CHAPTER 6

Continuous-Time Models and Their Applications

The price of a financial asset evolves over time and forms a *stochastic process*, which is a statistical term used to describe the evolution of a random variable over time. The observed prices are a realization of the underlying stochastic process. The theory of stochastic process is the basis on which the observed prices are analyzed and statistical inference is made.

There are two types of stochastic process for modeling the price of an asset. The first type is called the *discrete-time stochastic process*, in which the price changes at discrete time points. All the processes discussed in the previous chapters belong to this category. For example, the daily closing price of IBM stock on the New York Stock Exchange forms a discrete-time stochastic process. Here the price changes only at the closing of a trading day. Price movements within a trading day are not necessarily relevant to the observed daily price. The second type of stochastic process is the *continuous-time process*, in which the price changes continuously, even though the price is only observed at discrete time points. One can think of the price as the "true value" of the stock that always exists and is time varying.

For both types of process, the price can be continuous or discrete. A continuous price can assume any positive real number, whereas a discrete price can only assume a countable number of possible values. Assume that the price of an asset is a continuous-time stochastic process. If the price is a continuous random variable, then we have a continuous-time continuous process. If the price itself is discrete, then we have a continuous-time discrete process. Similar classifications apply to discrete-time processes. The series of price change in Chapter 5 is an example of a discrete-time discrete process.

In this chapter, we treat the price of an asset as a continuous-time continuous stochastic process. Our goal is to introduce the statistical theory and tools needed to model financial assets and to price options. We begin the chapter with some terminologies of stock options used in the chapter. In Section 6.2, we provide a brief

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introduction of Brownian motion, which is also known as a Wiener process. We then discuss some diffusion equations and stochastic calculus, including the well-known Ito lemma. Most option pricing formulas are derived under the assumption that the price of an asset follows a diffusion equation. We use the Black–Scholes formula to demonstrate the derivation. Finally, to handle the price variations caused by rare events (e.g., a profit warning), we also study some simple diffusion models with jumps.

If the price of an asset follows a diffusion equation, then the price of an option contingent to the asset can be derived by using hedging methods. However, with jumps the market becomes incomplete and there is no perfect hedging of options. The price of an option is then valued either by using diversifiability of jump risk or defining a notion of risk and choosing a price and a hedge that minimize this risk. For basic applications of stochastic processes in derivative pricing, see Cox and Rubinstein (1985) and Hull (2007).

6.1 OPTIONS

A stock option is a financial contract that gives the holder the right to trade a certain number of shares of a specified common stock by a certain date for a specified price. There are two types of options. A *call option* gives the holder the right to buy the underlying stock; see Chapter 3 for a formal definition. A *put option* gives the holder the right to sell the underlying stock. The specified price in the contract is called the *strike price* or *exercise price*. The date in the contract is known as the *expiration date* or *maturity*. *American options* can be exercised at any time up to the expiration date. *European options* can be exercised only on the expiration date.

The value of a stock option depends on the value of the underlying stock. Let K be the strike price and P be the stock price. A call option is *in-the-money* when $P > K$, *at-the-money* when $P = K$, and *out-of-the-money* when $P < K$. A put option is in-the-money when $P < K$, at-the-money when $P = K$, and out-of-the-money when $P > K$. In general, an option is in-the-money when it would lead to a positive cash flow to the holder if it were exercised immediately. An option is out-of-the-money when it would lead to a negative cash flow to the holder if it were exercised immediately. Finally, an option is at-the-money when it would lead to zero cash flow if it were exercised immediately. Obviously, only in-the-money options are exercised in practice. For more information on options, see Hull (2007).

6.2 SOME CONTINUOUS-TIME STOCHASTIC PROCESSES

In mathematical statistics, a continuous-time continuous stochastic process is defined on a probability space $(\Omega, F, \mathbf{P})$, where Ω is a nonempty space, F is a σ field consisting of subsets of Ω, and $\mathbf{P}$ is a probability measure; see Chapter 1 of Billingsley (1986). The process can be written as $\{x(\eta, t)\}$, where t denotes time and is continuous in $[0, \infty)$. For a given t, $x(\eta, t)$ is a real-valued continuous random variable (i.e., a mapping from Ω to the real line), and η is an element of Ω. For the price of an asset at time t, the range of $x(\eta, t)$ is the set of nonnegative real numbers. For a given η, $\{x(\eta, t)\}$ is a time series with values depending on the time t. For simplicity, we write a continuous-time stochastic process as $\{x_t\}$ with the understanding that, for a given t, x_t is a random variable. In the literature, some authors use $x(t)$ instead of x_t to emphasize that t is continuous. However, we use the same notation x_t, but call it a continuous-time stochastic process.

6.2.1 Wiener Process

In a discrete-time econometric model, we assume that the shocks form a white noise process, which is not predictable. What is the counterpart of shocks in a continuous-time model? The answer is the increments of a *Wiener process*, which is also known as a *standard Brownian motion*. There are many ways to define a Wiener process $\{w_t\}$. We use a simple approach that focuses on the small change $\Delta w_t = w_{t+\Delta t} - w_t$ associated with a small increment Δt in time. A continuous-time stochastic process $\{w_t\}$ is a Wiener process if it satisfies

1. $\Delta w_t = \epsilon\sqrt{\Delta t}$, where ϵ is a standard normal random variable; and
2. Δw_t is independent of w_j for all $j \le t$.

The second condition is a Markov property saying that conditional on the present value w_t, any past information of the process, w_j with $j < t$, is irrelevant to the future $w_{t+\ell}$ with $\ell > 0$. From this property, it is easily seen that for any two nonoverlapping time intervals Δ_1 and Δ_2, the increments $w_{t_1+\Delta_1} - w_{t_1}$ and $w_{t_2+\Delta_2} - w_{t_2}$ are independent. In finance, this Markov property is related to a weak form of efficient market.

From the first condition, Δw_t is normally distributed with mean zero and variance Δt. That is, $\Delta w_t \sim N(0, \Delta t)$, where $\sim$ denotes probability distribution. Consider next the process w_t. We assume that the process starts at $t = 0$ with initial value w_0, which is fixed and often set to zero. Then $w_t - w_0$ can be treated as a sum of many small increments. More specifically, define $T = t/\Delta t$, where Δt is a small positive increment. Then

$$w_t - w_0 = w_{T\Delta t} - w_0 = \sum_{i=1}^{T} \Delta w_i = \sum_{i=1}^{T} \epsilon_i \sqrt{\Delta t},$$

where $\Delta w_i = w_{i\Delta t} - w_{(i-1)\Delta t}$. Because the ϵ_i are independent, we have

$$E(w_t - w_0) = 0, \qquad \mathrm{Var}(w_t - w_0) = \sum_{i=1}^{T} \Delta t = T\,\Delta t = t.$$

Thus, the increment in w_t from time 0 to time t is normally distributed with mean zero and variance t. To put it formally, for a Wiener process w_t, we have

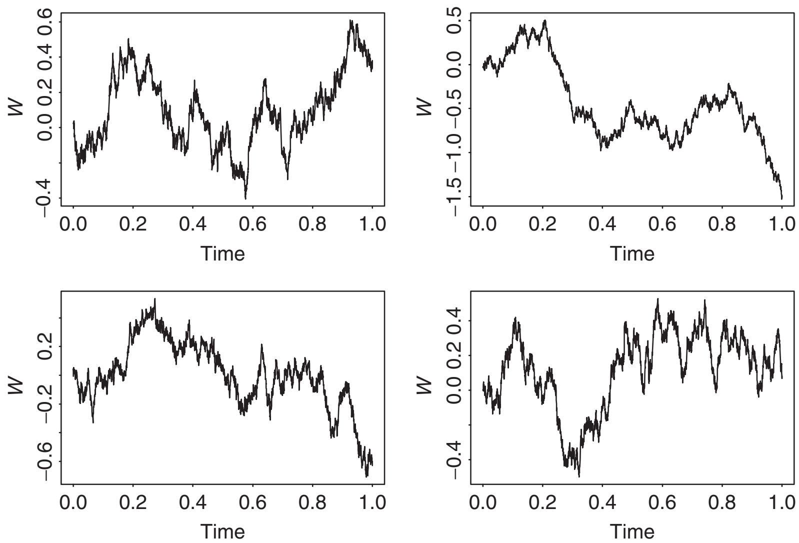

Figure 6.1 Four simulated Wiener processes.

that $w_t - w_0 \sim N(0, t)$. This says that the variance of a Wiener process increases linearly with the length of time interval.

Figure 6.1 shows four simulated Wiener processes on the unit time interval [0, 1]. They are obtained by using a simple version of Donsker's theorem in the statistical literature with $n = 3000$; see Donsker (1951) or Billingsley (1968). The four plots start with $w_0 = 0$ but drift apart as time increases, illustrating that the variance of a Wiener process increases with time. A simple time transformation from [0, 1) to $[0, \infty)$ can be used to obtain simulated Wiener processes for $t \in [0, \infty)$.

Donsker's Theorem

Assume that $\{z_i\}_{i=1}^{n}$ is a sequence of independent standard normal random variates. For any $t \in [0, 1]$, let $[nt]$ be the integer part of nt. Define $w_{n,t} = (1/\sqrt{n}) \sum_{i=1}^{[nt]} z_i$. Then $w_{n,t}$ converges in distribution to a Wiener process w_t on [0, 1] as n goes to infinity.

R or S-Plus Commands for Generating a Wiener Process

```
n = 3000
epsi = rnorm(n,0,1)
w=cumsum(epsi)/sqrt(n)
plot(w,type='l')
```

Remark. A formal definition of a Brownian motion w_t on a probability space $(\Omega, F, \mathbf{P})$ is that it is a real-valued, continuous stochastic process for $t \geq 0$ with independent and stationary increments. In other words, w_t satisfies the following:

1. Continuity: The map from t to w_t is continuous almost surely with respect to the probability measure $\mathbf{P}$.
2. Independent increments: If $s \le t$, $w_t - w_s$ is independent of w_v for all $v \le s$.
3. Stationary increments: If $s \le t$, $w_t - w_s$ and $w_{t-s} - w_0$ have the same probability distribution.

It can be shown that the probability distribution of the increment $w_t - w_s$ is normal with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$. Furthermore, for any given time indexes $0 \le t_1 < t_2 < \cdots < t_k$, the random vector $(w_{t_1}, w_{t_2}, \ldots, w_{t_k})$ follows a multivariate normal distribution. Finally, a Brownian motion is *standard* if $w_0 = 0$ almost surely, $\mu = 0$, and $\sigma^2 = 1$. □

Remark. An important property of Brownian motions is that their paths are not differentiable almost surely. In other words, for a standard Brownian motion w_t, it can be shown that dw_t/dt does not exist for all elements of Ω except for elements in a subset $\Omega_1 \subset \Omega$ such that $P(\Omega_1) = 0$. As a result, we cannot use the usual integration in calculus to handle integrals involving a standard Brownian motion when we consider the value of an asset over time. Another approach must be sought. This is the purpose of discussing Ito's calculus in the next section. □

6.2.2 Generalized Wiener Process

The Wiener process is a special stochastic process with zero drift and variance proportional to the length of the time interval. This means that the rate of change in expectation is zero and the rate of change in variance is 1. In practice, the mean and variance of a stochastic process can evolve over time in a more complicated manner. Hence, further generalization of a stochastic process is needed. To this end, we consider the *generalized Wiener process* in which the expectation has a drift rate μ and the rate of variance change is σ^2. Denote such a process by x_t and use the notation dy for a small change in the variable y. Then the model for x_t is

$$dx_t = \mu\, dt + \sigma\, dw_t, \tag{6.1}$$

where w_t is a Wiener process. If we consider a discretized version of Eq. (6.1), then

$$x_t - x_0 = \mu t + \sigma \epsilon \sqrt{t}$$

for increment from 0 to t. Consequently,

$$E(x_t - x_0) = \mu t, \qquad \mathrm{Var}(x_t - x_0) = \sigma^2 t.$$

The results say that the increment in x_t has a growth rate of μ for the expectation and a growth rate of σ^2 for the variance. In the literature, μ and σ of Eq. (6.1) are referred to as the drift and volatility parameters of the generalized Wiener process x_t.

