Solve the inequalities for conditions α_0, β_0, γ_0 have the same sign

*Note that these quantities s_1, s_2, s_3 here are all computed from the file **wr_lattice_sign_check_gram_matrix.ipynb**, corresponding to each basis.*

Shank's simplest cubic fields, basis $(1+\rho+\rho^2)/3$, ρ, $\rho+\rho^2$

In[1]:=
```
n = Symbol["n"];
s1 = (8 / 9) * (n^2 + 3 * n + 9) * (n^2 + 9 / 2 * n + 6);
s2 = (7 / 81) * (n^2 + 3 * n + 9) *
    (n^6 + 9 * n^5 + 236 / 7 * n^4 + 429 / 7 * n^3 + 234 / 7 * n^2 - 459 / 7 * n - 81);
s3 = (7 / 729) * (n^2 + 3 * n + 9) *
    (n^6 + 9 * n^5 + 180 / 7 * n^4 + 135 / 7 * n^3 - 108 / 7 * n^2 - 81 / 7 * n - 243 / 7);

cond1 = s1 * s3 > 0;
cond2 = s2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

Out[7]= $n \in \mathbb{Z}$ && $(n \leq -5 \,||\, n \geq 2)$

Washington's cyclic cubic fields, n even, basis ρ, $(\rho^2-1)/(n-1)-\rho$, ρ^2

In[15]:=
```
n = Symbol["n"];
s1 = (-2) * (n - 1)^2 * (n^2 - 3 * n + 3) * (n^2 + 3) * (n^4 - 2 * n^3 + 9 / 2 * n^2 - 13 / 2 * n + 9 / 2);
s2 = (n - 1)^4 * (n^2 - 3 * n + 3)^2 * (n^2 + 3)^2 *
    (n^8 - 4 * n^7 + 12 * n^6 - 26 * n^5 + 42 * n^4 - 52 * n^3 + 49 * n^2 - 30 * n + 11);
s3 = (-1) * (n - 1)^6 * (n^2 - 3 * n + 3)^2 * (n^2 + 3)^3;

cond1 = s1 * s3 > 0;
cond2 = s2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

Out[21]=

$n \in \mathbb{Z}$ && $(n \leq 0 \,||\, n \geq 2)$

Washington's cyclic cubic fields, n odd, 1st basis

In[22]:=
```
n = Symbol["n"];
s1 = (3 / 8) * (n^2 - 3 * n + 3) * (n^2 + 3) *
    (n^6 - 14 / 3 * n^5 + 21 / 2 * n^4 - 39 / 2 * n^3 + 74 / 3 * n^2 - 95 / 6 * n + 19 / 6);
s2 = (9 / 256) * (n^2 - 3 * n + 3) * (n^2 + 3) ^2 *
    (n^14 - 37 / 3 * n^13 + 221 / 3 * n^12 - 872 / 3 * n^11 + 7711 / 9 * n^10 -
      17 963 / 9 * n^9 + 33 878 / 9 * n^8 - 5807 * n^7 + 65 986 / 9 * n^6 - 67 409 / 9 * n^5 +
      54 266 / 9 * n^4 - 32 923 / 9 * n^3 + 13 826 / 9 * n^2 - 3307 / 9 * n + 79 / 3);
s3 = (9 / 4096) * (n^2 - 3 * n + 3) * (n^2 + 3) ^3 * (n^8 - 37 / 3 * n^7 + 188 / 3 * n^6 -
      171 * n^5 + 2410 / 9 * n^4 - 2045 / 9 * n^3 + 76 * n^2 + 37 / 3 * n - 19 / 9);

cond1 = s1 * s3 > 0;
cond2 = s2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

Out[28]=
$$n \in \mathbb{Z} \;\&\&\; (n \leq -1 \;||\; n \geq 4)$$

Washington's cyclic cubic fields, n odd, 2nd basis

In[29]:=
```
n = Symbol["n"];
s1 = (1 / 8) * (n^2 - 4 * n + 7) * (n^2 - 3 * n + 3) *
    (n^2 + 3) * (n^4 - 2 * n^3 + 9 / 2 * n^2 - 13 / 2 * n + 9 / 2);
s2 = (1 / 256) * (n^2 - 4 * n + 7) ^2 * (n^2 - 3 * n + 3) ^2 * (n^2 + 3) ^2 *
    (n^8 - 4 * n^7 + 12 * n^6 - 26 * n^5 + 42 * n^4 - 52 * n^3 + 49 * n^2 - 30 * n + 11);
s3 = (1 / 4096) * (n^2 - 3 * n + 3) ^2 * (n^2 - 4 * n + 7) ^3 * (n^2 + 3) ^3;

cond1 = s1 * s3 > 0;
cond2 = s2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

