Problem Set 4

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Problem 1.

1. We have the log-likelihood function:

$$\ell_n(p) = log(\prod_{i=1}^{n} p^{X_i}(1-p)^{1-X_i}) = log(p)\sum_{i=1}^{n} X_i + log(1-p)\sum_{i=1}^{n}(1-X_i)$$

Hence,

$$\frac{\partial \ell_n(p)}{\partial p} = \frac{\sum_{i=1}^{n} X_i}{p} - \frac{\sum_{i=1}^{n}(1-X_i)}{1-p}$$

Set this equals to 0, we get: $p = \frac{1}{n}\sum_{i=1}^{n} X_i$.
We have:

$$I(\theta) = \mathbb{E}[(\frac{\partial \ell_n(p)}{\partial p})^2] = \mathbb{E}\Big[(\frac{X}{p} - \frac{1-X}{1-p})^2\Big] = \mathbb{E}[\frac{X^2}{p^2}] - 2\mathbb{E}[\frac{X-X^2}{p(1-p)}] + \mathbb{E}[\frac{X^2-2X+1}{(1-p)^2}]$$

From Bernoulli distribution, we know: $\mathbb{E}[X] = p$ and $\mathbb{E}[X^2] = p$. Plugging those thing to equation above, we get:

$$I(\theta) = \frac{p}{p^2} - 2\frac{0-0}{p(1-p)} + \frac{p-2p+1}{(1-p)^2} = \frac{1}{p(1-p)}$$

2. We have the log-likelihood function:

$$\ell_n(\lambda) = log(\prod_{i=1}^{n} e^{-\lambda}\frac{\lambda^{X_i}}{X_i!}) = -n\lambda - \sum_{i=1}^{n} log(X_i!) + log(\lambda)\sum_{i=1}^{n} X_i$$

Hence,

$$\frac{\partial \ell_n(\lambda)}{\partial \lambda} = -n + \frac{1}{\lambda}\sum_{i=1}^{n} X_i$$

Set this equals to 0, we get: $\lambda = \frac{1}{n}\sum_{i=1}^{n} X_i$.
We have: $\frac{\partial^2 \ell_n(\lambda)}{\partial \lambda^2} = \frac{\partial(-1+\frac{X}{\lambda})}{\partial \lambda} = \frac{-X}{\lambda^2}$. Hence, $I(\lambda) = -\mathbb{E}[\frac{\partial^2 \ell_n(\lambda)}{\partial \lambda^2}] = -\mathbb{E}[-\frac{X}{\lambda^2}] = \frac{1}{\lambda}$

3. We have the log-likelihood function:

$$\ell_n(\lambda) = log(\lambda^n e^{-\lambda \sum_{i=1}^{n} X}) = nlog(\lambda) - \lambda\sum_{i=1}^{n} X_i$$

Hence,

$$\frac{\partial \ell_n(\lambda)}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} X_i$$

Set this equals to 0, we get: $\lambda = \frac{n}{\sum_{i=1}^{n} X_i}$.
We have: $\frac{\partial^2 \ell_n(\lambda)}{\partial \lambda^2} = \frac{\partial(\frac{1}{\lambda} - X)}{\partial \lambda} = \frac{-1}{\lambda^2}$. Hence, $I(\lambda) = -\mathbb{E}[\frac{\partial^2 \ell_n(\lambda)}{\partial \lambda^2}] = -\mathbb{E}[\frac{-1}{\lambda^2}] = \frac{1}{\lambda^2}$

4. We have this result from problem 2 in Problem 3:

$$\begin{cases} \mu = \bar{X_n} \\ \sigma^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X_n})^2}{n} \end{cases}$$

Recall from problem 2 in Problem 3:

$$\begin{cases} \frac{\partial \ell_n(\mu,\sigma^2)}{\partial \mu} = \frac{1}{\sigma^2}(X-\mu) \\ \frac{\partial \ell_n(\mu,\sigma^2)}{\partial \sigma^2} = \frac{-1}{2\sigma^2} + \frac{(x-\mu)^2}{2\sigma^4} \end{cases} \Leftrightarrow \begin{cases} \frac{\partial^2 \ell_n(\mu,\sigma^2)}{\partial \mu^2} = \frac{-1}{\sigma^2} \\ \frac{\partial^2 \ell_n(\mu,\sigma^2)}{\partial \mu \sigma^2} = \frac{\partial^2 \ell_n(\mu,\sigma^2)}{\partial \sigma^2 \mu} = \frac{-2(X-\mu)}{\sigma^3} \\ \frac{\partial^2 \ell_n(\mu,\sigma^2)}{\partial \sigma^2 \sigma^2} = \frac{1}{2\sigma^4} - \frac{(X-\mu)^2}{2\sigma^6} \end{cases}$$

