A Note On Reconfiguration Graphs of Cliques

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Introduction

- This report aim to study the structural properties of clique reconfiguration graph which has been researched from the algorithmic viewpoint [1].
- Our main findings include:
  - Maximum clique size, chromatic number of $\mathbf{TS}_k$ graph
  - Relationship of $\mathbf{TS}_{k-1}, \mathbf{TJ}_k$
  - Relationship of some graph classes produced by **TJ** graph

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Some Set and Graph Notation

Definition 1

For a set S and an element a, we use $S + a$ to denote $S \cup \{a\}$ and $S - a$ to denote $S - \{a\}$

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Definition 3

For a simple graph G. We use V_G, E_G to denote the set of vertices and edges of the graph, $N_G(v)$ to denote the set of neighbours of v, which is $\{w \in V_G : vw \in E_G\}$. We also use $\omega(G), \chi(G)$ to respectively denote the size of maximum clique and the chromatic number of the graph.

Johnson Graph

Definition 4 (Johnson Graph)

For integers n, k, we use $\mathbf{J}(n, k)$ to denote the graph G, whose set of vertices is all the subset with k elements of $\{1, 2, \cdots, n\}$ and each pair of vertices is adjacent if their intersection has exactly $k - 1$ elements.

Definition (Clique Reconfiguration Graph)

Definition 5

For a graph G, the Clique Reconfiguration Graph of G is the graph whose nodes are the cliques of G and some pairs of vertices are adjacent under some rules. Here, we refer a clique as a subset of V_G.

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Definition 6 (Rules)

- **Token Jumping (TJ):** Two cliques A, B are adjacent if $|A| = |B|$ and $|A \cap B| = |A| - 1$
- **Token Sliding (TS):** Two cliques A, B are adjacent if $|A| = |B|$ and $|A \cap B| = |A| - 1$ and $A \Delta B$ form an edge in G.
- **Token Addition/ Removal(TAR):** Two cliques A, B are adjacent if $|A \Delta B| = 1$

Definition (Clique Reconfiguration Graph)

Definition 7

We use $\mathbf{TJ(G)}$ to denote the clique reconfiguration graph with **TJ** rule and for an integer k we use $\mathbf{TJ_k(G)}$ for the induced subgraph of $\mathbf{TJ(G)}$ which the vertices are the clique with size k. Similarly, we define $\mathbf{TS(G)}$ and $\mathbf{TS_k(G)}$.

Definition (Clique Reconfiguration Graph)

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We use $\mathbf{TJ(G)}$ to denote the clique reconfiguration graph with **TJ** rule and for an integer k we use $\mathbf{TJ_k(G)}$ for the induced subgraph of $\mathbf{TJ(G)}$ which the vertices are the clique with size k. Similarly, we define $\mathbf{TS(G)}$ and $\mathbf{TS_k(G)}$.

Definition 8

We use $\mathbf{TAR(G)}$ to denote the clique reconfiguration graph with **TAR** rule and for an integer k we use $\mathbf{TAR_k(G)}$ for the induced subgraph of $\mathbf{TAR(G)}$ which the vertices are the clique with at least k.

Example

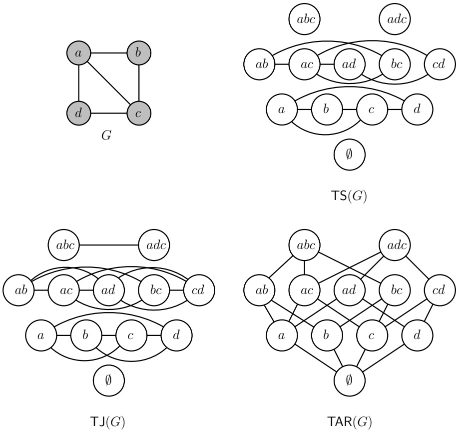

Figure: An example of a graph G and some reconfiguration graphs of cliques of G. Each label inside a white-colored vertex (e.g., abc) indicates a clique of G (e.g., $\{a, b, c\}$).

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Cliques K_n in Johnson graph

Lemma 9 (Clique in Johnson Graph)

Suppose that $\mathbf{J}(m, k)$ *contain a complete subgraph* K_n $(n \geq 3)$ *whose vertices* $A_1, \ldots, A_n$ *are k-elemnents subset of* $\{1, 2, \cdots, m\}$*.*
There exist either a subset $\mathrm{Uni} = (A_1 \cup A_2 \cup \cdots \cup A_n) \subseteq \{1, 2, \cdots, m\}$ *of size* $k + 1$ *and pairwise distinct number* $a_1, \ldots, a_n \in \mathrm{Uni}$ *such that* $A_i = \mathrm{Uni} - a_i$ *or a subset* $\mathrm{Int} = (A_1 \cap A_2 \cap \cdots \cap A_n) \subseteq \{1, 2, \cdots, m\}$ *of size* $k - 1$ *and pairwise distinct vertices* $a_1, \ldots, a_n \in \{1, 2, \cdots, m\} \setminus \mathrm{Int}$ *such that* $A_i = \mathrm{Int} + a_i$*, where* $1 \leq i \leq n$*.*

Proof of Lemma 9

- We prove by induction on n.
    - For the base case $n = 3$, we can prove that $k - 2 \leq |A_1 \cap A_2 \cap A_3| \leq k - 1$. There are 2 scenarios to consider:

Proof of Lemma 9

- We prove by induction on n.
  - For the base case $n = 3$, we can prove that $k - 2 \leq |A_1 \cap A_2 \cap A_3| \leq k - 1$. There are 2 scenarios to consider:
    - If $|A_1 \cap A_2 \cap A_3| = k - 1$, then we can find the set Int as required.
    - If $|A_1 \cap A_2 \cap A_3| = k - 2$, then we can prove that $|A_1 \cup A_2 \cup A_3| = k + 1$ and find Uni as required.

