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Mathematical Modelling

Part 1: Optimization Model

Content of part 1

Chapter 1: One-variable optimization

Chapter 2: Multivariable optimization

Chapter 3: Multi-objective optimization

Chapter 4: Computational methods for optimization

Outline

- **One-variable optimization**
- Multivariable optimization
- Linear programming
- Discrete optimization

One-variable optimization

- Real-life problem is usually messy.
- The task of locating global extreme points can be exceedingly difficult. Even the function is differentiable everywhere, the computation of the derivative is often complicated.
- Simple fact is that most equations cannot be solved analytically.
- The best we can do in most instances is to find an approximate solution by graphical or numerical techniques.

Illustrated example: A variant of Pig problem

- We reconsider the pig problem in the previous chapter, but now we take into account the fact that the growth rate is not constant. We assume that the pig is young, so that the growth rate is increasing by time.

Step 1: Ask a question

Variables:	t = time (days)
	w = weight of pig (lbs)
	p = price for pigs (\$/lb)
	C = cost of keeping pig t days (\$)
	R = revenue obtained by selling pig (\$)
	P = profit from sale of pig (\$)
Assumptions:	$\boxed{w = 200 + 5t}$
	$p = 0.65 - 0.01t$
	$C = 0.45t$
	$R = p \cdot w$
	$P = R - C$
	$t \geq 0$
Objective:	Maximize P

Step 1: Ask a question

Let assume that the growth rate of the pig is proportional to its weight. In other words,

$$\frac{dw}{dt} = cw$$

From the fact that $\frac{dw}{dt} = 5$ lbs/day when $w = 200$ lbs, we conclude that $c = 0.025$. So

$$\frac{dw}{dt} = 0.025w, w(0) = 200$$

Solving this we got $w = 200\, e^{0.025t}$

Step 2: Select a modelling approach

- We still model this problem as a one-variable optimization problem.
- However, computational method for this problem is much harder than for previous problem.

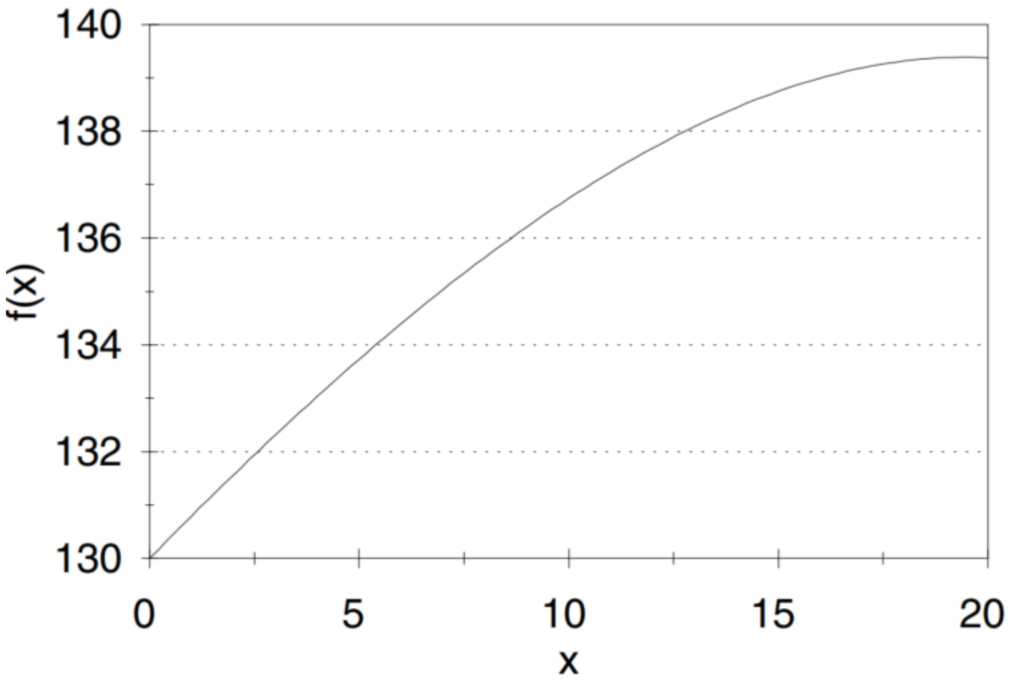
Step 3: Formulate the problem

- $P = R - C = pw - 0.45t = (0.65 - 0.01x)(200e^{0.025x}) - 0.45x$
- Or new objective function $y = f(x) = (0.65 - 0.01x)(200e^{0.025x}) - 0.45x$
- The problem is to maximize $f(x)$ over the set $S = \{x: x > 0\}$

Step 4: Solve the problem

We will use the graphical method.

