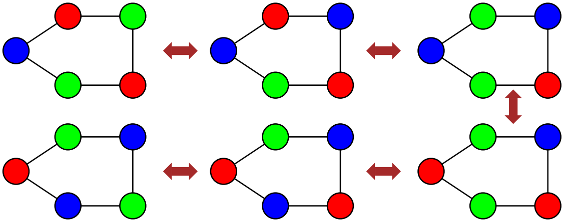
VIASM-KAIST Annual Meeting on Mathematics Sciences



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October 27–30, 2025

Distance Recoloring

in collaboration with
Niranka Banerjee (Kyoto University, Japan)
Christian Engels (National Institute of Informatics, Japan)

Contents

Introduction to Reconfiguration

Reconfiguration Setting

- A description of what *states* ($\equiv$ *configurations*) are
- One or more *allowed moves* between states ($\equiv$ *reconfiguration rule(s)*)

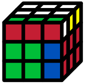
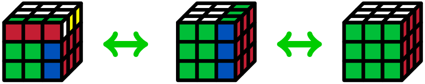
Reconfiguration Process	**Reconfiguration Graph**	**Example** RUBIK'S CUBE
state	node	
move *[reconfiguration rule]*	edge	
sequence of moves *[reconfiguration sequence]*	path	

Figure: Reconfiguration Setting

Introduction to Reconfiguration

Reconfiguration vs. Solution Space

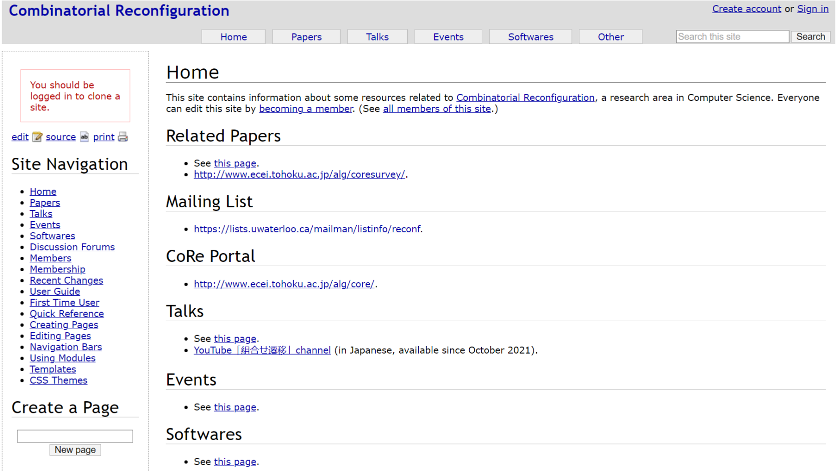
For a computational problem $\mathcal{P}$ (e.g., INDEPENDENT SET, DOMINATING SET, VERTEX-COLORING, etc.)

Reconfiguration	**Solution Space**
States/Configurations	Feasible solutions of $\mathcal{P}$
Allowed Moves	Slight modifications of a solution *without changing its feasibility*

Algorithmic Questions

- REACHABILITY: Given two states S and T, is there a sequence of moves that *transforms S into T*?
- SHORTEST TRANSFORMATION: Given two states S and T and a positive integer ℓ, is there a sequence of moves that *transforms S into T using at most ℓ moves*?
- CONNECTIVITY: Is there a sequence of moves *between any pair of states*?
- and so on

Introduction to Reconfiguration


Combinatorial Reconfiguration

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Site Navigation

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- Papers
- Talks
- Events
- Softwares
- Discussion Forums
- Members
- Membership
- Recent Changes
- User Guide
- First Time User
- Quick Reference
- Creating Pages
- Editing Pages
- Navigation Bars
- Using Modules
- Templates
- CSS Themes

Create a Page

New page

Home

This site contains information about some resources related to Combinatorial Reconfiguration, a research area in Computer Science. Everyone can edit this site by becoming a member. (See all members of this site.)

Related Papers

- See this page.
- http://www.ecei.tohoku.ac.jp/alg/coresurvey/.

