HA NOI UNIVERSITY OF SCIENCE AND TECHNOLOGY
SCHOOL OF INFORMATION AND COMMUNICATION TECHNOLOGY

FUNDAMENTALS OF OPTIMIZATION

Integer Linear Programming

CONTENT

- Relaxation and Bound
- Branch and Bound
- Cutting plan
- Gomory Cut
- Branch and Cut

Relaxation and Bound

- Given an Integer Program (IP)

$$z = \max\{cx: x \in X \subseteq Z^n\}$$

- Find decreasing sequence of upper bounds

$$\overline{z}_1 > \overline{z}_1 > \ldots > \overline{z}_s \geq z$$

- Find increasing sequence of lower bounds

$$\underline{z}_1 < \underline{z}_1 < \ldots < \underline{z}_t \leq z$$

- Algorithm stop when $\overline{z}_s - \underline{z}_t \leq \varepsilon$

Relaxation and Bound

- Primal bounds
  - Every feasible solution $x^* \in X$ provides a lower bound of the maximization problem: $\underline{z} = cx^* \leq z$
  - Example: in TSP, every close tour is a upper bound of the objective function (as TSP is a minimization problem)

Relaxation and Bound

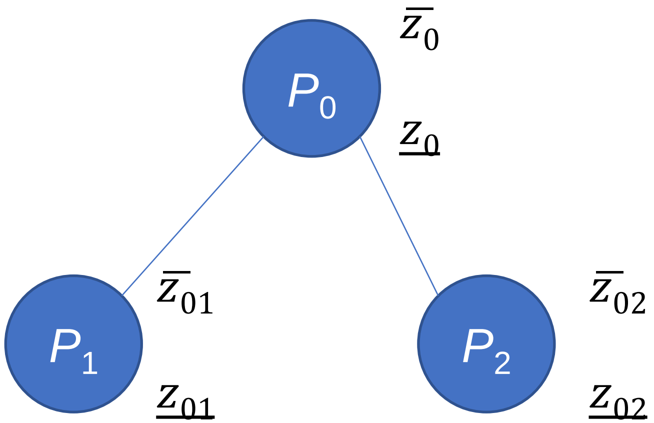
- Dual bounds
  - Finding upper bounds for a maximization problem (or lower bounds for a minimization problem) gives dual bounds of the objective
- **Definition** A problem (RP) $z^R = \max\{f(x): x \in T \subseteq R^n\}$ is a relaxation of (IP) $z = \max\{cx: x \in X \subseteq Z^n\}$ if:
  - $X \subseteq T$
  - $f(x) \geq cx, \forall x \in X$
- **Proposition** RP is a relaxation of IP, $z^R \geq z$

Relaxation and Bound

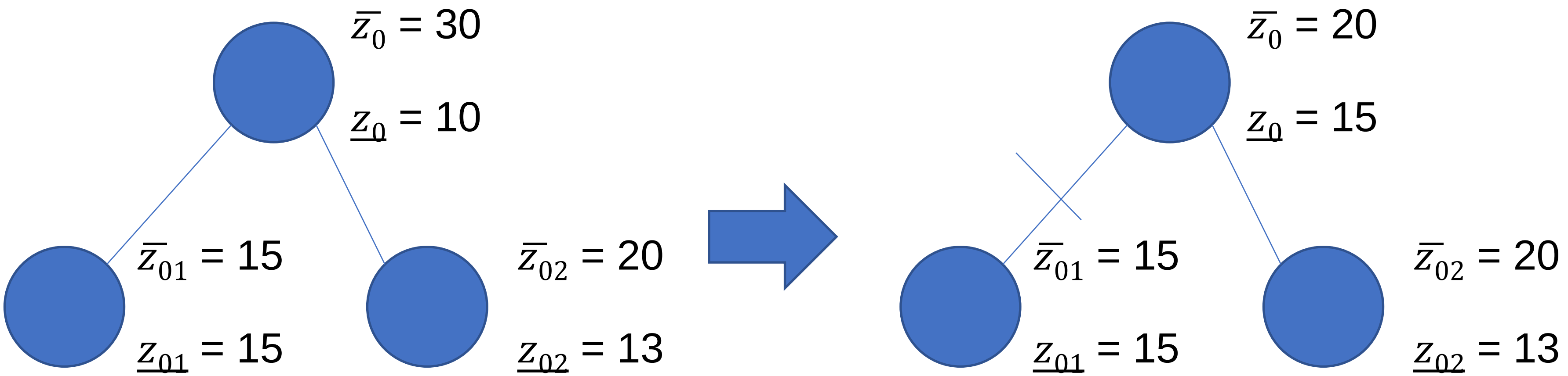
- Linear Relaxation
  - $Z^{LP} = \max\{cx: x \in P\}$ with $P = \{x \in R^n: Ax \leq b\}$ is a linear relaxation program of the *IP* $\max\{cx: x \in P \cap Z^n\}$

