Intractable Problems II

Summer 2017 • Lecture 08/10

A Few Notes

Homework 6

- Due 8/13 at 11:59 p.m. on Gradescope.

Final Logistics

Final

Will occur on 8/18 at 8:30-11:30 a.m.

Alternate Final

Will occur on 8/17 at 3:30-6:30 p.m.

Exam special cases

I sent confirmation emails on Tuesday.

Practice exam

Sorry we haven’t released this yet. Expect it later today!

The Rest of the Quarter

Lectures 1-11 covered the bulk of the material, congrats!

- This material will be emphasized on the final (~90% of points).

Lectures 12-14 will cover additional topics.

- Intractable problems, approximation algorithms, amortized analysis

Outline for Today

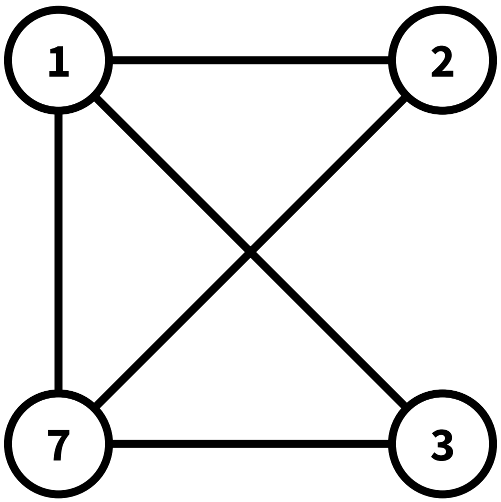
- Approximation algorithms
  - Vertex Cover
  - Set Cover
  - 0/1 Knapsack

Vertex Cover

Vertex Cover

Task Given an undirected graph G = (V, E), and a cost function on the vertices, find a minimum cost vertex cover, i.e. a set of V' ⊆ V such that every edge has at least one endpoint incident at V'.

Is this easier or harder than edge cover? 🤔



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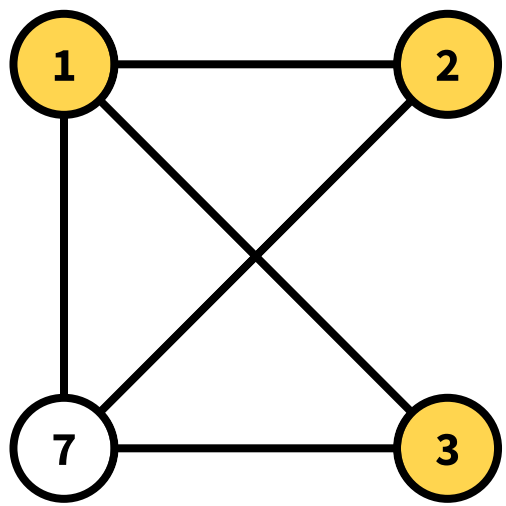

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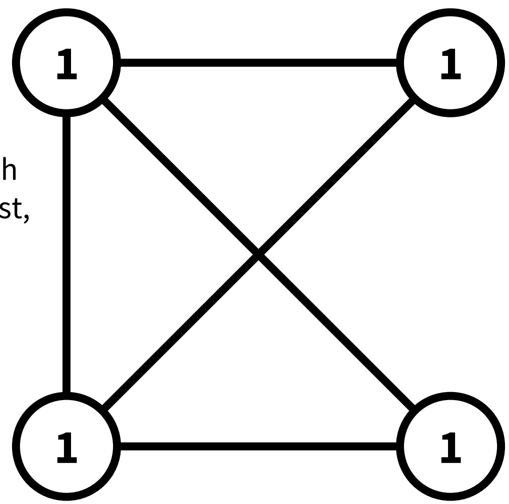

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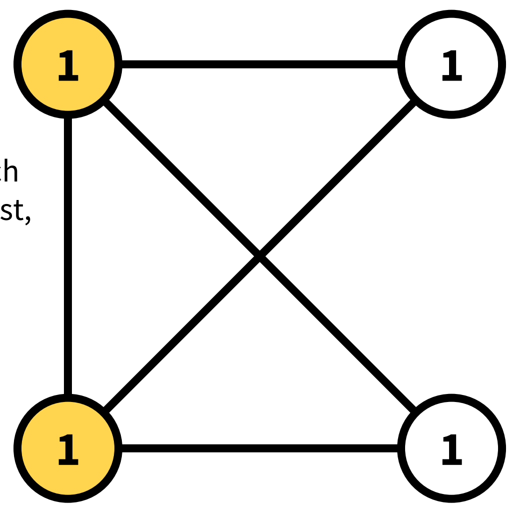

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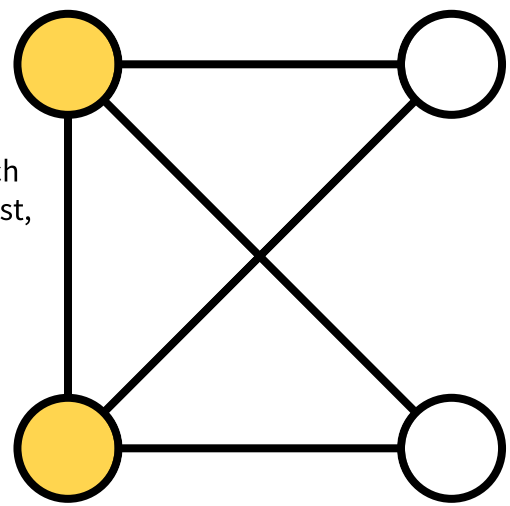
This will be represented with blank vertices.



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The cardinality vertex cover problem is **NP**-hard. How might we solve it??

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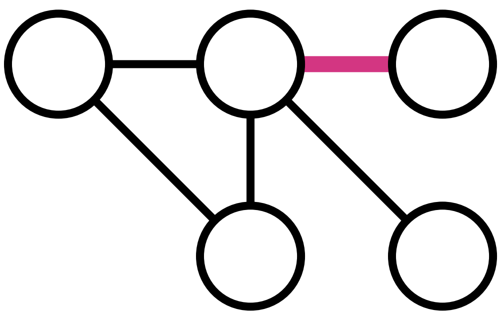
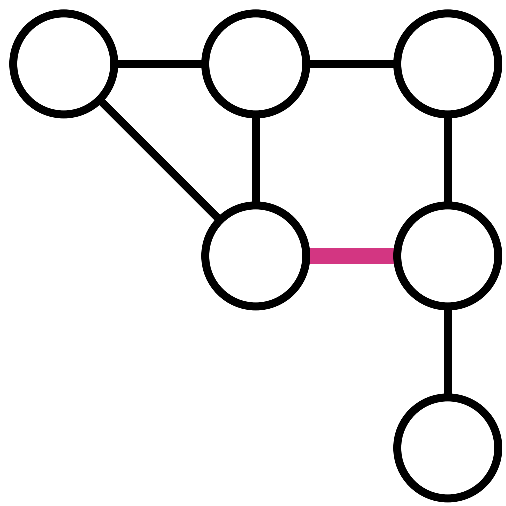
Let's introduce some useful terminology …

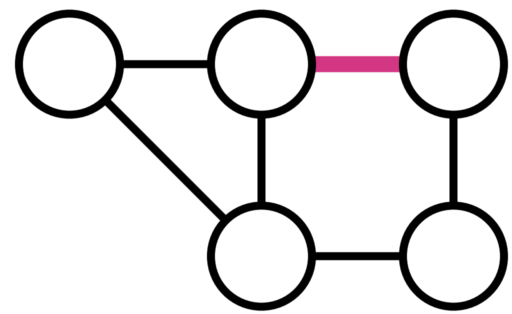
A **matching** is a set of edges such that no two edges share a common vertex.

A **maximal matching** is a matching such that if any if any edge is added to the matching, it's no longer a matching.

A **maximum matching** is a maximal matching that contains the largest possible
number of edges.







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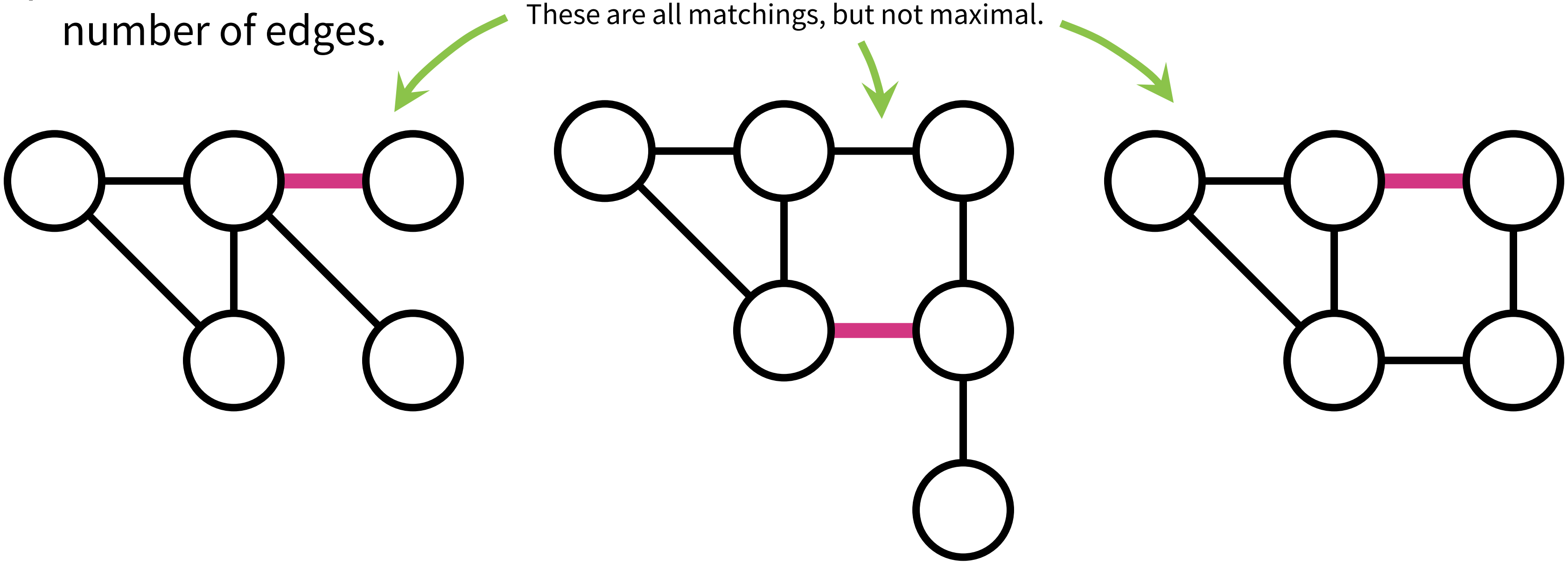

