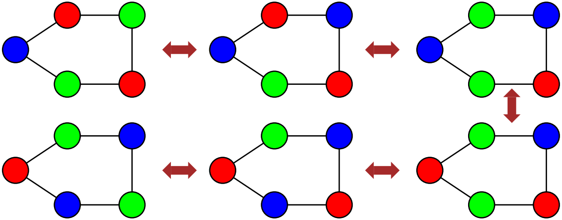
VIASM-KAIST Annual Meeting on Mathematics Sciences



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Distance Recoloring

in collaboration with
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Contents

Introduction to Reconfiguration

Reconfiguration Setting

- A description of what *states* (≡ *configurations*) are
- One or more *allowed moves* between states (≡ *reconfiguration rule(s)*)

Reconfiguration Process	Reconfiguration Graph	Example Rubik's Cube
state	node	
move *[reconfiguration rule]*	edge	
sequence of moves *[reconfiguration sequence]*	path	

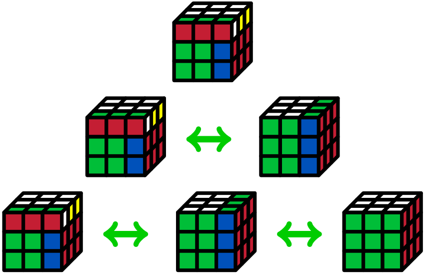


Figure: Reconfiguration Setting

Introduction to Reconfiguration

Reconfiguration vs. Solution Space

For a computational problem $\mathcal{P}$ (e.g., Independent Set, Dominating Set, Vertex-Coloring, etc.)

Reconfiguration	**Solution Space**
States/Configurations	Feasible solutions of $\mathcal{P}$
Allowed Moves	Slight modifications of a solution *without changing its feasibility*

Algorithmic Questions

- Reachability: Given two states S and T, is there a sequence of moves that *transforms S into T*?
- Shortest Transformation: Given two states S and T and a positive integer ℓ, is there a sequence of moves that *transforms S into T using at most ℓ moves*?
- Connectivity: Is there a sequence of moves *between any pair of states*?
- and so on

Introduction to Reconfiguration

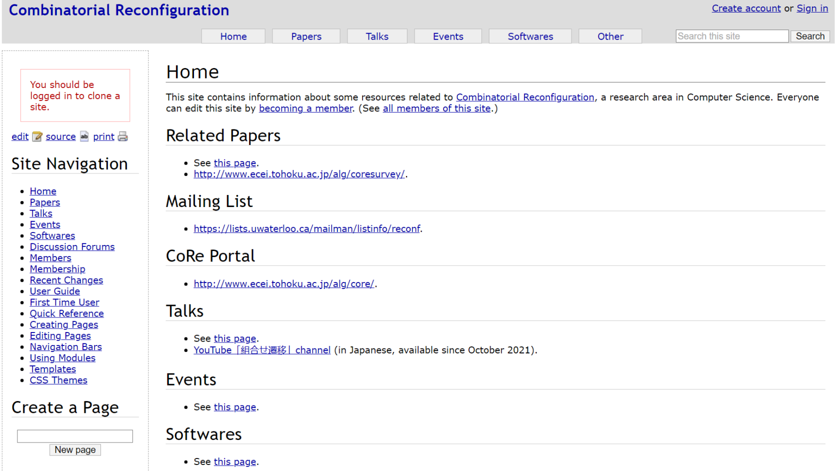


Figure: Online Reconfiguration Wiki Page (`https://reconf.wikidot.com/`)