Mailing List

- https://lists.uwaterloo.ca/mailman/listinfo/reconf.

CoRe Portal

- http://www.ecei.tohoku.ac.jp/alg/core/.

Talks

- See this page.
- YouTube「組合せ遷移」channel (in Japanese, available since October 2021).

Events

- See this page.

Softwares

- See this page.

Figure: Online Reconfiguration Wiki Page (`https://reconf.wikidot.com/`)

Introduction to Reconfiguration

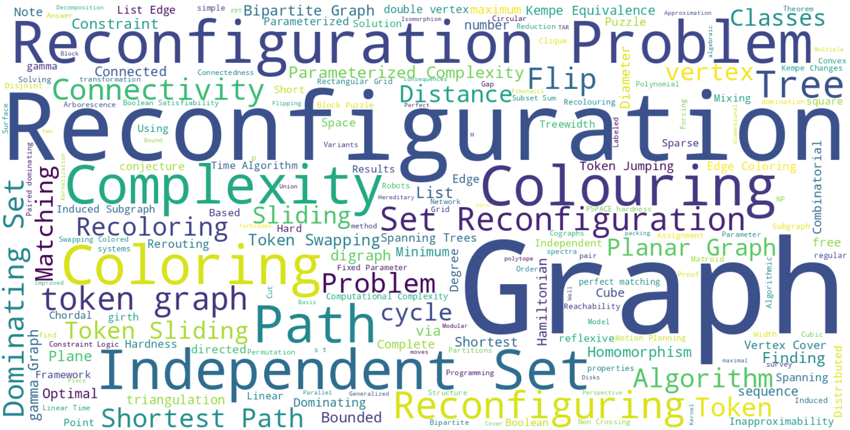
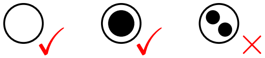


Figure: A word cloud of titles extracted from the current list of papers related to reconfiguration problems available at `https://reconf.wikidot.com/` (Accessed on October 26, 2025)

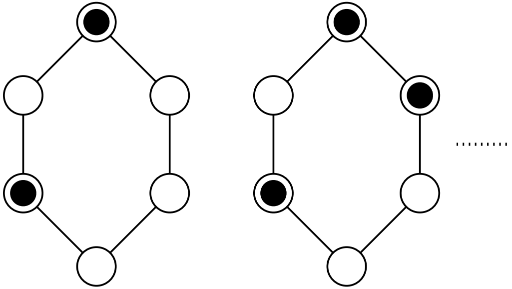
Some Examples

Token Reconfiguration

Each vertex has at most one token



Each state is a set of tokens satisfying certain property



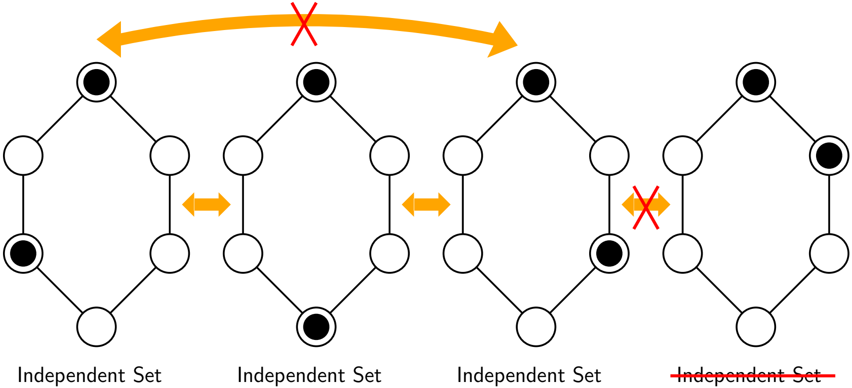
Independent Set   Dominating Set

Some Examples

[Hearn and Demaine 2005]

TOKEN RECONFIGURATION (Example: INDEPENDENT SET RECONFIGURATION)

Token Sliding (TS)



