Greedy Algorithms I

Summer 2017 • Lecture 07/25

A Few Notes

Homework 4

- Due Friday 7/28 at 11:59 p.m. on Gradescope.

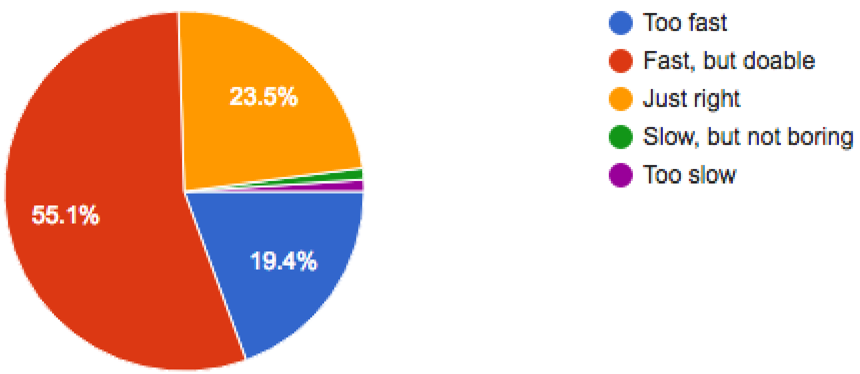
Homework 5

- Released Friday 7/28.

Week 4 Feedback

How do you find the pace of the course?

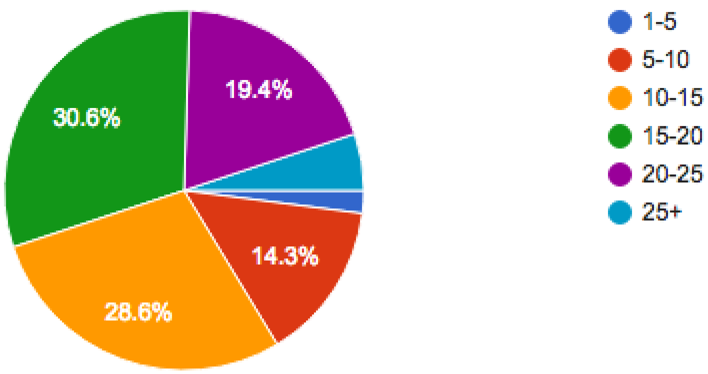
98 responses



Week 4 Feedback

How many hours did you spend on Homework 3?

98 responses



Week 4 Feedback

What's one thing that you noticed has improved in the course so far?

More reasonable workload with respect to homework. **Awesome!**

Office hours and the advice we receive from TAs is much better. **Great, the TA's are working super hard to make this happen!**

More examples provided in lectures. **This will continue this week!**

What's one thing that you wish was different about the course so far and hasn't improved?

More real-world examples of where these algorithms are useful in industrial or real-world context. **I've started an anthology of cool use-cases on Piazza!**

It would be great to receive feedback for the homework and solutions earlier. **Yes, we'll try to release feedback for homework by Fridays.**

Piazza response time. **We're doing our best, and switched up our internal processes. Also thank you to the upstanding Piazza citizens!**

Outline for Today

- Greedy algorithms
  - Frog Hopping
  - Activity Selection

Frog Hopping

Greedy Algorithms

Greedy algorithms construct solutions one step at a time, at each step choosing the locally best option.

- **Advantages:** simple to design, often efficient
- **Disadvantages:** difficult to verify correctness or optimality

Freddie the Frog

Freddie the Frog starts at position 0 along a river. His goal is to reach position n.

There are lilypads at various positions, including at position 0 and position n.

Freddie can hop at most r units at a time.

Task: Find the path Freddie should take to minimize hops, assuming such a path exists.

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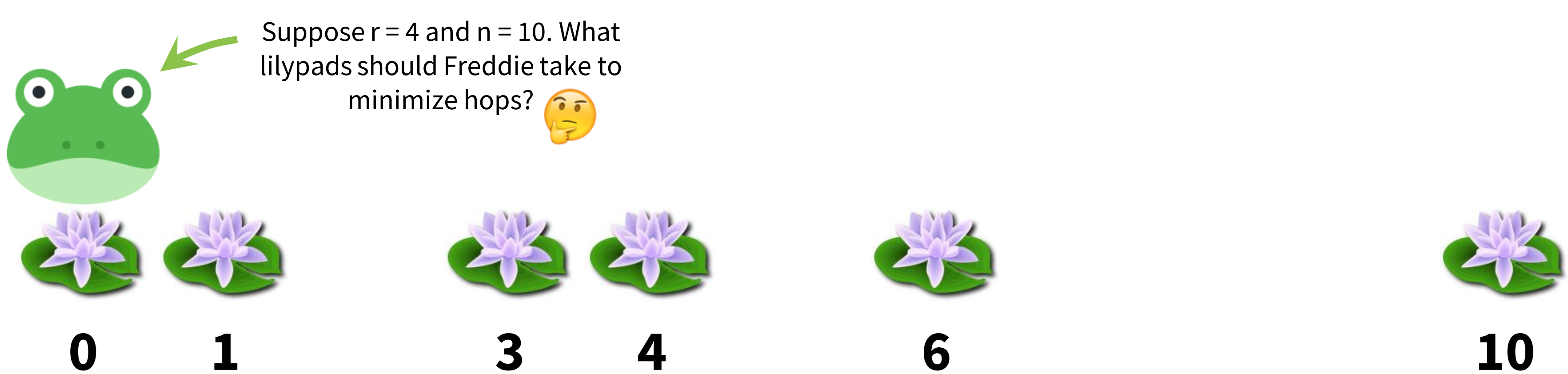

Frog Hopping

```
algorithm frog_hopping(lilys, r, n):
  # lilys = [0, 1, 3, 4, 6, 10] in the previous example
  H = [0]  # contains hops
  cur_lily = {"index": 0, "position": 0}
  while cur_lily["position"] < n:
    next_lily = furthest_reachable_lily(
      cur_lily, lilys, r
    )  # finds the furthest lilypad still reachable
       # from cur_lily
    H.append(next_lily["position"])
    cur_lily = next_lily
  return H
```

You should be able to implement this function yourself.

Runtime: $O(n)$

Frog Hopping

We need to prove two properties about the algorithm to guarantee correctness.

(1) **Feasibility.** The algorithm finds a feasible (aka legal) series of hops (i.e. it doesn't "get stuck" or break any rules).

