Mathematical Modelling

Part 2: Dynamic Model

Content of part 1

Chapter 1: Introduction to dynamic models

~~Chapter 2: Analysis of dynamic models~~

Chapter 3: Simulation of dynamic models

Dynamic models

- Many problems of practical interest involve process that evolve over time.
- Dynamic models are used to represent the changing behavior of these systems.
  - Space flight
  - Electrical circuits
  - Population growth
  - Military battles
  - Spread of disease

Outline

- **Steady state**
- Dynamical system
- Discrete time dynamic systems

Steady state analysis: Example

In an unmanaged tract of forest area, hardwood and softwood trees compete for the available land and water. The more desirable hardwood trees grow more slowly, but are more durable and produce more valuable timber. Softwood trees compete with the hardwoods by growing rapidly and consuming the available water and soil nutrients. Hardwoods compete by growing taller than the softwoods can and shading new seedlings. They are also more resistant to disease. Can these two types of trees coexist on one tract of forest land indefinitely, or will one type of tree drive the other to extinction?

Step 1: Ask a question

Variables:

H = hardwood population (tons/acre)
S = softwood population (tons/acre)
g_H = growth rate for hardwoods (tons/acre/year)
g_S = growth rate for softwoods (tons/acre/year)
c_H = loss due to competition for hardwoods (tons/acre/year)
c_S = loss due to competition for softwoods (tons/acre/year)

Assumptions:

$g_H = r_1 H - a_1 H^2$
$g_S = r_2 S - a_2 S^2$
$c_H = b_1 S H$
$c_S = b_2 S H$
$H \geq 0, S \geq 0$
r_1, r_2, a_1, a_2, b_1, b_2 are positive reals

Objective: Determine whether $H \to 0$ or $S \to 0$

Step 2: Select a modelling approach

We will model this problem as a dynamic model in steady state.

- We are given functions:

$$f_1(x_1, \ldots, x_n)$$
$$\vdots$$
$$f_n(x_1, \ldots, x_n)$$

Defined on a subset S of R^n. The functions $f_1, \ldots, f_n$ represent the rate of change of each variable $x_1, x_2, \ldots, x_n$ respectively.

Step 2: Select a modelling approach

A point $(x_1, \ldots, x_n)$ in the set S is called an equilibrium point provide that

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, \ldots, x_n) &= 0 \end{aligned}$$

At this point, the rate of change of each of the variable $x_1, \ldots, x_n$ is equal to zero. And the system is at rest.

Step 2: Select a modelling approach

- Variable $x_1, \dots, x_n$ are called state variables
- S is called the state space.
- Since the functions $f_1, \dots, f_n$ depend only on the current state $(x_1, \dots, x_n)$ of the system, knowledge of current state suffices to determine the entire feature of the system.
- We only need to know where we are now, not how we got here.
- When we are at an equilibrium point, we say that the system is in **steady state.**

Step 3: Formulate the problem

- Let $x_1 = H$ and $x_2 = S$ denote our two state variables, defined on the state space

$$\{(x_1, x_2): x_1 \geq 0, x_2 \geq 0\}$$

- The steady-state equations are

$$r_1 x_1 - a_1 x_1^2 - b_1 x_1 x_2 = 0$$
$$r_2 x_2 - a_2 x_2^2 - b_2 x_1 x_2 = 0.$$

Step 4: Solve the problem

$$r_1 x_1 - a_1 x_1^2 - b_1 x_1 x_2 = 0$$
$$r_2 x_2 - a_2 x_2^2 - b_2 x_1 x_2 = 0.$$

Solving this system, we have three points

$$(0,\, 0)$$
$$(0,\, r_2/a_2)$$
$$(r_1/a_1,\, 0)$$

and a point is intersection of two following lines

$$a_1 x_1 + b_1 x_2 = r_1$$
$$b_2 x_1 + a_2 x_2 = r_2.$$

Step 4: Solve the problem

Solving system

$$a_1 x_1 + b_1 x_2 = r_1$$
$$b_2 x_1 + a_2 x_2 = r_2.$$

We get

$$x_1 = \frac{r_1 a_2 - r_2 b_1}{a_1 a_2 - b_1 b_2}$$
$$x_2 = \frac{a_1 r_2 - b_2 r_1}{a_1 a_2 - b_1 b_2}.$$