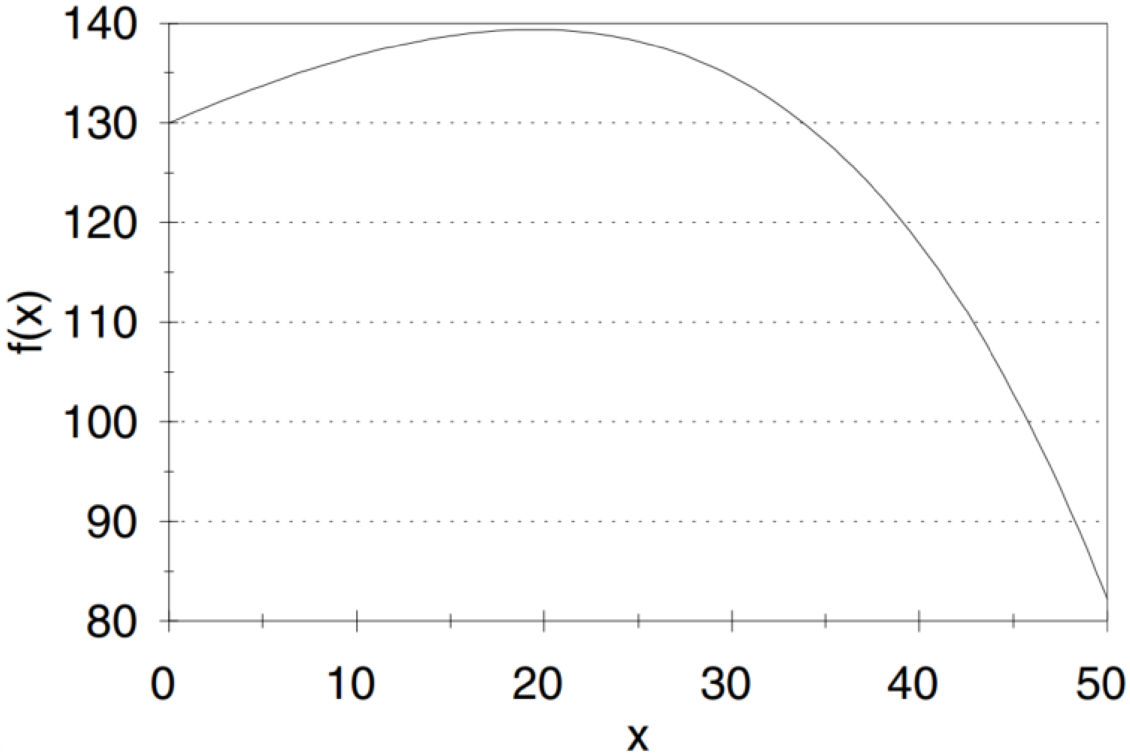
- Case 1: Not a complete graph



Step 4: Solve the problem

We will use the graphical method.