6.2.3 Ito Process

The drift and volatility parameters of a generalized Wiener process are time invariant. If one further extends the model by allowing μ and σ to be functions of the stochastic process x_t, then we have an Ito process. Specifically, a process x_t is an Ito process if it satisfies

$$dx_t = \mu(x_t, t)\,dt + \sigma(x_t, t)\,dw_t, \tag{6.2}$$

where w_t is a Wiener process. This process plays an important role in mathematical finance and can be written as

$$x_t = x_0 + \int_0^t \mu(x_s, s)\,ds + \int_0^t \sigma(x_s, s)\,dw_s,$$

where x_0 denotes the starting value of the process at time 0 and the last term on the right-hand side is a stochastic integral. Equation (6.2) is referred to as a stochastic diffusion equation with $\mu(x_t, t)$ and $\sigma(x_t, t)$ being the drift and diffusion functions, respectively.

The Wiener process is a special Ito process because it satisfies Eq. (6.2) with $\mu(x_t, t) = 0$ and $\sigma(x_t, t) = 1$.

6.3 ITO'S LEMMA

In finance, when using continuous-time models, it is common to assume that the price of an asset is an Ito process. Therefore, to derive the price of a financial derivative, one needs to use Ito's calculus. In this section, we briefly review Ito's lemma by treating it as a natural extension of the differentiation in calculus. Ito's lemma is the basis of stochastic calculus.

6.3.1 Review of Differentiation

Let $G(x)$ be a differentiable function of x. Using the Taylor expansion, we have

$$\Delta G \equiv G(x + \Delta x) - G(x) = \frac{\partial G}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(\Delta x)^2 + \frac{1}{6}\frac{\partial^3 G}{\partial x^3}(\Delta x)^3 + \cdots.$$

Taking the limit as $\Delta x \to 0$ and ignoring the higher order terms of Δx, we have

$$dG = \frac{\partial G}{\partial x}dx.$$

When G is a function of x and y, we have

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(\Delta x)^2 + \frac{\partial^2 G}{\partial x \partial y}\Delta x\,\Delta y + \frac{1}{2}\frac{\partial^2 G}{\partial y^2}(\Delta y)^2 + \cdots.$$

Taking the limit as $\Delta x \to 0$ and $\Delta y \to 0$, we have

$$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy.$$

6.3.2 Stochastic Differentiation

Turn next to the case in which G is a differentiable function of x_t and t, and x_t is an Ito process. The Taylor expansion becomes

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(\Delta x)^2 + \frac{\partial^2 G}{\partial x\ \partial t}\Delta x\ \Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}(\Delta t)^2 + \cdots. \tag{6.3}$$

A discretized version of the Ito process is

$$\Delta x = \mu\ \Delta t + \sigma\epsilon\sqrt{\Delta t}, \tag{6.4}$$

where, for simplicity, we omit the arguments of μ and σ, and $\Delta x = x_{t+\Delta t} - x_t$. From Eq. (6.4), we have

$$(\Delta x)^2 = \mu^2(\Delta t)^2 + \sigma^2\epsilon^2\ \Delta t + 2\mu\sigma\epsilon(\Delta t)^{3/2} = \sigma^2\epsilon^2\ \Delta t + H(\Delta t), \tag{6.5}$$

where $H(\Delta t)$ denotes higher order terms of Δt. This result shows that $(\Delta x)^2$ contains a term of order Δt, which cannot be ignored when we take the limit as $\Delta t \to 0$. However, the first term on the right-hand side of Eq. (6.5) has some nice properties:

$$E(\sigma^2\epsilon^2\ \Delta t) = \sigma^2\ \Delta t,$$
$$\mathrm{Var}(\sigma^2\epsilon^2\ \Delta t) = E[\sigma^4\epsilon^4(\Delta t)^2] - [E(\sigma^2\epsilon^2\ \Delta t)]^2 = 2\sigma^4(\Delta t)^2,$$

where we use $E(\epsilon^4) = 3$ for a standard normal random variable. These two properties show that $\sigma^2\epsilon^2\ \Delta t$ converges to a nonstochastic quantity $\sigma^2\ \Delta t$ as $\Delta t \to 0$. Consequently, from Eq. (6.5), we have

$$(\Delta x)^2 \to \sigma^2\, dt \quad \text{as} \quad \Delta t \to 0.$$

Plugging the prior result into Eq. (6.3) and using Ito's equation of x_t in Eq. (6.2), we obtain

$$\begin{aligned} dG &= \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\sigma^2\, dt \\ &= \left(\frac{\partial G}{\partial x}\mu + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\sigma^2\right) dt + \frac{\partial G}{\partial x}\sigma\, dw_t, \end{aligned}$$

which is the well-known Ito lemma in stochastic calculus.

Recall that we suppressed the argument (x_t, t) from the drift and volatility terms μ and σ in the derivation of Ito's lemma. To avoid any possible confusion in the future, we restate the lemma as follows.

Ito's Lemma
Assume that x_t is a continuous-time stochastic process satisfying

$$dx_t = \mu(x_t, t)\,dt + \sigma(x_t, t)\,dw_t,$$

where w_t is a Wiener process. Furthermore, $G(x_t, t)$ is a differentiable function of x_t and t. Then,

$$dG = \left[\frac{\partial G}{\partial x}\mu(x_t, t) + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\sigma^2(x_t, t)\right] dt + \frac{\partial G}{\partial x}\sigma(x_t, t)\,dw_t. \tag{6.6}$$

Example 6.1. As a simple illustration, consider the square function $G(w_t, t) = w_t^2$ of the Wiener process. Here we have $\mu(w_t, t) = 0$, $\sigma(w_t, t) = 1$, and

$$\frac{\partial G}{\partial w_t} = 2w_t, \qquad \frac{\partial G}{\partial t} = 0, \qquad \frac{\partial^2 G}{\partial w_t^2} = 2.$$

Therefore,

$$dw_t^2 = (2w_t \times 0 + 0 + \tfrac{1}{2} \times 2 \times 1)\,dt + 2w_t\,dw_t = dt + 2w_t\,dw_t. \tag{6.7}$$

6.3.3 An Application

Let P_t be the price of a stock at time t, which is continuous in $[0, \infty)$. In the literature, it is common to assume that P_t follows the special Ito process

$$dP_t = \mu P_t\,dt + \sigma P_t\,dw_t, \tag{6.8}$$

where μ and σ are constant. Using the notation of the general Ito process in Eq. (6.2), we have $\mu(x_t, t) = \mu x_t$ and $\sigma(x_t, t) = \sigma x_t$, where $x_t = P_t$. Such a special process is referred to as a *geometric Brownian motion*. We now apply Ito's lemma to obtain a continuous-time model for the logarithm of the stock price P_t. Let $G(P_t, t) = \ln(P_t)$ be the log price of the underlying stock. Then we have

$$\frac{\partial G}{\partial P_t} = \frac{1}{P_t}, \qquad \frac{\partial G}{\partial t} = 0, \qquad \frac{1}{2}\frac{\partial^2 G}{\partial P_t^2} = \frac{1}{2}\frac{-1}{P_t^2}.$$

Consequently, via Ito's lemma, we obtain

$$\begin{aligned} d\,\ln(P_t) &= \left(\frac{1}{P_t}\mu P_t + \frac{1}{2}\frac{-1}{P_t^2}\sigma^2 P_t^2\right) dt + \frac{1}{P_t}\sigma P_t\,dw_t \\ &= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\,dw_t. \end{aligned}$$

This result shows that the logarithm of a price follows a generalized Wiener process with drift rate $\mu - \sigma^2/2$ and variance rate σ^2 if the price is a geometric Brownian motion. Consequently, the change in logarithm of price (i.e., log return) between current time t and some future time T is normally distributed with mean $(\mu - \sigma^2/2)(T-t)$ and variance $\sigma^2(T-t)$. If the time interval $T - t = \Delta$ is fixed and we are interested in equally spaced increments in log price, then the increment series is a Gaussian process with mean $(\mu - \sigma^2/2)\,\Delta$ and variance $\sigma^2\,\Delta$.

6.3.4 Estimation of μ and σ

The two unknown parameters μ and σ of the geometric Brownian motion in Eq. (6.8) can be estimated empirically. Assume that we have $n+1$ observations of stock price P_t at equally spaced time interval Δ (e.g., daily, weekly, or monthly). We measure Δ in years. Denote the observed prices as $\{P_0, P_1, \ldots, P_n\}$ and let $r_t = \ln(P_t) - \ln(P_{t-1})$ for $t = 1, \ldots, n$.

Since $P_t = P_{t-1}\exp(r_t)$, r_t is the continuously compounded return in the tth time interval. Using the result of the previous section and assuming that the stock price P_t follows a geometric Brownian motion, we obtain that r_t is normally distributed with mean $(\mu - \sigma^2/2)\,\Delta$ and variance $\sigma^2\,\Delta$. In addition, the r_t are not serially correlated.

For simplicity, define $\mu_r = E(r_t) = (\mu - \sigma^2/2)\,\Delta$ and $\sigma_r^2 = \mathrm{var}(r_t) = \sigma^2\,\Delta$. Let $\bar{r}$ and s_r be the sample mean and standard deviation of the data—that is,

$$\bar{r} = \frac{\sum_{t=1}^{n} r_t}{n}, \qquad s_r = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}(r_t - \bar{r})^2}.$$

As mentioned in Chapter 1, $\bar{r}$ and s_r are consistent estimates of the mean and standard deviation of r_i, respectively. That is, $\bar{r} \to \mu_r$ and $s_r \to \sigma_r$ as $n \to \infty$. Therefore, we may estimate σ by

$$\hat{\sigma} = \frac{s_r}{\sqrt{\Delta}}.$$

Furthermore, it can be shown that the standard error of this estimate is approximately $\hat{\sigma}/\sqrt{2n}$. From $\hat{\mu}_r = \bar{r}$, we can estimate μ by

$$\hat{\mu} = \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = \frac{\bar{r}}{\Delta} + \frac{s_r^2}{2\Delta}.$$

When the series r_t is serially correlated or when the price of the asset does not follow the geometric Brownian motion in Eq. (6.8), then other estimation methods must be used to estimate the drift and volatility parameters of the diffusion equation. We return to this issue later.