Step 4: Solve the problem

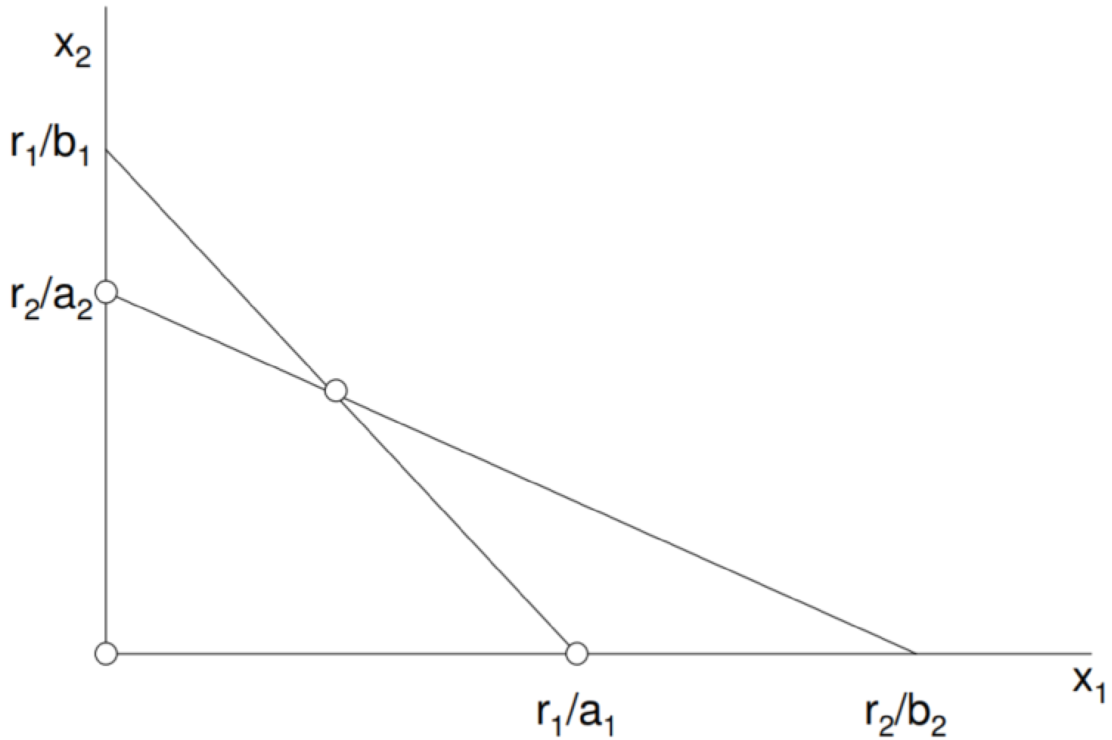


Figure 4.2: Graph of softwoods x_2 versus hardwoods x_1 showing equilibria for the tree problem.

Step 4: Solve the problem

- We have condition $x_1 > 0, x_2 > 0$.
- We consider additionally a reasonable assumption, the effect of competition between members of the same species should be stronger than the competition between species, $a_i > b_i$
- Condition for coexistence is

$$r_1 a_2 - r_2 b_1 > 0$$
$$a_1 r_2 - b_2 r_1 > 0,$$

Step 5: Answer the question

The parameters r_i measure growth tendency, and the parameters a_i and b_i measure the strength of competition within and between populations, respectively. Thus, the ratios r_i/a_i and r_i/b_i must measure the relative strength of growth versus competition. Let us try to go further. In the absence of competition between species, the growth rate is

$$r_i x_i - a_i x_i^2 = x_i(r_i - a_i x).$$

The ratio r_i/a_i represents the equilibrium population level in the absence of competition between species, or the level at which the population will stop growing of its own accord. Similarly, if we neglect the factor of competition within a population, the net growth rate is

$$r_i x_i - b_i x_i x_j = x_i(r_i - b_i x_j).$$

Step 5: Answer the question

The ratio r_i/b_i thus represents the level of population j necessary to put an end to growth of population i. In light of this, we can now give our analysis results the following concrete interpretation.

