VNU-HUS MAT1206E/3508: Introduction to AI

First-order Predicate Logic

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Motivation

Motivation

Many practical, relevant problems *cannot be* or *can only very inconveniently be formulated* in the language of *propositional logic*

Example 1

- Statement: "Robot 7 is situated at the xy position $(35, 79)$"
- Propositional variable:
  Robot_7_is_situated_at_xy_position_$(35, 79)$ (which is true if the statement holds and false otherwise)
- Assume that 100 robots can stop anywhere on a grid of 100×100 points. We need 10^6 propositional variables to describe all possible positions of the robots [Why?]

Motivation

Example 1 (cont.)

- Relationship between objects (robots): “Robot 7 is to the right of robot 12”
- Propositional variable: Robot_7_is_to_the_right_of_robot_12
- The relation can be represented by an ordered pair of x-coordinates (x_7, x_{12}) where $x_7 > x_{12}$
  - There are $99 + 98 + \cdots + 1 = (100 \cdot 99)/2 = 4950$ such ordered pairs
- There are 10^4 propositional variables to describe all possible relations and therefore 10^4 formulas of the type:

  Robot_7_is_to_the_right_of_robot_$12 \Leftrightarrow$

  Robot_7_is_situated_at_xy_position_$(35, 79) \wedge$

  Robot_12_is_situated_at_xy_position_$(10, 93) \vee \ldots$

  with total $4950 \cdot 10^4$ alternatives on the right-hand side of the formulas [Why?]

Motivation

- Given 100 robots, describing the relation "Robot A is to the right of robot B" for all pairs of robots requires a huge number of propositional variables
- In *first-order predicate logic (PL1)*, we can define a predicate *position*(*number*, *xPosition*, *yPosition*)
- Now the relation can be described abstractly with

$$\forall u \forall v \textit{is\_further\_right}(u, v) \Leftrightarrow$$
$$\exists x_u \exists y_u \exists x_v \exists y_v \textit{position}(u, x_u, y_u) \wedge \textit{position}(v, x_v, y_v) \wedge x_u > x_v$$

where $\forall u$ is read as "for every u" and $\exists v$ as "there exists v"

Syntax

- Set of *variables* V
- Set of *constants* K (which *stand for objects*)
- Set of *function symbols* F (which *stand for functions*)
- The sets V, K, F are *pairwise disjoint*

Terms

- A *term* is a logical expression that refers to an object
- All *variables* and *constants* are *(atomic) terms*
- If $t_1, \ldots, t_n$ are terms and f is an n-place function symbol, then $f(t_1, \ldots, t_n)$ is a *(complex) term*

Syntax

Example 2

- $f(\sin(\ln(3)), \exp(x))$ is a term
  - $V = \{x\}$, $K = \{3\}$, and $F = \{\sin, \ln, \exp, f\}$
- $g(g(g(x)))$ is a term
  - $V = \{x\}$, $K = \emptyset$, and $F = \{g\}$
- *LeftLeg*(*John*) is a term
  - $V = \emptyset$, $K = \{\textit{John}\}$, and $F = \{\textit{LeftLeg}\}$

- It is important to remember that *a complex term is just a complicated kind of name*. It is not a "subroutine call" that "returns a value."
  - Constant symbols are names for objects
  - It is not always convenient to have a distinct symbol for each object. Therefore, we also use function symbols to indicate names for objects
    - For example, instead of giving a constant symbol for "John's left leg", we use *LeftLeg*(*John*)

Syntax

To be able to establish *logical relationships between terms*, we build formulas from terms

Predicate Logic Formulas

Let P be a set of *predicate symbols* (which *stand for relations*)

- If $t_1, \ldots, t_n$ are terms and p is an n-place predicate symbol, then $p(t_1, \ldots, t_n)$ is an *(atomic) formula*
- If A and B are formulas, then $\neg A$, (A), $A \wedge B$, $A \vee B$, $A \Rightarrow B$, $A \Leftrightarrow B$ are also *formulas*
- If x is a variable and A a formula, then $\forall x\, A$ *and* $\exists x\, A$ are also *formulas*. $\forall$ is the *universal quantifier* and $\exists$ the *existential quantifier*
- $p(t_1, \ldots, t_n)$ and $\neg p(t_1, \ldots, t_n)$ are called *literals*
- Formulas in which every variable is in the scope of a quantifier are called *first-order sentences* or *closed formulas*. Variables which are not in the scope of a quantifier are called *free variables*
- Definitions of *CNF* and *Horn clauses* hold for formulas of predicate logic literals analogously

Syntax

Example 3

Formula	Description
$\forall x\ \mathit{frog}(x) \Rightarrow \mathit{green}(x)$	All frogs are green
$\forall x\ \mathit{frog}(x) \wedge \mathit{brown}(x) \Rightarrow \mathit{big}(x)$	All brown fogs are big
$\forall x\ \mathit{likes}(x, \mathit{cake})$	Everyone likes cake
$\neg\forall x\ \mathit{likes}(x, \mathit{cake})$	Not everyone likes cake
$\neg\exists x\ \mathit{likes}(x, \mathit{cake})$	No one likes cake
$\exists x\, \forall y\ \mathit{likes}(y, x)$	There is something that everyone likes
$\exists x\, \forall y\ \mathit{likes}(x, y)$	There is someone who likes everything
$\forall x\, \exists y\ \mathit{likes}(y, x)$	Everything is loved by someone
$\forall x\, \exists y\ \mathit{likes}(x, y)$	Everyone likes something

Syntax

Example 3 (cont.)

Formula	Description
$\forall x\ \mathit{customer}(x) \Rightarrow \mathit{likes}(\mathit{bob}, x)$	Bob likes every customer
$\exists x\ \mathit{customer}(x) \wedge \mathit{likes}(\mathit{bob}, x)$	There is a customer whom Bob likes
$\exists x\ \mathit{baker}(x) \wedge \forall y\ \mathit{customer}(y) \Rightarrow \mathit{likes}(x, y)$	There is a baker who likes all of his customers
$\forall x\ \mathit{older}(\mathit{mother}(x), x)$	Every mother is older than her child
$\forall x\ \mathit{older}(\mathit{mother}(\mathit{mother}(x)), x)$	Every grandmother is older than her daughter's child
$\forall x\, \forall y\, \forall z\ \mathit{rel}(x, y) \wedge \mathit{rel}(y, z) \Rightarrow \mathit{rel}(x, z)$	*rel* is a transitive relation

Semantics

In *propositional logic*, an *interpretation* (*assignment*) is a mapping that assigns a truth value (either t or f) to each variable. Analogously, in *first-order predicate logic*, we have

Interpretation

An *interpretation* (or *assignment*) $\mathbb{I}$ is defined as

- A mapping from the set of *constants and variables* $K \cup V$ to a set W of names of *objects* in the world
- A mapping from the set of *function symbols* to the set of *functions* in the world. Every n-place function symbol is assigned an n-place function
- A mapping from the set of *predicate symbols* to the set of *relations* in the world. Every n-place predicate symbol is assigned an n-place relation.

