Welcome to CS103!

Are there “laws of physics” in computer science?

For this, along with further course motivation and Day 1 logistics, see the Prezi included on today’s lecture page.

CS103 ACE

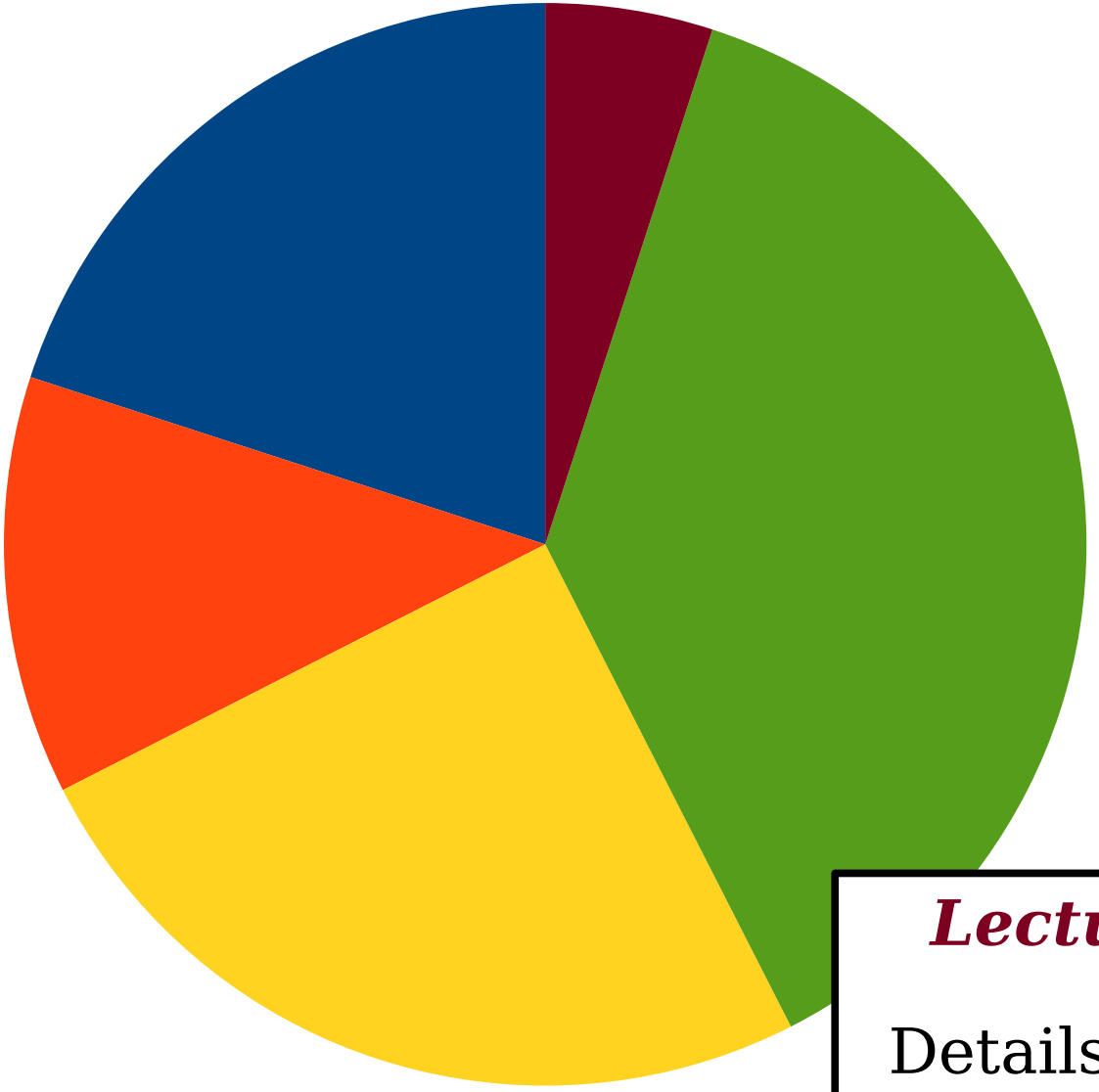
- ***CS103 ACE*** is an optional, one-unit companion course to CS103.
- CS103 ACE meets Tuesdays, 6:00PM – 7:50PM and provides additional practice with the course material in a small group setting.
- Interested? Apply online using ***this link.***
- The CS103 ACE materials are available to everyone. You can pull them up ***here***.

Elena Sierra
(ACE Instructor)

Problem Set 0

- Your first assignment, Problem Set 0, goes out today. It's due Friday at 1:00PM Pacific.
- This assignment requires you to set up your development environment and to get set up on Gradescope and EdStem.
- There's no coding involved, but it's good to start early in case you encounter any technical issues.

Grading



Lecture Participation

Details on Monday of next week!

Approaching this Course

- The material in this class is challenging, but you are well-equipped to succeed.
- Key recommendations:
  - ***Attend lecture in person***. This ensures you stay current with the concepts and gives time to digest the content.
  - ***Take notes by hand***. Explaining concepts in your own words improves learning and identifies questions to ask.
  - ***Ask questions!*** That's what we're here for. Feel free to ask in lecture, after class, on EdStem, or in office hours.
  - ***Read the Guides***. We post supplementary readings on the website in the form of "Guides to *X*." Those are the main readings for the course.
  - ***Focus on learning***. Problem sets are worth much less than the exams. Prioritize building skills over completion.
- CS103 is more like building a rocket than learning a language: there's less frequent feedback you'll need to review in more depth.

We've got a big journey ahead of us.

Let's get started!

Introduction to Set Theory

“CS103 students”

“Cool people”

“The chemical elements”

“Cute animals”

“US coins”

A ***set*** is an unordered collection of distinct objects, which may be anything, including other sets.

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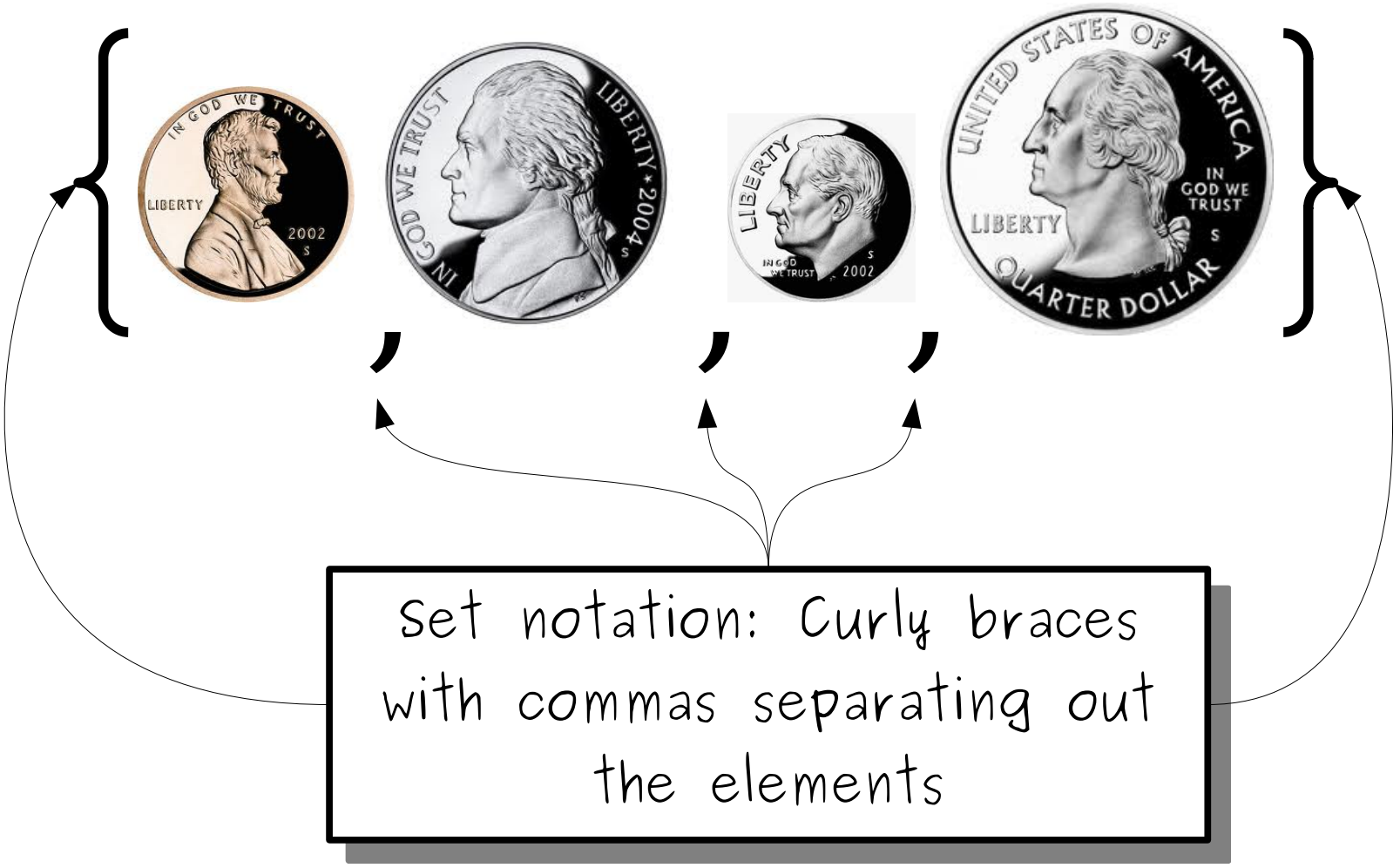

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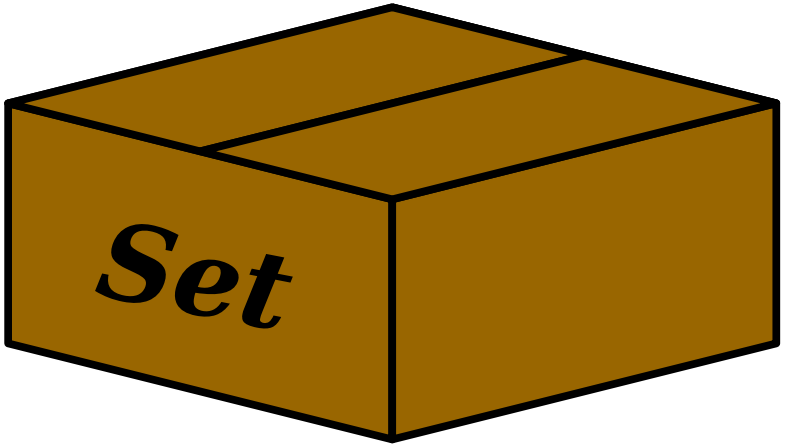
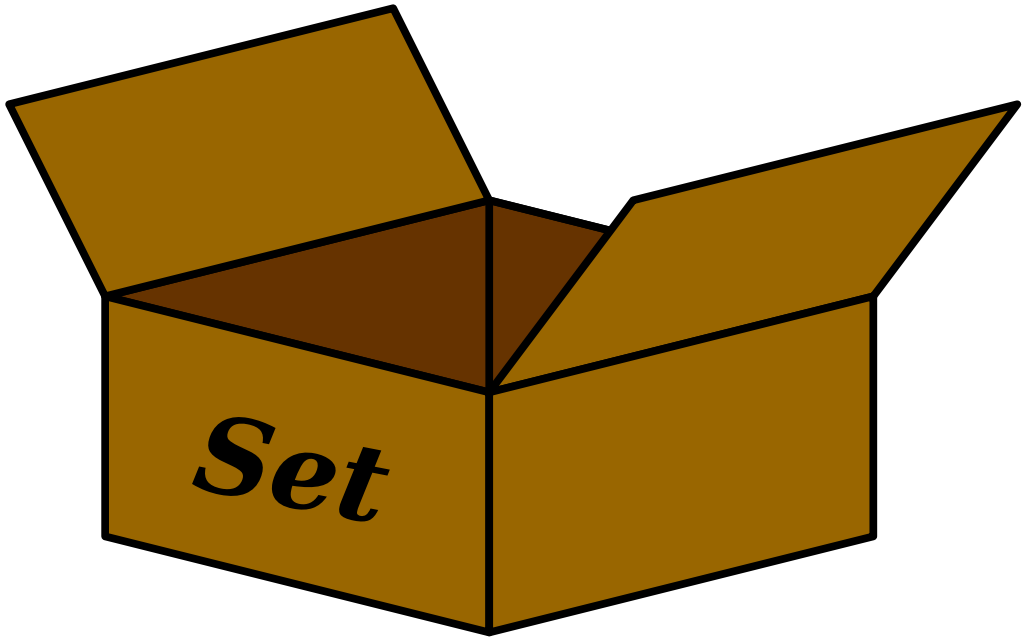


Two sets are equal when they have the same contents, ignoring order.



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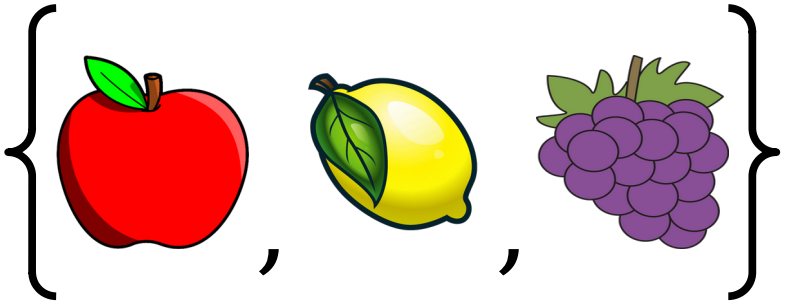

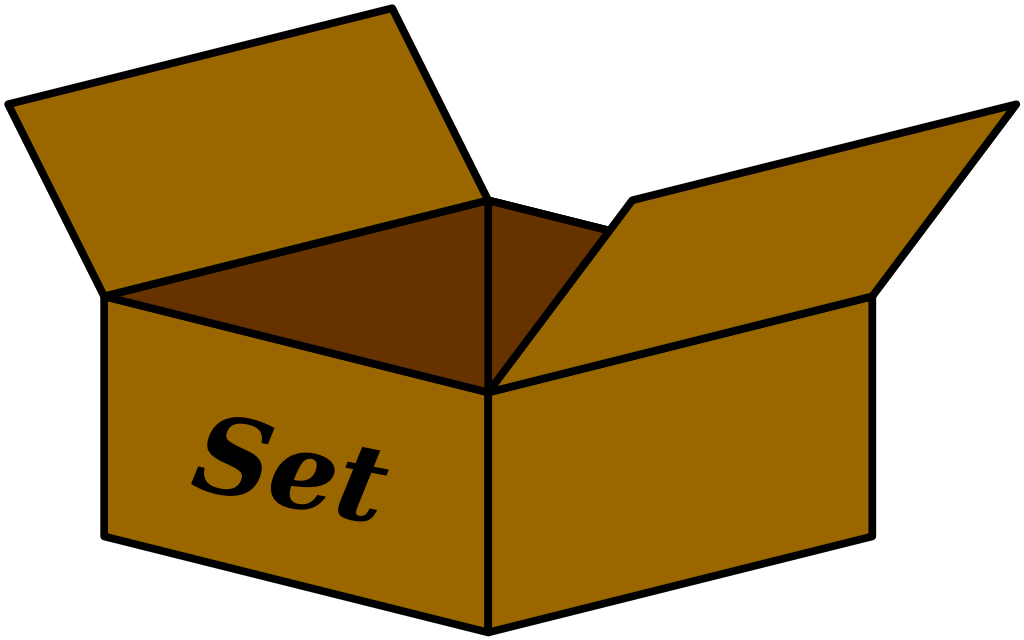

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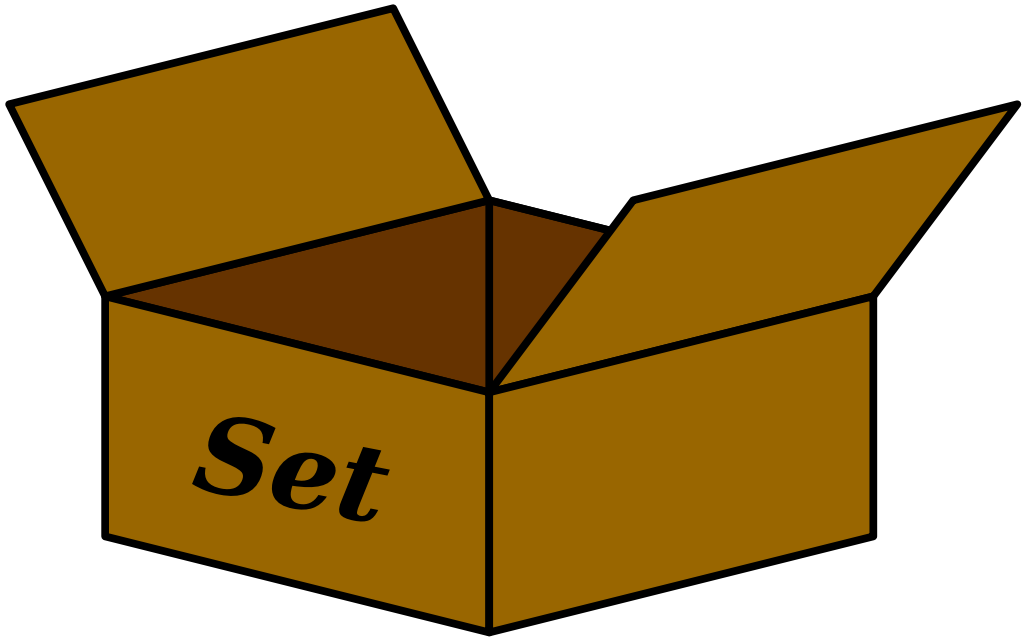

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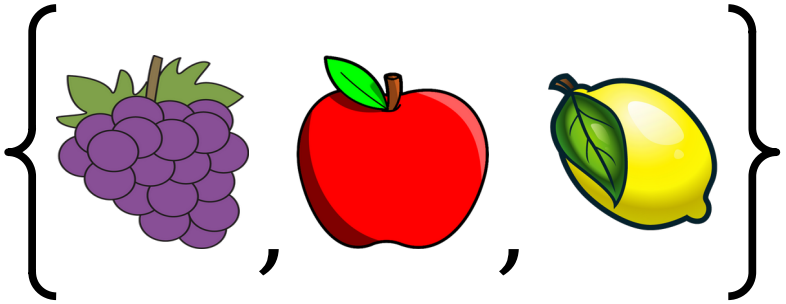

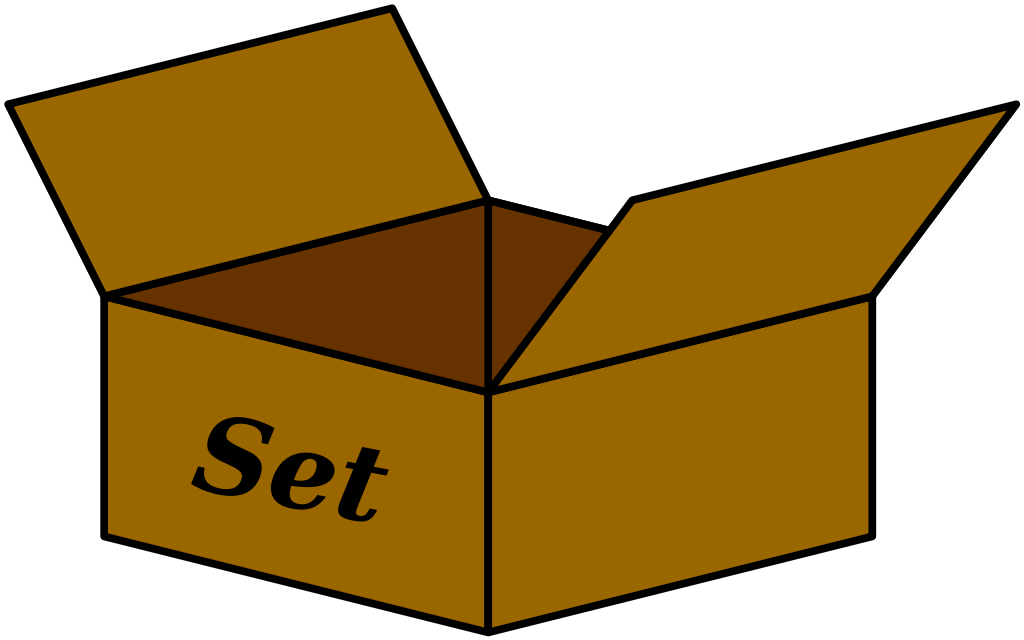

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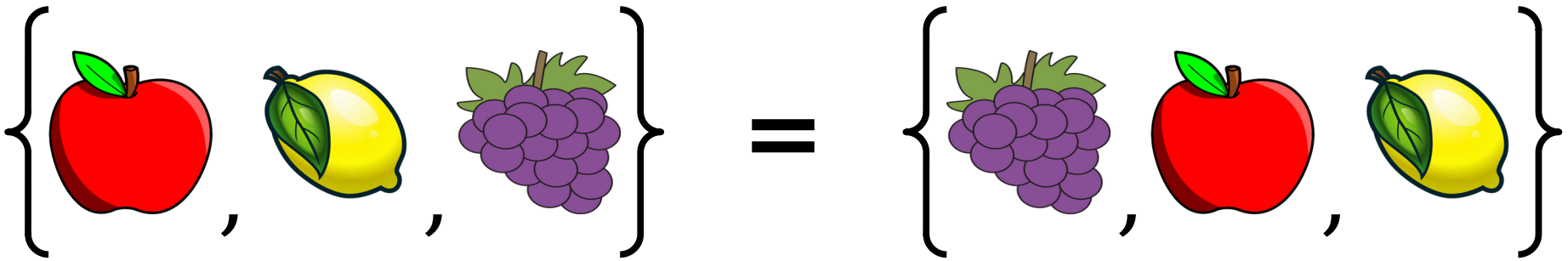

These are two descriptions of the same set.

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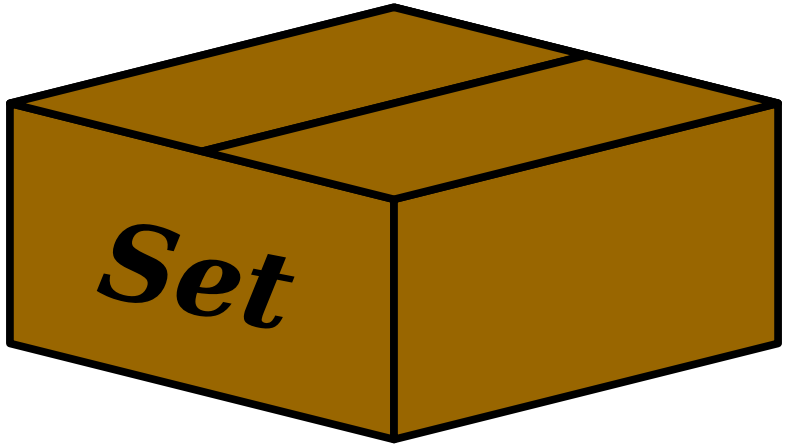
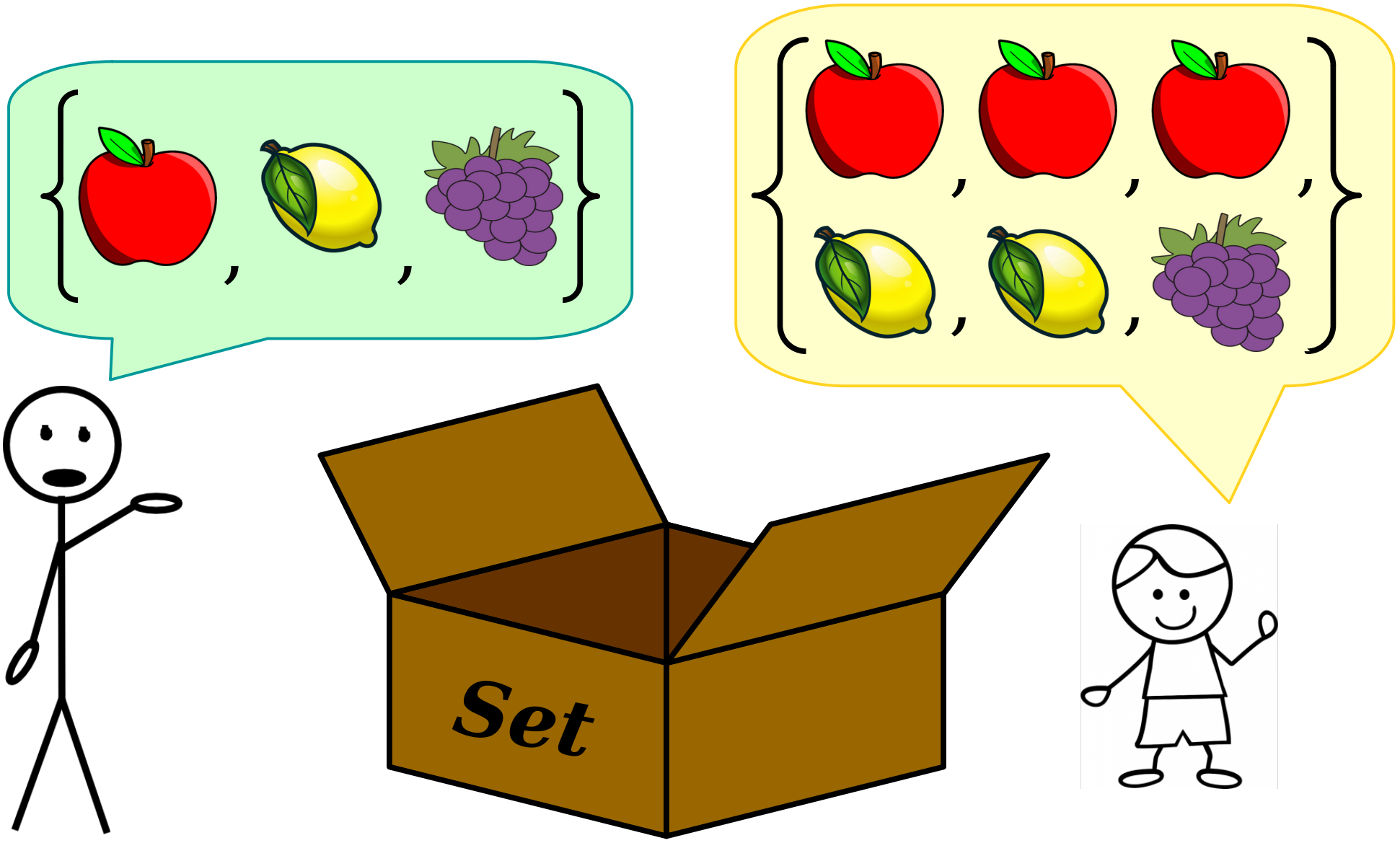


Sets cannot contain duplicate elements.
Any repeated elements are ignored.



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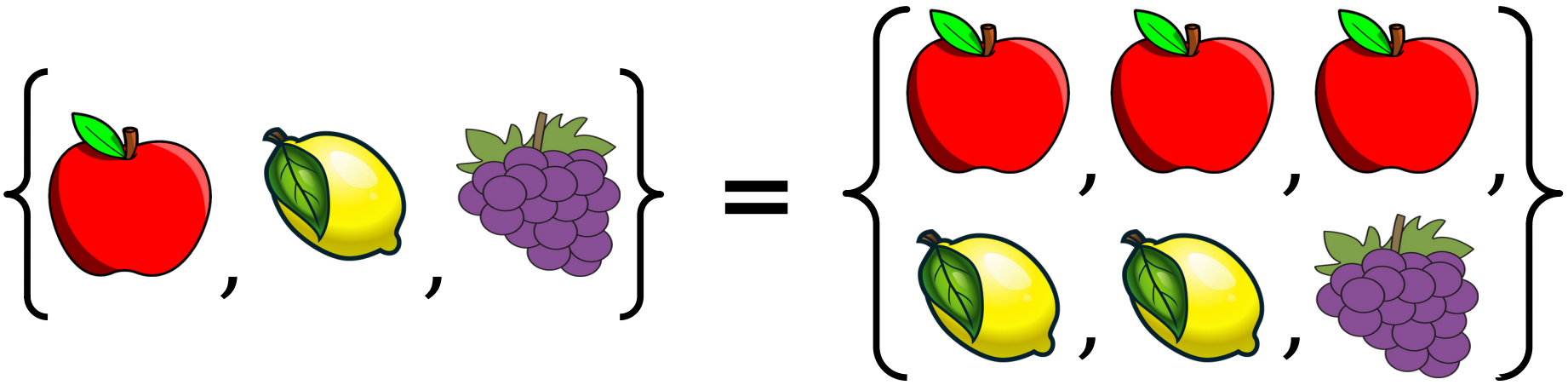

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The objects that make up a set are called the ***elements*** of that set.

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This symbol means "is an element of."

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$\notin \{$

,

,

,

$\}$

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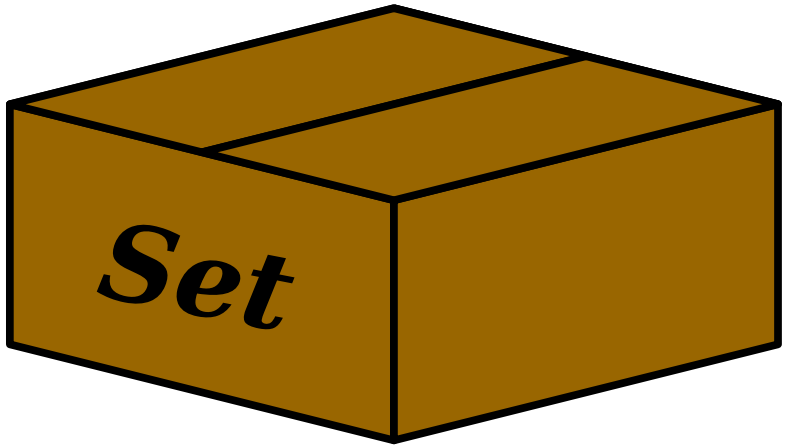
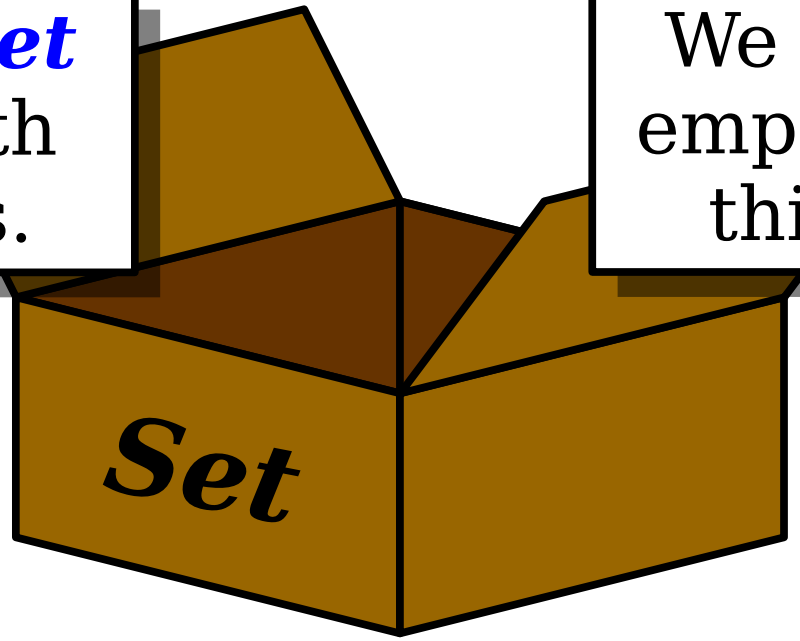


Sets can contain any number of elements.

$\{\ \} = \emptyset$

The ***empty set*** is the set with no elements.

We denote the empty set using this symbol.



Sets can contain any number of elements.

$$1 \stackrel{?}{=} \{1\}$$

Question: Are these objects equal?

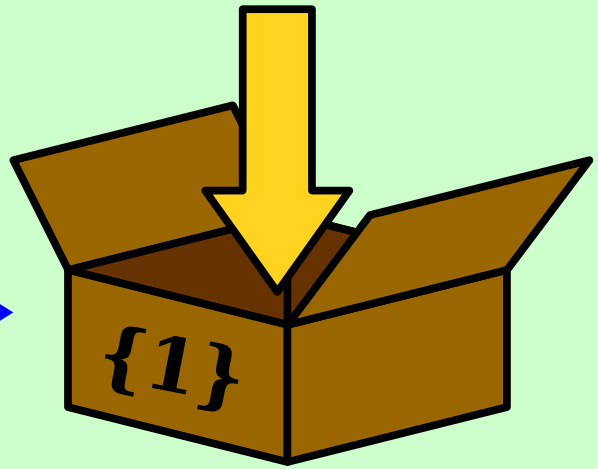
$$1 \stackrel{?}{=} \{1\}$$

This is a number.

1

1

This is a set.
It contains a number.



