Guide to State Elimination

Dear Mr. President: There are too many states nowadays. Please eliminate three. I am not a crackpot! (The Simpsons)

The state elimination algorithm transforms a DFA or NFA into regular expression. This is a big deal in Theoryland, as it shows that there's nothing you can do with DFA or NFA that you can't do with a regex. In practice, it's a useful tool to keep in your toolkit when designing regexes. While it probably shouldn't be your first line of attack when crafting regular expressions (the Guide to Regular Expressions outlines some techniques that are usually more effective), every now and then the state elimination algorithm ends up being one of the easiest ways to write a regex for a language.

In lecture, we did an example of the state elimination algorithm, which showed off the main technique. There's also a slide that details the full workings of how to do state elimination. This guide presents a further example of how to do state elimination so that the algorithm becomes easier to understand.

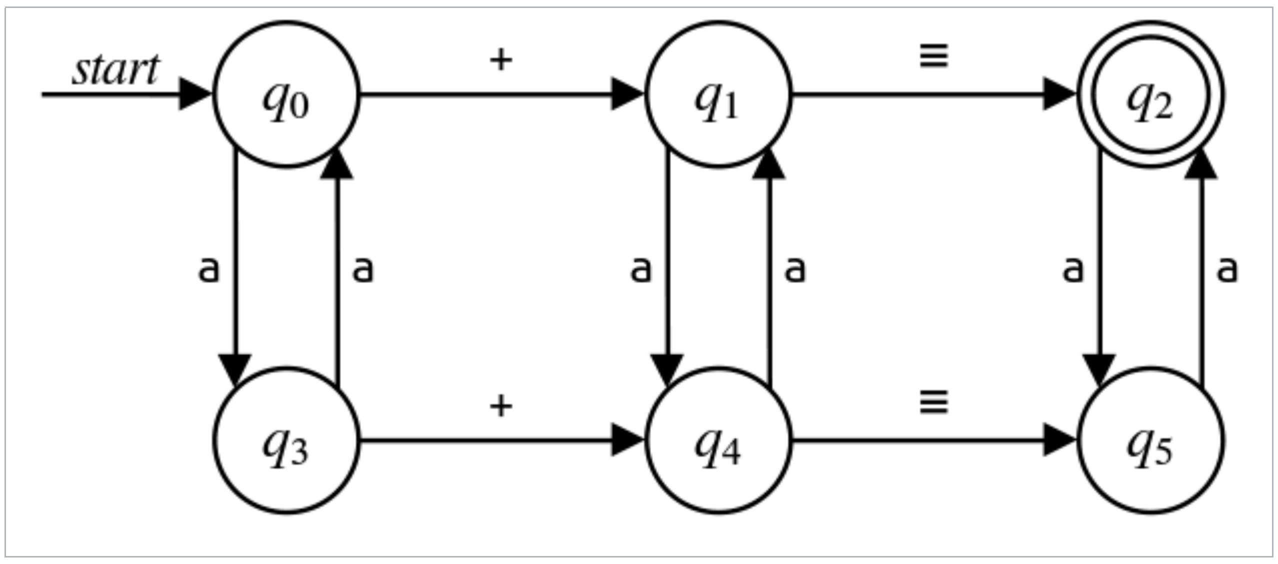
Example: Addition Parity

Let's begin with the following language L over $\Sigma = \{\texttt{a}, \texttt{+}, \equiv\}$:

$$L = \{\texttt{a}^m + \texttt{a}^n \equiv \texttt{a}^p \mid m, n, p \in \mathbb{N} \land m + n \equiv_2 p\}.$$

So, for example, we have $\texttt{aaa+aaa} \equiv \texttt{aaaaaa} \in L$ because there are three a's in the first group, three as in the second group, six a's in the third group, and it's true that $3 + 3 \equiv_2 6$. We also have $\texttt{aa+aaaa} \equiv \texttt{aa} \in L$, since there are two a's in the first group, four a's in the second group, and two a's in the third group, and $2 + 4 \equiv_2 2$. More generally, the format of the strings in the language is a group of a's of length m, a plus sign, then a second group of a's of length n, then an $\equiv$ symbol, and a third group of a's of length p such that $m + n$ and p have the same parity (they're either both even or both odd). Note that $+ \equiv$ is also in the language (zero a's in each group), as are $\texttt{a}+ \equiv \texttt{a}$ and $+\texttt{a} \equiv \texttt{a}$.

It's a worthwhile but challenging exercise to try to design a regex for this language given nothing more than the above. But let's suppose that, for some reason, you were getting stuck doing so and you weren't sure how to proceed. Your next line of attack might then be to design an NFA for this language and use the state elimination algorithm to turn that NFA into a regex. Here's one possible NFA for this language:



Before moving on, let's take a minute to see how this automaton works. Since all that matters to the language is the parity (evenness/oddness) of the counts of the as, the automaton consists of two parallel rows of states. On the top are states q_0, q_1, and q_2, which corresponding to being in the first, second, and third groups of a's. On the bottom are q_3, q_4, and q_5, which similarly track which group we're in. The transitions vertically between the two halves corresponding to changing the parity of the total number of a's read thus far. Every time we see an a in the top (even) set of states, we move to the bottom (odd) set of states and vice-versa.

Our goal is to turn this NFA into a regular expression for L using the state elimination algorithm.

