Hopf algebras and combinatorial comodules-bigebras

– Internship report –

Master of Computer Science

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Abstract

Hopf algebra is a special structure of bialgebra, which is simultaneously an algebra and a coalgebra. Hopf algebras has been ubiquitous in many fields of mathematics, including quantum field theory, number theory, representation theory, combinatorics... Furthermore, their application has extended into the realm of computer science, where Hopf algebraic structures enrich many combinatorial objects and machine learning models. This offers researchers more tools for establishing connections between disparate fields and gain deeper insights for algorithmic development. Recognizing the significance of them, this internship report presents fundamental principles of Hopf algebra and related terminology, before delving into details of some examples in combinatorics. They contain the well-known Hopf algebras of rooted forests, a more general version of directed acyclic graphs and bialgebras of finite topologies within species formalism.

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Contents

1 Introduction

Hopf algebras were first studied in the 1940s by topologists in algebraic topology. In the 1980s, it were intensively studied after some work of Alain Connes and Dirk Kreimer [2, 8] in order to explain the combinatorics of renormalization in the context of Quantum Field Theory. Not only that, Hopf algebras are ubiquitous in many fields of mathematics: number theory, algebraic geometry, locally compact group theory, combinatorics... There are many well written books about Hopf algebras, such that a very first one of M.E. Sweedler [11], a fundamental one of S. Dascalescu et al. [3], a quantum physics related one of Kassel [7]. Some applications of Hopf algebras to combinatorics are provided in the paper of Cartier [1] along with the relations between Hopf algebras and Lie groups.

The main scope of this internship report is presenting fundamental concepts of Hopf algebras, along with some construction of Hopf algebras on combinatorial objects. In details, Section 2 gives preliminaries of fundamental algebraic objects and Hopf algebras. Section 3 introduces an important combinatorial object named *species*, which is formulated by category theory. Besides, the algebra, coalgebra structures of species are also examined in this section. Section 4 presents a special structure named *comodule-bialgebras*, it contains two different bialgebras, where one of them is a comodule over the other. Three examples of this structure are discussed in Section 5:

- Rooted forests: this example bases on the notions from the paper of D. Manchon [10] and L. Foissy [5].
- Directed acyclic graphs: this example generalizes rooted forests to oriented Feynman graphs, and there are also many Hopf algebras are shown in the original paper of D. Manchon [9].
- Finite topologies: this example uses the species formalism from the paper of L. Foissy [6] and F. Fauvet [4].

From these combinatorial objects, we construct double bialgebras structures of them by introducing two procedures, one is similar to an extraction-contraction process, and the other behaves like a cutting process.

2 Algebras, bialgebras and Hopf algebras

This section provides some fundamental algebraic objects and a step-by-step construction of Hopf algebras based on the paper of D. Manchon [10].

2.1 Monoids and groups

Definition 2.1. *A **monoid** is a set M equipped with a product*

$$\begin{aligned} m : M \times M &\longrightarrow M \\ (x, y) &\longmapsto xy \end{aligned}$$

*such that it satisfies **associativity**:*

$$(xy)z = x(yz)$$

*for any $x, y, z \in M$ and has **unit element** $e \in M$ satisfying:*

$$ex = xe, \quad \forall x \in M.$$

In a monoid, the unit element is unique.

In extension to a group, a monoid requires all elements have their inverse element.

Definition 2.2. *A **group** is a monoid G together with a map*

$$\begin{aligned} \iota : G &\longrightarrow G \\ x &\longmapsto x^{-1} \end{aligned}$$

such that $x^{-1}x = xx^{-1} = e$ for any $x \in G$.

In a group, the inverse map is unique. More precisely, if x admits a left inverse y and a right inverse z, then x admits an inverse and $y = z = x^{-1}$.

2.2 Rings, fields and modules over a ring

Definition 2.3. *A **ring** is a triple $(R, +, \cdot)$ where:*

1. *$(R, +)$ is an abelian group, with unit element denoted by 0.*

2. $(R, \cdot)$ is a monoid, with unit element denoted by 1.

3. Multiplication is distributive with respect to addition for any $x, y, z \in R$:

$$x(y+z) = xy + xz \quad \textit{(left distributivity)},$$
$$(x+y)z = xz + yz \quad \textit{(right distributivity)}.$$

Definition 2.4. *A* ***field*** *is a ring $(R, +, \cdot)$ where $(R\backslash\{0\}, \cdot)$ is a group instead of a monoid.*

Definition 2.5. *A* ***module*** *over a commutative ring R is an abelian group $(M, +)$ together with a binary product*

$$\begin{aligned} m : R \times M &\longrightarrow M \\ (\lambda, x) &\longmapsto \lambda x \end{aligned}$$

such that

- $0x = 0_M$ *for any* $x \in M$,
- $1x = x$ *for any* $x \in M$,
- $\lambda(\mu x) = (\lambda\mu)x$ *for any* $\lambda, \mu \in R$ *and* $x \in M$,
- $\lambda(x - y) = \lambda x - \lambda y$ *for any* $\lambda \in R$ *and* $x, y \in M$.

2.3 Algebras and modules

Definition 2.6. *Let R be a commutative ring. An* ***associative R-algebra*** *(or* ***R-algebra****) is a ring $\mathcal{A}$ which is also R-module and satisfies the following scalar multiplication condition:*

$$\lambda(xy) = (\lambda x)y = x(\lambda y)$$

for any $\lambda \in R$ and $x, y \in \mathcal{A}$.

We denote the unit of $\mathcal{A}$ by $\mathbf{1}_{\mathcal{A}}$. The R-module addition is actually the ring addition of $\mathcal{A}$, and an R-algebra is nothing but a R-module equipped with a R-bilinear monoid structure of multiplication.

Proposition 2.7. *Let $\boldsymbol{k}$ be a field, and let $\mathcal{A}$ be a $\boldsymbol{k}$-algebra. Then*

- *$\mathcal{A}$ is a $\boldsymbol{k}$-vector space.*
- *The product of $\mathcal{A}$ defines a linear map $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ via $m(a \otimes b) := ab$.*
- *The map $u : \boldsymbol{k} \to \mathcal{A}$ defined by $u(\lambda) := \lambda \mathbf{1}_{\mathcal{A}}$ is linear.*
- *The associativity of the product and the unit property of $\mathbf{1}_{\mathcal{A}}$ are respectively equivalent to the commutativity of the two following diagrams:*

$$\begin{tikzcd}
\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \arrow[r, "m\otimes I"] \arrow[d, "I\otimes m"'] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "m"] \\
\mathcal{A} \otimes \mathcal{A} \arrow[r, "m"] & \mathcal{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\boldsymbol{k} \otimes \mathcal{A} \arrow[r, "u\otimes I"] \arrow[dr, "\sim"'] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "m"] & \mathcal{A} \otimes \boldsymbol{k} \arrow[l, "I\otimes u"'] \arrow[dl, "\sim"] \\
 & \mathcal{A} &
\end{tikzcd}$$

Example 2.8. Some examples:

- Any ring of matrices with coefficients in a commutative ring R forms an R-algebra, with the usual matrix addition and multiplication. In particular, $n \times n$ matrices with entries from a field $\boldsymbol{k}$ form an algebra $M_n(\boldsymbol{k})$ over $\boldsymbol{k}$.
- Any polynomial ring $R[x_1, ..., x_n]$ of n variables is an R-algebra.
- Group algebra: Let G be a multiplicative group and R be a commutative ring, an element of the group algebra $R[G]$ is a formal sum $\alpha = \sum_{g \in G} \alpha_g g$ with $\alpha_g \in R, \forall g \in G$. The addition and multiplication are defined by:

$$\alpha + \beta = \sum_{g \in G} \Big(\alpha_g + \beta_g \Big) g$$
$$\alpha\beta = \sum_{g \in G} \sum_{g' \in G} \Big(\alpha_g \beta_{g'} \Big) g g' = \sum_{h \in G} \Bigg(\sum_{k \in G} \Big(\alpha_{hk^{-1}} \beta_k \Big) \Bigg) h.$$

- Tensor algebra: Let V be a $\boldsymbol{k}$-vector space. Denote by $V^{\otimes 0} = \boldsymbol{k}, V^{\otimes 1} = V, V^{\otimes n} = V \otimes V \otimes \cdots \otimes V$ (n times, for all $n \geq 0$) as tensor product of n copies of vector space V. Then we can define **tensor algebra** of V as:

$$T(V) := \bigoplus_{n \geq 0} V^{\otimes n}.$$

The multiplication of two elements $x = x_1 \otimes x_2 \otimes \cdots \otimes x_p \in V^{\otimes p}$ and $y = y_1 \otimes y_2 \otimes \cdots \otimes y_q \in V^{\otimes q}$ is defined as follows:

$$x \cdot y = x_1 \otimes \cdots \otimes x_p \otimes y_1 \otimes \cdots \otimes y_q \in V^{\otimes p+q}.$$

- Symmetric algebra: Let V be a $\mathbf{k}$-vector space and J be the two-sided ideal generated by $\{x \cdot y - y \cdot x,\ x, y \in V\}$. The **symmetric algebra** of V is defined as the quotient algebra $S(V) = T(V)\big/J$.

Definition 2.9. *Let $\boldsymbol{k}$ be a field, and let $\mathcal{A}$ be an $\boldsymbol{k}$-algebra. A* ***left module*** *on $\mathcal{A}$ is a $\boldsymbol{k}$-vector space M together with a linear map*

$$\begin{aligned} \alpha : \mathcal{A} \otimes M &\longrightarrow M \\ a \otimes x &\longmapsto ax \end{aligned}$$

such that the two following diagrams commute:

$$\begin{tikzcd} \mathcal{A} \otimes \mathcal{A} \otimes M \arrow[r, "I\otimes\alpha"] \arrow[d, "m_{\mathcal{A}}\otimes I"'] & \mathcal{A} \otimes M \arrow[d, "\alpha"] \\ \mathcal{A} \otimes M \arrow[r, "\alpha"] & M \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{k} \otimes M \arrow[r, "u\otimes I"] \arrow[dr, "\sim"'] & \mathcal{A} \otimes M \arrow[d, "\alpha"] \\ & M \end{tikzcd}$$

2.4 Coalgebras and comodules

Definition 2.10. *Let $\boldsymbol{k}$ be a field. A* ***k-coalgebra*** *is a $\boldsymbol{k}$-vector space $\mathcal{C}$ together with two linear maps which are a* ***co-associative*** *comultiplication $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and* ***counit*** *$\varepsilon : \mathcal{C} \to \boldsymbol{k}$, such that two following diagrams commute:*

$$\begin{tikzcd} \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} & \mathcal{C} \otimes \mathcal{C} \arrow[l, "\Delta\otimes I"'] \\ \mathcal{C} \otimes \mathcal{C} \arrow[u, "I\otimes\Delta"] & \mathcal{C} \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{k} \otimes \mathcal{C} & \mathcal{C} \otimes \mathcal{C} \arrow[l, "\varepsilon\otimes I"'] \arrow[r, "I\otimes\varepsilon"] & \mathcal{C} \otimes \boldsymbol{k} \\ & \mathcal{C} \arrow[ul, "\sim"] \arrow[u, "\Delta"] \arrow[ur, "\sim"'] & \end{tikzcd}$$

Example 2.11. Some examples of coalgebras:

1. Coalgebra of a set: Let E be a non-empty set, the $\mathbf{k}$-vector space $\mathbf{k}[E]$ can be endowed with a $\mathbf{k}$-coalgebra with comultiplication Δ and counit ε defined by $\Delta(a) = a \otimes a, \varepsilon(a) = 1$ for any $a \in E$.

2. Divided power coalgebra: Let H be a $\mathbf{k}$-vector space with basis $\{c_m | m \in \mathbb{N}\}$. Then H is a divided power coalgebra with comultiplication Δ and counit ε defined by:

$$\Delta(c_m) = \sum c_i \otimes c_{m-i}, \quad \varepsilon(c_m) = \delta_{0,m}$$

for any $m \in \mathbb{N}$. Let $c_m = \frac{X^m}{m!}$, H becomes a coalgebra over a polynomial ring in one indeterminate X with comultiplication:

$$\Delta\left(\frac{X^m}{m!}\right) = \sum \frac{X^i}{i!} \otimes \frac{X^{m-i}}{(m-i)!},$$

which can be rewritten with binomial coefficients:

$$\Delta(X^n) = \sum_{k=0}^{n} \binom{n}{k} X^k \otimes X^{n-k}.$$

3. Tensor coalgebra: The tensor algebra $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ can also be a coalgebra by equipping it with the following deconcatenation comultiplication:

$$\Delta(v_1 \dots v_n) := \sum_{p=0}^{n} v_1 \dots v_p \otimes v_{p+1} \dots v_n$$

and the counit $\varepsilon(1) = 1,\ \varepsilon_{|V} = 0$.

Definition 2.12. *Let* $\boldsymbol{k}$ *be a field. A* ***left comodule*** *on the c-unital k-coalgebra* C *is a k-vector space* M *together with a* ***coaction map*** $\Phi : M \to \mathcal{C} \otimes M$ *such that two following diagrams commute:*

$$\begin{tikzcd} \mathcal{C} \otimes \mathcal{C} \otimes M & \mathcal{C} \otimes M \arrow[l, "\Delta \otimes I"'] \\ \mathcal{C} \otimes M \arrow[u, "I \otimes \Phi"] & M \arrow[l, "\Phi"] \arrow[u, "\Phi"'] \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{k} \otimes M & \mathcal{C} \otimes M \arrow[l, "\varepsilon \otimes I"'] \\ & M \arrow[ul, "\sim"] \arrow[u, "\Phi"'] \end{tikzcd}$$

2.5 Bialgebras and Hopf algebras

Definition 2.13. *A* ***bialgebra*** *is a vector space* $\mathcal{H}$ *endowed with a structure of algebra* $(\mathcal{H}, m, u)$ *and a structure of coalgebra* $(\mathcal{H}, \Delta, \varepsilon)$ *which are compatible. The compatibility requirement is that* Δ *is an algebra morphism (or equivalently that* m

is a coalgebra morphism), ε is an algebra morphism and u is a coalgebra morphism. It is expressed by the commutativity of the three following diagrams:

$$\begin{tikzcd}
\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H} \arrow[rr, "\tau_{23}"] & & \mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H} \arrow[d, "m\otimes m"] \\
\mathcal{H}\otimes\mathcal{H} \arrow[u, "\Delta\otimes\Delta"] \arrow[r, "m"'] & \mathcal{H} \arrow[r, "\Delta"'] & \mathcal{H}\otimes\mathcal{H}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{H}\otimes\mathcal{H} \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "m"'] & \boldsymbol{k}\otimes\boldsymbol{k} \arrow[d, "\sim"] \\
\mathcal{H} \arrow[r, "\varepsilon"'] & \boldsymbol{k}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{H}\otimes\mathcal{H} & \boldsymbol{k}\otimes\boldsymbol{k} \arrow[l, "u\otimes u"'] \\
\mathcal{H} \arrow[u, "\Delta"] & \boldsymbol{k} \arrow[l, "u"] \arrow[u, "\sim"']
\end{tikzcd}$$

Example 2.14. Some examples:

1. Group algebra: We can equip a coalgebra structure for group algebra in order to make it a bialgebra by the following comultiplication and counit:

$$\Delta(g) = g\otimes g, \quad \varepsilon(g) = 1, \quad \forall g\in G.$$

In details, Δ and ε transform formal sum $\alpha = \sum_{g\in G}\alpha_g g$ as follows:

$$\Delta\Big(\sum_{g\in G}\alpha_g g\Big) = \sum_{g\in G}\alpha_g(g\otimes g),$$
$$\varepsilon(\sum_{g\in G}\alpha_g g) = \sum_{g\in G}\alpha_g.$$

2. Tensor algebra: Although tensor algebra $T(V) = \bigoplus_{n\geq 0} V^{\otimes n}$ forms an algebra with the concatenation multiplication, and a coalgebra with the deconcatenation comultiplication, we can not combine them in order to make a bialgebra, since required bialgebra diagrams don't commute.
However, if we define the coproduct as the unique algebra morphism from $T(V)\longrightarrow T(V)\otimes T(V)$ such that:

$$\Delta(1) = 1\otimes 1, \qquad \Delta(x) = x\otimes 1 + 1\otimes x, \;\; x\in V,$$

and then homomorphically extend it to the whole algebra, we will obtain a tensor bialgebra. For example, coproduct on $x \cdot y \in V^{\otimes 2}$ is as follows:

$$\begin{aligned}\Delta(x \cdot y) &= (x \otimes 1 + 1 \otimes x) \cdot (y \otimes 1 + 1 \otimes y) \\ &= x \cdot y \otimes 1 + x \otimes y + y \otimes x + 1 \otimes x \cdot y.\end{aligned}$$

Definition 2.15. *A **Hopf algebra** is a bialgebra $\mathcal{H}$ together with a linear map $S : \mathcal{H} \to \mathcal{H}$ called the **antipode**, such that the following diagram commutes:*

$$\begin{tikzcd}
 & \mathcal{H} \otimes \mathcal{H} \arrow[rr, "S \otimes I"] & & \mathcal{H} \otimes \mathcal{H} \arrow[dr, "m"] & \\
\mathcal{H} \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & \boldsymbol{k} \arrow[rr, "u"] & & \mathcal{H} \\
 & \mathcal{H} \otimes \mathcal{H} \arrow[rr, "I \otimes S"] & & \mathcal{H} \otimes \mathcal{H} \arrow[ur, "m"'] &
\end{tikzcd}$$

Example 2.16. Some examples:

1. Group algebra: Examining the group algebra $\boldsymbol{k}[G]$ where $\boldsymbol{k}$ is a field. In order to make this bialgebra a Hopf algebra, we need to define its antipode map. Thanks to existence of inverse elements of group G, the antipode $S : \boldsymbol{k}[G] \to \boldsymbol{k}[G]$ can be directly defined as $S(g) = g^{-1}$ for any $g \in G$. The commutative diagram is satisfied by:

$$\begin{aligned} u \circ \varepsilon\Big(\sum_{g \in G} \alpha_g g\Big) &= u\Big(\sum_{g \in G} \alpha_g\Big) = \sum_{g \in G} \alpha_g \mathbf{1}_G, \\ m \circ (S \otimes I) \circ \Delta\Big(\sum_{g \in G} \alpha_g g\Big) &= m \circ (S \otimes I)\Big(\sum_{g \in G} \alpha_g (g \otimes g)\Big) \\ &= m\Big(\sum_{g \in G} \alpha_g (S(g) \otimes g)\Big) \\ &= \sum_{g \in G} \alpha_g g^{-1} g = \sum_{g \in G} \alpha_g \mathbf{1}_G. \end{aligned}$$

$m \circ (I \otimes S) \circ \Delta$ is similar.