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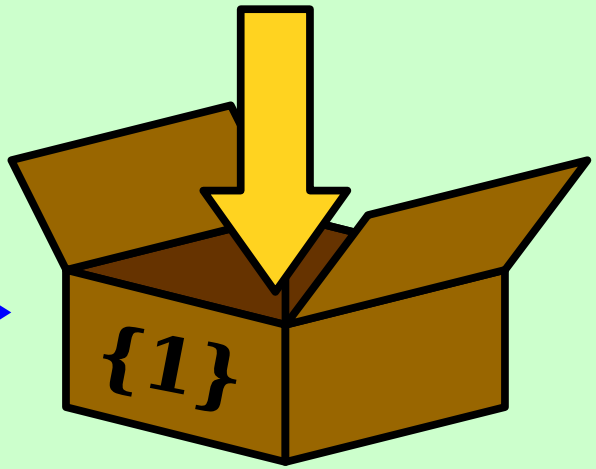

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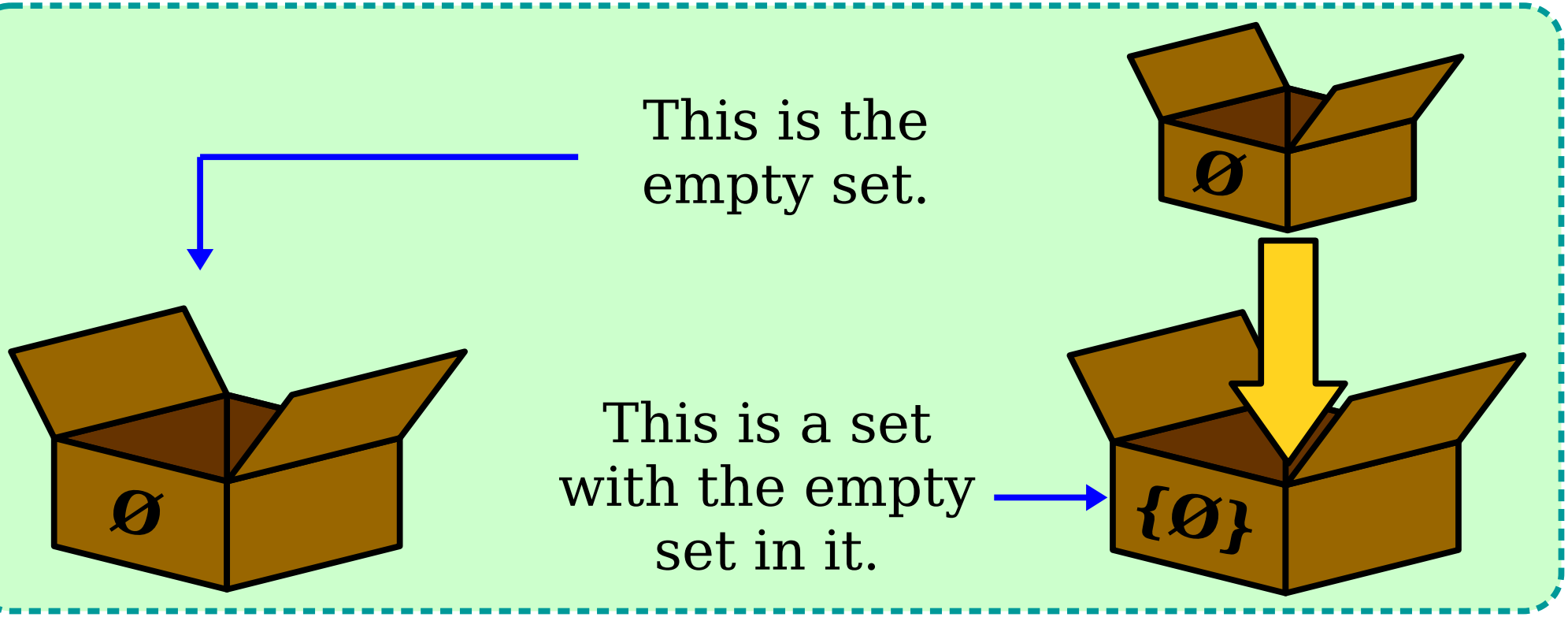

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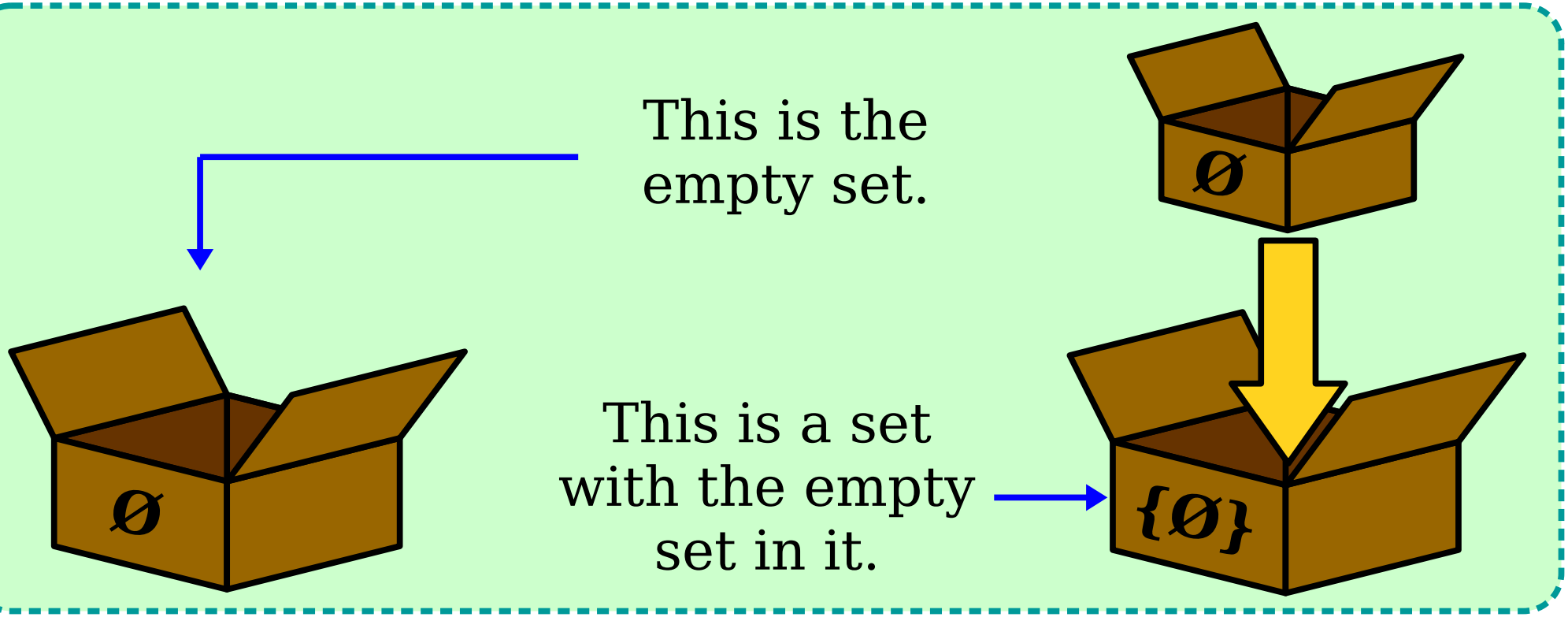
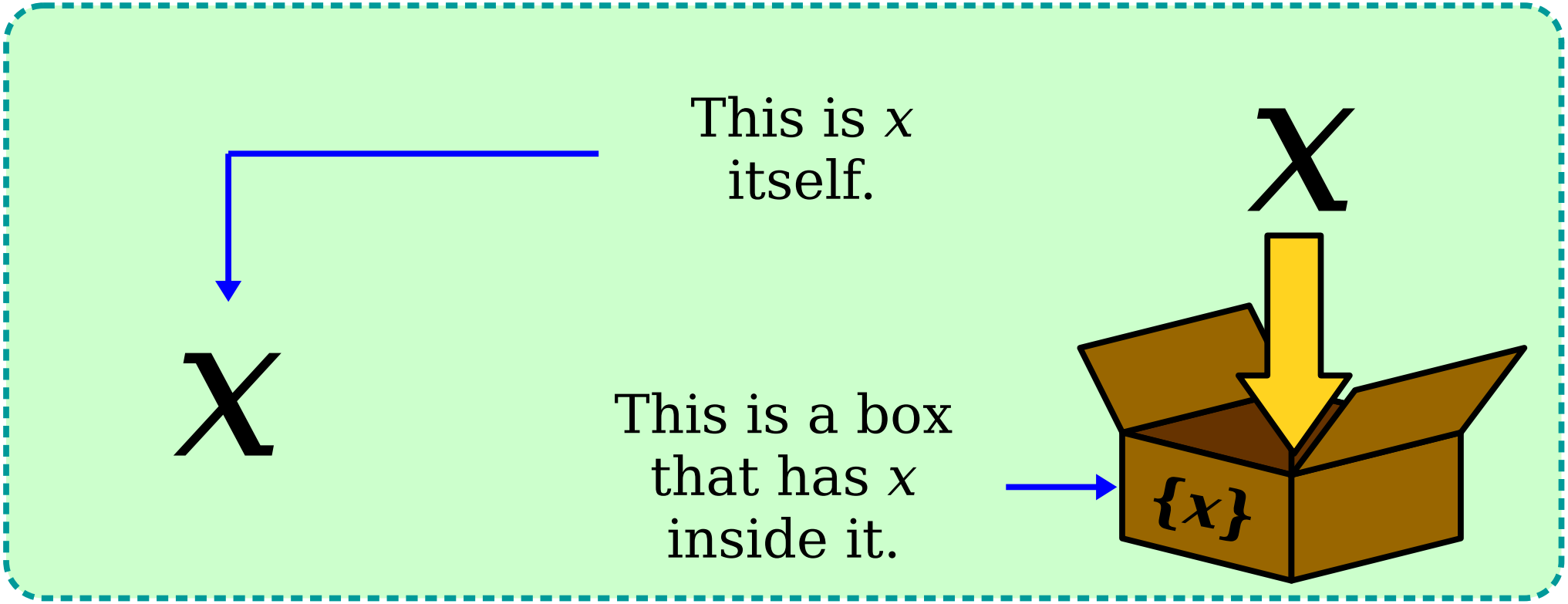


Question: Are these objects equal?

$$x \neq \{x\}$$



No object x is equal to the set containing x.

Infinite Sets

- Some sets contain *infinitely many* elements!
- The set $\mathbb{N} = \{ 0, 1, 2, 3, \ldots \}$ is the set of all the ***natural numbers***.
  - Some mathematicians don't include zero; in this class, assume that 0 is a natural number.
- The set $\mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \}$ is the set of all the ***integers***.
  - Z is from German “Zahlen.”
- The set $\mathbb{R}$ is the set of all ***real numbers***.
  - $e \in \mathbb{R}$, and $4 \in \mathbb{R}$, and $-137 \in \mathbb{R}$,

Describing Complex Sets

- Here are some English descriptions of infinite sets:

  “The set of all even natural numbers.”

  “The set of all real numbers less than 137.”

  “The set of all Python programs.”

- To describe complex sets like these mathematically, we'll use ***set-builder notation***.

Even Natural Numbers

$$\{\, n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \,\}$$

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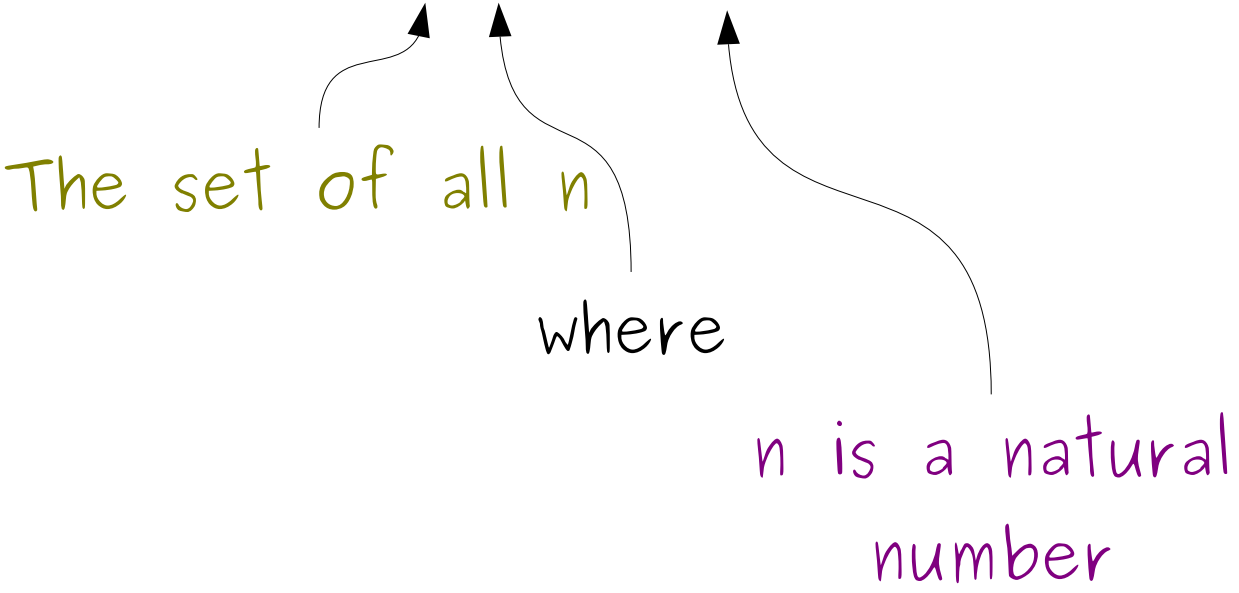

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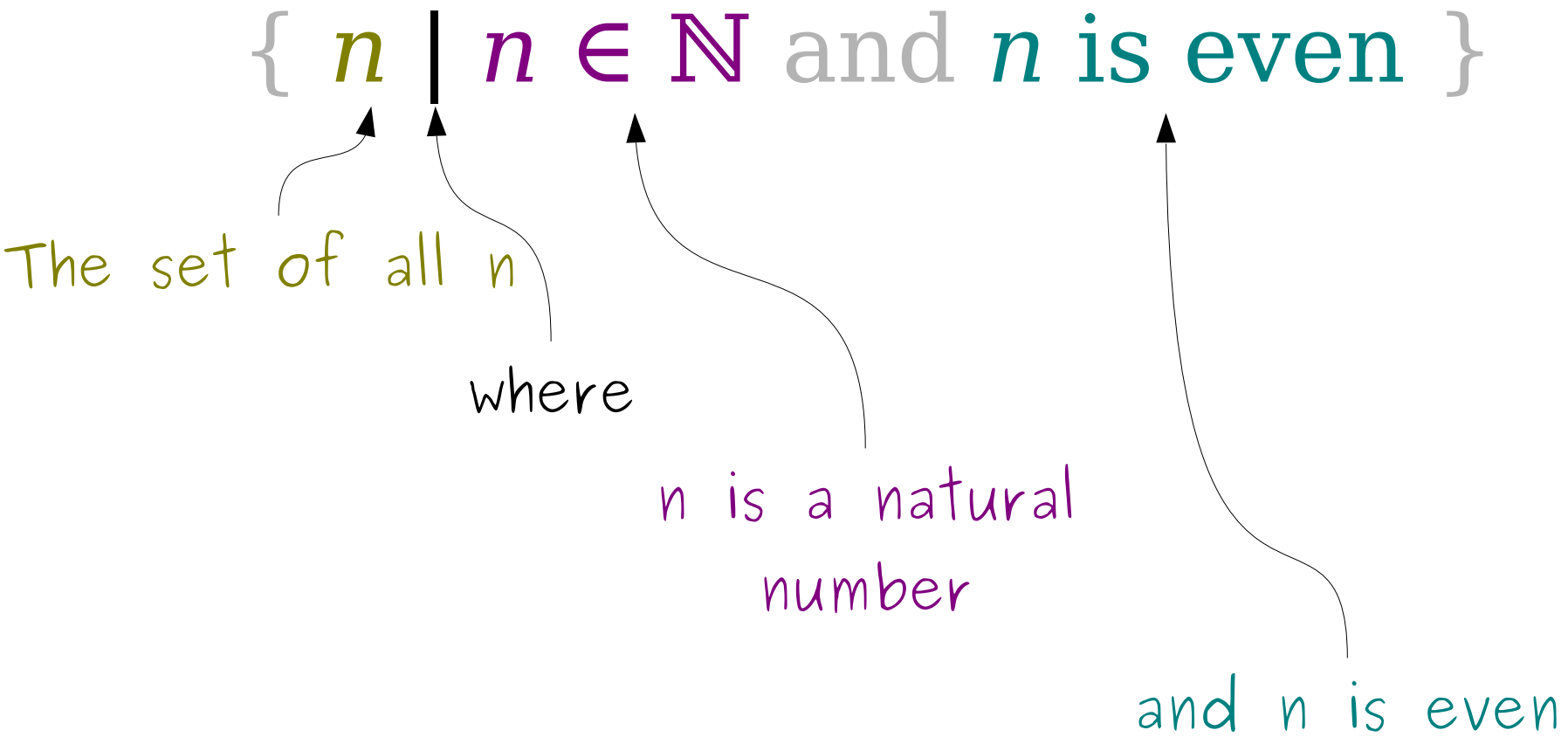

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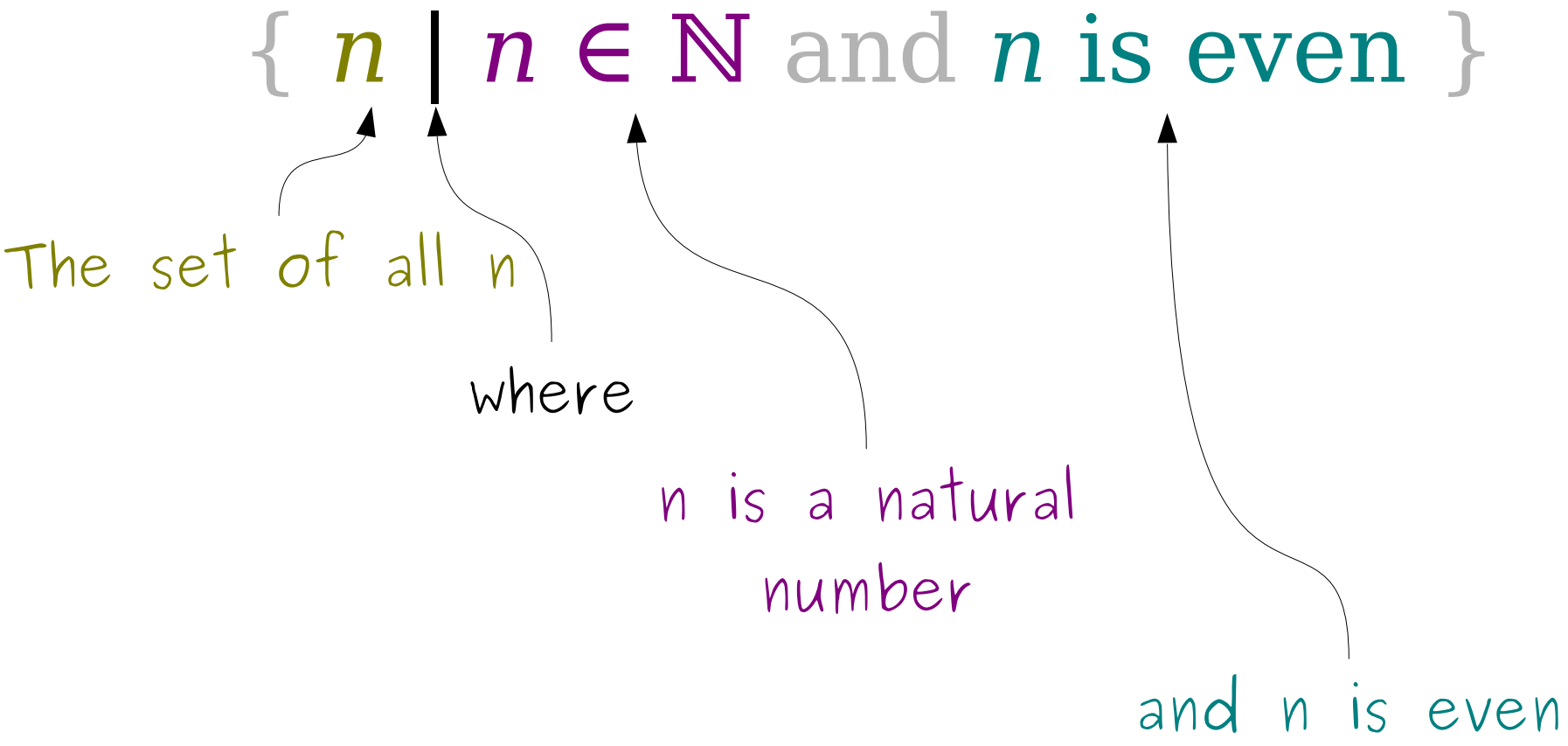


{ 0, 2, 4, 6, 8, 10, 12, 14, 16, ... }

Set Builder Notation

- A set may be specified in ***set-builder notation***:

$$\{\ x \mid \textit{some property } x \textit{ satisfies}\ \}$$

$$\{\ x \in S \mid \textit{some property } x \textit{ satisfies}\ \}$$

- For example:

$\{\ n \mid n \in \mathbb{N} \text{ and } n \text{ is even}\ \}$

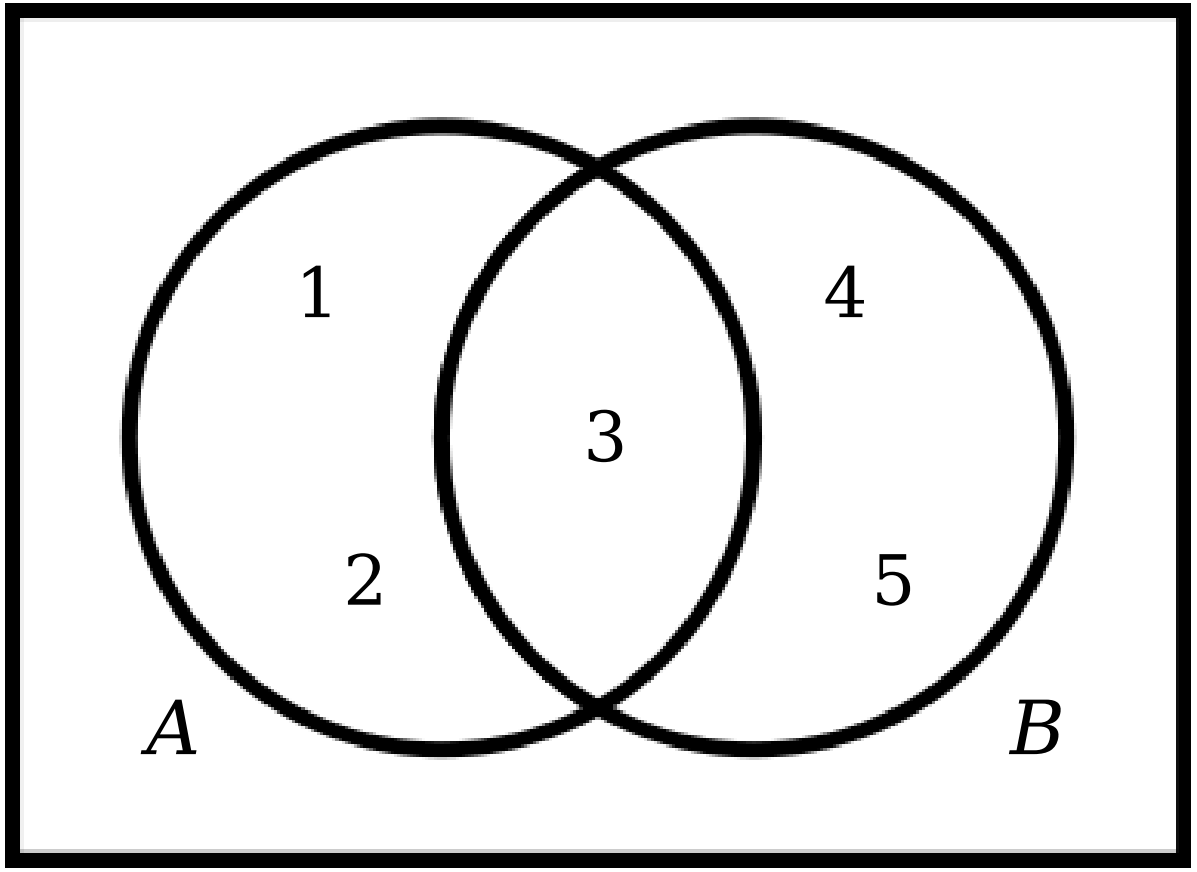
$\{\ C \mid C \text{ is a set of US coins}\ \}$

$\{\ r \in \mathbb{R} \mid r < 3\ \}$

$\{\ n \in \mathbb{N} \mid n < 3\ \}$ **(the set {0, 1, 2})**

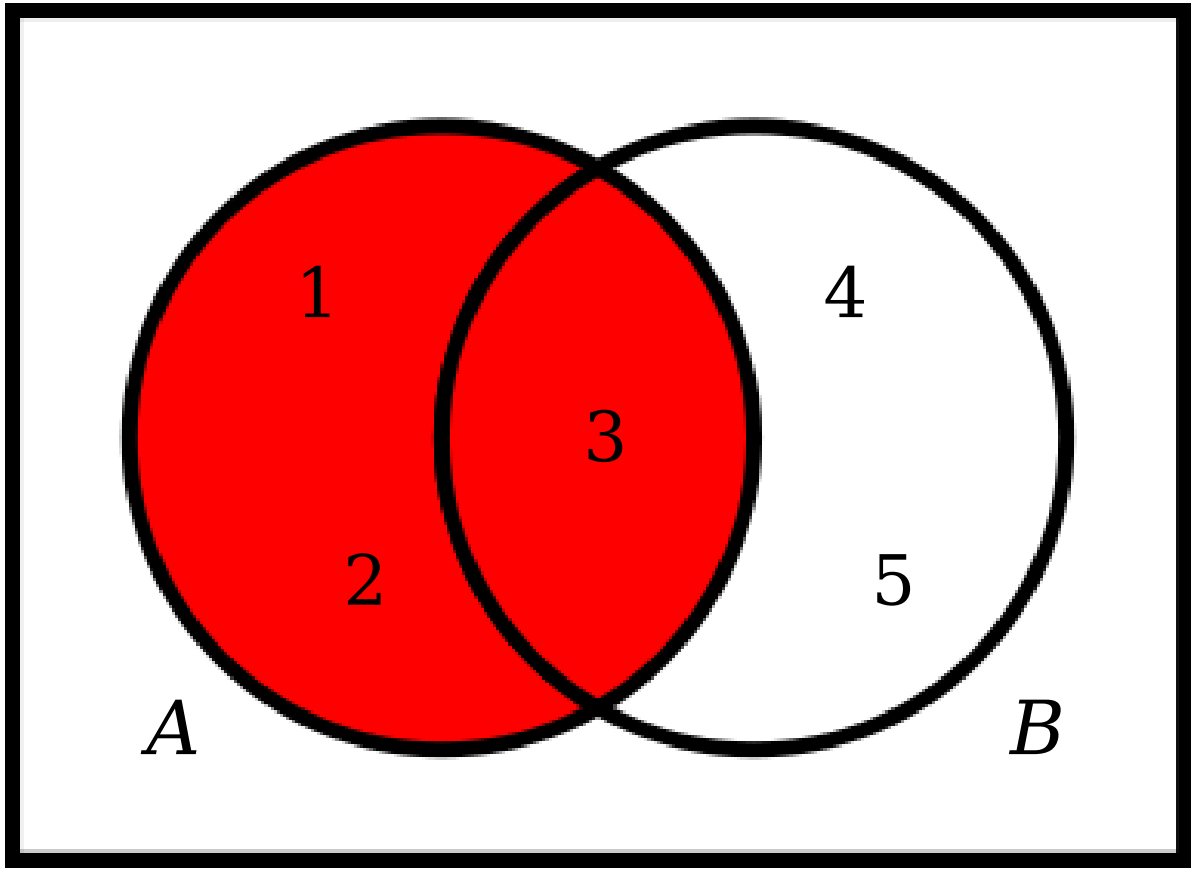
Combining Sets

Venn Diagrams



$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Venn Diagrams



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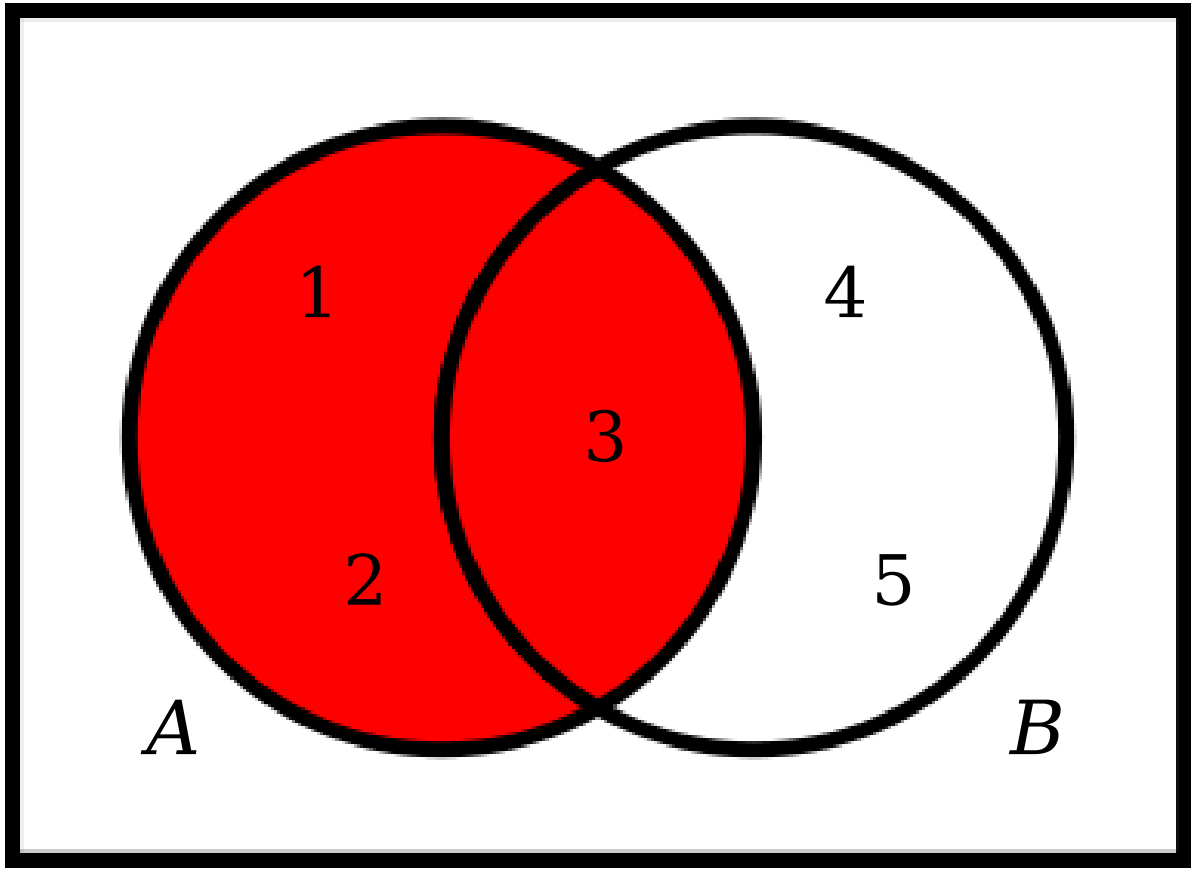

A

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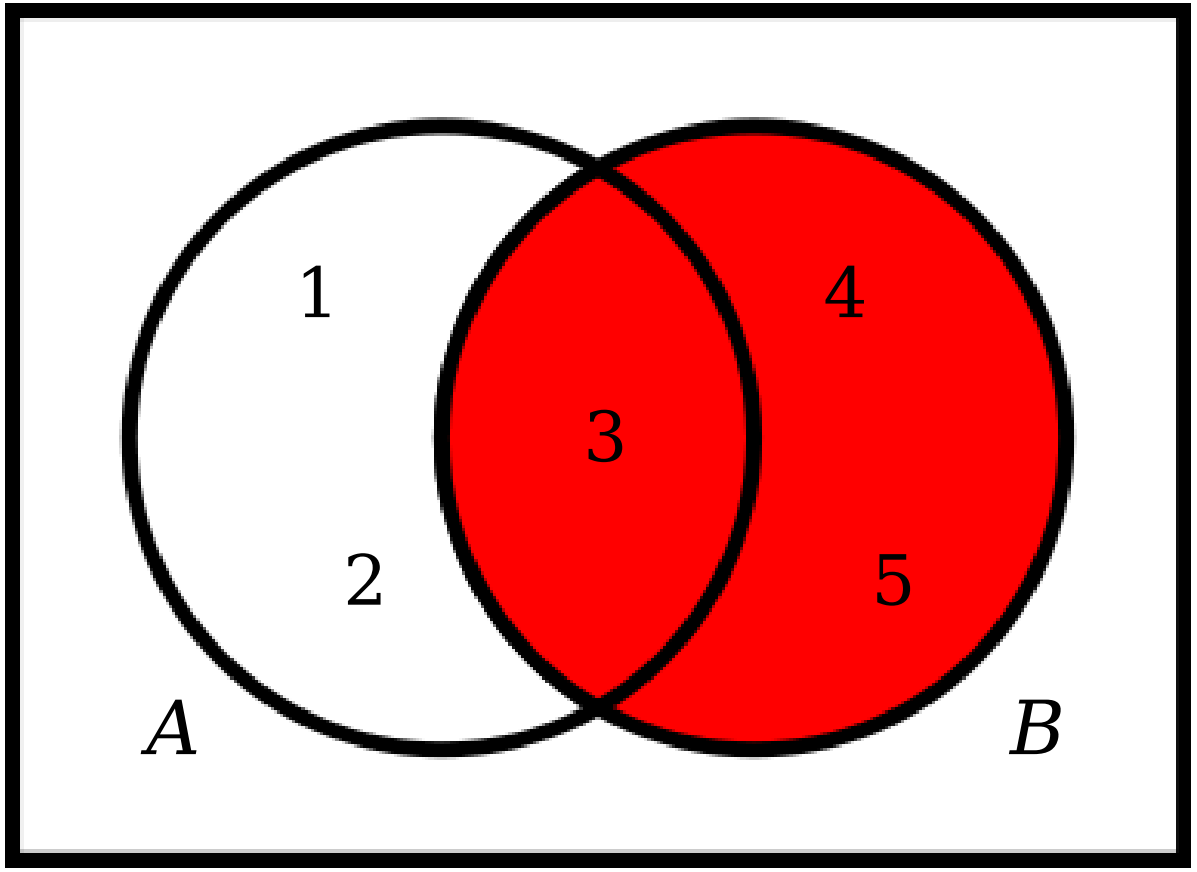

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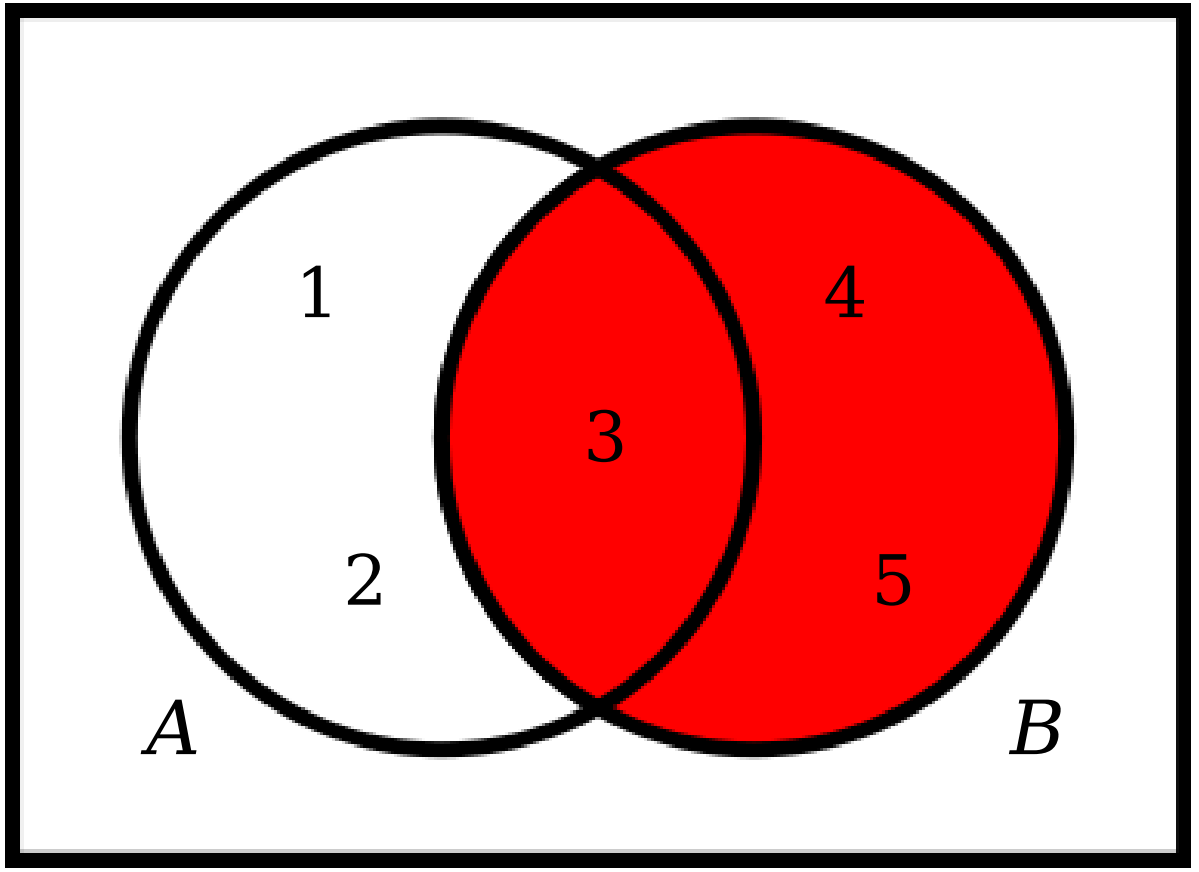

B

$A = \{ 1, 2, 3 \}$
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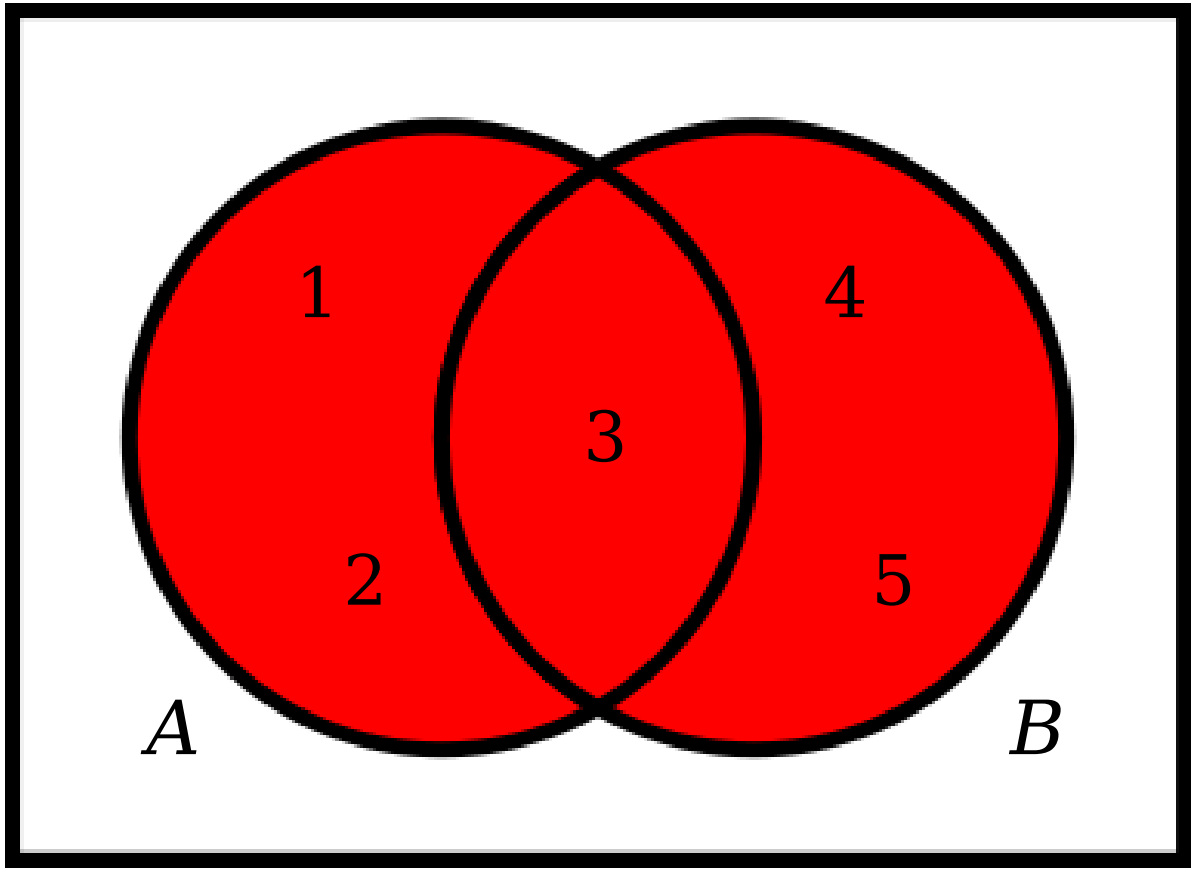
Venn Diagrams



