Problem Set 6

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Problem 1.

1. Because $X_1, X_2, ...X_n$ follow exponential, hence $\mathbb{E}[X] = \frac{1}{\lambda}$ and $\mathbb{V}[X] = \frac{1}{\lambda^2}$.
From CTT, we know: $\frac{\bar{X_n} - \frac{1}{\lambda}}{\frac{1}{\lambda}} \sim N(0,1)$.
Under H_0: $\frac{\bar{X_n} - \frac{1}{\lambda_0}}{\frac{1}{\lambda_0}} \sim N(0,1)$. Now we have test statistic: $T = |\frac{\bar{X_n} - \frac{1}{\lambda_0}}{\frac{1}{\lambda_0}}|$.
With asymptotic level α, $\mathbb{P}[T > c] = \alpha$, hence $c = 1 - \phi^{-1}(\frac{\alpha}{2})$
Conclusion: If $T > c$, we reject H_0. Otherwise, we fail to reject H_0.

2. Let denote $\hat{\lambda} = argmax_{\theta \in \{0;+\infty\}} l(\theta)$ and $\hat{\lambda^c} = argmax_{\theta \in H_0} l(\theta)$ where $l(\theta)$ is likehood funtion of θ. Using Likelood ratio test, we have test stattistic: $T = 2\Big(l(\hat{\lambda}) - l(\hat{\lambda^c})\Big)$.
The difference in the number of parameters for the two models is 1 because we only consider paramer $\lambda \in \mathbb{R}$. Hence $T \sim \chi_1^2$. With asymptotic level test α, $\mathbb{P}[T > c] = \alpha$, hence c is the $(1-\alpha)$-quantile of χ_1^2. Conclusion: If $T > c$, we reject H_0. Otherwise, we fail to reject H_0.

3. a) Because $X_1, X_2, ...X_n$ follow exponential, hence $l(\lambda) = nln(\lambda) - \lambda\sum_{i=1}^{n} X_i$. Because there are 50 consecutive calls and these observations have average value is 0.98, hence $\sum_{i=1}^{n} X_i = 0.98x50 = 49$.
Now we have log-likelihood function $l(\lambda) = 50ln(\lambda) - 49\lambda$. Taking derivate respect to λ, we get $l'(\lambda) = \frac{50}{\lambda} - 49$.
Setting this equals to zeros we get $\hat{\lambda} = \frac{50}{49}$, hence $l(\hat{\lambda}) = -48.989$
Because $l'(\lambda) > 0$ for all θ under H_0, hence $\hat{\lambda^c} = 1 => l(\hat{\lambda^c}) = -49$
Based on previous questions, we have $T = 2\Big(l(\hat{\lambda}) - l(\hat{\lambda^c})\Big) = 2(-48.989 + 49) = 0.022$. With asymptotic level test $\alpha = 0.05$, we get $c = 3.814$
Because $T = 0.022 < 3.814$, hence we fail to reject null hypothesis.
b) $p - value = \mathbb{P}[\chi_1^2 > 0.022] = 0.8829$

Problem 2.

1. Take derivative of Gaussian expression respect to μ_1, σ_1, and set it equals to 0, we get:
$\hat{\mu} = \bar{X_n}$ and $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{(\mu)})^2$.

2. Cochrans theorem implies that for $X_1, X_2, ...X_n$ follow Gaussian distribution with μ and σ^2, we have:

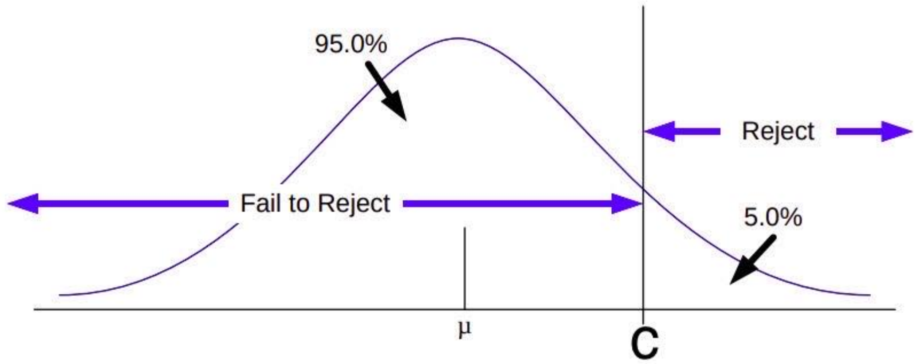
$$\sqrt{n-1}\frac{\bar{X_n} - \mu}{\sqrt{S_n}} \sim t_{n-1}$$

From previous question, we have $\hat{\mu} = \bar{X_n}$ and $S_n = \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$. Plugging this to the equation above, we get:

$$S = \sqrt{n-1}\frac{\hat{\mu} - \mu}{\sqrt{\hat{\sigma}^2}} \sim t_{n-1}$$

3 From previous question, we know $S \sim t_{n-1}$, so we will bulid Student test based on this intuition with test statistic S.
Let denote c is the $(1-\alpha)$-quantile of Student distribution with $n-1$ degrees of freedom. That means $\mathbb{P}[t_{n-1} > c] = \alpha$.
Based on the previous question, we have test statistic: $T = \sqrt{n-1}\frac{\hat{\mu}}{\sqrt{\hat{\sigma}^2}} = \sqrt{n-1}\frac{\bar{X_n}}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X_n})^2}} \sim t_{n-1}$.

Concretely, the graph below show the t-test diagram with Student test (n-1 dgrees of freedom) and level test $\alpha = 0.05$, for example.



Conclusion: If $T > c$ that means H_0 fall into reject region so we reject H_0,. Otherwise, we are fail to reject H_0

Problem 3.