For each type of tree (hardwood and softwood), there are two kinds of limits to growth. The first comes from competition with the other type of tree, and the second comes from competition between trees of the same type under crowded conditions. Thus, for each type of tree there is one point where growth will halt itself due to crowding, and another point where the growth of one type of tree will halt the growth of the other type due to competition. The condition for coexistence of both types is that each type reaches the point where it limits its own growth before it reaches the point where it limits the other's growth.

Outline

- Steady state
- **Dynamical system**
- Discrete time dynamic systems

Dynamical systems

The blue whale and fin whale are two similar species that inhabit the same areas. Hence, they are thought to compete. The intrinsic growth rate of each species is estimated at 5% per year for the blue whale and 8% per year for the fin whale. The environmental carrying capacity (the maximum number of whales that the environment can support) is estimated at 150,000 blues and 400,000 fins. The extent to which the whales compete is unknown. In the last 100 years intense harvesting has reduced the whale population to around 5,000 blues and 70,000 fins. Will the blue whale become extinct?

Step 1: Ask a question

Variables:

B = number of blue whales
F = number of fin whales
g_B = growth rate of blue whale population (per year)
g_F = growth rate of fin whale population (per year)
c_B = effect of competition on blue whales (whales per year)
c_F = effect of competition on fin whales (whales per year)

Assumptions:

$g_B = 0.05B(1 - B/150,000)$
$g_F = 0.08F(1 - F/400,000)$
$c_B = c_F = \alpha BF$
$B \geq 0, F \geq 0$
α is a positive real constant

Objective:

Determine whether dynamic system can reach stable equilibrium starting from $B = 5,000,\ F = 70,000$

Step 2: Select a modelling approach

We will model this problem as dynamic system.

- A dynamical system consists of n state variable $(x_1, \dots, x_n)$ and a system of differential equations

$$\begin{aligned}
\frac{dx_1}{dt} &= f_1(x_1, \dots, x_n) \\
\vdots \quad & \qquad \vdots \\
\frac{dx_n}{dt} &= f_n(x_1, \dots, x_n)
\end{aligned}$$

defined on the state space $(x_1, \dots, x_n\} \in S$, where S is a subset of R^n.

Step 2: Select a modelling approach

- The existence and uniqueness theorem of differential equation states that if $f_1, \ldots, f_n$ have continuous first partial derivatives in a neighborhood of a point $x_0 = (x_1^0, \ldots, x_n^0)$ the there exists a unique solution to this system of differential equations through this initial condition.
- It is best to think of a solution to a dynamical system as a path through the state space. As long as differentiability assumptions are satisfied, there is a path through each point, and paths cannot cross except at an equilibrium.

Step 2: Select a modelling approach

- Let $x = (x_1, \dots, x_n), F(x) = \big(f_1(x), \dots, f_n(x)\big)$.
- The dynamical system equations is $\frac{dx}{dt} = F(x)$
- For a path $x(t)$, the derivative dx/dt represents the velocity vector.
- Hence, for every solution curve $x(t)$, we have that $F(x(t))$ is the velocity vector at each point.
- The points where $F(x) = 0$ are the equilibria, and we will pay special attention to the vector field nearby these points.