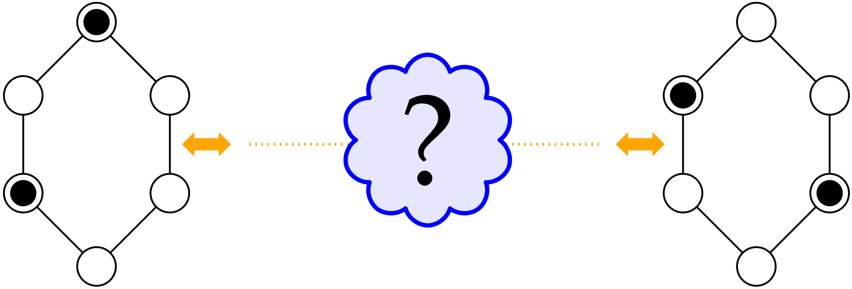
Some Examples

[Hearn and Demaine 2005]

Token Reconfiguration (Example: Independent Set Reconfiguration)

Token Sliding (TS)

PSPACE-complete on general graphs



Independent Set

Independent Set

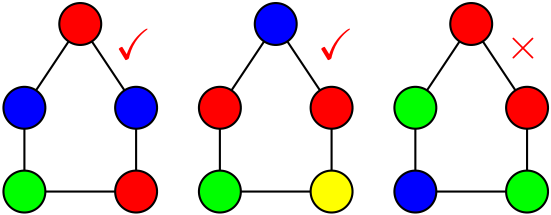
Some Examples

[Cereceda, van den Heuvel, and Johnson 2008]

VERTEX-COLORING RECONFIGURATION

Each vertex is colored by one of the k given colors

Each state is a k-coloring of all vertices such that
no two adjacent vertices share the same color



3-coloring $\quad$ 4-coloring $\quad$ ~~3-coloring~~

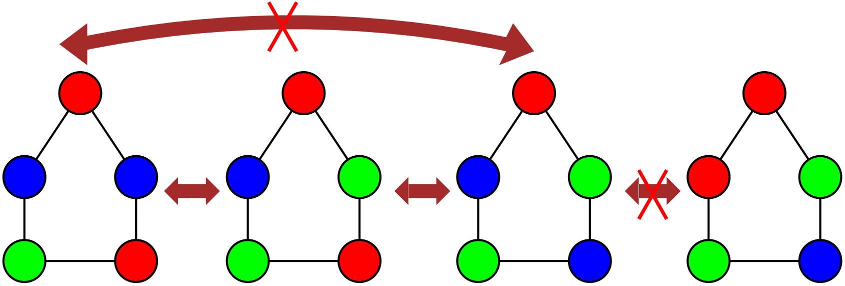
Some Examples

[Cereceda, van den Heuvel, and Johnson 2008]

Vertex-Coloring Reconfiguration

Recoloring using $\leq k$ colors

Example: $k = 3$



3-coloring   3-coloring   3-coloring   ~~3-coloring~~

Some Examples

[Cereceda, van den Heuvel, and Johnson 2008]

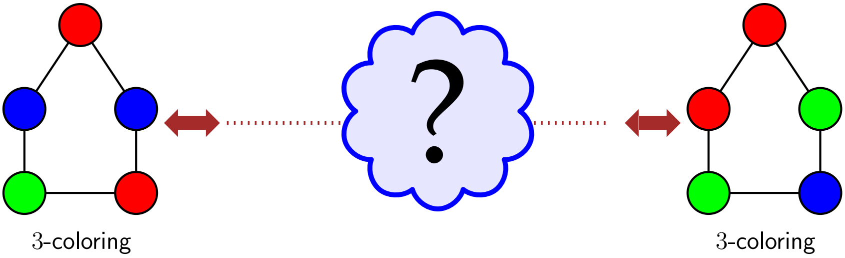
VERTEX-COLORING RECONFIGURATION

Recoloring using $\leq k$ colors

Example: $k = 3$

PSPACE-complete on general graphs for $k \geq 4$

P on general graphs for $k \leq 3$



Some Examples

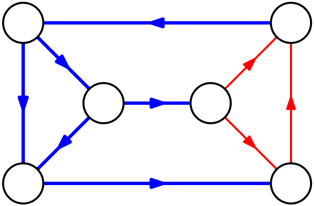
[Hearn and Demaine 2005]