- Case 2: A complete graph

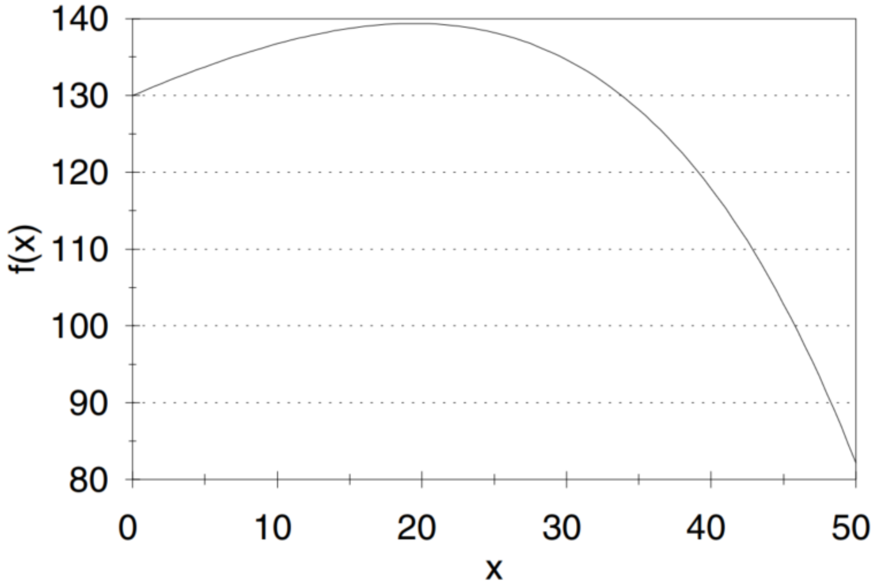


Step 4: Solve the problem

- The question: How do we know when we have a complete graph ?
- There is no simple answer to this question. It depends on problem at hand and experiences of modeler.

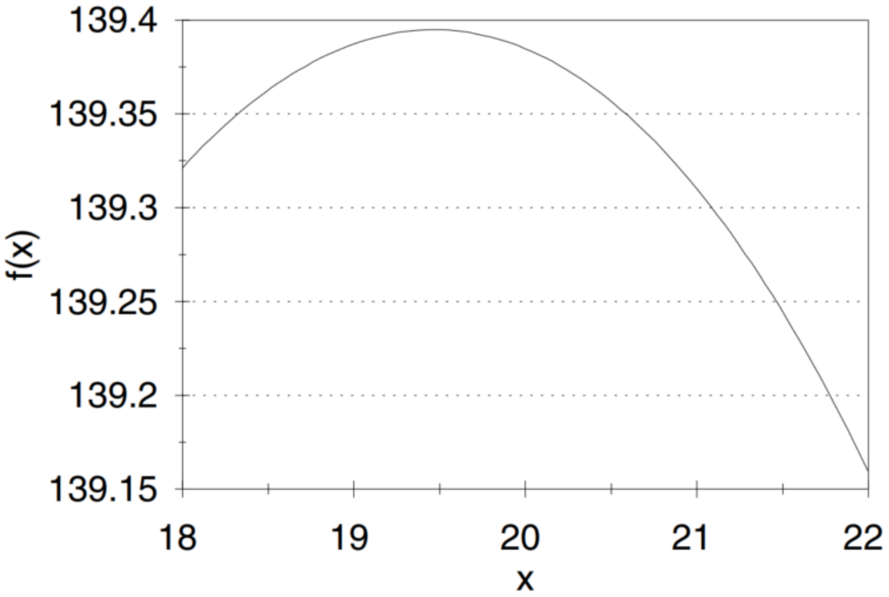
Step 4: Solve the problem

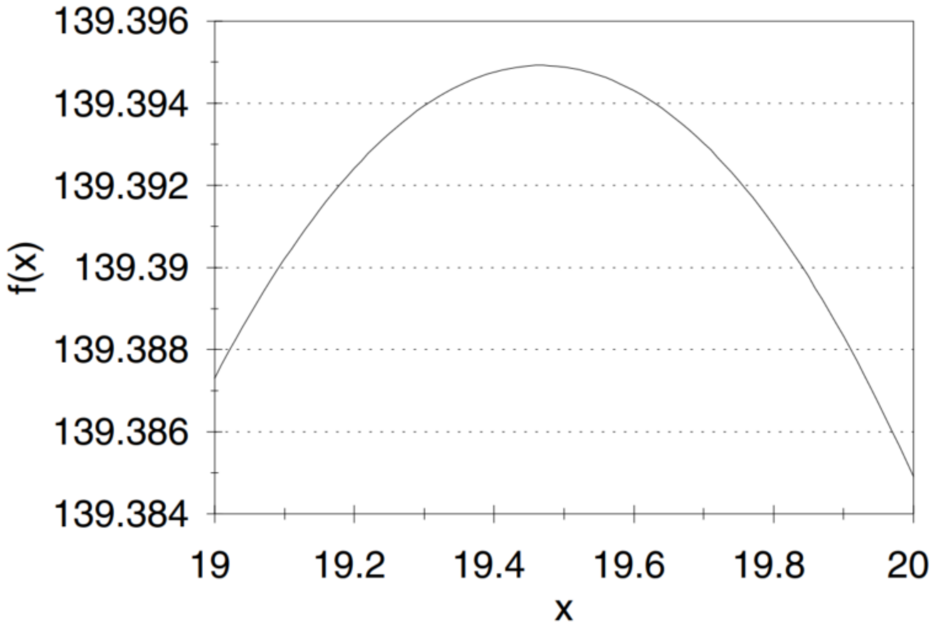
- From the graph, we see that the maximum occurs around $x = 20, y = f(x) = 140$. To obtain more accurate solution, we can zoom in on the maximum point of the graph



