

Randomized Algorithms II

Summer 2017 • Lecture 5

A Few Notes

Homework 2

Due tomorrow night at 11:59 p.m. on Gradescope.

Homework 3

Released tomorrow night.

Due Friday 7/21 at 11:59 p.m. on Gradescope.

Outline for Today

More randomized algorithms!

Hashing Basics

Universal Hash Functions

What's the Source of the Randomness?

Hashing Basics

Randomized Algorithms

A randomized algorithm is an algorithm that incorporates randomness as part of its operation.

Often aim for properties like ...

- Good average-case behavior

- Getting exact answers with high probability

- Getting answers that are close to the right answer

Data Structures

	Sorted linked lists	Sorted arrays	Balanced BSTs
Search	$O(n)$ expected & worst-case	$O(\log n)$ expected & worst-case	$O(\log n)$ expected & worst-case <small>$O(n)$ worst-case for generic BSTs</small>
Insert/Delete	$O(n)$ expected & worst-case <small>without a pointer to the element</small>	$O(n)$ expected & worst-case	$O(\log n)$ expected & worst case

Data Structures

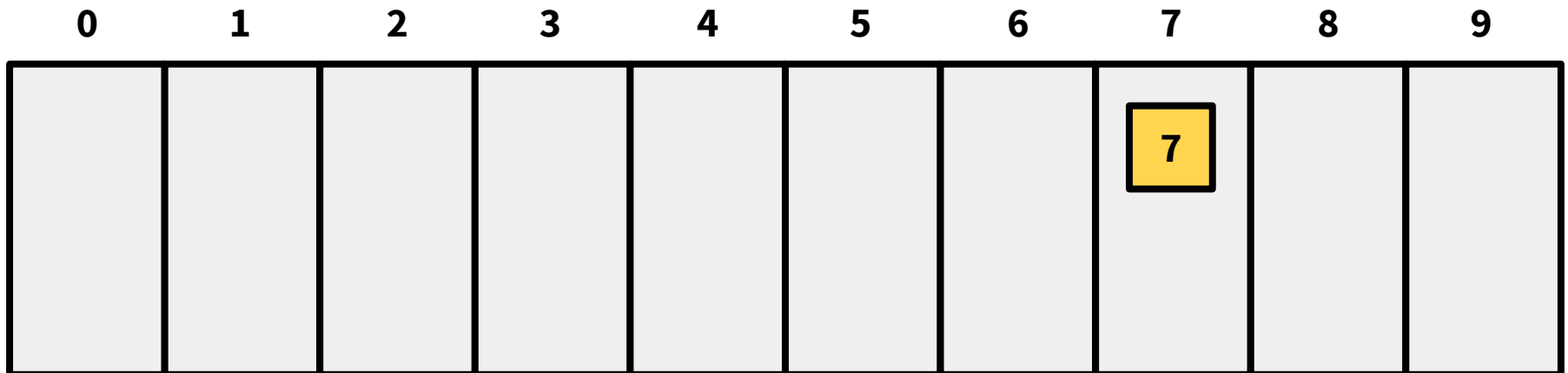
	Sorted linked lists	Sorted arrays	Balanced BSTs	Hash tables
Search	$O(n)$ expected & worst-case	$O(\log n)$ expected & worst-case	$O(\log n)$ expected & worst-case <small>$O(n)$ worst-case for generic BSTs</small>	$O(1)$ expected $O(n)$ worst-case
Insert/Delete	$O(n)$ expected & worst-case <small>without a pointer to the element</small>	$O(n)$ expected & worst-case	$O(\log n)$ expected & worst case	$O(1)$ expected $O(n)$ worst-case <small>without a pointer to the element</small>

Direct Addressing

How might we get **$O(1)$** -time? Try direct addressing!

One type of item per address.

`insert(7)`



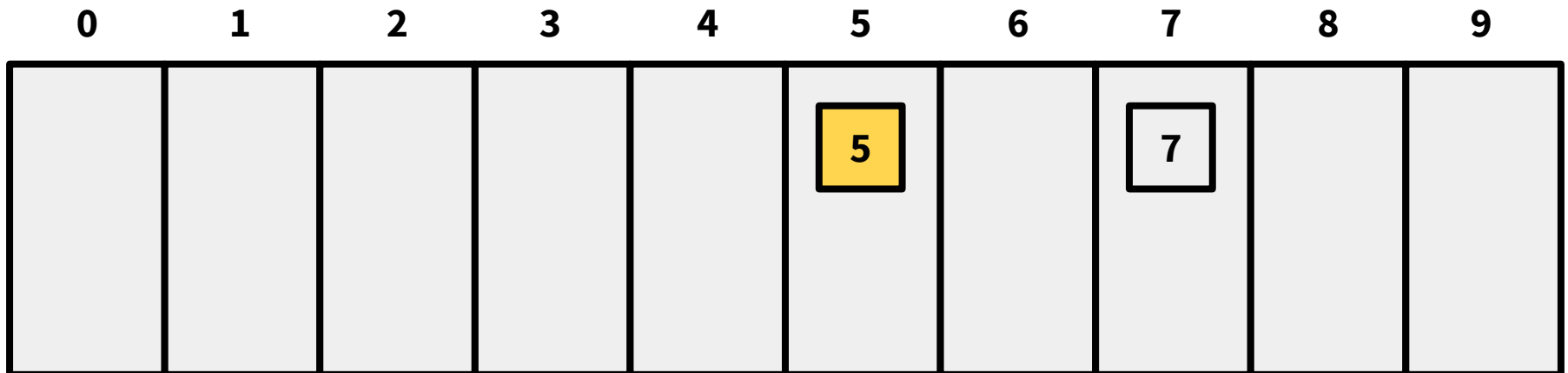
Direct Addressing

How might we get **$O(1)$** -time? Try direct addressing!

One type of item per address.

`insert(7)`

`insert(5)`



Direct Addressing

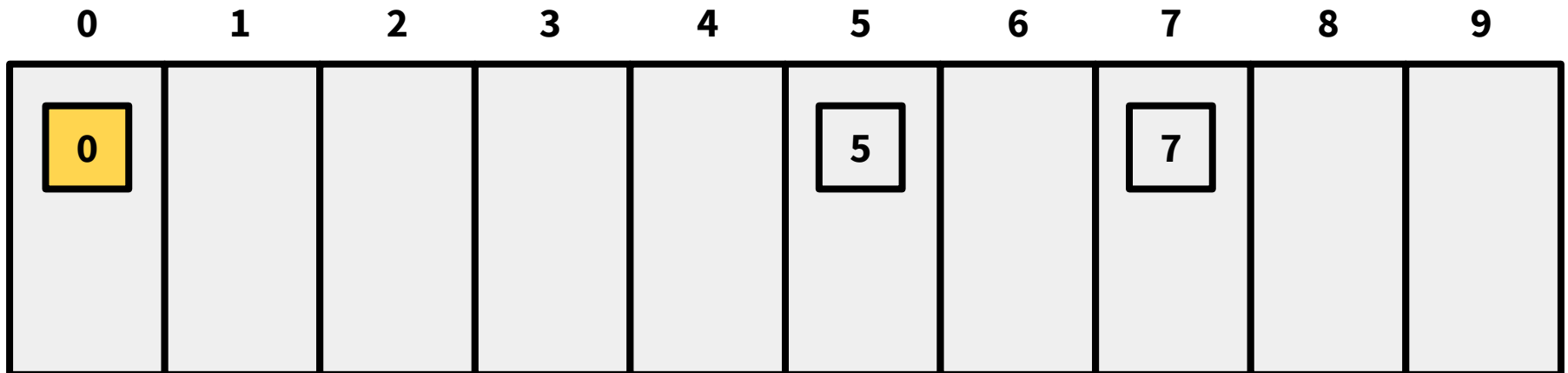
How might we get **0(1)**-time? Try direct addressing!

One type of item per address.

`insert(7)`

`insert(5)`

`insert(0)`



Direct Addressing

How might we get **0(1)**-time? Try direct addressing!