Out[35]=
$$n \in \mathbb{Z}$$

Kishi's cyclic cubic fields

$n \equiv 0, 2 \pmod 6$ or $n \equiv 4, 10 \pmod{18}$

In[43]:=
```
n = Symbol["n"];
s1 = (2) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) * (n^10 + 2 * n^9 + 9 * n^8 + 9 * n^7 +
      47 / 2 * n^6 + 15 / 2 * n^5 + 47 / 2 * n^4 - 17 / 2 * n^3 + 21 / 2 * n^2 - 10 * n + 5 / 2);
s2 = (n^2 + 1) * (n^2 + 3) * (n^3 + n^2 + 3 * n - 1) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^21 + 4 * n^20 + 24 * n^19 + 64 * n^18 + 212 * n^17 + 411 * n^16 + 950 * n^15 + 1379 * n^14 +
      2417 * n^13 + 2616 * n^12 + 3622 * n^11 + 2807 * n^10 + 3148 * n^9 + 1560 * n^8 +
      1429 * n^7 + 291 * n^6 + 176 * n^5 - 89 * n^4 - 105 * n^3 - 51 * n^2 - 34 * n - 8);
s3 = (n^2 + 3) * (n^2 + 1) ^2 * (n^3 + n^2 + 3 * n - 1) ^2 *
    (n^4 + n^3 + 4 * n^2 + 3) * (n^8 + n^7 + 6 * n^6 + 2 * n^5 + 8 * n^4 - 3 * n^3 + 2 * n^2 - 1);

cond1 = s1 * s3 > 0;
cond2 = s2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

Out[49]=
```
n ∈ ℤ && (n ≤ -1 || n ≥ 1)
```

$n \equiv 34, 52 \pmod{54}$

In[50]:=
```
n = Symbol["n"];
s1 = (10 / 9) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^10 + 14 / 5 * n^9 + 53 / 5 * n^8 + 93 / 5 * n^7 + 351 / 10 * n^6 + 423 / 10 * n^5 +
      463 / 10 * n^4 + 391 / 10 * n^3 + 221 / 10 * n^2 + 62 / 5 * n + 13 / 10);
s2 = (1 / 9) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^26 + 5 * n^25 + 232 / 9 * n^24 + 656 / 9 * n^23 + 509 / 3 * n^22 + 1762 / 9 * n^21 - 937 / 3 * n^20 -
      21 787 / 9 * n^19 - 27 752 / 3 * n^18 - 207 926 / 9 * n^17 - 473 455 / 9 * n^16 - 839 632 / 9 * n^15 -
      1 429 933 / 9 * n^14 - 1 960 073 / 9 * n^13 - 2 633 033 / 9 * n^12 - 2 841 769 / 9 * n^11 -
      1 018 988 / 3 * n^10 - 2 577 113 / 9 * n^9 - 2 211 563 / 9 * n^8 - 156 796 * n^7 - 944 612 / 9 * n^6 -
      426 257 / 9 * n^5 - 210 050 / 9 * n^4 - 57 856 / 9 * n^3 - 1947 * n^2 - 706 / 3 * n - 8);
s3 = (1 / 81) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^18 + 3 * n^17 + 97 / 9 * n^16 + 37 / 3 * n^15 - 16 * n^14 - 1103 / 9 * n^13 -
      478 * n^12 - 2647 / 3 * n^11 - 1898 * n^10 - 2165 * n^9 - 3576 * n^8 - 26 275 / 9 * n^7 -
      12 832 / 3 * n^6 - 3069 * n^5 - 32 074 / 9 * n^4 - 2383 * n^3 - 4469 / 3 * n^2 - 762 * n - 73);

cond1 = s1 * s3 > 0;
cond2 = s2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