- The *truth of a formula* in PL1 depends on the *interpretation*

Semantics

Example 4

- Constants $K = \{c_1, c_2, c_3\}$
- A two-place function symbols *plus*
- A two-place predicate symbol *gr*

Consider the formula $\varphi \equiv gr(plus(c_1, c_3), c_2)$

Interpretation $\mathbb{I}_1$

- Mapping constants: $c_1 \mapsto 1$, $c_2 \mapsto 2$, *and* $c_3 \mapsto 3$
- Mapping function symbol: *plus* $\mapsto +$
- Mapping predicate symbol: *gr* $\mapsto >$

- $\varphi \mapsto 1 + 3 > 2$ or equivalently $\varphi \mapsto 4 > 2$
- The greater-than relation $>$ on the set $\{1, 2, 3, 4\}$ is the set of pairs (x, y) of numbers in $\{1, 2, 3, 4\}$ with $x > y$, and $(4, 2)$ is *in* that set
- φ is *true* under the interpretation $\mathbb{I}_1$

Semantics

Example 4

- Constants $K = \{c_1, c_2, c_3\}$
- A two-place function symbols *plus*
- A two-place predicate symbol *gr*

Consider the formula $\varphi \equiv gr(plus(c_1, c_3), c_2)$

Interpretation $\mathbb{I}_2$

- Mapping constants: $c_1 \mapsto 2$, $c_2 \mapsto 3$, *and* $c_3 \mapsto 1$
- Mapping function symbol: *plus* $\mapsto -$
- Mapping predicate symbol: *gr* $\mapsto >$

- $\varphi \mapsto 2 - 1 > 3$ or equivalently $\varphi \mapsto 1 > 3$
- The greater-than relation $>$ on the set $\{1, 2, 3, 4\}$ is the set of pairs (x, y) of numbers in $\{1, 2, 3, 4\}$ with $x > y$, and $(1, 3)$ is *not in* that set
- φ is *false* under the interpretation $\mathbb{I}_2$

Semantics

The truth of a formula

- An *atomic formula* $p(t_1, \ldots, t_n)$ is *true* (or valid) *under the interpretation* $\mathbb{I}$ if, after interpretation and evaluation of all terms $t_1, \ldots, t_n$ and interpretation of the predicate p through the n-place relation r, it holds that $(\mathbb{I}(t_1), \ldots, \mathbb{I}(t_n)) \in r$
- *The truth of quantifierless formulas* follows from the *truth of atomic formulas through the semantics of the logical operators* $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$, like in propositional logic
- A *formula* $\forall x\, F$ is *true under the interpretation* $\mathbb{I}$ exactly when it is *true given an arbitrary change of the interpretation for the variable* x *(and only for* x*)*
- A *formula* $\exists x\, F$ is *true under the interpretation* $\mathbb{I}$ exactly when *there is an interpretation for* x *which makes the formula true*

The definitions of *semantic equivalence of formulas*, for the concepts *satisfiable*, *true*, *unsatisfiable*, and *model*, along with *semantic entailment* carry over *unchanged from propositional calculus to predicate logic*

Semantics

The semantics of logical operators holds analogously for PL1

A	B	(A)	$\neg A$	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
t	t	t	f	t	t	t	t
t	f	t	f	f	t	f	f
f	t	f	t	f	t	t	f
f	f	f	t	f	f	t	t

The following theorems hold analogously for PL1

Theorem 1 (Deduction theorem)

$A \models B$ *if and only if* $\models A \Rightarrow B$

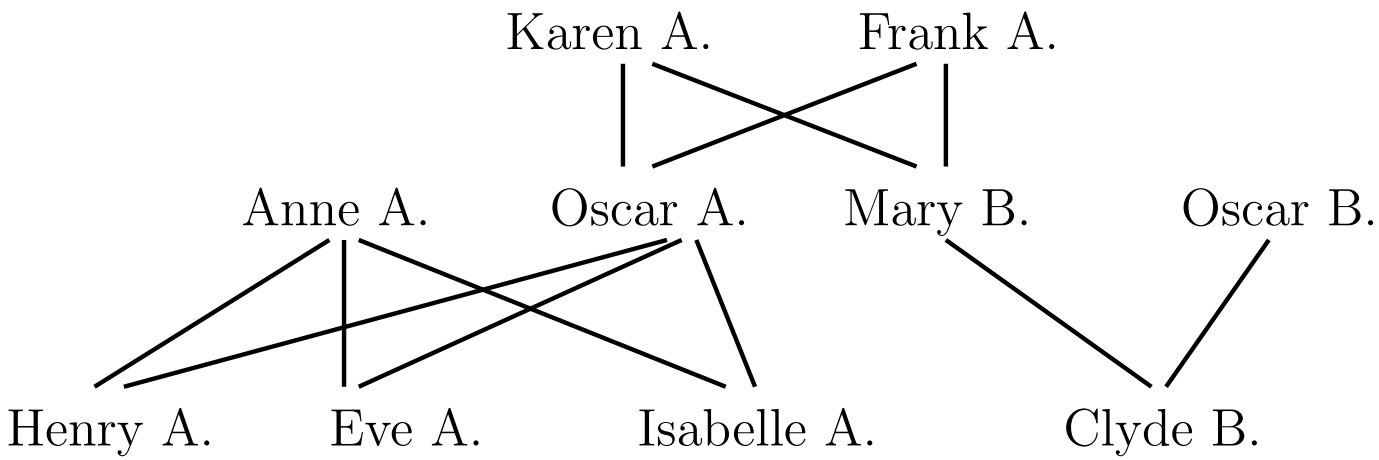
Theorem 2 (Proof by contradiction)

$KB \models Q$ *if and only if* $KB \wedge \neg Q$ *is unsatisfiable*

Semantics

Example 5

A family tree



- *Child* – A three-place relation (predicate)

  $Child = \{$(Oscar A., Karen A., Frank A.), (Mary B., Karen A., Frank A.), (Henry A., Anne A., Oscar A.), (Eve A., Anne A., Oscar A.), (Isabelle A., Anne A., Oscar A.), (Clyde B., Mary B., Oscar B.)$\}$

  - The triple (Oscar A., Karen A., Frank A.) stands for "Oscar A. is a child of Karen A. and Frank A."

Semantics

Example 5 (cont.)

- From the names, we read off the one-place relation (predicate) *Female* of the women
  $\mathit{Female} = \{\text{Karen A.}, \text{Anne A.}, \text{Mary B.}, \text{Eve A.}, \text{Isabelle A.}\}$
- We now want to *establish formulas for family relationships*
  - We define a *three-place predicate* $\mathit{child}(x, y, z)$ with the semantics $\mathbb{I}(\mathit{child}(x, y, z)) = w \equiv (\mathbb{I}(x), \mathbb{I}(y), \mathbb{I}(z)) \in \mathit{Child}$
  - If $\mathbb{I}$ *is the interpretation* $\mathit{oscar} \mapsto$ *Oscar A.,* $\mathit{eve} \mapsto$ *Eve A., and* $\mathit{anne} \mapsto$ *Anne A.*, then $\mathbb{I}(\mathit{child}(\mathit{eve}, \mathit{anne}, \mathit{oscar})) \equiv (\text{Eve A.}, \text{Anne A.}, \text{Oscar A.}) \in \mathit{Child}$. Thus, $\mathit{child}(\mathit{eve}, \mathit{anne}, \mathit{oscar})$ is *true under the interpretation* $\mathbb{I}$
  - Naturally, based on what we knew from the real world, we would *want* $\mathit{child}(\mathit{eve}, \mathit{oscar}, \mathit{anne})$ *also to be true*. To have this, we require $\forall x \, \forall y \, \forall z \; \mathit{child}(x, y, z) \Leftrightarrow \mathit{child}(x, z, y)$
  - We can also define a *two-place predicate* $\mathit{descendant}(x, y)$ recursively as

$$\forall x \, \forall y \; \mathit{descendant}(x, y) \Leftrightarrow \exists z \; \mathit{child}(x, y, z)$$
$$\vee \; (\exists u \, \exists v \; \mathit{child}(x, u, v) \wedge \mathit{descendant}(u, y))$$