Hence,

$$I(\mu,\sigma^2) = \mathbb{E}\begin{bmatrix} \frac{\partial \ell_n(\mu,\sigma^2)}{\partial \mu^2} & \frac{\partial \ell_n(\mu,\sigma^2)}{\partial \mu \sigma^2} \\ \frac{\partial \ell_n(\mu,\sigma^2)}{\partial \sigma^2 \mu} & \frac{\partial \ell_n(\mu,\sigma^2)}{\partial \sigma^2 \sigma^2} \end{bmatrix} = -\mathbb{E}\begin{bmatrix} -\frac{1}{\sigma^2} & \frac{-2(X-\mu)}{\sigma^3} \\ \frac{-2(X-\mu)}{\sigma^3} & \frac{1}{2\sigma^4} - \frac{(X-\mu)^2}{2\sigma^6} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{bmatrix}$$

5. We have the log-likelihood function:

$$\ell_n(\lambda, a) = log(\prod_{i=1}^{n} \lambda e^{-\lambda(X_i - a)} \mathbb{1}_{x \geq a}) = \sum_{i=1}^{n} = nlog(\lambda) - \lambda \sum_{i=1}^{n}(X_i - a) + \sum_{i=1}^{n} log(\mathbb{1}_{X_i \geq a})$$

Hence,

$$\frac{\partial \ell_n(\lambda, a)}{\partial a} = n\lambda$$

Because $n\lambda$ for all $\lambda > 0$, $\ell_n(\lambda, a)$ is an increasing function of a until $a > min(X_i)$. Hence $\ell_n(\lambda, a)$ is maximal with respect to a when a is made as large as possible without exceeding the minimum order statistic $\Longrightarrow a = min(X_i)$

Let's denote $y = x - a$, hence $f_y(y) = \frac{d}{dx}(x-a)f_x y = f_x(y)$ which is exponential distribution.

From question 3, we know $\lambda = \frac{n}{\sum_{i=1}^{n} Y_i} = \frac{n}{\sum_{i=1}^{n}(X_i - a)}$

We have:

$$\begin{cases} \frac{\partial \ell_n(\lambda,a)}{\partial \lambda} = \frac{1}{\lambda} - (X-a) \\ \frac{\partial \ell_n(\lambda,a)}{\partial a} = \lambda \end{cases} \Leftrightarrow \begin{cases} \frac{\partial^2 \ell_n(\lambda,a)}{\partial \lambda^2} = \frac{-1}{\lambda^2} \\ \frac{\partial^2 \ell_n(\lambda,a)}{\partial \lambda a} = \frac{\partial^2 \ell_n(\lambda,a)}{\partial a \lambda} = 1 \\ \frac{\partial^2 \ell_n(\lambda,a)}{\partial a^2} = 0 \end{cases}$$

Hence,

$$I(\lambda, a) = \mathbb{E}\begin{bmatrix} \frac{\partial \ell_n(\lambda,a)}{\partial \lambda^2} & \frac{\partial \ell_n(\lambda,a)}{\partial \lambda a} \\ \frac{\partial \ell_n(\lambda,a)}{\partial a \lambda} & \frac{\partial \ell_n(\lambda,a)}{\partial a^2} \end{bmatrix} = -\mathbb{E}\begin{bmatrix} -\frac{1}{\lambda^2} & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda^2} & 1 \\ 1 & 0 \end{bmatrix}$$

6. We have the log-likelihood function:

$$\ell_n(\mu, \sigma^2) = log(\prod_{i=1}^{n} \frac{1}{X_i\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(log(X_i) - \mu)^2}) = -\frac{n}{2}log(2\pi) - \frac{n}{2}log(\sigma^2) - \sum_{i=1}^{n} log(X_i) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(log(X_i) - \mu)^2$$

Hence,

$$\frac{\partial \ell_n(\mu, \sigma^2)}{\partial \mu} = \frac{1}{\sigma^2}\Big(\sum_{i=1}^{n} log(X_i) - n\mu\Big)$$

Set this equals to 0, we get: $\mu = \frac{1}{n}\sum_{i=1}^{n} log(X_i)$.