Step 4: Solve the problem

- Based on below graphs, we would estimate that the maximum occurs at $x = 19.5,\ y = f(x) = 139.395$





Step 5: Answer the question

- After taking into account the fact that the growth rate of the young pig is still increasing, now we recommend waiting 19 or 20 days to sell. This should result in a net profit of approximately $140.

Sensitivity analysis

- Let examine the sensitivity of the optimum value to the growth rate $c = 0.025$ for young pig.
- We should not repeat the graphical method for several different values of the parameter c.
- We need a more efficient method!

Sensitivity analysis

- Let us begin by generalizing the model.

$$\frac{dw}{dt} = cw, \;\; w(0) = 200,$$

so that

$$w = 200e^{ct}.$$

This leads to the objective function

$$f(x) = (0.65 - 0.01x)(200e^{cx}) - 0.45x.$$

Sensitivity analysis

- First part: Computing the derivative

$$f'(x) = 200ce^{cx}(0.65 - 0.01x) - 2e^{cx} - 0.45.$$

- Second part: Solve the equation

$$200ce^{cx}(0.65 - 0.01x) - 2e^{cx} - 0.45 = 0.$$

Not easy to solve by hand

Newton's method

We are given a differentiable function $F(x)$ and an approximation x_0 to the root $F(x) = 0$. Newton's method works by linear approximation. Near $x = x_0$ we have $F(x) \approx F(x_0) + F'(x_0)(x - x_0)$ by the usual tangent line approximation. To obtain a better estimate $x = x_1$ of the actual root to $F(x) = 0$, we set $F(x_0) + F'(x_0)(x - x_0) = 0$ and solve for $x = x_1$ to get $x_1 = x_0 - F(x_0)/F'(x_0)$. Geometrically, the tangent line to $y = F(x)$ at the point $x = x_0$ intersects the x–axis at the point $x = x_1$. As long as x_1 is not too far away from x_0, the tangent line approximation should be reasonably good, and so x_1 should be close to the actual root. Newton's method produces a sequence of increasingly accurate estimates $x_1, x_2, x_3, \ldots$, to the actual root by repeating this tangent line approximation. Once we are sufficiently close to the root, each successive approximation produced by Newton's method is accurate to about twice as many decimal places as the preceding estimate.

Newton's method

Variables: $x(n)$ = approximate location of root after n iterations.
N = number of iterations

Input: $x(0), N$

Process:

Begin
for $n = 1$ to N do
  Begin
  $x(n) \longleftarrow x(n-1) - F(x(n-1))/F'(x(n-1))$
  End
End

Output: $x(N)$

Sensitivity analysis

c	x
0.022	11.626349
0.023	14.515929
0.024	17.116574
0.025	19.468159
0.026	21.603681
0.027	23.550685
0.028	25.332247

Notes on step 5 (Solve the problem)

- Two stage of solving one-variable optimization
  - Stage 1: Apply a global method (graphing) to locate an approximate solution
  - Stage 2: Apply a fast local method (Newton’s method) to determine the exact solution to the desired accuracy.

Outline

- One-variable optimization
- **Multivariable optimization**
- Linear programming
- Discrete optimization

Multivariable optimization

- Graphical techniques for global method are not available in case the number of variables is larger than 3.
- Solving $\nabla f = 0$ becomes more complicated as the number of independent variables increases.
- Constrained optimization is also more difficult because the geometry of the feasible region can be more complicated.

Example: Fire station location

A suburban community intends to replace its old fire station with a new facility. The old station was located at the historical city center. City planners intend to locate the new facility more scientifically. A statistical analysis of response–time data yielded an estimate of $3.2 + 1.7r^{0.91}$minutes required to respond to a call r miles away from the station. Estimates of the frequency of calls from different areas of the city were obtained from the fire chief. Each block represents one square mile, and the numbers inside each block represent the number of emergency calls per year for that block. Find the best location for the new facility such that average response time is minimized.

