Deep Learning Enhanced Volatility Modeling with Covariates

Hien Thi Nguyen[(a)], Hoang Nguyen[(b)] and Minh-Ngoc Tran[(c)]

[(a)] Faculty of Mathematical Economics, Thuongmai University, Vietnam

[(b)] School of Business, Örebro University, Sweden

[(c)] Discipline of Business Analytics, the University of Sydney Business School, Australia

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November 8, 2023

Abstract

Exogenous information such as policy news and economic indicators can have the potential to trigger significant movements in financial asset volatility. This article presents a model, called the RECH-X model, that allows incorporating exogenous variables into a recurrent neural network for volatility modeling and forecasting. The RECH-X model can allow for abrupt changes in the volatility level and effectively capture the complex serial dependence structure in the volatility dynamics. We demonstrate in a wide range of applications that the RECH-X model consistently outperforms the benchmark models in terms of volatility modeling and forecasting.

Keywords: GARCH;GARCH-X; volatility forecast; realized measures; sequence Monte Carlo.

JEL Classification: C58, C53, C10.

1 Introduction

Prediction of asset volatility plays a vital role in the financial services industry, including but not limited to risk management, capital allocation, and financial engineering. Empirical studies suggest that the volatility of financial assets exhibits highly complex behavior including, for example, clustering (Mandelbrot, 1967), asymmetric volatility response (Black, 1976), and leptokurtosis (Carnero et al., 2004). These phenomena reflect the long-term autocorrelations and non-linear sequential dependencies, which make modeling and predicting volatility a challenging endeavor. The seminal volatility models such as the Generalized Autoregressive Conditional Heteroscedasticity (GARCH) (Engle, 1982; Bollerslev, 1986) and the Stochastic Volatility (SV) (Taylor, 1982) have gained popularity among scholars and become industrial standards for their ability to capture some of the intertemporal volatility dynamics. However, these models have yet to capture the external information sources on asset volatility. The exogenous information sources can be realized volatility or economic news that signal the changes in the financial market condition. For example, realized volatility (RV) estimated using high-frequency data carries far more information about the current level of volatility than the daily squared returns (Andersen and Bollerslev, 1998); (Andersen et al., 2001; Koopman et al., 2005; Hansen et al., 2012, 2014). Additionally, including economic news rather than just using the return time series itself in traditional econometric volatility models can result in increased predictive power (Engle and Rangel, 2008; Engle et al., 2013; Asgharian et al., 2013; Conrad and Kleen, 2020).

This paper introduces a new model, Recurrent Conditional Heteroscedasticity with exogenous variables, denoted as RECH-X, which builds upon the framework developed by Nguyen et al. (2022) and allows for the incorporation of multiple sources of exogenous data. In RECH-X, the long-term component of the volatility is governed by a recurrent neural network of exogenous variables to capture complex movements such as non-linearity and long-term dependence while the short-term component follows a GARCH-type process. Hence, the RECH-X model provides a flexible yet simple framework for incorporating a variety of exogenous data to improve the modeling and forecasting of volatility. Our work also contributes to the growing collection of evidence that it is possible to harness the predictive power of deep learning models in financial risk modeling while managing some of their shortcomings through the use of hybrid deep learning econometric models.

In many non-financial applications, such as language translation or speech synthesis, where sequential data exhibit long-range memory and complex dynamics, deep learning has demonstrated its impressive predictive performance. Recurrent Neural Networks (RNN) are the current state-of-the-art deep learning models for solving sequential learning problems. RNN can effectively process sequential data by retaining memory of the past inputs and capturing non-linear temporal dependencies within the data (Lipton et al., 2015; Goodfellow et al., 2016). The application of deep learning in financial econometrics has also emerged recently. Roh (2007) uses a feed-forward neural network (FNN) as another processing layer; i.e., an FNN is used to model the volatility estimates produced by an econometric model and produce the final estimate of the volatility. Kim and Won (2018) extend this idea by using an RNN rather than an FNN to depict more accurately the temporal effects. Hajizadeh et al. (2012) and Kristjanpoller et al. (2014) propose alternative approaches to combining FNNs and EGARCH, demonstrating improved predictive power. These hybrid models, which combine econometric models with deep learning, are in general superior to the former in terms of predictive performance. However, they suffer from an important shortcoming: unlike traditional econometric models, these hybrid models can be opaque and difficult to interpret and to audit. To harness the predictive power of RNN while retaining the interpretability of traditional econometric models, Nguyen et al. (2022) propose a hybrid RNN and GARCH-type model, called the Recurrent Conditional Heteroscedasticity (RECH) model. Nguyen et al. (2022) demonstrate using simulations and empirical studies on several stock market indices that their hybrid model exhibits superior predictive properties relative to the traditional econometric models while remaining transparent and interpretable.

