Problem Set 4

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Problem 1.

1. We have this result from problem 2 in Problem Set 3:

$$l(\theta) = \sum_{i=1}^{n} \Big(-\frac{1}{2}log(\theta) - \frac{1}{2\theta}X_i^2 \Big)$$

Taking deravite ans set to equal 0, we get:

$$\frac{\partial l(\theta)}{\partial \theta} = \sum_{i=1}^{n}(-\frac{1}{2\theta} + \frac{1}{2\theta^2}X_i^2) = 0 \Longleftrightarrow \theta^{MLE} = \frac{\sum_{i=1}^{n} X_i^2}{n} \tag{1}$$

2. From (1), we have the second deraviate of log-likelihood:

$$\frac{\partial l(\theta)}{\partial \theta^2} = \sum_{i=1}^{n} \Big(\frac{1}{2\theta^2} - \frac{1}{2\theta^3}X_i^2 \Big) \Leftrightarrow I(\theta) = -\sum_{i=1}^{n} \mathbb{E}[\frac{1}{2\theta^2} - \frac{1}{2\theta^3}X_i^2] = \frac{n}{2\theta^2}$$

Hence,

$$(\theta^{MLE} - \theta) \xrightarrow[n\to\infty]{(d)} N(0, \frac{2\theta^2}{n})$$

3.
a) From previous question, we know: $I(\theta) = \frac{n}{2\theta^2}$. Hence, $\pi(\theta) = c\sqrt{detI(\theta)} = c\sqrt{\frac{n}{2\theta^2}}$. This is improper prior
b) We have

$$\pi(\theta|X_1, .., X_n) = \frac{\pi(\theta)p_n(X_1, .., X_n|\theta))}{\int_0^{\infty} p_n(X_1, .., X_n|t)d\pi(t)} = \frac{c\sqrt{\frac{n}{2\theta^2}}\frac{1}{(2\pi\theta)^{\frac{n}{2}}}e^{-\frac{1}{2\theta}\sum_i^n X_i^2}}{\int_0^{\infty} p_n(X_1, .., X_n|t)d\pi(t)} = \frac{c\sqrt{n}}{(2\pi\theta)^{\frac{n}{2}}\sqrt{2\theta^2}}e^{-\frac{1}{2\theta}\sum_i^n X_i^2}$$

Hence, we can rewrite the posterio:

$$\frac{c\sqrt{n}}{(2\pi\theta)^{\frac{n}{2}}\sqrt{2\theta^2}}e^{-\frac{1}{2\theta}\sum_i^n X_i^2} = \frac{c\sqrt{n}}{(2\pi)^{\frac{n}{2}}\sqrt{2}}\frac{e^{-\frac{\frac{1}{2}\sum_{i=1}^n}{\theta}}}{\theta^{\frac{n}{2}+1}} = \frac{c\sqrt{n}}{(2\pi)^{\frac{n}{2}}\sqrt{2}}\frac{e^{-\frac{\beta}{\theta}}}{\theta^{\alpha+1}}$$

Where $\alpha = \frac{n}{2}$, $\beta = \frac{\sum_{i=1}^n X_i^2}{2}$ and c is the constant satisfied $\frac{c\sqrt{n}}{(2\pi)^{\frac{n}{2}}\sqrt{2}} = \frac{\beta^\alpha}{\tau(\alpha)}$. This posterior is Gamma distribution with parameters α and β. c)
We know $\theta^{\hat{(\pi)}} = \int_0^{\infty}\theta d\pi(\theta|X_1, .., X_n)$ is the expectation of Gamma distribution with parameters α and β. Hence, $\theta^{\hat{(\pi)}} = \frac{\beta}{\alpha-1} = \frac{\sum_{i=1}^n X_i^2}{n-2}$.

Problem 2.

Problem 3.