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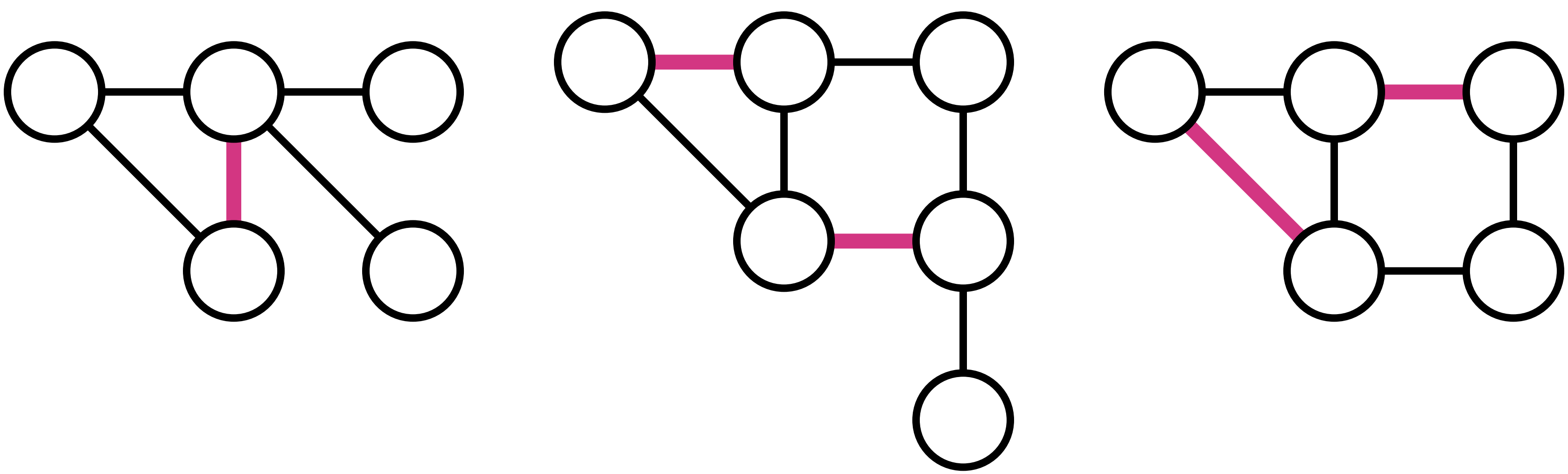

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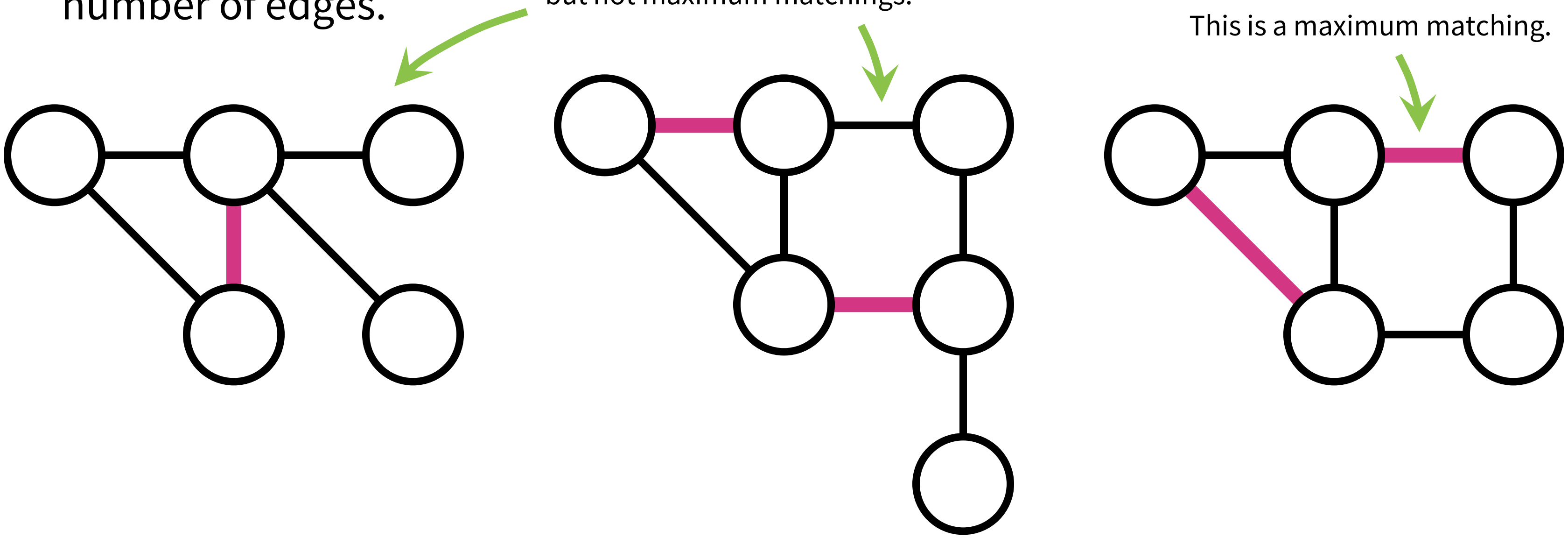
This is a maximum matching.



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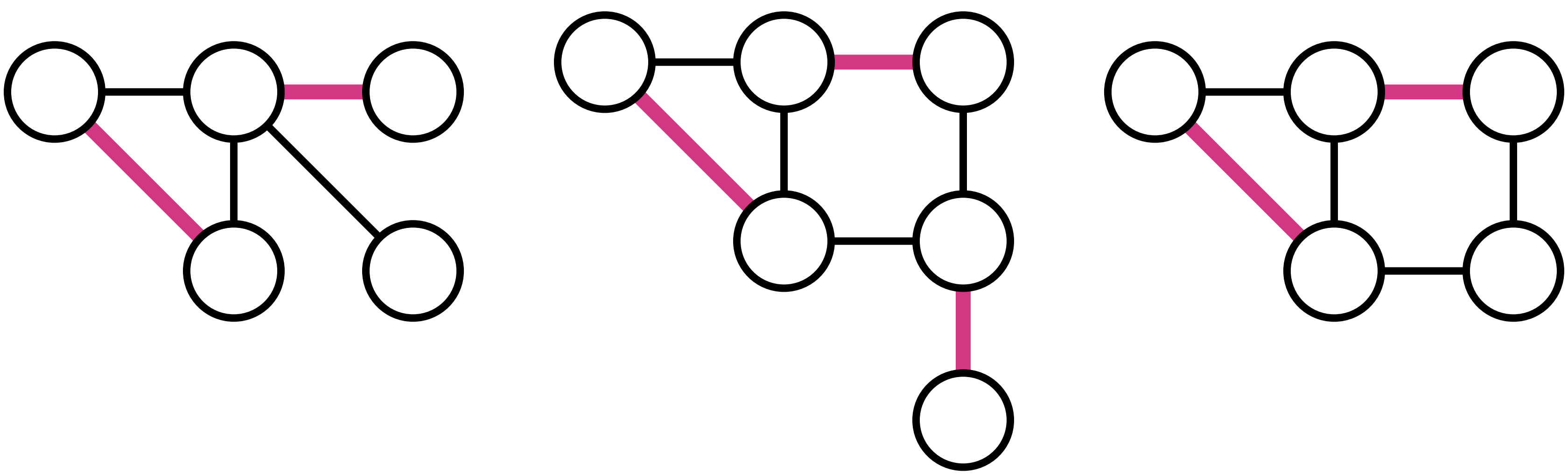

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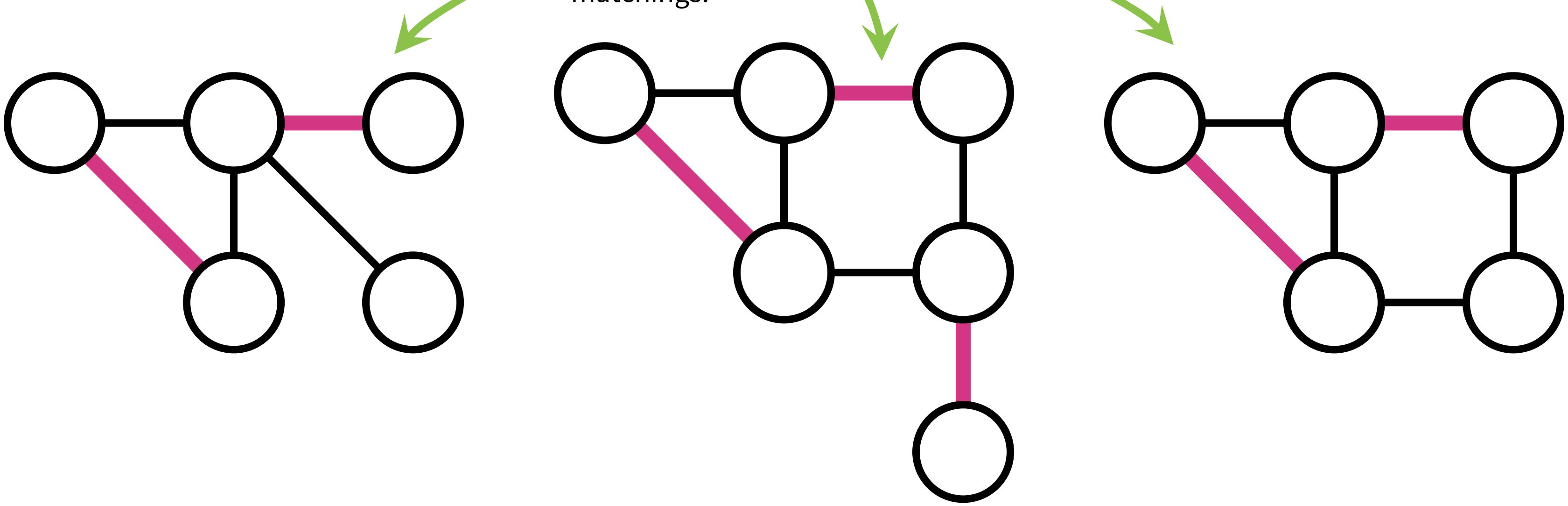


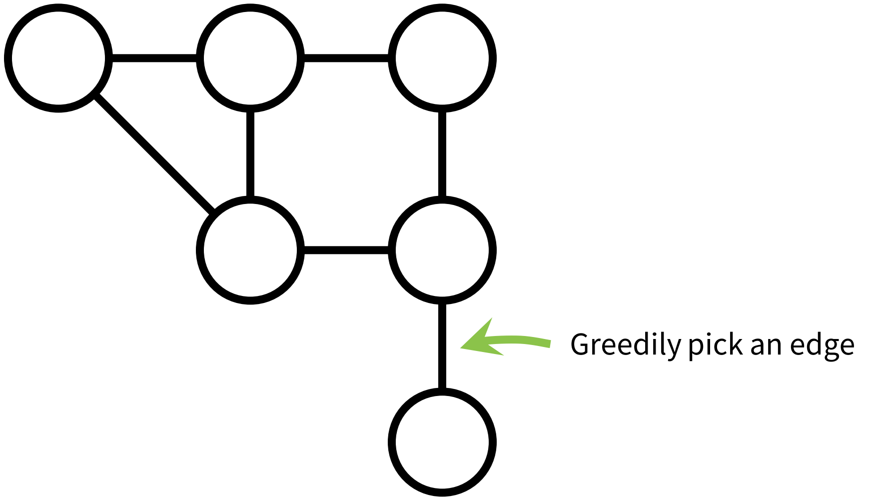
Vertex Cover

Let's attempt to design a greedy solution first.

Recall, greedy solutions are often the most natural algorithms to design, but sometimes it's difficult to prove their correctness.

Intuition Greedily pick an edge, add its endpoints to the vertex cover, and remove edges adjacent to the endpoints.

i.e. Find a maximal matching in G and output the set of matched vertices.