Branch and Bound

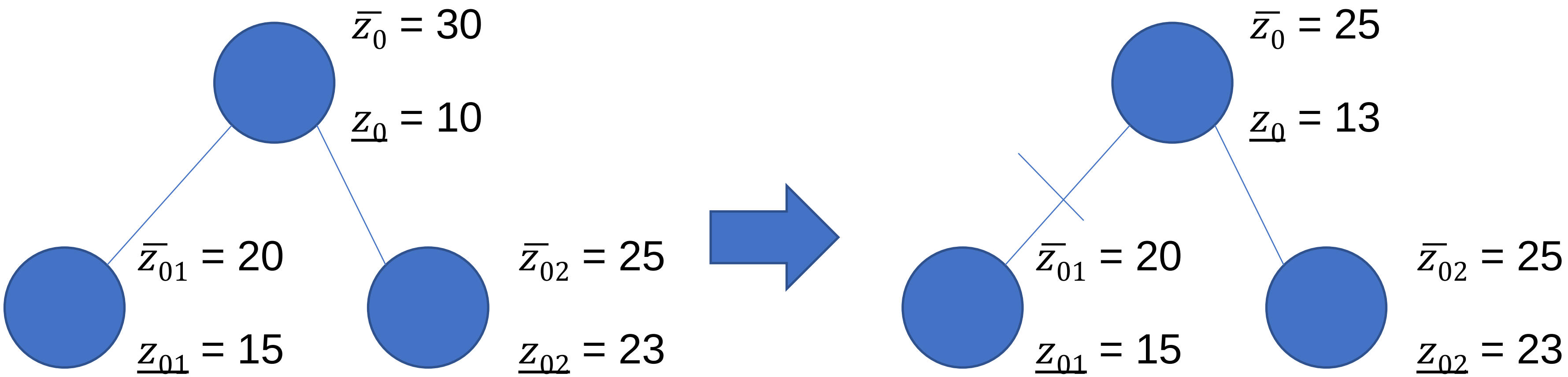
- Feasible region of P_0 is divided into feasible regions of P_1 and P_2: $X(P_0) = X(P_1) \cup X(P_2)$



Branch and Bound



Branch and Bound



LP-based Branch and Bound

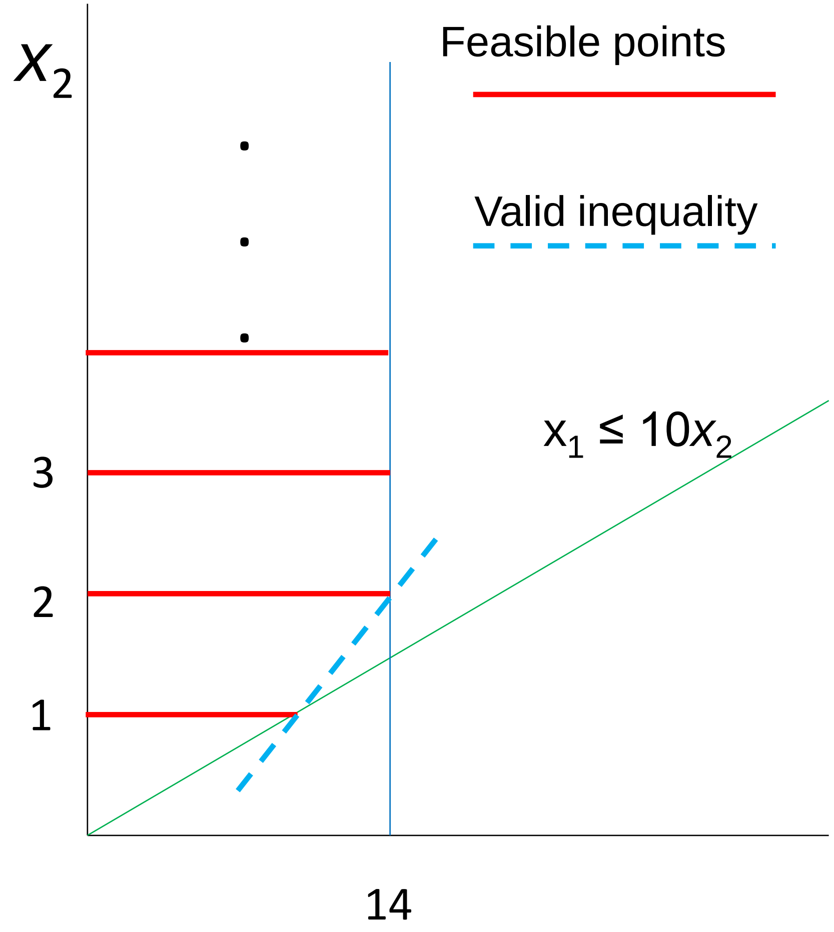
```
Initial problem S with formulation P on a list L
Incumbent x* is initialized with primal bound z = -INF
while L not empty do{
  Select a problem S^i with formulation P^i from L
  Solve LP relaxation over P^i got dual bound z̄^i and solution x^i(LP)
  if z̄^i ≤ z then continue; // prune by dual bound
  if x^i(LP) integer then{
     z = z̄^i
     x* = x^i(LP)
  }else{
     select a component x_i of x^i(LP) whose value λ_i is fractional
     P_1^i = P^i ∪ (x_i ≤ ⌊λ_i⌋), P_2^i = P^i ∪ (x_i ≥ ⌈λ_i⌉)
     add P_1^i and P_2^i to L
  }
}
Return x*
```

