

Problem Set 4

Hoang Nguyen, Huy Nguyen

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Problem 1.

1. We have the log-likelihood function:

$$\ell_n(p) = \log\left(\prod_{i=1}^n p^{X_i} (1-p)^{1-X_i}\right) = \log(p) \sum_{i=1}^n X_i + \log(1-p) \sum_{i=1}^n (1-X_i)$$

Hence,

$$\frac{\partial \ell_n(p)}{\partial p} = \frac{\sum_{i=1}^n X_i}{p} - \frac{\sum_{i=1}^n (1-X_i)}{1-p}$$

Set this equals to 0, we get: $p = \frac{1}{n} \sum_{i=1}^n X_i$.

We have:

$$I(\theta) = \mathbb{E}\left[\left(\frac{\partial \ell_n(p)}{\partial p}\right)^2\right] = \mathbb{E}\left[\left(\frac{X}{p} - \frac{1-X}{1-p}\right)^2\right] = \mathbb{E}\left[\frac{X^2}{p^2}\right] - 2\mathbb{E}\left[\frac{X-X^2}{p(1-p)}\right] + \mathbb{E}\left[\frac{X^2-2X+1}{(1-p)^2}\right]$$

From Bernoulli distribution, we know: $\mathbb{E}[X] = p$ and $\mathbb{E}[X^2] = p$. Plugging those thing to equation above, we get:

$$I(\theta) = \frac{p}{p^2} - 2\frac{0-0}{p(1-p)} + \frac{p-2p+1}{(1-p)^2} = \frac{1}{p(1-p)}$$

2. We have the log-likelihood function:

$$\ell_n(\lambda) = \log\left(\prod_{i=1}^n e^{-\lambda} \frac{\lambda^{X_i}}{X_i!}\right) = -n\lambda - \sum_{i=1}^n \log(X_i!) + \log(\lambda) \sum_{i=1}^n X_i$$

Hence,

$$\frac{\partial \ell_n(\lambda)}{\partial \lambda} = -n + \frac{1}{\lambda} \sum_{i=1}^n X_i$$

Set this equals to 0, we get: $\lambda = \frac{1}{n} \sum_{i=1}^n X_i$.

We have: $\frac{\partial^2 \ell_n(\lambda)}{\partial \lambda^2} = \frac{\partial(-1+\frac{X}{\lambda})}{\partial \lambda} = -\frac{X}{\lambda^2}$. Hence, $I(\lambda) = -\mathbb{E}\left[\frac{\partial^2 \ell_n(\lambda)}{\partial \lambda^2}\right] = -\mathbb{E}\left[-\frac{X}{\lambda^2}\right] = \frac{1}{\lambda}$

3. We have the log-likelihood function:

$$\ell_n(\lambda) = \log(\lambda^n e^{-\lambda \sum_{i=1}^n X_i}) = n\log(\lambda) - \lambda \sum_{i=1}^n X_i$$

Hence,

$$\frac{\partial \ell_n(\lambda)}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^n X_i$$

Set this equals to 0, we get: $\lambda = \frac{n}{\sum_{i=1}^n X_i}$.

We have: $\frac{\partial^2 \ell_n(\lambda)}{\partial \lambda^2} = \frac{\partial(\frac{1}{\lambda}-X)}{\partial \lambda} = -\frac{1}{\lambda^2}$. Hence, $I(\lambda) = -\mathbb{E}\left[\frac{\partial^2 \ell_n(\lambda)}{\partial \lambda^2}\right] = -\mathbb{E}\left[-\frac{1}{\lambda^2}\right] = \frac{1}{\lambda^2}$

4. We have this result from problem 2 in Problem 3:

$$\begin{cases} \mu = \bar{X}_n \\ \sigma^2 = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2}{n} \end{cases}$$

