Problem Set 3

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Problem 1.

1. We have :

$$\ell_n = \sum_{i=1}^{n} (\log\theta + \theta\log\tau - (\theta+1)\log X_i)\mathbf{1}(X_i \geqslant \tau)$$

$$\Rightarrow \frac{\partial \ell_n}{\partial\theta} = \sum_{i=1}^{n} (\frac{1}{\theta} + \log\tau - \log X_i)\mathbf{1}(X_i \geqslant \tau) = 0$$

$$\Leftrightarrow \theta = \frac{\sum_{i=1}^{n}(\mathbf{1}(X_i \geqslant \tau))}{\sum_{i=1}^{n}(\log X_i - \log\tau)\mathbf{1}(X_i \geqslant \tau)}$$

3.

$$\ell_n = \sum_{i=1}^{n} (\frac{1}{2}\log\theta + (\sqrt{\theta}-1)\log X_i)\mathbf{1}(0 \leqslant X_i \leqslant 1)$$

$$\Rightarrow \frac{\partial \ell_n}{\partial\theta} = (\frac{1}{2\theta} + \frac{\log X_i}{2\sqrt{\theta}})\mathbf{1}(0 \leqslant X_i \leqslant 1) = 0$$

$$\Leftrightarrow \theta = \Big(\frac{\sum_{i=1}^{n} 1}{\sum_{i=1}^{n}\log X_i}\Big)^2$$

4.

$$\ell_n = \sum_{i=1}^{n} (\log X_i - 2\log\theta - \frac{{X_i}^2}{2\theta^2})\mathbf{1}(X_i \geqslant 0)$$

$$\Leftrightarrow \frac{\partial \ell_n}{\partial\theta} = \sum_{i=1}^{n} (-\frac{2}{\theta} + \frac{{X_i}^2}{\theta^3})\mathbf{1}(X_i \geqslant 0) = 0$$

$$\Leftrightarrow \theta = \frac{\sum_{i=1}^{n}(X_i)\mathbf{1}(X_i \geqslant 0)}{\sum_{i=1}^{n}\sqrt{2}\mathbf{1}(X_i \geqslant 0)}$$

5.

$$\ell_n = \sum_{i=1}^{n} (\log\theta + \log\tau + (\tau-1)\log X_i - \theta {X_i}^\tau)\mathbf{1}(X_i \geqslant 0)$$

$$\Leftrightarrow \frac{\partial \ell_n}{\partial\theta} = \sum_{i=1}^{n} (\frac{1}{\theta} - {X_i}^\tau)\mathbf{1}(X_i \geqslant 0)$$

$$\Leftrightarrow \theta = \frac{\sum_{i=1}^{n} 1\mathbf{1}(X_i \geqslant 0)}{\sum_{i=1}^{n}({X_i}^\tau)\mathbf{1}(X_i \geqslant 0)}$$

Problem 2. We have :

$$\mathcal{L}(\mu,\sigma) = \prod_{i=1}^{n} \Big(\frac{1}{\sigma}\exp(-\frac{1}{2\sigma^2}(X_i-\mu)^2)\Big)$$

$$\ell_n = \log \mathcal{L}(\mu, \sigma) = \sum_{i=1}^{n} \Big(-\log \sigma - \frac{1}{2\sigma^2}(X_i - \mu)^2 \Big)$$

$$= \sum_{i=1}^{n} \Big(-\log \sigma - \frac{1}{2\sigma^2}(X_i - \bar{X}_n + \bar{X}_n - \mu)^2 \Big)$$

$$= \sum_{i=1}^{n} \Big(-\log \sigma - \frac{1}{2\sigma^2}\Big((X_i - \bar{X})^2 + 2(X_i - \bar{X})(\bar{X} - \mu) + (\bar{X} - \mu)^2 \Big) \Big)$$

$$-n\log \sigma - \frac{1}{2\sigma^2}\Big(\sum_{i=1}^{n}(X_i - \bar{X})^2 + 2(\bar{X} - \mu)\sum_{i=1}^{n}(X_i - \bar{X}) + n(\bar{X} - \mu)^2 \Big)$$

Because $\sum_{i=1}^{n}(X_i - \bar{X} = 0)$, we get :

$$\ell_n = -n\log \sigma - \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{2\sigma^2} - \frac{n(\bar{X} - \mu)}{2\sigma^2}$$

$$\Rightarrow \begin{cases} \frac{\partial \ell_n}{\partial \mu} = \frac{n(\bar{X} - \mu)}{\sigma^2} = 0 \\ \frac{\partial \ell_n}{\partial \sigma} = \frac{-n}{\sigma} + \frac{1}{\sigma^3}\Big(\sum_{i=n}^{n}(X_i - \bar{X}_n)^2 + n(\bar{X}_n - \mu)^2 \Big) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \mu = \bar{X}_n \\ \sigma^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}{n} \end{cases}$$

Prove this this is consitent:

To do this, we need to prove bias $\to 0$ and $se \to 0$ as n $\to \infty$.

