

The Complexity of Distance- r Dominating Set Reconfiguration

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- 1 Graph Reconfiguration
- 2 Distance- r Dominating Set Reconfiguration ($DrDSR$)
- 3 Planar Graphs
- 4 Split Graphs
- 5 Open Questions

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Graph Reconfiguration

Reconfiguration Setting

In a *reconfiguration variant* of a computational problem (e.g., SAT, INDEPENDENT SET, DOMINATING SET, VERTEX-COLORING, etc.), two *feasible solutions* S and T are given along with a *reconfiguration rule* that describes how to slightly modify one feasible solution to obtain a new one.

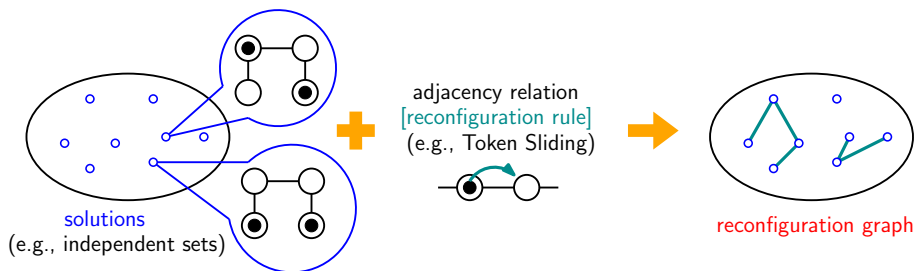


Figure: Reconfiguration.

Graph Reconfiguration

Main Questions

In the reconfiguration graph,

- REACHABILITY: Is there a path between two given solutions? Can we transform S into T via a sequence of feasible solutions?

Such a sequence, if exists, is called a *reconfiguration sequence*.

- SHORTEST PATH: If REACHABILITY is yes, can we find a shortest path between S and T ?

Graph Reconfiguration

Reconfiguration Rules

Tells us rules we need to follow to go from one feasible solution to another.

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- **Token Jumping (TJ)**: one can move a token to any unoccupied vertex as long as the resulting token-set forms a feasible solution.
- **Token Addition/Removal ($TAR(k)$)**: one can either add or remove a token as long as the resulting token-set forms a feasible solution of size at most some threshold $k \geq 0$.

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Distance- r Dominating Set Reconfiguration (D_rDSR)

Distance- r Dominating Set (D_rDS) (Meir and Moon [PJM 1975])

Given a fixed integer $r \geq 1$, a *distance- r dominating set* (D_rDS) of G is a vertex subset D where each vertex of G is within distance r from some member of D .

For $r = 1$, this is the classical dominating set concept.

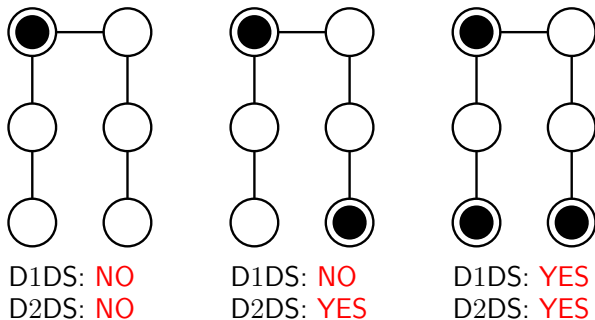


Figure: Examples of D_rDS s ($r = 1, 2$).

Distance- r Dominating Set Reconfiguration (D_r DSR)

Goal

We study **REACHABILITY** of D_r DSs ($r \geq 1$) under TS and TJ .

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(G, D_s, D_t) under R

- **Input:** Two Dr DSs ($r \geq 1$) D_s and D_t of a graph G and a reconfiguration rule $R \in \{TS, TJ\}$.

Distance- r Dominating Set Reconfiguration (DrDSR)

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We study **REACHABILITY** of DrDS s ($r \geq 1$) under TS and TJ .