2. Tensor algebra: As described, the tensor algebra $T(V) = \bigoplus_{n\geq 0} V^{\otimes n}$ has a bialgebra structure, therefore we need an appropriate antipode in order to make it a Hopf algebra. Consider the following anti-automorphism on $T(V)$:

$$S(x_1 \otimes \cdots \otimes x_n) = (-1)^n x_n \otimes \cdots \otimes x_1.$$

For $x \in V$ we have $S(x) = -x$, hence $S * I(x) = I * S(x) = 0 = u \circ \varepsilon(x)$. Then we can extend this property for higher-degree elements by induction, if $x, y \in T(V)$ satisfy the property, so do $x \cdot y$.

3 The species formalism

In this part, we present an important combinatorial object named *species* by using category theory, along with formulating a bialgebras structure over species.

3.1 Species

Definition 3.1. *Let* **Set** *be the category of finite sets with bijections as morphisms and* **Vect** *be the category of vector space over a field* $\boldsymbol{k}$ *with linear maps as morphisms. A* ***species*** $\mathcal{P}$ *is a functor from* **Set** *to* **Vect**.

As a functor, species $\mathcal{P}$ satisfies the following properties:

- For any finite set $A \in \mathbf{Set}$, $\mathcal{P}[A]$ is a vector space.
- For any bijection $\sigma : A \to B$ between two finite sets, $\mathcal{P}[\sigma] : \mathcal{P}[A] \to \mathcal{P}[B]$ is a linear map.
- Identity preserving: for any finite set A, $\mathcal{P}[\mathbf{1}_A] = \mathbf{1}_{\mathcal{P}[A]}$.
- Composition preserving: for any finite sets A, B, C and bijections $\sigma : A \to B$ and $\tau : B \to C$,
$$\mathcal{P}[\tau \circ \sigma] = \mathcal{P}[\tau] \circ \mathcal{P}[\sigma].$$

Furthermore, we can define morphisms between species by natural transformation on the functor category of species. A morphism of two species $f : \mathcal{P} \longrightarrow \mathcal{Q}$ assigns each finite set A a linear map $f[A] : \mathcal{P}[A] \longrightarrow \mathcal{Q}[A]$, such that for any bijection between two finite sets $\sigma : A \longrightarrow B$, the following diagram commutes

$$\begin{tikzcd}
\mathcal{P}[A] \arrow[r, "\mathcal{P}[\sigma]"] \arrow[d, "f[A]"'] & \mathcal{P}[B] \arrow[d, "f[B]"] \\
\mathcal{Q}[A] \arrow[r, "\mathcal{Q}[\sigma]"'] & \mathcal{Q}[B]
\end{tikzcd}$$

Example 3.2. Some combinatorial species:

1. Let $\boldsymbol{k}$ be a field. The functor **Com** is defined as follows:
   - For any finite set A, $\mathbf{Com}[A] = \boldsymbol{k}$. We denote $*_A$ the unit in $\mathbf{Com}[A]$.
   - For any bijection $\sigma : A \to B$, $\mathbf{Com}[\sigma] = \mathbf{1}_{\boldsymbol{k}}$.
2. Let A be any finite set, we denote by $\mathbf{Comp}[A]$ the set of set compositions of A, that is a set of finite disjoint union sequences $(A_1, ..., A_k)$. Let $\mathcal{C}omp[A]$ be the space generated by $\mathbf{Comp}[A]$. If σ is a bijection:

$$\mathcal{C}omp[\sigma] : \begin{cases} \mathcal{C}omp[A] & \longrightarrow \mathcal{C}omp[B] \\ (A_1, ..., A_k) \in \mathbf{Comp}[A] & \longrightarrow (\sigma(A_1), ..., \sigma(A_k)) \in \mathbf{Comp}[B]. \end{cases}$$

3. Species of graphs $\mathcal{G}r$ is defined by:
   - For any finite set A, $\mathcal{G}r[A]$ is the vector space generated by graphs G whose vertex sets $V(G)$ are equal to A.
   - For any bijection $\sigma : A \longrightarrow B$, $\mathcal{G}r[\sigma]$ sends the graph $G = (A, E(G))$ to the graph $\mathcal{G}r[\sigma](G) = G' = (B, E'(G))$ with:

$$E(G') = \Big\{ \{\sigma(a), \sigma(b)\}, \{a, b\} \in E(G) \Big\}.$$

   For example, if a, b are two elements of a set, we have:

$$\mathcal{G}r\big[\{a, b\}\big] = \mathrm{Vect}(\bullet_a \ \bullet_b, \begin{smallmatrix}\bullet_b \\ | \\ \bullet_a\end{smallmatrix}).$$

4. Species of posets $\mathcal{P}os$ is defined as:

- For any finite set A, $\mathcal{P}os[A]$ is the vector space generated by posets P whose vertex set is A, which means $P = (A, \leq_P)$ with $\leq_P$ being a partial order on A.
- For any bijection $\sigma : A \longrightarrow B$, $\mathcal{P}os[\sigma]$ sends each poset $P = (A, \leq_P)$ to poset $\mathcal{P}os[\sigma](P) = (B, \leq'_P)$, such that for any $a, b \in B$:

$$a \leq'_P b \quad \Longleftrightarrow \quad \sigma^{-1}(a) \leq \sigma^{-1}(b).$$

For example, if a, b are two elements of a set, we have:

$$\mathcal{P}os\big[\{a, b\}\big] = \mathrm{Vect}(\bullet_a \ \bullet_b, \begin{smallmatrix}\bullet_b\\ |\\ \bullet_a\end{smallmatrix}, \begin{smallmatrix}\bullet_a\\ |\\ \bullet_b\end{smallmatrix}).$$

Recall that a partition of a set A is a set of non-empty subsets of A, such that each element of A is in exactly one of these subsets. Then, we define the composition of two species $\mathcal{P}, \mathcal{Q}$ by:

$$\mathcal{P} \circ \mathcal{Q}[A] = \bigoplus_{I \text{ partition of } A} \mathcal{P}[I] \otimes \left(\bigotimes_{X \in I} \mathcal{Q}[X]\right).$$

A special case $\mathcal{Q} = \mathbf{Com}$ makes $\mathcal{P} \circ \mathbf{Com}[A] = \bigoplus_{I \text{ partition of } A} \mathcal{P}[I]$. For example, we denote $\mathcal{G}r' = \mathcal{G}r \circ \mathbf{Com}$, for any finite set A, $\mathcal{G}r'[A]$ is the vector space spanned by graphs having a partition of A as set of vertices. In details, if $A = \{a, b\}$:

$$\begin{aligned}
\mathcal{G}r'[A] &= \mathcal{G}r \circ \mathbf{Com}\big[\{a, b\}\big] \\
&= \bigoplus_{I \text{ partition of } \{a,b\}} \mathcal{G}r[I] \otimes \left(\bigotimes_{X \in I} \mathbf{Com}[X]\right) \\
&= \mathcal{G}r\Big[\big\{(a), (b)\big\}\Big] \otimes \Big(\mathbf{Com}\big[\{(a)\}\big] \cdot \mathbf{Com}\big[\{(b)\}\big]\Big) \\
&\quad \oplus \mathcal{G}r\Big[\big\{(a, b)\big\}\Big] \otimes \mathbf{Com}\big[\{(a, b)\}\big] \\
&= \mathrm{Vect}\big(\bullet_{(a)}\bullet_{(b)}, \begin{smallmatrix}\bullet_{(b)}\\ |\\ \bullet_{(a)}\end{smallmatrix}\big) \oplus \mathrm{Vect}\big(\bullet_{(a,\, b)}\big) \\
&= \mathrm{Vect}\big(\bullet_{(a)}\bullet_{(b)}, \begin{smallmatrix}\bullet_{(b)}\\ |\\ \bullet_{(a)}\end{smallmatrix}, \bullet_{(a,\, b)}\big).
\end{aligned}$$

Another example is the species $\mathcal{T}op$ of finite topologies defined as follows:

- For any finite set A, $\mathcal{T}op[A]$ is the vector space spanned by the set of topologies on A, which are subsets T of the power set of A with properties:

  - $\varnothing \in T, A \in T$.
  - If $X, Y \in T$, then $X \cap Y, X \cup Y \in T$.

- For any bijection $\sigma : A \longrightarrow B$ and T is a topology on A, then:

$$\mathcal{T}op[\sigma](T) = \{\sigma(X), X \in T\}$$

  is a topology on B.

A quasi-poset is a pair $(A, \leq_P)$ where A is a set and $\leq_P$ is a reflexive, transitive binary relation on A, or called by quasi-order for short. We have a natural bijection between finite topologies and quasi-posets on A. This bijection associates to a quasi-order $(A, \leq)$ the set of open sets of A which makes a topology, such that a set $X \subseteq A$ is $\leq$-open if:

$$\forall a, b \in A : \quad a \leq b, a \in X \Longrightarrow b \in X.$$

If $T = (A, \leq_T)$ is a quasi-poset, we define an equivalence on A by:

$$a \sim_T b \quad \text{if} \quad a \leq_T b \text{ and } b \leq_T a.$$

Then, the quotient space $A/\sim_T$ is endowed with a partial order $\overline{\leq}_T$:

$$X \overline{\leq}_T Y \quad \text{if} \quad \forall a \in X, \forall b \in Y : a \leq_T b.$$

with the antisymmetric property of $\leq_T$ in quotient space. Therefore, $(A/\sim_T, \overline{\leq}_T)$ is a poset, the species $\mathcal{T}op$ and $\mathcal{P}os \circ \mathbf{Com}$ are isomorphic. For example, if $A = \{a, b\}$, we represent finite topologies on A by the Hasse graph of $\overline{\leq}_T$, the vertices are equivalence classes of $\sim_T$:

$$\begin{aligned}
\mathcal{T}op[A] &= \mathrm{Vect}\Big(\bullet_{(a)}\bullet_{(b)}, \begin{smallmatrix}\bullet^{(b)}\\ |\\ \bullet_{(a)}\end{smallmatrix}, \begin{smallmatrix}\bullet^{(a)}\\ |\\ \bullet_{(b)}\end{smallmatrix}, \bullet_{(a,\,b)}\Big),\\
\mathcal{P}os \circ \mathbf{Com}[A] &= \bigoplus_{I \text{ partition of } \{a,b\}} \mathcal{P}os[I]\\
&= \mathcal{P}os\Big[\big\{(a), (b)\big\}\Big] \oplus \mathcal{P}os\Big[\big\{(a, b)\big\}\Big]\\
&= \mathrm{Vect}\Big(\bullet_{(a)}\bullet_{(b)}, \begin{smallmatrix}\bullet^{(b)}\\ |\\ \bullet_{(a)}\end{smallmatrix}, \begin{smallmatrix}\bullet^{(a)}\\ |\\ \bullet_{(b)}\end{smallmatrix}\Big) \oplus \mathrm{Vect}\Big(\bullet_{(a,\,b)}\Big)\\
&= \mathrm{Vect}\Big(\bullet_{(a)}\bullet_{(b)}, \begin{smallmatrix}\bullet^{(b)}\\ |\\ \bullet_{(a)}\end{smallmatrix}, \begin{smallmatrix}\bullet^{(a)}\\ |\\ \bullet_{(b)}\end{smallmatrix}, \bullet_{(a,\,b)}\Big).
\end{aligned}$$

3.2 Algebras in the category of species

First, we define the Cauchy tensor product of two species $\mathcal{P} \otimes \mathcal{Q}$ as:

- For any finite set:
$$\mathcal{P} \otimes \mathcal{Q}[A] = \bigoplus_{I \subseteq A} \mathcal{P}[I] \otimes \mathcal{Q}[A \backslash I].$$

- For any bijection $\sigma : A \to B$:

$$\mathcal{P} \otimes \mathcal{Q}[\sigma] : \begin{cases} \mathcal{P} \otimes \mathcal{Q}[A] & \longrightarrow \mathcal{P} \otimes \mathcal{Q}[B] \\ x \otimes y \in \mathcal{P}[I] \otimes \mathcal{Q}[A \backslash I] & \longrightarrow \mathcal{P}[\sigma_{|I}](x) \otimes \mathcal{Q}[\sigma_{|A \backslash I}](y). \end{cases}$$

For any species $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$, we have associativity $\mathcal{P} \otimes (\mathcal{Q} \otimes \mathcal{R}) = (\mathcal{P} \otimes \mathcal{Q}) \otimes \mathcal{R}$.

The unit is the species $\mathcal{I}$:

$$\mathcal{I}[A] = \begin{cases} \mathbb{K}, & \text{if } A = \varnothing, \\ (0), & \text{otherwise.} \end{cases}$$

which satisfies $\mathcal{P} \otimes \mathcal{I} = \mathcal{I} \otimes \mathcal{P}$ for any species $\mathcal{P}$.

Naturally, the product of two morphisms of species $f : \mathcal{P} \longrightarrow \mathcal{P}'$, $g : \mathcal{Q} \longrightarrow \mathcal{Q}'$ is $f \otimes g : \mathcal{P} \otimes \mathcal{Q} \longrightarrow \mathcal{P}' \otimes \mathcal{Q}'$ and also a morphism of species:

$$f \otimes g[A] : \begin{cases} \mathcal{P} \otimes \mathcal{Q}[A] & \longrightarrow \mathcal{P}' \otimes \mathcal{Q}'[A] \\ x \otimes y \in \mathcal{P}[I] \otimes \mathcal{Q}[A \backslash I] & \longrightarrow f[I](x) \otimes g[A \backslash I](y). \end{cases}$$

Moreover, a species morphism c between $\mathcal{P} \otimes \mathcal{Q}$ and $\mathcal{Q} \otimes \mathcal{P}$ (the flip) can be defined as:

$$c_{\mathcal{P},\mathcal{Q}}[A] : \begin{cases} \mathcal{P} \otimes \mathcal{Q}[A] & \longrightarrow \mathcal{Q} \otimes \mathcal{P}[A] \\ x \otimes y \in \mathcal{P}[I] \otimes \mathcal{Q}[A \backslash I] & \longrightarrow y \otimes x \in \mathcal{Q}[A \backslash I] \otimes \mathcal{Q}[I]. \end{cases}$$

Definition 3.3. *In the category of species, a* ***twisted algebra*** *is a structure* $(\mathcal{P}, m, \iota)$ *where* $\mathcal{P}$ *is a species,* $m : \mathcal{P} \otimes \mathcal{P} \longrightarrow \mathcal{P}$ *is the product,* $\iota : \mathcal{I} \longrightarrow \mathcal{P}$ *is the unit of the algebra, they are morphisms of species such that the following diagrams commute:*

$$\begin{tikzcd} \mathcal{P} \otimes \mathcal{P} \otimes \mathcal{P} \arrow[r, "m \otimes I"] \arrow[d, "I \otimes m"'] & \mathcal{P} \otimes \mathcal{P} \arrow[d, "m"] \\ \mathcal{P} \otimes \mathcal{P} \arrow[r, "m"] & \mathcal{P} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{I} \otimes \mathcal{P} \arrow[r, "\iota \otimes I"] \arrow[dr, "\sim"'] & \mathcal{P} \otimes \mathcal{P} \arrow[d, "m"] & \mathcal{P} \otimes \mathcal{I} \arrow[l, "I \otimes \iota"'] \arrow[dl, "\sim"] \\ & \mathcal{P} & \end{tikzcd}$$

We say that (P, m, ι) *is commutative if* $m \circ c_{\mathcal{P},\mathcal{P}} = m$.

