$$-\frac{i}{2}\sum_{m<n} J_{mn}\left[\hat{\sigma}^z_m\hat{\sigma}^z_n , \hat{\Delta}(\vec{\Theta},\vec{\Phi})\right]$$

$$= -\frac{i}{2}\sum_{m<n} J_{mn}\left[\hat{\sigma}^z_m\hat{\sigma}^z_n , \prod_{n'=1}^{N}\hat{\Delta}(\Theta_{n'},\Phi_{n'})\right]$$

$$\left[a, \prod_{n=1}^{N} b_n\right] = \left[a, \prod_{n=1}^{N-1} b_n\right] b_N + \prod_{n=1}^{N-1} b_n \left[a, b_N\right]$$

$$= \left[a, \prod_{n=1}^{N-2} b_n\right] b_{N-1} b_N + \prod_{n=1}^{N-2} b_n \left[a, b_{N-1}\right] b_N$$

$$+ \prod_{n=1}^{N-1} b_n \left[a, b_N\right]$$

$$= \left[a, \prod_{n=1}^{N-3} b_n\right] b_{N-2} b_{N-1} b_N + \prod_{n=1}^{N-3} b_n \left[a, b_{N-2}\right] b_{N-1} b_N$$

$$+ \prod_{n=1}^{N-2} b_n \left[a, b_{N-1}\right] b_N + \prod_{n=1}^{N-1} b_n \left[a, b_N\right]$$

$$= \left[a, b_1\right] \prod_{n=2}^{N} b_n + b_1\left[a, b_2\right] \prod_{n=3}^{N} b_n$$

$$+ b_1 b_2 \left[a, b_3\right] \prod_{n=4}^{N} b_n \quad \dots\,.$$

$$+ \prod_{n=1}^{N-1} b_n \left[a, b_N\right]$$

$$= \sum_{i=1}^{N}\left(\prod_{n=1}^{i-1} b_n\right)\left[a, b_i\right]\left(\prod_{n=i+1}^{N} b_n\right)$$

$$= -\frac{i}{2}\sum_{m<n} J_{mn} \sum_{l=1}^{N}\left(\prod_{n'=1}^{l-1}\hat{\Delta}_{n'}\right)\left[\hat{\sigma}^z_m\hat{\sigma}^z_n, \hat{\Delta}_l\right]\left(\prod_{n'=l+1}^{N}\hat{\Delta}_{n'}\right)$$

$$= -\frac{i}{2}\sum_{m<n} J_{mn} \sum_{l=1}^{N}\left(\prod_{n'=1}^{l-1}\hat{\Delta}_{n'}\right)\left(\hat{\sigma}^z_m\left[\hat{\sigma}^z_n, \hat{\Delta}_l\right] + \left[\hat{\sigma}^z_m, \hat{\Delta}_l\right]\hat{\sigma}^z_n\right)$$

$$\left(\prod_{n'=l+1}^{N}\hat{\Delta}_{n'}\right)$$

$$\hat\sigma^z_m[\hat\sigma^z_n, \hat\Delta_l] + [\hat\sigma^z_m, \hat\Delta_l]\hat\sigma^z_n$$

$$= \hat\sigma^z_m\left(\delta_{nl}\, i\sqrt{3}\{\cos\phi_l \sin\theta_l\, \hat\sigma^y_l + \sin\phi_l \sin\theta_l\, \hat\sigma^x_l\}\right)$$
$$+ \left(\delta_{ml}\, i\sqrt{3}\{\cos\phi_l \sin\theta_l\, \hat\sigma^y_l + \sin\phi_l \sin\theta_l\, \hat\sigma^x_l\}\right)\hat\sigma^z_n$$

$$= \frac{\sqrt3}{2}\sum_{m<n} J_{mn}\Big(\prod_{n'=1}^{n-1}\hat\Delta_{n'}\,\hat\sigma^z_m\{\cos\phi_n\sin\theta_n\,\hat\sigma^y_n$$
$$+\sin\phi_n\sin\theta_n\,\hat\sigma^x_n\}\prod_{n'=n+1}^{N}\hat\Delta_{n'}$$
$$+\prod_{n'=1}^{m-1}\hat\Delta_{n'}\{\cos\phi_m\sin\theta_m\,\hat\sigma^y_m$$
$$+\sin\phi_m\sin\theta_m\,\hat\sigma^x_m\}\,\hat\sigma^z_n\prod_{n'=m+1}^{N}\hat\Delta_{n'}\Big)$$