• Prove μ_{MLE} is consistent: $\mathbb{E}[\bar{X}_n] = \mathbb{E}[\frac{1}{n}\sum_{i=1}^{n} X_i] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \frac{1}{n}\sum_{i=1}^{n}\mu = \mu$. Hence, $bias(\mu_{MLE}) = \mathbb{E}[\mu_{MLE}] - \mu = \mathbb{E}[\bar{X}_n] - \mu = 0$. (1)

$\mathbb{V}[\bar{X}_n] = \mathbb{V}[\frac{1}{n}\sum_{i=1}^{n} X_i] = \frac{1}{n^2}\mathbb{V}[\sum_{i=1}^{n} X_i] = \frac{\sigma^2}{n} \longrightarrow 0$ as $n \longrightarrow \infty$. Hence, $se(\mu_{MLE}) = se(\bar{X}_n) = \sqrt{\mathbb{V}[\bar{X}_n]} \longrightarrow 0$ as $n \longrightarrow \infty$. (2)

Using (1) and (2), we conclude that μ_{MLE} is consistent.

• Prove σ^2_{MLE} is consistent:

We have $\mathbb{E}[\sigma^2_{MLE}] = \mathbb{E}\Big[\sum_{i=1}^{n}\frac{(X_i - \bar{X}_n)^2}{n}\Big] \Rightarrow n\mathbb{E}[\sigma^2_{MLE}] = \mathbb{E}\Big[\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\Big]$

$$n\mathbb{E}[\sigma^2_{MLE}] = \mathbb{E}\Big[\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\Big] = \mathbb{E}\Big[\sum_{i=1}^{n}((X_i - \mu) + (\mu + \bar{X}_n))^2\Big]$$

$$= \mathbb{E}\Big[\sum_{i=1}^{n}(X_i - \mu)^2 + \sum_{i=1}^{n}(\mu - \bar{X}_n)^2 + 2\sum_{i=1}^{n}(X_i - \mu)(\mu - \bar{X}_n)\Big]$$

$$= \mathbb{E}\Big[\sum_{i=1}^{n}(X_i - \mu)^2 + n(\mu - \bar{X}_n)^2 + 2(\mu - \bar{X}_n)\sum_{i=1}^{n}(X_i - \mu)\Big]$$

$$= \mathbb{E}\Big[\sum_{i=1}^{n}(X_i - \mu)^2 + n(\mu - \bar{X}_n)^2 + 2(\mu - \bar{X}_n)n(\bar{X}_n - \mu)\Big]$$

$$= \mathbb{E}\Big[\sum_{i=1}^{n}(X_i - \mu)^2 - n(\mu - \bar{X}_n)^2\Big] = \sum_{i=1}^{n}\mathbb{E}[(X_i - \mu)^2] - n\mathbb{E}[(\mu - \bar{X}_n)^2]$$

$$= n\sigma^2 - n\mathbb{V}[\bar{X}_n] = n\sigma^2 - n\frac{\sigma^2}{n} = n\sigma^2 - \sigma^2 = (n-1)\sigma^2$$

So, $\mathbb{E}[\sigma^2_{MLE}] = \frac{n-1}{n}\sigma^2 \longrightarrow 0$ as $n \longrightarrow \infty$. (3)

We have: $\frac{n\sigma^2_{MLE}}{\sigma^2} \sim \chi^2_{n-1}$. Hence $\mathbb{V}[\frac{n\sigma^2_{MLE}}{\sigma^2}] = \mathbb{V}[\chi^2_{n-1}] = 2(n-1)$, which equivalents to $\mathbb{V}[\sigma^2_{MLE}] = \frac{2(n-1)\sigma^4}{n^2}$.

This actually tends to 0 as n tends to infinity. (4)

Using (3) and (4), we conclude that σ^2_{MLE} is consistent.

Problem 3.

1. We have: $KL(N(a,\sigma^2), N(b,\sigma^2)) = \mathbb{E}[N(x|a,\sigma^2)] - \mathbb{E}[N(x|b,\sigma^2)] = a - b.$

2. We have: $KL(Ber(a), Ber(b)) = \mathbb{E}[Ber(x|a)] - \mathbb{E}[Ber(x|b)] = a - b.$

Problem 4.

1. We have: $TV(Uni(0,s), Uni(0,t)) = \frac{1}{2}\int_0^s |\frac{1}{s} - \frac{1}{t}|dx + \frac{1}{2}\int_s^t |-\frac{1}{t}|dx = \frac{t-s}{t}.$

2. We have: $TV(Ber(p), Ber(q)) = \frac{1}{2}\sum_{i=0}^{1} |Ber(i|p) - Ber(i|q)| = \frac{1}{2}(|p-q| + |1-p-(1-q)|) = |p-q|.$

3. From previous question, we get: $TV(Ber(\bar{X_n}), Ber(p)) = |\bar{X_n} - p|$. On the other hand, we know $\bar{X_n}$ converges to p in probability because $\mathbb{P}[|\bar{X_n} - p| > \epsilon] \longrightarrow 0$ (Law of large number). Hence, $Ber(\bar{X_n})$ and $Ber(p)$ converges to zero in probability.

4. I wonder what Dirac distribution is??