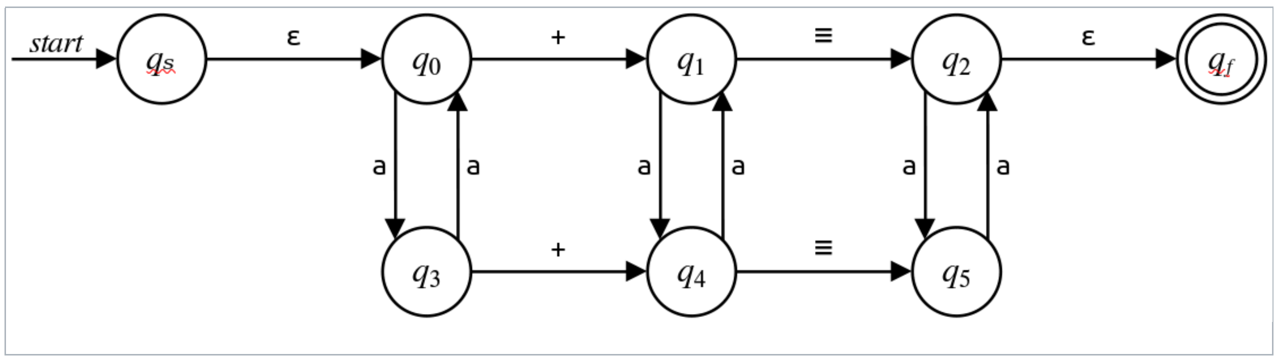
Initial Setup

We begin with the first step of the algorithm:

> ***Step One:*** Add in a new start state q_s with a ε transition to the old start state. Add in a new accepting state q_f and add ε transitions from each old accepting state to q_f. Finally, make only q_f the accepting state.

The purpose of this step is to set us up for success later on. We will ultimately remove all states in the automaton besides q_s and q_f, and when we're done, the transition from q_s to q_f will consist of the regular expression we want.

After doing this, we get the following automaton:



We next proceed to step two of the algorithm - which is the real meat of the algorithm:

> ***Step Two:*** Eliminate all states from the NFA other than q_s and q_f. To eliminate a state q_{gone}, do the following:
>
> 1. Find all pairs of states (q_{in}, q_{out}) where q_{in} has a transition to q_{gone}, where q_{gone} has a transition to q_{out}, and where neither q_{in} or q_{out} is q_{gone}.
> 2. For each pair (q_{in}, q_{out}), add a transition directly from q_{in} to q_{out}. Label that transition with a regex simulating the effect of transitioning from q_{in} to q_{gone}, following any self-loops on q_{gone}, then transitioning to q_{out}. (Details below.)
> 3. Remove q_{gone} from the automaton.
> 4. Consolidate parallel transitions. (Details below.)
> 5. (Optional) Simplify any complex regexes that arise in the process.

There is no requirement about which state we eliminate first or the order in which we eliminate the states. Any choice will work. However, some choices may work much, much better than others. It's something you pick up by feel as you get more comfortable with the algorithm. I've scouted this particular example out and found it easiest to begin by removing q_3 first, so that's where we'll begin.

Eliminating q_3

Our first task in eliminating q_3 is to find all pairs of states (q_{in}, q_{out}) where q_{in} transitions into q_3 and q_3 transitions into q_{out}. Looking at the automaton, we see that only one other state (q_0) has a transition into q_3, and that q_3 has transitions into two other states (q_0 and q_4). This means that there are two pairs of states for us to look at: (q_0, q_0) and (q_0, q_4).

The next step is to add new transitions into the automaton, one for each pair. We'll thus add a transition from q_0 to itself and a transition from q_0 to q_4.

What should go on these transitions? Let's begin with the new transition from q_0 to q_4. This new transition is designed to simulate the effect of taking the transition from q_0 to q_3 and from q_3 to q_4. The transition from q_0 to q_3 is an `a` transition, and the transition from q_3 to q_4 is a `+` transition, so we'll label this transition `a+`. (It's a slightly unfortunate choice of alphabet that we have `+` in this regex meaning "the character `+`" rather than "the Kleene plus operation." In retrospect, it would have been better to use a different symbol here, but `+` feels natural in the context of the language, so we'll just roll with it for this example.)

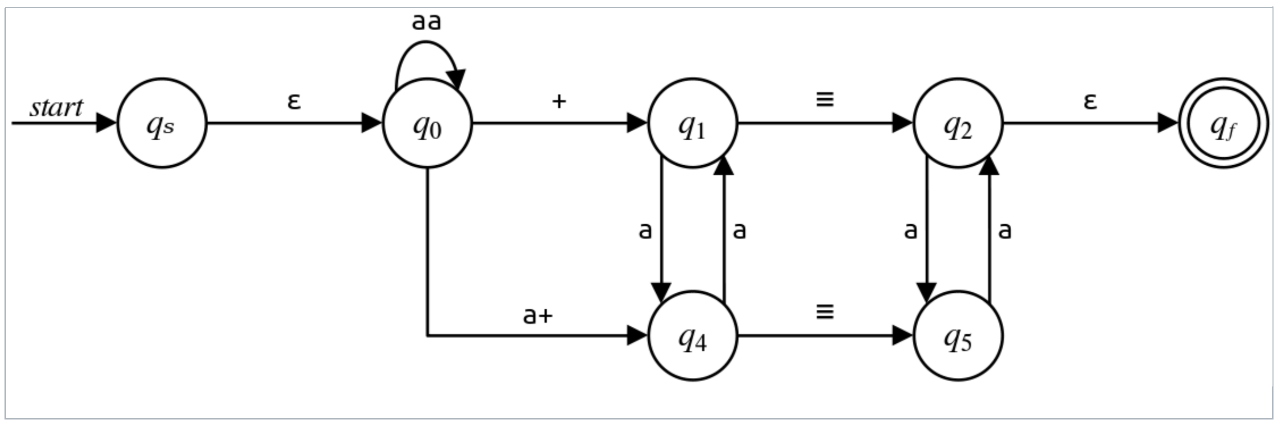
Similarly, let's look at the new transition we need to add from q_0 to itself. This should simulate the effect of going from q_0 to q_3, then from q_3 back to q_0. The transition from q_0

to q_3 is labeled `a` and the transition from q_3 to q_0 is labeled `a`, so the new transition will be labeled `aa`.

