Notes on Stochastic Volatility Models

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1 Stochastic Volatility Models

Let $\{r_t\}_t$ be a time series of financial returns.

All stochastic volatility (SV) models verify:

$$\mathrm{var}[r_t \mid \sigma_t] = \sigma_t^2 \tag{1}$$
$$\log \sigma_t^2 = \alpha + \beta \log \sigma_{t-1}^2 + \gamma \eta_t \tag{2}$$

with $\eta_0 \sim \mathcal{N}(0, \frac{\gamma^2}{1-\beta^2})$ (where $|\beta| < 1$ and $\gamma > 0$) and $\{\eta_t\}_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$.

$\sigma_t > 0$ is called the volatility.

2 Standard Stochastic Volatility Model

The observation equation is

$$r_t = \mu + \sigma_t \varepsilon_t \tag{3}$$

with $\{\varepsilon_t\}_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ and $\mathrm{cov}[\varepsilon_t, \eta_t] = 0$. This model is referred to as the SV model. It is has been introduced by Taylor in [7].

3 Asymmetric Stochastic Volatility Models

There are two types of asymmetric SV models.

In both types, the observation equation is the same as Equation (3) for the SV model.

In the first type, one has $\mathrm{cov}[\varepsilon_t, \eta_{t+1}] = \rho$ for all $t \geq 1$. This model is referred to as the ASV1 model. It is has been introduced by Harvey and Shepard in [4].

In the second type, one has $\mathrm{cov}[\varepsilon_t, \eta_t] = \rho$. This model is referred to as the ASV2 model. It is has been introduced by Jacquier et al. in [3].

4 Stochastic Volatility in Mean Model

In the stochastic volatility in mean model, the observation equation is

$$r_t = \mu + \lambda\sigma_t + \sigma_t\varepsilon_t \tag{4}$$

with $\{\varepsilon_t\}_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ and $\mathrm{cov}[\varepsilon_t, \eta_t] = 0$. This model is referred to as the SV-M model. It is has been introduced by Koopman and Hol Uspensky in [5].

5 Power Law Stochastic Volatility Model

In the power law stochastic volatility model, the observation equation is

$$r_t = \mu + \lambda\sigma_t^{\kappa} + \sigma_t\varepsilon_t \tag{5}$$

with $\{\varepsilon_t\}_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ and $\mathrm{cov}[\varepsilon_t, \eta_t] = 0$. This model is referred to as the PL-SV model. It is newly proposed by us.

6 Volatility Clustering

Volatility clustering is defined by Cont in [2] by the fact that "large price variations are more likely to be followed by large price variations".

The Markov process form of the volatility dynamics in Equation (2) accounts for volatility clustering in all types of SV models (SV, ASV1, ASV2, SV-M, PL-SV).

7 Leverage Effect

The leverage effect is defined by Cont in [2] by the fact that the "volatility of an asset [is] negatively correlated with the returns of that asset".

The SV model does not model the leverage effect. The ASV1, ASV2, SV-M and PL-SV models do take it into account, but differently.

In the ASV1 model, Yu showed in [8] that

$$\mathrm{E}[\log \sigma_{t+1}^2 \mid r_t] = \alpha + \frac{\alpha\beta}{1-\beta^2} + \rho\gamma \exp\left(-\frac{\gamma}{4(1-\phi^2)^2} + \frac{\gamma\alpha}{(1-\phi^2)(1-\phi)}\right) r_t. \quad (6)$$

When $\rho < 0$, a decrease of the return leads to an increase of the conditional expectation of next period's log-squared volatility given the return.

In the ASV2 model, Asai et al. showed in [1] that

$$\frac{\partial(\log \sigma_{t+1}^2)}{\partial r_t} = \frac{\rho\gamma\sigma_{t+1}/\sigma_t}{1 + 0.5\rho\gamma\varepsilon_{t+1}}. \quad (7)$$

This quantity is not ensured to be negative, even when $\rho < 0$. Hence Yu [8] and Asai et al. [1], followed by Men et al. in [6], argue that the ASV1 model describes better the leverage effect than the ASV2 model.

In the SV-M model, one has that

$$\mathrm{E}[r_t \mid \sigma_t] = \mu + \lambda\sigma_t. \quad (8)$$

When $\lambda < 0$, an increase of volatility leads to a decrease of the conditional expectation of same period's return given the volatility.

In the PL-SV model, one has that

$$\mathrm{E}[r_t \mid \sigma_t] = \mu + \lambda\sigma_t^{\kappa} \quad (9)$$

and we show that

$$\mathrm{cov}[r_t, \sigma_t] < 0 \quad (10)$$

when $\lambda > 0$ and $\kappa \in (0, -1)$. Here the volatility and the return are negatively correlated at each period.

References

[1] M. Asai, M. McAleer, J. Yu. Multivariate Stochastic Volatility: A Review. *Econometric Reviews*, 25, 2006.

[2] R. Cont. Empirical Properties of Asset Returns: Stylized Facts and Statistical Issues. *Quantitative Finance*, 1, 2001.

[3] E. Jacquier, N.G. Polson, and P.E. Rossi, Bayesian Analysis of Stochastic Volatility Models with Fat-Tails and Correlated Errors, *Journal of Econometrics*, 122, 2004.

[4] A.C. Harvey, N. Shepard. Estimation of an Asymmetric Stochastic Volatility Model for Asset Returns. *Journal of Business & Economic Statistics*, 14, 1996.

[5] S.J. Koopman, E. Hol Uspensky. The Stochastic Volatility in Mean Model: Empirical Evidence from International Stock Markets. *Journal of Applied Econometrics*, 17, 2002.

[6] Z. Men, D. McLeish, A.W. Kolkiewicz, T.S. Wirjanto. Comparison of Asymmetric Stochastic Volatility Models under Different Correlation Structures. *Journal of Applied Statistics*, 44, 2016.

[7] S.J. Taylor. *Modelling Financial Time Series*, John Wiley, 1986.

[8] J. Yu. On Leverage in a Stochastic Volatility Model. *Journal of Econometrics*, 127, 2005.