Cutting Plane

- Given a MIP max$\{cz: z \in X\}$
- Inequality $\pi z \leq \pi_0$ is called a valid inequality if $\pi z \leq \pi_0$ is true for all $z \in X$
- Finding valid inequalities allows us to narrow the search space, transform MIP to corresponding LP in which an optimal solution to LP is an optimal solution to the original MIP

Cutting Plane

- Example, consider a MIP with $X = \{(x_1,x_2): x_1 \le 10x_2,\ 0 \le x_1 \le 14,\ x_2 \in Z_+^1\}$
- Red lines represent X
- $x_1 \le 6 + 4x_2$ is a valid inequality (dashed line)



Example Integer Rounding

- Consider feasible region $X = P \cap Z^3$ where $P = \{x \in R_+^3 : 5x_1 + 9x_2 + 13x_3 \geq 19\}$
- From $5x_1 + 9x_2 + 13x_3 \geq 19$ we have $x_1 + \frac{9}{5}x_2 + \frac{13}{5}x_3 \geq \frac{19}{5}$

→ $x_1 + 2x_2 + 3x_3 \geq \frac{19}{5}$

- As x_1, x_2, x_3 are integers, so we have

$$x_1 + 2x_2 + 3x_3 \geq \left\lceil \frac{19}{5} \right\rceil = 4 \text{ (this is a valid inequality for } X\text{)}$$

Gomory Cut

- (*IP*) max {*cx*: *Ax* = *b*, *x* ≥ 0 and integer}
- Solve corresponding linear programming relaxation (*LP*) max {*cx*: *Ax* = *b*, *x* ≥ 0}
- Suppose with an optimal basis, the *LP* is rewritten in the form

$$\overline{a_{00}} + \sum_{j \in JN} \overline{a_{0j}} x_j \rightarrow \max$$

$$x_{B_u} + \sum_{j \in JN} \overline{a_{uj}} x_j = \overline{a_{u0}}, \; u = 1, 2, \ldots, m$$

$$x \geq 0 \text{ and integer}$$

with $\overline{a_{0j}} \leq 0$ (as these coefficients corresponds to a maximizer), and $\overline{a_{u0}} \geq 0$

Gomory Cut

- If the basic optimal solution *x** is not integer, then there exists some row u with $\overline{a_{u0}}$ is not integer

→ Create a Gomory cut $x_{B_u} + \sum_{j\in JN} \lfloor \overline{a_{uj}} \rfloor x_j \leq \lfloor \overline{a_{u0}} \rfloor$ (1)

→ Rewriting this inequality (as x_{B_u} is integer)

$$\sum_{j\in JN} (\overline{a_{uj}} - \lfloor \overline{a_{uj}} \rfloor) x_j \geq \overline{a_{u0}} - \lfloor \overline{a_{u0}} \rfloor$$

or

$$\sum_{j\in JN} f_{u,j} x_j \geq f_{u,0} \qquad (2)$$

with $f_{u,j} = \overline{a_{uj}} - \lfloor \overline{a_{uj}} \rfloor$ and $f_{u,0} = \overline{a_{u0}} - \lfloor \overline{a_{u0}} \rfloor$

- As $0 \leq f_{u,j} < 1$ and $0 < f_{u,0} < 1$ and $x_j^* = 0$, $\forall\, j \in J_N$ → (2) cuts off *x**.

