

Solve the inequalities for conditions $\alpha_0, \beta_0, \gamma_0$ have the same sign

Note that these quantities e_1, e_2, e_3 here are all computed from the file **wr_lattice_sign_check_gram_matrix.ipynb**, corresponding to each basis.

Shank's simplest cubic fields, basis $(1 + \rho + \rho^2)/3, \rho, \rho + \rho^2$

```
In[1]:= n = Symbol["n"];
e1 = (8 / 9) * (n^2 + 3 * n + 9) * (n^2 + 9 / 2 * n + 6);
e2 = (7 / 81) * (n^2 + 3 * n + 9) *
      (n^6 + 9 * n^5 + 236 / 7 * n^4 + 429 / 7 * n^3 + 234 / 7 * n^2 - 459 / 7 * n - 81);
e3 = (7 / 729) * (n^2 + 3 * n + 9) *
      (n^6 + 9 * n^5 + 180 / 7 * n^4 + 135 / 7 * n^3 - 108 / 7 * n^2 - 81 / 7 * n - 243 / 7);

cond1 = e1 * e3 > 0;
cond2 = e2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
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Out[7]= $n \in \mathbb{Z} \ \&\& \ (n \leq -5 \mid \mid n \geq 2)$

Washington's cyclic cubic fields, n even, basis
 $\rho, (\rho^2 - 1)/(n - 1) - \rho, \rho^2$

```
In[8]:= n = Symbol["n"];
e1 = (-2) * (n - 1)^2 * (n^2 - 3 * n + 3) * (n^2 + 3) * (n^4 - 2 * n^3 + 9 / 2 * n^2 - 13 / 2 * n + 9 / 2);
e2 = (n - 1)^4 * (n^2 - 3 * n + 3)^2 * (n^2 + 3)^2 *
      (n^8 - 4 * n^7 + 12 * n^6 - 26 * n^5 + 42 * n^4 - 52 * n^3 + 49 * n^2 - 30 * n + 11);
e3 = (-1) * (n - 1)^6 * (n^2 - 3 * n + 3)^2 * (n^2 + 3)^3;

cond1 = e1 * e3 > 0;
cond2 = e2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

Out[14]= $n \in \mathbb{Z} \ \&\& \ (n \leq 0 \mid \mid n \geq 2)$

Washington's cyclic cubic fields, n odd, 1st basis

```
In[15]:= n = Symbol["n"];
e1 = (3 / 8) * (n^2 - 3 * n + 3) * (n^2 + 3) *
      (n^6 - 14 / 3 * n^5 + 21 / 2 * n^4 - 39 / 2 * n^3 + 74 / 3 * n^2 - 95 / 6 * n + 19 / 6);
e2 = (9 / 256) * (n^2 - 3 * n + 3) * (n^2 + 3)^2 *
      (n^14 - 37 / 3 * n^13 + 221 / 3 * n^12 - 872 / 3 * n^11 + 7711 / 9 * n^10 -
        17 963 / 9 * n^9 + 33 878 / 9 * n^8 - 5807 * n^7 + 65 986 / 9 * n^6 - 67 409 / 9 * n^5 +
        54 266 / 9 * n^4 - 32 923 / 9 * n^3 + 13 826 / 9 * n^2 - 3307 / 9 * n + 79 / 3);
e3 = (9 / 4096) * (n^2 - 3 * n + 3) * (n^2 + 3)^3 * (n^8 - 37 / 3 * n^7 + 188 / 3 * n^6 -
        171 * n^5 + 2410 / 9 * n^4 - 2045 / 9 * n^3 + 76 * n^2 + 37 / 3 * n - 19 / 9);

cond1 = e1 * e3 > 0;
cond2 = e2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]

Out[21]=
n ∈ ℤ && (n ≤ -1 || n ≥ 4)
```