Introduction to Reconfiguration

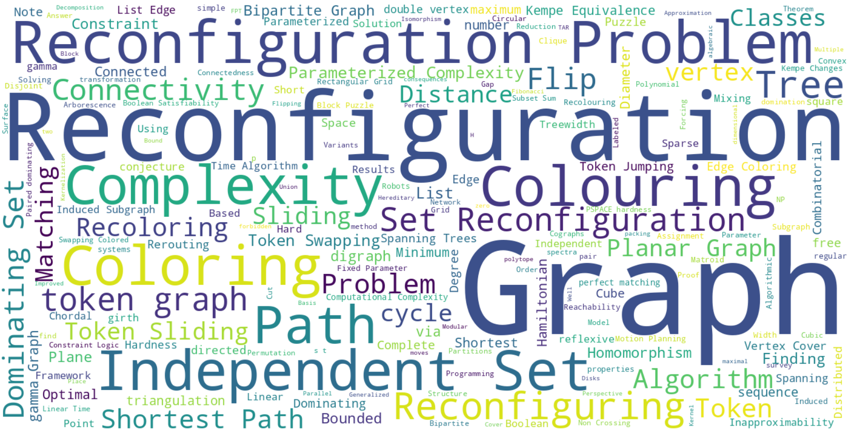
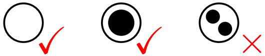


Figure: A word cloud of titles extracted from the current list of papers related to reconfiguration problems available at `https://reconf.wikidot.com/` (Accessed on October 26, 2025)

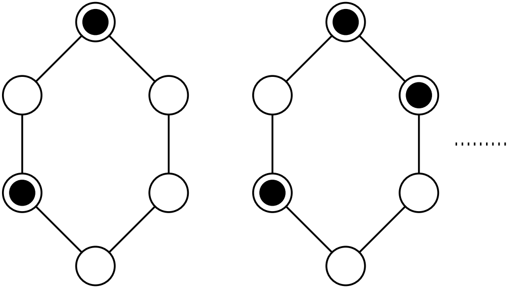
Some Examples

Token Reconfiguration

Each vertex has at most one token



Each state is a set of tokens satisfying certain property



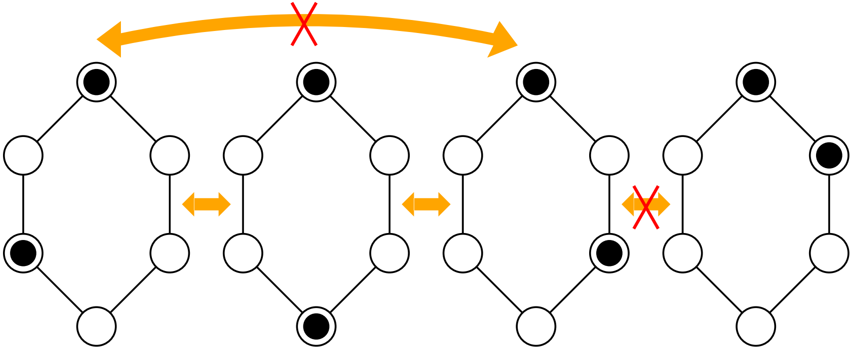
Independent Set Dominating Set

Some Examples

[Hearn and Demaine 2005]

Token Reconfiguration (Example: Independent Set Reconfiguration)

Token Sliding (TS)



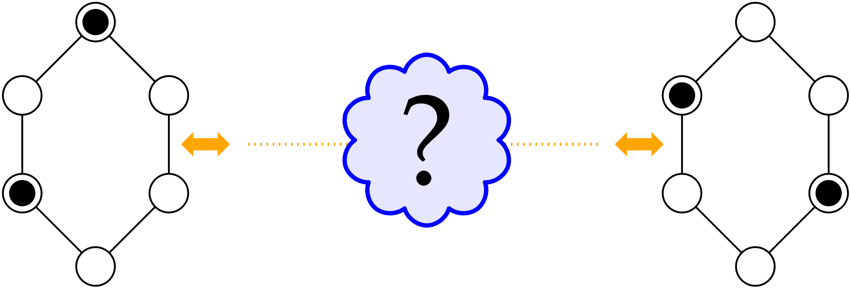
Some Examples

[Hearn and Demaine 2005]

Token Reconfiguration (Example: Independent Set Reconfiguration)

Token Sliding (TS)