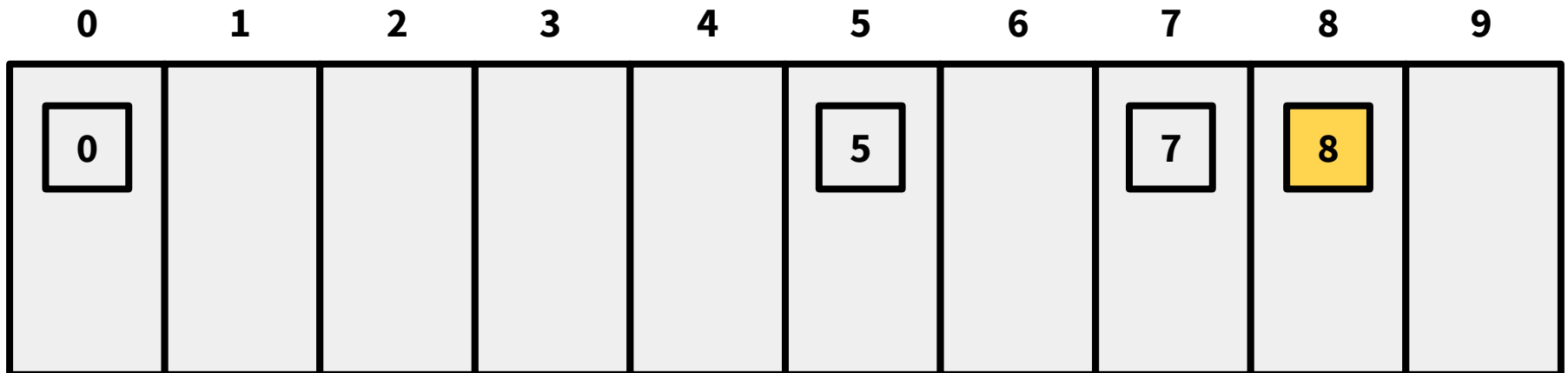
One type of item per address.

`insert(7)`

`insert(5)`

`insert(0)`

`insert(8)`



Direct Addressing

How might we get **0(1)**-time? Try direct addressing!

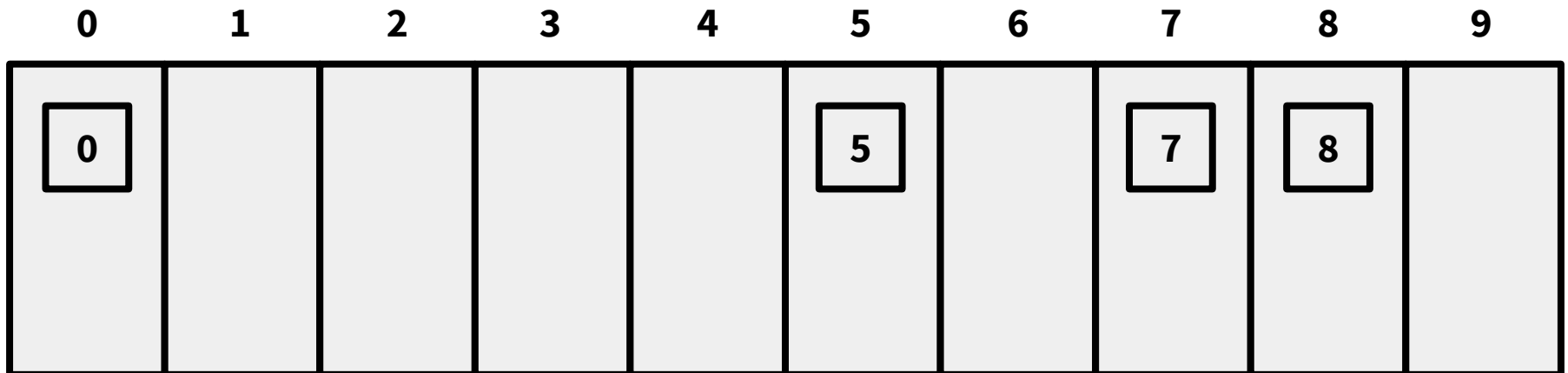
One type of item per address.

insert(7) search(7)

insert(5) search(2)

insert(0)

insert(8)



Direct Addressing

How might we get $O(1)$ -time? Try direct addressing!

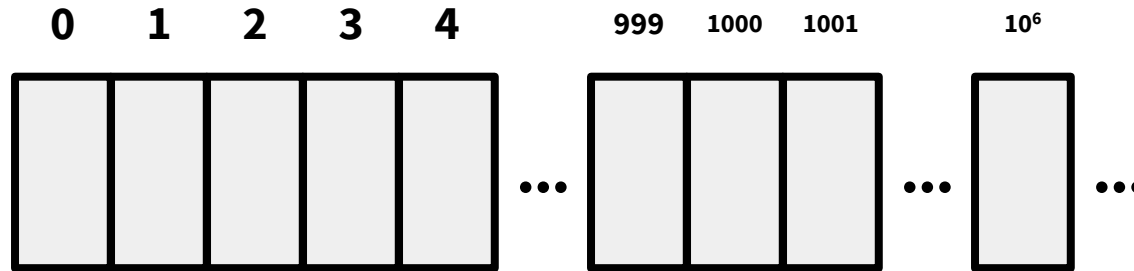
What's the issue with this approach? 🤔

Direct Addressing

How might we get $O(1)$ -time? Try direct addressing!

What's the issue with this approach? 🤔

Similar to counting_sort and bucket_sort (for $k \leq \text{num_buckets}$), if the set of items being inserted/deleted (e.g. $\{0, 1, 2, \dots, 999, 1000, \dots, 10^6, \dots\}$) is large, then the sheer space required to maintain this data structure becomes an issue.

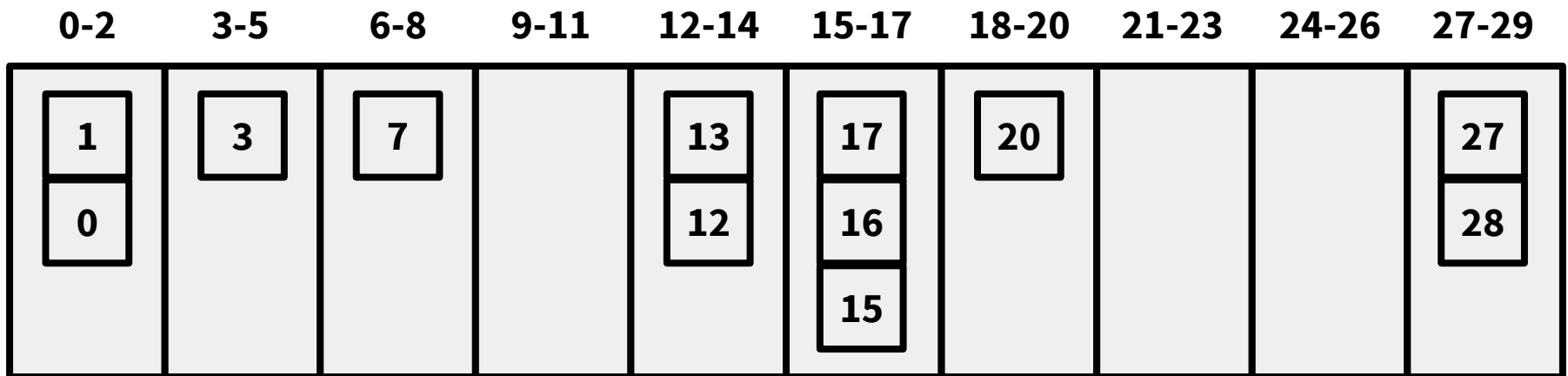


Direct Addressing

How might we get **$O(1)$** -time? Try direct addressing!

Can we fix this issue by assigning multiple types of item per address, like case (2) of bucket_sort?

Sometimes, this binning approach is useful. `search(12)` still runs pretty fast.