Example: Fire station location

Map showing the number of emergency calls per year in each one square mile area of the city

3	0	1	4	2	1
2	1	1	2	3	2
5	3	3	0	1	2
8	5	2	1	0	0
10	6	3	1	3	1
0	2	3	1	1	1

Step 1: Ask a question

- Represent city map as a grid, locations on the city is described by coordinates (x, y)
- For simplicity, the map is divided into nine 2×2 $-$mile squares and assume that each emergency is located at the center of the square.
- (x, y) is the location of new fire station, so the average response time to a call is $z = f(x, y)$
- $0 \leq x \leq 6, 0 \leq y \leq 6$

Step 2: Select a modelling approach

- We model this problem as a multiple variable optimization problem.

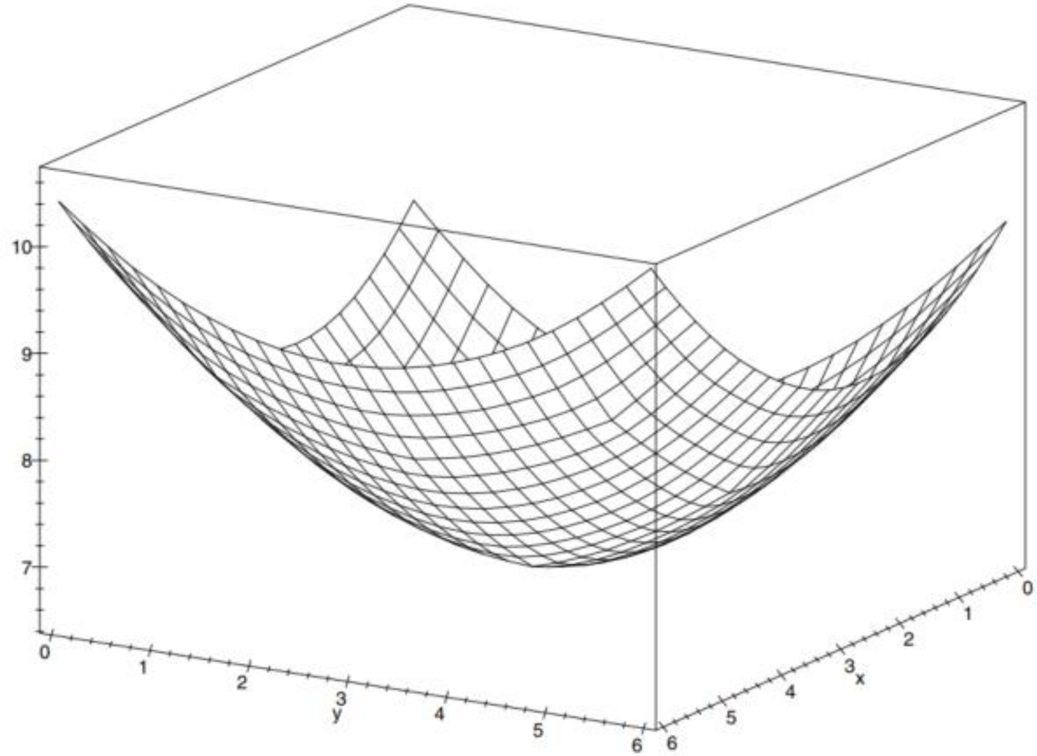
Step 3: Formulate the problem

$$
\begin{aligned}
z = 3.2 + 1.7\,[6&\sqrt{(x-1)^2+(y-5)^2}^{\,0.91} \\
&+ 8\sqrt{(x-3)^2+(y-5)^2}^{\,0.91} + 8\sqrt{(x-5)^2+(y-5)^2}^{\,0.91} \\
&+ 21\sqrt{(x-1)^2+(y-3)^2}^{\,0.91} + 6\sqrt{(x-3)^2+(y-3)^2}^{\,0.91} \\
&+ 3\sqrt{(x-5)^2+(y-3)^2}^{\,0.91} + 18\sqrt{(x-1)^2+(y-1)^2}^{\,0.91} \\
&+ 8\sqrt{(x-3)^2+(y-1)^2}^{\,0.91} + 6\sqrt{(x-5)^2+(y-1)^2}^{\,0.91}]/84.
\end{aligned}
$$