Washington's cyclic cubic fields, n odd, 2nd basis

```
In[22]:= n = Symbol["n"];
e1 = (1 / 8) * (n^2 - 4 * n + 7) * (n^2 - 3 * n + 3) *
      (n^2 + 3) * (n^4 - 2 * n^3 + 9 / 2 * n^2 - 13 / 2 * n + 9 / 2);
e2 = (1 / 256) * (n^2 - 4 * n + 7)^2 * (n^2 - 3 * n + 3)^2 * (n^2 + 3)^2 *
      (n^8 - 4 * n^7 + 12 * n^6 - 26 * n^5 + 42 * n^4 - 52 * n^3 + 49 * n^2 - 30 * n + 11);
e3 = (1 / 4096) * (n^2 - 3 * n + 3)^2 * (n^2 - 4 * n + 7)^3 * (n^2 + 3)^3;

cond1 = e1 * e3 > 0;
cond2 = e2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]

Out[28]=
n ∈ ℤ
```

Kishi's cyclic cubic fields

$$n \equiv 0, 2 \pmod{6} \text{ or } n \equiv 4, 10 \pmod{18}$$

```
In[36]:= n = Symbol["n"];
e1 = (2) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) * (n^10 + 2 * n^9 + 9 * n^8 + 9 * n^7 +
47 / 2 * n^6 + 15 / 2 * n^5 + 47 / 2 * n^4 - 17 / 2 * n^3 + 21 / 2 * n^2 - 10 * n + 5 / 2);
e2 = (n^2 + 1) * (n^2 + 3) * (n^3 + n^2 + 3 * n - 1) * (n^4 + n^3 + 4 * n^2 + 3) *
(n^21 + 4 * n^20 + 24 * n^19 + 64 * n^18 + 212 * n^17 + 411 * n^16 + 950 * n^15 + 1379 * n^14 +
2417 * n^13 + 2616 * n^12 + 3622 * n^11 + 2807 * n^10 + 3148 * n^9 + 1560 * n^8 +
1429 * n^7 + 291 * n^6 + 176 * n^5 - 89 * n^4 - 105 * n^3 - 51 * n^2 - 34 * n - 8);
e3 = (n^2 + 3) * (n^2 + 1)^2 * (n^3 + n^2 + 3 * n - 1)^2 *
(n^4 + n^3 + 4 * n^2 + 3) * (n^8 + n^7 + 6 * n^6 + 2 * n^5 + 8 * n^4 - 3 * n^3 + 2 * n^2 - 1);

cond1 = e1 * e3 > 0;
cond2 = e2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

```
Out[42]=
n ∈ ℤ && (n ≤ -1 || n ≥ 1)

n ≡ 34, 52 (mod 54)
```

```
In[43]:= n = Symbol["n"];
e1 = (10 / 9) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
(n^10 + 14 / 5 * n^9 + 53 / 5 * n^8 + 93 / 5 * n^7 + 351 / 10 * n^6 + 423 / 10 * n^5 +
463 / 10 * n^4 + 391 / 10 * n^3 + 221 / 10 * n^2 + 62 / 5 * n + 13 / 10);
e2 = (1 / 9) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
(n^26 + 5 * n^25 + 232 / 9 * n^24 + 656 / 9 * n^23 + 509 / 3 * n^22 + 1762 / 9 * n^21 - 937 / 3 * n^20 -
21787 / 9 * n^19 - 27752 / 3 * n^18 - 207926 / 9 * n^17 - 473455 / 9 * n^16 - 839632 / 9 * n^15 -
1429933 / 9 * n^14 - 1960073 / 9 * n^13 - 2633033 / 9 * n^12 - 2841769 / 9 * n^11 -
1018988 / 3 * n^10 - 2577113 / 9 * n^9 - 2211563 / 9 * n^8 - 156796 * n^7 - 944612 / 9 * n^6 -
426257 / 9 * n^5 - 210050 / 9 * n^4 - 57856 / 9 * n^3 - 1947 * n^2 - 706 / 3 * n - 8);
e3 = (1 / 81) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
(n^18 + 3 * n^17 + 97 / 9 * n^16 + 37 / 3 * n^15 - 16 * n^14 - 1103 / 9 * n^13 -
478 * n^12 - 2647 / 3 * n^11 - 1898 * n^10 - 2165 * n^9 - 3576 * n^8 - 26275 / 9 * n^7 -
12832 / 3 * n^6 - 3069 * n^5 - 32074 / 9 * n^4 - 2383 * n^3 - 4469 / 3 * n^2 - 762 * n - 73);

