Mini Project: Location Estimator with Compass Only

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Abstract

Location estimation is a critical task in numerous civil and military applications. While the Global Navigation Satellite System (GNSS) provides a highly accurate solution, its availability can be compromised in certain environments. This report presents a method for location estimation using only a compass and a set of known anchor points. We explore and compare two computational approaches: a Least Squares Estimator and a Maximum Likelihood Estimator (MLE). The MLE method is designed to account for uncertainties in both anchor positions and bearing measurements, including systematic bias. Through Monte Carlo simulations based on a real-world scenario in Hanoi, Vietnam, we demonstrate the effectiveness of the proposed estimators. The results indicate that the Maximum Likelihood Estimator generally outperforms the Least Squares method, especially in the presence of measurement bias, achieving a mean position error of under 100 meters in realistic conditions.

1 Introduction

1.1 Problem Statement

The ability to determine one's position is fundamental to navigation and has wide-ranging applications in both civilian and military domains. The advent of the Global Navigation Satellite System (GNSS), which includes constellations like GPS, GLONASS, BeiDou, and Galileo, has revolutionized positioning, providing solutions with remarkable accuracy. However, there are scenarios where GNSS signals may be unavailable or unreliable, such as in urban canyons, dense forests, underwater, or in environments subject to electronic interference. In such "extream cases," alternative methods for localization are necessary. This project addresses the challenge of estimating a user's location when the only available tool is a compass, and the positions of several known landmarks (anchors) are available.

1.2 Current Approach and Its Limitations

A classical method for this problem is the geometric approach, where bearings to two or three known landmarks are used to triangulate the observer's position. As illustrated in Figure 1, the intersection of the lines of bearing from the anchors indicates the user's location.

However, this traditional method has significant limitations:

- It is typically restricted to a small number of anchors, usually two or three.
- It fails to account for inherent uncertainties, such as errors in the known anchor positions and, more critically, noise and systematic bias in the bearing measurements from the compass.

These limitations necessitate a more robust and flexible approach that can handle an arbitrary number of anchors and explicitly model the various sources of error.

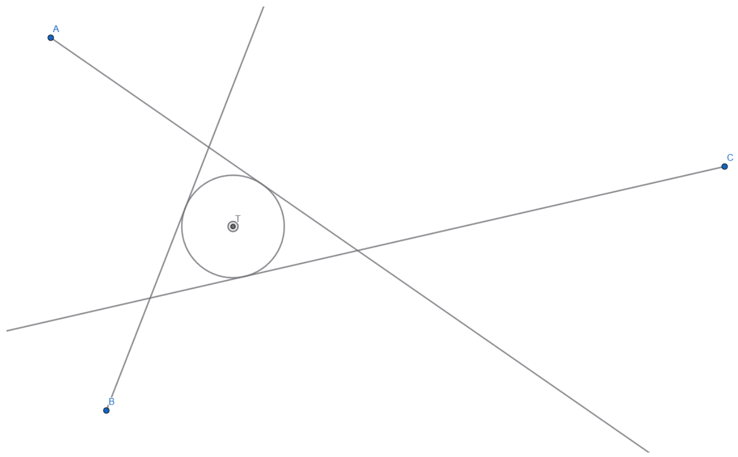


Figure 1: Classical geometric approach for location estimation.

2 Methods

To overcome the limitations of the classical geometric approach, we investigate two advanced estimation techniques. Both methods are capable of utilizing an arbitrary number of anchor points.

2.1 Least Squares Estimator

The Least Squares Estimator is an extension of the classical geometric method. Instead of finding a single intersection point, which may not exist in the presence of noisy measurements, this method seeks to find a point that minimizes the sum of the squared orthogonal distances to the lines of bearing from each anchor.

The objective function to be minimized is the sum of the L2 distances from the estimated user position, T, to each bearing line, l_i, as defined by the anchor positions and the measured bearings. The loss function is given by:

$$\mathcal{L}(T) = \sum_{i \in \text{anchors}} d(T, l_i)^2$$

where $d(T, l_i)$ is the perpendicular distance from point T to the line l_i.

While this method can handle more than three anchors, it inherits a key drawback from the classical approach: it does not explicitly model or account for the noise in anchor positions or the systematic bias in bearing measurements.