Nondeterministic Constraint Logic (NCL)

Each state is a weighted, oriented graph

weight 2 weight 1

Total incoming weight at each vertex ≥ 2



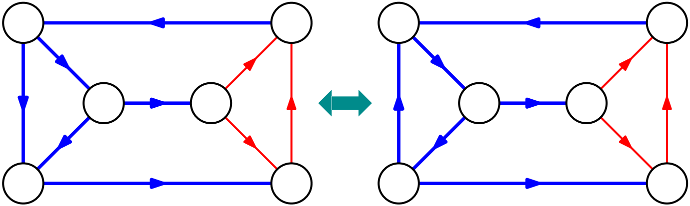
Some Examples

[Hearn and Demaine 2005]

NONDETERMINISTIC CONSTRAINT LOGIC (NCL)

Reversing edge direction

Total incoming weight at each vertex ≥ 2



Some Examples

[Hearn and Demaine 2005]

NONDETERMINISTIC CONSTRAINT LOGIC (NCL)

Reversing edge direction

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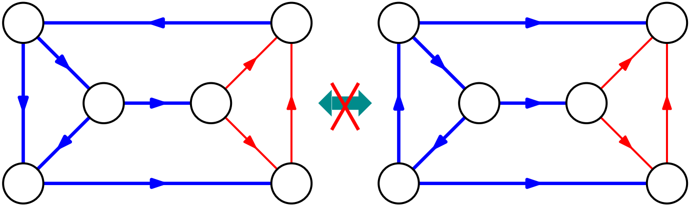


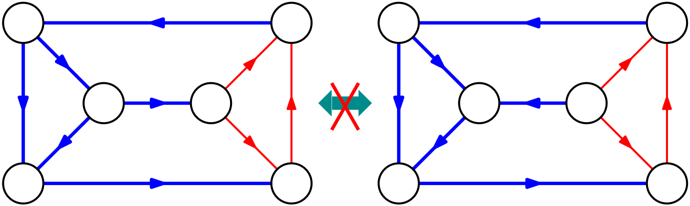
Some Examples

[Hearn and Demaine 2005]

NONDETERMINISTIC CONSTRAINT LOGIC (NCL)

Reversing edge direction

Total incoming weight at each vertex ≥ 2

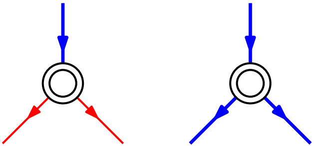


Some Examples

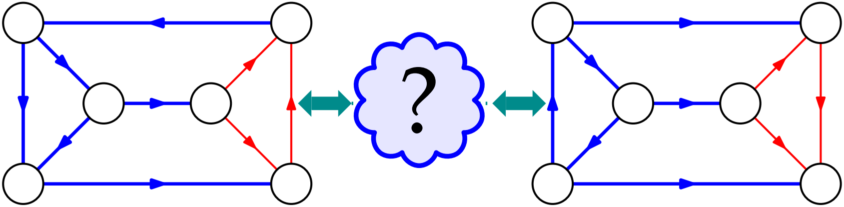
[van der Zanden 2015]

NONDETERMINISTIC CONSTRAINT LOGIC (NCL)

PSPACE-complete on
- planar graphs
- having only two types of vertices
- max degree 3
- bounded bandwidth



AND vertex   OR vertex

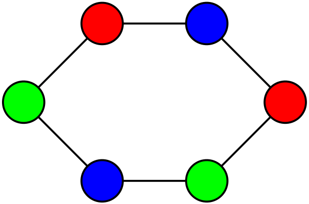


Distance Recoloring

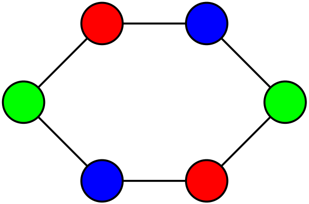
Distance constraints for *vertex colorings*

	k-Coloring	(d,k)-Coloring
Distance between two vertices having the same color	≥ 2	$\geq d+1$

