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Mathematical Modelling

Part 1: Optimization Model

Content of part 1

Chapter 1: One-variable optimization

Chapter 2: Multivariable optimization

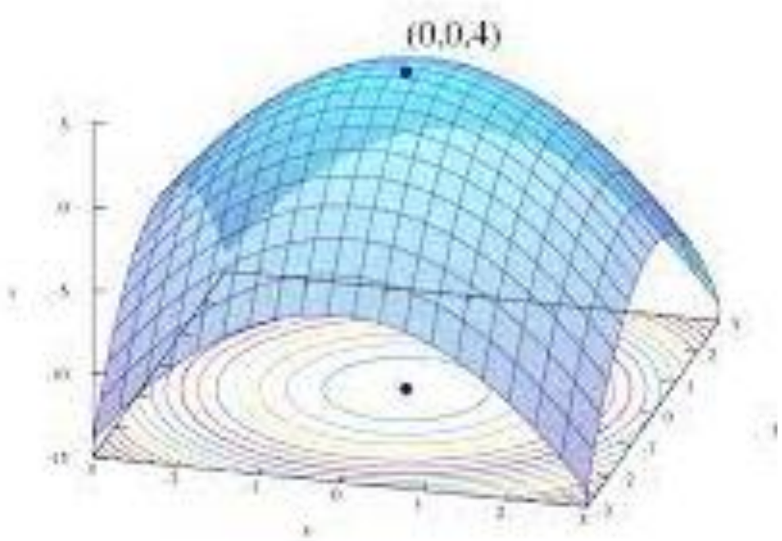
Chapter 3: Multi-objective optimization

Outline

- **Unconstrained optimization**
- Lagrange multipliers
- Exercises

Unconstrained optimization

- The simplest type of multivariable optimization problems involves finding the maximum or minimum of a differentiable function of several variables over a nice set.



Illustrated example

A manufacturer of TV sets is planning the introduction of two new products, a 19–inch LCD flat panel set with a manufacturer's suggested retail price (MSRP) of $339 and a 21–inch LCD flat panel set with an MSRP of $399. The cost to the company is $195 per 19–inch set and $225 per 21–inch set, plus an additional $400,000 in fixed costs. In the competitive market in which these sets will be sold, the number of sales per year will affect the average selling price. It is estimated that for each type of set, the average selling price drops by one cent for each additional unit sold. Furthermore, sales of the 19–inch set will affect sales of the 21–inch set, and vice–versa. It is estimated that the average selling price for the 19–inch set will be reduced by an additional 0.3 cents for each 21–inch set sold, and the price for the 21–inch set will decrease by 0.4 cents for each 19–inch set sold. How many units of each type of set should be manufactured?

Step 1: Ask a question

Variables:

s = number of 19–inch sets sold (per year)
t = number of 21–inch sets sold (per year)
p = selling price for a 19–inch set (\$)
q = selling price for a 21–inch set (\$)
C = cost of manufacturing sets (\$/year)
R = revenue from the sale of sets (\$/year)
P = profit from the sale of sets (\$/year)

Assumptions:

$$p = 339 - 0.01s - 0.003t$$
$$q = 399 - 0.004s - 0.01t$$
$$R = ps + qt$$
$$C = 400,000 + 195s + 225t$$
$$P = R - C$$
$$s \geq 0$$
$$t \geq 0$$

Objective: Maximize P

Step 2: Select a modelling approach

- We will solve this problem as a multivariable unconstrained optimization problem.
- General solution procedure for this kind of problem can be founded in previous courses such as mathematics, calculus and optimization

Step 2: Select the modelling approach

- Important theorem

We are given a function $y = f(x_1, \ldots, x_n)$ on a subset S of the n–dimensional space $\mathbb{R}^n$. We wish to find the maximum and/or minimum values of f on the set S. There is a theorem that states that if f attains its maximum or minimum at an interior point $(x_1, \ldots, x_n)$ in S, then $\nabla f = 0$ at that point, assuming that f is differentiable at that point. In other words, at the extreme point

$$\begin{aligned} \frac{\partial f}{\partial x_1}(x_1, \ldots, x_n) &= 0 \\ \frac{\partial f}{\partial x_n}(x_1, \ldots, x_n) &= 0. \end{aligned} \tag{2.1}$$