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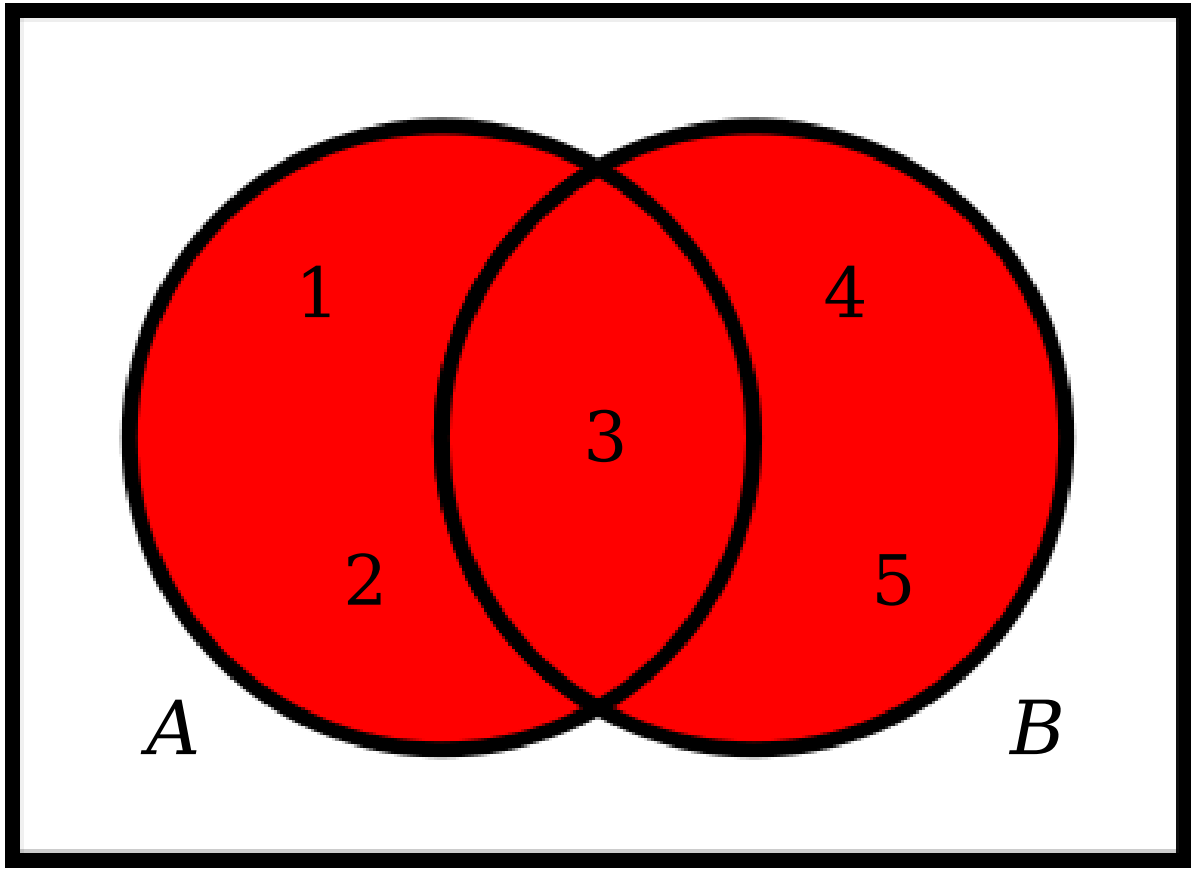


Union

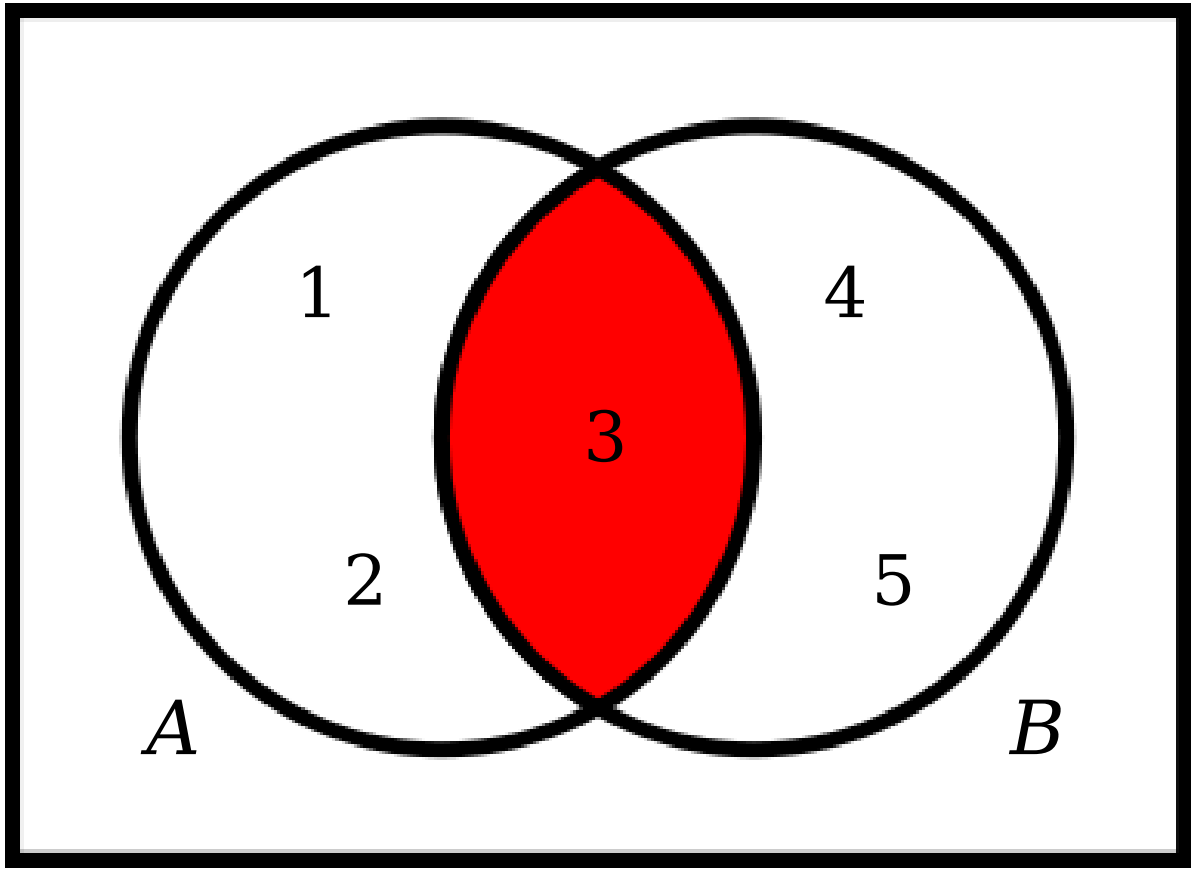
$A \cup B$

{ 1, 2, 3, 4, 5 }

$A = \{ 1, 2, 3 \}$

$B = \{ 3, 4, 5 \}$

Venn Diagrams



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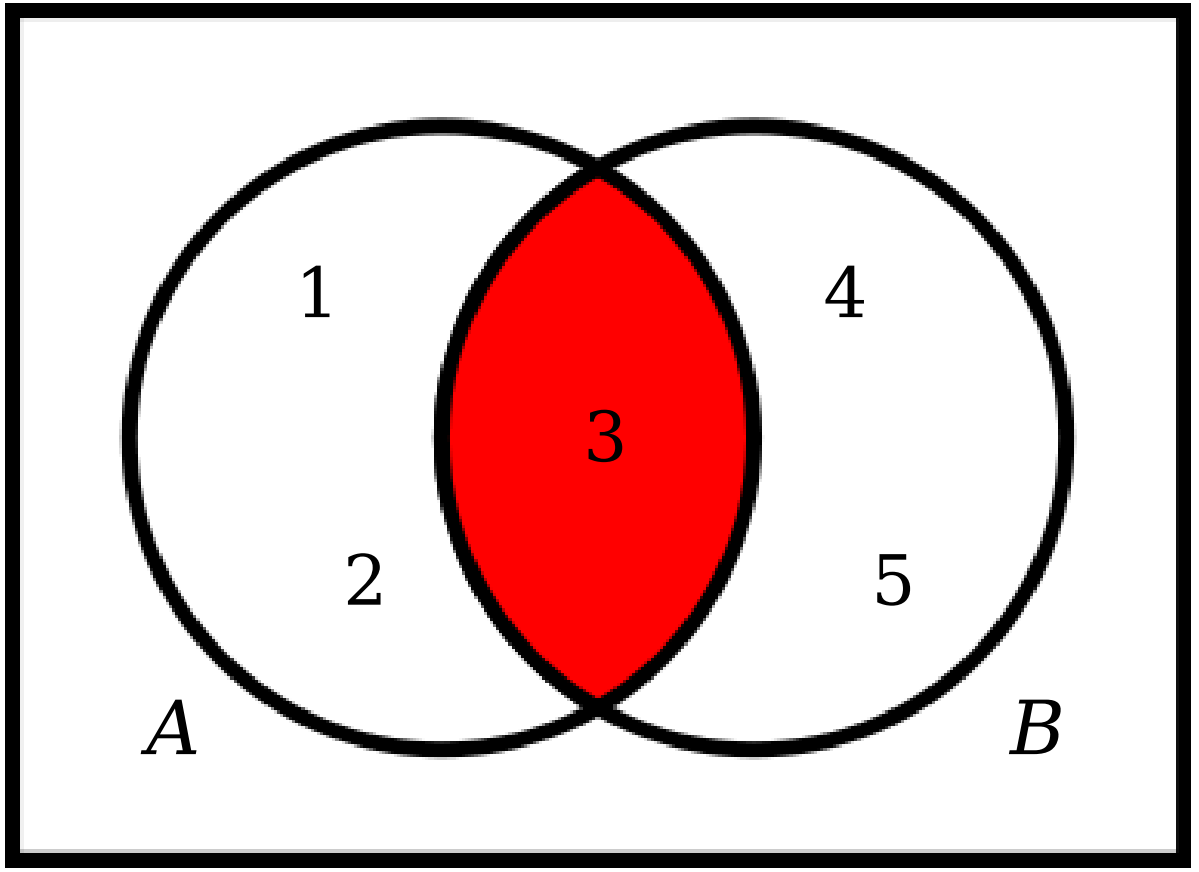


Intersection

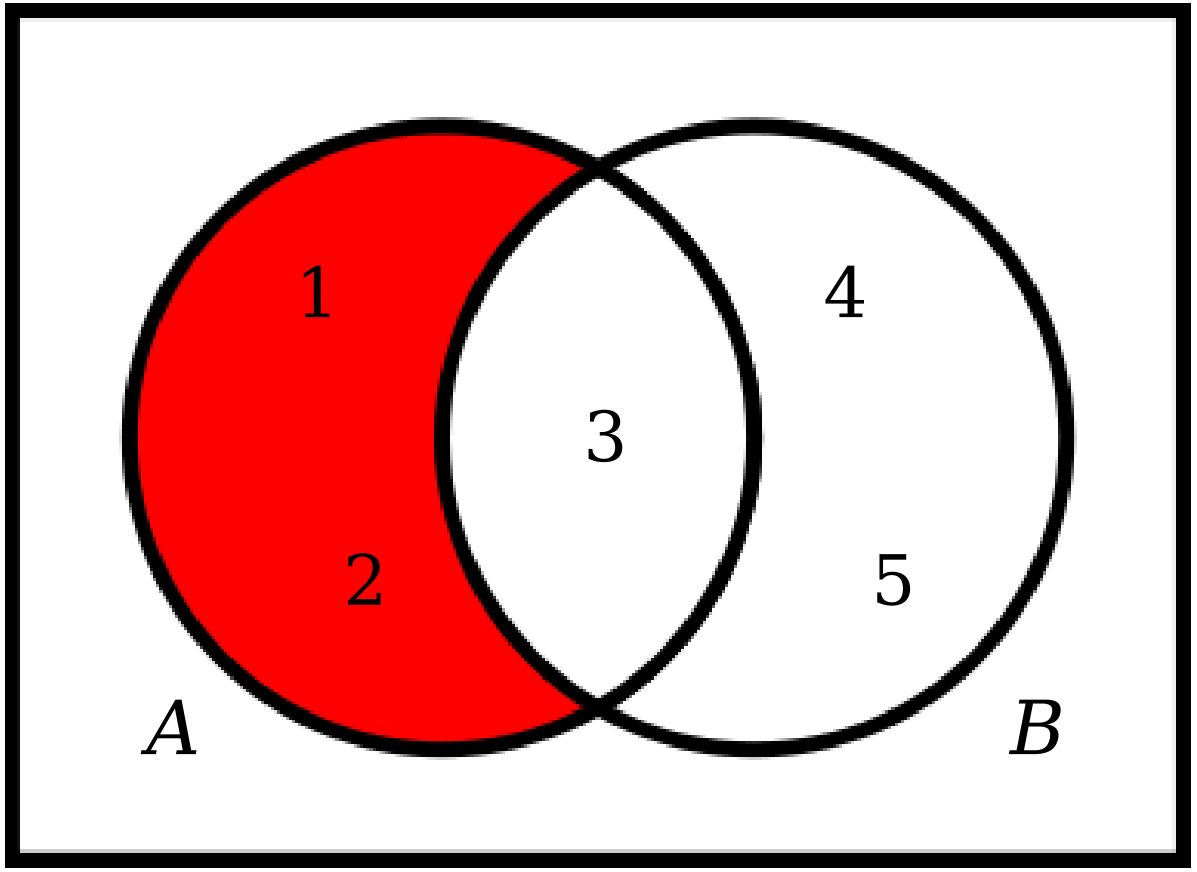
$A \cap B$

{ 3 }

$A = \{ 1, 2, 3 \}$

$B = \{ 3, 4, 5 \}$

Venn Diagrams



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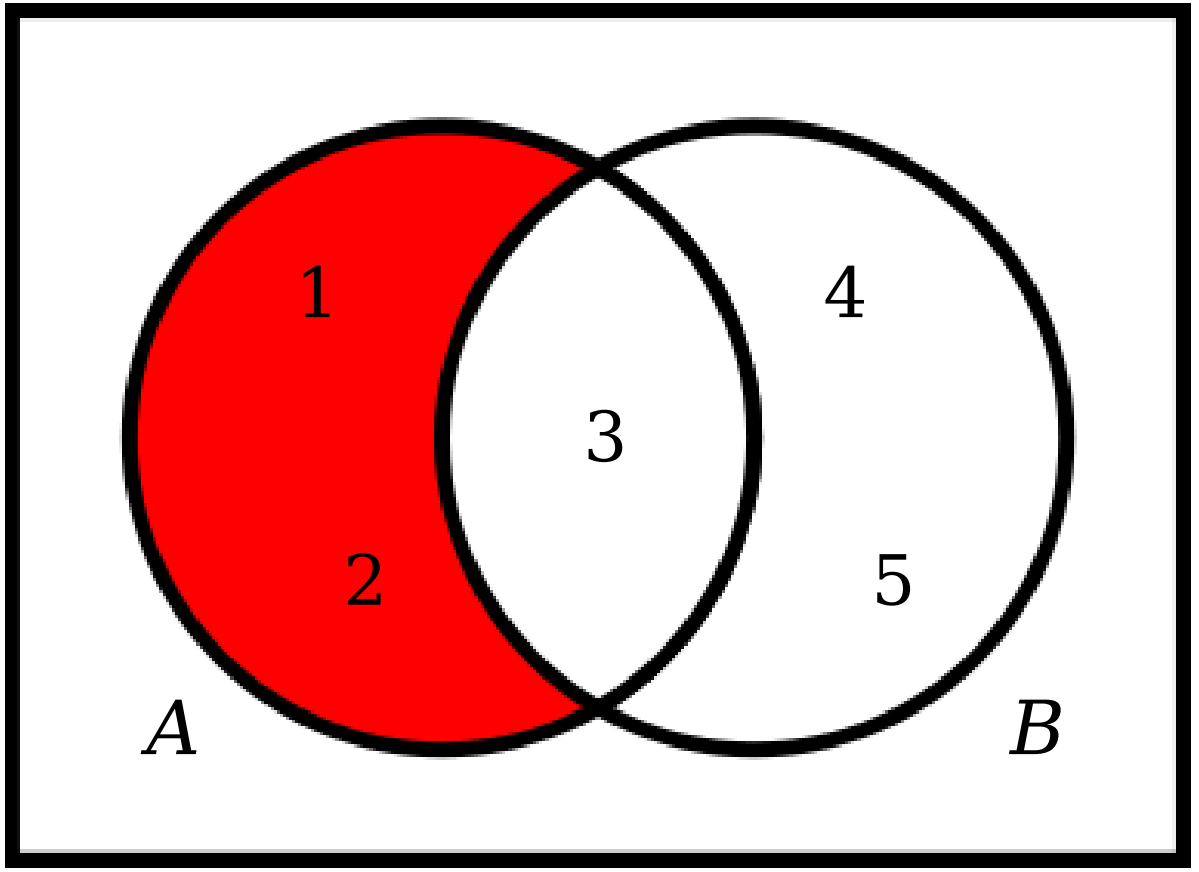


Difference

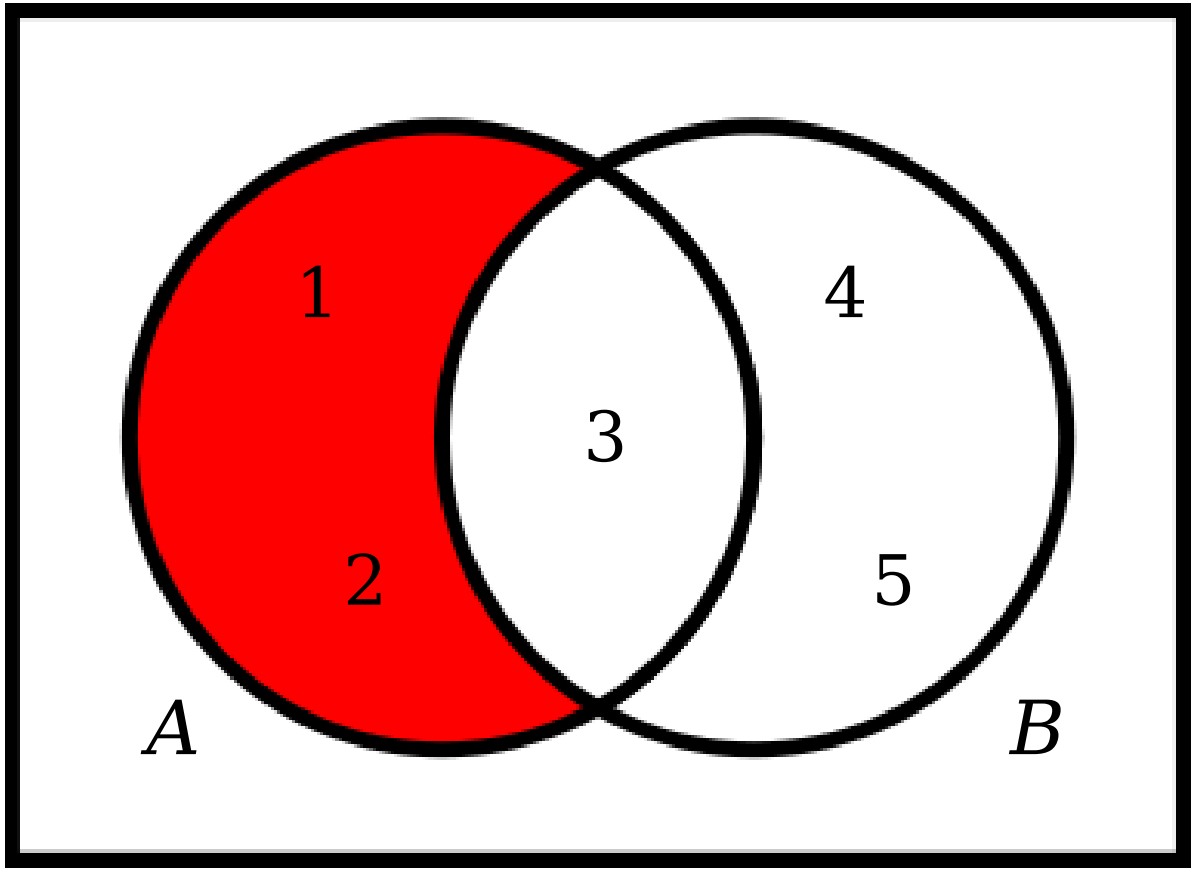
$A - B$

{ 1, 2 }

A = { 1, 2, 3 }

B = { 3, 4, 5 }

Venn Diagrams



Difference

$A \setminus B$

{ 1, 2 }

$A = \{ 1, 2, 3 \}$

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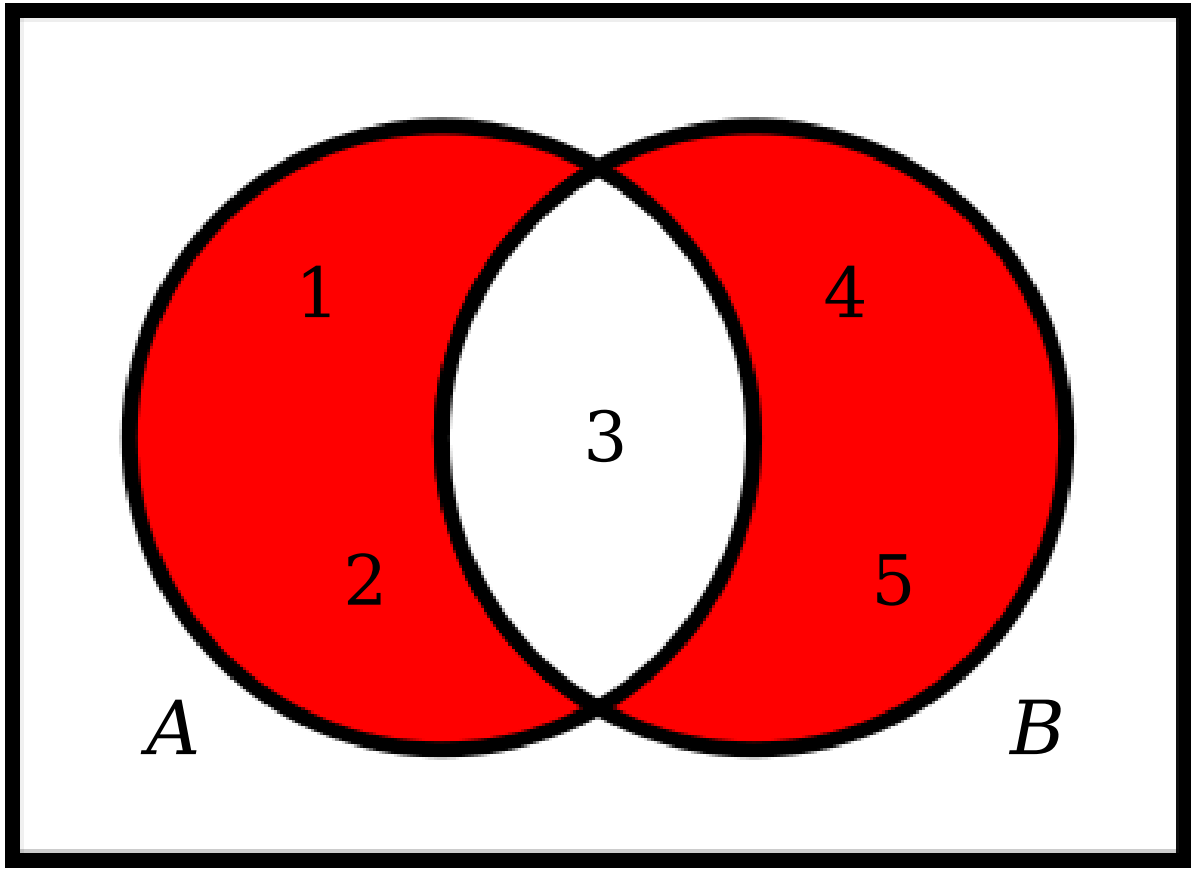
Venn Diagrams



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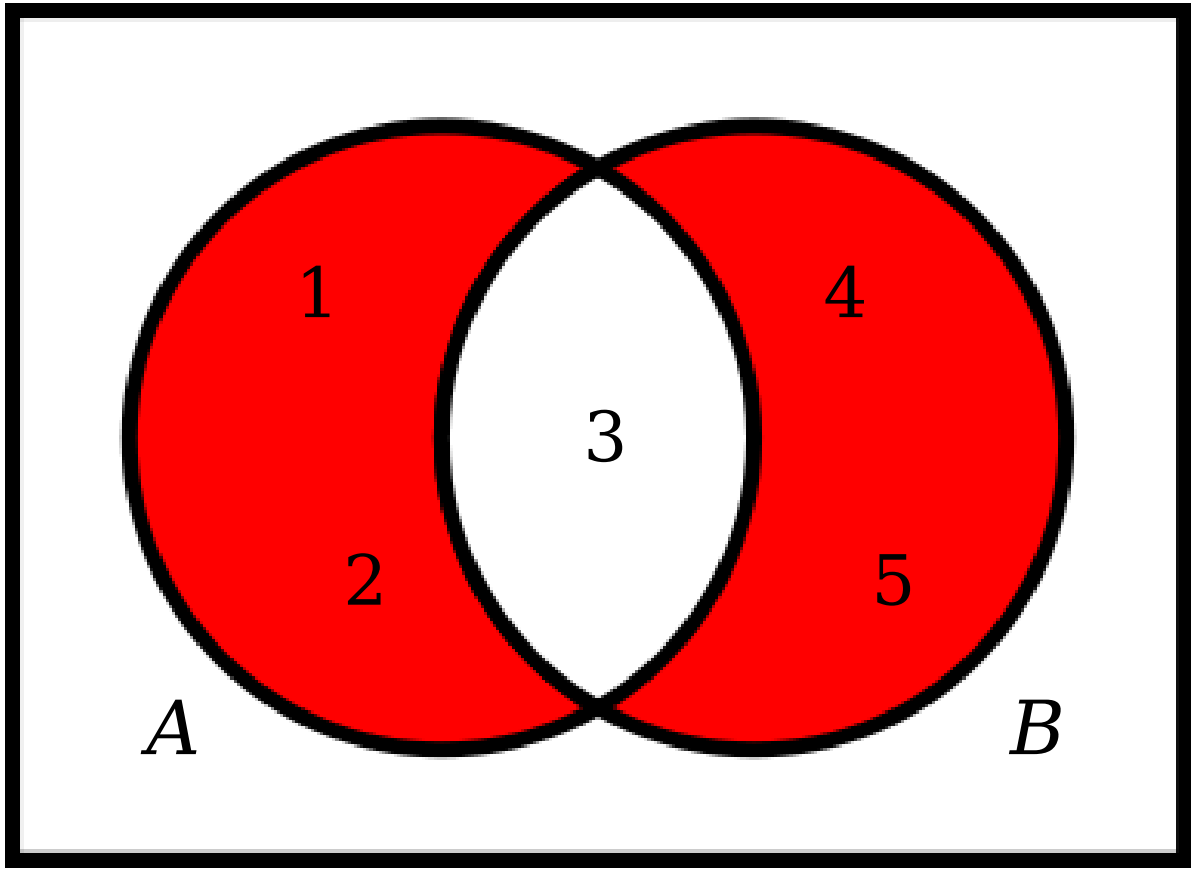

Symmetric Difference

$A \Delta B$

{ 1, 2, 4, 5 }

$A = \{ 1, 2, 3 \}$

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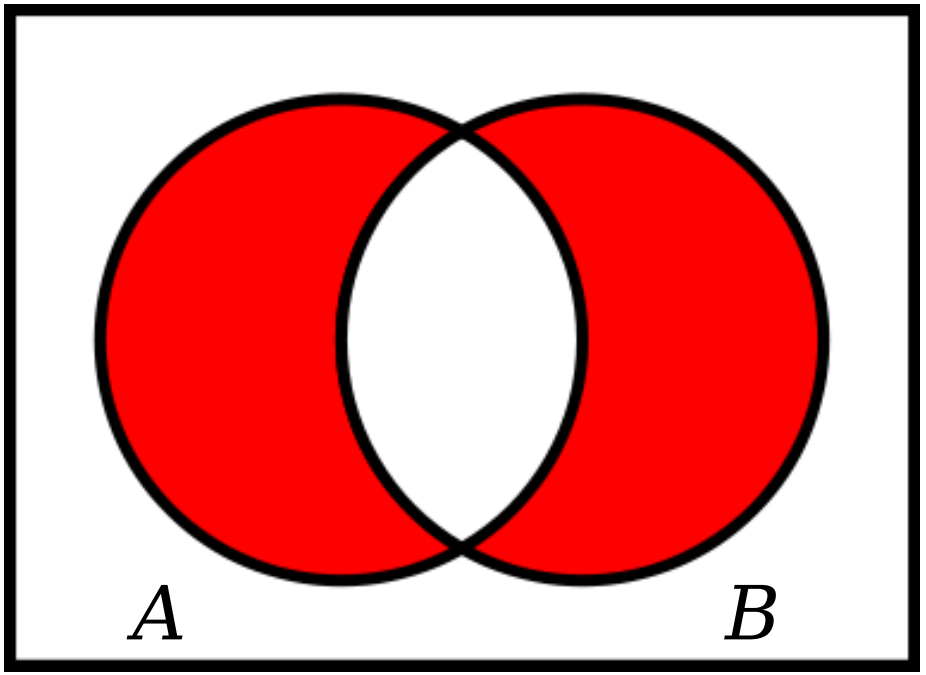
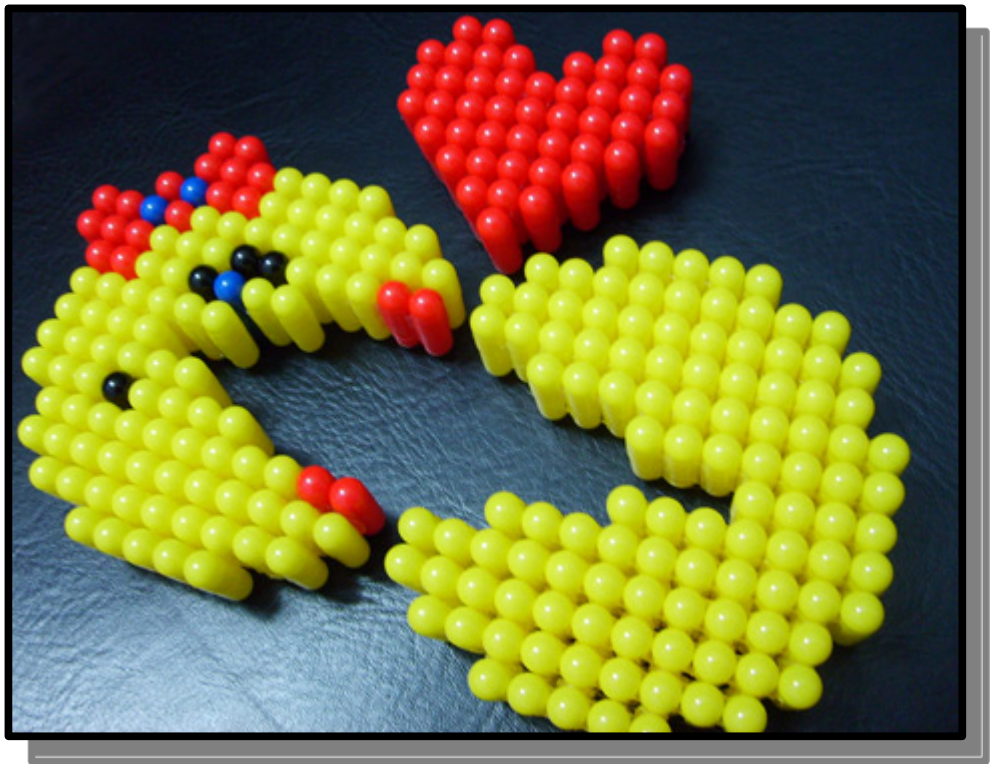
Venn Diagrams