Adding exogenous sources of information into a traditional econometric model can lead to improved predictive performance, but it may require ad-hoc modifications, such as the need of an additional measurement equation to model the dynamics of the noisy. The Multiplicative error model (MEM) (Engle, 2002) marked a significant milestone in the integration of realized measure as the covariate within the GARCH-X model framework. Barndorff-Nielsen and Shephard (2002) further refined the specification by incorporating both realized variance and bi-power variation into their model. Engle and Gallo (2006) also contributed to this line of research by leveraging intra-daily data as a covariate. In subsequent developments, Shephard and Sheppard (2010) introduced the High-frequency-based volatility (HEAVY) model, while Hansen et al. (2012) presented the Realized GARCH model. These models both specify the conditional variance within the GARCH-X framework.

In addition, the incorporation of exogenous economic variables has been observed in several studies. Notably, Glosten et al. (1993a); Gray (1996) and Engle and Patton (2001) employed interest rate levels as covariates within the GARCH-X model. Similarly, Hodrick (1989) and Hagiwara and Herce (1999) incorporated forward-spot spreads and interest rate spreads between countries as covariates in their analysis. Following, Han and Park (2008) used the yield spread between Aaa and Baa bonds as the covariate in the GARCH-X model. Other examples of exogenous economic variables include gold price (Smith, 2001), interest rate volatility (Brenner et al., 1996; Gray, 1996), exchange rate (Bollerslev and Melvin, 1994).

By extracting the related information from exogenous variables, we demonstrate the flexibility and efficiency of the RECH-X model in forecasting through a range of examples. In the first application, we show that the RECH-X model using realized volatility as an exogenous variable outperforms the benchmark models such as GARCH and RealGARCH for the index returns of ten major stock markets in the world. The second application employs four financial-economic indicators, including gold prices, oil prices, exchange rates, and the US stock market uncertainty, for improving volatility forecasting of five financial markets. Our results show that the RECH-X model consistently predicts volatility better than the GARCH model. The US stock market uncertainty has a strong positive effect, while the effects of gold prices, oil prices, and exchange rates are somewhat diverse.

This paper is organized as follows. The benchmark models GARCH and RealGARCH, together with RECH and its extension RECH-X, are presented in Section 2. Bayesian inference is presented in Section 3. Empirical results are presented in Section 4 and Section 5 concludes. The computer code is available upon contacting the authors.

2 Volatility models

This section first discusses the GARCH(1,1) model of Bollerslev (1986), with Student-t innovations. There exist a vast number of GARCH-type models, however, Hansen and Lunde (2005) find no evidence of significant outperformance over GARCH. We therefore use GARCH as the benchmark model in this paper. We also describe the RealGARCH model of Hansen et al. (2012) when realized measures are available. We then describe the RECH model of Nguyen et al. (2022), and discuss how to incorporate exogenous information into RECH to improve its volatility modeling and forecasting.