Step 2: Select a modelling approach

- To find the max-min points
  - First, we must solve simultaneously *n* equations mentioned in the theorem
  - Secondly, we must check any points on the boundary of interior set, as well as points where one or more the partial derivatives is undefined

Step 3: Formulate model

$$
\begin{aligned}
P &= R - C \\
&= ps + qt - (400,000 + 195s + 225t) \\
&= (339 - 0.01s - 0.003t)s \\
&\quad + (399 - 0.004s - 0.01t)t \\
&\quad - (400,000 + 195s + 225t).
\end{aligned}
$$

Step 3: Formulate model

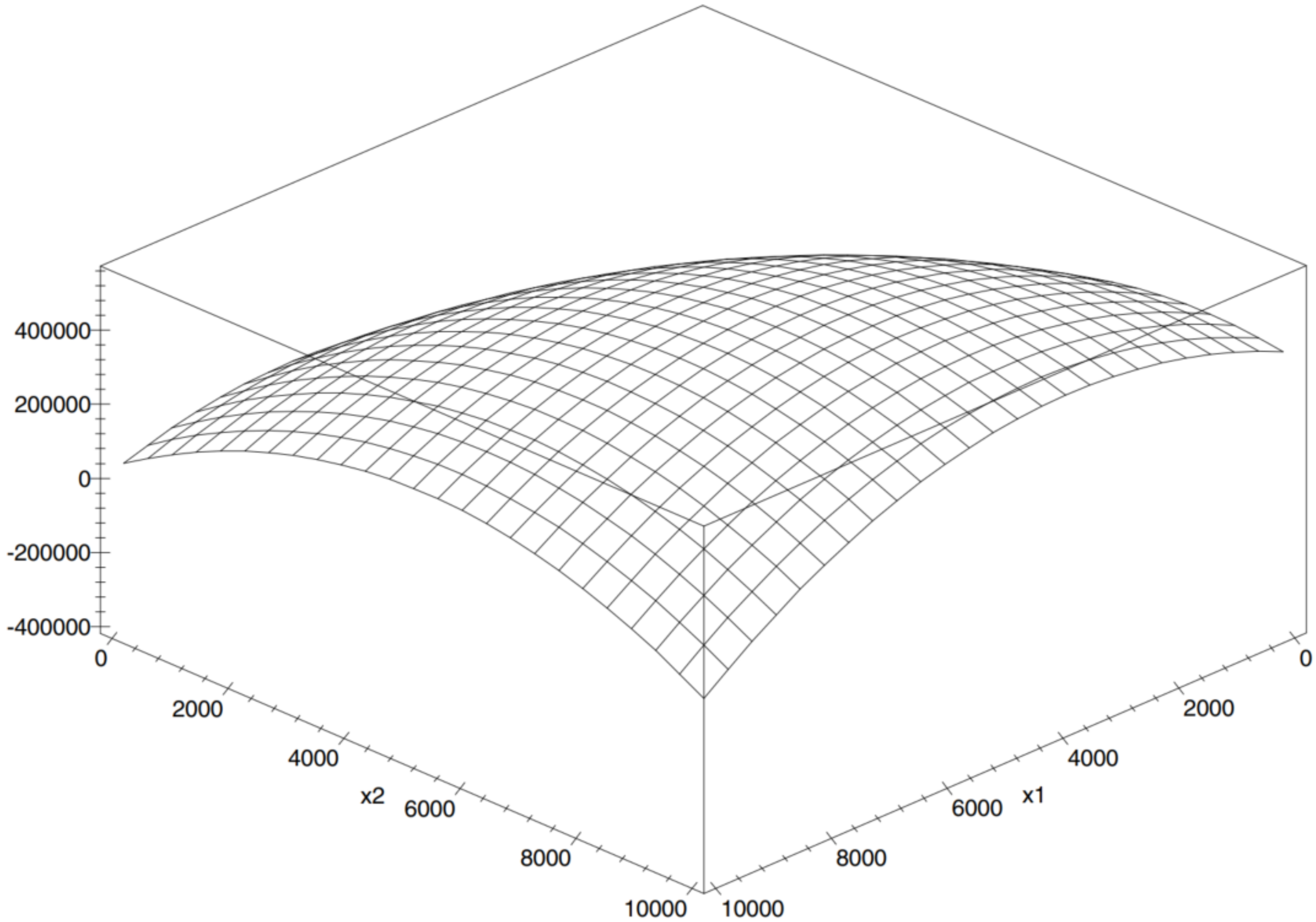
- Let $y = p$ and $x_1 = s, x_2 = t$

$$\begin{aligned} y &= f(x_1, x_2) \\ &= (339 - 0.01x_1 - 0.003x_2)x_1 \\ &\quad + (399 - 0.004x_1 - 0.01x_2)x_2 \\ &\quad - (400,000 + 195x_1 + 225x_2) \end{aligned}$$

over the set

$$S = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}.$$

Step 4: Solve the problem



Step 4: Solve the problem

$$\frac{\partial f}{\partial x_1} = 144 - 0.02x_1 - 0.007x_2 = 0$$

$$\frac{\partial f}{\partial x_2} = 174 - 0.007x_1 - 0.02x_2 = 0$$

at the point

$$x_1 = \frac{554,000}{117} \approx 4,735$$

$$x_2 = \frac{824,000}{117} \approx 7,043.$$

$$y = \frac{21,592,000}{39} \approx 553,641.$$

Step 5: Answer the question