2.2 Maximum Likelihood Estimator

The Maximum Likelihood Estimator (MLE) provides a robust probabilistic framework for this problem. Instead of minimizing geometric distances, it seeks to find the set of parameters—the user's position (x, y) and the compass bias b—that maximizes the likelihood of observing the measured bearings $\{\phi_{\text{obs},i}\}$. The total log-likelihood, which is maximized, is the sum of the log-likelihoods for each individual anchor observation:

$$\log \mathcal{L}(x, y, b | \{\phi_{\text{obs},i}\}) = \sum_{i \in \text{anchors}} \log P(\phi_{\text{obs},i} | x, y, b)$$

The core of the method lies in defining the probability $P(\phi_{\text{obs},i} | x, y, b)$, which requires a precise model of the measurement uncertainties.

2.2.1 Modeling Uncertainty with a Wrapped Normal Distribution Approximation

The observation model assumes that the measured bearing to an anchor is a combination of the true bearing, a systematic bias, and random noise. However, the "true" bearing is itself uncertain because the anchor's position is not known perfectly. The total uncertainty in a bearing measurement therefore has two independent sources:

1. **Bearing Measurement Uncertainty**: The compass has inherent random noise, which is modeled as a normal distribution with a standard deviation of σ_{bearing}.
2. **Anchor Position Uncertainty**: Each anchor's location is known with some uncertainty, represented by a 2D covariance matrix Σ_{anchor}.

Since angles are circular data (i.e., 360° is the same as 0°), the ideal probability distribution would be a Wrapped Normal or von Mises distribution. However, for computational efficiency and simplicity, the implementation uses a standard Normal distribution as an approximation. This approximation is highly effective when the total variance is small, causing the probability density to be concentrated over a narrow range of angles.

To make this approximation work, two key steps are taken in the code:

- The difference between the observed and expected angle is calculated as the shortest arc on the circle using a wrapping function ('wrap_angle' in the code, which maps the difference to $[-\pi, \pi]$).
- The variances from the two independent sources of error are combined to define the total variance of the Normal distribution.

2.2.2 Propagation of Anchor Uncertainty

A crucial step is converting the anchor's 2D positional uncertainty (in meters) into an angular uncertainty (in radians). This is achieved by linearizing the angle function around the anchor's mean position.

Let the user's estimated position be $\mathbf{p} = [x, y]^T$ and the anchor's uncertain position be $\mathbf{a} = [x_a, y_a]^T$ with mean $\boldsymbol{\mu}_a$ and covariance Σ_a. The angle from the user to the anchor is given by:

$$\theta(\mathbf{a}) = \arctan\left(\frac{y_a - y}{x_a - x}\right)$$

Using a first-order Taylor expansion, the variance of the angle due to the anchor's position uncertainty can be approximated by the formula for propagation of uncertainty:

$$\sigma^2_{\theta,\text{anchor}} \approx (\nabla_{\mathbf{a}}\theta)^T \Sigma_a (\nabla_{\mathbf{a}}\theta)$$

where $\nabla_{\mathbf{a}}\theta$ is the gradient of the angle function with respect to the anchor's position, evaluated at the anchor's mean position $\boldsymbol{\mu}_a$:

$$\nabla_{\mathbf{a}}\theta = \begin{bmatrix} \frac{\partial\theta}{\partial x_a} \\ \frac{\partial\theta}{\partial y_a} \end{bmatrix} = \frac{1}{(x_a - x)^2 + (y_a - y)^2} \begin{bmatrix} -(y_a - y) \\ x_a - x \end{bmatrix}$$

This calculation is performed by the 'angle_grad' and 'angle_var_anchor' variables in the code.

