

---

# OPTIMAL LIQUIDATION STRATEGY IN A MIXED AUCTION MARKET

---

A PREPRINT

**Hoang Hai Tran**

Department of Statistics & Applied Probability  
National University of Singapore  
Lower Kent Ridge Road 10  
119076 Singapore  
e0045275@nus.edu.sg

**Ying Chen**

Department of Mathematics and Risk Management Institute  
National University of Singapore  
Lower Kent Ridge Road 10  
119076 Singapore  
matcheny@nus.edu.sg

July 19, 2021

## ABSTRACT

We analyze the price impact of trading in a mixed auction market, in which the exchange employs both call auction and continuous auction. Based on a multiplayer game settings between a market maker and one or more liquidity takers, we derive a market impact model which characterizes both instantaneous and temporary price impact. The unified model shows how price discovery and market impact transit from the call auction to the continuous auction. Moreover, we provide an optimal trading strategy with closed-form solution to execute a large order utilizing both auctions. The optimal trading strategy is numerically demonstrated with the cost efficiency of trading in both auctions compared to alternative only in the continuous trading auction.

## 1 Introduction

In modern financial markets, double auction is a standard process of buying and selling shares adopted by exchanges. From an economist's perspective, an auction is a price discovery process where a competitive equilibrium price is found through interactions among bidders. The question of how the price reaches its equilibrium points, and how different players use their own information advantages to achieve a preferable price, has attracted attentions from various fields. [12] studies how market makers, insider traders and noise traders interact to form an equilibrium price in a continuous trading auction. [1] discusses how execution can be done optimally in a continuous trading phase under certain market impact assumptions. [2] derives an equilibrium price from the perspectives of a market maker to provide quotes in a continuous trading auction. This paper is in a similar vein, in which we discuss how price will reach its equilibrium price in a mixed auction market. By taking into account call auction, under different market impact assumptions, we derive the corresponding optimal liquidation strategy for a trader to unwind his position. We provide theoretical evidence that it is indeed cheaper to trade the call auction compared to the continuous auction. Specifically, we examine the market impact of trading in a market that operates under two auction models:

- **Call auction** Buyers set a maximum price that they want to buy, and sellers set a minimum price that they want to sell. The orders will then be batched together and matched at a price set by the exchange.
- **Continuous auction** Orders are matched continuously based on time and price priority. Participants in the auction can choose to either queue their orders to wait for an opposing trade, or cross the book to match against other's orders.

We seek to reduce market impact and cost of trading by utilizing both auctions in our liquidation algorithm. The aim is to significantly reduce the cost of trading by exploiting liquidity from both, which can be achieved from answering the following questions:

- What are the relevant factors that affect market impact under the mixed auction scenario?

- How does the transition of market impact from a call auction into a continuous auction evolve?
- How can we optimize for a liquidation trading strategy, taken into account those factors?

In general, the challenge lies in the different mechanisms of the auctions, and the modelling of market impact during both thereof. By modelling the interaction between different participants in the market, namely between market makers and liquidity takers, we derive the mechanism of market impact under both auctions. From the mechanism, we optimize for a liquidation problem that results in on average 50% reduction empirically in transaction cost against the classical TWAP solution during continuous trading phase only.

It is not a coincidence that by incorporating call auction into the optimal liquidation problem, we can reduce cost significantly. Historically, the adoption of the call auctions has resulted in a substantial shift of daily trading activity, with the call auctions responsible for a significant proportion of the average daily trading volumes. ([14]) found that the call auction in New York Stock Exchange (NYSE) accounts for roughly 10 percent of the average trade volume. ([5]) found similar phenomenon on the Paris Bourse, in which the opening auction accounts for about 10 percent of average daily trade value. ([6]) has a similar conclusion regarding the Australian Stock Exchange (ASX).

A call auction can lower execution costs for participants and improve price discovery process, as suggested in ([18]). The reduction in trading costs, such as market impact and adverse selection costs, is achieved through several channels, including ([6]):

- Temporal consolidation of order flow and execution of orders at a single price ([7]).
- Facilitation of price discovery process during times of market stress ([13]).

Various studies have found that there are significant strategic trading activities during the call auctions. ([5]) suggests that the fluctuation in the indicative price during the call auction period maybe due to strategic trading. ([20]) argues that large market makers may have incentives to trade during the opening call auction, to mitigate information leakage.

On the other hand, literature on optimal liquidation and market impact has rarely mentioned execution during call auctions, not to mention the mixed auction environment. These works are normally derived for continuous auction. One of the first study on optimal liquidation is ([9]), in which an optimal trading curve is derived under certain assumptions on stochastic trade arrival and price process. ([1]) extends the model further by incorporating a trader's risk aversion factor when pricing for an optimal liquidation strategy. ([16]) solves the problem under a different assumption: the impact of each trade recovers slowly instead of immediately. In terms of trading during a call auction, ([14]) derives a optimal liquidation strategy for a specialist to set an equilibrium price under the assumptions of multiple traders with private information.

Works on optimal execution today focuses more on minimizing the impact of trading during the continuous trading phase, which ignores the fact that the opening auction is usually very liquid and the matched volume represents a significant portion of daily trading volume. For executing a big block order with significant volume, including call auctions are usually more preferable since the execution cost is lower, as shown in our analysis later. Specifically, based on our theoretical framework, we found that:

- The call auction provides another venue to take liquidity from.
- Due to inventory management from the market maker, the transition period from the call auction will be characterized with a wide initial spread that will narrow over time. A mixed auction strategy takes into account this effect will reduce transition cost even further.
- There is an optimal liquidation strategy to execute trade during both the call auction and the continuous auction that results in better cost optimization than trading in the continuous auction only.

This paper is organized as follows: Section (2) will present the theoretical framework that we will use to characterize the call auction and optimize our trading strategy. Section (3) will perform empirical analysis based on the theoretical framework to examine their effects on the optimal liquidation strategy and its cost efficiency. Section (4) will conclude the paper with suggestions for further research.

## 2 The Analytical Framework

In order to derive an optimal liquidation strategy in a mixed auction market, we must first specify the auction mechanism for each session and how trading will impact the fair price of the market. We consider a market with two trading sessions in succession: a call auction session and a continuous trading session. The mechanism of each session, number of participants in each session and how their interaction affects the price of the underlying risky asset will be described, following by the market impact formula and the optimal liquidation strategy.

Total Buy	Buy	Price	Sell	Total Sell
		56	4000	16100
		55	10000	12100
3000	3000	54	2000	2100
3100	100	53	100	100
7100	4000	52		
10100	3000	53		

Table 1: A sample crossed orderbook during call auction. The orders will be matched at the equilibrium price of 54, which results in 2100 shares traded.

Buy	Price	Sell	Trade
	56	4000	
	55	10000	
900	54		2100
100	53		
4000	52		
3000	53		

Table 2: A sample orderbook at the end of the call auction.

## 2.1 Call auction mechanism and equilibrium price

In this section we will examine how different participants contribute to the price discovery process during the call auction. We assume the call auction is fully automated, and the matching mechanism is modelled after a Walrasian auction. Specifically, the equilibrium price is determined to maximize the volume traded in the call auction, and therefore has to satisfy the following requirements:

- All buy/sell orders at prices higher/lower than the equilibrium price must be executed.
- At the execution price, the entire either buy or sell quantity must be matched.

Traders can submit and amend orders anytime during the call auction. Those orders are not matched immediately, and placed into the book as "crossed", that is, the highest buy price is higher than the lowest sell price in the book (see Table (1) for an example book during the call auction). At the end of the auction period, the orders in the book are matched based on an equilibrium price calculated by the exchange. Table (2) gives an example of how a book can be "un-crossed" at the end of the call auctions. All the orders that are unmatched at the end of the call auction will be transferred to the continuous trading session. There is no specialist who can step in to modify the equilibrium price.

We assume that there are two types of participants:

- Market makers are those with private information about the value of the asset. This information is not necessary from possessing some insider knowledge about the asset, but from having access to superior information regarding the order flow of the book, such as real-time trades and quotes provided by the exchange and better data processing capability.
- Liquidity traders are those without private information and simply want to execute their orders. We assume the demands of liquidity traders are exogenous, and they trade using market orders.

Unlike ([14]), we do not assume that there is a specialist setting the equilibrium price, as most exchanges around the world has adopted automated call auction mechanism at the time of this writing<sup>1</sup>. We also do not assume that there are informed traders with information advantage, but a market maker who holds private information and make the market instead. There are several reasons for this setup:

- The model setup assumes that, in order to fully utilize the information advantage, the participant who holds it must have the ability to reprice his orders immediately after he receives a new signal. As we assume the private information is mainly from observing the order flow of the book, it means the participant is a high frequency trader (HFT), who can act immediately after each change in market order flow.

<sup>1</sup>This mechanism is adopted for the opening call auctions in Singapore Stock Exchange (SGX), Tokyo Stock Exchange (TSE) and Nasdaq Stock Market (NASDAQ). New York Stock Exchange (NYSE) is different, in which a specialist sets the equilibrium price that is not necessarily the same as the volume-maximizing Walrasian price.

- There are evidences that a significant portion of high frequency trading during is for liquidity provision. ([15]) studied the trading activity in Chi-X Europe, and found that a majority of high frequency trading activities is market making. Similarly, [3] found that there is a correlation between low-latency trading and the subsequent liquidity provision in the call auction in Tokyo Stock Exchange (TSE).

We further assume that there is a single market maker who makes the market for  $K$  liquidity traders. This can be either a designated market maker obligated to provide liquidity for the market, or an external agent who wants to capture the liquidity premium of the call auction. The market maker is then assumed to have a negative exponential expected utility function of the form

$$u(W_e) = -e^{-\lambda W_e}$$

where  $W$  is the terminal wealth of the trader at the end of the call auction. The terminal wealth of the market maker is

$$W_e = p_0(q + e_e) + (c_e - p_e q)$$

The notions are

- $\lambda > 0$  is the risk aversion factor of the market maker.
- $p_0$  is the stock's fair price at the end of the call auction.
- $p_e$  is the stock's equilibrium price at the end of the call auction.
- $c_e$  is the initial cash position of the market maker.
- $e_e$  is the overnight position of the market maker.
- $q$  is number of shares purchased by the market maker at the end of the call auction.