Semantics

The predicates we defined in Example 5 are based on our knowledge from the real world. Analogously to the propositional logic, we can *build a small knowledge base with rules and facts*, for example, like

$$KB \equiv \mathit{female}(\mathit{karen}) \wedge \mathit{female}(\mathit{anne}) \wedge \mathit{female}(\mathit{mary}) \wedge$$
$$\mathit{female}(\mathit{eve}) \wedge \mathit{female}(\mathit{isabelle}) \wedge \mathit{child}(\mathit{oscar}, \mathit{karen}, \mathit{frank}) \wedge$$
$$\mathit{child}(\mathit{mary}, \mathit{karen}, \mathit{frank}) \wedge \mathit{child}(\mathit{eve}, \mathit{anne}, \mathit{oscar}) \wedge$$
$$\mathit{child}(\mathit{henry}, \mathit{anne}, \mathit{oscar}) \wedge \mathit{child}(\mathit{isabelle}, \mathit{anne}, \mathit{oscar}) \wedge$$
$$\mathit{child}(\mathit{clyde}, \mathit{mary}, \mathit{oscarb}) \wedge (\forall x \, \forall y \, \forall z \; \mathit{child}(x, y, z) \Rightarrow \mathit{child}(x, z, y))$$
$$\wedge \; (\forall x \, \forall y \; \mathit{descendant}(x, y) \Leftrightarrow \exists z \; \mathit{child}(x, y, z)$$
$$\vee \; (\exists u \, \exists v \; \mathit{child}(x, u, v) \wedge \mathit{descendant}(u, y)))$$

and ask, for example, *whether child(eve, oscar, anne) or descendant(eve, frank) are (syntactically) derivable*? To that end, we require a *calculus*

Semantics

Exercise 1 ([Ertel 2025], Exercise 3.1, p. 65)

Let the three-place predicate "*child*" and the one-place predicate "*female*" from Example 5 be given. Define:

(a) A one-place predicate "*male*"

(b) A two-place predicate "*father*" and "*mother*"

(c) A two-place predicate "*siblings*"

(d) A predicate "*parents*(x, y, z)", which is true if and only if x is the father and y is the mother of z

(e) A predicate "*uncle*(x, y)", which is true if and only if x is the uncle of y (use the predicates that have already been defined)

(f) A two-place predicate "*ancestor*" with the meaning: *ancestors are parents, grandparents, etc. of arbitrarily many generations*

Exercise 2 ([Ertel 2025], Exercise 3.2, p. 65)

Formalize the following statements in predicate logic:

(a) Every person has a father and a mother

(b) Some people have children

(c) All birds fly

(d) There is an animal that eats (some) grain-eating animals

(e) Every animal eats plants or plant-eating animals which are much smaller than itself

Semantics

- To be able to compare terms, *equality* is a very important relation in predicate logic
- The *equality of terms in mathematics* is an *equivalence relation*, meaning it is *reflexive*, *symmetric* and *transitive*
- We define a *predicate "="* using infix notation as is customary in mathematics (that is, instead of writing "$eq(x, y)$", we write "$x = y$")

Equality Axioms

$$\forall x \quad x = x \quad \text{(reflexive)}$$

$$\forall x \forall y \quad x = y \Rightarrow y = x \quad \text{(symmetry)}$$

$$\forall x \forall y \forall z \quad x = y \wedge y = z \Rightarrow x = z \quad \text{(transitivity)}$$

To guarantee the uniqueness of functions, we additionally require that for any function symbol f and any predicate symbol p

$$\forall x \forall y \quad x = y \Rightarrow f(x) = f(y) \quad \text{(substitution axiom)}$$

$$\forall x \forall y \quad x = y \Rightarrow p(x) \Leftrightarrow p(y) \quad \text{(substitution axiom)}$$

Semantics

Exercise 3 ([Ertel 2025], Exercise 3.3, p. 65)

Adapt Exercise 1 by using one-place function symbols and equality instead of "*father*" and "*mother*"

Exercise 4 ([Ertel 2025], Exercise 3.4, p. 66)

Give predicate logic axioms for the two-place relation "$<$" as a total order. For a total order we must have (1) Any two elements are comparable. (2) It is symmetric. (3) It is transitive

Semantics

Often a variable must be replaced by a term

Example 6

Consider the formula $\forall x\, x = 5 \Rightarrow x = y$
Replace y by the term $\sin(x)$?

$$\forall x\, x = 5 \Rightarrow x = \sin(x) \quad \textbf{WRONG!}$$

(The "x" in the term $\sin(x)$ is different from the "x" in the original formula, as y does not depend on x in the original formula)

$$\forall x\, x = 5 \Rightarrow x = \sin(z) \quad \textbf{CORRECT!}$$

Replacing a variable by a term

We write $\varphi[x/t]$ for the formula that results when we *replace every free occurrence of the variable x in φ with the term t*. Thereby we *do not allow any variables in the term t that are quantified in φ*. In those cases *variables must be renamed* to ensure this

Semantics

Substitution

A *substitution* σ is *a map from variables to terms*.

- ϵ: the empty substitution
- $\sigma : x_1/t_1, x_2/t_2, \ldots, x_n/t_n$: a substitution that maps each variable x_i to the corresponding term t_i
- Also write $\sigma = \{x_1/t_1, x_2/t_2, \ldots, x_n/t_n\}$

- *Apply σ to a term t*, denoted by $\sigma(t)$ or $t[x_1/t_1, \ldots, x_n/t_n]$ to indicate $\sigma(t)$, means *simultaneously replacing every occurrence of each x_i in t by t_i*

Quantifiers and Normal Forms

- By definition, $\forall x\, p(x)$ is true if and only if $p(x)$ is true for all interpretations of the variable x. Instead, we can write

$$\forall x\, p(x) \equiv p(a_1) \wedge \cdots \wedge p(a_n),$$

  for all constants $a_1, \ldots, a_n$ in K
- Similarly, for $\exists x\, p(x)$

$$\exists x\, p(x) \equiv p(a_1) \vee \cdots \vee p(a_n),$$

  for all constants $a_1, \ldots, a_n$ in K
- From this, it follows with the De Morgan's law that

$$\forall x\, \varphi \equiv \neg \exists x\, \neg \varphi$$

  Through this equivalence, *universal and existential quantifiers are mutually replaceable*

Example 7

"Everyone wants to be loved" $\equiv$ "Nobody does not want to be loved"

Quantifiers and Normal Forms

Prenex Normal Form

A predicate logic formula φ is in *prenex normal form* if it holds that

- $\varphi = Q_1 x_1 \dots Q_n x_n \psi$
- ψ is a quantifierless formula
- $Q_i \in \{\forall, \exists\}$ for $i = 1, \dots, n$

Theorem 3

Every predicate logic formula can be transformed into an equivalent formula in prenex normal form

Example 8

- Be careful in case *a quantified variable appears outside the scope of its quantifier*
  - For example, consider $\forall x\,(p(x) \Rightarrow \exists x\, q(x))$
  - In this case, one of the two variables "x" (in $\forall x$ and $\exists x$) must be renamed, for example, like $\forall x\,(p(x) \Rightarrow \exists y\, q(y))$

Quantifiers and Normal Forms

Transformation into prenex normal form

- Transformation into conjunctive normal form
  - Elimination of equivalences
  - Elimination of implications
  - Repeated application of De Morgan's law and distributive law
- Renaming of variables if necessary
- Factoring out universal quantifiers

- The formula $\forall x\,(p(x) \Rightarrow \exists y\,q(y))$ can be written in prenex normal form by bringing the quantifier $\exists$ to the front, as in $\forall x\,\exists y\,(p(x) \Rightarrow q(y))$

Exercise 5

How about $(\forall x\,p(x)) \Rightarrow (\exists y\,q(y))$? Can you bring this formula to prenex normal form?

