compound statement. The table is filled in by considering all possible combinations of true and false for *P* and *Q* and then filling in the results for the various connectors mentioned above.

The truth table for Negation Statement

P	~P
T	F
F	T

If P is true, then its negation ~P is false and vice versa.

The "and" Truth Table

Below is the truth table for the conjunction of two simple statements *p* and *q*.

P	Q	P ∧ Q
T	T	T
T	F	F
F	T	F
F	F	F

Worked Examples

Use truth table to show whether each of the following statement is true or false.
i. 10 is divisible by 5 and 10 is a multiple of 2.
ii. 2 is a factor of 15 and 10 is a multiple of 2.
iii. 10 is a multiple of 2and 2 is a factor of 15
iv. 10 is not a multiple of 2 and 2 is a factor of 15

Solution

i. 10 is divisible by 5 and 10 is a multiple of 2.
P : 10 is divisible by 5 = T
Q : 10 is a multiple of 2 = T
The statement is true.

P	Q	$P \wedge Q$
T	T	T

ii. 2 is a factor of 15 and 10 is a multiple of 2.
P : 2 is a factor of 15 = F
Q : 10 is a multiple of 2 = T
The statement is false.

P	Q	$P \wedge Q$
F	T	F

iii. 10 is a multiple of 2 and 2 is a factor of 15.
P : 10 is a multiple of 2 = T
Q : 2 is a factor of 15 = F
The statement is false.

P	Q	$P \wedge Q$
T	F	F

iv. 10 is not a multiple of 2 and 2 is a factor of 15.

P : 10 is not a multiple of 2 = F
Q : 2 is a factor of 15 = F
The statement is false.

P	Q	$P \wedge Q$
F	F	F

Exercises 27.7

Use truth table to show whether each of the following statement is true or false.

1. 4 is a perfect square and four is the square root of 16.
2. x is odd, and x^2 is odd.
3. x is a prime number greater than 2 and x is odd.
4. x is a prime factor of 30 and x is a prime factor of 40.
5. x is a prime factor of 18 and x is a prime factor of 24.

The "or" Table

Below is the truth table for the disjunction of two simple statements *p* or *q*.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Worked Examples

Use truth table to show whether each of the following statement is true or false.

1. 2 is either an odd number or a factor of 15

Solution

2 is either an odd number or a factor of 15
P : 2 is an odd number = F
Q : 2 is a factor of 15 = F

P	Q	$P \vee Q$
F	F	F

The statement is **false**

2. A triangle has three sides, or a parallelogram has three sides.

Solution

P : A triangle has three sides = T
Q : A paralellogram has three sides = F

P	Q	$P \vee Q$
T	F	T

The statement is **true**

Exercises 27.8

Use truth table to show whether each of the following statement is true or false.

1. An acute angle is 90^0 or a right angle is between 90^0
2. The sum of interior angles of a triangle is less than 100^0, or an octagon has seven vertices
3. x is even or x^2 is odd.
4. 6 is a perfect square or eight is the square root of 16
5. The product of two numbers is even, or the sum of both numbers is even.
6. 83 is an odd number or 38 is a composite number.
7. 169 is a perfect square or 255 is a perfect square.

If, Only if, and If and Only if Table

P	Q	P ⇒Q	P ⇐Q	P⇔Q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note that the $P \Rightarrow Q$ column can be obtained from $(P \Rightarrow Q) \wedge (P \Leftarrow Q)$.

Worked Examples

Use truth table to show whether the statement is true or false.

"The product of two numbers is even if and only if both numbers are even"

Solution

P : the product of two numbers is even

Q : both numbers are even ,

P ⇒Q : If the product of two numbers is even then both numbers are even. T

P ⇐Q : If two numbers are even, then their product is even. T

P	Q	P ⇒Q	P ⇐Q	P⇔Q
T	T	T	T	T

The statement is true.

2. If 2 is an odd number then 2 is not a prime number.

Solution

P : 2 is an odd number F

Q : 2 is not a prime number F

The statement is true

3. If the number 21 is divisible by 10, then 10 is divisible by 21

Solution

P : 21 is divisible by 10 F

Q : 10 is divisible by 21 F

The statement is true

4. If the number 121 is divisible by 11, then 5 is divisible by 3

Solution

P : 121 is divisible by 11 T

Q : 5 is divisible by 3 F

The statement is false

5. If a triangle is a polygon, then a polygon is a triangle

Solution

P : A triangle is a polygon T

Q : A polygon is a triangle F

The statement is false

Exercises 27.9 A

Use true or false for each

1. A whole number is divisible by ten if and only if its decimal numeral ends with zero.
2. Two lines are parallel if and only if they do not meet, however far they are extended.
3. Two figures have the same area if they are congruent.
4. The product of two numbers is even if and only if both numbers are even.

B.1. If two acute angles are congruent and complementary then their measure i 45^0. T
2. If 8 is divisible by 5, then 8 is a factor of 5. T
3. If a rectangle is a parallelogram, then a square is parallelogram
4. If 6 is a factor of 36, then 6 is a perfect square
5. If 14 is a prime number, then 97 is a an odd number
6. If a kite has congruent angles, then a rectangle has congruent angles

Negation and Contrapositive Table

P	Q	¬P	¬Q	P⇒ Q	¬Q⇒ ¬P
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Exercises 27.9B

A. In the following exercises, *P*, *Q*, *R*, and *S* will represent truth statements.
Construct the truth tables for the following statements, and give their converse and contrapositive:
1. ¬P ⇒Q
2. P ⇒ ¬ Q
3. (P ∨ Q) ⇒ R
4. (P ∧ Q) ⇒ (R∨ S)
5. (P ⇒ Q) ⇐ (R⇒*S*)

B. Negate the following statements:
1. P ∧ Q
2. (P ∨ Q) ∧ (R∧ S)
2. (P ∧ ¬ Q) ∨ (¬R ∨ S)

Valid Arguments

A valid argument is a series of statements in which each statement follows logically from the preceding ones. Thus, an argument begins with a given statement and ends with a final statement (conclusion).

The student's duty is to determine whether the argument is valid/logical or invalid/ illogical by analyzing the given statement to see if it matches with the conclusion. This is easily determined by representing the statement in a Venn diagram.

In order to avoid making false deductions or conclusions, mathematicians assumes that all the premises (generalization, definitions..) are true

Worked Examples

1. Determine whether the following statement is valid or not:
If he is a good tennis player, he can run well.
Nana Amoah is a good tennis player. Therefore, Nana Amoah can run well.

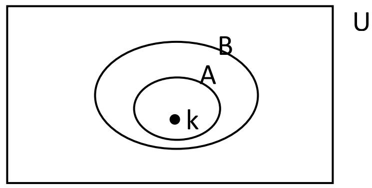
Solution

Rewrite the statement as follows:
P : All good tennis players can run well.
Q : Nana Amoah is a good tennis player.
Therefore, Nana Amoah can run well.

The statement P can be represented in the following Venn diagram:
∪ = {people}*A* = {good tennis players}
B = {goodrunners}

Nana Amoah = k

$k \in$ A,



A ⊂ B (because all good tennis players are good runners)

From the diagram, since $k \in A$, we conclude that $k \in$ B. This confirms that Nana Amoah is a good runner.

2. Obtain a valid conclusion from the following statements:

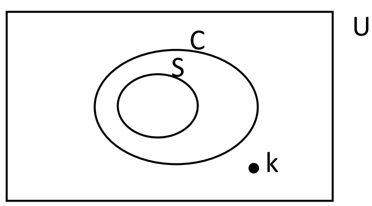
P : Every science student study Chemistry.

Q : Brown does not study chemistry.

Solution

U = {students}, C = {chemistry students}

S = {science students}, Brown = k, $k \notin$C,



From the Venn diagram, we conclude that 'Brown is not a science student'

3. i. Illustrate the statements P Q and R on a Venn diagram

P : All my friends are religious

Q : Religious people are honest people

R : Tom is an honest person since he is my friend

ii. Determine whether R is a valid conclusion from P and Q

Solution

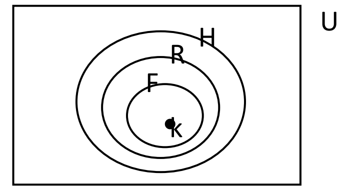
i. U = {people}

F = {my friends} R = {religious people}

H = {honest people}

b = Tom, b∈F,

F⊂ R⊂ H



Exercises 27.10

A. Study the arguments and state whether they are valid by using Venn diagrams to illustrate your answers;

1. All odd numbers are whole numbers.
3 is an odd number.
∴ 3 is a whole number.

2. All boys over 15 years acan wear long pants to school.
Ablo is 16 years old.
∴ Ablo can wear long pants to school

3. All flowers are beautiful to look at.
Roses are flowers.
∴ Roses are beautiful to look at

4. All locally manufactured goods are marked "Made in Ghana"
The portable radio is marked, "Made in China"
∴ the portable radio is not manufactured in Ghana

5. All squares are rhombuses
ABCD is not a rhombus
∴ *ABCD* is not a square

6. All triangles having at least two equal sides are isosceles triangles.
An equilateral triangle has three equal sides.
∴ an equilateral triangle is an isosceles triangle.

B. In each case,draw a Venn diagram to determine whether the statement, Q is a valid conclusion from the statement P.

1. P : Lazy students fail their examination.

Q : Aku is lazy since she failed her examination.

2. P : In Tanokrom, every old man uses a walking stick.

Q : Atiah is an oldman since he uses a walking stick.

3. P : My friend are all clever people

Q :Amuzu is a clever person since he is my friend.

4. P : People who lives in glass house donot throw stones.

Q : David lives in a glass since he does not throw stones.

5. P : When it rains , there is no school

Q : Yesterday it rained since there was no school.

C. Each case, determine by drawing Venn diagrams whether the statement R is a valid conclusion from the statements P and Q

1. P : Rich men live in big houses

Q : Mr. Owusu lives in a big house

R: Mr. Owusu is a rich man

2. P : Allboys like football

Q : Dzifa likes football

R : Dzifa is a boy

3. P : Basketball people are tall people.

Q : Neos is a basketball player.

R : Neos is tall.

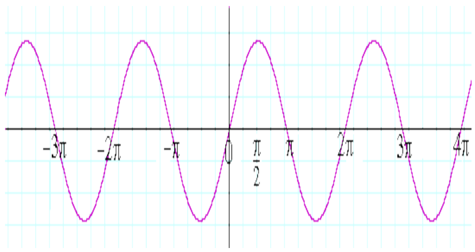
4. P : Vegetarians do not eat meat.

Q : Hindus are vegetarians.

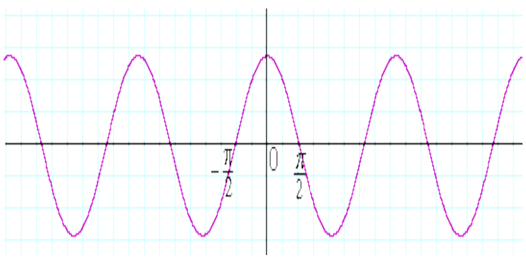
R : Hindus do not eat meat.