To model the asset price at the end of the call auction, we assume that the asset value  $p_0$  follows normal distribution with mean  $\mu_e$  and precision (the inverse of the variance)  $\zeta_e$ <sup>2</sup>. The private information of market maker  $\omega_e$  is then the normal variable

$$\omega_e \sim N(p_0, \frac{1}{\psi_e})$$

The posterior distribution of the asset according to market maker's information set  $\Omega_e$  is then normally distributed with mean

$$p_0 = E[p_0|\Omega_e] = \mu_e \alpha_e + \omega_e (1 - \alpha_e)$$

where

$$\alpha_e = \frac{\zeta_e}{\zeta_e + \psi_e}$$

and variance

$$\sigma_e^2 = \text{var}[p_0|\Omega_e] = \frac{1}{\zeta_e + \psi_e}$$

Maximizing expected utility of the market maker, we have the amount of shares submitted at price  $p$

$$q(p) = a_e - b_e p \tag{1}$$

where  $a_e = \frac{p_0}{\lambda \sigma_e^2} - e_e$  and  $b_e = \frac{1}{\lambda \sigma_e^2}$ . We assume the  $K$  liquidity traders will submit a total of  $\sum_{i=1}^K x_i$  shares into the call auction to be matched, with negative values implying selling and vice versa. The total shares transacted at a given price  $p$  is then

$$Q(p) = (p_0 - p)b_e + \sum_{i=1}^K x_i$$

At equilibrium, we set  $Q(p) = 0$ . This gives the Walrasian price as the matching price at the end of the call auction

$$p_e = p_0 + \frac{\sum_{i=1}^K x_i}{b_e} \tag{2}$$

The matching price is the fair price of the stock with a mark-up in proportion to the supply-demand imbalance in the market. The pricing function is plotted in Figure (1), which we will use to price our market impact of trading with the call auction in subsection (2.3) after we examine the continuous trading phase in the next section.

<sup>2</sup>While this means the stock may have negative price as time goes by, we can ignore this risk as we only need to account for one day of variance. Using an arithmetic stock price process instead of the classical logarithmic price process simplifies other calculation later in the study.

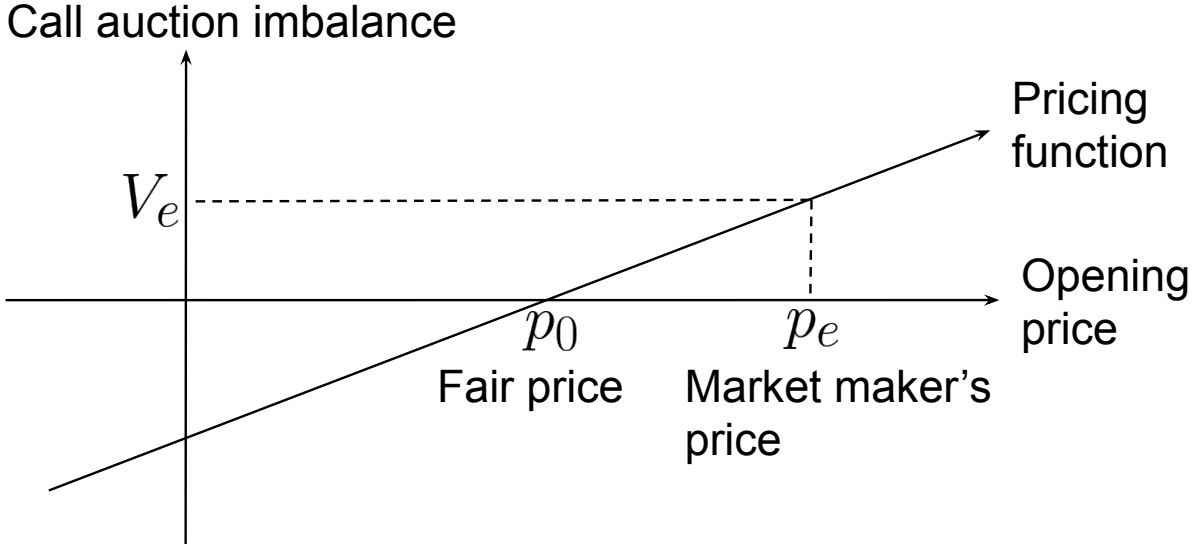


Figure 1: Pricing function of the market maker w.r.t call auction imbalance. If the imbalance is negative, implying that the sell demand is higher than buy demand, the market maker will price the opening price to be lower than his fair price, and vice versa. The slope of the pricing function is  $\frac{1}{b_e}$  in Equation (2).  $V_e$  is the imbalance at the end of the call auction, while  $p_e$  is the opening price.

## 2.2 Continuous auction mechanism and agents' behaviours

During the continuous trading session, orders can be posted into the market at anytime during the session to be matched. If a buy order has a price that is higher than the current lowest offer in the book, it will be matched immediately, and vice versa for a sell order. Otherwise, it will be added into the book.

Similar to the call auction, we model the continuous auction as a multi-round two-stage game. At the beginning of each round, we pick randomly a liquidity trader to act as the counterparty of the market maker. In the first stage of the round, the market maker places limit orders in the book according to his risk tolerance and private information. In the second stage, the liquidity trader will place a market order based on his exogenous demand. This setup is slightly different from the call auction in which:

- Unlike the call auction, only the market maker can submit limit orders during the first stage of the game. He has no knowledge of the liquidity trader's order, and only submit orders based on his own evaluation of the asset's value.
- The liquidity trader can only submit market orders during the final stage of the game, after the market maker has finished layering his orders in the book.

Under this mechanism, the market maker has a speed advantage with respect to the rest of the market, since he can evaluate the current fair pricing of the asset after each trade. After the reevaluation, he can modify his orders in the market to reflect the updated belief before any liquidity traders can place a new order. This grants the market maker distinct competitive advantage, in the sense that he can reprice his orders in the book before getting hit by any adverse selection event. This gives incentive to market makers to perform high frequency trading.

Similar to the call auction, we assume the market maker has private information about the asset from his superior analytical capability. Again, let assume the immediate asset value at the end of the continuous auction's round is a random variable  $v_c$  normally distributed with mean  $\mu_c$  and precision  $\zeta_c$ . The market maker deduces his own evaluation with a private signal

$$\omega_c \sim N\left(v_c, \frac{1}{\zeta_c}\right)$$

and forms a posterior belief on the asset value at the end of the round as a normal random variable with mean

$$v_c = \mu_c \alpha_c + \omega_c (1 - \alpha_c)$$

where

$$\alpha_c = \frac{\zeta_c}{\zeta_c + \psi_c}$$

and variance

$$\sigma_c^2 = \frac{1}{\zeta_c + \psi_c}$$

At the beginning of each round, the market maker has a position  $e_c$  that he wants to liquidate. Maximizing the market maker's utility function, we have his order quantity function in the book:

$$q_c(p) = a_c - b_c p_c \quad (3)$$

where  $a_c = \frac{v_c}{\lambda \sigma_c^2} - e_c$  and  $b_c = \frac{1}{\lambda \sigma_c^2}$ . Let assume the liquidity trader needs to transact  $y_0$  shares. The average price he gets after submitting his market order is

$$p_{c0} = \frac{a_c - y_0}{b_c} = p_0 - \frac{1}{b_c} y_0$$

where

$$p_0 = \frac{a_c}{b_c}$$

is the best price quoted by the market maker. Note that the position is from the perspective of the market maker, that is, the higher number of shares the liquidity trader wants to buy, the higher markup price he will have to pay to transact immediately. In other words, the markup price that the liquidity trader receives is

$$p_{c0} = p_0 + \frac{1}{b_c} y_0 \quad (4)$$

Comparing Equation (2) and Equation (4), we can see that, ceteris paribus, the instantaneous market impact of trading during the auction phase has a higher chance of getting offset by other liquidity traders in the market. On the other hand, during the continuous phase, the liquidity trader will always receive an adverse markup price implied in market maker's quotes. This is clearer once we examine the transition period immediate after market opening in the next subsection.

### 2.3 Price discovery at market opening

There is empirical evidence that the transition period immediate after market opening is crucial in determining the effectiveness of the opening call auction ([17]). In this subsection, we analyze how the market maker, and consequently the spread, may behave during this period, and the subsequent effects on the liquidity trader. Recalling from Equation (2), the opening price is marked up proportional to the buy-sell imbalance at the end of call auction. Let

$$V_e = \sum_{i=1}^K x_i$$

be the imbalance at the end of call auction. Based on the imbalance, the market maker will provide the liquidity necessary for the market to open at equilibrium price

$$p_e = p_0 + \frac{V_e}{b_e}$$

Without loss of generality, let  $V_e$  be positive, implying that the market has more buy interests than sell interests. During the call auction, the market maker will sell  $V_e$  shares at price  $p_e$  higher than his estimated fair value of  $p_0$ , and has an unrealized gain of

$$G_e = (p_e - p_0) * V_e$$

To capitalize this gain after market opening, the market maker will need to make a spread. Continuing from the call auction's we have the pricing function of the market maker during this period:

$$q(p) = a_e - b_e p$$

where  $a_e = \frac{p_0}{\lambda \sigma_e^2} - V_e$  and  $b_e = \frac{1}{\lambda \sigma_e^2}$ , based on his signal when pricing the call auction. As  $V_e$  is positive, this implies that the market maker is willing to sell at  $p_e$ . On the other hand, it is reasonable to assume that he will want to liquidate his position acquired from the opening call auction at  $p_0$ , his fair price of the asset at the end of the call auction.