1.
• Prove X and Y are independent implies $r = pq$:
From Bayes rule, we have $\mathbb{P}[X=1|Y=1] = \frac{\mathbb{P}[X=1,Y=1]}{\mathbb{P}[Y=1]}$ (1)
From X and Y are independent, we have $\mathbb{P}[X=1|Y=1] = \mathbb{P}[X=1]$. Plugging this to (1), we have: $\mathbb{P}[X=1]\mathbb{P}[Y=1] = \mathbb{P}[X=1,Y=1] \Leftrightarrow r = pq$.
• Prove $r = pq$ implies X and Y are independent:
By defination of independent event, we have if $\mathbb{P}[X=1]\mathbb{P}[Y=1] = \mathbb{P}[X=1,Y=1] \Leftrightarrow r = pq$, X and Y are independent.
Hence X and Y are independent iff $r = pq$.

2.
a)
• By Markov's inequality:

$$\mathbb{P}\Big[|\hat{p}-p| > \epsilon\Big] \leqslant \mathbb{P}\Big[\hat{p}-p > \epsilon\Big] \leqslant \frac{\mathbb{E}[\hat{p}-p]}{\epsilon} = \frac{\mathbb{E}[\hat{p}]-p}{\epsilon} = 0$$

Hence, $\hat{p}$ is consistent estimators of p.
Using similar method, we get $\hat{q}$ and $\hat{r}$ are consistent estimator of q and r, respectively.
b)
From CTT, we have $\sqrt{n}\frac{\bar{X_n}-p}{\sqrt{p(1-p)}} \sim N(0,1)$. Hence, $\sqrt{n}(\hat{p}-p) \sim N(0, p(1-p))$. Using similar method, we get $\sqrt{n}(\hat{q}-q) \sim N(0, q(1-q))$ and $\sqrt{n}(\hat{r}-r) \sim N(0, r(1-r))$.
Hence, $(\hat{p}, \hat{q}, \hat{r})$ is asymptotically normal with covariance matrix

$$\Sigma = \begin{bmatrix} p(1-p) & 0 & 0 \\ 0 & q(1-q) & 0 \\ 0 & 0 & r(1-r) \end{bmatrix}$$

c)
From previous question we have: $\sqrt{n}(\hat{\theta}-\theta) \sim N(0, \Sigma)$ where $\theta = \begin{bmatrix} r & p & q \end{bmatrix}^T$ and $\Sigma = \begin{bmatrix} r(1-r) & 0 & 0 \\ 0 & p(1-p) & 0 \\ 0 & 0 & q(1-q) \end{bmatrix}$
Let's denote $\theta = \begin{bmatrix} r & p & q \end{bmatrix}^T$, $\hat{\theta} = \begin{bmatrix} \hat{r} & \hat{p} & \hat{q} \end{bmatrix}^T$ and function $\mathbb{R}^3 \to \mathbb{R} : f(\theta) = \theta_1 - \theta_2\theta_3$.
We have $\sqrt{n}(\hat{r} - \hat{p}\hat{q}) = \sqrt{n}(f(\hat{\theta}) - f(\theta))$ where $\theta = \begin{bmatrix} r & p & q \end{bmatrix}^T$. Using Delta-method we have:

$$\sqrt{n}(f(\hat{\theta}) - f(\theta)) \sim N(0, \nabla_\theta^T \Sigma \nabla_\theta)$$

Where $\nabla_\theta = \frac{\partial f}{\partial \theta}(\theta)$ and Σ is the covariance matrix of asymptotically normal distributiion of θ. With $\theta = \begin{bmatrix} r & p & q \end{bmatrix}^T$ we have:

$$\nabla_\theta = \frac{\partial f}{\partial \theta}(\theta) = \theta = \begin{bmatrix} 1 & -q & -p \end{bmatrix}^T$$

Hence $V = \nabla_\theta^T \Sigma \nabla_\theta = \begin{bmatrix} 1 & -q & -p \end{bmatrix} \begin{bmatrix} r(1-r) & 0 & 0 \\ 0 & p(1-p) & 0 \\ 0 & 0 & q(1-q) \end{bmatrix} \begin{bmatrix} 1 \\ -q \\ -p \end{bmatrix} =.$

Let X, Y be two Bernoulli random variables and denote by $p = \mathbb{P}[X = 1]$, $q = \mathbb{P}[Y = 1]$ and $r = \mathbb{P}[X = 1, Y = 1]$.

1. Prove that X and Y are independent if and only if $r = pq$. 2. Let $(X_1, Y_1), ..., (X_n, Y_n)$ be a sample of n i.i.d copies of (X, Y). Based on this sample, we want to test whether X and Y are independent, i.e, where ther $r = pq$.

a) Define $\hat{p} = \frac{1}{n}\sum_{i=1}^n X_i$, $\hat{q} = \frac{1}{n}\sum_{i=1}^n Y_i$ and $\hat{r} = \frac{1}{n}\sum_{i=1}^n X_i Y_i$. Prrove taht these test are, respectively, consistent estimators of p, q and r.

b) Show that the vector $(\hat{p}, \hat{q}, \hat{r})$ is asymptotically normal and find the asymptotic covariance matrix.

c) Using the previous question combined with the Delta-method, prove that:

$$\sqrt{n}(\hat{r} - \hat{p}\hat{q} - (r - pq)) \longrightarrow N(0, V)$$

where V depends on p,q and r.

d) Consider the following hypotheses:

H_0: "X and Y are independent" vs. H_1: "X and Y are not independent".

Assuming that H_0 is true, show that $V = pq(1-p)(1-q)$ and propose a consistent estimator of V.

e) Using last two questions, propose a test with asymptotic α, for any $\alpha \in (0, 1)$.