Graphs of Trigonometric Functions

Here is the graph of $y = \sin \theta$:

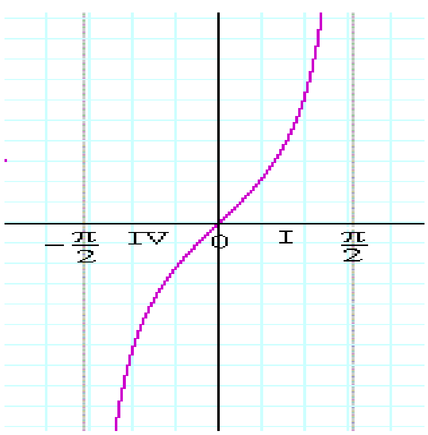


The graph of $y = \cos \theta$



The graph of $y = \cos x$ is the graph of $y = \sin \theta$ shifted, or translated

The graph of $y = \tan \theta$



Worked Examples

1. Copy and complete the table of values for $y = 1 - 4 \cos x$, to one decimal place for $0^0 \leq x \leq 300^0$

x	**0^0**	**30^0**	**60^0**	**90^0**	**120^0**	**150^0**
y	-3.0			1.0		
x	**180^0**	**210^0**	**240^0**	**270^0**	**300^0**	
y		4.5			-1.0	

c. Use the graph

i. to solve the equation $1 - 4 \cos x = 0$

ii. find the value of y when $x = 105^0$

iii. find x when $y = 1.5$

Solution

In $y = 1 - 4 \cos x$

When $x = 30^0$, $y = 1 - 4 \cos 30^0$

$y = - 2.5$

When $x = 60^0$, $y = 1 - 4 \cos 60^0$

$y = - 1.0$

When $x = 120^0$, $y = 1 - 4 \cos 120^0$

$y = 3.0$

When $x = 150^0$, $y = 1 - 4 \cos 150^0$

$y = 4.5$

When $x = 180^0$, $y = 1 - 4 \cos 30^0$

$y = 5.0$

When $x = 240^0$, $y = 1 - 4 \cos 240^0$

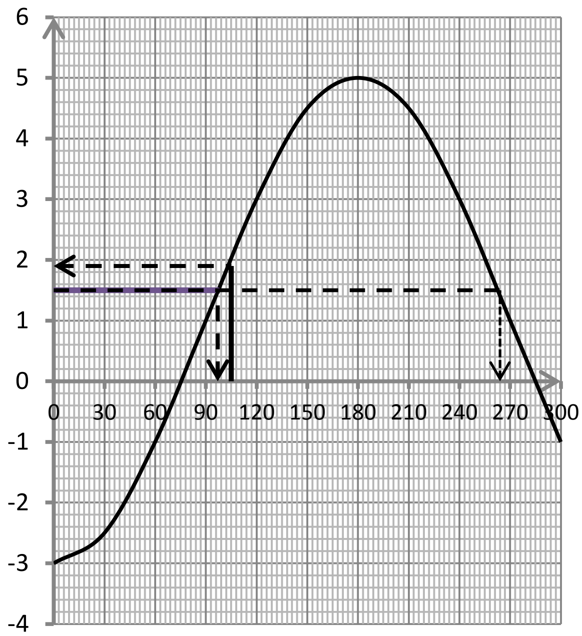
$y = 3.0$

When $x = 270^0$, $y = 1 - 4 \cos 270^0$

$y = - 1.0$

x	**0^0**	**30^0**	**60^0**	**90^0**	**120^0**	**150^0**
y	-3.0	-2.5	-1.0	1.0	3.0	4.5
x	**180^0**	**210^0**	**240^0**	**270^0**	**300^0**	
y	5.0	4.5	3.0	1.0	-1.0	

b.



Comparing 1 – 4 cos x = 0 with y = 1 – 4 cos x, y = 0

From the graph, when y = 0, $x = 75^0$ and $x = 285^0$

ii. From the graph, when $x = 105^0$, y = 1.9

iii. From the graph, when y = 1.5, $x = 97^0$ and $x = 264^0$

2. a. Copyand completethe table of values for y = 9 cos$x$ + 5 sin $x$to one decimal place

x	0^0	30^0	60^0	90^0	120^0	150^0
y		10.3			-0.2	

b. Using a scale of 2 cm to 30^0 on the x – axis and a scale of 2 cm to 2 units on the y – axis, draw the graph of y = 9 cos$x$ + 5 sin x for $0^0 \leq x \leq 150^0$

c. Use your graph to solve the equations

i. 9 cos$x$ + 5 sin x = 0

ii. 9 cos$x$ + 5 sin x = 3.5

d. Find, correct to one decimal place, the value of y for which $x = 72^0$

Solution

a. When $x = 0^0$, $y = 9 \cos 0^0 + 5 \sin 0^0$

y= 9.0

When $x = 60^0$, $y = 9 \cos 60^0 + 5 \sin 60^0$

y = 8.8

When $x = 90^0$, $y = 9 \cos 90^0 + 5 \sin 90^0$

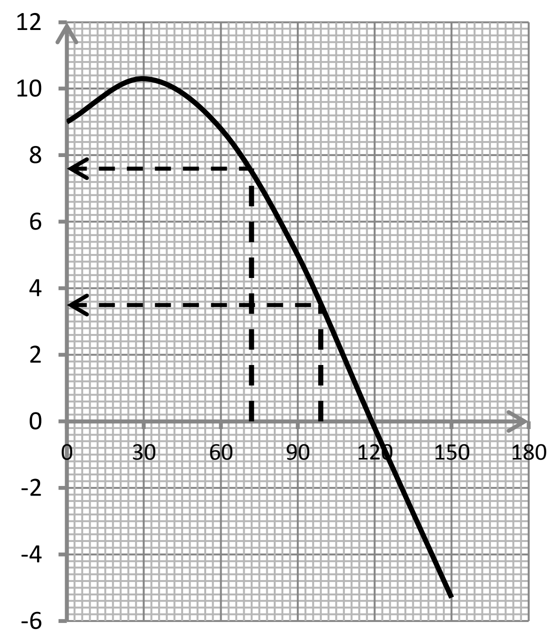
y = 5.0

When $x = 0^0$, $y = 9 \cos 150^0 + 5 \sin 150^0$

y = -5.3

x	0^0	30^0	60^0	90^0	120^0	150^0
y	9.0	10.3	8.8	5.0	-0.2	-5.3

b.



c. i. comparing 9 cos x + 5 sin x = 0 with y = 9 cos x + 5 sin x , y = 0

From the graph, when y = 0, $x = 119^0$

ii. comparing 9 cos x + 5 sin x = 3.5 with y = 9 cos x + 5 sin x , y = 3.5

From the graph, when $y = 3.5$, $x = 99^0$

d. From the graph, when $x = 72^0$, $y = 7.6$

Exercises 29

1. Construct a table of values for $y = 3 \sin x + 2 \cos x$ for $0^0 \leq x \leq 360^0$

b. Using a scale of 2 cm to 50^0 on the x –axis and 2 cm t 1 unit on the y – axis, draw the graph of for $y = 3 \sin x + 2 \cos x$ for $0^0 \leq x \leq 360^0$

c. Use your graph to solve the equations;

i. $3 \sin x + 2 \cos x = -1$

ii. $3 \sin x + 2 \cos x < 1.5$

2. a. Copy and complete the table for $y = \sin x + 2 \cos x$, correct to 1 decimal place

x	0^0	30^0	60^0	90^0	120^0	150^0	180^0	210^0	240^0
y		2.2				-1.2	-2.0		

b. Using a scale of 2cm to 30^0 on the x – axis and 2 cm to 0.5 units on the y axis, draw the graph of $y = \sin x + 2 \cos x$ for $0^0 \leq x \leq 240^0$

c. Use your graph to solve the equations;

i. $\sin x + 2 \cos x = 0$

ii. $\sin x = 2.1 - 2 \cos x$

d. Find y when $x = 171^0$.

ANSWERS TO EXERCISES

1. Sets and Properties of Sets

Ex 1.8
A. 1. {5, 10, 15.., 30} 2. {1, 2, 3…8} 3. {2, 6, 8, 10, 12} 4. {1, 3, 4, 5, 6, 7, 8}

B. 1. E ∪ F= {5, 7, 9} 2. E ∪ G = {1, 7, 9, 13}
3. F ∪ G = {1, 5, 7, 9, 13} 4. E ∪ (F ∪ G) = {1, 5, 7, 9, 13}

Ex 1.10
i. A^1 = {11, 12} ii. B^1 = {1, 2, 4, 5, 7, 8, 10, 11}
iii. C^1 = {3, 5, 6, 7, 9, 10, 11, 12}

Ex 1.11
6.i. μ = {1, 2, 3…18}, A = {2, 4, 6, …18}B = {3, 6, 9…18}A∩B = {6,12}, A∪B = {2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 15, 15, 16, 18}$(A\cup B)^1$ = {1, 5, 7, 11, 13, 17} 7. i. A = {5, 10, 15..95} B ={7, 14, 21, 28, 35, 42, 49, 56, 63,70, 77, 84, 91, 98}A ∩ B = {35, 70}ii. Multiples of 35 from 35 to 70iii. n(A) = 19, n(B) = 14 n(A∩B) = 2 , n(AUB) = n(A) + n(B) – n(A∩B)⇒31 = 19 + 14 – 2

Ex 1.12
1. U = {50}, E = {30} G = {17} i. 4 ii. 26 iii. 13 iv. 39
2. i. ii. 17 iii. 13 3. i. ii. 15% iii. 85% 4. x = 10 5. i. 39 ii. 676. 160 7. i. ii. 9 boys 8. i. 6 ii. 28 iii. 66 9. x = 13 10. x = 12

Ex 1.13
1. n(A∩B∩C) = 2 2…. 3. a. 20 b. 50 c. 10 4. 30 5. 20 6. 12 7. 43 8… 9. x = 26 10. 4 11. 24 12. n = 12 13. ii. 24 iii. 21 14. i. 270 ii. 125 iii. 0 iv. 205 v. 250 vi. 0 vii. 25 viii. 395

2. Real Numbers

Ex 2.13
1. $\frac{3}{2}$ 2. $\frac{12}{25}$ 3. $\frac{15}{8}$ 4. $\frac{36}{39}$ 5. $\frac{3}{5}$ 6. $\frac{230}{21}$ 7. $\frac{1647}{70}$8. $\frac{4}{15}$ 9. $\frac{9}{20}$

Ex 2.14
1. $\frac{63}{4}$ 2. $\frac{4}{3}$ 3. $\frac{5}{8}$ 4. $\frac{1}{5}$ 5. $\frac{7}{5}$ 6. 6

Ex 2.15
1. $\frac{35}{216}$ 2. 4 3. $\frac{19}{9}$ 4. $\frac{35}{3}$ 5. $\frac{95}{98}$

Ex 2.16
A. 1. $\frac{1}{6}$ 2. $\frac{3}{4}$ 3. $\frac{11}{15}$ 4. $\frac{5}{13}$ 5. $\frac{8}{19}$B. 1. Gh1,500.00 2. $\frac{7}{10}$ 3. $\frac{5}{11}$ 4. $\frac{11}{42}$ 5. $\frac{7}{20}$

Ex 2.17
1. i. $\frac{3}{7}$ ii. 14 2. i. $\frac{2}{3}$ ii. 16 3. i. $\frac{3}{16}$ ii. T = Gh2,100.00 , G = Gh1,800.00 P = Gh900.004. i. $\frac{13}{21}$ ii. Gh231.00 5. Gh60.00

Ex 2.18
1. 0.6 2. 0.2 3. 0.58 4. 1.25 5. 0.011 6. 0.06 7. 0.03 8. 0.009 9. 0.009

Ex 2.19
1. 1 2. 3 3. 4 4. 2 5. 3
Ex 2.20
1. 0.2 2. 0.064 3. 0.05 4. 0.11

Ex 2.23
A. 1. 1.9 2. 0.15 3. 3.49 4. 11.3 5. 31.0571
B. 1. 1.447 2. 90.093 3. 6.075 4. 1.145 5. 2.108
Ex 2.24
1. 0.46 2. 14.52 3. 3.762 4. 7.866 5. 3.1815 6. 0.0288

Ex 2.25
1. 2 2. 60 3. 20 4. 1.32 5. 0.9 6. 0.002

Ex 2.26
A. 1. 8.3×10^{3} 2. 1.4×10^{-3} 3. 9.8×10^{4} 4. 3.5607×10^{2} 5. 3.915×10^{-5} 6. 3.425×10^{4}7. 4.68702×10^{3} 8. 3.493×10^{2} 9. 4.3×10^{-3}
B. 1. 3.0×10^{-2} 2. 3×10^{2} 3. 6.0×10^{-5} 4. 4.0×10^{-2}

Ex 2.27
1. 34250 2. 0.1173 3. 0.0053 4. 7180300 5. 402000 6. 900

Ex 2.28
A. 1. 5.3×10^{5} 2. 3.95×10^{-2} 3. 87.525×10^{5} 4. 1.0×10^{-2}
B. 1. 7.3050×10^{4} 2. 4.405×10^{3} 3. 9.9×10^{-4} 4. 5.0×10^{-3}

Ex 2.29
1. 1.2×10^{5} 2. 1.8×10 3. 2.52×10^{3} 4. 6.24×10^{9}

Ex 2.30
A. 1. 8×10^{5} 2. 1.2×10^{8} 3. 3×10^{-3} B. 1. 4×10^{2} 2. 1.4×10^{3} 3. 6×10^{-3} C. 1.2×10^{2} 2. 2.8×10^{4}

Ex 2.35
1. -18 2. 6 3. 16 4. 18 5. 89 6. -27

Ex 2.36
1. $\frac{63}{5}$ 2. -$\frac{1}{25}$ 3. -$\frac{33}{10}$ 4. $\frac{10}{3}$ 5. $\frac{52}{10}$ 6. $\frac{52}{4}$

Ex 2.39
1.0. $\dot{4}$ 2. 0. $\dot{2}\dot{7}$ 3. 1. $\dot{3}$

Ex 2.40
1. $\frac{2}{3}$ 2. $\frac{140}{33}$ 3. $\frac{4}{33}$ 4. $\frac{73}{111}$ 5. $\frac{38}{33}$ 6. $\frac{3112}{990}$

Ex 2.41
1.i. a = 2, b = 5 ii. Associative

Ex 2.42
1.a. 23 b. 0 2. a. 32 b. 2 3. i. 39 ii. 55 iii. 5811 iv. P = 1 4. i. $\frac{28}{11}$ ii. $\frac{140}{83}$ iii. $\frac{14}{11}$ 5.a. $\frac{27}{2}$ b. $\frac{1}{6}$c. 80
6. i. a.