Quantifiers and Normal Forms

Example 9 (Transforming into prenex normal form)

- The convergence of a series $(a_n)_{n\in\mathbb{N}}$ to a limit a is defined by $\forall\epsilon > 0\,\exists n_0 \in \mathbb{N}\,\forall n > n_0\,|a_n - a| < \epsilon$
- We define the *functions* ***abs***(x) for x, $a(n)$ for a_n, ***minus***(x, y) for $x - y$, and the *predicates* ***el***(x, y) for $x \in y$, ***gr***(x, y) for $x > y$. The formula becomes

$$\forall\epsilon\,(gr(\epsilon, 0) \Rightarrow \exists n_0\,(\textbf{\textit{el}}(n_0, \mathbb{N}) \Rightarrow \forall n\,(\textbf{\textit{gr}}(n, n_0)$$
$$\Rightarrow \textbf{\textit{gr}}(\epsilon, \textbf{\textit{abs}}(\textbf{\textit{minus}}(a(n), a))))))$$

- To bring the formula to *prenex normal form*:
  - No variables need to be renamed
  - Eliminating the implications

$$\forall\epsilon\,(\neg gr(\epsilon, 0) \vee \exists n_0\,(\neg\textbf{\textit{el}}(n_0, \mathbb{N}) \vee \forall n\,(\neg\textbf{\textit{gr}}(n, n_0)$$
$$\vee\, \textbf{\textit{gr}}(\epsilon, \textbf{\textit{abs}}(\textbf{\textit{minus}}(a(n), a))))))$$

  - Move quantifiers to the front

$$\forall\epsilon\,\exists n_0\,\forall n\,(\neg gr(\epsilon, 0) \vee \neg\textbf{\textit{el}}(n_0, \mathbb{N}) \vee \neg\textbf{\textit{gr}}(n, n_0)$$
$$\vee\, \textbf{\textit{gr}}(\epsilon, \textbf{\textit{abs}}(\textbf{\textit{minus}}(a(n), a))))$$

Quantifiers and Normal Forms

Skolemization

- **Goal:**
  - Eliminate all existential quantifiers from a formula in prenex normal form
- **General Method:**
  - Replacement of existentially quantified variables by new Skolem functions
  - Deletion of resulting universal quantifiers

Note

- After Skolemization, *the resulting formula is no longer equivalent to the original one*
- However, *if the original formula is true, then the resulting formula is also true*
  - This is particularly useful when we want to show that a formula, say $KB \wedge \neg Q$, is not satisfied

Quantifiers and Normal Forms

Example 10 (Skolemization)

- We *skolemize* the following formula

$$\forall x_1\, \forall x_2\, \exists y_1\, \forall x_3\, \exists y_2\, p(f(x_1), x_2, y_1) \vee q(y_1, x_3, y_2)$$

- The variable y_1 apparently *depends on* x_1 *and* x_2. We *replace every occurrence of* y_1 *by a Skolem function* $g(x_1, x_2)$, where g is a new function symbol that has not yet appeared in the formula

$$\forall x_1\, \forall x_2\, \forall x_3\, \exists y_2\, p(f(x_1), x_2, g(x_1, x_2)) \vee q(g(x_1, x_2), x_3, y_2)$$

- Analogously, the variable y_2 apparently *depends on* x_1, x_2, *and* x_3. We *replace every occurrence of* y_2 *by a Skolem function* $h(x_1, x_2, x_3)$

$$\forall x_1\, \forall x_2\, \forall x_3\, p(f(x_1), x_2, g(x_1, x_2)) \vee q(g(x_1, x_2), x_3, h(x_1, x_2, x_3))$$

- Finally, as all variables are universally quantified, the universal quantifiers can be left out, giving us the resulting skolemized formula

$$p(f(x_1), x_2, g(x_1, x_2)) \vee q(g(x_1, x_2), x_3, h(x_1, x_2, x_3))$$

Quantifiers and Normal Forms

Example 11 (Skolemization)

- We *skolemize* the following formula

$$\forall \epsilon\, \exists n_0\, \forall n\, (\neg gr(\epsilon, 0) \vee \neg \mathit{el}(n_0, \mathbb{N}) \vee \neg \mathit{gr}(n, n_0) \vee \mathit{gr}(\epsilon, \mathit{abs}(\mathit{minus}(a(n), a))))$$

- The variable n_0 apparently *depends on* ϵ. We *replace every occurrence of* n_0 *by a Skolem function* $n_0(\epsilon)$.

$$\forall \epsilon\, \forall n\, (\neg gr(\epsilon, 0) \vee \neg \mathit{el}(n_0(\epsilon), \mathbb{N}) \vee \neg \mathit{gr}(n, n_0(\epsilon)) \vee \mathit{gr}(\epsilon, \mathit{abs}(\mathit{minus}(a(n), a))))$$

- Finally, we have the skolemized formula

$$(\neg gr(\epsilon, 0) \vee \neg \mathit{el}(n_0(\epsilon), \mathbb{N}) \vee \neg \mathit{gr}(n, n_0(\epsilon)) \vee \mathit{gr}(\epsilon, \mathit{abs}(\mathit{minus}(a(n), a))))$$

Quantifiers and Normal Forms

When skolemizing a formula in prenex normal form

- *All existential quantifiers are eliminated from the outside inward*. That is, a formula of the form $\forall x_1 \forall x_2 \ldots \forall x_n \exists y \varphi$ is replaced by $\forall x_1 \forall x_2 \ldots \forall x_n \varphi[y/f(x_1, x_2, \ldots, x_n)]$ where f is a new function that has not yet appeared in φ
- *If an existential quantifier is on the far outside*, such as *in* $\exists y\, p(y)$, then y *must be replaced by a constant*

Running time (in the number of literals)

- Transformation into prenex normal form
  - Exponential (naive)
  - Polynomial [Eder 1992]
- Skolemization
  - Polynomial

Proof Calculi

- For *reasoning in predicate logic*, various calculi of *natural reasoning* have been developed
- For example, *Gentzen calculus* or *sequent calculus*
  - *These calculi are meant to be applied by humans*: the inference rules are more or less intuitive, the calculi work on arbitrary PL1 formulas
- Based on Example 5, we give an example of a *small "natural" proof* using the following *inference rules*
  - *Modus Ponens (MP)*

$$\frac{A, A \Rightarrow B}{B}$$

  - *$\forall$-Elimination ($\forall$-E)*

$$\frac{\forall x\, A}{A[x/t]}$$

Note: t is a ground term that *contains no variables*

Proof Calculi

Example 12 (A natural deduction)

- In Example 5, we built a small knowledge base KB with rules and facts

  $KB \equiv \mathit{female}(\mathit{karen}) \wedge \mathit{female}(\mathit{anne}) \wedge \mathit{female}(\mathit{mary}) \wedge$
  $\mathit{female}(\mathit{eve}) \wedge \mathit{female}(\mathit{isabelle}) \wedge \mathit{child}(\mathit{oscar}, \mathit{karen}, \mathit{frank}) \wedge$
  $\mathit{child}(\mathit{mary}, \mathit{karen}, \mathit{frank}) \wedge \mathit{child}(\mathit{eve}, \mathit{anne}, \mathit{oscar}) \wedge$
  $\mathit{child}(\mathit{henry}, \mathit{anne}, \mathit{oscar}) \wedge \mathit{child}(\mathit{isabelle}, \mathit{anne}, \mathit{oscar}) \wedge$
  $\mathit{child}(\mathit{clyde}, \mathit{mary}, \mathit{oscarb}) \wedge (\forall x \forall y \forall z\ \mathit{child}(x, y, z) \Rightarrow \mathit{child}(x, z, y))$
  $\wedge (\forall x \forall y\ \mathit{descendant}(x, y) \Leftrightarrow \exists z\ \mathit{child}(x, y, z)$
  $\vee (\exists u \exists v\ \mathit{child}(x, u, v) \wedge \mathit{descendant}(u, y)))$

- Can $\mathit{child}(\mathit{eve}, \mathit{oscar}, \mathit{anne})$ be derived using the MP and $\forall$-E inference rules?

Step	**Proved by**
1. $\mathit{child}(\mathit{eve}, \mathit{anne}, \mathit{oscar})$	KB
2. $\forall x \forall y \forall z\ \mathit{child}(x, y, z) \Rightarrow \mathit{child}(x, z, y)$	KB
3. $\mathit{child}(\mathit{eve}, \mathit{anne}, \mathit{oscar}) \Rightarrow \mathit{child}(\mathit{eve}, \mathit{oscar}, \mathit{anne})$	$\forall$-E for 2: x/eve, y/anne, z/oscar
4. $\mathit{child}(\mathit{eve}, \mathit{oscar}, \mathit{anne})$	MP for 1 and 3

Proof Calculi

- The calculus with *just MP and $\forall$-E* is *not complete*
- It *can be extended to a complete calculus* by adding further inference rules

Theorem 4 (Gödel's completeness theorem [Godel 1931])

First-order predicate logic is complete. That is, there is a calculus with which every proposition that is a consequence of a knowledge base KB can be proved. If $KB \models \varphi$, then it holds that $KB \vdash \varphi$

- Every true proposition in first-order predicate logic is provable (= syntactically derivable)
- Is the reverse true? In other words, is everything we can derive syntactically actually true?