Step 3: Formulate the problem

- Let $x_1 = B, x_2 = F$, we write

$$x_1' = f_1(x_1, x_2)$$
$$x_2' = f_2(x_1, x_2),$$

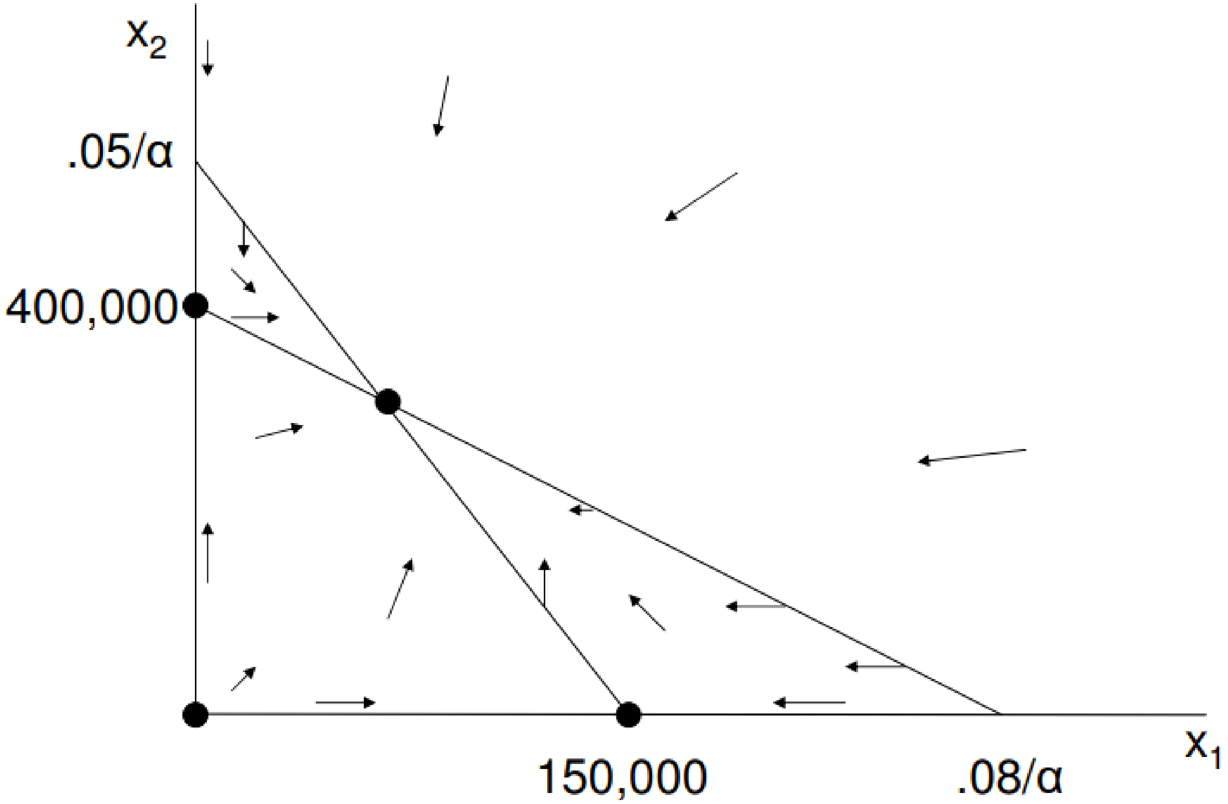
where

$$f_1(x_1, x_2) = 0.05x_1\left(1 - \frac{x_1}{150,000}\right) - \alpha x_1 x_2$$
$$f_2(x_1, x_2) = 0.08x_2\left(1 - \frac{x_2}{400,000}\right) - \alpha x_1 x_2.$$