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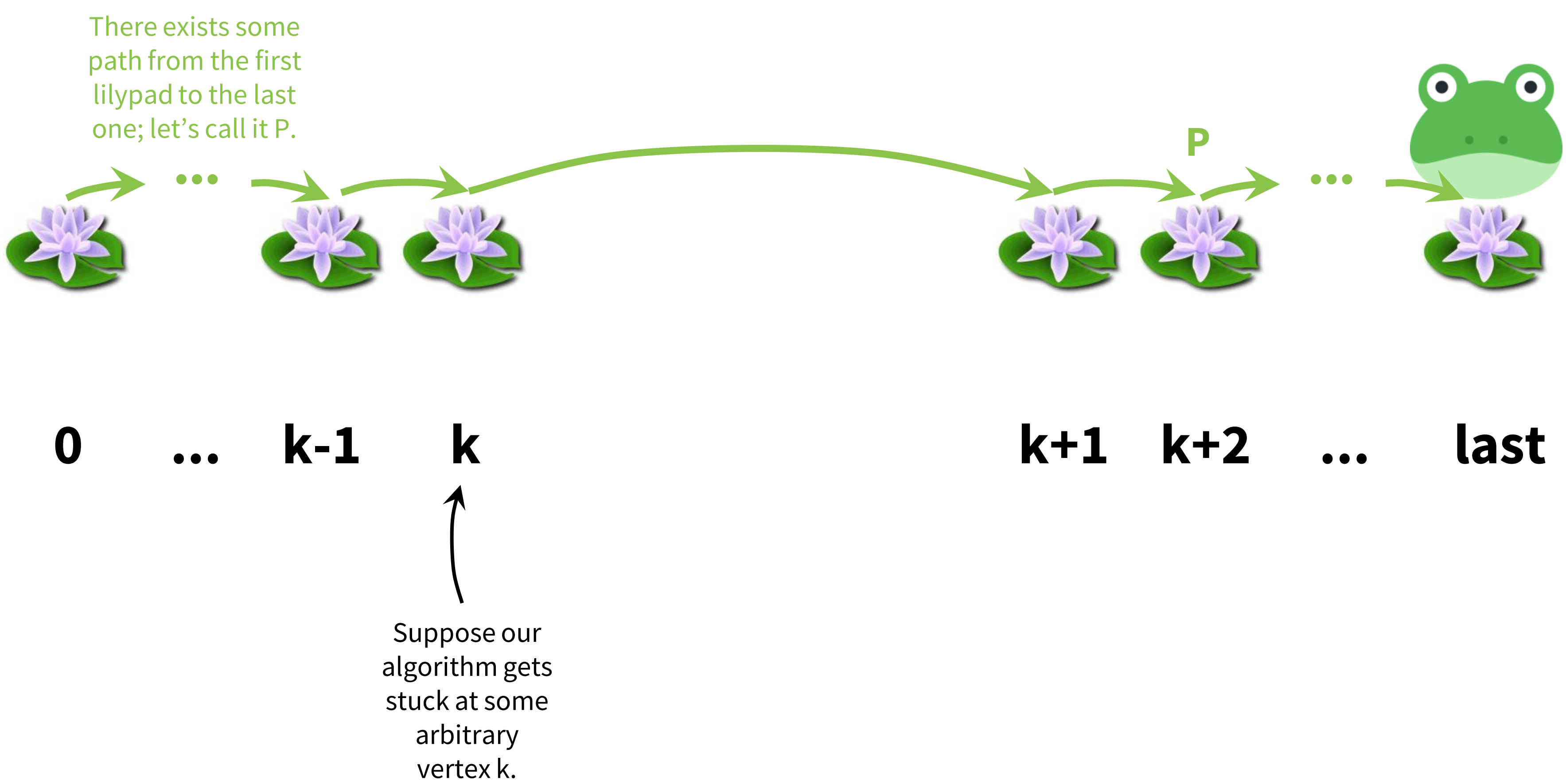

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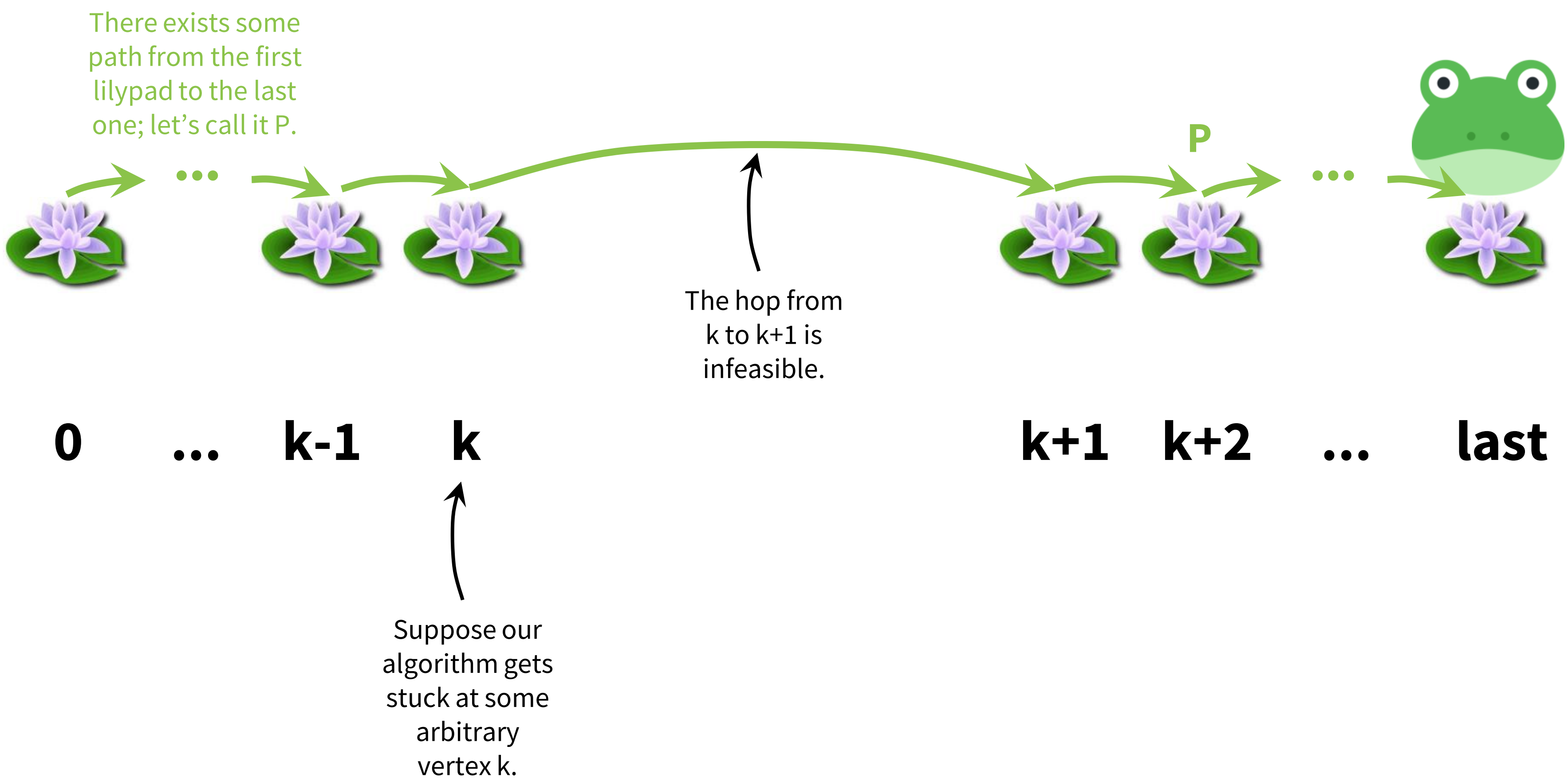

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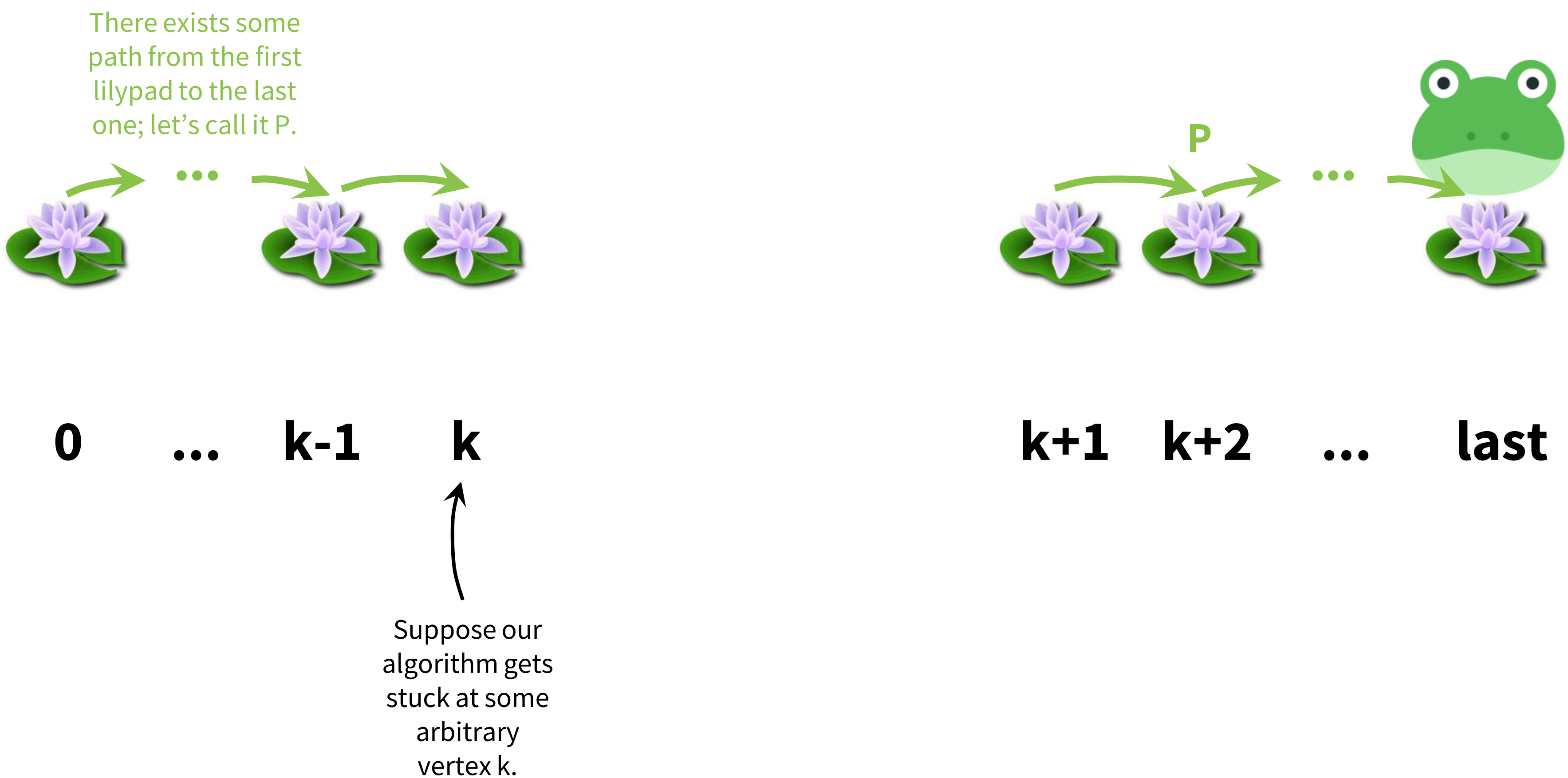