Here's a summary of the new transitions:

In	Out	Regex
q_0	q_4	`a+`
q_0	q_0	`aa`

Now that we know what the new transitions are, we will remove q_3 by adding in these new transitions, then deleting q_3 from the automaton. The result is shown here:



We now have one fewer state in our automaton. If we look at the rest of what we need to do to eliminate this state (consolidate parallel transitions and simplify regexes), we find that neither applies: there are no parallel transitions, and the regexes we have here are already as simple as possible. So let's move on to remove our next state.

Eliminating q_5

Let's next remove q_5. We begin, as before, by finding all other states with transitions into q_5 and all other states that q_5 transitions into. q_5 has two incoming transitions (one from q_4 and one from q_2) and one outgoing transition (to q_2). We therefore will need to fill in the following table:

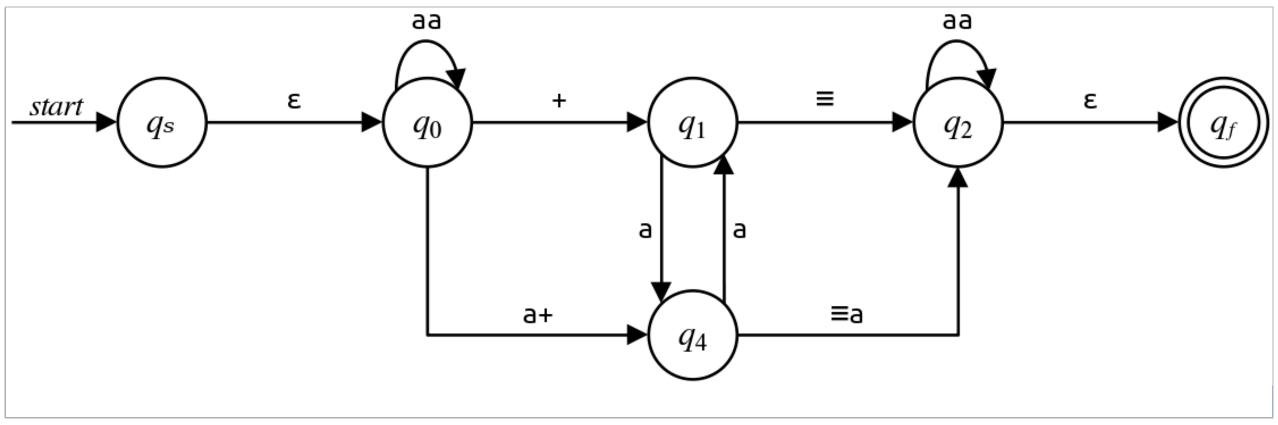
In	Out	Regex
q_4	q_2	______
q_2	q_2	______

Let's begin with the transition that will go from q_4 to q_2. As before, the idea is to simulate going from q_4 to q_5 and then from q_5 to q_2. The transition from q_4 to q_5 is labeled $\equiv$ and the transition from q_5 to q_2 is labeled `a`, so this regex will be labeled $\equiv$ `a`.

Next, let's look at the transition from q_2 back to itself. This transition simulates going from q_2 to q_5 and then from q_5 to q_2. Those transitions are each labeled with `a`, so the new transition gets the label `aa`. We've thus filled in our table:

In	Out	Regex
q_4	q_2	$\equiv$ `a`
q_2	q_2	`aa`

And we can update our automaton by removing q_5 and adding these new transitions in, as shown here:



You might notice a general pattern here for how we decide what goes on the transition. Specifically:

When eliminating state q_{gone} and adding a transition from q_{in} to q_{out}, where q_{gone} has no self-loops, determine the label on the transition as follows:

1. Let R_{in} be the regex on the transition from q_{in} to q_{gone}.
2. Let R_{out} be the regex on the transition from q_{gone} to q_{out}.
3. The resulting regex on the transition from q_{in} to q_{out} is $(R_{in})\,(R_{out})$, where the parentheses may be omitted if they don't change the meaning of the resulting regex.

The idea is to simulate entering the state and then leaving, hence the transition label.

Eliminating q_4

Let's now eliminate q_4. This one is a bit more complex because there are more incoming and outgoing transitions, but it's not too bad. We end up needing to fill in this table:

In	Out	Regex
q_0	q_2	_____
q_0	q_1	_____
q_1	q_2	_____
q_1	q_1	_____

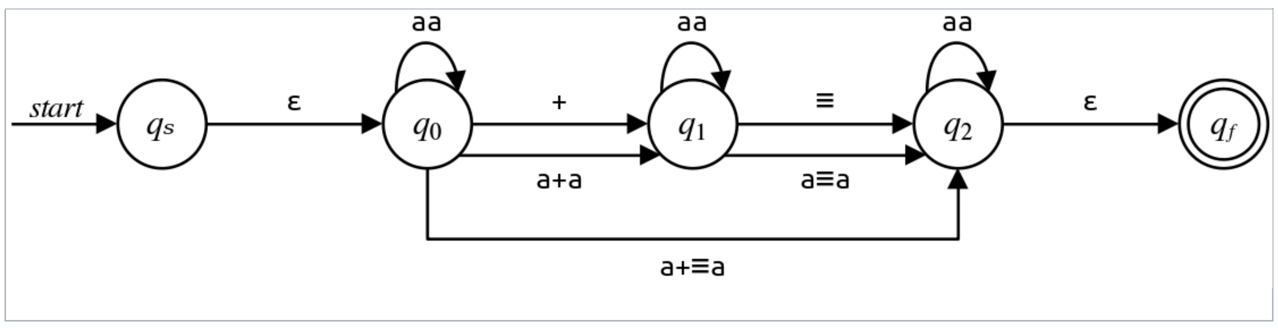
Using the information from the box above, we can determine the transitions like this:

- q_0 to q_2: The transition from q_0 to q_4 is labeled `a+` and the transition from q_4 to q_2 is labeled `≡ a`, so our new transition is labeled `a+ ≡ a`.
- q_0 to q_1: The transition from q_0 to q_4 is labeled `a+` and the transition from q_4 to q_1 is labeled `a`, so our new transition is labeled `a + a`.
- q_1 to q_2: The transition from q_1 to q_4 is labele `a` and the transition from q_4 to q_2 is labeled `≡ a`, so our new transition is labeled `a ≡ a`.
- q_1 to q_1: The transition from q_1 to q_4 is labele `a` and the transition from q_4 to q_1 is labeled `a`, so our new transition is labeled `aa`.