2.1 GARCH, GARCH-X and RealGARCH

Let $\{y_t, t = 1, \cdots, T\}$ be the daily return time series and $\{\text{RV}_t, t = 1, \cdots, T\}$ the corresponding realized volatilities. We consider the GARCH model with Student's t innovations:

$$y_t = \sigma_t \epsilon_t, \;\; \epsilon_t \overset{iid}{\sim} t_\nu, \;\; t = 1, 2, ..., T,$$
$$\sigma_t^2 = \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2, \;\; t = 2, ..., T.$$

As typical in the literature, we impose the condition that $\alpha > 0$, $\beta > 0$ and $\alpha \frac{\nu}{\nu-2} + \beta < 1$ for the stationarity of y_t. Also, we restrict $\nu > 2$ to ensure a finite variance of the y_t. We set σ_1^2 to be the sample variance of the training data following the common practice in the literature. A straightforward extension of the GARCH model is a GARCH-X which incorporates related exogenous information for volatility modeling:

$$y_t = \sigma_t \epsilon_t, \;\; \epsilon_t \overset{iid}{\sim} t_\nu, \;\; t = 1, 2, ..., T,$$

$$\sigma_t^2 = \omega + \alpha y_{t-1}^2 + \beta\sigma_{t-1}^2 + \pi^\top x_{t-1}, \;\; t = 2, ..., T.$$

Here, x_t is the vector of non-negative exogenous variables and π the vector of non-negative coefficients. For negative x_t, one often replaces $\pi^\top x_{t-1}$ with $\pi^\top x_{t-1}^2$ (Francq and Thieu, 2019).

When high-frequency data are available, one of the most successful econometric models that incorporate RV into volatility modeling is perhaps the non-exponential RealGARCH model of Hansen et al. (2012):

$$\begin{aligned} y_t &= \sigma_t\epsilon_t, \;\; \epsilon_t \overset{iid}{\sim} t_\nu, \;\; t = 1, 2, ..., T, \\ \sigma_t^2 &= \omega + \beta\sigma_{t-1}^2 + \gamma \mathrm{RV}_{t-1}, \;\; t = 2, ..., T, \\ \mathrm{RV}_t &= \xi + \varphi\sigma_t^2 + \tau_1\epsilon_t + \tau_2\big(\frac{\nu-2}{\nu}\epsilon_t^2 - 1\big) + u_t, \;\; t = 1, ..., T, \;\; u_t \overset{iid}{\sim} \mathcal{N}(0, \sigma_u^2). \end{aligned}$$

The model parameters include $\nu, \omega, \beta, \gamma, \xi, \varphi, \tau_1, \tau_2$ and σ_u^2. To ensure that σ_t^2 is finite and positive, we impose the following conditions: $\omega + \gamma\xi > 0$ and $0 < \beta + \gamma\varphi < 1$; see Hansen et al. (2012) and Gerlach and Wang (2016) for more details. The non-exponential RealGARCH of Hansen et al. (2012) can be considered as an extension of the GARCH(0,1)-X model (Han, 2015).

This paper employs the GARCH, the RealGARCH (when RV is available) and the GARCH-X as the benchmark models to assess the performance of the RECH-X model described in the next section.

2.2 Recurrent Conditional Heteroscedasticity

The RECH models of Nguyen et al. (2022) are a class of volatility models that incorporate an RNN within a GARCH-type model for flexible volatility modeling. Its key motivation is to use an additive component governed by an RNN to capture the complex serial dependence structure in the volatility dynamics which might be overlooked by the GARCH component.

The general RECH model is:

$$y_t = \sigma_t \epsilon_t, \;\; \epsilon_t \sim i.i.d, \;\; t = 1, 2, ..., T, \tag{1a}$$

$$\sigma_t^2 = g(\omega_t) + f(\sigma_{t-1}^2, ..., \sigma_{t-p}^2, y_{t-1}, ..., y_{t-q}), \;\; t = q+1, ..., T, \tag{1b}$$

$$\omega_t = \beta_0 + \beta_1 h_t, \;\; t = 1, ..., T, \tag{1c}$$

$$h_t = \text{RNN}(x_t, h_{t-1}), \;\; t = 2, ..., T, \;\; \text{with } h_1 \equiv 0. \tag{1d}$$

Here, $f(\sigma_{t-1}^2, ..., \sigma_{t-p}^2, y_{t-1}, ..., y_{t-q})$ is the GARCH-type component with p and q the lag orders of σ_t^2 and y_t respectively. The function $g(\omega_t)$ is called the recurrent component with ω_t governed by an RNN, where $g(\cdot)$ is an activation function; β_0 and β_1 are the coefficients. The input vector x_t in the calculation of the recurrent state $h_t = \text{RNN}(x_t, h_{t-1})$ is a vector of additional information sources whose choice is discussed shortly. The RECH model reduces to the GARCH model if $\beta_1 = 0$ and $g(\omega_t) = \omega_t$.

