ECE2810J

Data Structures and Algorithms

Minimum Spanning Tree

Learning Objectives:

- Know what a minimum spanning tree (MST) is
- Know the Prim's algorithm for finding the MST
- Know how the various choices of the supporting data structures affect the runtime of the Prim's algorithm
- Know the Kruskal's algorithm for finding the MST

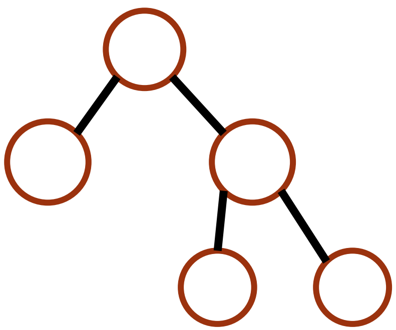
Outline

- Minimum Spanning Tree
  - Problem
  - Prim's Algorithm
  - Kruskal's Algorithm

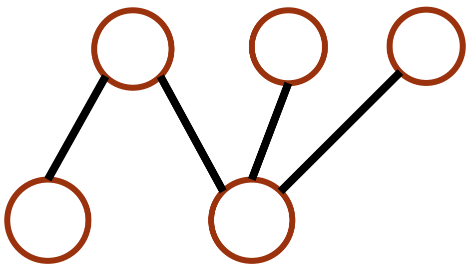
Tree and Graph

- A **tree** is an **acyclic, connected undirected** graph.

The tree we see before



However, this is also a tree

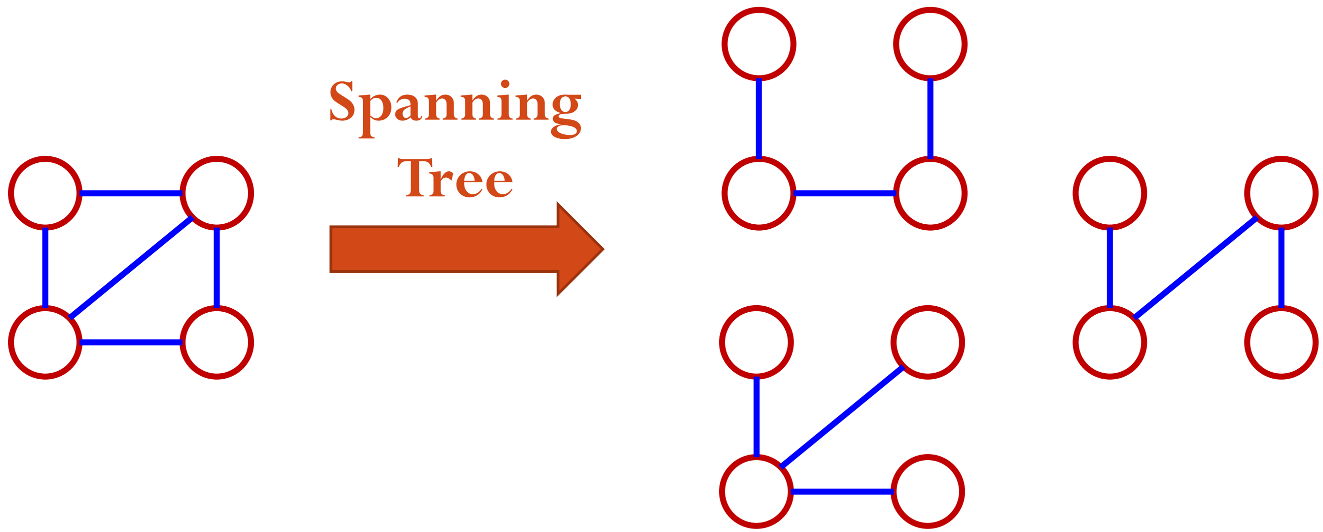


Any node can be the root of the tree.

- For a tree, $|E| = |V| - 1$.
- Claim: Any **connected** graph with N nodes and $N - 1$ edges is a tree.

Subgraph and Spanning Tree

- $G' = (V', E')$ is a **subgraph** of $G = (V, E)$ if and only if $V' \subseteq V$ and $E' \subseteq E$.
- A **spanning tree** of a **connected undirected** graph G is a subgraph of G that
  1. contains all the nodes of G;
  2. is a tree, i.e., connected and acyclic.

