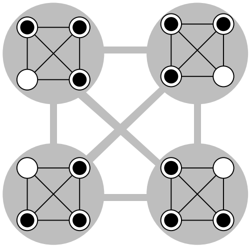
The 12th Annual International Conference on Algorithms and Discrete Applied Mathematics



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A Note on Reconfiguration Graphs of Cliques

in collaboration with
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Contents

Introduction to Reconfiguration

Reconfiguration Setting

- A description of what *states* (≡ *configurations*) are
- One or more *allowed moves* between states (≡ *reconfiguration rule(s)*)

Reconfiguration Process	Reconfiguration Graph	Example RUBIK'S CUBE
state	node	
move *[reconfiguration rule]*	edge	
sequence of moves *[reconfiguration sequence]*	path	

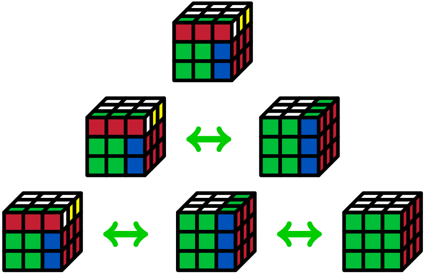

Figure: Reconfiguration Setting

Introduction to Reconfiguration

- **Reconfiguration vs. Solution Space**
  - For a computational problem $\mathcal{P}$ (e.g., Independent Set, Clique, Dominating Set, Vertex-Coloring, etc.)

Reconfiguration	**Solution Space**
States/Configurations	Feasible solutions of $\mathcal{P}$
Allowed Moves	Slight modifications of a solution *without changing its feasibility*

  - $\mathcal{P}$ is often called the *source problem* (of the reconfiguration setting)

Introduction to Reconfiguration

Reconfiguration vs. Solution Space

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Some Related Areas

- Recreational Math (puzzles, games, etc. involving reconfiguration)
- Enumeration (generate all solutions one by one, with small changes between "adjacent" solutions)
- Reoptimization (update an optimal solution after a small change to the input)
- Solution Sampling (randomly sample solutions via small changes)
- Solution Discovery (discover a solution "close enough" to a given initial state (which is not necessarily a solution)) (recently introduced by [Fellows et al. 2023])
- and so on

Introduction to Reconfiguration

- **Algorithmic Questions**
  - REACHABILITY: Given two states S and T, is there a sequence of moves that *transforms S into T*?
  - SHORTEST TRANSFORMATION: Given two states S and T and a positive integer ℓ, is there a sequence of moves that *transforms S into T using at most ℓ moves*?
  - CONNECTIVITY: Is there a sequence of moves *between any pair of states*?
  - and so on

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  - and so on
- **Graph-Theoretic Questions [Focus of this talk]**
  - *Structural properties* of the reconfiguration graph (e.g., connectivity, diameter, etc.)
  - *Classification* of the reconfiguration graph (e.g., which graphs can be realized as reconfiguration graphs under certain rules?)
  - *Original graph vs. reconfiguration graph* (e.g., how properties of the original graph relate to those of the reconfiguration graph?)
  - and so on

Introduction to Reconfiguration

- **General Surveys**
  - Jan van den Heuvel (2013). “The Complexity of Change”. In: *Surveys in Combinatorics*. Vol. 409. London Mathematical Society Lecture Note Series. Cambridge University Press, pp. 127–160. DOI: 10.1017/cbo9781139506748.005
  - Naomi Nishimura (2018). “Introduction to Reconfiguration”. In: *Algorithms* 11.4, p. 52. DOI: 10.3390/a11040052
  - C.M. Mynhardt and S. Nasserasr (2019). “Reconfiguration of Colourings and Dominating Sets in Graphs”. In: *50 years of Combinatorics, Graph Theory, and Computing*. Ed. by Fan Chung et al. 1st. CRC Press, pp. 171–191. DOI: 10.1201/9780429280092-10
  - Nicolas Bousquet, Amer E. Mouawad, Naomi Nishimura, and Sebastian Siebertz (2024). “A survey on the parameterized complexity of reconfiguration problems”. In: *Computer Science Review* 53. (article 100663). DOI: 10.1016/j.cosrev.2024.100663
- **Online Wiki:** https://reconf.wikidot.com/

Background and Motivation

Reconfiguration Setting

- **Source Problem:** Clique
  - Each clique is considered as a set of tokens placed on the vertices
- **Reconfiguration Rule:**
  - TS: Token Sliding
  - TJ: Token Jumping
  - TAR: Token Addition/Removal

Example 1 ($\mathsf{TS}_3(K_4)$)