The recurrent state h_t in (1d) can be modeled by any RNN framework. Nguyen et al. (2022) use a simple RNN (see Equation (2d)), and Liu et al. (2023) opt for the long-short term memory (LSTM) model of Hochreiter and Schmidhuber (1997) as an alternative. The LSTM uses gating mechanisms to control the flow of information through the network and enable the RNN to selectively retain and forget information over long time horizons. Liu et al. (2023) demonstrate that using the LSTM in general improves the predictive performance of RECH. However, the simple RNN has a simpler structure that allows for the interpretation of the influencing variables and requires less computational cost compared to LSTM. This paper focuses on testing the ability of the RECH model when incorporating exogenous information. To avoid possible confounding results when a sophisticated LSTM architecture is used, and to increase the interpretation, we opt for the simple RNN rather than LSTM. We consider GARCH(1,1) for the GARCH component and Student's t innovations. The model is fully written as follows:

$$y_t = \sigma_t \epsilon_t, \;\; \epsilon_t \overset{iid}{\sim} t_\nu, \;\; t = 1, 2, ..., T, \tag{2a}$$

$$\sigma_t^2 = \omega_t + \alpha y_{t-1}^2 + \beta\sigma_{t-1}^2, \;\; t = 2, ..., T, \tag{2b}$$

$$\omega_t = \beta_0 + \beta_1 h_t, \;\; t = 1, ..., T, \tag{2c}$$

$$h_t = \Psi(v^\top x_t + w_h h_{t-1} + b), \;\; t = 2, ..., T, \;\; \text{with} \;\; h_1 \equiv 0. \tag{2d}$$

Here, $\Psi(\cdot)$ is a non-linear activation function, which is chosen to be the ReLU $\Psi(x) = \max\{x, 0\}$. The GARCH parameters α and β are imposed the usual constraint that $\alpha, \beta > 0$ and $\alpha\frac{\nu}{\nu-2} + \beta < 1$, which guarantees that the volatility σ_t^2 is positive and finite for all t; see Nguyen et al. (2022). We set the initial σ_1^2 as before. The input vector $x_t = (\omega_{t-1}, y_{t-1}, \sigma_{t-1}^2)$ in Nguyen et al. (2022) contains the past information of the level of long-term volatility, return, and total volatility. However, it is easy to incorporate exogenous variables into its recurrent component whenever such inputs are available and useful in terms of modeling and predicting the distribution of y_t. Hence, with the availability of an exogenous variable z_t at time t, we propose to use $x_t = (\omega_{t-1}, y_{t-1}, \sigma_{t-1}^2, z_t)^\top$ and refer to the resulting model in (2a)-(2d) as RECH-X. For example, with the choice of z_t as RV_{t-1}, the vector v in (2d) has four coefficients and we denote by $v_\omega, v_y, v_\sigma, v_{\text{RV}}$ the coefficients with respect to $\omega_{t-1}, y_{t-1}, \sigma_{t-1}^2, \text{RV}_{t-1}$, respectively. The magnitude of v_{RV} characterizes the importance of realized measures for modeling the daily volatility σ_t^2. We demonstrate in Section 4.1 that v_{RV} is always significant across all the datasets considered. When multiple exogenous variables are used as in Section 4.2, z_t and its corresponding coefficients v_z become vectors. The flexible structure of RECH-X allows it to be able to capture the non-linear and long-term effects that the exogenous variables have on the volatility.