Proof Calculi

Indeed, the answer is “yes”

Theorem 5 (Correctness)

There are calculi with which only true propositions can be proved. That is, if $KB \vdash \varphi$, then $KB \models \varphi$

- *Provability and semantic consequence are therefore equivalent concepts, as long as correct and complete calculus is being used*. Thereby first-order predicate logic becomes a powerful tool for mathematics and AI
- The aforementioned calculi of natural deduction are rather *unsuited for automatization*
- Only *resolution calculus*, which was introduced in 1965 and essentially works with only one simple inference rule, *enabled the construction of powerful automated theorem provers*, which later were employed as inference machines for expert systems

Resolution

Recall: Resolution Rule

$$\frac{A \vee B, \neg B \vee C}{A \vee C} \quad \text{or} \quad \frac{A \vee B, B \Rightarrow C}{A \vee C}$$

Recall: Resolution Proof

- **Input:** Knowledge base KB
- **Goal:** Decide whether $KB \models Q$
- **Method:** Add $\neg Q$ to the knowledge base. If the empty clause can be derived, conclude $KB \models Q$. If there is no more resolvable pair of clauses (and the empty clause is not derived), conclude $KB \not\models Q$

Resolution

Example 13 (Resolution Proof of Example 12)

- $KB \equiv child(eve, anne, oscar) \wedge (\forall x \forall y \forall z\, child(x, y, z) \Rightarrow child(x, z, y))$
- $Q \equiv child(eve, oscar, anne)$
- Proof:

Step	Proved by
1. $child(eve, anne, oscar)$	KB
2. $\forall x \forall y \forall z\, child(x, y, z) \Rightarrow child(x, z, y)$	KB
3. $\neg child(eve, oscar, anne)$	$\neg Q$
4. $child(eve, anne, oscar) \Rightarrow child(eve, oscar, anne)$	Unification for 2: x/eve, $y/anne$, $z/oscar$
5. $\neg child(eve, anne, oscar) \vee child(eve, oscar, anne)$	Equivalent form of 4
6. $\neg child(eve, anne, oscar)$	Res$(3, 5)$
7. $()$	Res$(1, 6)$

Resolution

Example 14

- Everyone knows his own mother. Does Henry know anyone?
- We use a function symbol *mother* and a predicate symbol *knows*
- $KB \equiv \forall x\, \mathit{knows}(x, \mathit{mother}(x))$
- $Q \equiv \exists y\, \mathit{knows}(\mathit{henry}, y)$
- Proof:

Step	Proved by
1. $\forall x\, \mathit{knows}(x, \mathit{mother}(x))$	KB
2. $\forall y\, \neg \mathit{knows}(\mathit{henry}, y)$	$\neg Q$
3. $\mathit{knows}(\mathit{henry}, \mathit{mother}(\mathit{henry}))$	Uni. (σ) for 1: x/henry
4. $\neg \mathit{knows}(\mathit{henry}, \mathit{mother}(\mathit{henry}))$	Uni. (σ) for 2: $y/\mathit{mother}(\mathit{henry})$
5. $()$	Res$(3, 4)$

- The replacement step σ defined by x/henry and $y/\mathit{mother}(\mathit{henry})$ is called *unification*

Resolution

Unifier

- Two literals are called *unifiable* if *there is a substitution σ for all variables which makes the literals look identical*. Such a σ is called a *unifier*.
- A unifier is called the *most general unifier (MGU)* if *all other unifiers can be obtained from it by substitution of variables*.

Resolution

Example 15

- Two literals $knows(x, mother(x))$ and $knows(henry, y)$ in Example 14 are unifiable
  With the subtitution $\sigma: \quad x/henry, \quad y/mother(henry)$ both literals become $knows(henry, mother(henry))$.
- Two literals $knows(henry, x)$ and $knows(x, marry)$ are not unifiable.
  - The variable x cannot take on the values *henry* and *marry* at the same time.
  - However, $knows(henry, x)$ means that "Henry knows everyone". So we should be able to infer that Henry knows Marry
  - The problem arises because both literals use the same variable x, which can be avoided by renaming x in $knows(x, marry)$ to y (a new variable name) without changing its meaning. Now, $knows(henry, x)$ and $knows(y, marry)$ are unifiable by the substitution $\sigma: x/marry, y/henry$

Resolution

Example 16

- There are *different unifiers* for unifying the literals $p(f(g(x)), y, z)$ and $p(u, u, f(u))$

σ_1 :		$y/f(g(x))$,	$z/f(f(g(x)))$,	$u/f(g(x))$
σ_2 :	$x/h(v)$,	$y/f(g(h(v)))$,	$z/f(f(g(h(v))))$,	$u/f(g(h(v)))$
σ_3 :	$x/h(h(v))$,	$y/f(g(h(h(v))))$,	$z/f(f(g(h(h(v)))))$,	$u/f(g(h(h(v))))$
σ_4 :	$x/h(a)$,	$y/f(g(h(a)))$,	$z/f(f(g(h(a))))$,	$u/f(g(h(a)))$
σ_5 :	x/a,	$y/f(g(a))$,	$z/f(f(g(a)))$,	$u/f(g(a))$

- Among these unifiers, σ_1 *is the most general unifier (MGU)*: The other unifiers can be respectively obtained from σ_1 through the substitutions $x/h(v)$, $x/h(h(v))$, $x/h(a)$, and x/a

Resolution

- Need a procedure to find a MGU given a set of expressions
- Requirements:
  - stop after a finite number of steps
  - return an MGU if the set is unifiable
  - state that the set is not unifiable otherwise
- There are many possibilities
- We go for a recursive procedure

Resolution

Basic Idea

- Given a set of expressions $\{E_1, \ldots, E_k\}$.
- Find a *disagreement set* (which we will define later).
- Build a substitution that can eliminate the disagreement.

Example 17 (Disagreement Elimination)

- Consider the set $\{p(a), p(x)\}$ of expressions.
- They disagree because of the arguments a and x.
- The disagreement set here is $\{a, x\}$. Since x is a variable, we can eliminate this disagreement by using the substitution $\sigma : x/a$.
- $\sigma(p(a)) = \sigma(p(x)) = p(a)$.

Resolution

Disagreement Set

The *disagreement set* of a nonempty set of expressions W is obtained by *finding the first position (starting from the left) at which not all the expressions in the W have the same symbol*. We *then extract*, from each expression, *the sub-expression that begins with the symbol occupying that position*.
The set of these sub-expressions is the *disagreement set*.

Example 18

- Consider the set $W = \{p(x), p(a)\}$.
- The first position at which the string of symbols $p(a)$ and $p(x)$ differ is the position number 3.
- The sub-expressions starting from position 3 are a and x respectively.
- Disagreement set: $\{a, x\}$.

Resolution

Example 19

- Consider the set $W = \{p(x, f(y, z)), p(x, a), p(x, g(h(k(x))))\}$.
- The first position at which the string of symbols $p(x, f(y, z))$, $p(x, a)$, and $p(x, g(h(k(x))))$ differ is the positioin number 3.
- The sub-expressions starting from position 3 are $f(y, z)$, a, and $g(h(k(x)))$, respectively.
- Disagreement set: $\{f(y, z), a, g(h(k(x)))\}$

Resolution

Unification Algorithm:

(1) Set $k = 0$, $W_0 = W$, and $\sigma_0 = \epsilon$ (here ϵ denotes the empty substitution).