Recall from problem 2 in Problem 3:

$$\begin{cases} \frac{\partial \ell_n(\mu, \sigma^2)}{\partial \mu} = \frac{1}{\sigma^2} (X - \mu) \\ \frac{\partial \ell_n(\mu, \sigma^2)}{\partial \sigma^2} = \frac{-1}{2\sigma^2} + \frac{(x - \mu)^2}{2\sigma^4} \end{cases} \Leftrightarrow \begin{cases} \frac{\partial^2 \ell_n(\mu, \sigma^2)}{\partial \mu^2} = \frac{-1}{\sigma^2} \\ \frac{\partial^2 \ell_n(\mu, \sigma^2)}{\partial \mu \partial \sigma^2} = \frac{\partial^2 \ell_n(\mu, \sigma^2)}{\partial \sigma^2 \partial \mu} = \frac{-2(X - \mu)}{\sigma^3} \\ \frac{\partial^2 \ell_n(\mu, \sigma^2)}{\partial \sigma^2 \partial \sigma^2} = \frac{1}{2\sigma^4} - \frac{(X - \mu)^2}{2\sigma^6} \end{cases}$$

Hence,

$$I(\mu, \sigma^2) = \mathbb{E} \begin{bmatrix} \frac{\partial \ell_n(\mu, \sigma^2)}{\partial \mu^2} & \frac{\partial \ell_n(\mu, \sigma^2)}{\partial \mu \partial \sigma^2} \\ \frac{\partial \ell_n(\mu, \sigma^2)}{\partial \sigma^2 \partial \mu} & \frac{\partial \ell_n(\mu, \sigma^2)}{\partial \sigma^2 \partial \sigma^2} \end{bmatrix} = -\mathbb{E} \begin{bmatrix} -\frac{1}{\sigma^2} & \frac{-2(X - \mu)}{\sigma^3} \\ \frac{-2(X - \mu)}{\sigma^3} & \frac{1}{2\sigma^4} - \frac{(X - \mu)^2}{2\sigma^6} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{bmatrix}$$

5. We have the log-likelihood function:

$$\ell_n(\lambda, a) = \log\left(\prod_{i=1}^n \lambda e^{-\lambda(X_i - a)} \mathbb{1}_{x \geq a}\right) = \sum_{i=1}^n = n \log(\lambda) - \lambda \sum_{i=1}^n (X_i - a) + \sum_{i=1}^n \log(\mathbb{1}_{X_i \geq a})$$

Hence,

$$\frac{\partial \ell_n(\lambda, a)}{\partial a} = n\lambda$$

Because $n\lambda$ for all $\lambda > 0$, $\ell_n(\lambda, a)$ is an increasing function of a until $a > \min(X_i)$. Hence $\ell_n(\lambda, a)$ is maximal with respect to a when a is made as large as possible without exceeding the minimum order statistic $\implies a = \min(X_i)$

Let's denote $y = x - a$, hence $f_y(y) = \frac{d}{dx}(x - a)f_x y = f_x(y)$ which is exponential distribution.

From question 3, we know $\lambda = \frac{n}{\sum_{i=1}^n Y_i} = \frac{n}{\sum_{i=1}^n (X_i - a)}$

We have:

$$\begin{cases} \frac{\partial \ell_n(\lambda, a)}{\partial \lambda} = \frac{1}{\lambda} - (X - a) \\ \frac{\partial \ell_n(\lambda, a)}{\partial a} = \lambda \end{cases} \Leftrightarrow \begin{cases} \frac{\partial^2 \ell_n(\lambda, a)}{\partial \lambda^2} = \frac{-1}{\lambda^2} \\ \frac{\partial^2 \ell_n(\lambda, a)}{\partial \lambda \partial a} = \frac{\partial^2 \ell_n(\lambda, a)}{\partial a \partial \lambda} = 1 \\ \frac{\partial^2 \ell_n(\lambda, a)}{\partial a^2} = 0 \end{cases}$$

Hence,

$$I(\lambda, a) = \mathbb{E} \begin{bmatrix} \frac{\partial \ell_n(\lambda, a)}{\partial \lambda^2} & \frac{\partial \ell_n(\lambda, a)}{\partial \lambda \partial a} \\ \frac{\partial \ell_n(\lambda, a)}{\partial a \partial \lambda} & \frac{\partial \ell_n(\lambda, a)}{\partial a^2} \end{bmatrix} = -\mathbb{E} \begin{bmatrix} -\frac{1}{\lambda^2} & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda^2} & 1 \\ 1 & 0 \end{bmatrix}$$

6. We have the log-likelihood function:

$$\ell_n(\mu, \sigma^2) = \log\left(\prod_{i=1}^n \frac{1}{X_i \sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\log(X_i) - \mu)^2}\right) = -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log(\sigma^2) - \sum_{i=1}^n \log(X_i) - \frac{1}{2\sigma^2} \sum_{i=1}^n (\log(X_i) - \mu)^2$$

Hence,

$$\frac{\partial \ell_n(\mu, \sigma^2)}{\partial \mu} = \frac{1}{\sigma^2} \left(\sum_{i=1}^n \log(X_i) - n\mu \right)$$

Set this equals to 0, we get: $\mu = \frac{1}{n} \sum_{i=1}^n \log(X_i)$.