Example 6.2. Consider the daily log returns of IBM stock in 1998. Figure 6.2(a) shows the time plot of the data, which have 252 observations. Figure 6.2(b) shows the sample autocorrelations of the series. It is seen that

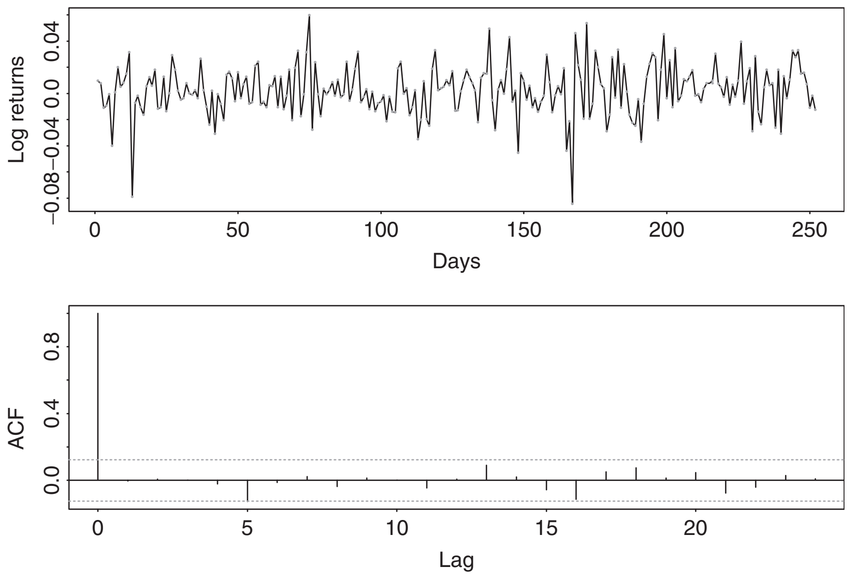

Figure 6.2 Daily returns of IBM stock in 1998: (a) log returns and (b) sample autocorrelations.

the log returns are indeed serially uncorrelated. The Ljung–Box statistic gives $Q(10) = 4.9$, which is highly insignificant compared with a chi-squared distribution with 10 degrees of freedom.

If we assume that the price of IBM stock in 1998 follows the geometric Brownian motion in Eq. (6.8), then we can use the daily log returns to estimate the parameters μ and σ. From the data, we have $\bar{r} = 0.002276$ and $s_r = 0.01915$. Since 1 trading day is equivalent to $\Delta = 1/252$ year, we obtain that

$$\hat{\sigma} = \frac{s_r}{\sqrt{\Delta}} = 0.3040, \qquad \hat{\mu} = \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = 0.6198.$$

Thus, the estimated expected return was 61.98% and the standard deviation was 30.4% per annum for IBM stock in 1998.

The normality assumption of the daily log returns may not hold, however. In this particular instance, the skewness −0.464(0.153) and excess kurtosis 2.396(0.306) raise some concern, where the number in parentheses denotes asymptotic standard error.

Example 6.3. Consider the daily log return of the stock of Cisco Systems, Inc. in 2007. There are 251 observations, and the sample mean and standard deviation are -3.81×10^{-5} and 0.0174, respectively. The log return series also shows no serial correlation with $Q(12) = 12.30$ with a p value of 0.42. Therefore, we have

$$\hat{\sigma} = \frac{s_r}{\sqrt{\Delta}} = \frac{0.0174}{\sqrt{1.0/251.0}} = 0.275, \qquad \hat{\mu} = \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = -0.0094.$$

Consequently, the estimated expected log return for Cisco Systems' stock was -0.94% per annum, and the estimated standard deviation was 27.5% per annum in 2007.

6.4 DISTRIBUTIONS OF STOCK PRICES AND LOG RETURNS

The result of the previous section shows that if one assumes that price of a stock follows the geometric Brownian motion

$$dP_t = \mu P_t \, dt + \sigma P_t \, dw_t,$$

then the logarithm of the price follows a generalized Wiener process

$$d \ln(P_t) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma \, dw_t,$$

where P_t is the price of the stock at time t and w_t is a Wiener process. Therefore, the change in log price from time t to T is normally distributed as

$$\ln(P_T) - \ln(P_t) \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right]. \tag{6.9}$$

Consequently, conditional on the price P_t at time t, the log price at time $T > t$ is normally distributed as

$$\ln(P_T) \sim N\left[\ln(P_t) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right]. \tag{6.10}$$

Using the result of lognormal distribution discussed in Chapter 1, we obtain the (conditional) mean and variance of P_T as

$$\begin{aligned} E(P_T) &= P_t \exp[\mu(T-t)], \\ \mathrm{Var}(P_T) &= P_t^2 \exp[2\mu(T-t)]\{\exp[\sigma^2(T-t)] - 1\}. \end{aligned}$$

Note that the expectation confirms that μ is the expected rate of return of the stock.

The prior distribution of stock price can be used to make inferences. For example, suppose that the current price of stock A is \$50, the expected return of the stock is 15% per annum, and the volatility is 40% per annum. Then the expected price of stock A in 6 months (0.5 year) and the associated variance are given by

$$\begin{aligned} E(P_T) &= 50 \exp(0.15 \times 0.5) = 53.89, \\ \mathrm{Var}(P_T) &= 2500 \exp(0.3 \times 0.5)[\exp(0.16 \times 0.5) - 1] = 241.92. \end{aligned}$$

The standard deviation of the price 6 months from now is $\sqrt{241.92} = 15.55$.

Next, let r be the continuously compounded rate of return per annum from time t to T. Then we have

$$P_T = P_t \exp[r(T-t)],$$

where T and t are measured in years. Therefore,

$$r = \frac{1}{T-t} \ln\left(\frac{P_T}{P_t}\right).$$

By Eq. (6.9), we have

$$\ln\left(\frac{P_T}{P_t}\right) \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right].$$

Consequently, the distribution of the continuously compounded rate of return per annum is

$$r \sim N\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T-t}\right).$$

The continuously compounded rate of return is, therefore, normally distributed with mean $\mu - \sigma^2/2$ and standard deviation $\sigma/\sqrt{T-t}$.

Consider a stock with an expected rate of return of 15% per annum and a volatility of 10% per annum. The distribution of the continuously compounded rate of return of the stock over 2 years is normal with mean $0.15 - 0.01/2 = 0.145$ or 14.5% per annum and standard deviation $0.1/\sqrt{2} = 0.071$ or 7.1% per annum. These results allow us to construct confidence intervals (CI) for r. For instance, a 95% CI for r is $0.145 \pm 1.96 \times 0.071$ per annum (i.e., 0.6%, 28.4%).

6.5 DERIVATION OF BLACK–SCHOLES DIFFERENTIAL EQUATION

In this section, we use Ito's lemma and assume no arbitrage to derive the Black–Scholes differential equation for the price of a derivative contingent to a stock valued at P_t. Assume that the price P_t follows the geometric Brownian motion in Eq. (6.8) and $G_t = G(P_t, t)$ is the price of a derivative (e.g., a call option) contingent on P_t. By Ito's lemma,

$$dG_t = \left(\frac{\partial G_t}{\partial P_t}\mu P_t + \frac{\partial G_t}{\partial t} + \frac{1}{2}\frac{\partial^2 G_t}{\partial P_t^2}\sigma^2 P_t^2\right) dt + \frac{\partial G_t}{\partial P_t}\sigma P_t \, dw_t.$$

The discretized versions of the process and previous result are

$$\Delta P_t = \mu P_t \, \Delta t + \sigma P_t \, \Delta w_t, \tag{6.11}$$

$$\Delta G_t = \left(\frac{\partial G_t}{\partial P_t}\mu P_t + \frac{\partial G_t}{\partial t} + \frac{1}{2}\frac{\partial^2 G_t}{\partial P_t^2}\sigma^2 P_t^2\right) \Delta t + \frac{\partial G_t}{\partial P_t}\sigma P_t \, \Delta w_t, \tag{6.12}$$

where ΔP_t and ΔG_t are changes in P_t and G_t in a small time interval Δt. Because $\Delta w_t = \epsilon\sqrt{\Delta t}$ for both Eqs. (6.11) and (6.12), one can construct a portfolio of the stock and the derivative that does not involve the Wiener process. The appropriate portfolio is short on derivative and long $\partial G_t/\partial P_t$ shares of the stock. Denote the value of the portfolio by V_t. By construction,

$$V_t = -G_t + \frac{\partial G_t}{\partial P_t} P_t. \tag{6.13}$$

The change in V_t is then

$$\Delta V_t = -\Delta G_t + \frac{\partial G_t}{\partial P_t} \Delta P_t. \tag{6.14}$$

Substituting Eqs. (6.11) and (6.12) into Eq. (6.14), we have

$$\Delta V_t = \left(-\frac{\partial G_t}{\partial t} - \frac{1}{2}\frac{\partial^2 G_t}{\partial P_t^2}\sigma^2 P_t^2 \right) \Delta t. \tag{6.15}$$

This equation does not involve the stochastic component Δw_t. Therefore, under the no arbitrage assumption, the portfolio V_t must be riskless during the small time interval Δt. In other words, the assumptions used imply that the portfolio must instantaneously earn the same rate of return as other short-term, risk-free securities. Otherwise there exists an arbitrage opportunity between the portfolio and the short-term, risk-free securities. Consequently, we have

$$\Delta V_t = r V_t \,\Delta t = (r\Delta t) V_t, \tag{6.16}$$

where r is the risk-free interest rate. By Eqs. (6.13)–(6.16), we have

$$\left(\frac{\partial G_t}{\partial t} + \frac{1}{2}\frac{\partial^2 G_t}{\partial P_t^2}\sigma^2 P_t^2 \right) \Delta t = r\left(G_t - \frac{\partial G_t}{\partial P_t} P_t \right) \Delta t.$$

Therefore,

$$\frac{\partial G_t}{\partial t} + r P_t \frac{\partial G_t}{\partial P_t} + \frac{1}{2}\sigma^2 P_t^2 \frac{\partial^2 G_t}{\partial P_t^2} = r G_t. \tag{6.17}$$

This is the Black–Scholes differential equation for derivative pricing. It can be solved to obtain the price of a derivative with P_t as the underlying variable. The solution so obtained depends on the boundary conditions of the derivative. For a European call option, the boundary condition is

$$G_T = \max(P_T - K, 0),$$

where T is the expiration time and K is the strike price. For a European put option, the boundary condition becomes

$$G_T = \max(K - P_T, 0).$$

Example 6.4. As a simple example, consider a forward contract on a stock that pays no dividend. In this case, the value of the contract is given by

$$G_t = P_t - K \exp[-r(T-t)],$$

where K is the delivery price, r is the risk-free interest rate, and T is the expiration time. For such a function, we have

$$\frac{\partial G_t}{\partial t} = -rK \exp[-r(T-t)], \qquad \frac{\partial G_t}{\partial P_t} = 1, \qquad \frac{\partial^2 G_t}{\partial P_t^2} = 0.$$

Substituting these quantities into the left-hand side of Eq. (6.17) yields

$$-rK \exp[-r(T-t)] + rP_t = r\{P_t - K \exp[-r(T-t)]\},$$

which equals the right-hand side of Eq. (6.17). Thus, the Black–Scholes differential equation is indeed satisfied.

6.6 BLACK–SCHOLES PRICING FORMULAS

Black and Scholes (1973) successfully solved their differential equation in Eq. (6.17) to obtain exact formulas for the price of European call-and-put options. In what follows, we derive these formulas using what is called *risk-neutral valuation* in finance.

6.6.1 Risk-Neutral World

The drift parameter μ drops out from the Black–Scholes differential equation. In finance, this means the equation is independent of risk preferences. In other words, risk preferences cannot affect the solution of the equation. A nice consequence of this property is that one can assume that investors are risk neutral. In a risk-neutral world, we have the following results:

- The expected return on all securities is the risk-free interest rate r.
- The present value of any cash flow can be obtained by discounting its expected value at the risk-free rate.