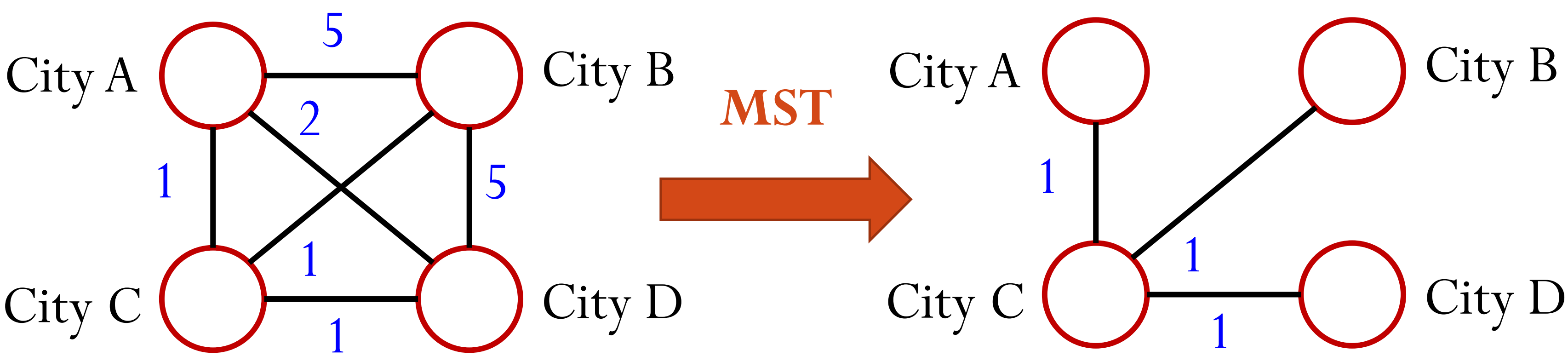


Minimum Spanning Tree (MST)

- Given a weighted, connected, undirected graph $G = (V, E)$, a **minimum spanning tree** T of G is a spanning tree of G whose sum of all edge weights is the minimal.

Application of MST

- A government planning a freeway system to connect all the cities.



- A power company planning where to lay down high-voltage power lines.

Minimum Spanning Tree

Algorithms

- Main idea: greedily select edges one by one and add to a growing sub-graph.

- Two standard algorithms:
  - Prim's algorithm
  - Kruskal's algorithm

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Prim's Algorithm

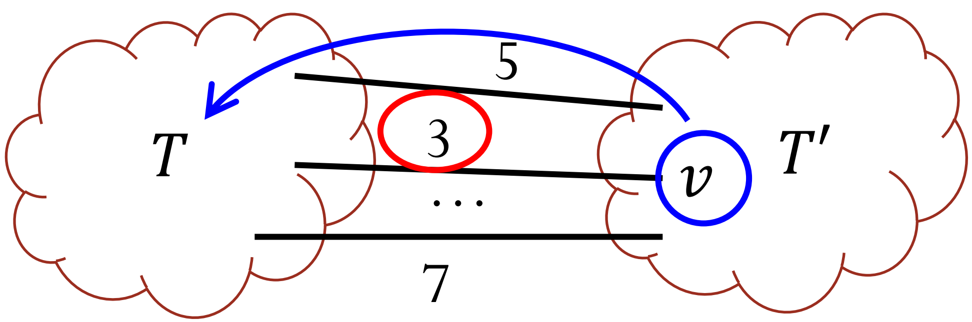
- Separate V into two sets:
  - T: the set of nodes that have been added to the MST.
  - T': those nodes that have not been added to the MST, i.e., $T' = V - T$.

- Prim's algorithm initially sets $T = \{s\}$, where s is an **arbitrarily** picked node, and $T' = V - \{s\}$. The algorithm moves one node from T' to T in each iteration. After the last iteration, $T = V$ and we have constructed the MST.

Prim's Algorithm

Basic Version

1. Arbitrarily pick one node s; set $T = \{s\}$ and $T' = V - \{s\}$.
2. While $T' \neq \emptyset$
   - Select an edge with the **smallest weight** that connects between a node in T and a node in T'. Suppose the edge connects with node v in T'. Move v from T' to T.



Selecting the Smallest Edge and Node

- For each node $v \in T'$, keep a measure $D(v)$, storing the "**current**" **smallest weight** over all edges that connect v to a node in T.
  - Will be updated later.

- To choose the edge with the smallest weight that connects between a node in T and a node in T', we pick the node $v \in T'$ with **the smallest** $D(v)$.
  - If edge (u, v) gives **the smallest** $D(v)$, then (u, v) is the edge with the smallest weight **across** set T and T'.

Updating *v*'s Neighbor

- If we move a node v from T' to T, then for each of v's neighbor u that is **still** in T', we update its $D(u)$ as follows:
  - If $D(u) > w(v, u)$, then let $D(u) = w(v, u)$.
  - I.e., update $D(u)$ if the weight of edge (v, u) is smaller than the weight of any other edge that connects a node in T to u.

Prim's Algorithm

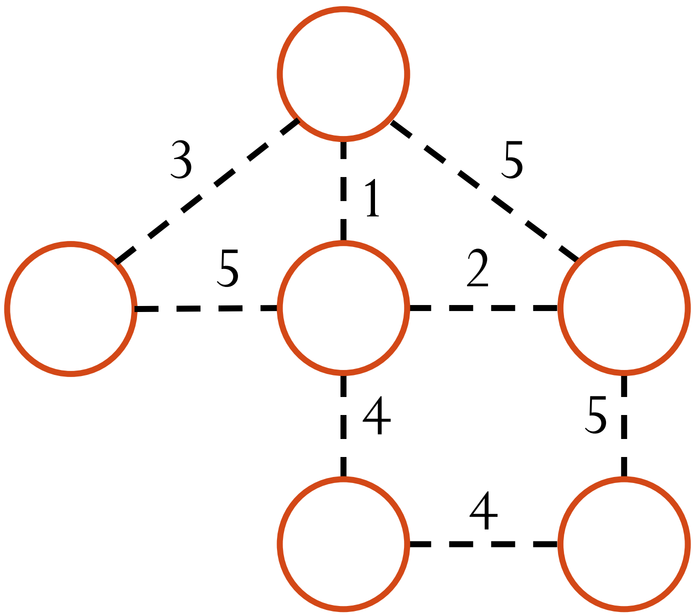
Full Version

- We keep $P(v)$ for each node v: $(P(v), v)$ is the edge chosen in the MST.

