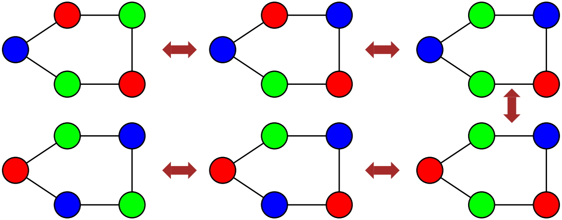
VIASM-KAIST Annual Meeting
on Mathematics Sciences



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Distance Recoloring

in collaboration with
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Christian Engels (National Institute of Informatics, Japan)

Contents

Introduction to Reconfiguration

Reconfiguration Setting

- A description of what *states* (≡ *configurations*) are
- One or more *allowed moves* between states (≡ *reconfiguration rule(s)*)

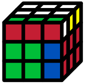
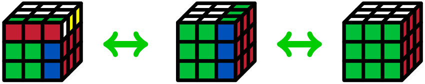
Reconfiguration Process	Reconfiguration Graph	Example Rubik's Cube
state	node	
move *[reconfiguration rule]*	edge	
sequence of moves *[reconfiguration sequence]*	path	

Figure: Reconfiguration Setting

Introduction to Reconfiguration

Reconfiguration vs. Solution Space

For a computational problem $\mathcal{P}$ (e.g., INDEPENDENT SET, DOMINATING SET, VERTEX-COLORING, etc.)

Reconfiguration	Solution Space
States/Configurations	Feasible solutions of $\mathcal{P}$
Allowed Moves	Slight modifications of a solution *without changing its feasibility*

Algorithmic Questions

- REACHABILITY: Given two states S and T, is there a sequence of moves that *transforms S into T*?
- SHORTEST TRANSFORMATION: Given two states S and T and a positive integer ℓ, is there a sequence of moves that *transforms S into T using at most ℓ moves*?
- CONNECTIVITY: Is there a sequence of moves *between any pair of states*?
- and so on

Introduction to Reconfiguration

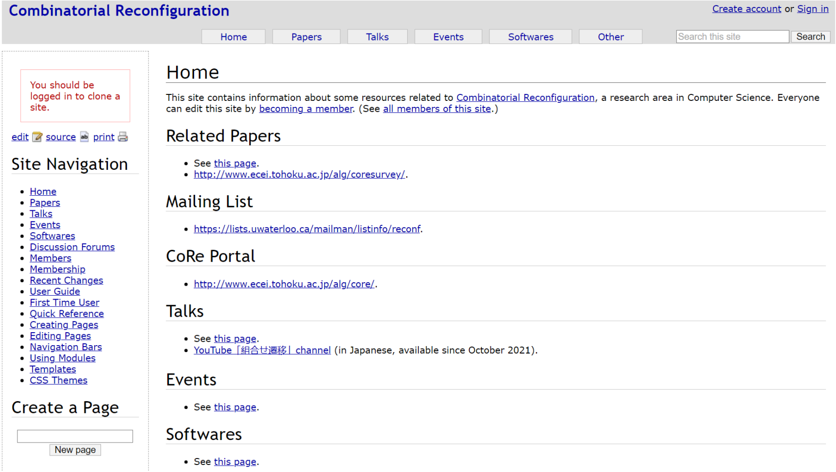


Figure: Online Reconfiguration Wiki Page (`https://reconf.wikidot.com/`)