Company can maximize profits by manufacturing 4,735 of the 19–inch sets and 7,043 of the 21–inch sets, resulting in a net profit of $553,641 for the year. The average selling price for a 19–inch set is $270.52 and $309.63 for a 21–inch set. The projected revenue is $3,461,590, resulting in a profit margin (profit/revenue) of 16 percent. These figures indicate a profitable venture, so we would recommend that the company proceed with the introduction of these new products.

Sensitivity analysis

We illustrate the procedure for sensitivity analysis by examining the sensitivity to price elasticity for 19–inch sets, which we will denote by the variable a. In the model $a = 0.01$

$$
\begin{aligned}
y &= f(x_1, x_2) \\
&= (339 - ax_1 - 0.003x_2)x_1 \\
&\quad + (399 - 0.004x_1 - 0.01x_2)x_2 \\
&\quad - (400,000 + 195x_1 + 225x_2).
\end{aligned}
\tag{2.7}
$$

When we compute partial derivatives and set them equal to zero, we obtain

$$
\begin{aligned}
\frac{\partial f}{\partial x_1} &= 144 - 2ax_1 - 0.007x_2 = 0 \\
\frac{\partial f}{\partial x_2} &= 174 - 0.007x_1 - 0.02x_2 = 0.
\end{aligned}
\tag{2.8}
$$

Sensitivity analysis

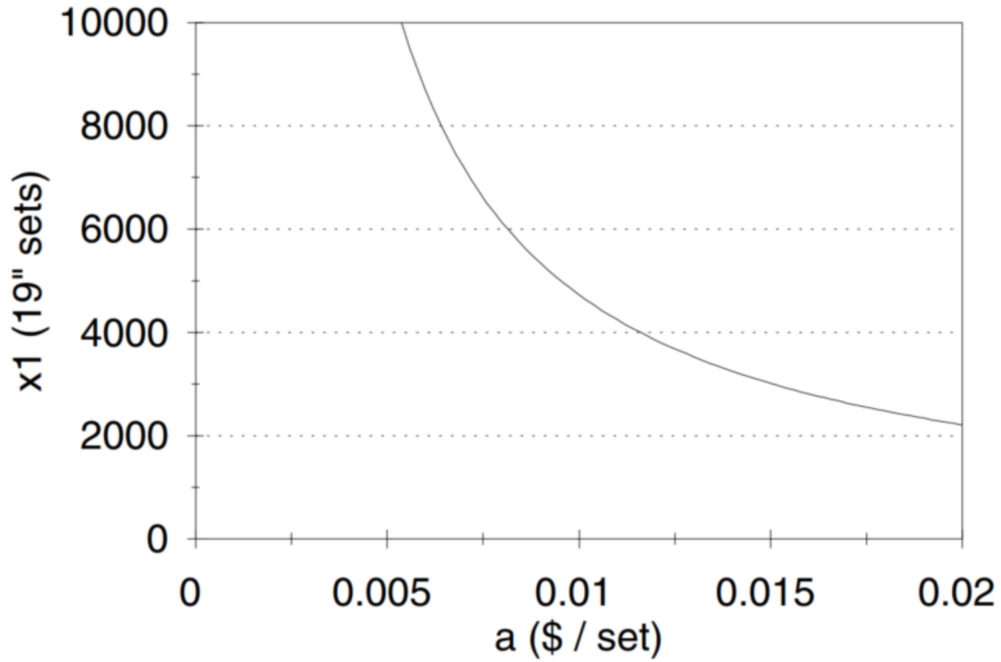
Solving for x_1 and x_2 as before yields