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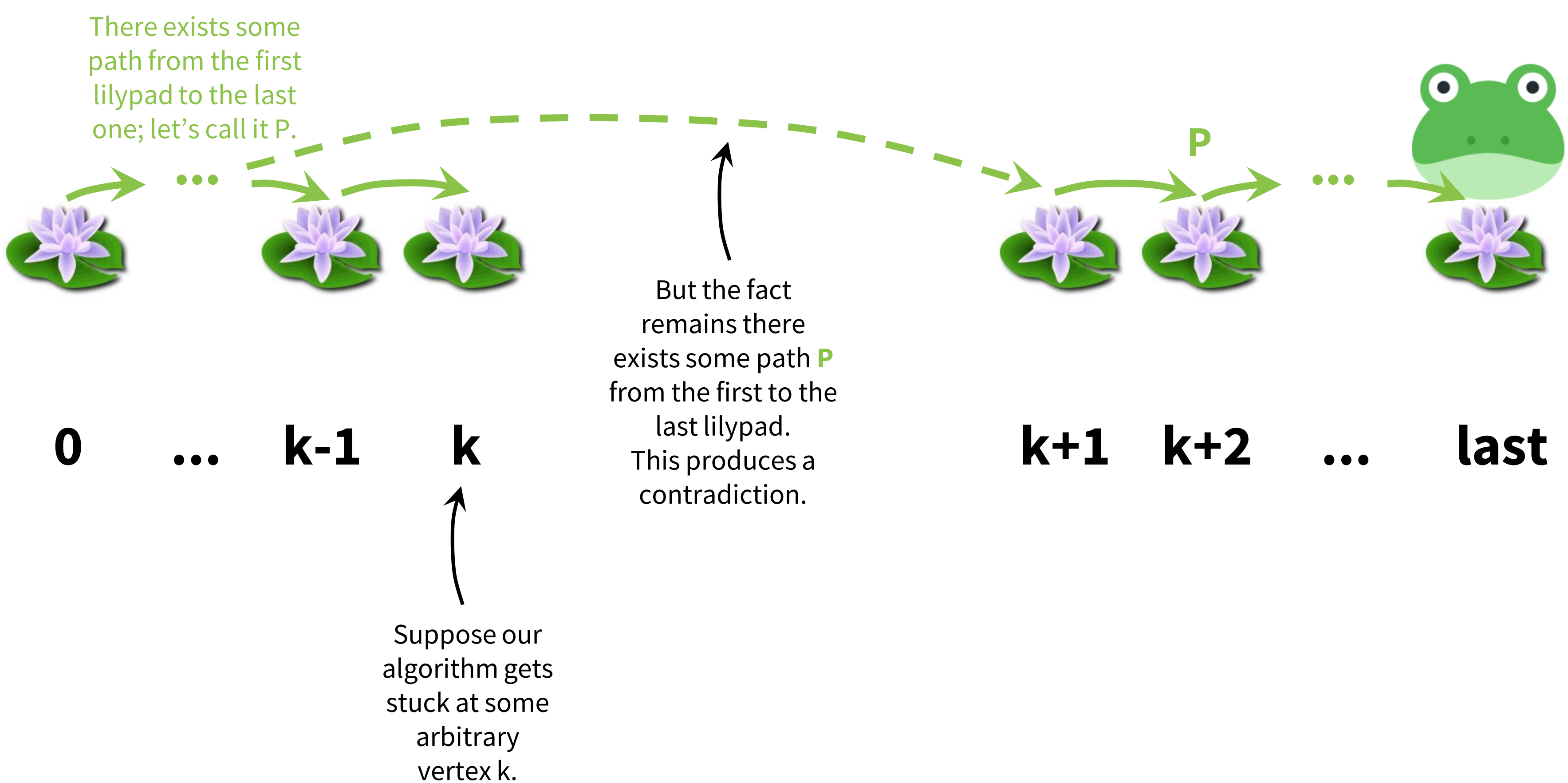

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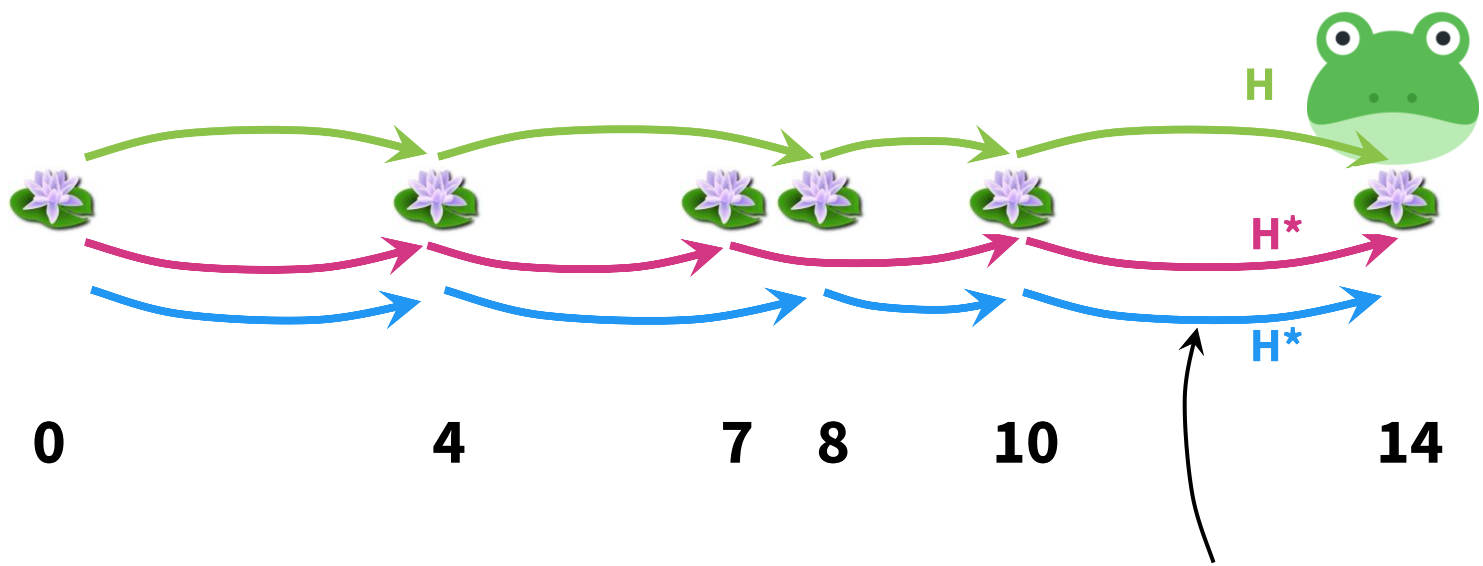
Intuition: Consider an arbitrary optimal series of hops H*, then show that our greedy algorithm produces a series of hops H no worse than H*.

