The Complexity of Distance-r Dominating Set Reconfiguration

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Outline

Graph Reconfiguration

Reconfiguration Setting

In a *reconfiguration variant* of a computational problem (e.g., SAT, INDEPENDENT SET, DOMINATING SET, VERTEX-COLORING, etc.), two *feasible solutions* S and T are given along with a *reconfiguration rule* that describes how to slightly modify one feasible solution to obtain a new one.

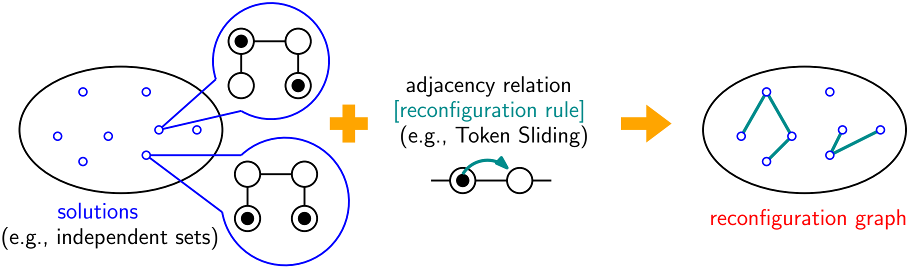

Figure: Reconfiguration.

Graph Reconfiguration

Main Questions

In the reconfiguration graph,

- REACHABILITY: Is there a path between two given solutions? Can we transform S into T via a sequence of feasible solutions?

  Such a sequence, if exists, is called a *reconfiguration sequence*.

- SHORTEST PATH: If REACHABILITY is yes, can we find a shortest path between S and T?

Graph Reconfiguration

Reconfiguration Rules

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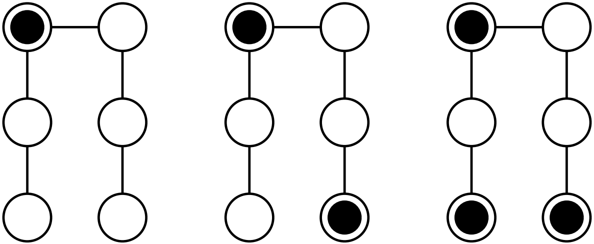
- **Token Jumping (TJ):** one can move a token to any unoccupied vertex as long as the resulting token-set forms a feasible solution.
- **Token Addition/Removal ($TAR(k)$):** one can either add or remove a token as long as the resulting token-set forms a feasible solution of size at most some threshold $k \geq 0$.

Distance-r Dominating Set Reconfiguration (DrDSR)

Distance-r Dominating Set (DrDS) (Meir and Moon [PJM 1975])

Given a fixed integer $r \geq 1$, a *distance-r dominating set (DrDS)* of G is a vertex subset D where each vertex of G is within distance r from some member of D.

For $r = 1$, this is the classical dominating set concept.



D1DS: NO
D2DS: NO

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Figure: Examples of DrDSs ($r = 1, 2$).

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(G, D_s, D_t) under R

- **Input:** Two DrDSs ($r \geq 1$) D_s and D_t of a graph G and a reconfiguration rule $R \in \{TS, TJ\}$.
- **Question:** Is there a reconfiguration sequence between D_s and D_t, i.e., a sequence $\langle D_0, D_1, \dots, D_\ell \rangle$ such that each D_i is a DrDS and D_{i+1} is obtained from D_i ($i \in \{0, \dots, \ell\}$) by applying R exactly once?

Distance-r Dominating Set Reconfiguration (DrDSR)

Haddadan et al. [TCS 2016] studied the problem for $r = 1$ under TAR. Later, Bonamy et al. [DAM 2021] observed that these results also hold under TJ.

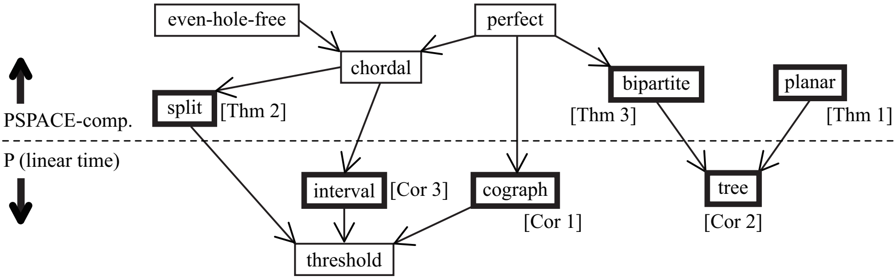

Figure: Complexity of D1DSR under TAR on some graphs, © Haddadan et al. [TCS 2016]. Arrows indicate inclusion.

