Problem Set 10

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Problem 1.

1. We have PDF: $p(x) = p^x(1-p)^{1-x} = e^{xlog(p)+(1-x)log(1-p)}$. Hence we have:

$$\begin{cases} \eta_1 = log(p) \\ \eta_2 = log(1-p) \\ T_1(x) = x \\ T_2(x) = 1-x \\ h(x) = 1 \\ B(p) = 0 \end{cases}$$

2. We have PDF $p(x) = e^{\mu x - \frac{\mu^2}{2}} \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$.Hence

$$\begin{cases} \eta = \mu \\ T(x) = x \\ h(x) = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} \\ B(p) = \frac{\mu^2}{2} \end{cases}$$

3. We have PDF $p(x) = e^{\frac{\mu}{\sigma^2} - \frac{1}{2\sigma^2}x^2 - \frac{\mu^2}{2\sigma^2}} \frac{1}{\sigma\sqrt{2\pi}}$. Hence

$$\begin{cases} \eta_1 = \frac{\mu}{\sigma^2} \\ \eta_2 = -\frac{1}{2\sigma^2} \\ T_1(x) = x \\ T_2(x) = x^2 \\ h(x) = 1 \\ B(p) = \frac{\mu^2}{2\sigma^2} + log(\sigma\sqrt{2\pi}) \end{cases}$$

4. We have $p(x) = \lambda e^{-\lambda x}$. Hence

$$\begin{cases} \eta = -\lambda \\ T(x) = x \\ h(x) = \lambda \\ B(p) = 0 \end{cases}$$

5. We have $p(x) = \frac{1}{\upsilon} = e^{-log(\upsilon)}$. Hence We have $p(x) = \lambda e^{-\lambda x}$. Hence

$$\begin{cases} \eta = -log(\upsilon) \\ T(x) = 1 \\ h(x) = 1 \\ B(p) = 0 \end{cases}$$

6. We have $p(x) = \frac{\beta^\alpha}{\mathbb{I}(\alpha)} x^{\alpha-1} e^{-\beta x} = e^{-\beta x + (\alpha-1) log(x)} \frac{\beta^\alpha}{\mathbb{I}(\alpha)}$. Hence

$$\begin{cases} \eta_1 = -\beta \\ \eta_2 = \alpha - 1 \\ T_1(x) = x \\ T_2(x) = log(x) \\ h(x) = 1 \\ B(p) = -log(\frac{\beta^\alpha}{\mathbb{I}(\alpha)}) \end{cases}$$

7. We have $p(x) = \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda + log(\lambda) x} \frac{1}{x!}$. Hence

$$\begin{cases} \eta_1 = log(\lambda) \\ T(x) = x \\ h(x) = \frac{1}{x!} \\ B(p) = -\lambda \end{cases}$$

Problem 2.

1.
a)
b) We have

$$\mathbb{E}[\frac{\partial log(f_\theta(x))}{\partial \theta}] = \mathbb{E}[\frac{\frac{\partial}{\partial \theta} f_\theta(x)}{f_\theta(x)}] = \int \frac{\frac{\partial}{\partial \theta} f_\theta(x)}{f_\theta(x)} f_\theta(x) = \frac{\partial}{\partial \theta} \int f_\theta(x) dx = 0$$

$$\mathbb{E}[\frac{\partial^2}{\partial \theta^2} log(f_\theta(x))] = \int \frac{[\frac{\partial^2}{\partial \theta^2} f_\theta(x)] f_\theta(x) - [\frac{\partial}{\partial \theta} f_\theta(x)]^2}{(f_\theta(x))^2} f_\theta(x) = \int \frac{\partial^2}{\partial \theta^2} f_\theta(x) + \int \Big(\frac{[\frac{\partial}{\partial \theta} f_\theta(x)]^2}{f_\theta(x)} dx \Big) = \mathbb{E}[(\frac{\frac{\partial}{\partial \theta} f_\theta(x)}{f_{\theta(x)}})^2]$$

We have $l(\theta) = -\theta x + b(\theta) + log(a(x))$. Hence $l'(\theta) = -x + b'(\theta)$ and $l''(\theta) = b''(\theta)$. From previous quantities, $\mathbb{E}[l'(\theta)] = -\mathbb{E}[x] + b'(\theta) = 0$
From previous quantities, $\mathbb{E}[\frac{\partial^2}{\partial \theta^2} log(f_\theta(x))] = \mathbb{E}[l''(\theta)] = \mathbb{E}[(l'(\theta))^2]$. Because $\mathbb{E}[l'(\theta)] = 0$, hence $\mathbb{E}[(l'(\theta))^2] = var(l'(\theta)) = -\mathbb{E}[l''(\theta)]$
c) Fisher Information.

2.
From previous questions, we have $l'(\theta) = -x + b'(\theta)$ and $l''(\theta) = b''(\theta)$

3.
From previous result $\mathbb{E}[l'(\theta)] - \mathbb{E}[x] + b'(\theta) = 0 \Rightarrow \mathbb{E}[x] = b'(\theta)$
From previous result $\mathbb{E}[(l'(\theta))^2] = \mathbb{E}[(-x + b'(\theta))^2] = \mathbb{E}[(-x + E[x])^2] = var(x) = -b''(\theta)$

4.
a) Sample space $X = (0, +\infty)$
b) $a(x) = x^{\alpha-1}$ and $b(\theta) = log(\frac{\theta^\alpha}{\mathbb{I}(\alpha)})$ c) From previous result, $\mathbb{E}[x] = b'(\theta) = \frac{\alpha}{\theta}$ and $Var(x) = -b''(\theta) = \frac{\alpha}{\theta^2}$

Problem 3.

1. Z_1 is either 0 or 1, hence Z_1 is Bernoulli distribution conditional on X_i with parameter $p = F(-X^T \beta)$.

2. We have $\mathbb{E}[Z] = p = F(-X^T \beta)$. Hence $g(F(-X^T \beta)) = X^T \beta \Rightarrow g(x) = -F^{-1}(x)$.

3. If ϵ is standard Gaussian, becasue CDF of standard Gaussian is even function, hence $F(x) = -F(x) \Rightarrow g(x) = -F^{-1}(x) = F^{-1}(x)$.

4.
a) $F(t) = \int \frac{e^{-t}}{(1+e^{-1})^2} = -\frac{e^{-t}}{1+e^{-t}}$
b) $g(x) = -F^{-1}(x) = log(\frac{x}{1-x})$
c) This link is called logit link.