On the other hand,

$$\frac{\partial \ell_n(\mu, \sigma^2)}{\partial \sigma^2} = \frac{-n}{2\sigma^2} + \frac{\sum_{i=1}^n (\log(X_i) - \mu)^2}{2\sigma^4}$$

Set this equals to 0, we get: $\sigma^2 = \frac{1}{n} \sum_{i=1}^n (\log(X_i) - \mu)^2$.

We have:

$$\begin{cases} \frac{\partial \ell_n(\mu, \sigma^2)}{\partial \mu} = \frac{1}{\sigma^2} (\log(X) - \mu) \\ \frac{\partial \ell_n(\mu, \sigma^2)}{\partial \sigma^2} = \frac{-1}{2\sigma^2} + \frac{(\log(X) - \mu)^2}{2\sigma^4} \end{cases} \Leftrightarrow \begin{cases} \frac{\partial^2 \ell_n(\mu, \sigma^2)}{\partial \mu^2} = \frac{-1}{\sigma^2} \\ \frac{\partial^2 \ell_n(\mu, \sigma^2)}{\partial \mu \partial \sigma^2} = \frac{\partial^2 \ell_n(\mu, \sigma^2)}{\partial \sigma^2 \partial \mu} = \frac{-2(\log(X) - \mu)}{\sigma^3} \\ \frac{\partial^2 \ell_n(\mu, \sigma^2)}{\partial \sigma^2 \partial \sigma^2} = \frac{1}{2\sigma^4} - \frac{(\log(X) - \mu)^2}{2\sigma^6} \end{cases}$$

Hence,

$$I(\mu, \sigma^2) = \mathbb{E} \begin{bmatrix} \frac{\partial \ell_n(\mu, \sigma^2)}{\partial \mu^2} & \frac{\partial \ell_n(\mu, \sigma^2)}{\partial \mu \partial \sigma^2} \\ \frac{\partial \ell_n(\mu, \sigma^2)}{\partial \sigma^2 \partial \mu} & \frac{\partial \ell_n(\mu, \sigma^2)}{\partial \sigma^2 \partial \sigma^2} \end{bmatrix} = -\mathbb{E} \begin{bmatrix} -\frac{1}{\sigma^2} & \frac{-2(\log(X) - \mu)}{\sigma^3} \\ \frac{-2(\log(X) - \mu)}{\sigma^3} & \frac{1}{2\sigma^4} - \frac{(\log(X) - \mu)^2}{2\sigma^6} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{bmatrix}$$

The last equation comes from the fact that $\ln(X) \sim N(\mu, \sigma^2)$ because of the following reason:

Let $y = \log(x)$

$$f_y(y) = f_x(e^y)e^y = \frac{e^y}{e^y \sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\log(e^y) - \mu)^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y - \mu)^2}$$

Hence, $y \sim N(\mu, \sigma^2) \Rightarrow \mathbb{E}[y] = \mathbb{E}[\log(x)] = \mu \Rightarrow \mathbb{E}[\log(x) - \mu] = 0$. There is an another better approach for the first part of this question (maximum likelihood). Let $y = \log(x)$, we have $y \sim N(\mu, \sigma^2)$ (which was proved a few lines above). Now, We come up with question 4.

Problem 2.