We therefore fill in our table like this:

In	Out	Regex
q_0	q_2	`a+ ≡ a`
q_0	q_1	`a + a`
q_1	q_2	`a ≡ a`
q_1	q_1	`aa`

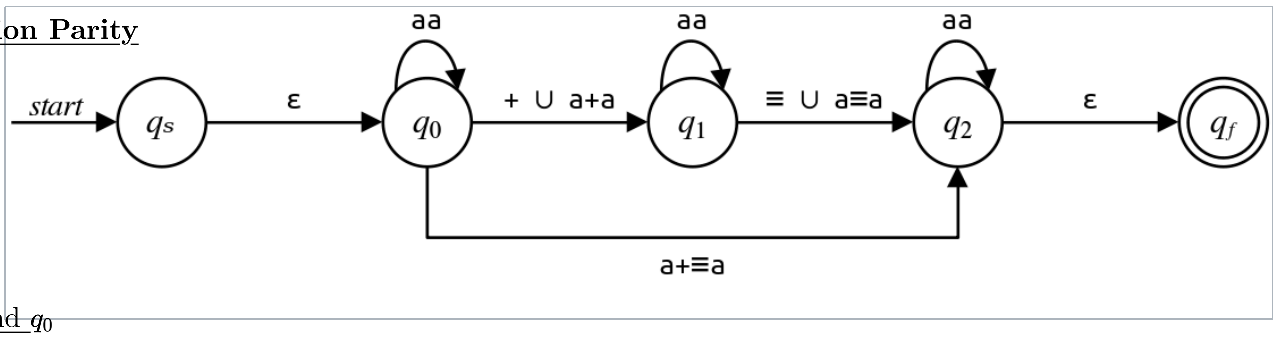
We can then introduce those transitions into our automaton, and remove q_4, to get the following:



At this point, we have a little bit more work to do. There are two parallel transitions between q_0 and q_1 and between $q1$ and q_2. We will need to consolidate these together.

Intuitively, you can think of these two parallel transitions as saying "there are two separate ways to transition between these states." We need to have a single transition between them, which we can create by taking the union of the regexes on the parallel transitions. That way, we can simulate the effect of taking the top transition by taking the first option in the regex, and we can simulate the effect of taking the bottom transition by taking the second option in the regex.

This is shown here:



Eliminating q_1

Let's now eliminate q_1 from this automaton. As before, we start with a table of in/out pairs, which is fairly short this time:

In	Out	Regex
q_0	q_2	____

The procedure for determining what goes on this transition is different than for the previous cases because q_1 has a transition back to itself. As before, we want to simulate with the new q_0 to q_2 transition the effect of entering q_1 from q_0 and then proceeding to q_2. However, we also need to account for the possibility that, once we're at q_1, we take its self-loop some number of times. We'll therefore use the following rule:

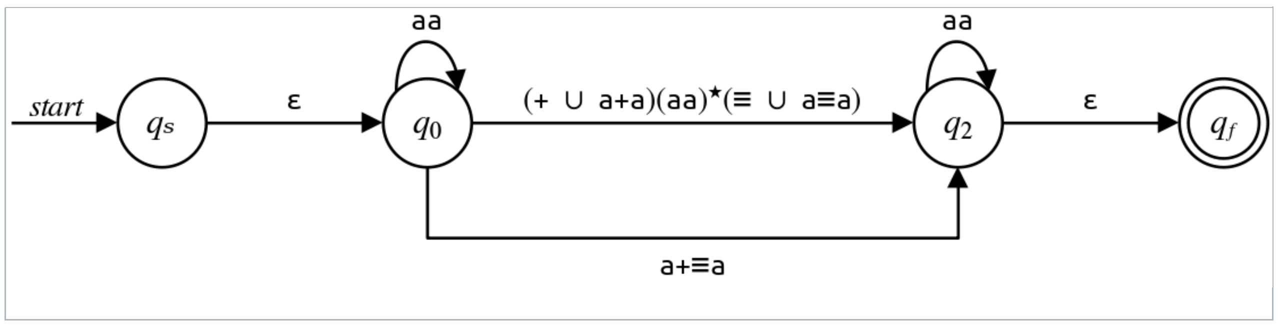
> When eliminating state q_{gone} and adding a transition from q_{in} to q_{out}, where q_{gone} has a self-loop, determine the label on the transition as follows:
>
> 1. Let R_{in} be the regex on the transition from q_{in} to q_{gone}.
> 2. Let R_{stay} be the regex on the transition from q_{gone} to itself.
> 3. Let R_{out} be the regex on the transition from q_{gone} to q_{out}.
> 4. The resulting regex on the transition from q_{in} to q_{out} is $(R_{in})(R_{stay})^\star(R_{out})$, where the parentheses may be omitted if they don't change the meaning of the resulting regex.