Introduction to Reconfiguration

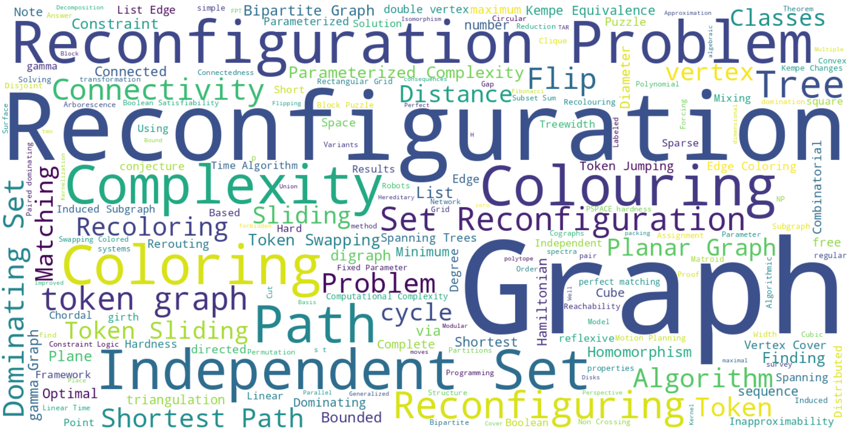
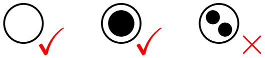


Figure: A word cloud of titles extracted from the current list of papers related to reconfiguration problems available at `https://reconf.wikidot.com/` (Accessed on October 26, 2025)

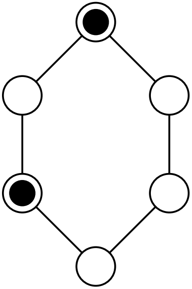
Some Examples

Token Reconfiguration

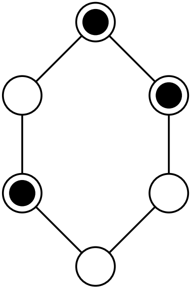
Each vertex has at most one token



Each state is a set of tokens satisfying certain property



Independent Set



Dominating Set

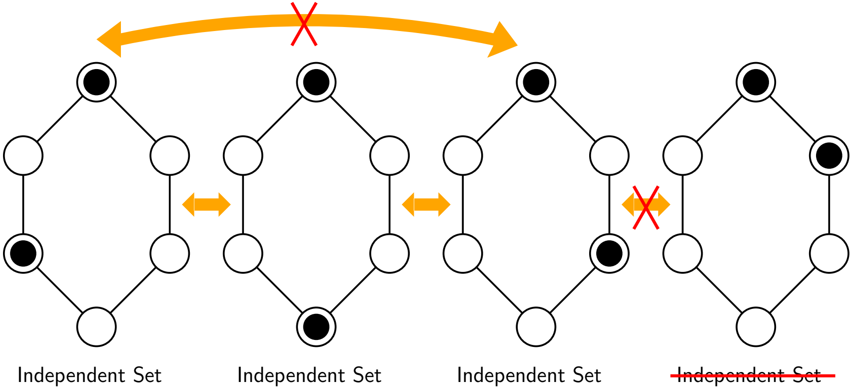
....

Some Examples

[Hearn and Demaine 2005]

Token Reconfiguration (Example: Independent Set Reconfiguration)

Token Sliding (TS)



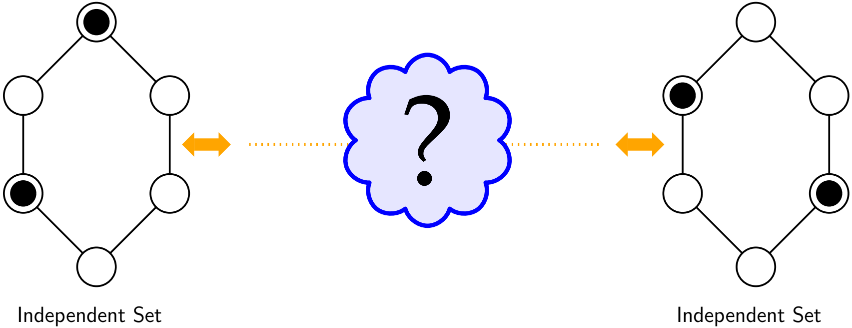
Some Examples

[Hearn and Demaine 2005]

Token Reconfiguration (Example: Independent Set Reconfiguration)