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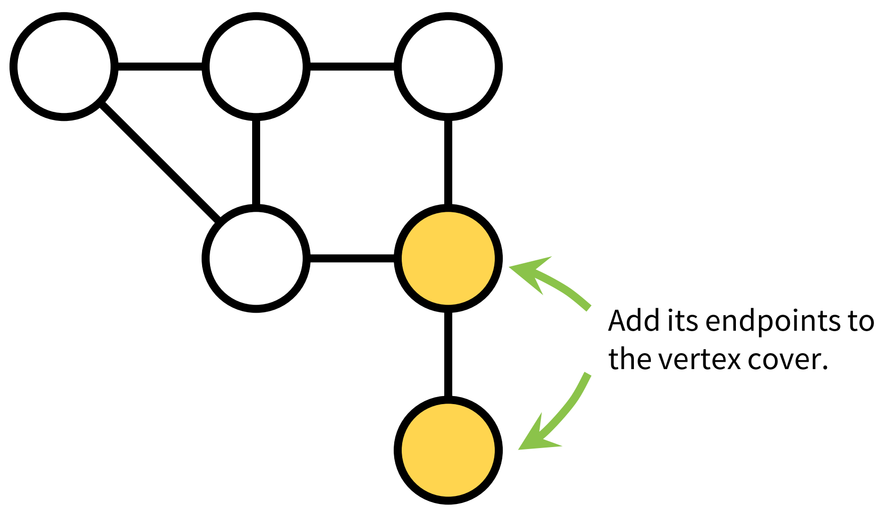

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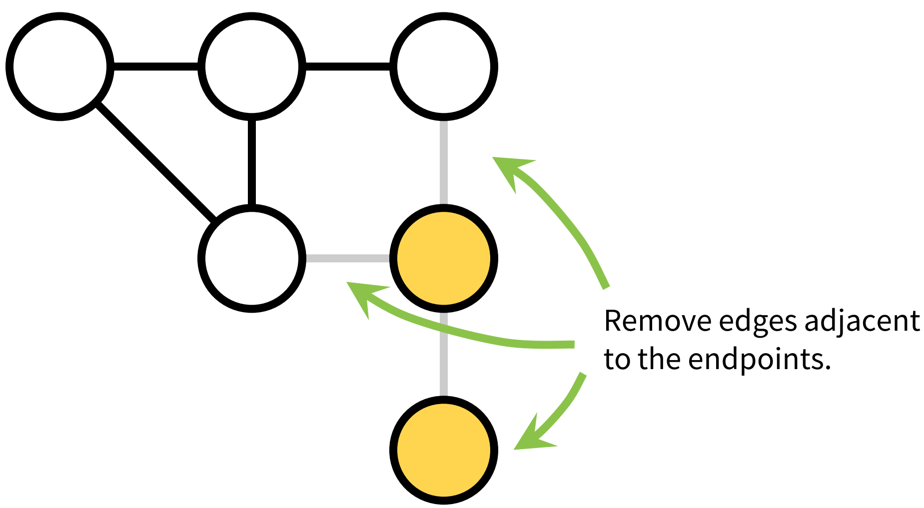

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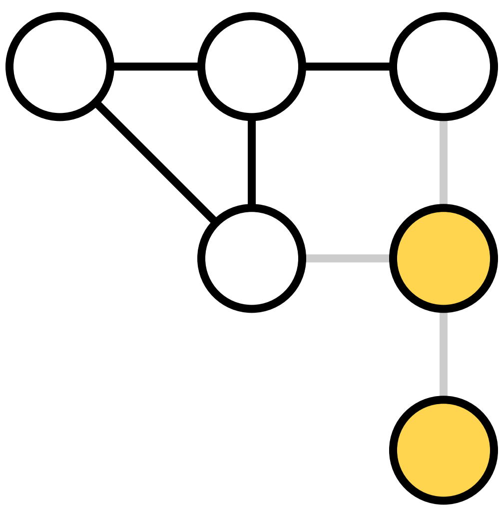

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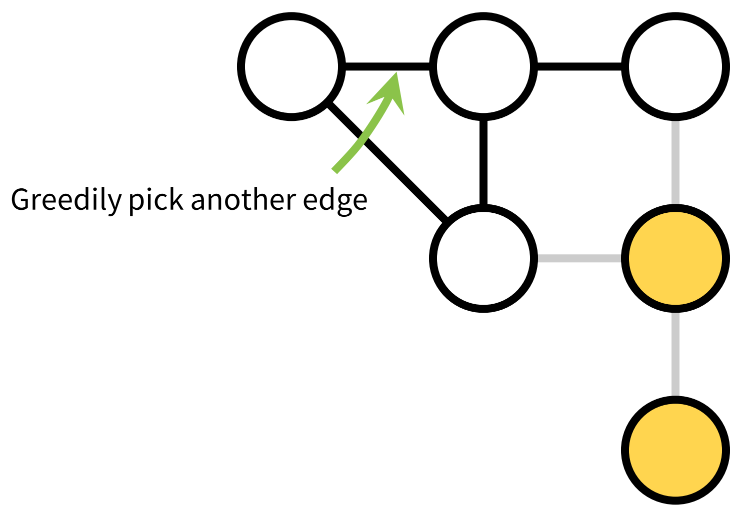

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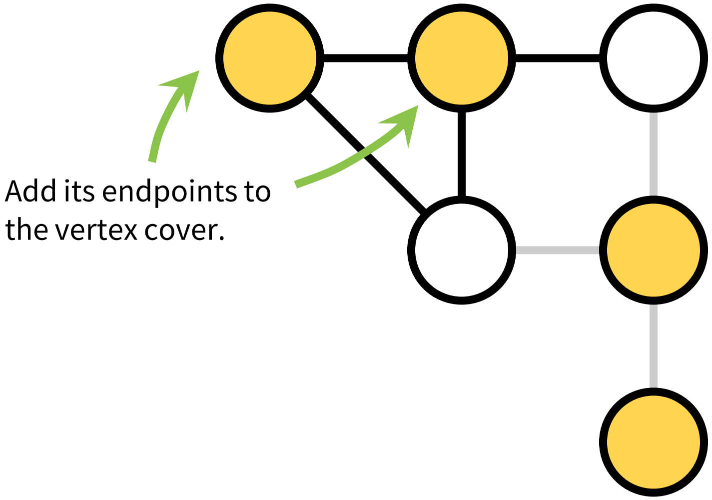

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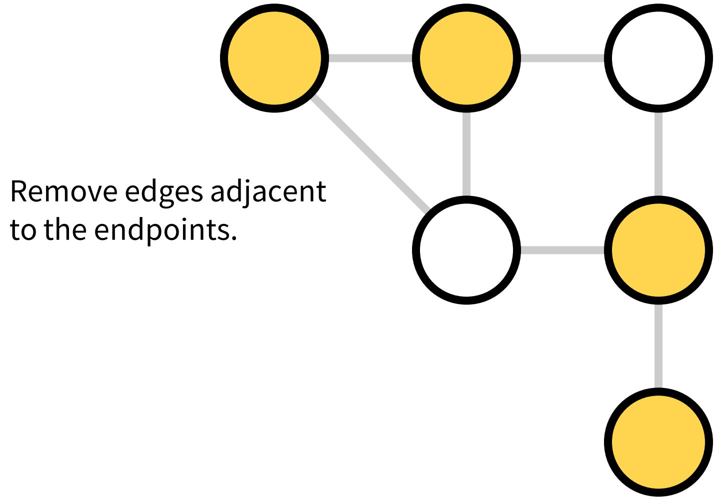

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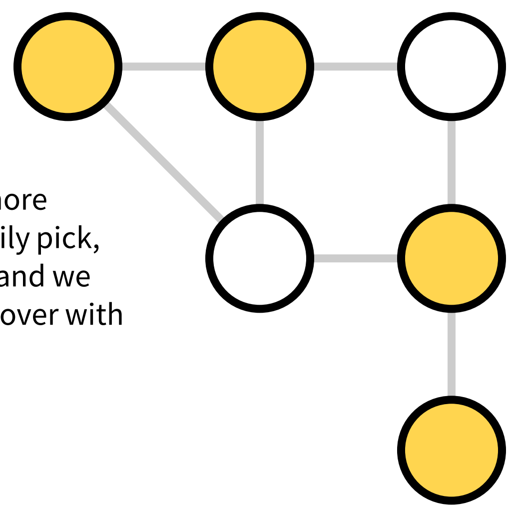
There are no more edges to greedily pick, so we're done and we have a vertex cover with cardinality 4!



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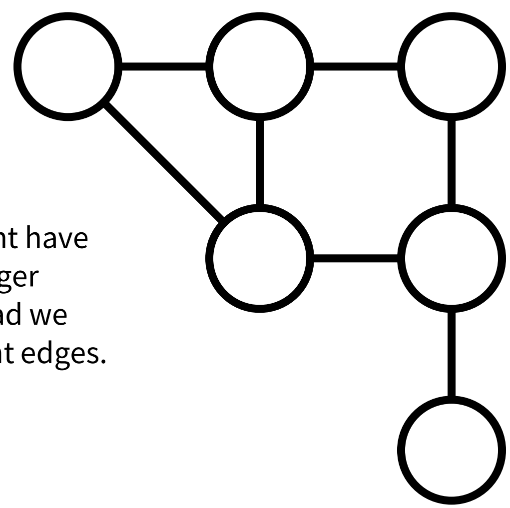
Obviously, this algorithm might have produced a larger vertex cover had we picked different edges.



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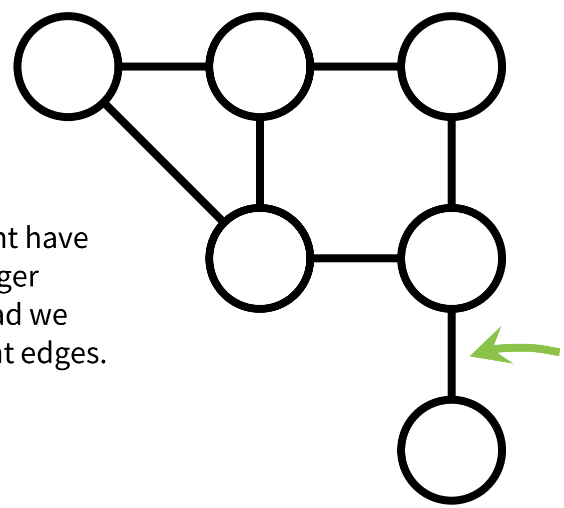

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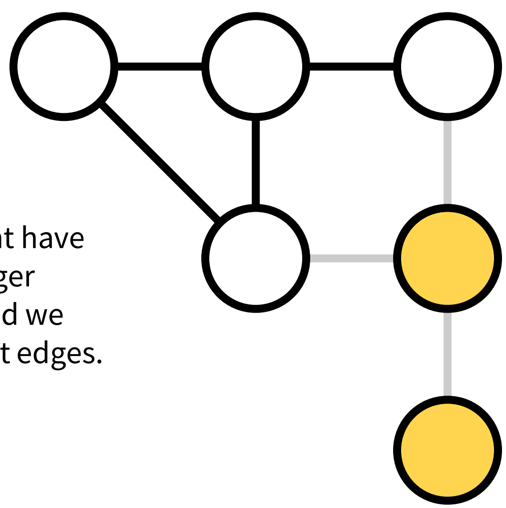

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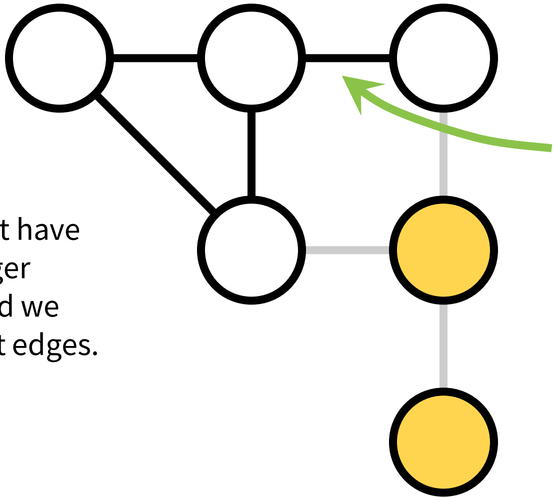

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Greedily pick another edge.