Out[56]=
```
n ∈ ℤ && (n ≤ -3 || n ≥ 3)
```

$n \equiv 3, 5 \pmod{6}$ or $n \equiv 1, 13 \pmod{18}$

In[57]:=
```
n = Symbol["n"];
s1 = (-1 / 4) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) * (n^10 + 3 / 2 * n^9 + 15 / 2 * n^8 + 4 * n^7 +
      59 / 4 * n^6 - 23 / 4 * n^5 + 19 / 4 * n^4 - 101 / 4 * n^3 - 53 / 4 * n^2 - 43 / 2 * n - 35 / 4);
s2 = (3 / 256) * n * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^25 + 7 / 3 * n^24 + 56 / 3 * n^23 + 18 * n^22 + 287 / 3 * n^21 - 358 / 3 * n^20 -
      111 * n^19 - 5825 / 3 * n^18 - 8416 / 3 * n^17 - 29198 / 3 * n^16 - 33635 / 3 * n^15 -
      74456 / 3 * n^14 - 19907 * n^13 - 97199 / 3 * n^12 - 30989 / 3 * n^11 -
      32849 / 3 * n^10 + 21510 * n^9 + 27807 * n^8 + 43201 * n^7 + 42060 * n^6 +
      93494 / 3 * n^5 + 70079 / 3 * n^4 + 8814 * n^3 + 12464 / 3 * n^2 + 1175 / 3 * n - 320);
s3 = (-9 / 4096) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^18 - 5 / 3 * n^17 + 9 * n^16 - 307 / 9 * n^15 + 392 / 9 * n^14 - 1751 / 9 * n^13 + 2408 / 9 * n^12 -
      3931 / 9 * n^11 + 9794 / 9 * n^10 - 4205 / 9 * n^9 + 17954 / 9 * n^8 - 1339 / 3 * n^7 +
      9664 / 9 * n^6 - 6181 / 9 * n^5 - 11584 / 9 * n^4 - 4417 / 9 * n^3 - 4145 / 3 * n^2 - 200 * n + 125);

cond1 = s1 * s3 > 0;
cond2 = s2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

Out[63]=
$$n \in \mathbb{Z} \;\&\&\; (n \leq -1 \;||\; n == 1 \;||\; n \geq 3)$$

In[64]:=
```
n = Symbol["n"];
s1 = (3 / 8) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) * (n^10 + 7 / 3 * n^9 + 28 / 3 * n^8 + 26 / 3 * n^7 +
      101 / 6 * n^6 - 55 / 6 * n^5 - 27 / 2 * n^4 - 301 / 6 * n^3 - 245 / 6 * n^2 - 35 * n - 109 / 6);
s2 = (9 / 256) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^26 + 17 / 3 * n^25 + 34 * n^24 + 346 / 3 * n^23 + 3443 / 9 * n^22 + 7970 / 9 * n^21 +
      17731 / 9 * n^20 + 27943 / 9 * n^19 + 41080 / 9 * n^18 + 3522 * n^17 + 6523 / 9 * n^16 -
      82732 / 9 * n^15 - 183119 / 9 * n^14 - 339619 / 9 * n^13 - 46905 * n^12 - 164279 / 3 * n^11 -
      419608 / 9 * n^10 - 337279 / 9 * n^9 - 180245 / 9 * n^8 - 85396 / 9 * n^7 - 932 * n^6 +
      12919 / 9 * n^5 + 12146 / 9 * n^4 + 8624 / 9 * n^3 + 815 / 9 * n^2 + 268 / 3 * n - 4);
s3 = (9 / 4096) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^18 + 11 / 3 * n^17 + 47 / 3 * n^16 + 83 / 3 * n^15 + 416 / 9 * n^14 - 47 / 9 * n^13 -
      944 / 9 * n^12 - 2855 / 9 * n^11 - 3398 / 9 * n^10 - 1981 / 9 * n^9 + 306 * n^8 + 6691 / 9 * n^7 +
      2392 / 3 * n^6 + 3331 / 9 * n^5 - 680 / 9 * n^4 - 1397 / 9 * n^3 - 121 / 3 * n^2 + 8 * n + 1);

cond1 = s1 * s3 > 0;
cond2 = s2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