6.6.2 Formulas

The expected value of a European call option at maturity in a risk-neutral world is

$$E_*[\max(P_T - K, 0)],$$

where E_* denotes expected value in a risk-neutral world. The price of the call option at time t is

$$c_t = \exp[-r(T-t)]E_*[\max(P_T - K, 0)]. \tag{6.18}$$

Yet in a risk-neutral world, we have $\mu = r$, and by Eq. (6.10), $\ln(P_T)$ is normally distributed as

$$\ln(P_T) \sim N\left[\ln(P_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right].$$

Let $g(P_T)$ be the probability density function of P_T. Then the price of the call option in Eq. (6.18) is

$$c_t = \exp[-r(T-t)]\int_K^\infty (P_T - K)g(P_T)\,dP_T.$$

By changing the variable in the integration and some algebraic calculations (details are given in Appendix A), we have

$$c_t = P_t\Phi(h_+) - K\ \exp[-r(T-t)]\Phi(h_-), \tag{6.19}$$

where $\Phi(x)$ is the cumulative distribution function (CDF) of the standard normal random variable evaluated at x,

$$h_+ = \frac{\ln(P_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$
$$h_- = \frac{\ln(P_t/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = h_+ - \sigma\sqrt{T-t}.$$

In practice, $\Phi(x)$ can easily be obtained from most statistical packages. Alternatively, one can use an approximation given in Appendix B.

The Black–Scholes call formula in Eq. (6.19) has some nice interpretations. First, if we exercise the call option on the expiration date, we receive the stock, but we have to pay the strike price. This exchange will take place only when the call finishes in-the-money (i.e., $P_T > K$). The first term $P_t\Phi(h_+)$ is the present value of receiving the stock if and only if $P_T > K$ and the second term $-K\ \exp[-r(T-t)]\Phi(h_-)$ is the present value of paying the strike price if and only if $P_T > K$. A second interpretation is particularly useful. As shown in the derivation of the Black–Scholes differential equation in Section 6.5, $\Phi(h_+) = \partial G_t/\partial P_t$ is the number of shares in the portfolio that does not involve uncertainty, the Wiener process. This quantity is known as the *delta* in hedging. We know that $c_t = P_t\Phi(h_+) + B_t$, where B_t is the dollar amount invested in risk-free bonds in the portfolio (or short on the derivative). We can then see that $B_t = -K\ \exp[-r(T-t)]\Phi(h_-)$

directly from inspection of the Black–Scholes formula. The first term of the formula, $P_t\Phi(h_+)$, is the amount invested in the stock, whereas the second term, $K\exp[-r(T-t)]\Phi(h_-)$, is the amount borrowed.

Similarly, we can obtain the price of a European put option as

$$p_t = K\exp[-r(T-t)]\Phi(-h_-) - P_t\Phi(-h_+). \tag{6.20}$$

Since the standard normal distribution is symmetric with respect to its mean 0.0, we have $\Phi(x) = 1 - \Phi(-x)$ for all x. Using this property, we have $\Phi(-h_i) = 1 - \Phi(h_i)$. Thus, the information needed to compute the price of a put option is the same as that of a call option. Alternatively, using the symmetry of normal distribution, it is easy to verify that

$$p_t - c_t = K\exp[-r(T-t)] - P_t,$$

which is referred to as the *put–call parity* and can be used to obtain p_t from c_t. The put–call parity can also be obtained by considering the following two portfolios:

1. *Portfolio A.* One European call option plus an amount of cash equal to $K\exp[-r(T-t)]$.
2. *Portfolio B.* One European put option plus one share of the underlying stock.

The payoff of these two portfolios is

$$\max(P_T, K)$$

at the expiration of the options. Since the options can only be exercised at the expiration date, the portfolios must have identical value today. This means

$$c_t + K\exp[-r(T-t)] = p_t + P_t,$$

which is the put–call parity given earlier.

Example 6.5. Suppose that the current price of Intel stock is \$80 per share with volatility $\sigma = 20\%$ per annum. Suppose further that the risk-free interest rate is 8% per annum. What is the price of a European call option on Intel with a strike price of \$90 that will expire in 3 months?

From the assumptions, we have $P_t = 80$, $K = 90$, $T - t = 0.25$, $\sigma = 0.2$, and $r = 0.08$. Therefore,

$$h_+ = \frac{\ln(80/90) + (0.08 + 0.04/2)\times 0.25}{0.2\sqrt{0.25}} = -0.9278,$$

$$h_- = h_+ - 0.2\sqrt{0.25} = -1.0278.$$

Using any statistical software (e.g., R or S-Plus) or the approximation in Appendix B, we have

$$\Phi(-0.9278) = 0.1767, \qquad \Phi(-1.0278) = 0.1520.$$

Consequently, the price of a European call option is

$$c_t = \$80\Phi(-0.9278) - \$90\Phi(-1.0278)\exp(-0.02) = \$0.73.$$

The stock price has to rise by \$10.73 for the purchaser of the call option to break even.

Under the same assumptions, the price of a European put option is

$$p_t = \$90\ \exp(-0.08 \times 0.25)\Phi(1.0278) - \$80\Phi(0.9278) = \$8.95.$$

Thus, the stock price can rise an additional \$1.05 for the purchaser of the put option to break even.

Example 6.6. The strike price of the previous example is well beyond the current stock price. A more realistic strike price is \$81. Assume that the other conditions of the previous example continue to hold. We now have $P_t = 80$, $K = 81$, $r = 0.08$, and $T - t = 0.25$, and the h_i become

$$\begin{aligned} h_+ &= \frac{\ln(80/81) + (0.08 + 0.04/2) \times 0.25}{0.2\sqrt{0.25}} = 0.125775, \\ h_- &= h_+ - 0.2\sqrt{0.25} = 0.025775. \end{aligned}$$

Using the approximation in Appendix B, we have $\Phi(0.125775) = 0.5500$ and $\Phi(0.025775) = 0.5103$. The price of a European call option is then

$$c_t = \$80\Phi(0.125775) - \$81\ \exp(-0.02)\Phi(0.025775) = \$3.49.$$

The price of the stock has to rise by \$4.49 for the purchaser of the call option to break even. On the other hand, under the same assumptions, the price of a European put option is

$$\begin{aligned} p_t &= \$81\ \exp(-0.02)\Phi(-0.025775) - \$80\Phi(-0.125775) \\ &= \$81\ \exp(-0.02) \times 0.48972 - \$80 \times 0.44996 = \$2.89. \end{aligned}$$

The stock price must fall \$1.89 for the purchaser of the put option to break even.

6.6.3 Lower Bounds of European Options

Consider the call option of a nondividend-paying stock. It can be shown that the price of a European call option satisfies

$$c_t \geq P_t - K \exp[-r(T-t)];$$

that is, the lower bound for a European call price is $P_t - K \exp[-r(T-t)]$. This result can be verified by considering two portfolios:

1. *Portfolio A*. One European call option plus an amount of cash equal to $K \exp[-r(T-t)]$.
2. *Portfolio B*. One share of the stock.

For portfolio A, if the cash is invested at the risk-free interest rate, it will result in K at time T. If $P_T > K$, the call option is exercised at time T and the portfolio is worth P_T. If $P_T < K$, the call option expires worthless and the portfolio is worth K. Therefore, the value of portfolio is

$$\max(P_T, K).$$

The value of portfolio B is P_T at time T. Hence, portfolio A is always worth more than (or, at least, equal to) portfolio B. It follows that portfolio A must be worth more than portfolio B today; that is,

$$c_t + K \exp[-r(T-t)] \geq P_t, \quad \text{or} \quad c_t \geq P_t - K \exp[-r(T-t)].$$

Furthermore, since $c_t \geq 0$, we have

$$c_t \geq \max(P_t - K \exp[-r(T-t)], 0).$$

A similar approach can be used to show that the price of a corresponding European put option satisfies

$$p_t \geq \max\{K \exp[-r(T-t)] - P_t, 0\}.$$

Example 6.7. Suppose that $P_t = \$30$, $K = \$28$, $r = 6\%$ per annum, and $T - t = 0.5$. In this case,

$$P_t - K \exp[-r(T-t)] = \$[30 - 28 \exp(-0.06 \times 0.5)] \approx \$2.83.$$

Assume that the European call price of the stock is \$2.50, which is less than the theoretical minimum of \$2.83. An arbitrageur can buy the call option and short the stock. This provides a new cash flow of $\$(30 - 2.50) = \$27.50$. If invested for 6 months at the risk-free interest rate, the \$27.50 grows to $\$27.50 \exp(0.06 \times 0.5) = \$28.34$. At the expiration time, if $P_T > \$28$, the arbitrageur exercises the option, closes out the short position, and makes a profit of \$(28.34 − 28) = \$0.34. On the other hand, if $P_T <$ \$28, the stock is bought in the market to close the short position. The arbitrageur then makes an even greater profit. For illustration, suppose that $P_T =$ \$27.00, then the profit is \$(28.34 − 27.00) = \$1.34.

6.6.4 Discussion

From the formulas, the price of a call or put option depends on five variables—namely, the current stock price P_t, the strike price K, the time to expiration $T - t$ measured in years, the volatility σ per annum, and the interest rate r per annum. It pays to study the effects of these five variables on the price of an option.

Marginal Effects

Consider first the marginal effects of the five variables on the price of a call option c_t. By marginal effects we mean changing one variable while holding the others fixed. The effects on a call option can be summarized as follows:

1. *Current Stock Price* P_t. c_t is positively related to $\ln(P_t)$. In particular, $c_t \to 0$ as $P_t \to 0$ and $c_t \to \infty$ as $P_t \to \infty$. Figure 6.3(a) illustrates the effects with $K = 80$, $r = 6\%$ per annum, $T - t = 0.25$ year, and $\sigma = 30\%$ per annum.

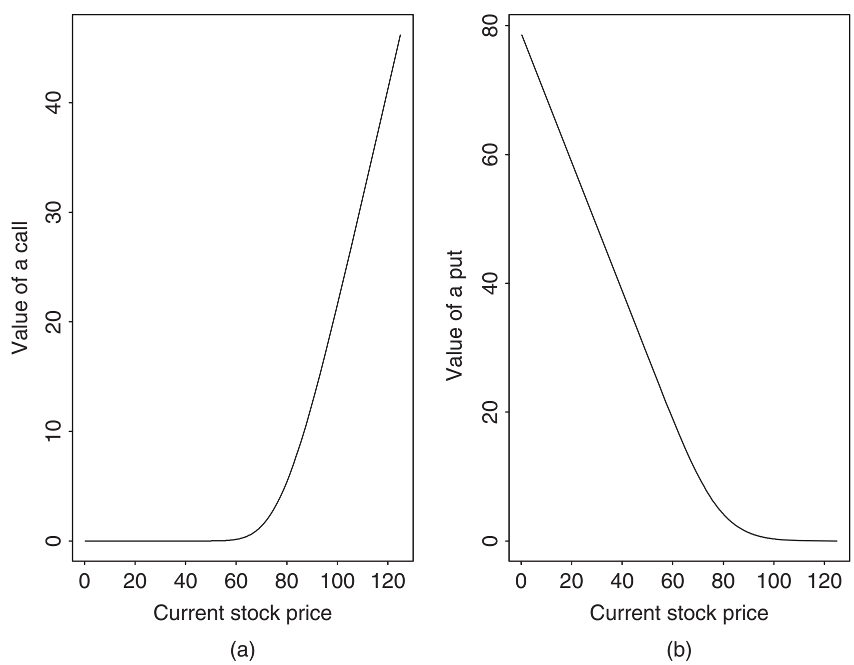

Figure 6.3 Marginal effects of current stock price on price of an option with $K = 80$, $T - t = 0.25$, $\sigma = 0.3$, and $r = 0.06$: (a) call option and (b) put option.