Ex 2.48
i. x^2y ii. $xy + xz$ iii.Distributive2. i. {3, 5} ii. {3, 5} iii. ∩ is distributive over∪

Ex 2.49
1. -2 2. 0 3. 2.25 4. i. 0 ii. commutative 5. i. ii. yes iii. b = 8

3. Algebraic Expressions

Ex 3.4

A. 1. $15a^3b^3$ 2. $80m^3n^4$ 3. $20a^3b^4$ 4. $12q^2b^2p$ 5. $50x^4y^5$ 6. $6a^3b^3b$ B. 1. $\frac{30x}{44}$ 2. $\frac{a^3b^2}{6}$ 3. 5t 4. $-\frac{2a^3b^4}{9}$ 5. $6b^2c^2$ 6. $\frac{3a^3b^2c}{10}$ 7. $-\frac{117m}{2}$

Ex 3.7

1. $x = 0$ or $x = 3$ 2. $x = -2$ or $x = -5$ 3. $x = 2$ or $x = 5$ 4. $x = 2$ 5. $x = -1$ or $x = 4$ 6. $x = 2$ or $x = 1$

Ex 3.8

1. $x = 9$ or $x = -9$ 2. $x = 4$ or $x = -4$ 3. $x = 0$ or $x = -5$ 4. $x = 1$ 5. $x = -\frac{5}{2}$ 6. $x =2$ or $x = 5$

Ex 3.9

Ex 3.10

A. 1. $m^2 - 14m + 40$ 2. $10a^2 - 4a + 6$ 3. 12ab + 3ad + 4bc + cd 4. Pq – 36 5. $3y^2 + 13y + 12$

B. $14y^2 - 44y + 35$ 2. $14m^2 + 2mn - 7n^2$ 3. $-3a^2 + ab - 5b^2$
4. $-6pq + 4q^2$ 5. 7qp

Ex 3.11

A.1. $2ny(1 + 3y)$ 2. $2y(2xy - 9)$ 3. $m(51 + 15m)$ 4. $12q(p + 3q)$ 5. $5u(2u + 7t)$ 6. $st(t - 28)$

B. 1. $4yt(3t - 4y)$ 2. $7pq(3q - 7p^2)$ 3. $6pr(1 + 7p)$ 4. $8rs(2 + 8s^2)$ 5. $9m(3k^2 - 1)$ 6. $5cd(3d + 5c)$

C. 1. $2xy(2x - 3y + 4t)$ 2. $3c(ab + 33c - 5a^2c)$ 3. $6km(2k + 3n + m^2)$ 4. $4pq(4pq - 5r - 2)$ 5. $px\left(\frac{x}{5} + \frac{1}{8}\right)$

Ex 3.12

1. $(a - b)(2p + q)$ 2. $(x + b)(x - a)$ 3. $(x + 2q)(y - 3c)$
4. $(p - q)(2r + 3s)$ 5. $(p - 2q)(r + 3s)$ 6. $(4y - 8d)(x - 2y)$

Ex 3.14

A. 1. $(x + 3)(3x + 2)$ 2. $(3x + 5)(2x - 1)$ 3. $(3x + 3)(3x + 1)$ 4. $(5x - 2)(x - 2)$ 5. $(x + 1)(7x + 2)$ 6. $(2x + 1)(5x + 3)$

B. 1. $(x - 4)(4x - 5)$ 2. $(3x - 1)(4x - 1)$ 3. $(x - 6)(3x - 2)$ 4. $(x - 3)(5x - 2)$ 5. $(2x - 1)(x - 3)$6. $(6x + 1)(x + 2)$

Ex 3.15

1. $(2x + 3y)(x + y)$ 2. $(6x - 2y)(x - y)$ 3. $(2x - 3y)(7x - y)$ 4. $(3x - 4y)(x - y)$ 5. $(5x + 5y)(x - 6y)$

Ex 3.16

A. 1. $(x + 5)^2$ 2. $(x + 2)^2$ 3. $(x + 9)^2$ 4. $(x - 16)^2$ 5. $(x - 13)^2$ 6. $(x - 4)^2$B. 1. $(3x - 2)^2$ 2. $(2x - 5)^2$ 3. $(2x - 9)^2$ 4. $(2x - 1)^2$

Ex 3.18A

1. $(x + y - 2)(x + y + 2)$ 2. $(a - 6 - 4b)(a - 6 + 4b$
3. $(2x + 9 - 5b)(2x + 9 + 5b)$ 4. $(2x - 5 - a)(2x- 5 + a)$ 5. $(-2b - 3)(-2b + 3)$

Ex 3.18B (Accept alternate answers)

1. $777^2 - 223^2$ 2. $674^2 - 326^2$ 3. $644^2 - 356^2$
4. $2356^2 - 2256^2$ 5. $792^2 - 692^2$ 6. $1867^2 - 1767^2$

Ex 3.19

1. $k = -3$ 2. $m = -1$

Ex 3.20

A.1. 39 2. 6 3. -28 4. 50 5. 97 B. 1. 14 2. -12 3. 27

4. Surds

Ex 4.2

1. $5\sqrt{6}$ 2. $5\sqrt{3}$ 3. $3\sqrt{6}$ 4. 22 5. $4\sqrt{17}$ 6. $4\sqrt{3}$ 7. $10\sqrt{3}$ 8. $7\sqrt{3}$ 9. $10\sqrt{10}$ 10. $3\sqrt{91}$ 11. $10\sqrt{2}$ 12. $6\sqrt{2}$ 13. $7\sqrt{2}$ 14. $11\sqrt{5}$ 15. $5\sqrt{29}$

Ex 4.7

A. 1. $-2 + \sqrt{2}$ 2. $2 + 5\sqrt{2}$ 3. $84 - 60\sqrt{2}$ 4. $24 - 18\sqrt{6}$ 5. $10\sqrt{338} - 5\sqrt{507}$ 6. $24 - 18\sqrt{6}$B. 1. 11 2. 23 3. $-9\sqrt{5} + 15\sqrt{2}$ 4. 23 5. $24\sqrt{147} + 21\sqrt{7}$ 6. 23

Ex 4.8

1. $8 - 2\sqrt{15}$ 2. $9 - 6\sqrt{2}$ 3. $59 - 24\sqrt{6}$

Ex 4.9

1. $3 + 2\sqrt{2}$ 2. $5 + 2\sqrt{6}$ 3. 50

Ex 4.10

1. $5\sqrt{3}$ 2. $6\sqrt{2}$ 3. 4 4. 45 5. $2\sqrt{3}$ 6. 6 7. $6\sqrt{6}$ 8. 40

Ex 4.11

1. 25.452 2. 0.2598

Ex 4.12

A. 1. $\frac{\sqrt{6}}{2}$ 2. $\frac{\sqrt{2}}{10}$ 3. $\frac{1}{2}$ 4. $\frac{\sqrt{2}}{4}$ 5. $\frac{5\sqrt{3}}{3}$6. $\frac{2\sqrt{3}}{3}$ 7. $\frac{3\sqrt{6}}{8}$
8. $\sqrt{2}$ 9. $\frac{\sqrt{10}}{2}$B. 1. $\frac{3\sqrt{10}}{10}$ 2. $\frac{\sqrt{7}}{7}$ 3. $\frac{\sqrt{15}}{5}$ 4. $\frac{\sqrt{11}}{11}$
C. 1. 0.707 2. 0.577 3. 0.466 4. 3.46 5. 4.476. 1.15 7. 1.34 8. 14.1 9. 0.671 10. 0.566D. 1. 19.48 2. 0.354 3. 0.3534 4. 3.732

Ex 4.13

A. 1. $\sqrt{3} - 1$ 2. $\sqrt{5} + 2$ 3. $\sqrt{7} + 3$ 4. $3 + \sqrt{2}$B. 1. $1 + \sqrt{2}$2. $-1 + \sqrt{5}$3. $24 + 12\sqrt{3}$4. $7 - 4\sqrt{3}$ 5. $\sqrt{3} + \sqrt{2}$ 6. $\sqrt{5} - \sqrt{2}$7. $3\sqrt{5} + 3\sqrt{2}$ 8. $\sqrt{7} - \sqrt{2}$ 9. $-1 + \sqrt{2}$ 10. $2 + \sqrt{3}$ 11. $-2 + \sqrt{6}$ 12. $-3\sqrt{6} - 3\sqrt{5}$C. 1. $-1 + \sqrt{7}$ 2. $\frac{6}{7} - \frac{2\sqrt{2}}{7}$ 3. $5 + 2\sqrt{6}$ 4. $\frac{1}{2} + \frac{\sqrt{2}}{3}$5. $-1 + \sqrt{2}$ 6. $-1 - \sqrt{2}$ 7. $9 + 4\sqrt{5}$ 8. $4 - 2\sqrt{3}$

5. Number Bases

Ex 5.4

A. 1. 3203 2. 11333 3. 21205 4. 22110 5. 1414
B. 1. 222 2. 43

Ex 5.5

A.1. 1002 2. 3140 3. 12 4. 2215 5. 1024 6. 3001
B. 13 2. 3401 3. 223 4. 323

Ex 5.6

1. 1011 2. 110000 3. 1111 4. 111000 5. 1010100

Ex 5.7

1. 1 2. 10110 3. 110 4. 101 5. 100 6. 111

Ex 5.8

A. 1. 20202 2. 4023 3. 204400 4. 22203 5. 1227 6. 6122
B. a. 10100 b. 1101100 c. 111100C. 1. 11001 2. 11110 3. 1111 4. 1001110

Ex 5.9

A. 1. 97 2. 543 3. 485 4. 110 5. 147 6. 1012B. 1. 11 2. 14 3. 5 4. 27 5. 12 6. 37C. 1.17 2. 67.641 3. 135 4. 8.519 5. 72 6. 448 7. x = 35, y = 45 z = 104 w = 202 8. a = 7, b = 5 c = 4D. 1. 1021 2. 2200 3. 11 4. 21 5. 20111
E. 1. 150 2. 405 3. 43 4. 701 5. 234 6. 2176F. 1. 4 2. 5 3. 5 4. 7 5. 5 6. 7 G. 1. 7.76 2. 154.192 3. 192.719
4. 22.5625

Ex 5.10
A. 1. 103 2. 1122 3. 304 4. 314 5. 402 6. 1132 7. 1434 8. 110B. 1. 111011 2. 1001101 3. 1100110 4. 1000010 C. 1. 11102 2. 11021 3. 11100 4. 452 5. 1300

Ex 5.11
A. 1. 123 2. 23 3. 101001 4. 2010 5. 111110
B. 1. 11011 2. 111001 3. 100011 4. 101110
C. 1. 24 2. 142 3. 41D. 1. 304 2. 212 3. 38 4. 2010

Ex 5.12
A. 1. 63 2. 108 3. 106 4. 1714 5. 5014 B. 1. 23 2. 84 3. 130 4. 6e4 5. 20teC. 1. 1219 2. 1t653 3. 3t 4. 2266

Ex 5.13
b. 14 c. i. n = 4 ii. n = 5
2. i. b. i. 31 ii.14 c. i. m = 3 ii. m = 4

Ex 5.14
A. 1. 1110_2 , 221_3 , 3011_4 2. 13_6, 40_5 , 10110_2 3. 232_4 , 230_5 , 2202_3 , 332_6 B. 1. 224_5 , 2002_3, 55_6 , 14_{10} 2. 11011_2 , 101_3 , 11_5 3. 403_6, 2201_3, 11_8

Ex 5.15
A.1. 7 2. 5 3. 5 4. 3 5. 2 6. 6 B. 1. 2 2. 3 3. 4 4. 2 5. 1 6. 1C. 1. 6. 2. 2 3. 5 4. 4 D. 1. 9 2. 8 3. 4 4. 3

Ex 5.16
1.