(G, D_s, D_t) under R

- **Input:** Two DrDS s ($r \geq 1$) D_s and D_t of a graph G and a reconfiguration rule $R \in \{TS, TJ\}$.
- **Question:** Is there a reconfiguration sequence between D_s and D_t , i.e., a sequence $\langle D_0, D_1, \dots, D_\ell \rangle$ such that each D_i is a DrDS and D_{i+1} is obtained from D_i ($i \in \{0, \dots, \ell\}$) by applying R exactly once?

Distance- r Dominating Set Reconfiguration (D_r DSR)

Haddadan et al. [TCS 2016] studied the problem for $r = 1$ under TAR . Later, Bonamy et al. [DAM 2021] observed that these results also hold under TJ .

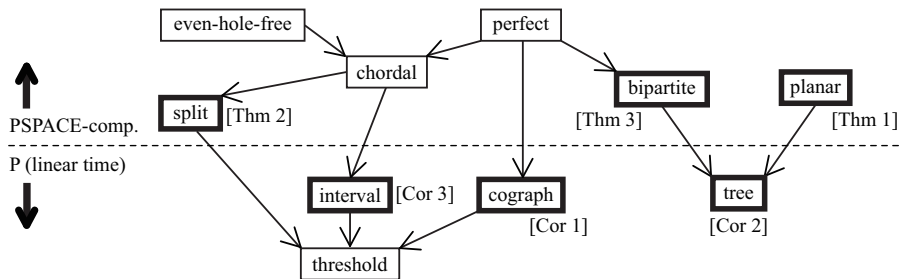


Figure: Complexity of D_1 DSR under TAR on some graphs, © Haddadan et al. [TCS 2016]

Distance- r Dominating Set Reconfiguration (D_r DSR)

Bonamy et al. [DAM 2021] showed that several results of Haddadan et al. [TCS 2016] also hold under TS and some can be generalized.

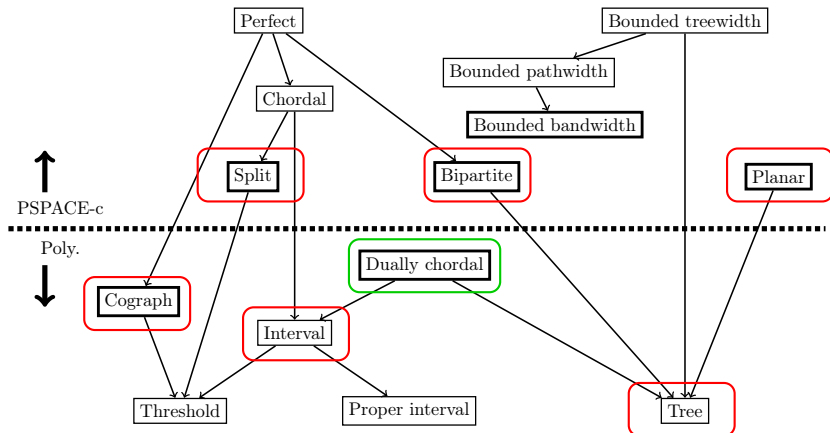


Figure: Complexity of D1DSR under TS on some graphs, © Bonamy et al. [DAM 2021]

Distance- r Dominating Set Reconfiguration (D_r DSR)

More results for $r = 1$ under TS :

- Bousquet and Joffard [FCT 2021]: PSPACE-complete on circle graphs, P on circular-arc graphs.
- Kříšt'an and Svoboda [FCT 2023]: Polynomial-time algorithms for **SHORTEST PATH variants** on trees and interval graphs.