For any bijections $\sigma : A \longrightarrow A'$ and $\tau : B \longrightarrow B'$ between finite sets, since m is a species morphism, $m_{A,B} : \mathcal{P}[A] \otimes \mathcal{P}[B] \longrightarrow \mathcal{P}[A \sqcup B]$ is actually a natural transformation between functors and makes the following diagram commute:

$$\begin{tikzcd} \mathcal{P}[A] \otimes \mathcal{P}[B] \arrow[r, "\mathcal{P}(\sigma)\otimes\mathcal{P}(\tau)"] \arrow[d, "m_{A,B}"'] & \mathcal{P}[A'] \otimes \mathcal{P}[B'] \arrow[d, "m_{A',B'}"] \\ \mathcal{P}[A \sqcup B] \arrow[r, "\mathcal{P}(\sigma \sqcup \tau)"'] & \mathcal{P}[A' \sqcup B'] \end{tikzcd}$$

The commutativity condition $m \circ c_{P,P} = m$ means that for any finite sets A and B, $m_{A,B}, m_{B,A}$ returns values in $\mathcal{P}[A \sqcup B]$ and satisfies:

$$m_{A,B}(x \otimes y) = m_{B,A}(y \otimes x), \quad \forall x \in \mathcal{P}[A], y \in \mathcal{P}[B].$$

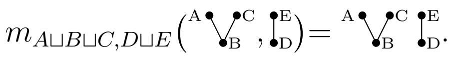
Example 3.4. Some examples of twisted algebras:

1. Species **Com** forms a twisted algebra, with multiplication induced by product over the field $\mathbb{K}$:
$$m_{A,B}(\lambda *_A \otimes \mu *_B) = \lambda\mu *_{A \sqcup B}$$
The unit $\iota : \mathcal{I}[\varnothing] \mapsto \mathbf{Com}[\varnothing]$ returns the identity value of $\mathbb{K}$.

2. Species $\mathcal{C}omp$ forms a twisted algebra by concatenation product on set compositions:
$$m_{A,B} : \mathcal{C}omp[A] \otimes \mathcal{C}omp[B] \longrightarrow \mathcal{C}omp[A \sqcup B]$$
$$(A_1, \ldots, A_k) \otimes (B_1, \ldots, B_l) \longmapsto (A_1, \ldots, A_k, B_1, \ldots, B_l)$$

The unit is the empty composition $\varnothing$. For example, let $A = \{x, y\}, B = \{u, v\}$ and set compositions $\big(\{x\}, \{y\}\big) \in \mathcal{C}omp[A]$, $\big(\{u, v\}\big) \in \mathcal{C}omp[B]$, we have the product $m\Big(\big(\{x\}, \{y\}\big), \big(\{u, v\}\big)\Big) = \big(\{x\}, \{y\}, \{u, v\}\big)$.

3. Species $\mathcal{G}r'$ is an algebra with the product given by disjoint union of graphs, the unit is the empty graph. For example, if A, B, C, D, E are finite sets, then:



4. The species $\mathcal{T}op$ is a twisted algebra with the disjoint union product.

3.3 Coalgebras, bialgebras in the category of species

Definition 3.5. *In the category of species, a* ***twisted coalgebra*** *is a structure* $(\mathcal{P}, \Delta, \varepsilon)$ *where* $\mathcal{P}$ *is a species,* $\Delta : \mathcal{P} \longrightarrow \mathcal{P} \otimes \mathcal{P}$ *is the coproduct constructed by* $\Delta_X : \mathcal{P}[X] \longmapsto (\mathcal{P} \otimes \mathcal{P})[X] = \bigoplus_{A \sqcup B = X} \mathcal{P}[A] \otimes \mathcal{P}[B]$*, the counit of the coalgebra is* $\varepsilon : \mathcal{P} \longrightarrow \mathcal{I}$*, they are morphisms of species such that the following diagrams commute:*

$$\begin{tikzcd}
\mathcal{P} \otimes \mathcal{P} \otimes \mathcal{P} & \mathcal{P} \otimes \mathcal{P} \arrow[l, "\Delta \otimes I"'] \\
\mathcal{P} \otimes \mathcal{P} \arrow[u, "I \otimes \Delta"] & \mathcal{P} \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{I} \otimes \mathcal{P} & \mathcal{P} \otimes \mathcal{P} \arrow[l, "\varepsilon \otimes I"'] \arrow[r, "I \otimes \varepsilon"] & \mathcal{P} \otimes \mathcal{I} \\
& \mathcal{P} \arrow[ul, "\sim"] \arrow[u, "\Delta"] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

We say that (P, Δ, ε) *is cocommutative if* $c_{\mathcal{P},\mathcal{P}} \circ \Delta = \Delta$*, and it is connected if* $\mathcal{P}[\varnothing]$ *is one-dimensional.*

Similarly, for any bijections $\sigma : A \longrightarrow A'$ and $\tau : B \longrightarrow B'$ between finite sets, since Δ is a species morphism, $\Delta_{A,B} : \mathcal{P}[A \sqcup B] \longrightarrow \mathcal{P}[A] \otimes \mathcal{P}[B]$ is actually a natural transformation between functors and makes the following diagram commute:

$$\begin{tikzcd}
\mathcal{P}[A \sqcup B] \arrow[r, "\mathcal{P}(\sigma \sqcup \tau)"] \arrow[d, "\Delta_{A,B}"'] & \mathcal{P}[A' \sqcup B'] \arrow[d, "\Delta_{A',B'}"] \\
\mathcal{P}[A] \otimes \mathcal{P}[B] \arrow[r, "\mathcal{P}(\sigma) \otimes \mathcal{P}(\tau)"'] & \mathcal{P}[A'] \otimes \mathcal{P}[B']
\end{tikzcd}$$

Example 3.6. Some examples of twisted coalgebras:

1. Species **Com** forms a twisted coalgebra, with comultiplication defined as follows:
$$\Delta_{A,B}(\lambda *_{A\sqcup B}) = \lambda(*_A \otimes *_B)$$
The counit $\varepsilon : \mathbf{Com}[\varnothing] \mapsto \mathcal{I}[\varnothing]$ returns the identity value of $\mathbb{K}$.

2. Species $\mathcal{C}omp$ forms a twisted coalgebra by comultiplication defined by set separation as follows:
$$\Delta_{A,B}(A_1,\dots,A_k) = \begin{cases} (A_1,\dots,A_p)\otimes(A_{p+1},\dots,A_k), \\ \qquad\qquad (\text{if } \exists p \text{ unique s.t. } A_1\sqcup\cdots\sqcup A_p = A), \\ 0, \quad \text{otherwise.} \end{cases}$$
The counit is given by $\varepsilon[\varnothing] = 1$.

Remark 3.7. *The Cauchy tensor product of species gives rise to the monoidal categories of twisted algebras and twisted coalgebras. That means if $(\mathcal{P}, m_{\mathcal{P}}, \iota_{\mathcal{P}})$ and $(\mathcal{Q}, m_{\mathcal{Q}}, \iota_{\mathcal{Q}})$ are two twisted algebras, then $\mathcal{P}\otimes\mathcal{Q}$ also has a structure of twisted algebra, with the multiplication $m_{\mathcal{P}\otimes\mathcal{Q}}$ defined as the following composition:*

$$\begin{tikzcd} \mathcal{P}\otimes\mathcal{Q}\otimes\mathcal{P}\otimes\mathcal{Q} \arrow[r, "I\otimes c_{\mathcal{Q},\mathcal{P}}\otimes I"] \arrow[dr, "m_{\mathcal{P}\otimes\mathcal{Q}}"'] & \mathcal{P}\otimes\mathcal{P}\otimes\mathcal{Q}\otimes\mathcal{Q} \arrow[d, "m_{\mathcal{P}}\otimes m_{\mathcal{Q}}"] \\ & \mathcal{P}\otimes\mathcal{Q} \end{tikzcd}$$

Similarly, if $(\mathcal{P}, \Delta_{\mathcal{P}}, \varepsilon_{\mathcal{P}})$ and $(\mathcal{Q}, \Delta_{\mathcal{Q}}, \varepsilon_{\mathcal{Q}})$ are two twisted coalgebras, then $\mathcal{P}\otimes\mathcal{Q}$ also has a structure of twisted coalgebra, with the comultiplication $\Delta_{\mathcal{P}\otimes\mathcal{Q}}$ defined as the following composition:

$$\begin{tikzcd} \mathcal{P}\otimes\mathcal{Q} \arrow[r, "\Delta_{\mathcal{P}}\otimes\Delta_{\mathcal{Q}}"] \arrow[dr, "\Delta_{\mathcal{P}\otimes\mathcal{Q}}"'] & \mathcal{P}\otimes\mathcal{P}\otimes\mathcal{Q}\otimes\mathcal{Q} \arrow[d, "I\otimes c_{\mathcal{P},\mathcal{Q}}\otimes I"] \\ & \mathcal{P}\otimes\mathcal{Q}\otimes\mathcal{P}\otimes\mathcal{Q} \end{tikzcd}$$

The unit and counit are defined naturally by tensor product.

Definition 3.8. *Let $\mathcal{P}$ be a species, both a twisted algebra $(\mathcal{P}, m, \iota)$ and a twisted coalgebra $(\mathcal{P}, \Delta, \varepsilon)$. The following conditions are equivalent:*

1. $\varepsilon : \mathcal{P} \longrightarrow \mathcal{I}$ *and* $\Delta : \mathcal{P} \longrightarrow \mathcal{P} \otimes \mathcal{P}$ *are algebra morphisms.*

2. $\iota : \mathcal{I} \longrightarrow \mathcal{P}$ *and* $m : \mathcal{P} \otimes \mathcal{P} \longrightarrow \mathcal{P}$ *are coalgebra morphisms.*

If this holds, we say that $(\mathcal{P}, m, \iota, \Delta, \varepsilon)$ *is a bialgebra in the category of species or a **twisted bialgebra**.*

For instance, the species **Com** with its algebra and coalgebra structures defined in previous examples forms a twisted bialgebra.

4 Comodule-bialgebras

In this part, we define the comodule-bialgebras structure based on the paper of D. Manchon [10]. Besides, species formalism of this structure is also presented in the paper of L. Foissy [6].

4.1 Comodule-bialgebras

Definition 4.1. *Let* $(\mathcal{A}, m_{\mathcal{A}}, u_{\mathcal{A}}), (\mathcal{B}, m_{\mathcal{B}}, u_{\mathcal{B}})$ *be two* $\boldsymbol{k}$*-algebras. The* $\boldsymbol{k}$*-linear map* $f : \mathcal{A} \longrightarrow \mathcal{B}$ *is a morphism of algebras if the following diagrams are commutative:*

$$\begin{tikzcd} \mathcal{A} \otimes \mathcal{A} \arrow[r, "f \otimes f"] \arrow[d, "m_{\mathcal{A}}"'] & \mathcal{B} \otimes \mathcal{B} \arrow[d, "m_{\mathcal{B}}"] \\ \mathcal{A} \arrow[r, "f"'] & \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[rr, "f"] & & \mathcal{B} \\ & \boldsymbol{k} \arrow[ul, "u_{\mathcal{A}}"] \arrow[ur, "u_{\mathcal{B}}"'] & \end{tikzcd}$$

Definition 4.2. *Let* $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}}), (\mathcal{D}, \Delta_{\mathcal{D}}, \varepsilon_{\mathcal{D}})$ *be two* $\boldsymbol{k}$*-coalgebras. The* $\boldsymbol{k}$*-linear map* $g : \mathcal{C} \longrightarrow \mathcal{D}$ *is a morphism of coalgebras if the following diagrams are commutative:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "g"] \arrow[d, "\Delta_{\mathcal{C}}"'] & D \arrow[d, "\Delta_{\mathcal{D}}"] \\ \mathcal{C} \otimes \mathcal{C} \arrow[r, "g \otimes g"'] & \mathcal{D} \otimes \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[rr, "g"] \arrow[dr, "\varepsilon_{\mathcal{C}}"'] & & \mathcal{D} \arrow[dl, "\varepsilon_{\mathcal{D}}"] \\ & \boldsymbol{k} & \end{tikzcd}$$

Definition 4.3. *Let $\mathcal{C}$ be a $\boldsymbol{k}$-coalgebra, and $(M,\rho),(N,\phi)$ two left $\mathcal{C}$-comodules. The $\boldsymbol{k}$-linear map $g : M \longrightarrow N$ is called a morphism of $\mathcal{C}$-comodules if the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\rho"'] & N \arrow[d, "\phi"] \\ \mathcal{C}\otimes M \arrow[r, "I\otimes g"'] & \mathcal{C}\otimes N \end{tikzcd}$$

Definition 4.4. *Let $\mathcal{B}$ be a unital counital bialgebra over a field $\boldsymbol{k}$. A **comodule-bialgebra** on $\mathcal{B}$ is a unital counital bialgebra in the category of $\mathcal{B}$-comodules.*

In particular, a comodule-bialgebra on $\mathcal{B}$ is a unital counital bialgebra $\mathcal{H}$ endowed with a linear map

$$\Phi : \mathcal{H} \longrightarrow \mathcal{B}\otimes\mathcal{H}$$

such that:

- Φ is a left coaction, i.e. the following diagrams commute:

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\Phi"] \arrow[d, "\Phi"'] & \mathcal{B}\otimes\mathcal{H} \arrow[d, "\Delta_{\mathcal{B}}\otimes I"] \\ \mathcal{B}\otimes\mathcal{H} \arrow[r, "I\otimes\Phi"] & \mathcal{B}\otimes\mathcal{B}\otimes\mathcal{H} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H} \arrow[r, "\Phi"] \arrow[dr, "\sim"'] & \mathcal{B}\otimes\mathcal{H} \arrow[d, "\varepsilon_{\mathcal{B}}\otimes I"] \\ & \boldsymbol{k}\otimes\mathcal{H} \end{tikzcd}$$

- The coproduct $\Delta_{\mathcal{H}}$ and the counit $\varepsilon_{\mathcal{H}}$ are morphisms of left $\mathcal{B}$-comodules, where the comodule structure on $\boldsymbol{k}$ is given by the unit map $u_{\mathcal{B}}$, and the comodule structure on $\mathcal{H}\otimes\mathcal{H}$ is given by $\widetilde{\Phi} = (m_{\mathcal{B}}\otimes I\otimes I)\circ\tau_{23}\circ(\Phi\otimes\Phi)$. This amounts to the commutativity of the two following diagrams:

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\Phi"] \arrow[d, "\Delta_{\mathcal{H}}"'] & \mathcal{B}\otimes\mathcal{H} \arrow[d, "I\otimes\Delta_{\mathcal{H}}"] \\ \mathcal{H}\otimes\mathcal{H} \arrow[r, dashed, "\widetilde{\Phi}"] \arrow[d, "\Phi\otimes\Phi"'] & \mathcal{B}\otimes\mathcal{H}\otimes\mathcal{H} \\ \mathcal{B}\otimes\mathcal{H}\otimes\mathcal{B}\otimes\mathcal{H} \arrow[r, "\tau_{23}"] & \mathcal{B}\otimes\mathcal{B}\otimes\mathcal{H}\otimes\mathcal{H} \arrow[u, "m_{\mathcal{B}}\otimes I\otimes I"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H} \arrow[r, "\Phi"] \arrow[d, "\varepsilon_{\mathcal{H}}"'] & \mathcal{B}\otimes\mathcal{H} \arrow[d, "I\otimes\varepsilon_{\mathcal{H}}"] \\ \boldsymbol{k} \arrow[r, "u_{\mathcal{B}}"'] & \mathcal{B} \end{tikzcd}$$

where τ_{23} stands for the flip of the two middle factors.

- $m_{\mathcal{H}}$ and $u_{\mathcal{H}}$ are morphisms of left $\mathcal{B}$-comodules. This means that Φ is a unital algebra morphism, which makes two following diagrams commutes:

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "\Phi"] & \mathcal{B}\otimes\mathcal{H} \\
\mathcal{H}\otimes\mathcal{H} \arrow[u, "m_{\mathcal{H}}"] \arrow[r, dashed, "\widetilde{\Phi}"] \arrow[d, "\Phi\otimes\Phi"'] & \mathcal{B}\otimes\mathcal{H}\otimes\mathcal{H} \arrow[u, "I\otimes m_{\mathcal{H}}"'] \\
\mathcal{B}\otimes\mathcal{H}\otimes\mathcal{B}\otimes\mathcal{H} \arrow[r, "\tau_{23}"] & \mathcal{B}\otimes\mathcal{B}\otimes\mathcal{H}\otimes\mathcal{H} \arrow[u, "m_{\mathcal{B}}\otimes I\otimes I"'] \arrow[uu, bend right, "m_{\mathcal{B}}\otimes m_{\mathcal{H}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{H} \arrow[r, "\Phi"] & \mathcal{B}\otimes\mathcal{H} \\
\boldsymbol{k} \arrow[u, "u_{\mathcal{H}}"] \arrow[r, "u_{\mathcal{B}}"'] & \mathcal{B} \arrow[u, "I\otimes u_{\mathcal{H}}"']
\end{tikzcd}$$

These two diagrams can be shortened as follows:

$$\begin{tikzcd}
\mathcal{H}\otimes\mathcal{H} \arrow[r, "\Phi\otimes\Phi"] \arrow[d, "m_{\mathcal{H}}"'] & \mathcal{B}\otimes\mathcal{H}\otimes\mathcal{B}\otimes\mathcal{H} \arrow[d, "(m_{\mathcal{B}}\otimes m_{\mathcal{H}})\circ\tau_{23}"] \\
\mathcal{H} \arrow[r, "\Phi"'] & \mathcal{B}\otimes\mathcal{H}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{H} \arrow[rr, "\Phi"] & & \mathcal{B}\otimes\mathcal{H} \\
& \boldsymbol{k} \arrow[ul, "u_{\mathcal{H}}"] \arrow[ur, "u_{\mathcal{B}}\otimes u_{\mathcal{H}}"'] &
\end{tikzcd}$$

which amounts to the fact that Φ is a unital algebra morphism.