What Does Arbitrary H* Mean?



There could be many optimal H* (this series of lilypads has 2); this proof relies on an arbitrary choice from among this H*.

Suppose we choose **H***.

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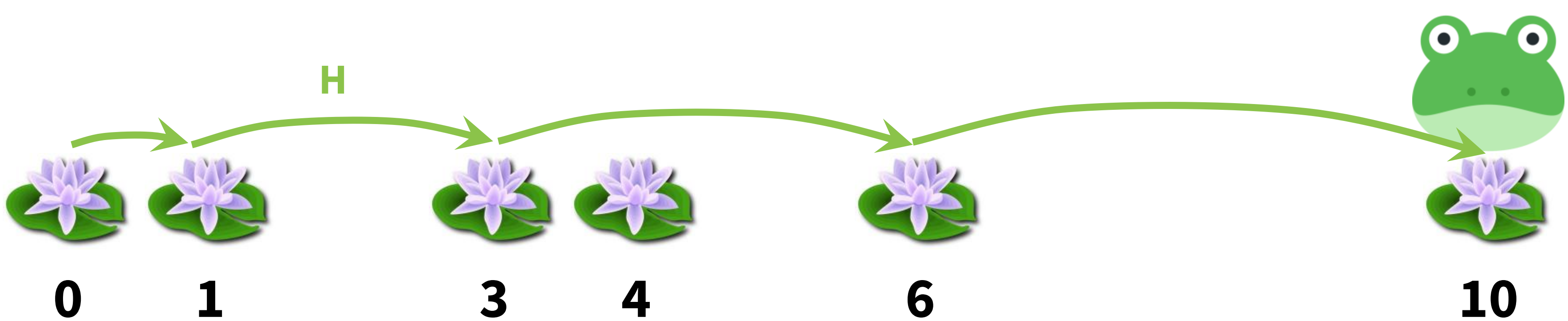

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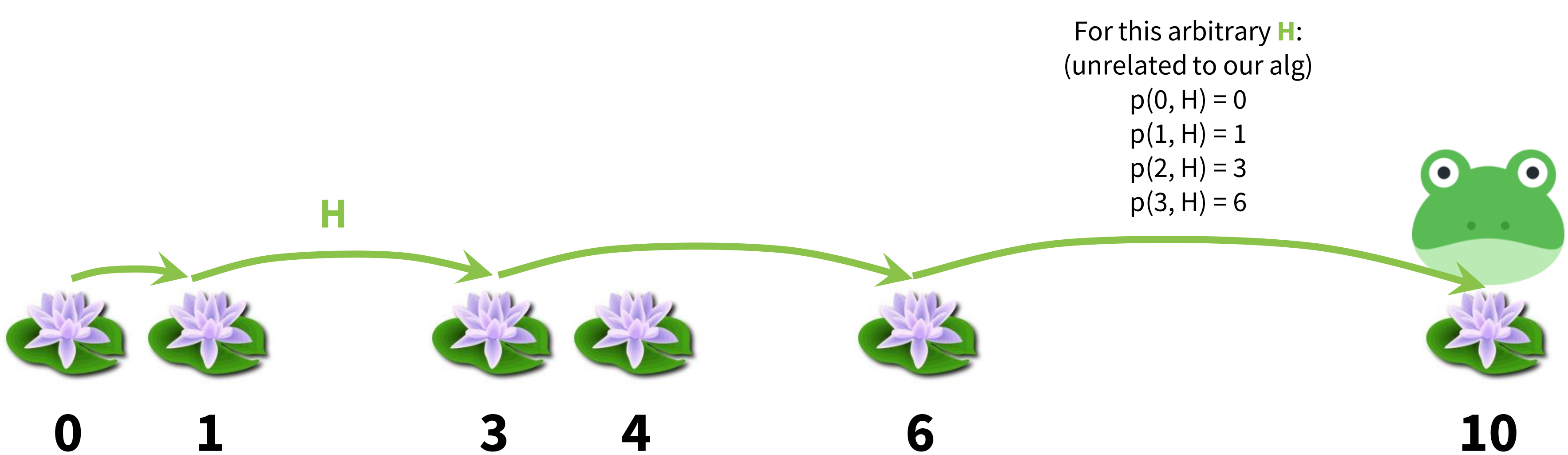

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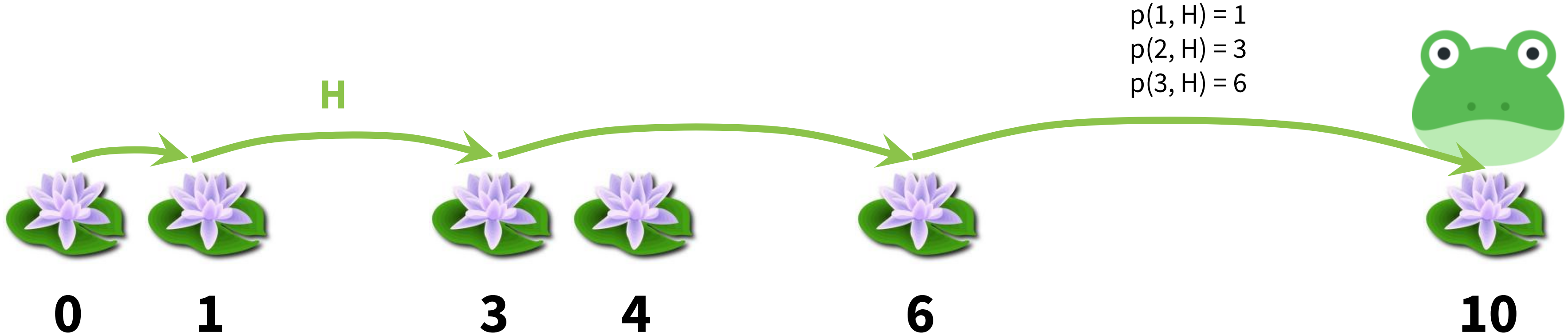
Let's formalize this using induction.

For this arbitrary **H**:
(unrelated to our alg)
$p(0, H) = 0$
$p(1, H) = 1$
$p(2, H) = 3$
$p(3, H) = 6$



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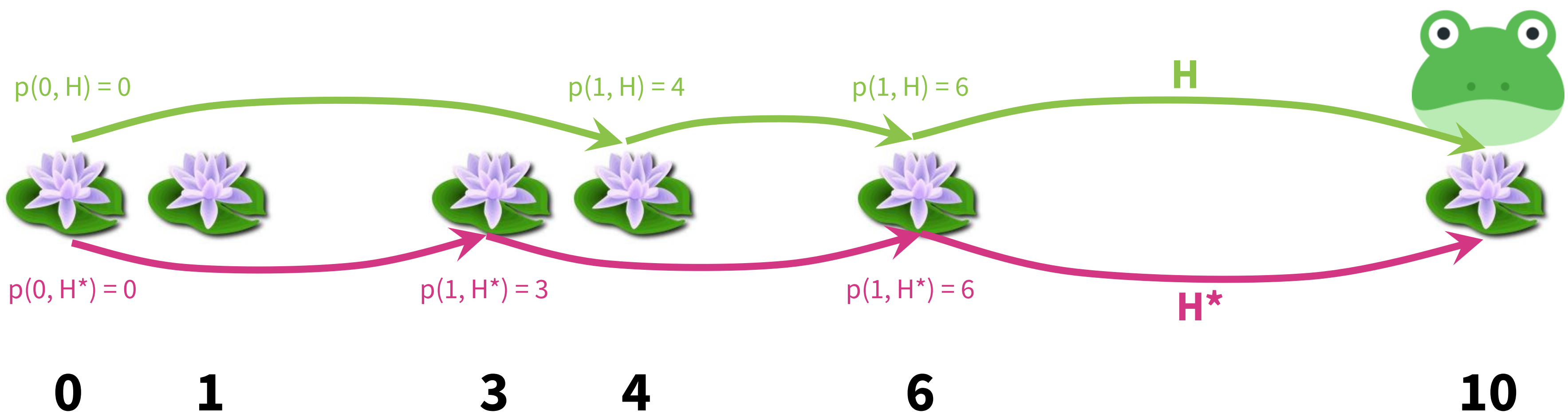
Lemma: For any i in 0 ≤ i ≤ |H*|, we have p(i, H) ≥ p(i, H*), constructing H from `frog_hopping`.