- Suppose that the lemma holds for $n-1$. We claim that it also holds for $n \geq 4$. There are 2 scenarios to consider:
  - There is a set $\text{Uni}_{n-1} = (A_1 \cup A_2 \cup \cdots \cup A_{n-1})$ with $k+1$ elements.
    - If $A_n \not\subset \text{Uni}_{n-1}$, there is an element $x \in A_n$ which is not in A_i for $i < n$. A_i has to contain $A_n - x$ and hence $|(A_1 \cap A_2 \cap \cdots \cap A_{n-1})| \geq |A_n - x| = k-1$ which contradicts since we can prove that this set has $k-(n-1)$ elements.
    - If $A_n \subset \text{Uni}_{n-1}$ then $\text{Uni} = \text{Uni}_{n-1}$ is the set we need to find

Proof of Lemma 9 (cont.)

- Suppose that the lemma holds for $n-1$. We claim that it also holds for $n \geq 4$. There are 2 scenarios to consider:
  - There is a set $\mathrm{Int}_{n-1} = (A_1 \cap A_2 \cap \cdots \cap A_{n-1})$ with $k-1$ elements.
    - We can consider the sets $B_i = \{1, 2, \cdots, m\} - A_i$ for $1 \leq i \leq n$ that form a clique in graph $\mathbf{J}(n, n-k)$ to achieve the first scenario.

Corolary for **TS** and **TJ** graph

Corollary 10 (Clique in *TJ*, *TS* Graph)

The theorem also apply to $\mathbf{TS_k(G)}$ *and* $\mathbf{TJ_k(G)}$ *graph.*

Proof.

Since we can assume $V_G = \{1, 2, \cdots, m\} (m = |V_G|)$, the graph $\mathbf{TS_k(G)}$ and $\mathbf{TJ_k(G)}$ are actually subgraphs of $\mathbf{J(m, k)}$ and hence the theorem also apply to them. $\square$

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Some properties of **TS** graph

Theorem 11 (Maximum clique in TS)

Let G be a graph.

- *If $k > \omega(G)$ then $\omega(\mathbf{TS}_k(G)) = 0$.*
- *If $k = \omega(G)$ then $\omega(\mathbf{TS}_k(G)) = 1$.*
- *If $k < \omega(G)$ then $\omega(\mathbf{TS}_k(G)) = \max\{k+1, \omega(G) - k + 1\}$.*

Theorem 12 (Lower and upper bound for chromatic number of TS)

Given an arbitrary graph G and number k, $\chi(\mathbf{TS}_k(G)) \leq \chi(\mathbf{J}(\chi(G), k))$ and $\chi(\mathbf{TS}_k(G)) \geq \chi(\mathbf{J}(\omega(G), k))$

Proof of theorem 11

- The case $k > \omega(G)$ is trivial.
- The case $k = \omega(G)$, we have to prove that there is no edge in $\mathbf{TS}_k(G)$. Suppose $IJ \in E(\mathbf{TS}_k(G))$. Thus, $I \cup J$ forms a clique of size $k + 1$ in G, which contradicts the assumption $k = \omega(G)$.

Proof of theorem 11

- The case $k < \omega(G)$.
  - We first show that $\omega(\mathbf{TS}_k(G)) \geq \max\{k+1, \omega(G)-k+1\}$ by constructing a $(k+1)$-clique and a $(\omega(G)-k+1)$-clique.
    - Let $A = \{a_1, \ldots, a_{k+1}\}$ be any clique of size exactly $k+1$ in G. Clearly, the vertices $A - a_i$ $(1 \leq i \leq k+1)$ form a $(k+1)$-clique of $\mathbf{TS}_k(G)$.
    - Let $B = \{b_1, \ldots, b_{\omega(G)}\}$ be any clique of size exactly $\omega(G)$ in G and let $C = \{b_1, \ldots, b_{k-1}\} \subseteq B$. Clearly, the vertices $C + b_j$ $(k \leq j \leq \omega(G))$ form a $(\omega(G)-k+1)$-clique of $\mathbf{TS}_k(G)$.

