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Mathematical Modelling

Part 1: Optimization Model

Content of part 1

Chapter 1: One-variable optimization

Chapter 2: Multivariable optimization

Chapter 3: Multi-objective optimization

Chapter 4: Computational methods for optimization

Outline

- **Five-step method**
- Sensitivity analysis
- Sensitivity and Robustness

Optimization problem

- Optimization problems are the most common applications.
- Objectives is to maximize profit or to minimize cost.
- Business managers attempt to manipulate values of control variables to achieve desired objective.
- Examples:
  - Farming
  - Manufacturing
  - Finance
  - Education

Five-step method

Five-step method is a general procedure that can be used to solve problems using mathematical modeling.

1. Ask the question
2. Select the modelling approach
3. Formulate the model
4. Solve the model
5. Answer the question

Illustrated example

- A pig weighing 200 pounds gains 5 pounds per day and costs 45 cents a day to keep. The market price for pigs is 65 cents per pound but is falling 1 cent per day. When should the pig be sold?

Step 1: Ask a question

- Question must be phrased in the mathematical terms:
- Start with a set of assumptions or suppositions about the way things really are.
- Problem components:
  - List of variables (appropriate units)
  - List of relations between variables (equations/ inequalities)
  - List of objectives (mathematical terms)
- Note that
  - Be careful not to confuse variables and parameters
  - Check units to make sure that your assumptions makes sense

Step 1: Ask a question

In Example 1.1 the weight w of the pig (in lbs), the number of days t until we sell the pig, the cost C of keeping the pig t days (in dollars), the market price p for pigs (\$/lb), the revenue R obtained when we sell the pig (\$), and our resulting net profit P (\$) are all variables. There are other numerical quantities involved in the problem, such as the initial weight of the pig (200 lbs). However, these are not variables. It is important at this stage to separate variables from those quantities that will remain constant.

Ask a question in illustrated example

$$(w \text{ lbs}) = (200 \text{ lbs}) + \left(\frac{5 \text{ lbs}}{\text{day}}\right)(t \text{ days}).$$

The other assumptions inherent in our problem are as follows:

$$\left(\frac{p \text{ dollars}}{\text{lb}}\right) = \left(\frac{0.65 \text{ dollars}}{\text{lb}}\right) - \left(\frac{0.01 \text{ dollars}}{\text{lb} \cdot \text{day}}\right)(t \text{ days})$$

$$(C \text{ dollars}) = \left(\frac{0.45 \text{ dollars}}{\text{day}}\right)(t \text{ days})$$

$$(R \text{ dollars}) = \left(\frac{p \text{ dollars}}{\text{lb}}\right)(w \text{ lbs})$$

$$(P \text{ dollars}) = (R \text{ dollars}) - (C \text{ dollars})$$

Ask a question in illustrated example

Variables:

t = time (days)
w = weight of pig (lbs)
p = price for pigs (\$/lb)
C = cost of keeping pig t days (\$)
R = revenue obtained by selling pig (\$)
P = profit from sale of pig (\$)

Assumptions:

$w = 200 + 5t$
$p = 0.65 - 0.01t$
$C = 0.45t$
$R = p \cdot w$
$P = R - C$
$t \geq 0$

Objective: Maximize P

Figure 1.1: Results of step 1 of the pig problem.

Step 2: Select the modelling approach

- We have already a problem stated in mathematical language. We need to select a mathematical approach to use to get an answer to the research question.
- How to select approaches:
  - Characteristics of variables
  - Characteristics of relations
  - Complexity of problem

  => Experiences, Skill and Familiarity with the relevant literature
- Illustrated example: One-variable optimization

Step 3: Formulate model

- We need to take the question exhibited in step 1 and reformulate it in the standard form selected in step 2, so that we can apply the standard general solution procedure.

$$\begin{aligned} P &= R - C \\ &= p \cdot w - 0.45t \\ &= (0.65 - 0.01t)(200 + 5t) - 0.45t. \end{aligned}$$

Let $y = P$ be the quantity we wish to maximize and $x = t$ the independent variable. Our problem now is to maximize

$$\begin{aligned} y &= f(x) \\ &= (0.65 - 0.01x)(200 + 5x) - 0.45x \end{aligned} \tag{1.1}$$

over the set $S = \{x : x \geq 0\}$.