Token Sliding (TS)

PSPACE-complete on general graphs



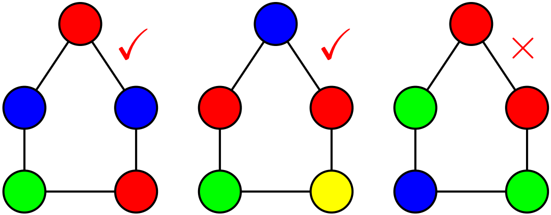
Some Examples

[Cereceda, van den Heuvel, and Johnson 2008]

Vertex-Coloring Reconfiguration

Each vertex is colored by one of the k given colors

Each state is a k-coloring of all vertices such that
no two adjacent vertices share the same color



3-coloring $\quad$ 4-coloring $\quad$ ~~3-coloring~~

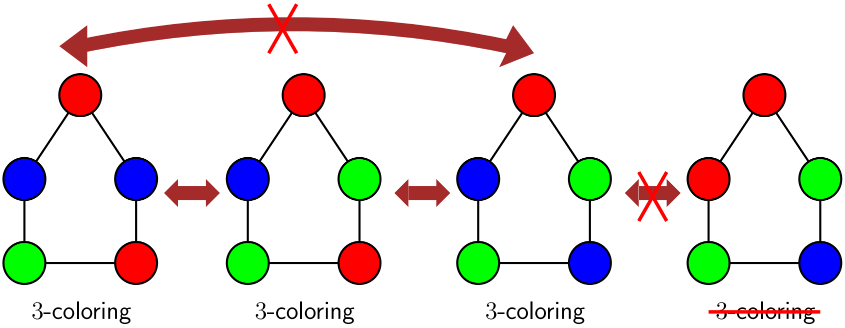
Some Examples

[Cereceda, van den Heuvel, and Johnson 2008]

VERTEX-COLORING RECONFIGURATION

Recoloring using $\leq k$ colors

Example: $k = 3$



Some Examples

[Cereceda, van den Heuvel, and Johnson 2008]

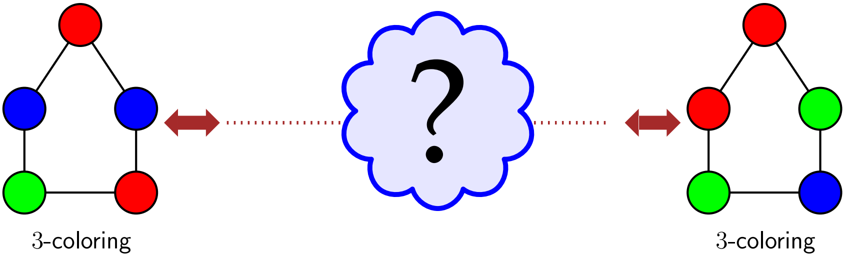
VERTEX-COLORING RECONFIGURATION

Recoloring using $\leq k$ colors

Example: $k = 3$

PSPACE-complete on general graphs for $k \geq 4$

P on general graphs for $k \leq 3$

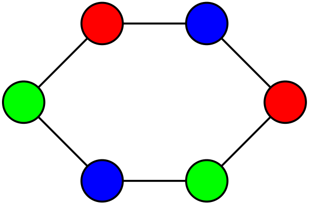


Distance Recoloring

Distance constraints for *vertex colorings*

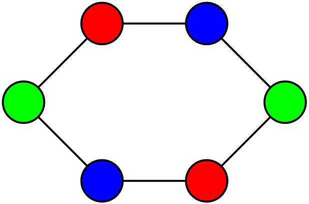
	k-Coloring	(d, k)-Coloring
Distance between two vertices having the same color	≥ 2	$\geq d + 1$

- (d, k)-coloring was first studied in [F. Kramer and H. Kramer 1969]
- Has applications in *frequency assignment problem* (or radio channel assignment) [F. Kramer and H. Kramer 2008]