The comodule-bialgebra $\mathcal{H}$ is a **comodule-Hopf algebra** if $\mathcal{H}$ is a Hopf algebra wit antipode S such that the following diagram commute:

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "\Phi"] \arrow[d, "S"'] & \mathcal{B}\otimes\mathcal{H} \arrow[d, "I\otimes S"] \\
\mathcal{H} \arrow[r, "\Phi"'] & \mathcal{B}\otimes\mathcal{H}
\end{tikzcd}$$

4.2 Double twisted bialgebras

Definition 4.5. *We define the* ***Hadamard product*** *of species, which is also known as element-wise product.*

1. *Let $\mathcal{P}, \mathcal{Q}$ be two species.*
   - *For any finite set A, we put $\mathcal{P} \boxtimes \mathcal{Q}[A] = \mathcal{P}[A] \otimes \mathcal{Q}[A]$.*
   - *For any bijection $\sigma : A \longrightarrow B$, we put $\mathcal{P} \boxtimes \mathcal{Q}[\sigma] = \mathcal{P}[\sigma] \otimes \mathcal{Q}[\sigma]$.*

2. *The product of two morphisms of species* $\phi_1 : \mathcal{P}_1 \longrightarrow \mathcal{Q}_1$, $\phi_2 : \mathcal{P}_2 \longrightarrow \mathcal{Q}_2$ *is* $\phi_1 \boxtimes \phi_2 : \mathcal{P}_1 \boxtimes \mathcal{Q}_1 \longrightarrow \mathcal{P}_2 \boxtimes \mathcal{Q}_2$ *and also a morphism of species:*

$$\phi_1 \boxtimes \phi_2[A] = \phi_1[A] \otimes \phi_2[A] : \mathcal{P}_1 \boxtimes \mathcal{P}_2[A] \longrightarrow \mathcal{Q}_1 \boxtimes \mathcal{Q}_2[A]$$

3. *A species morphism* τ *between* $\mathcal{P} \boxtimes \mathcal{Q}$ *and* $\mathcal{Q} \boxtimes \mathcal{P}$ *(the flip) can be defined as:*

$$\tau_{\mathcal{P},\mathcal{Q}}[A] : \begin{cases} \mathcal{P} \boxtimes \mathcal{Q}[A] & \longrightarrow \mathcal{Q} \boxtimes \mathcal{P}[A] \\ x \otimes y & \longrightarrow y \otimes x \end{cases}$$

The species **Com** is the identity for this tensor product, let $\mathcal{P}$ be any species:

$$\mathcal{P} \boxtimes \mathbf{Com} = \mathbf{Com} \boxtimes \mathcal{P} = \mathcal{P}.$$

Similar to Cauchy tensor product, we can define twisted bialgebras with Hadamard product.

Definition 4.6. *A twisted bialgebra of second kind is a family* $(\mathcal{P}, m, \delta)$ *where* $(\mathcal{P}, m)$ *is a twisted algebra and* $\delta : \mathcal{P} \longrightarrow \mathcal{P} \boxtimes \mathcal{P}$ *is a morphism of species such that the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{P} \boxtimes \mathcal{P} \boxtimes \mathcal{P} & \mathcal{P} \boxtimes \mathcal{P} \arrow[l, "\delta\boxtimes I"'] \\
\mathcal{P} \boxtimes \mathcal{P} \arrow[u, "I\boxtimes\delta"] & \mathcal{P} \arrow[l, "\delta"] \arrow[u, "\delta"']
\end{tikzcd}$$

There exists a morphism of species $\varepsilon' : \mathcal{P} \longrightarrow \mathbf{Com}$ *such that the following diagram commutes:*

$$\begin{tikzcd}
\mathbf{Com} \boxtimes \mathcal{P} & \mathcal{P} \boxtimes \mathcal{P} \arrow[l, "\varepsilon'\boxtimes I"'] \arrow[r, "I\otimes\varepsilon'"] & \mathcal{P} \boxtimes \mathbf{Com} \\
& \mathcal{P} \arrow[ul, "\sim"] \arrow[u, "\delta"] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

Moreover, δ is an algebra morphism from $\mathcal{P}$ to $\mathcal{P}\boxtimes\mathcal{P}$, that is to say, for any finite sets A, B, the following diagrams commute:

$$\begin{tikzcd}
\mathcal{P}[A]\otimes\mathcal{P}[B] \arrow[r, "\delta_A\otimes\delta_B"] \arrow[d, "m_{A,B}"'] & \mathcal{P}\boxtimes\mathcal{P}[A]\otimes\mathcal{P}\boxtimes\mathcal{P}[B] \arrow[r, "I\otimes c_{A,B}\otimes I"] & \mathcal{P}[A]\otimes\mathcal{P}[B]\otimes\mathcal{P}[A]\otimes\mathcal{P}[B] \arrow[d, "m_{A,B}\otimes m_{A,B}"] \\
\mathcal{P}[A\sqcup B] \arrow[rr, "\delta_{A\sqcup B}"'] & & \mathcal{P}[A\sqcup B]\otimes\mathcal{P}[A\sqcup B]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{P}[\varnothing] \arrow[rr, "\delta_\varnothing"] & & \mathcal{P}\boxtimes\mathcal{P}[\varnothing] \\
& \mathcal{I}[\varnothing] \arrow[ul, "\iota"] \arrow[ur, "\iota'"'] &
\end{tikzcd}$$

which can be written as:

$$\delta_{A\sqcup B}\circ m_{A,B} = (m_{A,B}\otimes m_{A,B})\circ(I\otimes c_{A,B}\otimes I)\circ(\delta_A\otimes\delta_B)$$

and $\delta_\varnothing(\mathbf{1}_\mathcal{P}) = \mathbf{1}_\mathcal{P}\otimes\mathbf{1}_\mathcal{P}$.

In other words, for any finite set A, there exists a coproduct $\delta_A : \mathcal{P}[A] \longrightarrow \mathcal{P}[A]\otimes\mathcal{P}[A]$, making $\mathcal{P}[A]$ a coalgebra of counit ε'. Also, if $\sigma : A \longrightarrow B$ is a bijection between finite sets, $\mathcal{P}[\sigma]$ is a coalgebra isomorphism from $(\mathcal{P}[A], \delta_A)$ to $(\mathcal{P}[B], \delta_B)$.

Definition 4.7 (L. Foissy [6])**.** *A* ***double twisted bialgebra*** *is a family $(\mathcal{P}, m, \Delta, \delta)$ such that:*

1. *$(\mathcal{P}, m, \Delta)$ is a twisted bialgebra with its counit denoted by ε.*
2. *$(\mathcal{P}, m, \delta)$ is a twisted bialgebra of the second kind with its counit denoted by ε'.*
3. *Δ is a right comodule morphism, that is, for any finite sets A, B:*

$$(\Delta_{A,B}\otimes I)\circ\delta_{A\sqcup B} = m_{1,3,24}\circ(\delta_A\otimes\delta_B)\circ\Delta_{A,B},$$

where

$$m_{1,3,24} : \begin{cases}\mathcal{P}[A]\otimes\mathcal{P}[A]\otimes\mathcal{P}[B]\otimes\mathcal{P}[B] & \longrightarrow \mathcal{P}[A]\otimes\mathcal{P}[B]\otimes\mathcal{P}[A\sqcup B]\\ x\otimes y\otimes z\otimes t & \longrightarrow x\otimes z\otimes m_{A,B}(y\otimes t).\end{cases}$$

which means the following diagram commutes:

$$\begin{tikzcd}
\mathcal{P}[A \sqcup B] \arrow[rr, "\Delta_{A,B}"] \arrow[dd, "\delta_{A\sqcup B}"'] & & \mathcal{P}[A] \otimes \mathcal{P}[B] \arrow[d, "\delta_A \otimes \delta_B"] \\
& & \mathcal{P}[A] \otimes \mathcal{P}[A] \otimes \mathcal{P}[B] \otimes \mathcal{P}[B] \arrow[d, "m_{1,3,24}"] \\
\mathcal{P}[A \sqcup B] \otimes \mathcal{P}[A \sqcup B] \arrow[rr, "\Delta_{A,B}\otimes I"'] & & \mathcal{P}[A] \otimes \mathcal{P}[B] \otimes \mathcal{P}[A \sqcup B]
\end{tikzcd}$$

4. *The counit* $\varepsilon : \mathcal{P} \longrightarrow \mathcal{I}$ *is a right comodule morphism, that is, for any* $x \in \varnothing$*:*

$$(\varepsilon \otimes I) \circ \delta_{\varnothing}(x) = \varepsilon(x)\mathbf{1}_{\mathcal{P}} \otimes \mathbf{1}_{\mathcal{P}}.$$

An example of double bialgebra is the species **Com** with the second kind twisted bialgebra, defined with the following coproduct on $\mathbf{Com}[A] = \boldsymbol{k}$:

$$\delta_A(1) = 1 \otimes 1.$$

5 Three important Hopf algebras

5.1 Rooted forests

In this part, we firstly examine the Hopf algebra of rooted forests which was studied by A. Connes [2], in the context of Quantum Field Theory. The associated coproduct Δ is a cutting procedure on forest. Then, we consider another bialgebra structure of rooted forests with coproduct Γ, which is an extraction-contraction procedure. After that, we examine the paper of D. Manchon [10], which investigates these two bialgebras simultaneously in a comodule-Hopf algebra structure. Some notation of this part also relies on the paper of L. Foissy [5].

5.1.1 Hopf algebra with cutting coproduct

A *rooted tree* is a class of oriented (non-planar) graphs with a finite number of vertices, these graphs are connected and contain no cycles. Also, there is a special vertex called *root* which has no outgoing edges, but any other vertex admits exactly one outgoing edge. Any tree yields a poset structure on the set of its vertices: two vertices x and y satisfy $x \leq y$ if and only if there is a path from a root to y passing through x.

A *rooted forest* is a finite collection of rooted trees. The empty set is the forest containing no trees and is denoted by $\mathbf{1}$.

Let $\mathbf{T}$ be the set of non-empty rooted trees and $\mathcal{H} = \boldsymbol{k}[\mathbf{T}]$ is the free commutative unitary algebra generated by $\mathbf{T}$, whose product is the concatenation of rooted forests. Also, we denote by $\mathbf{F}$ the set of rooted forests. To make $\mathcal{H}$ a bialgebra, we now define the coproduct on a rooted forest by a cutting procedure described as follows: the set $\mathcal{V}(u)$ of vertices of a forest u is endowed with the partial order defined as above. Any subset W of the set of vertices $\mathcal{V}(u)$ defines a *subforest* $u_{|W}$, which limits u to edges connecting vertices of W. The coproduct is then defined by the unique algebra morphism from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{H}$ as follows:

$$\Delta(u) = \sum_{V \sqcup W = \mathcal{V}(u), W < V} u_{|V} \otimes u_{|W}.$$

The notation $W < V$ means that $y \nleq x$ for any vertices $x \in W, y \in V$. Each pair of (V, W) is called an *admissible cut* with *crown* $u_{|V}$ and *trunk* $u_{|W}$. The counit is $\varepsilon(\mathbf{1}) = 1$ and $\varepsilon(u) = 0$ for any non-empty forest u. We have for example:

$$\begin{aligned} \Delta(\bullet) &= \bullet \otimes \mathbf{1} + \mathbf{1} \otimes \bullet, \\ \Delta(\vee) &= \vee \otimes \mathbf{1} + \mathbf{1} \otimes \vee + 2 \bullet \otimes \mathbf{|} + \bullet\ \bullet \otimes \bullet. \end{aligned}$$

Lemma 5.1. *The* ***grafting operator*** $B_+ : \mathcal{H} \longmapsto \mathcal{H}$ *takes any forest and returns the tree obtained by grafting all components onto a common root, and* $B_+(\mathbf{1}) = \bullet$*. Then, for all* $x \in \mathcal{H}$*:*

$$\Delta \circ B_+(x) = B_+(x) \otimes \mathbf{1} + (I \otimes B_+) \circ \Delta(x).$$

Proof. Assume that $x = t_1 \dots t_k$ and let $t = B_+(x) \in \mathbf{F}$. Consider coproduct on this tree, in the case of total cut, we obtain the tensor product $t \otimes 1$ or $B_+(x) \otimes 1$. For a non-total cut (V, W) on t, we can restrict it to the cut (V_i, W_i) on each tree

$t_i \in x$, they are actually admissible cuts on t_i correspondingly. Therefore, we can merge these cuts and have $x_{|V} = t_{1|V_1} \dots t_{k|V_k}$, $x_{|W} = B_+(t_{1|W_1} \dots t_{k|W_k})$. Hence:

$$\begin{aligned}
\Delta(t) &= t \otimes 1 + \sum_{V_i \sqcup W_i = \mathcal{V}(t_i), W_i < V_i, 1 \le i \le k} t_{1|V_1} \dots t_{k|V_k} \otimes B_+(t_{1|W_1} \dots t_{k|W_k}) \\
&= B_+(x) \otimes 1 + (I \otimes B_+) \circ \left(\prod_{i=1}^{k} \sum_{V_i \sqcup W_i = \mathcal{V}(t_i), W_i < V_i} t_{i|V_i} \otimes t_{i|W_i} \right) \\
&= B_+(x) \otimes 1 + (I \otimes B_+) \circ \big(\Delta(t_1) \dots \Delta(t_k)\big) \\
&= B_+(x) \otimes 1 + (I \otimes B_+) \circ \Delta(x).
\end{aligned}$$

□

Theorem 1. *The algebra $\mathcal{H}$ with the defined coproduct Δ and counit $\varepsilon : \mathcal{H} \longrightarrow \mathbf{k}$ which makes $\varepsilon(F) = \delta_{\mathbf{1},F}$ for any $F \in \mathbf{F}$, is a bialgebra.*

Proof. Firstly, we need to prove the coassociativity of Δ. Consider the set:

$$A = \{x \in \mathcal{H} / (\Delta \otimes I) \circ \Delta(x) = (I \otimes \Delta) \circ \Delta(x)\}.$$

$(\Delta \otimes I) \circ \Delta$ and $(I \otimes \Delta) \circ \Delta(x)$ are two algebra morphisms from $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, which makes A is a subalgebra of $\mathcal{H}$. Now we prove that for any forest $x \in A$, then the tree $B_+(x) \in A$ by showing that:

$$(\Delta \otimes I) \circ \Delta(B_+(x)) = (I \otimes \Delta) \circ \Delta(B_+(x)).$$

This is true since the left-hand side is:

$$\begin{aligned}
(\Delta \otimes I) \circ \Delta(B_+(x)) &= (\Delta \otimes I)\big(B_+(x) \otimes \mathbf{1} + (I \otimes B_+) \circ \Delta(x)\big) \\
&= \Delta \circ B_+(x) \otimes \mathbf{1} + (\Delta \otimes I) \circ (I \otimes B_+) \circ \Delta(x) \\
&= B_+(x) \otimes \mathbf{1} \otimes \mathbf{1} + (I \otimes B_+) \circ \Delta(x) \otimes \mathbf{1} + \\
&\qquad (\Delta \otimes I) \circ (I \otimes B_+) \circ \Delta(x) \\
&= B_+(x) \otimes \mathbf{1} \otimes \mathbf{1} + \sum x_1 \otimes B_+(x_2) \otimes \mathbf{1} + \\
&\qquad (\Delta \otimes I) \circ (I \otimes B_+) \left(\sum x_0 \otimes x_3 \right) \\
&= B_+(x) \otimes \mathbf{1} \otimes \mathbf{1} + \sum x_1 \otimes B_+(x_2) \otimes \mathbf{1} + \sum \Delta(x_0) \otimes B_+(x_3) \\
&= B_+(x) \otimes \mathbf{1} \otimes \mathbf{1} + \sum x_1 \otimes B_+(x_2) \otimes \mathbf{1} + \sum x_1 \otimes x_2 \otimes B_+(x_3).
\end{aligned}$$

which is equal to the right-hand side:

$$\begin{aligned}
(I \otimes \Delta) \circ \Delta(B_+(x)) &= (I \otimes \Delta)\big(B_+(x) \otimes \mathbf{1} + (I \otimes B_+) \circ \Delta(x)\big) \\
&= B_+(x) \otimes \mathbf{1} \otimes \mathbf{1} + (I \otimes \Delta) \circ (I \otimes B_+) \circ \Delta(x) \\
&= B_+(x) \otimes \mathbf{1} \otimes \mathbf{1} + (I \otimes \Delta) \circ (I \otimes B_+)\left(\sum x_1 \otimes x_2\right) \\
&= B_+(x) \otimes \mathbf{1} \otimes \mathbf{1} + \sum x_1 \otimes \Delta(B_+(x_2)) \\
&= B_+(x) \otimes \mathbf{1} \otimes \mathbf{1} + \sum x_1 \otimes \left(B_+(x_2) \otimes \mathbf{1} + (I \otimes B_+) \circ \Delta(x_2)\right) \\
&= B_+(x) \otimes \mathbf{1} \otimes \mathbf{1} + \sum x_1 \otimes B_+(x_2) \otimes \mathbf{1} + \sum x_1 \otimes x_2 \otimes B_+(x_3).
\end{aligned}$$

Therefore, A is stable under operator B_+. Now we prove that any forest $F \in \mathbf{F}$ belongs to A by induction on $n = weight(F)$ (number of vertices).