$$= \frac{\sqrt3}{2}\sum_{m<n} J_{mn}\Big(\{\cos\phi_n\sin\theta_n\,\hat\sigma^y_n+\sin\phi_n\sin\theta_n\,\hat\sigma^x_n\}$$
$$\prod_{n'=1}^{n-1}\hat\Delta_{n'}\,\hat\sigma^z_m\prod_{n'=n+1}^{N}\hat\Delta_{n'}$$
$$+\{\cos\phi_m\sin\theta_m\,\hat\sigma^y_m+\sin\phi_m\sin\theta_m\,\hat\sigma^x_m\}$$
$$\prod_{n'=1}^{m-1}\hat\Delta_{n'}\,\hat\sigma^z_m\prod_{n'=m+1}^{N}\hat\Delta_{n'}\Big)$$

contains $\hat\Delta_m$ because $m<n$

$$= -\frac{i}{2}\sum_{m<n} J_{mn}\Big([\hat\sigma^z_n,\hat\Delta_n]\boxed{\Big\{\prod_{n'=1}^{n-1}\hat\Delta_{n'}\Big\}}\,\hat\sigma^z_m\Big\{\prod_{n'=n+1}^{N}\hat\Delta_{n'}\Big\} \qquad 1$$
$$+[\hat\sigma^z_m,\hat\Delta_m]\Big\{\prod_{n'=1}^{m-1}\hat\Delta_{n'}\Big\}\,\hat\sigma^z_n\boxed{\Big\{\prod_{n'=m+1}^{N}\hat\Delta_{n'}\Big\}}\Big) \qquad 2$$

contains $\hat\Delta_n$ because $m<n$

1 calculate $\hat\Delta_m\hat\sigma^z_m$

2 calculate $\hat\sigma_n\hat\Delta_n$

$$[\hat{\sigma}_n^{z}, \hat{\Delta}_n] = -2i \frac{\partial}{\partial\phi_n}\hat{\Delta}_n$$

$$[\hat{\sigma}_m^{z}, \hat{\Delta}_m] = -2i \frac{\partial}{\partial\phi_m}\hat{\Delta}_m$$

$$\hat{\Delta}_m \hat{\sigma}_m^z = \left(-2\sqrt{3}\cos\theta_m + \frac{-4\csc\theta_m + 6\sin\theta_m}{\sqrt{3}}\frac{\partial}{\partial\theta_m} + 2i\frac{\partial}{\partial\phi_m} - \frac{4\cot\theta_m\csc\theta_m}{\sqrt{3}}\frac{\partial^2}{\partial\phi_m^2}\right)\hat{\Delta}_m$$

$$\hat{\sigma}_n^z \hat{\Delta}_n = \left(-2\sqrt{3}\cos\theta_n + \frac{-4\csc\theta_n + 6\sin\theta_n}{\sqrt{3}}\frac{\partial}{\partial\theta_n} - 2i\frac{\partial}{\partial\phi_n} - \frac{4\cot\theta_n\csc\theta_n}{\sqrt{3}}\frac{\partial^2}{\partial\phi_n^2}\right)\hat{\Delta}_n$$

$$\hat{\Delta}_m \hat{\sigma}_m^z [\hat{\sigma}_n^z, \hat{\Delta}_n] = \left(-2\sqrt{3}\cos\theta_m\frac{\partial}{\partial\phi_n} + \frac{-4\csc\theta_m + 6\sin\theta_m}{\sqrt{3}}\frac{\partial^2}{\partial\theta_m\partial\phi_n} \boxed{+2i\frac{\partial^2}{\partial\phi_n\partial\phi_m}} - \frac{4\cot\theta_m\csc\theta_m}{\sqrt{3}}\frac{\partial^3}{\partial\phi_m^2\partial\phi_n}\right)\left(-4i\,\hat{\Delta}_n\hat{\Delta}_m\right)$$