(2) If W_k is a singleton, STOP, σ_k is a MGU. Otherwise, find the disagreement set D_k for W_k.

(3) If there is a pair (v_k, t_k) such that $v_k, t_k \in D_k$, v_k is a variable that does not occur in the term t_k, go to step (4); Otherwise STOP, W is not unifiable.

(4) Appply the substitution v_k/t_k to σ_k and then add v_k/t_k to the resulting set to obtain σ_{k+1}. Apply the substitution v_k/t_k to W_k to obtain W_{k+1}.

(5) Set $k = k + 1$ and go to step (2).

Resolution

Example 20

- $W_0 = W = \{p(f(g(x)), y, z), p(u, u, f(u))\}$, $\sigma_0 = \epsilon$.
- $D_0 = \{f(g(x)), u\} \Rightarrow \sigma_1 = \{u/f(g(x))\}$,
  $W_1 = \{p(f(g(x)), y, z), p(f(g(x)), f(g(x)), f(f(g(x))))\}$.
- $D_1 = \{y, f(g(x))\} \Rightarrow \sigma_2 = \{u/f(g(x)), y/f(g(x))\}$, $W_2 = \{p(f(g(x)), f(g(x)), z), p(f(g(x)), f(g(x)), f(f(g(x))))\}$.
- $D_2 = \{z, f(f(g(x)))\} \Rightarrow$
  $\sigma_3 = \{u/f(g(x)), y/f(g(x)), z/f(f(g(x)))\}$,
  $W_3 = \{p(f(g(x)), f(g(x)), f(f(g(x))))\} \Rightarrow$ STOP and output σ_3.
- MGU: $u/f(g(x)), y/f(g(x)), z/f(f(g(x)))$. Term: $p(f(g(x)), f(g(x)), f(f(g(x))))$.

Resolution

Example 20

- $W_0 = W = \{p(f(g(x)), y, z), p(u, u, f(u))\}$, $\sigma_0 = \epsilon$.
- $D_0 = \{f(g(x)), u\} \Rightarrow \sigma_1 = \{u/f(g(x))\}$,
  $W_1 = \{p(f(g(x)), y, z), p(f(g(x)), f(g(x)), f(f(g(x))))\}$.
- $D_1 = \{y, f(g(x))\} \Rightarrow \sigma_2 = \{u/f(g(x)), y/f(g(x))\}$, $W_2 = \{p(f(g(x)), f(g(x)), z), p(f(g(x)), f(g(x)), f(f(g(x))))\}$.
- $D_2 = \{z, f(f(g(x)))\} \Rightarrow$
  $\sigma_3 = \{u/f(g(x)), y/f(g(x)), z/f(f(g(x)))\}$,
  $W_3 = \{p(f(g(x)), f(g(x)), f(f(g(x))))\} \Rightarrow$ STOP and output σ_3.
- MGU: $u/f(g(x)), y/f(g(x)), z/f(f(g(x)))$. Term: $p(f(g(x)), f(g(x)), f(f(g(x))))$.

Example 21

- $W_0 = W = \{p(f(x), g(y)), p(z, z)\}$, $\sigma_0 = \epsilon$.
- $D_0 = \{f(x), z\} \Rightarrow \sigma_1 = \{z/f(x)\}$,
  $W_1 = \{p(f(x), g(y)), p(f(x), f(x))\}$.
- $D_1 = \{g(y), f(x)\} \Rightarrow$ STOP, W is not unifiable.

Resolution

- Complexity of unification
  - The simplest unification algorithms are very fast in most cases
  - Worst case: the computation time grows exponentially with the size of the terms.
  - In practice, nearly all unification attempts fail, in most cases the worst case complexity has no dramatic effect
  - The fastest unification algorithms have nearly linear complexity [Bibel 1982]

Exercise 6 ([Ertel 2025], Exercise 3.5, p. 66)

Unify (if possible) the following terms and give the MGU and the resulting terms

(a) $p(x, f(y))$ and $p(f(z), u)$

(b) $p(x, f(x))$ and $p(y, y)$

(c) $x = 4 - 7 \cdot x$ and $\cos y = z$

(d) $x < 2 \cdot x$ and $3 < 6$

(e) $q(f(x, y, z), f(g(w, w), g(x, x), g(y, y)))$ and $q(u, u)$

Resolution

Generalized Resolution Rule for PL1

The resolution rule for two clauses in conjunctive normal form reads

$$\frac{A_1 \vee \ldots A_m \vee B, \neg B' \vee C_1 \vee \cdots \vee C_n \quad \sigma(B) = \sigma(B')}{\sigma(A_1) \vee \cdots \vee \sigma(A_m) \vee \sigma(C_1) \vee \cdots \vee \sigma(C_n)},$$

where σ *is the MGU of* B *and* B'

What does this definition mean?

- Premise 1: $A_1 \vee \ldots A_m \vee B$
- Premise 2: $\neg B' \vee C_1 \vee \cdots \vee C_n$
- $\sigma(B) = \sigma(B')$ means that B and B' are matched by applying the MGU σ
- Apply σ for every literal in each premise
- Now, Premise 1 becomes $\sigma(A_1) \vee \ldots \sigma(A_m) \vee \sigma(B)$ and Premise 2 becomes $\neg\sigma(B') \vee \sigma(C_1) \vee \cdots \vee \sigma(C_n)$
- The usual resolution rule can now be applied

Theorem 6 (Correctness)

The resolution rule is correct. That is, the resolvent is a semantic consequence of the two parent clauses

Resolution

For Completeness, we need an additional inference rule

Factorization Rule for PL1

Factorization of a clause is accomplished by

$$\frac{A_1 \vee A_2 \vee \cdots \vee A_n, \quad \sigma(A_1) = \sigma(A_2)}{\sigma(A_2) \vee \cdots \vee \sigma(A_n)},$$

where σ is the MGU of A_1 and A_2

What does this definition mean?

- Premise: $A_1 \vee A_2 \vee \cdots \vee A_n$
- $\sigma(A_1) = \sigma(A_2)$ means that A_1 and A_2 are matched after their MGU σ is applied
- Apply σ for every literal in the premise
- Now, the Premise becomes $\sigma(A_1) \vee \sigma(A_2) \vee \ldots \sigma(A_n)$
- As $p \vee p \equiv p$, the Conclusion $\sigma(A_2) \vee \ldots \sigma(A_n)$ is derived
- Intuitively, after σ is applied, as $\sigma(A_1) = \sigma(A_2)$, one of these literals becomes "redundant" and can be "removed"

Resolution

Example 22

- Russell's paradox: *There is a barber who shaves everyone who does not shave himself*
- $Q \equiv \forall x\, \mathit{shaves}(\mathit{baber}, x) \Leftrightarrow \neg \mathit{shaves}(x, x) \equiv \forall x\, (\neg \mathit{shaves}(\mathit{baber}, x) \vee \neg \mathit{shaves}(x, x)) \wedge (\mathit{shaves}(x, x) \vee \mathit{shaves}(\mathit{baber}, x))$
- Proof (that Q is contradictory):

Step	**Proved by**
1. $\neg \mathit{shaves}(\mathit{baber}, x) \vee \neg \mathit{shaves}(x, x)$	Q
2. $\mathit{shaves}(x, x) \vee \mathit{shaves}(\mathit{baber}, x)$	Q
3. $\neg \mathit{shaves}(\mathit{baber}, \mathit{barber})$	Fak$(1, \sigma : x/\mathit{barber})$
4. $\mathit{shaves}(\mathit{baber}, \mathit{barber})$	Fak$(2, \sigma : x/\mathit{barber})$
5. $()$	Res$(3, 4)$

Note: Without the Factorization Rule, from the two clauses 1 and 2, we can derive several tautologies, but no contradiction

Resolution

Exercise 7 ([Ertel 2025], Exercise 3.6, p. 66)

(a) Transform Russell's Paradox from Example 22 into CNF

(b) Show that the empty clause cannot be derived using resolution without factorization. Try to understand this intuitively