- (d,k)-coloring was first studied in [F. Kramer and H. Kramer 1969]
- Has applications in *frequency assignment problem* (or radio channel assignment) [F. Kramer and H. Kramer 2008]



3-coloring
~~$(2,3)$-coloring~~

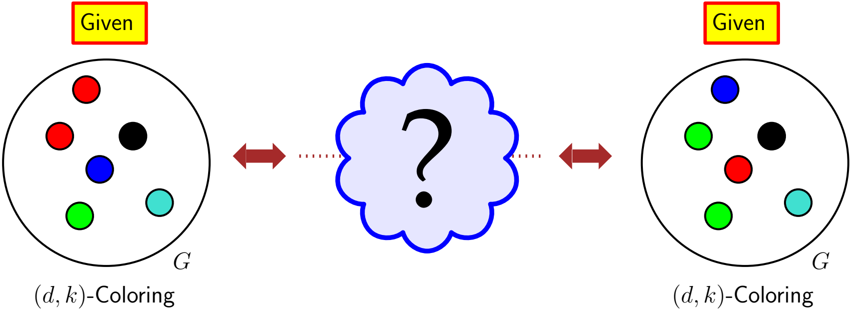


3-coloring
$(2,3)$-coloring

Distance Recoloring

(d, k)-Coloring Reconfiguration ((d, k)-CR)

Reconfiguration Rule: Recoloring a vertex



- The case $d = 1$ (k-Coloring Reconfiguration (k-CR)) has been well-studied [Mynhardt and Nasserasr 2019]; [Heuvel 2013]
- We focus on the case $d \geq 2$

Distance Recoloring

Graph	k-CR $(= (1,k)$-CR)	(d,k)-CR $(d \geq 2)$
general	PSPACE-C $(k \geq 3)$ [Cereceda, Heuvel, and Johnson 2011]	PSPACE-C $(k = \Omega(d^2))$
planar	PSPACE-C $(4 \leq k \leq 6)$ P $(k \geq 7)$ [Bonsma and Cereceda 2009]	
bipartite	PSPACE-C $(k \geq 4)$ [Bonsma and Cereceda 2009]	
planar ∩ bipartite	PSPACE-C $(k = 4)$ P $(k \geq 5)$ [Bonsma and Cereceda 2009]	
2-degenerate	P [Hatanaka, Ito, and Zhou 2019]	
planar ∩ bipartite ∩ 2-degenerate	P (⊆ 2-degenerate)	
path	P (⊆ planar ∩ bipartite ∩ 2-degenerate)	P $(k \geq d+1)$
split	P [Hatanaka, Ito, and Zhou 2019]	PSPACE-C $(d = 2$, large $k)$ P $(d \geq 3)$

Table: Our Results for $d \geq 2$. We provide the status for $d = 1$ for comparison. Here PSPACE-C stands for PSPACE-complete [Banerjee, Engels, and **Hoang** 2024]

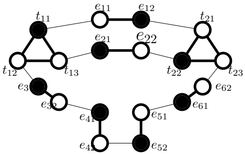
Distance Recoloring

Theorem (Banerjee, Engels, and Hoang 2024)

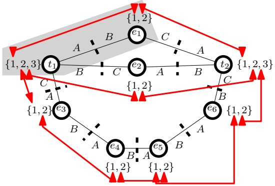
(d,k)-CR is PSPACE*-complete for $d \geq 2$ and $k = \Omega(d^2)$ on graphs which are planar, bipartite, and 2-degenerate.*

Proof Sketch.