2. *Strike Price K.* c_t is negatively related to $\ln(K)$. In particular, $c_t \to P_t$ as $K \to 0$ and $c_t \to 0$ as $K \to \infty$.
3. *Time to Expiration.* c_t is related to $T - t$ in a complicated manner, but we can obtain the limiting results by writing h_+ and h_- as

$$h_+ = \frac{\ln(P_t/K)}{\sigma\sqrt{T-t}} + \frac{(r+\sigma^2/2)\sqrt{T-t}}{\sigma},$$
$$h_- = \frac{\ln(P_t/K)}{\sigma\sqrt{T-t}} + \frac{(r-\sigma^2/2)\sqrt{T-t}}{\sigma}.$$

If $P_t < K$, then $c_t \to 0$ as $(T-t) \to 0$. If $P_t > K$, then $c_t \to P_t - K$ as $(T-t) \to 0$ and $c_t \to P_t$ as $(T-t) \to \infty$. Figure 6.4(a) shows the marginal effects of $T - t$ on c_t for three different current stock prices. The fixed variables are $K = 80$, $r = 6\%$, and $\sigma = 30\%$. The solid, dotted, and dashed lines of the plot are for $P_t = 70$, 80, and 90, respectively.
4. *Volatility σ.* Rewriting h_+ and h_- as

$$h_+ = \frac{\ln(P_t/K) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{\sigma}{2}\sqrt{T-t},$$

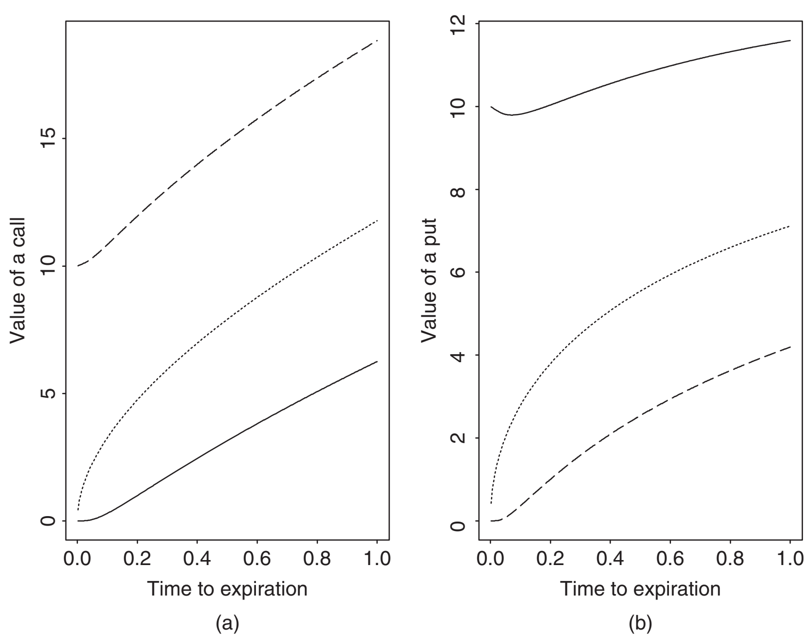

Figure 6.4 Marginal effects of time to expiration on price of an option with $K = 80$, $\sigma = 0.3$, and $r = 0.06$: (a) call option and (b) put option. Solid, dotted, and dashed lines are for current stock price $P_t = 70$, 80, and 90, respectively.

$$h_- = \frac{\ln(P_t/K) + r(T-t)}{\sigma\sqrt{T-t}} - \frac{\sigma}{2}\sqrt{T-t},$$

we obtain that (a) if $\ln(P_t/K) + r(T-t) < 0$, then $c_t \to 0$ as $\sigma \to 0$, and (b) if $\ln(P_t/K) + r(T-t) \geq 0$, then $c_t \to P_t - Ke^{-r(T-t)}$ as $\sigma \to 0$ and $c_t \to P_t$ as $\sigma \to \infty$. Figure 6.5(a) shows the effects of σ on c_t for $K = 80$, $T - t = 0.25$, $r = 0.06$, and three different values of P_t. The solid, dotted, and dashed lines are for $P_t = 70$, 80, and 90, respectively.

5. *Interest Rate.* c_t is positively related to r such that $c_t \to P_t$ as $r \to \infty$.

The marginal effects of the five variables on a put option can be obtained similarly. Figures 6.3(b), 6.4(b), and 6.5(b) illustrates the effects for some selected cases.

Some Joint Effects

Figure 6.6 shows the joint effects of volatility and strike price on a call option, where the other variables are fixed at $P_t = 80$, $r = 0.06$, and $T - t = 0.25$. As expected, the price of a call option is higher when the volatility is high and the strike price is well below the current stock price. Figure 6.7 shows the effects on

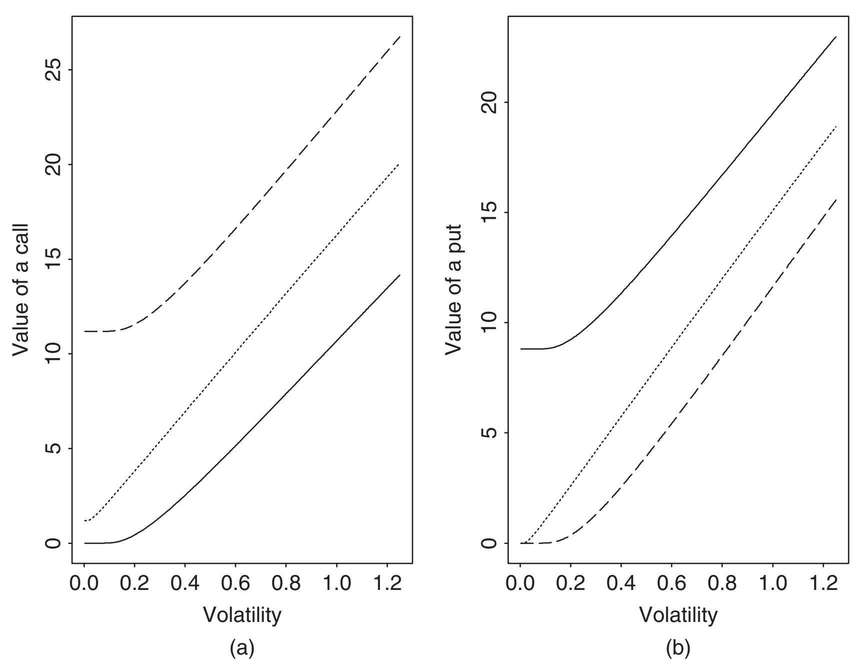

Figure 6.5 Marginal effects of stock volatility on price of an option with $K = 80$, $T - t = 0.25$, and $r = 0.06$: (a) call option and (b) put option. Solid, dotted, and dashed lines are for current stock price $P_t = 70$, 80, and 90, respectively.

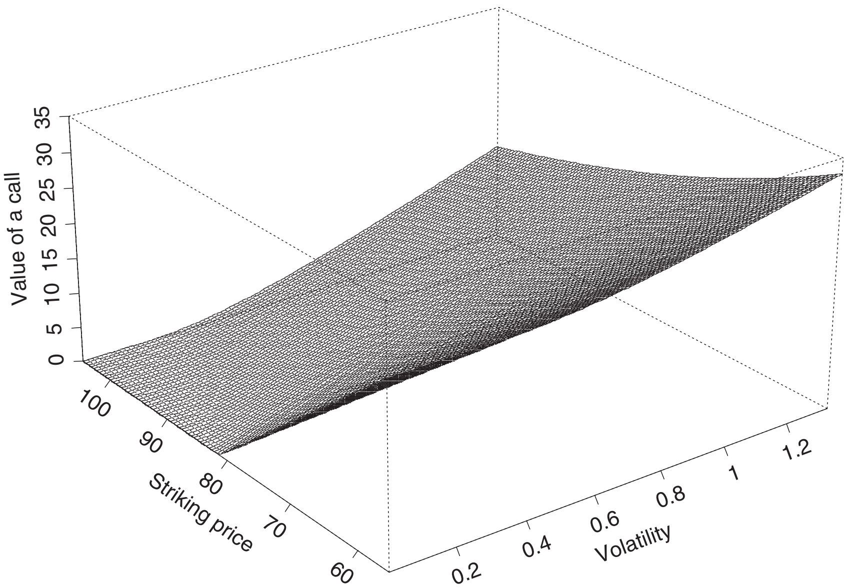

Figure 6.6 Joint effects of stock volatility and strike price on call option with $P_t = 80$, $r = 0.06$, and $T - t = 0.25$.

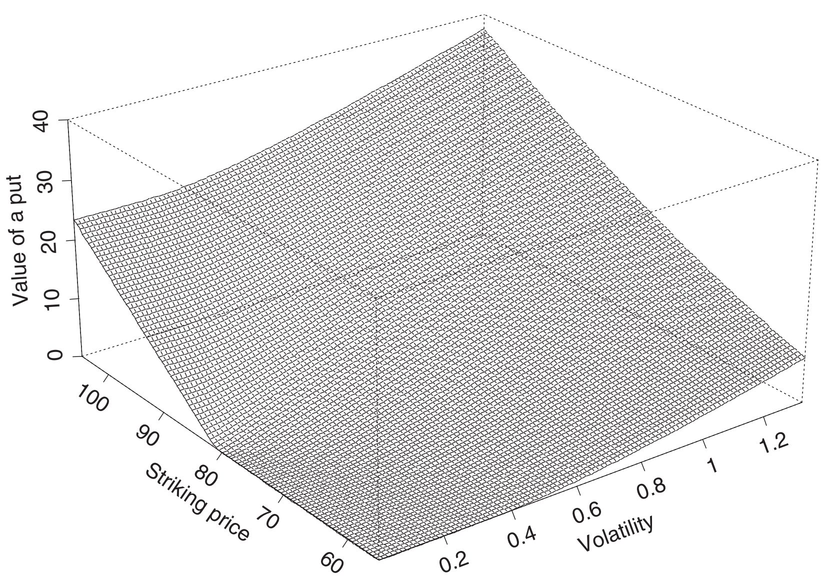

Figure 6.7 Joint effects of stock volatility and strike price on put option with $K = 80$, $T - t = 0.25$, and $r = 0.06$.

a put option under the same conditions. The price of a put option is higher when the volatility is high and the strike price is well above the current stock price. Furthermore, the plot also shows that the effects of a strike price on the price of a put option becomes more linear as the volatility increases.

6.7 EXTENSION OF ITO'S LEMMA

In derivative pricing, a derivative may be contingent on multiple securities. When the prices of these securities are driven by multiple factors, the price of the derivative is a function of several stochastic processes. The two-factor model for the term structure of interest rate is an example of two stochastic processes. In this section, we briefly discuss the extension of Ito's lemma to the case of several stochastic processes.