Out[70]=
$$n \in \mathbb{Z} \;\&\&\; (n \leq -1 \;||\; n \geq 2)$$

In[71]:=
```
n = Symbol["n"];
s1 = (3 / 8) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) * (n^10 + 5 / 3 * n^9 + 22 / 3 * n^8 + 16 / 3 * n^7 +
      101 / 6 * n^6 + 7 / 2 * n^5 + 143 / 6 * n^4 + 23 / 6 * n^3 + 175 / 6 * n^2 + 13 / 3 * n + 95 / 6);
s2 = (9 / 256) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^26 + 13 / 3 * n^25 + 82 / 3 * n^24 + 238 / 3 * n^23 + 2531 / 9 * n^22 + 1826 / 3 * n^21 +
      13 907 / 9 * n^20 + 23 827 / 9 * n^19 + 47 720 / 9 * n^18 + 7582 * n^17 + 12 683 * n^16 +
      145 556 / 9 * n^15 + 209 989 / 9 * n^14 + 28 469 * n^13 + 35 263 * n^12 + 381 427 / 9 * n^11 +
      129 640 / 3 * n^10 + 445 981 / 9 * n^9 + 39 055 * n^8 + 40 468 * n^7 + 203 392 / 9 * n^6 +
      60 949 / 3 * n^5 + 20 630 / 3 * n^4 + 15 736 / 3 * n^3 + 6407 / 9 * n^2 + 1348 / 3 * n - 4);
s3 = (9 / 4096) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^18 + 7 / 3 * n^17 + 35 / 3 * n^16 + 43 / 3 * n^15 + 368 / 9 * n^14 + 209 / 9 * n^13 +
      188 / 3 * n^12 + 193 / 9 * n^11 + 458 / 9 * n^10 + 491 / 9 * n^9 - 290 / 9 * n^8 + 329 / 3 * n^7 -
      1352 / 9 * n^6 - 509 / 9 * n^5 - 2684 / 9 * n^4 - 2845 / 9 * n^3 - 553 / 3 * n^2 - 308 * n - 71);

cond1 = s1 * s3 > 0;
cond2 = s2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

Out[77]=
$$n \in \mathbb{Z} \;\&\&\; (n \le -2 \;||\; n \ge 2)$$

In[78]:=
```
n = Symbol["n"];
s1 = (-1 / 4) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) * (n^10 + 7 / 2 * n^9 + 23 / 2 * n^8 + 23 * n^7 +
      147 / 4 * n^6 + 201 / 4 * n^5 + 191 / 4 * n^4 + 195 / 4 * n^3 + 115 / 4 * n^2 + 37 / 2 * n + 21 / 4);
s2 = (3 / 256) * n * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^25 + 23 / 3 * n^24 + 136 / 3 * n^23 + 190 * n^22 + 661 * n^21 + 5806 / 3 * n^20 +
      4881 * n^19 + 32 875 / 3 * n^18 + 21 632 * n^17 + 117 062 / 3 * n^16 + 62 623 * n^15 +
      279 124 / 3 * n^14 + 123 757 * n^13 + 458 897 / 3 * n^12 + 169 641 * n^11 +
      521 671 / 3 * n^10 + 479 266 / 3 * n^9 + 400 541 / 3 * n^8 + 99 281 * n^7 + 195 296 / 3 * n^6 +
      110 446 / 3 * n^5 + 51 943 / 3 * n^4 + 18 274 / 3 * n^3 + 1536 * n^2 - 97 / 3 * n - 96);
s3 = (-9 / 4096) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^18 + 23 / 3 * n^17 + 37 * n^16 + 1189 / 9 * n^15 + 3272 / 9 * n^14 + 7529 / 9 * n^13 + 4820 / 3 *
       n^12 + 8015 / 3 * n^11 + 3906 * n^10 + 14 921 / 3 * n^9 + 51 406 / 9 * n^8 + 50 423 / 9 * n^7 +
      43 904 / 9 * n^6 + 32 443 / 9 * n^5 + 6644 / 3 * n^4 + 9847 / 9 * n^3 + 309 * n^2 - 36 * n - 27);

cond1 = s1 * s3 > 0;
cond2 = s2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