- The case $k < \omega(G)$.
  - It remains to show that $\omega(\mathbf{TS}_k(G)) \leq \max\{k+1, \omega(G)-k+1\}$. We show that if $m = \omega(\mathbf{TS}_k(G)) > k+1$ then $m \leq \omega(G) - k + 1$.
  Let C be a m-clique of $\mathbf{TS}_k(G)$ whose vertices $A_1, \ldots, A_m$ are k-cliques of G.
  By corollary 10 about clique in **TS** graph, there exist a clique Int of size $k-1$ such that $A_i = \mathrm{Int} + a_i$, where $1 \leq i \leq m$ (the union can not exist since $m > k+1$)

Proof of theorem 11

- The case $k < \omega(G)$.
  - It remains to show that $\omega(\mathbf{TS}_k(G)) \leq \max\{k+1, \omega(G) - k + 1\}$. We show that if $m = \omega(\mathbf{TS}_k(G)) > k+1$ then $m \leq \omega(G) - k + 1$.
    Let C be a m-clique of $\mathbf{TS}_k(G)$ whose vertices $A_1, \ldots, A_m$ are k-cliques of G.
    By corollary 10 about clique in **TS** graph, there exist a clique Int of size $k-1$ such that $A_i = \text{Int} + a_i$, where $1 \leq i \leq m$ (the union can not exist since $m > k+1$)
  - We can prove that $a_i a_j \in E(G)$ for any pairwise distinct pair $i, j \in \{1, \ldots, m\}$,Thus, $\text{Int} + \{a_1, a_2, \ldots, a_m\}$ is a $(m+k-1)$-clique of G which mean $(m+k-1) \leq \omega(G)$. Our proof is complete.

Sketch Proof of theorem 12

- For the lowerbound, let H be a clique of size $\omega(G)$ in G. We know that $\mathbf{TS}_k(H)$ is the subgraph of $\mathbf{TS}_k(G)$ and $J(\omega(G), k) \simeq \mathbf{TS}_k(H)$. Hence $\chi(\mathbf{TS}_k(G)) \geq \chi(J(\omega(G), k))$.

Sketch Proof of theorem 12

- For the upperbound, let $n = \chi(G)$, we consider a coloring of graph G(the colors are $\{1, \cdots n\}$) . The main idea is to create a homomorphism map $f : V_{\mathbf{TS}_k(G)} \to V_{\mathbf{J}(n,k)}$ such that $uv \in E_{\mathbf{TS}_k(G)}$ then $f(u)f(v) \in E_{\mathbf{J}}(n, k)$.

Sketch Proof of theorem 12

• For the upperbound, let $n = \chi(G)$, we consider a coloring of graph G(the colors are $\{1, \cdots n\}$) . The main idea is to create a homomorphism map $f : V_{\mathbf{TS}_k(G)} \to V_{\mathbf{J}(n,k)}$ such that $uv \in E_{\mathbf{TS}_k(G)}$ then $f(u)f(v) \in E_{\mathbf{J}}(n,k)$.

• We can observe that in a clique , all the vertices must have different colors so we can assign $f(u)(u \in V_{\mathbf{TS}_k(G)})$ to be the set of colors of all vertices in clique u. We can prove that this is a homomorphism.

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Definition $k-$good graph

Definition 13

We call G a *k-good* graph if it satisfies the following *k-good conditions*: for every vertex u of G, the set $N_G(u)$ can be partitioned into k disjoint subsets (possibly empty) $S_1(u), S_2(u), \ldots, S_k(u)$ such that each $S_i(u)$ $(1 \leq i \leq k)$, if it is not empty, is a clique in G and there is no edge joining a vertex of $S_i(u)$ and that of $S_j(u)$, for distinct pair $i, j \in \{1, \ldots, k\}$.

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- In this section, we will talk about:
  - The properties of k-good graph
  - If $k = \omega(G)$ then $\mathbf{TJ}_k(G)$ is a $k-$ good graph
  - Relation between $\mathbf{TS}_{k-1}(G), \mathbf{TJ}_k(G)$
  - The algorithm to obtain $\mathbf{TS}_{k-1}(G)$ from $\mathbf{TJ}_k(G)$ if $k = \omega(G)$

Properties of $k-$good graph

Lemma 14

For a k-good graph $T = (V, E)$, one can find a multiset **Msets** *of subsets of V such that the following conditions are satisfied.*

- *For each pair of different sets $X, Y \in$* **Msets**, $|X \cap Y| < 2$.
- *For each $X \in$* **Msets**, *X is a clique in T.*
- *For each $u \in V(T)$, there are exactly k sets $X \in$* **Msets** *such that $u \in X$.*
- *For each $uv \in E(T)$, there exists exactly one set $X \in$* **Msets** *such that $u, v \in X$.*

Sketch Proof of lemma 14

- We build the sets $M_i(u) = S_i(u) + u(1 \leq i \leq k, u \in V)$. Then we can prove that if $|M_i(u) \cap M_j(v)| \geq 2$ then $M_i(u) = M_j(v)$

Sketch Proof of lemma 14

- We build the sets $M_i(u) = S_i(u) + u (1 \leq i \leq k, u \in V)$. Then we can prove that if $|M_i(u) \cap M_j(v)| \geq 2$ then $M_i(u) = M_j(v)$
- Now, we can build **Msets** to be the multiset of $M_i(u)(1 \leq i \leq k, u \in V)$ after eliminating all duplicate copies of set with at least 2 elements and prove the 4 desired properties.

Properties of $\mathbf{TJ}_{\omega(G)}(G)$ graph

Lemma 15

Suppose that the $\omega(G)$-cliques A, B, C of a graph G form a triangle in $\mathbf{TJ}_{\omega(G)}(G)$. Then because of corollary 10 about clique in TJ, $A \cap B = B \cap C = A \cap C$.

Lemma 16

Let $k = \omega(G)$. The graph $\mathbf{TJ}_k(G)$ is $k-$ good graph.