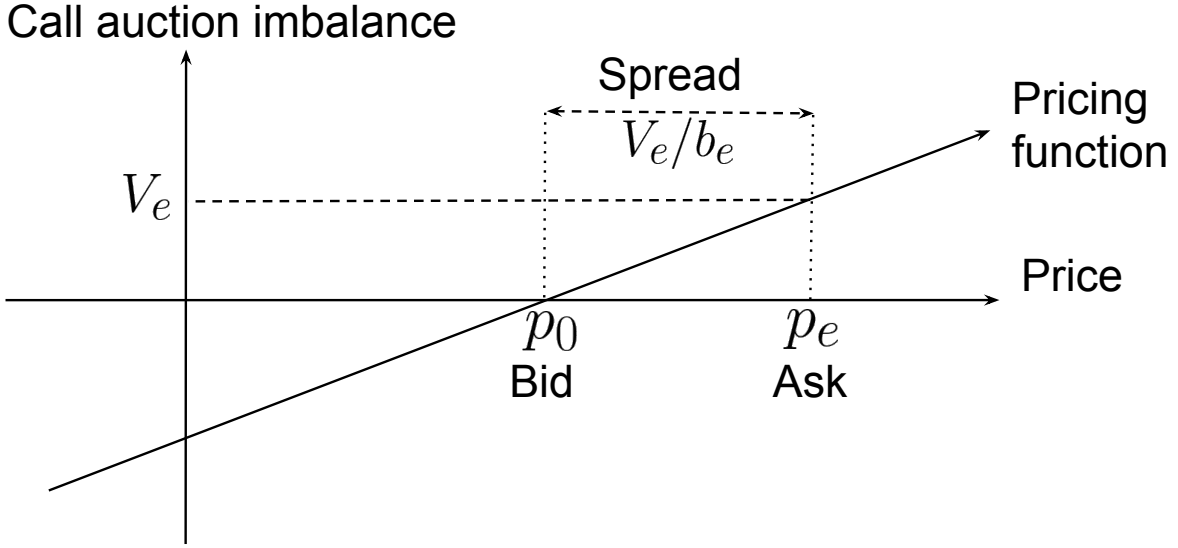


Figure 2: Market maker's quotes immediately after opening w.r.t call auction imbalance. Upon market opening, the market maker will short  $V_e$  shares at  $p_e$ . His evaluation of the asset's fair value is at  $p_0$ . However, due to his inventory, he will quote the ask side further away at  $p_e$ , and bid side at  $p_0$ . The bid-ask spread is then  $\frac{V_e}{b_e}$ , based on the market maker's pricing function (4).

Therefore, the market maker will make a spread of  $\frac{V_e}{b_e}$  immediately after call auction. The quotes of the market maker are illustrated in Figure (2).

Interestingly, the bid-ask spread after market opens reveals the market maker's pricing in the call auction, as we can calculate the implicit factor  $b_e$  as

$$b_e = \frac{V_e}{\zeta_e} \quad (5)$$

where

- $V_e$  is the matching volume at the open of continuous trading session.
- $\zeta_e$  is the spread at the open of continuous trading session.

From the similarity of the pricing function during the call auction in Equation (2) and the continuous auction in Equation (4), we can see that, ceteris paribus, it is advantageous to trade in both sessions to consume liquidity from both to lower his market impact. This will be clearer once we derive the optimal trading strategy in the next section. Moreover, as the call auction compresses the liquidity over a long period of time in one single price, there is usually a much larger quantity traded than an average market order during continuous auction. Consequently, the market maker will need to unwind the position acquired from the call auction over a transition period immediately after the matching. This will affect the pricing for the liquidity trader during the continuous auction period, in addition to the normal pricing in the continuous session. Specifically, the liquidity trader will potentially face one of the following two scenarios during the call auction:

- **Scenario 1: The liquidity trader trades in the same direction as the market imbalance** As  $V_e$ , the buy-sell imbalance at the end of the call auction, is assumed to be positive, we have the liquidity trader wants to buy in this case. As a result, his block trade size of  $B_0$  will push the opening ask up, from  $p_e$  to  $p_e^+$

$$p_e^+ = p_0 + \frac{V_e + B_0}{b_e}$$

As he is buying, this is equivalent to lifting the offer during the continuous phase. In this case, ceteris paribus, there is no difference between trading during the auction and trading during the continuous phase.

- **Scenario 2: The liquidity trader trades against the market imbalance** In this case, the liquidity trader will sell to the market, similar to the market maker. Therefore, he is providing liquidity to the market, reducing the bid-ask spread immediately after market opening. Consequently, the opening price bid will improve from  $p_e$  to  $p_e^-$

$$p_e^- = p_0 + \frac{V_e - B_0}{b_e}$$

As he is selling, this is equivalent to placing his offer into the bid-ask spread during the continuous phase and getting filled immediately.

### Call auction imbalance

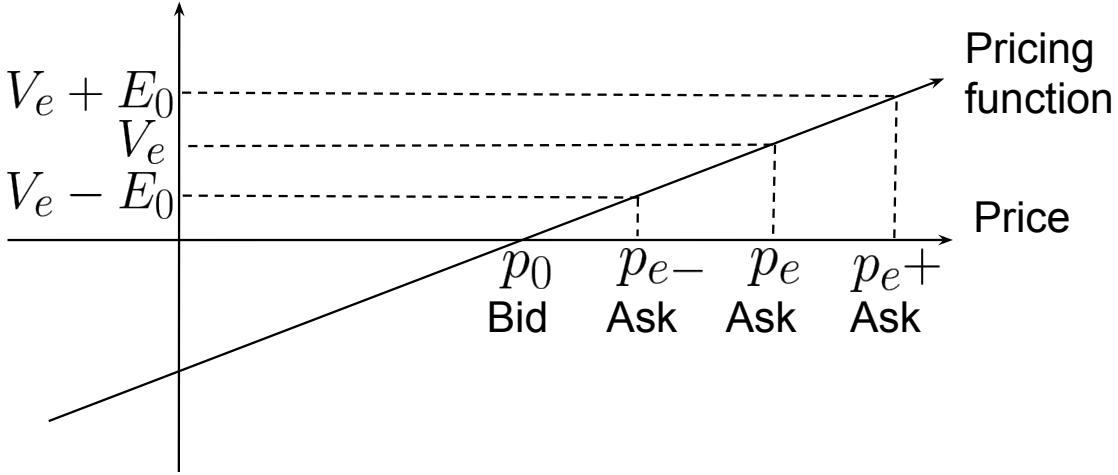


Figure 3: Bid-ask spread during the transition period

As the market maker is buying at the fair price of  $p_0$  and selling at a discounted price  $p_e$ , it is reasonable to assume that he will get filled more on the bid side than the ask side. This allows the market maker to reduce his inventory over time, and therefore tighten the bid-ask spread. Depending on which scenario the liquidity trader falls into, the market maker's behaviour may affect his cost of liquidation. Specifically, we have:

- Under Scenario 1, trading in the call auction results in an ongoing penalty cost to the liquidity trader that will decay over time, as he will need to lift the offer to buy. Let the speed of liquidation of the market maker be  $\rho$  and  $V_t$  be the remaining position of the market maker acquired from the call auction in Scenario 1. Consequently, we have

$$dV_t = -\rho V_t$$

$$V_t[0] = (V_e + B_0)$$

$$\Leftrightarrow V_t = (V_e + B_0)e^{-\rho t} \quad (6)$$

where  $B_0$  is the liquidity trader's block trade size during auction. The resulting spread due to our trading during the call auction at time  $t$  will be

$$\zeta_t = \frac{e^{-\rho t}}{b_e}(V_e + B_0)$$

Therefore, we would expect the spread to be wide initially, and slowly narrowing due to the inventory management effects from the market maker. This is in agreement with current empirical evidence that the bid-ask spread narrows shortly after market opens.

- Under Scenario 2, trading in the call auction has no "spillover" effect on the liquidity trader, as the liquidity trader will keep hitting the bid to sell while the market maker will try to improve the offer.



This "spillover" cost only incurs half of the time, under Scenario 1, while under Scenario 2 there is no additional cost for the liquidity trader during the continuous phase. Assuming both scenarios are equally likely, the expected markup cost to the liquidity trader during this transition period is:

$$\zeta_t = \frac{e^{-\rho t}}{2b_e}(V_e + B_0) \quad (7)$$

which we use in the next section to derive the optimal liquidation strategy for the liquidity trader.

## 2.4 Optimal liquidation strategy in a mixed auction market

Based on the behaviors of market makers in the previous subsections, we can derive the market impact of trading for the liquidity trader across the sessions. Let the fundamental price process  $S_t$  at time  $t$  be

$$S_t = S_0 + \int_0^t \sigma dZ_s$$

that is,  $S_t$  follows a classical arithmetic random walk without any drift. The actual price process  $P_t$  is then the combination of the fundamental price process  $S_t$  and a short term deviation process  $D_t$  due to the market maker repricing his book, as described in Subsection (2.1) and (2.2):

$$P_t = S_t + D_t$$

From Equation (2) and Equation (4), we can add into  $D_t$  a linear cost component proportional to the trading sizes under each session:

$$D_t = \alpha B_0 + E_t \beta \quad (8)$$

where

- $B_0$  is the block trade size at the end of the call auction
- $E_t$  is the continuous trading rate of the liquidity trader
- $\alpha$  and  $\beta$  are the cost penalty factor for aggrsessing the book during the call auction and continuous auction, respectively. We have  $\alpha = \frac{1}{b_e}$  and  $\beta = \frac{1}{b_c}$  based on Equation (2) and Equation (4), respectively.