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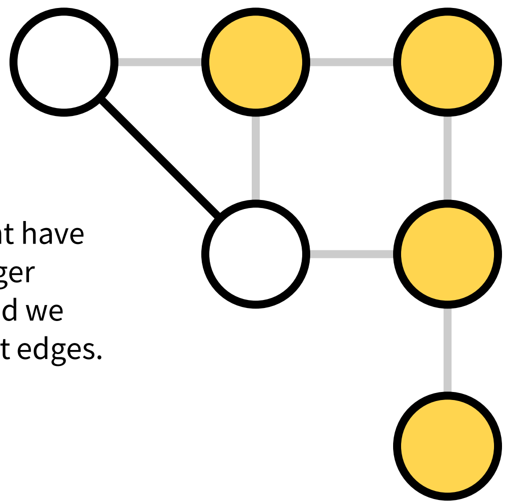

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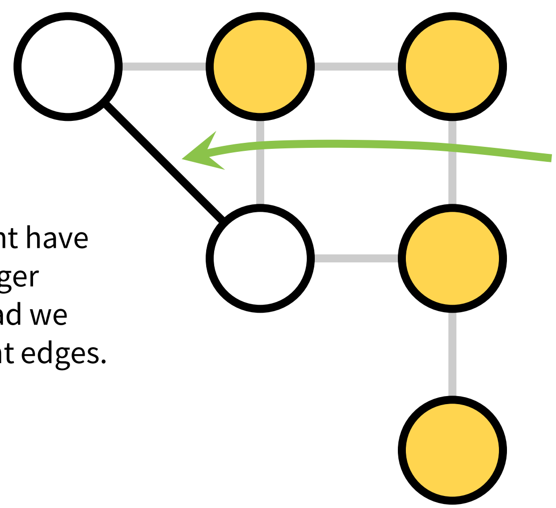

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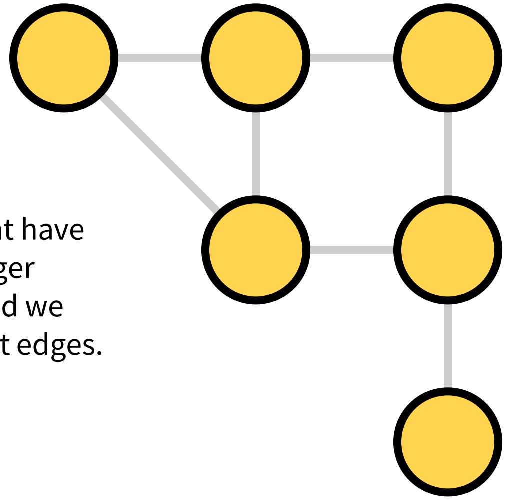

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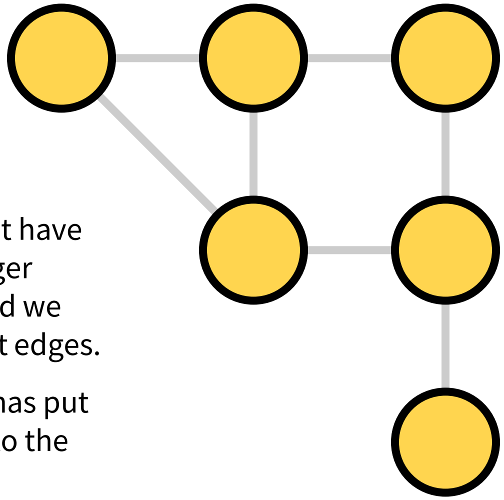

Obviously, this algorithm might have produced a larger vertex cover had we picked different edges.

The algorithm has put every vertex into the vertex cover!

Vertex Cover

```
algorithm find_vertex_cover(G):
  matching, vc = find_maximal_matching(G)
  return vc

algorithm find_maximal_matching(G):
  matching = {}  # a set of edges
  matched_vertices = {}  # a set of vertices
  ce = {}  # stands for "covered edges"
  while ce ≠ G.E:
    e = pick an edge from G.E - ce at random
    add e to matching
    add e.v1 and e.v2 to matched_vertices
    add edges adjacent to v1, v2 to ce
  return matching, matched_vertices
```

Runtime: $O(|V|+|E|)$

Vertex Cover

(1) How do we establish an approximation guarantee for our algorithm?

Theorem: `find_vertex_cover` is a factor 2-approximation algorithm for the cardinality vertex cover problem.

Proof:

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Let OPT be the optimal cardinality vertex cover in G. Notice that the size of a maximal matching M in G provides a lower bound for OPT since any vertex cover has to pick at least one endpoint of each matched edge. Thus, $|M| \leq OPT$.

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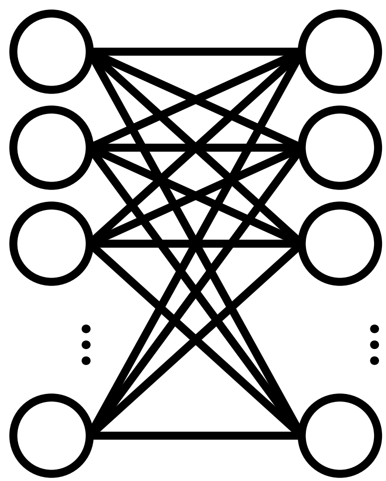
The algorithm picks a cover with cardinality $2 \cdot |M|$, which is at most $2 \cdot OPT$. ∎

Vertex Cover

(2) Can the approximation guarantee of `find_vertex_cover` be improved by a better analysis?

No, our analysis is tight. Consider an infinite complete bipartite graph, called a **tight example** since the approximation guarantee is tight

i.e. `find_vertex_cover` produces a solution twice the optimal.



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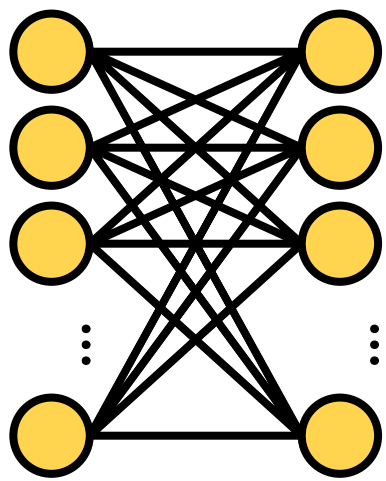

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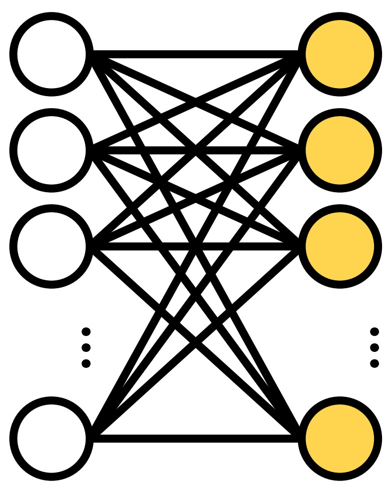

`find_vertex_cover` will pick all 2n vertices.

But picking one side of the bipartition gives a cover of size n.

Vertex Cover

(3) Is there some other lower bounding method that can lead to an improved approximation guarantee for vertex cover?

This is an open problem!

Set Cover

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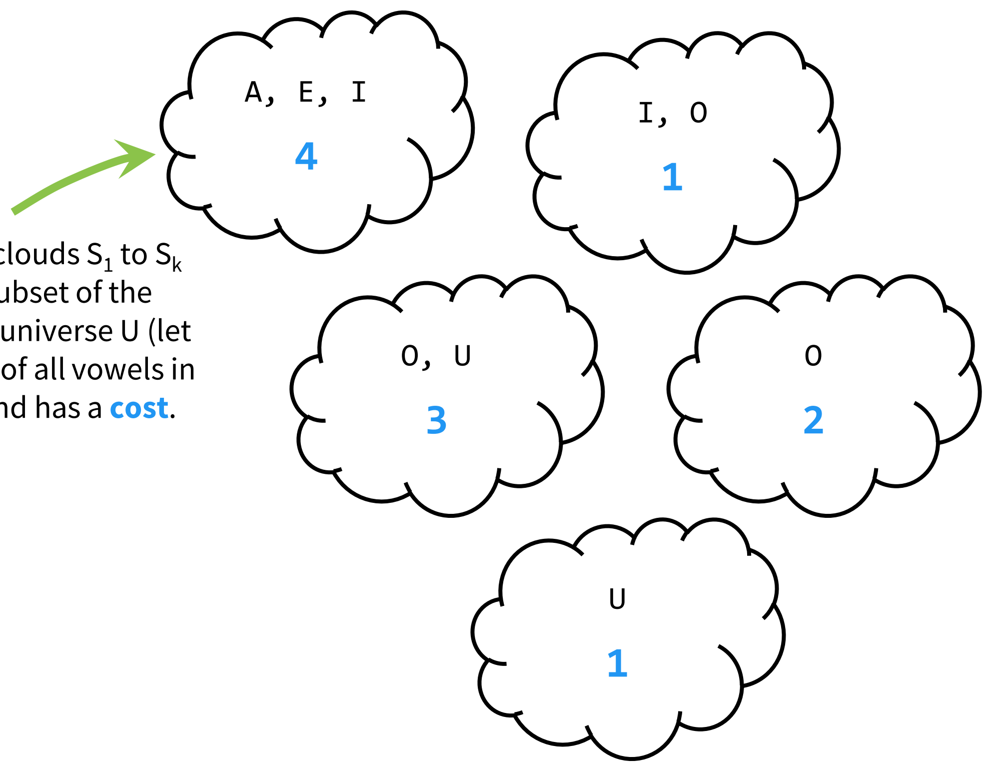
Task Given a universe U of n elements, a collection of subsets $S = \{S_1, \ldots, S_k\}$ of U each of which has a cost, find a min cost subcollection of S that covers all elements of U.



Each of the clouds S_1 to S_k contains a subset of the elements in universe U (let U be the list of all vowels in this case), and has a **cost**.

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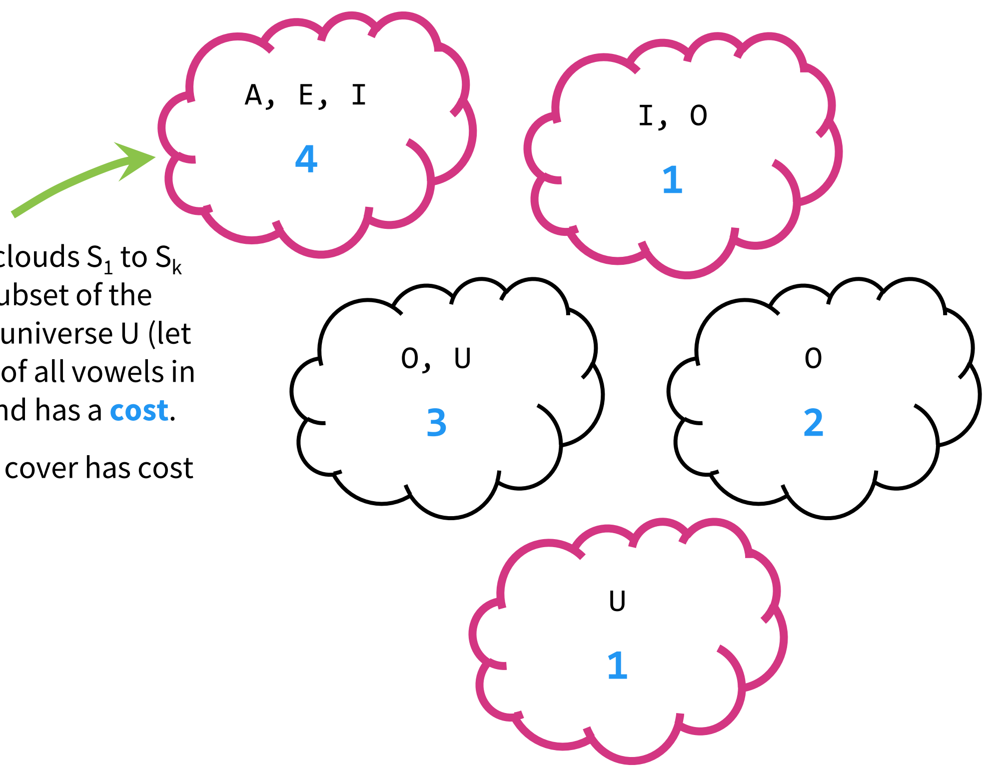

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The optimal cover has cost $4 + 1 + 1 = 6$.