Sketch proof of lemma 16

- For each $U \in V_{\mathbf{TJ}_k(G)}$, we need to prove that we can split $N_{\mathbf{TJ}_k(G)}(U)$ into k disjoint clique $S_1(U), \cdots, S_k(U)$ and there is no edge joining a vertex between 2 different sets.

Sketch proof of lemma 16

- For each $U \in V_{\mathbf{TJ}_k(G)}$, we need to prove that we can split $N_{\mathbf{TJ}_k(G)}(U)$ into k disjoint clique $S_1(U), \cdots, S_k(U)$ and there is no edge joining a vertex between 2 different sets.
- For two neighbors V, W of U in $\mathbf{TJ}_k(G)$, we know that if $U \cap V = U \cap W$ if and only if V and W are adjacent in $\mathbf{TJ}_k(G)$(because of lemma 15). Hence, we can split each vertex $V \in N_{\mathbf{TJ}_k(G)}(U)$ according to $(U \cap V)$, there are at most k set of intersection since there are k subset of U with $k-1$ elements of U.

Properties of $\mathbf{TJ}_{\omega(G)}(G)$ graph)

Corollary 17

Let G be any graph and let $k = \omega(G)$. There exists a multiset $\mathbf{Msets}(\mathbf{TJ}_k(G))$ containing subsets of $V(\mathbf{TJ}_k(G))$ such that the following conditions are satisfied.

- *For each different sets $X, Y \in \mathbf{Msets}(\mathbf{TJ}_k(G))$ $|X \cap Y| < 2$*
- *X form a clique in $TJ_k(G)(\forall X \in \mathbf{Msets}(\mathbf{TJ}_k(G)))$*
- *For each $u \in V(\mathbf{TJ}_k(G))$, there are exactly k sets $X \in \mathbf{Msets}(\mathbf{TJ}_k(G))$ such that $u \in X$*
- *For each $uv \in E(\mathbf{TJ}_k(G))$, there exists one set $X \in \mathbf{Msets}(\mathbf{TJ}_k(G))$ such that $u, v \in X$*

Relation with $\mathbf{TS}_{k-1}(G)$

Lemma 18

Let G be a graph and k be an integer satisfying $2 \leq k \leq \omega(G)$. Let $A, B \in V(\mathbf{TS}_{k-1}(G))$. Then, $AB \in E(\mathbf{TS}_{k-1}(G))$ if and only if $A \cup B \in V(\mathbf{TJ}_k(G))$.

Relation with $\mathbf{TS}_{k-1}(G)$

Lemma 18

Let G be a graph and k be an integer satisfying $2 \leq k \leq \omega(G)$. Let $A, B \in V(\mathbf{TS}_{k-1}(G))$. Then, $AB \in E(\mathbf{TS}_{k-1}(G))$ if and only if $A \cup B \in V(\mathbf{TJ}_k(G))$.

Definition 19

For each $v \in V(\mathbf{TS}_{k-1}(G))$, let $\mathbf{Expand}(v)$ be the set of u in $V_{\mathbf{TJ}_k(G)}$ such that $v \subset u$.

Relation with $\mathbf{TS}_{k-1}(G)$

Lemma 18

Let G be a graph and k be an integer satisfying $2 \leq k \leq \omega(G)$. Let $A, B \in V(\mathbf{TS}_{k-1}(G))$. Then, $AB \in E(\mathbf{TS}_{k-1}(G))$ if and only if $A \cup B \in V(\mathbf{TJ}_k(G))$.

Definition 19

For each $v \in V(\mathbf{TS}_{k-1}(G))$, let $\mathbf{Expand}(v)$ be the set of u in $V_{\mathbf{TJ}_k(G)}$ such that $v \subset u$.

Lemma 20

For each w, r in $V(\mathbf{TS}_{k-1}(G))$: $\mathbf{Expand}(w), \mathbf{Expand}(r)$ has a common element if and only if wr is in $E(\mathbf{TS}_{k-1}(G))$ (the common element is $w \cup r$).

Relation with $\mathbf{TS}_{k-1}(G)$

Corollary 21

The vertices $w \in V(TS_{k-1}(G))$ such that $\mathbf{Expand}(w) = \emptyset$ are isolated vertices.

Relation with $\mathbf{TS}_{k-1}(G)$

Corollary 21

The vertices $w \in V(TS_{k-1}(G))$ such that $\mathbf{Expand}(w) = \emptyset$ are isolated vertices.

Lemma 22

*Given $k = \omega(G)$. Let **Msets** be the multiset of non-empty sets $\mathbf{Expand}(w) (\forall w \in V(TS_{k-1}(G))$ then $\mathbf{Msets} = \mathbf{Msets}(TJ_k(G))$.*

Sketch proof of lemma 22

- We will prove that for every x in $V(TJ_k(G))$, the number of occurrences of x in **Msets** and **Msets**$(TJ_k(G))$ equals.