cond1 = e1 * e3 > 0;
cond2 = e2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

```
Out[49]=
n ∈ ℤ && (n ≤ -3 || n ≥ 3)

n ≡ 3, 5 (mod 6) or n ≡ 1, 13 (mod 18)
```

```
In[50]:= n = Symbol["n"];
e1 = (-1/4) * (n^2 + 3) * (n^4 + n^3 + 4*n^2 + 3) * (n^10 + 3/2*n^9 + 15/2*n^8 + 4*n^7 +
59/4*n^6 - 23/4*n^5 + 19/4*n^4 - 101/4*n^3 - 53/4*n^2 - 43/2*n - 35/4);
e2 = (3/256) * n * (n^2 + 3) * (n^4 + n^3 + 4*n^2 + 3) *
(n^25 + 7/3*n^24 + 56/3*n^23 + 18*n^22 + 287/3*n^21 - 358/3*n^20 -
111*n^19 - 5825/3*n^18 - 8416/3*n^17 - 29198/3*n^16 - 33635/3*n^15 -
74456/3*n^14 - 19907*n^13 - 97199/3*n^12 - 30989/3*n^11 -
32849/3*n^10 + 21510*n^9 + 27807*n^8 + 43201*n^7 + 42060*n^6 +
93494/3*n^5 + 70079/3*n^4 + 8814*n^3 + 12464/3*n^2 + 1175/3*n - 320);
e3 = (-9/4096) * (n^2 + 3) * (n^4 + n^3 + 4*n^2 + 3) *
(n^18 - 5/3*n^17 + 9*n^16 - 307/9*n^15 + 392/9*n^14 - 1751/9*n^13 + 2408/9*n^12 -
3931/9*n^11 + 9794/9*n^10 - 4205/9*n^9 + 17954/9*n^8 - 1339/3*n^7 +
9664/9*n^6 - 6181/9*n^5 - 11584/9*n^4 - 4417/9*n^3 - 4145/3*n^2 - 200*n + 125);
```

```
cond1 = e1 * e3 > 0;
cond2 = e2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

```
Out[56]=
n ∈ ℤ && (n ≤ -1 || n == 1 || n ≥ 3)
```

```
In[57]:= n = Symbol["n"];
e1 = (3/8) * (n^2 + 3) * (n^4 + n^3 + 4*n^2 + 3) * (n^10 + 7/3*n^9 + 28/3*n^8 + 26/3*n^7 +
101/6*n^6 - 55/6*n^5 - 27/2*n^4 - 301/6*n^3 - 245/6*n^2 - 35*n - 109/6);
e2 = (9/256) * (n^2 + 3) * (n^4 + n^3 + 4*n^2 + 3) *
(n^26 + 17/3*n^25 + 34*n^24 + 346/3*n^23 + 3443/9*n^22 + 7970/9*n^21 +
17731/9*n^20 + 27943/9*n^19 + 41080/9*n^18 + 3522*n^17 + 6523/9*n^16 -
82732/9*n^15 - 183119/9*n^14 - 339619/9*n^13 - 46905*n^12 - 164279/3*n^11 -
419608/9*n^10 - 337279/9*n^9 - 180245/9*n^8 - 85396/9*n^7 - 932*n^6 +
12919/9*n^5 + 12146/9*n^4 + 8624/9*n^3 + 815/9*n^2 + 268/3*n - 4);
e3 = (9/4096) * (n^2 + 3) * (n^4 + n^3 + 4*n^2 + 3) *
(n^18 + 11/3*n^17 + 47/3*n^16 + 83/3*n^15 + 416/9*n^14 - 47/9*n^13 -
944/9*n^12 - 2855/9*n^11 - 3398/9*n^10 - 1981/9*n^9 + 306*n^8 + 6691/9*n^7 +
2392/3*n^6 + 3331/9*n^5 - 680/9*n^4 - 1397/9*n^3 - 121/3*n^2 + 8*n + 1);
```

