Problem Set 1

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Problem 1.

1. We have:

$$\mathbb{E}[X_n] = (1 - \frac{1}{n^2}).\frac{1}{n} + \frac{1}{n^2}.n$$
$$= \frac{1}{n} - \frac{1}{n^3} + \frac{1}{n}$$
$$= \frac{2}{n} - \frac{1}{n^3}$$

On the other hand:

$$lim_{n\to\infty}\mathbb{P}[|X_n| > \epsilon] = lim_{n\to\infty}\mathbb{P}[X_n > \epsilon] \leq \frac{lim_{n\to\infty}\mathbb{E}[X_n]}{\epsilon} = \frac{lim_{n\to\infty}(\frac{2}{n} - \frac{1}{n^3})}{\epsilon} = 0$$

Hence X_n converge in probability

$$\mathbb{E}[X_n^2] = (1 - \frac{1}{n^2}).\frac{1}{n^2} + \frac{1}{n^2}.n^2$$
$$= \frac{1}{n^2} - \frac{1}{n^4} + 1$$
$$\Rightarrow lim_{n\to\infty}\mathbb{E}[X_n^2] = 1$$

We have:

$$lim_{n\to\infty}\mathbb{E}[(X_n - X)^2] = lim_{n\to\infty}(\mathbb{E}[X_n^2] - 2\mathbb{E}[X_n]\mathbb{E}[X] + \mathbb{E}[X^2])$$
$$= 1 + \mathbb{E}[X^2] \qquad (because \quad \mathbb{E}[X_n] = 0)$$

If X_n converge in $\mathbb{L}^2 \Rightarrow 1 + \mathbb{E}[X^2] = 0 \Rightarrow \mathbb{E}[X^2] = -1 \quad (imposible)$

Hence, X_n does not converge in $\mathbb{L}^2$

2. We have:

$$\mathbb{E}[X_n] = p$$
$$\mathbb{V}[X_n] = \frac{p(1-p)}{n}$$

In addition:

$$n\bar{X_n} = \sum_{i=1}^{n} X_i$$
$$\Rightarrow \frac{n\bar{X_n} - \mu_n}{\sigma_n} = \frac{\sum_{i=1}^{n} X_i - p}{\frac{p(1-p)}{n}} \sim \mathbb{N}(0,1) \qquad (Central \quad Limit \quad Theorem)$$

$$\Rightarrow \mathbb{P}[\frac{n\bar{X_n} - p}{\frac{p(1-p)}{n}} \lneq t] = \Phi(t)$$

$$\Rightarrow \mathbb{P}\Big[n\bar{X_n} \lneq \frac{tp(1-p)}{n} + p\Big] = \Phi(t)$$

$$\Rightarrow \mathbb{P}[n\bar{X_n} \lneq x] = \Phi(\frac{(x-p)n}{p(1-p)}$$

$$\Rightarrow F_n(x) = \Phi(\frac{(x-p)n}{p(1-p)} \qquad (1)$$

(1) is the CDF of distribution of n$\bar{X_n}$
We have:

$$\mathbb{E}[(\bar{X_n} - p)^2] = \mathbb{E}[\bar{X_n}^2] - 2p\mathbb{E}[\bar{X_n}] + p^2$$
$$= \sigma_n + (\mathbb{E}[\bar{X_n}])^2 - 2p\mathbb{E}[\bar{X_n}] + p^2$$
$$= \frac{p(1-p)}{n} + p^2 - 2p^2 + p^2$$
$$= \frac{p(1-p)}{n}$$
$$\Rightarrow lim_{n\to\infty}\mathbb{E}[(\bar{X_n} - p)^2] = 0$$

$\Rightarrow \bar{X_n}$ converges to p in L

3. a) From Markov's Inequality:

$$\mathbb{P}[|X_n| > \epsilon] = \mathbb{P}[X_n > \epsilon] < \frac{\mathbb{E}[X_n]}{\epsilon} = \frac{\lambda}{\epsilon} = \frac{1}{n\epsilon}$$
$$\Rightarrow \lim_{n\to\infty} \mathbb{P}[|X_n| > \epsilon] = 0$$
$$\Rightarrow X_n \xrightarrow{\mathbb{P}} 0$$

b) We have:

$$\mathbb{P}[|nX_n| < \epsilon] = \mathbb{P}[X_n < \frac{\epsilon}{n}] = \sum_{i=0}^{\frac{\epsilon}{n}} e^{-\lambda}\frac{\lambda^i}{i!}$$

Problem 2.

1. True

2.True

3.

4.

We get the formula for the interval confident of Bernulli random variables:

$$\mathbb{L} = \Big[\bar{X_n} - \frac{1.96\sqrt{p(1-p)}}{\sqrt{n}}, \bar{X_n} + \frac{1.96\sqrt{p(1-p)}}{\sqrt{n}}\Big]$$

In addition:

$$p = 0.43$$

$$n = 100$$
$$\Rightarrow \mathbb{L} = [0.33, 0.53]$$

Problem 3.