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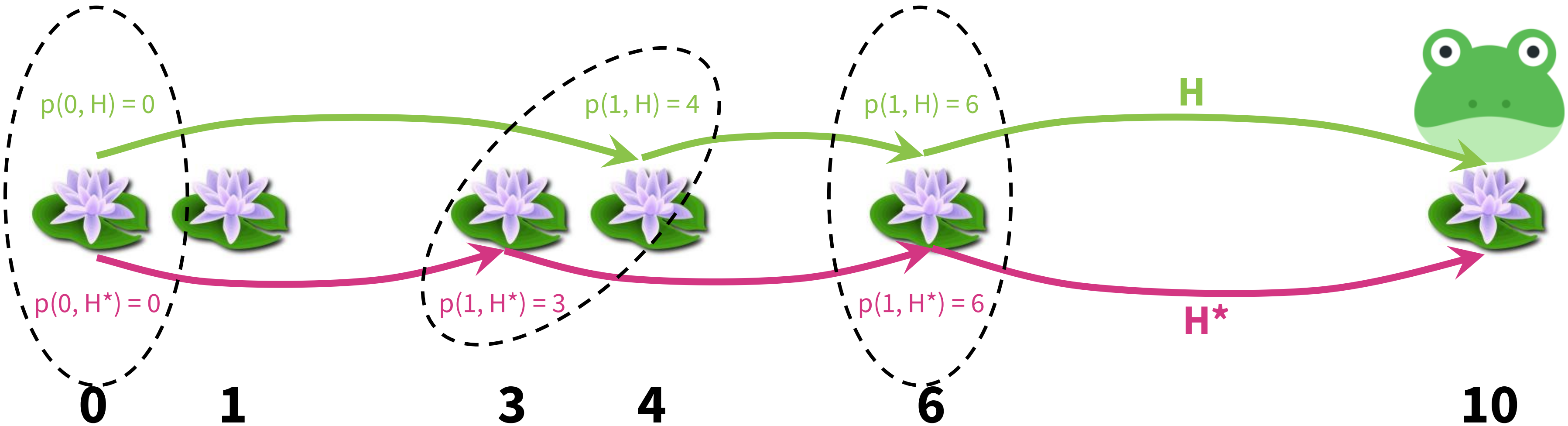

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Lemma 2: For all $0 \leq i \leq |H^*|$, we have $p(i, H) \geq p(i, H^*)$, constructing H from `frog_hopping`.

Proof: We proceed by induction.

As a base case, if $i = 0$, then $p(0, H) = 0 \geq 0 = p(0, H^*)$ since the frog hasn't moved.

For the inductive step, assume that the claim holds for some $0 \leq i < |H^*|$. We'll prove the claim holds for $i + 1$ by considering two cases:

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Case 2: $p(i, H) < p(i+1, H^*)$. Each hop is of size at most r, so $p(i+1, H^*) \leq p(i, H^*) + r$. By our inductive hypothesis, we know $p(i, H) \geq p(i, H^*)$, so $p(i+1, H^*) \leq p(i, H) + r$; i.e. position $p(i+1, H^*)$ is reachable from position $p(i, H)$.

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For the inductive step, assume that the claim holds for some $0 \leq i < |H^*|$. We'll prove the claim holds for $i + 1$ by considering two cases:

Case 1: $p(i, H) \geq p(i+1, H^*)$. Since each hop moves forward, we have $p(i+1, H) \geq p(i, H)$, so we have $p(i+1, H) \geq p(i+1, H^*)$.

Case 2: $p(i, H) < p(i+1, H^*)$. Each hop is of size at most r, so $p(i+1, H^*) \leq p(i, H^*) + r$. By our inductive hypothesis, we know $p(i, H) \geq p(i, H^*)$, so $p(i+1, H^*) \leq p(i, H) + r$; i.e. position $p(i+1, H^*)$ is reachable from position $p(i, H)$. Since the greedy algorithm hops to the furthest lilypad still reachable from position $p(i, H)$, it hops to at least position $p(i+1, H^*)$. Therefore, $p(i+1, H) \geq p(i+1, H^*)$.

Frog Hopping

Lemma 2: For all $0 \leq i \leq |H^*|$, we have $p(i, H) \geq p(i, H^*)$, constructing H from `frog_hopping`.

Proof: We proceed by induction.

As a base case, if $i = 0$, then $p(0, H) = 0 \geq 0 = p(0, H^*)$ since the frog hasn't moved.

For the inductive step, assume that the claim holds for some $0 \leq i < |H^*|$. We'll prove the claim holds for $i + 1$ by considering two cases:

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So $p(i+1, H) \geq p(i+1, H^*)$, completing the induction. ■

Frog Hopping

Now for the theorem: `frog_hopping` produces an optimal solution for Freddie.

Frog Hopping

Theorem: `frog_hopping` produces an optimal solution for Freddie.

Proof:

Since H^* is an optimal solution, we know that $|H^*| \leq |H|$. We will prove $|H^*| = |H|$.

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Let $k = |H^*|$. By **Lemma 2**, we have $p(k, H) \geq p(k, H^*)$.

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Theorem: `frog_hopping` produces an optimal solution for Freddie.

Proof:

Since H^* is an optimal solution, we know that $|H^*| \leq |H|$. We will prove $|H^*| = |H|$.

Let $k = |H^*|$. By **Lemma 2**, we have $p(k, H) \geq p(k, H^*)$. Since Freddie arrives at position n after k hops along series H^*, we know that $p(k, H) \geq p(k, H^*) = n$.

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The greedy algorithm arrives at position n after k hops, so $|H| = k$.

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The greedy algorithm arrives at position n after k hops, so $|H| = k$. Importantly, it's impossible to reach position n in fewer than k hops since doing so would contradict the optimality of H^*.

Frog Hopping

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The greedy algorithm arrives at position n after k hops, so $|H| = k$. Importantly, it's impossible to reach position n in fewer than k hops since doing so would contradict the optimality of H^*. Thus, $|H| = k = |H^*|$, so `frog_hopping` produces an optimal solution. ■

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The greedy algorithm arrives at position n after k hops, so $|H| = k$. Importantly, it's impossible to reach position n in fewer than k hops since doing so would contradict the optimality of H^*. Thus, $|H| = k = |H^*|$, so `frog_hopping` produces an optimal solution. ■

Here, we proved this step using a direct proof. You should be able to structure the proof by contradiction here too.

Frog Hopping

We need to prove two properties about the algorithm to guarantee correctness.

(1) **Feasibility.** The algorithm finds a feasible (aka legal) series of hops (i.e. it doesn't "get stuck" or break any rules). 👌

(2) **Optimality.** The algorithm finds an optimal series of hops (i.e. there isn't a better path available).

Frog Hopping

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Greedy Stays Ahead

The style of proof we just wrote is an example of a **greedy stays ahead** proof.

(1) Find intermediate values that evaluate the solution produced by any algorithm, including the greedy one.

What's our values in `frog_hopping`? 🤔 The position after i hops.

(2) Show the greedy algorithm produces values at least as good as any solution's (using induction).

(3) Prove that since the greedy algorithm produces values at least as good as any solution's, it must be optimal (using direct proof or proof by contradiction).