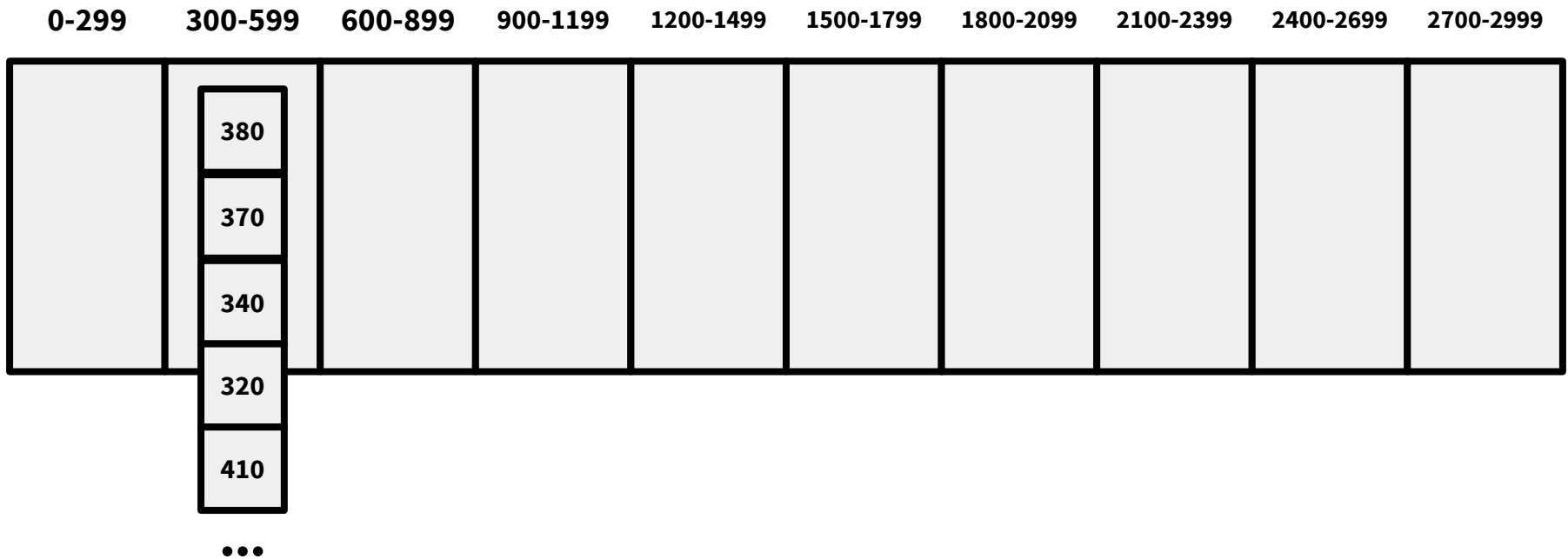


Direct Addressing

How might we get **$O(1)$** -time? Try direct addressing!

Can we fix this issue by assigning multiple types of item per address, like case (2) of bucket_sort?

Other times, it causes an issue. `search(432)` is slow.

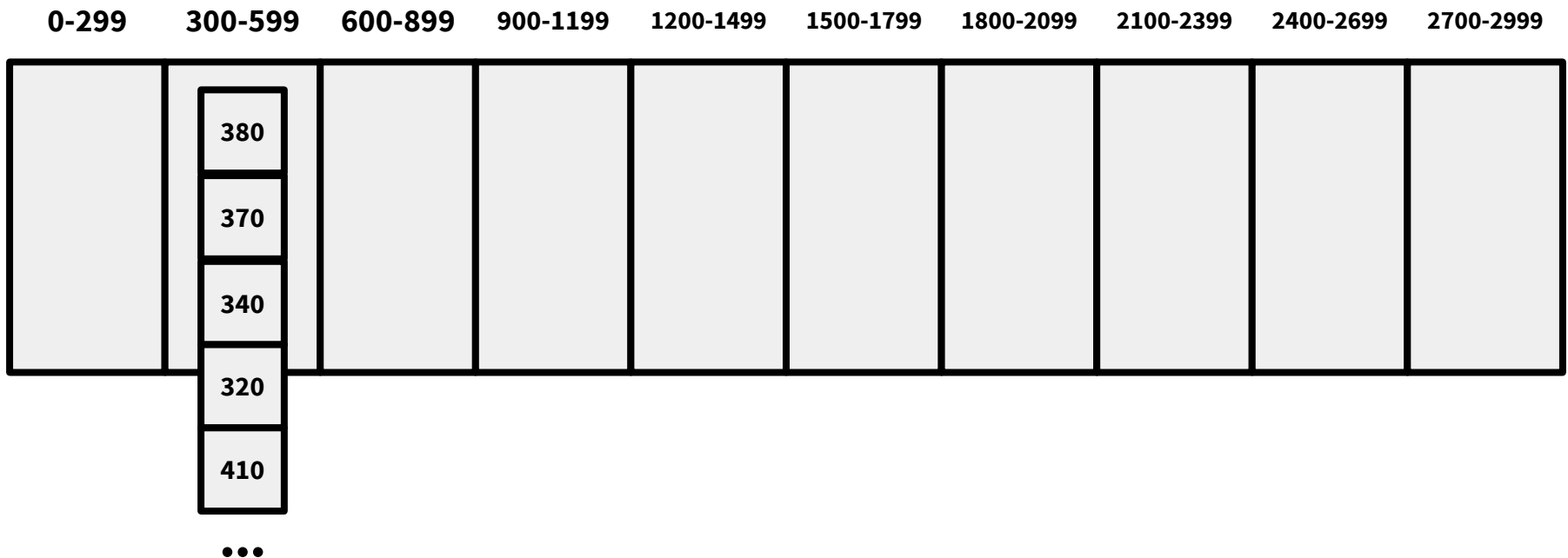


Direct Addressing

This is an example of a hash table.

Albeit one with a basic bucketing scheme.

Can we do better?



Terminology

There exists a universe U of keys, size $|U|$.

$|U|$ is really big.

What is $|U|$ if U is the set of ASCII strings of length 16? 🤔

Terminology

There exists a universe U of keys, size $|U|$.

$|U|$ is really big.

What is $|U|$ if U is the set of ASCII strings of length 16? 🤔 $|U| = 128^{16}$.

Terminology

There exists a universe U of keys, size $|U|$.

$|U|$ is really big.

What is $|U|$ if U is the set of ASCII strings of length 16? 🤔 $|U| = 128^{16}$.

We hash the keys to n buckets.

$|U| \gg n$; i.e. $|U|$ is a lot bigger than n .

We don't know which of the $|U|$ possible keys we'll need to store;
Unless otherwise stated, let's assume $\leq n$.

Terminology

There exists a universe U of keys, size $|U|$.

$|U|$ is really big.

What is $|U|$ if U is the set of ASCII strings of length 16? 🤔 $|U| = 128^{16}$.

We hash the keys to n buckets.

$|U| \gg n$; i.e. $|U|$ is a lot bigger than n .

We don't know which of the $|U|$ possible keys we'll need to store;

Unless otherwise stated, let's assume $\leq n$.

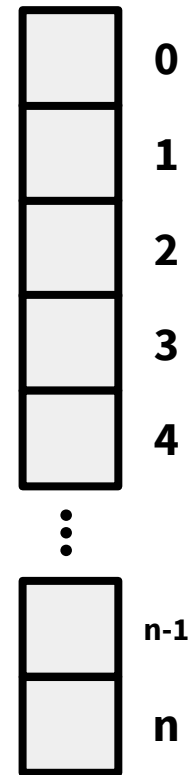
There's a hash function $h: U \rightarrow \{1, \dots, n\}$ that maps keys to buckets.