PSPACE-complete on planar graphs



Independent Set

Independent Set

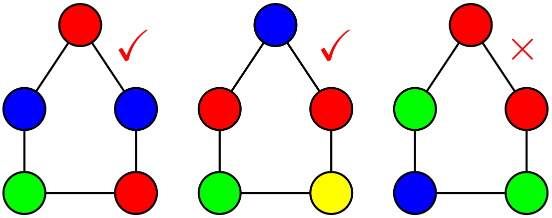
Some Examples

[Cereceda, van den Heuvel, and Johnson 2008]

Vertex-Coloring Reconfiguration

Each vertex is colored by one of the k given colors

Each state is a k-coloring of all vertices such that
no two adjacent vertices share the same color



3-coloring 4-coloring ~~3-coloring~~

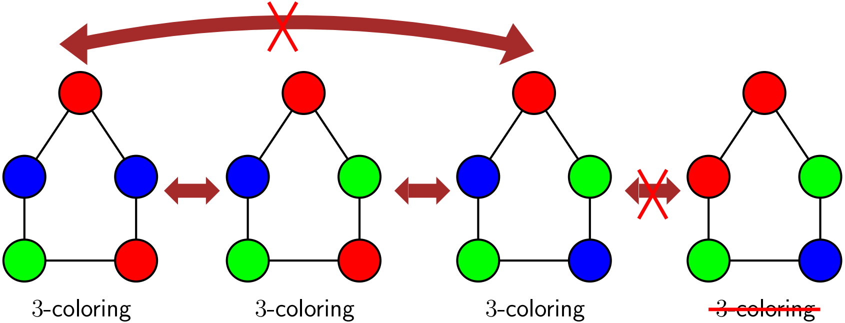
Some Examples

[Cereceda, van den Heuvel, and Johnson 2008]

VERTEX-COLORING RECONFIGURATION

Recoloring using $\leq k$ colors

Example: $k = 3$



Some Examples

[Cereceda, van den Heuvel, and Johnson 2008]

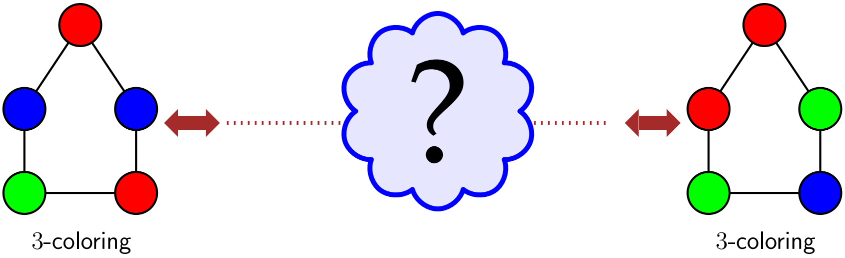
VERTEX-COLORING RECONFIGURATION

Recoloring using $\leq k$ colors

Example: $k = 3$

PSPACE-complete on general graphs for $k \geq 4$

P on general graphs for $k \leq 3$

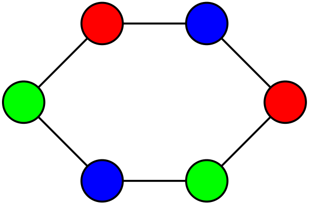


Distance Recoloring

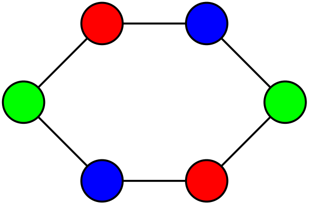
Distance constraints for *vertex colorings*

	k-Coloring	(d, k)-Coloring
Distance between two vertices having the same color	≥ 2	$\geq d + 1$

- (d, k)-coloring was first studied in [F. Kramer and H. Kramer 1969]
- Has applications in *frequency assignment problem* (or radio channel assignment) [F. Kramer and H. Kramer 2008]