The state space $S = \{(x_1, x_2): x_1 > 0, x_2 > 0\}$

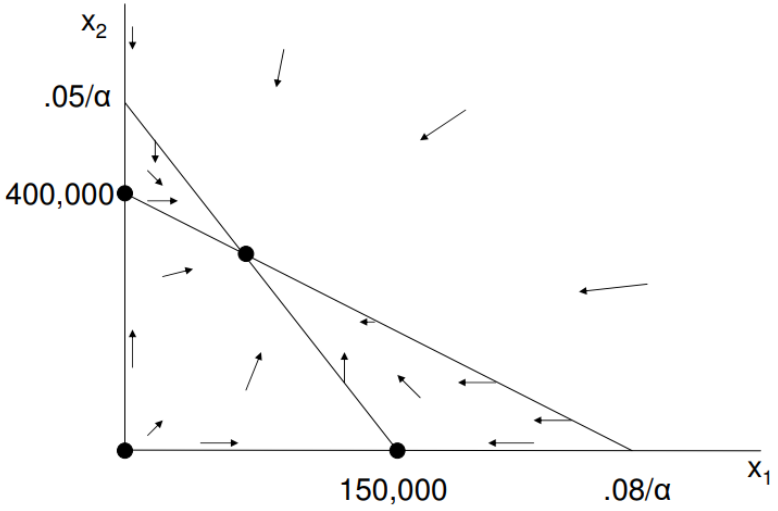
Step 4: Solve the model

- From the dynamical system we sketch above graph



Step 4: Solve the model

- From the graph, we have 4 equilibrium solutions
  - (0,0)
  - (150000, 0)
  - (0, 400000)
  - And a point whose coordinate depend on α (note that the graph assume $400000 < 0.05/\alpha$
- Due to the initial condition $x_1(0) = 5000, x_2(0) = 70000$, solution tends to the fourth equilibrium point.



Step 5: Answer the question

- If $40000 > 0.05/\alpha$, then two species could not coexist.
- In the absence of further harvesting, the whale populations will grow back to their natural levels, and the ecological system will remain in stable equilibrium.

Sensitivity analysis

Let do sensitivity analysis for parameter α

- For any value $\alpha < 1.25 \times 10^{-7}$ there is a equilibrium $x_1 > 0, x_2 > 0$ at

$$x_1 = \frac{150,000(8,000,000\alpha - 1)}{D}$$
$$x_2 = \frac{400,000(1,875,000\alpha - 1)}{D}$$

where

$$D = 15,000,000,000,000\alpha^2 - 1,$$

Sensitivity analysis

which we found by Cramer's Rule. For example, if $\alpha = 10^{-7}$, then

$$x_1 = \frac{600,000}{17} \approx 35,294$$
$$x_2 = \frac{6,500,000}{17} \approx 382,353.$$

The sensitivities at this point are

$$S(x_1, \alpha) = -\frac{21,882,352,927}{6,000,000,000} \approx -3.6$$

and

$$S(x_2, \alpha) = \frac{27}{221} \approx 0.122.$$

Outline

- Steady state
- Dynamical system
- **Discrete time dynamic systems**

Discrete time dynamic systems

Example: Astronauts in training are required to practice a docking maneuver under manual control. As a part of this maneuver, it is required to bring an orbiting spacecraft to rest relative to another orbiting craft. The hand controls provide for variable acceleration and deceleration, and there is a device on board that measures the rate of closing between the two vehicles. The following strategy has been proposed for bringing the craft to rest.

Discrete time dynamic systems

First, look at the closing velocity. If it is zero, we are done. Otherwise, remember the closing velocity and look at the acceleration control. Move the acceleration control so that it is opposite to the closing velocity (i.e., if closing velocity is positive, we slow down, and we speed up if it is negative) and proportional in magnitude (i.e., we brake twice as hard if we find ourselves closing twice as fast). After a time, look at the closing velocity again and repeat the procedure. Under what circumstances will this strategy be effective?

Discrete time dynamic systems

We will use the five-step method. Let v_n denote the closing velocity observed at time t_n, the time of the nth observation. Let

$$\Delta v_n = v_{n+1} - v_n$$

denote the change in closing velocity as a result of our adjustments. We will denote the time between observations of the velocity indicator by

$$\Delta t_n = t_{n+1} - t_n.$$

This time interval naturally divides into two parts: the time it takes to adjust the velocity controls and the time between adjustment and the next observation of the velocity indicator. Write

$$\Delta t_n = c_n + w_n,$$

Discrete time dynamic systems

This time interval naturally divides into two parts: the time it takes to adjust the velocity controls and the time between adjustment and the next observation of the velocity indicator. Write

$$\Delta t_n = c_n + w_n,$$

where c_n is the time to adjust the controls and w_n is the waiting time until the next observation. The parameter c_n is a function of astronaut response time, and we are free to choose w_n.