Equation (8) is similar to what has been described as the "temporary impact" term in the pioneering works of [4] and [1]. On the other hand, from Equation (7), we have the expected spillover cost of trading during call auction to continuous phase is:

$$R_t = \frac{\alpha(B_0 + \bar{V}_e)e^{-\rho t}}{2}$$

The short term deviation term  $D_t$  is then

$$D_t = \underbrace{\alpha B_0}_{\text{Cost of the call auction}} + \underbrace{\frac{\alpha(B_0 + \bar{V}_e)e^{-\rho t}}{2}}_{\text{Spillover cost}} + \underbrace{\beta E_t}_{\text{Cost of the continuous auction}}$$

and the execution cost is

$$C_t = \alpha B_0^2 + \int_0^t \frac{\alpha(B_0 + \bar{V}_e)e^{-\rho s}}{2} E_s ds + \int_0^t \beta E_s^2 ds \quad (9)$$

Based on the execution cost equation, we can derive an optimal strategy for the liquidity trader. Let assume the liquidity trader wants to unwind a large position, and he has the option to execute his trades in both the call auction and the continuous auction. We have the following notations regarding his trading strategy:

- $Q_t$  is the total number of shares of his portfolio at time  $t$ .
- $E_t$  is the continuous liquidation strategy during the continuous trading auction.
- $X_t = \int_0^t E_s ds$  is the number of total executed shares at time  $t$ .
- $B_0 = Q_0 - \int_0^T E_s ds$  is the size of the block trade shares executed at the end of the opening call auction.

Based on the Proof (5) in the Appendix, we have the optimal size of block trade in the call auction as:

$$B_0 = \frac{Q_0 F_{11} + V_e F_{12}}{F_2}$$

where

$$\begin{aligned} F_{11} &= 8e^{T\rho}\beta\rho[\alpha - e^{T\rho}(\alpha - 4\beta\rho)] \\ F_{12} &= (e^{T\rho} - 1)\alpha[\alpha(2 + T\rho) + e^{T\rho}(8\beta\rho + \alpha(T\rho - 2))] \\ F_2 &= \alpha^2(2 + T\rho) - 4e^{T\rho}\alpha(\alpha + 2\beta\rho) + e^{2T\rho}[8\beta\alpha\rho + 8\beta(T\alpha + \beta)\rho^2 + \alpha^2(2 - T\rho)] \end{aligned}$$

The optimal trading strategy during continuous session is:

$$\begin{aligned} E_t &= A - B_0 \frac{\alpha e^{-t\rho}}{4\beta} \\ X_t &= At - B_0 \frac{\alpha(1 - e^{-t\rho})}{4\beta\rho} \end{aligned}$$

where

$$A = \frac{4(Q_0 - B_0) + \frac{(1 - e^{-T\rho})}{\beta\rho} B_0 \alpha}{4T}$$

The optimal trading strategy during auction phase is a modification of a TWAP strategy with an exponentially decayed component to account for the spillover effect. This is expected as a TWAP is proven to be optimal for a strategy with the cost of aggressing the orderbook only in ([9]). A sample optimal allocation between the auction's block trade and the continuous trading strategy's size is in Figure (4). The corresponding optimal liquidation rate curve of the strategy is plotted in Figure (5).

## 2.5 Summary

The main findings of this section can be summarized as follows:

- The presence of informed trading from the market maker is required for the transition to continuous trading, as the market maker will provide the necessary liquidity to offset the buy-sell imbalance at the end of the call auction.
- The transition period from the call auction will be characterized with a wide initial spread narrowing over time due to inventory management from the market maker, as shown in Equation (9).
- Trading in the call auction lower the total execution cost compared to trading in continuous auction only, as the call auction provides an additional venue for liquidity consumption.
- There is an optimal liquidation strategy to execute trade during both the call auction and the continuous auction. The mixing ratio of the strategy is determined by the efficiency of each auction, in terms of immediate cost of aggressing the book and the speed of spread recovery after market opening.

Before moving to empirical analysis, it is imperative to address some stylized facts of our model. We assume that the liquidity traders are price takers who submit market orders to signify his intention to trade regardless of opening price, and the market maker will post limit orders to counter the imbalance. In most modern stock exchanges, all participants can post limit orders in the order book to be "crossed" during the opening match. Limit orders can also be modified or cancelled before the opening match, allowing participants to update their pricing. However, as long as the market maker in our model can place his trade after all the liquidity traders has submitted their orders, the pricing schedule based on the market imbalance still holds.

We also assume that there is only one market maker and only he or she trades with private information. In practice, there can be multiple agents trading with information. However, as long as the private information is derived from public data, that is, no insider trading, we can assume that all informed agent will have the same "signal" regarding the fair price of the stock, and are only different in their risk aversion factor. Therefore, we can assume that all informed agents are trading with same risk aversion factor, which are represented by the market maker in our model. The principal agent who provides liquidity in the call auction is assumed to be a market maker because, as shown in our model, he will need to make a spread in the continuous auction to capture the expected profit from his signal.

The source of private information is also a vital assumption in our model. We assume that the market maker derives his valuation from public information, which include, but not limited to, the following factors:

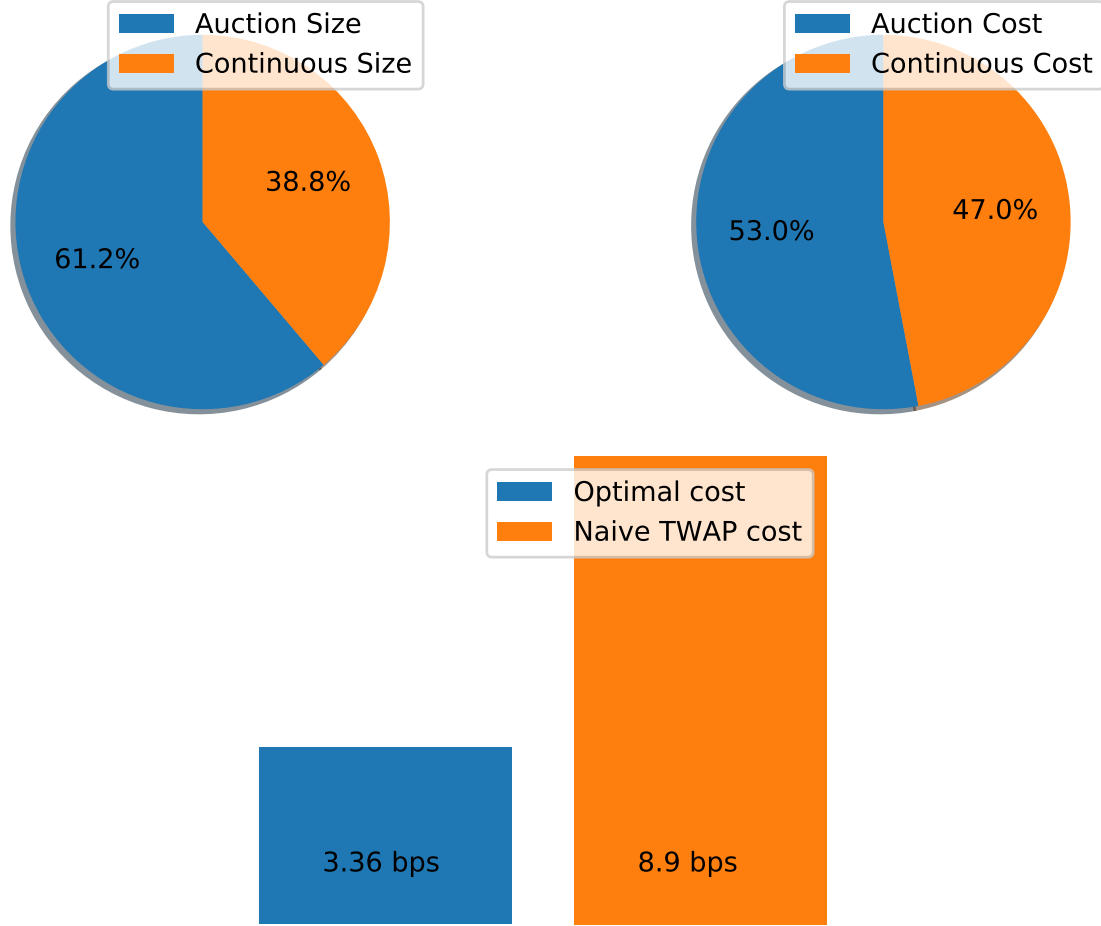


Figure 4: Sample allocation between auction's block trade and the continuous trading strategy's size, using data from stock ACGI in October 2019, trading 5 percents of average trading volume of the first half of trading day. Compared to a pure TWAP trading strategy, the optimal strategy allocates 61.19 percents of the total position to be executed during the auction phase. The cost of trading in the auction phase accounts for 53.00 percents of the total cost of the strategy, while reducing 62.26 percents of total cost compared to a pure TWAP strategy, from 8.90 bps per share to 3.36 bps per share.

- Public order flow, with or without identity.
- General market information, such as index futures movement before equity market opening.
- Price continuity from previous day's closing price. In absence of any fundamental news, it is reasonable to expect the opening price to be close to the previous day's closing price.

Another deviation is the inventory management effects of the market maker. In our model, there is market maker's inventory management effects on the spread due to the position acquired from the opening auction, but not in the continuous auction. While it would be more complete to account for the inventory management effects in both auctions, our empirical analysis suggests that the inventory management effects of the continuous auction is less prominent compared to the opening auction. Specifically, we observed that

- There is little evidence that the bid-ask spread is changed following a trade except near the opening and closing auctions.
- The auction matching volume is usually much larger than a normal trade during continuous phase, accounted for a significant fraction of the daily trading volume.