```
cond1 = e1 * e3 > 0;
cond2 = e2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

```
Out[63]=
n ∈ ℤ && (n ≤ -1 || n ≥ 2)
```

```
In[64]:= n = Symbol["n"];
e1 = (3/8) * (n^2 + 3) * (n^4 + n^3 + 4*n^2 + 3) * (n^10 + 5/3*n^9 + 22/3*n^8 + 16/3*n^7 +
101/6*n^6 + 7/2*n^5 + 143/6*n^4 + 23/6*n^3 + 175/6*n^2 + 13/3*n + 95/6);
e2 = (9/256) * (n^2 + 3) * (n^4 + n^3 + 4*n^2 + 3) *
(n^26 + 13/3*n^25 + 82/3*n^24 + 238/3*n^23 + 2531/9*n^22 + 1826/3*n^21 +
13907/9*n^20 + 23827/9*n^19 + 47720/9*n^18 + 7582*n^17 + 12683*n^16 +
145556/9*n^15 + 209989/9*n^14 + 28469*n^13 + 35263*n^12 + 381427/9*n^11 +
129640/3*n^10 + 445981/9*n^9 + 39055*n^8 + 40468*n^7 + 203392/9*n^6 +
60949/3*n^5 + 20630/3*n^4 + 15736/3*n^3 + 6407/9*n^2 + 1348/3*n - 4);
e3 = (9/4096) * (n^2 + 3) * (n^4 + n^3 + 4*n^2 + 3) *
(n^18 + 7/3*n^17 + 35/3*n^16 + 43/3*n^15 + 368/9*n^14 + 209/9*n^13 +
188/3*n^12 + 193/9*n^11 + 458/9*n^10 + 491/9*n^9 - 290/9*n^8 + 329/3*n^7 -
1352/9*n^6 - 509/9*n^5 - 2684/9*n^4 - 2845/9*n^3 - 553/3*n^2 - 308*n - 71);

cond1 = e1 * e3 > 0;
cond2 = e2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

```
Out[70]=
n ∈ ℤ && (n ≤ -2 || n ≥ 2)
```

```
In[71]:= n = Symbol["n"];
e1 = (-1/4) * (n^2 + 3) * (n^4 + n^3 + 4*n^2 + 3) * (n^10 + 7/2*n^9 + 23/2*n^8 + 23*n^7 +
147/4*n^6 + 201/4*n^5 + 191/4*n^4 + 195/4*n^3 + 115/4*n^2 + 37/2*n + 21/4);
e2 = (3/256) * n * (n^2 + 3) * (n^4 + n^3 + 4*n^2 + 3) *
(n^25 + 23/3*n^24 + 136/3*n^23 + 190*n^22 + 661*n^21 + 5806/3*n^20 +
4881*n^19 + 32875/3*n^18 + 21632*n^17 + 117062/3*n^16 + 62623*n^15 +
279124/3*n^14 + 123757*n^13 + 458897/3*n^12 + 169641*n^11 +
521671/3*n^10 + 479266/3*n^9 + 400541/3*n^8 + 99281*n^7 + 195296/3*n^6 +
110446/3*n^5 + 51943/3*n^4 + 18274/3*n^3 + 1536*n^2 - 97/3*n - 96);
e3 = (-9/4096) * (n^2 + 3) * (n^4 + n^3 + 4*n^2 + 3) *
(n^18 + 23/3*n^17 + 37*n^16 + 1189/9*n^15 + 3272/9*n^14 + 7529/9*n^13 + 4820/3*
n^12 + 8015/3*n^11 + 3906*n^10 + 14921/3*n^9 + 51406/9*n^8 + 50423/9*n^7 +
43904/9*n^6 + 32443/9*n^5 + 6644/3*n^4 + 9847/9*n^3 + 309*n^2 - 36*n - 27);