+	**0**	**1**	**2**
0	**0**	**1**	**2**
1	**1**	**1**	**10**
2	**2**	**10**	**11**

×	**0**	**1**	**2**
0	**0**	**0**	**0**
1	**0**	**1**	**2**
2	**0**	**2**	**11**

2. 100000_{two} cm ; 100111_{two} 3. 4 4. x = 3, y = 7 5. 0, 2, 3, 4, 5, 6, 7, 8, 9
C.P:
2.

Base	12	10	8	5	3	2
	23	27	33	102	1000	11011
	2*e*	35	43	120	1022	100011
	25	29	35	104	1002	11101

3.110111; 55 4. 2020_{three} mm, 22100_{three} mm^2

Ex 5.17
1. i. 44 ii. 30
2. a. b. i. 204 ii. 413 iii. 23 c. i. 4 ii. 3

*	1	2	3	4
1	11	23	40	102
2	22	101	130	204
3	33	124	220	311
4	44	202	310	413

3. a. b. i. 8 ii. 13 c. i. 6 ii. 3

6. Relations and Functions

Ex 6.6
1. y = - 3x + 15 2. $y = \frac{5}{2}x + \frac{15}{2}$ 3. y = 3x + 17
4. $y = \frac{1}{3x+5}$ 5. $y = \frac{3}{3x-10}$ 6. $y = \frac{2}{\frac{2}{3}x+9}$7. $y = \frac{1}{2}x + \frac{1}{2}$ 8. y = - 2x + 3
9. y = -2x – 4

Ex 6.7
1. $y = 2^x$ 2. $y = \left(\frac{1}{2}\right)^x$ 3. $y = 2^x$ 4. $y = 2(4)^{x-1}$ 5. $y = \left(\frac{1}{3}\right)^x$

Ex 6.8
1. $y = x^2 + 2x$ 2. $y = 2x^2 + 5x – 3$ 3. $y = 3x^2 – 5x + 2$ 4. $y = - 5x^2 – 3x – 5$ 5. $y = x^2 – 6$

Ex 6.9
1. $y = \frac{2}{3}x - 2$, k = 20 2.$y = 3x^2 – x + 4$, 3044 3.$y = 5(3)^{x - 1}$, 32805
4. $y = \frac{3}{-x+5}$, $\frac{1}{25}$

Ex 6.11
A. 1. 126 2. $\frac{91}{64}$ 3. – 7 4. $\frac{9}{8}$ 5. 15 6. $\frac{57}{2}$
B. 1 . a. 2, $\frac{3}{4}$, - 8 b. 2 2. a. – 10 b. 0 c. 7 d. $\frac{20}{3}$3. a. i. 2 ii. 5 b. 1 c. $\pm$ 10 4. i. 22 ii. – 3 iii. – 2 iv. – 3 5. i. 24 ii. 3 6. 60 7. - $\frac{11}{21}$
8. $\frac{6}{5}$ 9. 15, 1 10. 15, 6 11. 8 or 9 12. -2 <x< 3 13. i. -2, $\frac{-8}{3}$, - 1.27 ii. 2 14. i. Q = {2, -1, -2, -1, 2}

C. 1. {-7, - 4,-1, 2} 2. {- 8, - 5, 0, 7, 32} 3. 20, 24, 27, 30, 32}
Ex 6.12
1. a = 5, b = - 8 2. b = 3 c = 5, x = -2 or x = $\frac{1}{3}$ 3. a = -3, b = 2
4. b = $-\frac{2}{3}$ 5. b = -2

Ex 6.13
1. 0 2. -$\frac{5}{2}$ 3. 3 4. 3 5. 5 or 2 6. 4 or 3

Ex 6. 14
1. $-\frac{2}{5}$ 2. 2 or -5 3. 3 4. 3 or 4 5. 6 6. – 15

Ex 6.17
A(9,6), B (3,8) C-3,8) D(-1, 4) E(4, -10), F(-5, -8),G(-3,-5), H(0, -6) I(-2, -4) J(8, -2)

Ex 6.21
1. a. b. i. (2, - 4) ii. – 4 iii. x = 2
2. a. b. i. (a, b) = (1,5) ii. (0, 3) iii min : 3 iv. No zeroes v. 3≤$f(x)$ ≤ 19 vi. x = 0 3. a. b. i. P(2, 6) and P(4, 4) ii. (2.5, 6.25) iii. 6.25 iv. x = 0 or x = 5 v. -6≤$f(x)$ ≤ 6.5 vi. x = 2.5
4. a. b. (-3, 0) and (1, 4)

Ex 6.23
1. (3, 2) 2. (-7, - 6) 3. (3, 0) 4. (0, 3.5) 5. – 2, -1) 6. (9.2, - 4.5)

Ex 6.24

1. (-8, 3) 2. (7, 6) 3. (8, 5) 4. (- 11, 12)

Ex 6.25
A. 1. 5 2. 4.47 3. 3.61 4. 13.34B. 1. 9.23 2. 4 3. 17
C.p: 2. i. (-1, -1), (0, 0), (1, 1) etc ii. $y = c$ iii. (2, 2) 3. t = 2 or t = -2
4. $k = 14$ or $k = -2$

Ex 6.26
1. PQ = PR = $\sqrt{137}$, QR = $\sqrt{98}$, M = (2.5, 4.5) 2. AB = AC = 5, BC = 7 3. AB = AC = $\sqrt{73}$, BC = $\sqrt{50}$4. AB = AC = 13, BC = 14 5. AB = AC = 5, BC = 3

Ex 6.34
1. $4y + 3x - 27 = 0$ 2. $4y + 5x - 7 = 0$ 3. $11y + 3x - 35 = 0$
4. $5y - x + 19 = 0$ 5. $5y + 12x - 1 = 0$

Ex 6.35
1. $y - 3x - 11 = 0$ 2. $3y + x - 13 = 0$ 3. $y - 2x + 3 = 0$ 4. $3y - x + 17 = 0$ 5. $y + 5x + 22 = 0$C.P. 1. i. 3y – x – 1 = 0 ii. y + 2x – 12 = 0 iii. Q(5, 2) and R(1, - 3) 2.i. $3y = -4x$ ii. OA = OB = 5, A = 10 sq. units iii. 20 sq. units

Ex 6.36
1. $y = 3x + 2$ 2. $y = -\frac{1}{5}x + 4$ 3. $y = 4x + 6$ 4. $y = \frac{2}{7}x - 4$ 5. $y = -3x - 2$ 6. $y = \frac{1}{5}x + 7$

Ex 6.37
A. 1. $4y - 3x - 1 = 0$ 2. $y - 7x + 28 = 0$ 3. $5y + 2x + 14 = 0$ 4. $2y - 5x - 16 = 0$ 5. $4y - 3x - 1 = 0$
B. 1. i. $2y - 5x + 19$ ii. $5y + 2x + 4 = 0$ 2. i. $3y - 2x + 14 = 0$ ii. $2y + 3x - 8 = 0$ 3. $3y - 4x + 13 = 0$ 4. $x^2 + y^2 = 25$

Ex 6.38
1. Not 2. Yes 3. Yes 4. NotC. P: 1. i. 6.32 ii. (3, 1) iii. m = 3; y – 3x + 8 = 0 iv. r = -14 2. p = 3, q =1

Ex 6.39
1. (7, -7) 2. (1, 5) 3. (- 0.5, - 3.5) 4. (-1.5, – 5.5) 5. (4, -7) C.P. 1. i. (2, - 1) ii. 2y – 5x– 8 = 0 iii. yes 3. (-1, -1) 4. (5, 4)

7. Plane Geometry 1
Ex 7.5
1. … 2. 85^0

Ex 7.6
A. 1. z = 58^0 , y = 58^0, x = 118^0 2. e = 82^0 3. y = 118^0, x = 62^0 4. a = 40^0, b = 120^0, c = 20^0B. 1. <PRQ = <PQR = 80^0 2. x = 58^0, y = 22^0, z = 88^0 3. x= 80^0, y = 60^0

Ex 7.7
A. 1. r = 50^0, t =70^0 2. a = 70^0, b = 70^0, c = 40^0 3. y = 60^0, 4. 50^0 5. k = 56^0, m = 54^0 6. b = 108^0, c = 55^0, a= 43^0 7. d = 102^0 8. x = 20^0 9. a = 130^0 10. b = 110^0 11. d = 30^0 , e = 90^0 12. k = 30^0B. 1. y = 280^0 2. a = 73^0 3. x = 75^0 , y = 30^0 4. x =30^0 , y = 20^0 5. d = 65^0 , e = 48^0 , f = 67^0

Ex 7.8 (every answer is in cm)
1. 12 2. 24 3. 36 4. 36 5. 6.7 6. 7 7. $x = 20$
B. 1. 5.2 2. 23.05

Ex 7.11
1. 13m 2. 17.89m 3. 7.42m 4. 6.92ft 5. 15.49ft
C.P. 1. 5m 2. i. 8m ii. 4m

Ex 7.12
1. 23cm 2. 60m 3. 10 inches 4. 7cm 5. 5 cm 6. A = 72cm^2,
P = 33.96cm 7. 0.8cm
C.P. 1. $8\sqrt{2}$ 2. H = $\sqrt{3a^2}$, A = $\sqrt{3a}$ 3. i. 5 ii. 60cm^2

Ex 7.13
1. 100cm 2. 26.8cm 3. 10.58cm 4. 8.05 cm

Ex 7.14
1. 26cm 2. 8 cm 3. 12.12cm 4. 7.6cm 5. h = 50cm, LM = 50.35 cm 6. 40cm 7. a. 4 b. 8

Ex 7.15
1. x = 5cm, a = 30cm^2 2. 4:1 3. 17cm 4. 3cm 5. 1.6cm and 2.6 cm 6. 3.95 cm and 6.95 cm 7. 6cm, 8cm, 10cm

Ex 7.16
1. 22 ft 2. 4.2m 3. 9 miles 4. 93ft 5. 5 cm 6. 5m 7. 6.38m 8. i. 6.40km ii. a. 6.40km b. 12.8km 9. 25ft 10. i. /AB/ = 22.5m ii. /BD/ = 33.1m 11. AB = 28.7m, BD = 44.6m 12. 4ft

Ex 7.17
A. 1. 60^0 2. 60^0 3. x = 65^0, y = 115^0 4. x =70^0, y = 110^0
B. 1. 22^0 2. 85^0, 58^0, 95^0 3. x = 98^0, 49^0, 147^0, 65.3^0

Ex 7.19
A. 1. 2160^0 2. 3600^0 3. 7200^0 4. 5940^0 5. 2700^0 6. 3640^0
B. 1. 120^0 2. 88^0 3. 159^0 4. 48^0

Ex 7.20
1. 8 2. 10 3. 18 4. 22 5. 19 6. 29

Ex 7.21
A. (f, e) (g, h), (l, k) and (m, n)B. $x = 115^0$, $y = 10^0$z = 170^0

Ex 7.22
1. 152.30^0 2. 154.28^0 3. 158.82^0 4. 161.05^0 5. 162.85^0

Ex 7.23
1. 35^0 2. $x = 37.5^0$, $y = 67.5^0$ 3. 24^0 4. 46.7^0

Ex 7.24
1. 12 2. 6 3. i. Ext < = 30^0, Int < = 150^0 ii.1800^0

8. Equations and Inequalities
Ex 8.15
A. 1. 4 < x < 7 or (4, 7) 2. -3 ≤ x < 2 or [-3, 2) 3. -3 ≤ x < 2 or (-3, 2) 4. -1 < x < 3 or (-1, 3) 5. -4 < x ≤ 2 or (-4, 2] 6. 2 ≤ x ≤ 3 or [2, 3] B. 1. -20 ≤ x < -4 or [-20, -4) 2. 0 < x < 1 or (0, 1) 3. -5.5 < x < 1.5 or [-5.5, 1.5) 4. -1 < x < 5 or (-1, 5) 5. $-\frac{7}{3} \leq m \leq 2$ or ($-\frac{7}{3}$, 3]

