Solve the inequalities for WR lattice (in R^3) conditions

Shank's simplest cubic fields, basis $(1+\rho+\rho^2)/3,\ \rho,\ \rho+\rho^2$

In[1]:=
```
n = Symbol["n"];
a = (1/9) * (n^2 + 3 n + 3) * (n^2 + 3 n + 9);
b = (-1/27) * (n^2 + 9 n + 9) * (n^2 + 3 n + 9);
c = (1/27) * (n^2 - 3 n - 9) * (n^2 + 3 n + 9);
d = (2/3) * (n^2 + 3 n + 9);
cond1 = Max[Abs[b], Abs[c], Abs[d]] ≤ a/2;
cond2 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond1 && cond2 && Element[n, Integers], n]
```

Out[8]= $n \in \mathbb{Z}$ && $(n \le -13 \mid\mid n \ge 10)$

Washington's cyclic cubic fields, n even, basis $\rho,\ (\rho^2-1)/(n-1)-\rho,\ \rho^2$

In[17]:=
```
n = Symbol["n"];
a = n^2 - n + 3;
b = n;
c = -n;
d = n;
cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a/2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[24]=

$n == 0 \mid\mid (n \in \mathbb{Z}$ && $(n \le 1 \mid\mid n \ge 3))$

Washington’s cyclic cubic fields, n odd, 1st basis

In[25]:=
```
n = Symbol["n"];
a = (1 / 16) * (n^2 - 3 * n + 3) * (n^2 + 3) * (n^4 - 5 * n^3 + 10 * n^2 - 11 * n + 1);
b=  (1/32)*(n^2-4*n+7)*(n^2-3*n+3)*(n^2-2*n-1)*(n^2+3);
c=  (1/32)*(n^2-3*n+3)*(n^2+3)*(n^4-8*n^3+16*n^2-16*n-1);
d=  (1/64)*(n-1)*(n^2-3*n+3)*(n^2+3)*(n^3-11*n^2+19*n-1);

cond2=Max[Abs[b],Abs[c],Abs[d]]<=a/2;
cond3=Max[-b+c+d,b-c+d,b+c-d,-b-c-d]<=a;
 Reduce[cond2&&cond3&&Element[n,Integers],n]
```

Out[26]=

```
(n∈ℤ&&n≥5)
```

Washington’s cyclic cubic fields, n odd, 2nd basis

In[27]:=
```
n = Symbol["n"];
a = (1 / 32) * (n^2 - 4 * n + 7) * (n^2 - 3 * n + 3) * (n^2 - 2 * n + 3) * (n^2 + 3);
b = (1 / 64) * (n - 3) * (n - 1) * (n^2 - 4 * n + 7) * (n^2 - 3 * n + 3) * (n^2 + 3);
c =  (1 / 64) * (n - 3) * (n - 1) * (n^2 - 4 * n + 7) * (n^2 - 3 * n + 3) * (n^2 + 3);
d = (1 / 64) * (n - 3) * (n - 1) * (n^2 - 4 * n + 7) * (n^2 - 3 * n + 3) * (n^2 + 3);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[34]=

```
n == 1 || n == 3 || (n ∈ ℤ && n ≥ 0)
```

Kishi’s cyclic cubic fields

$n \equiv 0, 2 \pmod{6}$ or $n \equiv 4, 10 \pmod{18}$

In[35]:=
```
n = Symbol["n"];
a = (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 + n^5 + 5 n^4 + n^3 + 5 n^2 - 2 n + 1);
b = (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) n;
c = (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^5 + n^4 + 3 n^3 - 1);
d = - (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^4 + n^3 + 2 n^2 - n + 1);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[42]=

```
n ∈ ℤ && (n ≤ -1 || n ≥ 1)
```

$n \equiv 34, 52 \pmod{54}$

In[43]:=
```
n = Symbol["n"];
a = (1 / 9) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) * (n^6 + n^5 + 5 * n^4 + n^3 + 5 * n^2 - 2 * n + 1);
b = (-8 / 27) * n * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) * (n^4 + n^3 + 39 / 8 * n^2 + 2 * n + 37 / 8);
c = (1 / 27) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) * (n^5 + n^4 + 3 * n^3 + 8 * n^2 + 8 * n + 15);
d = (-1 / 9) * (n^2 + 3) * (n^4 + n^3 - 6 * n^2 - n - 7) * (n^4 + n^3 + 4 * n^2 + 3);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[50]=