For $n = 0$ and $n = 1$, it is easy to check that $F = \mathbf{1} \in A$ and $F = \bullet \in A$. We assume that the hypothesis holds for $\forall n \leq k$ and consider the case $n = k + 1$.

- If F is not a tree, then $F = t_1 \dots t_m$ with $m \geq 2$. Since the hypothesis holds for $\forall n \leq k$, it holds for $t_1, \dots, t_m$, which means $t_i \in A, \forall i = 1, \dots, m$. In addition, A is a subalgebra of $\mathcal{H}$, hence, the product $t_1 \dots t_m$ also belongs to A, or $F \in A$.

- If F is a tree, then $F = B_+(F')$ with $F' \in \mathbf{F}$ and $weight(F') = k$. Since the hypothesis holds for $n = k$, we have $F' \in A$, and due to stability of A under B_+, we obtain $F = B_+(F') \in A$.

By induction, the hypothesis is true, and we conclude that $A = \mathcal{H}$ or the coassociativity satisfies for any forest in H with the coproduct Δ.

Secondly, we need to prove that ε is a counit of Δ, for any tree $t \in \mathbf{T}$ we have:

$$\begin{aligned}
(\varepsilon \otimes I) \circ \Delta(t) &= (\varepsilon \otimes I)\Bigg(\sum_{V \sqcup W = \mathcal{V}(t), W < V} t_{|V} \otimes t_{|W} \Bigg) \\
&= \sum_{V \sqcup W = \mathcal{V}(t), W < V} \varepsilon(t_{|V}) \otimes t_{|W} \\
&= \sum_{V \sqcup W = \mathcal{V}(t), W < V} \delta_{\mathbf{1},(t_{|V})} t_{|W} \\
&= \delta_{\mathbf{1},\mathbf{1}} t + \sum_{V \sqcup W = \mathcal{V}(t), W < V, V \neq \varnothing} 0 t_{|W} \\
&= t.
\end{aligned}$$

Similarly, we can prove that $(I \otimes \varepsilon) \circ \Delta(t) = t$, therefore $(\varepsilon \otimes I) \circ \Delta(t) = (I \otimes \varepsilon) \circ \Delta(t) = t$ for any $t \in \mathbf{T}$. Since $\mathbf{T}$ generates $\mathcal{H}$, so this property preserves for $\mathcal{H}$ also and hence ε is a counit of Δ. $\square$

In order to make bialgebra $\mathcal{H}$ a Hopf algebra, we need to define its antipode. In a more general sense, we prove that any connected graded bialgebra of this kind has an antipode. We first look at gradation of $\mathcal{H}$.

Intuitively, we can make a gradation based on the weight of forests, formally we put $\mathcal{H}(n) = \mathrm{Vect}(\mathbf{F}(n))$ and have $\mathcal{H} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}(n)$, which mean $\mathcal{H}$ is a graded bialgebra. Since $\mathcal{H}(0) = \mathbf{k}$ has dimension 1, we say that $\mathcal{H}$ is *connected*.

Lemma 5.2. *Let A be a connected graded bialgebra. Then A has an antipode.*

Proof. The proof contains two parts, the first one proves the counit map ε is trivial for all elements except first gradation $A(0)$ and one provides in the second part an inductive formula for the antipode.

First, we need to prove that $\varepsilon(x) = 0, \forall x \in A(n), n \geq 1$. We assume that this is false and we take minimal $x \in A(n), n \geq 1$ such that $\varepsilon(x) \neq 0$. Since A is connected, we can express the coproduct of x as:

$$\Delta(x) = x_1 \otimes \mathbf{1} + \mathbf{1} \otimes x_2 + \sum x' \otimes x'',$$

where $\sum x' \otimes x'' \in \sum_{i=1}^{n-1} A(i) \otimes A(n-i)$. Due to the assumption of minimality of x, elements of $A(i), A(n-i)$ are in $Ker(\varepsilon)$. Therefore, thanks to property of coproduct and counit, we have $x = (I \otimes \varepsilon) \circ \Delta(x) = x_1 + \mathbf{1}\varepsilon(x_2)$. By homogeneity, we have $x_1 = x, \varepsilon(x_2) = 0$. Similiarly, we also have $x_2 = x, \varepsilon(x_1) = 0$, which makes a contradiction $0 = \varepsilon(x_1) = \varepsilon(x) \neq 0$. Hence, the assumption if false and we have $Ker(\varepsilon) = \bigoplus_{n\geq 1} A(n)$, and for any $x \in A(n), n \geq 1$:

$$\Delta(x) - x \otimes \mathbf{1} - \mathbf{1} \otimes x \in \sum_{i=1}^{n-1} A(i) \otimes A(n-i).$$

Second, we define an inductive map as follows:

$$S_l : \begin{cases} A & \longrightarrow A \\ \mathbf{1} & \longrightarrow \mathbf{1} \\ x \in A(n) & \longrightarrow -x - \sum S_l(x')x''. \end{cases}$$

We prove that S_l satisfies antipode property by showing $m \circ (S_l \otimes I) \circ \Delta(x) = u \circ \varepsilon(x)$ for all $x \in A$. For $x = \mathbf{1}$, we have $m \circ (S_l \otimes I) \circ \Delta(\mathbf{1}) = m \circ (S_l \otimes I)(\mathbf{1} \otimes \mathbf{1}) = m(S_l(\mathbf{1}) \otimes I(\mathbf{1})) = \mathbf{1} = u \circ \varepsilon(\mathbf{1})$.

For $x \in A(n), n \geq 1$, this is true due to the following deduction:

$$\begin{aligned} m \circ (S_l \otimes I) \circ \Delta(x) &= m \circ (S_l \otimes I)\Big(x \otimes \mathbf{1} + \mathbf{1} \otimes x + \sum x' \otimes x''\Big) \\ &= m\Big(\big(-x - \sum S_l(x')x''\big)\otimes\mathbf{1} + \mathbf{1} \otimes x + \sum S_l(x') \otimes x''\Big) \\ &= -x - \sum S_l(x')x'' + x + \sum S_l(x')x'' \\ &= \mathbf{1} = u(0) = u \circ \varepsilon(x). \end{aligned}$$

Therefore, S_l is a left antipode of A. Similarly, we can define a right antipode S_r. However, these two antipodes actually are the same. Consider the convolution product on the linear maps space from $\mathcal{H}$ to itself, defined by:

$$\phi * \psi = m \circ (\phi \otimes \psi) \circ \Delta.$$

The associativity is a direct consequence of both associativity and coassociativity of $\mathcal{H}$. We prove that $S_r = (S_l * I) * S_r$ by the following deduction:

$$\begin{aligned}
(S_l * I) * S_r(x) &= m \circ \big((S_l * I) \otimes S_r\big) \circ \Delta(x) \\
&= m \circ \big(u \circ \varepsilon \otimes S_r\big)\big(\sum x' \otimes x''\big) \\
&= \sum u \circ \varepsilon(x') S_r(x'') \\
&= u \circ \varepsilon(\mathbf{1}) S_r(x) \\
&= S_r(x).
\end{aligned}$$

Similarly, we can show that $S_l = S_l * (I * S_r)$ and have:

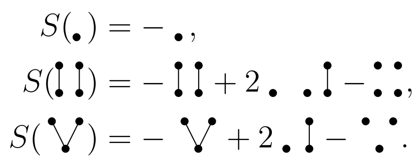
$$S_r = (S_l * I) * S_r = S_l * (I * S_r) = S_l.$$

So, we can conclude that A has an antipode $S = S_l = S_r$. $\square$

According to the above lemma, we can define the antipode for the rooted forests bialgebra in order to make it a Hopf algebra. For any $t \in \mathbf{T}$, we have:

$$S(t) = -t - \sum_{V \sqcup W = \mathcal{V}(t), W < V, V \neq \varnothing, W \neq \varnothing} (-1)^{n_{V,W}+1} t_{|V} t_{|W},$$

where $n_{V,W}$ is the number of cut edges splitting t into (V, W). For examples:



5.1.2 Bialgebra with contraction coproduct

We now define the second coproduct Γ which is an extraction-contraction procedure.

Definition 5.3. *A* ***covering subforest*** *of a rooted forest* u *is a collection of disjoint subtrees of* u *such that any vertex of* u *belongs to only one tree of the collection. For any covering subforest* s *of* u*, the* ***contracted forest*** u/s *is obtained from* u *by shrinking each tree to a single vertex.*

We denote the notation $s \subseteq u$ for "s is a covering subforest of u".

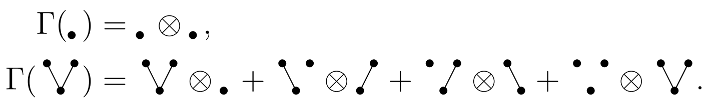
Lemma 5.4. *Let u be a rooted forest and t is a covering subforest of u. Then*

1. *There is a bijection $s \mapsto s/t$ from the set of covering subforests s such that $t \subseteq s$ onto covering subforest of u/t.*
2. *$(u/t)/(s/t) = u/s$.*

As mentioned, there is a second coproduct Γ defined as follows:

$$\Gamma(u) \coloneqq \sum_{s \subseteq u} s \otimes u/s.$$

Based on the algebra of rooted forests $\mathcal{H}$, the coproduct Γ also gives rise to a bialgebra structure. For examples:



Theorem 2. $(\mathcal{H}, \cdot, \Gamma)$ *is a graded bialgebra, the grading being given by the number of edges.*

Proof. The main part of the proof is the coassociativity, which is the equation:

$$(\Gamma \otimes I) \circ \Gamma(u) = (I \otimes \Gamma) \circ \Gamma(u).$$

We transform two sides of the above equation as follows:

$$(\Gamma \otimes I) \circ \Gamma(u) = (\Gamma \otimes I)\left(\sum_{s \subseteq u} s \otimes u/s\right) = \sum_{s \subseteq u} \Gamma(s) \otimes u/s = \sum_{t \subseteq s \subseteq u} t \otimes s/t \otimes u/s,$$
$$(I \otimes \Gamma) \circ \Gamma(u) = (I \otimes \Gamma)\left(\sum_{t \subseteq u} t \otimes u/t\right) = \sum_{t \subseteq u} t \otimes \Gamma(u/t) = \sum_{t \subseteq u, \widetilde{s} \subseteq u/t} t \otimes \widetilde{s} \otimes (u/t)/\widetilde{s}.$$

According to the Lemma 5.4, there is a bijection between covering subforests s of u such that $t \subseteq s$ and covering subforests of u/t. Therefore, since $\widetilde{s}$ is a covering subforest of u/t, there exists a covering subforest s of u such that $t \subseteq s$ and $s/t = \widetilde{s}$.

Moreover, we also have the equation $(u/t)/(s/t) = u/s$, which makes the above right-hand side becomes:

$$
\begin{aligned}
(I \otimes \Gamma) \circ \Gamma(u) &= \sum_{t \subseteq u, \widetilde{s} \subseteq u/t} t \otimes \widetilde{s} \otimes (u/t)/\widetilde{s}. \\
&= \sum_{t \subseteq s \subseteq u, s/t \subseteq u/t} t \otimes s/t \otimes (u/t)/(s/t) \\
&= \sum_{t \subseteq s \subseteq u, s/t \subseteq u/t} t \otimes s/t \otimes u/s.
\end{aligned}
$$

This claims that the coassociativity is satisfied and $(\mathcal{H}, \cdot, \Gamma)$ is a graded bialgebra. □

However, the bialgebra of rooted forests $\mathcal{H}$ with the second product Γ is not a Hopf algebra. An example to see why, assuming that there exists an antipode S, we need to have $m \circ (S \otimes I) \circ \Gamma(\bullet) = u \circ \varepsilon(\bullet)$, but:

$$
u \circ \varepsilon(\bullet) = \mathbf{1} \neq S(\bullet)\bullet = m \circ (S \otimes I)(\bullet \otimes \bullet) = m \circ (S \otimes I) \circ \Gamma(\bullet)
$$

for any function S. Hence, $(\mathcal{H}, \cdot, \Gamma)$ only has structure of a bialgebra. Although $(\mathcal{H}, \cdot, \Gamma)$ is a graded bialgebra with gradation on number of edges, Lemma 5.2 requires the bialgebra to be connected in order to have an antipode. $(\mathcal{H}, \cdot, \Gamma)$ does not satisfy this because its first gradation $\mathcal{H}(0) = \text{Vect}(\mathbf{1}, \bullet, \bullet\,\bullet, \dots)$ has more than 1 dimension. But if we identify all rooted forest of $\mathcal{H}(0)$ by the unit $\mathbf{1}$, we can construct a Hopf algebra, we will do this for bialgebra of oriented Feynman graphs in a later section.

5.1.3 Double bialgebra of rooted forests

The mentioned Hopf algebra $(\mathcal{H}, \cdot, \Delta)$ and bialgebra $(\mathcal{H}, \cdot, \Gamma)$ form a double bialgebra structure on rooted forests, which is a comodule-Hopf algebra introduced by the next theorem.

Theorem 3. *The Hopf algebra $(\mathcal{H}, \cdot, \Delta)$ of rooted forests is a comodule-Hopf algebra over extraction-contraction bialgebra $(\mathcal{H}, \cdot, \Gamma)$. The coaction map Φ is given by the coproduct Γ.*

Proof. ■ Firstly, we need to prove that Γ is a left coaction, which means the following equation holds:

$$
(\Delta_{\mathcal{B}} \otimes I) \circ \Phi = (I \otimes \Phi) \circ \Phi.
$$

This is equivalent to $(\Gamma \otimes I) \circ \Gamma = (I \otimes \Gamma) \circ \Gamma$ which is true due to coassociativity of $(\mathcal{H}, \cdot, \Gamma)$.

■ Secondly, we need Δ and counit ε of coalgebra $(\mathcal{H}, \Delta)$ are morphisms of left $\mathcal{H}$-comodules, which means the following equations hold:

$$(I \otimes \Delta) \circ \Gamma = m^{1,3} \circ (\Gamma \otimes \Gamma) \circ \Delta,$$
$$(I \otimes \varepsilon) \circ \Gamma = u_\Gamma \circ \varepsilon$$

where $m^{1,3} : \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ making $m^{1,3}(a \otimes b \otimes c \otimes d) = ac \otimes b \otimes d$ and u_Γ is unit of bialgebra $(\mathcal{H}, \cdot, \Gamma)$. The below equation is easy, so we consider first equation, for any forest u, we have:

$$(I \otimes \varepsilon) \circ \Gamma(u) = (I \otimes \varepsilon)\left(\sum_{s \subseteq u} s \otimes u/s\right) = \sum_{s \subseteq u} s \otimes \varepsilon(u/s) = \sum_{s \subseteq u} s \otimes 0 = \mathbf{1} = u \circ \varepsilon(u).$$

In order to prove the equation $(I \otimes \Delta) \circ \Gamma = m^{1,3} \circ (\Gamma \otimes \Gamma) \circ \Delta$, let u be any forest, the left-hand side equals to:

$$(I \otimes \Delta) \circ \Gamma(u) = (I \otimes \Delta)\left(\sum_{s \subseteq u} s \otimes u/s\right) = \sum_{s \subseteq u} \sum_{V \sqcup W = \mathcal{V}(u/s), W < V} s \otimes (u/s)_{|V} \otimes (u/s)_{|W}.$$

Besides, the right-hand side can be expressed as follows:

$$
\begin{aligned}
m^{1,3} \circ (\Gamma \otimes \Gamma) \circ \Delta(u) &= m^{1,3} \circ (\Gamma \otimes \Gamma) \Bigg(\sum_{V' \sqcup W' = \mathcal{V}(u), W' < V'} u_{|V'} \otimes u_{|W'} \Bigg) \\
&= m^{1,3} \Bigg(\sum_{V' \sqcup W' = \mathcal{V}(u), W' < V'} \sum_{s' \subseteq V'} \sum_{s'' \subseteq W'} s' \otimes V'/s' \otimes s'' \otimes W'/s'' \Bigg) \\
&= \sum_{V' \sqcup W' = \mathcal{V}(u), W' < V'} \sum_{s' \subseteq V'} \sum_{s'' \subseteq W'} s's'' \otimes V'/s' \otimes W'/s'' \\
&= \sum_{V' \sqcup W' = \mathcal{V}(u), W' < V'} \sum_{s' \subseteq V'} \sum_{s'' \subseteq W'} s's'' \otimes (V'W')_{|V'}/(s's'')_{|V'} \otimes (V'W')_{|W'}/(s's'')_{|W'} \\
&= \sum_{V' \sqcup W' = \mathcal{V}(u), W' < V'} \sum_{s \subseteq u} s \otimes u_{|V'}/s_{|V'} \otimes u_{|W'}/s_{|W'} \\
&= \sum_{V' \sqcup W' = \mathcal{V}(u), W' < V'} \sum_{s \subseteq u} s \otimes (u/s)_{|(V'/s)} \otimes (u/s)_{|(W'/s)} \\
&= \sum_{V \sqcup W = \mathcal{V}(u/s), W < V} \sum_{s \subseteq u} s \otimes (u/s)_{|V} \otimes (u/s)_{|W} \\
&= \sum_{s \subseteq u} \sum_{V \sqcup W = \mathcal{V}(u/s), W < V} s \otimes (u/s)_{|V} \otimes (u/s)_{|W},
\end{aligned}
$$

which equals to its left-hand side. The meaning of this equation is that two following processes are equivalent. The first one is that we take a covering subforest s of a forest u and split the corresponding contracted forest u/s into two parts. The second one splitting the original forest u first into (V, W), then we take covering subforests of two parts before combining them to form a bigger one, two contracted forests are the same as split contracted forests in the first process. These two processses are equivalent due to the mergeability of covering subforests and contracted forests between separated components of an admissibly cut forest.

