Problem Set 5

Hoang Nguyen, Huy Nguyen

December 4, 2019

Problem 1.

1. From exercise 3 in Problem 2, If $Z \sim N(0,1)$, we get: $\mathbb{P}\Big[|\frac{\sqrt{n}(\bar{X_i}-\mu)}{\sigma}| \leqslant t\Big] = 2\mathbb{P}[Z \leqslant t] - 1$.
$X_1, X_2, ..., X_n$ follow Poisson distribution. Hence, $\mathbb{E}[X_i] = \lambda$ and $\mathbb{V}[X_i] = \lambda$
From CTL: $\frac{\sqrt{n}(\bar{X_n}-\lambda)}{\sqrt{\lambda}} \sim N(0,1)$. Hence, $\mathbb{P}\Big[|\frac{\sqrt{n}(\bar{X_n}-\lambda)}{\sqrt{\lambda}}| \leqslant t\Big] = 2\mathbb{P}\Big[\frac{\sqrt{n}(\bar{X_n}-\lambda)}{\sqrt{\lambda}} \leqslant t\Big] - 1 = 1 - \alpha$.
Hence, $\mathbb{P}\Big[\frac{\sqrt{n}(\bar{X_n}-\lambda)}{\sqrt{\lambda}} \leqslant t\Big] = 1 - \frac{\alpha}{2}$. Therefore, $t = \phi^{-1}(1-\frac{\alpha}{2})$.
We have $= \mathbb{P}\Big[|\frac{\sqrt{n}(\bar{X_n}-\lambda)}{\sqrt{\lambda}}| \leqslant \phi^{-1}(1-\frac{\alpha}{2})\Big] \to (1-\alpha)$ as n tends to infinity. This equivalent to:

$$\mathbb{P}\Big[-\frac{\sqrt{n}(\bar{X_n}-\lambda)}{\sqrt{\lambda}} \leqslant \phi^{-1}(1-\frac{\alpha}{2}) \leqslant \frac{\sqrt{n}(\bar{X_n}-\lambda)}{\sqrt{\lambda}}\Big] \to 1-\alpha$$

$$\mathbb{P}\Big[\bar{X_n} - \frac{\phi^{-1}(1-\frac{\alpha}{2})\sqrt{\lambda}}{\sqrt{n}} \leqslant \lambda \leqslant \bar{X_n} + \frac{\phi^{-1}(1-\frac{\alpha}{2})\sqrt{\lambda}}{\sqrt{n}}\Big] \to 1-\alpha$$

We know: $\bar{X_n} \xrightarrow{P} \mathbb{E}[\bar{X_n}] = \lambda$
Hence,

$$\mathbb{P}\Big[\bar{X_n} - \frac{\phi^{-1}(1-\frac{\alpha}{2})\sqrt{\bar{X_n}}}{\sqrt{n}} \leqslant \lambda \leqslant \bar{X_n} + \frac{\phi^{-1}(1-\frac{\alpha}{2})\sqrt{\bar{X_i}}}{\sqrt{n}}\Big] \geqslant 1-\alpha$$

$\Rightarrow$ L=$[\bar{X_n} - \frac{\phi^{-1}(1-\frac{\alpha}{2})\sqrt{\bar{X_i}}}{\sqrt{n}}, \bar{X_n} + \frac{\phi^{-1}(1-\frac{\alpha}{2})\sqrt{\bar{X_i}}}{\sqrt{n}}]$

2. From previous result, if $\lambda_0 \in L$, we do not reject H_0. Otherwise, there is evidence to reject H_0.

Problem 2.

1. Take derivative of Gaussian expression respect to $\mu_1, \sigma_1, \mu_2, \sigma_2$ and set it equals to 0, we get:
$\hat{\mu_1} = \bar{X_{n1}}$, $\hat{\mu_2} = \bar{Y_{n2}}$, $\hat{\sigma_1}^2 = \frac{1}{n1}\sum_{i=1}^{n1}(X_i - \mu_1)^2$ and $\hat{\sigma_2}^2 = \frac{1}{n1}\sum_{i=1}^{n2}(Y_i - \mu_2)^2$.