Out[84]=
$$n \in \mathbb{Z} \;\&\&\; (n \le -3 \;||\; n \ge 1)$$

$n \equiv 7, 25 \pmod{54}$

In[85]:=
```
n = Symbol["n"];
s1 = (5 / 24) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^10 + 13 / 5 * n^9 + 124 / 15 * n^8 + 62 / 5 * n^7 + 533 / 30 * n^6 + 553 / 30 * n^5 +
      271 / 30 * n^4 + 371 / 30 * n^3 - 149 / 30 * n^2 + 59 / 15 * n - 109 / 30);
s2 = (1 / 256) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^26 + 43 / 9 * n^25 + 614 / 27 * n^24 + 1642 / 27 * n^23 + 3473 / 27 * n^22 + 4262 / 27 * n^21 -
      13 117 / 81 * n^20 - 29 395 / 27 * n^19 - 344 248 / 81 * n^18 - 684 926 / 81 * n^17 -
      1 477 837 / 81 * n^16 - 1 982 540 / 81 * n^15 - 3 162 047 / 81 * n^14 - 2 988 995 / 81 * n^13 -
      1 280 059 / 27 * n^12 - 2 396 645 / 81 * n^11 - 2 681 456 / 81 * n^10 - 853 247 / 81 * n^9 -
      993 277 / 81 * n^8 + 32 956 / 81 * n^7 - 40 828 / 27 * n^6 + 105 287 / 81 * n^5 +
      26 962 / 81 * n^4 + 22 256 / 81 * n^3 + 1861 / 27 * n^2 + 332 / 27 * n - 4 / 9);
s3 = (1 / 36 864) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^18 + 25 / 9 * n^17 + 167 / 27 * n^16 + 17 / 9 * n^15 - 800 / 27 * n^14 -
      1709 / 27 * n^13 - 13 336 / 81 * n^12 - 4375 / 81 * n^11 - 2998 / 9 * n^10 -
      18 341 / 81 * n^9 - 87 014 / 81 * n^8 - 43 549 / 81 * n^7 - 13 576 / 27 * n^6 -
      4727 / 27 * n^5 - 3584 / 81 * n^4 + 257 / 3 * n^3 + 815 / 27 * n^2 + 16 / 3 * n + 1 / 9);

cond1 = s1 * s3 > 0;
cond2 = s2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

Out[91]=
$$n \in \mathbb{Z} \;\&\&\; (n \leq -3 \;||\; n \geq 3)$$

In[92]:=
```
n = Symbol["n"];
s1 = (-1 / 12) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) * (n^10 + 13 / 6 * n^9 + 9 * n^8 + 32 / 3 * n^7 +
      283 / 12 * n^6 + 169 / 12 * n^5 + 91 / 4 * n^4 + 9 / 4 * n^3 + 27 / 4 * n^2 - 25 / 6 * n - 13 / 12);
s2 = (1 / 768) * (n^2 + 3) * (n^3 + n^2 + 3 * n - 1) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^23 + 8 / 3 * n^22 + 155 / 9 * n^21 + 604 / 27 * n^20 + 1958 / 27 * n^19 - 685 / 9 * n^18 -
      7006 / 27 * n^17 - 43 964 / 27 * n^16 - 91 907 / 27 * n^15 - 76 495 / 9 * n^14 -
      365 131 / 27 * n^13 - 624 113 / 27 * n^12 - 779 528 / 27 * n^11 - 1 001 239 / 27 * n^10 -
      984 061 / 27 * n^9 - 107 603 / 3 * n^8 - 735 835 / 27 * n^7 - 181 516 / 9 * n^6 -
      305 845 / 27 * n^5 - 52 982 / 9 * n^4 - 59 563 / 27 * n^3 - 5915 / 9 * n^2 - 1090 / 9 * n - 26 / 3);
s3 = (-1 / 36 864) * (n^2 + 3) * (n^3 + n^2 + 3 * n - 1)^2 * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^12 - n^11 + 40 / 9 * n^10 - 124 / 9 * n^9 - 311 / 81 * n^8 - 3916 / 81 * n^7 - 62 / 3 * n^6 -
      3730 / 81 * n^5 - 47 / 81 * n^4 + 383 / 27 * n^3 + 590 / 27 * n^2 + 74 / 3 * n + 37 / 9);

cond1 = s1 * s3 > 0;
cond2 = s2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