$$x_1 = \frac{1,662,000}{40,000a - 49}$$

$$x_2 = 8,700 - \frac{581,700}{40,000a - 49}.$$

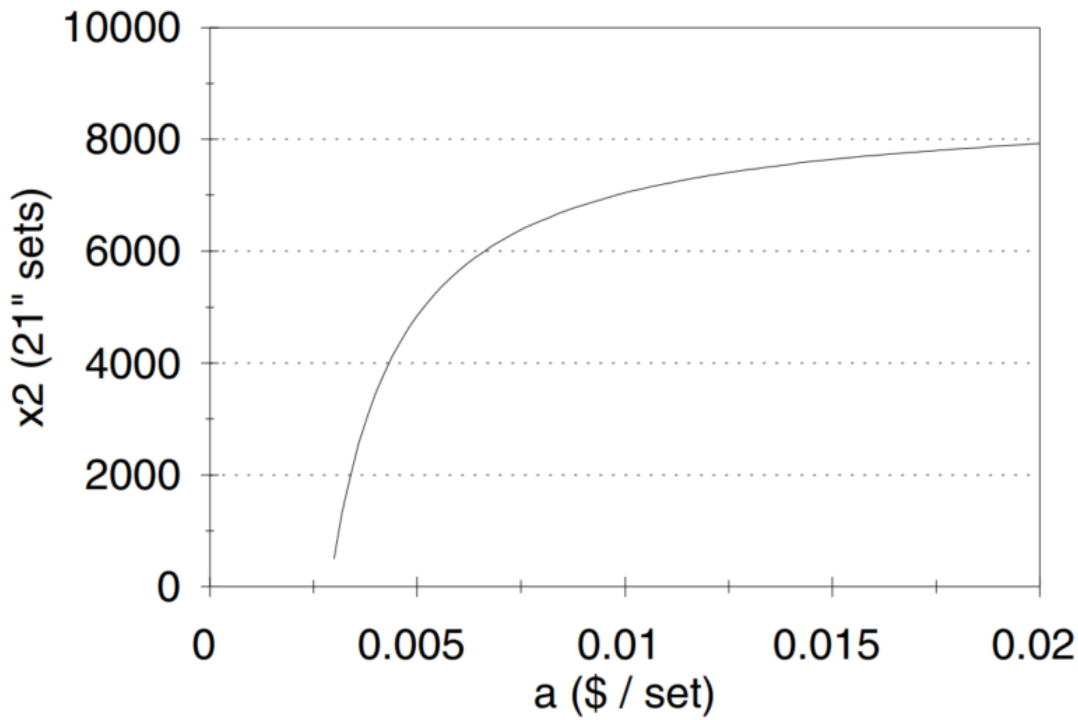
Sensitivity analysis

- Graph of optimum production level x_1 of 19–inch sets versus price elasticity a for the color TV problem



Sensitivity analysis

- Graph of optimum production level x_2 of 21–inch sets versus price elasticity a for the color TV problem.