2. We have: $\frac{n_1\hat{\sigma_1}^2}{\sigma_1^2} = \frac{n_1\frac{1}{n1}\sum_{i=1}^{n_1}(X_i-\mu_1)^2}{\sigma_1^2} = \sum_{i=1}^{n_1}(\frac{X_i-\mu_1}{\sigma_1})^2$. Since $\frac{X_i-\mu_1}{\sigma_1} \sim N(0,1)$, hence $\frac{n_1\hat{\sigma_1}^2}{\sigma_1^2} \sim \chi^2_{n1}$.
By similar method, we get $\frac{n_2\hat{\sigma_2}^2}{\sigma_2^2} \sim \chi^2_{n2}$

3. From previous result, $\frac{n_1\hat{\sigma_1}^2}{\sigma_1^2} + \frac{n_2\hat{\sigma_2}^2}{\sigma_2^2} \sim \chi^2_{n1} + \chi^2_{n2} \sim \chi^2_{n1+n2}$.

4. From CTT, $\bar{X_{n1}} \sim N(\mu_1, \sigma_1^2)$, $\bar{X_{n2}} \sim N(\mu_2, \sigma_2^2)$. Hence, $\Delta = \hat{\mu_1} - \hat{\mu_2} = \bar{X_{n1}} - \bar{X_{n2}} \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$.

5. From the previous question, we get $\Delta = \hat{\mu_1} - \hat{\mu_2} = \bar{X_{n1}} - \bar{X_{n2}} \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$.
Under H_0: $\Delta \sim N(0, \sqrt{\sigma_1^2 + \sigma_2^2})$. Hence, $\frac{\Delta}{\sigma_1^2+\sigma_2^2} \to \frac{\Delta}{\sqrt{\hat{\sigma_1}^2+\hat{\sigma_2}^2}} \sim N(0,1)$.
Let's denote test statistic $T = \Big|\frac{\Delta}{\sqrt{\hat{\sigma_1}^2+\hat{\sigma_2}^2}}\Big|$. We now $\mathbb{P}[T > c] = \alpha$, hence $c = 1 - \phi^{-1}(\frac{\alpha}{2})$.
Conclusion: If $T > c$, we reject H_0, otherwise, we fail to reject H_0.

6. In this case, from the previous qeuestion, we have $T = \Big|\frac{\Delta}{\sqrt{\hat{\sigma_1}^2+\hat{\sigma_2}^2}}\Big| = \frac{8.43-8.07}{\sqrt{0.22^2+0.17^2}} = 1.295$. Morever, $c = \phi^{-1}(1-\frac{\alpha}{2}) = \phi^{-1}(0.975) = 1.96$. Because $T < c$, we fail to reject H_0 which means we can conclude two machines are significantly identical.
p-value=$\mathbb{P}[|N(0,1)| > 1.295] = 1 - \mathbb{P}[|N(0,1)| \leqslant 1.295] = 1 - (2\mathbb{P}[N(0,1) \leqslant 1.295] - 1) = 2 - 2\phi(1.295) = 0.3$.

Problem 3.

1. By central limit theorem (CLT), we have:

$$\frac{\sqrt{n}(\bar{X}_i - \mu)}{\sigma} \sim (N(0,1))$$

Hence, $(a_n)_{n\in\mathbb{N}}$ can be $\frac{\sqrt{n}}{\sigma}$ and $(b_n)_{n\in\mathbb{N}}$ can be μ.

2. We have: $Z \backsim N(0,1)$

Hence, $\mathbb{P}[|Z| \leqslant t] = \mathbb{P}[-t \leqslant Z \leqslant t] = \phi(t) - \phi(-t) = \phi(t) - (1 - \phi(t)) = 2\phi(t) - 1 = 2\mathbb{P}[Z \leqslant t] - 1$.