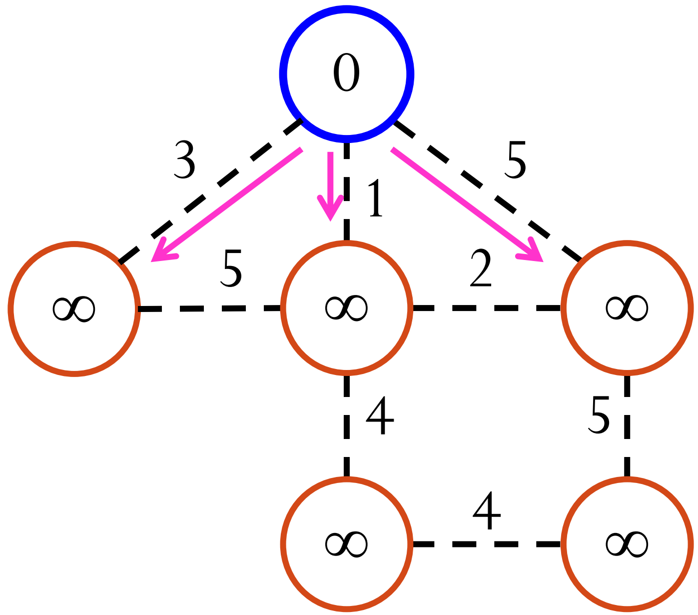
1. Arbitrarily pick one node s. Set $D(s) = 0$. For any other node v, set $D(v)$ as infinite and $P(v)$ as unknown.
2. Set $T' = V$.
3. While $T' \neq \emptyset$
    1. Choose node v in T' such that $D(v)$ is the smallest. Remove v from the set T'.
    2. For each of v's **neighbors** u that is **still** in T',
       if $D(u) > w(v, u)$, then update $D(u)$ as $w(v, u)$ and $P(u)$ as v.