Last but not least, although it is immaterial to our cost analysis, it is not obvious that the market maker will be profitable after placing his "stabilization" trade to counter the opening auction's imbalance. Even when his initial signal is correct up to the end of the opening auction, new information may come and he has to change his valuation of the stock before

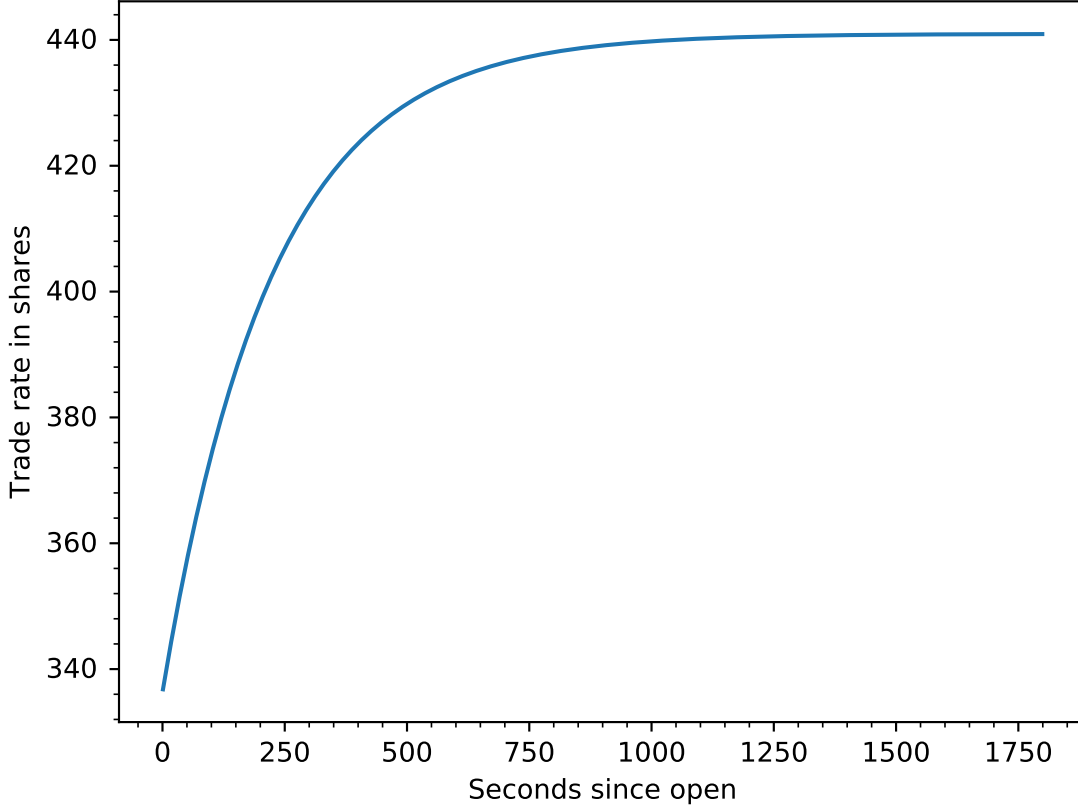


Figure 5: Sample optimal trading strategy's execution curve 30 minutes after market opens from auction phase. The shape of the optimal trading curve reflects the nature of the cost, in which the rate of execution is speed up over time as the market impact of the auction subsidizing and the spread narrowing.

realize the signal from the call auction. This is evident in the high volatility immediately after market opening. However, the continuous book is only one of many execution venue for the market maker to realize his signal. For example, the market maker may attempt to "lock-in" his profit by hedging into the index futures immediately. He then can slowly unwind both legs of the hedge over the course of the day, resulting in a tightening spread over time.

With those caveats in mind, we can proceed to the empirical analysis of NASDAQ data to estimate the liquidation cost of trading in both auctions compared to only during the continuous auction.

### 3 Application: optimal execution for a portfolio of stocks in mixed auction market

In this section, we apply our model to the case of transaction cost analysis. We consider a liquidity trader who wants to transact in both the opening auction and the subsequent continuous trading auction of the NASDAQ stock exchange. While theoretically reasonable to expect significant reduction in execution cost compared to trading in continuous auction only, the uncertainty from the estimation of market impact factors may affect the efficiency of the optimal strategy. Consequently, the trader may want to assess this uncertainty if it is too large compared to the benefits of trading in both auctions. The trader's job is therefore to perform sensitivity analysis of his portfolio against different underlying factors. Specifically, the transaction cost analysis will need to answer the following questions:

- What is the transaction cost of the portfolio for execution during the continuous auction only?
- How large the difference of the transaction cost when transact in both the call auction and the continuous auction compared to the continuous auction only?

- How accurate the estimation of the cost factors? How would this affect the accuracy of the execution cost estimation?

This section starts with the descriptive statistics of the stocks in the sample portfolio, along with the trading mechanism of the NASDAQ exchange. We then proceed to present the estimation method from public data, along with corresponding estimates and their variances. We perform sensitivity analysis with respect to three main cost factors:

- Cost of trading in the opening auction.
- Cost of trading in the continuous auction.
- The recovery rate of market impact of auction trading in continuous phase.

We conclude the section with the summary of the sensitivity analysis, and the optimal course of action for the liquidity trader given his risk appetite.

### 3.1 Data and market mechanism

We use equity data from NASDAQ exchange in this study, which has operating mechanism resembling our theoretical model. Opening auction in NASDAQ is called **Opening Cross**. While similar to the general call auction described in our model, there are certain subtle difference between our model and the NASDAQ's Opening Cross:

- Before the opening bell at 9:30 EST, NASDAQ allows investors to submit orders to two different books: a continuous book and an opening cross book. Orders submitted to the continuous book will be matched immediately and the matching information will be visible in the market data feed. Orders submitted to the opening cross book will be withheld to be matched later and are not visible in the market data feed.
- At 9:30 EST, the opening cross book and the continuous book will be merged together to form a single book, and a Opening Cross price will be calculated based on the following three rules, similar to our Walrasian matching rule:
  - Maximize the number of shares executed.
  - Minimize the imbalance of Cross orders.
  - Minimize the distance from the Nasdaq inside bid-ask midpoint.

Matched orders and total matching quantity from the Opening Cross will then be disseminated in the market data feed. Market is considered to be in the continuous trading phase at this point of time.

While the presence of the continuous book during the call auction period deviates from our assumption, it makes little difference in practice due to limited volume transacted during this period. On the other hand, the hidden nature of the Opening Cross orders mean that we cannot directly derive the cost of aggressing the book during the call auction. However, based on the arguments in subsection (2.3), we can infer the cost of trading during the call auction from the spread and matching volume immediate upon market opening.

The continuous auction in the NASDAQ exchange, on the other hand, is similar to our model.

Our data is captured directly from NASDAQ's ITCH-TotalView market data feed, provided by the National University of Singapore. The data contains all the add, modification, trade and delete for each order submitted to the continuous book, with fields indicating whether it is a buy or sell order. The data also contains the matching quantity and the Nasdaq Official Opening Price (NOOP) for each stock. Given the dataset of 3500 NASDAQ-listed stocks, we pick 200 random stocks with the following constraints:

- Since we do not account for the value of the queue priority in our market maker's modelling, we avoid selecting stocks with tick size higher than 10 basis points in our sample.
- We avoid low liquidity stock by selecting ones higher than 10M daily traded notional, which is roughly the first quartile of our sample.

Given those constraints, we use 1 week of data, from 01/10/2019 to 07/10/2019, in our empirical analysis. Since we only account for the market impact of the opening auction in our analysis, we use the data from 9:30 AM to 12:30 AM in our derivation. Due to the limited number of observations per stock, with only one opening auction a day, we group stocks by their relative volatility to improve the accuracy of the estimations. As shown in subsequent analysis, those metrics capture the variance in our cost factors. The descriptive statistics of the data is in Table (3). From the statistics, we can see that the sample represents a reasonable range of stocks in terms of traded volume and tick size.

Table 3: Summary statistics for each group of stocks based on daily volatility as percentage of average price. The diverse range of traded notional and tick size indicates that the sample includes stocks from various regimes in the population. The standard deviation is in the brackets.

Volatility (%/Px)	Count	Volume (million \$/day)	Tick (bps)
< 0.50	32	50.88 (17.24)	5.90 (2.78)
< 1.00	43	47.00 (19.73)	2.91 (1.48)
< 1.50	34	53.44 (21.25)	2.47 (1.35)
$\geq 1.50$	41	54.07 (21.65)	1.69 (0.69)

### 3.2 Estimation method

In order to estimate the cost of trading, we need to estimate three factors:

- The temporary book aggressing cost factor  $\alpha$  before market opening in Equation (1)
- The temporary book aggressing cost factor  $\beta$  during the continuous phase in Equation (3)
- The spread recovery factor  $\rho$  post opening auction in Equation (6)

In the following, we describe how those factors can be estimated from market data.

#### 3.2.1 Estimation of opening auction's cost factor $\alpha$

As mentioned, NASDAQ does not publish order imbalance before market opening, but we can derive the cost factor  $\alpha$  from the matching quantity and the opening bid-ask spread. Recall from Subsection (2.3) that the market maker will be incentivized to provide the offsetting liquidity  $V_e$  for market to open, with price sensitivity coefficient  $b_e$ . The market maker will then need to make a spread of  $\zeta_e$  to realize his expected profit from  $V_e$ . Assuming his price sensitivity coefficient  $b_e$  does not change after market opening, we can estimate  $b_e$  based on Equation (5) as

$$\hat{b}_e = \frac{\bar{V}_e}{\bar{\zeta}_e}$$

where

- $\bar{V}_e$  is the average matching volume at the open.
- $\bar{\zeta}_e$  is the average spread at the open.