3-coloring
~~$(2, 3)$-coloring~~

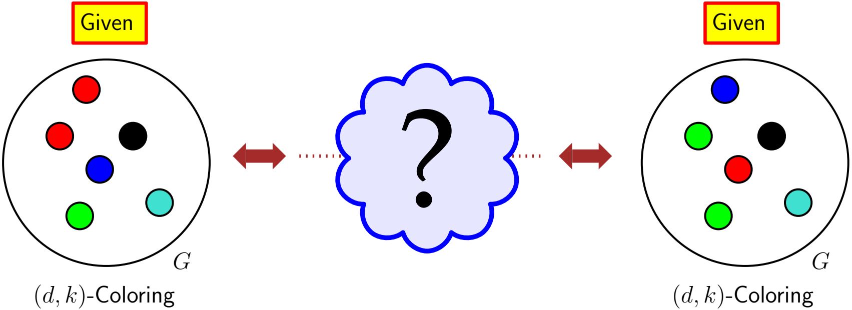


3-coloring
$(2, 3)$-coloring

Distance Recoloring

(d, k)-Coloring Reconfiguration ((d, k)-CR)

Reconfiguration Rule: Recoloring a vertex



- The case $d = 1$ (k-Coloring Reconfiguration (k-CR)) has been well-studied [Mynhardt and Nasserasr 2019]; [Heuvel 2013]
- We focus on the case $d \geq 2$

Distance Recoloring

<table>
<tr><th>Graph</th><th>k-CR ($=(1,k)$-CR)</th><th>(d,k)-CR ($d \geq 2$)</th></tr>
<tr><td>general</td><td>PSPACE-C ($k \geq 3$)
[Cereceda, Heuvel, and Johnson 2011]</td><td rowspan="6">PSPACE-C ($k = \Omega(d^2)$)</td></tr>
<tr><td>planar</td><td>PSPACE-C ($4 \leq k \leq 6$)
P ($k \geq 7$)
[Bonsma and Cereceda 2009]</td></tr>
<tr><td>bipartite</td><td>PSPACE-C ($k \geq 4$)
[Bonsma and Cereceda 2009]</td></tr>
<tr><td>planar ∩ bipartite</td><td>PSPACE-C ($k = 4$)
P ($k \geq 5$)
[Bonsma and Cereceda 2009]</td></tr>
<tr><td>2-degenerate</td><td>P
[Hatanaka, Ito, and Zhou 2019]</td></tr>
<tr><td>planar ∩ bipartite ∩ 2-degenerate</td><td>P
(⊆ 2-degenerate)</td></tr>
<tr><td>path</td><td>P
(⊆ planar ∩ bipartite ∩ 2-degenerate)</td><td>P ($k \geq d+1$)</td></tr>
<tr><td rowspan="2">split</td><td rowspan="2">P
[Hatanaka, Ito, and Zhou 2019]</td><td>PSPACE-C ($d = 2$, large k)</td></tr>
<tr><td>P ($d \geq 3$)</td></tr>
</table>

Table: Our Results for $d \geq 2$. We provide the status for $d = 1$ for comparison. Here PSPACE-C stands for PSPACE-complete [Banerjee, Engels, and **Hoang** 2024]

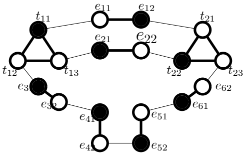
Distance Recoloring

Theorem (Banerjee, Engels, and Hoang 2024)

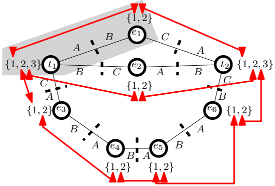
(d,k)-CR *is* PSPACE*-complete for* $d \geq 2$ *and* $k = \Omega(d^2)$ *on graphs which are planar, bipartite, and* 2*-degenerate.*

Proof Sketch.

