hw1

February 2, 2018

1 Homework 1

This notebook includes both coding and written questions. Please hand in this notebook file with all the outputs and your answers to the written questions.

This assignment covers linear filters, convolution and correlation

```
In [2]: # Setup
        import numpy as np
        import matplotlib.pyplot as plt
        from time import time
        from skimage import io

        from __future__ import print_function

        %matplotlib inline
        plt.rcParams['figure.figsize'] = (10.0, 8.0) # set default size of plots
        plt.rcParams['image.interpolation'] = 'nearest'
        plt.rcParams['image.cmap'] = 'gray'

        # for auto-reloading extenrnal modules
        %load_ext autoreload
        %autoreload 2

        print("Huynh Bao Quoc (B1400516)")
        print("Nguyen Trung Hau (B1400493)")
```

```
Huynh Bao Quoc (B1400516)
Nguyen Trung Hau (B1400493)
```

1.1 Part 1: Convolutions

1.1.1 1.1 Commutative Property (10 points)

Recall that the convolution of an image $f : \mathbb{R}^2 \to \mathbb{R}$ and a kernel $h : \mathbb{R}^2 \to \mathbb{R}$ is defined as follows:

$$(f * h)[m, n] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f[i, j] \cdot h[m - i, n - j]$$

Or equivalently,

$$(f * h)[m,n] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h[i,j] \cdot f[m-i, n-j] \tag{1}$$
$$= (h * f)[m,n] \tag{2}$$

Show that this is true (i.e. prove that the convolution operator is commutative: $f * h = h * f$).
Chng minh rng phép tích chp có tính giao hoán: $f * h = h * f$.

Your Answer: *Write your solution in this markdown cell. Please write your equations in LaTex equations.*

$$V1 : (f * h)[m,n] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f[i,j] \cdot h[m-i, n-j]$$

$$t :$$

$$u = m - i$$

$$v = n - j$$

$$Suyra :$$

$$i \to -\infty \Rightarrow u \to \infty, j \to -\infty \Rightarrow v \to \infty$$

$$i \to \infty \Rightarrow u \to -\infty, j \to \infty \Rightarrow v \to -\infty$$

$$\Leftrightarrow (f * h)[m,n] = \sum_{u=\infty}^{-\infty} \sum_{v=\infty}^{-\infty} f[m-u, n-v] \cdot h[u,v]$$

$$Thay : u = i, v = j$$

$$V2 : (f * h)[m,n] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f[m-i, n-j] \cdot h[i,j]$$

1.1.2 1.2 Linear and Shift Invariance (10 points)

Let f be a function $\mathbb{R}^2 \to \mathbb{R}$. Consider a system $f \xrightarrow{s} g$, where $g = (f * h)$ with some kernel $h : \mathbb{R}^2 \to \mathbb{R}$. Show that S defined by any kernel h is a Linear Shift Invariant (LSI) system. In other words, for any h, show that S satisfies both of the following: - $S[a \cdot f_1 + b \cdot f_2] = a \cdot S[f_1] + b \cdot S[f_2]$ - If $f[m,n] \xrightarrow{s} g[m,n]$ then $f[m-m_0, n-n_0] \xrightarrow{s} g[m-m_0, n-n_0]$

Cho f là mt hàm $\mathbb{R}^2 \to \mathbb{R}$. Xét mt h thng x lý $f \xrightarrow{s} g$, trong ó $g = (f * h)$ vi mt kernel (mt n chp) $h : \mathbb{R}^2 \to \mathbb{R}$. Hãy chng minh rng h thng S vi kernel h bt k là mt h thng Linear Shift Invariant (LSI). Nói cách khác, vi h bt k, hãy ch ra rng S tho c hai iu kin sau: - $S[a \cdot f_1 + b \cdot f_2] = a \cdot S[f_1] + b \cdot S[f_2]$ - If $f[m,n] \xrightarrow{s} g[m,n]$ then $f[m-m_0, n-n_0] \xrightarrow{s} g[m-m_0, n-n_0]$

Your Answer: *Write your solution in this markdown cell. Please write your equations in LaTex equations.*

Nhp câu tr li ca bn ây.

1.1.3 1.3 Implementation (30 points)

In this section, you will implement two versions of convolution: - `conv_nested` - `conv_fast`
First, run the code cell below to load the image to work with.

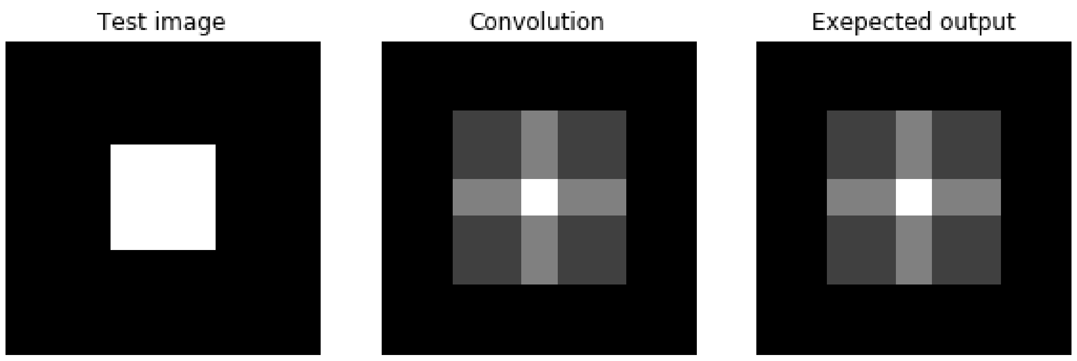
In [18]:
```
# Open image as grayscale
img = io.imread('dog.jpg', as_grey=True)

# Show image
plt.imshow(img)
plt.axis('off')
plt.title("Isn't he cute?")
plt.show()
```

Isn't he cute?