3. From part 1 we get:

$$\frac{\sqrt{n}(\bar{X}_i - \mu)}{\sigma} \sim (N(0,1))$$

from part 2 we get:

$$\mathbb{P}[|Z| \leqslant t] = 2\mathbb{P}[Z \leqslant t] - 1$$

Substitution:

$$\mathbb{P}\Big[|\frac{\sqrt{n}(\bar{X}_i - \mu)}{\sigma}| \leqslant t\Big] = 2\mathbb{P}[Z \leqslant t] - 1$$

We have $2\mathbb{P}[Z \leqslant t] - 1 = 0.95 \Rightarrow t = \phi^{-1}(\frac{0.95+1}{2}) = 1.96$.

Hence,

$$\mathbb{P}\Big[|\frac{\sqrt{n}(\bar{X}_i - \mu)}{\sigma}| \leqslant 1.96\Big] = 0.95$$

$$\mathbb{P}\Big[-\frac{\sqrt{n}(\bar{X}_i - \mu)}{\sigma} \leqslant 1.96 \leqslant \frac{\sqrt{n}(\bar{X}_i - \mu)}{\sigma}\Big] = 0.95$$

Because X_i is Poisson random variable with parameter λ, so $\mu = \lambda$ and $\sigma = \sqrt{\lambda}$

We get:

$$\mathbb{P}\Big[-\frac{\sqrt{n}(\bar{X}_i - \lambda)}{\sqrt{\lambda}} \leqslant 1.96 \leqslant \frac{\sqrt{n}(\bar{X}_i - \lambda)}{\sqrt{\lambda}}\Big] = 0.95$$

$$\mathbb{P}\Big[\bar{X}_i - \frac{1.96\sqrt{\lambda}}{\sqrt{n}} \leqslant \lambda \leqslant \bar{X}_i + \frac{1.96\sqrt{\lambda}}{\sqrt{n}}\Big] = 0.95$$

We know: $\bar{X}_i \xrightarrow{P} \mathbb{E}[\bar{X}_i] = \lambda$

Hence,

$$\mathbb{P}\Big[\bar{X}_i - \frac{1.96\sqrt{\bar{X}_i}}{\sqrt{n}} \leqslant \lambda \leqslant \bar{X}_i + \frac{1.96\sqrt{\bar{X}_i}}{\sqrt{n}}\Big] \geqslant 0.95$$

$\Rightarrow$ L=$[\bar{X}_i - \frac{1.96\sqrt{\bar{X}_i}}{\sqrt{n}}, \bar{X}_i + \frac{1.96\sqrt{\bar{X}_i}}{\sqrt{n}}]$

4. We can easily see min$(X_i)\leqslant \bar{X}_i \leqslant$ max(X_i). Hence, a new interval can be:

$$L = [min(X_i) - \frac{1.96\sqrt{\bar{X}_i}}{\sqrt{n}}, max(X_i) + \frac{1.96\sqrt{\bar{X}_i}}{\sqrt{n}}]$$

Problem 4. We have X_i is IID. Hence, $\mathbb{P}(M_n \leqslant t) = \prod_{n=1}^{n} \mathbb{P}(X_i \leqslant t)$.

By uniform distribution, the CDF of M_n:

$$\mathbb{P}(M_n \leqslant t) = F(t) = \Big(\frac{t}{\theta}\Big)^n$$

Hence, the PDF of M_n is:

$$f(t) = \frac{dF}{dt} = n\theta^{-n}t^{n-1}$$

We can easily get:

$$\mathbb{E}[M_n] = \int_0^\theta tn\theta^{-n}t^{n-1}dt = \frac{n}{n+1}\theta \to \theta \textbf{ as n} \to \infty$$

By Markov's Inequality:

$$\mathbb{P}\Big[|M_n - \theta| > \epsilon\Big] \leqslant \mathbb{P}[M_n - \theta > \epsilon] \leqslant \frac{\mathbb{E}[M_n - \theta]}{\epsilon} = \frac{\mathbb{E}[M_n] - \theta}{\epsilon} \to 0$$

Hence, M_n converages in probility to θ.