Prim’s Algorithm

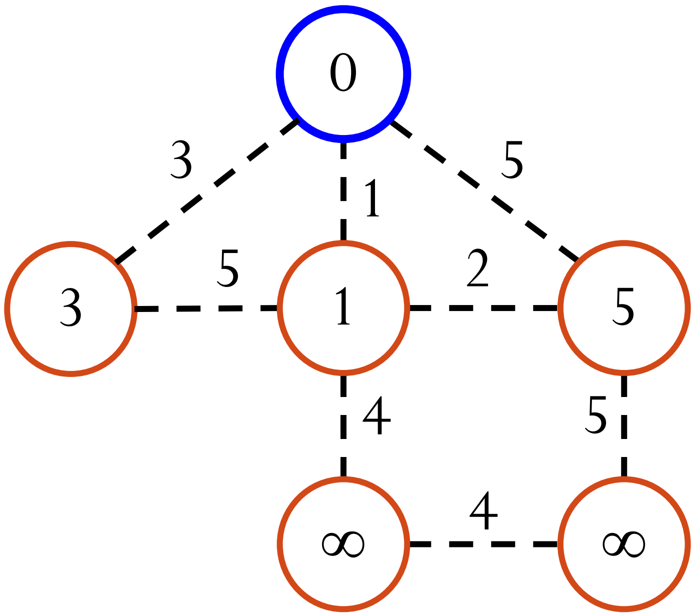
Example



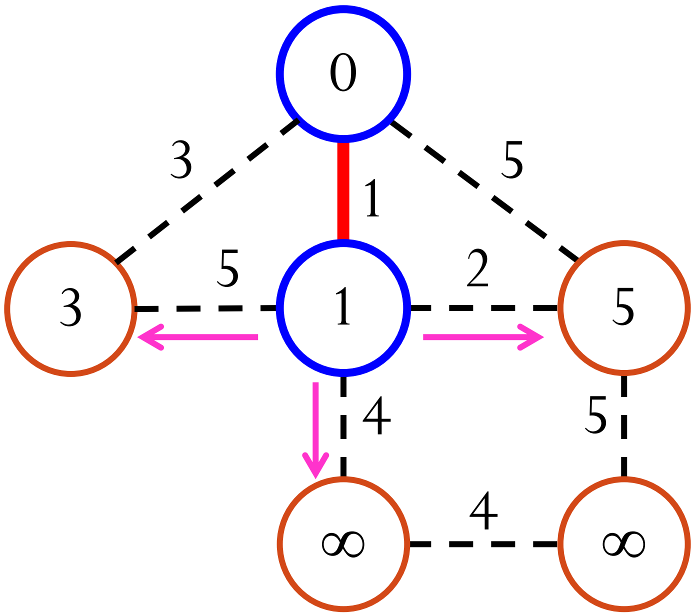
Prim's Algorithm
Example



Prim's Algorithm
Example

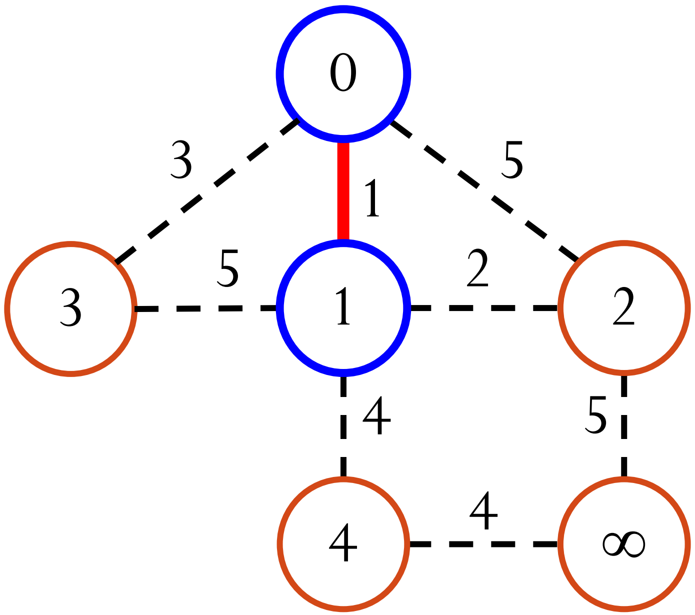


Prim’s Algorithm
Example

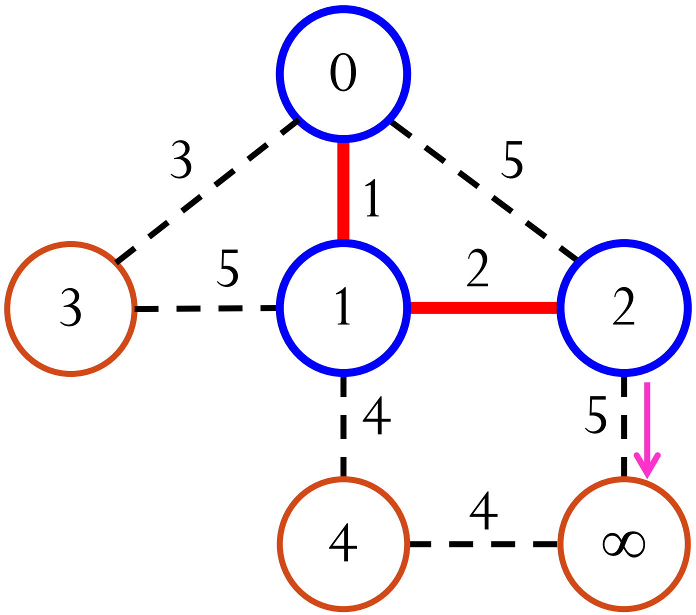


Prim’s Algorithm

Example

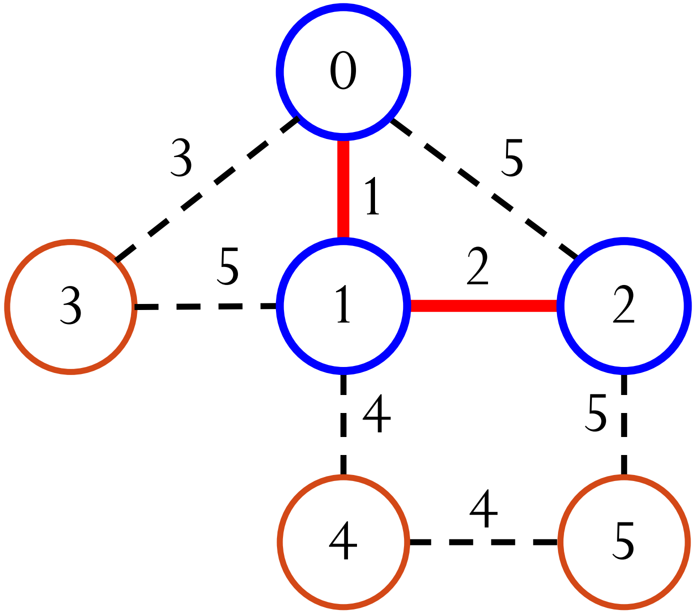


Prim's Algorithm
Example



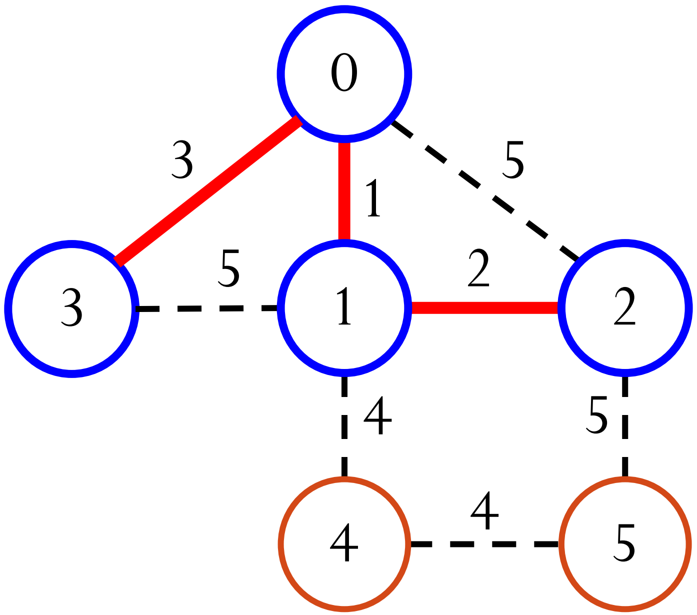
Prim's Algorithm

Example



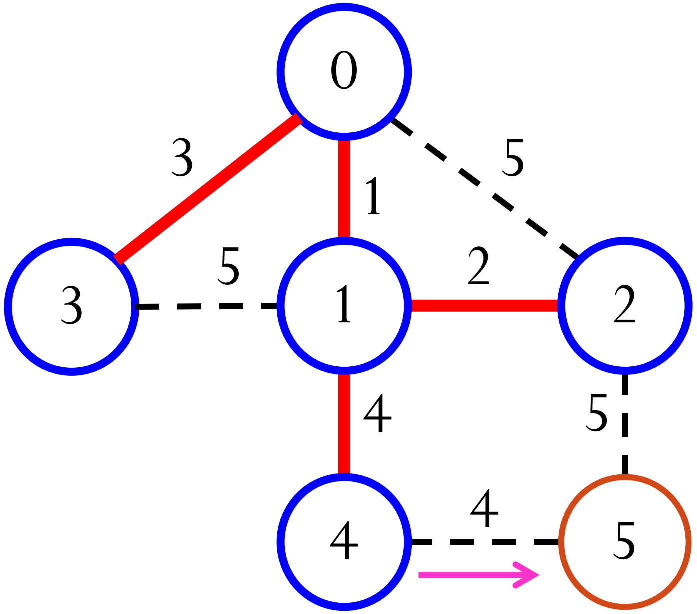
Prim’s Algorithm

Example

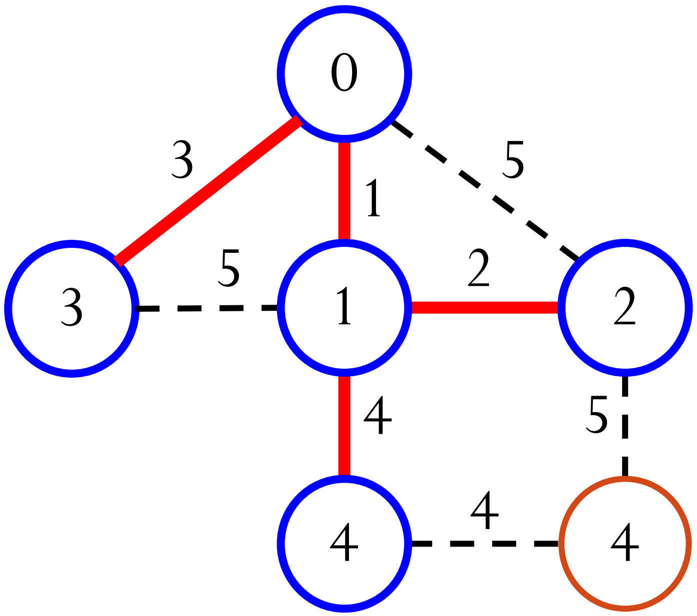


Prim's Algorithm

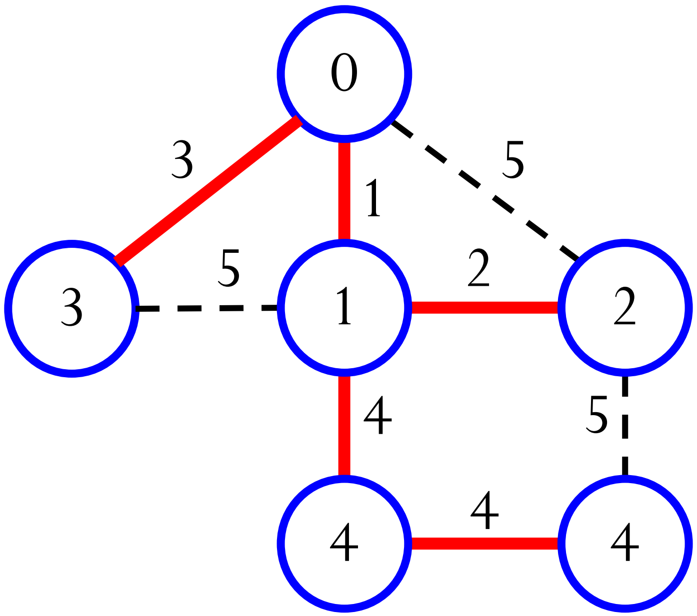
Example



Prim's Algorithm
Example

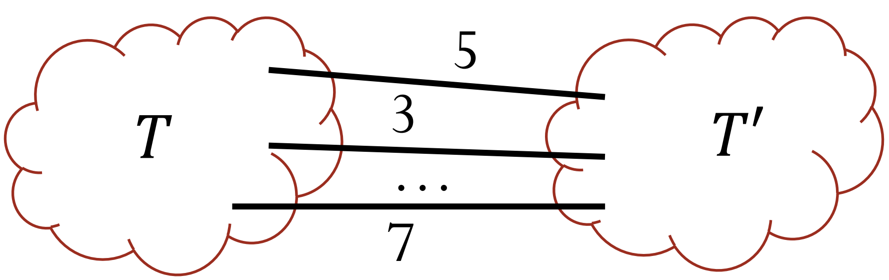


Prim's Algorithm
Example



Prim's Algorithm
Justification

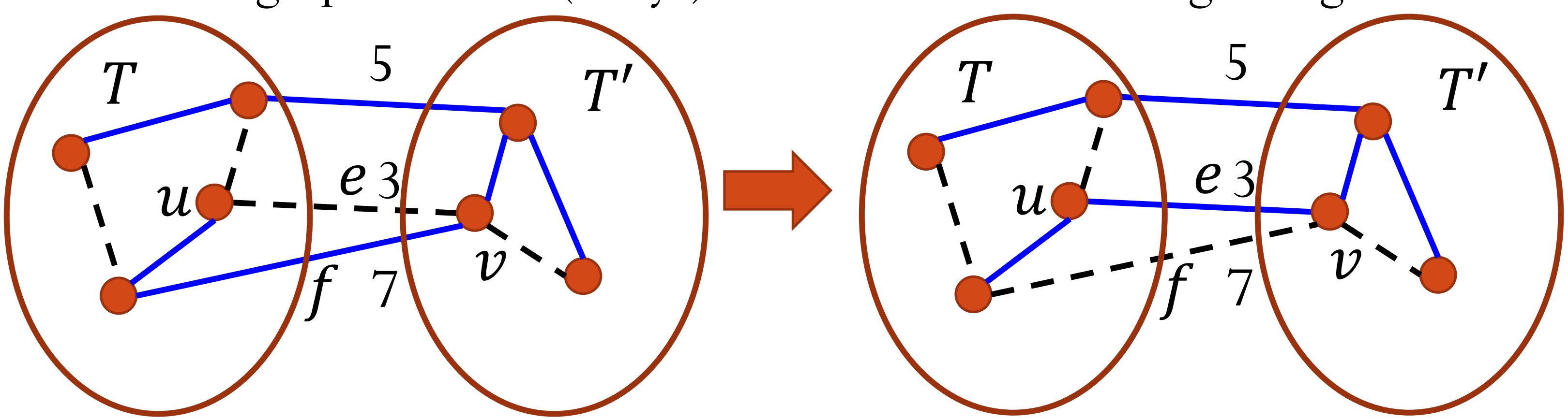