Resolution

Theorem 7

The resolution rule together with the factorization rule is refutation complete. That is, by application of factorization and resolution steps, the empty clause can be derived from any unsatisfiable formula in conjunctive normal form

Note

- The *search for a proof* can be *very frustrating in practice*
  - Even when $KB \wedge \neg Q$ has only a few clauses, *every resolution step generates a new clause* which increases the number of possible resolution steps in the next iteration

Resolution Strategies

- Unit Resolution
- Set of Support (SOS) Strategy
- Input Resolution
- Pure Literal Rule
- Subsumption

Resolution

Unit Resolution

- Prioritizes resolution steps in which one of the two clauses *consists of only one literal*, called a *unit clause*
- Complete
- Heuristic (search space reduction not guaranteed)

Set of Support (SOS) $\subset KB \wedge \neg Q$

- *Every resolution step* must *involve a clause from the SOS*
- The *resolvent* is *added to the SOS*
- Search space reduction guaranteed
- Not complete
- Complete, if $(KB \wedge \neg Q) \setminus$ SOS is satisfiable
- Often, the negated query $\neg Q$ is used as the initial SOS

Resolution

Input Resolution

- *A clause from the input set* $KB \wedge \neg Q$ must be *involved in every resolution step*
- Search space reduction guaranteed
- Not complete

Pure Literal Rule

- All clauses that contain *literals for which there are no complementary literals in other clauses* are *deleted*
- Search space reduction guaranteed
- Complete
- Is used by practically all resolution provers

Resolution

Subsumption

- If *the literals of a clause K_1 represent a subset of the literals of the clause K_2*, then *K_2 can be deleted*
  - For example, the clause $(raining(today) \Rightarrow street\_wet(today))$ is redundant if $street\_wet(today)$ is already valid
- Search space reduction guaranteed
- Complete
- Is used by practically all resolution provers

Resolution

Exercise 8 ([Ertel 2025], Exercise 3.7, p. 66)

(a) Why is resolution with the set of support strategy incomplete?

(b) Justify (without proving) why the set of support strategy becomes complete if $(KB \wedge \neg Q) \setminus \text{SOS}$ is satisfiable

(c) Why is resolution with the pure literal rule complete?

Resolution

Equality is an especially inconvenient cause of *explosive growth of the search space*

Example 23

- Knowledge Base KB includes
  - Equality Axioms
    - $\forall x\ x = x$
    - $\forall x \forall y\ x = y \Rightarrow y = x$
    - $\forall x\ \forall y \forall z\ x = y \wedge y = z \Rightarrow x = z$
  - Substitution Axioms
    - $\forall x \forall y\ x = y \Rightarrow f(x) = f(y)$, for every function symbol f
    - $\forall x \forall y\ x = y \Rightarrow p(x) \Leftrightarrow p(y)$, for every predicate symbol p
- The symmetry clause $\neg x = y \vee x = y$ *can be unified with every positive or negated equation*. This leads to the *derivation of new clauses and equations upon which equality axioms can again be applied, and so on*. (Most of these new clauses are not helpful to a proof)

Resolution

- *Special inference rules for equality have been developed* which get by without explicit equality axioms and, in particular, reduce the search space
  - **Demodulation:** Take a clause $t_1 = t_2$ and some clause α that contains the term t_1, and yields a new clause formed by substituting t_2 for t_1 within α. Note that *demodulation is directional*; *given $t_1 = t_2$, the t_1 always gets replaced with t_2, never vice versa*

$$\frac{t_1 = t_2, (\dots t \dots), \quad \sigma(t_1) = \sigma(t)}{(\dots \sigma(t_2) \dots)}$$

    - Premise 1: $t_1 = t_2$
    - Premise 2: A formula $\alpha = (\dots t \dots)$
    - $\sigma(t_1) = \sigma(t)$ means that the terms t_1 and t are matched after σ is applied
    - The substitution σ is applied for α, t_1, and t_2
    - Now, α becomes $(\dots \sigma(t_1) \dots)$
    - Replace $\sigma(t_1)$ by $\sigma(t_2)$ everywhere in α, we get the Conclusion $(\dots \sigma(t_2) \dots)$
  - **Paramodulation:** More general rule, works with conditional equations

Resolution

Example 24 (Demodulation)

- KB contains
  - *father*(*father*(x)) = *grandfather*(x)
  - *birthdate*(*father*(*father*(John)), 1945)
- By demodulation, we get
  - *birthdate*(*grandfather*(John), 1945)

Automated Theorem Provers

Theorem Provers: Implementations of proof calculi on computers.

- Otter, 1984
  - One of the oldest resolution provers (with equality handling).
  - L. Wos, W. McCune: Argonne National Laboratory, Chicago.
  - Otter was successfully applied in specialized areas of mathematics: “Currently, the main application of Otter is research in abstract algebra and formal logic. Otter and its predecessors have been used to answer many open questions in the areas of finite semigroups, ternary Boolean algebra, logic calculi, combinatory logic, group theory, lattice theory, and algebraic geometry.”
- SETHEO, 1987
  - PROLOG technology.
  - Warren Abstract Machine.
  - W. Bibel, J. Schumann, R. Letz: Munich Technical University.
  - PARTHEO: implementation on parallel computers.

Automated Theorem Provers

- E, 2000
  - Modern equation prover.
  - S. Schulz: Munich Technical University.
  - Homepage of E: “E is a purely equational theorem prover for clausal logic. That means it is a program that you can stuff a mathematical specification (in clausal logic with equality) and a hypothesis into, and which will then run forever, using up all of your machines resources. Very occasionally it will find a proof for the hypothesis and tell you so ;-).”
- Vampire
  - Resolution with equality handling.
  - A. Voronkov: University of Manchester, England.
  - Winner of CADE (CADE = Conference on Automated Deduction) ATP (Automated Theorem Prover) System Competition from 2001 to 2006.
- Isabelle, 2002
  - Interactive prover for higher-order predicate logic
  - T. Nipkov, L. Paulson, M. Wenzel: Univ. Cambridge, Munic Techn. Univ.

Mathematical Examples

We now wish to demonstrate the application of an automated prover with the aforementioned prover E. E is a specialized equality prover which greatly shrinks the search space through an optimized treatment of equality.

Definition

A structure $(M, \cdot)$ consisting of a set M with a two-place inner operation "$\cdot$" is called a semigroup if the law of associativity

$$\forall x \, \forall y \, \forall z \, (x \cdot y) \cdot z = x \cdot (y \cdot z)$$

holds. An element $e \in M$ is called *left-neutral* (*right-neutral*) if $\forall x \, e \cdot x = x$ ($\forall x \, x \cdot e = x$).

Theorem 8

If a semigroup has a left-neutral element e_l and a right-neutral element e_r, then $e_1 = e_r$.

Mathematical Examples

First, we prove by intuitive mathematical reasoning.

Proof of Theorem 8.

For every $x \in M$, it holds that

$$e_l \cdot x = x \tag{1}$$

$$x \cdot e_r = x \tag{2}$$

Replacing $x = e_r$ in Eq. (1) and $x = e_l$ in Eq. (2), we have

$$e_l \cdot e_r = e_r \tag{3}$$

$$e_l \cdot e_r = e_l \tag{4}$$

Combining Eq. (3) and Eq. (4), we have $e_l = e_l \cdot e_r = e_r$. □

Mathematical Examples

Next, we carry out the resolution proof manually. The function $m(x, y) = x \cdot y$.