Distance-r Dominating Set Reconfiguration (DrDSR)

Bonamy et al. [DAM 2021] showed that several results of Haddadan et al. [TCS 2016] also hold under TS.

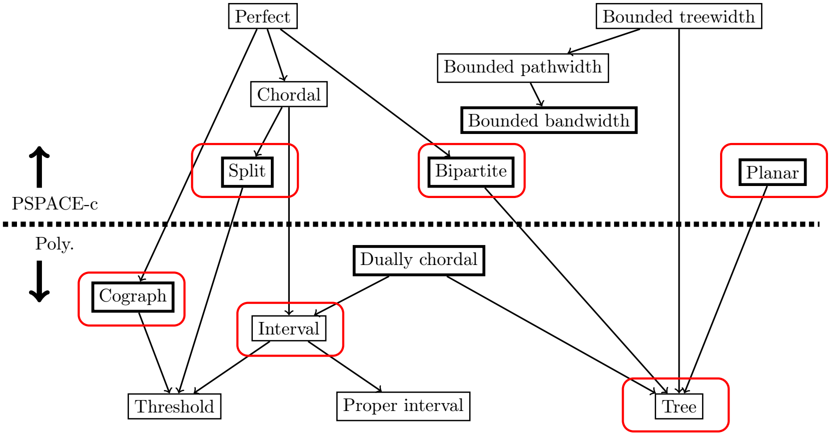

Figure: Complexity of D1DSR under TS on some graphs, © Bonamy et al. [DAM 2021]. Arrows indicate inclusion.

Distance-r Dominating Set Reconfiguration (DrDSR)

More results for $r = 1$ under TS:

- Bousquet and Joffard [FCT 2021]: PSPACE-complete on circle graphs, P on circular-arc graphs.
- Křišt'an and Svoboda [FCT 2023]: Polynomial-time algorithms for SHORTEST PATH variants on trees and interval graphs.

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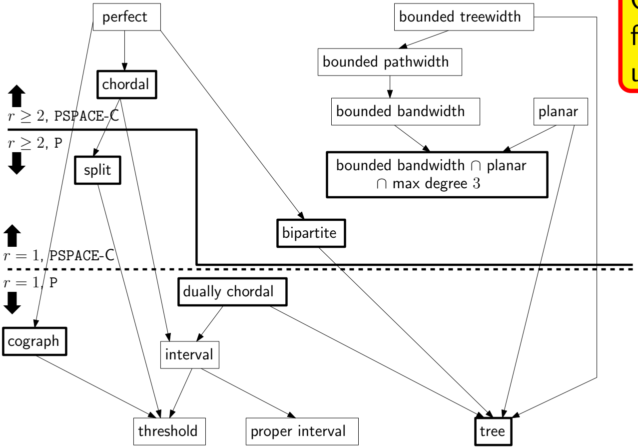
- Bousquet et al. [CSR 2024]: FPT parameterized by ℓ on any graph class where first-order model-checking is in FPT.

Distance-r Dominating Set Reconfiguration (DrDSR)

Our Results

We prove several (classic) complexity results for $r \geq 2$ under TJ and TS on different graph classes. (Arrows indicate inclusion.)

Our Results for $r \geq 2$ under TJ



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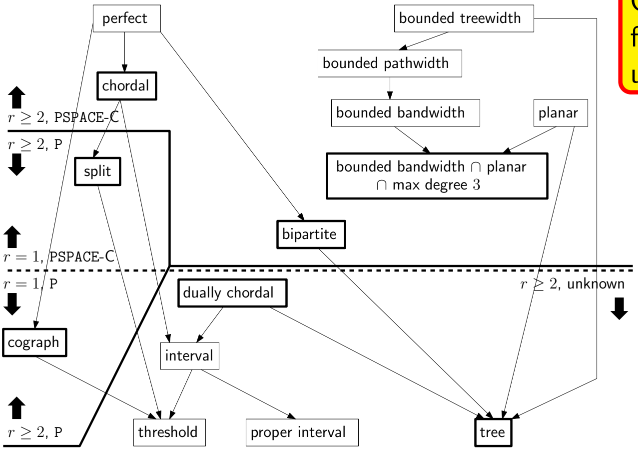

Distance-r Dominating Set Reconfiguration (DrDSR)

Main Result 1

DrDSR ($r \geq 1$) is PSPACE-complete on planar graphs of maximum degree 3 and bounded bandwidth.