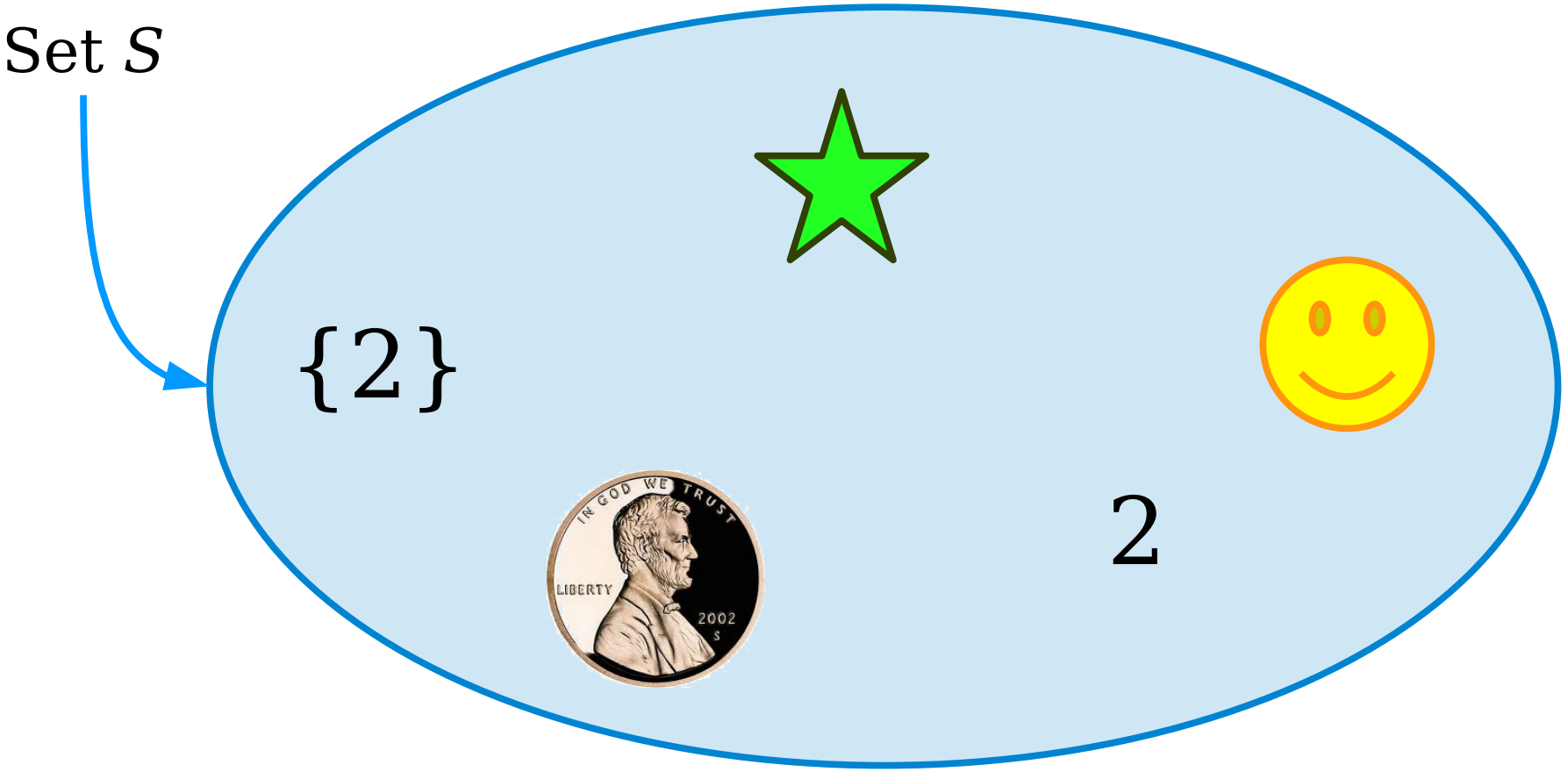
$A \Delta B$

Subsets and Power Sets

Subsets

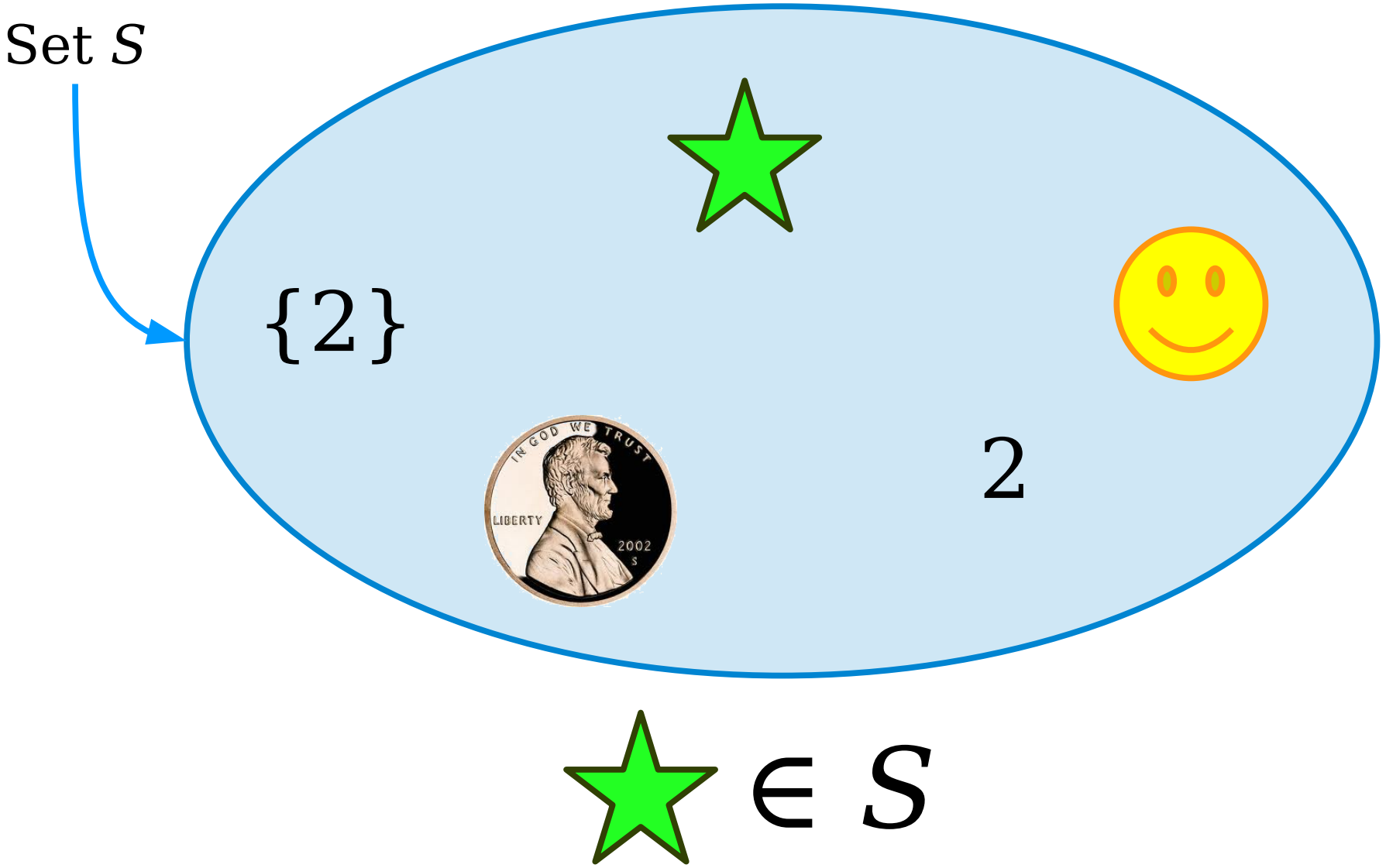
- A set S is called a ***subset*** of a set T (denoted $\boldsymbol{S \subseteq T}$) when all elements of S are also elements of T.
- Examples:
  - $\{ 1, 2, 3 \} \subseteq \{ 1, 2, 3, 4 \}$
  - $\{ b, c \} \subseteq \{ a, b, c, d \}$
  - $\{ \mathrm{H}, \mathrm{He}, \mathrm{Li} \} \subseteq \{ \mathrm{H}, \mathrm{He}, \mathrm{Li} \}$
  - $\mathbb{N} \subseteq \mathbb{Z}$ (*every natural number is an integer*)
  - $\mathbb{Z} \subseteq \mathbb{R}$ (*every integer is a real number*)

Subsets and Elements



$S = \{\ 2,$ $, \{2\},$ $,$ $\}$

Subsets and Elements



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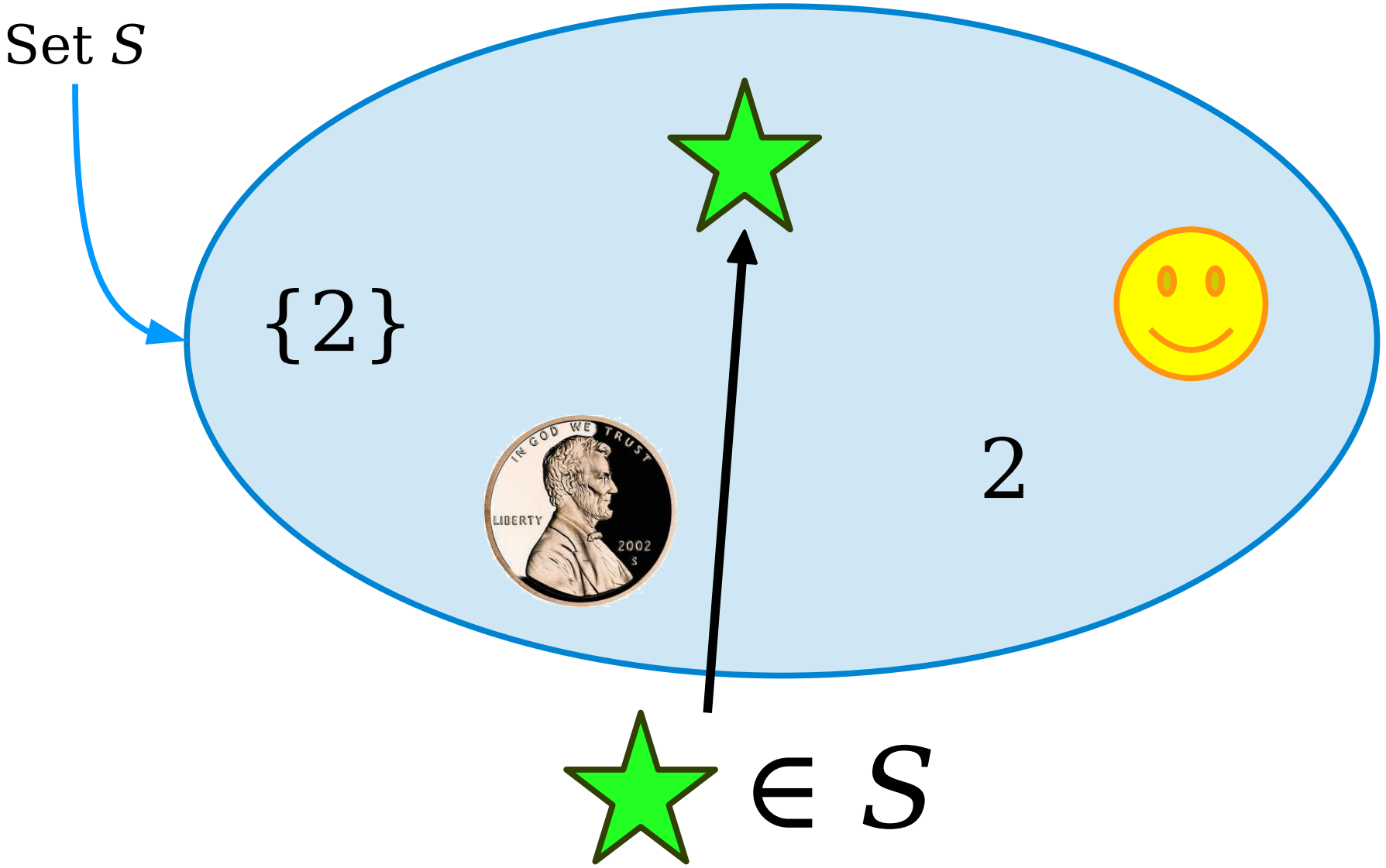

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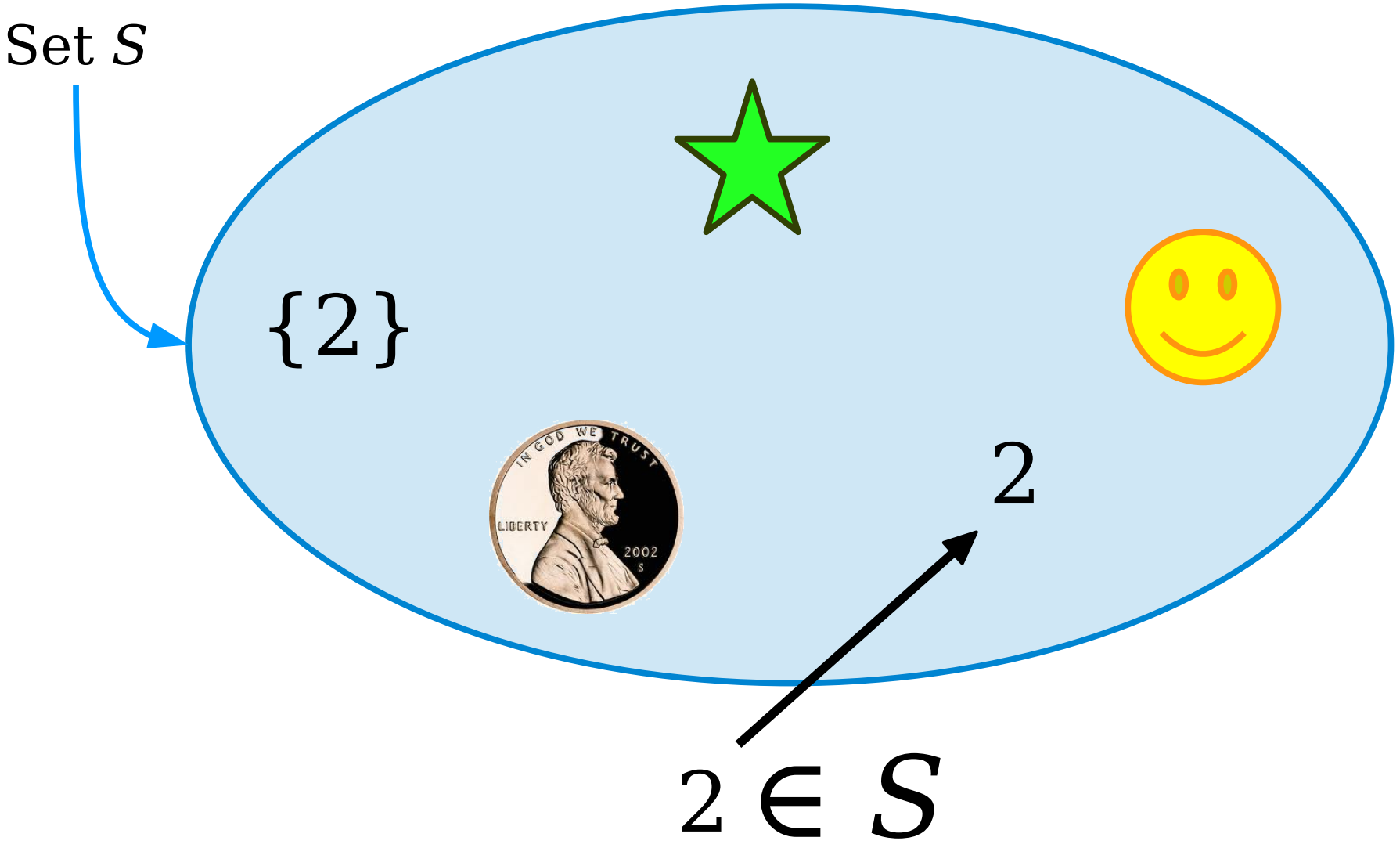

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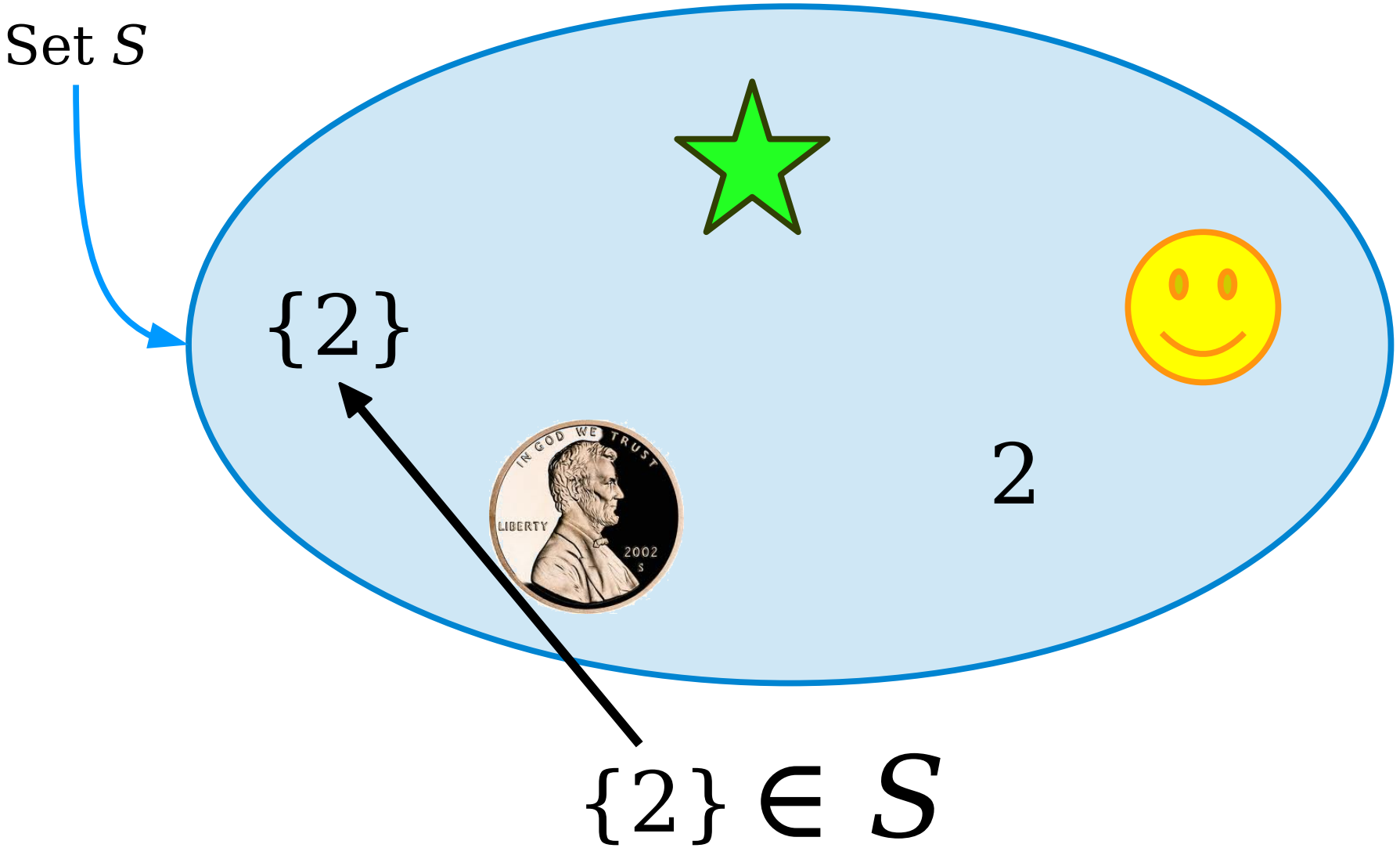

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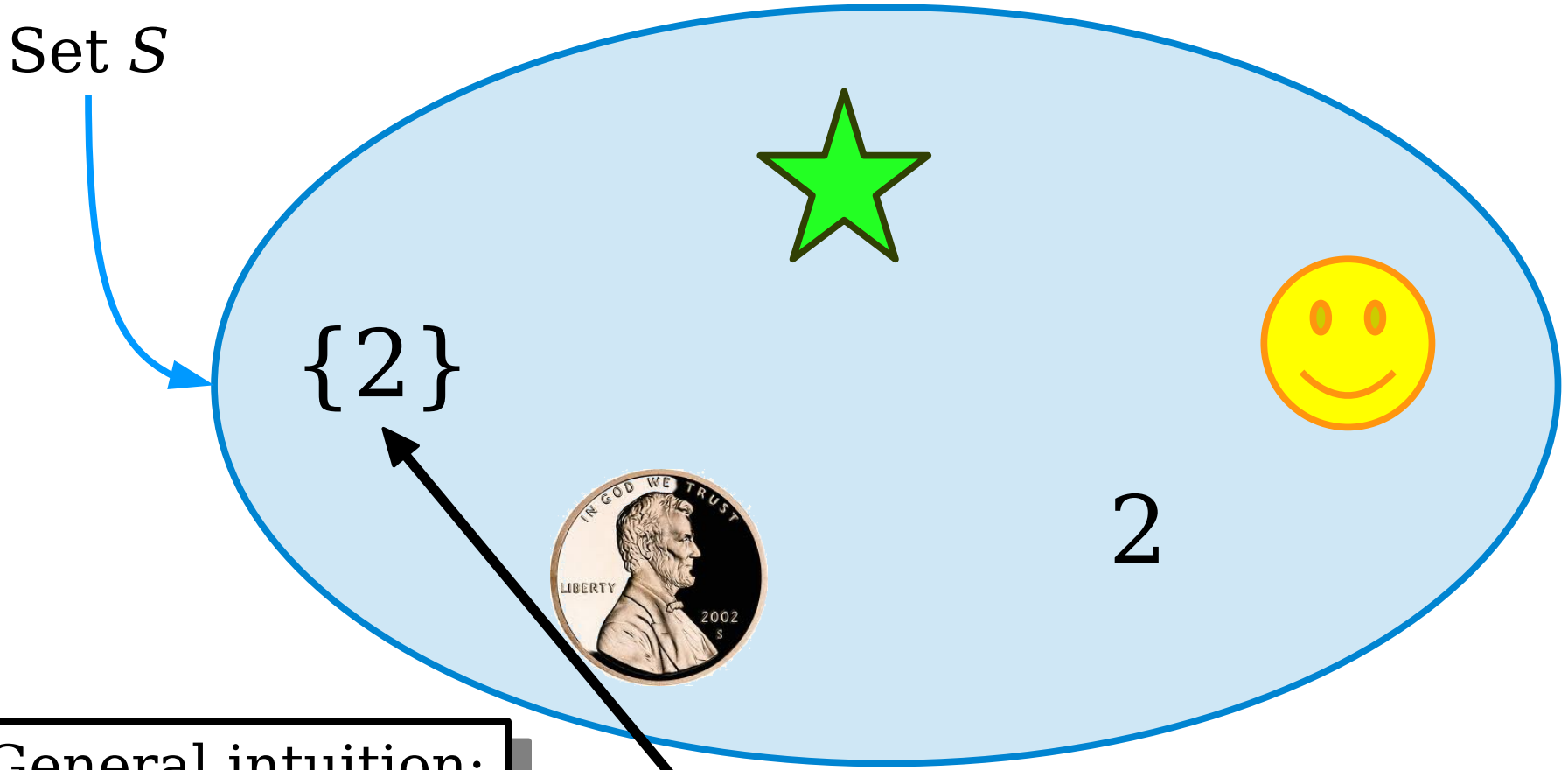
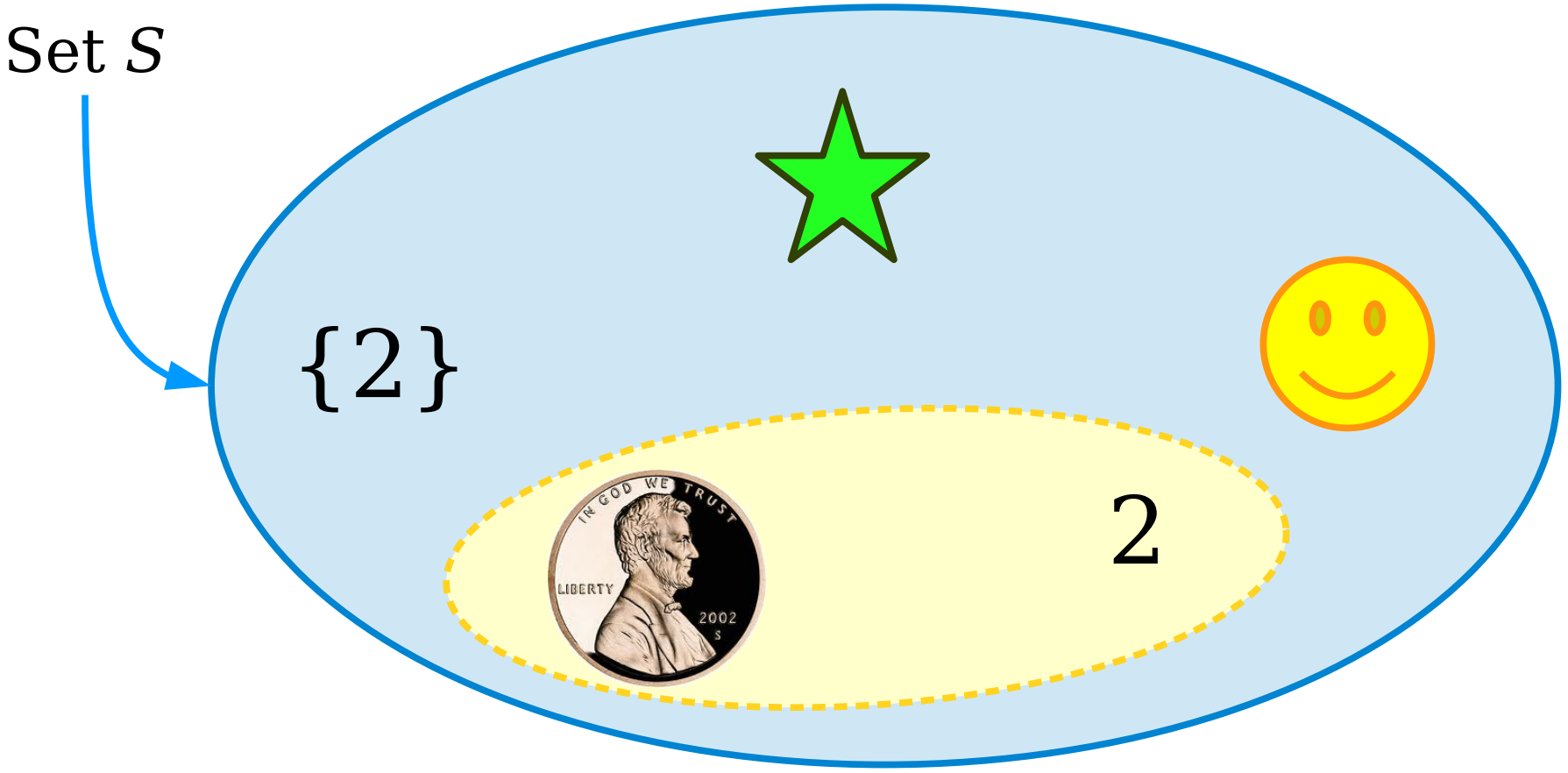


$\{2\} \in S$

General intuition: $x \in S$ means you can ***point at x inside of S***.

Subsets and Elements



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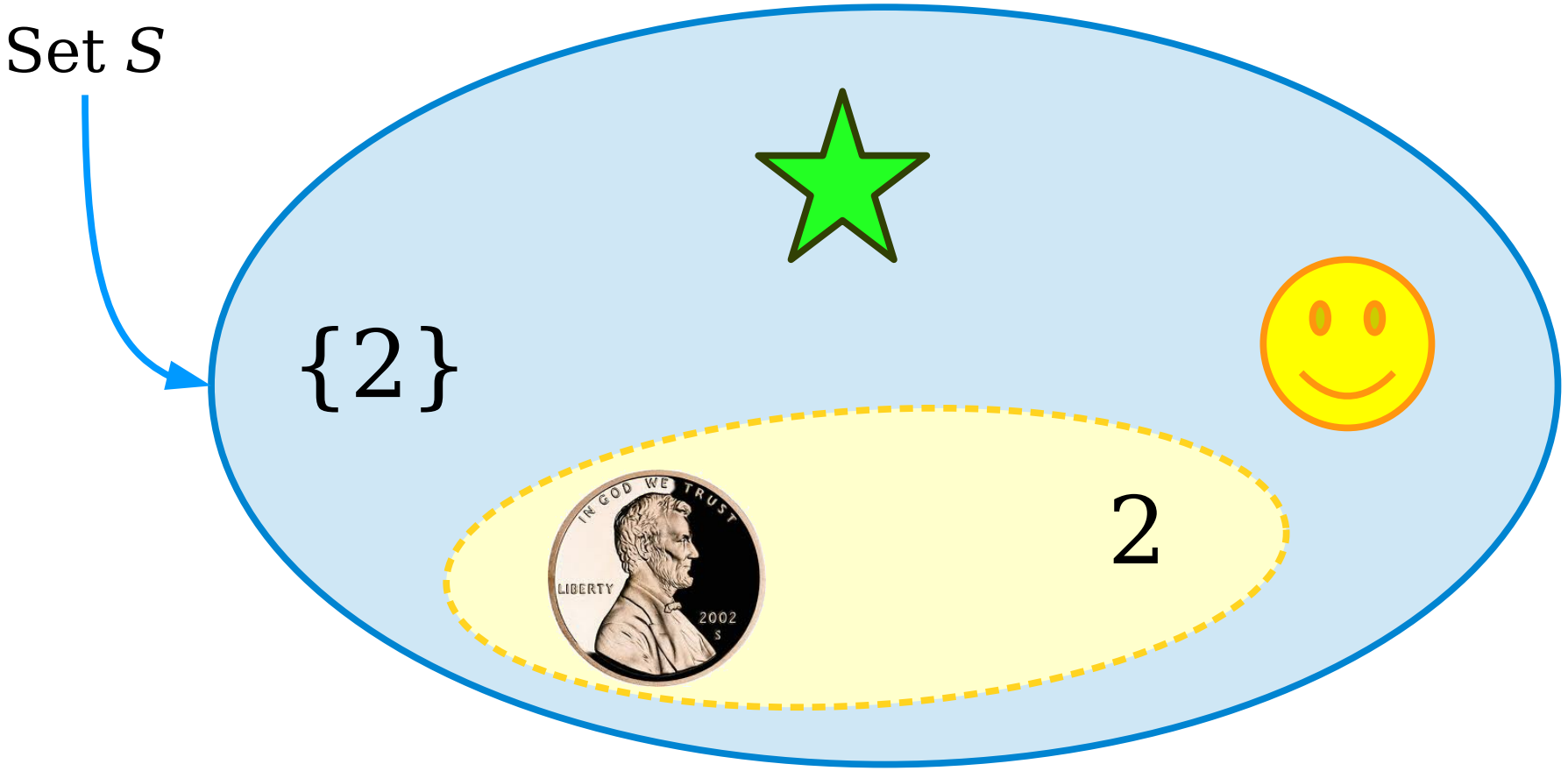
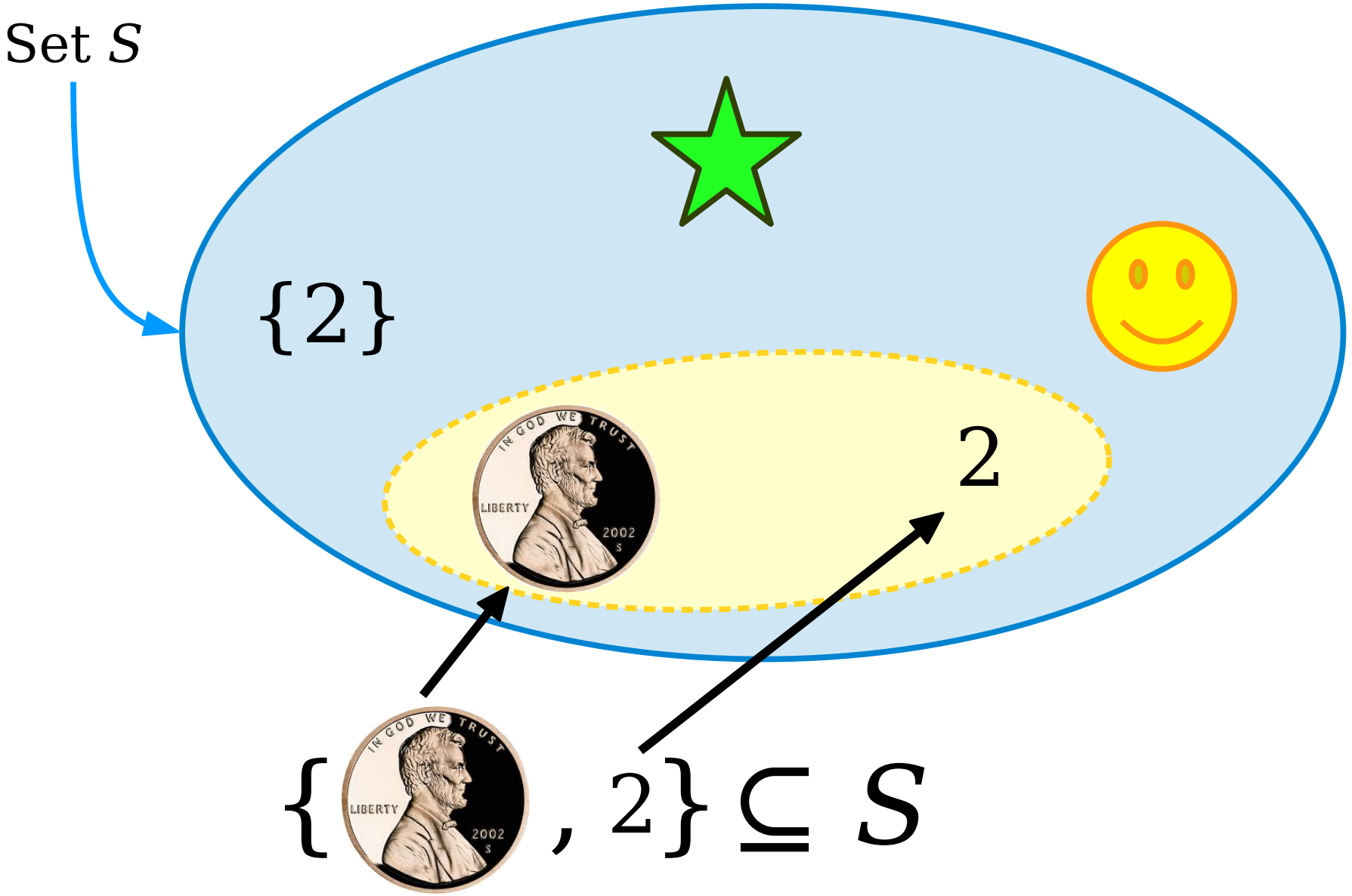
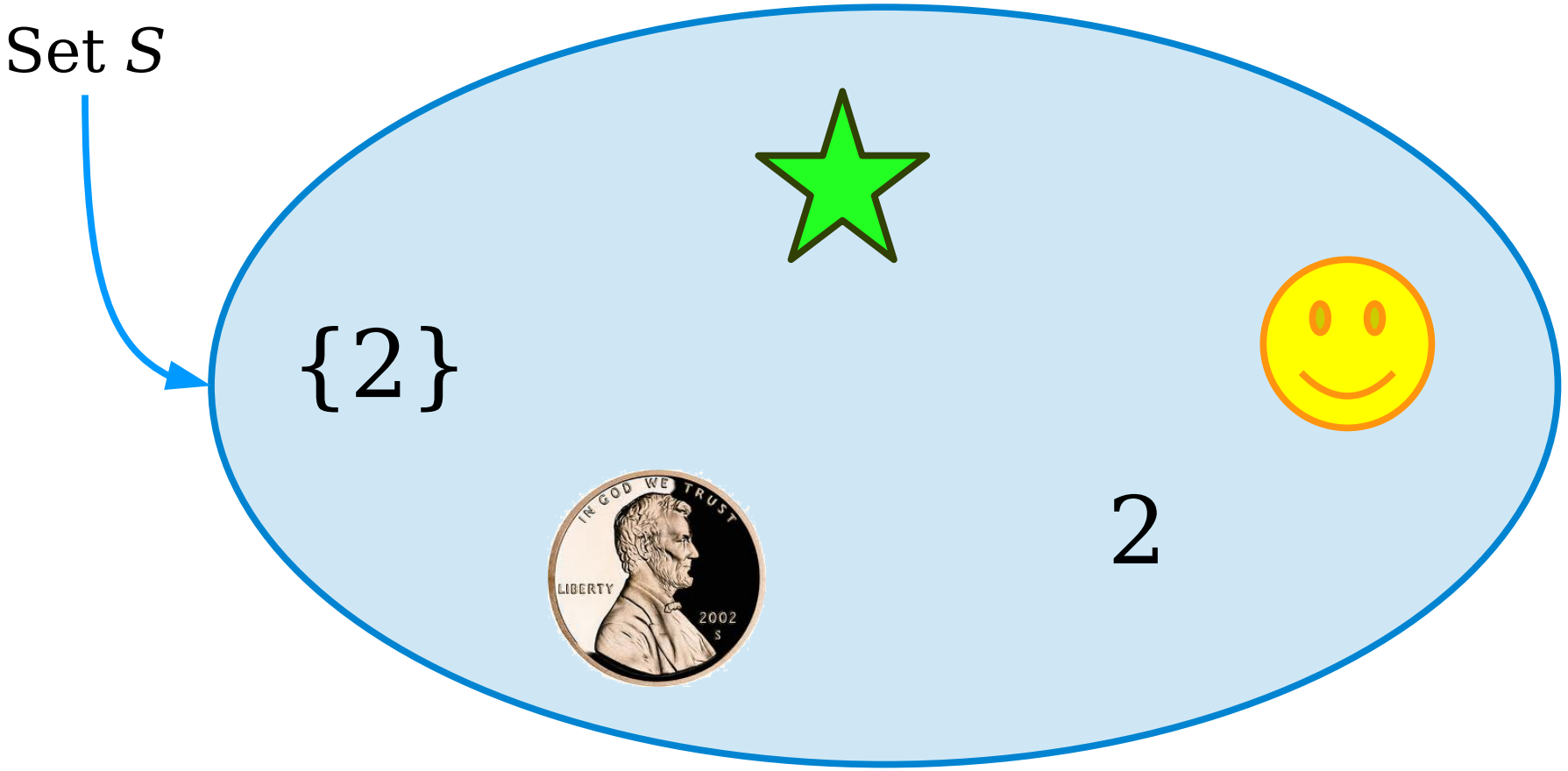