- Negated query $(\neg e_l = e_r)_1$
- Knowledge Base KB
  - Definitions of semi-groups and left-/right- neutrals.

$$(m(m(x, y), z) = m(x, m(y, z)))_2 \quad \text{associativity}$$
$$(m(e_l, x) = x)_3 \quad \text{left-neutral}$$
$$(m(x, e_r) = x)_4 \quad \text{right-neutral}$$

  - Equality axioms (for comparing terms)

$$(x = x)_5 \quad \text{reflexive}$$
$$(\neg x = y \vee y = x)_6 \quad \text{symmetry}$$
$$(\neg x = y \vee \neg y = z \vee x = z)_7 \quad \text{transitivity}$$
$$(\neg x = y \vee m(x, z) = m(y, z))_8 \quad \text{substitution for } m$$
$$(\neg x = y \vee m(z, x) = m(z, y))_9 \quad \text{substitution for } m$$

Mathematical Examples

A resoution proof may be as follows.

Step	Proved by
10. $x = m(e_l, x)$	Res$(3, 6, x_6/m(e_l, x_3), y_6/x_3)$
11. $\neg m(e_l, x) = z \vee x = z$	Res$(7, 10, x_7/x_{10}, y_7/m(e_l, x_{10}))$
12. $e_r = e_l$	Res$(4, 11, x_4/e_l, x_{11}/e_r, z_{11}/e_l)$
13. $()$	Res$(1, 12, \emptyset)$

For example, Res$(3, 6, x_6/m(e_l, x_3), y_6/x_3)$ means that in the resolution of clause 3 with clause 6, the x in clause 6 is replaced by $m(e_l, x)$ where the variable x is from clause 3 and y from clause 6 is replaced by x from clause 3.

Mathematical Examples

Now we want to apply the prover E (`https://wwwlehre.dhbw-stuttgart.de/~sschulz/E/E.html`) to the problem.
Transformation in clause normal form language LOP (The syntax of LOP represents an extension of the PROLOG syntax for non Horn clauses.)

$$(\neg A_1 \vee \cdots \vee \neg A_m \vee B_1 \vee \cdots \vee B_n) \mapsto \mathtt{B_1;\ldots;B_n<\text{-}A_1,\ldots,A_m}$$

An input file for E

`halbgr1.lop`

```
1  <- eq(el,er). % query
2  eq( m(m(X,Y),Z), m(X,m(Y,Z)) ). % associativity of m
3  eq( m(el,X), X ). % left-neutral element of m
4  eq( m(X,er), X ). % right-neutral element of m
5  eq(X,X). % equality: reflexivity
6  eq(Y,X) <- eq(X,Y). % equality: symmetry
7  eq(X,Z) <- eq(X,Y), eq(Y,Z). % equality: transitivity
8  eq( m(X,Z), m(Y,Z) ) <- eq(X,Y). % equality: substitution in m
9  eq( m(Z,X), m(Z,Y) ) <- eq(X,Y). % equality: substitution in m
```

Mathematical Examples

Calling the prover with `eprover --proof-object halbgr1.lop | epclextract`. The output:

```
0 :  : (eq(X1,X2)|~(eq(X2,X1))) : initial("halbgr1.lop", at_line_6_column
1 :  : (eq(X1,X2)|(~(eq(X1,X3))|~(eq(X3,X2)))) : initial("halbgr1.lop", a
2 :  : eq(m(el,X1),X1) : initial("halbgr1.lop", at_line_3_column_1)
3 :  : ~(eq(el,er)) : initial("halbgr1.lop", at_line_1_column_1)
4 :  : eq(m(X1,er),X1) : initial("halbgr1.lop", at_line_4_column_1)
5 :  : (eq(X1,X2)|~(eq(X2,X1))) : fof_simplification(0)
6 :  : (eq(X1,X2)|(~(eq(X1,X3))|~(eq(X3,X2)))) : fof_simplification(1)
7 :  : [++eq(X1,X2),--eq(X2,X1)] : 5
8 :  : [++eq(m(el,X1),X1)] : 2
9 :  : ~(eq(el,er)) : fof_simplification(3)
10 :  : [++eq(X1,X2),--eq(X1,X3),--eq(X3,X2)] : 6
11 :  : [++eq(X1,m(el,X1))] : pm(7,8)
12 :  : [--eq(el,er)] : 9
13 :  : [++eq(X1,X2),--eq(m(el,X1),X2)] : pm(10,11)
14 :  : [++eq(m(X1,er),X1)] : 4
15 :  : [--eq(er,el)] : pm(12,7)
16 :  : [] : sr(pm(13,14),15) : 'proof'
```

Mathematical Examples

- Lines 0–4: marked with `initial`, the clauses from the input data are listed again.
- Positive literals are identified by ++ and negative literals by `--`.
- `pm(a, b)` stands for a resolution step between clause `a` and clause `b`. (`pm` means Paramodulation.)
- The proof found by E is somewhat similar to the manually created proof.

Mathematical Examples

Because we explicitly model the equality by the predicate `eq`, the particular strengths of E do not come into play. Now we omit the equality axioms and obtain

halbgr2.lop

```
1  <- el = er. % query
2  m(m(X,Y),Z) = m(X,m(Y,Z)) . % associativity of m
3  m(el,X) = X . % left-neutral element of m
4  m(X,er) = X . % right-neutral element of m
```

as the input.
The proof also becomes more compact.

```
0 :  : ~(equal(el,er)) : initial("halbgr2.lop", at_line_1_column_1)
1 :  : equal(m(X1,er),X1) : initial("halbgr2.lop", at_line_4_column_1)
2 :  : equal(m(el,X1),X1) : initial("halbgr2.lop", at_line_3_column_1)
3 :  : ~(equal(el,er)) : fof_simplification(0)
4 :  : [++equal(m(X1,er), X1)] : 1
5 :  : [++equal(m(el,X1), X1)] : 2
6 :  : [--equal(el, er)] : 3
7 :  : [++equal(er, el)] : pm(4,5)
8 :  : [] : cn(rw(6,7)) : 'proof'
```

From the above output, the proof consists essentially of a single inference step on the two relevant clauses 4 and 5.

Mathematical Examples

Exercise 9 ([Ertel 2025], Exercise 3.8, p. 66)

Formalize and prove with resolution that in a semigroup with at least two different elements a, b, a left-neutral element e (i.e., for every element x, we have $e \cdot x = e$), and a left null element n (i.e., for every element x, we have $n \cdot x = n$), these two elements have to be different, that is, that $n \neq e$. Use demodulation.

Exercise 10 ([Ertel 2025], Exercise 3.9, p. 66)

Obtain the theorem prover E (`https://wwwlehre.dhbw-stuttgart.de/~sschulz/E/E.html`) or another prover and prove the following statements. Compare these proofs with those in the text.

(a) The claim from Example 7 in Lecture “Propositional Logic”.

(b) Russell’s paradox from Example 22.

(c) The claim from Exercise 9.

Applications

Some applications of Automated Theorem Provers.

- Four color theorem was first proved in 1976 with the help of a special prover.
- Inference in expert systems.
  - Little use nowadays due to the problems of predicate logic in modeling uncertainty
- Automated program verification.
  - For example, in sofware engineering, a proof of certain security characteristics of a program is required.
  - Such a proof cannot be brought about through testing of a finished program, for in general it is impossible to apply a program to all possible inputs. This is therefore an ideal domain for general or even specialized inference systems.
- Software reuse.

Applications

Example 25 (Software reuse)

- **Specification of the query**

  PRE_Q: the preconditions, which must hold before the desired program is applied.

  POST_Q: the postconditions, which must hold after the desired program is applied.

- **Task:** Search a software database for modules, which fulfil these requirements. For each module M, the software database contains a description of the preconditions PRE_M and the postconditions POST_M.
  - Thus it must hold: $\text{PRE}_Q \Rightarrow \text{PRE}_M$.
    - If, for example, a module in the database only accepts lists of integers, then lists of integers as input must also appear as preconditions in the query.
    - An additional requirement in the query that, for example, only even numbers appear, doesn't cause a problem.
  - It must also hold: $\text{POST}_M \Rightarrow \text{POST}_Q$.
    - After the application of the module, all properties required by the query must be fulfilled.

Summary

- The proof of mathematical theorems can be automated.
- Automated provers can be used for verification tasks.
- PL1 is not suitable for reasoning in everyday life.

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