$n \in \mathbb{Z}$ && $(n \le -6 \mid\mid n \ge 6)$

$n \equiv 3, 5 \pmod 6$ or $n \equiv 1, 13 \pmod{18}$

In[51]:=
```
n = Symbol["n"];
a = 1 / 16 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 - n^5 + 3 n^4 - 9 n^3 + n^2 - 10 n - 5);
b = -1 / 64 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 - 3 n^4 - 8 n^3 - 21 n^2 - 8 n - 25);
c = -1 / 32 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 - 3 n^5 + n^4 - 16 n^3 - n^2 - 9 n - 5);
d = 1 / 32 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 - n^5 + 2 n^4 - 8 n^3 + 3 n^2 - 3 n - 10);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[58]=

$n \in \mathbb{Z}$ && $n \ge 3$

In[59]:=
```
Symbol["n"];
a = 1 / 16 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 + n^5 + 3 n^4 - n^3 - 3 n^2 - 4 n - 1);
b = 1 / 64 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n + 1) (n^5 - n^4 + 2 n^3 - 10 n^2 - 3 n - 5);
c = -1 / 32 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n + 1) (n^5 - n^4 + 3 * n^3 - 3 * n^2 + 2 * n + 2);
d = -1 / 32 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n + 1) (n^5 - 4 n^2 - 5 n + 4);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[66]=

$n == -1 \mid\mid n == -1 \mid\mid n == -1 \mid\mid (n \in \mathbb{Z}$ && $(n == -1 \mid\mid n \ge 2))$

In[67]:=
```
Symbol["n"];
a = 1 / 16 * (n^2 + 3) (n^4 + n^3 + 4 * n^2 + 3) (n^6 + n^5 + 3 n^4 - n^3 + n^2 - 1);
b = 1 / 64 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 + 2 n^5 + 3 n^4 + 4 n^3 - 5 n^2 + 2 n - 23);
c = 1 / 32 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 + 2 n^5 + 4 n^4 - 2 n^3 - n^2 - 8 n - 4);
d = 1 / 32 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 + n^5 - 4 n^3 - 5 n^2 - n - 8);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[74]=

$n \in \mathbb{Z}$ && $n \leq -2$

In[75]:=
```
Symbol["n"];
a = 1 / 16 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 + 3 n^5 + 7 n^4 + 11 n^3 + 9 n^2 + 10 n + 3);
b = -1 / 64 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 + 2 n^5 - n^4 - 4 n^3 - 21 n^2 - 6 n - 3);
c = 1 / 32 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n^6 + 5 n^5 + 9 n^4 + 18 n^3 + 11 n^2 + 9 n + 3);
d = -1 / 32 * (n^2 + 3) (n^4 + n^3 + 4 n^2 + 3) (n + 1) (n^5 + 2 n^4 + 4 n^3 + 4 n^2 + 3 n + 6);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[82]=