Step 4: Solve model

- Apply the standard solution procedure identified in step 2 to solve model.
- Check again your model
- Use appropriate tools/tech

$$y = f(x)$$
$$= (0.65 - 0.01x)(200 + 5x) - 0.45x$$

$$f'(x) = \frac{(8 - x)}{10}$$

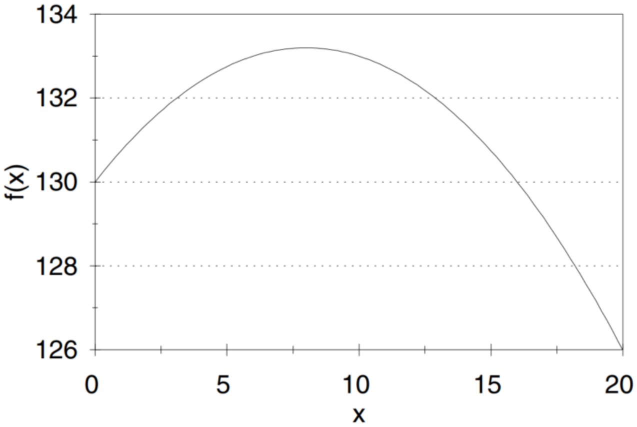


Figure 1.2: Graph of net profit $f(x) = (0.65 - 0.01x)(200 + 5x) - 0.45x$ versus time to sell x for the pig problem.

Step 5: Answer the question

- Explain results from mathematical models in nontechnical terms such that anyone who can understand the question be able to understand your answer.
- Illustrated example: Sell the pig after 8 days to get a net profit $133.2.
- Dealing with real-life problem, there exists uncertainty in assumptions/suppositions, so we should investigate several alternatives. This is called by sensitivity analysis.

Summary (1)

Step 1. Ask the question.

- Make a list of all the variables in the problem, including appropriate units.
- Be careful not to confuse variables and constants.
- State any assumptions you are making about these variables, including equations and inequalities.
- Check units to make sure that your assumptions make sense.
- State the objective of the problem in precise mathematical terms.

Summary (2)

Step 2. Select the modeling approach.

- Choose a general solution procedure to be followed in solving this problem.
- Generally speaking, success in this step requires experience, skill, and familiarity with the relevant literature.
- In this book we will usually specify the modeling approach to be used.

Summary (3)

Step 3. Formulate the model.

- Restate the question posed in step 1 in the terms of the modeling approach specified in step 2.
- You may need to relabel some of the variables specified in step 1 in order to agree with the notation used in step 2.
- Note any additional assumptions made in order to fit the problem described in step 1 into the mathematical structure specified in step 2.

Summary (4)

Step 4. Solve the model.

- Apply the general solution procedure specified in step 2 to the specific problem formulated in step 3.
- Be careful in your mathematics. Check your work for math errors. Does your answer make sense?
- Use appropriate technology. Computer algebra systems, graphics, and numerical software will increase the range of problems within your grasp, and they also help reduce math errors.

Summary (5)

Step 5. Answer the question.

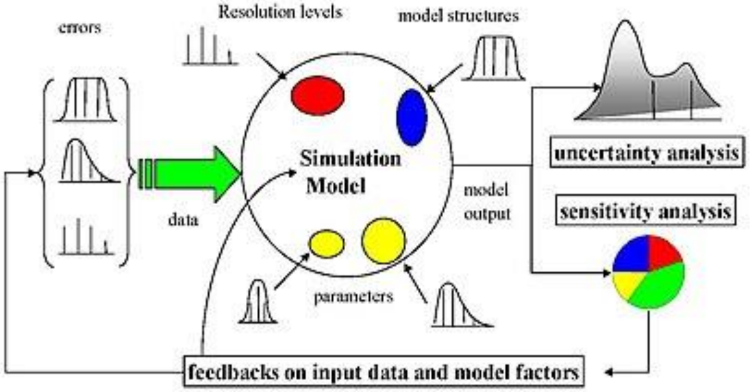
- Rephrase the results of step 4 in nontechnical terms.
- Avoid mathematical symbols and jargon.
- Anyone who can understand the statement of the question as it was presented to you should be able to understand your answer.