Sketch proof of lemma 22

- We will prove that for every x in $V(TJ_k(G))$, the number of occurrences of x in **Msets** and $\textbf{Msets}(TJ_k(G))$ equals.
    - If $x = \emptyset$ then they are both 0.
    - If $|x| \geq 2$, we prove that the occurrence for both multiset is 0 or 1 and then prove that $x \in \textbf{Msets} \longleftrightarrow x \in \textbf{Msets}(TJ_k(G))$
    - If $x = \{u\}$, we can calculate the number of set $U \in \textbf{Msets}$ such that U contain u which is same as that of $\textbf{Msets}(TJ_k(G))$. Hence, the occurrences of $\{u\}$ have to be equal.

Algorithm to obtain $\mathbf{TS}_{k-1}(G)$ from $\mathbf{TJ}_k(G)(k = \omega(G))$

Lemma 23

Given $k = \omega(G)$ and $\mathbf{TJ}_k(G)$ as input without knowing the set label of $\mathbf{TJ}_k(G)$, one can construct a graph H in $k^2 \cdot poly(\mathbf{TJ}_k(G))$ such that $\mathbf{TS}_{k-1}(G) \cong H + cK_1$ for some integer $c \geq 0$(i.e. adding some independent vertices to H).

Example

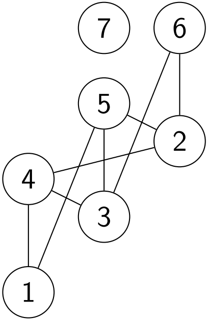

Figure: graph G with $\omega(G)$=2

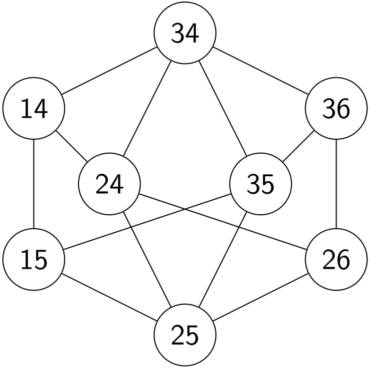

Figure: $TJ_2(G)$

Example

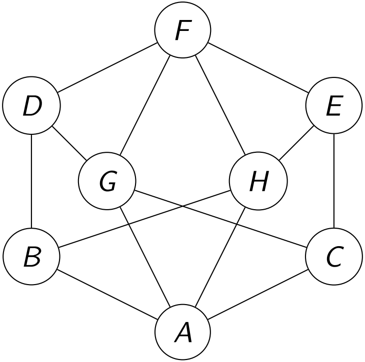

Figure: Input graph T for Algorithm

Example

- First phase of algorithm will split the neighbour sets as follow:
  - $N_T(A) \to \{B, H\}, \{G, C\}$
  - $N_T(B) \to \{A, H\}, \{D\}$
  - $N_T(C) \to \{A, G\}, \{E\}$
  - $N_T(D) \to \{B\}, \{F, G\}$
  - $N_T(E) \to \{C\}, \{F, H\}$
  - $N_T(F) \to \{D, G\}, \{E, H\}$
  - $N_T(G) \to \{D, F\}, \{A, C\}$
  - $N_T(H) \to \{E, F\}, \{A, B\}$

Example

- First phase of algorithm will split the neighbour sets as follow:
  - $N_T(A) \to \{B, H\}, \{G, C\}$
  - $N_T(B) \to \{A, H\}, \{D\}$
  - $N_T(C) \to \{A, G\}, \{E\}$
  - $N_T(D) \to \{B\}, \{F, G\}$
  - $N_T(E) \to \{C\}, \{F, H\}$
  - $N_T(F) \to \{D, G\}, \{E, H\}$
  - $N_T(G) \to \{D, F\}, \{A, C\}$
  - $N_T(H) \to \{E, F\}, \{A, B\}$
- After that we can obtain

Msets $= \{BD, CE, ABH, ACG, DFG, EFH\}$

Example

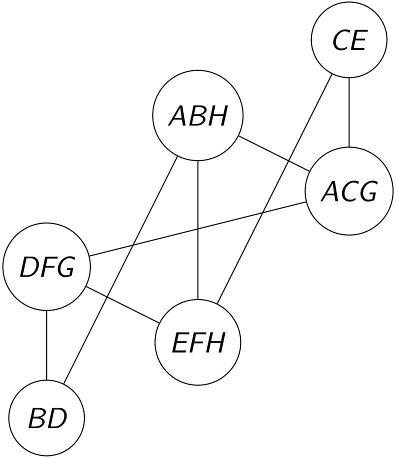

Figure: Output of the algorithm(reassemble $TS_1(G) = G$)

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Definition of $\mathbf{TJ}_k$-graph and maximum $\mathbf{TJ}_k$-graph

Definition 24

For some integer k, we call a graph T satisfying that $T \cong \mathbf{TJ}_k(G)$ is called a $\mathbf{TJ}_k$*-graph*. Additionally, we call T a *maximum* $\mathbf{TJ}_k$*-graph* if it is a $\mathbf{TJ}_k$-graph for some graph G satisfying $k = \omega(G)$.

Definition of $\mathbf{TJ}_k$-graph and maximum $\mathbf{TJ}_k$-graph

Definition 24

For some integer k, we call a graph T satisfying that $T \cong \mathbf{TJ}_k(G)$ is called a $\mathbf{TJ}_k$*-graph*. Additionally, we call T a *maximum* $\mathbf{TJ}_k$*-graph* if it is a $\mathbf{TJ}_k$-graph for some graph G satisfying $k = \omega(G)$.

