Solve the inequalities for WR lattice (in R^3) conditions

Shank's simplest cubic fields, basis $(1+\rho+\rho^2)/3$, ρ, $\rho+\rho^2$

In[1]:=
```
n = Symbol["n"];
a = (1 / 9) * (n^2 + 3 n + 3) * (n^2 + 3 n + 9);
b = (-1 / 27) * (n^2 + 9 n + 9) * (n^2 + 3 n + 9);
c = (1 / 27) * (n^2 - 3 n - 9) * (n^2 + 3 n + 9);
d = (2 / 3) * (n^2 + 3 n + 9);
cond1 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond2 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

Out[8]= n ∈ ℤ && (n ≤ -13 || n ≥ 10)

Washington's cyclic cubic fields, n even, basis ρ, $(\rho^2-1)/(n-1)-\rho$, ρ^2

Out[◦]=

n ∈ ℤ && (n ≤ -13 || n ≥ 10)

In[17]:=
```
n = Symbol["n"];
a = n^2 - n + 3;
b = n;
c = -n;
d = n;
cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[24]=

n == 0 || (n ∈ ℤ && (n ≤ 1 || n ≥ 3))

Washington's cyclic cubic fields, n odd, 1st basis

In[27]:=
```
n = Symbol["n"];
a = (1 / 16) * (n^2 - 3 * n + 3) * (n^2 + 3) * (n^4 - 5 * n^3 + 10 * n^2 - 11 * n + 1);
b= (1/32)*(n^2-4*n+7)*(n^2-3*n+3)*(n^2-2*n-1)*(n^2+3);
c= (1/32)*(n^2-3*n+3)*(n^2+3)*(n^4-8*n^3+16*n^2-16*n-1);
d= (1/64)*(n-1)*(n^2-3*n+3)*(n^2+3)*(n^3-11*n^2+19*n-1);

cond2=Max[Abs[b],Abs[c],Abs[d]]<=a/2;
cond3=Max[-b+c+d,b-c+d,b+c-d,-b-c-d]<=a;
 Reduce[cond2&&cond3&&Element[n,Integers],n]
```

Out[28]=
```
(n∈ℤ&&n≥5)
```

Washington's cyclic cubic fields, n odd, 2nd basis

In[29]:=
```
n = Symbol["n"];
a = (1 / 32) * (n^2 - 4 * n + 7) * (n^2 - 3 * n + 3) * (n^2 - 2 * n + 3) * (n^2 + 3);
b = (1 / 64) * (n - 3) * (n - 1) * (n^2 - 4 * n + 7) * (n^2 - 3 * n + 3) * (n^2 + 3);
c = (1 / 64) * (n - 3) * (n - 1) * (n^2 - 4 * n + 7) * (n^2 - 3 * n + 3) * (n^2 + 3);
d = (1 / 64) * (n - 3) * (n - 1) * (n^2 - 4 * n + 7) * (n^2 - 3 * n + 3) * (n^2 + 3);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[36]=
```
n == 1 || n == 3 || (n ∈ ℤ && n ≥ 0)
```

Kishi's cyclic cubic fields

In[37]:=
```
n = Symbol["n"];
a = 1 / 16 * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) * (n^6 + n^5 + 3 * n^4 - n^3 - 3 * n^2 - 4 * n - 1);
b = 1 / 64 * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) * (n + 1) * (n^5 - n^4 + 2 * n^3 - 10 * n^2 - 3 * n - 5);
c = -1 / 32 * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) * (n + 1) * (n^5 - n^4 + 2 * n^3 - 10 * n^2 - 3 * n - 5);
d = -1 / 32 * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) * (n + 1) * (n^5 - 4 * n^2 - 5 * n + 4);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[44]=
```
n == -1 || n == -1 || n == -1 || n == -1 || n == -1 ||
 n == -1 || n == -1 || n == -1 || n == -1 || (n ∈ ℤ && (n == -1 || n ≥ 2))
```