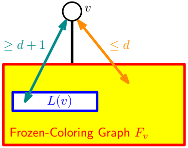
Restricted PSPACE-Complete Variant of ISR under TS (Restricted Sliding Tokens)



List $(d, \Omega(d))$-Coloring Reconfiguration



$(d, \Omega(d^2))$-Coloring Reconfiguration



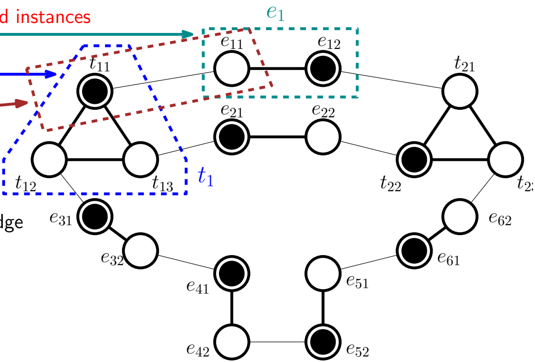
Distance Recoloring

First Phase

RESTRICTED SLIDING TOKENS $\Rightarrow$ LIST $(d, \Omega(d))$-COLORING RECONFIGURATION

RESTRICTED SLIDING TOKENS

- PSPACE-complete on very restricted instances
  - Three types of gadgets
    - Token triangle
    - Token edge
    - Link edge
  - Planar
  - Max degree 3, min degree 2
  - No two token triangles are directly joined by a link edge
  - Each token triangle/token edge has exactly one token



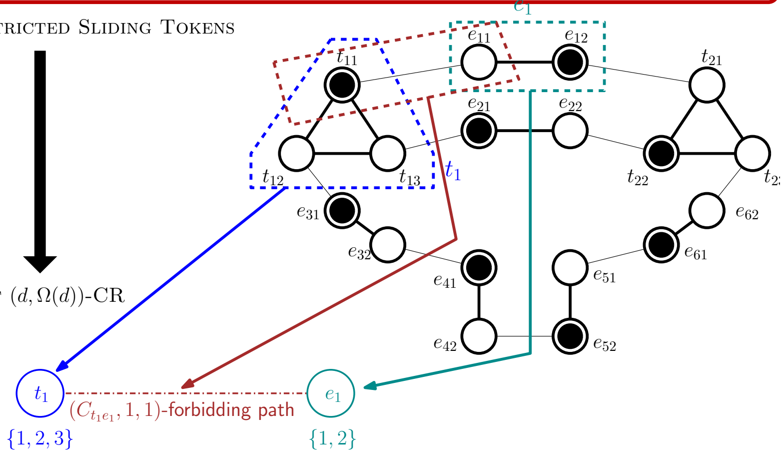
Distance Recoloring

First Phase

Restricted Sliding Tokens $\Rightarrow$ List $(d, \Omega(d))$-Coloring Reconfiguration

Restricted Sliding Tokens

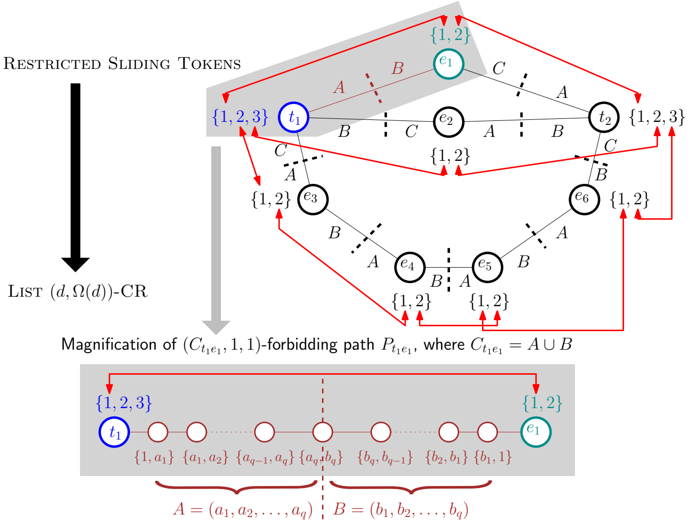
List $(d, \Omega(d))$-CR



Distance Recoloring

First Phase

Restricted Sliding Tokens $\Rightarrow$ List $(d, \Omega(d))$-Coloring Reconfiguration