Consider a k-dimensional continuous-time process $\boldsymbol{x}_t = (x_{1t}, \ldots, x_{kt})'$, where k is a positive integer and x_{it} is a continuous-time stochastic process satisfying

$$dx_{it} = \mu_i(\boldsymbol{x}_t)\,dt + \sigma_i(\boldsymbol{x}_t)\,dw_{it}, \qquad i = 1, \ldots, k, \tag{6.21}$$

where w_{it} is a Wiener process. It is understood that the drift and volatility functions $\mu_i(x_{it})$ and $\sigma_i(x_{it})$ are functions of time index t as well. We omit t from their arguments to simplify the notation. For $i \neq j$, the Wiener processes w_{it} and w_{jt} are different. We assume that the correlation between dw_{it} and dw_{jt} is ρ_{ij}. This means that ρ_{ij} is the correlation between the two standard normal random variables ϵ_i and ϵ_j defined by $\Delta w_{it} = \epsilon_i\,\Delta t$ and $\Delta w_{jt} = \epsilon_j\,\Delta t$. Assume that $G_t = G(\boldsymbol{x}_t, t)$ is a function of the stochastic processes x_{it} and time t. The Taylor expansion gives

$$\begin{aligned}\Delta G_t = {} & \sum_{i=1}^{k} \frac{\partial G_t}{\partial x_{it}} \Delta x_{it} + \frac{\partial G_t}{\partial t} \Delta t + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{\partial^2 G_t}{\partial x_{it} \partial x_{jt}} \Delta x_{it}\,\Delta x_{jt} \\ & + \frac{1}{2} \sum_{i=1}^{k} \frac{\partial^2 G_t}{\partial x_{it} \partial t} \Delta x_{it}\,\Delta t + \cdots .\end{aligned}$$

The discretized version of Eq. (6.21) is

$$\Delta w_{it} = \mu_i(\boldsymbol{x}_t)\,\Delta t + \sigma_i(\boldsymbol{x}_t)\,\Delta w_{it}, \qquad i = 1, \ldots, k.$$

Using a similar argument as that of Eq. (6.5) in Section 6.3, we can obtain that

$$\lim_{\Delta t \to 0} (\Delta x_{it})^2 \to \sigma_i^2(\boldsymbol{x}_t)\,dt, \tag{6.22}$$

$$\lim_{\Delta t \to 0} (\Delta x_{it}\,\Delta x_{jt}) \to \sigma_i(\boldsymbol{x}_t)\sigma_j(\boldsymbol{x}_t)\rho_{ij}\,dt. \tag{6.23}$$

Using Eqs. (6.21)–(6.23), taking the limit as $\Delta t \to 0$, and ignoring higher order terms of Δt, we have

$$dG_t = \left[\sum_{i=1}^{k} \frac{\partial G_t}{\partial x_{it}} \mu_i(\boldsymbol{x}_t) + \frac{\partial G_t}{\partial t} + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{\partial^2 G_t}{\partial x_{it} \partial x_{jt}} \sigma_i(\boldsymbol{x}_t) \sigma_j(\boldsymbol{x}_t) \rho_{ij} \right] dt + \sum_{i=1}^{k} \frac{\partial G_t}{\partial x_{it}} \sigma_i(\boldsymbol{x}_t)\, dw_{it}. \tag{6.24}$$

This is a generalization of Ito's lemma to the case of multiple stochastic processes.

6.8 STOCHASTIC INTEGRAL

We briefly discuss stochastic integration so that the price of an asset can be obtained under the assumption that it follows an Ito process. We deduce the integration result using Ito's formula. For a rigorous treatment on the topic, readers may consult textbooks on stochastic calculus. First, like the usual integration of a deterministic function, integration is the opposite of differentiation so that

$$\int_0^t dx_s = x_t - x_0$$

continues to hold for a stochastic process x_t. In particular, for the Wiener process w_t, we have $\int_0^t dw_s = w_t$ because $w_0 = 0$. Next, consider the integration $\int_0^t w_s\, dw_s$. Using the prior result and taking integration of Eq. (6.7), we have

$$w_t^2 = t + 2\int_0^t w_s\, dw_s.$$

Therefore,

$$\int_0^t w_s\, dw_s = \frac{1}{2}(w_t^2 - t).$$

This is different from the usual deterministic integration for which $\int_0^t y\, dy = (y_t^2 - y_0^2)/2$.

Turn to the case that x_t is a geometric Brownian motion—that is, x_t satisfies

$$dx_t = \mu x_t\, dt + \sigma x_t\, dw_t,$$

where μ and σ are constant with $\sigma > 0$; see Eq. (6.8). Applying Ito's lemma to $G(x_t, t) = \ln(x_t)$, we obtain

$$d\, \ln(x_t) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dw_t.$$

Performing the integration and using the results obtained before, we have

$$\int_0^t d\ \ln(x_s) = \left(\mu - \frac{\sigma^2}{2}\right)\int_0^t ds + \sigma\int_0^t dw_s.$$

Consequently,

$$\ln(x_t) = \ln(x_0) + (\mu - \sigma^2/2)t + \sigma w_t$$

and

$$x_t = x_0 \exp[(\mu - \sigma^2/2)t + \sigma w_t].$$

Changing the notation x_t to P_t for the price of an asset, we have a solution for the price under the assumption that it is a geometric Brownian motion. The price is

$$P_t = P_0 \exp[(\mu - \sigma^2/2)t + \sigma w_t]. \tag{6.25}$$

6.9 JUMP DIFFUSION MODELS

Empirical studies have found that the stochastic diffusion model based on Brownian motion fails to explain some characteristics of asset returns and the prices of their derivatives [e.g., the "volatility smile" of implied volatilities; see Bakshi, Cao, and Chen (1997) and the references therein]. Volatility smile is referred to as the convex function between the implied volatility and strike price of an option. Both out-of-the-money and in-the-money options tend to have higher implied volatilities than at-the-money options especially in the foreign exchange markets. Volatility smile is less pronounced for equity options. The inadequacy of the standard stochastic diffusion model has led to the developments of alternative continuous-time models. For example, jump diffusion and stochastic volatility models have been proposed in the literature to overcome the inadequacy; see Merton (1976) and Duffie (1995).

Jumps in stock prices are often assumed to follow a probability law. For example, the jumps may follow a Poisson process, which is a continuous-time discrete process. For a given time t, let X_t be the number of times a special event occurs during the time period $[0, t]$. Then X_t is a Poisson process if

$$\Pr(X_t = m) = \frac{\lambda^m t^m}{m!}\exp(-\lambda t), \qquad \lambda > 0.$$

That is, X_t follows a Poisson distribution with parameter λt. The parameter λ governs the occurrence of the special event and is referred to as the *rate* or *intensity* of the process. A formal definition also requires that X_t be a right-continuous homogeneous Markov process with left-hand limit.

In this section, we discuss a simple jump diffusion model proposed by Kou (2002). This simple model enjoys several nice properties. The returns implied by the model are leptokurtic and asymmetric with respect to zero. In addition, the model can reproduce volatility smile and provide analytical formulas for the prices of many options. The model consists of two parts, with the first part being continuous and following a geometric Brownian motion and the second part being a jump process. The occurrences of jump are governed by a Poisson process, and the jump size follows a double exponential distribution. Let P_t be the price of an asset at time t. The simple jump diffusion model postulates that the price follows the stochastic differential equation

$$\frac{dP_t}{P_t} = \mu\, dt + \sigma\, dw_t + d\left[\sum_{i=1}^{n_t}(J_i - 1)\right], \tag{6.26}$$

where w_t is a Wiener process, n_t is a Poisson process with rate λ, and $\{J_i\}$ is a sequence of independent and identically distributed nonnegative random variables such that $X = \ln(J)$ has a double exponential distribution with probability density function

$$f_X(x) = \frac{1}{2\eta} e^{-|x-\kappa|/\eta}, \qquad 0 < \eta < 1. \tag{6.27}$$

The double exponential distribution is also referred to as the *Laplacian distribution*. In model (6.26), n_t, w_t, and J_i are independent so that there is no relation between the randomness of the model. Notice that n_t is the number of jumps in the time interval $[0, t]$ and follows a Poisson distribution with parameter λt, where λ is a constant. At the ith jump, the proportion of price jump is $J_i - 1$.

The double exponential distribution can be written as

$$X - \kappa = \begin{cases} \xi & \text{with probability } 0.5, \\ -\xi & \text{with probability } 0.5, \end{cases} \tag{6.28}$$

where ξ is an exponential random variable with mean η and variance η^2. The probability density function of ξ is

$$f(x) = \frac{1}{\eta} e^{-x/\eta}, \qquad 0 < x < \infty.$$

Some useful properties of the double exponential distribution are

$$E(X) = \kappa, \qquad \mathrm{Var}(X) = 2\eta^2, \qquad E(e^X) = \frac{e^\kappa}{1-\eta^2}.$$

For finite samples, it is hard to distinguish a double exponential distribution from a Student-t distribution. However, a double exponential distribution is more tractable analytically and can generate a higher probability concentration (e.g., higher peak) around its mean value. As stated in Chapter 1, histograms of observed asset returns tend to have a higher peak than the normal density. Figure 6.8 shows the probability

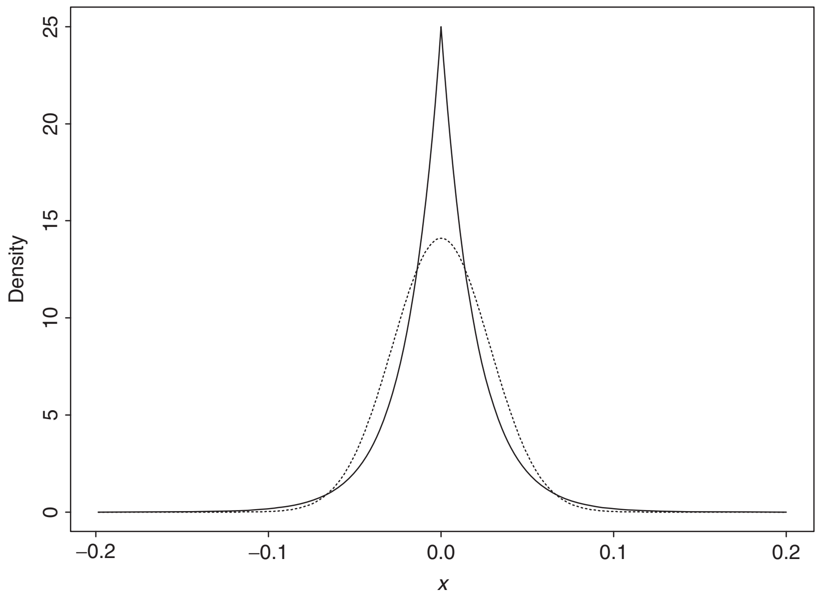

Figure 6.8 Probability density functions of double exponential and normal random variable with mean zero and variance 0.0008. Solid line denotes the double exponential distribution. Dotted line is the normal distribution.

density function of a double exponential random variable in the solid line and that of a normal random variable in the dotted line. Both variables have mean zero and variance 0.0008. The high peak of the double exponential density is clearly seen.

Solving the stochastic differential equation in Eq. (6.26), we obtain the dynamics of the asset price as

$$P_t = P_0 \exp\left[\left(\frac{\mu - \sigma^2}{2}\right) t + \sigma w_t\right] \prod_{i=1}^{n_t} J_i, \tag{6.29}$$

where it is understood that $\prod_{i=1}^{0} = 1$. This result is a generalization of Eq. (6.25) by including the stochastic jumps. It can be obtained as follows. Let t_i be the time of the ith jump. For $t \in [0, t_1)$, there is no jump and the price is given in Eq. (6.25). Consequently, the left-hand price limit at time t_1 is

$$P_{t_1^-} = P_0 \exp[(\mu - \sigma^2/2)t_1 + \sigma w_{t_1}].$$

At time t_1, the proportion of price jump is $J_1 - 1$ so that the price becomes

$$P_{t_1} = (1 + J_1 - 1)P_{t_1^-} = J_1 P_{t_1^-} = P_0 \exp[(\mu - \sigma^2/2)t_1 + \sigma w_{t_1}]J_1.$$

For $t \in (t_1, t_2)$, there is no jump in the interval $(t_1, t]$ so that

$$P_t = P_{t_1} \exp[(\mu - \sigma^2/2)(t - t_1) + \sigma(w_t - w_{t_1})].$$

Plugging in P_{t_1}, we have

$$P_t = P_0 \exp[(\mu - \sigma^2/2)t + \sigma w_t] J_1.$$

Repeating the scheme, we obtain Eq. (6.29).