- <u>Claim</u>: the obtained subgraph is a tree
- Proof:
  - The nodes in set T are connected (can be shown by induction)
  - Furthermore, $|V| = |E| + 1$
  - <u>Claim</u>: Any **connected** graph with N nodes and $N - 1$ edges is a tree



Prim's Algorithm

Justification

- Claim: the obtained subgraph is an MST
- Proof by contradiction:
  - Assume the MST does not contain the cheapest edge e between T and T'
  - Assume $e = (u, v)$. Its weight is w
  - In the MST, there exists a unique path between u and v. On this path, there is an edge f across T and T'. Its weight $> w$
  - We replace f by e in original MST.
  - The new graph is a tree (Why?) with smaller sum of edge weights



Prim's Algorithm

Time Complexity

1. Arbitrarily pick one node s. Set $D(s) = 0$. For any other node v, set $D(v)$ as infinite and $P(v)$ as unknown.
2. Set $T' = V$.
3. While $T' \neq \emptyset$
   1. Choose node v in T' such that $D(v)$ is the smallest. Remove v from the set T'.
   2. For each of v's **neighbors** u that is **still** in T', if $D(u) > w(v, u)$, then update $D(u)$ as $w(v, u)$ and $P(u)$ as v.

What is the time complexity of Prim's algorithm?

Prim's Algorithm
Time Complexity

- Method 1: linear scan the set T' to find the smallest $D(v)$.
- Number of times to find the smallest $D(v)$: $|V|$.
  - Each cost: $O(|V|)$.

- **Maximal** number of times to update the neighbors: $|E|$.
  - Since each neighbor of each node could be **potentially** updated.
  - Each cost: $O(1)$.

- Total running time is $O(|E| + |V|^2) = O(|V|^2)$.