vanishes due to addition of $[\hat{\sigma}_m^z, \hat{\Delta}_m]\hat{\Delta}_n\hat{\sigma}_n^z$

$$= 4\sum_n \sum_{m\neq n} J_{mn}\left(+\sqrt{3}\cos\theta_m\frac{\partial}{\partial\phi_n} + \frac{2\csc\theta_m - 3\sin\theta_m}{\sqrt{3}}\frac{\partial^2}{\partial\theta_m\partial\phi_n} + \frac{2\cot\theta_m\csc\theta_m}{\sqrt{3}}\frac{\partial^3}{\partial\phi_m^2\partial\phi_n}\right)\hat{\Delta}(\vec{\theta},\vec{\phi})$$

$$[\hat{\sigma}_m^z \hat{\sigma}_n^z, \hat{\Delta}(\vec{\Theta}, \vec{\Phi})] = \hat{\sigma}_m^z \hat{\sigma}_n^z \hat{\Delta}(\vec{\Theta}, \vec{\Phi}) - \hat{\Delta}(\vec{\Theta}, \vec{\Phi}) \hat{\sigma}_m^z \hat{\sigma}_n^z$$

$$\hat{\sigma}_m^z \hat{\sigma}_n^z \prod_i \hat{\Delta}(\Theta_i, \Phi_i) = \hat{\sigma}_m^z \prod_i^{n-1} \hat{\Delta}(\Theta_i, \Phi_i) \hat{\sigma}_n^z \prod_{i=n}^{N} \hat{\Delta}(\Theta_i, \Phi_i)$$

$$\hat{\Delta}_i = \frac{1}{2}\left(1 \, \hat{\sigma}_i^0 + \sqrt{3} \cos\Phi_i \sin\Theta_i \, \hat{\sigma}_i^x - \sqrt{3} \sin\Phi_i \sin\Theta_i \, \hat{\sigma}_i^y - \sqrt{3} \cos\Theta_i \hat{\sigma}_i^z \right)$$

$[\hat{\sigma}_n^z, \hat{\sigma}_i^0] = 0$ $\qquad$ $[a, b] = c$

$[\hat{\sigma}_n^z, \hat{\sigma}_i^x] = \delta_{ni} \, 2i \, \hat{\sigma}_i^y$ $\qquad$ $ab - ba = c$

$[\hat{\sigma}_n^z, \hat{\sigma}_i^y] = -\delta_{ni} \, 2i \, \hat{\sigma}_i^x$ $\qquad$ $ab = ba + c$

$[\hat{\sigma}_n^z, \hat{\sigma}_i^z] = 0$

$$\begin{aligned} \hat{\sigma}_n^z \hat{\Delta}(\Theta_i, \Phi_i) = {} & \delta_{ni} \, i\sqrt{3} \cos\Phi_i \sin\Theta_i \, \hat{\sigma}_i^y \\ & + \delta_{ni} \, i\sqrt{3} \sin\Phi_i \sin\Theta_i \, \hat{\sigma}_i^x \\ & + \hat{\Delta}_i \hat{\sigma}_m^z \end{aligned}$$

$$\hat{\sigma}_m^z \prod_i^{m-1} \hat{\Delta}_i \prod_{i=m}^{N} \hat{\Delta}_i \left(i\sqrt{3} \{\cos\Phi_m \sin\Theta_n \cdot \hat{\sigma}_n^y + \sin\Phi_m \sin\Theta_n \hat{\sigma}_m^x\} \right.$$
$$\left. \hat{\sigma}_m^{1-z} \right)$$

$$= \prod_i^N \hat{\Delta}_i \left(i\sqrt{3} \{\cos\Phi_m \sin\Theta_m \hat{\sigma}_m^y + \sin\Phi_m \sin\Theta_m \hat{\sigma}_m^x\} + \hat{\sigma}_m^z \right)$$
$$\left(i\sqrt{3} \{\cos\Phi_n \sin\Theta_n \hat{\sigma}_n^y + \sin\Phi_n \sin\Theta_n \hat{\sigma}_n^x\} + \hat{\sigma}_m^z \right)$$