Out[98]=
$$n \in \mathbb{Z} \;\&\&\; (n \leq -1 \;||\; n \geq 3)$$

$n \equiv 16 \pmod{54}$

In[99]:=
```
n = Symbol["n"];
s1 = (80 / 729) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^10 + 109 / 40 * n^9 + 981 / 80 * n^8 + 1917 / 80 * n^7 + 4357 / 80 * n^6 + 5987 / 80 * n^5 +
      8551 / 80 * n^4 + 951 / 10 * n^3 + 7227 / 80 * n^2 + 1661 / 40 * n + 537 / 20);
s2 = (64 / 59 049) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^26 + 59 / 12 * n^25 + 659 / 24 * n^24 + 1981 / 24 * n^23 + 32 477 / 144 * n^22 + 25 775 / 72 * n^21 +
      27 737 / 192 * n^20 - 359 449 / 192 * n^19 - 5 144 783 / 576 * n^18 - 5 272 549 / 192 * n^17 -
      3 195 091 / 48 * n^16 - 26 675 711 / 192 * n^15 - 987 627 / 4 * n^14 - 231 448 417 / 576 * n^13 -
      80 129 731 / 144 * n^12 - 422 215 133 / 576 * n^11 - 76 696 781 / 96 * n^10 - 165 480 799 / 192 * n^9 -
      104 751 971 / 144 * n^8 - 370 019 797 / 576 * n^7 - 231 398 095 / 576 * n^6 - 2 583 077 / 9 * n^5 -
      1 938 993 / 16 * n^4 - 12 954 955 / 192 * n^3 - 247 687 / 16 * n^2 - 93 119 / 16 * n - 20 751 / 64);
s3 = (64 / 43 046 721) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^18 + 35 / 12 * n^17 + 117 / 8 * n^16 + 199 / 8 * n^15 + 4475 / 144 * n^14 - 1649 / 24 * n^13 -
      223 505 / 576 * n^12 - 77 377 / 64 * n^11 - 843 817 / 288 * n^10 - 859 339 / 144 * n^9 -
      6 735 383 / 576 * n^8 - 6 620 831 / 288 * n^7 - 205 021 / 6 * n^6 - 5 938 507 / 96 * n^5 -
      4 263 909 / 64 * n^4 - 5 247 661 / 64 * n^3 - 2 362 641 / 32 * n^2 - 1 135 899 / 32 * n - 2 222 163 / 64);

cond1 = s1 * s3 > 0;
cond2 = s2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

Out[105]=
$$n \in \mathbb{Z} \&\& (n \leq -2 \,||\, n \geq 3)$$

In[106]:=
```
n = Symbol["n"];
s1 = (-17 / 729) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^10 + 175 / 51 * n^9 + 631 / 51 * n^8 + 1145 / 51 * n^7 + 2086 / 51 * n^6 +
      2195 / 51 * n^5 + 2464 / 51 * n^4 + 429 / 17 * n^3 + 941 / 51 * n^2 + 44 / 51 * n - 13 / 17);
s2 = (8 / 59 049) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^26 + 31 / 4 * n^25 + 3505 / 72 * n^24 + 44 675 / 216 * n^23 + 488 983 / 648 * n^22 +
      1 443 425 / 648 * n^21 + 470 893 / 81 * n^20 + 8 425 397 / 648 * n^19 + 5 638 345 / 216 * n^18 +
      7 457 569 / 162 * n^17 + 47 566 765 / 648 * n^16 + 8 420 294 / 81 * n^15 + 86 398 459 / 648 * n^14 +
      32 907 647 / 216 * n^13 + 33 921 205 / 216 * n^12 + 10 352 579 / 72 * n^11 + 8 449 051 / 72 * n^10 +
      27 291 661 / 324 * n^9 + 5 663 725 / 108 * n^8 + 18 207 575 / 648 * n^7 + 7 936 513 / 648 * n^6 +
      2 801 255 / 648 * n^5 + 36 637 / 36 * n^4 + 2953 / 24 * n^3 - 299 / 24 * n^2 - 161 / 18 * n - 9 / 8);
s3 = (-64 / 43 046 721) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^18 + 77 / 12 * n^17 + 2305 / 72 * n^16 + 22 933 / 216 * n^15 + 23 647 / 81 * n^14 +
      2 446 555 / 3888 * n^13 + 2 977 769 / 2592 * n^12 + 4 452 331 / 2592 * n^11 +
      33 900 115 / 15 552 * n^10 + 11 879 333 / 5184 * n^9 + 10 464 721 / 5184 * n^8 +
      22 804 333 / 15 552 * n^7 + 243 665 / 288 * n^6 + 1 971 911 / 5184 * n^5 +
      16 507 / 144 * n^4 + 28 249 / 1728 * n^3 - 1031 / 576 * n^2 - 23 / 24 * n - 17 / 192);