That is, the new regex corresponds to entering q_1, looping there as much as we'd like, and then leaving q_1. In this case, that ends up being the following regex:

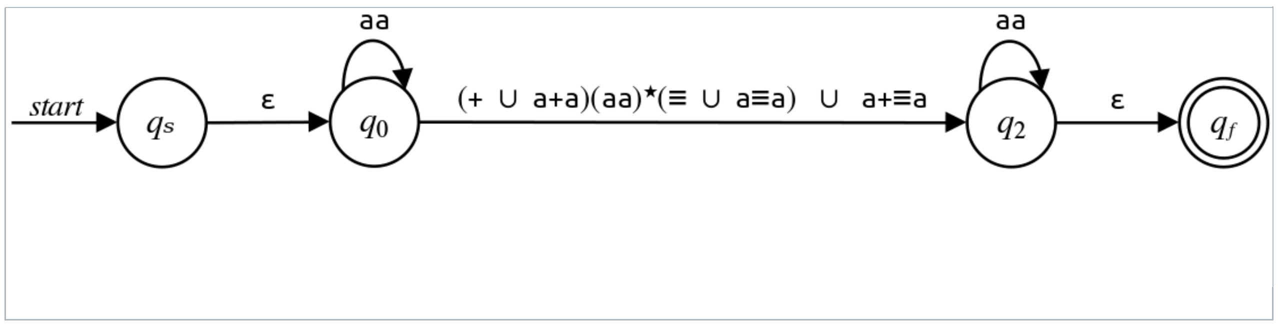
In	Out	Regex
q_0	q_2	$(+ \cup \mathtt{a+a})(\mathtt{aa})^\star(\equiv \cup \mathtt{a} \equiv \mathtt{a})$

We need all the parentheses here so that we have the right operator precedence. It's first choose whether you want + or a + a, then do some number of copies of aa, and finally choose whether we want $\equiv$ or a $\equiv$ a.

Having computed the regex, let's see what happens when we add that transition in and delete q_1:



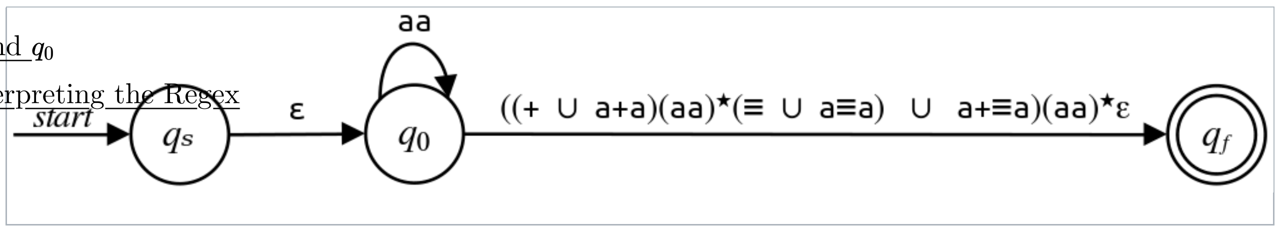
As before, we now have parallel edges to consider, so we'll combine them by making one giant transition with the union of the two input transitions:



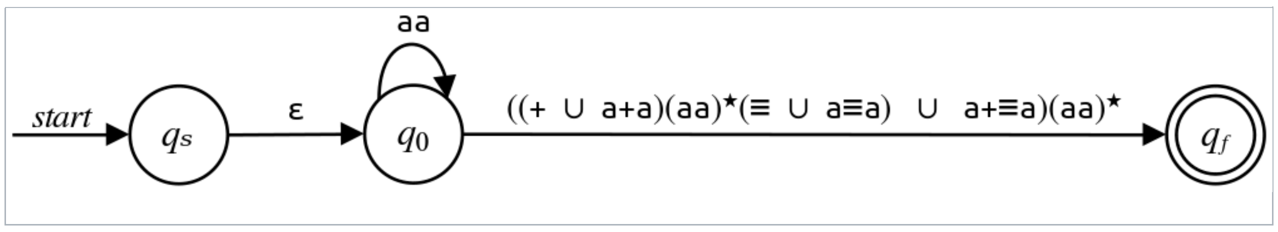
As you can see, the regexes are starting to get a lot longer as the number of states drop. That's normal - state elimination often produces very large regular expressions!

Eliminating q_2 and q_0

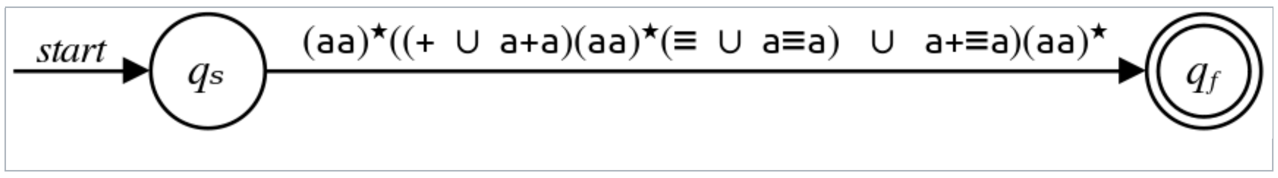
Let's now eliminate q_2. We only have one input/output state pair (we have to enter from q_0 and end up in q_f after taking the transition). We also see that we have a self-loop on q_2, so we'll build the new regex by gluing together the regex on the q_0 to q_2 transition, the star of the q_2 self-loop, and the regex on the q_2 to q_f transition. Here's what that looks like:



Now, we could proceed from this point, but we might want to do some brief cleanup first. The transition from q_0 to q_f ends with a concatenated ε, which has no effect. Let's simply drop that ε to keep our very large regex from being a very *very* large regex. That's shown here:



We can now eliminate q_0. We'll follow the same procedure as before: glue together the regex on the transition from q_s to q_0, then the star of the transition from q_0 to itself, and finally the transition from q_0 to q_f. If we look at what this asks us to do, we can see that we're supposed to glue a ε on the front of this regex from the q_s to q_0 transition, but that has no effect. We'll therefore skip that part and just glue together the self-loop and the transition, giving this final automaton:



We can't eliminate any more states, or we wouldn't have a start state or an accepting state. So we're done with the hard part!