1. We have the first moment $m_1 = \mathbb{E}[X] = p$. Hence, $p_{MM} = \frac{1}{n} \sum_{i=1}^n X_i$.
2. We have the first moment $m_1 = \mathbb{E}[X] = \lambda$. Hence, $\lambda_{MM} = \frac{1}{n} \sum_{i=1}^n X_i$.
3. We have the first moment $m_1 = \mathbb{E}[X] = \frac{1}{\lambda}$. Hence, $\frac{1}{\lambda_{MM}} = \frac{1}{n} \sum_{i=1}^n \frac{1}{X_i} \Leftrightarrow \lambda_{MM} = \frac{n}{\sum_{i=1}^n X_i}$.
4. We have:

$$\begin{cases} m_1 = \mathbb{E}[X] = \mu \\ m_2 = \mathbb{E}[X^2] = \mu^2 + \sigma^2 \end{cases}$$

Hence,

$$\begin{cases} \mu_{MM} = \frac{1}{n} \sum_{i=1}^n X_i \\ \mu_{MM}^2 + \sigma_{MM}^2 = \frac{1}{n} \sum_{i=1}^n X_i^2 \end{cases} \Leftrightarrow \begin{cases} \mu_{MM} = \frac{1}{n} \sum_{i=1}^n X_i \\ \sigma_{MM}^2 = \frac{1}{n} \sum_{i=1}^n X_i^2 - \mu_{MM}^2 \end{cases}$$

5. Let's denote $y = x - a$, from question 5 problem 1 we have $y \sim \text{Exp}(\lambda)$. Hence:

$$\begin{cases} \mathbb{E}[X] = \mathbb{E}[Y] + a = \frac{1}{\lambda} + a \\ \mathbb{E}[X^2] = \mathbb{E}[(Y + a)^2] = \frac{2}{\lambda^2} + \frac{2a}{\lambda} + a^2 \end{cases}$$

. We have system equation:

$$\begin{cases} \frac{1}{\lambda_{MM}} + a_{MM} = \bar{X}_n \\ \frac{2}{\lambda_{MM}^2} + \frac{2a_{MM}}{\lambda_{MM}} + a_{MM}^2 = \bar{X}_n^2 \end{cases} \Leftrightarrow \begin{cases} \lambda_{MM} = \frac{1}{\sqrt{\bar{X}_n^2 - \bar{X}_n}} \\ a = \bar{X}_n - \frac{1}{\lambda_{MM}} \end{cases}$$

6. Let's denote $Y = \log(X)$. From question 6 Problem 1 we know $Y \sim N(\mu, \sigma)$
Form question 4, we have

$$\begin{cases} \mu_{MM} = \frac{1}{n} \sum_{i=1}^n Y_i = \frac{1}{n} \sum_{i=1}^n \log(X_i) \\ \sigma_{MM}^2 = \frac{1}{n} \sum_{i=1}^n Y_i^2 - \mu_{MM}^2 = \frac{1}{n} \sum_{i=1}^n \log(X_i)^2 - \mu_{MM}^2 \end{cases}$$

Problem 3.

1. $\mathbb{P}[X = 1] = \mathbb{P}[Exp(\lambda) > z] = 1 - e^{-\lambda z}$. Hence X_i 's follow Bernoulli distribution with parameter $p = 1 - e^{-\lambda z}$.
2. We have $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$. By CLL, $\bar{X}_n \sim N(\mathbb{E}[X], \frac{\mathbb{V}[X]}{n})$. From question 1, we have $\mathbb{E}[X] = p = 1 - e^{-\lambda z}$ and $\mathbb{V}[X] = p(1 - p) = e^{-\lambda z}(1 - e^{-\lambda z})$. Hence, $\bar{X}_n \sim N(1 - e^{-\lambda z}, \frac{e^{-\lambda z}(1 - e^{-\lambda z})}{n})$.
3. From question 3 problem 1, we have $\lambda_{MLE} = \frac{1}{\sum_{i=1}^n X_i} = \frac{1}{\frac{1}{n} \sum_{i=1}^n X_i} = \frac{1}{\bar{X}_n}$. We already know this is actually a consistent estimator of true λ . Hence, $f(\bar{X}_n) = \frac{1}{\bar{X}_n}$ is a consistent estimator of λ .
4. By Fisher Information and result from question 3 problem 1:

$$\sqrt{n}(\lambda_{MLE} - \lambda) \xrightarrow{d} N(0, \lambda^2) \Leftrightarrow \sqrt{n}(f(\bar{X}_n) - \lambda) \xrightarrow{d} N(0, \lambda^2) \Leftrightarrow f(\bar{X}_n) \sim N(\lambda, \frac{\lambda^2}{n})$$