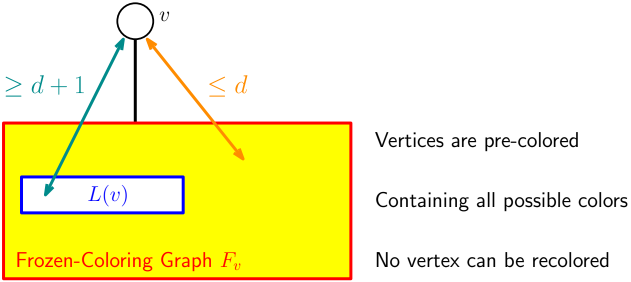
Distance Recoloring

Second Phase

LIST $(d, \Omega(d))$-COLORING RECONFIGURATION $\Rightarrow$ $(d, \Omega(d^2))$-COLORING RECONFIGURATION

Key Ideas

1. List Coloring $\equiv$ Coloring with constraints on which colors can be used for each vertex
2. *Frozen-Coloring Graphs:* Pre-colored graphs where *no vertex can be recolored*



Distance Recoloring

Second Phase

List $(d, \Omega(d))$-Coloring Reconfiguration $\Rightarrow$ $(d, \Omega(d^2))$-Coloring Reconfiguration

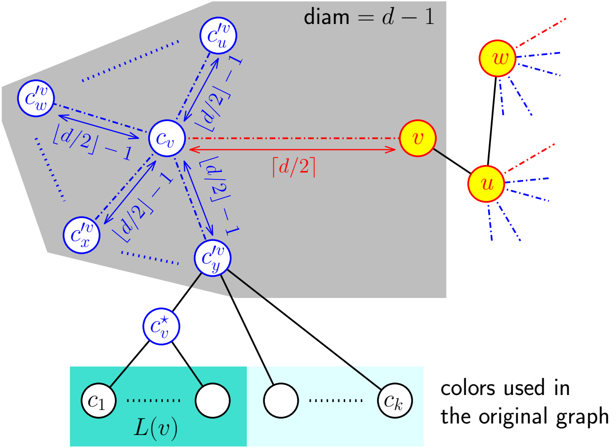


Figure: Construction of the frozen-coloring subgraph F_v for a vertex v. The colors used in this figure are just for illustration of paths.

Distance Recoloring

Theorem (Banerjee, Engels, and Hoang 2024)

On split graphs,

	(d,k)-*Coloring Reconfiguration*
$d = 1$	$\mathtt{P}$ *for any* k *[Hatanaka, Ito, and Zhou 2019]*
$d = 2$	$\mathtt{PSPACE}$-*complete for large* k
$d \geq 3$	$\mathtt{P}$ *for any* k

Proof Sketch.

$d \geq 3$ Trivial. (Any connected split graph has diameter $\leq 3 \Rightarrow$ Reconfiguration is easy!)

$d = 2$ Reduction from the problem for $d = 1$ and $k \geq 4$ on general graphs (which is known to be $\mathtt{PSPACE}$-complete [Bonsma and Cereceda 2009])

$\square$

Concluding Remarks

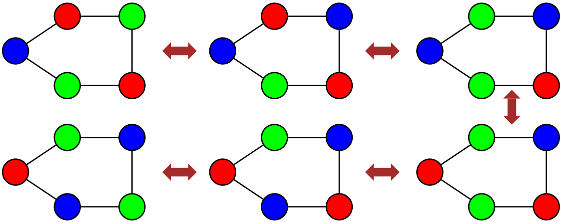
Take-Home Messages

1. *Reconfiguration* studies the "solution space" of a problem
   - Moving from one solution to another *without violating feasibility*
2. Under certain *distance constraints*,
   - *Reconfiguration problems* can be *hard* for very restricted graph classes
   - *Problems on graphs whose diameters are bounded by some constant c* (e.g., split graphs) are interesting when *restricted to distances close to c*
3. *Nondeterministic Constraint Logic* is a powerful tool for *hardness reductions*

Open Problems

The complexities of the following problem remain *open* for *trees*:

- (d, k)-CR $(d \geq 2)$



Partially supported by

Came to my talk, you did.
Thank you, I must!

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