■ Thirdly, we need to prove that the product m and unit u_Δ of $(\mathcal{H}, \cdot, \Delta)$ Hopf algebra are morphisms of left $\mathcal{H}$-comodules (note that the products of both bialgebras are the same), which means the following equations hold:

$$
\begin{aligned}
\Gamma \circ m &= (m \otimes m) \circ \tau_{23} \circ (\Gamma \otimes \Gamma), \\
\Gamma \circ u_\Delta &= (I \otimes u_\Delta) \circ u_\Gamma,
\end{aligned}
$$

where τ_{23} stands for the flip of the two middle factors. The above equation is obvious, since one Γ action on a big forest can be separated into two different Γ actions on two subforests of the big forest. For the below equation, let k be any value in the base field $\mathbf{k}$, the left and right-hand sides actions on k are the same because we have:

$$\Gamma \circ u_\Delta(k) = \Gamma(k\mathbf{1}) = k\mathbf{1} \otimes \mathbf{1},$$
$$(I \otimes u_\Delta) \circ u_\Gamma(k) = (I \otimes u_\Delta)(k\mathbf{1}) = \mathbf{1} \otimes k\mathbf{1} = k\mathbf{1} \otimes \mathbf{1}.$$

■ Fourthly, we need the antipode S of Hopf algebra $(\mathcal{H}, \cdot, \Delta)$ satisfying the following equation:

$$(I \otimes S) \circ \Gamma = \Gamma \circ S.$$

For any forest u, the left and right-hand sides act on u as follows:

$$\begin{aligned}
&(I \otimes S) \circ \Gamma(u) = (I \otimes S)\Big(\sum_{s \subseteq u} s \otimes u/s\Big) \\
&\quad = \sum_{s \subseteq u} s \otimes \Big(-u/s - \sum_{V \sqcup W = \mathcal{V}(u/s), W < V, V \neq \varnothing, W \neq \varnothing} (-1)^{n_{V,W}+1} (u/s)_{|V} (u/s)_{|W}\Big) \\
&\Gamma \circ S(u) = \Gamma\Big(-u - \sum_{V' \sqcup W' = \mathcal{V}(u), W' < V', V' \neq \varnothing, W' \neq \varnothing} (-1)^{n_{V',W'}+1} u_{|V'} u_{|W'}\Big) \\
&\quad = \sum_{s \subseteq u} s \otimes (-u/s) - \sum_{V' \sqcup W' = \mathcal{V}(u), W' < V', V' \neq \varnothing, W' \neq \varnothing} \sum_{s \subseteq u} s \otimes (-1)^{n_{V',W'}+1} \big(u_{|V'} u_{|W'}\big)/s \\
&\quad = -\sum_{s \subseteq u} s \otimes u/s - \sum_{V' \sqcup W' = \mathcal{V}(u), W' < V', V' \neq \varnothing, W' \neq \varnothing} \sum_{s \subseteq u} s \otimes (-1)^{n_{V',W'}+1} \big(u_{|V'}/s_{|V'}\big)\big(u_{|W'}/s_{|W'}\big) \\
&\quad = -\sum_{s \subseteq u} s \otimes u/s - \sum_{V' \sqcup W' = \mathcal{V}(u), W' < V', V' \neq \varnothing, W' \neq \varnothing} \sum_{s \subseteq u} s \otimes (-1)^{n_{V',W'}+1} (u/s)_{|(V'/s)} (u/s)_{|(W'/s)} \\
&\quad = -\sum_{s \subseteq u} s \otimes u/s - \sum_{s \subseteq u} \sum_{V \sqcup W = \mathcal{V}(u/s), W < V, V \neq \varnothing, W \neq \varnothing} s \otimes (-1)^{n_{V,W}+1} (u/s)_{|V} (u/s)_{|W},
\end{aligned}$$

which proves the required equation and the whole theorem. □

5.2 Directed acyclic graphs

In this part, we examine several Hopf algebras on directed acyclic graphs introduced by D. Manchon [9], by generalizing ideas from rooted forests algebras.

5.2.1 Preliminaries

In graph theory, a **directed graph** is a graph, which has an assignment of direction for all edges. An **oriented graph** is a directed graph if there is no pair of vertices linked by two symmetric edges. An **oriented Feynman graph** is an oriented graph with a finite number of vertices and edges, but each edge can be internal or external:

- **internal edge**: an edge connected at both ends to a vertex.
- **external edge**: an edge with one open end, one connects to a vertex.

A **cycle** in an oriented Feynman graph is a finite collection $(e_1, \dots, e_n)$ of oriented internal edges such that the target of e_k and the source of e_{k+1} are the same for any $k = 1, \dots, n$(modulo n). The **loop number** of a graph Γ is given by:

$$L(\Gamma) = I(\Gamma) - V(\Gamma) + 1,$$

where $I(\Gamma)$ is the number of internal edges of Γ and $V(\Gamma)$ is the number of vertices.

One important class of oriented Feynman graph is **cycle-free** graphs, these graphs endow a poset structure on the set of vertices described as: $v < w$ if and only if there is an oriented path $(e_1, \dots, e_n)$ from v to w formed by internal edges such that $e_1 = v, e_n = w$ and the target of e_k is also the source of e_{k+1} for any $k = 1, \dots, n-1$.

Let Γ be an oriented Feynman graph with its vertices set $\mathcal{V}(\Gamma)$. If P is a subset of $\mathcal{V}(\Gamma)$, then a **subgraph** $\Gamma(P)$ of Γ is the restriction of Γ to a graph formed by P vertices with:

- internal edges, which are internal edges of Γ with both ends in P.
- external edges, which are edges of Γ (internal or external), with only one end in P.

The subgraph $\Gamma(P)$ is **connected** if we can connect two distinct vertices by following internal edges in $\Gamma(P)$ without considering their orientation. Similar to rooted tree algebras, we denote the empty graph by $\mathbf{1}$.

A **covering subgraph** of Γ is an oriented Feynman graph γ, given by a collection $\{\Gamma(P_1), \dots, \Gamma(P_n)\}$ of connected subgraphs such that $\{P_1, \dots, P_n\}$ is a partition of $\mathcal{V}(\Gamma)$. Similarly to contracted forest, the **contracted graph** Γ/γ is defined by shrinking all connected components of γ inside Γ onto a point.

5.2.2 Hopf algebra with contraction coproduct

In this part, we only consider the full bialgebra of oriented Feynman graphs and its subalgebra of cycle-free graphs which makes a comodule-bialgebra structure or double bialgebra. More subalgebras endowed with Hopf algebra structure are included in paper of D. Manchon [9].

Bialgebra of oriented Feynman graphs

Let $\widetilde{\mathcal{H}}$ be the vector space spanned by oriented Feynman graphs. The product is given by concatenation, hence $\widetilde{\mathcal{H}} = S(V)$, where V is the vector space spanned by connected oriented Feynman graphs and $S(V)$ is the symmetric algebra of V. By denoting $\gamma \subseteq \Gamma$ for γ being a covering subgraph of Γ, the coproduct is given by the following formula:

$$\Delta(\Gamma) = \sum_{\gamma \subseteq \Gamma} \gamma \otimes \Gamma/\gamma.$$

We want this coproduct to satisfy the coassociativity:

$$(\Delta \otimes I) \circ \Delta(\Gamma) = (I \otimes \Delta) \circ \Delta(\Gamma).$$

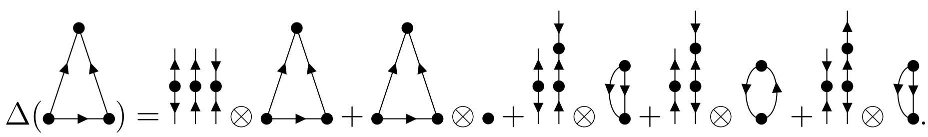
Two sides of the above equation can be transformed into:

$$(\Delta \otimes I) \circ \Delta(\Gamma) = \sum_{\delta \subseteq \gamma \subseteq \Gamma} \delta \otimes \gamma/\delta \otimes \Gamma/\gamma,$$
$$(I \otimes \Delta) \circ \Delta(\Gamma) = \sum_{\delta \subseteq \Gamma, \widetilde{\gamma} \subseteq \Gamma/\gamma} \delta \otimes \widetilde{\gamma} \otimes (\Gamma/\delta)/\widetilde{\gamma}.$$

Similar to the case of rooted forests, we have a bijection between covering subgraph γ of Γ containing δ and covering subgraphs $\widetilde{\gamma} = \gamma/\delta$ of Γ/δ. We also have the property $\Gamma/\gamma = (\Gamma/\delta)/(\gamma/\delta)$, which makes coassociativity to hold.

The counit is defined by $\varepsilon(\Gamma) = \delta_{\mathbf{1},\Gamma}$ for any non-empty graph $\Gamma \in \widetilde{\mathcal{H}}$. The constructed bialgebra $\widetilde{\mathcal{H}}$ is graded by the number of internal edges.

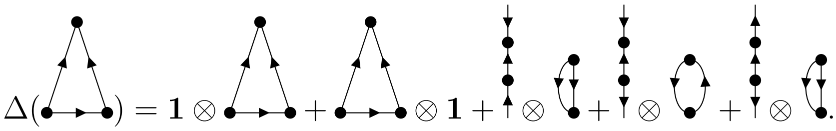
An example of coproduct computation is as follows:



In order to make $\widetilde{\mathcal{H}}$ a Hopf algebra, we merge all graphs without internal edges by unit $\mathbf{1}$, formally:

$$\mathcal{H} = \widetilde{\mathcal{H}}/\mathcal{J},$$

where $\mathcal{J}$ is the bi-ideal generated by elements of $\Gamma - \mathbf{1}$ with Γ being any graph containing no internal edges. Thanks to this formulation, $\mathcal{H}$ becomes a connected graded bialgebra, which is a Hopf algebra according to Lemma 5.2. Therefore, the above example of coproduct computation can be rewritten in $\mathcal{H}$ as follows:

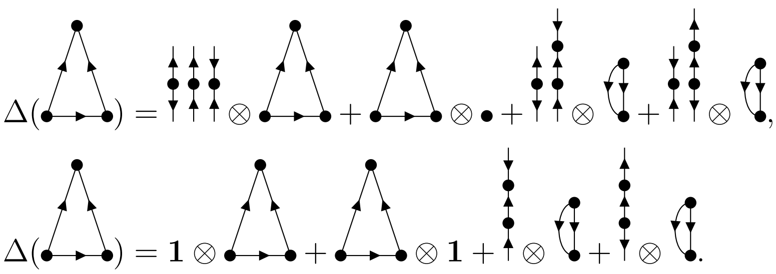


Subalgebra of cycle-free graphs

Let Γ be a cycle-free oriented Feynman graph, a covering subgraph γ of Γ is a **poset-compatible** if the contracted graph Γ/γ is cycle-free, and we denote $\gamma \sqsubseteq \Gamma$. Besides, if we denote the vector space spanned by connected oriented cycle-free Feynman graphs by V_{CF}, then bialgebra $\widetilde{\mathcal{H}}_{\mathrm{CF}} = S(V_{\mathrm{CF}})$ with the coproduct:

$$\Delta(\Gamma) = \sum_{\gamma \sqsubseteq \Gamma} \gamma \otimes \Gamma/\gamma.$$

The counit is defined similar to $\widetilde{\mathcal{H}}$, the coassociativity can be proved in the same way. The associated Hopf algebra $\mathcal{H}_{\mathrm{CF}}$ is built up correspondingly to $\mathcal{H}$ by identifying all elements of degree zero with the unit $\mathbf{1}$. The example computed with $\widetilde{\mathcal{H}}$ and $\mathcal{H}$ is now presented in $\widetilde{\mathcal{H}}_{\mathrm{CF}}$ and $\mathcal{H}_{\mathrm{CF}}$ respectively:



5.2.3 Hopf algebra with cutting coproduct

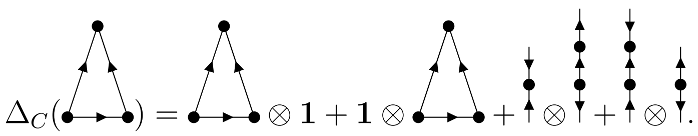
As for rooted forests, we constructed two bialgebra structures, including cutting and contraction coproducts, we want to do it for oriented cycle-free Feynman graphs also. Considering the bialgebra $\widetilde{\mathcal{H}}_{\mathrm{CF}}$, we define the cutting coproduct as follows:

$$\Delta_C(\Gamma) = \sum_{V_1 \sqcup V_2 = \mathcal{V}(\Gamma), V_2 < V_1} \Gamma(V_1) \otimes \Gamma(V_2),$$

where $V_2 < V_1$ means that $v_2 < v_1$ for any $v_1 \in V_1, v_2 \in V_2$ since there is a poset structure on vertices set $\mathcal{V}(\Gamma)$ of cycle-free graph Γ. This is actually similar to admissible cut of cutting coproduct on rooted forests as described in previous section. The coassociativity is straightforward:

$$
\begin{aligned}
(\Delta_C \otimes I) \circ \Delta_C(\Gamma) &= (\Delta_C \otimes I)\Big(\sum_{V_0 \sqcup V_3 = \mathcal{V}(\Gamma), V_3 < V_0} \Gamma(V_0) \otimes \Gamma(V_3) \Big) \\
&= \sum_{V_1 \sqcup V_2 = \mathcal{V}\big(\Gamma(V_0)\big), V_2 < V_1, V_0 \sqcup V_3 = \mathcal{V}(\Gamma), V_3 < V_0} \Gamma(V_0)(V_1) \otimes \Gamma(V_0)(V_2) \otimes \Gamma(V_3) \\
&= \sum_{V_1 \sqcup V_2 \sqcup V_3 = \mathcal{V}(\Gamma), V_3 < V_2 < V_1} \Gamma(V_1) \otimes \Gamma(V_2) \otimes \Gamma(V_3), \\
(I \otimes \Delta_C) \circ \Delta_C(\Gamma) &= (I \otimes \Delta_C)\Big(\sum_{V_1 \sqcup V_0 = \mathcal{V}(\Gamma), V_0 < V_1} \Gamma(V_1) \otimes \Gamma(V_0) \Big) \\
&= \sum_{V_1 \sqcup V_0 = \mathcal{V}(\Gamma), V_0 < V_1, V_2 \sqcup V_3 = \mathcal{V}\big(\Gamma(V_0)\big), V_3 < V_2} \Gamma(V_1) \otimes \Gamma(V_0)(V_2) \otimes \Gamma(V_0)(V_3) \\
&= \sum_{V_1 \sqcup V_2 \sqcup V_3 = \mathcal{V}(\Gamma), V_3 < V_2 < V_1} \Gamma(V_1) \otimes \Gamma(V_2) \otimes \Gamma(V_3).
\end{aligned}
$$

This coproduct is also an algebra morphism, we can easily see that it creates a graded bialgebra, with the grading being the number of vertices of graphs. By this grading, the first gradation has dimension 1 due to the empty graph $\mathbf{1}$, hence it is a connected graded bialgebra, and therefore a Hopf algebra. We denote this Hopf algebra by $\mathcal{H}_{\mathrm{CF}_C}$, which is actually isomorphic to $\widetilde{\mathcal{H}}_{\mathrm{CF}}$, and we can compute the familiar example as follows:



5.2.4 Double bialgebra of oriented cycle-free graphs

Similarly to double bialgebra of rooted forests, we can construct a comodule-Hopf algebra structure with Δ and Δ_C on oriented cycle-free Feynman graphs.