An Example

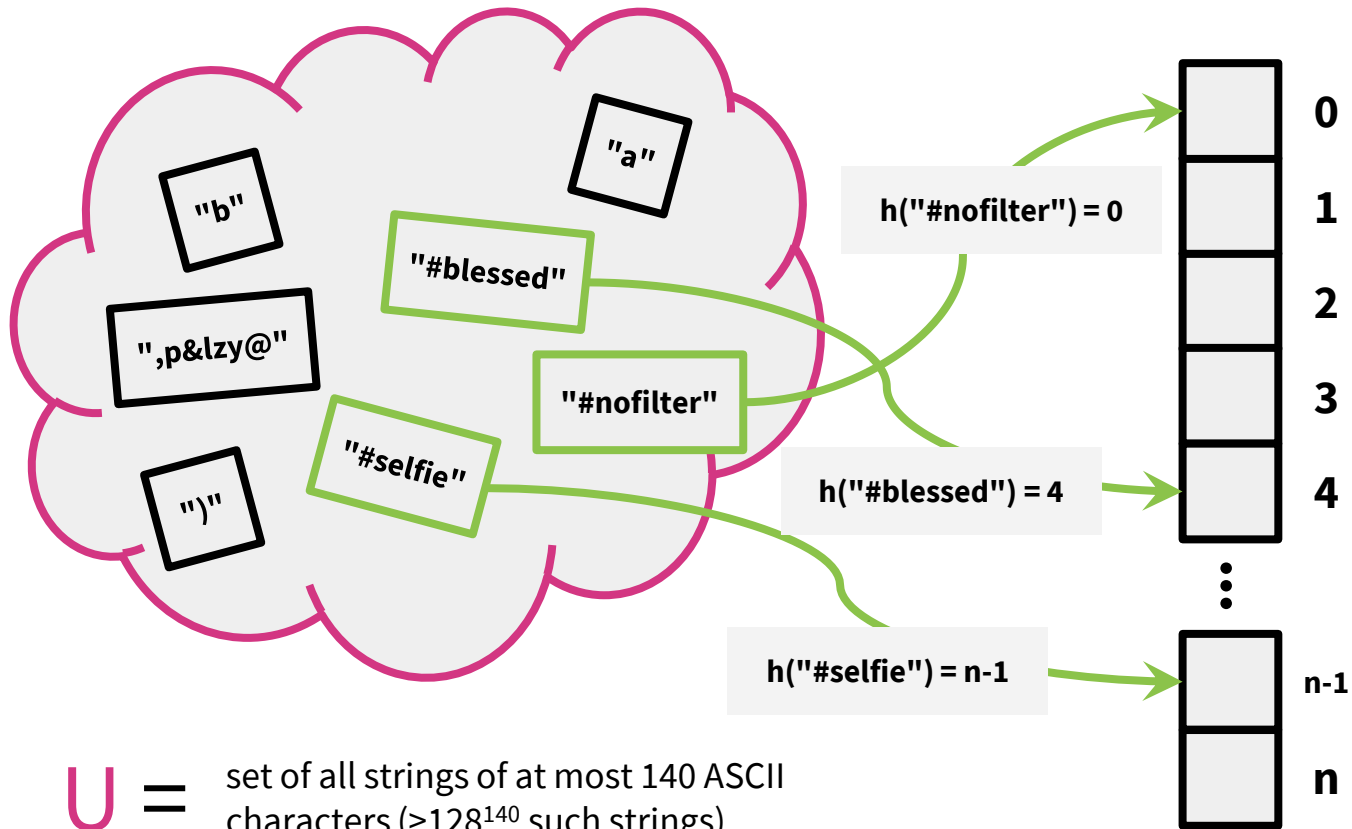


U = set of all strings of at most 140 ASCII characters ($>128^{140}$ such strings)

And we'll need to store a small subset of U (say, the ones that might be trending hashtags on Twitter); we're assuming the number of hashtags $\leq n$, the number of buckets.



An Example



And we'll need to store a small subset of U (say, the ones that might be trending hashtags on Twitter); we're assuming the number of hashtags $\leq n$, the number of buckets.

Hash Tables (with chaining)

List of n buckets.

Each bucket stores an unsorted linked list.

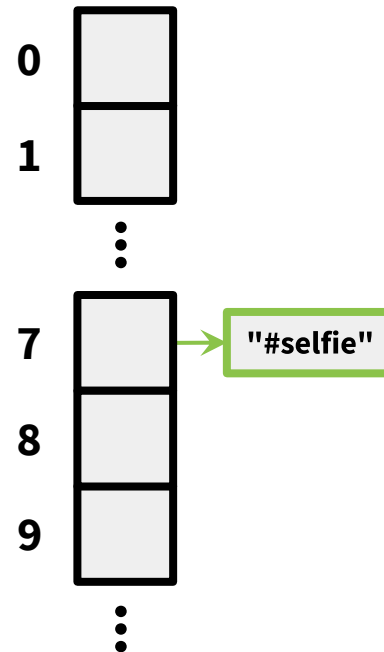
insert in $O(1)$ since it's unsorted; search in $O(n)$.

$h: U \rightarrow \{1, \dots, n\}$ can be any function

For concreteness, suppose it's length.

Suppose we insert a bunch of keys and then search.

`insert("#selfie")`



Hash Tables (with chaining)

List of n buckets.

Each bucket stores an unsorted linked list.

insert in $O(1)$ since it's unsorted; search in $O(n)$.

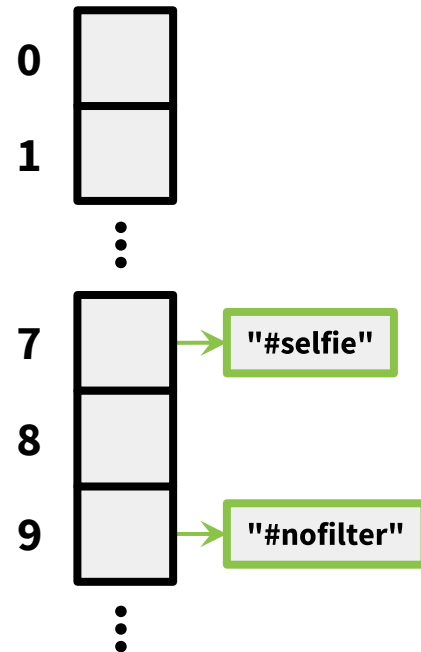
$h: U \rightarrow \{1, \dots, n\}$ can be any function

For concreteness, suppose it's length.

Suppose we insert a bunch of keys and then search.

`insert("#selfie")`

`insert("#nofilter")`



Hash Tables (with chaining)

List of n buckets.

Each bucket stores an unsorted linked list.

insert in $O(1)$ since it's unsorted; search in $O(n)$.

$h: U \rightarrow \{1, \dots, n\}$ can be any function

For concreteness, suppose it's length.

Suppose we insert a bunch of keys and then search.

```
insert("#selfie")
```

```
insert("#nofilter")
```

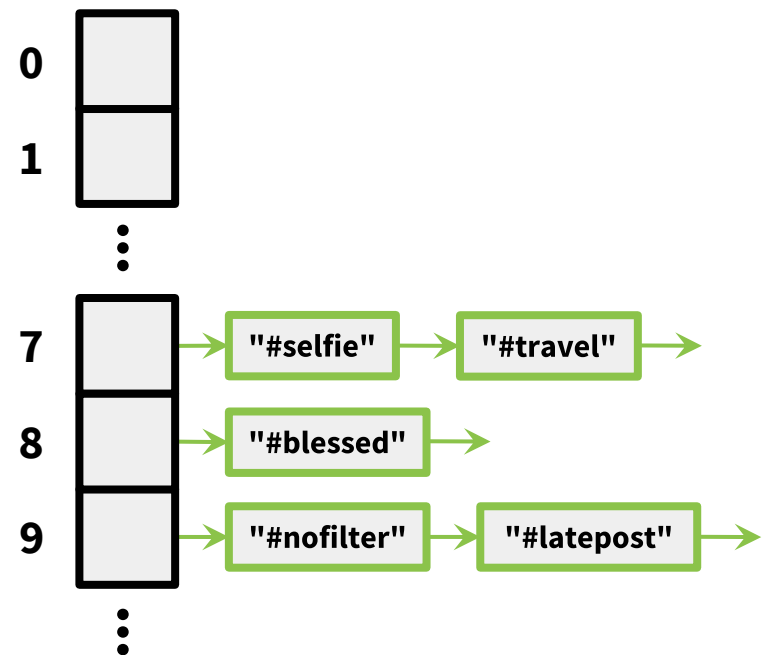
```
insert("#blessed")
```

```
insert("#travel")
```

```
insert("#latepost")
```

```
search("#travel")
```

Scans through
all elements in
bucket
 $h("#travel")$



Hash Tables (with chaining)

Is choosing $h: U \rightarrow \{1, \dots, n\}$ to be length a good idea? 🤔

Hash Tables (with chaining)

Is choosing $h: U \rightarrow \{1, \dots, n\}$ to be length a good idea? 🤔

Not really. In fact, it's quite terrible since there are a lot of hashtags that share the same lengths.

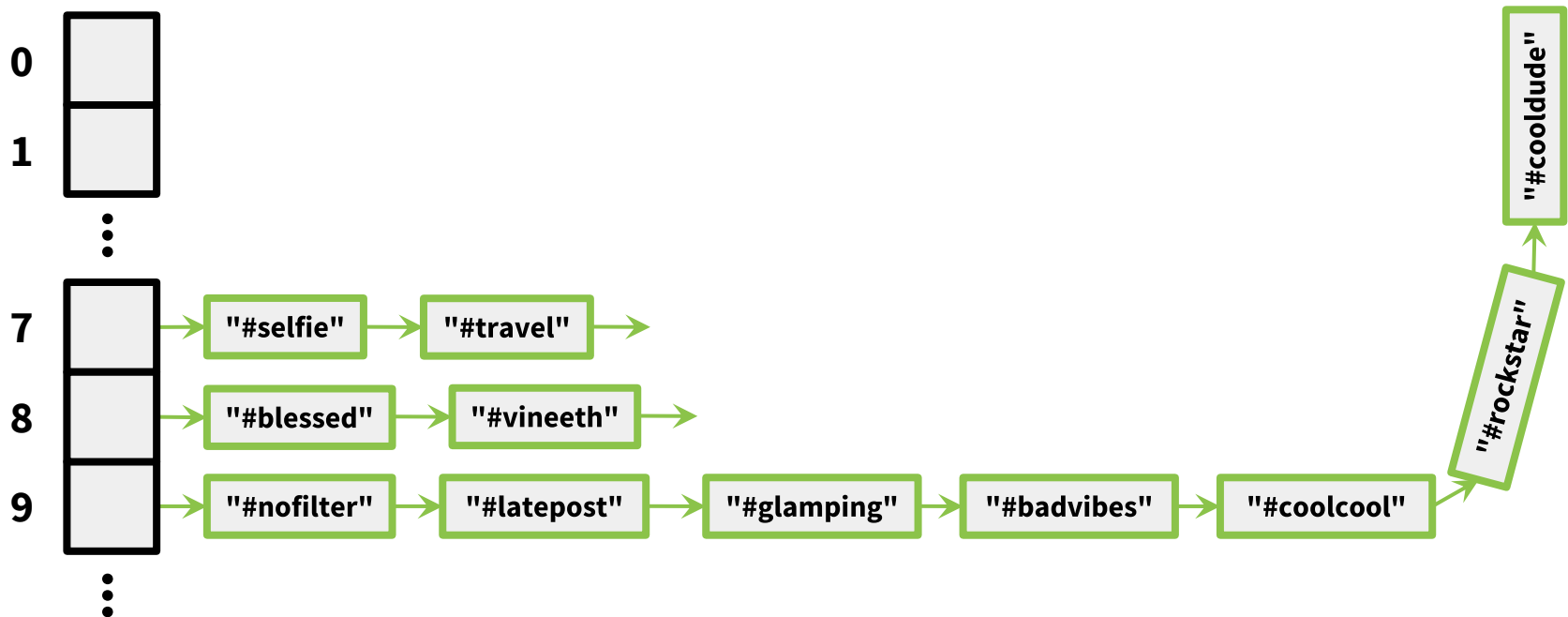
Hash Tables (with chaining)

