Randomized Algorithms II

Summer 2017 • Lecture 5

A Few Notes

Homework 2

- Due tomorrow night at 11:59 p.m. on Gradescope.

Homework 3

- Released tomorrow night.
- Due Friday 7/21 at 11:59 p.m. on Gradescope.

Outline for Today

More randomized algorithms!

- Hashing Basics
- Universal Hash Functions
- What's the Source of the Randomness?

Hashing Basics

Randomized Algorithms

A randomized algorithm is an algorithm that incorporates randomness as part of its operation.

Often aim for properties like …

- Good average-case behavior
- Getting exact answers with high probability
- Getting answers that are close to the right answer

Data Structures

	Sorted linked lists	Sorted arrays	Balanced BSTs
Search	`O(n)` **expected & worst-case**	`O(log n)` **expected & worst-case**	`O(log n)` **expected & worst-case** O(n) worst-case for generic BSTs
Insert/ Delete	`O(n)` **expected & worst-case** without a pointer to the element	`O(n)` **expected & worst-case**	`O(log n)` **expected & worst case**

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Insert/ Delete	`O(n)` expected & worst-case without a pointer to the element	`O(n)` expected & worst-case	`O(log n)` expected & worst case	`O(1)` expected `O(n)` worst-case without a pointer to the element

Direct Addressing

How might we get **`O(1)`**-time? Try direct addressing!

One type of item per address.

`insert(7)`

0	1	2	3	4	5	6	7	8	9
							7		

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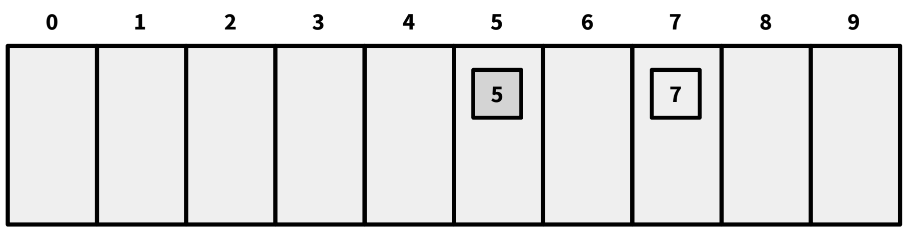

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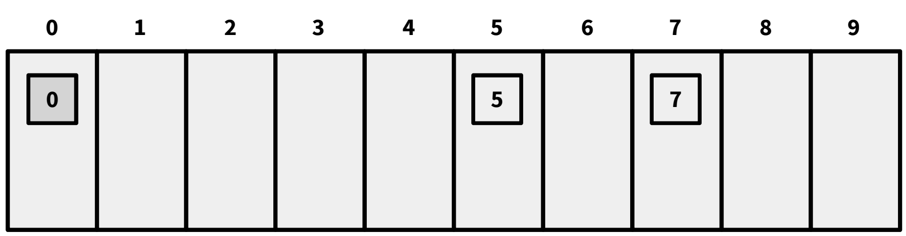

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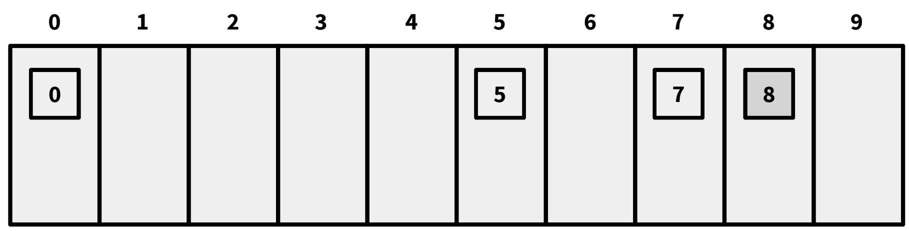