When high-frequency data are available, Liu et al. (2023) extend the RealGARCH model by incorporating an RNN, similar to the approach used in RECH, and name their model RealRECH. The RECH-X model differs from the RealRECH model in a significant way. Specifically, RECH-X adopts a distinct strategy by directly incorporating the exogenous information into the volatility equation, rather than introducing a measurement equation for the realized volatility as in RealRECH. This modification is made to accommodate the inclusion of other exogenous variables, which may not be feasible to include in the measurement equation. As a result, RECH-X offers greater flexibility in allowing market participants to model the volatility by making use of timely available related variables.

3 Bayesian inference

We adopt the Bayesian approach for inference and prediction in this paper. We follow Nguyen et al. (2022) and use their priors for the RECH-X parameters. In particular, the priors of the parameters in the GARCH component are $\alpha \sim U(0,1)$ and $\beta \sim U(0,1)$; the priors of the coefficients β_0 and β_1 are $\beta_0 \sim U(0,0.5)$ and $\beta_1 \sim U(0,0.5)$; and we use a normal prior with a zero mean and variance 0.1 for the recurrence parameters of the RNN component, $v_\omega, v_y, v_\sigma, w_h, b$, as they are often small. The prior for the regressors coefficients v_z in (2d) is $N(0,0.5)$. Lastly, the prior of the degrees of freedom is $\nu \sim \text{Gamma}(1,0.1)$. For the parameters of the GARCH and the RealGARCH model, we employ the commonly used priors in the literature; see, e.g., Gerlach and Wang (2016).

Due to the complex nature of the RECH-X model and its challenging posterior distribution, we use the likelihood-annealing SMC method (Duan and Fulop, 2015) to sample from the posterior. Besides, the likelihood-annealing SMC also provides an estimation of the log marginal likelihood for the model comparison.

Let $y = \{y_t, t = 1, \cdots T\}$ be the training data and θ the set of model parameters. The likelihood-annealing SMC samples sequentially from the prior distribution $p(\theta)$ to the posterior distribution $p(\theta|y) \propto p(y|\theta)p(\theta)$ through a sequence of annealed distributions,

$$\pi_k(\theta) \propto p(\theta)p(y|\theta)^{a_k}, \quad k = 1, ..., K \tag{3}$$

with $0 = a_1 < a_2 < \cdots < a_K = 1$. The samples drawn from the last annealed distribution $\pi_K(\theta)$ are from the posterior $p(\theta|y)$.

For expanding-window out-of-sample prediction that updates the posterior each time a new observation arrives, we employ the data-annealing SMC method that samples from the sequence

$$\pi_t(\theta) \propto p(\theta)p(y_1, ..., y_{T+t}|\theta), \quad t = 1, 2, ... \tag{4}$$

We refer the reader to Nguyen et al. (2022) for the detailed implementation of the likelihood-annealing and data-annealing SMC samplers for RECH, and to Gunawan et al. (2022) for a general discussion of these SMC samplers.

4 Empirical illustrations

This section demonstrates the performance of the RECH-X model in comparison with the benchmark models using stock index and exogenous data from various countries. The first example tests the performance of RECH-X using a realized measure as the exogenous data. We use daily index returns from ten major stock markets including the S&P500 index and the Japanese Nikkei 225 index. The data are obtained from the Oxford-Man Institute. Table 9 in Appendix B provides the detailed description of the data. Each series consists of 2001 closed prices, leading to 2000 return values where the first 1500 observations were used for training and the rest for out-of-sample testing. In order to assess the predictive performance, we compare the one-step-ahead point and density forecasts of the last 500 observations using five predictive scores: the partial predictive score (PPS), the quantile score (QS), the mean squared forecast error (MSE), the mean absolute forecast error (MAE) and the R2LOG; see the definition in Appendix A.