Is choosing $h: U \rightarrow \{1, \dots, n\}$ to be length a good idea? 🤔

Not really. In fact, it's quite terrible since there are a lot of hashtags that share the same lengths.

So how do we choose a better h ?

The items need to be spread out in the buckets.



One h to Rule Them All?

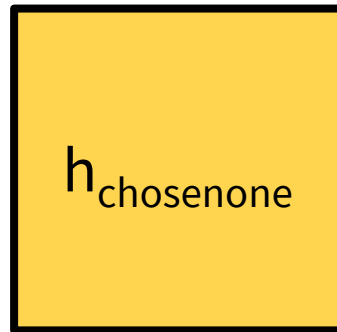
(1) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have size $O(1)$ after hashing any n items?

One h to Rule Them All?

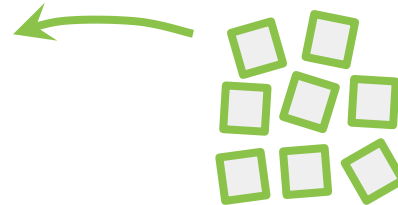
(1) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?

One h to Rule Them All?

(1) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?



1. You choose your
hash function
 $h_{\text{chosenone}}$



2. An adversary gives
your hash function n
items to hash.

Is it possible to construct $h_{\text{chosenone}}$ such that you're guaranteed that all buckets will have size $O(1)$? This would be ideal. 🤔

One h to Rule Them All?

(1) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?

You probably couldn't think of how. Why not? It's impossible!

One h to Rule Them All?

(1) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?

You probably couldn't think of how. Why not? It's impossible!

$h_{\text{chosenone}}$ is defined from a domain of $|U|$ items to a range of n buckets.

One h to Rule Them All?

(1) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?

You probably couldn't think of how. Why not? It's impossible!

$h_{\text{chosenone}}$ is defined from a domain of $|U|$ items to a range of n buckets. By the pigeonhole principle, at least one of the buckets receives at least $|U|/n$ items.

One h to Rule Them All?

(1) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?

You probably couldn't think of how. Why not? It's impossible!

$h_{\text{chosenone}}$ is defined from a domain of $|U|$ items to a range of n buckets. By the pigeonhole principle, at least one of the buckets receives at least $|U|/n$ items. Recall that $|U| \gg n$, so $|U|/n > n$; therefore at least one of the buckets receives at least n items.

One h to Rule Them All?

(1) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?

You probably couldn't think of how. Why not? It's impossible!

$h_{\text{chosenone}}$ is defined from a domain of $|U|$ items to a range of n buckets. By the pigeonhole principle, at least one of the buckets receives at least $|U|/n$ items. Recall that $|U| \gg n$, so $|U|/n > n$; therefore at least one of the buckets receives at least n items.

Let's call the set of items that get hashed to this bucket $U_{\text{bigbucket}}(h_{\text{chosenone}})$ where $U_{\text{bigbucket}} \subset U$. The adversary could choose to hash n items from $U_{\text{bigbucket}}$. This is a valid set of n items, and results in one bucket with all n items, by construction. Therefore, **(1)** is impossible.

One h to Rule Them All?

(1) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?

You probably couldn't think of how. Why not? It's impossible!

$h_{\text{chosenone}}$ is defined from a domain of $|U|$ items to a range of n buckets. By the pigeonhole principle, at least one of the buckets receives at least $|U|/n$ items. Recall that $|U| \gg n$, so $|U|/n > n$; therefore at least one of the buckets receives at least n items.

Notation indicating $U_{\text{bigbucket}}$ is a function of $h_{\text{chosenone}}$



Let's call the set of items that get hashed to this bucket $U_{\text{bigbucket}}(h_{\text{chosenone}})$ where $U_{\text{bigbucket}} \subset U$. The adversary could choose to hash n items from $U_{\text{bigbucket}}$. This is a valid set of n items, and results in one bucket with all n items, by construction. Therefore, **(1)** is impossible.

One h to Rule Them All?

~~(1) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have size $O(1)$ after hashing any n items?~~

 Impossible!

(2) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have **expected** size $O(1)$ after hashing any n items?

One h to Rule Them All?

~~(1) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have size $O(1)$ after hashing any n items?~~

 Impossible!

(2) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have **expected** size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?

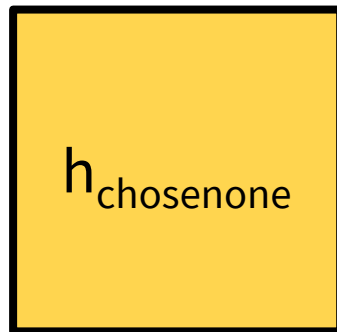
One h to Rule Them All?

~~(1) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have size $O(1)$ after hashing any n items?~~

Impossible!

(2) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have **expected** size $O(1)$ after hashing any n items? after an adversary chooses n items to hash?

1. You choose your
hash function
 $h_{\text{chosenone}}$



2. An adversary gives
your hash function n
items to hash.

Is it possible to construct $h_{\text{chosenone}}$ such that you're guaranteed that all buckets will have **expected** size $O(1)$? This would be good. 🤔

One h to Rule Them All?

~~(1) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have size $O(1)$ after hashing any n items?~~

 Impossible!

(2) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have **expected** size $O(1)$ after hashing any n items? after an adversary chooses n items to hash?

Can you think of such an $h_{\text{chosenone}}$? 🤔


Probably not. This is the same question as **(1)**! Since the adversary is choosing the n items, there's no randomness anywhere in the process. As a result, the **expected** size of a bucket is trivially just the size.

One h to Rule Them All?

~~(1) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have size $O(1)$ after hashing any n items?~~

 Impossible!

~~(2) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have **expected** size $O(1)$ after hashing any n items?~~

 Impossible as stated. But if we introduce randomness ...

In order for all buckets to have **expected** size $O(1)$ after hashing any n items, we need to introduce randomness.

One h to Rule Them All?

~~(1) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have size $O(1)$ after hashing any n items?~~

Impossible!

~~(2) Can we design a single $h_{\text{chosenone}}: U \rightarrow \{1, \dots, n\}$ such that all buckets will have **expected** size $O(1)$ after hashing any n items?~~

Impossible as stated. But if we introduce randomness ...

In order for all buckets to have **expected** size $O(1)$ after hashing any n items, we need to introduce randomness.

Where? Well there's only one option ... in our choice of hash function.

We will randomly choose h from a large set of hash functions!

(There won't be an h to rule them all).



Lots of h's?

(3) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , all buckets will have **expected** size $O(1)$ after hashing any n items?

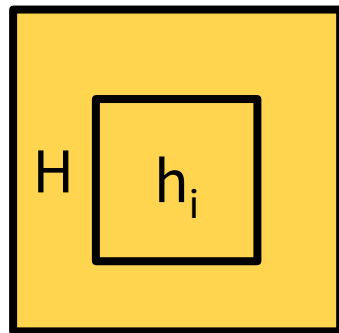
Lots of h's?

(3) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , all buckets will have **expected** size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?

Lots of h's?

(3) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , all buckets will have **expected** size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?