3-coloring

~~$(2, 3)$-coloring~~



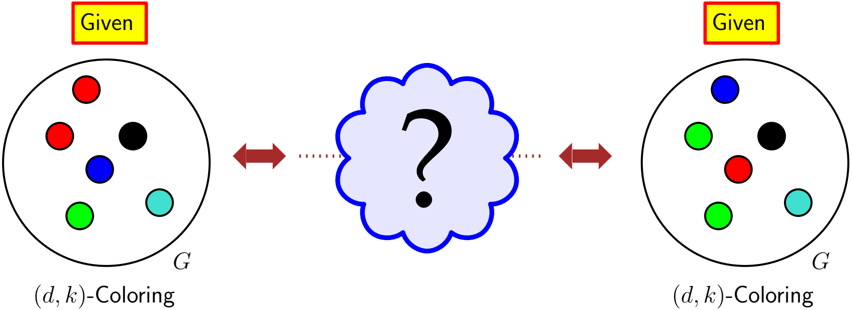
3-coloring

$(2, 3)$-coloring

Distance Recoloring

(d,k)-Coloring Reconfiguration ((d,k)-CR)

Reconfiguration Rule: Recoloring a vertex



- The case $d = 1$ (k-Coloring Reconfiguration (k-CR)) has been well-studied [Mynhardt and Nasserasr 2019]; [Heuvel 2013]
- We focus on the case $d \geq 2$

Distance Recoloring

Graph	k-CR $(= (1,k)$-CR$)$	(d,k)-CR $(d \geq 2)$
general	PSPACE-C $(k \geq 3)$ [Cereceda, Heuvel, and Johnson 2011]	
planar	PSPACE-C $(4 \leq k \leq 6)$ P $(k \geq 7)$ [Bonsma and Cereceda 2009]	
bipartite	PSPACE-C $(k \geq 4)$ [Bonsma and Cereceda 2009]	
planar $\cap$ bipartite	PSPACE-C $(k = 4)$ P $(k \geq 5)$ [Bonsma and Cereceda 2009]	PSPACE-C $(k = \Omega(d^2))$
2-degenerate	P [Hatanaka, Ito, and Zhou 2019]	
planar $\cap$ bipartite $\cap$ 2-degenerate	P $(\subseteq$ 2-degenerate$)$	
path	P $(\subseteq$ planar $\cap$ bipartite $\cap$ 2-degenerate$)$	P $(k \geq d + 1)$
split	P [Hatanaka, Ito, and Zhou 2019]	PSPACE-C $(d = 2$, large $k)$ P $(d \geq 3)$

Table: Our Results for $d \geq 2$. We provide the status for $d = 1$ for comparison. Here PSPACE-C stands for PSPACE-complete [Banerjee, Engels, and **Hoang** 2024]

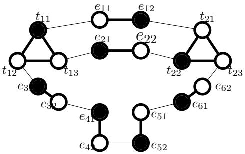
Distance Recoloring

Theorem (Banerjee, Engels, and Hoang 2024)

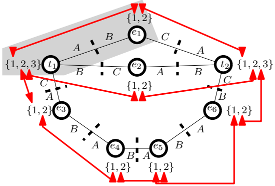
(d, k)-CR *is* PSPACE*-complete for* $d \geq 2$ *and* $k = \Omega(d^2)$ *on graphs which are planar, bipartite, and* 2*-degenerate.*

Proof Sketch.