The estimation of cost factor  $\alpha$  will then be

$$\hat{\alpha} = \frac{1}{\hat{b}_e} = \frac{\bar{\zeta}_e}{\bar{V}_e}$$

It should be noted that both  $V_e$  and  $\zeta_e$  are random variables, hence makes the measurement of the cost factor  $\alpha$  follow a ratio distribution. Ratio distribution often is heavy-tailed and has high variance. For example, assuming that  $V_e$  and  $\zeta_e$  follow independent normal distribution with zero means, then  $\alpha$  will follow Cauchy distribution, with both its expected value and variance undefined ([8]). In reality, we expect  $V_e$  and  $\zeta_e$  to be both correlated and noncentral, making the exact distribution of  $\alpha$  even more complicated. To mediate this, we use the log transformation of the ratio in our subsequent analysis. The rationale, according to ([11]), is as follows:

Let assume  $V_e$  and  $\zeta_e$  to be correlated noncentral normal random variables. The cost ratio  $\alpha$  then can be expressed as:

$$\alpha \sim \frac{\mu_{\zeta_e} + \mathbb{N}(0, \sigma_{\zeta_e}^2)}{\mu_{V_e} + \mathbb{N}(0, \sigma_{V_e}^2)} = \frac{\mu_{\zeta_e} + \zeta_e}{\mu_{V_e} + V_e} = \frac{\mu_{\zeta_e}}{\mu_{V_e}} \frac{1 + \frac{\zeta_e}{\mu_{\zeta_e}}}{1 + \frac{V_e}{\mu_{V_e}}}$$

Take logs to get

$$\log(\alpha) = \log\left(\frac{\mu_{\zeta_e}}{\mu_{V_e}}\right) + \log\left(1 + \frac{\zeta_e}{\mu_{\zeta_e}}\right) - \log\left(1 + \frac{V_e}{\mu_{V_e}}\right)$$

Since  $\log(1 + \delta) = \delta - \frac{\delta^2}{2} + \frac{\delta^3}{3} + \dots$  then asymptotically

$$\log(\alpha) \approx \log\left(\frac{\mu_{\zeta_e}}{\mu_{V_e}}\right) + \frac{\zeta_e}{\mu_{\zeta_e}} - \frac{V_e}{\mu_{V_e}} \sim \log\left(\frac{\mu_{\zeta_e}}{\mu_{V_e}}\right) + \mathbb{N}\left(0, \frac{\sigma_{\zeta_e}^2}{\mu_{\zeta_e}^2} + \frac{\sigma_{V_e}^2}{\mu_{V_e}^2}\right)$$

In other words, the log transformation allows us to consider the cost factor  $\alpha$  approximately a Gaussian distributed random variable.

### 3.2.2 Estimation of continuous auction's cost factor $\beta$

Equation (4) allows us to estimate cost factor  $\beta$  directly from the liquidity posted in the book, with a slight adjustment. Since Equation (4) is in terms of average price, we need to convert the price of the book to average price from the best bid-offer level, as illustrated in Table (4). Given the cumulative sum quantity as the independent variable  $X$  and the average price as dependent variable  $Y$ , we have the  $\beta$  estimate in a linear regression equation

$$Y_t = \text{const} + \beta * X_t + e_t$$

One caveat is the period of observation for the regression. Given the discussion in Subsection (2.3), we would want to avoid the transition period immediately after the opening. To identify this period, we perform two-sample t-test on consecutive 5-minute bin of data to determine if the average spread has stabilize. Specifically, given  $\zeta_t$  as the average spread of the current bin and  $\zeta_{t+1}$  as the average spread of the next 5-minute bin, the null hypothesis is:

$$H_0 : \zeta_t = \zeta_{t+1}$$

where the sample statistics of the spread mean is calculated as

$$\bar{\zeta}_t = \frac{\sum_{i=1}^N \sum_{j=1}^5 \zeta_{ij}}{N * 5}$$

where  $N$  is the number of days in our sample, and we measure the spread  $\zeta_{ij}$  every 1 minute in each of the 5-minute bin. When we find the first 5-minute bin that we cannot reject the null hypothesis at 5% confidence level, we use the data in the bin as data for the regression. The sample statistics for  $X$  and  $Y$  is estimated similarly to the spread mean, where we bootstrap the average of 1-minute interval spread and cumulative sum quantity of the bin as sample data point. Furthermore, we use the data in the identified transition period to measure the recovery speed of the spread in the next subsection.

Buy	Price	Sell	Sum	Average Price
	56	4000	14000	55.286
	55	10000	10000	55
900	54		900	54
100	53		1000	53.9
4000	52		5000	52.317

Table 4: A sample orderbook with average price calculated for each level. For example, the average price of the offer level 56 is  $(55 * 10000 + 56 * 4000) / 14000 = 55.286$

### 3.2.3 Estimation of spread recovery rate $\rho$

Given the transition period, we measure the average spread on a 1-minute interval:

$$\bar{\zeta}_j = \frac{\sum_{i=1}^N p_i}{N}$$

where  $j$  is the index of 1-minute interval, and  $N$  is number of days in our dataset. From the time series  $p_j$  we can perform an exponential linear regression to estimate  $\rho$ . We consider the following exponential model:

$$y = \rho e^{-\rho t}$$

where the independent variable  $t$  is number of seconds since market opening, and the dependent variable  $y$  represented the average spread  $\bar{\zeta}_j$ .

### 3.2.4 Data binning by volatility group

While the estimation method is straightforward, there is one observation of each factor per stock per day, and our data span only 1 week of trading. Consequently, the estimates for each stock may not be representative of the cost factors' expected values. To improve the accuracy of the estimates, we group the data by each stock's daily volatility,

Table 5: The average log of parameters per volatility group. The standard deviation is in the brackets. (\*\*) indicates significant at 10%, and (\*) indicates significant at 5%. In general, the sample means in each group converge to normal distribution, so we can assume that the sample means approximate expected values of the parameters per group.

Volatility	$\log\alpha$ (bps/shr)	$\log\beta$ (bps/shr)	$\log\rho$	$\log V_e$	$\log Q_0$
< 0.50	-6.10 (1.24)**	-5.23 (1.07)	4.27 (0.87)**	10.41 (0.80)**	7.32 (1.09)**
< 1.00	-4.92 (1.26)**	-4.22 (0.88)**	3.81 (0.84)**	9.26 (1.34)**	6.45 (1.18)**
< 1.50	-4.64 (1.09)**	-3.62 (0.65)**	3.80 (0.80)**	8.63 (1.29)**	6.28 (0.96)**
$\geq 1.50$	-4.26 (1.25)**	-3.17 (0.66)**	4.39 (0.78)**	8.50 (1.05)**	6.02 (0.94)**

normalized as percentage of its average price. The correlation between volatility and  $\alpha$ ,  $\beta$  and  $\rho$  are 28%, 48% and 24%, respectively. From the correlation, we hypothesize that the stocks' parameters within the same volatility bin will have similar distribution. Formally, let  $F_\alpha$ ,  $F_\beta$  and  $F_\rho$  be the log distribution of  $\alpha$ ,  $\beta$  and  $\rho$ , respectively:

$$\log^*\alpha \sim F_\alpha(\mu_\alpha, \sigma_\alpha|v)$$

$$\log^*\beta \sim F_\beta(\mu_\beta, \sigma_\beta|v)$$

$$\log^*\rho \sim F_\rho(\mu_\rho, \sigma_\rho|v)$$

where  $(\mu_\alpha, \sigma_\alpha)$ ,  $(\mu_\beta, \sigma_\beta)$  and  $(\mu_\rho, \sigma_\rho)$  are the means and standard deviations of the log of cost factors  $\alpha$ ,  $\beta$  and  $\rho$  per volatility group  $v$ , respectively. The ratios are expected to be all positive and skewed towards zero, so we use log-transformation to make the sample more symmetric. Let  $\log\alpha^*$ ,  $\log\beta^*$  and  $\log\rho^*$  be the sample mean per stock:

$$\log\alpha^* = \sum_{i=1}^m \alpha_i / m$$

$$\log\beta^* = \sum_{i=1}^m \beta_i / m$$

$$\log\rho^* = \sum_{i=1}^m \rho_i / m$$

where  $m$  is number of observations per stock. Based on the Central Limit Theorem, we have  $\log\alpha^*$ ,  $\log\beta^*$  and  $\log\rho^*$  converge to normal distribution with means approximating expected values of cost factors per volatility group. We test the convergence using the Jarque–Bera test ([10]), reported along with the log estimates in the subsequent result summary.

### 3.3 Transaction cost summary

Based on the cost factor estimates per volatility group, table (5) presents the estimates of the model for the group of stock based on daily volatility in our sample. In general, the estimates of cost factors per volatility group follows normal distribution with 5% significance level. Therefore, we can assume that the estimates are representative of the expected cost factors per group.

The estimates in each group are positive and within a reasonable range to each other. This suggests that the estimation methods are robust to the choice of instruments. We also observe that:

- The estimates of  $\log\alpha$  and  $\log\beta$  are in similar range. This suggests that our estimation of the auction cost factor from spread and auction matching quantity concurs with the cost implied in the liquidity posted in the continuous trading book. In other words, we expect the market maker to have consistent estimation of fair values across trading sessions, so  $\alpha$  and  $\beta$  should have similar values across volatility groups.
- The estimates of  $\log\rho$  are similar across volatility groups. This suggests that the recovery rate of the market impact during the auction matching is consistent across stocks and volatility bands.

We then proceed to calculate the execution cost of each group using the following cost measures:

- Square-root cost ( $SR$ ): the standard method of market impact is the square-root formula

$$SR = c\sigma\sqrt{\frac{n}{v}}$$



where  $SR$  is the price change from executing a trade for  $n$  shares, with market volatility  $\sigma$ , average daily volume  $v$  and some constant  $c$  ([19]). We use this as a benchmark to our cost formula to ensure that our estimates are reasonable. Constant  $c$  is set to one in our estimates. Note that the square-root cost only applies to cost of trading during continuous auction.