1. You choose your set of hash functions H .



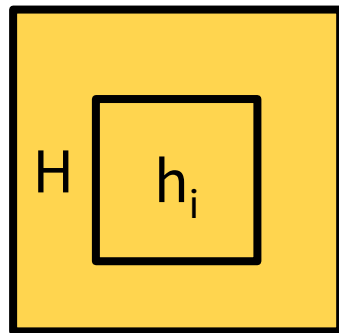
2. An adversary gives your hash function n items to hash.

3. You randomly pick a hash function h_i from H to hash the n items.

Lots of h's?

(3) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , all buckets will have **expected** size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?

1. You choose your set of hash functions H .



2. An adversary gives your hash function n items to hash.

3. You randomly pick a hash function h_i from H to hash the n items.

Is it possible to construct H such that you're guaranteed that all buckets will have **expected** size $O(1)$? This would be good.

Lots of h's?

(3) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , all buckets will have **expected** size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?

Yes! But it's not very useful.

Lots of h's?

(3) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , all buckets will have **expected** size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?

Yes! But it's not very useful.

Let H be the set of n hash functions where h_i hashes all keys in the entire universe to bucket i .

Lots of h's?

(3) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , all buckets will have **expected** size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?

Yes! But it's not very useful.

Let H be the set of n hash functions where h_i hashes all keys in the entire universe to bucket i . With probability $1/n$, a bucket b will have all the keys that the adversary chose get hashed to it. Otherwise, the bucket will be empty.

Lots of h's?

(3) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , all buckets will have **expected** size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?

Yes! But it's not very useful.

Let H be the set of n hash functions where h_i hashes all keys in the entire universe to bucket i . With probability $1/n$, a bucket b will have all the keys that the adversary chose get hashed to it. Otherwise, the bucket will be empty.

$$\begin{aligned} E[\text{size_of}(b)] &= P(\text{all keys hashed to it}) \cdot n + P(0 \text{ keys hashed to it}) \cdot 0 \\ &= (1/n) \cdot n \\ &= 1 \end{aligned}$$

But $P(\text{lots of keys get hashed to one bucket}) = 1$.

Lots of h's?

(3) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , all buckets will have **expected** size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?

Yes! But it's not very useful.

Let H be the set of n hash functions where h_i hashes all keys in the entire universe to bucket i . With probability $1/n$, a bucket b will have all the keys that the adversary chose get hashed to it. Otherwise, the bucket will be empty.

$$\begin{aligned} E[\text{size_of}(b)] &= P(\text{all keys hashed to it}) \cdot n + P(0 \text{ keys hashed to it}) \cdot 0 \\ &= (1/n) \cdot n \\ &= 1 \end{aligned}$$

But $P(\text{lots of keys get hashed to one bucket}) = 1$.

This is not good. Maybe we should be using a different metric.

Lots of h's?

(3) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , all buckets will have **expected** size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?



Not useful!

(4) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , after an adversary chooses n items $\{u_1, \dots, u_n\}$ to hash, the **expected** number of items in u_x 's bucket is $O(1)$?

Lots of h's?

(3) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , all buckets will have **expected** size $O(1)$ ~~after hashing any n items?~~ after an adversary chooses n items to hash?

 Not useful!

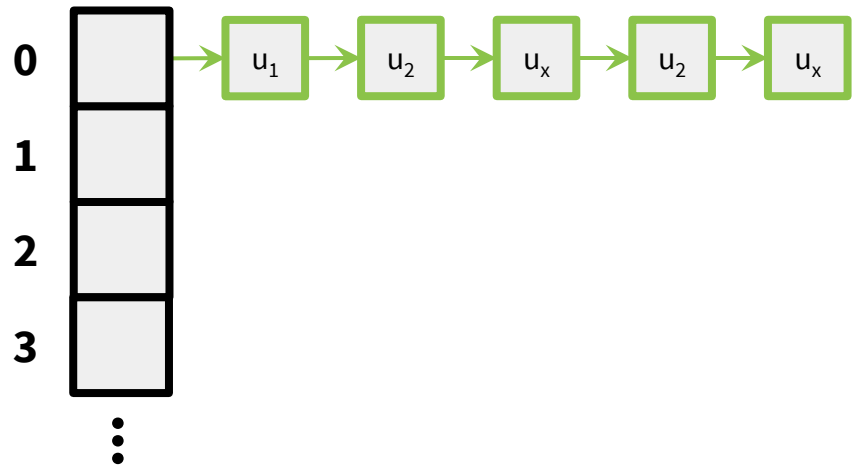
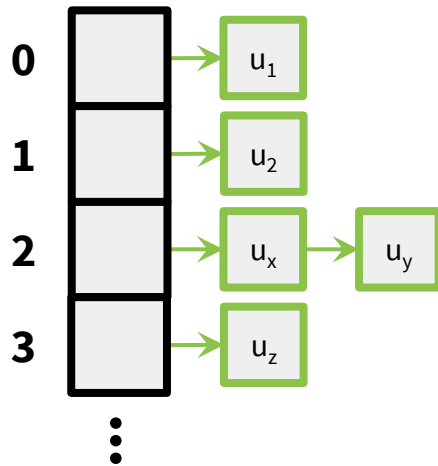
(4) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , after an adversary chooses n items $\{u_1, \dots, u_n\}$ to hash, the **expected** number of items in u_x 's bucket is $O(1)$?

As an analogy for the difference between **(3)** and **(4)**, consider the “small classes illusion.” Suppose a university offers 10 classes, 9 of which have 1 person in them and the last of which has 500 persons in them. Using reasoning from **(3)**, the university might tout average class sizes of ~ 50 , when in reality, it should report much class sizes experienced by the average student, as in **(4)**.

Lots of h's?

(4) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , after an adversary chooses n items $\{u_1, \dots, u_n\}$ to hash, the **expected** number of items in u_x 's bucket is $O(1)$?

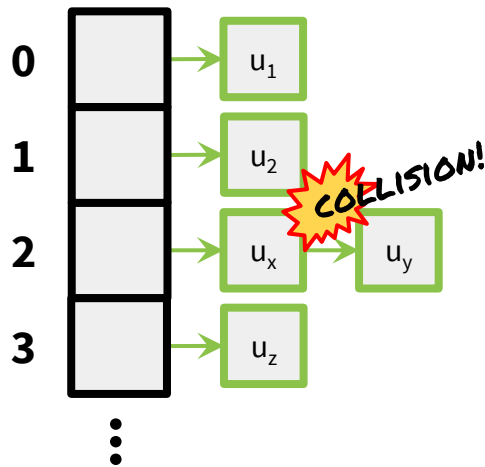
We can think of this statement in terms of minimizing the expected number of collisions.



Lots of h's?

(4) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , after an adversary chooses n items $\{u_1, \dots, u_n\}$ to hash, the **expected** number of items in u_x 's bucket is $O(1)$?

We can think of this statement in terms of minimizing the expected number of collisions.



Lots of h's?

(4) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , after an adversary chooses n items $\{u_1, \dots, u_n\}$ to hash, the **expected** number of items in u_x 's bucket is $O(1)$?

Yes! This time it's possible.

	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
"a"	0	0	0	0	1	1	1	1
"b"	0	0	1	1	0	0	1	1
"c"	0	1	0	1	0	1	0	1

The 0's and 1's represent the buckets i.e. h_8 hashes "b" to bucket 1.

Lots of h's?

(4) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , after an adversary chooses n items $\{u_1, \dots, u_n\}$ to hash, the **expected** number of items in u_x 's bucket is $O(1)$?

Yes! This time it's possible.

Let H be the exhaustive set of all hash functions that map elements in the universe U to buckets 1 to n , which has size $|H| = n^{|U|}$.

	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
"a"	0	0	0	0	1	1	1	1
"b"	0	0	1	1	0	0	1	1
"c"	0	1	0	1	0	1	0	1

The 0's and 1's represent the buckets i.e. h_8 hashes "b" to bucket 1.

Lots of h's?

(4) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , after an adversary chooses n items $\{u_1, \dots, u_n\}$ to hash, the **expected** number of items in u_x 's bucket is $O(1)$?

Yes! This time it's possible.

Let H be the exhaustive set of all hash functions that map elements in the universe U to buckets 1 to n , which has size $|H| = n^{|U|}$.

e.g. Suppose $U = \{\text{"a"}, \text{"b"}, \text{"c"}\}$ and $n = 2$ (there are 2 buckets). H would be a set of 8 hash functions. One h would map "a", "b", and "c" all to bucket 0. Another h would map "a" and "b" to bucket 0 and "c" to bucket 1. etc. etc.