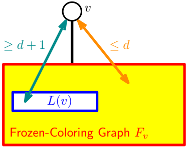
Restricted PSPACE-Complete Variant of ISR under TS
(Restricted Sliding Tokens)



List $(d, \Omega(d))$-Coloring Reconfiguration



$(d, \Omega(d^2))$-Coloring Reconfiguration



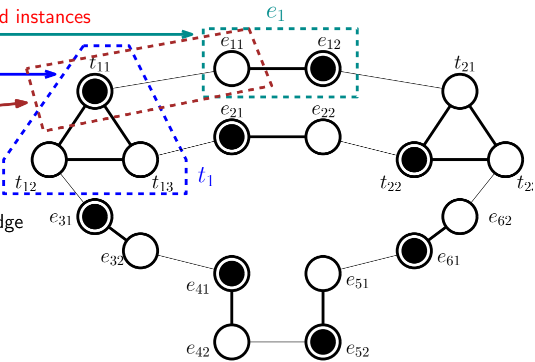
Distance Recoloring

First Phase

RESTRICTED SLIDING TOKENS $\Rightarrow$ LIST $(d, \Omega(d))$-COLORING RECONFIGURATION

RESTRICTED SLIDING TOKENS

- PSPACE-complete on very restricted instances
  - Three types of gadgets
    - Token triangle
    - Token edge
    - Link edge
  - Planar
  - Max degree 3, min degree 2
  - No two token triangles are directly joined by a link edge
  - Each token triangle/token edge has exactly one token



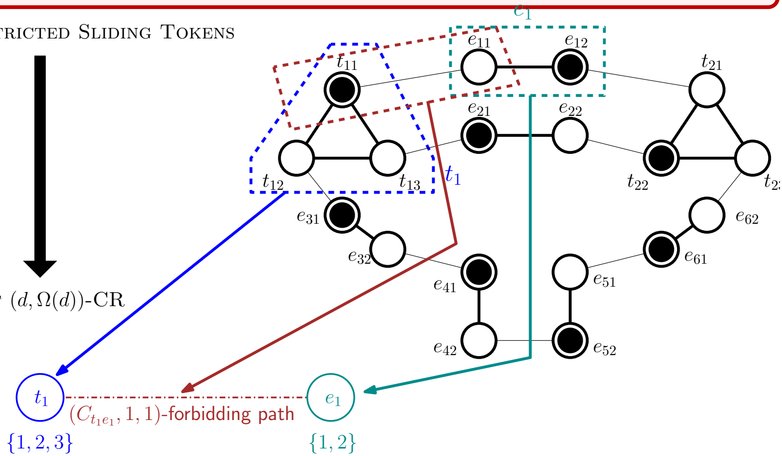
Distance Recoloring

First Phase

RESTRICTED SLIDING TOKENS $\Rightarrow$ LIST $(d, \Omega(d))$-COLORING RECONFIGURATION

RESTRICTED SLIDING TOKENS

LIST $(d, \Omega(d))$-CR



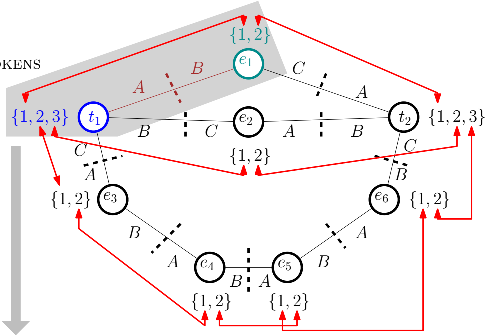
Distance Recoloring

First Phase

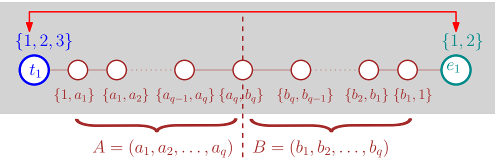
RESTRICTED SLIDING TOKENS $\Rightarrow$ LIST $(d, \Omega(d))$-COLORING RECONFIGURATION

RESTRICTED SLIDING TOKENS

LIST $(d, \Omega(d))$-CR



Magnification of $(C_{t_1e_1}, 1, 1)$-forbidding path $P_{t_1e_1}$, where $C_{t_1e_1} = A \cup B$