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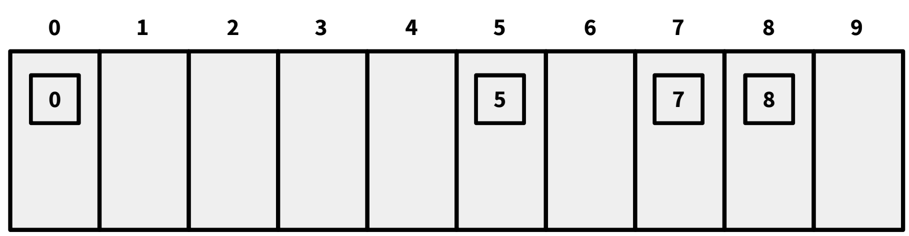
insert(7)    search(7)
insert(5)    search(2)
insert(0)
insert(8)
```



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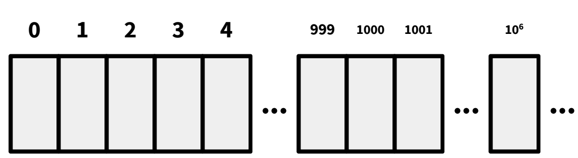
Similar to `counting_sort` and `bucket_sort` (for k ≤ num_buckets), if the set of items being inserted/deleted (e.g. {0, 1, 2, …, 999, 1000, …, 10^6, …}) is large, then the sheer space required to maintain this data structure becomes an issue.



Direct Addressing

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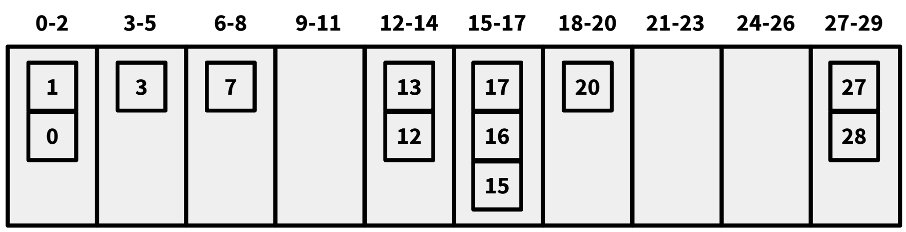
- Can we fix this issue by assigning multiple types of item per address, like case (2) of `bucket_sort`?
- Sometimes, this binning approach is useful. `search(12)` still runs pretty fast.



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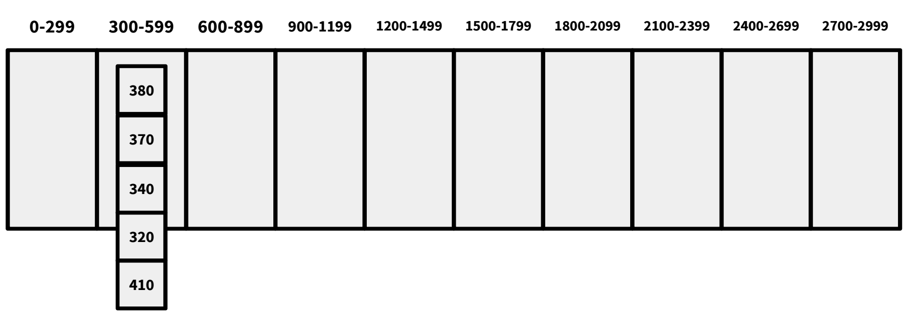
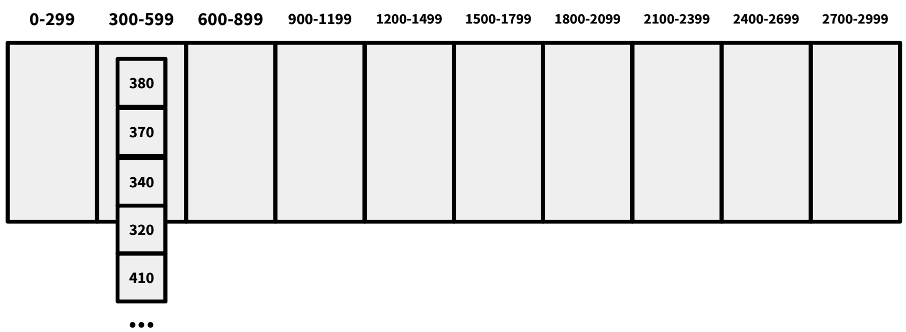
- Can we fix this issue by assigning multiple types of item per address, like case (2) of `bucket_sort`?
- Other times, it causes an issue. `search(432)` is slow.



Direct Addressing

This is an example of a hash table.

- Albeit one with a basic bucketing scheme.
- Can we do better?



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- |U| >>> n; i.e. |U| is a lot bigger than n.
- We don't know which of the |U| possible keys we'll need to store;
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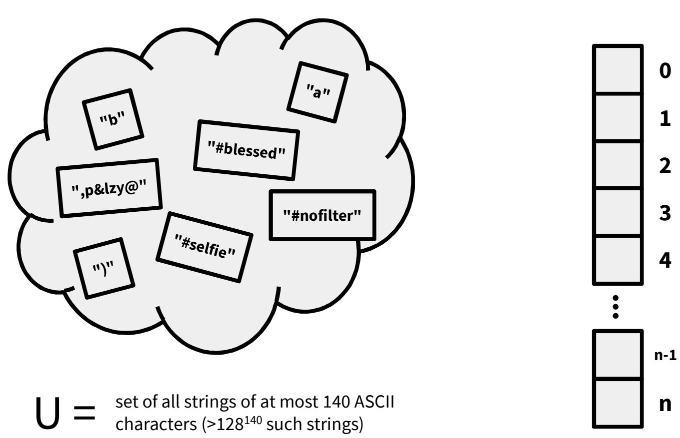
There's a hash function h: U → {1, …, n} that maps keys to buckets.

An Example



U = set of all strings of at most 140 ASCII characters ($>128^{140}$ such strings)

And we'll need to store a small subset of U (say, the ones that might be trending hashtags on Twitter); we're assuming the number of hashtags ≤ n, the number of buckets.

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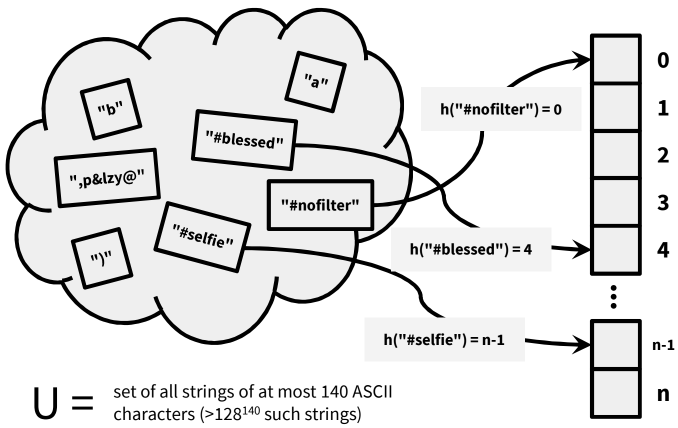

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Hash Tables (with chaining)

List of n buckets.

Each bucket stores an unsorted linked list.

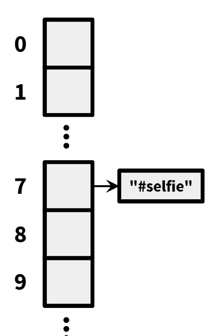
- `insert` in **`O(1)`** since it's unsorted; `search` in **`O(n)`**.

h: U → {1, …, n} can be any function

- For concreteness, suppose it's `length`.

Suppose we `insert` a bunch of keys and then `search`.

`insert("#selfie")`



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insert("#nofilter")
```