Let a_n denote the acceleration setting after the nth adjustment. Elementary physics yields

$$\Delta v_n = a_{n-1}c_n + a_n w_n.$$

The control law says to set acceleration proportional to $(-v_n)$; hence

$$a_n = -kv_n.$$

Step 1: Ask a question

Variables:

t_n = time of nth velocity observation (sec)
v_n = velocity at time t_n (m/sec)
c_n = time to make nth control adjustment (sec)
a_n = acceleration after nth adjustment (m/sec^2)
w_n = wait before $(n+1)$th observation (sec)

Assumptions:

$$t_{n+1} = t_n + c_n + w_n$$
$$v_{n+1} = v_n + a_{n-1}c_n + a_n w_n$$
$$a_n = -kv_n$$
$$c_n > 0$$
$$w_n \geq 0$$

Objective: Determine whether $v_n \to 0$

Step 2: Select a modelling approach

- We will model this problem as a discrete-time dynamical system.

A discrete–time dynamical system consists of a number of state variables $(x_1, \ldots, x_n)$ defined on the state space $S \subseteq \mathbb{R}^n$ and a system of difference equations

$$\begin{aligned} \Delta x_1 &= f_1(x_1, \ldots, x_n) \\ &\vdots \\ \Delta x_n &= f_n(x_1, \ldots, x_n). \end{aligned} \tag{4.9}$$

Step 2: Select a modelling approach

Here Δx_n represents the change in x_n over one time step. It is common to take time steps of length 1, which just amounts to selecting appropriate units. If time steps are of variable length, or if the dynamics of the system vary over time, then we include time as a state variable. If we let

$$x = (x_1, \dots, x_n)$$
$$F = (f_1, \dots, f_n),$$

then the equations of motion can be written in the form

$$\Delta x = F(x).$$

A solution to this difference equation model is a sequence of points

$$x(0), x(1), x(2), \dots$$

Step 2: Select a modelling approach

in the state space with

$$\begin{aligned}\Delta x(n) &= x(n+1) - x(n) \\ &= F(x(n))\end{aligned}$$

for all n. An equilibrium point x_0 is characterized by

$$F(x_0) = 0,$$

and the equilibrium is stable if

$$x(n) \to x_0$$

whenever $x(0)$ is sufficiently close to x_0. As in the continuous time case, many other difference equation models can be reduced to the form (4.9) by introducing additional state variables.

Step 2: Select a modelling approach

A solution to this difference equation model is a sequence of points

$$x(0),\ x(1),\ x(2),\ \ldots$$

in the state space with

$$\begin{aligned}\Delta x(n) &= x(n+1) - x(n)\\ &= F(x(n))\end{aligned}$$

for all n. An equilibrium point x_0 is characterized by

$$F(x_0) = 0,$$

and the equilibrium is stable if

$$x(n) \to x_0$$

whenever $x(0)$ is sufficiently close to x_0.