$\{$ $, 2\} \subseteq S$

Subsets and Elements

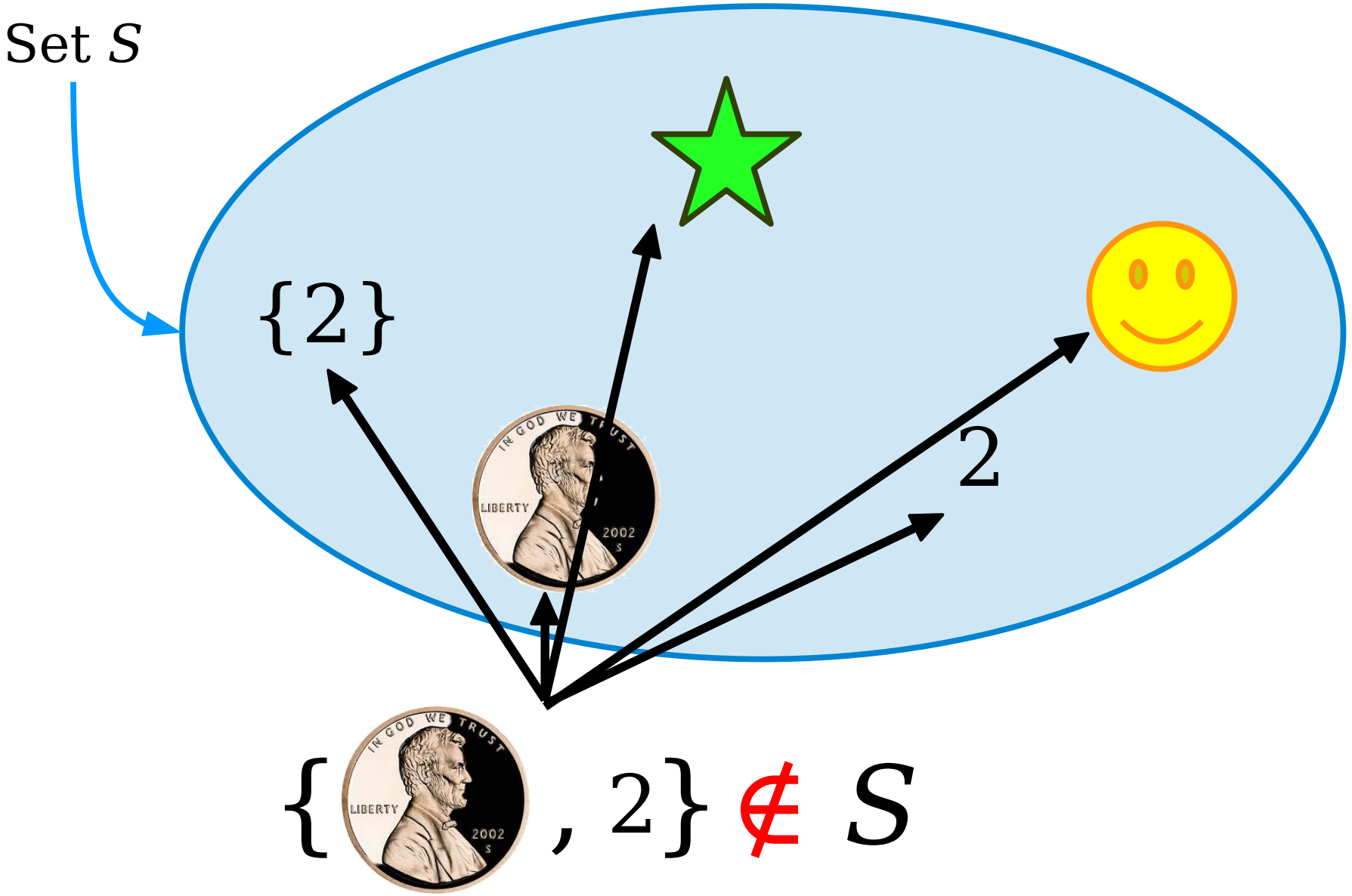


Subsets and Elements



$\{$ $, 2\} \notin S$

Subsets and Elements



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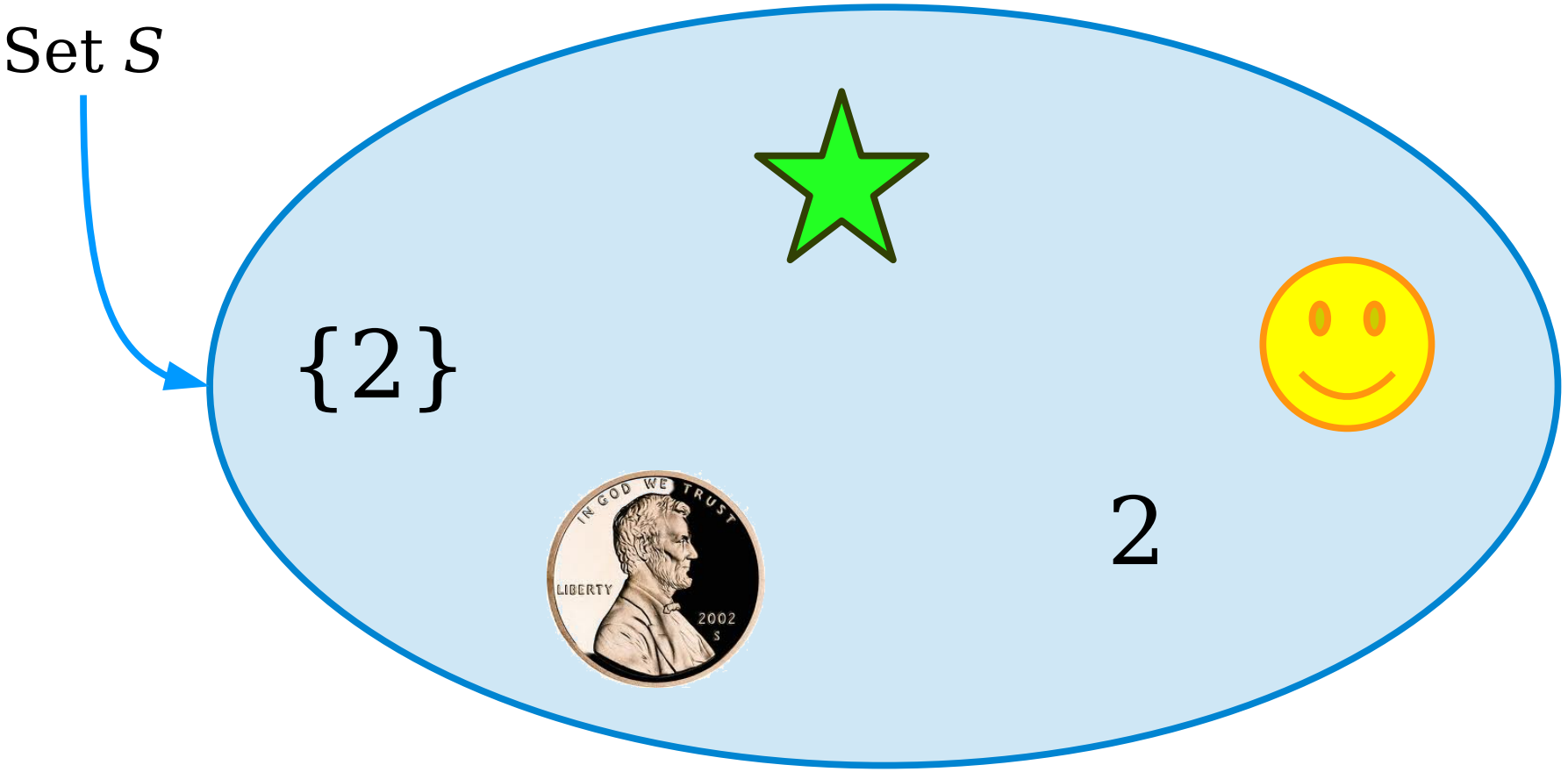
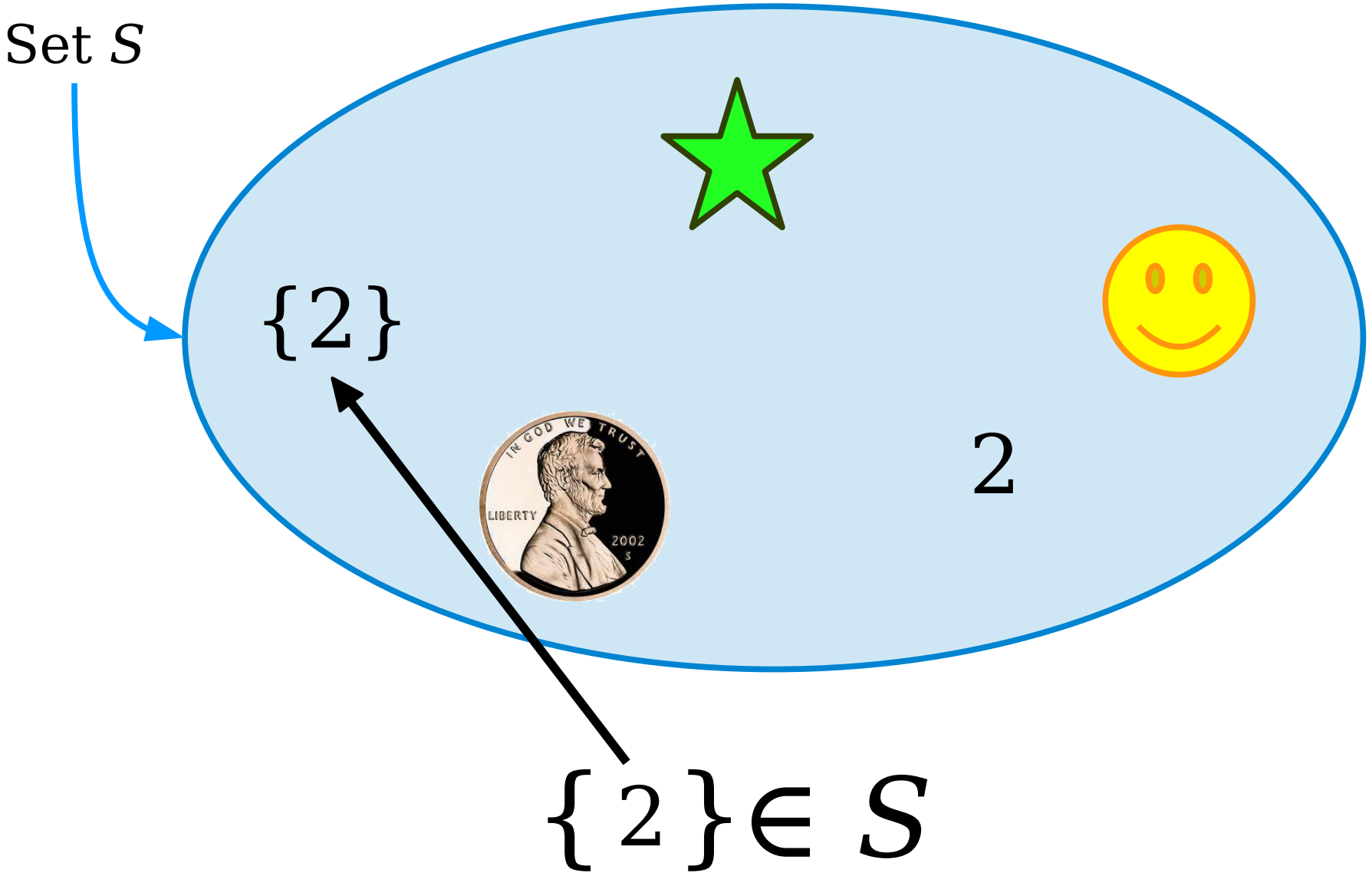


$$\{2\} \in S$$

Subsets and Elements



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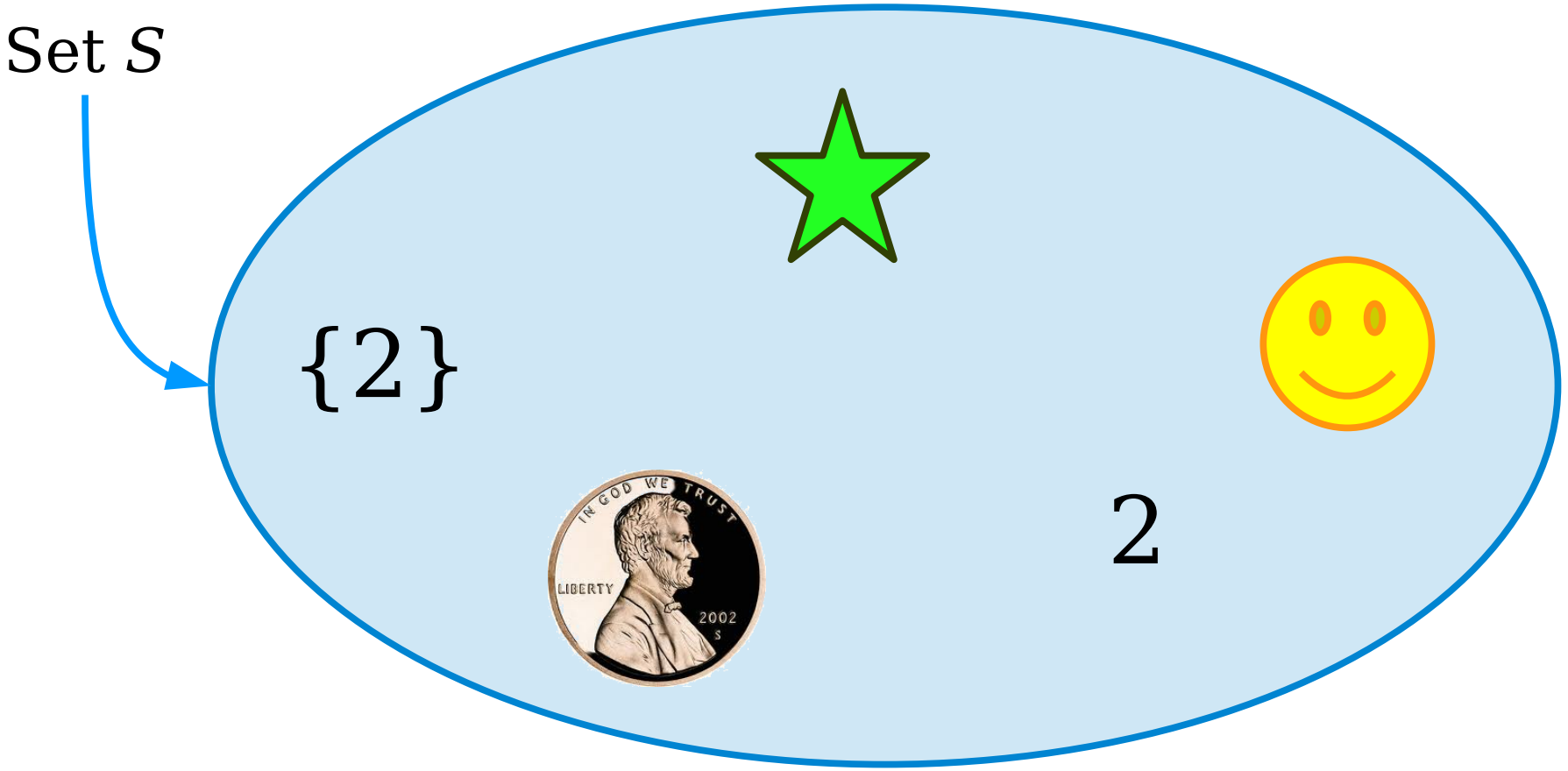

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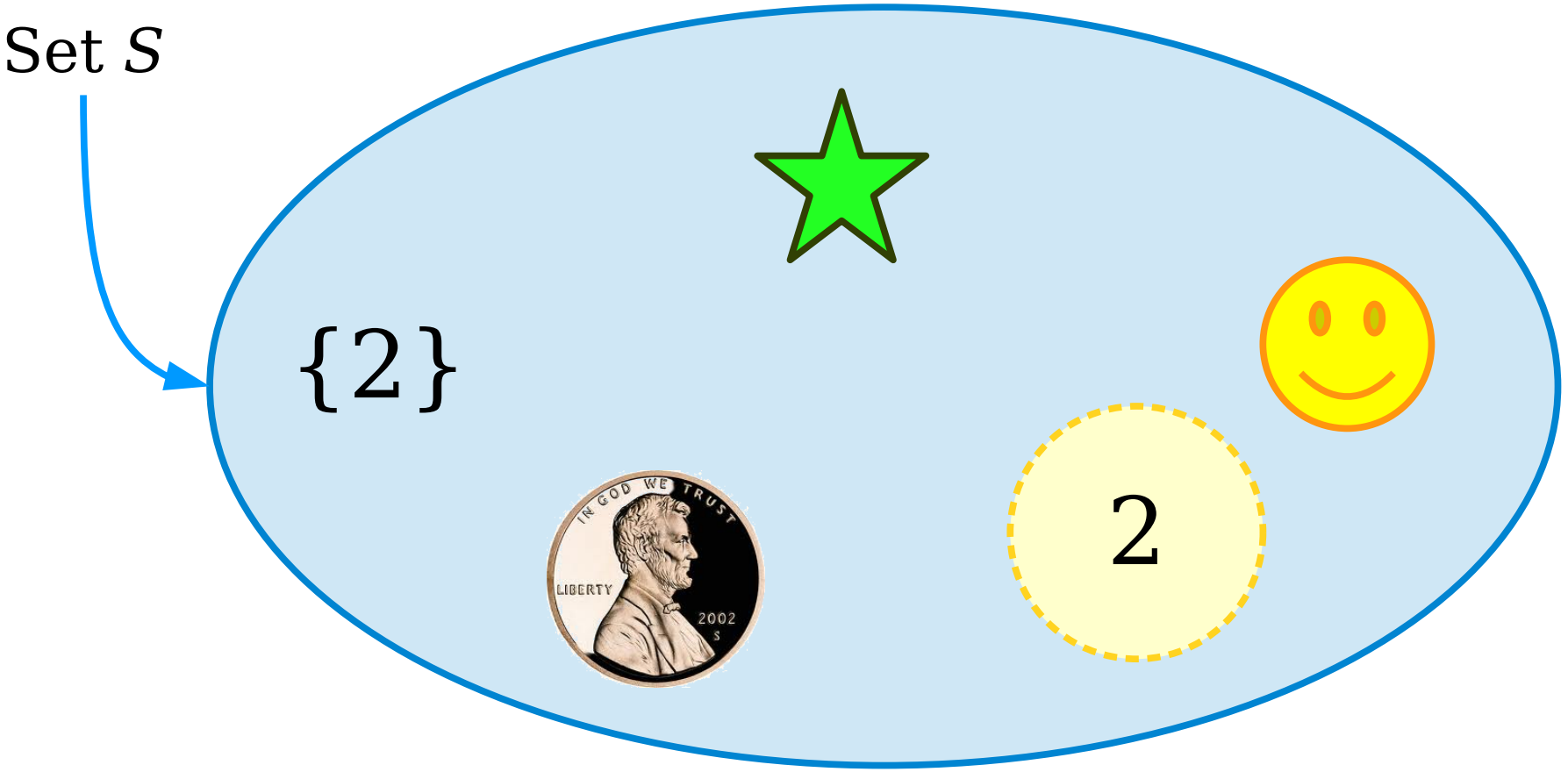
Subsets and Elements



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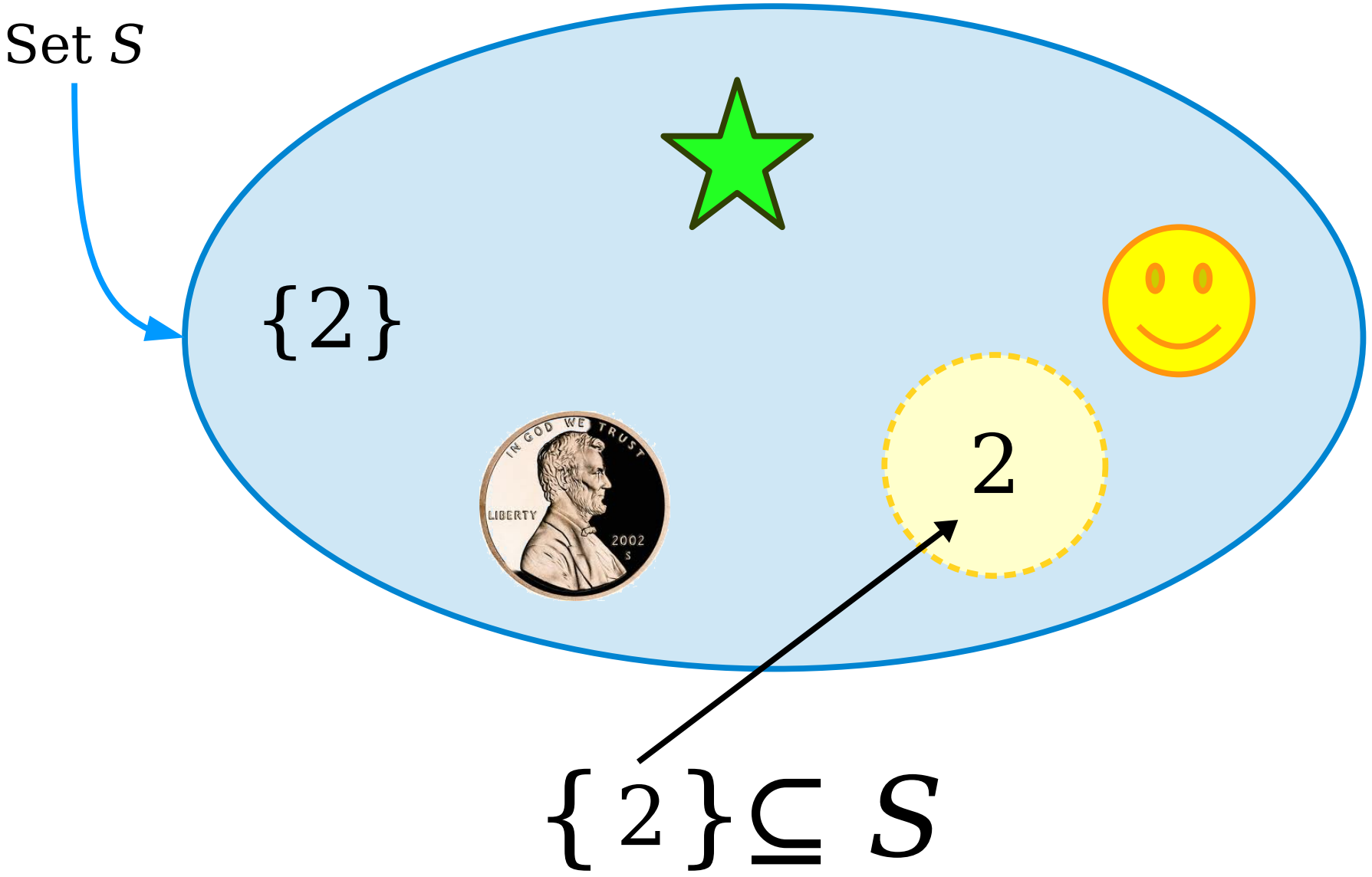
Subsets and Elements



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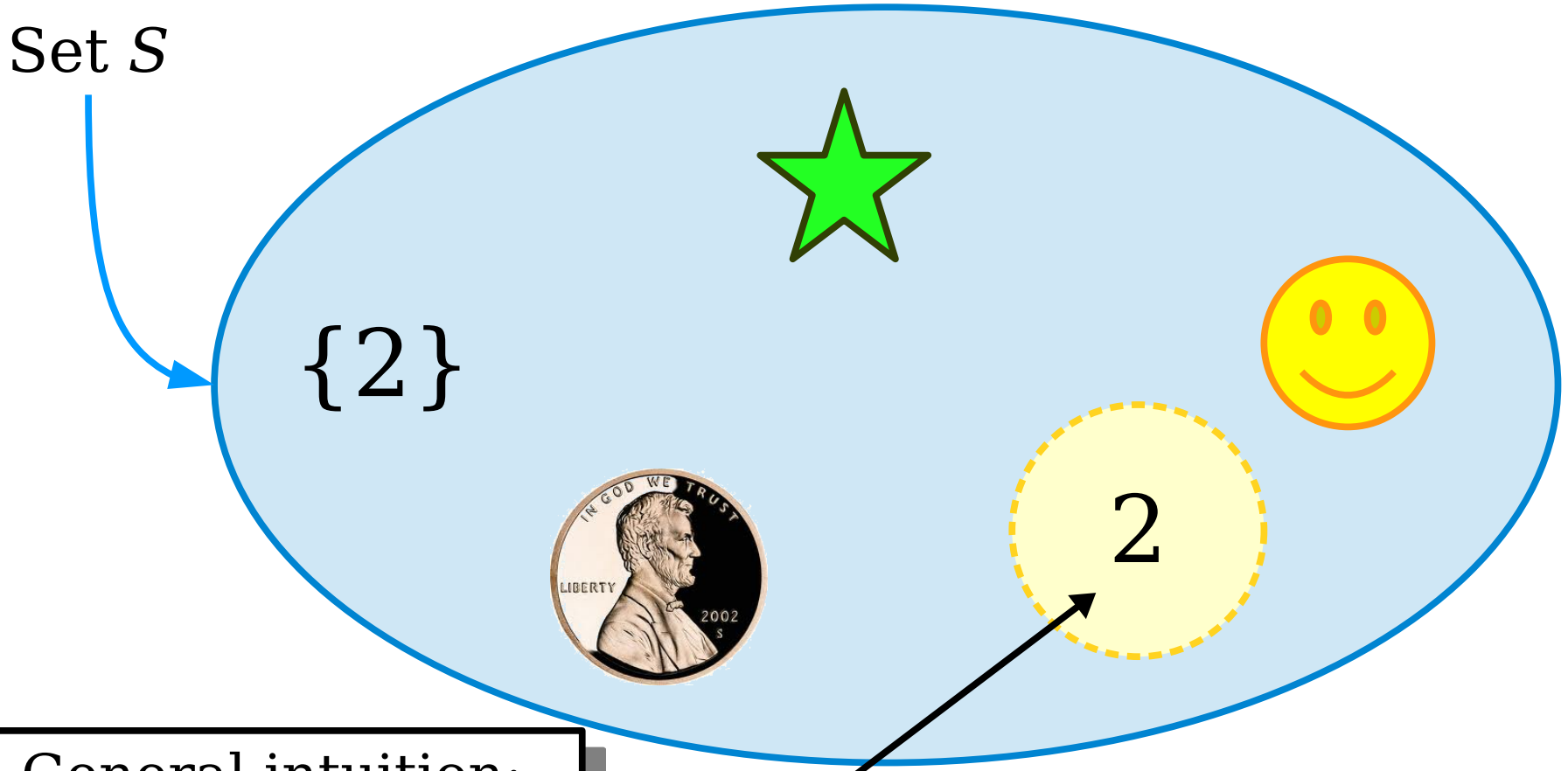
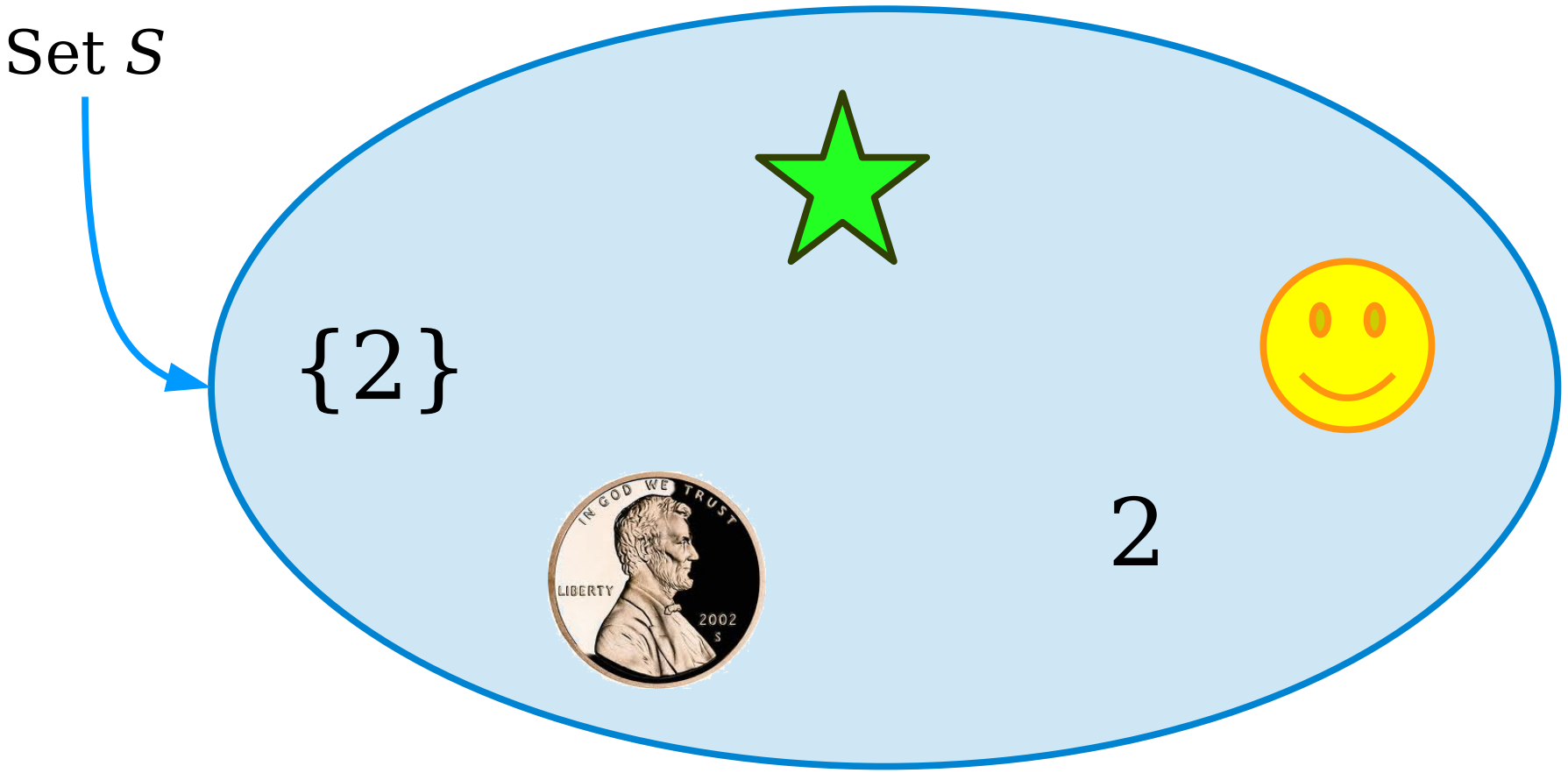


General intuition:
$A \subseteq B$ if you can form A by ***circling elements of B***.

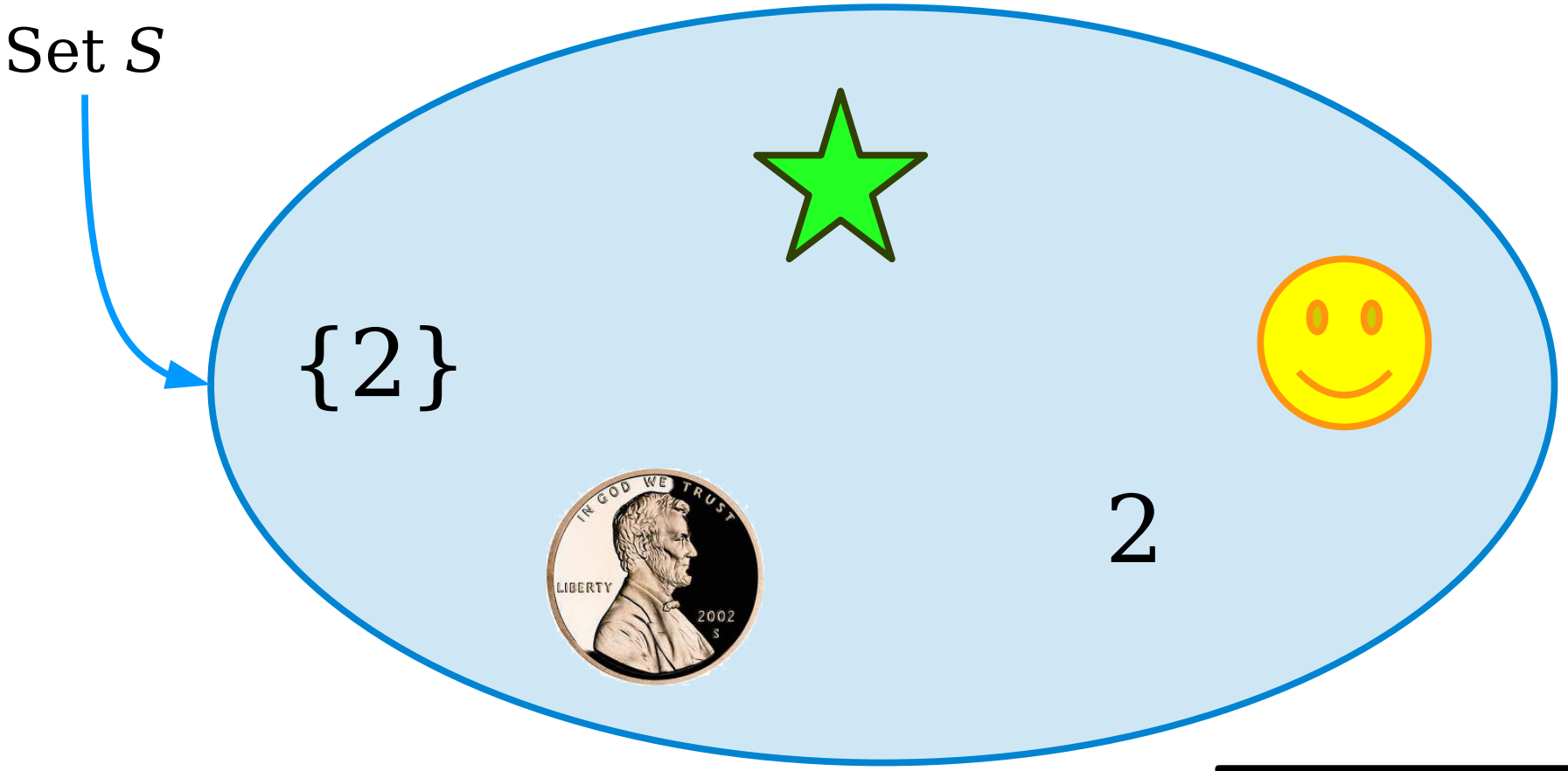
$\{2\} \subseteq S$

Subsets and Elements



$$2 \notin S$$

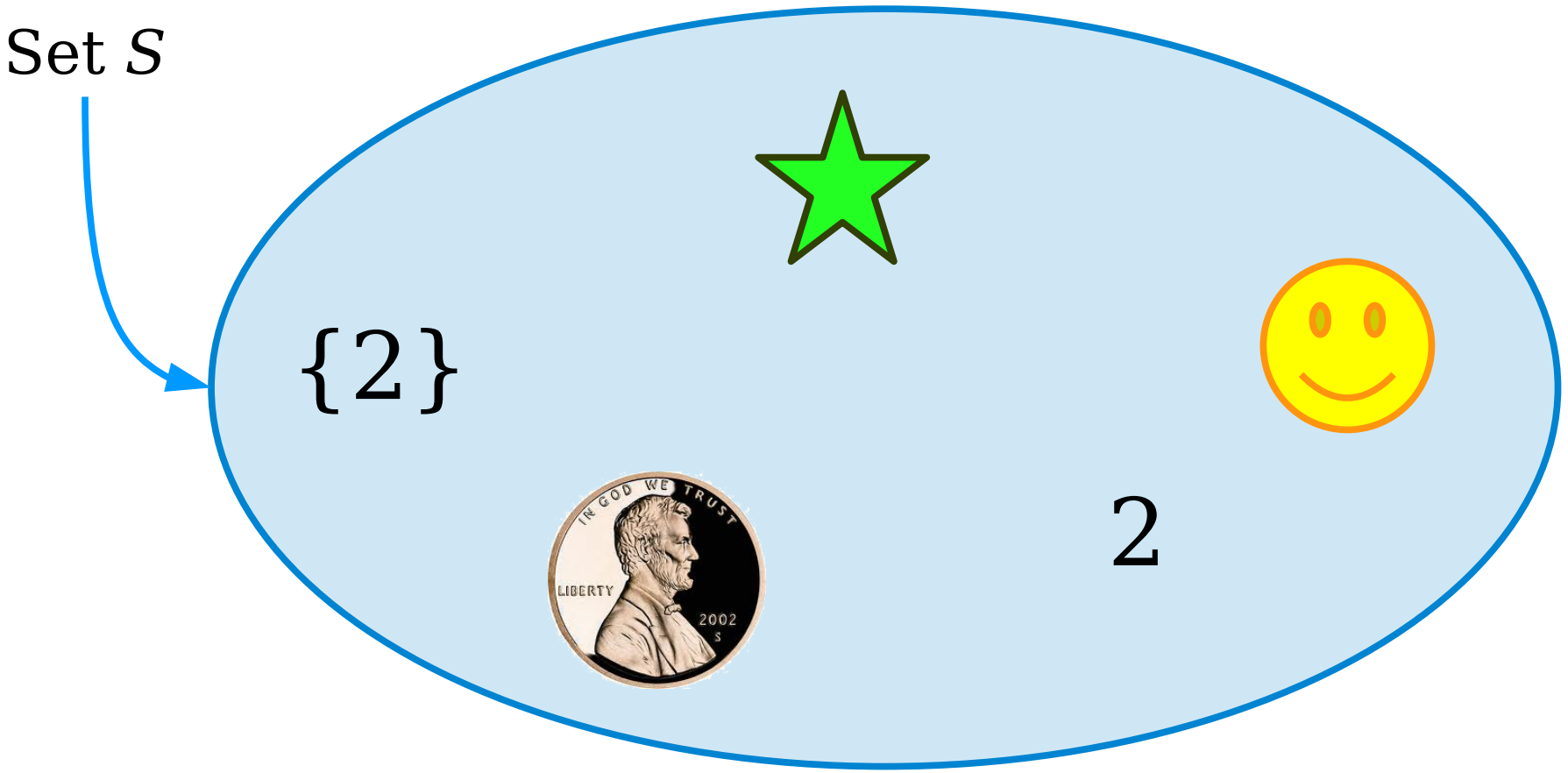
Subsets and Elements



$2 \notin S$

(Since 2 isn't a set.)