Sensitivity analysis

$$\frac{dx_1}{da} = \frac{-66,480,000,000}{(40,000a - 49)^2}$$
$$= \frac{-22,160,000,000}{41,067}$$

at $a = 0.01$, so that

$$S(x_1, a) = \left(\frac{-22,160,000,000}{41,067}\right)\left(\frac{0.01}{554,000/117}\right)$$
$$= -\frac{400}{351} \approx -1.1.$$

Sensitivity analysis

A similar calculation yields $S(x_2, a)$

$$S(x_2,\, a) = \frac{9,695}{36,153} \approx 0.27.$$

So, If the price elasticity for 19–inch sets were to increase by 10%, then we should make 11% fewer 19–inch sets and 2.7% more 21–inch sets.

Sensitivity analysis

- What is $S(y, a)$

$$S(y,a) = -\left(\frac{554,000}{117}\right)^2 \frac{0.01}{(21,592,000/39}$$

$$= -\frac{383,645}{947,349} \approx -0.40.$$

Outliner

- Unconstrained optimization
- **Lagrange multipliers**
- Exercises

Lagrange multipliers

- Given a function $f(x_1, x_2, \dots, x_n)$ and a set of constraints which are expressed as k equations

$$\begin{aligned} g_1(x_1, \dots, x_n) &= c_1 \\ g_2(x_1, \dots, x_n) &= c_2 \\ &\vdots \\ g_k(x_1, \dots, x_n) &= c_k. \end{aligned}$$

Lagrange multipliers

There is a theorem that states that at an extreme point $x \in S$, we must have

$$\nabla f = \lambda_1 \nabla g_1 + \cdots + \lambda_k \nabla g_k.$$

We call $\lambda_1, \ldots, \lambda_k$ the *Lagrange multipliers*. This theorem assumes that $\nabla g_1, \ldots, \nabla g_k$ are linearly independent vectors (see Edwards (1973), p. 113). Then in order to locate the max–min points of f on the set S, we must solve the n Lagrange multiplier equations

Lagrange multipliers

$$\frac{\partial f}{\partial x_1} = \lambda_1 \frac{\partial g_1}{\partial x_1} + \cdots + \lambda_k \frac{\partial g_k}{\partial x_1}$$

$$\vdots$$

$$\frac{\partial f}{\partial x_n} = \lambda_1 \frac{\partial g_1}{\partial x_n} + \cdots + \lambda_k \frac{\partial g_k}{\partial x_n}$$

together with the k constraint equations

$$g_1(x_1, \ldots, x_n) = c_1$$

$$\vdots$$

$$g_k(x_1, \ldots, x_n) = c_k$$

Lagrange multipliers

- Note that
  - We have variables $x_1, x_2, \dots, x_n$ and $\lambda_1, \dots, \lambda_k$
  - We must check any exceptional points where gradient vectors $\nabla g_1, \nabla g_2, \dots, \nabla g_k$ are not linearly independent.

Lagrange multipliers

Example 1: Maximize $\boldsymbol{x+2y+3z}$ over the set $\boldsymbol{x^2+y^2+z^2=3}$.

Solution:

- Let $f(x,y,z) = x + 2y + 3z$ denote the objective function and $g(x,y,z) = x^2 + y^2 + z^2$ denote the constraint function.
- We compute $\nabla f = (1,2,3), \nabla g = (2x, 2y, 2z)$.
- At the maximum, $\nabla f = \lambda \nabla g$, in other words, $1 = 2x\lambda, 2 = 2y\lambda, 3 = 2z\lambda$.

Lagrange multipliers

- Solving in term of λ we have $x = \frac{1}{2\lambda}, y = \frac{1}{\lambda}, z = \frac{3}{2\lambda}$
- Using the fact that $x^2 + y^2 + z^2 = 3$, we have two candidates $\lambda_1 = \frac{\sqrt{42}}{6}$ and $\lambda_2 = -\frac{\sqrt{42}}{6}$.
- If $\lambda = \frac{\sqrt{42}}{6}$ so the point $a = (\frac{\sqrt{42}}{14}, \frac{\sqrt{42}}{7}, \frac{3\sqrt{42}}{14})$ is a candidate for the maximum
- If $\lambda = -\frac{\sqrt{42}}{6}$ so $b = -a$ is another candidate for maximum.

Lagrange multipliers

- Since $\nabla g \neq 0$ due to constraint set $g = 3$, so a and b are the only two candidates for the maximum.
- We have $f(a) = \sqrt{42}$ and $f(b) = -\sqrt{42}$
- In conclusion, a is the maximum point.