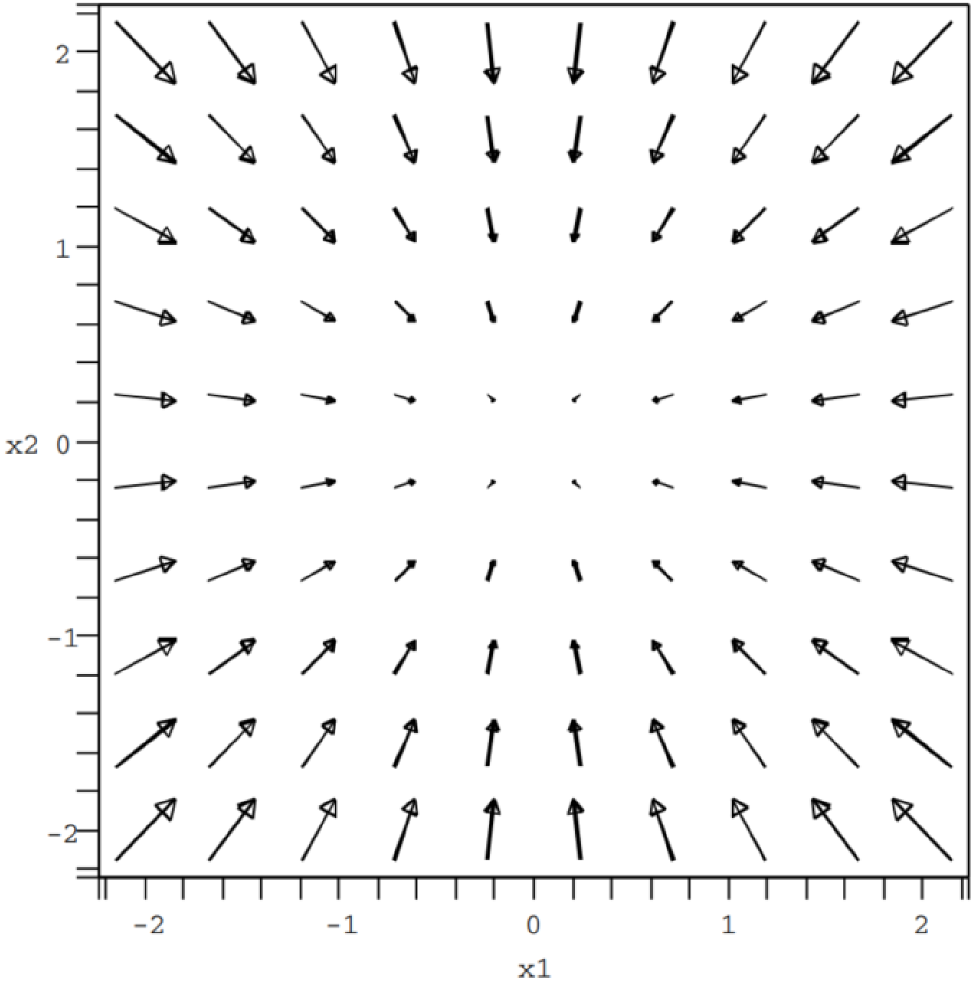
Step 2: Select a modelling approach

Example 4.4. Let $x = (x_1, x_2)$, and consider the difference equation

$$\Delta x = -\lambda x, \tag{4.10}$$

where $\lambda > 0$. What is the behavior of solutions near the equilibrium point $x_0 = (0, 0)$?

Step 2: Select a modelling approach



Step 2: Select a modelling approach

Figure 4.8 shows a graph of the vector field $F(x) = -\lambda x$ in the case where $0 < \lambda < 1$. It is clear that $x_0 = (0, 0)$ is a stable equilibrium. Each step moves closer to x_0. Now let us consider what happens when λ becomes larger. Each of the vectors in Fig. 4.8 will stretch as λ increases. For $\lambda > 1$ the vectors are so long that they overshoot the equilibrium. For $\lambda > 2$ they are so long that the terminal point $x(n+1)$ is actually farther away from $(0, 0)$ than the starting point $x(n)$. In this case x_0 is an unstable equilibrium.

Step 3: Formulate the problem

We return now to the docking problem of Example 4.3. Step 3 of the five-step method is to formulate the model. We are modeling the docking problem as a discrete–time dynamical system. From Fig. 4.7 we obtain

$$(v_{n+1} - v_n) = -k\, v_{n-1}\, c_n - k\, v_n w_n.$$

Hence, the change in velocity over the nth time step depends on both v_n and v_{n-1}. To simplify the analysis, let us assume that $c_n = c$ and $w_n = w$ for all n. Then the length of each time step is

$$\Delta t = c + w$$

seconds, and we do not need to include time as a state variable. We do, however, need to include both v_n and v_{n-1}. Let

$$\begin{aligned} x_1(n) &= v_n \\ x_2(n) &= v_{n-1}. \end{aligned}$$

Step 3: Formulate the problem

$$x_1(n) = v_n$$
$$x_2(n) = v_{n-1}.$$

Compute

$$\Delta x_1 = -kwx_1 - kcx_2$$
$$\Delta x_2 = x_1 - x_2.$$

The state space is $(x_1, x_2) \in \mathbb{R}^2$.