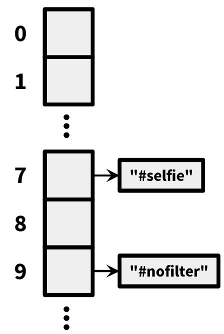

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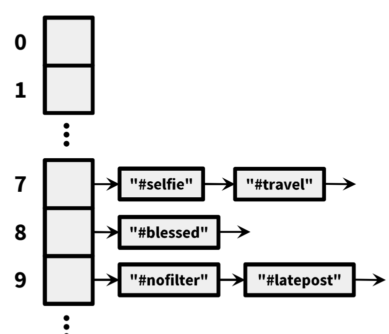
Suppose we `insert` a bunch of keys and then `search`.

```
insert("#selfie")
insert("#nofilter")
insert("#blessed")
insert("#travel")
insert("#latepost")
search("#travel")
```

Scans through all elements in bucket h("#travel")



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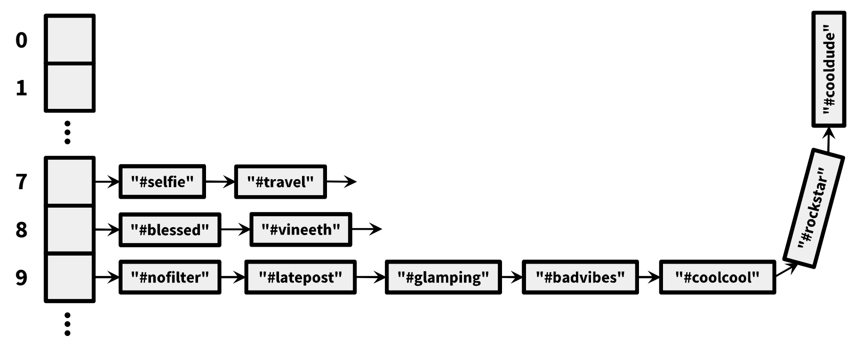
So how do we choose a better h?