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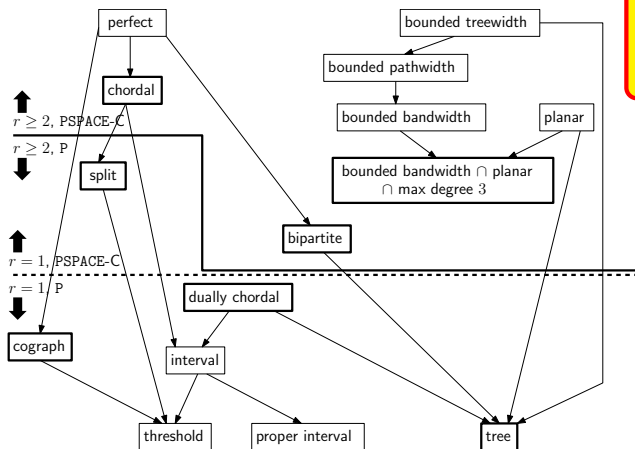
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- **Bousquet et al. [CSR 2024]:** FPT parameterized by ℓ on any graph class where first-order model-checking is in FPT.

Distance- r Dominating Set Reconfiguration (D_r DSR)

Our Results

We prove several (classic) complexity results for $r \geq 2$ under TJ and TS on different graph classes.

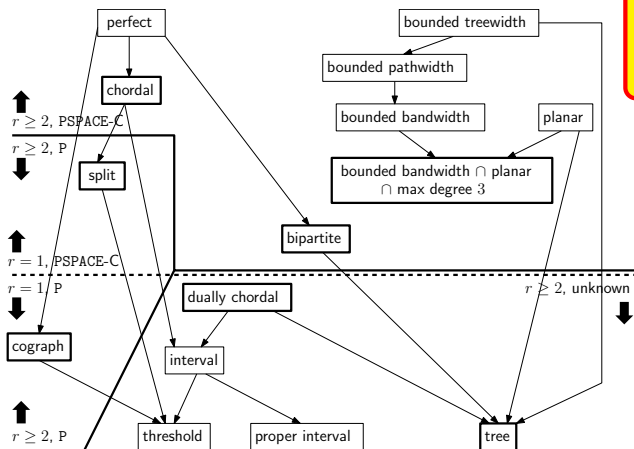


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Main Result 1

D_r DSR ($r \geq 1$) is PSPACE-complete on planar graphs of maximum degree 3 and bounded bandwidth.

- Previously known results ($r = 1$) are only for “maximum degree 6”.
- We improve the known results and extend for $r \geq 2$.

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Main Result 2

D_r DSR ($r \geq 1$) on split graphs: PSPACE-complete when $r = 1$ (which is already known) but in P when $r \geq 2$ (which we prove).

- An interesting complexity dichotomy.
- We further establish some non-trivial bounds on the length of a shortest reconfiguration sequence when $r = 2$. (The case $r \geq 3$ is trivial and boring.)

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Idea

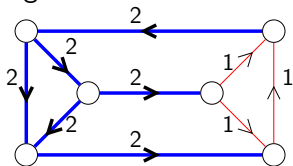
Reduction from [Nondeterministic Constraint Logic \(NCL\)](#), a powerful tool introduced by Hearn and Demaine [TCS 2008]

Nondeterministic Constraint Logic (NCL)

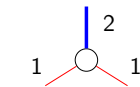
• Input:

- Each *state/configuration* involves a graph having **red** (weight 1) and **blue** (weight 2) edges where each edge is oriented such that **(*) the sum of weights of in-coming arcs at each vertex is at least 2.**
- Reconfiguration Rule:** Each *move* involves re-orienting an edge such that **(*)** is satisfied.

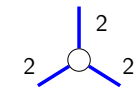
- Question:** Is there a sequence of moves that transforms one given configuration into another?



(a) An NCL configuration.



(b) AND vertex.

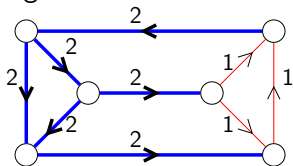


(c) OR vertex.

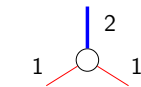
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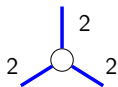
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PSPACE-complete even on *planar graphs* having only *two types of vertices*.