Now, implement the function **`conv_nested`** in **`filters.py`**. This is a naive implementation of convolution which uses 4 nested for-loops. It takes an image f and a kernel h as inputs and outputs the convolved image $(f * h)$ that has the same shape as the input image. This implementation should take a few seconds to run.

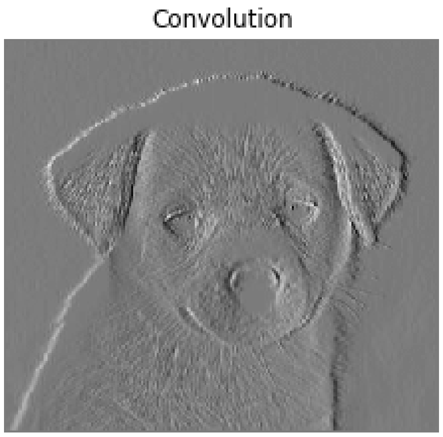
- Hint: It may be easier to implement $(h * f)$
We'll first test your `conv_nested` function on a simple input.

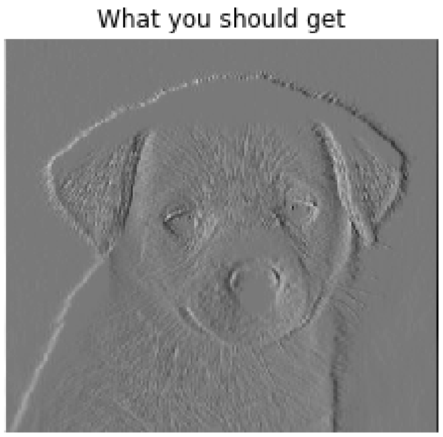
```
In [19]: from filters import conv_nested

         # Simple convolution kernel.
         kernel = np.array(
         [
             [1,0,1],
             [0,0,0],
             [1,0,1]
         ])

         # Create a test image: a white square in the middle
         test_img = np.zeros((9, 9))
         test_img[3:6, 3:6] = 1

         # Run your conv_nested function on the test image
         test_output = conv_nested(test_img, kernel)

         # Build the expected output
         expected_output = np.zeros((9, 9))
         expected_output[2:7, 2:7] = 1
         expected_output[4, 2:7] = 2
         expected_output[2:7, 4] = 2
         expected_output[4, 4] = 4

         # Plot the test image
         plt.subplot(1,3,1)
         plt.imshow(test_img)
         plt.title('Test image')
         plt.axis('off')

         # Plot your convolved image
         plt.subplot(1,3,2)
         plt.imshow(test_output)
         plt.title('Convolution')
         plt.axis('off')

         # Plot the exepected output
         plt.subplot(1,3,3)
         plt.imshow(expected_output)
         plt.title('Exepected output')
         plt.axis('off')
         plt.show()

         # Test if the output matches expected output
         assert np.max(test_output - expected_output) < 1e-10, "Your solution is not correct."
```



Now let's test your `conv_nested` function on a real image.

```
In [20]: from filters import conv_nested

         # Simple convolution kernel.
         # Feel free to change the kernel and to see different outputs.
         kernel = np.array(
         [
             [1,0,-1],
             [2,0,-2],
             [1,0,-1]
         ])

         out = conv_nested(img, kernel)

         # Plot original image
         plt.subplot(2,2,1)
         plt.imshow(img)
         plt.title('Original')
         plt.axis('off')

         # Plot your convolved image
         plt.subplot(2,2,3)
         plt.imshow(out)
         plt.title('Convolution')
         plt.axis('off')

         # Plot what you should get
         solution_img = io.imread('convoluted_dog.jpg', as_grey=True)
         plt.subplot(2,2,4)
         plt.imshow(solution_img)
         plt.title('What you should get')
         plt.axis('off')
plt.show()
```