cond1 = e1 * e3 > 0;
cond2 = e2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

```
Out[77]=
n ∈ ℤ && (n ≤ -3 || n ≥ 1)

n ≡ 7, 25 (mod 54)
```

```
In[78]:= n = Symbol["n"];
e1 = (5 / 24) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
      (n^10 + 13 / 5 * n^9 + 124 / 15 * n^8 + 62 / 5 * n^7 + 533 / 30 * n^6 + 553 / 30 * n^5 +
        271 / 30 * n^4 + 371 / 30 * n^3 - 149 / 30 * n^2 + 59 / 15 * n - 109 / 30);
e2 = (1 / 256) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
      (n^26 + 43 / 9 * n^25 + 614 / 27 * n^24 + 1642 / 27 * n^23 + 3473 / 27 * n^22 + 4262 / 27 * n^21 -
        13 117 / 81 * n^20 - 29 395 / 27 * n^19 - 344 248 / 81 * n^18 - 684 926 / 81 * n^17 -
        1 477 837 / 81 * n^16 - 1 982 540 / 81 * n^15 - 3 162 047 / 81 * n^14 - 2 988 995 / 81 * n^13 -
        1 280 059 / 27 * n^12 - 2 396 645 / 81 * n^11 - 2 681 456 / 81 * n^10 - 853 247 / 81 * n^9 -
        993 277 / 81 * n^8 + 32 956 / 81 * n^7 - 40 828 / 27 * n^6 + 105 287 / 81 * n^5 +
        26 962 / 81 * n^4 + 22 256 / 81 * n^3 + 1861 / 27 * n^2 + 332 / 27 * n - 4 / 9);
e3 = (1 / 36 864) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
      (n^18 + 25 / 9 * n^17 + 167 / 27 * n^16 + 17 / 9 * n^15 - 800 / 27 * n^14 -
        1709 / 27 * n^13 - 13 336 / 81 * n^12 - 4375 / 81 * n^11 - 2998 / 9 * n^10 -
        18 341 / 81 * n^9 - 87 014 / 81 * n^8 - 43 549 / 81 * n^7 - 13 576 / 27 * n^6 -
        4727 / 27 * n^5 - 3584 / 81 * n^4 + 257 / 3 * n^3 + 815 / 27 * n^2 + 16 / 3 * n + 1 / 9);

cond1 = e1 * e3 > 0;
cond2 = e2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]

Out[84]=
n ∈ ℤ && (n ≤ -3 || n ≥ 3)
```

```
In[85]:= n = Symbol["n"];
e1 = (-1 / 12) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) * (n^10 + 13 / 6 * n^9 + 9 * n^8 + 32 / 3 * n^7 +
      283 / 12 * n^6 + 169 / 12 * n^5 + 91 / 4 * n^4 + 9 / 4 * n^3 + 27 / 4 * n^2 - 25 / 6 * n - 13 / 12);
e2 = (1 / 768) * (n^2 + 3) * (n^3 + n^2 + 3 * n - 1) * (n^4 + n^3 + 4 * n^2 + 3) *
      (n^23 + 8 / 3 * n^22 + 155 / 9 * n^21 + 604 / 27 * n^20 + 1958 / 27 * n^19 - 685 / 9 * n^18 -
        7006 / 27 * n^17 - 43 964 / 27 * n^16 - 91 907 / 27 * n^15 - 76 495 / 9 * n^14 -
        365 131 / 27 * n^13 - 624 113 / 27 * n^12 - 779 528 / 27 * n^11 - 1 001 239 / 27 * n^10 -
        984 061 / 27 * n^9 - 107 603 / 3 * n^8 - 735 835 / 27 * n^7 - 181 516 / 9 * n^6 -
        305 845 / 27 * n^5 - 52 982 / 9 * n^4 - 59 563 / 27 * n^3 - 5915 / 9 * n^2 - 1090 / 9 * n - 26 / 3);
e3 = (-1 / 36 864) * (n^2 + 3) * (n^3 + n^2 + 3 * n - 1)^2 * (n^4 + n^3 + 4 * n^2 + 3) *
      (n^12 - n^11 + 40 / 9 * n^10 - 124 / 9 * n^9 - 311 / 81 * n^8 - 3916 / 81 * n^7 - 62 / 3 * n^6 -
        3730 / 81 * n^5 - 47 / 81 * n^4 + 383 / 27 * n^3 + 590 / 27 * n^2 + 74 / 3 * n + 37 / 9);

cond1 = e1 * e3 > 0;
cond2 = e2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]

Out[91]=
n ∈ ℤ && (n ≤ -1 || n ≥ 3)

n ≡ 16 (mod 54)
```

```
In[92]:= n = Symbol["n"];
e1 = (80 / 729) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