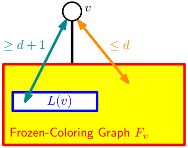
Restricted PSPACE-Complete Variant of ISR under TS
(Restricted Sliding Tokens)



List $(d, \Omega(d))$-Coloring Reconfiguration



$(d, \Omega(d^2))$-Coloring Reconfiguration



Distance Recoloring

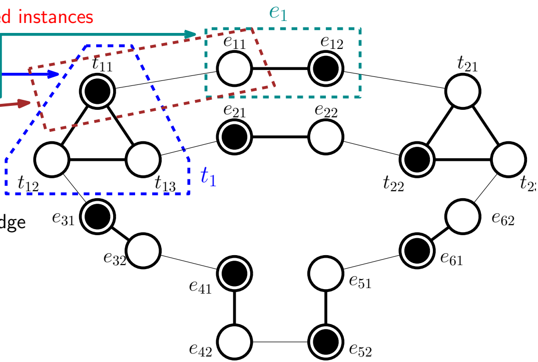
First Phase

RESTRICTED SLIDING TOKENS $\Rightarrow$ LIST $(d, \Omega(d))$-COLORING RECONFIGURATION

RESTRICTED SLIDING TOKENS

PSPACE-complete on very restricted instances

- Three types of gadgets
  - Token triangle
  - Token edge
  - Link edge
- Planar
- Max degree 3, min degree 2
- No two token triangles are directly joined by a link edge
- Each token triangle/token edge has exactly one token



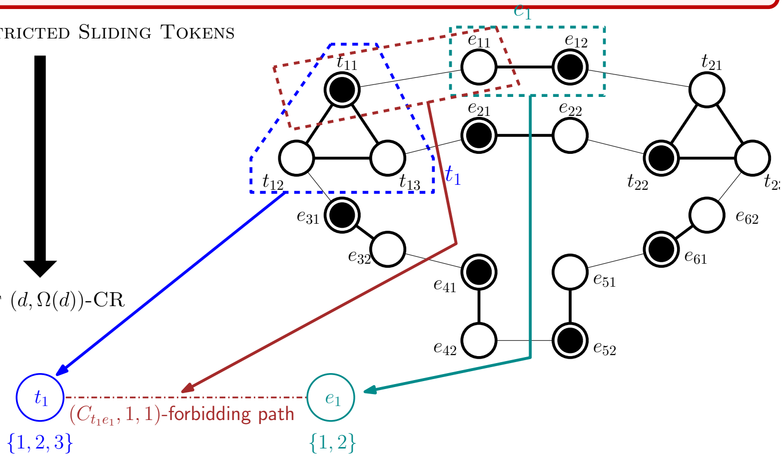
Distance Recoloring

First Phase

RESTRICTED SLIDING TOKENS $\Rightarrow$ LIST $(d, \Omega(d))$-COLORING RECONFIGURATION

RESTRICTED SLIDING TOKENS

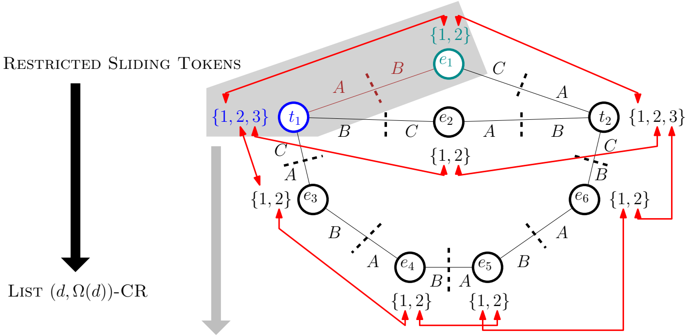
LIST $(d, \Omega(d))$-CR



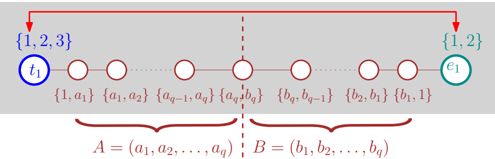
Distance Recoloring

First Phase

RESTRICTED SLIDING TOKENS $\Rightarrow$ LIST $(d, \Omega(d))$-COLORING RECONFIGURATION



Magnification of $(C_{t_1e_1}, 1, 1)$-forbidding path $P_{t_1e_1}$, where $C_{t_1e_1} = A \cup B$