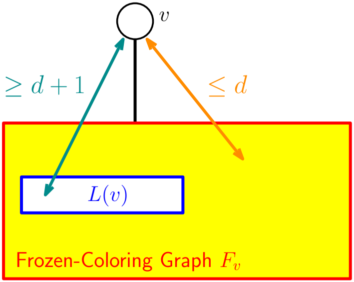
Distance Recoloring

Second Phase

List $(d, \Omega(d))$-Coloring Reconfiguration $\Rightarrow$ $(d, \Omega(d^2))$-Coloring Reconfiguration

Key Ideas

1. List Coloring $\equiv$ Coloring with constraints on which colors can be used for each vertex
2. *Frozen-Coloring Graphs:* Pre-colored graphs where *no vertex can be recolored*



Vertices are pre-colored

Containing all possible colors

No vertex can be recolored

Distance Recoloring — Duc A. Hoang

Distance Recoloring

Second Phase

LIST $(d, \Omega(d))$-COLORING RECONFIGURATION $\Rightarrow$ $(d, \Omega(d^2))$-COLORING RECONFIGURATION

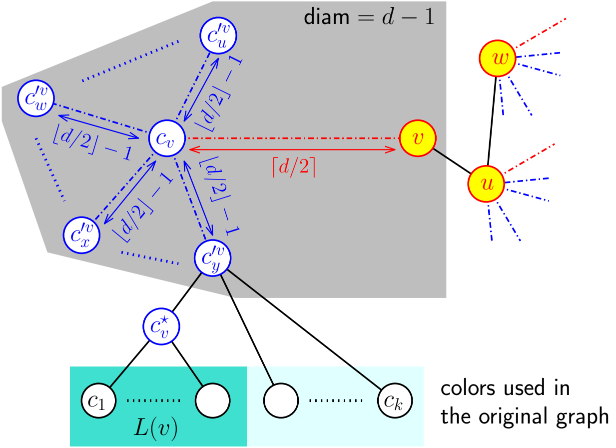


Figure: Construction of the frozen-coloring subgraph F_v for a vertex v. The colors used in this figure are just for illustration of paths.

Distance Recoloring

Theorem (Banerjee, Engels, and Hoang 2024)

On split graphs,

	(d, k)-*Coloring Reconfiguration*
$d = 1$	`P` *for any* k *[Hatanaka, Ito, and Zhou 2019]*
$d = 2$	`PSPACE`-*complete for large* k
$d \geq 3$	`P` *for any* k

Proof Sketch.

$d \geq 3$ Trivial. (Any connected split graph has diameter $\leq 3 \Rightarrow$ Reconfiguration is easy!)

$d = 2$ Reduction from the problem for $d = 1$ and $k \geq 4$ on general graphs (which is known to be `PSPACE`-complete [Bonsma and Cereceda 2009])

□

Concluding Remarks

Take-Home Messages

1. *Reconfiguration* studies the "solution space" of a problem
   - Moving from one solution to another *without violating feasibility*
2. Under certain *distance constraints*,
   - *Reconfiguration problems* can be *hard* for very restricted graph classes
   - *Problems on graphs whose diameters are bounded by some constant c* (e.g., split graphs) are interesting when *restricted to distances close to c*
3. *Nondeterministic Constraint Logic* is a powerful tool for *hardness reductions*

Open Problems

The complexities of the following problem remain *open* for *trees*:

- (d, k)-CR $(d \geq 2)$

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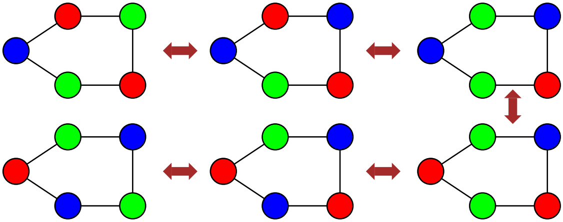

Came to my talk, you did.
Thank you, I must!

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