On the other hand,

$$\frac{\partial \ell_n(\mu, \sigma^2)}{\partial \sigma^2} = \frac{-n}{2\sigma^2} + \frac{\sum_{i=1}^{n}(log(X_i) - \mu)^2}{2\sigma^4}$$

Set this equals to 0, we get: $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(log(X_i) - \mu)^2$.

We have:

$$\begin{cases} \frac{\partial \ell_n(\mu,\sigma^2)}{\partial \mu} = \frac{1}{\sigma^2}(log(X)-\mu) \\ \frac{\partial \ell_n(\mu,\sigma^2)}{\partial \sigma^2} = \frac{-1}{2\sigma^2} + \frac{(log(x)-\mu)^2}{2\sigma^4} \end{cases} \Leftrightarrow \begin{cases} \frac{\partial^2 \ell_n(\mu,\sigma^2)}{\partial \mu^2} = \frac{-1}{\sigma^2} \\ \frac{\partial^2 \ell_n(\mu,\sigma^2)}{\partial \mu \sigma^2} = \frac{\partial^2 \ell_n(\mu,\sigma^2)}{\partial \sigma^2 \mu} = \frac{-2(log(X)-\mu)}{\sigma^3} \\ \frac{\partial^2 \ell_n(\mu,\sigma^2)}{\partial \sigma^2 \sigma^2} = \frac{1}{2\sigma^4} - \frac{(log(X)-\mu)^2}{2\sigma^6} \end{cases}$$

Hence,

$$I(\mu,\sigma^2) = \mathbb{E}\begin{bmatrix} \frac{\partial \ell_n(\mu,\sigma^2)}{\partial \mu^2} & \frac{\partial \ell_n(\mu,\sigma^2)}{\partial \mu \sigma^2} \\ \frac{\partial \ell_n(\mu,\sigma^2)}{\partial \sigma^2 \mu} & \frac{\partial \ell_n(\mu,\sigma^2)}{\partial \sigma^2 \sigma^2} \end{bmatrix} = -\mathbb{E}\begin{bmatrix} -\frac{1}{\sigma^2} & \frac{-2(log(X)-\mu)}{\sigma^3} \\ \frac{-2(log(X)-\mu)}{\sigma^3} & \frac{1}{2\sigma^4} - \frac{(log(X)-\mu)^2}{2\sigma^6} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{bmatrix}$$

The last equation comes from the fact that $ln(X) \sim N(\mu, \sigma^2)$ because of the following reason:
Let $y = log(x)$

$$f_y(y) = f_x(e^y)e^y = \frac{e^y}{e^y\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(log(e^y)-\mu)^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-\mu)^2}$$

Hence, $y \sim N(\mu,\sigma^2) \implies \mathbb{E}[y] = \mathbb{E}[log(x)] = \mu \implies \mathbb{E}[\log(x)-\mu] = 0$. There is an another better approach for the first part of this question (maximum likelihood). Let $y = log(x)$, we have $y \sim N(\mu,\sigma^2)$ (which was proved a few lines above). Now, We come up with question 4.

Problem 2.

1. We have the first moment $m_1 = \mathbb{E}[X] = p$. Hence, $p_{MM} = \frac{1}{n}\sum_{i=1}^n X_i$.

2. We have the first moment $m_1 = \mathbb{E}[X] = \lambda$. Hence, $\lambda_{MM} = \frac{1}{n}\sum_{i=1}^n X_i$.

3. We have the first moment $m_1 = \mathbb{E}[X] = \frac{1}{\lambda}$. Hence, $\frac{1}{\lambda_{MM}} = \frac{1}{n}\sum_{i=1}^n \Leftrightarrow \lambda_{MM} = \frac{n}{\sum_{i=1}^n X_i}$.