Lagrange multipliers

Example 2: Maximize $\boldsymbol{x + 2y + 3z}$ over the set $\boldsymbol{x^2 + y^2 + z^2 = 3}$ and $\boldsymbol{x = 1}$.

Solution:

- Let $f(x, y, z) = x + 2y + 3z$ denote the objective function and $g_1(x, y, z) = x^2 + y^2 + z^2, g_2(x, y, z) = x$ denote the constraint functions.
- We compute $\nabla f = (1, 2,3), \nabla g_1 = (2x, 2y, 2z), \nabla g_2(x, y, z) = (1,0,0)$.
- At the maximum, $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$, in other words, $1 = 2x\lambda_1 + \lambda_2, 2 = 2y\lambda_1, 3 = 2z\lambda_1$.

Lagrange multipliers

- Solving in term of λ_1, λ_2, we have $x = \frac{1-\lambda_2}{2\lambda_1}, y = \frac{1}{\lambda_1}, z = \frac{3}{2\lambda_1}$
- Using the fact that $x^2 + y^2 + z^2 = 3, x = 1$, we have two candidates $\lambda_1 = \pm\frac{\sqrt{26}}{4}$ and $\lambda_2 = 1 - 2\lambda_1$.
- Substituting back yields the two following solutions $c = (1, \frac{2\sqrt{26}}{13}, \frac{3\sqrt{26}}{13})$, $d = (1, \frac{-2\sqrt{26}}{13}, \frac{-3\sqrt{26}}{13})$

Lagrange multipliers

- Due to ∇g_1 and ∇g_2 are linearly independent everywhere on the constraint set, the points c and d are the only candidates for a maximum.
- So $f(c) = 1 + \sqrt{26}$ is maximum point.

Lagrange multipliers

- In the illustrated example, we assume that the company can produce any number of product (no limitation of capacity). This is not too much realistic. If we consider additionally constraints on maximum numbers of products that the company can product per year

Step 1: Ask a question

Variables:

s = number of 19–inch sets sold (per year)
t = number of 21–inch sets sold (per year)
p = selling price for a 19–inch set (\$)
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C = cost of manufacturing sets (\$/year)
R = revenue from the sale of sets (\$/year)
P = profit from the sale of sets (\$/year)

Step 1: Ask a question

Assumptions:

$$
\begin{aligned}
p &= 339 - 0.01s - 0.003t \\
q &= 399 - 0.004s - 0.01t \\
R &= ps + qt \\
C &= 400,000 + 195s + 225t \\
P &= R - C \\
s &\le 5000 \\
t &\le 8000 \\
s + t &\le 10,000 \\
s &\ge 0 \\
t &\ge 0
\end{aligned}
$$

Objective: Maximize P

Step 2: Select a modelling approach

- Due to additional constraints, we select multivariable constrained optimization problem.