2.2.3 The Complete Log-Likelihood Function

Since the bearing measurement noise and the anchor position uncertainty are independent, mathematically, we need to use integral to calculate the distribution of their sum. In our implemented code we approximated the value by simply adding their variances. The total variance for the bearing to anchor i is:

$$\sigma^2_{\text{total},i} = \sigma^2_{\theta,\text{anchor},i} + \sigma^2_{\text{bearing}}$$

The expected (mean) value of the observed angle, in mathematical angle space, is the calculated angle to the anchor's mean position plus the mathematical angle bias (which is the negative of the bearing bias):

$$\mu_{\text{expected},i} = \theta(\boldsymbol{\mu}_{a,i}) - b$$

The log-likelihood for a single observation $\phi_{\text{obs},i}$ is then the log-PDF of the Normal distribution $N(\mu_{\text{expected},i}, \sigma^2_{\text{total},i})$:

$$\log P(\phi_{\text{obs},i}|\cdot) = -\frac{1}{2}\log(2\pi\sigma^2_{\text{total},i}) - \frac{(\text{wrap}(\phi_{\text{obs},i} - \mu_{\text{expected},i}))^2}{2\sigma^2_{\text{total},i}}$$

This is precisely the formula implemented in the ‘analytical_log_likelihood‘ function. The optimizer then adjusts (x, y, b) to find the values that make the sum of these log-likelihoods as large as possible.

3 Implementation Details

The estimators were implemented in Python using the NumPy and SciPy libraries.

3.1 Coordinate System

For computational purposes, latitude and longitude coordinates are converted to a local Cartesian coordinate system in meters. A simple equirectangular projection is used, which is sufficiently accurate for the relatively small geographical areas considered in this project. The first anchor point is used as the reference for this conversion.

3.2 Angle and Bearing Conversion

A distinction is made between navigational bearings (0° North, clockwise) and mathematical angles (0° East, counter-clockwise). The implementation includes helper functions to convert between these two representations, as trigonometric functions in programming libraries typically use mathematical angles in radians.

3.3 Optimization

The core of the estimation process involves minimizing the respective loss functions. For the Maximum Likelihood Estimator, the ‘scipy.optimize.minimize‘ function is used. To enhance robustness and avoid local minima, the optimization is run multiple times from different starting points. The initial guess for the first attempt is the centroid of the anchor points with zero bias. Subsequent attempts introduce random perturbations to the initial guess. The 'L-BFGS-B' optimization method was found to be effective, with bounds placed on the parameters to ensure stability.

4 Simulation and Results

4.1 Experimental Setup

To evaluate and compare the performance of the two estimators, a Monte Carlo simulation was conducted. A realistic scenario was set up using a location in West Lake, Hanoi, as the true user position.