2. From part 1 we get: M_n: $\mathbb{P}[M_n \leqslant t] = \Big(\frac{t}{\theta}\Big)^n$. Hence, CDF of $n(1 - \frac{M_n}{\theta})$ is:

$$P\Big[n(1 - \frac{M_n}{\theta}) \leqslant t\Big] = \mathbb{P}\Big[M_n \geqslant \frac{(n-t)\theta}{n}\Big] = 1 - \Big(\frac{n-t}{n}\Big)^n \to 1 - e^{-t} \textbf{ as n} \to \infty$$

Hence, $n(1 - \frac{M_n}{\theta})$ converages in distribution to an exponential random variable with parameter 1.

3. Let A is an exponential random variable with parameter 1. Because $n(1 - \frac{M_n}{\theta})$ converages in distribution to X, we have:

$$\mathbb{P}\Big[n(1 - \frac{M_n}{\theta}) \leqslant t\Big] \to \mathbb{P}[X \leqslant t] = 1 - e^{-t}$$

$1 - e^{-t} = 0.95 \Rightarrow t = 3$. We have:

$$\mathbb{P}\Big[n(1 - \frac{M_n}{\theta}) \leqslant 3\Big] \to 0.95$$

which is:

$$\mathbb{P}\Big[\theta \leqslant \frac{nM_n}{n-3}\Big] \to 0.95$$

On the other hand, we always have $\theta \geqslant M_n$ (uniform distribution). Hence, we get:

$$\mathbb{P}\Big[M_n \leqslant \theta \leqslant \frac{nM_n}{n-3}\Big] \to 0.95 \textbf{ as n} \to \infty$$

We conclude $L = \Big[M_n, \frac{nM_n}{n-3}\Big] = \Big[M_n, M_n + \frac{3M_n}{n-3}\Big]$.

4. $bias(M_n) = \mathbb{M}_\ltimes - \theta = \frac{n}{n+1}\theta - \theta \neq 0$. Hence, M_n is biased.

As described in the doc, $\boldsymbol{y}$ is a one-hot vector with a 1 for the true outside word o, that means y_i is 1 if and only if $i == o$. so the proof could be below: ¡!– $-\sum_{w \in Vocab} y_w \log(\hat{y}_o) = -$¿

$$\begin{aligned} -\sum_{w \in Vocab} y_w \log(\hat{y}_w) &= -[y_1 \log(\hat{y}_1) + \cdots + y_o \log(\hat{y}_o) + \cdots + y_w \log(\hat{y}_w)] \\ &= -y_o \log(\hat{y}_o) \\ &= -\log(\hat{y}_o) \\ &= -\log \mathrm{P}(O = o | C = c) \end{aligned}$$

(b) we know this deravatives:

$$\because J = CE(y, \hat{y})\hat{y} = softmax(\theta) \therefore \frac{\partial J}{\partial \theta} = (\hat{y} - y)^T$$

y is a column vector in the above equation. So, we can use chain rules to solve the deravitive:

$$\begin{aligned} \frac{\partial J}{\partial v_c} &= \frac{\partial J}{\partial \theta}\frac{\partial \theta}{\partial v_c} \\ &= (\hat{y} - y)\frac{\partial U^T v_c}{\partial v_c} \\ &= U^T(\hat{y} - y)^T \end{aligned}$$

(c) similar to the equation above.

$$
\begin{aligned}
\frac{\partial J}{\partial v_c} &= \frac{\partial J}{\partial \theta}\frac{\partial \theta}{\partial U} \\
&= (\hat{y} - y)\frac{\partial U^T v_c}{\partial U} \\
&= v_c(\hat{y} - y)^T
\end{aligned}
$$