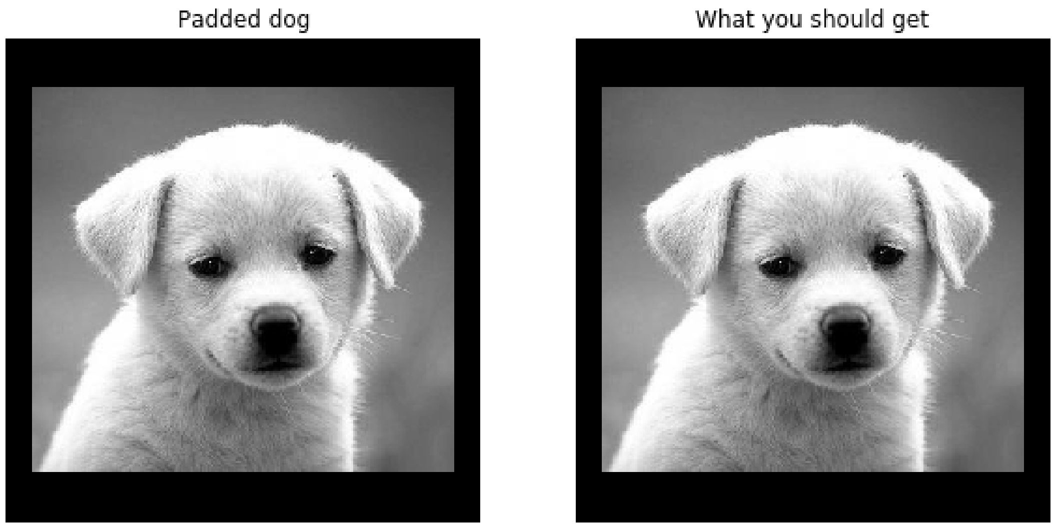




Let us implement a more efficient version of convolution using array operations in numpy. As shown in the lecture, a convolution can be considered as a sliding window that computes sum of the pixel values weighted by the flipped kernel. The faster version will i) zero-pad an image, ii) flip the kernel horizontally and vertically, and iii) compute weighted sum of the neighborhood at each pixel.

First, implement the function **`zero_pad`** in **`filters.py`**.

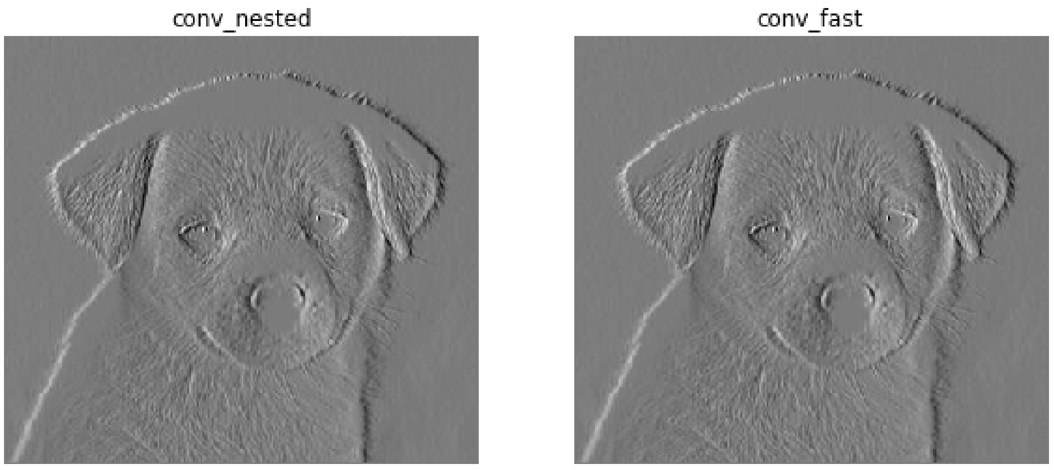
```
In [5]: from filters import zero_pad

        pad_width = 20 # width of the padding on the left and right
        pad_height = 40 # height of the padding on the top and bottom

        padded_img = zero_pad(img, pad_height, pad_width)

        # Plot your padded dog
        plt.subplot(1,2,1)
        plt.imshow(padded_img)
        plt.title('Padded dog')
        plt.axis('off')

        # Plot what you should get
        solution_img = io.imread('padded_dog.jpg', as_grey=True)
        plt.subplot(1,2,2)
        plt.imshow(solution_img)
        plt.title('What you should get')
        plt.axis('off')

        plt.show()
```



Next, complete the function **`conv_fast`** in **`filters.py`** using `zero_pad`. Run the code below to compare the outputs by the two implementations. `conv_fast` should run significantly faster than `conv_nested`.
Depending on your implementation and computer, `conv_nested` should take a few seconds and `conv_fast` should be around 5 times faster.

```
In [6]: from filters import conv_fast

        t0 = time()
        out_fast = conv_fast(img, kernel)
        t1 = time()
        out_nested = conv_nested(img, kernel)
        t2 = time()

        # Compare the running time of the two implementations
        print("conv_nested: took %f seconds." % (t2 - t1))
        print("conv_fast: took %f seconds." % (t1 - t0))

        # Plot conv_nested output
        plt.subplot(1,2,1)
        plt.imshow(out_nested)
        plt.title('conv_nested')
        plt.axis('off')

        # Plot conv_fast output
        plt.subplot(1,2,2)
        plt.imshow(out_fast)
        plt.title('conv_fast')
        plt.axis('off')

        # Make sure that the two outputs are the same
        if not (np.max(out_fast - out_nested) < 1e-10):
            print("Different outputs! Check your implementation.")
```

```
conv_nested: took 1.054778 seconds.
conv_fast: took 0.505025 seconds.
```