Step 4: Solve the problem

Step 4 is to solve the model. There is one equilibrium point (0, 0) found at the intersection of the two lines

$$\begin{aligned} kwx_1 + kcx_2 &= 0 \\ x_1 - x_2 &= 0. \end{aligned}$$

Step 4: Solve the problem

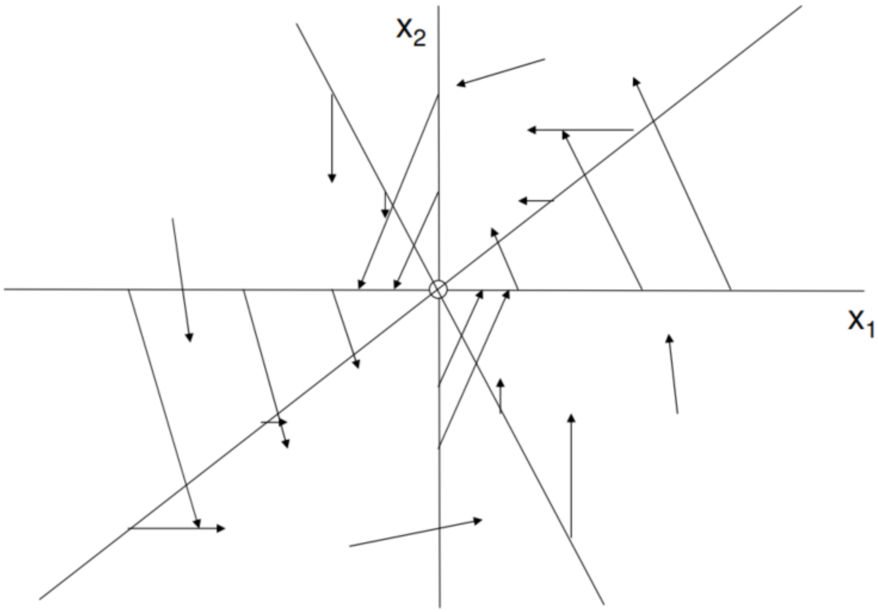


Figure 4.9: Graph of previous velocity x_2 versus current velocity x_1 showing vector field for the docking problem.

It appears as though solutions will tend toward equilibrium, but it is hard to be sure. If $k, c,$ and w are large, then the equilibrium is probably unstable, but once again it is difficult to tell.

Step 5: Answer the question

Step 5 is to answer the analysis question in plain English. Maybe we should just say in plain English that we don't know the answer. However, we probably can do better than that. Let us report that a completely satisfactory solution is not obtainable by elementary graphical methods. In other words, it will take more work using more sophisticated methods to determine exactly the conditions under which the proposed control strategy will work. It does seem that the strategy will be effective in most cases as long as the time interval between control adjustments is not too long and the magnitude of those adjustments is not too large. The problem is complicated by the fact that there is a time delay between reading the velocity indicator and adjusting the controls. Since the actual closing velocity may change during this interval, we are acting on dated and inaccurate information. This adds an element of uncertainty to our calculations. If we ignore the effects of this time delay (which may be permissible if the delay is small), we can then draw some general conclusions, which are as follows.

Step 5: Answer the question

The control strategy will work so long as the control adjustments are not too violent. Furthermore, the longer the interval between adjustments, the lighter those adjustments must be. In addition, the relationship is one of proportion. If we go twice as long between adjustments, we can only use half as much control. To be specific, if we adjust the controls once every 10 seconds, then we can only set the acceleration controls at 1/10 of the velocity setting to avoid overshooting the target velocity of zero. In order to allow for human and equipment error, we should actually set the controls somewhat lower, say 1/15 or 1/20 of velocity. More frequent adjustments require more frequent observations of the closing velocity indicator and more concentration on the part of the operator, but they do allow for the successful administration of more thrusting power under control. Presumably, this would be advantageous.

Exercise

Reconsider the tree problem of Example 4.1. Assume that

$$\frac{r_2}{a_2} < \frac{r_1}{b_1} \text{ and } \frac{r_1}{a_1} < \frac{r_2}{b_2}$$

so that the situation is as pictured in Fig. 4.2.

(a) Draw the vector field for this model.

(b) Classify each of the four equilibrium points as stable or unstable.

(c) Can the two species of trees coexist in stable equilibrium?

(d) Suppose that a logging operation removes all but a few of the valuable hardwood trees in this stand of forest. What does this model predict about the future of the two species of trees?