9. Vector and Bearings
Ex 9.1
1. A = 355^0 , B = 017^0 , C = 057^0, D = 152^0, E = 090^0 , F = 285^0

Ex 9.2
A. 1. 213^0 2. 335^0 3. 300^0 4. 15^0 5. 107^0B. S 50^0 W 2. N 38^0 E 3. S 25^0 E 4. S 75^0 E 5. N 57^0 W 6. N 5^0 W
Ex 9.3
B. 1. (7km, 105^0) 2. (6km, 315^0) 3. (11km, 065^0) 4. (3km, 250^0)

Ex 9.4
A1. 225^0 2. 145^0 3. 120^0 4. 305^0B. 1. 297^0 2. 040^0 3. 135^0 4. 027^0C. i. 290 ii. 042 iii. 035^0 iv. 147^0

Ex 9.5

1. (20km, 342^0) 2. (46km, 263^0) 3. i. 63km ii. (63km, 234^0) 4. i. 23 km ii. (23km, 266^0) 5.i. 146 km ii. (1146km, 287^0) 6. i. 116km ii. (120km, 260^0) b. 113km 7.i. 19km ii. 21km, 087^0 iii 4km 8. a. i. 47m ii. 17m b. i. 108m ii. 34m c. 128^0 d. 113.6m

Ex 9.6

1. $\binom{-3}{-2}$ 2. $\binom{6}{-4}$ 3. $\binom{-10}{7}$ 4. $\binom{1}{11}$

Ex 9.7

1. (- 4, 1) 2. (2, 5) 3. (- 6, 7) 4. (2, 5) 5. (- 3, 5) 6. (-4 , 1)

Ex 9.8

A. i. c and f ii. a = (2, -3), (18, - 27), (8, - 12) … b = (20, 14), (30, 12), (40, 28)…c = (- 1 , - 3), (- 8, - 24), (- 5 , - 15)

B. i. (20, 0) ii. (- 8, - 10) 2.i. a. b – a b. 2b c. 2a d. 2(a – b) ii. $\overrightarrow{BR}$= -2 $\overrightarrow{BC}$ 3. ii. N is the mid point of AC

Ex. 9.9

1. 6 2. 7.8 3. 8.1 4. 5 5. 8.5 6. 10

C.P. /u/ + /v/ = 2, /u + v/ = 1

Ex 9.15

1. i. $\binom{-15}{-12}$ ii. $\binom{5}{4}$ iii. 6.4 units 2. $\binom{6}{12}$ 3. i. $\binom{11}{1}$ ii. $\binom{-5}{-2}$ iii. $\binom{1.5}{4}$ iv. $\sqrt{29}$ 4. i. $a = \binom{-1}{1}$, $b = \binom{3}{4}$ ii. $\binom{4}{3}$ iii. 5 5. a . B = (1, 4), C = (8, 4) b. (0.5, 3) 6. a. $\overrightarrow{AB} = \binom{4}{-4}$, $\overrightarrow{DC} = \binom{5-x}{0-y}$, D= (6, -1) b. / $\overrightarrow{DB}$ / = 4 7. i. Q = $\binom{2}{2}$ S = $\binom{5}{-12}$ ii. 13.6 units 8. a. i. D = $\binom{1}{-1}$, B = $\binom{2}{6}$ ii. $\overrightarrow{BC} = \binom{4}{-3}$, $\overrightarrow{AD} = \binom{4}{-3}$ b. equal vectors 9.i. $\overrightarrow{PQ} = \binom{6}{8}$, $\overrightarrow{QR} = \binom{-10}{0}$, $\overrightarrow{PR} = \binom{-4}{8}$ ii. /PQ/ = /QR/ = 10 10. Q = (10, 2)

C.P. 1. h = 2, k = - 1 2. S(4, 2) or S(2, -6) 3. C(0, 4) D((-5, 3)

Ex 9.16

1. (7, 7) 2. V = 18, U = 8 3. a = 9, b = - 2 4. X = 3, y = - 1 5. (1, 2) 6. i. 13.5 units ii. $\binom{-14}{17}$

Ex 9.17

1. (p, q) = (2, 3) 2. i. (m, n) = (6, -1) ii. 6.4 units 3. i. $\binom{0}{1}$ ii. m = 3.5, n = - 1 .5 4. a. m = 3 , n = - 1 b. 8.5 units

Ex 9.18

A. 1. $\binom{8E}{10\ N}$ 2. $\binom{5\ E}{2\ S}$ 3. $\binom{6W}{12\ N}$ 4. $\binom{9W}{3\ S}$

B. 1. $\binom{-7}{4}$ 2. $\binom{2}{-9}$ 3. $\binom{-5}{-5}$ 4. $\binom{3}{4}$ 5. $\binom{12}{-19}$ 6. $\binom{0}{-7}$

Ex 9.19

1. 54^0 2. 30^0 3. 59^0 4. 13^0 5. 69^0 6. 30^0

Ex 9.20

A. 1. $\binom{7.188}{5.416}$ 2. $\binom{9.613}{-2.756}$ 3. $\binom{-8.425}{-11.181}$ 4. $\binom{-3}{5.196}$ 5. $\binom{-4.636}{-1.873}$

B. 1. $\binom{-3.473}{-3.716}$ 2. $\binom{10.392}{4.392}$ 3. $\binom{-7.608}{2.472}$ 4. $\binom{8.387}{5.446}$ 5. $\binom{-10.199}{-4.121}$

Ex 9.21

A. 1. (14 units, 060^0) 2. (9 units, 198^0) 3. (10 units, 307^0) 4. (12 units, 180^0) 5. (9 units, 32^0)6. (11 units, 243^0) B. 1. (10 units, 191^0) 2. (18 units, 038^0) 3. (15 units, 127^0) 4. (22 units, 117^0) 5. (10 units, 45^0)6. (19 units, 298^0)

Ex 9.22

1. a. 20.52km b. 063^0 2. a. 1252km b. 138^0 3. a. 22.67km b. 243^0

10. Statistics I

Ex 10.2

1. -4 2. 6 3. 48 4. i. 3 ii. 3 iii . 3.33 5. 15

Ex 10.3

1. 3 2. 38 3. 23yrs 3 months 4. -4 5. 408 6. Gh¢4800 7. 12

Ex 10.6

1. $\frac{2}{15}$ 2. a. $\frac{1}{6}$ b. $\frac{1}{3}$ 3. a. $\frac{1}{10}$ b. $\frac{4}{5}$

Ex 10.8

1. i. 120^0 ii. 45mins 2. i. 40^0 ii. Gh¢1260.00 iii. $\frac{1}{9}$ 3. i. 120^0 ii 30 iii. 90 iv. 11% 4. i. 50 ii.… 5. i. 36^0 ii. Gh¢900.00 iii. Gh¢60.00

Ex 10.12

1. 27.4 2. i. 21 iii. $\frac{10}{89}$

Ex 10.13 assumed mean

1. 68.07 2. 73.7 3. 30.25 4. 33.25

11. Rigid Motion

Ex 11.1

1. P_1(-3, 12), Q_1(-6, - 4), R_1(2, 9) 2. A_1(8, 11) 3. (4, - 5) 4. R_1(-5, 8), S_1(2, 9), T_1(4, 0) 5. (-1, 4)

Ex 11.2

1. A_1(-4, 8), B_1(-7, 11), C_1(-9, -2) 2. (4, - 2) 3. (-3, - 3) 4. (6, - 4) 5. (5, 3)

Ex 11.3

1. (-8, - 3) 2. (6, - 1) 3. (-3, - 10) 4. A (1, 4), B (7, 9), C (1, 9)

Ex 11.4

1. (-6 , - 14) 2. ($\frac{-7}{4}$, $\frac{-11}{4}$) 3. A_1 (15, 12) , B_1 (3, 9) , C_1 (- 6, 6), D_1 (3, 6)

Ex 11.5

1. (40, -28) 2. A_1 (0, - 10), B_1 (8, -14), C_1 (4, 6) 3. R_1 ($\frac{-5}{3}$, 1) S_1 ($\frac{-7}{3}$, $\frac{-4}{3}$), T_1 (2, $\frac{1}{3}$) 4. (6, - 8)

Ex 11.6

1. A_1 (1, - 1), B_1 (-4, 9) , C_1 (-9, 14) 2. A_1 (1, 1) , B_1 (3, 0), 3. P_1 (0, 2) . Q_1 (4, 2), R_1 (4, -6) , S_1 (0, - 2) 4. i. (3, 2) ii. (12, 11)

Ex 11.7

1. (5, 2) 2. (3, 1) 3. P(- 3, 1) , Q(0, $\frac{1}{4}$), R(6, 8)

Ex 11.8

1. (-3, 0) 2. U(- 6, 1) V(-3, 3) W(- 4, 7)

Ex 11.9

1. A(- 11, 1) B(- 5, -4) C(- 8, - 4) D(- 13, - 2) 2. i. P_1 (- 3, 8) , R_1 (0, 12), T_1 (3, 8) ii. O_2 (4, 0) , P_2 (7, 8), R_2 (4, 12), T_2 (1, 8) 3. A_1(1, -3), B_1(4, -5), C_1(9, -5) , D_1(6, -3), A_2(-5, -3), B_2(- 8, - 5) , C_2(-13, -5), D_2(-10, -3) 4. A_1(2, - 6), B_1(- 8, - 8) , C_1(- 5, - 11), A_2(6, - 8) , B_2(11, -8) , C_2(- 3, 9)

Ex 11.10

1. A_1(4, -3), B_1(-2, -5), C_1(-3, 9), 2. A_2(- 3, - 4), B_2(5, -2), C_2(9, 3), A_3(3, 4), B_3(5, - 2), C_3(- 3, 9), 4. i. P_1(3, 4), Q_1(2, -1), R_1(5, -3), ii. P_2(4, -3), ii. P_2(4, -3), Q_2(1, -2), R_2(-3, -5),

Ex 11.11

1. Rotation through 90^0 about the origin 2. 1. Rotation through - 90^0 about the origin 3. 1. Rotation through 180^0 about the origin 4. 1. Rotation through 90^0 about the origin

12. Ratio and Rates

Ex 12.3
A. 1. Proportion 2. Not a proportion 3. Proportion 4. ProportionB. 1. $\frac{1}{5}$ 2. $\frac{1}{4}$ 3. $\frac{3}{25}$ C. 1. $\frac{21}{40}$, $\frac{19}{40}$ 2. $\frac{7}{10}$ 3. $\frac{2}{15}$

Ex 12.4
A. 1. 20 2. 5 3. 6 B. 1. 9 2. 12 3. 3 4. 30 5. 6 6. 3C. 1. 8 2. 4 3. 8 4. 15 5. $\frac{21}{5}$D. 1. -1 2. 1 3. 4 4. – 4 5. 16

Ex 12.5
1. 120 2. 32.5 3. 30 4. 80 5. 504

Ex 12.6
1. 45 2. 36 mins 3. 24hrs 4. 4 days 5. i. 14 ii. 50 mins

Ex 12.7
A.1. i. Gh¢400, Gh¢200 ii. Gh¢360, Gh¢240 iii. Gh¢480, Gh¢120 iv. Gh¢350, Gh¢250 v. Gh¢375, Gh¢225 2. i. Gh¢1,960, Gh¢280 ii. Gh¢1, Gh¢540, Gh¢700 iii. Gh¢1,400 , Gh¢560, 280 iv. Gh¢1,120, Gh¢840, Gh¢280
B. 1. Shamo = Gh¢8,000.00, Bako = Gh¢20,000.00, Aspa = Gh¢12, 000.00 ii. Gh¢12, 000.002. Kate = Gh¢7,470.00, Comfort = Gh¢9, 960.00 3. Gh¢20,000.00 4. Gh¢20,000.00 5. Gh¢18,700.00, Gh¢22,100 6. 8.75 7. i. Okra = Gh¢55,000.00, Harry = Gh¢77,500.00 , Regina = Gh¢31,000 ii. Gh¢22,500.00 8. i. Naa = Gh¢24, 000.00, Ayeley = Gh¢40,000.00 ii. Gh¢16,000.00 9. i. Gh¢ 8,000.00, Gh¢8,500.00, Gh¢9,500.00 ii. Gh¢,600.00, Gh¢5,950.00, Gh¢6,650.00 10. i. Tom = Gh¢100,000.00, Ben = Gh¢30,000.00, Pak = Gh¢50,000.00 ii. Gh¢100,000.00 iii. Gh¢50,000.00

Ex 12.8
1. 2102. 233. Gh¢400,000.00 4. 1125. 13,155 C. P. i. Gh¢39,000.00ii. An = Gh¢ 7,800, Nt = Gh¢ 5,200, As = Gh¢ 26,000 iii. Gh¢ 18,200.00

Ex 12.9
1. Gh¢ 300.00 2. Gh¢ 100.00 3. i. Gh¢ 32.00 ii. Gh¢ 64.00 4. Gh¢ 267.00 5. Gh¢70.006. Gh¢2.07 7. 22words8. 23 deaths 9. 6 mins C.P. 1. 4,270.00 2. Gh¢47.00

Ex 12.10
1. i. 350 km/h ii. 97.2 m/s 2. i. 3.33m/s ii.0.2 km/min 3. 1110km/h 4. i.125km/h ii. 34,7m/s 5. i.48km/h ii. 40km/h iii. 38km/h

Ex 12.11
1. 1.8 hr 2. 3 hr 3. 300 mins 4. 21,600s

Ex 12.12
1. 720km 2. 337km 3. 4800 m 4. 360 km 930km

Ex 12.13
1. i. 8,000km ii. 3,500km iii. 10,300km iv. 2,800km2. a. 1 : 50 b. i. 3 cm ii. 5.6cm 3. 1,200km 4. L = 7200, B = 4500, A = 32,400,0005. i. 1 : 2 ii. 25km 6. i. 220km ii. 7. 8.