The items need to be spread out in the buckets.



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(1) Can we design a single $h_{chosenone}$: U → {1, …, n} such that all buckets will have size O(1) after hashing any n items?

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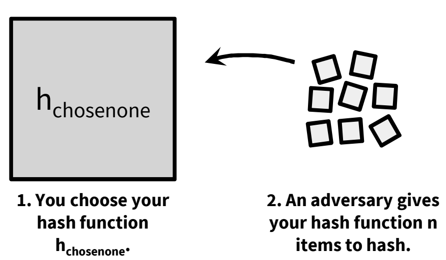

Is it possible to construct $h_{chosenone}$ such that you're guaranteed that all buckets will have size O(1)? This would be ideal. 🤔

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Let's call the set of items that get hashed to this bucket $U_{bigbucket}(h_{chosenone})$ where $U_{bigbucket} \subset U$. The adversary could choose to hash n items from $U_{bigbucket}$. This is a valid set of n items, and results in one bucket with all n items, by construction. Therefore, **(1)** is impossible.

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Notation indicating $U_{bigbucket}$ is a function of $h_{chosenone}$ ↘

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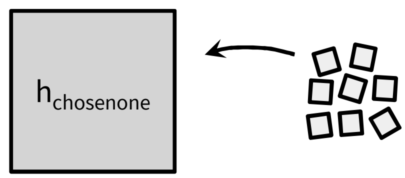

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Can you think of such an $h_{chosenone}$? 🤔

Probably not. This is the same question as **(1)**! Since the adversary is choosing the n items, there's no randomness anywhere in the process. As a result, the **expected** size of a bucket is trivially just the size.

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Where? Well there's only one option … in our choice of hash function.

We will randomly choose h from a large set of hash functions!
(There won't be an h to rule them all).

Lots of h's?

(3) Can we design a set $H = \{h_1, \dots, h_k\}$ where $h: U \to \{1, \dots, n\}$, such that if we chose a random h in H, all buckets will have **expected** size O(1) after hashing any n items?

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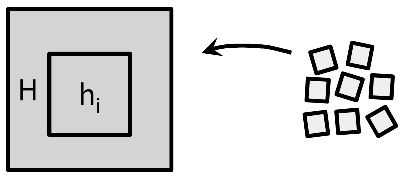

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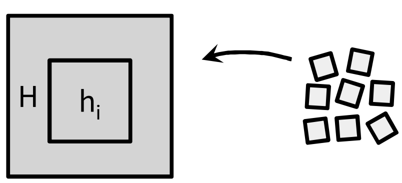

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$$
\begin{aligned}
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&= (1/n) \cdot n \\
&= 1
\end{aligned}
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But P(lots of keys get hashed to one bucket) = 1.

This is not good. Maybe we should be using a different metric.

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Not useful!

(4) Can we design a set $H = \{h_1, \ldots, h_k\}$ where $h: U \rightarrow \{1, \ldots, n\}$, such that if we chose a random h in H, after an adversary chooses n items $\{u_1, \ldots, u_n\}$ to hash, the **expected** number of items in u_x's bucket is O(1)?

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As an analogy for the difference between **(3)** and **(4)**, consider the "small classes illusion." Suppose a university offers 10 classes, 9 of which have 1 person in them and the last of which has 500 persons in them. Using reasoning from **(3)**, the university might tout average class sizes of ~50, when in reality, it should report much class sizes experienced by the average student, as in **(4)**.

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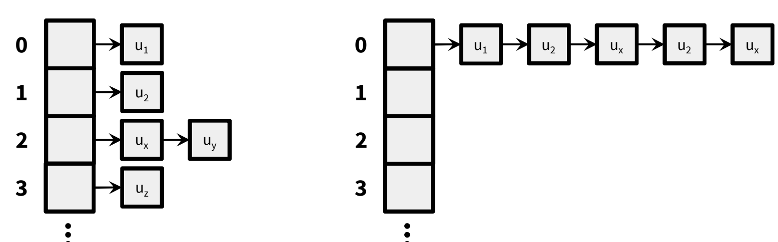
(4) Can we design a set H = {h_1, … , h_k} where h: U → {1, …, n}, such that if we chose a random h in H, after an adversary chooses n items {u_1, …, u_n} to hash, the **expected** number of items in u_x's bucket is O(1)?