- Solution approach is method of Lagrange Multipliers

Step 3: Formulate model

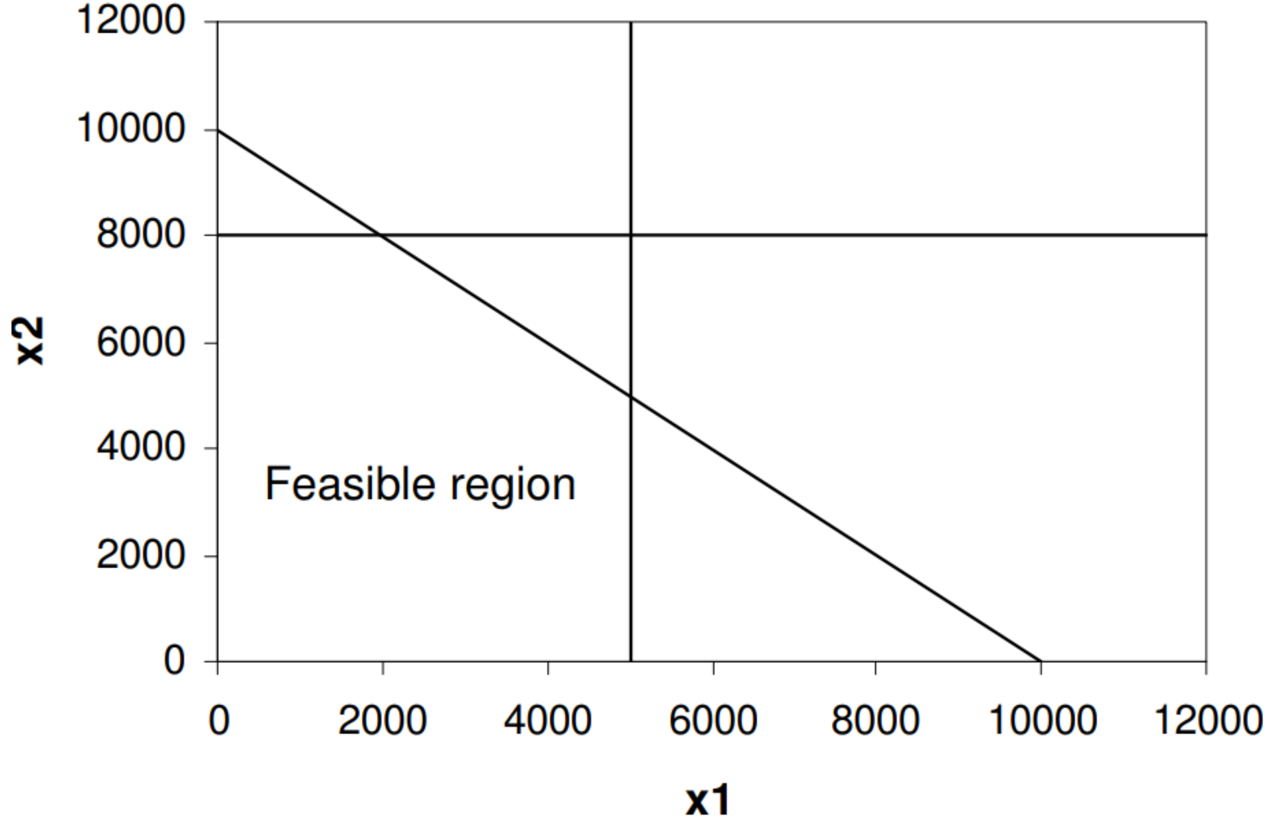
$$
\begin{aligned}
y &= f(x_1, x_2) \\
&= (339 - 0.01x_1 - 0.003x_2)x_1 + (399 - 0.004x_1 - 0.01x_2)x_2 \\
&\quad - (400,000 + 195x_1 + 225x_2).
\end{aligned}
$$

We wish to maximize f over the set S consisting of all x_1 and x_2 satisfying the constraints

$$
\begin{aligned}
x_1 &\leq 5,000 \\
x_2 &\leq 8,000 \\
x_1 + x_2 &\leq 10,000 \\
x_1 &\geq 0 \\
x_2 &\geq 0.
\end{aligned}
$$

Step 3: Formulate model

- Feasible region



Step 3: Formulate model

- We apply Lagrange multiplier methods to fin the maximum of $y = f(x_1, x_2)$ over the set S
- We compute
  - $\nabla f = (144 - 0.02x_1 - 0.007x_2, 174 - 0.007x_1 - 0.02x_2)$
  - $g(x_1, x_2) = x_1 + x_2 = 10000, \ \nabla g = (1,1)$
- So, the Lagrange multipliers equations are $144 - 0.02x_1 - 0.007x_2 = \lambda$ and $174 - 0.007x_1 - 0.02x_2 = \lambda$
- Solving these equations together with the constraint $x_1 + x_2 = 10000$

Step 3: Formulate model

- We have

$$x_1 = \frac{50,000}{13} \approx 3,846$$
$$x_2 = \frac{80,000}{13} \approx 6,154$$
$$\lambda = 24.$$

- Substituting back the equation, we have $y = 532308$ at the maximum

Outliner

- Unconstrained optimization
- Lagrange multipliers
- **Exercises**

Exercise

- Calculate the following values for illustrated examples:
  - $S(x_1, a)$
  - $S(x_2, a)$
  - $S(y, a)$

in which, a is price elasticity for 19-inch sets