Ex 12.14
1. 1 : 20,000 2. 1 : 30,000 3. 1 : 50,000 4. 1 : 85 5. 1 : 4,000,000
Ex 12.15
A. 1. Gh¢ 7,611.94 2. Gh¢2,537.31 3. Gh¢ 15,223.88 4. Gh¢101,492.54B. 1. Gh¢7,515.79 2. Gh¢25,052.63 3. Gh¢3,364.21 4. Gh¢16,821.05

Ex 12.17
i. 18 mins ii 8 : 20 am iii. 1 hr iv. 8 : 02 am v. 0.375 km/h

Ex 12.18
1. 20 2. 8 3. 113,520 4. 109,020

13. Percentages 1

Ex 13.44
1. 2% 2. 50% 3. 5.3% 4. 7% 5. 20% 6. 6.3%

Ex 13.5
A. 1. 60 2. 1,300 3. 1,206 4. 18,000 5. 10, 000
B. 1.Gh¢6,100 2. Gh¢4,787.33 3. Gh¢6,000 4. Gh¢250

Ex 13.6
1. 40% 2. 75% 3. 75% 4. 7.5% 5. 40% 6. 100

Ex 13.7
1. Gh¢22.22 2. Gh¢91 3. Gh¢160 4. i. 20c. ii. 16 cm
5. Gh¢13,110 6. Gh¢2,944 7. 23% 8. 3,600 votes

Ex 13.8
A. 1. Gh¢7,000 2. Gh¢1,700 3. Gh¢1,800 4. Gh¢20,900
B. 1. Gh¢1,650 2. Gh¢25,900 3. Gh¢18,300 4. Gh¢387,000
C. 1. Gh¢1,402.50 2. i. Gh¢1,640 ii. Gh¢2,772 iii. Gh¢4,412 3. Gh¢3,840 ii. Gh¢6,8160

Ex 13.9
1. 87.5

Ex 13.10
1. Gh¢ 3,176.47 2. Gh¢50,000

Ex 13.11
A. 1. Gh¢1,414.40 2. Gh¢91.65 3. Gh¢18.36 4. Gh¢704
5. Gh¢ 20.90B. 1. Gh¢12,500 2. Gh¢1,083.90 3. Gh¢990
4. Gh¢4,743 5. Gh¢1,665.6

Ex 13.12
A. Gh¢40,000 2. Gh¢240,000 3. Gh¢350,000 4. Gh¢23,000

Ex 13.13
1. 2% 2. 30% 3. 12.5% 4. 15% C.P 1 . i. Gh¢42 ii. Gh¢1250 2. Gh¢41.25
Ex 13.14
1. 3.5 2. 59.34 3. 1711 4. 7639 5. 230

Ex 13.15
1. 84% 2. 35.5% 3. 54.5% 4. 25% 5. i. 16 ii. 50% 6. 25% 7. 9% 8. 27.8% 9. 30%, 23.1% C.P: 1. 12%

Ex 13.16
A. 1. Gh¢ 4,800.00 2. Gh¢270.00 3. a. Gh¢ 3.22 b. Gh¢.38
4. Gh¢1,800.005. Gh¢440.80B. 1. Gh¢ 200.00 2. Gh¢ 50,090.91 3. Gh¢ 15,000.00 4. Gh¢32, 579.19

Ex 13.17
1. Gh¢13,750.00 2. Gh¢1,100.00 3. Gh¢27,000.00
4. Gh¢330,000.00 5. a. Gh¢59,888.00 b. Gh¢1,247.67

Ex 13.18
1. Gh¢3,000.00 2. Gh¢23,809.52 3. Gh¢5,000.00 4. Gh¢8,500.00

Ex 13.19
1. 8% 2. 60% 3. 30% 4. 8%

Ex 13 .20
1. 4yrs 2. 10yrs 3. 2yrs 4. 40 yrs 5. Gh¢3,369.50

Ex 13.21
1. Gh¢3,200.00 2. a. Gh¢400x b. 1.12x c. 12%

Ex 13.22
1. i. Gh¢ 23,000.00 ii. Gh¢ 3,000.00 iii. 68.58% 2. 6.15% 3. i. Gh¢17,200.00 ii. Gh¢3,200.00 iii. 156% 4. 37.5% 5. i. Gh¢1,243.55 ii. Gh¢10, 878.30 iii. 23.62% 6. i. Gh¢1,475.00 ii. Gh¢27,700.00 iii. 10.8% 7. 27.27% 8. 34.66% 9. i. Gh¢8093.25 ii. Gh¢83,875.50 iii. 6.88% 10. i. Gh¢22,800.00 ii. 14% iii. 32% 11. 25.85%

14. Modulo

Ex 14.2B
1. a. Feb b. Apr 2. a. June b. Nov 3. a. Sat b. Wed c. Wed d. Mon 4. A. Mon b. Thu c. Sat d. Sun 5. a. 116 b. 562

Ex 14.3
1. 1 pm 2. i. 9 days ii. 36 days iii. 27 days 3. i. wed ii. 7 wks (35 days) 4. i. mon, fri, tue, sat, wed, sun, thu ii. sat iii. sun iv. sat v. 28 days5. sat 6. Wed

Ex 14.5
1. 2 2. 3 3. 0 4. 27 5. 8 6. 2 7. 31 8. 5

Ex 14.7
A. 1. 1 2. 6 3. 10 4. 8 5. 8 6. 4B. 1. 5 2. 4 3. 13 4. 22 5. 13 6. 1C. 1. 3 2. 24 3. 8 4. 48 5. 19 6. 9

Ex 14.8
1. 5, 10 2. 7, 21 3. 4, 44 4. 30 5. 11, 33 6. 45

Ex 14.9
1. 1, 3 2. 0 3. 2, 4 4. 2, 3 5. 3 6. 5 7. 2 8. 1, 3, 5, 7

15. Indices and Logarithms

Ex15.5
A. 1. 3 2. $\frac{1}{2}$ 3. -3 4. $\frac{1}{3}$ 5. 1 6. $\frac{3}{2}$ 7. $\frac{4}{3}$ 8. 3B. 1. 4 2. 2 3. 4 4. 2 5. $\frac{5}{2}$ 6. $\frac{5}{2}$ 7. -1 8. 7 9. $\frac{1}{2}$10. $-\frac{21}{5}$C. 1. 2 2. -1 3. -4 4. -3 5. 3 6. -3 7. 0 8. -3

Ex 15.6
A. 1. $\log_5 125 = 3$ 2. $\log_{343} 7 = \frac{1}{3}$ 3. $\log_{256} 64 = \frac{3}{4}$ 4. $\log_{10} 0.01 = 2$ 5. $\log_{25} 5 = \frac{1}{2}$ 6. $\log_2 \frac{1}{8}$ = -3B. 1 = 9^0 2. 2 = $4^{\frac{1}{2}}$ 3. 625 = 5^4 4. $3^{-2} = \frac{1}{9}$ 5. 3 = $27^{\frac{1}{3}}$ 6. 0.1 × 10^8 = 10^7

Ex 15.7
1. $\log 10$ 2. $\log 6$ 3. $\log 2$ 4. $\log_6 1296$ 5. $\log 625$

Ex 15.8
1. 2 + 3 $\log_3 2$ 2. 2 + $\log_3 7$ 3. 2 + 3 $\log_5 3$ + $\log_5 2$ 4. 1 + $\log_8 3$ + $\log_8 5$ 5. $\frac{1}{2}$+ $\log_2 7$ 6. $\frac{2}{3}\log 3 + \log 2 + \frac{1}{3}$

Ex 15.9
1. 1 + $\log_5 24$ 2. $\log 0$ 3. $\log 10$ 4. 2 + $\log_3 4$5. 2 + 3 $\log_7 4$

Ex 15.10
A. 1. 7 2. 0 3. 3 4. 5 5. $\frac{1}{3}$ 6. $-\frac{1}{2}$ 7. $\frac{1}{3}$ 8. 2 9. 5B. 1. 2 2. 1 3. $-\frac{1}{3}$ 4. 4 5. $\frac{1}{4}$ 6. -2 7. 1 8. -1 9. $\frac{3}{4}$C. 1. 0.67 2. 0.85 3. 0.51 4. 1.5

Ex 15.11
1. $\frac{1}{2}$ 2. 2 3. 2.712 4. 1 5. 0.3010

Ex 15.12
1. 2 2. -2 3. $\frac{2}{3}$ 4. 1 5. $-\frac{9}{2}$ 6. 3

Ex 15.13
A. 1. 1.465 2. 0.6309 3. 0.1981 4. 1.161 5. -2.585 6. 4.419
B. 1. 2.161 2. 7.122 3. 4.052 4. 4.954 5. 3.878 6. 4.358 7. 1.757 8. 1.635

Ex 15.14
A.1. 1.079 2. 0.5927 3. 1.255 4. 0.1761 5. 0.6532 6. 1.6019 7. 2.68 8. -2.372B. 1. 1.113 2. 2.046 3. 1.293 4. 1.544 5. 0.341 6. -0.251 7. -0.862 8. 2.862 9. 0.251

Ex 15.15
A. 1. 2 2. 4 3. -2 4. 1 B. 1. $\frac{\log_3 45}{\log_3 6}$ 2. $\frac{\log_2 30}{\log_2 2}$ 3. $\frac{\log_2 128}{\log_2 4}$ 4. $\frac{\log_6 72}{\log_6 12}$ 5. $\frac{1}{3\ \log_2 2}$ 6. $\frac{\log_2 100}{\log_2 30}$

Ex 15.16
A. 1. 0 2. 3 3. 2.5 4. $\frac{2}{3}$ 5. 4 6. 8 7. 7 or 9 8. 6.5 or 2.4 9. 10 or -1 B. 1. 1 2. 3 3. 13 4. $\frac{30}{9}$ 5. 0.24 or -4 25 C. 1. 10 2. 2 or -2.5 3. 3 4. -$\frac{27}{7}$ 5. 3.69 or 0.81

Ex 15.19
1. -1.3923 2. 1.3923 3. 3.3923 4. -5.6021 5. 0.3586 6. 0.8597

Ex 15.20
1. 4.914 2. 1.251 3. 3167 4. 0.0001596 5. 0.02721 6. 0.7902

Ex 15.21
1. 3.31 2. 1.127 3. 0.002825

Ex 15.22
A. 1. 0.3865 2. 1.374 3. 25.41 4. 29016 5. 0.2802 6. 0.0107B. 1. 20.21 2. 3.075 3. 0.6559 4. 0.2724 5. 0.0357
C. 1. 1.447 2. 1961 3. 8.932 4. 932.8 5. 0.04152

16. Simultaneous Equations
Ex 16.7
1. (15, 11) 2. (19, 11) 3. 30 and 12 4. (-5, - 4) 5. P = 5p and R = 15p 6. 92 people 7. 21, 50p and 33, 20p 8. 11. i. 400 ii. 200 9. 320 10. p = 6, q = 4 C.P.