From Eq. (6.29), the simple return of the underlying asset in a small time increment Δt becomes

$$\frac{P_{t+\Delta t} - P_t}{P_t} = \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) \Delta t + \sigma(w_{t+\Delta t} - w_t) + \sum_{i=n_t+1}^{n_{t+\Delta t}} X_i\right] - 1,$$

where it is understood that a summation over an empty set is zero and $X_i = \ln(J_i)$. For a small Δt, we may use the approximation $e^x \approx 1 + x + x^2/2$ and the result $(\Delta w_t)^2 \approx \Delta t$ discussed in Section 6.3 to obtain

$$\begin{aligned}\frac{P_{t+\Delta t} - P_t}{P_t} &\approx \left(\mu - \frac{1}{2}\sigma^2\right) \Delta t + \sigma\, \Delta w_t + \sum_{i=n_t+1}^{n_{t+\Delta t}} X_i + \frac{1}{2}\sigma^2(\Delta w_t)^2 \\ &\approx \mu\, \Delta t + \sigma\epsilon\sqrt{\Delta t} + \sum_{i=n_t+1}^{n_{t+\Delta t}} X_i,\end{aligned}$$

where $\Delta w_t = w_{t+\Delta t} - w_t$ and ϵ is a standard normal random variable.

Under the assumption of a Poisson process, the probability of having one jump in the time interval $(t, t + \Delta t]$ is $\lambda\, \Delta t$ and that of having more than one jump is $o(\Delta t)$, where the symbol $o(\Delta t)$ means that if we divide this term by Δt then its value tends to zero as Δt tends to zero. Therefore, for a small Δt, by ignoring multiple jumps, we have

$$\sum_{i=n_t+1}^{n_{t+\Delta t}} X_i \approx \begin{cases} X_{n_t+1} & \text{with probability } \lambda\Delta t, \\ 0 & \text{with probability } 1 - \lambda\, \Delta t. \end{cases}$$

Combining the prior results, we see that the simple return of the underlying asset is approximately distributed as

$$\frac{P_{t+\Delta t} - P_t}{P_t} \approx \mu\, \Delta t + \sigma\epsilon\sqrt{\Delta t} + I \times X, \tag{6.30}$$

where I is a Bernoulli random variable with $\Pr(I = 1) = \lambda\, \Delta t$ and $\Pr(I = 0) = 1 - \lambda\, \Delta t$, and X is a double exponential random variable defined in Eq. (6.28). Equation (6.30) reduces to that of a geometric Brownian motion without jumps.

Let $G = \mu\, \Delta t + \sigma\epsilon\sqrt{\Delta t} + I \times X$ be the random variable on the right-hand side of Eq. (6.30). Using the independence between the exponential and normal distributions used in the model, Kou (2002) obtains the probability density function of G as

$$g(x) = \frac{\lambda\,\Delta t}{2\eta} e^{\sigma^2 \Delta t/(2\eta^2)} \left[e^{-\omega/\eta} \Phi\left(\frac{\omega\eta - \sigma^2\,\Delta t}{\sigma\eta\sqrt{\Delta t}}\right) + e^{\omega/\eta} \Phi\left(\frac{\omega\eta + \sigma^2\,\Delta t}{\sigma\eta\sqrt{\Delta t}}\right)\right] + (1 - \lambda\,\Delta t)\frac{1}{\sigma\sqrt{\Delta t}} f\left(\frac{x - \mu\,\Delta t}{\sigma\sqrt{\Delta t}}\right), \tag{6.31}$$

where $\omega = x - \mu\,\Delta t - \kappa$, and $f(\cdot)$ and $\Phi(\cdot)$ are, respectively, the probability density and cumulative distribution functions of the standard normal random variable. Furthermore,

$$E(G) = \mu\,\Delta t + \kappa\lambda\,\Delta t, \qquad \mathrm{Var}(G) = \sigma^2\,\Delta t + \lambda\,\Delta t[2\eta^2 + \kappa^2(1 - \lambda\,\Delta t)].$$

Figure 6.9 shows some comparisons between probability density functions of a normal distribution and the distribution of Eq. (6.31). Both distributions have mean zero and variance 2.0572×10^{-4}. The mean and variance are obtained by assuming that the return of the underlying asset satisfies $\mu = 20\%$ per annum, $\sigma = 20\%$ per annum, $\Delta t = 1$ day $= 1/252$ year, $\lambda = 10$, $\kappa = -0.02$, and $\eta = 0.02$. In other words, we assume that there are about 10 daily jumps per year with average jump size -2%, and the jump size standard error is 2%. These values are reasonable for a U.S. stock. From the plots, the leptokurtic feature of the distribution derived from the jump diffusion process in Eq. (6.26) is clearly shown. The distribution has a higher peak and fatter tails than the corresponding normal distribution.

6.9.1 Option Pricing under Jump Diffusion

In the presence of random jumps, the market becomes incomplete. In this case, the standard hedging arguments are not applicable to price an option. But we can still derive an option pricing formula that does not depend on attitudes toward risk by assuming that the number of securities available is very large so that the risk of the sudden jumps is diversifiable and the market will therefore pay no risk premium over the risk-free rate for bearing this risk. Alternatively, for a given set of risk premiums, one can consider a risk-neutral measure P^* such that

$$\begin{aligned}\frac{dP_t}{P_t} &= [r - \lambda E(J - 1)]\,dt + \sigma\,dw_t + d\left[\sum_{i=1}^{n_t}(J_i - 1)\right]\\ &= (r - \lambda\psi)\,dt + \sigma\,dw_t + d\left[\sum_{i=1}^{n_t}(J_i - 1)\right],\end{aligned}$$

where r is the risk-free interest rate, $J = \exp(X)$ such that X follows the double exponential distribution of Eq. (6.27), $\psi = e^{\kappa}/(1 - \eta^2) - 1$, $0 < \eta < 1$, and the parameters κ, η, ψ, and σ become risk-neutral parameters taking consideration of

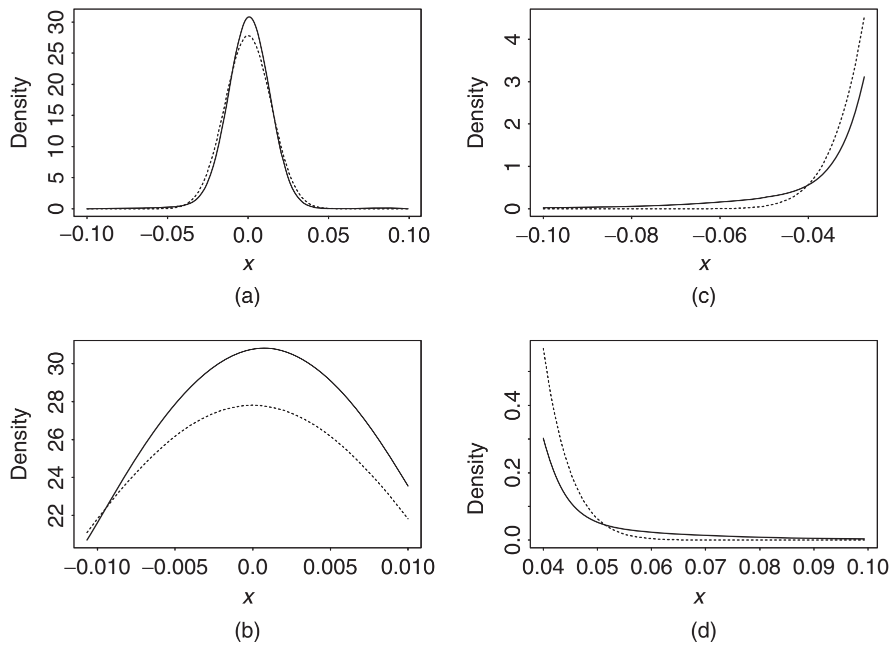

Figure 6.9 Density comparisons between normal distribution and distribution of Eq. (6.31). Dotted line denotes the normal distribution. Both distributions have mean zero and variance 2.0572×10^{-4}. (a) Overall comparison, (b) comparison of peaks, (c) left tails, and (d) right tails.

the risk premiums; see Kou (2002) for more details. The unique solution of the prior equation is given by

$$P_t = P_0 \exp\left[\left(r - \frac{\sigma^2}{2} - \lambda\psi\right) t + \sigma w_t\right] \prod_{i=1}^{n_t} J_i.$$

To price a European option in the jump diffusion model, it remains to compute the expectation, under the measure P^*, of the discounted final payoff of the option. In particular, the price of a European call option at time t is given by

$$\begin{aligned} c_t &= E_*[e^{-r(T-t)}(P_T - K)_+] \\ &= E_*\left(e^{-r(T-t)}\left\{P_t \exp\left[\left(\frac{r-\sigma^2}{2} - \lambda\psi\right)(T-t)\right.\right.\right. \\ &\qquad \left.\left.\left. + \sigma\sqrt{T-t}\epsilon\right] \prod_{i=1}^{n_T} J_i - K\right\}_+\right), \end{aligned} \tag{6.32}$$

where T is the expiration time, $(T-t)$ is the time to expiration measured in years, K is the strike price, $(y)_+ = \max(0, y)$, and ϵ is a standard normal random variable. Kou (2002) shows that c_t is analytically tractable as

$$c_t = \sum_{n=1}^{\infty} \sum_{j=1}^{n} e^{-\lambda(T-t)} \frac{\lambda^n (T-t)^n}{n!} \frac{2^j}{2^{2n-1}} \begin{pmatrix} 2n-j-1 \\ n-1 \end{pmatrix} (A_{1,n,j} + A_{2,n,j} + A_{3,n,j})$$
$$+ \; e^{-\lambda(T-t)} \left[P_t e^{-\lambda\psi(T-t)} \Phi(h_+) - K e^{-r(T-t)} \Phi(h_-) \right], \tag{6.33}$$

where $\Phi(\cdot)$ is the CDF of the standard normal random variable,

$$\begin{aligned}
A_{1,n,j} &= P_t e^{-\lambda\psi(T-t)+n\kappa} \frac{1}{2} \left[\frac{1}{(1-\eta)^j} + \frac{1}{(1+\eta)^j} \right] \Phi(b_+) - e^{-r(T-t)} K \Phi(b_-), \\
A_{2,n,j} &= \frac{1}{2} e^{-r(T-t)-\omega/\eta+\sigma^2(T-t)/(2\eta^2)} K \\
&\quad \times \sum_{i=0}^{j-1} \left[\frac{1}{(1-\eta)^{j-i}} - 1 \right] \left(\frac{\sigma\sqrt{T-t}}{\eta} \right)^i \frac{1}{\sqrt{2\pi}} Hh_i(c_-), \\
A_{3,n,j} &= \frac{1}{2} e^{-r(T-t)+\omega/\eta+\sigma^2(T-t)/(2\eta^2)} K \\
&\quad \times \sum_{i=0}^{j-1} \left[1 - \frac{1}{(1+\eta)^{j-i}} \right] \left(\frac{\sigma\sqrt{T-t}}{\eta} \right)^i \frac{1}{\sqrt{2\pi}} Hh_i(c_+), \\
b_{\pm} &= \frac{\ln(P_t/K) + (r \pm \sigma^2/2 - \lambda\psi)(T-t) + n\kappa}{\sigma\sqrt{T-t}}, \\
h_{\pm} &= \frac{\ln(P_t/K) + (r \pm \sigma^2/2 - \lambda\psi)(T-t)}{\sigma\sqrt{T-t}}, \\
c_{\pm} &= \frac{\sigma\sqrt{T-t}}{\eta} \pm \frac{\omega}{\sigma\sqrt{T-t}}, \\
\omega &= \ln\left(\frac{K}{P_t}\right) + \lambda\psi(T-t) - \left(r - \frac{\sigma^2}{2}\right)(T-t) - n\kappa, \\
\psi &= \frac{e^{\kappa}}{1-\eta^2} - 1,
\end{aligned}$$

and the $Hh_i(\cdot)$ functions are defined as

$$Hh_n(x) = \frac{1}{n!} \int_x^{\infty} (s-x)^n e^{-s^2/2} \, ds, \qquad n = 0, 1, \ldots, \tag{6.34}$$

and $Hh_{-1}(x) = \exp(-x^2/2)$, which is $\sqrt{2\pi} f(x)$ with $f(x)$ being the probability density function of a standard normal random variable; see Abramowitz and Stegun

(1972). The $Hh_n(x)$ functions satisfy the recursion

$$nHh_n(x) = Hh_{n-2}(x) - xHh_{n-1}(x), \qquad n \geq 1, \tag{6.35}$$

with starting values $Hh_{-1}(x) = e^{-x^2/2}$ and $Hh_0(x) = \sqrt{2\pi}\,\Phi(-x)$.