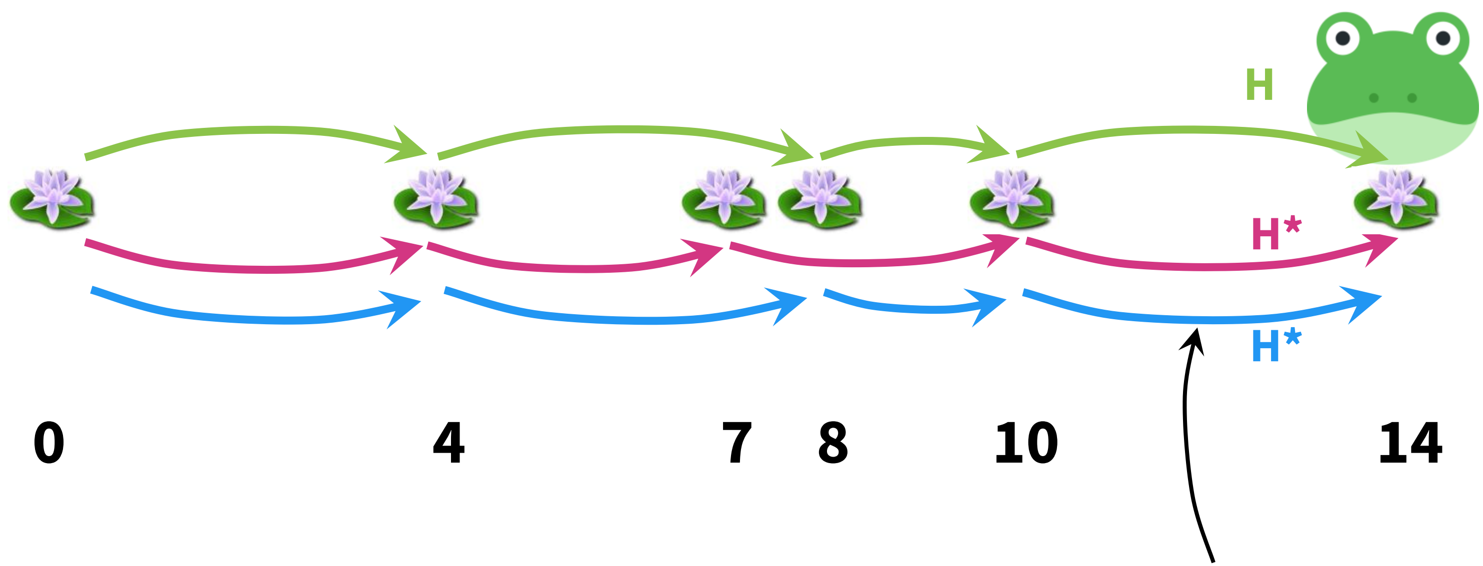
Greedy Exchange Argument

There's another style of proof that uses **greedy exchange argument**.

- If we swap an optimal solution out for the greedy solution, argue that we're still optimal.

Greedy Exchange Argument

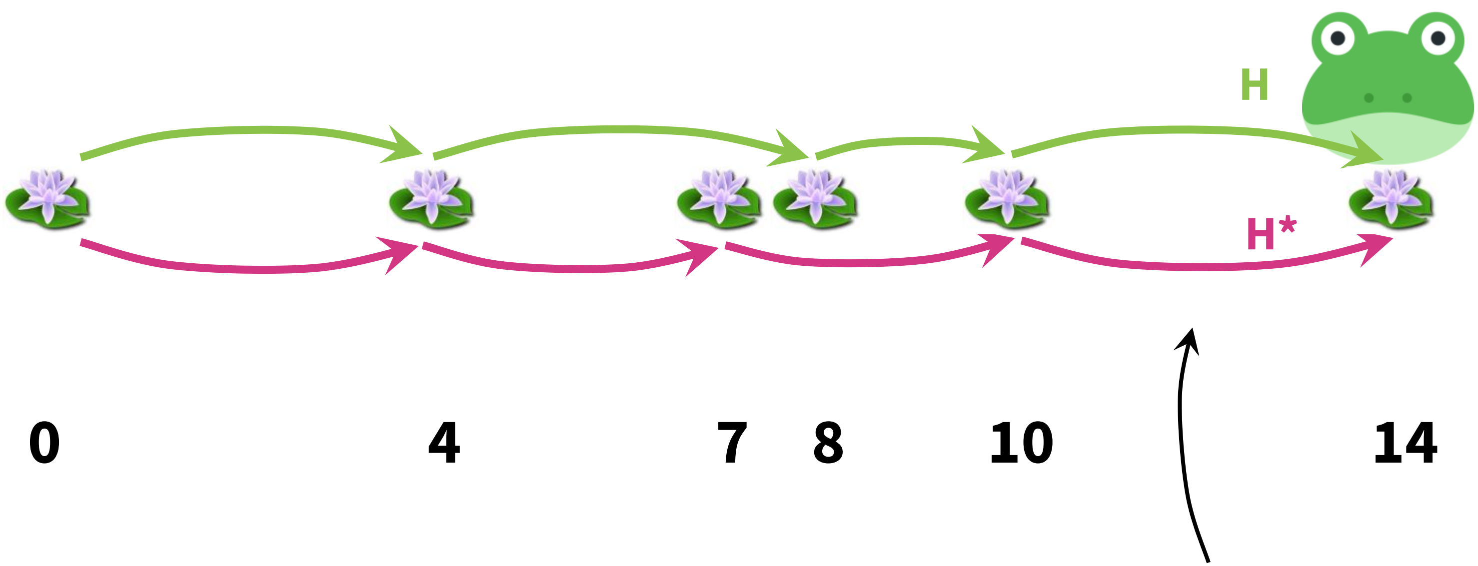
Again, this proof will rely on an arbitrary choice of H*.



There could be many optimal H* (this series of lilypads has 2); this proof relies on an arbitrary choice from among this H*.

Suppose we choose **H***.

Greedy Exchange Argument



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Greedy Exchange Argument

Theorem: `frog_hopping` produces an optimal solution.

Proof: We proceed by induction.

As a base case, we initialize H to `[0]` and all feasible hops H* must have H*[0] = 0.

For the inductive step, assume that after hop j has been added to H, there exists an optimal feasible series of hops H* such that H*[0..j] = H[0..j].

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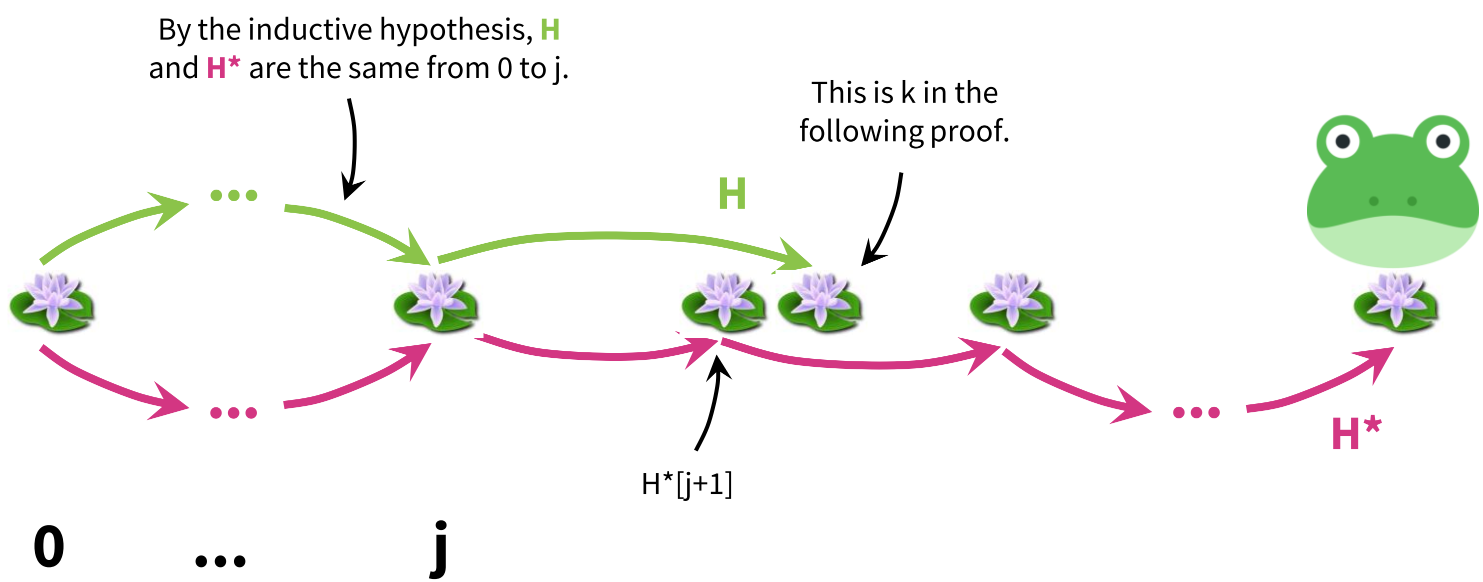
Greedy Exchange Argument