1.

We have:

$$\mu_n = \mathbb{E}_n[\bar{X_n}] = p$$
$$\sigma_n = \frac{p(1-p)}{n}$$

By using Central Limit Theorem:

$$\Rightarrow \bar{X_n} \sim \mathbb{N}(\mu, \frac{\sigma_n^2}{n})$$
$$\sqrt{n}\frac{\bar{X_n} - p}{\sqrt{p(1-p)}} \sim \mathbb{N}(0,1)$$

2.

We have: $Z \backsim N(0,1)$

Hence, $\mathbb{P}[|Z| \leqslant t] = \mathbb{P}[-t \leqslant Z \leqslant t] = \phi(t) - \phi(-t) = \phi(t) - (1 - \phi(t)) = 2\phi(t) - 1 = 2\mathbb{P}[Z \leqslant t] - 1.$

3.

We have:

$$\mathbb{P}[p \in I_t] = \mathbb{P}\Big[\bar{X_n} - \frac{t\sqrt{p(1-p)}}{\sqrt{n}} \leqslant p \leqslant \bar{X_n} + \frac{t\sqrt{p(1-p)}}{\sqrt{n}}\Big]$$
$$= \mathbb{P}\Big[\Big|\sqrt{n}\frac{\bar{X_n} - p}{\sqrt{p(1-p)}}\Big| \leqslant t\Big]$$

Because:

$$\sqrt{n}\frac{\bar{X_n} - p}{\sqrt{p(1-p)}} \sim \mathbb{N}(0,1)$$
$$\mathbb{P}[|Z| \leqslant t] = 2\mathbb{P}[Z \leqslant t] - 1 \quad (Z = \mathbb{N}(0,1))$$

Hence,

$$\mathbb{P}[p \in I_t] \to 2\phi(t) - 1, n \to \infty \quad (using \quad the \quad previous \quad questions)$$

4.

We have:

$$\mathbb{P}[p \in I_t] \to 2\phi(t) - 1 = 0.95$$
$$\Rightarrow t_0 = 1.96$$

Hence,

$$I_{t_0} = \Big[\bar{X_n} - \frac{t\sqrt{p(1-p)}}{\sqrt{n}}, \bar{X_n} + \frac{t\sqrt{p(1-p)}}{\sqrt{n}}\Big]$$

$$= \left[\bar{X}_n - \frac{1.96\sqrt{p(1-p)}}{\sqrt{n}}, \bar{X}_n + \frac{1.96\sqrt{p(1-p)}}{\sqrt{n}}\right]$$

Using the fact that:

$$0 \leqslant \sqrt{p(1-p)} \leqslant \frac{1}{4}$$

We get:

$$I_t = \left[\right]$$

$$\Rightarrow I_t = \left[\bar{X}_n - \frac{0.49}{\sqrt{n}}, \bar{X}_n + \frac{0.49}{\sqrt{n}}\right]$$

5.

a) Using Cauchy's inequality:

$$p(1-p) \leqslant \frac{q+1-p}{4} = \frac{1}{4}$$

b)From the previous question:

$$0 \leqslant \sqrt{p(1-p)} \leqslant \frac{1}{2}$$

We get:

$$I_t = \left[\right]$$

$$\Rightarrow I_t = \left[\bar{X}_n - \frac{0.98}{\sqrt{n}}, \bar{X}_n + \frac{0.98}{\sqrt{n}}\right]$$

c) Because $p \leqslant 0.3 \Rightarrow p(1-p) \leqslant 0.21$

$$\Rightarrow J_1 = \left[\bar{X}_n - \frac{1.96\sqrt{0.21}}{\sqrt{n}}, \bar{X}_n + \frac{1.96\sqrt{02.1}}{\sqrt{n}}\right] = \left[\bar{X}_n - \frac{0.89}{\sqrt{n}}, \bar{X}_n + \frac{0.89}{\sqrt{n}}\right]$$

6.

a) We have:

$$p \in I_{t_0} \iff \bar{X}_n - \frac{t_0\sqrt{p(1-p)}}{\sqrt{n}} \leqslant p \leqslant \bar{X}_n + \frac{t_0\sqrt{p(1-p)}}{\sqrt{n}}$$

$$\iff p^2(1 + \frac{{t_0}^2}{n}) - (2\bar{X}_n + \frac{{t_0}^2}{n})p + {\bar{X}_n}^2 \leqslant 0$$

b) Solving this inequality we get the value of p in form:

$$\left[\frac{-b-\sqrt{\Delta}}{2a}, \frac{-b+\sqrt{\Delta}}{2a}\right]$$

With:

$$b = -(2\bar{X}_n + \frac{{t_0}^2}{n})$$

$$\Delta = \frac{{t_0}^4}{n^2} + 4\bar{X}_n\frac{{t_0}^2}{n} - \frac{4{\bar{X}_n}^2{t_0}^2}{n}$$