- TWAP cost ( $TW$ ): the total cost of executing a TWAP strategy during continuous trading session only. The formula is:

$$TW = \frac{Q_0(2Q_0T\beta + \frac{1-e^{-T\rho}TV_e\alpha}{\rho})}{2T^2}$$

- Optimal cost ( $OC$ ): the total cost of executing the optimal strategy in both the opening auction and the continuous session. The formula is:

$$OC = B_0^3\alpha + A^2T\beta + \frac{A(1-e^{-T\rho})V_e\alpha}{2\rho} - \frac{B_0e^{-T\rho}(B_0+2V_e)\alpha^2 \sinh(T\rho)}{16\beta\rho}$$

where

$$A = \frac{4(Q_0 - B_0) + \frac{(1-e^{-T\rho})}{\beta\rho}B_0\alpha}{4T}$$

- Auction optimal cost ( $Auc$ ): the cost of executing the optimal strategy in the opening auction. The formula is:

$$Auc = B_0^2\alpha$$

- Continous optimal cost ( $Cont$ ): the cost of executing the optimal strategy in the continuous session. The formula is:

$$Cont = OC - C4$$

- Optimal cost ratio ( $Ratio$ ): the ratio of the difference between cost of TWAP and the optimal strategy. The formula is:

$$Ratio = \frac{TWAP - C3}{TWAP}$$

The results are reported in Table (6). We can observe that:

- The square-root law cost ( $SR$ ), the TWAP strategy's cost ( $TWAP$ ) and the optimal strategy's cost are on similar scale. This suggests our cost measures concurs with the square-root law of market impact. We can see that  $TWAP$  increases proportionally with the square-root cost, as the volatility increases. This suggests the more volatile a stock is, the higher its cost of aggreasing the book.
- The difference between the square-root law cost ( $SR$ ) and the TWAP strategy's cost ( $TWAP$ ) widens significantly as volatility increases, even with constant  $c$  in the square-root law adjusted. This implies that there is an additional component in the square-root law cost that is not captured in our formula. We hypothesizes that this is the permanent impact that moves the fundamental price up as we buy and down as we sell (See, for example, [12]). However, as shown in ([1]), this component is not optimizable.
- The improvement is significant across the volatility groups. This makes sense as the opening auction's pricing is similar to the continuous auction's pricing. Therefore, the opening auction provides an alternative venue to take liquidity from. By balancing his trades across two auctions, the liquidity trader can get cheaper pricing for his trades.
- The optimal cost does not increase as much as the TWAP cost when volatility increases. Examining the auction proportion of the optimal cost, we can see the reason: the optimal strategy allocates the rising cost to the auction trade. The cost of trading in auction is consistently lower than continuous trading phase in (5). Therefore, by allocate more shares to be execute in the auction, the optimal strategy lowers the rate of increase in cost compared to the TWAP strategy. The continuous portion of the optimal cost stays almost constant across volatility groups.

From those observations, we can see that the optimal strategy does not only lower the cost of trading compared to the TWAP strategy, but also help stabilize cost among different volatility regimes. Therefore, the liquidity trader can achieve better control of his cost, as the optimal strategy is balanced against volatility.

## 4 Conclusion

We shows that when the cost of trading in the call auction is comparable to the continuous trading phase, our strategy yields significant cost savings. Subsequent research on different forms of market impact under both auctions, and extending to multi-phase auction market, can be conducted in the future to yield more interesting findings.

Table 6: The average of cost measures grouped by volatility. The standard deviation is in the brackets.

Volatility	$SR$	$TW$	$OC$	$Auc$	$Cont$	$Ratio$
$< 0.50$	11.18	5.88	2.35	1.39	0.97	59.94
$< 1.00$	22.36	8.20	3.43	1.96	1.47	58.15
$< 1.50$	33.54	7.50	3.05	1.77	1.27	59.38
$\geq 1.50$	44.72	9.67	3.45	2.20	1.26	64.29

## 5 Appendix

*Derivation of the optimal trading strategy in a mixed auction market.* The cost of trading in both auction given the notations in the paper is:

$$C = \alpha B_0^2 + \alpha \int_0^T \frac{B_0 + V_e}{2} e^{-\rho s} E_s ds + \beta \int_0^T E_s^2 ds$$

We need to find a tuple of  $(B_0, E_t, X_t)$  as a stationary point (function) of the functional. First we assume  $B_0$  is static, and solve the cost of trading during continuous phase:

$$C_c = \int_0^T \frac{B_0 + V_e}{2} \alpha e^{-\rho s} E_s ds + \int_0^T \beta E_s^2 ds$$

which has the following Euler-Lagrange equation

$$\begin{aligned} \frac{\alpha}{4\beta} B_0 e^{-\rho s} + E_s &= A \\ \Leftrightarrow \hat{E}_s &= A - \frac{\alpha}{4\beta} B_0 e^{-\rho s} \\ \Leftrightarrow \hat{X}_t &= At + B - \frac{\alpha}{4\beta\rho} B_0 (1 - e^{-\rho t}) \end{aligned}$$

Using the boundary conditions:

$$\hat{X}_0 = 0$$

$$\hat{X}_T = Q_0 - B_0$$

we have

$$B = 0$$

$$A = \frac{4(Q_0 - B_0) + \frac{(1 - e^{-T\rho})}{\beta\rho} B_0 \alpha}{4T}$$

Taking the first derivative of the cost function  $C$  w.r.t.  $B_0$  and solve for  $B_0$ , we have

$$B_0 = \frac{Q_0 F_{11} + V_e F_{12}}{F_2}$$

where

$$F_{11} = 8e^{T\rho} \beta \rho [\alpha - e^{T\rho} (\alpha - 4\beta\rho)]$$

$$F_{12} = (e^{T\rho} - 1) \alpha [\alpha (2 + T\rho) + e^{T\rho} (8\beta\rho + \alpha (T\rho - 2))]$$

$$F_2 = \alpha^2 (2 + T\rho) - 4e^{T\rho} \alpha (\alpha + 2\beta\rho) + e^{2T\rho} [8\beta\alpha\rho + 8\beta(T\alpha + \beta)\rho^2 + \alpha^2 (2 - T\rho)]$$

□

Table 7: The parameters per symbol (from index 1 to 55).

symbol	Volatility	$\alpha$	$\beta$	$\rho$	$V_e$	$Q_0$
PBCT	0.262	0.001	0.002	187.784	46192.000	2078.158
JBLU	0.330	0.000	0.004	156.842	81111.000	5720.204
ACWX	0.353	0.004	0.000	60.079	49226.000	128.624
NWL	0.373	0.000	0.005	102.016	92250.000	6355.374
VXUS	0.392	0.000	0.000	32.658	57961.000	3086.921
WEN	0.393	0.001	0.006	48.198	38962.000	2188.756
LNT	0.418	0.004	0.008	50.331	15855.000	937.718
IXUS	0.424	0.004	0.000	23.165	30854.000	136.641
BBBY	0.428	0.000	0.006	256.791	119468.000	8905.098
DBX	0.473	0.001	0.011	71.372	24285.000	2552.126
LBTYA	0.497	0.004	0.009	117.349	27312.000	1146.043
HDS	0.561	0.010	0.009	8.945	2212.000	230.153
HOLX	0.621	0.001	0.005	20.176	89421.000	4012.866
LKQ	0.629	0.006	0.008	33.474	17593.000	788.238
AAXJ	0.634	0.002	0.003	48.560	30887.000	1160.355
XRAY	0.671	0.012	0.008	32.383	23608.000	596.277
STLD	0.677	0.001	0.007	45.271	41339.000	3668.282
SFIX	0.750	0.001	0.008	95.788	56736.000	4611.269
WMGI	0.767	0.009	0.017	94.362	44209.000	1815.915
DISH	0.774	0.006	0.009	70.232	35289.000	1327.218
NTNX	0.826	0.005	0.012	101.202	33167.000	1867.387
HSIC	0.872	0.015	0.014	39.336	9850.000	507.461
TRIP	0.909	0.007	0.012	16.455	9654.000	466.742
PFG	0.928	0.005	0.005	24.847	10567.000	543.217
MOMO	0.964	0.003	0.011	45.980	68470.000	3543.795
MXIM	1.074	0.001	0.011	9.218	3383.000	1547.115
EXPD	1.101	0.010	0.016	13.809	1313.000	219.949
WB	1.208	0.007	0.021	30.026	4624.000	460.635
STNE	1.240	0.015	0.038	152.562	10732.000	700.141
NKTR	1.252	0.003	0.010	39.413	49235.000	2727.206
QRVO	1.284	0.002	0.029	143.486	17952.000	1369.904
GRMN	1.305	0.003	0.011	13.927	13491.000	1307.241
YY	1.314	0.013	0.004	39.774	32198.000	452.550
BMRN	1.361	0.002	0.014	50.288	63047.000	3575.019
WW	1.366	0.004	0.038	102.494	15153.000	1521.545
FSLR	1.395	0.013	0.014	29.445	10397.000	532.029
IONS	1.483	0.021	0.030	69.634	8131.000	335.101
OSTK	1.496	0.002	0.026	96.990	5802.000	2831.484
INCY	1.551	0.006	0.015	19.241	1252.000	1038.562
BZUN	1.575	0.017	0.047	46.729	4377.000	352.583
LITE	1.579	0.003	0.020	46.067	15913.000	1976.926
CREE	1.671	0.003	0.027	57.839	25027.000	1726.270
DDOG	1.715	0.002	0.012	83.875	20707.000	1727.238
ALNY	1.830	0.020	0.046	69.586	13669.000	728.295
NBIX	1.870	0.039	0.054	55.244	3428.000	193.383
HQY	1.966	0.008	0.052	269.299	9532.000	784.383
ACAD	1.975	0.001	0.020	45.134	24787.000	3132.673
IRBT	2.112	0.010	0.030	52.925	6677.000	433.722
TNDM	2.149	0.001	0.032	271.474	15299.000	1247.559
NVCR	2.367	0.015	0.074	230.628	3957.200	423.687
PTC	2.370	0.007	0.021	39.242	5255.000	406.057
SGEN	2.628	0.002	0.020	64.998	32010.000	2152.309
OLLI	2.771	0.005	0.010	13.472	2105.000	497.339
SEDG	3.055	0.010	0.050	86.744	7448.000	652.610
GH	3.429	0.009	0.032	309.184	11162.000	826.191

Table 8: The parameters per symbol (from index 56 to 110).