Theorem 4. *The Hopf algebra $(\mathcal{H}_{CF_C}, \cdot, \Delta_C)$ of oriented cycle-free graphs is a comodule-Hopf algebra over bialgebra $(\widetilde{\mathcal{H}}_{CF}, \cdot, \Delta)$. The coaction map Φ is given by the coproduct $\Delta : \mathcal{H}_{CF_C} \longrightarrow \widetilde{\mathcal{H}}_{CF} \otimes \mathcal{H}_{CF_C}$.*

Proof. We prove the most important condition of comodule-Hopf algebra, which is the following equation:

$$(I \otimes \Delta_C) \circ \Delta = m^{1,3} \circ (\Delta \otimes \Delta) \circ \Delta_C.$$

For any non-empty graph Γ, the left-hand side becomes:

$$\begin{aligned}(I \otimes \Delta_C) \circ \Delta(\Gamma) &= (I \otimes \Delta_C)\Big(\sum_{\gamma \sqsubseteq \Gamma} \gamma \otimes \Gamma/\gamma\Big) \\ &= \sum_{\gamma \sqsubseteq \Gamma} \sum_{V_1 \sqcup V_2 = \mathcal{V}(\Gamma/\gamma), V_2 < V_1} \gamma \otimes (\Gamma/\gamma)(V_1) \otimes (\Gamma/\gamma)(V_2).\end{aligned}$$

Similarly to the case of rooted forests, the right-hand side can be transformed as:

$$\begin{aligned}m^{1,3} \circ (\Delta \otimes \Delta) \circ \Delta_C(\Gamma) &= m^{1,3} \circ (\Delta \otimes \Delta)\Big(\sum_{V_1' \sqcup V_2' = \mathcal{V}(\Gamma), V_2' < V_1'} \Gamma(V_1') \otimes \Gamma(V_2')\Big) \\ &= m^{1,3}\Big(\sum_{V_1' \sqcup V_2' = \mathcal{V}(\Gamma), V_2' < V_1'} \sum_{\gamma' \sqsubseteq \Gamma(V_1')} \sum_{\gamma'' \sqsubseteq \Gamma(V_2')} \gamma' \otimes \Gamma(V_1')/\gamma' \otimes \gamma'' \otimes \Gamma(V_2')/\gamma''\Big) \\ &= \sum_{V_1' \sqcup V_2' = \mathcal{V}(\Gamma), V_2' < V_1'} \sum_{\gamma' \sqsubseteq \Gamma(V_1')} \sum_{\gamma'' \sqsubseteq \Gamma(V_2')} \gamma'\gamma'' \otimes \Gamma(V_1')/\gamma' \otimes \Gamma(V_2')/\gamma'' \\ &= \sum_{V_1' \sqcup V_2' = \mathcal{V}(\Gamma), V_2' < V_1'} \sum_{\gamma' \sqsubseteq \Gamma(V_1')} \sum_{\gamma'' \sqsubseteq \Gamma(V_2')} \gamma'\gamma'' \otimes \Gamma(V_1')/\gamma' \otimes \Gamma(V_2')/\gamma'' \\ &= \sum_{V_1' \sqcup V_2' = \mathcal{V}(\Gamma), V_2' < V_1'} \sum_{\gamma \sqsubseteq \Gamma} \gamma \otimes \Gamma(V_1')/\gamma(V_1') \otimes \Gamma(V_2')/\gamma(V_2') \\ &= \sum_{V_1' \sqcup V_2' = \mathcal{V}(\Gamma), V_2' < V_1'} \sum_{\gamma \sqsubseteq \Gamma} \gamma \otimes \big(\Gamma/\gamma\big)(V_1'/\gamma) \otimes \big(\Gamma/\gamma\big)(V_2'/\gamma) \\ &= \sum_{\gamma \sqsubseteq \Gamma} \sum_{V_1 \sqcup V_2 = \mathcal{V}(\Gamma), V_2 < V_1} \gamma \otimes \big(\Gamma/\gamma\big)(V_1) \otimes \big(\Gamma/\gamma\big)(V_2).\end{aligned}$$

which equals to its left-hand side. The other condition for comodule-Hopf algebra can be proved similarly as the rooted forests case. □

5.3 Finite topologies

In this part, we consider double bialgebra structure of finite topologies in terms of species from the paper of L. Foissy [6]. More precisely, we endow the species of finite topologies spaces with a double bialgebra structure.

5.3.1 Preliminaries

Definition 5.5. *Let $T = (A, \leq)$ be a finite topology and $\sim$ be an equivalence on A.*

1. *We define a second quasi-order $\leq_{T|\sim}$ on A by the relation:*

$$\forall x, y \in A : \quad x \leq_{T|\sim} \quad \textit{if} \quad (x \leq y \textit{ and } x \sim y).$$

2. *We define a third quasi-order $\leq_{T/\sim}$ on A as the transitive closure of the relation R defined by:*

$$\forall x, y \in A : \quad xRy \quad \textit{if} \quad (x \leq y \textit{ or } x \sim y).$$

   In other words, $x \leq_{T/\sim} y$ if there exists $x_1, y_1, \ldots x_n, y_n \in A$, such that:

$$x = x_1 \sim y_1 \leq_T \cdots \leq_T x_k \sim y_k = y.$$

3. *We say that $\sim$ is T-compatible, and we write $\sim \lhd T$ if the two following conditions are satisfied:*

   - *The restriction of T to any equivalence class of $\sim$ is connected.*
   - *The equivalence $\sim_{/T\sim}$ and $\sim$ are equal. In other words:*

$$\forall x, y \in A : \quad (x \leq_{T/\sim} y \textit{ and } y \leq_{T/\sim} x) \Longrightarrow x \sim y.$$

     Meanwhile, the converse assertion is trivial.

   The set of T-compatible equivalences is denoted by $\mathrm{CE}(T)$.

Lemma 5.6. *Let T be a topology on a finite set A and $\sim, \sim'$ be equivalences on A such that:*

$$\forall x, y \in A : \quad x \sim y \Longrightarrow x \sim' y.$$

Then we have:

$$(T|\sim')|\sim = T|\sim, \quad (T/\sim)/\sim' = T/\sim'.$$

Moreover, if $\sim' \in \mathrm{CE}(T)$, then:

$$(T|\sim')/\sim = (T/\sim)|\sim'.$$

Proof.

- We prove that: $(T|\sim')|\sim = T|\sim$.
  Let $x, y \in A$, we have:

$$\begin{aligned} x \leq_{(T|\sim')|\sim} y &\iff x \leq_{T|\sim'} y \text{ and } x \sim y \\ &\iff x \leq_T y \text{ and } x \sim' y \text{ and } x \sim y \\ &\iff x \leq_T y \text{ and } x \sim y \\ &\iff x \leq_{T|\sim} y. \end{aligned}$$

  Therefore, $(T|\sim')|\sim = T|\sim$, which mean that restricting T by a finer equivalence relation $\sim$ after a coarser one $\sim'$ only depends on the finer one $\sim$.

- We prove that: $(T/\sim)/\sim' = T/\sim'$.
  The relation $\leq_{(T/\sim)/\sim'}$ is the transitive closure of the relation given by:

$$\begin{aligned} &x \leq_T y \text{ or } x \sim y \text{ or } x \sim' y \\ \iff &x \leq_T y \text{ or } x \sim' y. \end{aligned}$$

  Therefore, $(T/\sim)/\sim' = T/\sim'$, which means that saturating T by a coarser equivalence relation $\sim'$ after a finer one $\sim$ only depends on the coarser one $\sim$.

- We prove that: $(T|\sim')/\sim = (T/\sim)|\sim'$.
  ■ $x \leq_{(T|\sim')/\sim} y \Longrightarrow x \leq_{(T/\sim)|\sim'} y$.
  Let $x, y \in A$, if $x \leq_{(T|\sim')/\sim} y$, there exist $x_1, y_1, \ldots, x_k, y_k \in A$ such that:

$$x = x_1 \sim y_1 \leq_{T|\sim'} x_2 \sim \cdots \leq_{T|\sim'} x_k \sim y_k = y.$$

  Since $a \leq_{T|\sim'} b$ means $a \leq_T b$ and $a \sim' b$, we have the two consequences:

$$\begin{aligned} &x = x_1 \sim y_1 \leq_T x_2 \sim \cdots \leq_T x_k \sim y_k = y \Longrightarrow x \leq_{T/\sim} y, \\ &x = x_1 \sim y_1 \sim' x_2 \sim \cdots \sim' x_k \sim y_k = y \\ \Longrightarrow &x = x_1 \sim' y_1 \sim' x_2 \sim' \cdots \sim' x_k \sim' y_k = y \\ \Longrightarrow &x \sim' y. \end{aligned}$$

  By combining $x \leq_{T/\sim} y$ and $x \sim' y$, we deduce that $x \leq_{(T/\sim)|\sim'} y$.

■ $x \leq_{(T/\sim)|\sim'} y \Longrightarrow x \leq_{(T|\sim')/\sim} y$.
Since $x \leq_{(T/\sim)|\sim'} y$, we have $x \leq_{T/\sim} y$ and $x \sim' y$. Hence, there exists $x_1, y_1, \ldots, x_k, y_k \in A$ such that:

$$x = x_1 \sim y_1 \leq_T x_2 \sim y_2 \leq_T \cdots \leq_T x_k \sim y_k = y.$$

Because $x \sim y \Longrightarrow x \sim' y$ and $(x \leq_T y$ or $x \sim' y) \iff x \leq_{T/\sim'} y$, we have the following deduction:

$$\begin{aligned}
&x = x_1 \sim y_1 \leq_T x_2 \sim y_2 \leq_T \cdots \leq_T x_k \sim y_k = y\\
\Longrightarrow &x = x_1 \sim' y_1 \leq_T x_2 \sim' y_2 \leq_T \cdots \leq_T x_k \sim' y_k = y\\
\Longrightarrow &x = x_1 \leq_{T/\sim'} y_1 \leq_{T/\sim'} x_2 \leq_{T/\sim'} y_2 \leq_{T/\sim'} \cdots \leq_{T/\sim'} x_k \leq_{T/\sim'} y_k = y.
\end{aligned}$$

So all these elements are $\sim_{T/\sim'}$-equivalent. Since $\sim' \in \mathrm{CE}(T)$, $\sim_{T/\sim'}$-equivalent deduces $\sim'$-equivalent, which makes $y_i \sim' x_{i+1}$ for $i = \overline{1, k-1}$. By combining this with $y_i \leq_T x_{i+1}$, we have $y_i \leq_{T|\sim'} x_{i+1}$ for $i = \overline{1, k-1}$. Hence, we have the following deduction:

$$\begin{aligned}
&x = x_1 \sim y_1 \leq_T x_2 \sim y_2 \leq_T \cdots \leq_T x_k \sim y_k = y\\
\Longrightarrow &x = x_1 \sim y_1 \leq_{T|\sim'} x_2 \sim y_2 \leq_{T|\sim'} \cdots \leq_{T|\sim'} x_k \sim y_k = y\\
\Longrightarrow &x \leq_{(T|\sim')/\sim} y.
\end{aligned}$$

□

We will examine how to construct Hasse graph of $T/\sim$ and $T|\sim$ from Hasse graph of T by the following proposition.

Proposition 5.7. *Let T be a finite topology on a set A and $\sim \in \mathrm{CE}(T)$.*

1. *The Hasse graph of $T|\sim$ is obtained from the Hasse graph of T by deleting the edges of the graph of T between non $\sim$-equivalent vertices.*

2. *The Hasse graph of $T/\sim$ is obtained from the Hasse graph of T by:*

   - *Contracting any equivalence class of $\sim$ to a single vertex.*
   - *Deleting the superfluous edges created in this process.*

Proof. 1. Firstly, we have the equivalence of the two relations $\sim_{T|\sim} = \sim_T$. If $x \sim_{T|\sim} y$, we obviously have $x \sim_T y$. Conversely, if $x \sim_T y$, since $\sim \in \mathrm{CE}(T)$, we would have $x \sim_T y \Longrightarrow x \sim y$.

Therefore, vertices of Hasse graph of $T| \sim$ are vertices of Hasse graph of T, which are equivalent classes of $\sim_T$. For any $x \in A$, we denote $cl_T(x)$ its equivalence class for $\sim_T$. Now we need to prove that all edges of Hasse graph of $T| \sim$ are of the form $(cl_T(x), cl_T(y))$ with $x \sim y$.

- If $(cl_T(x), cl_T(y))$ is an edge of Hasse graph of $T| \sim$, then $x \leq_{T|\sim} y$, so $x \leq_T y, x \sim y$, hence $(cl_T(x), cl_T(y))$ is an edge of Hasse graph of T.
If $\exists z \in A : x \leq_T z \leq_T y$ on T, we prove that z is in the same $\sim_T$-equivalence class of x or y, which is merged with x or y.

$$\begin{aligned}
& x \leq_T z \leq_T y \\
\Longrightarrow & x \leq_{T/\sim} z \leq_{T/\sim} y \leq_{T/\sim} x \qquad \text{(since } x \sim y\text{)} \\
\Longrightarrow & x \sim_{T/\sim} z \sim_{T/\sim} y \\
\Longrightarrow & x \sim z \sim y \qquad \text{(since } \sim \in \mathrm{CE}(T)\text{)} \\
\Longrightarrow & x \leq_{T|\sim} z \leq_{T|\sim} y \qquad \text{(combine with } x \leq_T z \leq_T y\text{)} \\
\Longrightarrow & x \sim_T z \quad \text{or} \quad y \sim_T z \qquad \text{(since } (cl_T(x), cl_T(y)) \text{ is an edge of } T| \sim\text{)}.
\end{aligned}$$

- If $(cl_T(x), cl_T(y))$ is an edge of Hasse graph of T and $x \sim y$, then $x \leq_{T|\sim} y$, hence $(cl_T(x), cl_T(y))$ is an edge of Hasse graph of $T| \sim$.
If $\exists z \in A : x \leq_{T|\sim} z \leq_{T|\sim} y$ on $T| \sim$, then $x \leq_T z \leq_T y$, which deduces $x \sim z$ or $y \sim z$ and therefore $x \sim_T z$ or $y \sim_T z$.

2. Because $\sim \in \mathrm{CE}(T)$, the vertices of the Hasse graph of $T/ \sim$ are the classes of $\sim$. Now we prove that for any edge $(cl_\sim(x), cl_\sim(y))$ of the Hasse graph of $T/ \sim$, there exists $x' \sim x, y' \sim y$ such that $(cl_T(x'), cl_T(y'))$ is an edge of T, this means that $cl_T(x), cl_T(y)$ are respectively contracted to x', y' and superfluous edges are removed except for (x', y').

As $x \leq_T y$, there exists $x_1, y_1, \ldots, x_k, y_k \in A$ such that:

$$x = x_1 \sim y_1 \leq_T \cdots \leq_T x_k \sim y_k = y.$$

Therefore, we have:

$$x = x_1 \leq_{T/\sim} y_1 \leq_{T/\sim} \cdots \leq_{T/\sim} x_k \leq_{T/\sim} y_k = y.$$

Since $(cl_\sim(x), cl_\sim(y))$ of the Hasse graph of $T/\sim$, there exists i such that:

$$x = x_1 \sim y_1 \sim \cdots \sim y_i \leq_T x_{i+1} \sim \cdots \sim x_k \sim y_k = y,$$

which proves that the following set is non-empty:

$$X = \{x' \in A, x' \sim x, \exists y' \in A, y' \sim y \text{ and } x' \leq_T y'\}.$$

Let $x' \in X$, which is maximal for $\leq_T$. We now consider:

$$Y = \{y' \in A, y' \sim y \text{ and } x' \leq_T y'\}.$$

Due to the existence of x', Y is non-empty, hence let $y' \in Y$, which is minimal for $\leq_T$. Therefore, we have $x \sim x' \leq_T y' \sim y$. If $\exists z \in A : x' \leq_T z \leq_T y'$, then $x' \leq_{T/\sim} z \leq_{T/\sim} y'$, which deduces $x' \sim z$ or $y' \sim z$.

Thanks to maximality of x' and minimality of y', this makes $x' \sim_T z$ or $y' \sim_T z$.

□

Corollary 5.8. *Let T be a finite topology on a set A and let $\sim \in \mathrm{CE}(T)$.*

1. *Let $X \subseteq A$, $\sim$-saturated. Then $T_{|X}$ is connected if and only if $(T/\sim)_{|X}$ is connected.*

2. *Let $X \subseteq A$ included in a class of $\sim$. Then $T_{|X}$ is connected if and only if $(T|\sim)_{|X}$ is connected.*

Proof. 1. $\Longrightarrow$. Let $x, y \in X$. There exists a path from $cl_T(x)$ to $cl_T(y)$ in the Hasse graph of T, with all its vertices in X. By construction of the Hasse graph of $T/\sim$, there exists a path from $cl_\sim(x)$ to $cl_\sim(y)$ in the Hasse graph of $T/\sim$, with all its vertices in X. So X is $T/\sim$-connected.

$\Longleftarrow$. Let $x, y \in X$. There exists a path from $cl_\sim(x)$ to $cl_\sim(y)$ in the Hasse graph of $T/\sim$, with all its internal vertices $x = x_0, x_1, \ldots, x_k = y$ be classes of elements of X.