- Previously known results ($r = 1$) are only for "maximum degree 6".
- We improve the known results and extend for $r \geq 2$.

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Main Result 2

DrDSR ($r \geq 1$) on split graphs: PSPACE-complete when $r = 1$ (which is already known) but in P when $r \geq 2$ (which we prove).

- An interesting complexity dichotomy.
- We further establish some non-trivial bounds on the length of a shortest reconfiguration sequence when $r = 2$. (The case $r \geq 3$ is trivial and boring.)

Planar Graphs

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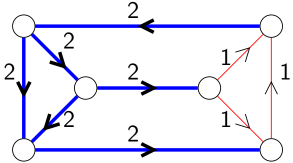
Idea

Reduction from Nondeterministic Constraint Logic (NCL), a powerful tool introduced by Hearn and Demaine [TCS 2008]

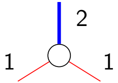
Planar Graphs

Nondeterministic Constraint Logic (NCL)

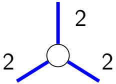
- **Input:**
  - Each *state/configuration* involves a graph having red (weight 1) and blue (weight 2) edges where each edge is oriented such that $(\star)$ the sum of weights of in-coming arcs at each vertex is at least 2.
  - **Reconfiguration Rule:** Each *move* involves re-orienting an edge such that $(\star)$ is satisfied.
- **Question:** Is there a sequence of moves that transforms one given configuration into another?



(a) An NCL configuration.



(b) AND vertex.



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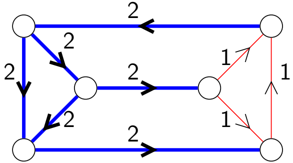

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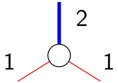

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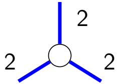

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PSPACE-complete even on *planar graphs* having only *two types of vertices*.

Planar Graphs

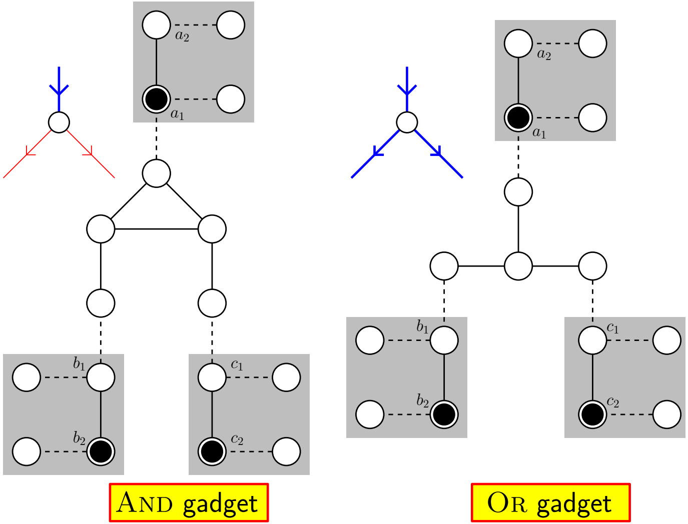
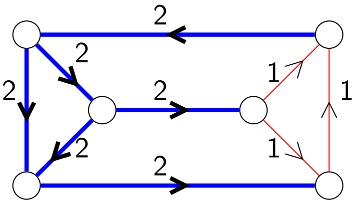


Figure: Our gadgets for DrDSR. Each dashed edge represents a path of length $r-1$. The gray boxes indicate the *link components* in a gadget.

Planar Graphs

Gadgets are joined together at their link components.



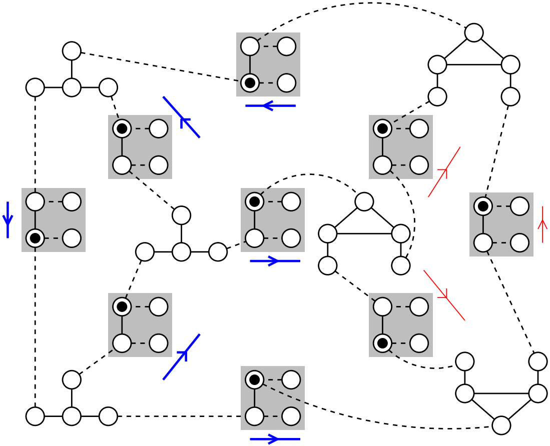


Figure: A NCL AND/OR constraint graph and its configuration (Left) and the corresponding graph and token-set (Right). Each dashed edge represents a path of length $r-1$. The gray boxes indicate the *link components* in a gadget.