- In this section, we will discuss about these graph classes are equipvalent
  - The class of graphs that are $\mathbf{TJ}_k$-graph for infinitely many k and the class of graphs that are maximum $\mathbf{TJ}_k$ -graph for some number k.
  - For small $k \leq 3$, The class of Diamond-free graphs that are $\mathbf{TJ}_k$ -graph and the class maximum $\mathbf{TJ}_k$ -graphs.

First Link

- A graph that is maximum $\mathbf{TJ}_k$ -graph for some number k then it is also $\mathbf{TJ}_n$-graph for infinitely n.

Lemma 25

For an integer k, if T is a maximum $\mathbf{TJ}_k$-graph then it is a (maximum) $\mathbf{TJ}_n$-graph for any $n \geq k$.

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- A graph that is maximum $\mathbf{TJ}_k$ -graph for some number k then it is also $\mathbf{TJ}_n$-graph for infinitely n.

Lemma 25

For an integer k, if T is a maximum $\mathbf{TJ}_k$-graph then it is a (maximum) $\mathbf{TJ}_n$-graph for any $n \geq k$.

- A graph is $\mathbf{TJ}_n$-graph for infinitely n then there exist k so that it is maximum $\mathbf{TJ}_k$-graph

Lemma 26

If for infinitely many integers n, a non-empty graph T is a $\mathbf{TJ}_n$-graph then there exists k so that T is a maximum $\mathbf{TJ}_k$-graph.

Sketch Proof of lemma 25

- For lemma 25, Let G be the graph such that $\omega(G) = k$ and $\mathbf{TJ}_k(G) \cong T$.

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    - For a number $n \geq k$, let the graph G_n be the graph G but add a clique K_{n-k} and all the edges between the vertices in G and the clique.

Sketch Proof of lemma 25

- For lemma 25, Let G be the graph such that $\omega(G) = k$ and $\mathbf{TJ}_k(G) \cong T$.
  - For a number $n \geq k$, let the graph G_n be the graph G but add a clique K_{n-k} and all the edges between the vertices in G and the clique.
  - We can prove that $\mathbf{TJ}_n(G_n) \cong \mathbf{TJ}_k(G) \cong T$.

Sketch Proof of lemma 26

- For lemma 26, Let $m = \omega(T)$. Since T is a $\mathbf{TJ}_n$-graph for infinitely many integers n, there must be a number $k > m$ such that T is a $\mathbf{TJ}_k$-graph.

Sketch Proof of lemma 26

- For lemma 26, Let $m = \omega(T)$. Since T is a $\mathbf{TJ}_n$-graph for infinitely many integers n, there must be a number $k > m$ such that T is a $\mathbf{TJ}_k$-graph.
  - Let G be a graph such that $T \cong \mathbf{TJ}_k(G)$.

Sketch Proof of lemma 26

- For lemma 26, Let $m = \omega(T)$. Since T is a $\mathbf{TJ}_n$-graph for infinitely many integers n, there must be a number $k > m$ such that T is a $\mathbf{TJ}_k$-graph.
    - Let G be a graph such that $T \cong \mathbf{TJ}_k(G)$.
    - We can prove $\omega(G) = k$ by using the theorem 11 about $\omega(\mathbf{TS}_k(G))$.

First Link

- Because of lemma 25, 26, we have the following theorem:

Theorem 27

A non-empty graph T is a $\mathbf{TJ}_n$-graph for infinitely many integers n if and only if there exists k so that graph T is a maximum $\mathbf{TJ}_k$-graph.

Second Link

- Follow from lemma 15, we have this lemma:

Lemma 28

For any graph G, the graph $\mathbf{TJ}_{\omega(G)}(G)$ does not contain any diamond as an induced subgraph.

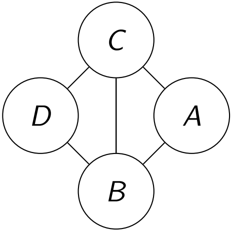

Figure: Diamond graph $K_4 - e$

Second Link

Lemma 29

For $k \leq 3$, a graph non-empty T is maximum $\mathbf{TJ}_k$-graph then it is $\mathbf{TJ}_k$-graph and is diamond-free.

Second Link

Lemma 29

For $k \leq 3$, a graph non-empty T is maximum $\mathbf{TJ}_k$-graph then it is $\mathbf{TJ}_k$-graph and is diamond-free.

- We prove the other way around.

Lemma 30

For $k \leq 3$, if a non-empty graph T is a $\mathbf{TJ}_k$-graph and T does not contain any diamond $(K_4 - e)$ as an induced subgraph then T is a maximum $\mathbf{TJ}_k$-graph.

Sketch proof of lemma 30

- We consider the graph G such that $T \cong \mathbf{TJ}_k(G)$ and produce the union $\cup_{v \in V(\mathbf{TJ}_k(G))}\{v\}$ with maximum number of elements (this set cardinality is bounded by $k \cdot |V_T|$).

Sketch proof of lemma 30

- We consider the graph G such that $T \cong \mathbf{TJ}_k(G)$ and produce the union $\cup_{v \in V(\mathbf{TJ}_k(G))}\{v\}$ with maximum number of elements (this set cardinality is bounded by $k \cdot |V_T|$).
- We prove that the graph G has $\omega(G) = k$. Suppose the contrary, let $A_1, \cdots A_{k+1}$ be the clique of size $k+1$ in G.