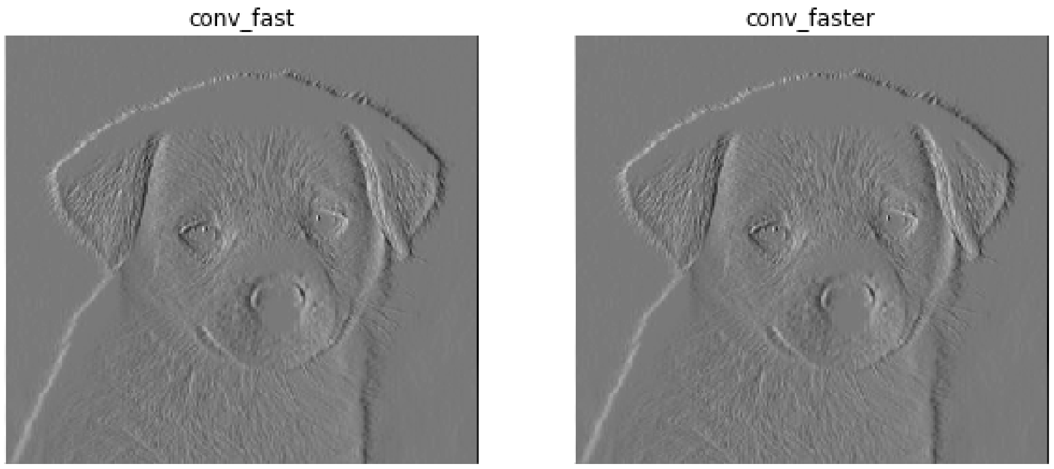


1.1.4 Extra Credit 1 (1% of final grade)

Devise a faster version of convolution and implement `conv_faster` in `filters.py`. You will earn extra credit only if the `conv_faster` runs faster (by a fair margin) than `conv_fast` **and** outputs the same result.

```
In [7]: from filters import conv_faster
        kernel = np.array(
        [
            [1,0,-1],
            [2,0,-2],
            [1,0,-1]
        ])

        t0 = time()
        out_fast = conv_fast(img, kernel)
        t1 = time()
        out_faster = conv_faster(img, kernel)
        t2 = time()
        out_fast=out_fast-(out_fast-out_faster)

        # Compare the running time of the two implementations
        print("conv_fast: took %f seconds." % (t1 - t0))
        print("conv_faster: took %f seconds." % (t2 - t1))

        # Plot conv_nested output
        plt.subplot(1,2,1)
        plt.imshow(out_fast)
        plt.title('conv_fast')
        plt.axis('off')

        # Plot conv_fast output
        plt.subplot(1,2,2)
        plt.imshow(out_faster)
        plt.title('conv_faster')
        plt.axis('off')

        # Make sure that the two outputs are the same
        if not (np.max(out_fast - out_faster) < 1e-10):
            print("Different outputs! Check your implementation.")
```

```
conv_fast: took 0.501648 seconds.
conv_faster: took 0.008152 seconds.
```



1.2 Part 2: Cross-correlation

Cross-correlation of two 2D signals f and g is defined as follows:

$$(f \star g)[m,n] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f[i,j] \cdot g[i-m, j-n]$$

1.2.1 2.1 Template Matching with Cross-correlation (12 points)

Suppose that you are a clerk at a grocery store. One of your responsibilites is to check the shelves periodically and stock them up whenever there are sold-out items. You got tired of this laborious task and decided to build a computer vision system that keeps track of the items on the shelf.

Luckily, you have learned in CS131 that cross-correlation can be used for template matching: a template g is multiplied with regions of a larger image f to measure how similar each region is to the template.

The template of a product (`template.jpg`) and the image of shelf (`shelf.jpg`) is provided. We will use cross-correlation to find the product in the shelf.

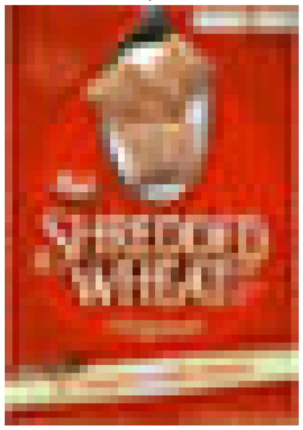
Implement **`cross_correlation`** function in **`filters.py`** and run the code below.

- Hint: you may use the `conv_fast` function you implemented in the previous question.