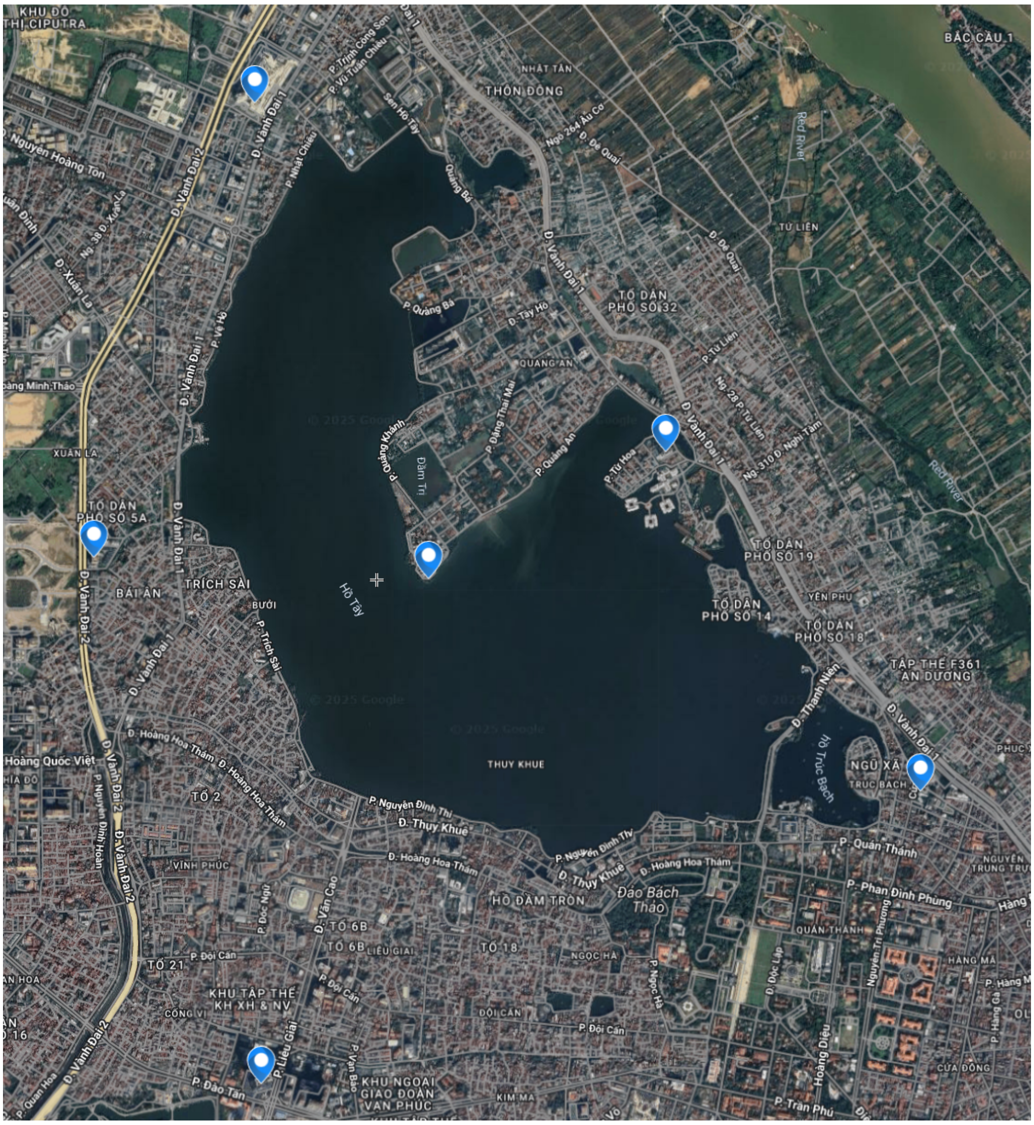


Figure 2: Map of the simulation area in West Lake, Hanoi, with five anchor points.

Five prominent landmarks around the lake were chosen as anchor points:

- Sheraton Hanoi Hotel
- EVN Twin Tower
- Lotte Hotel Hanoi
- Moon Tower
- Lotte World Aquarium Hanoi

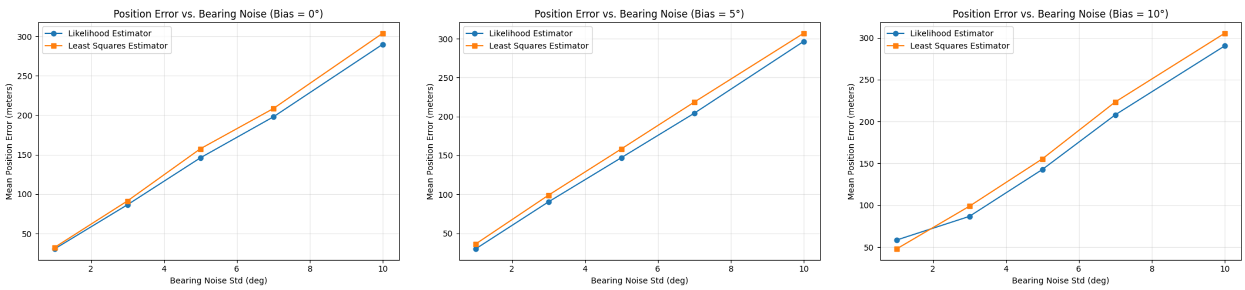
The simulation process for each run is as follows:

1. For each anchor, a true position is sampled from a normal distribution defined by its known mean position and an uncertainty diameter.
2. The true bearing from the true user position to each sampled anchor position is calculated.
3. A true compass bias is added to this true bearing.
4. Random measurement noise, sampled from a normal distribution, is added to the biased bearing to generate the final observed bearing.
5. Both the Least Squares and Maximum Likelihood estimators are used to estimate the user's position (and bias, for the MLE) based on the observed bearings and the mean anchor positions.
6. The estimation error (distance between the estimated and true user position) and bias error are calculated.

This process was repeated for 1000 simulations for various levels of bearing noise and bias to obtain statistically significant results.