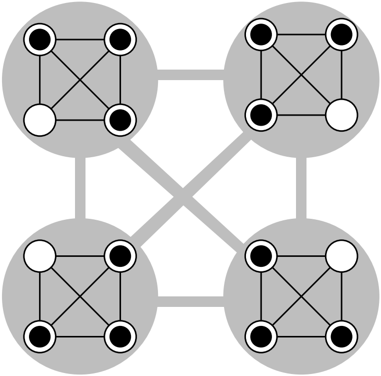
Reconfiguration graph of 3-Cliques in K_4 under TS

Reconfiguration rule

$$\mathsf{TS}_3(K_4)$$

Original graph

Size of token-set
(Lower bound under TAR)



Background and Motivation

- Algorithmic Perspective:
  - Reachability under TS, TJ, and TAR is PSPACE-complete on perfect graphs and in P on even-hole-free graphs and cographs [Ito, Ono, and Otachi 2023]
  - Shortest Transformation is in P when the input graph is chordal, bipartite, planar, or has bounded treewidth
- Graph-Theoretic Perspective:
  - Under TAR: studied since 1989 under the name *simplex graphs* [Bandelt and van de Vel 1989]
  - Under TS and TJ: no systematic study except for $\mathsf{TS}_k(K_n)$ which relates to *token graphs* [Fabila-Monroy et al. 2012] and *Johnson graphs* [Holton and Sheehan 1993]

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This Talk

We study structural properties of reconfiguration graphs of cliques mainly under TS and TJ

Our Results

- We derived a number of structural properties of $\mathsf{TS}_k(G)$ and $\mathsf{TJ}_k(G)$
- We mention some interesting results in this talk

Our Results

Clique number relationship with original graph

Theorem 2

$$\omega(\mathsf{TS}_k(G)) = \begin{cases} 0, & k > \omega(G), \\ 1, & k = \omega(G), \\ \max\{k+1,\ \omega(G)-k+1\}, & k < \omega(G). \end{cases}$$

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Proof Sketch.

- First two cases are straigtforward
- Last case:
  - Lower bound: from $(k+1)$-clique A and a maximum clique B give cliques of sizes $k+1$ and $\omega(G) - k + 1$ in $\mathsf{TS}_k(G)$
    - Imagine A has k tokens and one empty slot $\Rightarrow K_{k+1}$ in $\mathsf{TS}_k(G)$
    - Imagine B (of size $\omega(G)$) has $k-1$ fixed tokens and one movable token which can move around "empty slots" in $B \Rightarrow K_{\omega(G)-k+1}$ in $\mathsf{TS}_k(G)$
  - Upper bound: any m-clique $A_1, \dots, A_m$ in $\mathsf{TS}_k(G)$ with $m > k+1$ implies a clique of size $m+k-1$ in G
    - When $m = \omega(\mathsf{TS}_k(G)) > k+1$, there must be a $(k-1)$-clique C such that $A_i = C + a_i$ for $1 \leq i \leq m$ and distinct $a_1, \dots, a_m$ in $G \Rightarrow C + \{a_1, \dots, a_m\}$ is a clique of size $m + k - 1 \geq \omega(G)$ in G

Our Results

Chromatic number relationship with Johnson graph

Theorem 3

- *Upper bound:* $\chi(\mathsf{TS}_k(G)) \leq \chi(J(\chi(G), k))$.
- *Lower bound:* $\chi(\mathsf{TS}_k(G)) \geq \chi(J(\omega(G), k))$.

- A *Johnson graph* $J(n, k)$ is a graph whose vertices are size-k subsets of an n-element set and two vertices are adjacent if their intersection is of size exactly $k - 1$.

Our Results

Chromatic number relationship with Johnson graph

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Proof Sketch.

- Upper bound: a k-clique uses k distinct vertex-colors, and a slide changes exactly one vertex, hence changing exactly one used color; therefore $\mathsf{TS}_k(G)$ is a subgraph of the Johnson graph on color-sets, so any coloring of that Johnson graph induces a coloring of $\mathsf{TS}_k(G)$.
- Lower bound: For a maximum clique H in G, every k-clique in H is a k-subset of $V(H)$, and a slide replaces exactly one chosen vertex, so the induced reconfiguration graph $\mathsf{TS}_k(H)$ is $J(\omega(G), k)$. Since this Johnson graph appears as a subgraph of $\mathsf{TS}_k(G)$, $\mathsf{TS}_k(G)$ must need at least as many colors as $\chi(J(\omega(G), k))$.

Our Results

Relating $\mathsf{TJ}_{\omega(G)}(G)$ to $\mathsf{TS}_{\omega(G)-1}(G)$

Theorem 4

Given $k = \omega(G)$ and the graph $\mathsf{TJ}_k(G)$ without any "predefined vertex-labels", one can construct a graph H in $k^2 \cdot \mathrm{poly}(|\mathsf{TJ}_k(G)|)$ time such that

$$\mathsf{TS}_{k-1}(G) \;\cong\; H + cK_1,$$

i.e., one can recover $\mathsf{TS}_{k-1}(G)$ (from $\mathsf{TJ}_k(G)$) up to isolated vertices.