	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
"a"	0	0	0	0	1	1	1	1
"b"	0	0	1	1	0	0	1	1
"c"	0	1	0	1	0	1	0	1

The 0's and 1's represent the buckets i.e. h_8 hashes "b" to bucket 1.

Lots of h's?

(4) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , after an adversary chooses n items $\{u_1, \dots, u_n\}$ to hash, the **expected** number of items in u_x 's bucket is $O(1)$?

$$E[\text{number of items in } u_x\text{'s bucket}] = \sum_{y=1}^n P[h(u_x) = h(u_y)]$$

Lots of h's?

(4) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , after an adversary chooses n items $\{u_1, \dots, u_n\}$ to hash, the **expected** number of items in u_x 's bucket is $O(1)$?

$$E[\text{number of items in } u_x\text{'s bucket}] = \sum_{y=1}^n P[h(u_x) = h(u_y)]$$

$$= 1 + \sum_{y \neq x} P[h(u_x) = h(u_y)]$$


Lots of h's?

(4) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , after an adversary chooses n items $\{u_1, \dots, u_n\}$ to hash, the **expected** number of items in u_x 's bucket is $O(1)$?

$$E[\text{number of items in } u_x\text{'s bucket}] = \sum_{y=1}^n P[h(u_x) = h(u_y)]$$

$$= 1 + \sum_{y \neq x} P[h(u_x) = h(u_y)]$$

$$= 1 + \sum_{y \neq x} 1/n$$



You will prove this is the case for the exhaustive set H in your Homework.


Lots of h's?

(4) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , after an adversary chooses n items $\{u_1, \dots, u_n\}$ to hash, the **expected** number of items in u_x 's bucket is $O(1)$?

$$E[\text{number of items in } u_x\text{'s bucket}] = \sum_{y=1}^n P[h(u_x) = h(u_y)]$$

$$= 1 + \sum_{y \neq x} P[h(u_x) = h(u_y)]$$

$$= 1 + \sum_{y \neq x} 1/n$$

 You will prove this is the case for the exhaustive set H in your Homework.

$$= 1 + (n-1/n)$$


Lots of h's?

(4) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , after an adversary chooses n items $\{u_1, \dots, u_n\}$ to hash, the **expected** number of items in u_x 's bucket is $O(1)$?

$$E[\text{number of items in } u_x\text{'s bucket}] = \sum_{y=1}^n P[h(u_x) = h(u_y)]$$

$$= 1 + \sum_{y \neq x} P[h(u_x) = h(u_y)]$$

$$= 1 + \sum_{y \neq x} 1/n$$

 You will prove this is the case for the exhaustive set H in your Homework.

$$= 1 + (n-1/n)$$

$$\leq 2$$

The Good News


(4) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \rightarrow \{1, \dots, n\}$, such that if we chose a random h in H , after an adversary chooses n items $\{u_1, \dots, u_n\}$ to hash, the **expected** number of items in u_x 's bucket is $O(1)$?

Yes! This is great news! It means that we can choose H to be the exhaustive set of all hash functions, and the `insert`, `delete`, `search` operations on any n elements will have an expected runtime of $O(1)$ per operation.

The Bad News

The exhaustive set of all hash functions is HUGEEEE!!!

How many bits would it take to write down the name of one of the $n^{|U|}$ hash functions in this H ? 🤔

 Like really huge.

The Bad News

The exhaustive set of all hash functions is HUGEEE!!!

How many bits would it take to write down the name of one of the $n^{|U|}$ hash functions in this H ? 🤔 $\log n^{|U|} = |U| \log n$.

Like really huge.

To see why, consider how much memory it would take to write down the name of one of the 8 hash functions from earlier. You could assign h_1 the id 000, h_2 the id 001, etc. So 8 hash functions requires $\log 8 = 3$ bits to write down.

$|U| \log n$ bits is enough to do direct addressing!

H Is Too Big

How can we fix this issue of the size of H?

3 Min Break

Universal Hash Functions

H Is Too Big

How can we fix this issue of the size of H?

Pick from a smaller set H, that still guarantees **(4)**.

Recall the bound that allowed us to achieve this guarantee:

$$\begin{aligned} E[\text{number of items in } u_x \text{'s bucket}] &= \sum_{y=1}^n P[h(u_x) = h(u_y)] \\ &= 1 + \sum_{y \neq x} P[h(u_x) = h(u_y)] \\ &= 1 + \sum_{y \neq x} 1/n \quad \swarrow \text{This step!} \\ &= 1 + (n-1)/n \\ &\leq 2 \end{aligned}$$

Universal Hash Family

This bound is so important, there's a special name for sets H that satisfy it.

A **hash family** is a fancy name for a set of hash functions.

Universal Hash Family

This bound is so important, there's a special name for sets H that satisfy it.

A **hash family** is a fancy name for a set of hash functions.

A **universal hash family** describes a set of hash functions that satisfy the bound: $P_{h \in H}[h(u_x) = h(u_y)] \leq 1/n$.

The exhaustive set of hash functions is an example of a universal hash family but, as discussed previously, it's too big to be practical.

A Smaller Universal Hash Family

Identifying new smaller universal hash families is an active field of research in computer science.

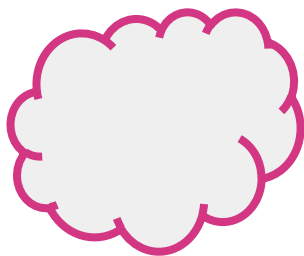
One of the more well-studied universal hash families:

To hash an integer x in $\{0, \dots, |U|-1\}$ to a bucket $\{1, \dots, n\}$:

$$h_{a,b}(x) = ax + b \bmod p \bmod n$$

for some prime $p \geq |U|$ and $a \in \{1, \dots, p-1\}$ and $b \in \{0, \dots, p-1\}$

To select an $h_{a,b}$ from this family:



1. Determine $|U|$.

p

2. Find the smallest prime $p \geq |U|$.

a

3. Let **a** be a random number in $\{1, \dots, p-1\}$.

b

3. Let **b** be a random number in $\{0, \dots, p-1\}$.

How Small Is This H?

There are $p-1$ choices for **a** and p choices for **b**,
so $|H| = p(p-1) = O(p^2) = O(|U|^2)$.

That's much better than $n^{|U|}$.

The space need to store h is $\log |U|^2 = O(\log |U|) \ll O(|U| \log n)$.



1. Determine $|U|$.

p

2. Find the smallest
prime **p** $\geq |U|$.

a

3. Let **a** be a
random number in
 $\{1, \dots, p-1\}$.

b

3. Let **b** be a
random number in
 $\{0, \dots, p-1\}$.

Another Universal Hash Family

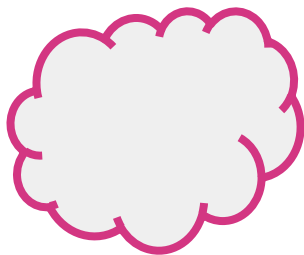
Another of the more well-studied universal hash families (using matrix multiplication!):

To hash a u -bit string x (i.e. bit string of length u) to a bucket $\{1, \dots, n\}$ (i.e. bit string of length $b = \log(n)$):

$$h_A(x) = Ax$$

for some $b \times u$ matrix A of 0's and 1's, using binary (modulo 2) arithmetic.

To select an h_A from this family:



1. Determine $|U|$.

u

2. $u = \log(|U|)$.

b

3. $b = \log(n)$.

A

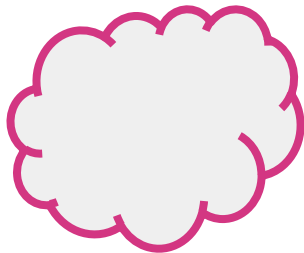
3. Let A be a $b \times u$ random matrix of 0's and 1's.

How Small Is This H?

How many possible binary matrices of size $b \times u$ for **A**?

$$2^{ub} = O(|U|^{\log(n)}).$$

That's much better than $n^{|U|}$, but larger than the other universal hash family $O(|U|^2)$.



1. Determine $|U|$.

u

2. $u = \log(|U|)$.

b

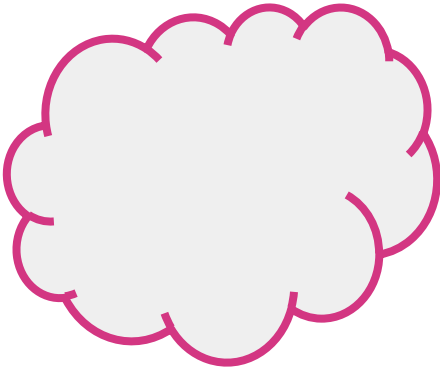
3. $b = \log(n)$.