4.2 Results

The performance of the Likelihood Estimator and the Least Squares Estimator was evaluated under varying conditions of bearing noise and systematic compass bias. The mean position error in meters served as the primary performance metric for the comparison. The results, presented in Figure 3 and Figure 4, reveal several key findings.



(a) Bias = 0° (b) Bias = 5° (c) Bias = 10°

Figure 3: Comparison of Mean Position Error vs. Bearing Noise for different bias values. The Likelihood Estimator (blue) consistently outperforms the Least Squares Estimator (orange) as systematic bias increases.

Figure 3 illustrates the impact of increasing random bearing noise at three fixed levels of systematic bias (0°, 5°, and 10°). The key observations are:

- For both estimators, the mean position error increases as the bearing noise standard deviation becomes larger.
- When there is no systematic bias (Figure 3a), the two estimators perform very similarly.
- As a systematic bias is introduced and increased (Figure 3b and 3c), the Maximum Likelihood Estimator consistently outperforms the Least Squares Estimator, and the performance gap widens.

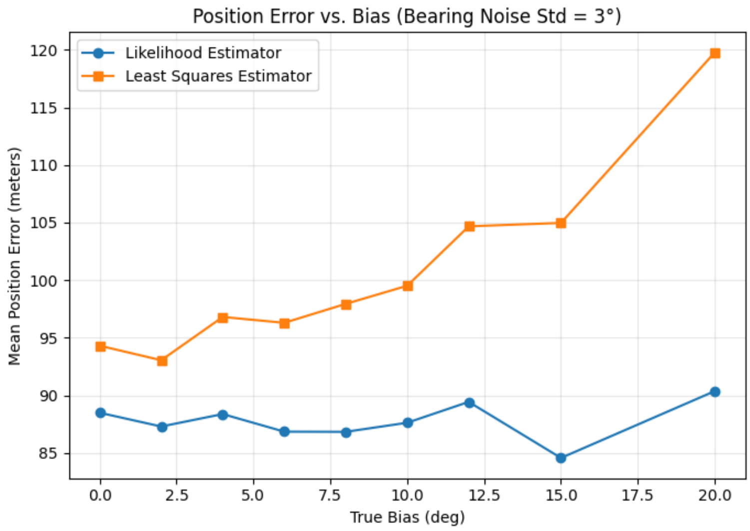


Figure 4: Comparison of Mean Position Error vs. Bias when sd of Bearing Noise = 3°

To further isolate the effect of systematic bias, Figure 4 shows the performance of both estimators as the true bias is varied from 0° to 20°, while the bearing noise is held at a constant, realistic level of 3° standard deviation. This analysis demonstrates:

- **Likelihood Estimator:** The performance of the Likelihood Estimator remains remarkably stable and robust. Its mean position error stays relatively flat, generally below 90 meters, across the entire range of tested biases. This indicates its effectiveness in correctly identifying and compensating for the unknown bias.
- **Least Squares Estimator:** In contrast, the error for the Least Squares Estimator shows a distinct upward trend as the bias increases. While its performance is comparable at zero bias, it degrades significantly as the bias grows, with the error climbing from under 95 meters to over 115 meters.

Collectively, these results show that while both methods are sensitive to random noise, the Maximum Likelihood Estimator is significantly more robust against the systematic errors common in real-world compass measurements. Its ability to model and correct for bias makes it the superior method for this application.

5 Conclusion and Future Work

5.1 Conclusion

This project successfully developed and evaluated two methods for location estimation using only a compass and a set of known anchor points. The Maximum Likelihood Estimator, which accounts for uncertainties in both anchor positions and bearing measurements (including bias), demonstrated superior performance compared to a simpler Least Squares approach, especially in more realistic scenarios where measurement bias is present. With minimal equipment, the MLE method provides a viable way to achieve location estimation with a respectable accuracy of under 100 meters. This work highlights the power of probabilistic modeling in overcoming the limitations of classical geometric methods in navigation.

5.2 Future Work

While the current hand-crafted Maximum Likelihood Estimator performs well, there is potential for further improvement. Future research will focus on exploring deep learning models to enhance the estimation process. We are currently working on a deep learning model that could potentially learn the complex relationships and non-linearities in the data more effectively than the analytical MLE, possibly leading to even more accurate and robust location estimates.