Prim's Algorithm
Time Complexity

- Method 2: use a binary heap to store $D(v)$'s.
- Number of times to extract the smallest $D(v)$: $|V|$.
  - Each cost: $O(\log |V|)$.

- <u>**Maximal**</u> number of times to update the neighbors: $|E|$.
  - Each cost is $O(\log |V|)$, since after updating $D(v)$, we should percolate up new $D(v)$ into right location of binary heap.

- Total running time is $O(|V| \log|V| + |E| \log|V|)$ $= O\big((|V| + |E|) \log|V|\big)$.

Prim's Algorithm

Time Complexity

- Method 1: linear scan the set T' to find the smallest $D(v)$
  - Total runtime: $O(|V|^2)$
- Method 2: use a binary heap to store $D(v)$'s
  - Total runtime: $O((|V| + |E|) \log |V|)$

- Which one is better?
  - Answer: it depends.
  - For sparse graphs, i.e., $|E| \approx \Theta(|V|)$, use binary heap. The runtime is $O(|V| \log|V|)$.
  - For dense graphs, i.e., $|E| \approx \Theta(|V|)$, use linear scan. The runtime is $O(|V|^2)$.

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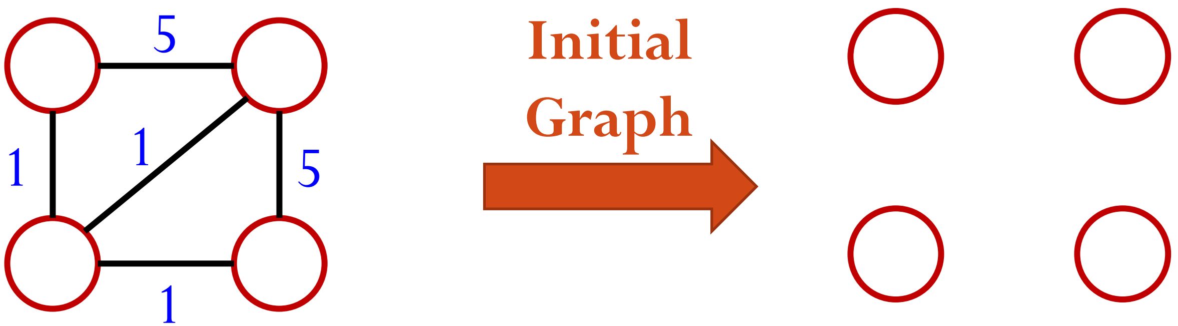
Kruskal's Algorithm

- Start with a graph containing $|V|$ nodes and no edges

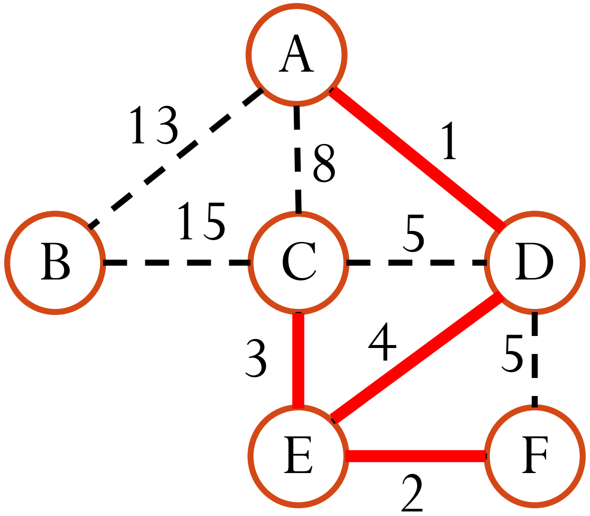


- This initial graph can be viewed as a **forest** of trees.
  - Each tree has only a single node.
- Main idea: repeatedly add the edge with the **smallest weight** that **does not cause a cycle** until no such edges exist.
  - Each added edge performs a **union** on two trees in the forest.
  - After adding $|V| - 1$ edges, there is only one tree. This tree is the MST.

Kruskal's Algorithm

Example

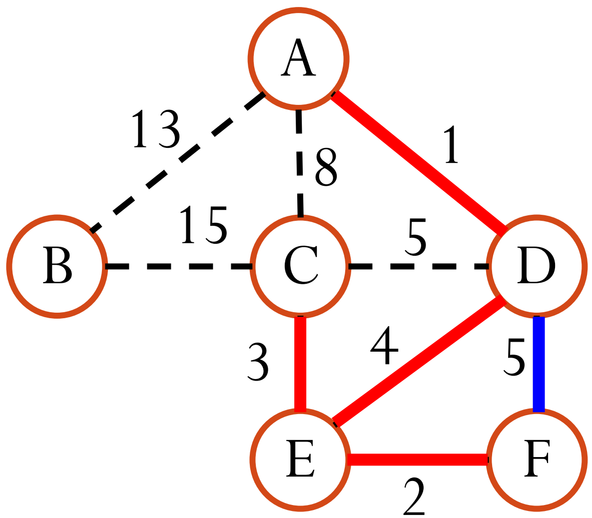
Repeatedly add the edge with the **smallest weight** that **does not cause a cycle** until no such edges exist.