$$a = 1 + \frac{{t_0}^2}{n}$$

c) From the previous question, we have :

$$J_2 = \Big[\frac{2\bar{X}_n + \frac{{t_0}^2}{n} - \sqrt{\frac{{t_0}^4}{n^2} + 4\bar{X}_n\frac{{t_0}^2}{n} - \frac{4{\bar{X}_n}^2{t_0}^2}{n}}}{2 + \frac{2{t_0}^2}{n}}, \frac{2\bar{X}_n + \frac{{t_0}^2}{n} + \sqrt{\frac{{t_0}^4}{n^2} + 4\bar{X}_n\frac{{t_0}^2}{n} - \frac{4{\bar{X}_n}^2{t_0}^2}{n}}}{2 + \frac{2{t_0}^2}{n}}\Big]$$

7.

Replace p with $\bar{X}_n$, we get:

$$J_3 = \Big[\bar{X}_n - \frac{t_0\sqrt{\bar{X}_n(1-\bar{X}_n)}}{\sqrt{n}}, \bar{X}_n + \frac{t_0\sqrt{\bar{X}_n(1-\bar{X}_n)}}{\sqrt{n}}\Big]$$

As n goes to infinity, we have:

$$\bar{X}_n \xrightarrow{\mathbb{P}} \mu = p$$
$$\Rightarrow J_3 \longrightarrow \Big[\bar{X}_n - \frac{t_0\sqrt{p(1-p)}}{\sqrt{n}}, \bar{X}_n + \frac{t_0\sqrt{p(1-p)}}{\sqrt{n}}\Big] \longrightarrow 0.95$$

8.

a) We have from the question:

$$n = 10000$$
$$\bar{X}_n = 1 - 0.7341 = 0.2659$$

$$J_1 = \Big[\bar{X}_n - \frac{0.89}{\sqrt{n}}, \bar{X}_n + \frac{0.89}{\sqrt{n}}\Big] = \Big[0.2659 - \frac{0.89}{\sqrt{100}}, 0.2659 + \frac{0.89}{\sqrt{100}}\Big] = \Big[0.1769, 0.3549\Big]$$

$$J_2 = \Big[\frac{2\bar{X}_n + \frac{{t_0}^2}{n} - \sqrt{\frac{{t_0}^4}{n^2} + 4\bar{X}_n\frac{{t_0}^2}{n} - \frac{4{\bar{X}_n}^2{t_0}^2}{n}}}{2 + \frac{2{t_0}^2}{n}}, \frac{2\bar{X}_n + \frac{{t_0}^2}{n} + \sqrt{\frac{{t_0}^4}{n^2} + 4\bar{X}_n\frac{{t_0}^2}{n} - \frac{4{\bar{X}_n}^2{t_0}^2}{n}}}{2 + \frac{2{t_0}^2}{n}}\Big]$$

$$= \Big[\frac{2\times 0.2659 + \frac{1.96^2}{100} - \sqrt{\frac{1.96^4}{100^2} + 4\times 0.2659 \times \frac{1.96^2}{100} - \frac{4\times 0.2659^2\times 1.96^2}{100}}}{2 + \frac{2\times 1.96^2}{100}}, \frac{2\times 0.2659 + \frac{1.96^2}{100} + \sqrt{\frac{1.96^4}{100^2} + 4\times 0.2659 \times \frac{1.96^2}{100} - 4\times}}{2 + \frac{2\times 1.96^2}{100}}$$

$$= \Big[0.189, 0.365\Big]$$

$$J_3 = \Big[\bar{X}_n - \frac{t_0\sqrt{\bar{X}_n(1-\bar{X}_n)}}{\sqrt{n}}, \bar{X}_n + \frac{t_0\sqrt{\bar{X}_n(1-\bar{X}_n)}}{\sqrt{n}}\Big]$$
$$= \Big[0.2659 - \frac{1.96\sqrt{0.2659\times(1-0.2659)}}{\sqrt{100}}, 0.2659 + \frac{1.96\sqrt{0.2659\times(1-0.2659)}}{\sqrt{100}}\Big]$$
$$= \Big[0.1793, 0.3524\Big]$$

b) From the previous question, we get the tightest interval of p is J_3, so we obtain J_3 to solve this problem. We have:

The length of the interval is at most 0.05:

$$\Rightarrow \frac{t_0\sqrt{\bar{X}_n(1-\bar{X}_n)}}{\bar{X}_n\sqrt{n}} \leqslant 0.025$$

$$\iff \Big(\frac{t_0\sqrt{\bar{X_n}(1-\bar{X_n})}}{\bar{X_n}\times 0.025}\Big)^2 \leqslant n$$
$$\iff 16970 \leqslant n$$

So, the minimal of n is 16970.