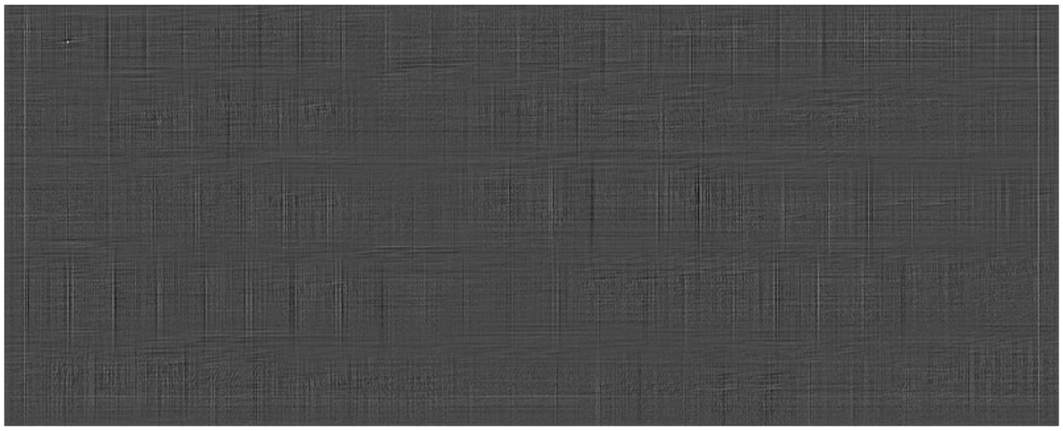
```
In [11]: from filters import cross_correlation

         # Load template and image in grayscale
         img = io.imread('shelf.jpg')
         img_grey = io.imread('shelf.jpg', as_grey=True)
         temp = io.imread('template.jpg')
         temp_grey = io.imread('template.jpg', as_grey=True)
```

```
# Perform cross-correlation between the image and the template
out = cross_correlation(img_grey, temp_grey)

# Find the location with maximum similarity
y,x = (np.unravel_index(out.argmax(), out.shape))

# Display product template
plt.figure(figsize=(25,20))
plt.subplot(3, 1, 1)
plt.imshow(temp)
plt.title('Template')
plt.axis('off')

# Display cross-correlation output
plt.subplot(3, 1, 2)
plt.imshow(out)
plt.title('Cross-correlation (white means more correlated)')
plt.axis('off')

# Display image
plt.subplot(3, 1, 3)
plt.imshow(img)
plt.title('Result (blue marker on the detected location)')
plt.axis('off')

# Draw marker at detected location
plt.plot(x, y, 'bx', ms=40, mew=10)
plt.show()
```

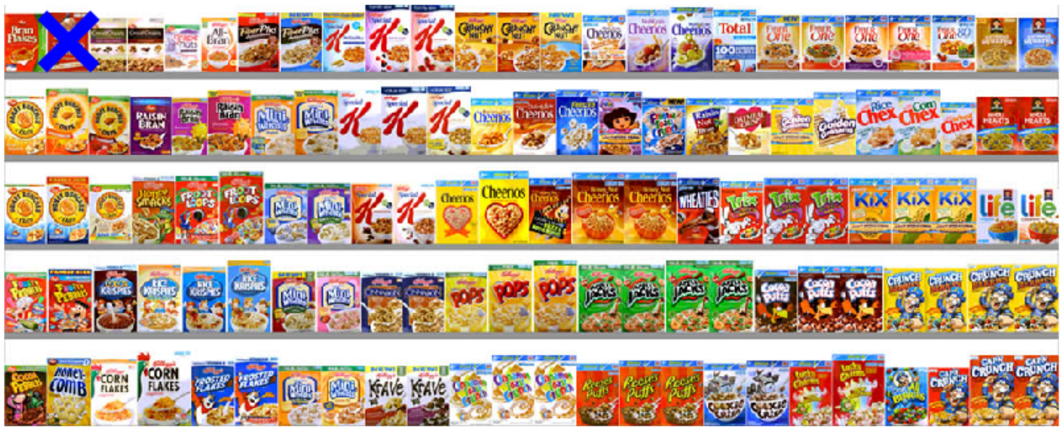
Template



Cross-correlation (white means more correlated)



Result (blue marker on the detected location)



Interpretation How does the output of cross-correlation filter look like? Was it able to detect the product correctly? Explain what might be the problem with using raw template as a filter.

1.2.2 2.2 Zero-mean cross-correlation (6 points)

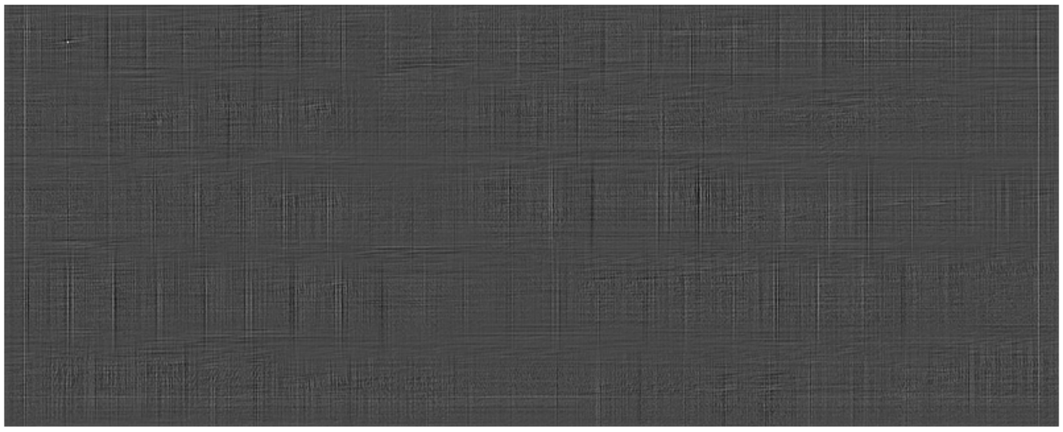
A solution to this problem is to subtract off the mean value of the template so that it has zero mean.

Implement `zero_mean_cross_correlation` function in `filters.py` and run the code below.