      (n^10 + 109 / 40 * n^9 + 981 / 80 * n^8 + 1917 / 80 * n^7 + 4357 / 80 * n^6 + 5987 / 80 * n^5 +
      8551 / 80 * n^4 + 951 / 10 * n^3 + 7227 / 80 * n^2 + 1661 / 40 * n + 537 / 20);
e2 = (64 / 59049) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
      (n^26 + 59 / 12 * n^25 + 659 / 24 * n^24 + 1981 / 24 * n^23 + 32477 / 144 * n^22 + 25775 / 72 * n^21 +
      27737 / 192 * n^20 - 359449 / 192 * n^19 - 5144783 / 576 * n^18 - 5272549 / 192 * n^17 -
      3195091 / 48 * n^16 - 26675711 / 192 * n^15 - 987627 / 4 * n^14 - 231448417 / 576 * n^13 -
      80129731 / 144 * n^12 - 422215133 / 576 * n^11 - 76696781 / 96 * n^10 - 165480799 / 192 * n^9 -
      104751971 / 144 * n^8 - 370019797 / 576 * n^7 - 231398095 / 576 * n^6 - 2583077 / 9 * n^5 -
      1938993 / 16 * n^4 - 12954955 / 192 * n^3 - 247687 / 16 * n^2 - 93119 / 16 * n - 20751 / 64);
e3 = (64 / 43046721) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
      (n^18 + 35 / 12 * n^17 + 117 / 8 * n^16 + 199 / 8 * n^15 + 4475 / 144 * n^14 - 1649 / 24 * n^13 -
      223505 / 576 * n^12 - 77377 / 64 * n^11 - 843817 / 288 * n^10 - 859339 / 144 * n^9 -
      6735383 / 576 * n^8 - 6620831 / 288 * n^7 - 205021 / 6 * n^6 - 5938507 / 96 * n^5 -
      4263909 / 64 * n^4 - 5247661 / 64 * n^3 - 2362641 / 32 * n^2 - 1135899 / 32 * n - 2222163 / 64);

cond1 = e1 * e3 > 0;
cond2 = e2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]

Out[98]=
n ∈ ℤ && (n ≤ -2 || n ≥ 3)
```

```
In[99]:= n = Symbol["n"];
e1 = (-17 / 729) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
      (n^10 + 175 / 51 * n^9 + 631 / 51 * n^8 + 1145 / 51 * n^7 + 2086 / 51 * n^6 +
      2195 / 51 * n^5 + 2464 / 51 * n^4 + 429 / 17 * n^3 + 941 / 51 * n^2 + 44 / 51 * n - 13 / 17);
e2 = (8 / 59049) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
      (n^26 + 31 / 4 * n^25 + 3505 / 72 * n^24 + 44675 / 216 * n^23 + 488983 / 648 * n^22 +
      1443425 / 648 * n^21 + 470893 / 81 * n^20 + 8425397 / 648 * n^19 + 5638345 / 216 * n^18 +
      7457569 / 162 * n^17 + 47566765 / 648 * n^16 + 8420294 / 81 * n^15 + 86398459 / 648 * n^14 +
      32907647 / 216 * n^13 + 33921205 / 216 * n^12 + 10352579 / 72 * n^11 + 8449051 / 72 * n^10 +
      27291661 / 324 * n^9 + 5663725 / 108 * n^8 + 18207575 / 648 * n^7 + 7936513 / 648 * n^6 +
      2801255 / 648 * n^5 + 36637 / 36 * n^4 + 2953 / 24 * n^3 - 299 / 24 * n^2 - 161 / 18 * n - 9 / 8);
e3 = (-64 / 43046721) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
      (n^18 + 77 / 12 * n^17 + 2305 / 72 * n^16 + 22933 / 216 * n^15 + 23647 / 81 * n^14 +
      2446555 / 3888 * n^13 + 2977769 / 2592 * n^12 + 4452331 / 2592 * n^11 +
      33900115 / 15552 * n^10 + 11879333 / 5184 * n^9 + 10464721 / 5184 * n^8 +
      22804333 / 15552 * n^7 + 243665 / 288 * n^6 + 1971911 / 5184 * n^5 +
      16507 / 144 * n^4 + 28249 / 1728 * n^3 - 1031 / 576 * n^2 - 23 / 24 * n - 17 / 192);

cond1 = e1 * e3 > 0;
cond2 = e2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]

Out[105]=
n ∈ ℤ && (n ≤ -1 || n ≥ 1)
```