Gomory Cut

- Difference between the left-hand side (LHS) and right-hand side (RHS) of (1) is integral (as x is integral) → the difference between LHS and RHS of (2) is also integral

→ rewrite (2) in the form s = $\sum_{j \in JN} f_{u,j} x_j$ - $f_{u,0}$ where the slack variable s is nonnegative integer

Gomory Cut

Consider the Integer Program (IP)

$$f(x_1, x_2, x_3, x_4, x_5) = x_1 + x_2 \rightarrow \max$$

$$2x_1 + x_2 + x_3 = 8$$

$$3x_1 + 4x_2 + x_4 = 24$$

$$x_1 - x_2 + x_5 = 2$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0 \text{ and integer}$$

Gomory Cut

Solve the LP relaxation

$$
\begin{aligned}
f(x_1, x_2, x_3, x_4, x_5) \quad = \quad & x_1 + x_2 \rightarrow \max \\
& 2x_1 + x_2 + x_3 = 8 \\
& 3x_1 + 4x_2 + x_4 = 24 \\
& x_1 - x_2 + x_5 = 2 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

→ Optimal solution (1.6, 4.8, 0, 0, 5.2) with JB = (1,2,5) and JN = (3, 4)

Rewrite the original IP

$$
\begin{aligned}
f(x_1, x_2, x_3, x_4, x_5) \quad = \quad & 6.4 - 0.2x_3 - 0.2x_4 \rightarrow \max \\
& x_1 + 0.8x_3 - 0.2x_4 = 1.6 \\
& x_2 - 0.6x_3 + 0.4x_4 = 4.8 \\
& x_5 - 1.4x_3 + 0.6x_4 = 5.2 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0 \text{ and integer}
\end{aligned}
$$

Gomory Cut

We obtain an equivalent IP

$$f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = x_1 + x_2 \rightarrow \max$$

$$\begin{aligned}
2x_1 + x_2 + x_3 &= 8 \\
3x_1 + 4x_2 + x_4 &= 24 \\
x_1 - x_2 + x_5 &= 2 \\
0.8x_3 + 0.8x_4 - x_6 &= 0.6 \\
0.4x_3 + 0.4x_4 - x_7 &= 0.8 \\
0.6x_3 + 0.6x_4 - x_8 &= 0.2 \\
x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 &\geq 0 \text{ and integer}
\end{aligned}$$

Gomory Cut

Solve the corresponding LP relaxation

$$f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = x_1 + x_2 \rightarrow \max$$

$$\begin{aligned}
2x_1 + x_2 + x_3 &= 8 \\
3x_1 + 4x_2 + x_4 &= 24 \\
x_1 - x_2 + x_5 &= 2 \\
0.8x_3 + 0.8x_4 - x_6 &= 0.6 \\
0.4x_3 + 0.4x_4 - x_7 &= 0.8 \\
0.6x_3 + 0.6x_4 - x_8 &= 0.2 \\
x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 &\geq 0
\end{aligned}$$

→ Optimal solution $x^* = (0, 6, 2, 0, 8, 1, 0, 1)$ which is integer.

→ So (0,6,2,0,8) is an optimal solution to the original problem with the objective value = 0+6 = 6

Branch and Cut [Wolsey, 98]

INITIALIZATION z = max{cx: x∈X} with formulation P, $\underline{z}$ = -INF, incumbent x^* empty, init Nodelist

NODE if Nodelist empty, goto EXIT else choose and remove node *i* from Nodelist and goto RESTORE

RESTORE formulation P_i of the set X^i, set $k = 1$ and $P^{i,1} = P^i$

LP RELAXATION Iteration k. Solve $\bar{z}^{i,k} = \max\{cx: x \in P^{i,k}\}$. If infeasible, prune and goto NODE Else solution $x^{i,k}$ and goto CUT

CUT Iteration k. Try to cut off $x^{i,k}$. If not cuts found, goto PRUNE Else add cuts to $P^{i,k}$ giving $P^{i,k+1}$. Increase k by 1, and goto LP RELAXATION

PRUNE If $\bar{z}^{i,k} \le \underline{z}$, goto NODE. If x∈X, set $\underline{z} = \bar{z}^{i,k}$, update incumbent $x^* = x^{i,k}$ and goto NODE Else goto BRANCHING

BRANCHING Create two or more new problems and add them to the Nodelist

EXIT Incumbent x* with optimal value $\underline{z}$

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Thank you for your attentions!

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