The pricing formula involves an infinite series, but its numerical value can be approximated quickly and accurately through truncation (e.g., the first 10 terms). Also, if $\lambda = 0$ (i.e., there are no jumps), then it is easily seen that c_t reduces to the Black–Scholes formula for a call option discussed before.

Finally, the price of a European put option under the jump diffusion model considered can be obtained by using the put$f$-call parity; that is,

$$p_t = c_t + Ke^{-r(T-t)} - P_t.$$

Pricing formulas for other options under the jump diffusion model in Eq. (6.26) can be found in Kou (2002).

Example 6.8. Consider the stock of Example 6.6, which has a current price of \$80. As before, assume that the strike price of a European option is $K = \$81$ and other parameters are $r = 0.08$ and $T - t = 0.25$. In addition, assume that the price of the stock follows the jump diffusion model in Eq. (6.26) with parameters $\lambda = 10$, $\kappa = -0.02$, and $\eta = 0.02$. In other words, there are about 10 jumps per year with average jump size -2% and jump size standard error of 2%. Using the formula in Eq. (6.33), we obtain $c_t = \$3.92$, which is higher than the \$3.49 of Example 6.6 when there are no jumps. The corresponding put option assumes the value $p_t = \$3.31$, which is also higher than what we had before. As expected, adding the jumps while keeping the other parameters fixed increases the prices of both European options. Keep in mind, however, that adding the jump process to the stock price in a real application often leads to different estimates for the stock volatility σ.

6.10 ESTIMATION OF CONTINUOUS-TIME MODELS

Next, we consider the problem of estimating directly the diffusion equation (i.e., Ito process) from discretely sampled data. Here the drift and volatility functions $\mu(x_t, t)$ and $\sigma(x_t, t)$ are time varying and may not follow a specific parametric form. This is a topic of considerable interest in recent years. Details of the available methods are beyond the scope of this chapter. Hence, we only outline the approaches proposed in the literature. Interested readers can consult the corresponding references and Lo (1988).

There are several approaches available for estimating a diffusion equation. The first approach is the quasi-maximum-likelihood approach, which makes use of the fact that for a small time interval dw_t is normally distributed; see Kessler (1997) and the references therein. The second approach uses methods of moments; see

Conley, Hansen, Luttmer, and Scheinkman (1997) and the references therein. The third approach uses nonparametric methods; see Ait-Sahalia (1996, 2002). The fourth approach uses semiparametric and reprojection methods; see Gallant and Long (1997) and Gallant and Tauchen (1997). Recently, many researchers have applied Markov chain Monte Carlo methods to estimate the diffusion equation; see Eraker (2001) and Elerian, Chib, and Shephard (2001).

APPENDIX A: INTEGRATION OF BLACK–SCHOLES FORMULA

In this appendix, we derive the price of a European call option given in Eq. (6.19). Let $x = \ln(P_T)$. By changing variable and using $g(P_T)\,dP_T = f(x)\,dx$, where $f(x)$ is the probability density function of x, we have

$$\begin{aligned} c_t &= \exp[-r(T-t)] \int_K^{\infty} (P_T - K) g(P_T)\,dP_t \\ &= e^{-r(T-t)} \int_{\ln(K)}^{\infty} (e^x - K) f(x)\,dx \\ &= e^{-r(T-t)} \left[\int_{\ln(K)}^{\infty} e^x f(x)\,dx - K \int_{\ln(K)}^{\infty} f(x)\,dx \right]. \end{aligned} \tag{6.36}$$

Because $x = \ln(P_T) \sim N[\ln(P_t) + (r - \sigma^2/2)(T-t), \sigma^2(T-t)]$, the integration of the second term of Eq. (6.36) reduces to

$$\begin{aligned} \int_{\ln(K)}^{\infty} f(x)\,dx &= 1 - \int_{-\infty}^{\ln(K)} f(x)\,dx \\ &= 1 - \text{CDF}[\ln(K)] \\ &= 1 - \Phi(-h_-) = \Phi(h_-), \end{aligned}$$

where $\text{CDF}[\ln(K)]$ is the cumulative distribution function (CDF) of $x = \ln(P_T)$ evaluated at $\ln(K)$, $\Phi(\cdot)$ is the CDF of the standard normal random variable, and

$$\begin{aligned} -h_- &= \frac{\ln(K) - \ln(P_t) - (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{-\ln(P_t/K) - (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

The integration of the first term of Eq. (6.36) can be written as

$$\int_{\ln(K)}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2(T-t)}} \exp\left\{ x - \frac{[x - \ln(P_t) - (r - \sigma^2/2)(T-t)]^2}{2\sigma^2(T-t)} \right\} dx,$$

where the exponent can be simplified to

$$x - \frac{\{x - [\ln(P_t) + (r - \sigma^2/2)(T - t)]\}^2}{2\sigma^2(T - t)}$$
$$= -\frac{\{x - [\ln(P_t) + (r + \sigma^2/2)(T - t)]\}^2}{2\sigma^2(T - t)} + \ln(P_t) + r(T - t).$$

Consequently, the first integration becomes

$$\int_{\ln(K)}^{\infty} e^x f(x)\,dx = P_t e^{r(T-t)} \int_{\ln(K)}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2(T-t)}} \times \exp\left(-\frac{\{x - [\ln(P_t) + (r + \sigma^2/2)(T - t)]\}^2}{2\sigma^2(T - t)}\right) dx,$$

which involves the CDF of a normal distribution with mean $\ln(P_t) + (r + \sigma^2/2)(T - t)$ and variance $\sigma^2(T - t)$. By using the same techniques as those of the second integration shown before, we have

$$\int_{\ln(K)}^{\infty} e^x f(x)\,dx = P_t e^{r(T-t)} \Phi(h_+),$$

where h_+ is given by

$$h_+ = \frac{\ln(P_t/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}.$$

Putting the two integration results together, we have

$$c_t = e^{-r(T-t)}[P_t e^{r(T-t)}\Phi(h_+) - K\Phi(h_-)] = P_t\Phi(h_+) - Ke^{-r(T-t)}\Phi(h_-).$$

APPENDIX B: APPROXIMATION TO STANDARD NORMAL PROBABILITY

The CDF $\Phi(x)$ of a standard normal random variable can be approximated by

$$\Phi(x) = \begin{cases} 1 - f(x)[c_1k + c_2k^2 + c_3k^3 + c_4k^4 + c_5k^5] & \text{if } x \geq 0, \\ 1 - \Phi(-x) & \text{if } x < 0, \end{cases}$$

where $f(x) = \exp(-x^2/2)/\sqrt{2\pi}$, $k = 1/(1 + 0.2316419x)$, $c_1 = 0.319381530$, $c_2 = -0.356563782$, $c_3 = 1.781477937$, $c_4 = -1.821255978$, and $c_5 = 1.330274429$.

For illustration, using the earlier approximation, we obtain $\Phi(1.96) = 0.975002$, $\Phi(0.82) = 0.793892$, and $\Phi(-0.61) = 0.270931$. These probabilities are very close to that obtained from a typical normal probability table.

EXERCISES

6.1. Assume that the log price $p_t = \ln(P_t)$ follows a stochastic differential equation

$$dp_t = \gamma\, dt + \sigma\, dw_t,$$

where w_t is a Wiener process. Derive the stochastic equation for the price P_t.

6.2. Considering the forward price F of a nondividend-paying stock, we have

$$F_{t,T} = P_t e^{r(T-t)},$$

where r is the risk-free interest rate, which is constant, and P_t is the current stock price. Suppose P_t follows the geometric Brownian motion $dP_t = \mu P_t\, dt + \sigma P_t\, dw_t$. Derive a stochastic diffusion equation for $F_{t,T}$.

6.3. Assume that the price of IBM stock follows the Ito process

$$dP_t = \mu P_t\, dt + \sigma P_t\, dw_t,$$

where μ and σ are constant and w_t is a standard Brownian motion. Consider the daily log returns of IBM stock in 1997. The average return and the sample standard deviation are 0.00131 and 0.02215, respectively. Use the data to estimate the parameters μ and σ assuming that there were 252 trading days in 1997.

6.4. Suppose that the current price of a stock is \$120 per share with volatility $\sigma = 50\%$ per annum. Suppose further that the risk-free interest rate is 7% per annum and the stock pays no dividend. (a) What is the price of a European call option contingent on the stock with a strike price of \$125 that will expire in 3 months? (b) What is the price of a European put option on the same stock with a strike price of \$118 that will expire in 3 months? If the volatility σ is increased to 80% per annum, then what are the prices of the two options?

6.5. Derive the limiting marginal effects of the five variables K, P_t, $T - t$, σ, and r on a European put option contingent on a stock.

6.6. A stock price is currently \$60 per share and follows the geometric Brownian motion $dP_t = \mu P_t\, dt + \sigma P_t\, dt$. Assume that the expected return μ from the stock is 20% per annum and its volatility is 40% per annum. What is the probability distribution for the stock price in 2 years? Obtain the mean and standard deviation of the distribution and construct a 95% confidence interval for the stock price.

6.7. A stock price is currently \$60 per share and follows the geometric Brownian motion $dP_t = \mu P_t\, dt + \sigma P_t\, dt$. Assume that the expected return μ from the stock is 20% per annum and its volatility is 40% per annum. What is the probability distribution for the continuously compounded rate of return of the stock over 2 years? Obtain the mean and standard deviation of the distribution.

6.8. Suppose that the current price of stock A is \$70 per share and the price follows the jump diffusion model in Eq. (6.26). Assume that the risk-free interest rate is 8% per annum, the stock pays no dividend, and its volatility (σ) is 30% per annum. In addition, the price on average has about 15 jumps per year with average jump size −2% and jump standard error 3%. What is the price of a European call option with strike price \$75 that will expire in 3 months? What is the price of the corresponding European put option?

6.9. Consider the European call option of a nondividend-paying stock. Suppose that $P_t = \$20$, $K = \$18$, $r = 6\%$ per annum, and $T - t = 0.5$ year. If the price of a European call option of the stock is \$2.10, what opportunities are there for an arbitrageur?

6.10. Consider the put option of a nondividend-paying stock. Suppose that $P_t = \$44$, $K = \$47$, $r = 6\%$ per annum, and $T - t = 0.5$ year. If the European put option of the stock is selling at \$1.00, what opportunities are there for an arbitrageur?

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