$$n \equiv 43 \pmod{54}$$

In[106]:=

```

n = Symbol["n"];
e1 = (-175 / 5832) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
      (n^10 + 487 / 175 * n^9 + 2178 / 175 * n^8 + 864 / 35 * n^7 + 3917 / 70 * n^6 + 3887 / 50 * n^5 +
        38791 / 350 * n^4 + 34851 / 350 * n^3 + 32931 / 350 * n^2 + 1531 / 35 * n + 1947 / 70);
e2 = (1225 / 15116544) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
      (n^26 + 523 / 105 * n^25 + 2918 / 105 * n^24 + 8794 / 105 * n^23 + 2513279 / 11025 * n^22 +
        3878578 / 11025 * n^21 + 342953 / 3675 * n^20 - 7893979 / 3675 * n^19 -
        108233348 / 11025 * n^18 - 110613382 / 3675 * n^17 - 267030247 / 3675 * n^16 -
        559136672 / 3675 * n^15 - 331757799 / 1225 * n^14 - 4874523559 / 11025 * n^13 -
        1356514529 / 2205 * n^12 - 8949390341 / 11025 * n^11 - 93647096 / 105 * n^10 -
        3531292573 / 3675 * n^9 - 1811538397 / 2205 * n^8 - 7944908056 / 11025 * n^7 -
        5075444956 / 11025 * n^6 - 3565651541 / 11025 * n^5 - 173636478 / 1225 * n^4 -
        39972508 / 525 * n^3 - 23007601 / 1225 * n^2 - 228832 / 35 * n - 77928 / 175);
e3 = (-1225 / 176319369216) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
      (n^18 + 313 / 105 * n^17 + 519 / 35 * n^16 + 883 / 35 * n^15 + 42524 / 11025 * n^14 -
        117797 / 735 * n^13 - 4616324 / 11025 * n^12 - 151957 / 175 * n^11 - 15069482 / 11025 * n^10 -
        20940781 / 11025 * n^9 - 106383986 / 11025 * n^8 - 354029533 / 11025 * n^7 -
        260705516 / 3675 * n^6 - 589902707 / 3675 * n^5 - 54527124 / 245 * n^4 -
        324998833 / 1225 * n^3 - 382148103 / 1225 * n^2 - 152625456 / 1225 * n - 199306827 / 1225);

cond1 = e1 * e3 > 0;
cond2 = e2 > 0;
Reduce[cond1 && cond2 && Element[n, Integers], n]

```

Out[112]=

$$n \in \mathbb{Z} \ \&\& \ (n \leq -2 \mid \mid n \geq 3)$$