4. We have:

$$\begin{cases} m_1 = \mathbb{E}[X] = \mu \\ m_2 = \mathbb{E}[X^2] = \mu^2 + \sigma^2 \end{cases}$$

Hence,

$$\begin{cases} \mu_{MM} = \frac{1}{n}\sum_{i=1}^n X_i \\ \mu_{MM}^2 + \sigma_{MM}^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 \end{cases} \Leftrightarrow \begin{cases} \mu_{MM} = \frac{1}{n}\sum_{i=1}^n X_i \\ \sigma_{MM}^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 - \mu_{MM}^2 \end{cases}$$

5. Let's denote $y = x - a$, from question 5 problem 1 we have $y \sim Exp(\lambda)$. Hence:

$$\begin{cases} \mathbb{E}[X] = \mathbb{E}[Y] + a = \frac{1}{\lambda} + a \\ \mathbb{E}[X^2] = \mathbb{E}[(Y+a)^2] = \frac{2}{\lambda^2} + \frac{2a}{\lambda} + a^2 \end{cases}$$

. We have system equation:

$$\begin{cases} \frac{1}{\lambda_{MM}} + a_{MM} = \bar{X_n} \\ \frac{2}{\lambda_{MM}^2} + \frac{2a_{MM}}{\lambda_{MM}} + a_{MM}^2 = \bar{X_n}^2 \end{cases} \Leftrightarrow \begin{cases} \lambda_{MM} = \frac{1}{\sqrt{\bar{X_n}^2 - \bar{X_n}}} \\ a = \bar{X_n} - \frac{1}{\lambda_{MM}} \end{cases}$$

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6. Let's denote $Y = log(X)$. From question 6 Problem 1 we know $Y \sim N(\mu,\sigma)$
Form question 4, we have

$$\begin{cases} \mu_{MM} = \frac{1}{n}\sum_{i=1}^n Y_i = \frac{1}{n}\sum_{i=1}^n log(X_i) \\ \sigma_{MM}^2 = \frac{1}{n}\sum_{i=1}^n Y_i^2 - \mu_{MM}^2 = \frac{1}{n}\sum_{i=1}^n log(X_i)^2 - \mu_{MM}^2 \end{cases}$$

Problem 3.

1. $\mathbb{P}[X=1] = \mathbb{P}[Exp(\lambda) > z] = e^{-\lambda z}$. Hence X_i's follow Bernoulli distribution with parameter $p = e^{-\lambda z}$.

2. We have $\bar{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$. By CLL, $\bar{X_n} \sim N(\mathbb{E}[X], \frac{\mathbb{V}[X]}{n})$. From question 1, we have $\mathbb{E}[X] = p = e^{-\lambda z}$ and $\mathbb{V}[X] = p(1-p) = e^{-\lambda z}(1-e^{-\lambda z})$. Hence, $\bar{X_n} \sim N(e^{-\lambda z}, \frac{e^{-\lambda z}(1-e^{-\lambda z})}{n})$.

3. From question 1 problem 1 we have $\bar{X_n}$ is a consistent estimator of $p = e^{-\lambda z}$. Hence, $\frac{-log(\bar{X_n})}{z}$ is a consistent estimator of λ. Hence, $f(\bar{X_n}) = \frac{-log(\bar{X_n})}{z}$

4. By CLT, we have $\sqrt{n}(\bar{X_n} - p) \longrightarrow N(0, p(1-p))$
On the other hand, let $f(x) = \frac{-log(x)}{z} \Rightarrow \frac{\partial f(x)}{\partial x} = \frac{-1}{zx} \Rightarrow \frac{\partial f(x)}{\partial x}(p) = \frac{-1}{zp}$. Using Delta-method, we know:

$$\sqrt{n}(f(\bar{X_n}) - f(p)) \longrightarrow N(0, (\frac{-1}{zp})^2 p(1-p)) \Leftrightarrow \sqrt{n}(f(\bar{X_n}) - \frac{-log(p)}{z}) \longrightarrow N(0, \frac{1-p}{z^2 p})$$

5. Let denote $f(p) = \frac{1-p}{z^2 p}$. Hence, $\frac{\partial f(p)}{\partial p} = \frac{-1}{z^2 p^2} \leqslant 0 \Rightarrow argmin f(p) = 1$. Hence, $e^{-\lambda z} = 1 \Leftrightarrow g_\lambda(z) = z e^{-\lambda z} = z$

6.
a) The statistical model is $(\mathbb{R}+, (Exp(\lambda))_{\lambda > 0})$.
b) From question 3 problem 1, we know $I_Y(\lambda) = \frac{1}{\lambda^2}$.
c) From question 1 problem 1, we know $I_X(\lambda) = \frac{1}{p(1-p)} = \frac{1}{e^{-\lambda z}(1-e^{-\lambda z})}$.
d) I wonder what value of threshold z is?