We can think of this statement in terms of minimizing the expected number of collisions.



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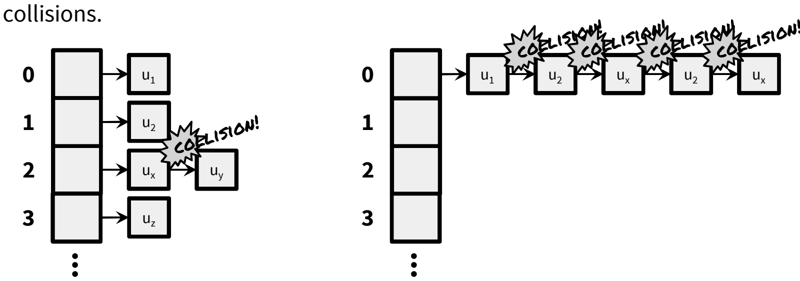

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Yes! This time it's possible.

	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
"a"	0	0	0	0	1	1	1	1
"b"	0	0	1	1	0	0	1	1
"c"	0	1	0	1	0	1	0	1

The 0's and 1's represent the buckets i.e. h_8 hashes "b" to bucket 1.

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e.g. Suppose U = {"a", "b", "c"} and n = 2 (there are 2 buckets). H would be a set of 8 hash functions. One h would map "a", "b", and "c" all to bucket 0. Another h would map "a" and "b" to bucket 0 and "c" to bucket 1. etc. etc.

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$$\leq 2$$

The Good News

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Yes! This is great news! It means that we can choose H to be the exhaustive set of all hash functions, and the `insert`, `delete`, `search` operations on any n elements will have an expected runtime of O(1) per operation.

The Bad News

The exhaustive set of all hash functions is HUGEEE!!!

How many bits would it take to write down the name of one of the $n^{|U|}$ hash functions in this H? 🤔

Like really huge.

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How many bits would it take to write down the name of one of the $n^{|U|}$ hash functions in this H? 🤔 $\log n^{|U|} = |U| \log n$.

Like really huge.

To see why, consider how much memory it would take to write down the name of one of the 8 hash functions from earlier. You could assign h_1 the id 000, h_2 the id 001, etc. So 8 hash functions requires log 8 = 3 bits to write down.

|U| log n bits is enough to do direct addressing!

H Is Too Big

How can we fix this issue of the size of H?

3 Min Break

Universal Hash Functions

H Is Too Big

How can we fix this issue of the size of H?

- Pick from a smaller set H, that still guarantees **(4)**.
- Recall the bound that allowed us to achieve this guarantee:

$$
\begin{aligned}
E[\text{number of items in } u_x\text{'s bucket}] &= \sum_{y=1}^{n} P[h(u_x) = h(u_y)] \\
&= 1 + \sum_{y \neq x} P[h(u_x) = h(u_y)] \\
&= 1 + \sum_{y \neq x} 1/n \quad \textbf{This step!} \\
&= 1 + (n-1/n) \\
&\leq 2
\end{aligned}
$$

Universal Hash Family

This bound is so important, there's a special name for sets H that satisfy it.

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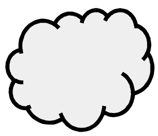
- A **hash family** is a fancy name for a set of hash functions.
- A **universal hash family** describes a set of hash functions that satisfy the bound: $P_{h \in H}[h(u_x) = h(u_y)] \leq 1/n$.
  - The exhaustive set of hash functions is an example of a universal hash family but, as discussed previously, it's too big to be practical.

A Smaller Universal Hash Family

Identifying new smaller universal hash families is an active field of research in computer science.

