Solve the inequalities for WR lattice (in R^3) conditions

Shank's simplest cubic fields, basis $(1+\rho+\rho^2)/3, \rho, \rho+\rho^2$

In[ ]:=
```
n = Symbol["n"];
a = (1 / 9) * (n^2 + 3 n + 3) * (n^2 + 3 n + 9);
b = (-1 / 27) * (n^2 + 9 n + 9) * (n^2 + 3 n + 9);
c = (1 / 27) * (n^2 - 3 n - 9) * (n^2 + 3 n + 9);
d = (2 / 3) * (n^2 + 3 n + 9);
cond1 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond2 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

Out[ ]=

n ∈ ℤ && (n ≤ -13 || n ≥ 10)

Washington's cyclic cubic fields, n even, basis $\rho, (\rho^2-1)/(n-1)-\rho, \rho^2$

Out[ ]=

n ∈ ℤ && (n ≤ -13 || n ≥ 10)

In[ ]:=
```
n = Symbol["n"];
a = n^2 - n + 3;
b = n;
c = -n;
d = n;
cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[ ]=

n == 0 || (n ∈ ℤ && (n ≤ 1 || n ≥ 3))

Washington's cyclic cubic fields, n odd, 1st basis

In[ ]:=
```
n = Symbol["n"];
a = (1 / 16) * (n^2 - 3 * n + 3) * (n^2 + 3) * (n^4 - 5 * n^3 + 10 * n^2 - 11 * n + 1);
b=  (1/32)*(n^2-4*n+7)*(n^2-3*n+3)*(n^2-2*n-1)*(n^2+3);
c=  (1/32)*(n^2-3*n+3)*(n^2+3)*(n^4-8*n^3+16*n^2-16*n-1);
d=  (1/64)*(n-1)*(n^2-3*n+3)*(n^2+3)*(n^3-11*n^2+19*n-1);

cond2=Max[Abs[b],Abs[c],Abs[d]]<=a/2;
cond3=Max[-b+c+d,b-c+d,b+c-d,-b-c-d]<=a;
 Reduce[cond2&&cond3&&Element[n,Integers],n]
```

Out[ ]=

```
(n∈ℤ&&n≥5)
```

Washington's cyclic cubic fields, n odd, 2nd basis

In[ ]:=
```
n = Symbol["n"];
a = (1 / 32) * (n^2 - 4 * n + 7) * (n^2 - 3 * n + 3) * (n^2 - 2 * n + 3) * (n^2 + 3);
b = (1 / 64) * (n - 3) * (n - 1) * (n^2 - 4 * n + 7) * (n^2 - 3 * n + 3) * (n^2 + 3);
c =  (1 / 64) * (n - 3) * (n - 1) * (n^2 - 4 * n + 7) * (n^2 - 3 * n + 3) * (n^2 + 3);
d = (1 / 64) * (n - 3) * (n - 1) * (n^2 - 4 * n + 7) * (n^2 - 3 * n + 3) * (n^2 + 3);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[ ]=

```
n == 1 || n == 3 || (n ∈ ℤ && n ≥ 0)
```

Kishi's cyclic cubic fields

$n \equiv 0, 2 \pmod 6$ or $n \equiv 4, 10 \pmod{18}$

In[41]:=
```
n = Symbol["n"];
a = (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 + n^5 + 5 n^4 + n^3 + 5 n^2 - 2 n + 1);
b = (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) n;
c = (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^5 + n^4 + 3 n^3 - 1);
d = - (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^4 + n^3 + 2 n^2 - n + 1);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[48]=

```
n ∈ ℤ && (n ≤ -1 || n ≥ 1)
```

$n \equiv 3, 5 \pmod 6$ or $n \equiv 1, 13 \pmod{18}$

In[105]:=

```
n = Symbol["n"];
a = 1 / 16 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 - n^5 + 3 n^4 - 9 n^3 + n^2 - 10 n - 5);
b = -1 / 64 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 - 3 n^4 - 8 n^3 - 21 n^2 - 8 n - 25);
c = -1 / 32 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 - 3 n^5 + n^4 - 16 n^3 - n^2 - 9 n - 5);
d = 1 / 32 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 - n^5 + 2 n^4 - 8 n^3 + 3 n^2 - 3 n - 10);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[112]=

n ∈ ℤ && n ≥ 3

In[113]:=

```
Symbol["n"];
a = 1 / 16 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 + n^5 + 3 n^4 - n^3 - 3 n^2 - 4 n - 1);
b = 1 / 64 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n + 1) (n^5 - n^4 + 2 n^3 - 10 n^2 - 3 n - 5);
c = -1 / 32 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n + 1) (n^5 - n^4 + 2 n^3 - 10 n^2 - 3 n - 5);
d = -1 / 32 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n + 1) (n^5 - 4 n^2 - 5 n + 4);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[120]=

n == -1 || n == -1 || n == -1 || n == -1 || n == -1 ||
 n == -1 || n == -1 || n == -1 || n == -1 || (n ∈ ℤ && (n == -1 || n ≥ 2))

In[121]:=

```
Symbol["n"];
a = 1 / 16 * (n^2 + 3) (n^4 + n^3 + 4 * n^2 + 3) (n^6 + n^5 + 3 n^4 - n^3 + n^2 - 1);
b = 1 / 64 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 + 2 n^5 + 3 n^4 + 4 n^3 - 5 n^2 + 2 n - 23);
c = 1 / 32 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 + 2 n^5 + 4 n^4 - 2 n^3 - n^2 - 8 n - 4);
d = 1 / 32 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 + n^5 - 4 n^3 - 5 n^2 - n - 8);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[128]=

n ∈ ℤ && n ≤ -2

In[129]:=

```
Symbol["n"];
a = 1 / 16 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 + 3 n^5 + 7 n^4 + 11 n^3 + 9 n^2 + 10 n + 3);
b = -1 / 64 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 + 2 n^5 - n^4 - 4 n^3 - 21 n^2 - 6 n - 3);
c = 1 / 32 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 + 5 n^5 + 9 n^4 + 18 n^3 + 11 n^2 + 9 n + 3);
d = -1 / 32 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n + 1) (n^5 + 2 n^4 + 4 n^3 + 4 n^2 + 3 n + 6);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[136]=

$n \in \mathbb{Z}$ && $n \leq -3$