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- We consider the graph G such that $T \cong \mathbf{TJ}_k(G)$ and produce the union $\cup_{v \in V(\mathbf{TJ}_k(G))}\{v\}$ with maximum number of elements (this set cardinality is bounded by $k \cdot |V_T|$).
- We prove that the graph G has $\omega(G) = k$. Suppose the contrary, let $A_1, \cdots A_{k+1}$ be the clique of size $k+1$ in G.
    - Here the condition T being diamond free and the bound of k will be used to prove that any clique of size k that contain A_1, A_2 is a subset of $\{A_1, \cdots A_{k+1}\}$.

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- We consider the graph G such that $T \cong \mathbf{TJ}_k(G)$ and produce the union $\cup_{v \in V(\mathbf{TJ}_k(G))}\{v\}$ with maximum number of elements (this set cardinality is bounded by $k \cdot |V_T|$).
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    - Here the condition T being diamond free and the bound of k will be used to prove that any clique of size k that contain A_1, A_2 is a subset of $\{A_1, \cdots A_{k+1}\}$.
    - Next, we build a new graph G' from G by deleting the edge A_1, A_2 and add some more point $\{B_1, \cdots B_{k-1}\}$ and all the edges between those points and $\{A_3, \cdots A_{k+1}\}$.

Sketch proof of lemma 30

- We consider the graph G such that $T \cong \mathbf{TJ}_k(G)$ and produce the union $\cup_{v \in V(\mathbf{TJ}_k(G))}\{v\}$ with maximum number of elements (this set cardinality is bounded by $k \cdot |V_T|$).
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    - Here the condition T being diamond free and the bound of k will be used to prove that any clique of size k that contain A_1, A_2 is a subset of $\{A_1, \cdots A_{k+1}\}$.
    - Next, we build a new graph G' from G by deleting the edge A_1, A_2 and add some more point $\{B_1, \cdots B_{k-1}\}$ and all the edges between those points and $\{A_3, \cdots A_{k+1}\}$.
    - It remains to prove $T \cong \mathbf{TJ}_k(G')$ and $\cup_{v \in V(\mathbf{TJ}_k(G'))}\{v\}$ has more elements

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6. $\mathbf{TJ}_k$-graph class
7. Some Properties of **TAR** Graph

TAR graph

In this section, we mention some properties of the TAR graphs, which is known as *the simplex graphs*. Recall that a *gear graph* (also known as *bipartite wheel graph*) is a graph obtained by inserting an extra vertex between each pair of adjacent vertices on the perimeter of a wheel graph. A *Fibonacci cube* [2] Γ_n is a subgraph of a hypercube Q_n induced by vertices (binary strings of length n) not having two consecutive 1s.

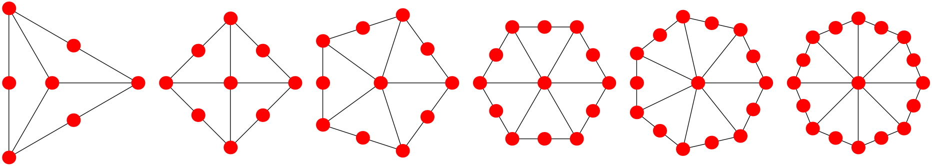

Figure: Illustration of gear graphs G_3 to G_8, notice that a gear graph G_n has $2n + 1$ nodes and $3n$ edges.

TAR graph

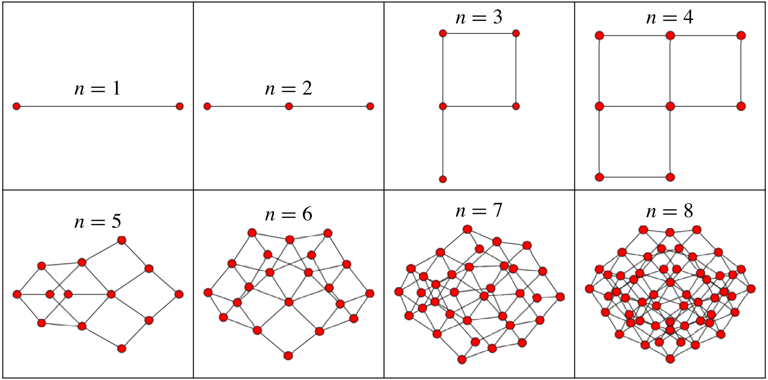

Figure: Fibonacci cube graph Γ_n from $n = 1$ to $n = 8$

Proof of Lemma 31

Lemma 31

(a) Simplex graph of a complete graph is a hypercube.

(b) Simplex graph of a cycle of length $n \geq 4$ is a gear graph.

(c) Simplex graph of the complement of a path is a Fibonacci cube.

Proof.

(a) Given a complete graph K_n, every subsets of the vertex set $\{v_1, v_2, ..., v_n\}$ of K_n forms a clique, including from 0-clique, which is the empty set, to $n-$clique. Hence, we can see that the simplex graph of K_n has 2^n vertices. Similar to the hypercube Q_n, the vertices of Q_n can be represented by binary strings of length n, which correspond to the 2^n subsets of the vertex set of Q_n. Then, we define a bijection

$$f : V(\mathrm{TAR}(K_n)) \to V(Q_n)$$
$$C \mapsto f(C), \quad \text{where } C \text{ is a clique in } K_n.$$

Proof of Lemma 31

Proof.