Reading and Interpreting the Regex

The final step of the state elimination algorithm is to read off the overall regex. Here's how that works:

> ***Step Three:*** The final automaton now just has q_s and q_f. The regex is given as follows:
>
> 1. If there is no transition from q_s to q_f, the final regex is $\emptyset$.
> 2. Otherwise, the final regex is the label on the transition from q_s to q_f.

We are in Case 2 here, so our final regex ends up being

$$(\mathtt{aa})^\star\big((+\cup \mathtt{a}+\mathtt{a})(\mathtt{aa})^\star(\equiv\cup\,\mathtt{a}\equiv\mathtt{a})\ \cup\ \mathtt{a}{+}\equiv\mathtt{a}\big)(\mathtt{aa})^\star$$

This is a very complex regex - how do we know that it's correct? The answer is "we don't unless we test it," the same way that if you solve for x in an equation it's never a bad idea to go back and plug in your final answer to make sure you didn't make any silly math errors. In this case, we've checked the regex, and we're pretty sure it's correct.

Any time you finish doing state elimination, I recommend that you take a few minutes to analyze the regex you got back to see how it works. After all, perhaps the way that it works contains some clever technique or approach that would not have occurred to you earlier. (I've been writing regexes for a long time and am often surprised by what drops out of state elimination!) In this case, how do we interpret the regex?

To make it a bit easier to see what's going on, let's use some color-coding. I'll mark the part of the regex that contributes to the first group of a's in gold, the parts contributing to the second group in teal, and the parts contributing to the final group in purple:

$$(\mathtt{aa})^\star\left((\mathtt{+} \cup \mathtt{a+a})(\mathtt{aa})^\star(\equiv \cup \mathtt{a} \equiv \mathtt{a}) \;\cup\; \mathtt{a+} \equiv \mathtt{a}\right)(\mathtt{aa})^\star$$

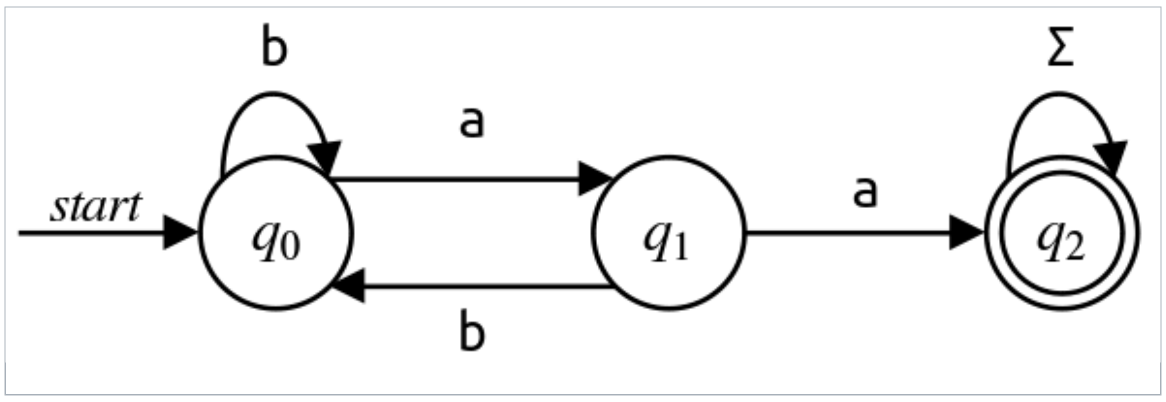
It's interesting to see how the groups are built up. All of them are built up two `a`'s at a time. The cases where one group is odd is handled by placing an `a` on both sides of one of the separator symbols.

When I first read this I was mystified by why there needed to be an `a+` $\equiv$ `a` case. I would definitely recommend taking some time to think this one over - as a hint, how does the regex handle the case where the first and last group have an odd number of `a`'s while the middle group has an even number of `a`'s?

Exercises

Here are some exercises you can use to get more practice with state elimination.

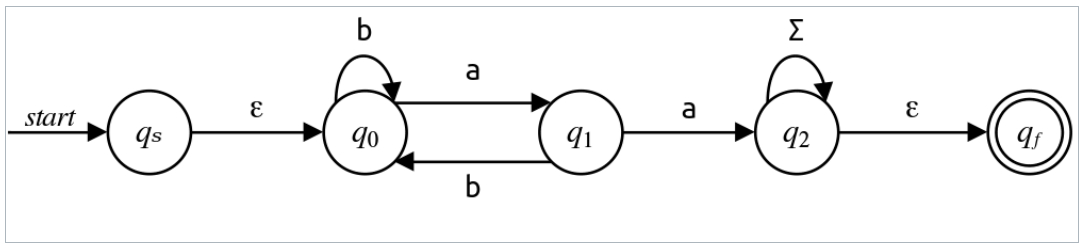
i. Here is a DFA that acccepts all strings that contain `aa` as a substring:



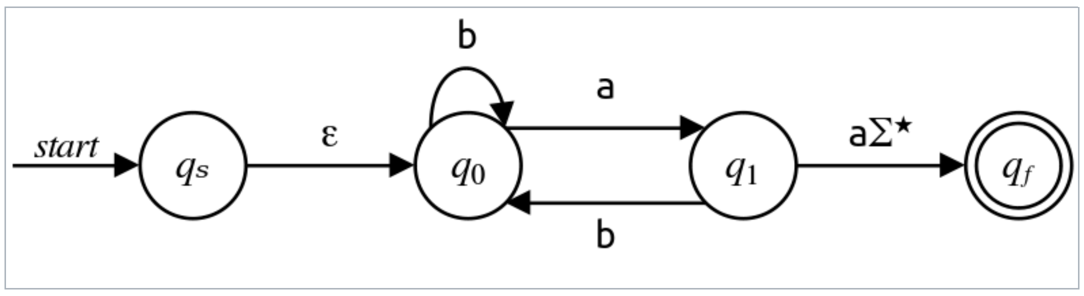
Perform the state-elimination algorithm on this DFA, removing the states in the order q_2, q_1, q_0. What is the resulting regular expression?