The second example evaluates the performance of the RECH-X model using multiple financial-economic indicators as exogenous variables. We consider five stock markets including Vietnam (VN100), Japan (N225), France (CAC40), Australia (ASX) and Brazil (BVSP). The data were collected from the financial portal `https://investing.com` for the period from January 2015 to April 2023 that includes the Covid-19 period, with about 2000 observations. Each dataset is divided into two parts: the in-sample data comprises the first 1000 observations, while the remaining data is utilized for evaluating out-of-sample prediction performance. This example examines the predictive ability of financial-economic indicators for volatility modeling and forecasting. Four indicators are considered in this paper: the currency exchange rates of the above countries against the USD, the uncertainty of the US stock market via the VIX index, Gold price and Crude oil price (West Texas Intermediate - WTI).

4.1 The effect of realized measures on volatility forecast

We discuss the results for the selected markets S&P500 and N225 in detail in this section, and the results for the other markets can be found in Appendix B.

4.1.1 S&P500 data

The in-sample estimation results of the Standard & Poor's 500 index returns are summarized in Table 1; only the main interested parameters are shown. The posterior mean of the coefficient with respect to the realized volatility v_{RV} is more than two standard deviations from zero, indicating the significant effect of the realized volatility on the volatility dynamic. The degrees of freedom in GARCH and RealGARCH models are smaller than those in the RECH-X model meaning that it is less likely to observe extreme innovations in the RECH-X model. The last column in Table 1 lists the log marginal likelihood estimates, which show that the S&P500 data strongly support the RECH-X model.

Model	α/γ	β	ν	φ/β_1	v_{RV}	Mar.llh
GARCH	0.118 (0.021)	0.764 (0.037)	4.892 (0.644)			−1556.3 (0.21)
RealGARCH	0.001 (0.000)	0.997 (0.000)	3.376 (0.316)	1.281 (0.204)		−1595.0 (0.18)
RECH-X	0.018 (0.013)	0.185 (0.164)	6.594 (1.000)	0.375 (0.122)	0.412 (0.158)	**-1487.4** (0.19)

Table 1: S&P500 data: Posterior means, with the posterior standard deviations in brackets, of the main model parameters. The last column shows the log marginal likelihood estimates with the Monte Carlo standard errors in brackets, across 10 different runs of the likelihood-annealing SMC sampler. The number in bold indicates the best value.

Table 2 shows the summary statistics of the *normalized* residuals for in-sample data, i.e.

$\widehat{\epsilon}_t = \Phi^{-1}\big(F_{t_\nu}(y_t/\widehat{\sigma}_t)\big)$, where $\Phi(\cdot)$ and $F_{t_\nu}(\cdot)$ are the cumulative probability distribution functions of the standard normal distribution and Student's t distribution t_ν, respectively. These normalized residuals are expected to follow the standard normal distribution. As shown, for each of the three models, the residuals' standard deviation and kurtosis are close to 1 and 3 - the quantities of the standard normal distribution, respectively. However, all three models produce residuals that are still slightly skewed to the left; perhaps, the RECH specification that combines RNN with the GJR model (Glosten et al., 1993b) can increase the in-sample fit further.

Model	mean	Std	skew	kurtosis
GARCH	0.050	0.995	−0.242	2.953
RealGARCH	0.065	1.009	−0.172	2.821
RECH-X	0.044	1.037	−0.296	2.943

Table 2: S&P500 data: summary statistics of normalized residuals.

Figure 1 plots the return indices in the out-of-sample data together with 95% one-step-ahead forecast intervals from the three models. The forecast intervals generated by RECH-X can quickly expand to accommodate extreme observations and decay at a faster rate compared to the benchmark models. The predictive performance of the three models is summarized in Table 3, which shows that the RECH-X model has the best performance across all the five predictive scores. The RealGARCH model performs better the GARCH model in terms of MSE and MAE, but not PPS, QS and R2LOG.

Model	PPS	QS	MSE	MAE	R2LOG
GARCH	1.225	0.035	0.138	0.304	1.079
RealGARCH	1.236	0.037	0.120	0.283	1.129
RECH-X	**1.139**	**0.032**	**0.095**	**0.234**	**0.670**

Table 3: S&P500 data: Predictive performance. The numbers in bold indicate the best values.