By construction of the Hasse graph of $T/\sim$, for any i, there exists x_i', x_i'' such that there is an edge between $cl_T(x_i'), cl_T(x_i'')$ in the Hasse graph of T and $x_i \sim x_i', x_i'' \sim x_{i+1}$. Besides, by definition of $\sim$ CE(T), classes of $\sim$ are connected, which deduces that there exists a path in Hasse graph of T between $cl_T(x_i), cl_T(x_i')$ and $cl_T(x_i''), cl_T(x_{i+1})$ also.

Because X is $\sim$-saturated, vertices of $cl_T(x_i'), cl_T(x_i'')$ are equivalent to x_i, x_{i+1} respectively and also in X. Therefore, there will exist a path in the Hasse graph of T between $cl_T(x)$ and $cl_T(y)$, so X is T-connected.

2. $\Longrightarrow$. There exists a path from $cl_T(x)$ to $cl_T(y)$ in the Hasse graph of T, with all its vertices in X. Because X is included in a single class of $\sim$, all edges between them are also in $T|\sim$ by definition, so X is $T|\sim$-connected.

   $\Longleftarrow$. Due to the fact that Hasse graph of $(T|\sim)_{|X}$ is a subgraph of Hasse graph of $T_{|X}$.

□

5.3.2 Double twisted bialgebra of finite topologies

Similarly to double bialgebra of rooted forests and oriented cycle-free Feynman graphs, we construct a double bialgebra of finite topologies using the species formalism. We also have a cutting coproduct Δ and a contraction coproduct δ.

Theorem 5. *The species $\mathcal{T}op$ is a double twisted bialgebra with the two following coproducts:*

1. *For any quasi-poset $T \in \mathcal{T}op[A \sqcup B]$,*

$$\Delta_{A,B}(T) = \begin{cases} T_{|A} \otimes T_{|B}, & \text{if } B \text{ is an open set of } T, \\ 0, & \text{otherwise.} \end{cases}$$

   For any quasi-poset $T_A \in \mathcal{T}op[A]$, the counit ε of Δ is given by:

$$\varepsilon_A(T_A) = \begin{cases} 1, & \text{if } T_A = \varnothing, \\ 0, & \text{otherwise.} \end{cases}$$

2. *For any quasi-poset $T \in \mathcal{T}op[A]$,*

$$\delta_A(T) = \sum_{\sim \in \mathrm{CE}(T)} T/\sim \otimes T|\sim .$$

   The counit ε' of δ is given by:

$$\varepsilon'_A(T) = \begin{cases} 1, & \text{if } \leq_T \text{ is an equivalence}, \\ 0, & \text{otherwise.} \end{cases}$$

The product is defined as follows, for any quasi-posets $T_A \in \mathcal{T}op[A], T_B \in \mathcal{T}op[B]$:

$$m_{A,B}(T_A \otimes T_B) = T_A T_B,$$

where $T_A T_B \in \mathcal{T}op[A \sqcup B]$ is disjoint union topologies of T_A and T_B. An open set $C \in T_A T_B$ if and only if $C \cap A \in T_A$ and $C \cap B \in T_B$.

Proof. ■ We first prove that $(\mathcal{T}op, m, \Delta)$ is a twisted bialgebra.
Let T be a quasi-poset on a set E, and let $E = A \sqcup B \sqcup C$, we prove that Δ is coassociative satisfying the following equation:

$$(\Delta_{A,B} \otimes I) \circ \Delta_{A \sqcup B, C}(T) = (I \otimes \Delta_{B,C}) \circ \Delta_{A, B \sqcup C}(T).$$

We have the following deduction:

$$\begin{aligned}
&(\Delta_{A,B} \otimes I) \circ \Delta_{A \sqcup B, C}(T) \\
&= \begin{cases} (\Delta_{A,B} \otimes I)\big(T_{|A \sqcup B} \otimes T_C\big), & \text{if } C \text{ is an open set of } T, \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} T_{|A} \otimes T_{|B} \otimes T_{|C}, & \text{if } C \text{ is an open set of } T,\ B \text{ is an open set of } T_{|A \sqcup B}, \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} T_{|A} \otimes T_{|B} \otimes T_{|C}, & \text{if } C \text{ is an open set of } T,\ B \sqcup C \text{ is an open set of } T, \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} T_{|A} \otimes T_{|B} \otimes T_{|C}, & \text{if } B \sqcup C \text{ is an open set of } T,\ C \text{ is an open set of } T_{|B \sqcup C}, \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} (I \otimes \Delta_{B,C})\big(T_{|A} \otimes T_{|B \sqcup C}\big), & \text{if } B \sqcup C \text{ is an open set of } T, \\ 0, & \text{otherwise}, \end{cases} \\
&= (I \otimes \Delta_{B,C}) \circ \Delta_{A, B \sqcup C}(T).
\end{aligned}$$

Therefore, Δ is coassociative, the other commutative diagram of counit is easy to prove. Hence, $(\mathcal{T}op, m, \Delta)$ is a twisted bialgebra.

■ We now prove that $(\mathcal{T}op, m, \delta)$ is also a twisted bialgebra.
Let T be a quasi-poset on A, we need to prove that δ is coassociative satisfying the following equation:

$$(\delta_A \otimes I) \circ \delta(T) = (I \otimes \delta_A) \circ \delta(T).$$

Two hand sides are:

$$\begin{aligned}
(\delta_A \otimes I) \circ \delta_A(T) &= (\delta_A \otimes I)\Big(\sum_{\sim \in \mathrm{CE}(T)} T/\sim \otimes T|\sim\Big)\\
&= \sum_{\sim \in \mathrm{CE}(T), \sim' \in \mathrm{CE}(T/\sim)} (T/\sim)/\sim' \otimes (T/\sim)|\sim' \otimes T|\sim,\\
(I \otimes \delta_A) \circ \delta(T) &= (I \otimes \delta_A)\Big(\sum_{\sim' \in \mathrm{CE}(T)} T/\sim' \otimes T|\sim'\Big)\\
&= \sum_{\sim' \in \mathrm{CE}(T), \sim \in \mathrm{CE}(T|\sim')} T/\sim' \otimes (T|\sim')/\sim \otimes (T|\sim')|\sim .
\end{aligned}$$

Consider two following sets:

$$\begin{aligned}
\mathcal{X} &= \big\{(\sim, \sim')|\sim \in \mathrm{CE}(T), \sim' \in \mathrm{CE}(T/\sim)\big\},\\
\mathcal{Y} &= \big\{(\sim, \sim')|\sim' \in \mathrm{CE}(T), \sim \in \mathrm{CE}(T|\sim')\big\}.
\end{aligned}$$

We prove that both $\mathcal{X}, \mathcal{Y}$ have the relation $\sim$ being finer than the relation $\sim'$. This should be true because the following intuition: saturating T by $\sim$ makes $T/\sim$ coarser than T, and restricting T by $\sim'$ makes $T|\sim'$ finer than T.

For a pair $(\sim, \sim') \in \mathcal{X}$, if $x \sim y$, then $x \sim_{T/\sim}$ and $x \sim_{(T/\sim)/\sim'}$ (by definition of $\mathrm{CE}(T)$). Also, since $\sim' \in \mathrm{CE}(T/\sim)$, $x \sim_{(T/\sim)/\sim'}$ deduces $x \sim' y$.

For a pair $(\sim, \sim') \in \mathcal{Y}$, $\sim \in \mathrm{CE}(T|\sim')$ deduces that any equivalence class of $\sim$ is $T|\sim'$-connected. Hence, if $x \sim y$, then $x \sim_{T|\sim'} y$ which makes $x \sim' y$.

Therefore, we have $\sim$ is finer than $\sim'$ and can apply Lemma 5.6 to have:

$$\begin{aligned}
(\delta_A \otimes I) \circ \delta_A(T) &= \sum_{\sim \in \mathrm{CE}(T), \sim' \in \mathrm{CE}(T/\sim)} (T/\sim)/\sim' \otimes (T/\sim)|\sim' \otimes T|\sim\\
&= \sum_{(\sim,\sim') \in \mathcal{X}} T/\sim' \otimes (T|\sim')/\sim \otimes T|\sim,\\
(I \otimes \delta_A) \circ \delta(T) &= \sum_{\sim' \in \mathrm{CE}(T), \sim \in \mathrm{CE}(T|\sim')} T/\sim' \otimes (T|\sim')/\sim \otimes (T|\sim')|\sim\\
&= \sum_{(\sim,\sim') \in \mathcal{Y}} T/\sim' \otimes (T|\sim')/\sim \otimes T|\sim .
\end{aligned}$$

Hence, the coassociativity of the coproduct δ can be proved if $\mathcal{X} = \mathcal{Y}$.

- $\mathcal{X} \subseteq \mathcal{Y}$. Let $(\sim, \sim') \in \mathcal{X}$, we prove that $\sim' \in \mathrm{CE}(T)$ and $\sim \in \mathrm{CE}(T|\sim')$.

Let X' be an equivalence class of $\sim'$, then it is $\sim'$-saturated, so also $\sim$-saturated. By Corollary 5.8, as X' is $T/\sim$-connected, it is also T-connected.

Let X be an equivalence class of $\sim$, then it is included in a single class of $\sim'$, which deduces that $\forall x, y \in A : x \sim y \Longrightarrow x \sim' y$. Besides, by Corollary 5.8, as X is T-connected (by definition of $\sim$), it is also $T|\sim'$-connected.

By Lemma 5.6, we have

$$\sim_{T/\sim'} = \sim_{(T/\sim)/\sim'},$$

while $\sim_{(T/\sim)/\sim'} = \sim'$ as $\sim' \in \mathrm{CE}(T/\sim)$, therefore $\sim_{T/\sim'} = \sim'$. Combining this with the fact that X' is T-connected, we deduce that $\sim' \in \mathrm{CE}(T)$. Again, thanks to Lemma 5.6, we would have $\sim_{(T|\sim')/\sim} = \sim_{(T/\sim)|\sim'}$.

For any $x, y \in A$, we have:

$$\begin{aligned} x \sim_{(T/\sim)|\sim'} y &\iff x \sim' y \text{ and } x \sim_{T/\sim} y \\ &\iff x \sim' y \text{ and } x \sim y \\ &\iff x \sim y. \end{aligned}$$

Hence, $\sim_{(T|\sim')/\sim} = \sim$, we can combine this with the fact that X is $T|\sim'$-connected to obtain that $\sim \in \mathrm{CE}(T|\sim')$. Therefore, $\mathcal{X} \subseteq \mathcal{Y}$.

- $\mathcal{Y} \subseteq \mathcal{X}$. Let $(\sim, \sim') \in \mathcal{Y}$. Let X be a class of $\sim$, then X is $T|\sim'$-connected. By Corollary 5.8, it is T-connected. Let X' be a class of $\sim'$, then it is $\sim$-saturated and T-connected. By Corollary 5.8, it is $T/\sim$-connected.

  By Lemma 5.6, $\sim_{(T/\sim)/\sim'} = \sim_{T/\sim'} = \sim'$ as $\sim' \in \mathrm{CE}(T)$, so $\sim'$ $\mathrm{CE}(T/\sim)$. Let $x, y \in A$, if $x \sim_{T/\sim} y$, as $\sim \in \mathrm{CE}(T|\sim')$, x and y are in the same connected component of $T|\sim'$, which deduces that they are also in same class of $\sim'$ as $\sim' \in \mathrm{CE}(T)$, so $x \sim' y$. By combining this with $x \sim_{T/\sim} y$, we have $x \sim_{(T/\sim)|\sim'} y$, so $x \sim_{(T|\sim')/\sim} y$ and $x \sim y$ as $\sim \in \mathrm{CE}(T|\sim')$. Hence, $\sim \in \mathrm{CE}(T)$ and $\mathcal{Y} \subseteq \mathcal{X}$ as a consequence.

■ We prove that Δ is a right comodule morphism, that is, for any quasi-poset $T \in \mathcal{T}op[A]$ and $A = U \sqcup V$:

$$(\Delta_{U,V} \otimes I) \circ \delta_A(T) = m_{1,3,24} \circ (\delta_U \otimes \delta_V) \circ \Delta_{U,V}(T).$$

We have the following deduction:

$$
\begin{aligned}
&(\Delta_{U,V} \otimes I) \circ \delta_A(T) \\
&= (\Delta_{U,V} \otimes I)\Bigg(\sum_{\sim \in \mathrm{CE}(T)} T/\sim \otimes T|\sim \Bigg) \\
&= \sum_{\sim \in \mathrm{CE}(T), V \in O(T/\sim)} (T/\sim)_{|U} \otimes (T/\sim)_{|V} \otimes T|\sim \\
&= \sum_{\sim \in \mathrm{CE}(T), V \in O(T), \sim\text{-saturated}} (T/\sim)_{|U} \otimes (T/\sim)_{|V} \otimes T|\sim \\
&= \begin{cases} \sum_{\sim' \in \mathrm{CE}(T_{|U}), \sim'' \in \mathrm{CE}(T_{|V})} (T_{|U})/\sim' \otimes (T_{|V})/\sim'' \otimes T|(\sim' \sqcup \sim''), & \text{if } V \in O(T), \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} \sum_{\sim' \in \mathrm{CE}(T_{|U}), \sim'' \in \mathrm{CE}(T_{|V})} (T_{|U})/\sim' \otimes (T_{|V})/\sim'' \otimes (T_{|U})|\sim' \, (T_{|V})|\sim'', & \text{if } V \in O(T), \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} m_{1,3,24}\Bigg(\sum_{\sim' \in \mathrm{CE}(T_{|U}), \sim'' \in \mathrm{CE}(T_{|V})} (T_{|U})/\sim' \otimes (T_{|U})|\sim' \otimes (T_{|V})/\sim'' \otimes (T_{|V})|\sim'' \Bigg), \\ \qquad\qquad \text{if } V \in O(T), \\ 0, \quad \text{otherwise}, \end{cases} \\
&= \begin{cases} m_{1,3,24} \circ (\delta_U \otimes \delta_V)\big(T_{|U} \otimes T_{|V}\big), & \text{if } V \in O(T), \\ 0, & \text{otherwise}, \end{cases} \\
&= m_{1,3,24} \circ (\delta_U \otimes \delta_V) \circ \Delta_{U,V}(T).
\end{aligned}
$$

By combining three proved statements, we conclude that $(\mathcal{T}op, m, \Delta, \delta)$ is a double twisted bialgebra of finite topologies. $\square$

Example 5.9. If A, B, C are finite sets:

$$
\begin{aligned}
\Delta(\bullet_A) &= \bullet_A \otimes \mathbf{1} + \mathbf{1} \otimes \bullet_A, \\
\Delta({}^{B}\vee^{C}_{A}) &= {}^{B}\vee^{C}_{A} \otimes \mathbf{1} + \mathrm{l}^{B}_{A} \otimes \bullet_C + \mathrm{l}^{C}_{A} \otimes \bullet_B + \bullet_A \otimes \bullet_B \bullet_C + \mathbf{1} \otimes {}^{B}\vee^{C}_{A}, \\
\Delta(\vdots^{C}_{B,A}) &= \vdots^{C}_{B,A} \otimes \mathbf{1} + \mathrm{l}^{B}_{A} \otimes \bullet_C + \bullet_A \otimes \mathrm{l}^{C}_{B} + \mathbf{1} \otimes \vdots^{C}_{B,A}, \\
\delta(\bullet_A) &= \bullet_A \otimes \bullet_A, \\
\delta({}^{B}\vee^{C}_{A}) &= {}^{B}\vee^{C}_{A} \otimes \bullet_A \bullet_B \bullet_C + \mathrm{l}^{C}_{A \sqcup B} \otimes \mathrm{l}^{B}_{A} \bullet_C + \mathrm{l}^{B}_{A \sqcup C} \otimes \mathrm{l}^{C}_{A} \bullet_B + \bullet_{A \sqcup B \sqcup C} \otimes {}^{B}\vee^{C}_{A}, \\
\delta(\vdots^{C}_{B,A}) &= \vdots^{C}_{B,A} \otimes \bullet_A \bullet_B \bullet_C + \mathrm{l}^{C}_{A \sqcup B} \otimes \mathrm{l}^{B}_{A} \bullet_C + \mathrm{l}^{B \sqcup C}_{A} \otimes \bullet_A \mathrm{l}^{C}_{B} + \bullet_{A \sqcup B \sqcup C} \otimes \vdots^{C}_{B,A}.
\end{aligned}
$$

6 Conclusion

The Hopf algebra structures appear naturally in three considered examples of combinatorial objects. They are visualizable and provide us the intuition of how algebra and coalgebra structures of the objects interact and satisfy compatibility relation. This internship report accomplishes its primary objective of presenting pivotal Hopf algebra concepts, accompanied by intuitive instances, and furnishes a bedrock of underlying knowledge to facilitate future researches.

Along with its ubiquity in all fields of mathematics, Hopf algebras recently appeared in machine learning researches, provided more tools for computer scientists in order to have better understanding of models and develop new algorithms. Discussing the emerging field of Hopf algebras and machine learning is out of the scope of this internship report, so it will be left for further investigation.

In the future, I would like to delve deeper into related topics of this internship report, such as combinatorial species and operads, application of shuffle Hopf algebras and locally compact group theory.

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