Solution

We begin by adding in a new start and accepting state, as shown here:



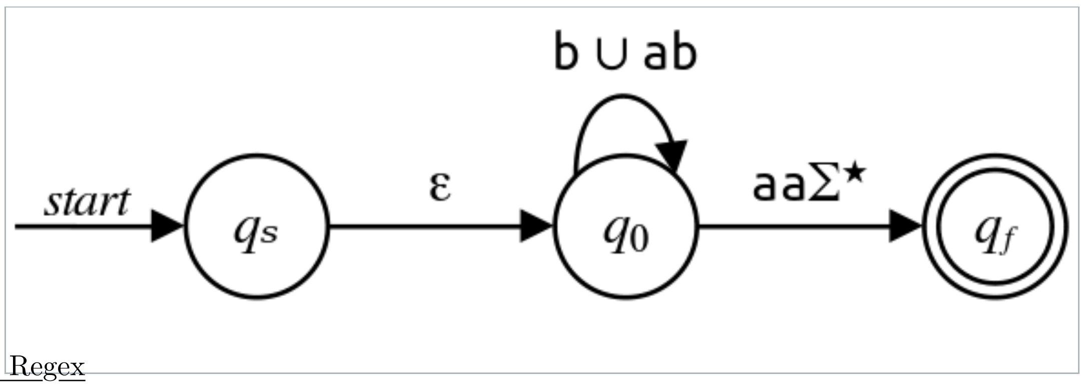
To eliminate q_2, we need to add a transition from q_1 to q_f labeled $\mathtt{a}\Sigma^\star\varepsilon$. Concatenating with ε has no effect here, so we label that transition $\mathtt{a}\Sigma^\star$ instead:



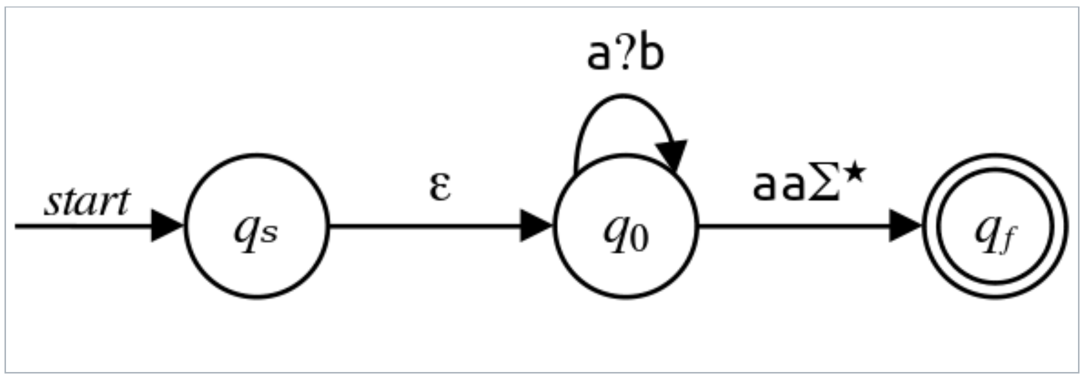
Next, we'll eliminate q_1. There are two pairs of in/out states to consider, and the necessary transitions are listed below:

In	Out	Regex
q_0	q_0	`ab`
q_0	q_f	$\mathtt{aa}\Sigma^\star$

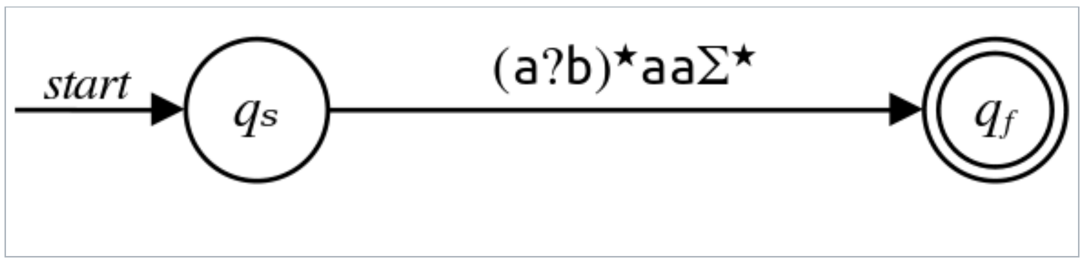
This is shown here, after collapsing the parallel transitions from q_0 to itself:



An optional but nice step: we can notice that the regex $\mathtt{b} \cup \mathtt{ab}$ is equivalent to the regex **a?b** and simplify the automaton a bit:



Finally, removing q_0 gives us the regex $\varepsilon(\mathtt{a?b})^\star\mathtt{aa}\Sigma^\star$, which we can simplify by dropping the unnecessary concatenated ε:

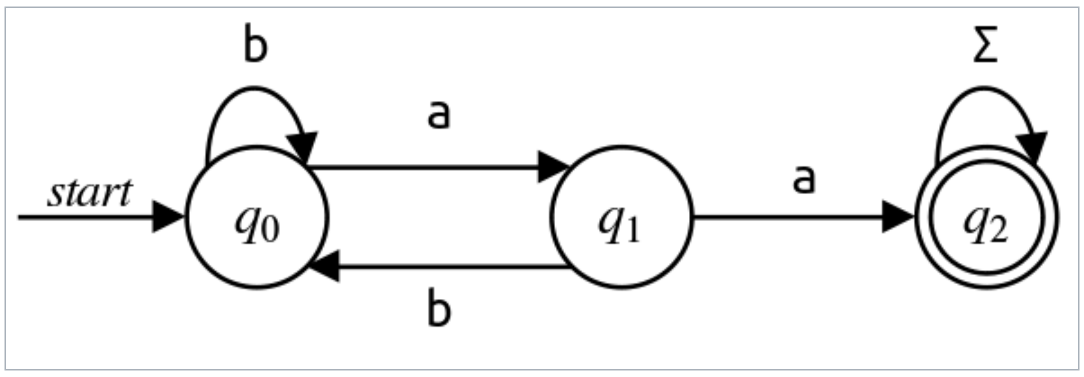


This means that our final regex is $(\mathtt{a?b})^\star\mathtt{aa}\Sigma^\star$. It's worth taking a minute to see why this works.

ii. Perform state elimination on the same automaton as in part (i), but this time using the order q_0, then q_1, and then q_2. What regex do you get back now? This shows that the order in which states are eliminated can influence the form of the final regex. (The resulting regex will always have the same language as the original automaton, though.)