Subsets and Elements



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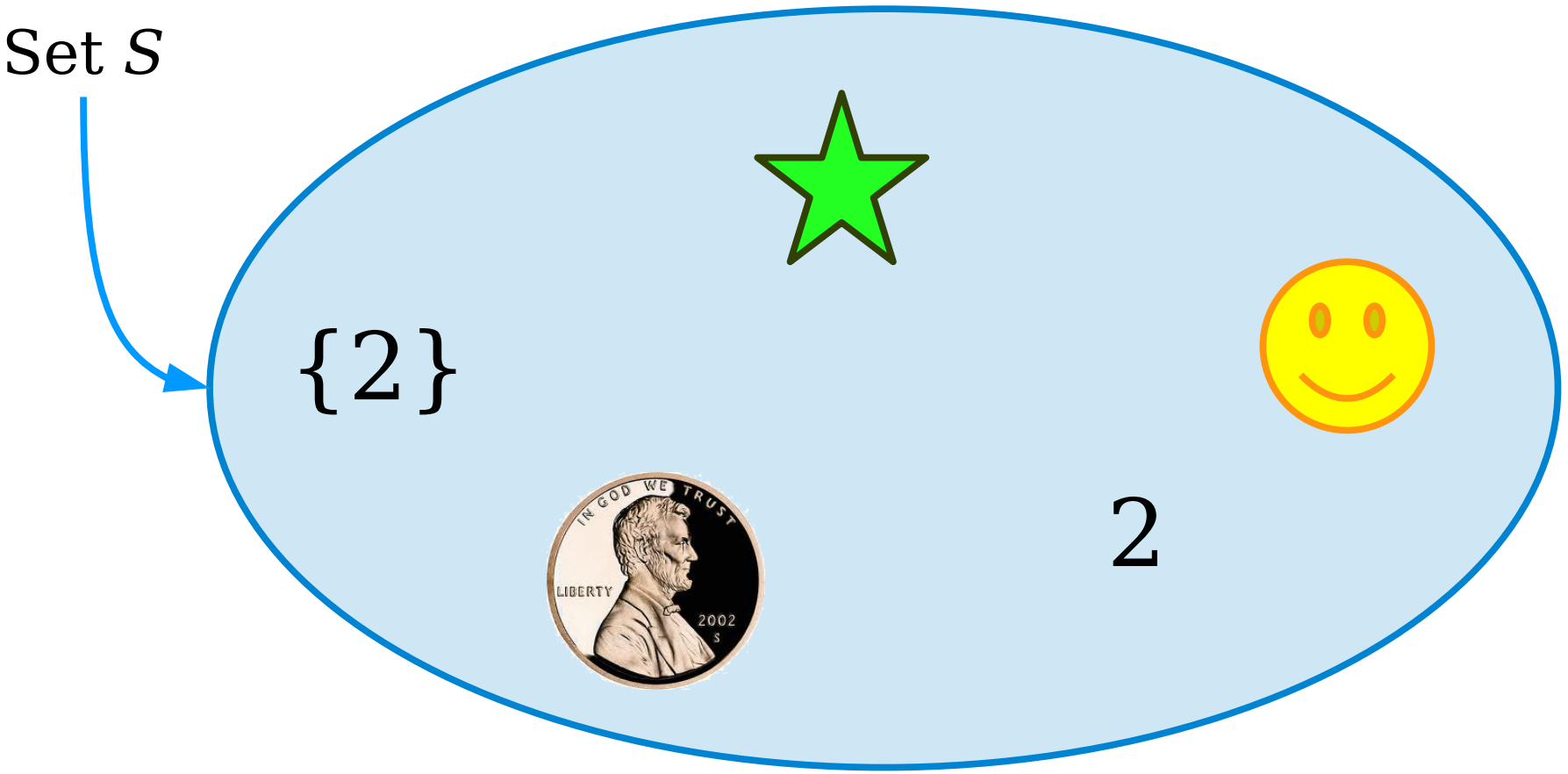
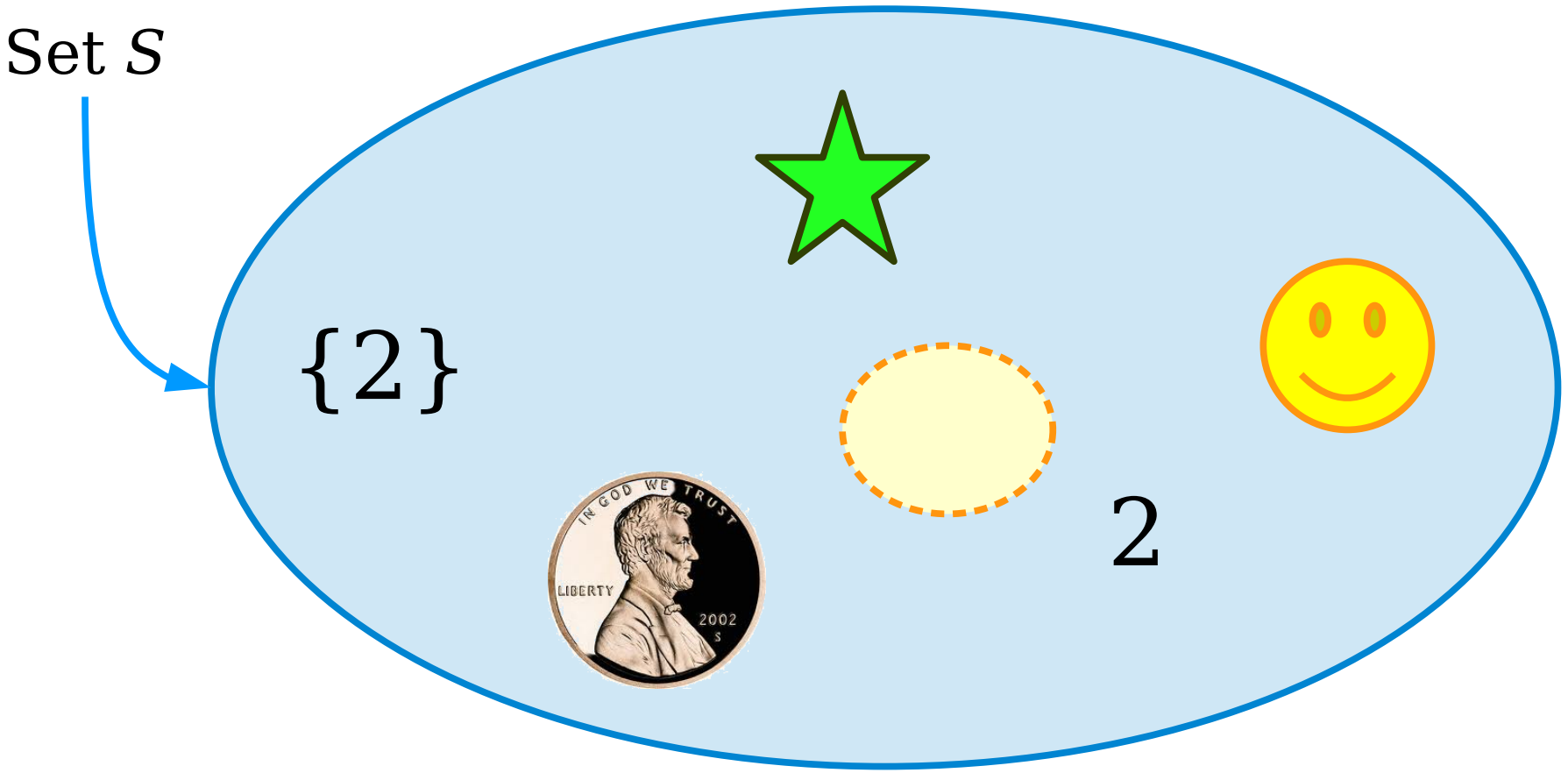


$$\emptyset \subseteq S$$

Subsets and Elements



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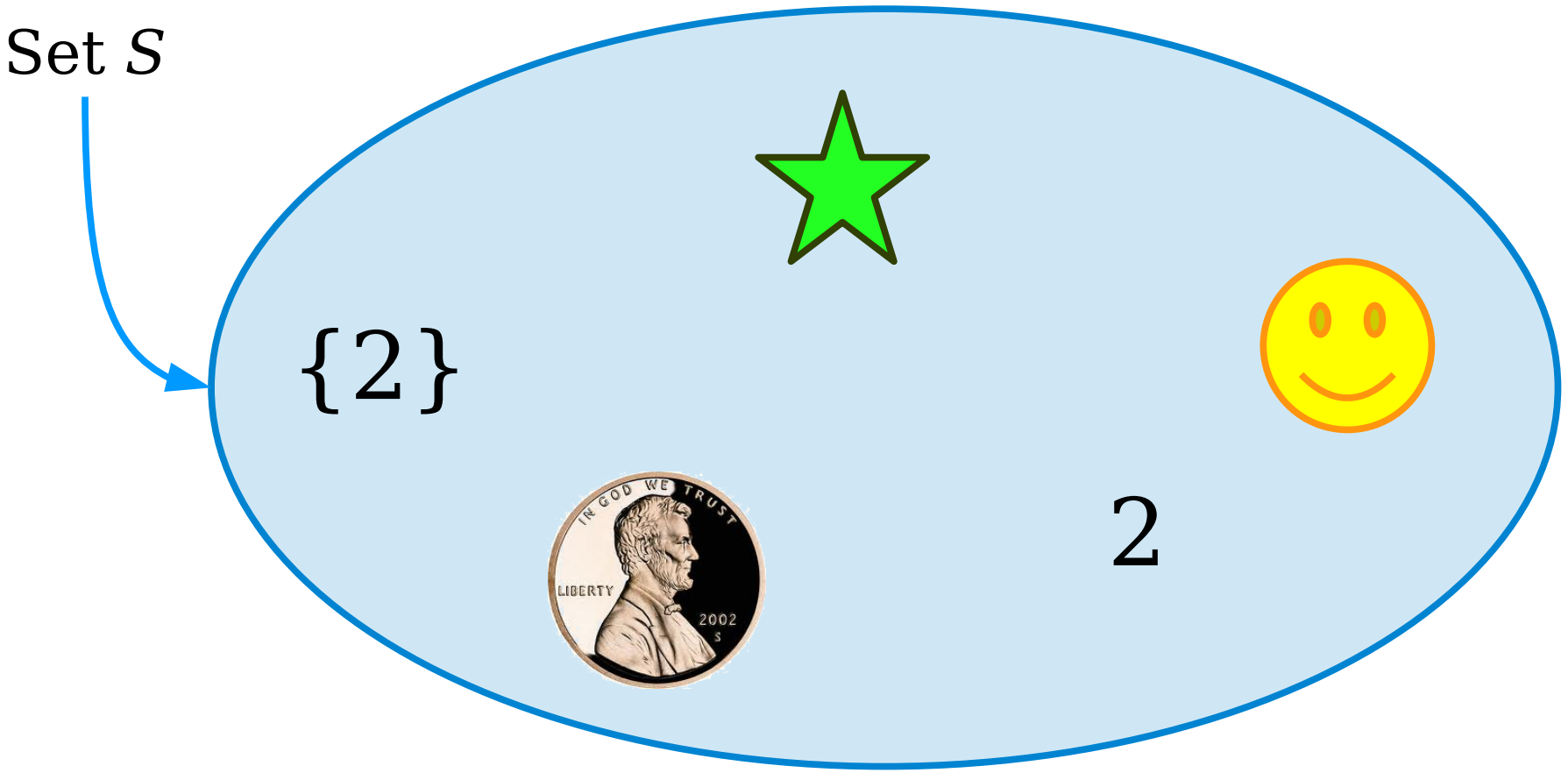

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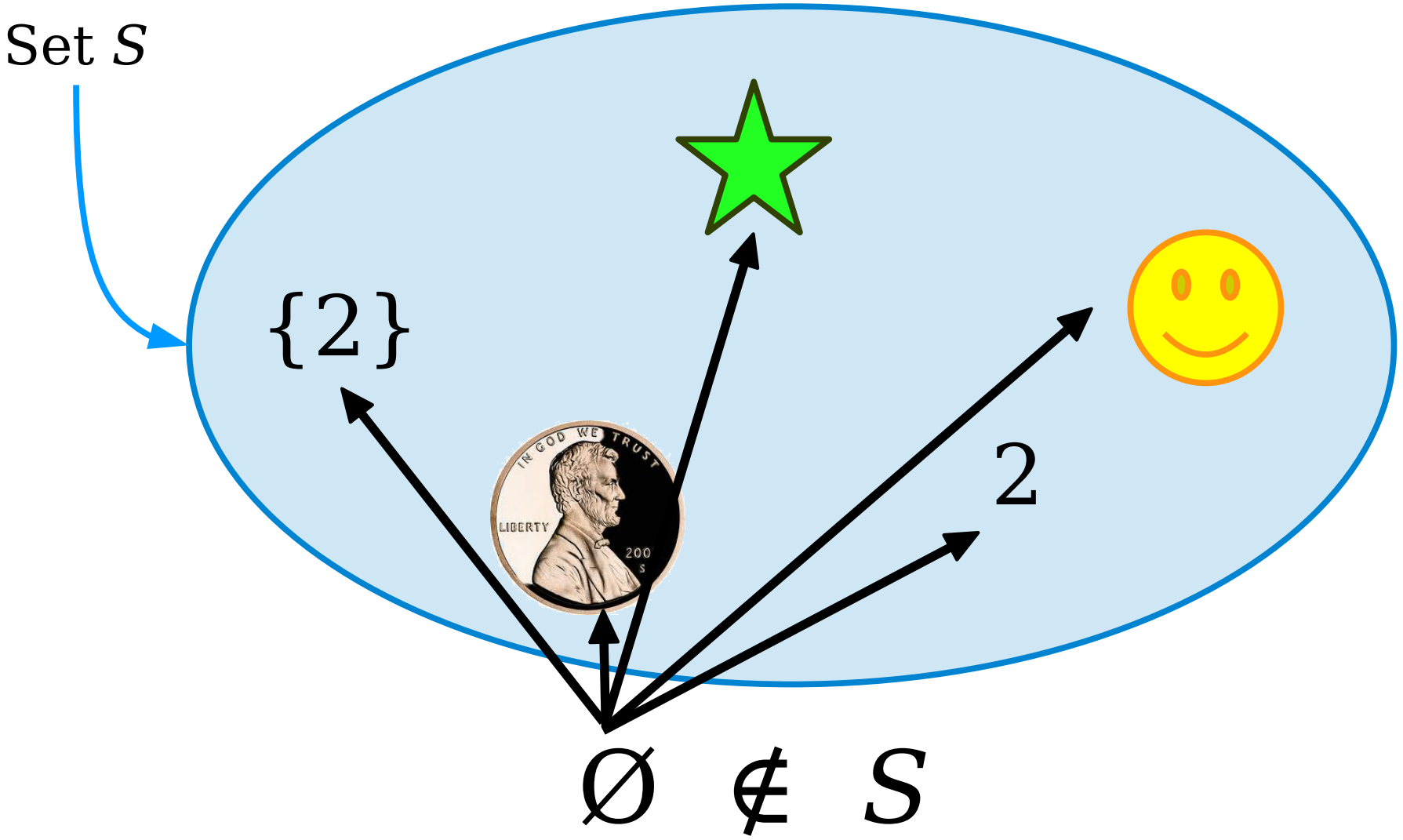
Subsets and Elements



Subsets and Elements

- We say that $S \in T$ when, among the elements of T, one of them is *exactly* the object S.
- We say that $S \subseteq T$ when S is a set and every element of S is also an element of T. (S has to be a set for the statement $S \subseteq T$ to be true.)
- Although these concepts are similar, ***they are not the same!*** Not all elements of a set are subsets of that set and vice-versa.
- We have a resource on the course website, the Guide to Elements and Subsets, that explores this in more depth.

$$S = \left\{ \text{ }, \text{ } \right\}$$

$$\wp(S) = \left\{ \varnothing, \{\text{ }\}, \{\text{ }\}, \{\text{ }, \text{ }\} \right\}$$

This is the ***power set*** of S, the set of all subsets of S. We write the power set of S as $\wp(S)$.

Formally, $\wp(S) = \{ T \mid T \subseteq S \}$.

(Do you see why?)

What is $\wp(\emptyset)$?

Answer: $\{\emptyset\}$

Remember that $\emptyset \neq \{\emptyset\}$!

Cardinality

Cardinality

Cardinality

- The ***cardinality*** of a set is the number of elements it contains.
- If S is a set, we denote its cardinality as $\boldsymbol{|S|}$.
- Examples:
  - $|\{whimsy, mirth\}| = 2$
  - $|\{\{a, b\}, \{c, d, e, f, g\}, \{h\}\}| = 3$
  - $|\{1, 2, 3, 3, 3, 3, 3\}| = 3$
  - $|\{\ n \in \mathbb{N} \mid n < 4\ \}| = |\{0, 1, 2, 3\}| = 4$
  - $|\ \varnothing\ | = 0$
  - $|\ \{\varnothing\}\ | = 1$

The Cardinality of ℕ

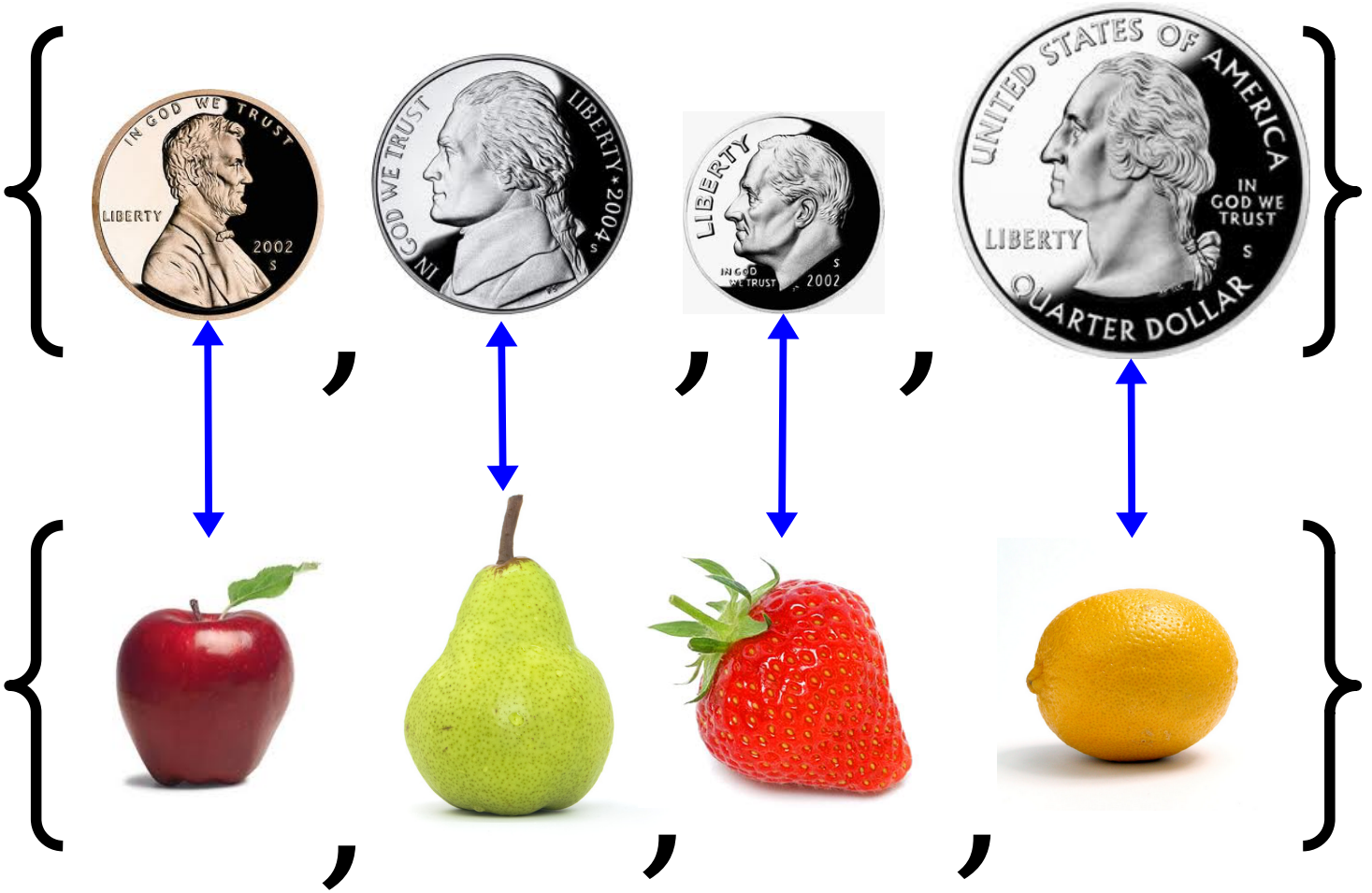
- What is |ℕ|?
  - There are infinitely many natural numbers.
  - |ℕ| can't be a natural number, since it's infinitely large.
- We need to introduce a new term.
- Let's define **ℵ₀** = |ℕ|.
  - ℵ₀ is pronounced “aleph-zero,” “aleph-nought,” or “aleph-null.”
- ***Question:*** Why don’t we say |ℕ| = ∞?

Astonishing Fact: Not all infinite sets have the same cardinality. Some infinite sets are bigger than others!

More Astonishing Fact: This has practical consequences!

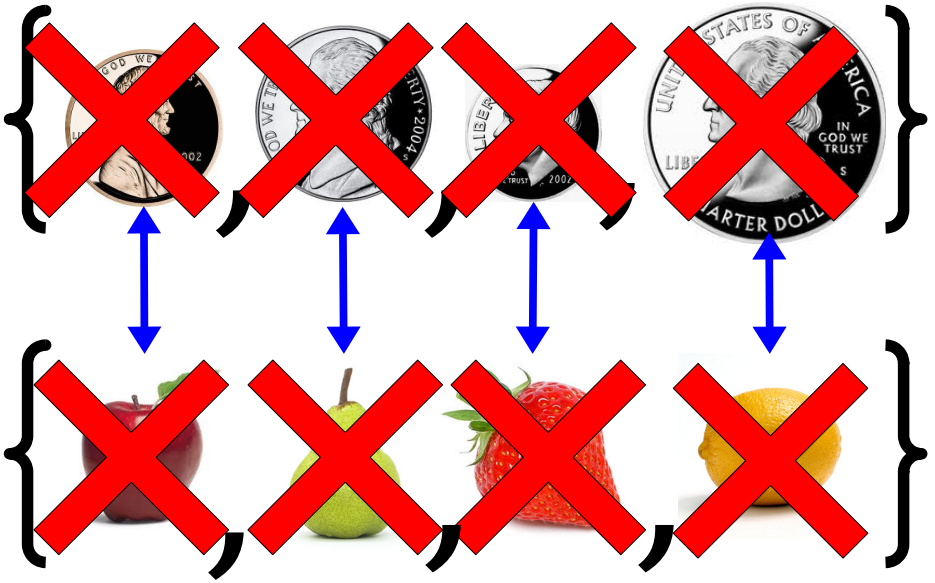
What does it mean for one set to be “bigger” than another?

How Big Are These Sets?



Comparing Cardinalities

- If S and T are sets, we say that $|S| = |T|$ when there is a way of pairing off the elements of S and T without leaving anything uncovered.



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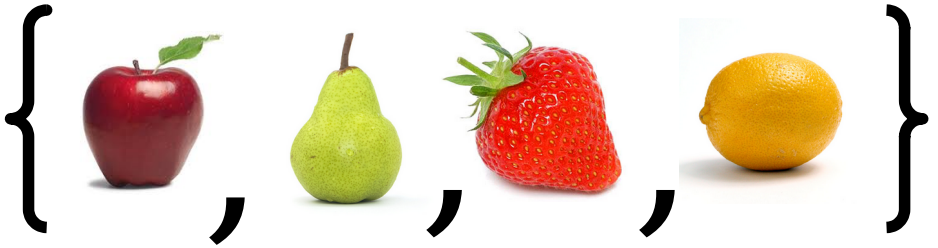
- If S and T are sets, we say $|S| < |T|$ when, no matter how you pair off the elements of S and T, there's always at least one element of T left uncovered.



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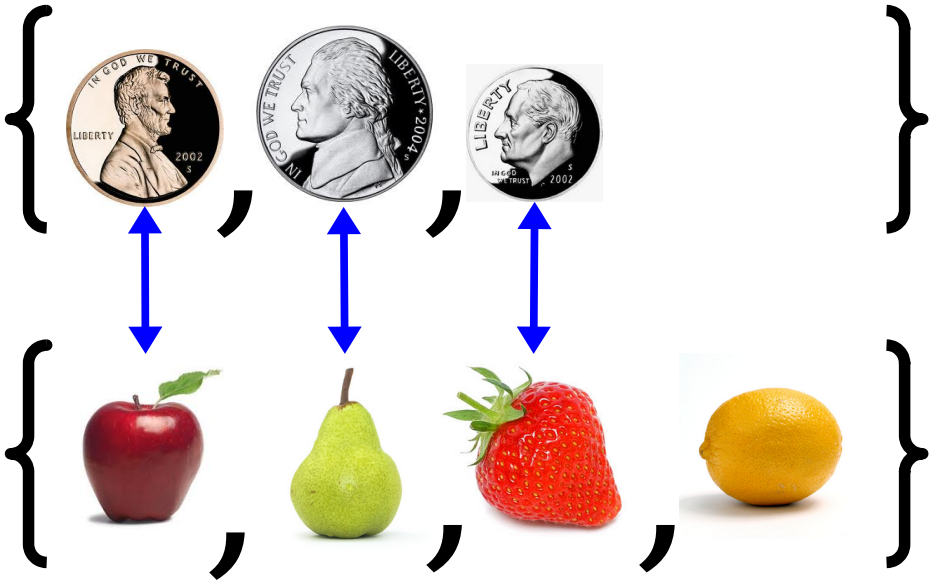

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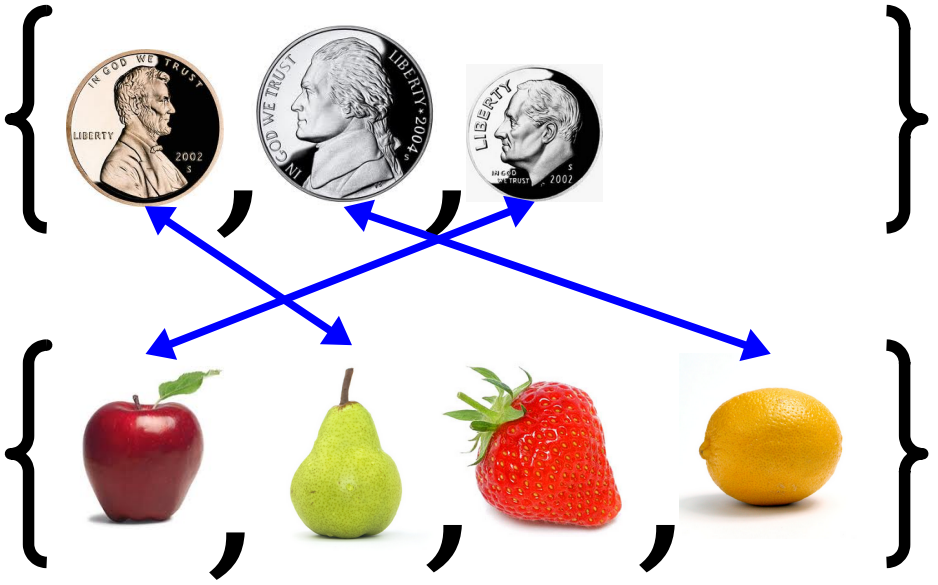

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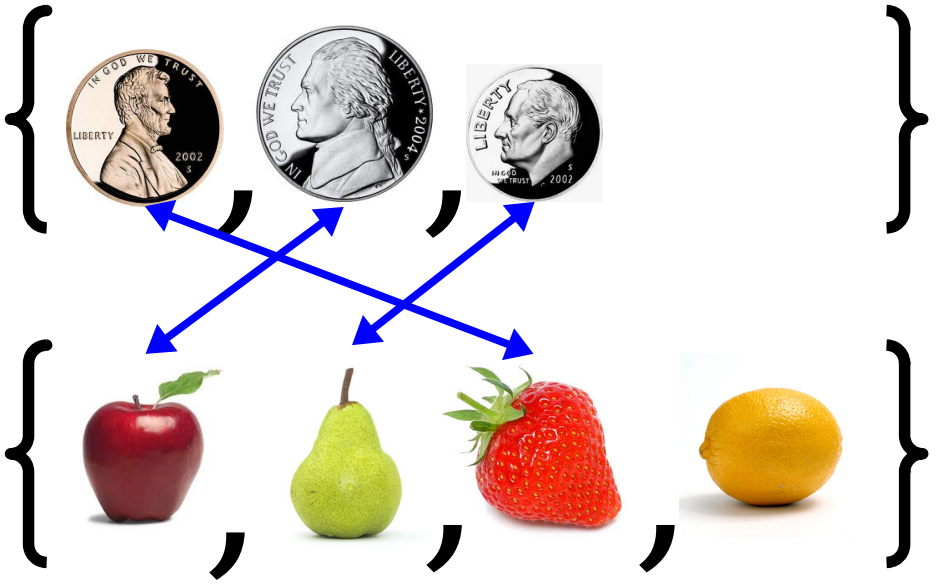

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
$\mathbb{Z}$	...	-3	-2	-1	0	1	2	3	4	...

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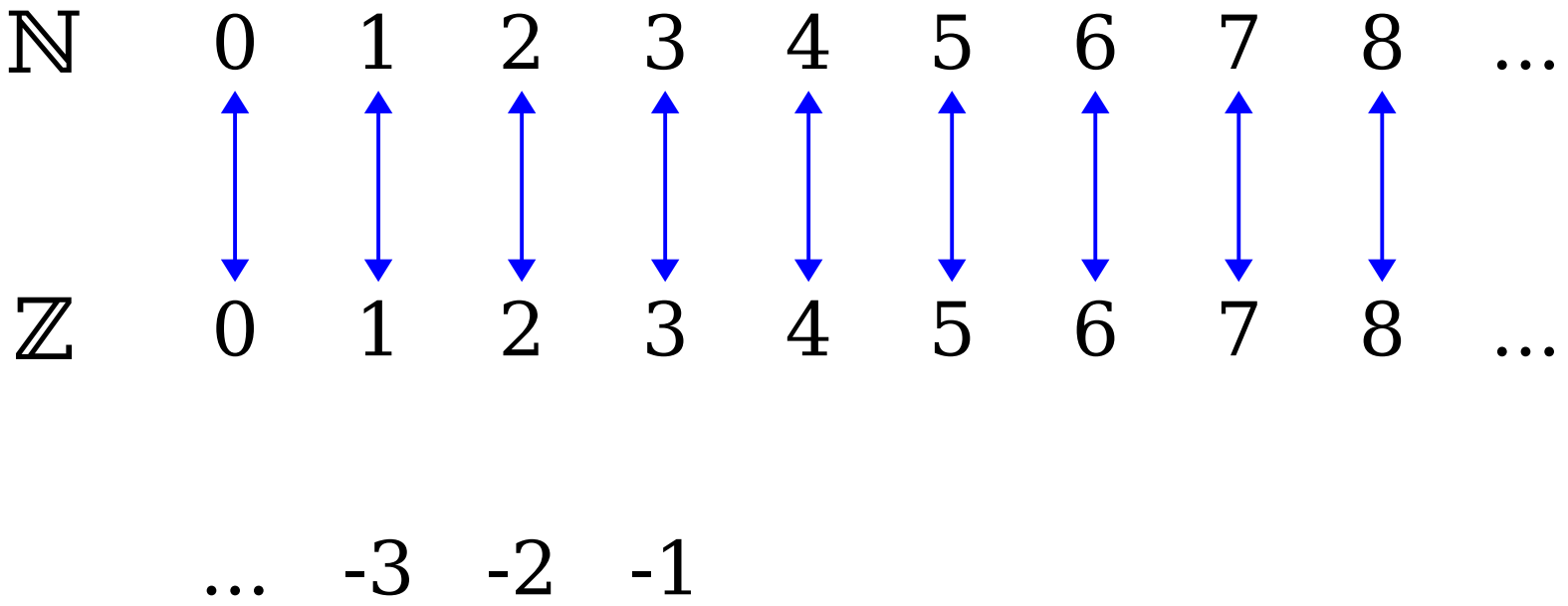

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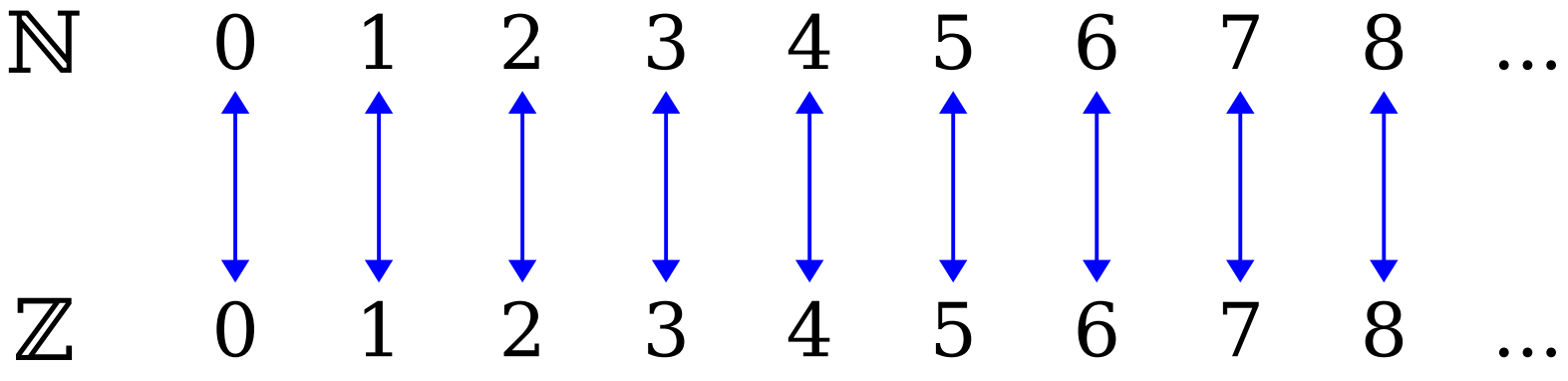

One set is smaller than another when *every way* of pairing the sets off leaves an element of the second set uncovered.

Two sets have the same size when *there is a way* to pair their elements off without leaving any elements uncovered

Infinite Cardinalities

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	...
$\mathbb{Z}$	...	-3	-2	-1	0	1	2	3	4	...