A

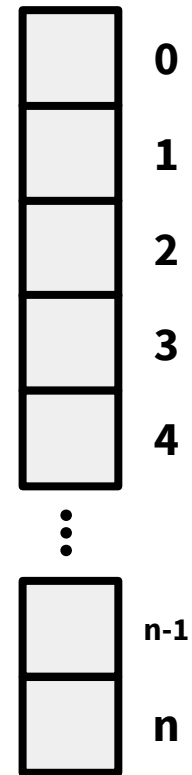
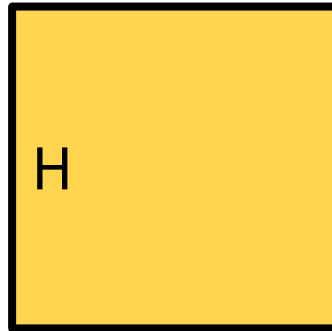
3. Let **A** be a $b \times u$ random matrix of 0's and 1's.

Hash Tables

Let's say you wanted to implement a hash table ...

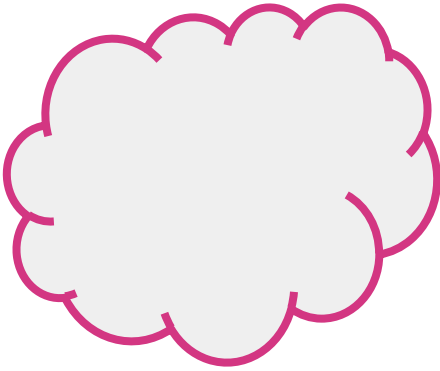


1. You choose your set of hash functions H , likely a universal hash family like $H = \text{mod } p \text{ mod } n$.



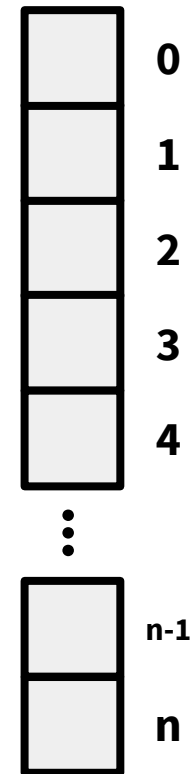
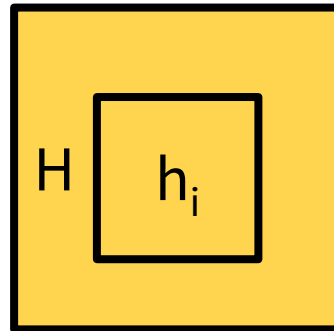
Hash Tables

Let's say you wanted to implement a hash table ...



1. You choose your set of hash functions H , likely a universal hash family like $H = \text{mod } p \text{ mod } n$.

2. When the client initializes a hash table, you randomly pick a hash function h_i from H to use in the hash table to hash the items.



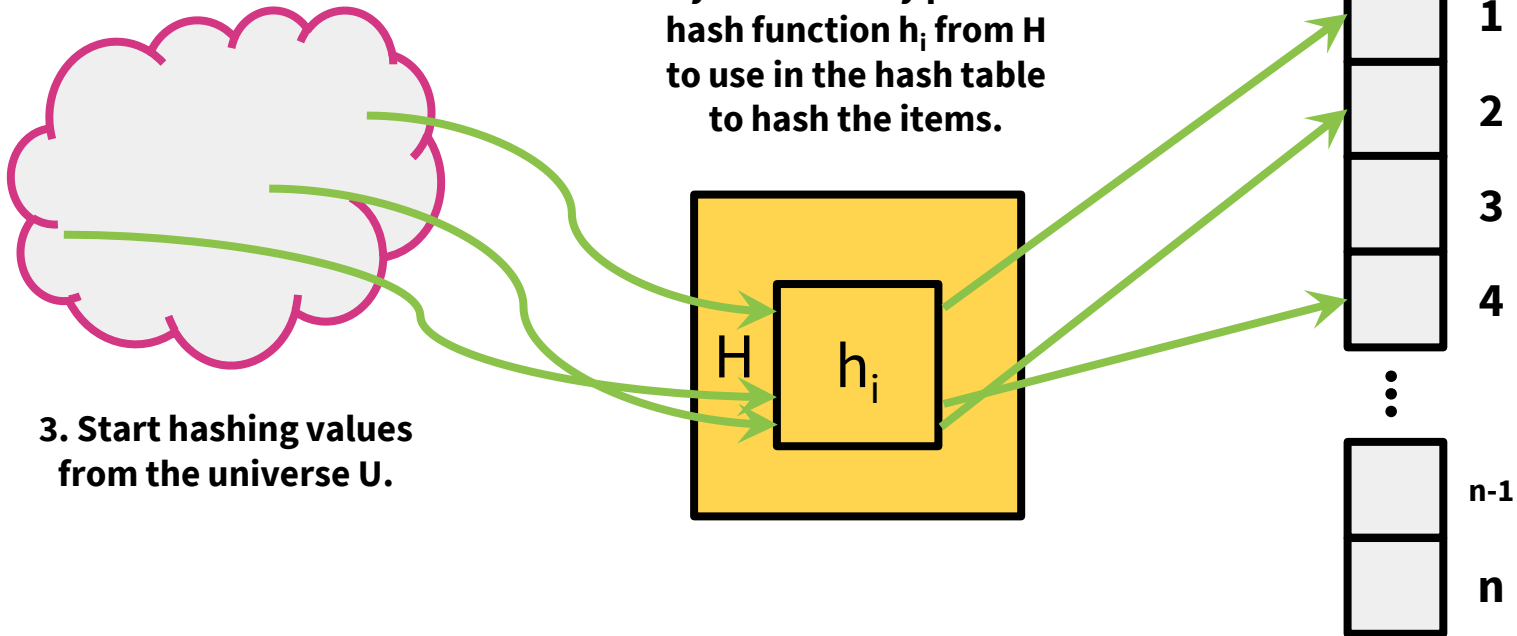
Hash Tables

Let's say you wanted to implement a hash table ...

1. You choose your set of hash functions H , likely a universal hash family like $H = \text{mod } p \text{ mod } n$.

2. When the client initializes a hash table, you randomly pick a hash function h_i from H to use in the hash table to hash the items.

3. Start hashing values from the universe U .



What's the Source of the Randomness?

As was the case in quicksort, we want the average-case runtime for a specific input to be low.

This is why it was important for us to select our pivot randomly as opposed to select, say, the first element in the sublist in quicksort.

What's the Source of the Randomness?

As was the case in quicksort, we want the average-case runtime for a specific input to be low.

This is why it was important for us to select our pivot randomly as opposed to select, say, the first element in the sublist in quicksort.

If we selected the pivot as the first element, then the expected runtime over all of the inputs would be low, but for certain inputs (namely, the one in reverse order), runtime would be guaranteed to be high.

What's the Source of the Randomness?

As was the case in quicksort, we want the average-case runtime for a specific input to be low.

This is why it was important for us to select our pivot randomly as opposed to select, say, the first element in the sublist in quicksort.

If we selected the pivot as the first element, then the expected runtime over all of the inputs would be low, but for certain inputs (namely, the one in reverse order), runtime would be guaranteed to be high.

Instead, since our algorithm supplied the randomness, for any specific input (even the one in reverse order), the expected runtime is low.

What's the Source of the Randomness?

As was the case in quicksort, we want the average-case runtime for a specific input to be low.

This is why it was important for us to select our pivot randomly as opposed to select, say, the first element in the sublist in quicksort.

If we selected the pivot as the first element, then the expected runtime over all of the inputs would be low, but for certain inputs (namely, the one in reverse order), runtime would be guaranteed to be high.

Instead, since our algorithm supplied the randomness, for any specific input (even the one in reverse order), the expected runtime is low.

Note this does not say anything the worst-case runtime, which you can think of as the case in which the adversary controls the randomness we're using. This remains the same even though we introduced randomness into our algorithm.

What's the Source of the Randomness?

As was the case in quicksort, we want the average-case runtime for a specific input to be low.

This is why it was important for us to select our pivot randomly as opposed to select, say, the first element in the sublist in quicksort.

If we selected the pivot as the first element, then the expected runtime over all of the inputs would be low, but for certain inputs (namely, the one in reverse order), runtime would be guaranteed to be high.

Instead, since our algorithm supplied the randomness, for any specific input (even the one in reverse order), the expected runtime is low.

Note this does not say anything the worst-case runtime, which you can think of as the case in which the adversary controls the randomness we're using. This remains the same even though we introduced randomness into our algorithm.

Same thing here with hash tables.