Solution

We begin, as above, by adding in q_s and q_f:

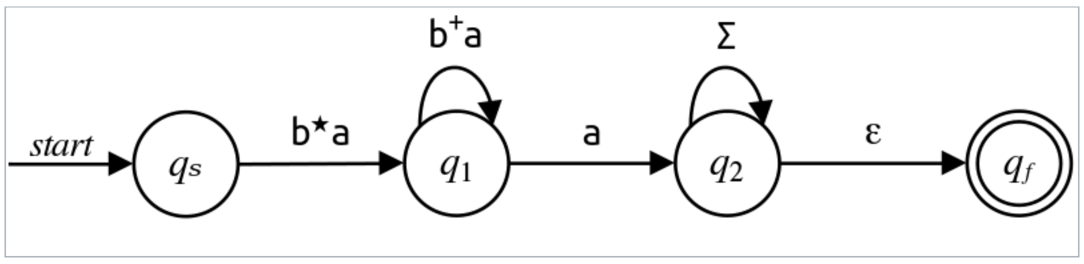


To eliminate q_0, we fill in the following table to determine all the transitions we need to add. I've simplified the table here by removing all unnecessary concatenated ε's:

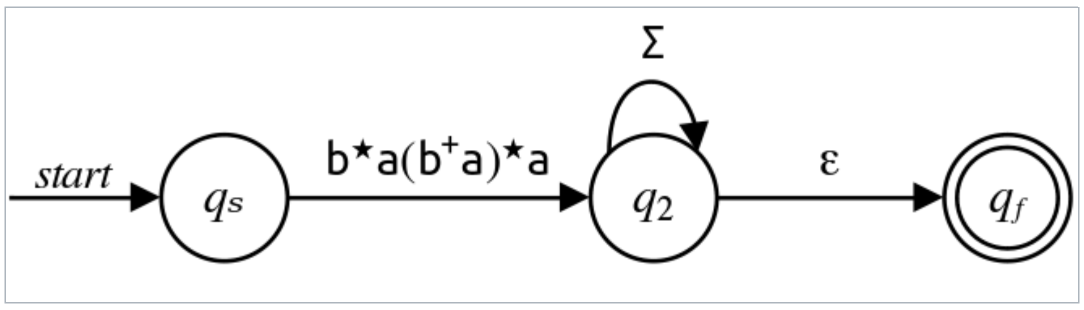
In	Out	Regex
q_s	q_1	$\mathtt{b}^\star\mathtt{a}$
q_1	q_1	$\mathtt{bb}^\star\mathtt{a}$

We can further simplify this by noticing that $\mathtt{bb}^\star$ is more compactly expressed as $\mathtt{b}^+$.

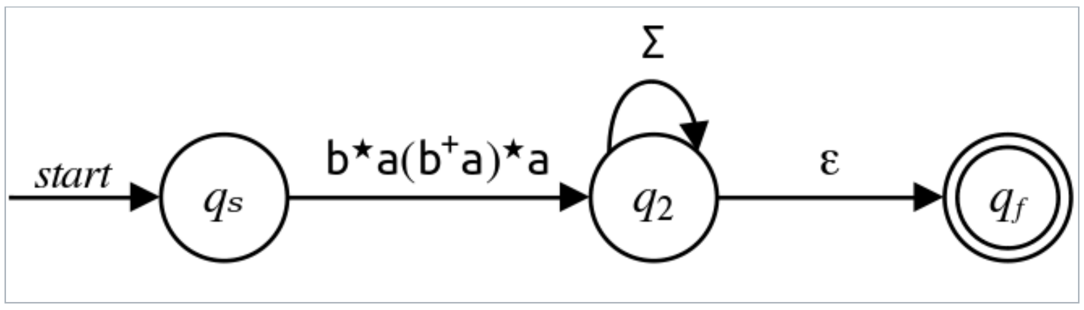
Here's what the automaton looks like after removing q_0:



To eliminate q_1, we add a transition labeled $\mathtt{b}^\star\mathtt{a}(\mathtt{b}^+\mathtt{a})^\star\mathtt{a}$ from q_s to q_2, as shown here:



Finally, we eliminate q_2 here by adding a transition labeled $\mathtt{b}^\star\mathtt{a}(\mathtt{b}^+\mathtt{a})^\star\mathtt{a}\Sigma^\star$ from q_s to q_f. (I've already removed the redundant concatenation with ε here.)



This gives us our final regex, $\mathtt{b}^\star\mathtt{a}(\mathtt{b}^+\mathtt{a})^\star\mathtt{a}\Sigma^\star$. It's again worth taking a minute to see why this regex works.

iii. Write a much simple regex for the language of strings that contain `aa` as a substring than you obtained through either parts (i) or (ii) of these exercises. This shows that while state elimination is guaranteed to always convert a DFA or NFA into a regular expression, the regex you get back might not be the nicest or simplest.

Solution

Here's a very simple regex for this language:

$$\Sigma^\star\mathtt{aa}\Sigma^\star$$

This says "match anything, then `aa`, then anything."