The problem is to minimize z over the feasible regions $0 \le x \le 6, 0 \le y \le 6$

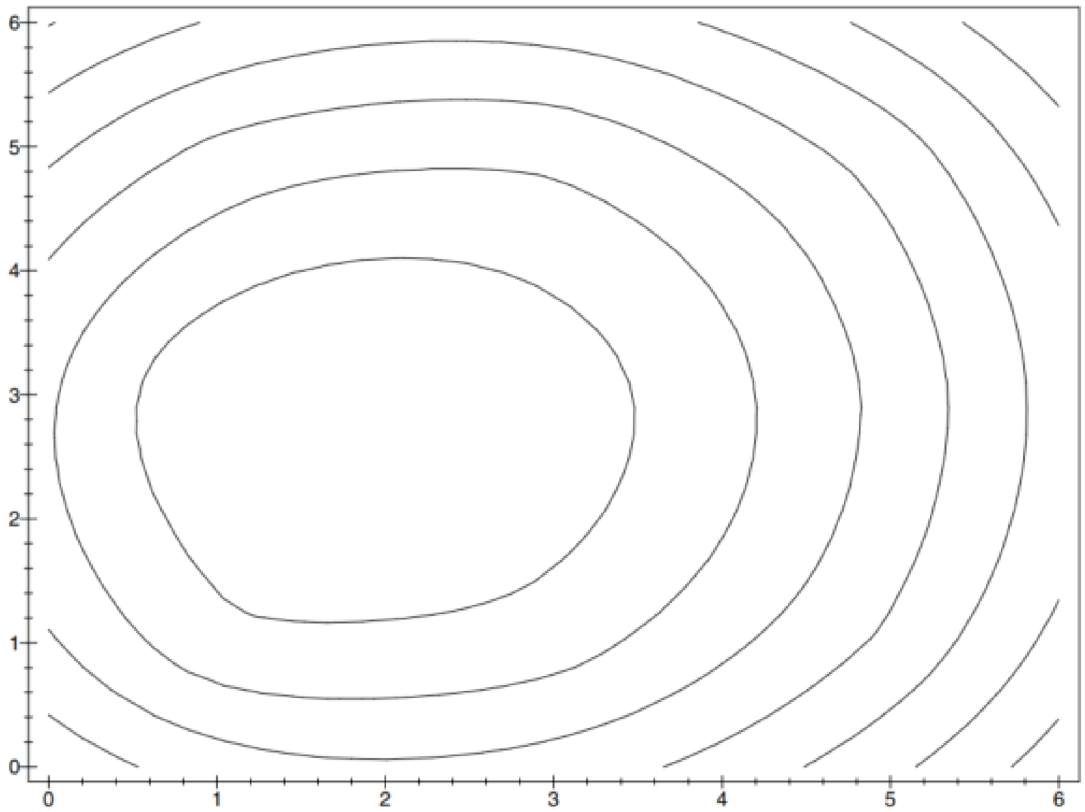
Step 4: Solve the problem

3-D graph of average response time



Step 4: Solve the problem

Contour plot of average response time



Step 4: Solve the problem

- From 3-D graph of average response time, we see that f attains its minimum at the unique interior point at which $\nabla f = 0$
- The contour plot indicates that $\nabla f = 0$ near the point $x = 2, y = 3$.

Step 4: Solve the problem

Algorithm: METHOD OF RANDOM SEARCH

Variables:

$a =$ lower limit on x
$b =$ upper limit on x
$c =$ lower limit on y
$d =$ upper limit on y
$N =$ number of iterations
$x\,min =$ approximate x coordinate of minimum
$y\,min =$ approximate y coordinate of minimum
$z\,min =$ approximate value of $F(x, y)$ at minimum

Step 4: Solve the problem

Process:

Begin
$x \longleftarrow$ Random $\{[a,\, b]\}$
$y \longleftarrow$ Random $\{[c,\, d]\}$
$z\,min \longleftarrow F(x,\, y)$
for $n = 1$ to N do
  Begin
  $x \longleftarrow$ Random $\{[a,\, b]\}$
  $y \longleftarrow$ Random $\{[c,\, d]\}$
  $z \longleftarrow F(x,\, y)$
  if $z < z\,min$ then
    $x\,min \longleftarrow x$
    $y\,min \longleftarrow y$
    $z\,min \longleftarrow z$
  End
End