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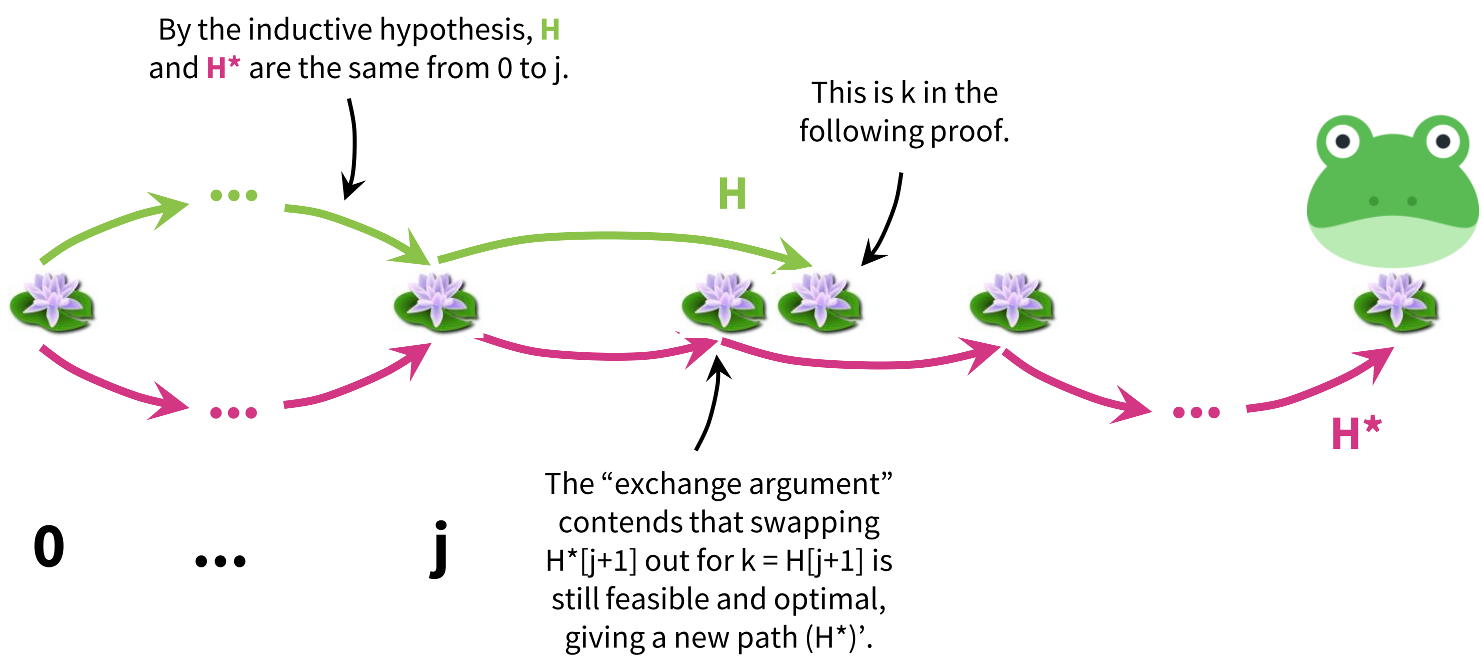

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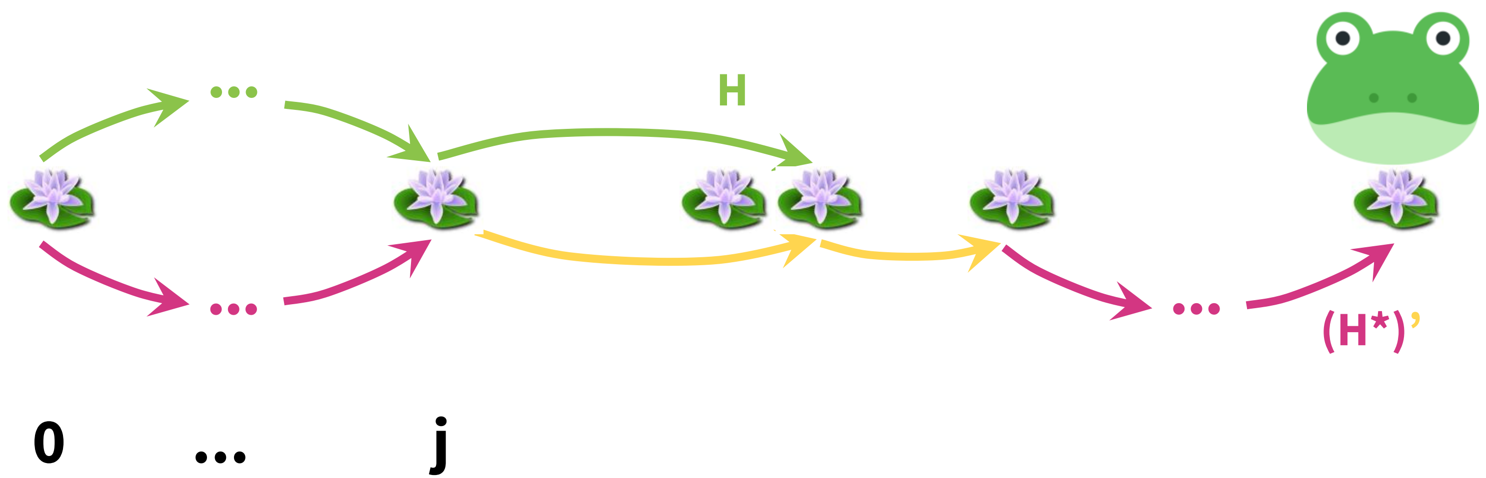

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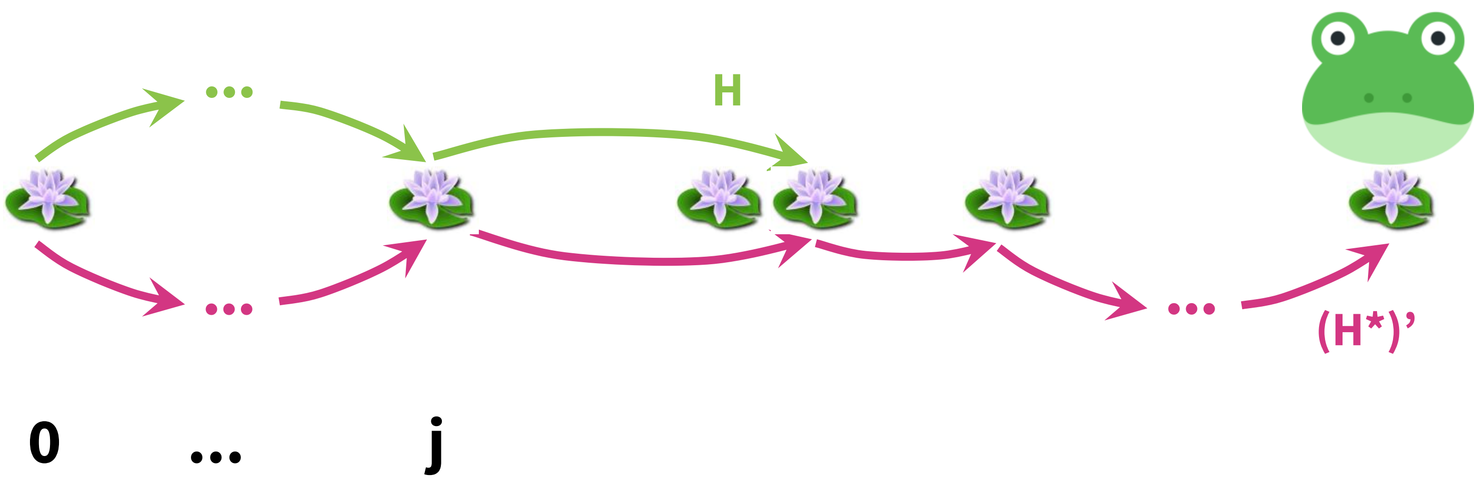

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Let H* be an optimal series of hops such that H*[0..j] = H[0..j]. Suppose we add k as H[j+1].

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For the inductive step, assume that after hop j has been added to H, there exists an optimal feasible series of hops H* such that H*[0..j] = H[0..j]. We'll prove that after hop j+1 has been added to H, there still exists an optimal series of hops H_{new}* such that H_{new}*[0..j+1] = H[0..j+1].