Outline

- Five-step method
- **Sensitivity analysis**
- Sensitivity and Robustness

Sensitivity analysis

- Five-step method starts with making some assumptions about the problem. However, we are rarely certain enough abouts things to be able expect all these assumptions.
- Therefore, we need to consider hov sensitive our conclusions are to each of the assumption we have made.
- This plays an important role in mathematical model.



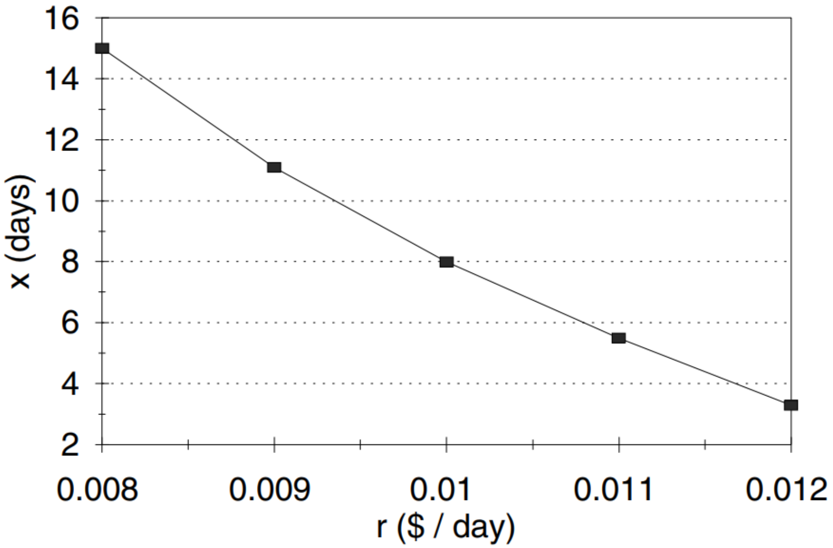
Assumption of the illustrated example

- Assumptions:
  - Current weight of the pig
  - Current price for pigs
  - Cost per day of keeping the pig
  - Rate of growth of the pig
  - Rate of falling price
- Uncertain levels of assumptions are different to each other due to nature of data

Illustrated example: Sensitivity of decrease price rate

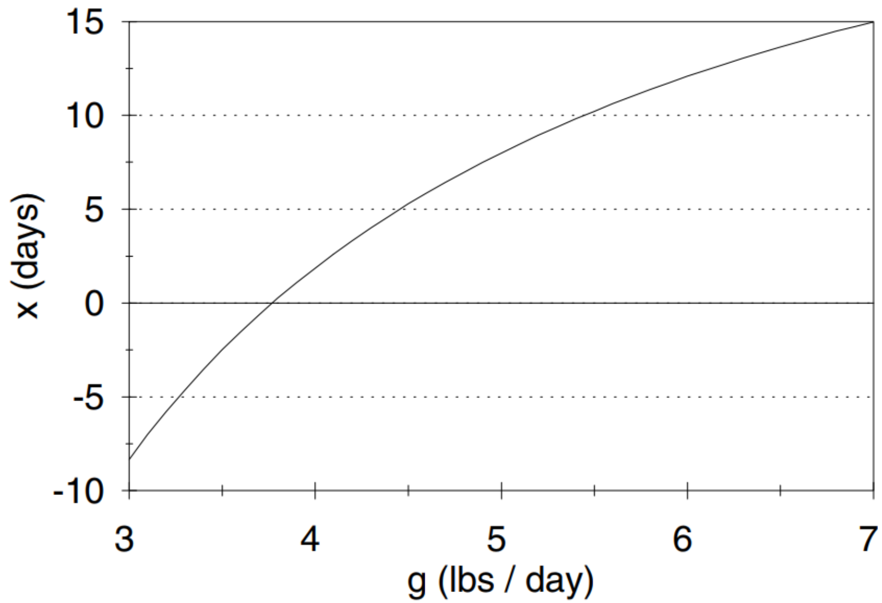
- Original assumption, rate at which the price is falling, *r* = 0.01 dollars per day. We try to change this value in different scenarios.

