Problem Set 10

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Problem 1.

1. We have this result from problem 2 in Problem Set 3:

$$l(\theta) = \sum_{i=1}^{n} \Big(-\frac{1}{2} log(\theta) - \frac{1}{2\theta} X_i^2 \Big)$$

Taking deravite ans set to equal 0, we get:

$$\frac{\partial l(\theta)}{\partial \theta} = \sum_{i=1}^{n} (-\frac{1}{2\theta} + \frac{1}{2\theta^2} X_i^2) = 0 \Longleftrightarrow \theta^{MLE} = \frac{\sum_{i=1}^{n} X_i^2}{n} \tag{1}$$

2. From (1), we have the second deraviate of log-likelihood:

$$\frac{\partial l(\theta)}{\partial \theta^2} = \sum_{i=1}^{n} \Big(\frac{1}{2\theta^2} - \frac{1}{2\theta^3} X_i^2 \Big) \Leftrightarrow I(\theta) = -\sum_{i=1}^{n} \mathbb{E}[\frac{1}{2\theta^2} - \frac{1}{2\theta^3} X_i^2] = \frac{n}{2\theta^2}$$

Hence,

$$(\theta^{MLE} - \theta) \xrightarrow[n \to \infty]{\text{(d)}} N(0, \frac{2\theta^2}{n})$$

3.
a) From previous question, we know: $I(\theta) = \frac{n}{2\theta^2}$. Hence, $\pi(\theta) = c\sqrt{detI(\theta)} = c\sqrt{\frac{n}{2\theta^2}}$. This is improper prior
b) We have

$$\pi(\theta|X_1, .., X_n) = \frac{\pi(\theta) p_n(X_1, .., X_n|\theta))}{\int_0^\infty p_n(X_1, .., X_n|t) d\pi(t)} = \frac{c\sqrt{\frac{n}{2\theta^2}} \frac{1}{(2\pi\theta)^{\frac{n}{2}}} e^{-\frac{1}{2\theta}\sum_i^n X_i^2}}{\int_0^\infty p_n(X_1, .., X_n|t) d\pi(t)} = \frac{c\sqrt{n}}{(2\pi\theta)^{\frac{n}{2}}\sqrt{2\theta^2}} e^{-\frac{1}{2\theta}\sum_i^n X_i^2}$$

Hence, we can rewrite the posterio:

$$\frac{c\sqrt{n}}{(2\pi\theta)^{\frac{n}{2}}\sqrt{2\theta^2}} e^{-\frac{1}{2\theta}\sum_i^n X_i^2} = \frac{c\sqrt{n}}{(2\pi)^{\frac{n}{2}}\sqrt{2}} \frac{e^{-\frac{\frac{1}{2}\sum_{i=1}^n}{\theta}}}{\theta^{\frac{n}{2}+1}} = \frac{c\sqrt{n}}{(2\pi)^{\frac{n}{2}}\sqrt{2}} \frac{e^{-\frac{\beta}{\theta}}}{\theta^{\alpha+1}}$$

Where $\alpha = \frac{n}{2}$, $\beta = \frac{\sum_{i=1}^n X_i^2}{2}$ and c is the constant satisfied $\frac{c\sqrt{n}}{(2\pi)^{\frac{n}{2}}\sqrt{2}} = \frac{\beta^\alpha}{\tau(\alpha)}$. This posterior is Gamma distribution with parameters α and β. c)
We know $\hat{\theta^{(\pi)}} = \int_0^\infty \theta d\pi(\theta|X_1, .., X_n)$ is the expectation of Gamma distribution with parameters α and β. Hence, $\hat{\theta^{(\pi)}} = \frac{\beta}{\alpha-1} = \frac{\sum_{i=1}^n X_i^2}{n-2}$.

Problem 2.

1. Conditionally on β, we have $Y - X\beta \sim N_p(0, \sigma^2 I_n)$. Hence, $Y \sim N_p(X\beta, \sigma^2 I_n)$.

2.
a) We know:

$$\pi(\beta|X_1,..,X_n) \propto \pi(\beta)p_n(X_1,..,X_n|\beta)) \propto e^{\frac{1}{\sigma^2}||Y-X\beta||_2^2} e^{\frac{1}{\tau^2}||\beta||_2^2} = e^{\frac{1}{\sigma^2}\left(||Y-X\beta||_2^2 + \frac{\sigma^2}{\tau^2}||\beta||_2^2\right)}$$

b) Let $\pi(\beta|X_1,..,X_n) = Ke^{\frac{1}{\sigma^2}\left(||Y-X\beta||_2^2 + \frac{\sigma^2}{\tau^2}||\beta||_2^2\right)}$
We know:

$$\frac{1}{\sigma^2}\Big(||Y-X\beta||_2^2 + \frac{\sigma^2}{\tau^2}||\beta||_2^2\Big) = \frac{1}{\sigma^2}\Big((Y-X\beta)^T(Y-X\beta) + \frac{\sigma^2}{\tau^2}\beta^T\beta\Big)$$

$$= \frac{1}{\sigma^2}\Big(Y^TY - 2\beta^TX^TY + \beta^TX^TX\beta + \frac{\sigma^2}{\tau^2}\beta^T\beta\Big) = \frac{1}{\sigma^2}\Big(Y^TY - 2\beta^TX^TY\Big) + \beta^T(\frac{1}{\sigma^2}X^TX + \frac{1}{\tau^2}I)\beta$$

Hence,

$$\pi(\beta|X_1,..,X_n) = Ke^{\frac{1}{\sigma^2}\left(Y^TY-2\beta^TX^TY\right)+\beta^T(\frac{1}{\sigma^2}X^TX+\frac{1}{\tau^2}I)\beta} = K_1e^{\frac{-2}{\sigma^2}\beta^TX^TY+\beta^T(\frac{1}{\sigma^2}X^TX+\frac{1}{\tau^2}I)\beta}$$

$$= K_1e^{\beta^T(\frac{1}{\sigma^2}X^TX+\frac{1}{\tau^2}I)\beta-\frac{2}{\sigma^2}\beta^TX^TY}$$

Let $\sum^{-1} = \frac{1}{\sigma^2}X^TX + \frac{1}{\tau^2}I$ and $\mu = (X^TX + \frac{\sigma^2}{\tau^2}I)^{-1}X^Ty$. We can see that $\frac{1}{\sigma^2}X^Ty = \sum^{-1}\mu$. Therefore, we can rewrite

$$\pi(\beta|X_1,..,X_n) = K_1e^{\beta^T\sum^{-1}\beta-2\beta\sum^{-1}\mu} = K_1e^{\beta^T\sum^{-1}\beta-2\beta\sum^{-1}\mu} = K_1e^{\beta^T\sum^{-1}\beta-2\beta\sum^{-1}\mu+\mu^T\sum^{-1}\mu-\mu^T\sum^{-1}\mu}$$

We see that $\mu^T\sum^{-1}\mu$ depends on X and Y which are constants, hence we can get rid this constant by rewrite:

$$\pi(\beta|X_1,..,X_n) = K_2e^{\beta^T\sum^{-1}\beta-2\beta\sum^{-1}\mu+\mu^T\sum^{-1}\mu} = K_2e^{(\beta-\mu)^T\sum^{-1}(\beta-\mu)}$$

c)) The posterior mean of β is the expectation of $g(\beta)$ which is $\mu = (X^TX + \frac{\sigma^2}{\tau^2}I)^{-1}X^Ty$

Problem 3.