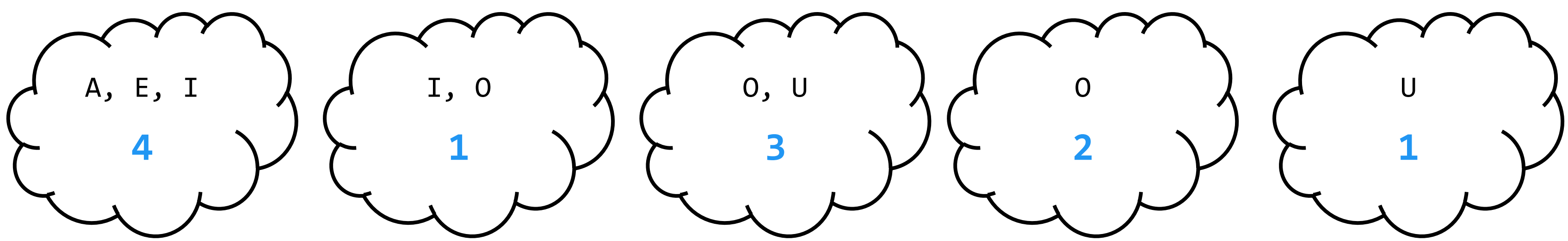
Set Cover

Let's attempt to design a greedy solution first.

Recall, greedy solutions are often the most natural algorithms to design, but sometimes it's difficult to prove their correctness.

Intuition Iteratively pick the most "cost-effective" set and remove the covered elements until all elements are covered.

Let the **cost-effectiveness score** of a set S_i be the average cost at which It covers new elements i.e. $\text{cost}(S_i) / |S_i - C|$ where C is the set of elements covered already.



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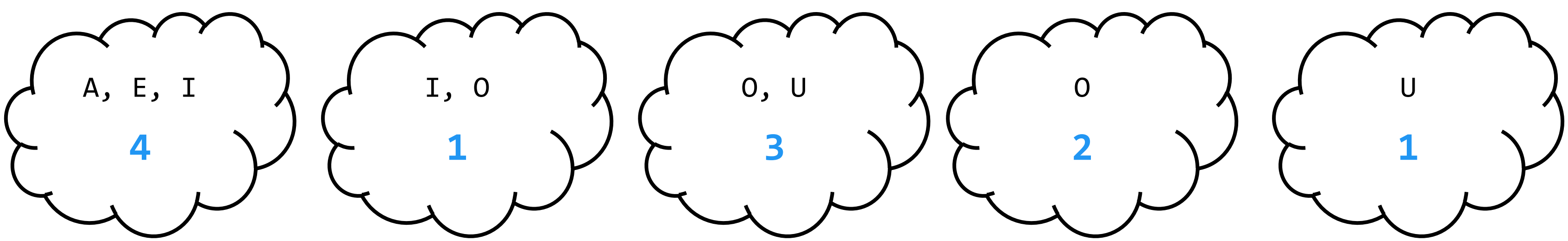

$4 / |\{A, E, I\}| = 4/3$ $1 / |\{I, O\}| = 1/2$ $3 / |\{O, U\}| = 3/2$ $2 / |\{O\}| = 2$ $1 / |\{U\}| = 1$

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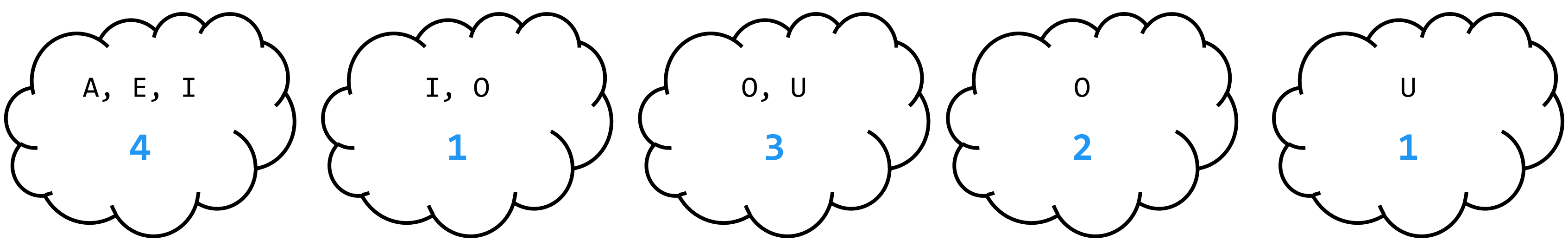

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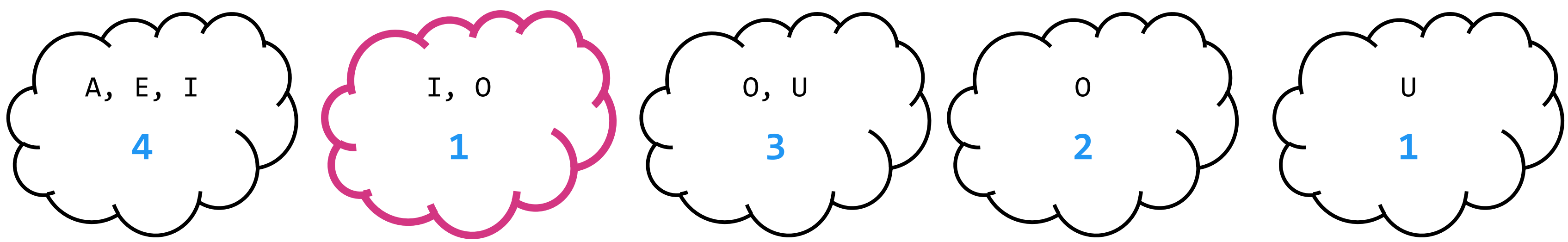

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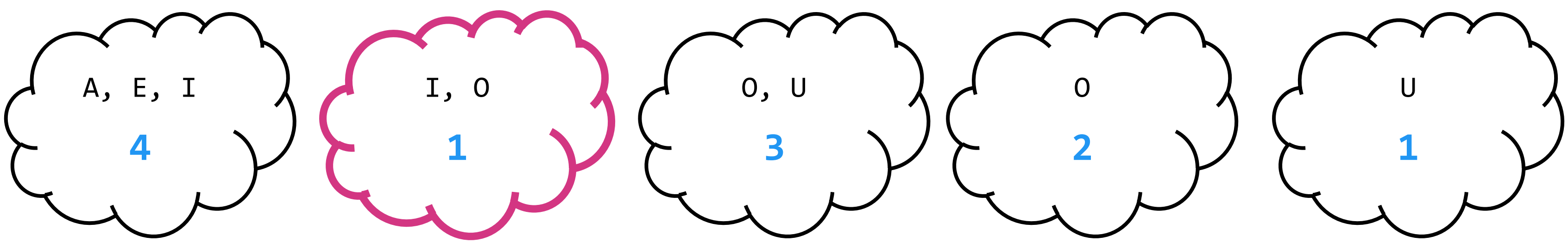

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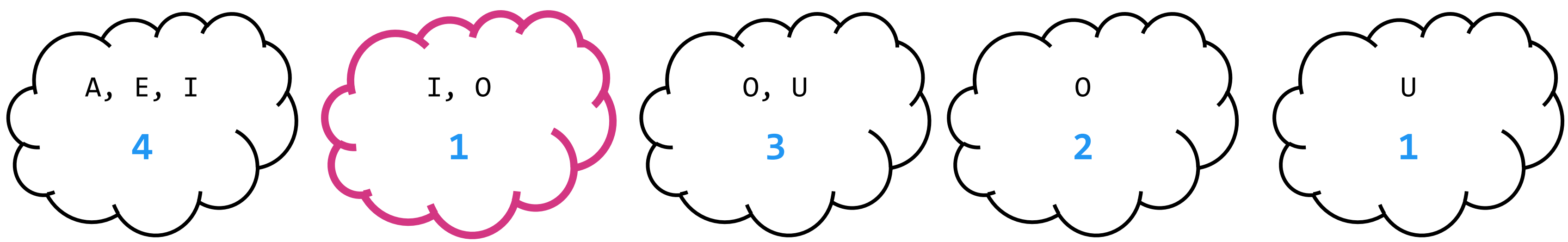

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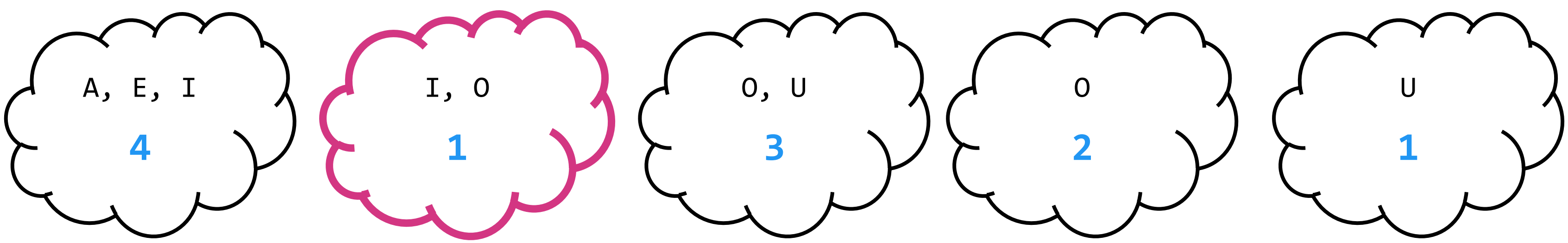

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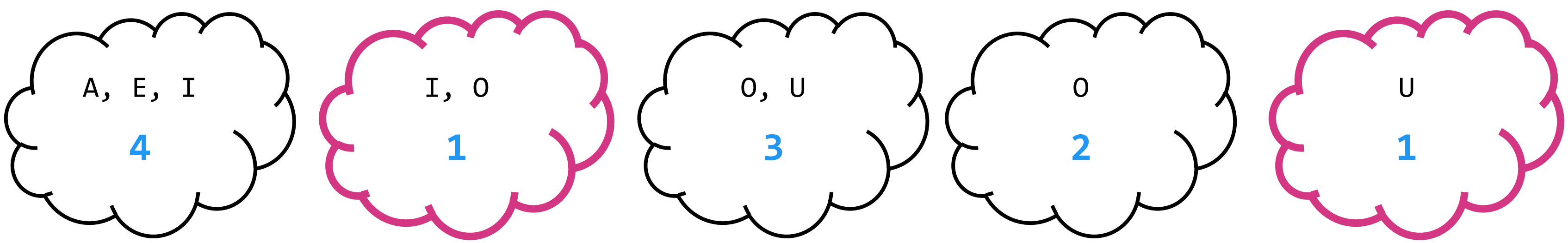

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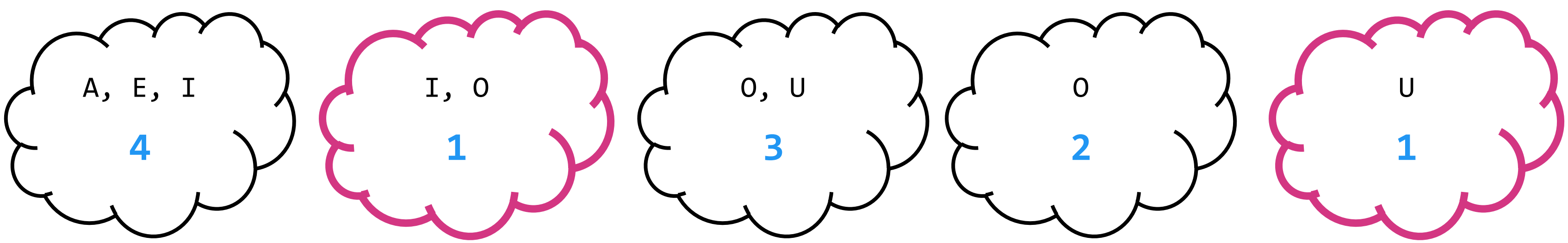