It suffices to show that f is an isomorphism since for all $C \in V(K_n)$, $f(C)$ is the corresponding binary string of length n, where the ith bit 1 if the ith vertex is included in C, and 0 otherwise. Furthermore, two cliques in the simplex graph of K_n are adjacent if they differ by one element, corresponding to Q_n since two vertices in Q_n are adjacent if their binary strings differ by one bit. Then we can conclude that K_n is isomorphic the hypercube Q_n. □

Proof of Lemma 31

Proof.

(b) Given a cycle graph $C_n, n \geq 4$. Since the cliques in C_n include all subsets of vertices that form complete subgraphs, we can conclude that the simplex graph of C_n only has 0-clique, which is the empty set, 1-cliques, and 2-cliques. We can see that the number of 1-cliques and the number of 2-cliques in C_n is equal to n, because the number of vertices in C_n is equal to the number of edges. Together with the empty set, the total number of vertices in the simplex graph of C_n is $2n + 1$. Then, every 1-cliques is adjacent with the empty set, and each 2-cliques is adjacent with 2 1-cliques which are adjacent in C_n, hence the number of edges in the simplex graph of C_n is $3n$. Since the simplex graph of C_n has the equivalent number of vertices and edges to a general gear graph, we can define a bijection

$$f : V(\mathrm{TAR}(C_n)) \to V(G_n), \quad \text{where } G_n \text{ is the gear graph.}$$

Proof of Lemma 31

Proof.

It suffices to show that f is an isomorphism since for any 1-clique and 2-clique vertices in the $\mathrm{TAR}(C_n)$ can be seen as the outer cycle of the gear graph G_n, and the empty set vertex is the central vertex of G_n. Besides, each 1-clique vertex in $\mathrm{TAR}(C_n)$ is connected to the empty set vertex, and each 2-clique vertex is connected to 1-clique vertices, which form a gear graph G_n. □

Proof.

(c) A proof of this statement appeared in [2]. □

Cartesian product of simplex graphs

Definition

Recall that the join $G \oplus H$ of two graphs G and H is the graph obtained from the disjoint union of G and H by joining every vertex of G with every vertex of H. The Cartesian product of two graphs G and H, denoted by $G \Box H$, is the graph with vertex-set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G \Box H)$ whenever either $ab \in E(G)$ and $x = y$ or $a = b$ and $xy \in E(H)$.

Proposition 32

Let G and H be two disjoint graphs. Then, $\mathrm{TAR}(G \oplus H) = \mathrm{TAR}(G) \Box \mathrm{TAR}(H)$. In other words, the Cartesian product of two simplex graphs is also a simplex graph.

Some properties on Eulerian and Hamiltonian graphs

Definition

Recall that the graph G has an Eulerian cycle if and only if all the degrees are even order, i.e. $\deg(v) \equiv 0 \pmod 2$, or G has exactly 2 vertices with odd orders. G is Hamiltonian if it has Hamilton cycle, which visits all the vertices exactly once. Then we have some properties on the Eulerian and Hamiltonian Simplex Graphs.

Some properties on Eulerian and Hamiltonian graphs

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Property 1

Bipartite graphs with odd order are not Hamiltonian.

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Property 1

Bipartite graphs with odd order are not Hamiltonian.

This property comes from the fact that bipartite graphs do not have odd cycles.

Some properties on Eulerian and Hamiltonian graphs

Property 2

If a graph G has an odd number of cliques, $\mathrm{TAR}(G)$ is not Hamiltonian.

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$\mathsf{TAR}(G)$ with a vertex of degree 1 corresponding to isolated vertices in G are not Hamiltonian.

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Property 3

$\mathrm{TAR}(G)$ with a vertex of degree 1 corresponding to isolated vertices in G are not Hamiltonian.

It suffices to show that a graph G has a vertex of degree 1, or more than 1 vertices of degree 1, is not Hamiltonian, since we visit the vertices with degree 1, we cannot find a way to get back from it, therefore the Hamilton cycle does not exist.

Some properties on Eulerian and Hamiltonian graphs

Property 4

If a graph G has at least one vertex of even degree, simplex graph of G is not Eulerian.

We can see that any clique of size one in G connects to the vertex of empty set in $\mathrm{TAR}(G)$, therefore the degree of the $\mathrm{TAR}(G)$ vertex corresponding to the 1-clique is one more than that in G, i.e. $\deg(\{v\})$ in $\mathrm{TAR}(G) \equiv \deg(v) + 1$, even if that vertex in G is isolated.

Some properties on Eulerian and Hamiltonian graphs

Property 5

If a graph G has odd number of vertices, simplex graph of G is not Eulerian.

the degree of vertex of empty set in $\mathrm{TAR}(G)$ is equal to the number of vertices in G, in that case vertex of empty set in $\mathrm{TAR}(G)$ is odd, then no Eulerian cycle exists. In the general case, the maximal size of clique in a graph G is odd implies that the number of neighbors of the vertex of maximal clique of G in $\mathrm{TAR}(G)$ is also odd, then $\mathrm{TAR}(G)$ is not Eulerian.

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