$n \in \mathbb{Z}$ && $n \leq -3$

$n \equiv 7, 25 \pmod{54}$

In[ ]:=
```
Symbol["n"];
a =
  (1 / 144) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) * (n^6 + n^5 + 3 * n^4 - n^3 + 5 * n^2 + 4 * n - 1);
b = (1 / 576) * (n - 1) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
   (n^5 - 17 / 3 * n^4 - 34 / 3 * n^3 - 106 / 3 * n^2 - 17 / 3 * n - 7);
c = (1 / 288) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
   (n^6 + 4 / 3 * n^5 + 10 / 3 * n^4 - 4 / 3 * n^3 + 37 / 3 * n^2 + 28 / 3 * n - 2);
d = (1 / 288) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) * (n^6 + n^5 - 4 * n^3 + 23 * n^2 - n + 4);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[ ]=

$n \in \mathbb{Z}$ && $n \leq -8$

In[83]:=
```
Symbol["n"];
a = (1 / 144) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^6 + n^5 + 41 / 9 * n^4 - 1 / 9 * n^3 + 23 / 9 * n^2 - 4 * n - 1);
b = (-1 / 576) * (n^2 + 3) * (n^2 + 2 * n + 3) *
    (n^4 + n^3 + 4 * n^2 + 3) * (n^4 + 2 * n^3 + 38 / 9 * n^2 - 26 / 9 * n - 5 / 3);
c = (-1 / 288) * (n^2 + 3) * (n^2 + 2 * n + 3) *
    (n^4 + 17 / 9 * n^2 - 14 / 9 * n + 4 / 3) * (n^4 + n^3 + 4 * n^2 + 3);
d = (1 / 288) * (n^2 + 3) * (n^2 + 2 * n + 3) *
    (n^4 + n^3 + 26 / 9 * n^2 + 13 / 9 * n + 1 / 3) * (n^4 + n^3 + 4 * n^2 + 3);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[90]=

n ∈ ℤ && n ≤ −2

$n \equiv 16 \pmod{54}$

In[91]:=
```
Symbol["n"];
a = (1 / 729) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^6 + n^5 + 7 * n^4 + 3 * n^3 + 17 * n^2 + 4 * n + 27);
b = (1 / 2187) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^6 + n^5 + 15 * n^4 + 11 * n^3 - 31 * n^2 + 9 * n + 12);
c = (-1 / 6561) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^6 + 68 / 3 * n^5 + 104 / 3 * n^4 + 488 / 3 * n^3 + 88 * n^2 + 222 * n + 177);
d = (-1 / 2187) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^6 + 4 / 3 * n^5 + 22 / 3 * n^4 + 7 / 3 * n^3 + 39 * n^2 + 22 * n + 119);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[98]=

n ∈ ℤ && (n ≤ −6 || n ≥ 7)

In[99]:=
```
Symbol["n"];
a = (1 / 729) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^6 + 7 / 3 * n^5 + 61 / 9 * n^4 + 67 / 9 * n^3 + 67 / 9 * n^2 + 8 / 3 * n + 1 / 3);
b = (1 / 6561) * n^2 * (n^2 + 3) *
    (n^4 + n^3 + 4 * n^2 + 3) * (n^4 + 14 / 3 * n^3 + 8 / 3 * n^2 - 4 / 3 * n - 19);
c = (-1 / 2187) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^6 + 5 * n^5 + 101 / 9 * n^4 + 155 / 9 * n^3 + 116 / 9 * n^2 + 3 * n - 1 / 3);
d = (-1 / 2187) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^6 + 2 * n^5 + 56 / 9 * n^4 + 74 / 9 * n^3 + 53 / 9 * n^2 + 7 * n + 5 / 3);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[106]=

$n \in \mathbb{Z}$ && ($n \leq -2$ || $n \geq 5$)

$n \equiv 43 \pmod{54}$

In[107]:=
```
Symbol["n"];
a = (1 / 11664) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^6 + n^5 + 7 * n^4 + 3 * n^3 + 17 * n^2 + 4 * n + 27);
b = (-1 / 69984) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^6 + n^5 + 42 * n^4 + 38 * n^3 - 193 * n^2 + 9 * n - 42);
c = (-1 / 69984) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^6 + 4 / 3 * n^5 + 22 / 3 * n^4 - 20 / 3 * n^3 + 111 * n^2 + 76 * n + 434);
d = (1 / 419904) * (n^2 + 3) * (n^4 + n^3 + 4 * n^2 + 3) *
    (n^6 + 284 / 3 * n^5 + 401 / 3 * n^4 + 2108 / 3 * n^3 + 367 * n^2 + 960 * n + 717);

cond2 = Max[Abs[b], Abs[c], Abs[d]] ≤ a / 2;
cond3 = Max[-b + c + d, b - c + d, b + c - d, -b - c - d] ≤ a;
Reduce[cond2 && cond3 && Element[n, Integers], n]
```

Out[114]=

$n \in \mathbb{Z}$ && ($n \leq -5$ || $n \geq 6$)