symbol	Volatility	$\alpha$	$\beta$	$\rho$	$V_e$	$Q_0$
FOLD	0.283	0.005	0.012	189.793	73109.000	3027.321
FLEX	0.291	0.002	0.004	124.083	46660.000	1813.574
FEYE	0.296	0.004	0.008	310.920	46157.000	2147.224
BILI	0.301	0.004	0.005	81.062	84873.000	2666.016
SABR	0.319	0.005	0.007	23.262	17420.000	739.117
QRTEA	0.342	0.001	0.010	180.352	51848.000	3771.041
GT	0.379	0.001	0.008	97.525	63333.000	4871.249
COMM	0.416	0.002	0.007	279.932	51108.000	3034.006
PTEN	0.419	0.001	0.009	106.529	68508.000	5224.320
GNTX	0.420	0.006	0.006	21.282	12237.000	790.205
SCZ	0.438	0.001	0.001	15.689	13017.000	583.856
MLCO	0.450	0.015	0.011	31.807	45356.000	1007.001
MAT	0.455	0.002	0.007	136.604	55938.000	2969.368
ACGL	0.462	0.002	0.006	26.238	12203.800	1071.419
EXEL	0.481	0.002	0.009	27.420	28402.000	1738.561
DOX	0.550	0.021	0.031	12.245	1626.000	130.558
HZNP	0.570	0.005	0.005	26.146	22740.000	1002.931
LSCC	0.614	0.003	0.017	121.403	30809.000	2051.033
SPWR	0.617	0.001	0.011	313.460	59631.000	4985.407
ENTG	0.758	0.006	0.040	39.927	7543.000	650.119
TRMB	0.758	0.005	0.019	21.644	7930.000	792.500
PLAY	0.821	0.022	0.035	42.957	10702.000	486.451
AVT	0.839	0.012	0.029	16.661	2370.000	215.740
FLIR	0.842	0.007	0.023	15.207	2097.000	516.391
BHF	0.844	0.010	0.020	149.685	28077.000	1167.994
IIVI	0.852	0.007	0.015	158.290	48830.000	1926.326
CROX	0.877	0.005	0.003	60.644	9917.000	636.217
CDK	0.918	0.006	0.021	26.331	200.000	161.208
ARWR	1.023	0.003	0.060	43.647	5705.000	1013.855
SRCL	1.027	0.013	0.021	21.489	1233.000	155.275
PDCE	1.034	0.011	0.022	32.136	21056.000	993.216
DNKN	1.043	0.008	0.028	28.094	5960.000	446.554
EWBC	1.059	0.022	0.041	35.728	6797.000	391.876
PZZA	1.076	0.006	0.030	30.718	9006.000	574.689
CGNX	1.090	0.003	0.021	22.262	3040.000	486.943
SBGI	1.137	0.051	0.023	96.357	14132.000	258.144
GBT	1.293	0.022	0.047	49.402	1210.000	266.248
PS	1.401	0.009	0.012	22.933	1235.000	287.696
UTHR	1.411	0.052	0.051	59.692	6191.000	206.916
TXRH	1.438	0.010	0.033	82.531	10335.000	652.685
TLRY	1.542	0.007	0.059	141.574	10773.000	943.479
AMBA	1.550	0.014	0.022	77.601	18643.000	778.969
CRUS	1.653	0.020	0.030	260.614	7699.200	466.551
JACK	1.663	0.052	0.041	66.968	4948.000	153.030
PRAH	1.673	0.049	0.073	67.331	3336.000	169.558
WING	1.845	0.041	0.049	62.408	6421.000	270.949
ACIA	1.890	0.003	0.023	52.753	1964.000	331.712
ICPT	1.892	0.070	0.111	103.082	3052.000	196.029
MANH	1.962	0.082	0.057	122.253	4397.000	118.740
BPMC	2.293	0.042	0.176	333.385	2637.000	202.884
MKSI	2.645	0.057	0.071	35.333	1208.000	112.398
MYGN	2.820	0.024	0.057	56.501	4757.200	440.228
MRTX	3.309	0.041	0.060	207.044	3159.000	155.410
POOL	3.321	0.037	0.067	269.752	1814.000	102.456
ALLK	4.652	0.264	0.146	226.001	3771.800	88.201

Table 9: The parameters per symbol (from index 111 to 150).

symbol	Volatility	$\alpha$	$\beta$	$\rho$	$V_e$	$Q_0$
VIAV	0.270	0.002	0.010	187.200	36657.000	2289.531
SFM	0.335	0.003	0.007	52.594	13722.000	1182.277
MIK	0.393	0.005	0.014	171.741	56094.000	2506.546
PENN	0.413	0.012	0.018	24.905	4226.000	349.253
CG	0.461	0.041	0.011	24.777	10821.000	133.221
MRNA	0.484	0.010	0.019	44.149	6702.000	495.116
IMMU	0.551	0.004	0.015	27.192	10686.000	1068.192
RUN	0.554	0.004	0.002	83.362	63977.000	1114.887
IUSG	0.573	0.002	0.002	36.386	5398.000	275.948
PPC	0.581	0.012	0.021	158.202	11097.000	575.414
PACW	0.649	0.015	0.027	43.345	8768.000	448.545
CARG	0.729	0.012	0.017	293.187	23756.000	971.628
CAR	0.768	0.036	0.032	27.515	4374.000	169.327
VGLT	0.770	0.001	0.033	112.759	30187.000	1746.401
SEIC	0.784	0.024	0.044	17.999	1728.000	285.360
BPOP	0.790	0.054	0.039	19.092	3972.000	119.640
CAKE	0.819	0.008	0.027	19.108	1787.000	258.343
BRKR	0.847	0.037	0.035	131.963	4494.800	249.500
GDS	0.979	0.074	0.032	68.474	5457.000	106.138
RP	0.981	0.076	0.033	28.053	1561.000	50.979
SIGI	0.986	0.171	0.061	40.622	1184.000	34.491
TW	0.996	0.002	0.054	49.807	12038.000	2192.693
SINA	1.085	0.013	0.042	279.643	10714.600	748.878
REAL	1.093	0.004	0.025	57.011	17364.000	1317.058
CVET	1.144	0.009	0.020	26.633	7062.000	1788.976
ZG	1.256	0.021	0.029	17.255	991.000	81.859
WTFC	1.256	0.048	0.057	297.570	4289.800	274.840
NUVA	1.257	0.067	0.070	57.743	1482.000	125.589
TCBI	1.273	0.029	0.066	62.542	3927.000	245.646
RPD	1.323	0.012	0.052	36.362	687.000	392.854
LECO	1.413	0.051	0.080	25.165	121.000	80.500
LIVN	1.515	0.040	0.082	69.738	6187.000	289.638
APPN	1.642	0.021	0.028	35.981	533.000	127.745
GTLS	1.646	0.024	0.038	78.980	4539.000	335.853
LVGO	1.659	0.009	0.017	128.050	22324.000	1049.087
BL	1.816	0.005	0.038	31.756	759.000	454.329
MEDP	1.818	0.022	0.077	55.975	5530.000	359.094
QURE	1.827	0.015	0.132	166.881	2302.000	483.529
MMSI	1.882	0.038	0.044	28.511	2235.000	220.300
STMP	2.696	0.025	0.073	53.200	342.000	75.481

## References

- [1] Robert Almgren and Neil Chriss. Optimal execution of portfolio transactions. *Journal of Risk*, pages 5–39, 2000.
- [2] Marco Avellaneda and Sasha Stoikov. High Frequency Trading in a Limit Order Book. *Quantitative Finance*, 8:217–224, 04 2008.
- [3] Mario Bellia, Lorian Pelizzon, Marti G. Subrahmanyam, Jun Uno, and Darya Yuferova. Low-latency trading and price discovery: Evidence from the Tokyo Stock Exchange in the pre-opening and opening periods. (144), 2017.
- [4] D. Bertimas, Andrew Lo, and P. Hummel. Optimal Control of Execution Costs for Portfolios. *Computing in Science & Engineering*, 1:40 – 53, 12 1999.
- [5] Bruno Biais, Pierre Hillion, and Chester Spatt. Price Discovery and Learning during the Preopening Period in the Paris Bourse. *Journal of Political Economy*, 107(6):1218–1248, 1999.
- [6] Carole Comerton-Forde and James Rydge. The influence of call auction algorithm rules on market efficiency. *Journal of Financial Markets*, 9(2):199 – 222, 2006.
- [7] Nicholas Economides and Robert A. Schwartz. Electronic Call Market Trading. *The Journal of Portfolio Management*, 21(3):10–18, 1995.
- [8] R. C. Geary. The frequency distribution of the quotient of two normal variates. *Journal of the Royal Statistical Society*, 93(3):442–446, 1930.
- [9] Thomas Ho and Hans R. Stoll. Optimal dealer pricing under transactions and return uncertainty. *Journal of Financial Economics*, 9(1):47 – 73, 1981.
- [10] Carlos M. Jarque and Anil K. Bera. Efficient tests for normality, homoscedasticity and serial independence of regression residuals. *Economics Letters*, 6(3):255 – 259, 1980.
- [11] D. Katz, J. Baptista, S. P. Azen, and M. C. Pike. Obtaining confidence intervals for the risk ratio in cohort studies. *Biometrics*, 34(3):469–474, 1978.
- [12] Albert S. Kyle. Continuous Auctions and Insider Trading. *Econometrica*, 53(6):1315–1335, 1985.
- [13] Ananth Madhavan. Trading Mechanisms in Securities Markets. *The Journal of Finance*, 47(2):607–641, 1992.
- [14] Ananth Madhavan and Venkatesh Panchapagesan. Price Discovery in Auction Markets: A Look Inside the Black Box. *The Review of Financial Studies*, 13(3):627–658, 06 2015.
- [15] Albert J. Menkveld. High frequency trading and the new market makers. *Journal of Financial Markets*, 16(4):712 – 740, 2013. High-Frequency Trading.
- [16] Anna A. Obizhaeva and Jiang Wang. Optimal trading strategy and supply/demand dynamics. *Journal of Financial Markets*, 16(1):1 – 32, 2013.
- [17] Michael S. Pagano, Lin Peng, and Robert A. Schwartz. A call auction’s impact on price formation and order routing: Evidence from the nasdaq stock market. *Journal of Financial Markets*, 16(2):331–361, 2013.
- [18] Michael S. Pagano and Robert A. Schwartz. A closing call’s impact on market quality at Euronext Paris. *Journal of Financial Economics*, 68(3):439 – 484, 2003.
- [19] Bence Tóth, Yves Lempriere, Cyril Deremble, Joachim Lataillade, Julien Kockelkoren, and Jean-Philippe Bouchaud. Anomalous price impact and the critical nature of liquidity in financial markets. *Physical Review X*, 1, 05 2011.
- [20] Xavier Vives and Luis Medrano. Strategic Behavior and Price Discovery. *RAND Journal of Economics*, 32:221–48, 02 2001.