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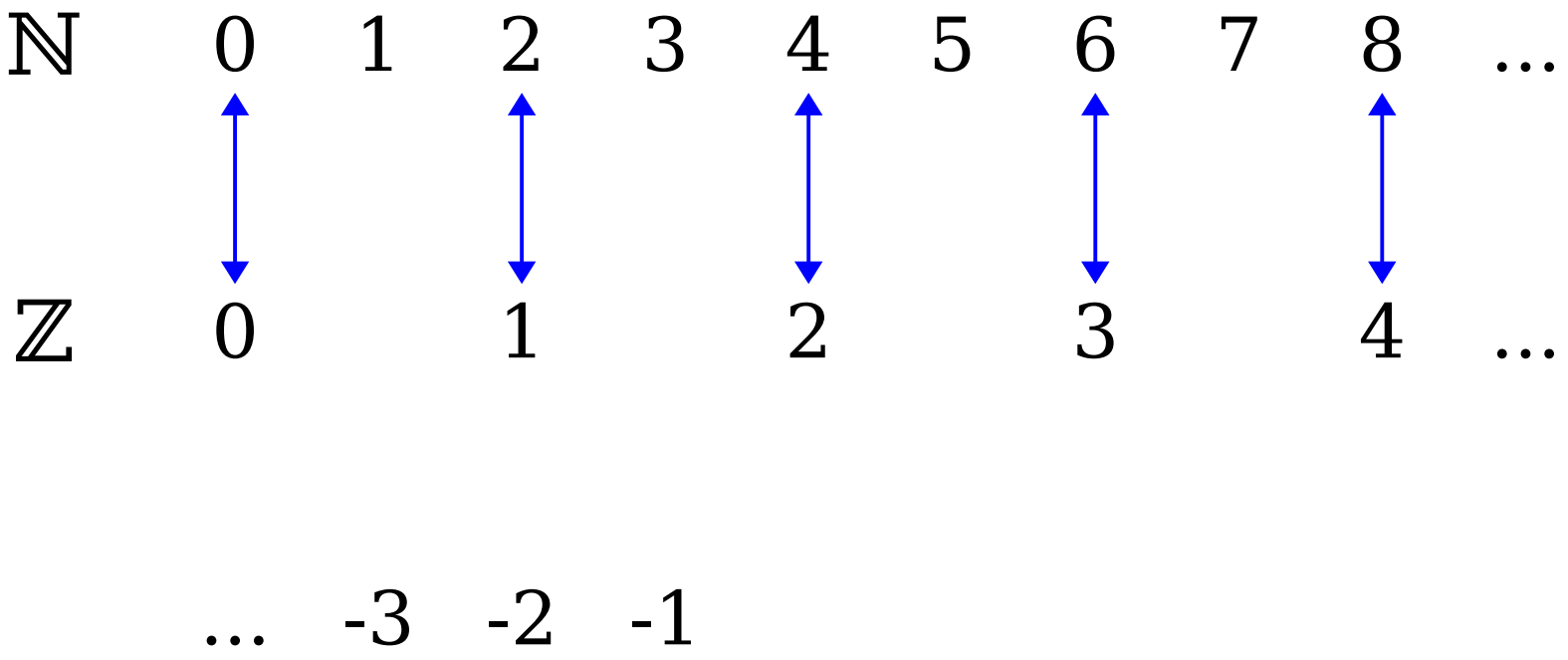

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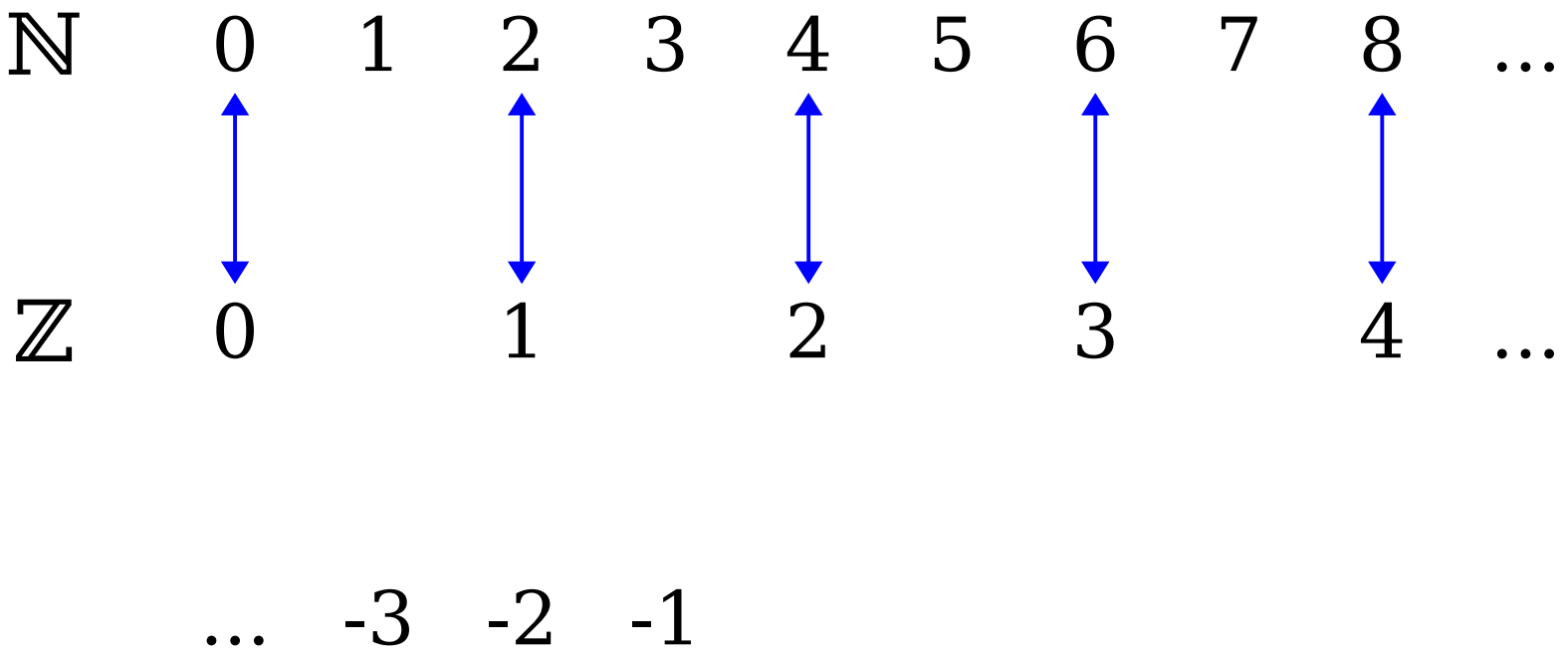
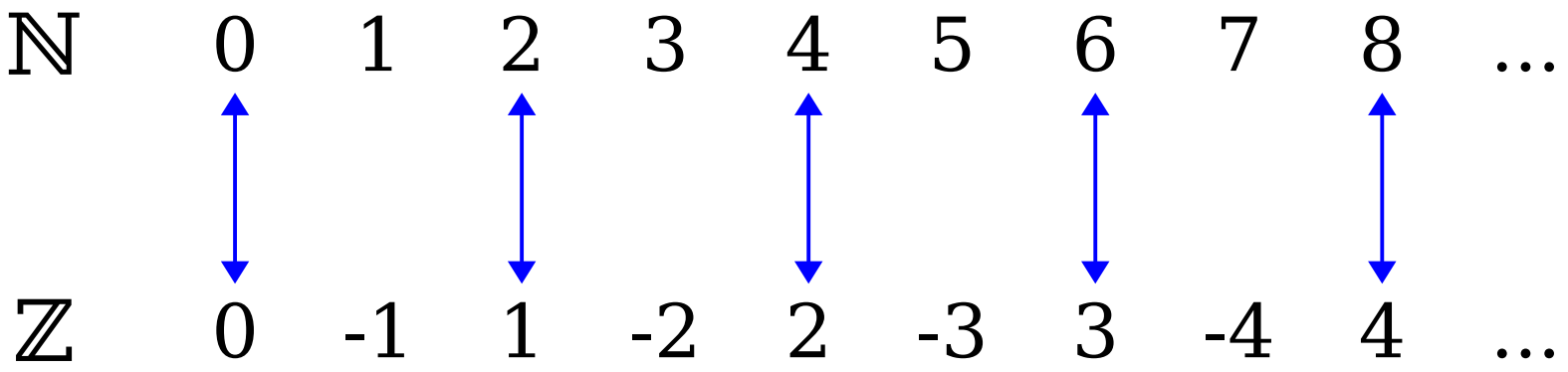


Pair nonnegative integers with even natural numbers.

Infinite Cardinalities



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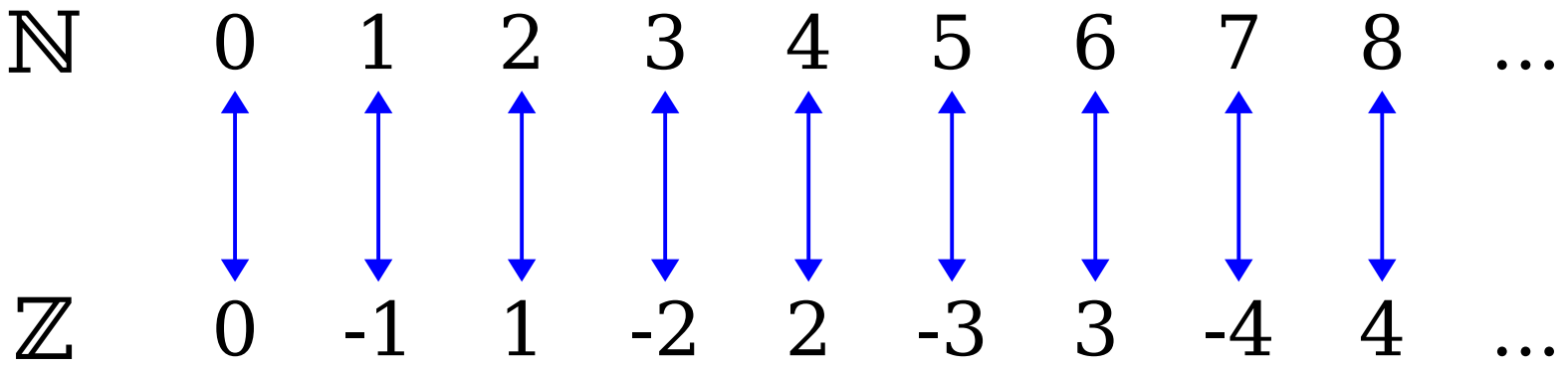

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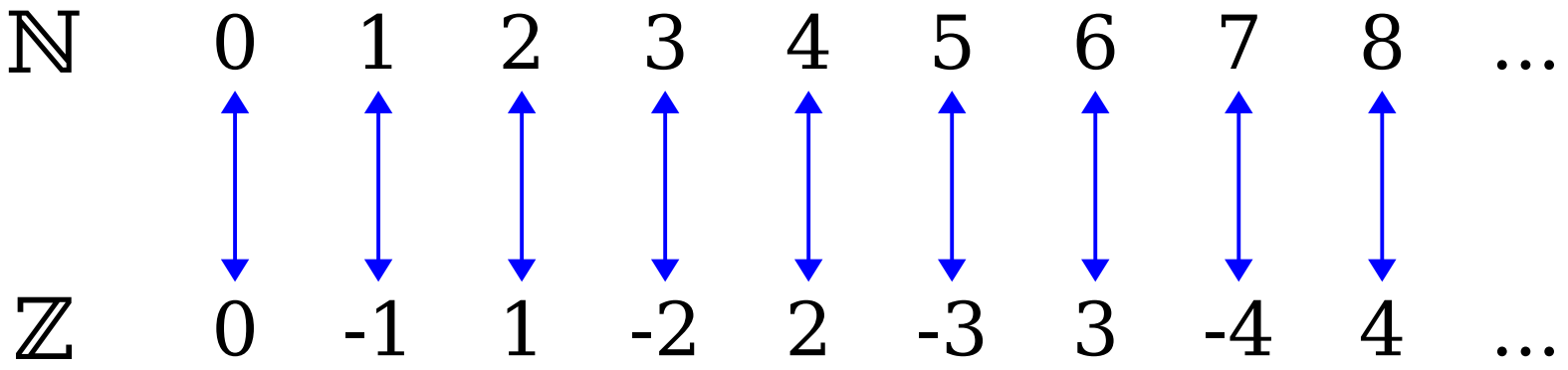
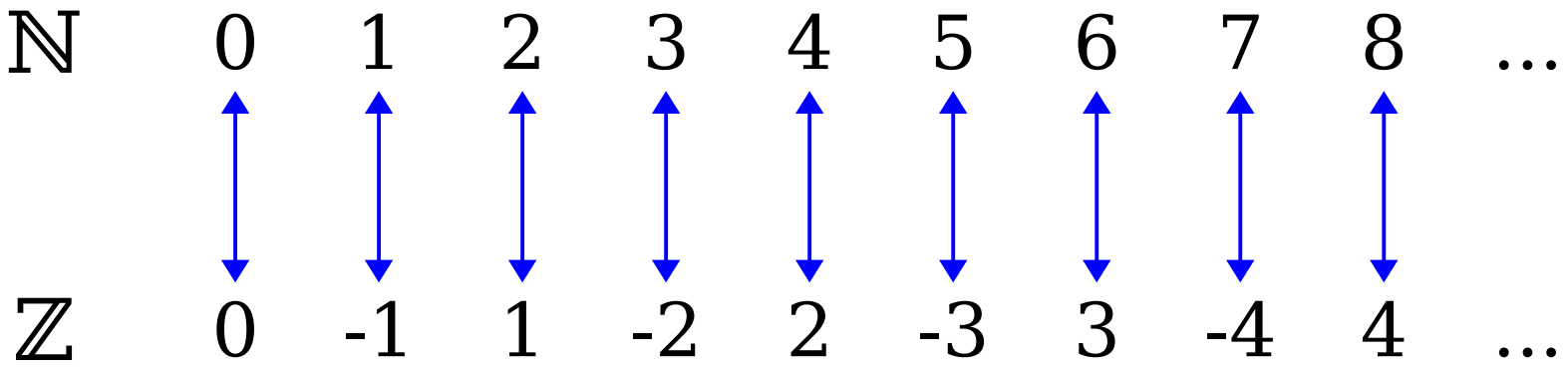


Pair nonnegative integers with even natural numbers.
Pair negative integers with odd natural numbers.

Infinite Cardinalities



$$|\mathbb{N}| = |\mathbb{Z}| = \aleph_0$$

Pair nonnegative integers with even natural numbers.
Pair negative integers with odd natural numbers.

A Beautiful Result: ***Cantor's Theorem***

$$S = \left\{ \text{ }, \text{ } \right\}$$

$$\wp(S) = \left\{ \varnothing, \left\{ \text{ } \right\}, \left\{ \text{ } \right\}, \left\{ \text{ }, \text{ } \right\} \right\}$$

$$\boldsymbol{|S| < |\wp(S)|}$$

$$S = \{ \text{penny}, \text{nickel}, \text{button} \}$$

$$\wp(S) = \{ \emptyset, \{\text{penny}\}, \{\text{nickel}\}, \{\text{button}\}, \{\text{penny}, \text{nickel}\}, \{\text{penny}, \text{button}\}, \{\text{nickel}, \text{button}\}, \{\text{penny}, \text{nickel}, \text{button}\} \}$$

$$|S| < |\wp(S)|$$

$$S = \{a, b, c, d\}$$

$$\wp(S) = \{$$
$$\emptyset,$$
$$\{a\}, \{b\}, \{c\}, \{d\},$$
$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$$
$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\},$$
$$\{a, b, c, d\}$$
$$\}$$

$$|S| < |\wp(S)|$$

Cantor's Theorem: If S is a set, then $|S| < |\wp(S)|$.

Stated differently: no matter how you pair off the elements of a set S with the elements of $\wp(S)$, there is always some element of $\wp(S)$ left uncovered.

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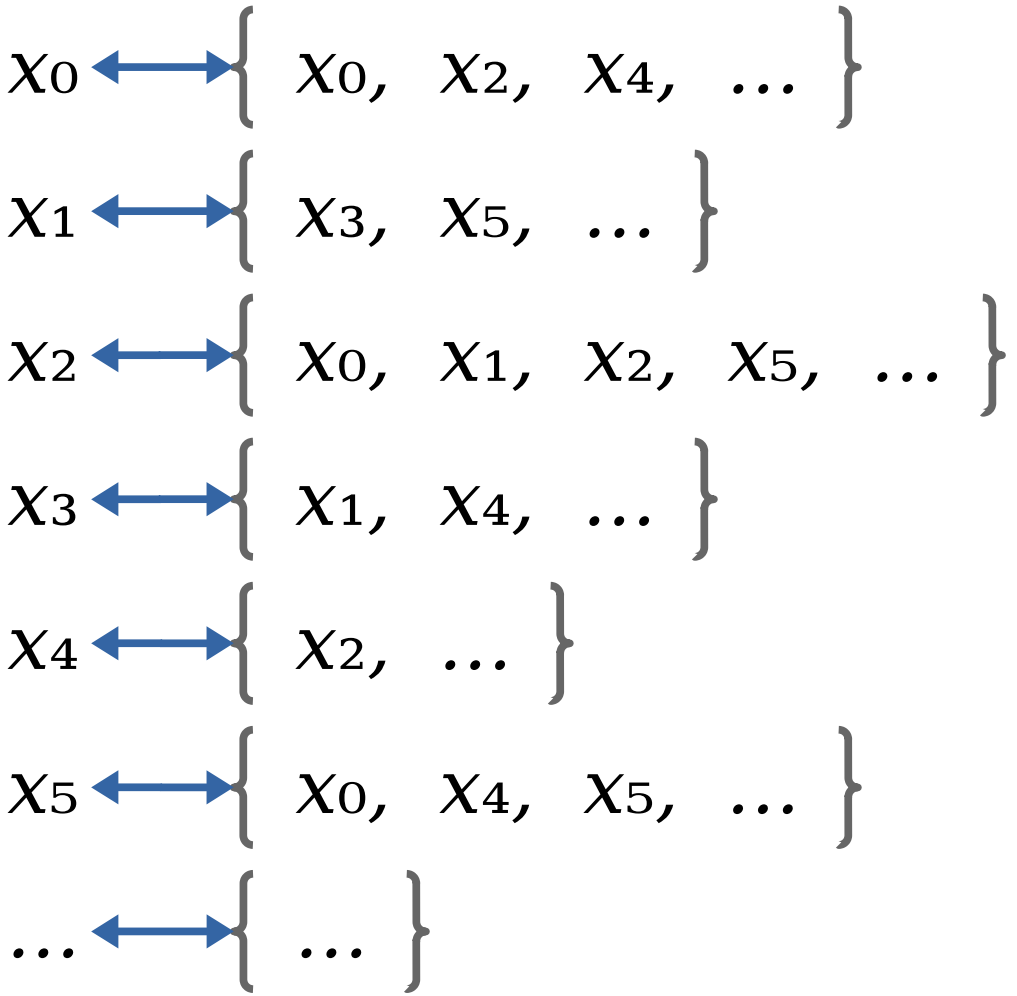
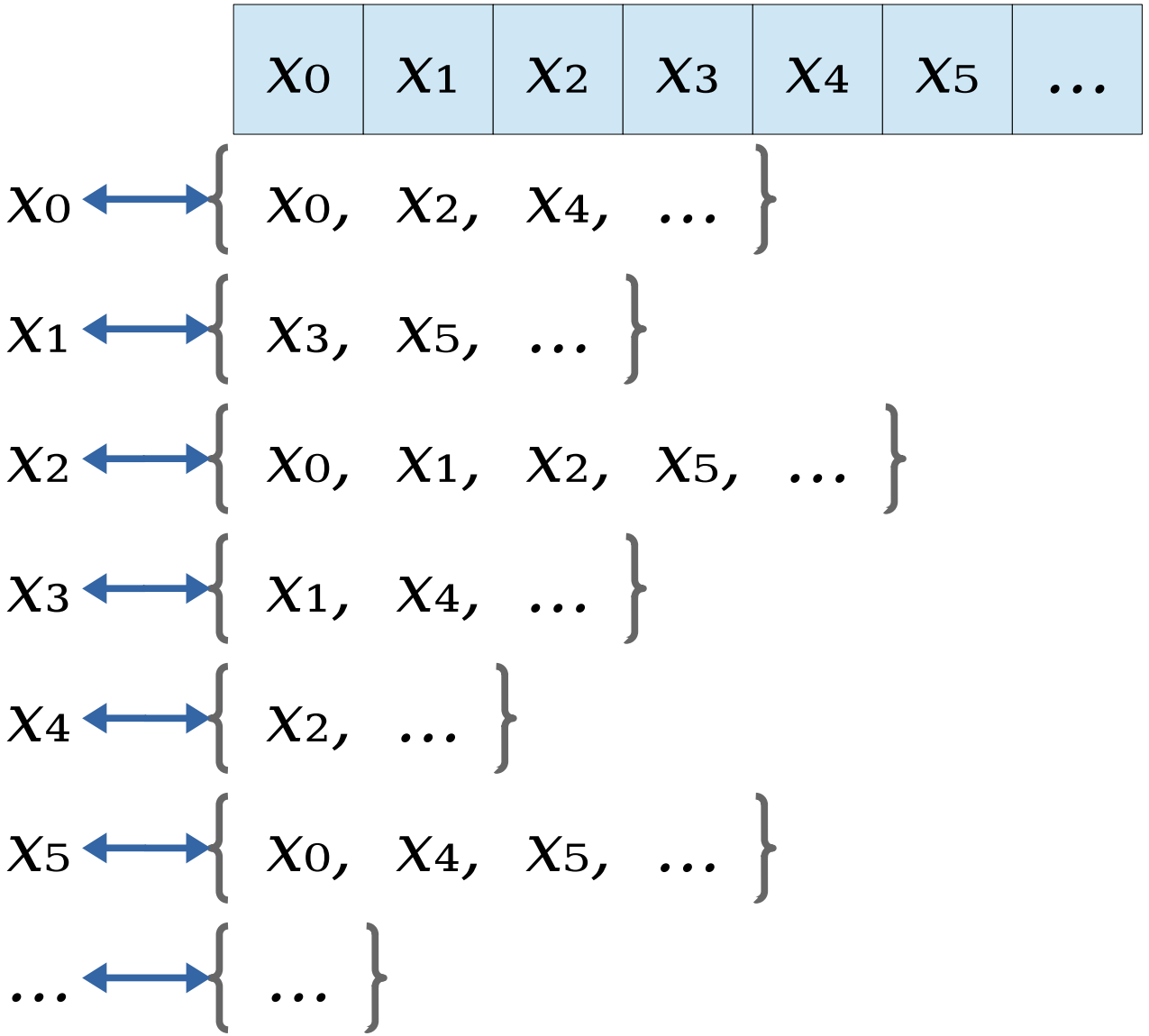
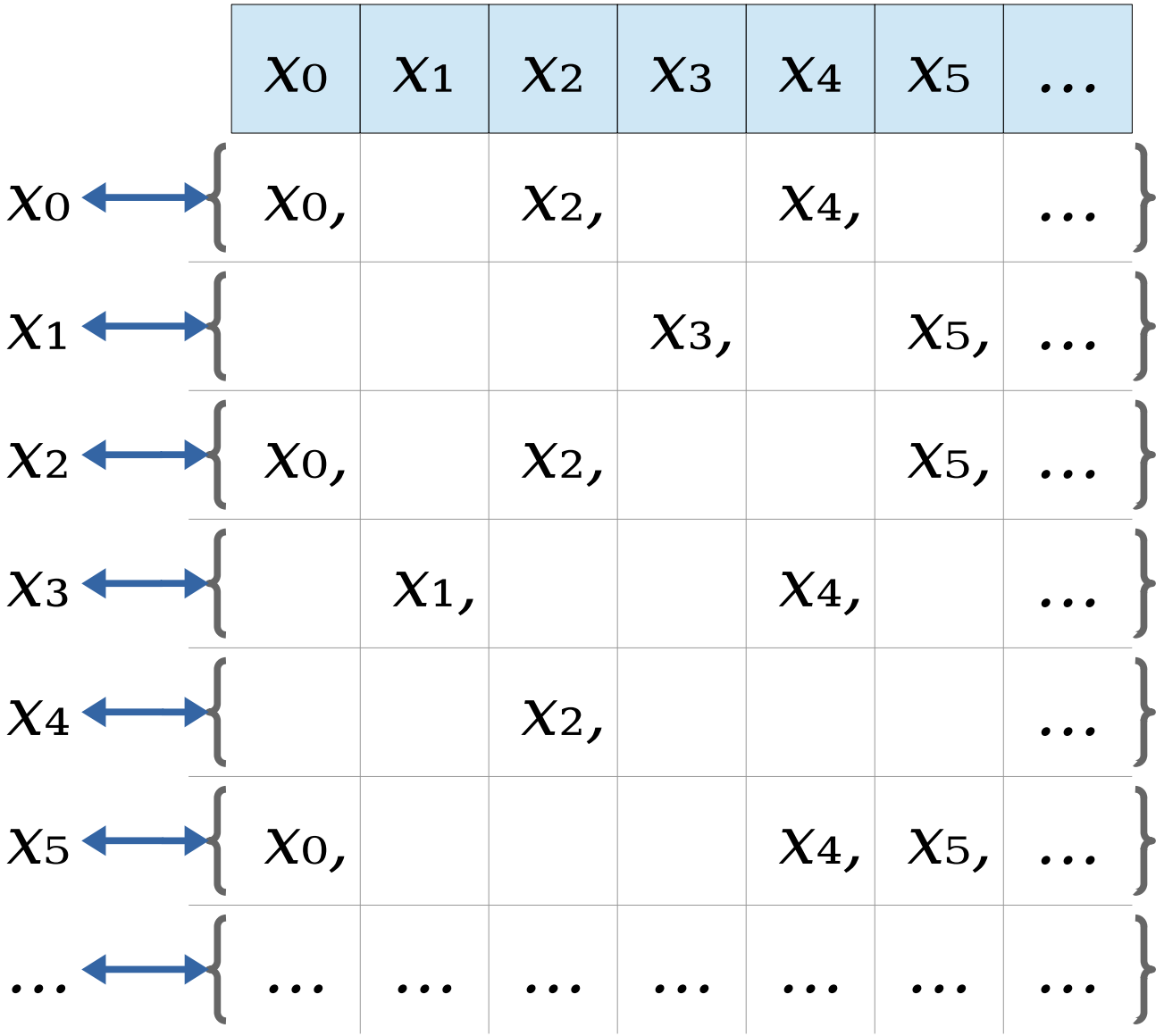
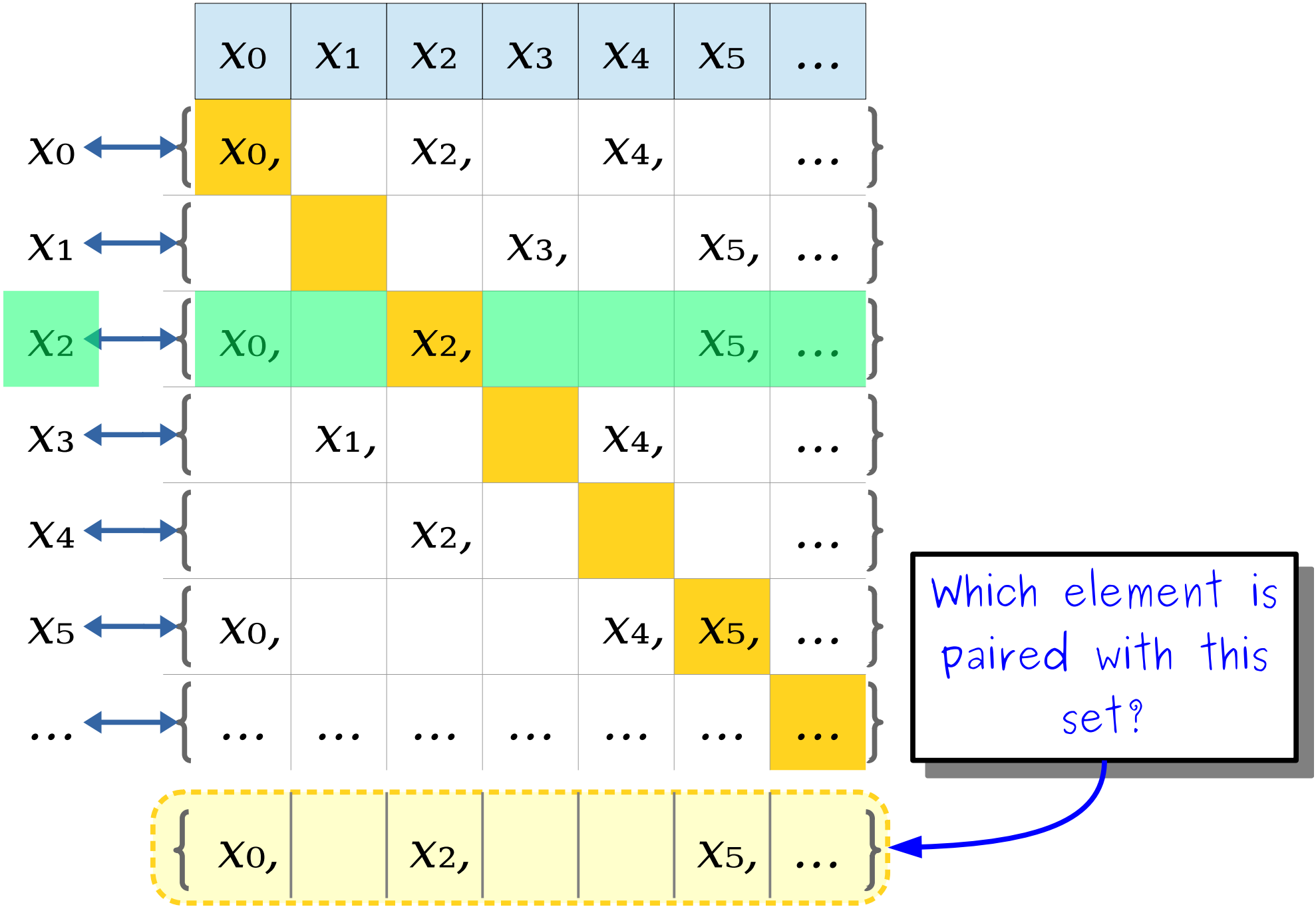
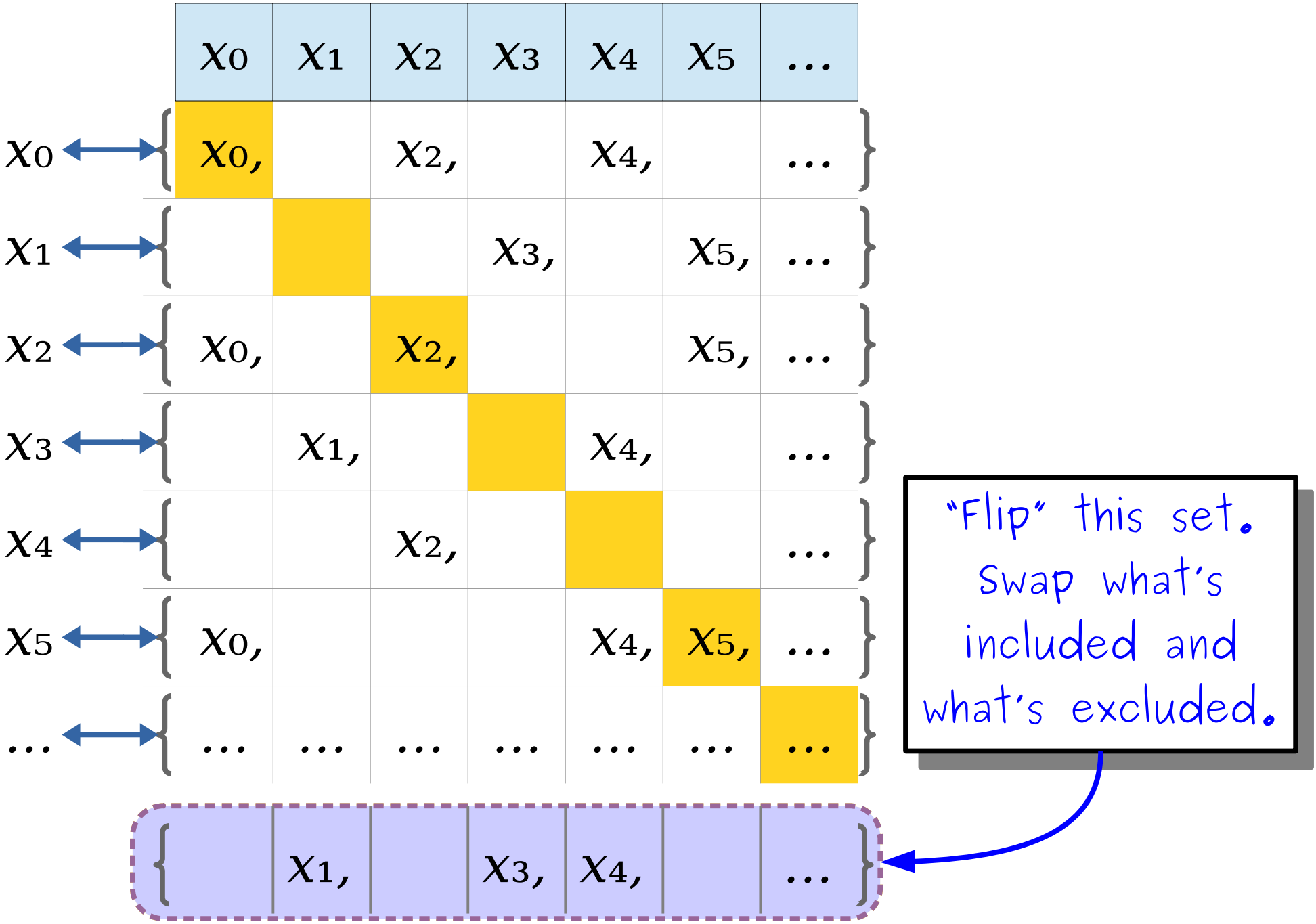
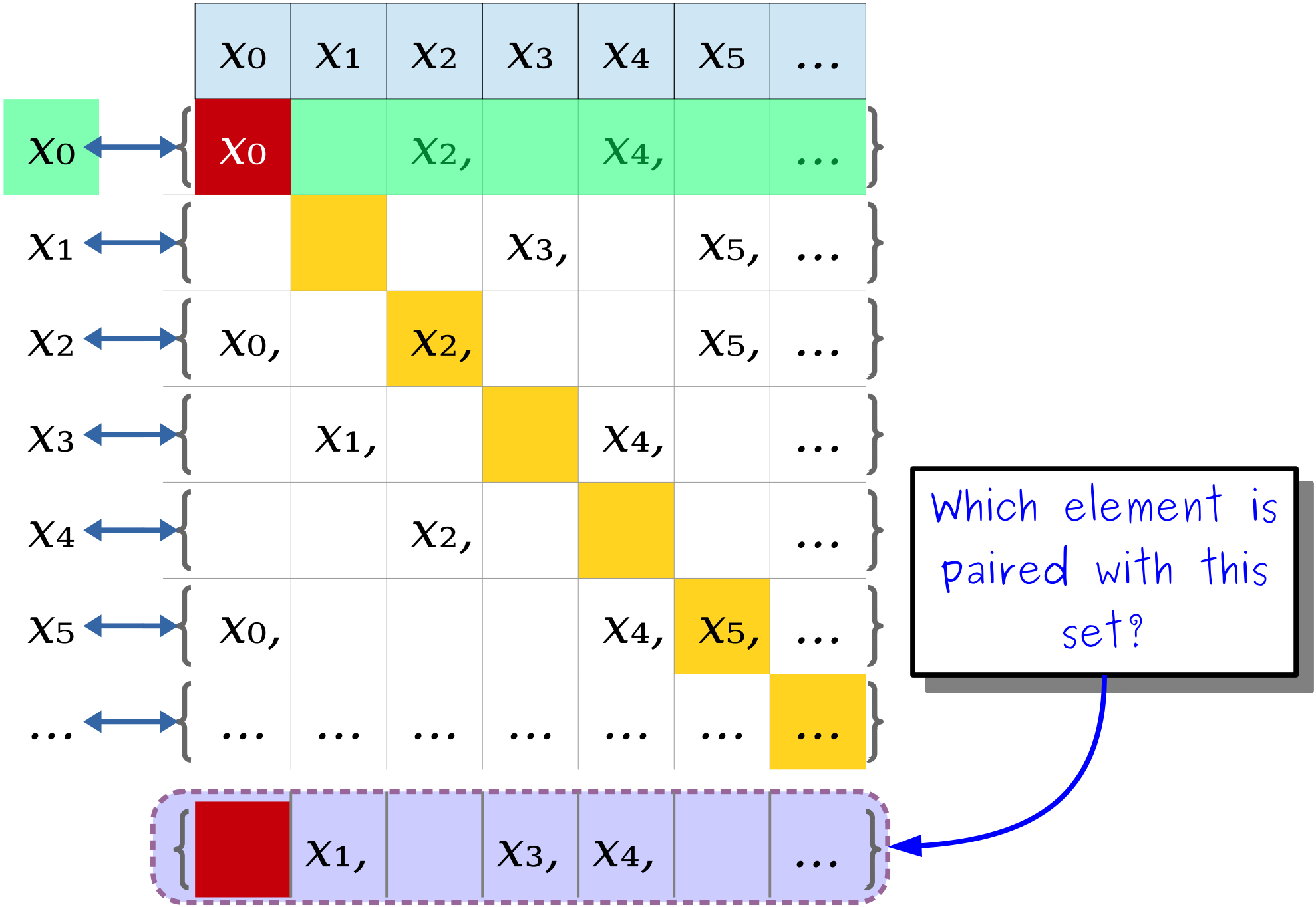

x_0 $\{x_0, x_2, x_4, \ldots\}$
x_1 $\{x_3, x_5, \ldots\}$
x_2 $\{x_0, x_1, x_2, x_5, \ldots\}$
x_3 $\{x_1, x_4, \ldots\}$
x_4 $\{x_2, \ldots\}$
x_5 $\{x_0, x_4, x_5, \ldots\}$
$\ldots$ $\{\ldots\}$