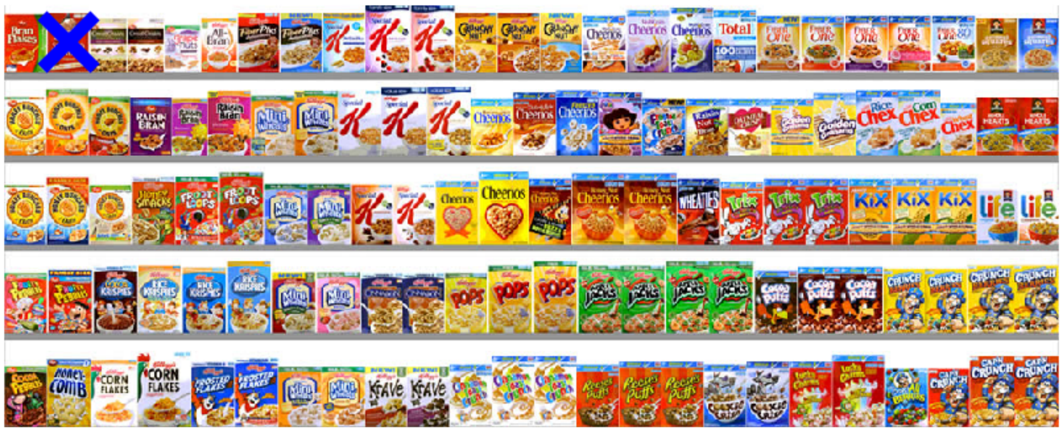
```
In [12]: from filters import zero_mean_cross_correlation

         # Perform cross-correlation between the image and the template
         out = zero_mean_cross_correlation(img_grey, temp_grey)

         # Find the location with maximum similarity
         y,x = (np.unravel_index(out.argmax(), out.shape))

         # Display product template
         plt.figure(figsize=(30,20))
         plt.subplot(3, 1, 1)
         plt.imshow(temp)
         plt.title('Template')
         plt.axis('off')

         # Display cross-correlation output
         plt.subplot(3, 1, 2)
         plt.imshow(out)
         plt.title('Cross-correlation (white means more correlated)')
         plt.axis('off')

         # Display image
         plt.subplot(3, 1, 3)
         plt.imshow(img)
         plt.title('Result (blue marker on the detected location)')
         plt.axis('off')

         # Draw marker at detcted location
         plt.plot(x, y, 'bx', ms=40, mew=10)
         plt.show()
```

Template

Cross-correlation (white means more correlated)



Result (blue marker on the detected location)



You can also determine whether the product is present with appropriate scaling and thresholding.

```
In [13]: def check_product_on_shelf(shelf, product):
        out = zero_mean_cross_correlation(shelf, product)

        # Scale output by the size of the template
        out = out / float(product.shape[0]*product.shape[1])

        # Threshold output (this is arbitrary, you would need to tune the threshold for a
        out = out > 0.025

        if np.sum(out) > 0:
            print('The product is on the shelf')
        else:
            print('The product is not on the shelf')

    # Load image of the shelf without the product
    img2 = io.imread('shelf_soldout.jpg')
    img2_grey = io.imread('shelf_soldout.jpg', as_grey=True)

    plt.imshow(img)
    plt.axis('off')
    plt.show()
    check_product_on_shelf(img_grey, temp_grey)

    plt.imshow(img2)
    plt.axis('off')
    plt.show()
    check_product_on_shelf(img2_grey, temp_grey)
```

```
The product is on the shelf
```

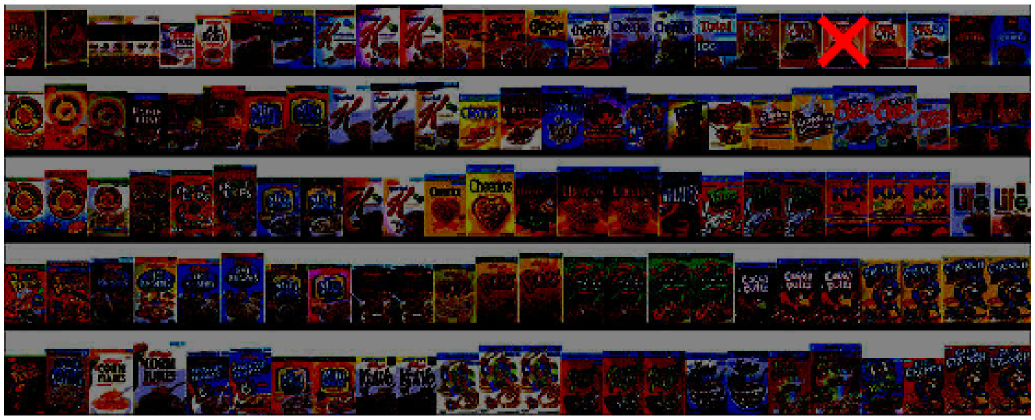
```
The product is not on the shelf
```

1.2.3 2.3 Normalized Cross-correlation (12 points)

One day the light near the shelf goes out and the product tracker starts to malfunction. The `zero_mean_cross_correlation` is not robust to change in lighting condition. The code below demonstrates this.

```
In [29]: from filters import normalized_cross_correlation

         # Load image
         img = io.imread('shelf_dark.jpg')
         img_grey = io.imread('shelf_dark.jpg', as_grey=True)
         temp_grey = io.imread('template.jpg', as_grey=True)
         # Perform cross-correlation between the image and the template
         out = zero_mean_cross_correlation(img_grey, temp_grey)

         # Find the location with maximum similarity
         y,x = (np.unravel_index(out.argmax(), out.shape))

         # Display image
         plt.imshow(img)
         plt.title('Result (red marker on the detected location)')
         plt.axis('off')

         # Draw marker at detcted location
         plt.plot(x, y, 'rx', ms=25, mew=5)
         plt.show()
```

Result (red marker on the detected location)



A solution is to normalize the pixels of the image and template at every step before comparing them. This is called **normalized cross-correlation**.