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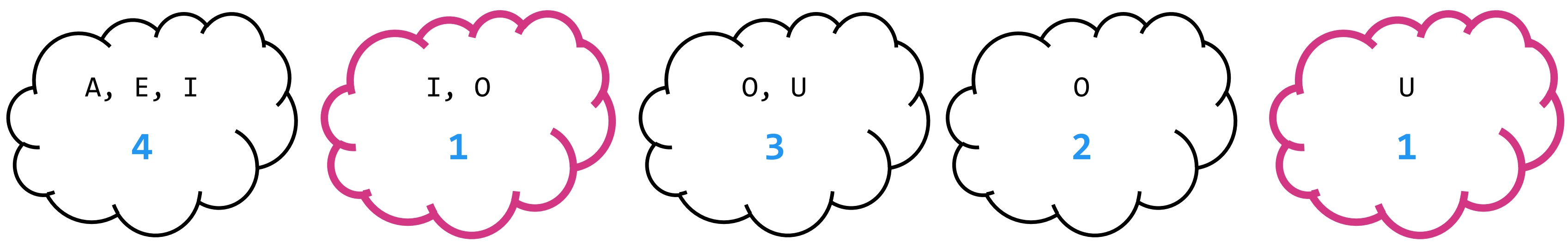

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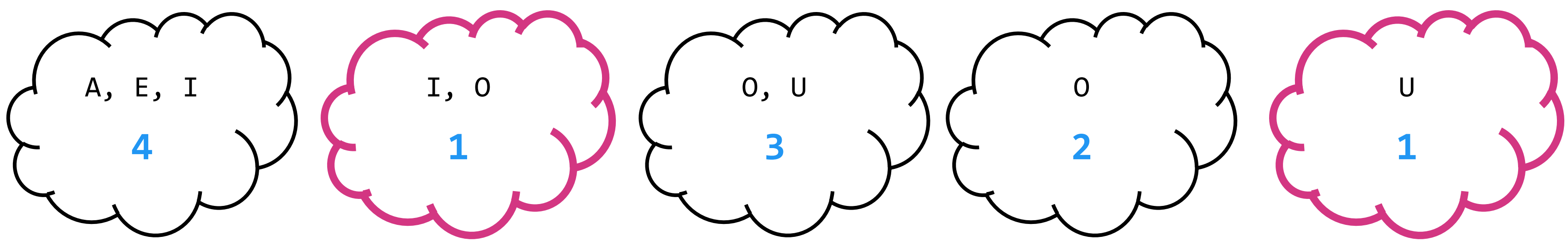

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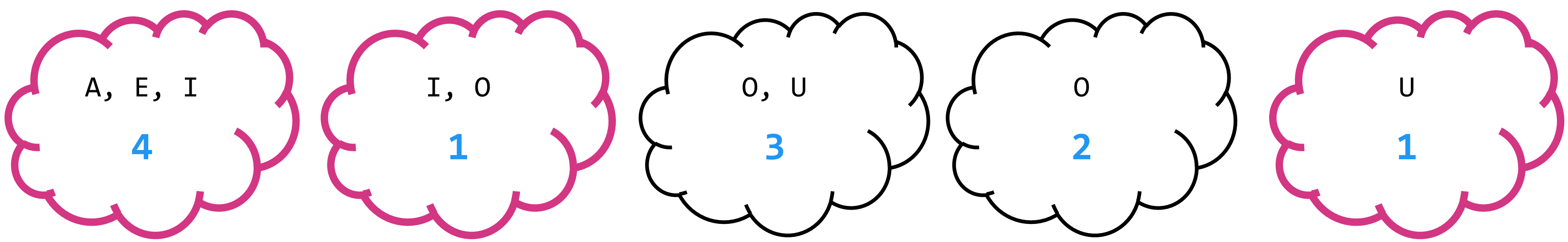

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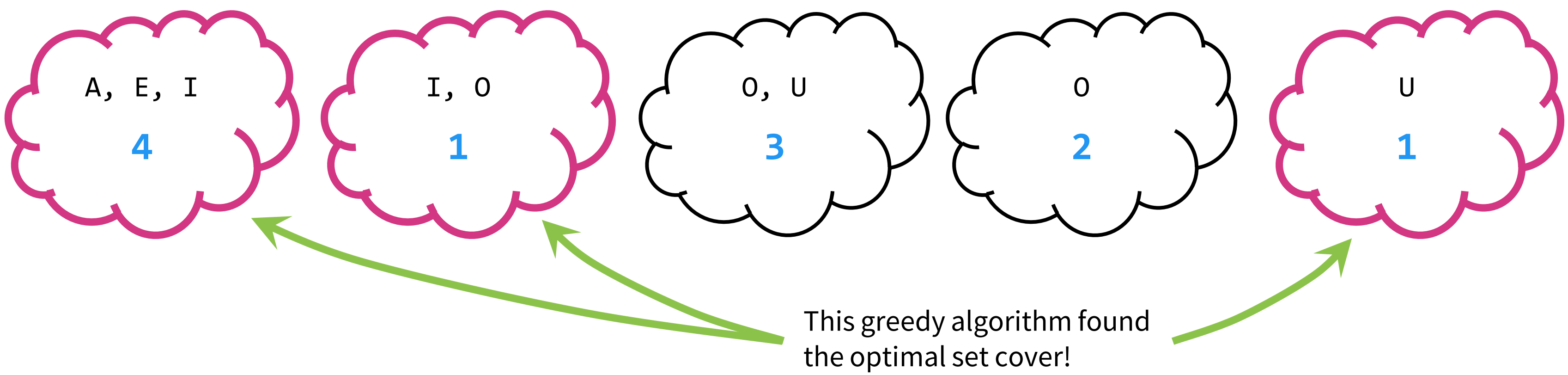

Set Cover

```
algorithm find_set_cover(S):
  C = {}  # stands for "covered"
  sc = {}
  prices = dict()
  while C ≠ U:
    # α is the cost effectiveness of the most
    # cost-effective set, S_i
    S_i, α = get_most_cost_effective(S, C)
    sc.add(S_i)
    prices[e] = α for e in S_i - C
    C = C ∪ S_i
  return sc
```

i.e. The elements in S_i that haven't been covered by a previous set.

Runtime: $O(|S|^2)$

We can improve this runtime by using fancy data structures.

Set Cover

(1) How do we establish an approximation guarantee for our algorithm?

Number the elements in U in the order in which they were covered by the algorithm, resolving ties arbitrarily. Let $e_1, \ldots, e_n$ be this numbering. Note $n = |U|$.

Lemma: For each $k \in \{1, \ldots, n\}$, $\text{price}[e_k] \leq \text{OPT} / (n - k + 1)$.

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In the iteration in which element e_k was covered, C^C contained at least $n - k + 1$ elements. Since e_k was covered by the most cost-effective set in this iteration, it follows that $\text{price}[e_k] \leq \text{OPT}/|C^C| \leq \text{OPT} / (n - k + 1)$. ∎

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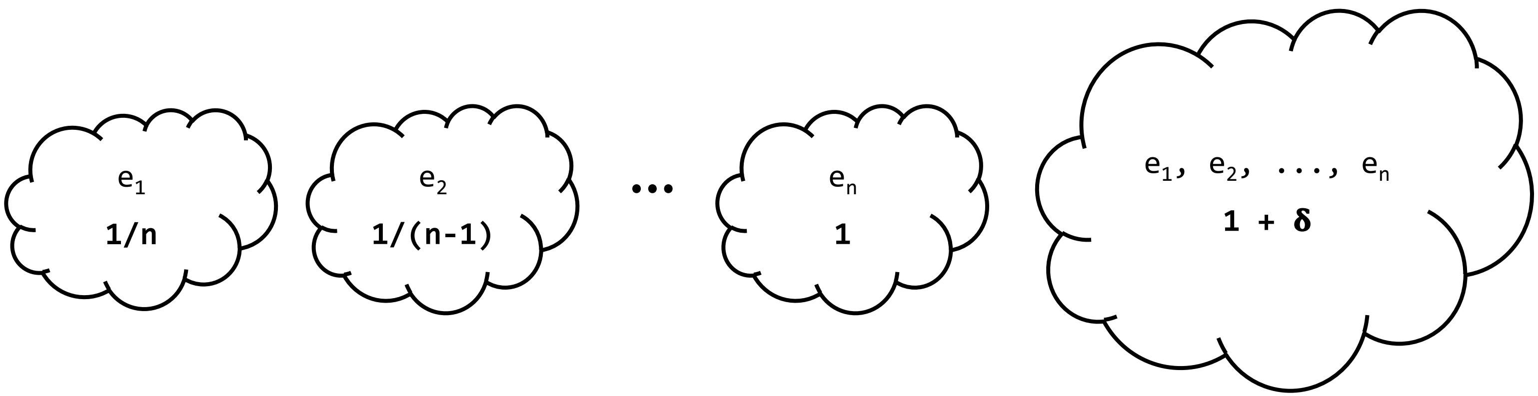
By the lemma, $\sum_k \text{price}[e_k] \leq (1 + 1/2 + \ldots + 1/n) \cdot \text{OPT} \leq \log(n) \cdot \text{OPT}$. ∎

Set Cover

(2) Can the approximation guarantee of `find_set_cover` be improved by a better analysis?

No. Let's find a **tight example** to convince ourselves that our approximation guarantee is tight i.e. `find_set_cover` produces a solution log(n) times the optimal.

Consider a set of singleton sets and one exhaustive set, each with specific costs.



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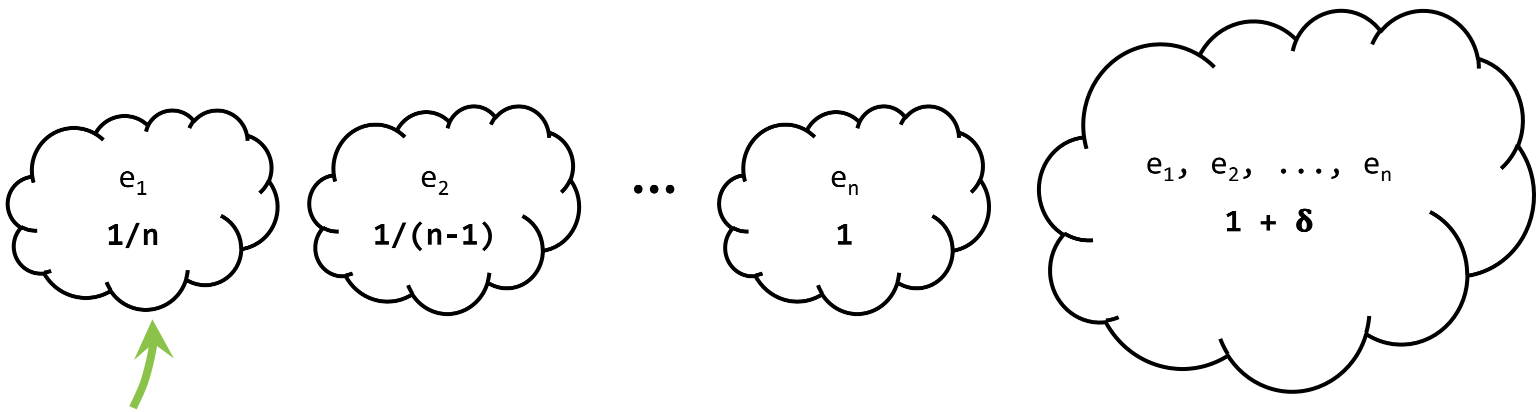