Ex 16.8
1. 35 2. 53 3. 86 4. 45

17. Percentages 2
Ex 17.1
A. 1. Gh¢1295.77 2. Gh¢2174.70 3. Gh¢26147.93
B. 1. Gh¢112,394.24 2. I = Gh¢4,210.89, A = Gh¢6,710.89 3. Gh¢ 821.35 4. Gh¢ 80,767

Ex 17.2
A. 1. Gh¢ 443.94 2. Gh¢ 883.22 3. Gh¢ 18,316.80
B. 1. Gh¢ 40320.00 2. Gh¢ 228,131.25 3. Gh¢ 164,115.72 4. Gh¢343,637.24 5.Gh¢595,444.90

Ex 17.3
1. Gh¢ 29,864 2. Gh¢ 3762.00 3. Gh¢ Gh¢1,613.66 4. i. Gh¢1,504.03 ii. Gh¢1,575.97 5. Gh¢5,200.93

Ex 17.4
1. Gh¢1,734.00 2. Gh¢145,800.00 3. Gh¢ 851.84 4. Gh¢547.00 5. i. Gh¢1,500.00 ii. Gh¢5,742.19 6. Gh¢19,000, Gh¢17,100.00, Gh¢14,022.00, Gh¢9,815.40 7. i. Gh¢12,609.92 ii. Gh¢954.6 iii. Gh¢13,564.52

Ex 17.5
1. i. D = ¢31,500, G = ¢10,500 ii. D = 75%, G = 25% 2. i. B = Gh¢10.5 million, W = Gh¢4.5 million ii. 300% 3. S = ¢3,876.92, D = Gh¢4,123.07 4. Gh¢2,660.00 5. i. 16,000.00 ii. Gh¢1,190.00 iii. 25% 6. Gh¢14,700.00 7. Gh¢3,300.008. 45 9. 8 months 10. Gh¢7,500

Ex 17.6
1. Gh¢1,500.00 2. a. Gh¢ 10,975,610 b. 12.9%

Ex 17.7A
1. Gh¢9,000 2. i. Gh¢385.00 ii. Gh¢2,585.003. i. Gh¢7,320.00 ii. Gh¢43,920.004.i. Gh¢225.00 ii. Gh¢2,100.00

Ex 17.7B
1. i. Gh¢355.50 ii. Gh¢44.50 2. i. Gh¢700.00 ii. Gh¢100 3. Gh¢435.55 4. Gh¢3,888.005. Gh¢9,900

Ex 17.8
A. 1. Gh¢200.00 2. Gh¢115.00 3. Gh¢45.00 4. Gh¢25.00 5. Gh¢117.50 B. 1. Gh¢422.00 2. Gh¢2,321.00 3. Gh¢1,181.60 4. Gh¢2,532.00

Ex 17.9
1. i. Gh¢4,117.07 ii. Gh¢102.93 2. Gh¢720.00 3.Gh¢1333.76 4. i. Gh¢27,203.80 ii. Gh¢1,496.20

Ex 17.10
1. Gh¢918.00 2. Gh¢2,283.00 3. Gh¢1,440.00 4. Gh¢945.00 5.Gh¢1,200.00 6. Gh¢1,300.00 7. 5% 8. 9%

Ex 17.11
A.Gh¢75.50, Gh¢150.00, Gh¢1504.00, Gh¢5940.00B. 1. Gh¢ 6,000.002. i. Gh¢ 1,200.00 ii. Gh¢1,020.003.i. Gh¢4,700.00 ii. Gh¢658.00 iii. Gh¢4,342.004. i. Gh¢540.00 ii. Gh¢1,188.00iii. Gh¢3,312.005. 15% 6. 6. i. 15,700.00 ii. 1,000.00iii. 17,000.00 7. a. 9000.00 b. 122.92 c. 12.3% 8. a. 6752.00 b. 30148.00

Ex 17.12
1. 13,750.00 2. 4,000.00 3. 9,000.00 4. 10,080.00

Ex 17.13
1. Gh¢ 71,000.00 2. i. Gh¢8082.50 ii. Gh¢710.00, iii. 900 units 3.i. Gh¢236.00 ii. 560 units 4. i. 34,400.00 ii. 54,800 liters C.P. 265.48

18. Variations
Ex 18.1
B. 1. i. 64 ii. $\frac{1}{2}$ 2. 6.75, y = $2x^3$ 3. 31.5 4. y = 3x, 12 5. a. A = 44.1cm^2 b. r = 3 6. x = 4, y = 2 7. i. 1.5 ii. 16 8. 55 9. 128 10. 7 11. 4 12 . s = $\sqrt{\frac{w}{2}}$ 13. 216 14. 23.35liters 15. i. 15 16. 2.60

Ex 18.2
B. 1. i. 1.5 ii. $\frac{9}{8}$ 2. y = $\frac{3}{x}$ 3. $\frac{20}{9}$ 4. $\frac{3}{4}$ 5. i. y = $\frac{6}{x^2}$ ii. $\frac{2}{3}$ 6. $\frac{5}{4}$ 7. 7.5 8. $\frac{32}{343}$ 9. 3.6 10. 0.04

Ex 18.3
A. 1. i. Z = kx^2 y ii. V = khr^2 iii. R = $\frac{k\sqrt{x}}{y^2}$ B. 1. Z = $\frac{2x^2}{3\sqrt{y}}$, $\frac{4}{3}$ 2. $i. Q = \frac{196\,x}{3z^2}$ ii. $\frac{1}{9}$ 3. 913 4. 189 5. C = $\frac{3m^3}{2n}$, $\frac{3}{4}$ 6. y = $\frac{x^2z^3}{3}$, 48 7. i. 1 ii. 4.9 8. 6, 3

Ex 18.4
B. 1. i. T = 15 + 10r ii. a. 6 b. 20 2. 12 3. y = 4x - $\frac{1}{2}$ 4. a. k = 1 and c = -1, y = 1 – x b. 1 5. i. c = 6500 + $\frac{5}{12}$d ii. 7,000 6. Gh1, 090,000 7. S = 2000 + 15n 8. i. C = 6 + $\frac{2000}{n}$ ii. 7.3 9. Gh195 10. i. C = 10000 + 100nw ii. Gh130,000 11. i. C = $8d^3$ + $\frac{150}{T}$ii. GH512,000 iii., 42 miles 12. i. C = 830 + $\frac{n}{5}$ ii. Gh93013. i. V = $\frac{49}{2}t$ - $\frac{21}{2}t^2$ ii. – 21

20. Probability
Ex 20.10
1.i. $\frac{2}{5}$ ii. $\frac{31}{40}$ 2. I, Not ii. Ind 3. i. $\frac{5}{24}$ ii. $\frac{7}{8}$ iii. $\frac{2}{3}$ 4. $\frac{1}{8}$ 5. $\frac{4}{5}$ 6. $\frac{2}{3}$
7. i. $\frac{1}{6}$ ii. $\frac{1}{3}$ iii.$\frac{1}{2}$ 8. i. 0.6 ii. 0.05 iii. 53 9. 0.57 10. i. $\frac{8}{27}$ ii. $\frac{2}{27}$

Ex 20.11
1. $\frac{1}{21}$ 2. $\frac{1}{32}$ 3. i. $\frac{1}{8}$ ii .$\frac{1}{8}$ iii. $\frac{3}{4}$

Ex 20.12
1. $\frac{3}{10}$ 2. $\frac{1}{22}$ 3. i. a. $\frac{7}{9}$ b. $\frac{5}{9}$ ii. a. $\frac{1}{12}$ b. $\frac{4}{81}$ 4. i. $\frac{2}{9}$ ii. $\frac{5}{55}$ 5 . i. $\frac{1}{2}$ ii. $\frac{3}{10}$
6. i. $\frac{7}{15}$ ii. $\frac{1}{10}$ iii. $\frac{1}{5}$ 7. i. $\frac{1}{4}$ ii. $\frac{2}{9}$ 8. $\frac{1}{30}$ 9. i. $\frac{1}{2}$ ii $\frac{5}{18}$ 10. i. $\frac{44}{91}$ ii. $\frac{36}{455}$

21. Quadratic equation
Ex 21.7
1. 2 or -3 2. 17 3. 7 or -12 4. b = 10 5. 13 or – 6 6. 2 or 9 7. 16yrs C.P 1. 46 or 57 2. 60km/h 3. 5

Ex 21.9
1.c. i. x = 1 ii. x = - 0.7 or x = 2.7 iii. - 0.7 <x< 2.7 2. b. i. x = -1 ii. 2 iii. -5 ≤x< -1 iv. -1 ≤x< 3 3. b. i. x = -1 ii. y = - 4 iii. -5 ≤x< -1 iv. -1 ≤x< 3 4.b. ii. 1.2<x≤ 5 iii. - 4 <x≤ 1.2iv. -4 < x< 1.2 v. - 4 >x> 1.2 5. c. i. x = -1 or x = 3 ii. y = 4 iii. x = 1 iv. 1 <x≤ 5

22. Mensuration 1
Ex 22.1
1. i. 2.10 ii. 3.14 iii. 8.10 2. i. 18.84 ii. 9.42 iii. 17.42 3. i. 23.9cm ii. 33.9cm^2 4. i. 32.97cm ii. 230.79cm^2 iii. 60.97cm 5. 35.5

Ex 22.2
1. 11^0 2. 94^0 3. 158^0

Ex 22.3

1.

Time	1:00	2:30	7:00	8:45	10:30	11:20
Angle	30^0	105^0	210^0	262.5^0	315^0	340^0

2. 8.18cm^2, 3.27cm

Ex 22.4

1. 0.03 2. 220.87 3. i. 14 ii. 29.3 iii. 44.3 4. i. 7.3 ii. 21.3 iii. 4.45 5. 14.1

Ex 22.5

A. 1. 108 2. 64 3. 76 4. 560 5. 84 6. 160 B. 1. 26 2. 42 3. 18 4. 37 5. 71 6. 25C. 1. 100 2. 49 3. 484 4. 1225 5. 81 6. 121

Ex 22.6

1. 37 2. 36 3. 72 4. 52

Ex 22.7

1. 351.68 2. 44 3. 176 4. 262 5. 66 6. 62.86

Ex 22.7B

1. 24cm^2 2. 96cm^2 3. 70cm^2 4. 60cm^2

Ex 22.8

1. 1930.5 2. 225 3. 1287 4. 704 CP; 1. 123 2. 22 3. 140

Ex 22.9

1. 140 2. 9 3. 15 4. 18 5. 30

Ex 22.10

A. 1. 25 2. 144 3. 256 4. 625 5. 529 6. 1681

B. 1. 9 2. 7 3. 15 4. 13 5. 20 6. 11

Ex 22.11

A.1. 40 2. 391 3. 27B. 1. 28 2. 1200 3. 240 4. 418

Ex 22.12

A. 1. 504 2. 1288 3. 220 4. 289.71 B. 1. 30.86 2. 150 3. 506 4. 99C. 1. 586 2. 214

Ex 22.13

1. 5300 2. 51350 3. 11414 4. 7982 5. 7391

23. Plane Geometry I

Ex 23.1

1. a = 50^0, b = 90^0 c = 40^0 2. <OCP= <OCQ= 90^0, <AOB=140, <OAB = <OBA = 20, <OAC = <OCA = 40, <OCB = <OBC = 30, <ACP = 50, <BCQ = 60 3. DG = 10, EF = 2.5 4. 5cm

Ex 23.2

1. i. ORT, OST ii. TR = TS, OR = OS iii. < ORT = < OST , < ROT = < SOT, < RTO = < STO iv. < ORT , <OST v. 7.4 2. CBD, ADB,; < CBD = < CDB = 70^0, < ABD = < ADB = 20^0, < BAD = 140^0 3. i. OQ: P↔R ii. < SRQ = < OPS = < ORS = 20^0; four right angles at S iii. 6.7cm 4. i. 15 ii. 28

CP: 1. i. AY = AZ, BZ = BX, CX = CY ii. AB = 7cm, BC = 9cm, CA = 8cm iii. AB = 14cm, AC = 15cm iv. 15cm v. 18cm 2. ii. BC = 7cm, CA = 8cm, AB = 9cm iii. 5cm, 3cm, 3cm 3. i. AF = AE, BD = BF, CE = CD ii. 10cm iii. 5 cm 4. i. < PTQ = 54^0, < POQ = 126^0 ii. < PTQ = (180 – 2*x*)^0, <POQ = 2*x* 5. 12cm

Ex 23.3

1. i. < BAC, < BEC, < BDC ii. < DBE, < DCE iii. < ABD, < ACD 2. 220^0 3. 76^0 4. 26^0 5. i. 38^0 ii. 107^0 7. i. 60^0, 30^0 ii. 300^0, 150^0 8. i. 80 ii. (180 – 2*x*)^0 C.P. 1. <YOZ = 120^0, < OYZ = < OZY = 30^0 2. < P = 65^0, < Q = 60^0, < R = 55^0 3. Δ AOB is equilateral