The mathematical definition for normalized cross-correlation of f and template g is:

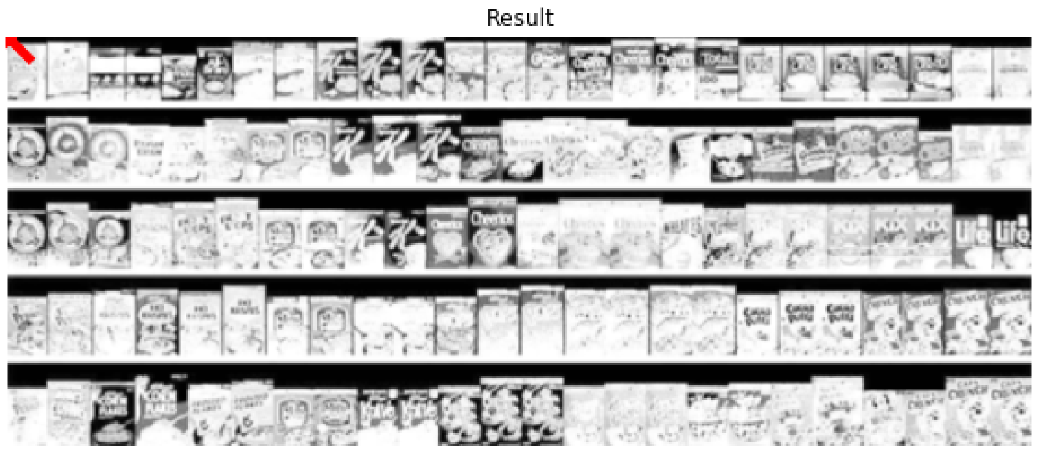
$$(f \star g)[m,n] = \sum_{i,j} \frac{f[i,j] - \overline{f_{m,n}}}{\sigma_{f_{m,n}}} \cdot \frac{g[i-m,j-n] - \overline{g}}{\sigma_g}$$

where: - $f_{m,n}$ is the patch image at position (m,n) - $\overline{f_{m,n}}$ is the mean of the patch image $f_{m,n}$ - $\sigma_{f_{m,n}}$ is the standard deviation of the patch image $f_{m,n}$ - $\overline{g}$ is the mean of the template g - σ_g is the standard deviation of the template g

Implement **`normalized_cross_correlation`** function in **`filters.py`** and run the code below.

```
In [34]: from filters import normalized_cross_correlation
         img_grey = io.imread('shelf_dark.jpg', as_grey=True)
         temp_grey = io.imread('template.jpg', as_grey=True)



         # Perform normalized cross-correlation between the image and the template
         out = normalized_cross_correlation(img_grey, temp_grey)
         # Find the location with maximum similarity
         y,x=(np.unravel_index(out.argmax(), out.shape))
         # Display image
         plt.imshow(out)
         plt.title('Result')
         plt.axis('off')

         # Draw marker at detcted location
         plt.plot(x, y, 'rx', ms=25, mew=5)
         plt.show()
```

```
[[ 1.   1.   1.]
 [ 1.   2.   1.]
 [ 1.   1.   1.]]
[[-0.11111111 -0.11111111 -0.11111111]
 [-0.11111111 -0.11111111 -0.11111111]
 [-0.11111111 -0.11111111 -0.11111111]]
```



1.3 Part 3: Separable Filters

1.3.1 3.1 Theory (10 points)

Consider a $M_1 \times N_1$ image I and a $M_2 \times N_2$ filter F. A filter F is **separable** if it can be written as a product of two 1D filters: $F = F_1 F_2$.

For example,

$$F = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

can be written as a matrix product of

$$F_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, F_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Therefore F is a separable filter.

Prove that for any separable filter $F = F_1 F_2$,

$$I * F = (I * F_1) * F_2$$

Your Answer: *Write your solution in this markdown cell. Please write your equations in LaTex equations.*

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1.3.2 3.2 Complexity comparison (10 points)

(i) How many multiplications do you need to do a direct 2D convolution (i.e. $I * F$?)
(ii) How many multiplications do you need to do 1D convolutions on rows and columns (i.e. $(I * F_1) * F_2$)
(iii) Use Big-O notation to argue which one is more efficient in general: direct 2D convolution or two successive 1D convolutions?