Output: $x\,min,\ y\,min,\ z\,min$

Step 4: Solve the problem

- Result

$$x\,min = 1.66$$

$$y\,min = 2.73$$

$$z\,min = 6.46.$$

Outline

- One-variable optimization
- Multivariable optimization
- **Linear programming**
- Discrete optimization

Linear programming

- The simplest type of multivariable constrained optimization where both the objective function and the constraint functions are linear.
- Study of computational methods for such problems is called linear programming.
- Huge number of software packages for linear programming.

Example: Planting problem

A family farm has 625 acres available for planting. The crops the family is considering are corn, wheat, and oats. It is anticipated that 1,000 acres-ft of water will be available for irrigation, and the armers will be able to devote 300 hours of labor per week. Additional data are presented in the next slide. Find the amount of each crop that should be planted for maximum profit.

Example: Planting problem

Requirements per acre	Corn	Wheat	Oats
Irrigation (acre–ft)	3.0	1.0	1.5
Labor (person–hrs/week)	0.8	0.2	0.3
Yield ($)	400	200	250

Step 1: Ask a question

Variables:

x_1 = acres of corn planted
x_2 = acres of wheat planted
x_3 = acres of oats planted
w = irrigation required (acre–ft)
l = labor required (person–hrs/wk)
t = total acreage planted
y = total yield (\$)

Step 1: Ask a question

Assumptions:

$$
\begin{aligned}
& w = 3.0x_1 + 1.0x_2 + 1.5x_3 \\
& l = 0.8x_1 + 0.2x_2 + 0.3x_3 \\
& t = x_1 + x_2 + x_3 \\
& y = 400x_1 + 200x_2 + 250x_3 \\
& w \leq 1,000 \\
& l \leq 300 \\
& t \leq 625 \\
& x_1 \geq 0;\ \ x_2 \geq 0;\ \ x_3 \geq 0
\end{aligned}
$$

Objective: Maximize y

Step 2: Select a modelling approach

- We model this problem as a linear programming problem.

Step 3: Formulate the problem

- Decision variables are the number of acres of each crop x_1, x_2
- Objective function: Maximize the total yield, $y = 400x_1 + 200x_2 + 250x_3$
- Constraint sets:

$$
\begin{aligned}
3.0x_1 + 1.0x_2 + 1.5x_3 &\leq 1,000 \\
0.8x_1 + 0.2x_2 + 0.3x_3 &\leq \ \ 300 \\
x_1 + \quad x_2 + \quad x_3 &\leq \ \ 625
\end{aligned}
$$

Step 4: Solving the problem

- Using software package, the optimal solution is $Z = 162500$ at $x_1 = 41.667, x_2 = 0, x_3 = 583.333$.

Step 5: Answer the question

- The optimal solution is to plant 187.5 acres of corn, 437.5 acres of wheat, and no oats. This should yield $162,500.
- The optimal crop mixture we found uses all 625 acres and all 1,000 acre–ft of irrigation water, but only 237.5 of the available 300 person–hours of labor per week. Thus, there will be 62.5 person–hours per week that may be devoted to other profitable activities, or to leisure.

Sensitivity analysis

We will begin our sensitivity analysis by considering the amount of water available for irrigation. This amount will vary as a result of rainfall and temperature, which determine the status of the farm’s irrigation pond. It would also be possible to purchase additional irrigation water from a nearby farm. The result illustrates the effect of one additional acre–ft of irrigation water on our optimal solution. Now we can plant an additional half–acre of corn (a more profitable crop), and in fact we save a little bit of labor (0.3 person–hours per week). The net result is an additional $100 in yield.

Outliner

- One-variable optimization
- Multivariable optimization
- Linear programming
- **Discrete optimization**

Discrete optimization

- In many real-world problems, we must deal with variables that are discrete, like the integers.
- As in the continuous case, there are still no universally effective methods for solving discrete optimization problems.
- This section, we concentrate on one type of discrete optimization problem called integer programming.