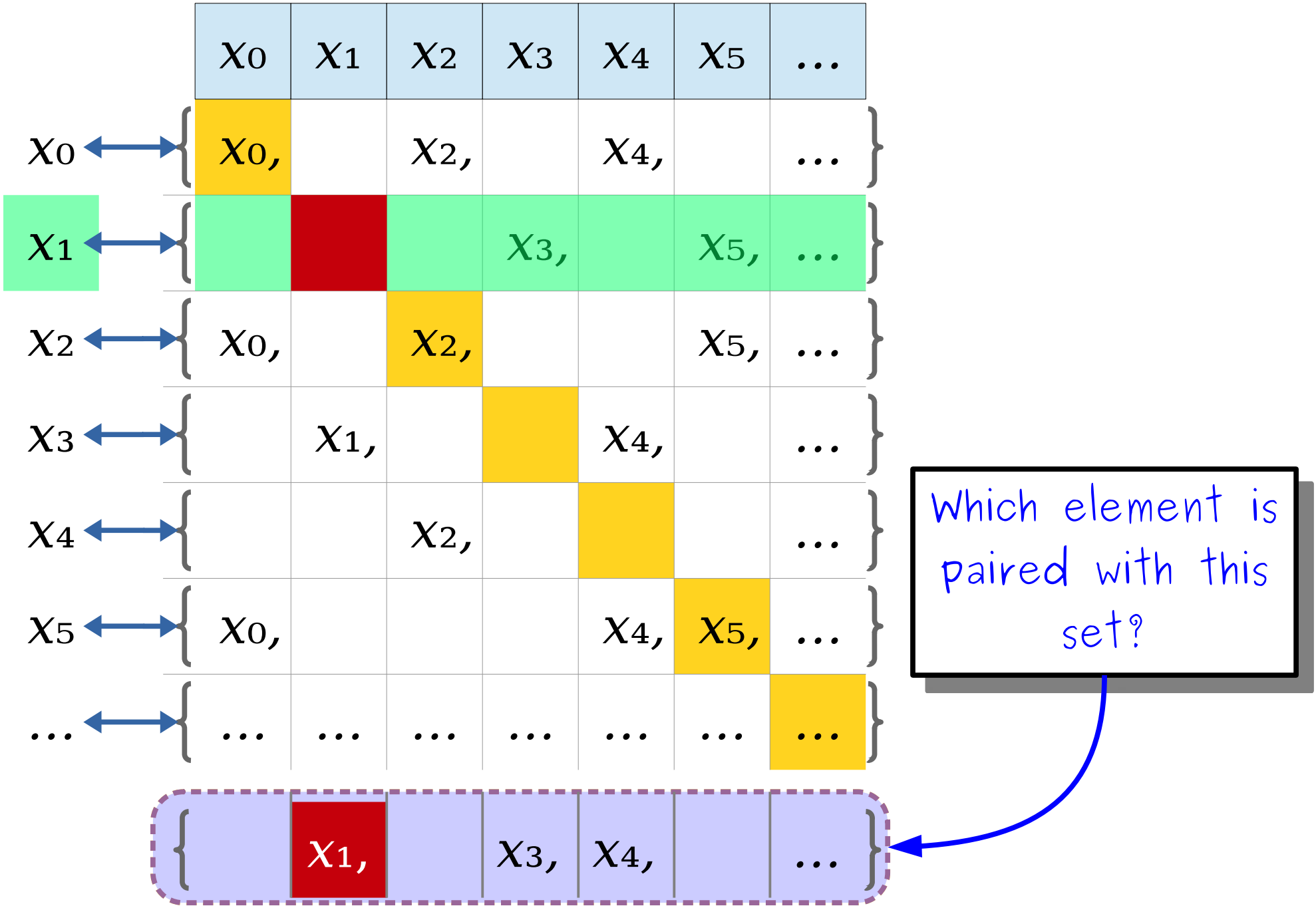

x_0 x_1 x_2 x_3 x_4 x_5 ...
x_0 { x_0, x_2, x_4, ... }
x_1 { x_3, x_5, ... }
x_2 { x_0, x_1, x_2, x_5, ... }
x_3 { x_1, x_4, ... }
x_4 { x_2, ... }
x_5 { x_0, x_4, x_5, ... }
... { ... }

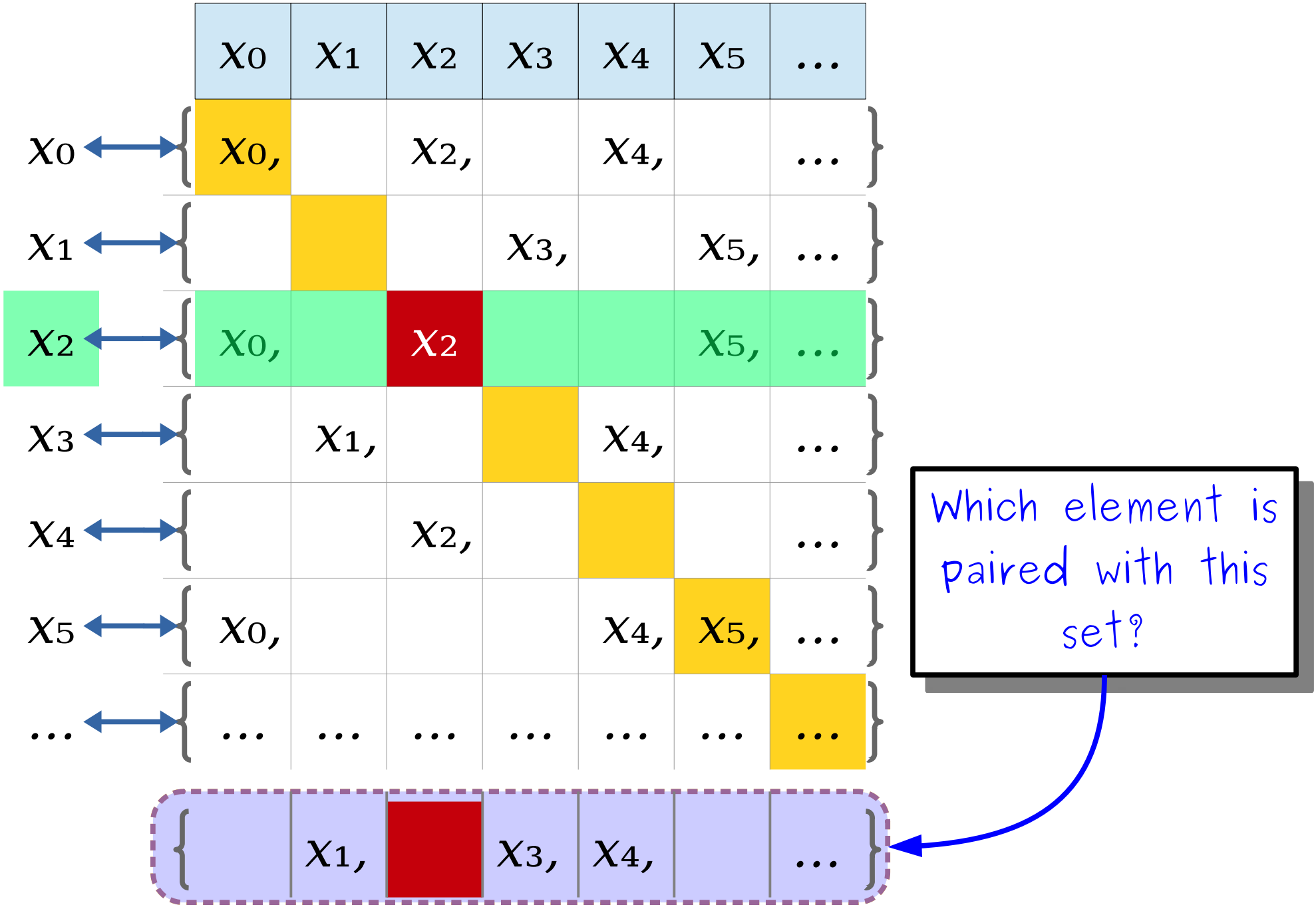

x0 x1 x2 x3 x4 x5 ...
x0 x0, x2, x4, ...
x1 x3, x5, ...
x2 x0, x2, x5, ...
x3 x1, x4, ...
x4 x2, ...
x5 x0, x4, x5, ...
... ...

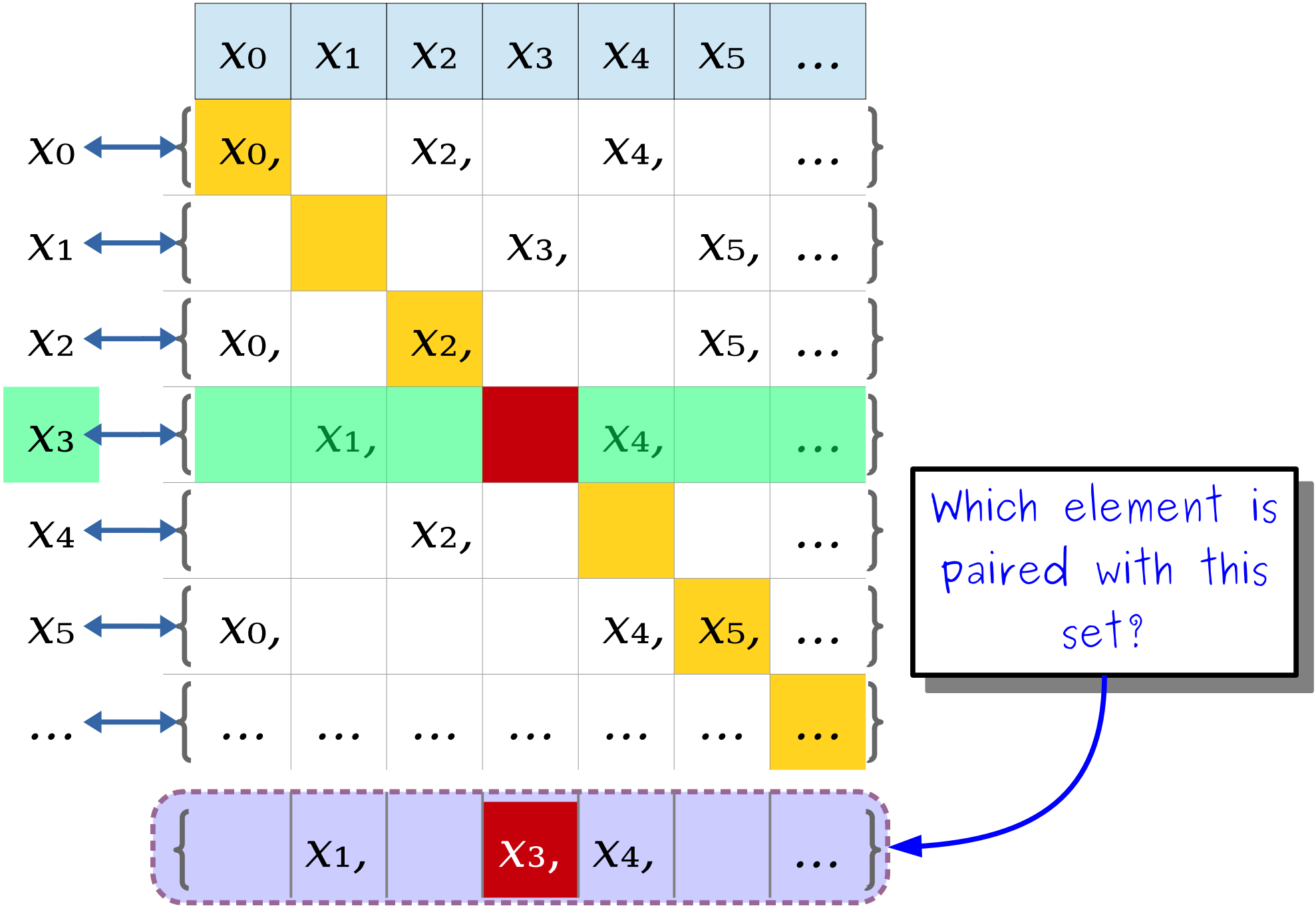

x0 x1 x2 x3 x4 x5 ...
x0 x0, x2, x4, ...
x1 x3, x5, ...
x2 x0, x2, x5, ...
x3 x1, x4, ...
x4 x2, ...
x5 x0, x4, x5, ...
... ...
x0, x2, x5, ...
Which element is paired with this set?

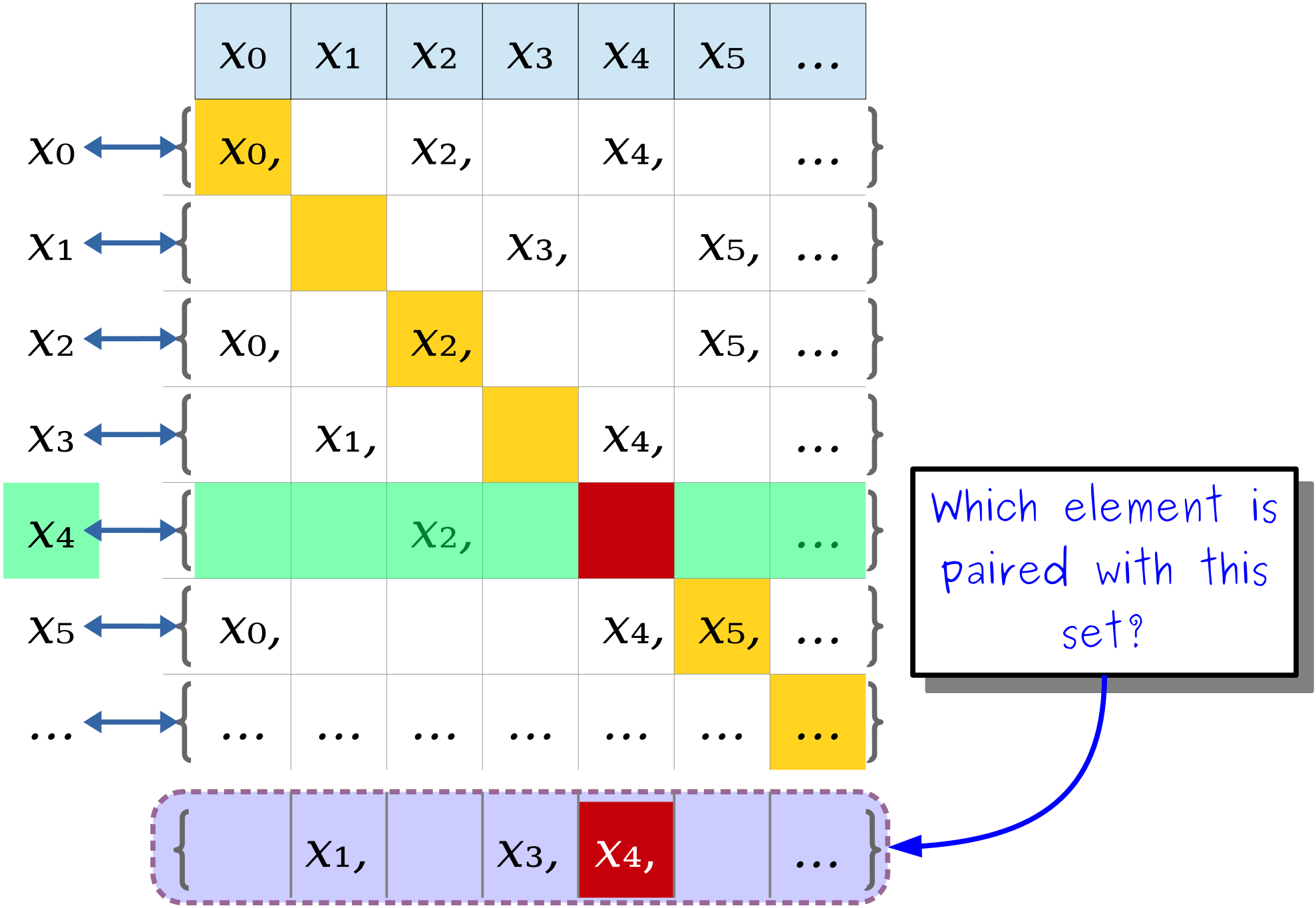

x0 x1 x2 x3 x4 x5 ...
x0 x0, x2, x4, ...
x1 x3, x5, ...
x2 x0, x2, x5, ...
x3 x1, x4, ...
x4 x2, ...
x5 x0, x4, x5, ...
... ...
x1, x3, x4, ...
"Flip" this set. Swap what's included and what's excluded.

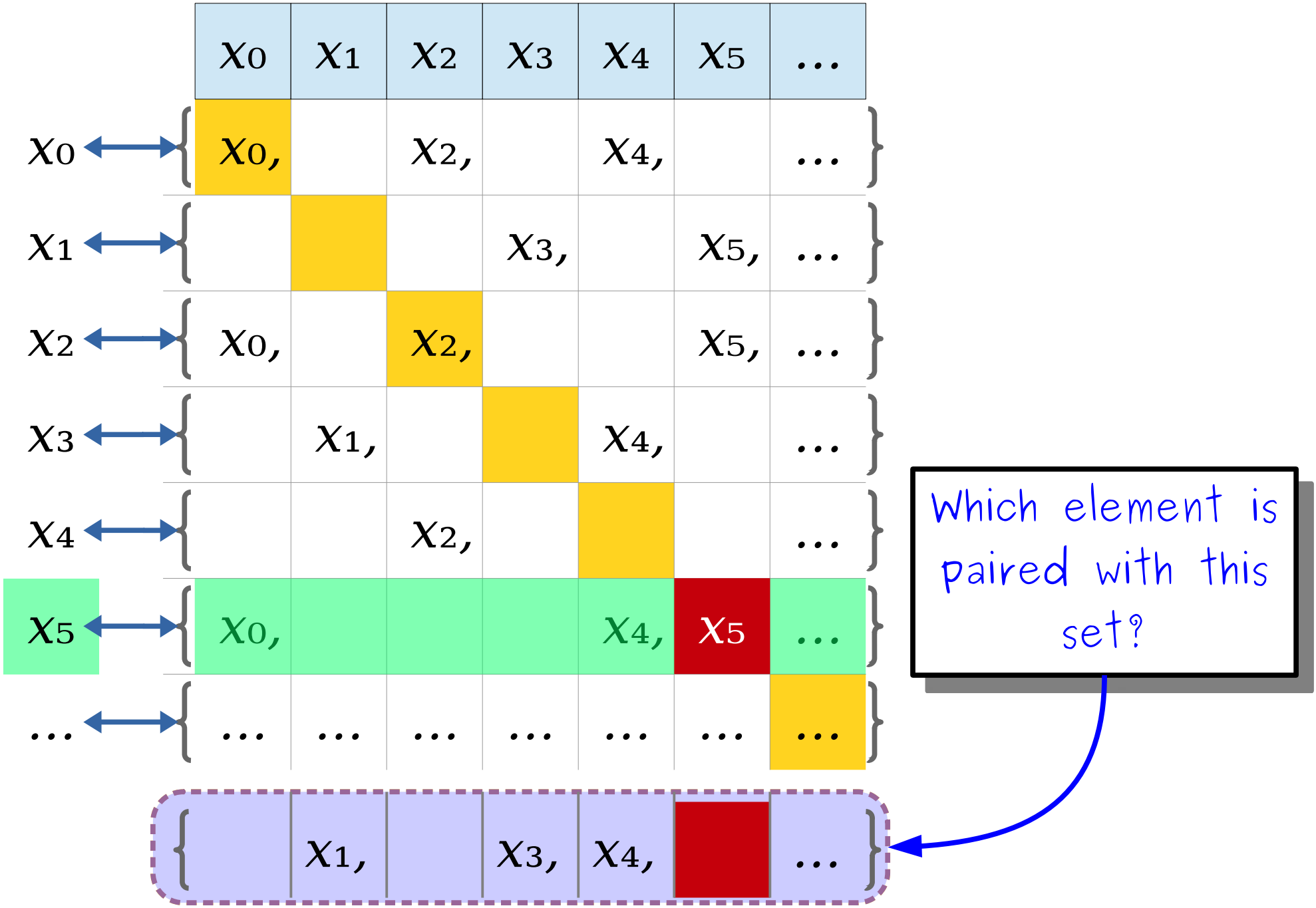

x0 x1 x2 x3 x4 x5 ...
x0
x0 x2, x4, ...
x1
x3, x5, ...
x2
x0, x2, x5, ...
x3
x1, x4, ...
x4
x2, ...
x5
x0, x4, x5, ...
...
... ...
x1, x3, x4, ...
Which element is paired with this set?

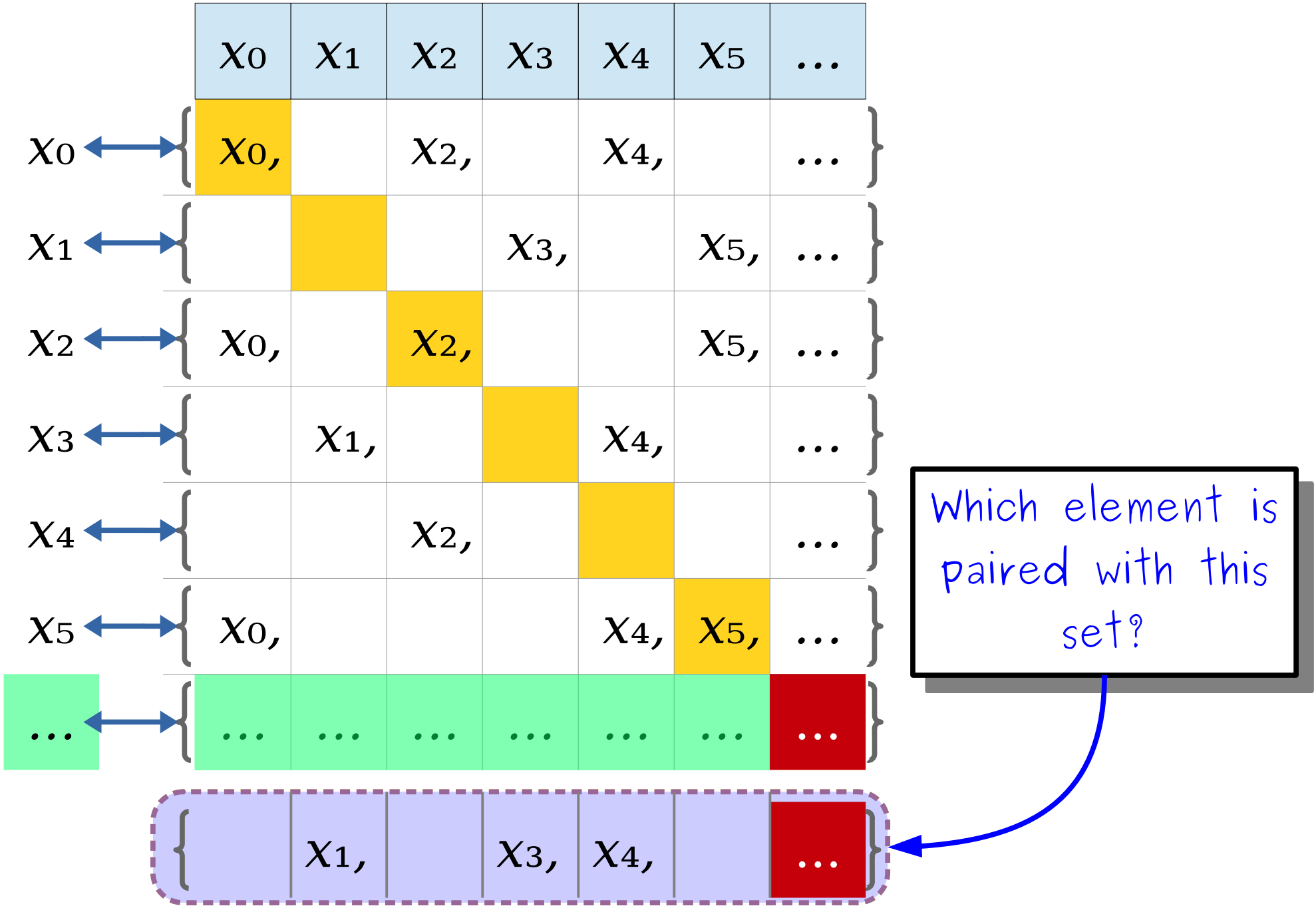

x0 x1 x2 x3 x4 x5 ...
x0 x0, x2, x4, ...
x1 x3, x5, ...
x2 x0, x2, x5, ...
x3 x1, x4, ...
x4 x2, ...
x5 x0, x4, x5, ...
... ...
x1, x3, x4, ...
Which element is paired with this set?


x0 x1 x2 x3 x4 x5 ...
x0 x0, x2, x4, ...
x1 x3, x5, ...
x2 x0, x2 x5, ...
x3 x1, x4, ...
x4 x2, ...
x5 x0, x4, x5, ...
... ...
x1, x3, x4, ...
Which element is paired with this set?


x0 x1 x2 x3 x4 x5 ...
x0 x0, x2, x4, ...
x1 x3, x5, ...
x2 x0, x2, x5, ...
x3 x1, x4, ...
x4 x2, ...
x5 x0, x4, x5, ...
... ...
x1, x3, x4, ...
Which element is paired with this set?


x0 x1 x2 x3 x4 x5 ...
x0 x0, x2, x4, ...
x1 x3, x5, ...
x2 x0, x2, x5, ...
x3 x1, x4, ...
x4 x2, ...
x5 x0, x4, x5, ...
... ...
x1, x3, x4, ...
Which element is paired with this set?


x0 x1 x2 x3 x4 x5 ...
x0 x0, x2, x4, ...
x1 x3, x5, ...
x2 x0, x2, x5, ...
x3 x1, x4, ...
x4 x2, ...
x5 x0, x4, x5 ...
... ...
x1, x3, x4, ...
Which element is paired with this set?


x0 x1 x2 x3 x4 x5 ...
x0 x0, x2, x4, ...
x1 x3, x5, ...
x2 x0, x2, x5, ...
x3 x1, x4, ...
x4 x2, ...
x5 x0, x4, x5, ...
... ...
x1, x3, x4, ...
Which element is paired with this set?

Two Infinities…

- By Cantor's Theorem:

$$|\mathbb{N}| < |\wp(\mathbb{N})|$$

…And Beyond!

- By Cantor's Theorem:

$$|\mathbb{N}| < |\wp(\mathbb{N})|$$
$$|\wp(\mathbb{N})| < |\wp(\wp(\mathbb{N}))|$$
$$|\wp(\wp(\mathbb{N}))| < |\wp(\wp(\wp(\mathbb{N})))|$$
$$|\wp(\wp(\wp(\mathbb{N})))| < |\wp(\wp(\wp(\wp(\mathbb{N}))))|$$
$$\ldots$$

- ***Not all infinite sets have the same size!***
- ***There is no biggest infinity!***
- ***There are infinitely many infinities!***

How does this have any practical consequences?

What does this have to do with computation?

“The set of all computer programs”

“The set of all problems to solve”

Every computer program is a string.

So, the number of programs is at most the number of strings.

From Cantor's Theorem, we know that there are more sets of strings than strings.

There are at least as many problems as there are sets of strings *(see appendix!)*.

$$|\text{Programs}| \leq |\text{Strings}| < |\wp(\text{Strings})| \leq |\text{Problems}|$$

Every computer program is a string.

So, the number of programs is at most the number of strings.

From Cantor's Theorem, we know that there are more sets of strings than strings.

There are at least as many problems as there are sets of strings *(see appendix!)*.

|Programs| < |Problems|

There are more problems to solve than there are programs to solve them.

|Programs| < |Problems|

It Gets Worse

- Using more advanced set theory, we can show that there are *infinitely more* problems than solutions.
- In fact, if you pick a totally random problem, the probability that you can solve it is *zero*.
- ***More troubling fact:*** We've just shown that *some* problems are impossible to solve with computers, but we don't know *which* problems those are!

We need to develop a more nuanced understanding of computation.

Where We're Going

- ***What makes a problem impossible to solve with computers?***
  - Is there a deep reason why certain problems can't be solved with computers, or is it completely arbitrary?
  - How do you know when you're looking at an impossible problem?
  - Are these real-world problems, or are they highly contrived?
- ***How do we know that we're right?***
  - How can we back up our pictures with rigorous proofs?
  - How do we build a mathematical framework for studying computation?

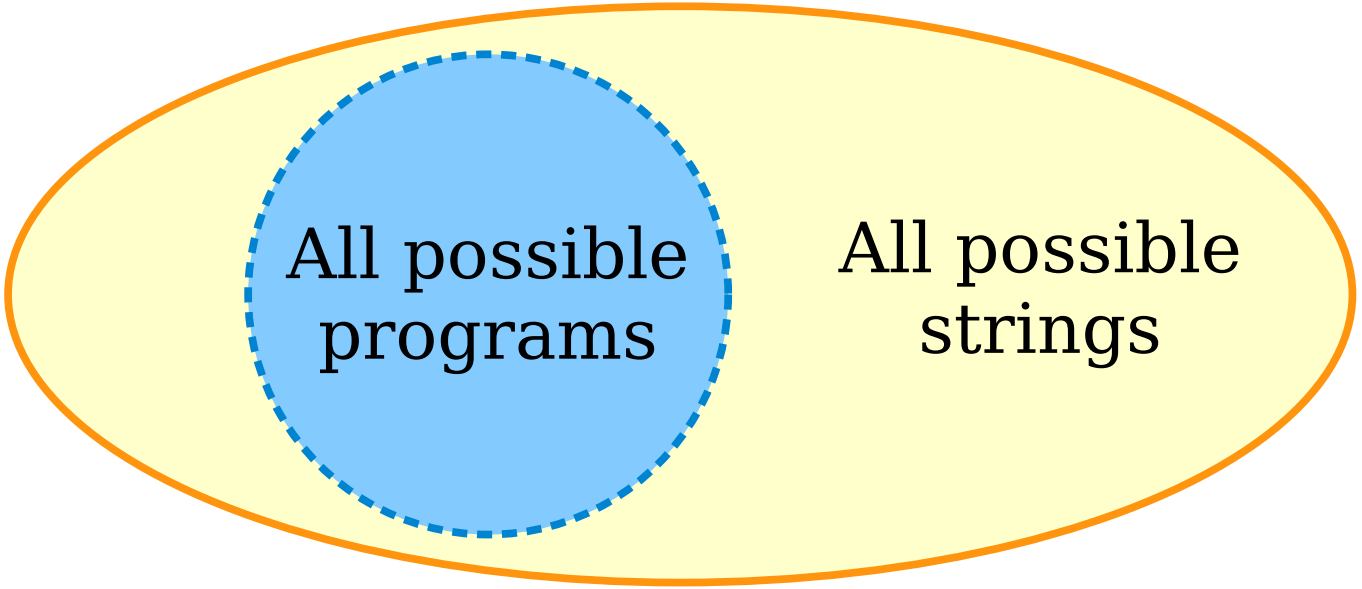
Next Time

- ***Mathematical Proof***
  - What is a mathematical proof?
  - How can we prove things with certainty?

Appendix: Stringy Thingies

Strings and Programs

- The source code of a computer program is just a (long, structured, well-commented) string of text.
- All programs are strings, but not all strings are necessarily programs.



|Programs| ≤ |Strings|

Strings and Problems

- There is a connection between the number of sets of strings and the number of problems to solve.
- Let S be any set of strings. This set S gives rise to a problem to solve:

Given a string w, determine whether $w \in S$.

Strings and Problems

Given a string *w*, determine whether *w* ∈ *S*.

- Suppose that S is the set

$$S = \{ \text{"}a\text{"}, \text{"}b\text{"}, \text{"}c\text{"}, \ldots, \text{"}z\text{"} \}$$

- From this set S, we get this problem:

Given a string *w*, determine whether *w* is a single lower-case English letter.

Strings and Problems

Given a string *w*, determine whether *w* ∈ *S*.

- Suppose that S is the set

  S = { "0", "1", "2", …, "9", "10", "11", … }

- From this set S, we get this problem:

  Given a string *w*, determine whether *w* represents a natural number.

Strings and Problems

Given a string *w*, determine whether *w* ∈ *S*.

- Suppose that S is the set

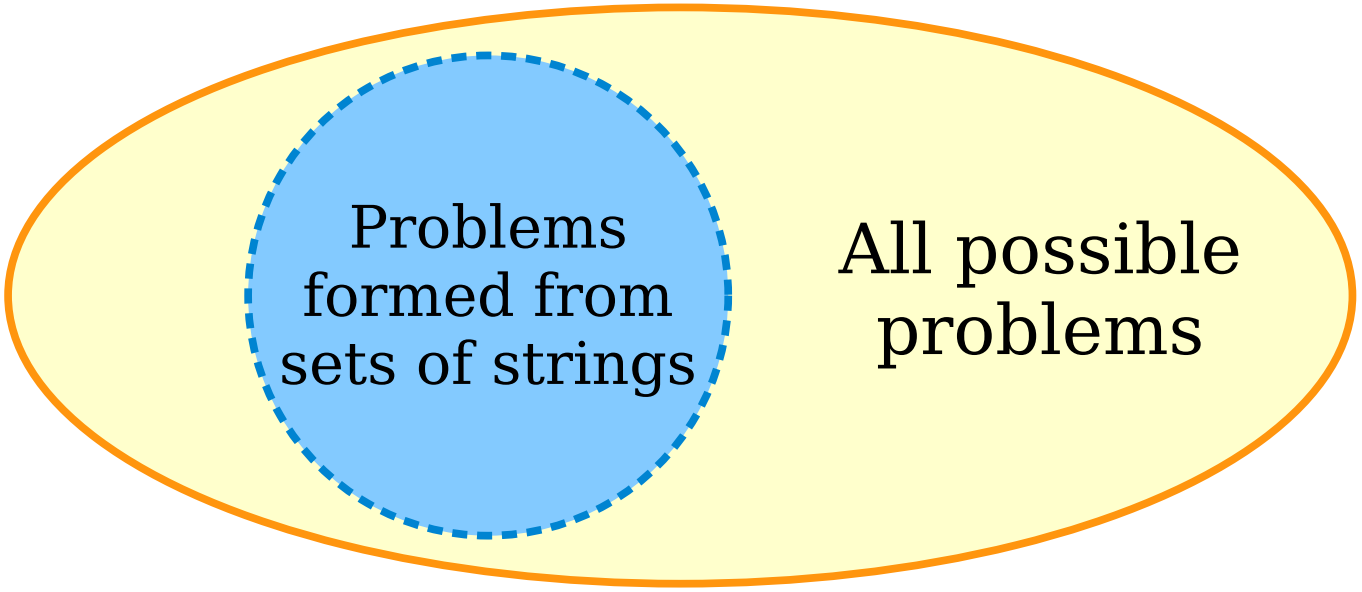
$$S = \{ p \mid p \text{ is a legal C++ program} \}$$

- From this set S, we get this problem:

Given a string *w*, determine whether *w* is a legal C++ program.

Strings and Problems

- Every set of strings gives rise to a unique problem to solve.
- Other problems exist as well.



$|\wp$(Strings)| ≤ |Problems|