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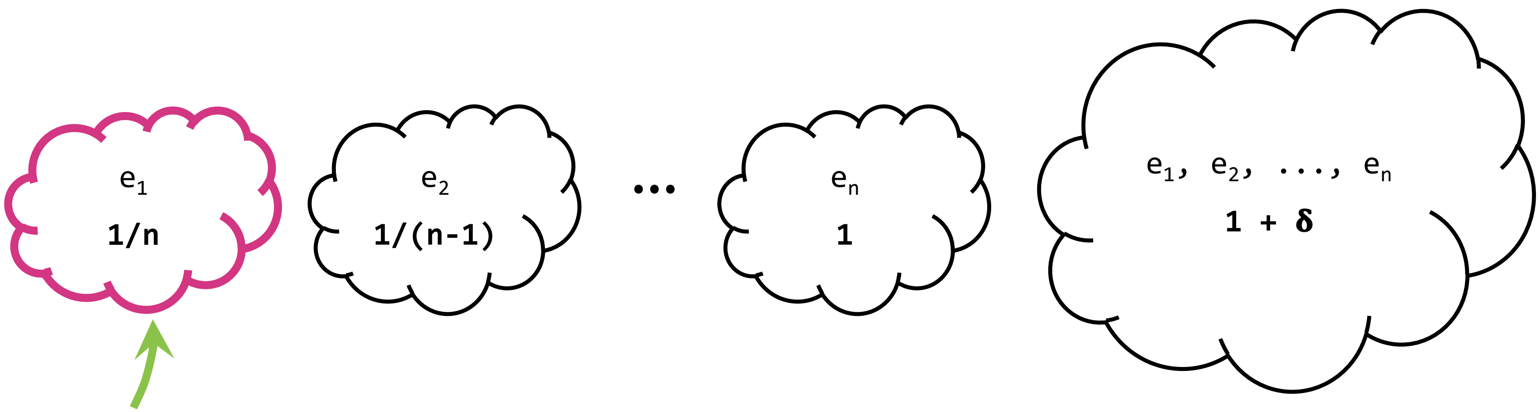

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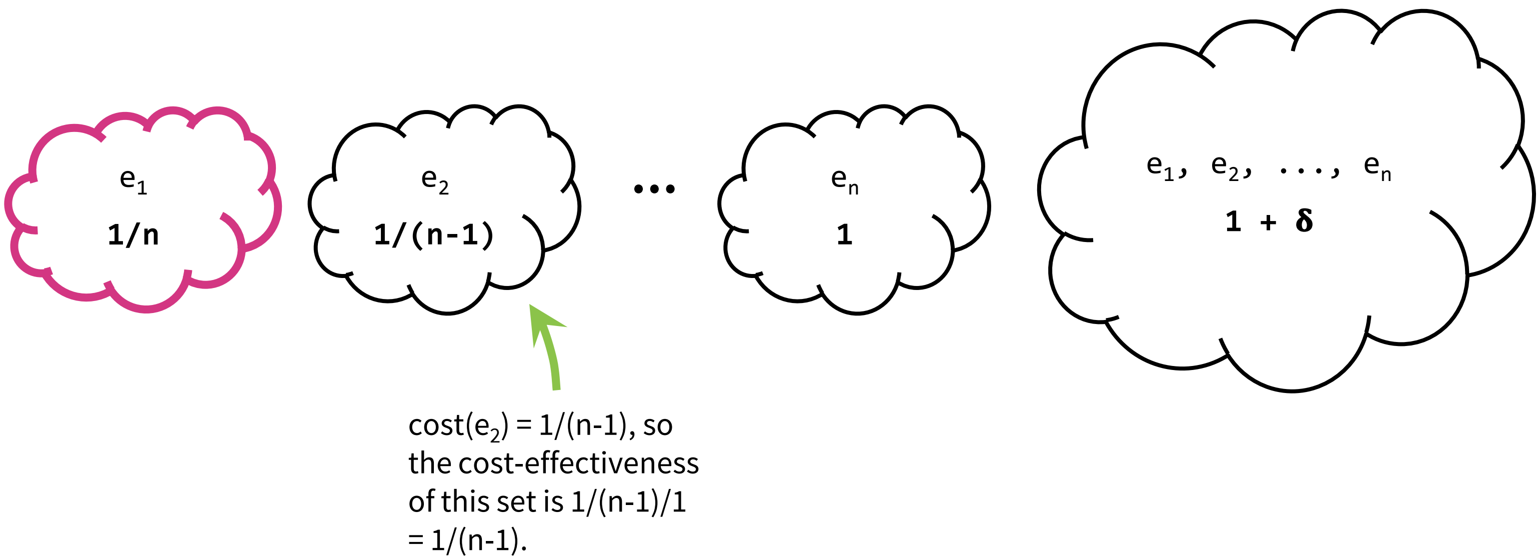

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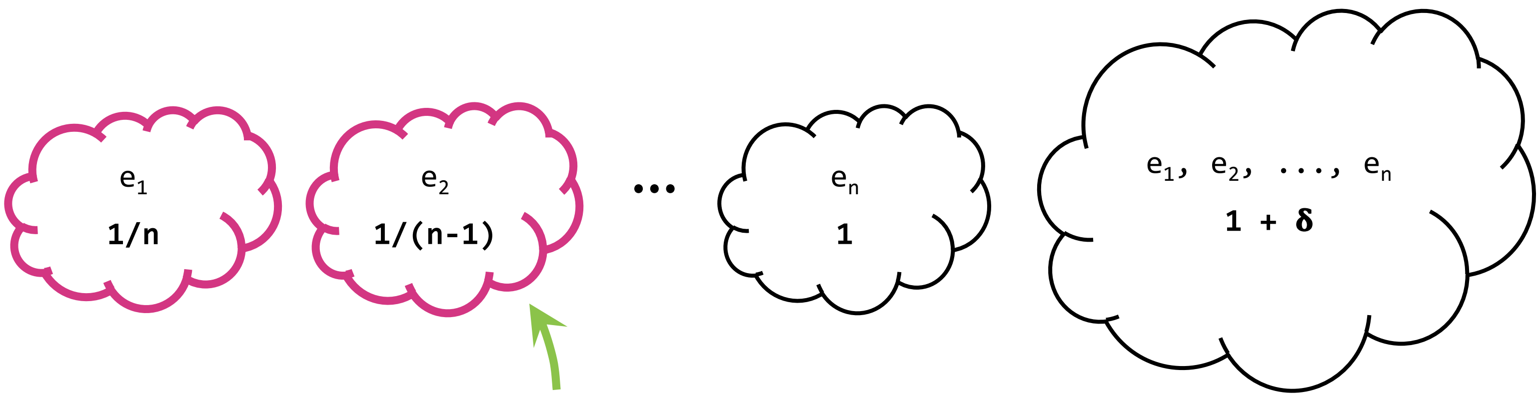

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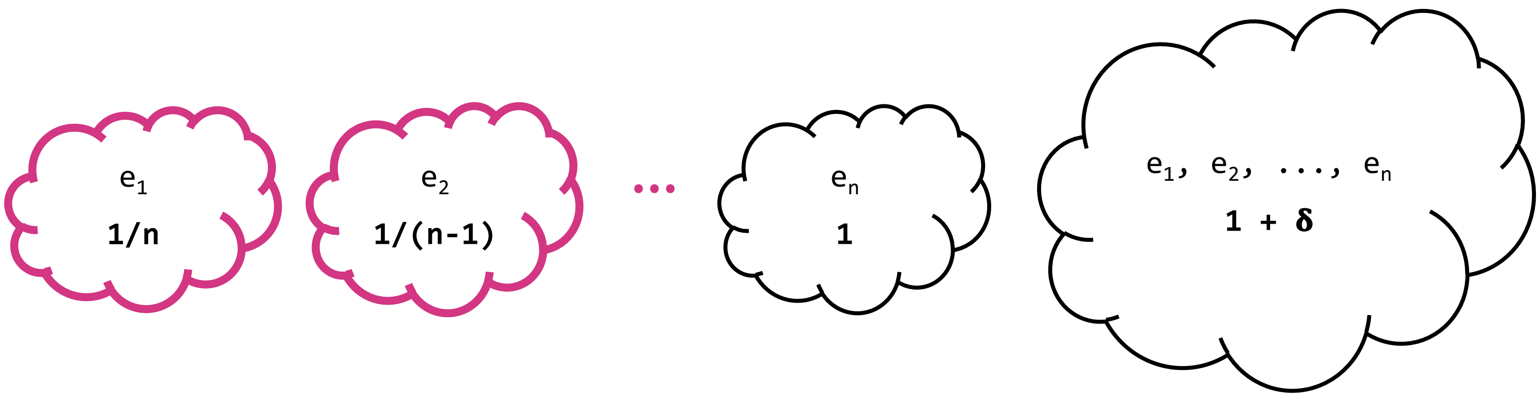

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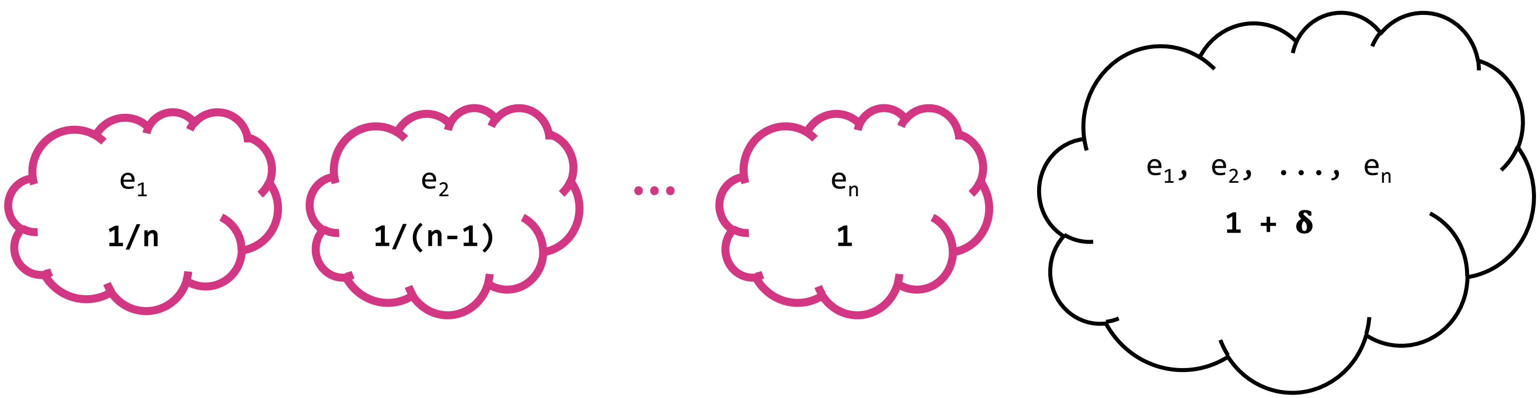