In[45]:=
```
Symbol["n"];
a = -1 / 16 * n^12 - 9 / 16 * n^10 + 5 / 8 * n^9 - 25 / 16 * n^8 + 39 / 8 * n^7 -
   19 / 16 * n^6 + 27 / 2 * n^5 + 23 / 16 * n^4 + 123 / 8 * n^3 + 33 / 8 * n^2 + 45 / 8 * n + 45 / 16
b = 1 / 64 * n^12 + 1 / 64 * n^11 + 1 / 16 * n^10 - 1 / 8 * n^9 - 35 / 64 * n^8 - 47 / 32 * n^7 -
   15 / 4 * n^6 - 33 / 8 * n^5 - 541 / 64 * n^4 - 267 / 64 * n^3 - 141 / 16 * n^2 - 9 / 8 * n - 225 / 64
c = 1 / 32 * n^12 - 1 / 16 * n^11 + 5 / 32 * n^10 - 33 / 32 * n^9 - 1 / 8 * n^8 - 41 / 8 * n^7 -
   45 / 32 * n^6 - 169 / 16 * n^5 - 17 / 8 * n^4 - 147 / 16 * n^3 - 21 / 8 * n^2 - 81 / 32 * n - 45 / 32
d = -1 / 32 * n^12 - 1 / 4 * n^10 + 5 / 16 * n^9 - 21 / 32 * n^8 + 65 / 32 * n^7 - 23 / 32 * n^6 +
   151 / 32 * n^5 + 1 / 2 * n^4 + 147 / 32 * n^3 + 123 / 32 * n^2 + 27 / 32 * n + 45 / 16
a = -a
b = -b
c = -c
d = -d

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[46]=
$$\frac{45}{16}+\frac{45 n}{8}+\frac{33 n^2}{8}+\frac{123 n^3}{8}+\frac{23 n^4}{16}+\frac{27 n^5}{2}-\frac{19 n^6}{16}+\frac{39 n^7}{8}-\frac{25 n^8}{16}+\frac{5 n^9}{8}-\frac{9 n^{10}}{16}-\frac{n^{12}}{16}$$

Out[47]=
$$-\frac{225}{64}-\frac{9 n}{8}-\frac{141 n^2}{16}-\frac{267 n^3}{64}-\frac{541 n^4}{64}-\frac{33 n^5}{8}-\frac{15 n^6}{4}-\frac{47 n^7}{32}-\frac{35 n^8}{64}-\frac{n^9}{8}+\frac{n^{10}}{16}+\frac{n^{11}}{64}+\frac{n^{12}}{64}$$

Out[48]=
$$-\frac{45}{32}-\frac{81 n}{32}-\frac{21 n^2}{8}-\frac{147 n^3}{16}-\frac{17 n^4}{8}-\frac{169 n^5}{16}-\frac{45 n^6}{32}-\frac{41 n^7}{8}-\frac{n^8}{8}-\frac{33 n^9}{32}+\frac{5 n^{10}}{32}-\frac{n^{11}}{16}+\frac{n^{12}}{32}$$

Out[49]=
$$\frac{45}{16}+\frac{27 n}{32}+\frac{123 n^2}{32}+\frac{147 n^3}{32}+\frac{n^4}{2}+\frac{151 n^5}{32}-\frac{23 n^6}{32}+\frac{65 n^7}{32}-\frac{21 n^8}{32}+\frac{5 n^9}{16}-\frac{n^{10}}{4}-\frac{n^{12}}{32}$$

Out[50]=
$$-\frac{45}{16}-\frac{45 n}{8}-\frac{33 n^2}{8}-\frac{123 n^3}{8}-\frac{23 n^4}{16}-\frac{27 n^5}{2}+\frac{19 n^6}{16}-\frac{39 n^7}{8}+\frac{25 n^8}{16}-\frac{5 n^9}{8}+\frac{9 n^{10}}{16}+\frac{n^{12}}{16}$$

Out[51]=
$$\frac{225}{64}+\frac{9 n}{8}+\frac{141 n^2}{16}+\frac{267 n^3}{64}+\frac{541 n^4}{64}+\frac{33 n^5}{8}+\frac{15 n^6}{4}+\frac{47 n^7}{32}+\frac{35 n^8}{64}+\frac{n^9}{8}-\frac{n^{10}}{16}-\frac{n^{11}}{64}-\frac{n^{12}}{64}$$

Out[52]=
$$\frac{45}{32}+\frac{81 n}{32}+\frac{21 n^2}{8}+\frac{147 n^3}{16}+\frac{17 n^4}{8}+\frac{169 n^5}{16}+\frac{45 n^6}{32}+\frac{41 n^7}{8}+\frac{n^8}{8}+\frac{33 n^9}{32}-\frac{5 n^{10}}{32}+\frac{n^{11}}{16}-\frac{n^{12}}{32}$$

Out[53]=
$$-\frac{45}{16}-\frac{27 n}{32}-\frac{123 n^2}{32}-\frac{147 n^3}{32}-\frac{n^4}{2}-\frac{151 n^5}{32}+\frac{23 n^6}{32}-\frac{65 n^7}{32}+\frac{21 n^8}{32}-\frac{5 n^9}{16}+\frac{n^{10}}{4}+\frac{n^{12}}{32}$$

Out[56]=
$n \in \mathbb{Z}$ && $n \geq 3$