Your Answer: *Write your solution in this markdown cell. Please write your equations in LaTex equations.*

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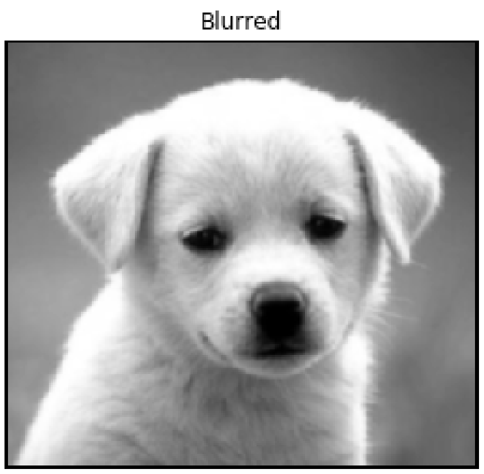
Now, we will empirically compare the running time of a separable 2D convolution and its equivalent two 1D convolutions. Gaussian kernel, widely used for blurring images, is one example of a separable filter. Run the code below to see its effect.

In [23]:
```
# Load image
img = io.imread('dog.jpg', as_grey=True)

# 5x5 Gaussian blur
kernel = np.array(
[
    [1,4,6,4,1],
    [4,16,24,16,4],
    [6,24,36,24,6],
    [4,16,24,16,4],
    [1,4,6,4,1]
])

t0 = time()
out = conv_nested(img, kernel)
t1 = time()
t_normal = t1 - t0

# Plot original image
plt.subplot(1,2,1)
plt.imshow(img)
plt.title('Original')
plt.axis('off')

# Plot convolved image
plt.subplot(1,2,2)
plt.imshow(out)
plt.title('Blurred')
plt.axis('off')

plt.show()
```



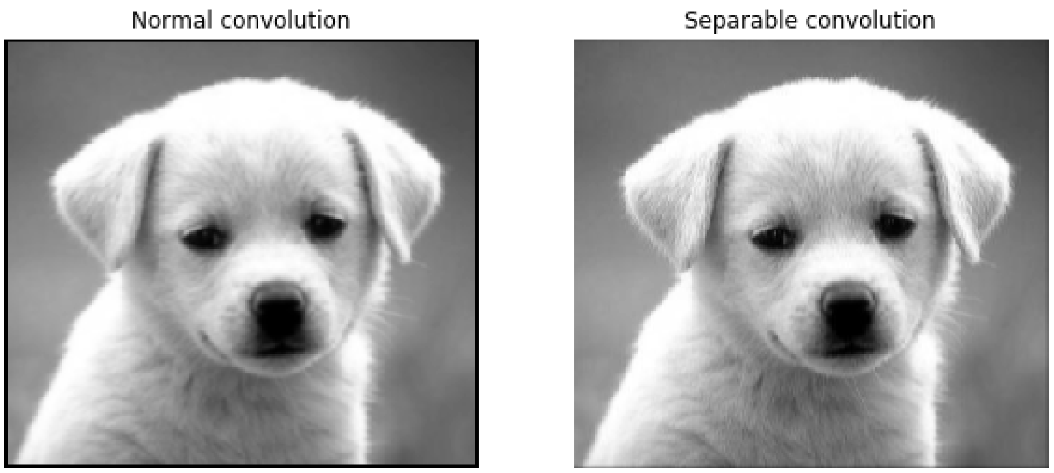
In the below code cell, define the two 1D arrays (`k1` and `k2`) whose product is equal to the Gaussian kernel.

```
In [25]: import math
         # The kernel can be written as outer product of two 1D filters
         k1 = None  # shape (5, 1)
         k2 = None  # shape (1, 5)

         ### YOUR CODE HERE
         u,s,v=np.linalg.svd(kernel)
         sqrt_value=math.sqrt(s[0])
         k1=u[:,0]*sqrt_value
         k2=v[:,0]*sqrt_value

         k1=np.array([k1]).T
         k2=np.array([k2])
         print(k1)
         print(k2)
         ### END YOUR CODE

         # Check if kernel is product of k1 and k2
         if not  np.all(k1 * k2 == kernel):
             print('k1 * k2 is not equal to kernel')

         assert k1.shape == (5, 1), "k1 should have shape (5, 1)"
         assert k2.shape == (1, 5), "k2 should have shape (1, 5)"
```

```
[[-1.]
 [-4.]
 [-6.]
 [-4.]
 [-1.]]
[[-1.         -0.68935454 -8.27797018  0.          -0.        ]]
k1 * k2 is not equal to kernel
```

We now apply the two versions of convolution to the same image, and compare their running time. Note that the outputs of the two convolutions must be the same.

```
In [271]: # Perform two convolutions using k1 and k2
          t0 = time()
          out_separable = conv_faster(img, k1)
          out_separable = conv_faster(out_separable, k2)
          t1 = time()
          t_separable = t1 - t0

          # Plot normal convolution image
          plt.subplot(1,2,1)
          plt.imshow(out)
          plt.title('Normal convolution')
          plt.axis('off')

          # Plot separable convolution image
          plt.subplot(1,2,2)
          plt.imshow(out_separable)
          plt.title('Separable convolution')
          plt.axis('off')

          plt.show()

          print("Normal convolution: took %f seconds." % (t_normal))
          print("Separable convolution: took %f seconds." % (t_separable))
```



```
Normal convolution: took 2.928843 seconds.
Separable convolution: took 0.011029 seconds.
```

```
In [ ]: # Check if the two outputs are equal
        assert np.max(out_separable - out) < 1e-10
```