r (\$/day)	x (days)
0.008	15.0
0.009	11.1
0.010	8.0
0.011	5.5
0.012	3.3



Illustrated example: Sensitivity of growth rate

- Original assumption, growth rate *g* is 5 lbs per day. We try to change this value in different scenarios.



Sensitivity analysis

- We should interpret sensitivity data in terms of relative change or percent change, rather than in absolute terms.
- Illustrated example:
  - 10% decrease in falling price rate *r* leads to a 39% increase in optimal value
  - 10% decrease in growth rate leads to 34% decrease in optimal value

Sensitivity analysis

If x changes by an amount Δx, the relative change in x is given by $\Delta x/x$. If r changes by Δr, resulting in the change Δx in x, then the ration between the relative changes is $\Delta x/x$ divided by $\Delta r/r$. Let $\Delta r \to 0$ and using the definition of the derivative, we obtain

$$\frac{\Delta x/x}{\Delta r/r} \to \frac{dx}{dr} \cdot \frac{r}{x}.$$

We call this limiting quantity the sensitivity of x to r, and we will denote it by $S(x, r)$.

Sensitivity analysis

In the example we have $\frac{dx}{dr} = -\frac{7}{25r^2} = -2800$. With $r = 0.01, x = 8$, thus

$$\begin{aligned} S(x, r) &= \frac{dx}{dr} \cdot \frac{r}{x} \\ &= (-2,800)\left(\frac{.01}{8}\right) \\ &= \frac{-7}{2}. \end{aligned}$$

It means that if r goes up by 2%, then x goes down by 7%

Sensitivity analysis

We have

$$\frac{dx}{dg} = \frac{245}{2g^2}$$
$$= 4.9,$$

And

$$S(x,\, g) = \frac{dx}{dg} \cdot \frac{g}{x}$$
$$= (4.9)\left(\frac{5}{8}\right)$$
$$= 3.0625,$$

So that a 1% increase in the growth rate of the pig would cause us to wat about 3% longer to sell the pig

Sensitivity analysis

- Question: What is $S = (y, g)$

Outline

- Five-step method
- Sensitivity analysis
- **Sensitivity and Robustness**

Sensitivity and Robustness

- A mathematical model is robust if the conclusions it leads to remain true even through the model is not completely accurate.
- In real problems, we will never have perfect information, and even if it were possible to construct a perfectly accurate model, we might be better off with a simpler and more tractable approximation.
- For this reason, a consideration of robustness is a necessary ingredient in any mathematical modelling project.

Illustrated example

- In assumption, both the weight of the pig and the selling price per pound are linear functions of time. These are not exactly valid in real life.
- According this assumptions, if we keep the pig a year, so

$$\begin{aligned} w &= 200 + 5(365) \\ &= 2,025 \,\mathrm{lbs} \end{aligned}$$

$$\begin{aligned} p &= 0.65 - 0.01(365) \\ &= -3.00 \,\mathrm{dollars/lb.} \end{aligned}$$

- Assumptions should be as much realistic as possible!

Exercises

An automobile manufacturer makes a profit of \$1,500 on the sale of a certain model. It is estimated that for every \$100 of rebate, sales increase by 15%.

(a) What amount of rebate will maximize profit? Use the five-step method, and model as a one–variable optimization problem.

(b) Compute the sensitivity of your answer to the 15% assumption. Consider both the amount of rebate and the resulting profit.

(c) Suppose that rebates actually generate only a 10% increase in sales per \$100. What is the effect? What if the response is somewhere between 10 and 15% per \$100 of rebate?

(d) Under what circumstances would a rebate offer cause a reduction in profit?