Let H* be an optimal series of hops such that H*[0..j] = H[0..j]. Suppose we add k as H[j+1]. Then k ≥ H*[j+1] since H*[j+1] ≤ r + H*[j] = r + H[j] and, by construction, k is the furthest lilypad such that k ≤ r + H[j].

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Theorem: `frog_hopping` produces an optimal solution.

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For the inductive step, assume that after hop j has been added to H, there exists an optimal feasible series of hops H* such that H*[0..j] = H[0..j]. We'll prove that after hop j+1 has been added to H, there still exists an optimal series of hops H_{new}* such that H_{new}*[0..j+1] = H[0..j+1].

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Consider (H*)' obtained from H* and setting H*[j+1] = k.

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For the inductive step, assume that after hop j has been added to H, there exists an optimal feasible series of hops H* such that H*[0..j] = H[0..j]. We'll prove that after hop j+1 has been added to H, there still exists an optimal series of hops H_{new}* such that H_{new}*[0..j+1] = H[0..j+1].

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Consider (H*)' obtained from H* and setting H*[j+1] = k.

This is still feasible since (H*)'[j+1] = k ≤ r + (H*)'[j] and (H*)'[j+2] = H*[j+2] ≤ r + H*[j+1] ≤ r + k = r + (H*)'[j].

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Theorem: `frog_hopping` produces an optimal solution.

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As a base case, we initialize H to `[0]` and all feasible hops H* must have H*[0] = 0.

For the inductive step, assume that after hop j has been added to H, there exists an optimal feasible series of hops H* such that H*[0..j] = H[0..j]. We'll prove that after hop j+1 has been added to H, there still exists an optimal series of hops H_{new}* such that H_{new}*[0..j+1] = H[0..j+1].

Let H* be an optimal series of hops such that H*[0..j] = H[0..j]. Suppose we add k as H[j+1]. Then k ≥ H*[j+1] since H*[j+1] ≤ r + H*[j] = r + H[j] and, by construction, k is the furthest lilypad such that k ≤ r + H[j].

Consider (H*)' obtained from H* and setting H*[j+1] = k.

This is still feasible since (H*)'[j+1] = k ≤ r + (H*)'[j] and (H*)'[j+2] = H*[j+2] ≤ r + H*[j+1] ≤ r + k = r + (H*)'[j].

Since H* and (H*)' are the same except at position j+1.

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Theorem: `frog_hopping` produces an optimal solution.

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As a base case, we initialize H to `[0]` and all feasible hops H* must have H*[0] = 0.

For the inductive step, assume that after hop j has been added to H, there exists an optimal feasible series of hops H* such that H*[0..j] = H[0..j]. We'll prove that after hop j+1 has been added to H, there still exists an optimal series of hops H_{new}* such that H_{new}*[0..j+1] = H[0..j+1].

Let H* be an optimal series of hops such that H*[0..j] = H[0..j]. Suppose we add k as H[j+1]. Then k ≥ H*[j+1] since H*[j+1] ≤ r + H*[j] = r + H[j] and, by construction, k is the furthest lilypad such that k ≤ r + H[j].

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This is still feasible since (H*)'[j+1] = k ≤ r + (H*)'[j] and (H*)'[j+2] = H*[j+2] ≤ r + H*[j+1] ≤ r + k = r + (H*)'[j].

Since H* and (H*)' are the same except at position j+1.

This is still optimal since (H*)' has the same number of hops as H*.

3 min break

Activity Selection

Planning Your Life

You have a list of activities $(s_1, e_1), (s_2, e_2), \ldots, (s_n, e_n)$ denoted by their start and end times.

All activates are equally attractive to you, and you want to maximize the number of activities you do.

Task: Choose the largest number of non-overlapping activities possible.

Greedy Stays Ahead

What are a few ways of picking activities greedily? 🤔

- **Be impulsive:** choose activities in ascending order of start times.
- **Avoid commitment:** choose activities in ascending order of length.
- **Finish fast:** Choose activities in ascending order of end times.

Only the third one seems to work.

Activity Selection

```
algorithm activity_selection(activities):
  sort activities into ascending order by end time
  U = set of activities
  while U not empty:
    choose any activity with the earliest finishing time
    add that activity to S
    remove other activities that overlap with it from U
  return U
```

Runtime: $O(n^2)$

Activity Selection

We need to prove two properties about the algorithm to guarantee correctness.

(1) **Legality.** The algorithm finds a legal schedule of activities (i.e. it doesn't "schedule conflicting activities").

(2) **Optimality.** The algorithm finds an optimal schedule of activities (i.e. there isn't a better schedule available).

Activity Selection

Lemma: The schedule produced by `activity_selection` is a legal schedule.

Intuition: Use induction to show that at each step, the set U only contains activities that don't conflict with activities selected from S.

Activity Selection

We need to prove two properties about the algorithm to guarantee correctness.

(1) **Legality.** The algorithm finds a legal schedule of activities (i.e. it doesn't "schedule conflicting activities"). 👌

(2) **Optimality.** The algorithm finds an optimal schedule of activities (i.e. there isn't a better schedule available). 🤔

Activity Selection

To prove that the schedule S produced by the algorithm is optimal, we will use another "greedy stays ahead" argument.

(1) Find intermediate values that evaluate the solution produced by any algorithm, including the greedy one. **Here, the end_time of the kth activity chosen.**

(2) Show the greedy algorithm produces values at least as good as any solution's (using induction).

(3) Prove that since the greedy algorithm produces values at least as good as any solution's, it must be optimal (using direct proof or proof by contradiction).

Activity Selection

How might we prove that `activity_selection` finds an optimal schedule of activities?

- Let's introduce notation to talk with greater precision about the algorithm …
- Let S be the schedule produced by our algorithm and S^* be **an arbitrary** (not necessarily **the only**) optimal schedule. Then $|S|$ and $|S^*|$ denote the number of activities in S and S^*, respectively.
- Note that $|S| \le |S^*|$. Why? 🤔
- We want to prove that $|S| = |S^*|$. How?

Intuition: Consider an arbitrary optimal schedule S^*, then show that our greedy algorithm produces a schedule S no worse than S^*.

Activity Selection

Let f(i, S) denote the time that the ith activity finishes in schedule S.

Lemma: For any $1 \leq i \leq |S|$, we have $f(i, S) \leq f(i, S^*)$.

- i.e. After scheduling i activities according to the greedy algorithm, you will be at most as late as if you scheduled i activities according to an optimal solution.

Let's formalize this using induction!

Proving Optimality

Lemma: For all $1 \leq i \leq |S|$, we have $f(i, S) \leq f(i, S^*)$.

Proof: We proceed by induction.

As a base case, the first activity the greedy algorithm selects must be an activity that ends no later than any other activity, so $f(1, S) \leq f(1, S^*)$.

For the inductive step, assume that the claim holds for some $1 \leq i < |S|$. We will prove the claims holds for $i + 1$. Since $f(i, S) \leq f(i, S^*)$, the ith activity in S finishes before the ith activity in S*. Since the (i+1)st activity in S* must start after the ith activity in S* ends, the (i+1)st activity in S* must start after the ith activity in S ends.

Therefore, the (i+1)st activity in S* must be in U when the greedy algorithm selects the activity in U with the lowest end time, we have $f(i+1, S) \leq f(i+1, S^*)$, completing the induction.

Proving Optimality

Theorem: `activity_selection` produces an optimal solution.

Proof: Since S^* is optimal, we have $|S| \leq |S^*|$. We will prove $|S| = |S^*|$.

We proceed by contradiction. Suppose that $|S| < |S^*|$. Let $k = |S|$. By our lemma, we know $f(k, S) \leq f(k, S^*)$, so the kth activity in S finishes no later than the kth activity in S^*. Since $|S| < |S^*|$, there is a (k+1)st activity in S^*, and its start time must be after $f(k, S^*)$ and therefore after $f(k, S)$. Thus after the greedy algorithm added its kth activity to S, the (k+1)st activity from S^* would still belong to U. But the greedy algorithm ended after k activities, so U must have been empty.

We have reached a contradiction, so our assumption was wrong and $|S^*| = |S|$, so the greedy algorithm produces an optimal solution. ■

In Frog Hopping, we proved this step using a direct proof. Here, we use a proof by contradiction. You should be able to structure the direct proof here too.

Activity Selection

We need to prove two properties about the algorithm to guarantee correctness.

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