- $AB \in E(\mathsf{TS}_{k-1}(G))$ if and only if $A \cup B \in V(\mathsf{TJ}_k(G))$
- Consequently,
  - Constructing $\mathsf{TJ}_k(G)$ from $\mathsf{TS}_{k-1}(G)$ is easy: we have vertices of $\mathsf{TJ}_k(G)$ from this equivalence and then their adjcencies can be inferred easily
  - If vertices of $\mathsf{TJ}_k(G)$ are labeled by the corresponding k-cliques of G, then recovering $\mathsf{TS}_{k-1}(G)$ up to isolated vertices is also easy
- Without labels, recovering $\mathsf{TS}_{k-1}(G)$ from $\mathsf{TJ}_k(G)$ up to isolated vertices is non-trivial

Our Results

Relating $\mathsf{TJ}_{\omega(G)}(G)$ to $\mathsf{TS}_{\omega(G)-1}(G)$

What information is hidden in $\mathsf{TJ}_k(G)$ when $k = \omega(G)$?

- Each vertex of $T = \mathsf{TJ}_k(G)$ represents a maximum k-clique of G. Two vertices U, V are adjacent in T iff the corresponding k-cliques differ by exactly one vertex (they overlap in $k-1$ vertices).
- Locally, from a maximum clique U, every neighbor (in $\mathsf{TJ}_k(G)$) is obtained by: *Choosing one vertex of U to throw away, and replace it with some other vertex to get another maximum clique.*
- There are exactly k choices of which vertex of U got thrown away $\Rightarrow$ Expect k neighbor-types.

Our Results

Relating $\mathsf{TJ}_{\omega(G)}(G)$ to $\mathsf{TS}_{\omega(G)-1}(G)$

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- There are exactly k choices of which vertex of U got thrown away $\Rightarrow$ Expect k neighbor-types.

Two-phase Strategy to recover $\mathsf{TS}_{k-1}(G)$

- **Phase 1:** Detect the k "swap positions" around each vertex (the k-good test)
- **Phase 2:** Group maximum cliques that share the same "$(k-1)$-core", and then connects these cores in $\mathsf{TS}_{k-1}(G)$ when they coexist inside a maximum clique
  - Allow rebuilding $\mathsf{TS}_{k-1}(G)$ except for $(k-1)$-cliques that never extend to a maximum clique (which become isolated vertices).

Our Results

Planarity Preservation

Theorem 5

- *If G is planar then* $\mathsf{TS}_k(G)$ *is planar* ($1 \leq k \leq \omega(G) \leq 4$).
- *Let* F_3, F_4 *be respectively the numbers of* K_3, K_4 *in* G. *Then,* $F_3 \leq |E| - 2$, $F_3 \leq 3|V| - 8$, *and* $2F_4 \leq F_3 - 2$

- The theorem is "tight": planar graphs have max. clique size ≤ 4
- The reverse does not hold
  - Take $G = K_{3,3}$ ($\omega(G) = 2$) then $\mathsf{TS}_2(G) \cong 9K_1$ is planar but G is not
- We re-prove the known upper bounds on the number of triangles in planar graphs (e.g., see [Wood 2007]) using properties of reconfiguration graphs. There exists a better bound ($F_4 \leq |V| - 3$) on the number of K_4 in planar graphs (see [Wood 2007])

Our Results

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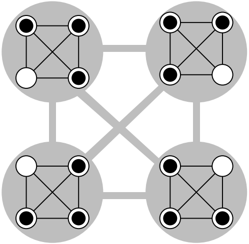
Proof Sketch.

- $k = 1$ and $k = 4$ are trivial
- $k = 2$: We create a planar drawing of $\mathsf{TS}_2(G)$ from that of G
- $k = 3$: $\mathsf{TJ}_4(G)$ is acyclic + max. degree ≤ 4. Use Theorem 4 to recover $\mathsf{TS}_3(G)$ (up to isolated vertices) from $\mathsf{TJ}_4(G)$
- Counting k-cliques in G is $\Leftrightarrow$ counting vertices in $\mathsf{TS}_k(G)$

Open Questions

- More structural properties: e.g., Eulerianity, Hamiltonicity, Connectivity, etc.?
- More classification: which graphs can be realized as $\mathsf{TS}_k(G)$ or $\mathsf{TJ}_k(G)$ for some G and k?
- Proving properties of original graphs using properties of corresponding reconfiguration graphs?



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Thank you for your attention!

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