Example: Planting problem

The family has 625 acres available for planting. There are 5 plots of 120 acres each and another plot of 25 acres. The family wants to plant each plot with only one crop: corn, wheat, or oats. As before, 1,000 acre–ft of water will be available for irrigation, and the farmers will be able to devote 300 hours of labor per week. Additional data are presented in the next slide. Find the crop that should be planted in each plot for maximum profit.

Example: Planting problem

Requirements per acre	Corn	Wheat	Oats
Irrigation (acre–ft)	3.0	1.0	1.5
Labor (person–hrs/week)	0.8	0.2	0.3
Yield ($)	400	200	250

Step 1: Ask a question

Variables:

x_1 = number of 120 acre plots of corn planted
x_2 = number of 120–acre plots of wheat planted
x_3 = number of 120–acre plots of oats planted
x_4 = number of 25–acre plots of corn planted
x_5 = number of 25–acre plots of wheat planted
x_6 = number of 25–acre plots of oats planted
w = irrigation required (acre–ft)
l = labor required (person–hrs/wk)
t = total acreage planted
y = total yield ($)

Step 1: Ask a question

Assumptions:

$$
\begin{aligned}
& w = 120(3.0x_1 + 1.0x_2 + 1.5x_3) \\
& \qquad + 25(3.0x_4 + 1.0x_5 + 1.5x_6) \\
& l = 120(0.8x_1 + 0.2x_2 + 0.3x_3) \\
& \qquad + 25(0.8x_4 + 0.2x_5 + 0.3x_6) \\
& t = 120(x_1 + x_2 + x_3) + 25(x_4 + x_5 + x_6) \\
& y = 120(400x_1 + 200x_2 + 250x_3) \\
& \qquad + 25(400x_4 + 200x_5 + 250x_6) \\
& w \le 1,000 \\
& l \le 300 \\
& t \le 625 \\
& x_1 + x_2 + x_3 \le 5 \\
& x_4 + x_5 + x_6 \le 1 \\
& x_1, \ldots, x_6 \text{ are nonnegative integers}
\end{aligned}
$$

Objective: Maximize y

Step 2: Select a modelling approach

- We will model this problem as an integer programming problem.

Step 3: Formulate the problem

- Decision variables are the number of 120-arce plots and the number of 25-arce plots to plant with corn, wheat, or oats ($x_1, x_2, x_3, x_4, x_5, x_6$).
- Objective function: Maximize $y = 48000x_1 + 24000x_2 + 30000x_3 + 10000x_4 + 5000x_5 + 6250x_6$
- Constraint set:
  - $x_1, \ldots, x_6 \in Z^+$

Step 3: Formulate the problem

- Constraint set:
  - $x_1, \dots, x_6 \in Z^+$

$$
\begin{aligned}
375x_1 + 125x_2 + 187.5x_3 + 75x_4 + 25x_5 + 37.5x_6 &\leq 1000 \\
100x_1 + 25x_2 + 37.5x_3 + 20x_4 + 5x_5 + 7.5x_6 &\leq 300 \\
x_1 + x_2 + x_3 &\leq 5 \\
x_4 + x_5 + x_6 &\leq 1
\end{aligned}
$$

Step 4: Solve the problem

- Using a software package, the optimal solution is $y = 156250$ which occurs when $x_3 = 5, x_6 = 1$ and other decision variable are all zero.

Step 5: Answer the question

- The best plan is to plant oats in every plot. This results in an expected total yield of $156250 for the season.

Sensitivity analysis

- Sensitivity analysis can be very time-consuming for integer programming problem due to integer programming take so much longer to solve than linear programming.
- We consider the amount of irrigation water available. If we add more 100 acre-feet of water. This leads to another plan: planting one 120-acre plot of corn, one 120-arc plot of wheat, and planting oats everywhere else. The expected revenue in this new plan is higher $12000 than the old plan.

Bài tập

A retired engineer has \$250,000 to invest and is willing to spend five hours per week managing her investments. Municipal bonds earn 6% per year and require no management. Real estate investments are expected to appreciate at 8% per year and require one hour of management per \$100,000 invested. Blue chip stocks earn 10% per year and require 1.5 hours of management. Junk bonds earn 12% and require 2.5 hours, while grain futures earn 15% and require five hours per \$100,000 invested.

(a) How should the retiree invest her money in order to maximize her expected earnings? Use the five-step method and solve as a linear programming problem.