One of the more well-studied universal hash families:

- To hash an integer x in {0, …, |U|-1} to a bucket {1, …, n}:
- $h_{a,b}(x) = ax + b \; mod \; p \; mod \; n$
  - for some prime $p \geq |U|$ and $a \in \{1, …, p - 1\}$ and $b \in \{0, …, p - 1\}$
- To select an $h_{a,b}$ from this family:

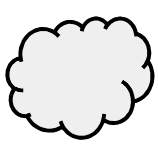


	p	**a**	**b**
1. Determine |U|.	**2. Find the smallest prime p ≥ |U|.**	**3. Let a be a random number in {1, …, p - 1}.**	**3. Let b be a random number in {0, …, p - 1}.**

How Small Is This H?

There are p-1 choices for **a** and p choices for **b**, so $|H| = p(p-1) = O(p^2) = O(|U|^2)$.

- That's much better than $n^{|U|}$.
- The space need to store h is $\log |U|^2 = O(\log |U|) << O(|U| \log n)$.



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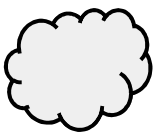
Another Universal Hash Family

Another of the more well-studied universal hash families (using matrix multiplication!):

- To hash a u-bit string x (i.e. bit string of length u) to a bucket {1, …, n} (i.e. bit string of length b = log(n)):
- $h_A(x) = Ax$
- for some $b \times u$ matrix A of 0's and 1's, using binary (modulo 2) arithmetic.
- To select an h_A from this family:



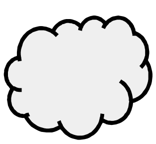
u **b** **A**

1. Determine |U|. **2. u = log(|U|).** **3. b = log(n).** **3. Let A be a $b \times u$ random matrix of 0's and 1's.**

How Small Is This H?

How many possible binary matrices of size $b \times u$ for **A**?

- $2^{ub} = O(|U|^{\log(n)})$.
- That's much better than $n^{|U|}$, but larger than the other universal hash family $O(|U|^2)$.



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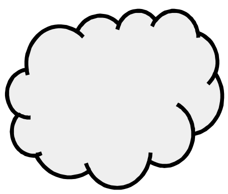
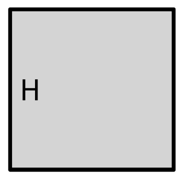
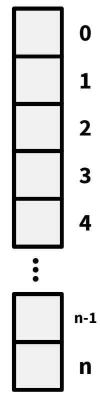
Hash Tables

Let's say you wanted to implement a hash table ...

1. You choose your set of hash functions H, likely a universal hash family like H = mod p mod n.







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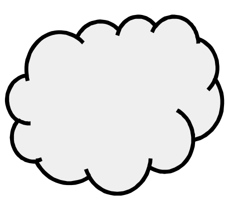
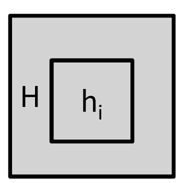
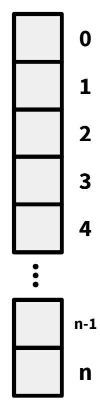
1. You choose your set of hash functions H, likely a universal hash family like H = mod p mod n.

2. When the client initializes a hash table, you randomly pick a hash function h_i from H to use in the hash table to hash the items.







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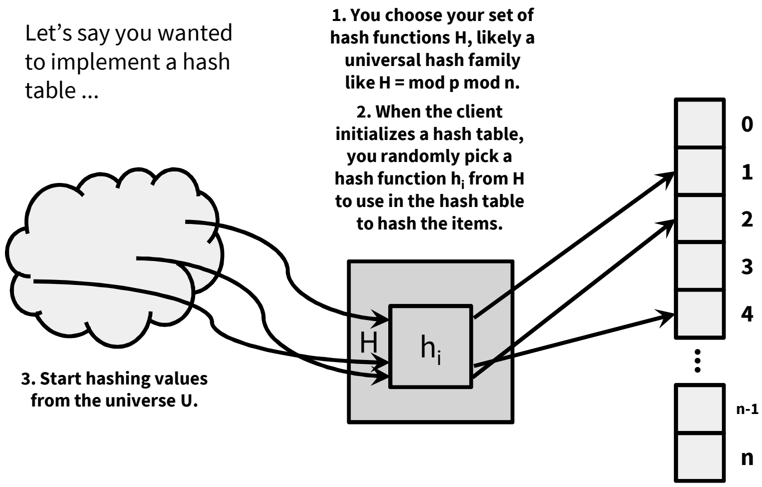

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As was the case in `quicksort`, we want the average-case runtime for a specific input to be low.

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Same thing here with hash tables.