cond1 = s1 * s3 > 0;
cond2 = s2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

Out[112]=
$$n \in \mathbb{Z} \&\& (n \leq -1 \,||\, n \geq 1)$$

$n \equiv 43 \pmod{54}$

In[113]:=

```
n = Symbol["n"];
s1 = (-175 / 5832) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^10 + 487 / 175 * n^9 + 2178 / 175 * n^8 + 864 / 35 * n^7 + 3917 / 70 * n^6 + 3887 / 50 * n^5 +
      38 791 / 350 * n^4 + 34 851 / 350 * n^3 + 32 931 / 350 * n^2 + 1531 / 35 * n + 1947 / 70);
s2 = (1225 / 15 116 544) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^26 + 523 / 105 * n^25 + 2918 / 105 * n^24 + 8794 / 105 * n^23 + 2 513 279 / 11 025 * n^22 +
      3 878 578 / 11 025 * n^21 + 342 953 / 3675 * n^20 - 7 893 979 / 3675 * n^19 -
      108 233 348 / 11 025 * n^18 - 110 613 382 / 3675 * n^17 - 267 030 247 / 3675 * n^16 -
      559 136 672 / 3675 * n^15 - 331 757 799 / 1225 * n^14 - 4 874 523 559 / 11 025 * n^13 -
      1 356 514 529 / 2205 * n^12 - 8 949 390 341 / 11 025 * n^11 - 93 647 096 / 105 * n^10 -
      3 531 292 573 / 3675 * n^9 - 1 811 538 397 / 2205 * n^8 - 7 944 908 056 / 11 025 * n^7 -
      5 075 444 956 / 11 025 * n^6 - 3 565 651 541 / 11 025 * n^5 - 173 636 478 / 1225 * n^4 -
      39 972 508 / 525 * n^3 - 23 007 601 / 1225 * n^2 - 228 832 / 35 * n - 77 928 / 175);
s3 = (-1225 / 176 319 369 216) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^18 + 313 / 105 * n^17 + 519 / 35 * n^16 + 883 / 35 * n^15 + 42 524 / 11 025 * n^14 -
      117 797 / 735 * n^13 - 4 616 324 / 11 025 * n^12 - 151 957 / 175 * n^11 - 15 069 482 / 11 025 * n^10 -
      20 940 781 / 11 025 * n^9 - 106 383 986 / 11 025 * n^8 - 354 029 533 / 11 025 * n^7 -
      260 705 516 / 3675 * n^6 - 589 902 707 / 3675 * n^5 - 54 527 124 / 245 * n^4 -
      324 998 833 / 1225 * n^3 - 382 148 103 / 1225 * n^2 - 152 625 456 / 1225 * n - 199 306 827 / 1225);

cond1 = s1 * s3 > 0;
cond2 = s2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

Out[119]=

$n \in \mathbb{Z} \;\&\&\; (n \leq -2 \;||\; n \geq 3)$