Kruskal's Algorithm

Example

Repeatedly add the edge with the **smallest weight** that **<u>does not cause a cycle</u>** until no such edges exist.



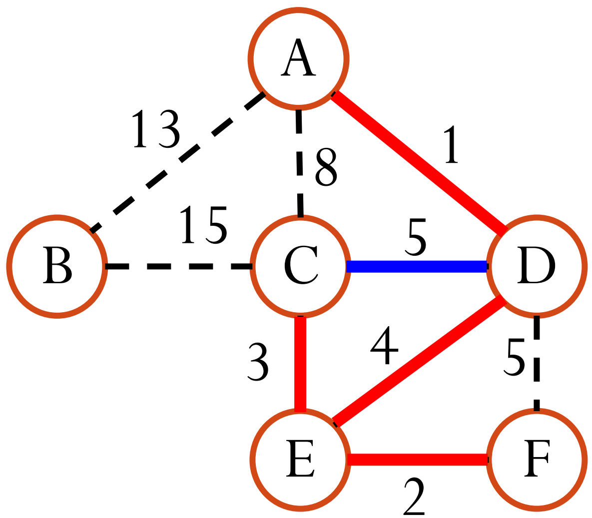
The next edge with the smallest weight is (D, F).

However, adding it causes a cycle. So it is discarded.

Kruskal's Algorithm

Example

Repeatedly add the edge with the **smallest weight** that **<u>does not cause a cycle</u>** until no such edges exist.



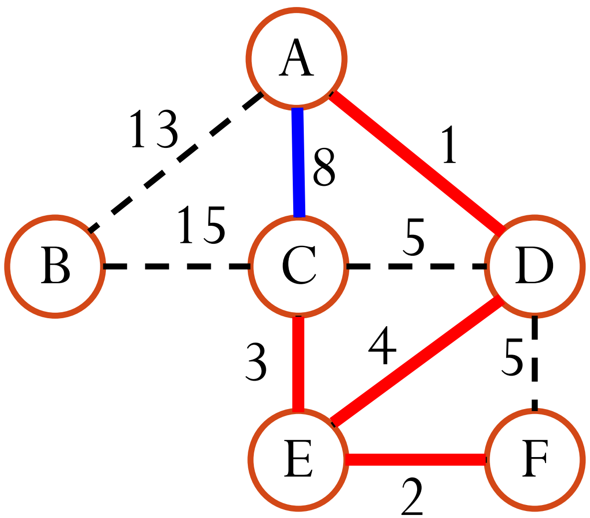
The next edge with the smallest weight is (C, D).

However, adding it causes a cycle. So it is discarded.

Kruskal's Algorithm

Example

Repeatedly add the edge with the **smallest weight** that **does not cause a cycle** until no such edges exist.



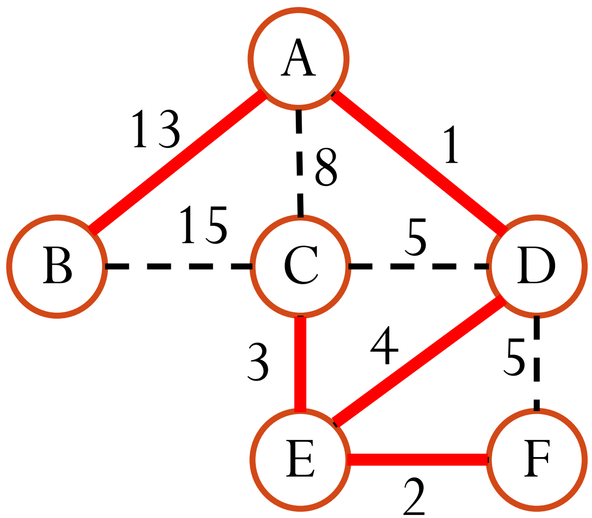
The next edge with the smallest weight is (A, C).

However, adding it causes a cycle. So it is discarded.

Kruskal's Algorithm

Example

Repeatedly add the edge with the **smallest weight** that **<u>does not cause a cycle</u>** until no such edges exist.



The next edge with the smallest weight is (A, B).

MST construction done.

Detecting Cycles

- Not simple.
- Connected nodes form a **component**.
- Detecting cycle: an edge (u, v) causes a cycle if nodes u and v are in the same component.
- If the edge does not cause a cycle, we add the edge and **make union** on the two different components connected by the edge.
  - This updates the set of components for later detecting cycle purpose.

Kruskal's Algorithm

Implementation and Time Complexity

- Sorting the edges by weights
  - Time complexity: $O(|E| \log |E|)$.
- Detecting cycle. If no cycle, add edge and merge two trees.
  - Time complexity: $O(\log |V|)$. (Not covered)
  - In the worst case, we detect cycles for all edges. The time complexity is $O(|E| \log |V|)$.

- Since $|E| = O(|V|^2)$, the total running time is $O(|E| \log |V|)$.