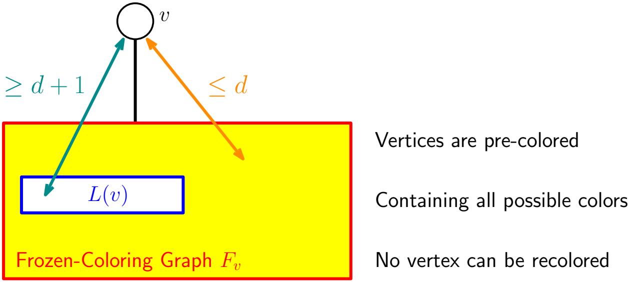
Distance Recoloring

Second Phase

LIST $(d, \Omega(d))$-COLORING RECONFIGURATION $\Rightarrow$ $(d, \Omega(d^2))$-COLORING RECONFIGURATION

Key Ideas

1. List Coloring $\equiv$ Coloring with constraints on which colors can be used for each vertex
2. *Frozen-Coloring Graphs:* Pre-colored graphs where *no vertex can be recolored*



Distance Recoloring

Second Phase

LIST $(d, \Omega(d))$-COLORING RECONFIGURATION $\Rightarrow$ $(d, \Omega(d^2))$-COLORING RECONFIGURATION

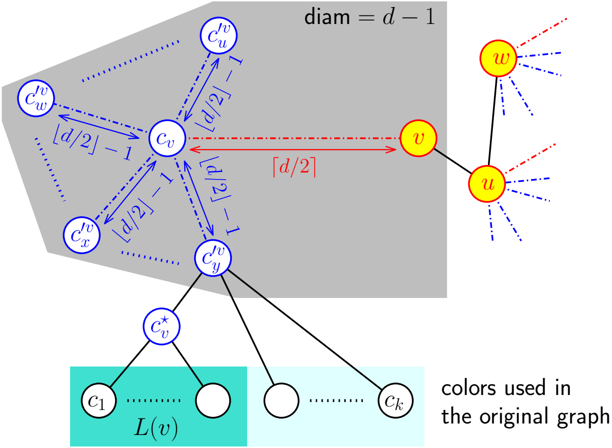


Figure: Construction of the frozen-coloring subgraph F_v for a vertex v. The colors used in this figure are just for illustration of paths.

Distance Recoloring

Theorem (Banerjee, Engels, and Hoang 2024)

On split graphs,

	(d,k)-*Coloring Reconfiguration*
$d=1$	P *for any* k *[Hatanaka, Ito, and Zhou 2019]*
$d=2$	PSPACE-*complete for large* k
$d\geq 3$	P *for any* k

Proof Sketch.

$d\geq 3$ Trivial. (Any connected split graph has diameter $\leq 3 \Rightarrow$ Reconfiguration is easy!)

$d=2$ Reduction from the problem for $d=1$ and $k\geq 4$ on general graphs (which is known to be PSPACE-complete [Bonsma and Cereceda 2009])

□

Concluding Remarks

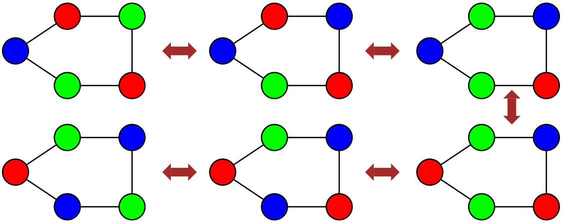
Take-Home Messages

1. *Reconfiguration* studies the "solution space" of a problem
   - Moving from one solution to another *without violating feasibility*
2. Under certain *distance constraints*,
   - *Reconfiguration problems* can be *hard* for very restricted graph classes
   - *Problems on graphs whose diameters are bounded by some constant c* (e.g., split graphs) are interesting when *restricted to distances close to c*

Open Problems

The complexities of the following problem remain *open* for *trees*:

- (d, k)-CR $(d \geq 2)$



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Came to my talk, you did.
Thank you, I must!

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