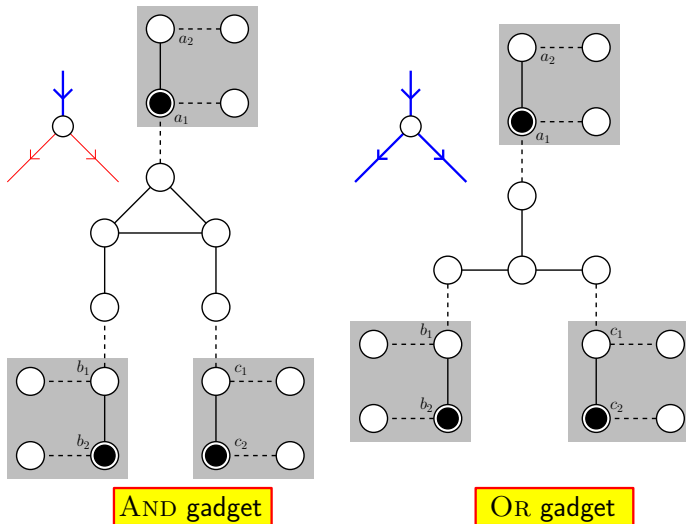


Figure: Our gadgets for DrDSR. Each dashed edge represents a path of length $r - 1$. The gray boxes indicate the *link components* in a gadget.

Gadgets are joined together at their link components

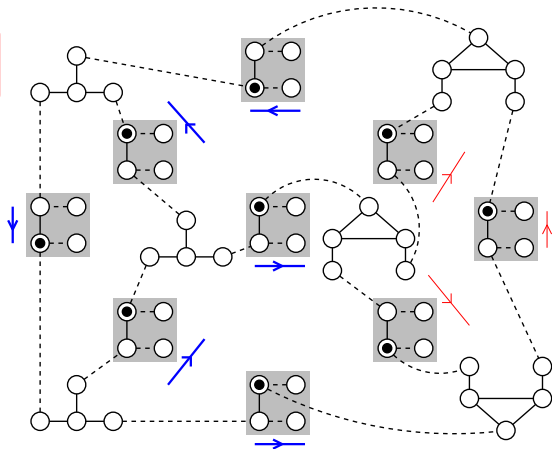
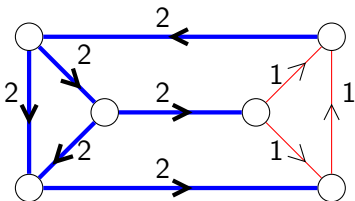


Figure: A NCL AND/OR constraint graph and its configuration (Left) and the corresponding graph and token-set (Right). Each dashed edge represents a path of length $r - 1$. The gray boxes indicate the link components in a gadget.

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Idea (for $r \geq 2$)

- $r \geq 3$: trivial. (Any connected split graph has diameter $\leq 3 \Rightarrow$ Any non-empty token-set is a $DrDS$ \Rightarrow Reconfiguration is easy!)
- $r = 2$: when doing reconfiguration, *always keep at least one token in the clique side* of the split graph.

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Bounds on Length ℓ_R ($R \in \{TS, TJ\}$) of a Shortest R -Sequence Between Two D2DSs D_s and D_t of a Split Graph

$$\min_{\substack{\text{bijection} \\ f: D_s \rightarrow D_t}} \sum_{s \in D_s} \text{dist}(s, f(s)) \underset{\text{trivial}}{\leq} \ell_{TS} \underset{\text{non-trivial}}{\leq} \min_{\substack{\text{bijection} \\ f: D_s \rightarrow D_t}} \sum_{s \in D_s} \text{dist}(s, f(s)) + 2.$$

$$\frac{|D_s \Delta D_t|}{2} \underset{\text{trivial}}{\leq} \ell_{TJ} \underset{\text{non-trivial}}{\leq} \frac{|D_s \Delta D_t|}{2} + 1.$$

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Open Questions

Question 1

What is the complexity of DrDSR ($r \geq 2$) on *trees* under TS?

Question 2

What is the complexity of DrDSR ($r \geq 2$) on *interval graphs* under TS?