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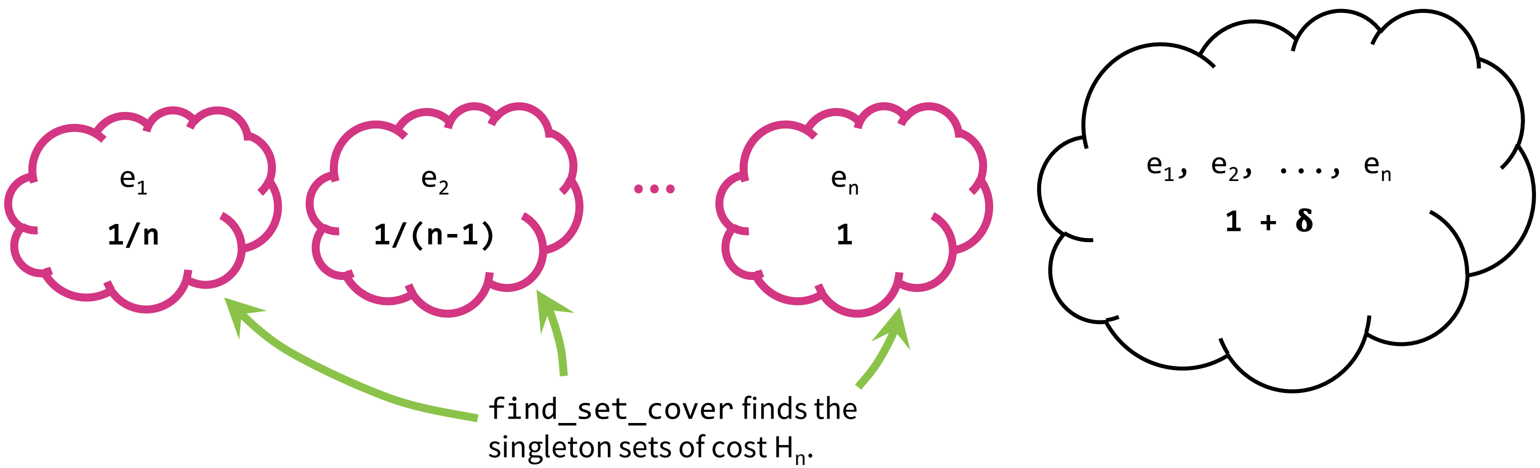

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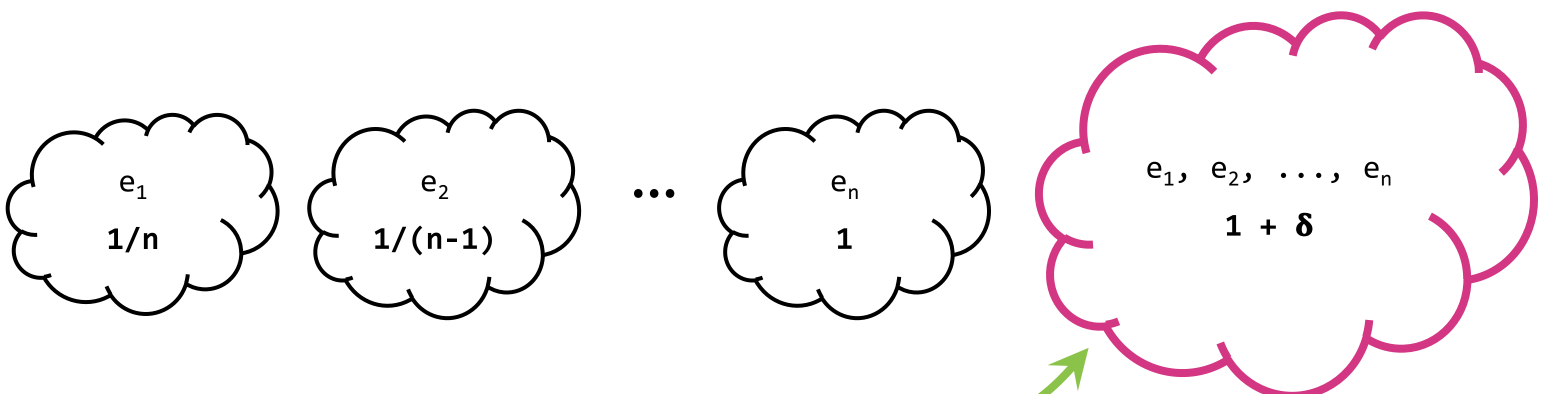

`find_set_cover` finds the singleton sets of cost H_n.

But the optimal cover has cost $1 + \delta$.

Set Cover

Study of the **set cover** problem led to the development of fundamental techniques for the entire field of approximation algorithms.

This approximation algorithm is widely (and wildly) useful!

- The human DNA can be represented as a very long string over a four-letter Alphabet. Since it's very long, several overlapping short segments of this string get deciphered, but the locations of these segments on the original remains unknown.
- We can use set cover to find the shortest string which contains these segments as substrings to approximate the original DNA string.

0/1 Knapsack III

Approximation Schemes

Approximation schemes

Let P be an **NP**-hard optimization problem.

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What is f_P in 0/1 Knapsack? 🤔 The total value derived from the items taken.

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A is an **approximation scheme** if it outputs a solution s such that:

$f_P(s) \le (1 + \varepsilon) \cdot OPT$ if P is a minimization problem

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FPTAS is the best you can get for an **NP**-hard optimization problem, assuming **P** ≠ **NP**. In other words, it's the holy grail solution for **NP**-hard problems!

Fixed Parameter Tractability

Suppose that the input to a problem P can be characterized by two parameters, n and k.

P is called fixed-parameter tractable iff there is some algorithm that solves P in time `O(f(k)p(n))`.

- f(k) is an arbitrary function and p(n) is a polynomial in n.
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Which parameter gets “fixed” depends on your perspective.

- If you’re the robber designing an algorithm to help you steal items, you’ll fix the capacity of the knapsack and reason about variable sets of items.
- If you’re the victim designing an algorithm to predict which items the robbers with variable size knapsacks will steal, you’ll fix the value of the items.

Pseudo-Polynomial Time

- Recall that to be considered efficient, an algorithm must have runtime polynomial in the size of its input.
  - An instance of 0/1 Knapsack requires log(n) bits to represent a number n in binary (e.g. adding one more bit to the end of the representation of W doubles its size and doubles the runtime).

Pseudo-Polynomial Time

Recall that to be considered efficient, an algorithm must have runtime polynomial in the size of its input.

- An instance of 0/1 Knapsack requires log(n) bits to represent a number n in binary (e.g. adding one more bit to the end of the representation of W doubles its size and doubles the runtime).

To be considered weakly efficient, an algorithm must have runtime pseudo-polynomial in the of its input.

- What is polynomial vs. pseudo-polynomial runtime? Polynomial runtime assumes inputs represented in binary. Pseudo-polynomial runtime assumes inputs represented in unary.
- An instance of 0/1 Knapsack requires n bits to represent a number n in unary (e.g. adding one more bit to the end of the representation of W doesn’t really affect its size or its runtime).

$$6 = 1\ 1\ 1\ 1\ 1\ 1$$

0/1 Knapsack

First DP alg Previously, we described a fixed-parameter polynomial time algorithm for 0/1 Knapsack.

Our original solution from Lecture 11 answers "What is the maximum value that fits in X capacity given just the first k items?" and fills this table.

First solve the problem for small knapsacks

Then larger knapsacks

Then larger knapsacks

First solve the problem for few items

Then more items

Then more items

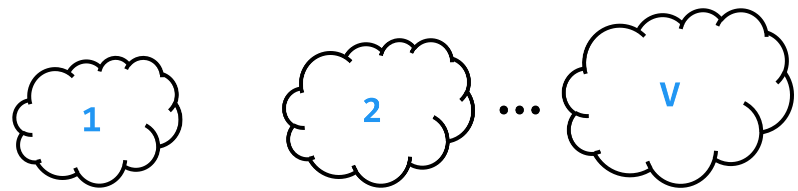
Fill in maximum values here.

0/1 Knapsack

New DP alg Here, we'll describe a pseudo-polynomial time algorithm for 0/1 Knapsack.

Our new solution answers "What is the min capacity needed to make X value with the first k items?" and fills this table.

Let V be the maximum possible value obtainable: $V = v_1 + v_2 + \ldots + v_n$.



First solve the problem for small values

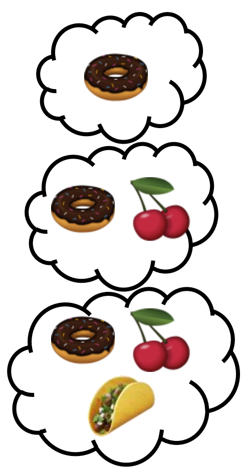
Then larger values

Then larger values

First solve the problem for few items

Then more items

Then more items



Fill in minimum capacities here.

0/1 Knapsack

First DP alg

Let **OPT[c,i]** be the optimal (max) value for capacity c with i items.

$$\texttt{OPT[c,i]} = \begin{cases} 0 & \text{if } \mathbf{c} \text{ or } \mathbf{i} \text{ are } 0 \\ \max\{\texttt{OPT[c,i-1]},\ \texttt{OPT}[c-w_i, i-1] + v_i\} & \text{otherwise} \end{cases}$$

$O(nW)$ Dynamic programming solution

New DP alg

Let **OPT[i,v]** be the optimal (min) capacity for value v with i items.

$$\texttt{OPT[i,v]} = \begin{cases} 0 & \text{if } \mathbf{i} \text{ and } \mathbf{v} \text{ are } 0 \\ \infty & \text{if } \mathbf{i} \text{ is } 0,\ \mathbf{v} > 0 \\ \texttt{OPT[i-1, v]} & \text{if } v_i > v \\ \min\{\texttt{OPT[i-1,v]},\ \texttt{OPT}[i-1, v-v_k] + w_i\} & \text{otherwise} \end{cases}$$

$O(nV)$ Dynamic programming solution

0/1 Knapsack

	Brute-force	First DP solution	New DP solution
Runtime	$O(n2^n)$ worst-case	$O(nW)$ worst-case	$O(nV)$ worst-case
Usage	Don't do it.	You're the robber! Capacity is fixed and the number of items might grow large.	You're the victim! Total value is fixed and the number of items might grow large.
Analysis	Exponential, anyway you look at it.	Fixed-parameter polynomial.	Pseudo-polynomial.

0/1 Knapsack

Let's extend on our idea!

Intuition If the values of objects were small numbers bounded by a polynomial in n, then **New DP alg** would be a regular polynomial-time algorithm, so let's coerce the values of the items to be small.

0/1 Knapsack

```
algorithm zero_one_knapsack(capacity, weights, values):
  k = εv_max/n
  v_i' = ⌊v_i/k⌋ for v_i in values
  S', value = use the value-based DP algorithm to find the
    most valuable items using values v_i' and weights.
  return S', value * k
```

Runtime: $O(nV)$

0/1 Knapsack

(1) How do we establish an approximation guarantee for our algorithm?

Let A be the set output by `zero_one_knapsack`.

Lemma: value(A) $\geq (1 - \varepsilon)\cdot$OPT

Proof:

Let O be the optimal set. For any object a, because of rounding down, $k\cdot v_a'$ can be smaller than v_a but by not more than k. Thus,

$$\text{value}(O) - k\cdot\text{value}'(O) \leq nk$$

The DP step must return a set at least as good as O under the new values. Therefore,

$$\text{value}(S') \geq k\cdot\text{value}'(O) \geq \text{value}(O) - nk = \text{OPT} - \varepsilon v_{max} \geq (1 - \varepsilon)\cdot\text{OPT}$$

where the last inequality follows from the fact that OPT $\geq v_{max}$.∎

0/1 Knapsack

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Lemma: value(A) $\geq (1 - \varepsilon) \cdot$OPT

Theorem: `zero_one_knapsack` is a FPTAS for Knapsack.

Proof:

By the lemma, the solution found is within $(1 - \varepsilon)$ factor of OPT. Since the running time of the algorithm is $O(n) = O(n^2 \lfloor v_{max}/k \rfloor) = O(n^2 \lfloor n/\varepsilon \rfloor)$, which is polynomial in n and $1/\varepsilon$, the theorem follows. ∎

We have the holy grail of approximation algorithms!

0/1 Knapsack

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We have the holy grail of approximation algorithms!

Conclusion

	Vertex Cover	Set Cover	0/1 Knapsack
Runtime	$O(\|V\|+\|E\|)$ worst-case	$O(\|S\|^2)$ worst-case	$O(nV)$ worst-case
Approximation	2	$\log(\|U\|)$	$1 - \varepsilon$
Analysis	-	-	FPTAS.