Split Graphs

Main Result 2

DrDSR ($r \geq 1$) on split graphs: PSPACE-complete when $r = 1$ (which is already known) but in P when $r \geq 2$ (which we prove).

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Idea (for $r \geq 2$)

- $r \geq 3$: trivial. (Any connected split graph has diameter $\leq 3 \Rightarrow$ Any non-empty token-set is a DrDS $\Rightarrow$ Reconfiguration is easy!)
- $r = 2$: when doing reconfiguration, *always keep at least one token in the clique side* of the split graph.

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Bounds on Length ℓ_R ($R \in \{TS, TJ\}$) of a Shortest R-Sequence Between Two D2DSs D_s and D_t of a Split Graph

$$\min_{\substack{\text{bijection}\\ f: D_s \to D_t}} \sum_{s \in D_s} \mathsf{dist}(s, f(s)) \underset{\text{trivial}}{\leq} \ell_{TS} \underset{\text{non-trivial}}{\leq} \min_{\substack{\text{bijection}\\ f: D_s \to D_t}} \sum_{s \in D_s} \mathsf{dist}(s, f(s)) + 2.$$

$$\frac{|D_s \Delta D_t|}{2} \underset{\text{trivial}}{\leq} \ell_{TJ} \underset{\text{non-trivial}}{\leq} \frac{|D_s \Delta D_t|}{2} + 1.$$

Split Graphs

(G, D_s, D_t) under TS with $\ell_{TS} = \min_{\substack{\text{bijection} \\ f:D_s \to D_t}} \sum_{s \in D_s} \text{dist}(s, f(s)) + 2$.

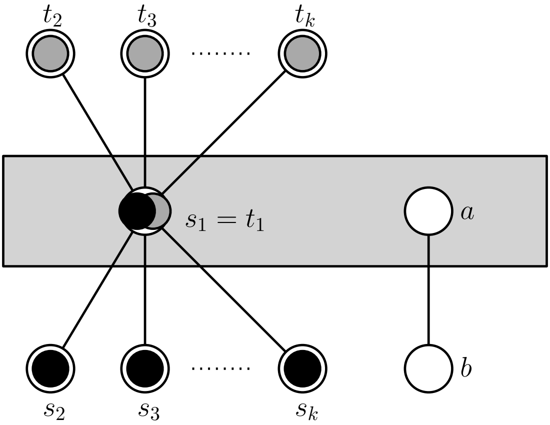
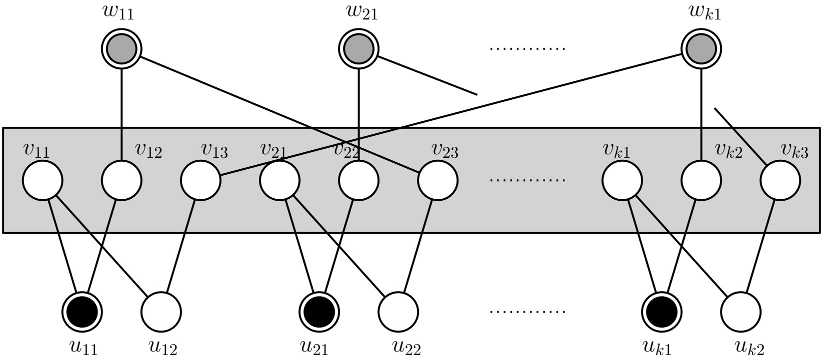


Figure: A D2DSR's instance on split graphs where ℓ_{TS} achieves the (non-trivial) upper bound. Black/gray tokens are respectively in D_s/D_t.

Split Graphs

(G, D_s, D_t) under TJ with $\ell_{TJ} = \dfrac{|D_s \Delta D_t|}{2} + 1$.



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Open Questions

Question 1

What is the complexity of DrDSR ($r \geq 2$) on *trees* under TS?

Question 2

What is the complexity of DrDSR ($r \geq 2$) on *interval graphs* under TS?