Ex 23.4

A. 1. x = 70^0, y = 25^0 2. x = 45^0, y = 45^0, z = 55^0 3. x = 30^0, y =30^0, z = 120^0 B. 1. i. <ACB = 90^0, < ABC = 45^0 2. 25^0 3. 80^0 C. P. i. 12cm ii. 192 cm^2 iii. 192 cm^2

Ex 23.5

2. i. all 40^0 ii. all 36^0 iii. all x^0 *iv. all equal* 3. 120^0 4. i. 55^0 ii. 25^0 5. x = 40^0, y = 60^0, p = q = 20^0 8. <XYZ = <YXZ = < ZAB = 35^0, < XZY = < AZB = 110^0 C.P. 1. i. they are equiangular ii. DE = 5 cm, DC = 6 cm 2. 5 cm 3 …. 4. <A = 40^0, < B = 60^0, < C = 80^0

Ex 23.6

1. i. 280^0 ii. 40^0 iii. 140^0 iv. 180^0 2. m = 50^0, n = 130^0 3. i. 45^0 ii. 22.5^0 iii. 112.5^0 4. x = 80^0, y = 60^0 5. a = 115^0, b = 65^0 6. x = 36^{0}7. i. 50^0 ii. 130^0 C.P 1. 58^0 ii. 41^0 2. i. x = y = z ii. 121^0 3. 85^0**4.** *< ABE = < EFC=105^0, <DEB= < BCF = 80^0, < EBC = 75^0, < BEF = 100^0*

Ex 23.7

1. 70^0 2. < P = 75^0, < R = 105^0, < PQR = 100^0, < PSR = 80^0

Ex 23.8

i. 63^0 ii. 27^0 iii. 82^0 iv. 90^0 2. r = 63^0, q = 88^0, p = 29^0 3. 85^0

Review ex

1. 130^0, 50^0 2. x = 44^0 3. y = 48^0 4. y = 25^0 5. a = 25^0, b = 55^0 and c = 62^0 6. x = 25.7^0 y = 115.7^0 7. 40^0, 39^0

24. Trigonometry 1

Ex 24.2

1. $\frac{\sqrt{3}}{3}$ 2. - $\frac{\sqrt{3}}{3}$ 3. $\frac{1}{2}$ 4. $\frac{\sqrt{2}}{2}$ 5. 1 6. - $\frac{\sqrt{2}}{2}$

Ex 24.3

A. 1. $\frac{-1-\sqrt{2}}{2}$2. $\frac{4\sqrt{3}}{3}$ 3. $\frac{5}{4}$ 4. - $\frac{5}{12}$B. 1. $\frac{\sqrt{6}+\sqrt{2}}{4}$ 2. $\frac{7}{4}$ 3. 4 + 2$\sqrt{3}$ 4. 20 – 10 $\sqrt{3}$ 5. 2 - $\sqrt{3}$ 6. 2 + $\sqrt{3}$C. 1. $\frac{3}{4}$ 2. 3 3. 2 4. $\frac{2+2\sqrt{3}}{3}$ 5. 1 + $\sqrt{3}$

Ex 24.4

1. 120^0 2. 53.13^0 3. – 49^0 4. 36.87^0

Ex 24.5

A. 1. 12^0 2. –24^0 3. 68^0 4. 37^0 5. 21^0 6. 41^0 7. 90^0 8. 81^0

B. 1. 75^0 2. 30^0 3. 0^0,180^0, 30^0 150^0 4. ± 60^0, ±90^0

Ex 24.6

1. 14^0 2. 15^0 3. 18^0 4. 45^0

Ex 24.7

1. $\frac{\sqrt{6}-\sqrt{2}}{4}$ 2. $\frac{\sqrt{6}-\sqrt{2}}{4}$ 3. $\frac{1}{2}$ 4. $\frac{-\sqrt{6}-\sqrt{2}}{4}$ 5. $\frac{\sqrt{6}+\sqrt{2}}{4}$ 6. $\frac{\sqrt{6}+\sqrt{2}}{4}$

Ex 24.8

A. 1. 37^0 2. 53^0 3. 24^0 4. 34^{0}B. 1. 38^0, 2. 51^0 3. 31^0 4. 49^{0}5.a. 45^{0}b. 6cm c. 1

Ex 24.9

A. 1. 22cm, 27 cm 2. a. 7m b. 4m 3. i. 9.6m ii. 7.2 m

B. 1. 13 2. 67^0 3. P = 11, m = 38^0 4. r = 17, u = 60^0 5. t = 22m, q = 23m

Ex 24.10

A. 1. $\cos\theta = \frac{7}{25}$, $\tan\theta = \frac{-24}{7}$ 2. $\cos\theta = \frac{12}{13}$, $\tan\theta = \frac{12}{15}$ 3. i. $\sin\theta = \frac{13}{15}$, $\cos\theta = \frac{8}{15}$ ii. 36 iii. 90^0, 60^0, 30^0 B. 1. 0.51 2. $\frac{5}{4}$**3.** $\frac{17}{13}$4. $\frac{15}{32}$ 5.i. 0.85 ii.1.15 iii 1.92 iv. 11.56 6. i. 41^0 ii. $\frac{13}{15}$ iii. $\frac{45}{52}$ iv. 0.75 7. 0.28 8. 7.5

Ex 24.11

A. 1. R = 66^0, q = 13.92 r = 12.79 2. 9.17cm3. $61.5^0$4. 79.8^0, 52.6^0, 47.6^0, $135cm^2$5.i. $38.2cm^2$ ii. 10.7cm iii. 58.3^0, 48.8^0B. 1. 7.5 sq. units 2. 3 sq. units 3. 15 sq. units 4. 10 sq. unitsC.1. 1. 28.96^0, 46.57^0, 104.48^0 2. 27 sq. units 3. 4 4. 8.8cm 5. i. 4.77m ii. $8.34m^2$

Ex 24.12

1 . 15.1m 2. 53^0 3. 18.43m 4. 71m 5. 69m6. 12,222 cm

Ex 24.13

1. i. 15km, ii. 3 km 2. 23m 3. 68.7m,50m 4. 1011m
5. a. 71m, 64m b. 26m 6. i. 21m, ii. 17^0 iii. 27^0

Revision Ex

1. B : 25.9km, 96.6 km, C : 217.1km, 38.2km
2. 6.4km3. a. 113^0 b. 36.69 km4. a. 021^0b. 14km 5. a. 4.9 km b. i. 4.1km iii. 1.5km c. (4.4km, 020^0)

Ex 24.14

1. a. 4.9m 2. 7m b. 62^0 3. i. 29^0ii. 2.2m 4. i.4m ii. 8m.

Ex 24.15

1.a. 79m, b. 4740m 2. 7cm 3. 23^0 5. i. 50m ii. 100m

25. Series and Sequence

Ex 25.1

Ex 25.5

1. i. a = 1, d = 2, 1 + 2(n – 1) ii. a = 0 , d = 4, 4(n – 1)
2. i. a = 14, d = 12 ii. $U_n = 14 + 12(n-1)$ 3. 99 4. 126 5. a = 30, d = -2, $U_{10} = 12$

Ex 25.6

1. 2550 2. -300 3. 15150 4. 18 5. 910 6. 98 7. 125 8. 6, 1, - 4, - 9, - 14 9. *a* = 3, *d* = 4; 3, 7, 11, 15, 19 10. 8 11. 3.5, 2 12. i. *a* = 9, d = 3 ii. 351

Ex 25.7

A. 1. r = 3, 708,588 2. r = $\frac{3}{2}$, 1,108 3. r = 4, 65, 536 4. - $\frac{1}{2}$, - $\frac{3}{512}$ 5. 7B.1. $U_n = 3(2)^{n-1}$ 2. 128 3. $U_n = 192\left(-\frac{1}{4}\right)^{n-1}$ 4. 327,680

Ex 25.7B

1. 7. 2. 8 3. 13

Ex 25.8

1. 243 2. - $\frac{1}{2}$ 3. $\frac{243}{8}$ 4. $\frac{10}{243}$ 5. - $\frac{625}{162}$

Ex 25.9

1. i. a = 1, r = 2 ii. 32 2. $\frac{1}{3}$ 3. -3, -2

Ex 25.10

1. – 349,525 2. 18,662 3. $\frac{4}{19683}$ 4. a = $\frac{5}{2}$, r = 2, S = 157.5

Ex 25.11

1. 2160 2. 7.8 3. 1,024,000 4. Gh 3,360 5. Gh16,384

6. 75,712 7. i. Gh25,000 ii. Gh164,500 8. 1085 9. Gh7,000
10. 8. i. Gh¢10,740.00 ii. Gh¢91,20011. b

27. Mensuration 2

Ex 27.1

A. 1. 972 2. 542 2 3. 358 4. 2628 B.1 920 2. 132 3. 34 4. 606

Ex 27.2

1. 54 2. 294 3. 216 4. 600 5. 150 6. 384

Ex 27.3

A. 93.5 2. 1171.5 3. 1842.5 4. 1276 5. 96.7 6. 487.08
B. 1. 66 2. 528 3. 113 4. 3.5 5. 7

Ex 27.4

1. 304 2. 63 3. 160 4. 9600 5. 925 6. 2075

Ex 27.5

A. 1. 2145 2. 209 3. 5808 4. 13, 420 5. 1, 012
B. 1. 528 2. 1980 3. 150.72 C. 1. 2 2. 5 3. 4.8

Ex 27.6

1. 616 2. 314.3 3. 1018.3 4. 3850 5. 1386 6. 2464

Ex 27.8

A. 1. $108cm^3$ 2. $1584cm^3$ 3. $840m^3$ 4. $160cm^3$
5. $18000cm^3$B. 1. 32cm 2. 5.5cm 3. 3cm 4. 19cm

Ex 27.9

A. 1. 1520.88 2. 1728 3. 8000 B. 1. 7 2. 14 3. 11
C. 1. 4 2. 3 3. 2744

Ex 27.10

1) $49896cm^3$ 2) $249480cm^3$ 3) $15400cm^3$4) 3.5cm 5) 14cm

Ex 27.11

1. $125cm^3$ 2. $213.33cm^3$ 3. a. $300cm^2$b. $900cm^3$ 4. a. 15 b. 633

Ex 27. 12

1. 36cm 2. 21cm 3. $468.91cm^3$ 4. 366.33

Ex 27. 12B

1. 3.84cm 2. 3cm 3. $489.54cm^3$ 4. 2.1cm 5. 126 6. $353.57cm^2$

Ex 27.13

1. 949 2. 1037, 1804, 5338.7 3. 146.67 $cm^3$4. i. 1006 ii. 1659 iii. 4777

Ex 27.14

A. 1. $268cm^3$ 2. $524cm^3$ 3. $905cm^3$ 4. $1437cm^3$ 5. $2146cm^3$ 6. $4190cm^3$B. 1. 2.04cm 2. 2.93cm 3. 4.57cm 4. 2.77cm 5. 8cm 6. 5.3cm

Ex 27.15

1. 858 $cm^2$2. 266 cm^3 3. 572 $cm^2$4. 220 cm^2

Ex 27.16

1. i. 6.9 ii. 373.86 iii. 207 2. 300 3. a. 12 b. 96 4. 72

Ex 27.17

1. $32{,}153.60cm^3$2. $761.14cm^3$ 3. 1330 cm^3 4. 188.57 5. V = 1.713 cm^3 A = 26.47 $cm^2$6. i. $44m^2$ ii. Gh¢22,000.00

TABLE 1: LOGARITHMS OF NUMBERS

x	0	1	2	3	4	5	6	7	8	9	Differences								
											1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

TABLE 1 *(contd.)*: LOGARITHMS OF NUMBERS

x	0	1	2	3	4	5	6	7	8	9	Differences								
											1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8159	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9590	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

TABLE 2 *(contd.)*: ANTILOGARITHMS OF NUMBERS

x	0	1	2	3	4	5	6	7	8	9	Differences 1	2	3	4	5	6	7	8	9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	6	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	6	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	5	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	8	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	6	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	11	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	4	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7491	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	14	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

TABLE 2: ANTILOGARITHMS OF NUMBERS

x	0	1	2	3	4	5	6	7	8	9	Differences 1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	1	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	1	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	[illegible]	[illegible]	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	[illegible]	[illegible]	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6
x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

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