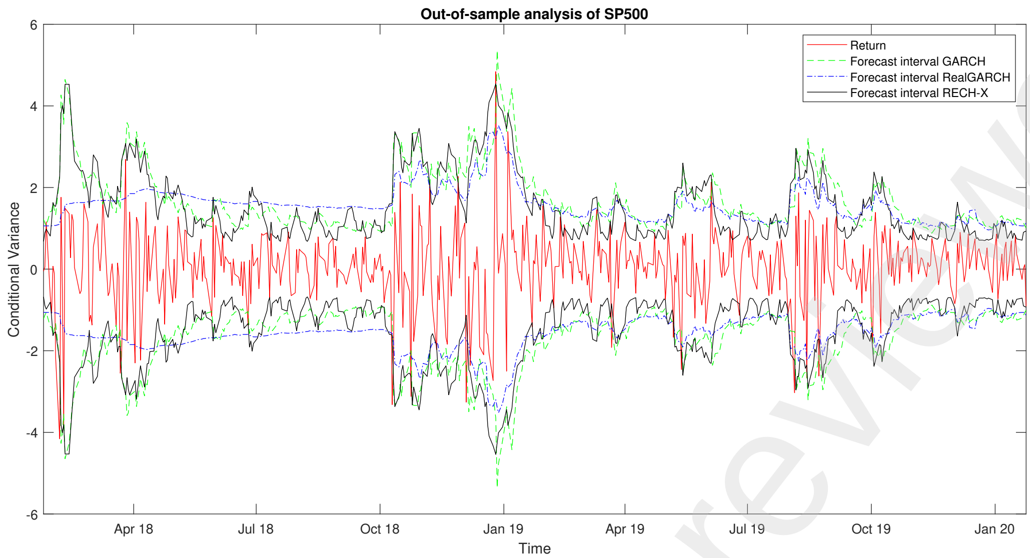

Figure 1: S&P500: plots of return indices in the out-of-sample data together with 95% one-step-ahead forecast intervals from GARCH (dashed green), RealGARCH (dash-dotted blue) and RECH-X (solid black).

4.1.2 N225 data

The analysis of the Japanese Nikkei 225 index returns shows similar results to those of the S&P500 data. The in-sample estimation results are summarized in Table 4. Even that the degrees of freedom are similar among the models, the posterior mean of v_{RV} is still more than two standard deviations from zero, indicating the significant forecast ability of the realized volatility on the underlying volatility dynamic. The last column lists the log marginal likelihood estimates, which shows that the data strongly support the RECH-X model.

Model	α/γ	β	ν	φ/β_1	v_{RV}	Mar.llh
GARCH	0.072 (0.015)	0.872 (0.026)	5.494 (0.783)			-2399.8 (0.18)
RealGARCH	0.538 (0.069)	0.368 (0.055)	5.567 (0.780)	0.631 (0.065)		-2382.6 (0.23)
RECH-X	0.031 (0.014)	0.497 (0.211)	5.687 (0.772)	0.399 (0.114)	0.617 (0.186)	**-2371.0** (0.20)

Table 4: N225 data: Posterior means, with the posterior standard deviations in brackets, of the main model parameters. The last column shows the log marginal likelihood estimates with the Monte Carlo standard errors in brackets, across 10 different runs of the SMC sampler. The number in bold indicates the best value.

Figure 2: N225: plots of return indices in the testing data together with 95% one-step-ahead forecast intervals from GARCH (dashed green), RealGARCH (dashed-dotted blue) and RECH-X (solid black).

Table 5 summarizes the predictive performance of the three models, which, similar to the case with the S&P500 data, shows that the RECH-X model has the best performance across all the five predictive scores. Figure 2 plots of the return indices in the testing data together with 95% one-step-ahead forecast intervals from the three models. As shown, the RECH-X forecasts trace the data well and be responsive to abrupt changes in the returns.

We also examine eight other stock indices; detailed results can be found in Appendix B. These results consistently confirm the superiority of RECH-X over the benchmark models GARCH and RealGARCH.

Model	PPS	QS	MSE	MAE	R2LOG
GARCH	1.459	0.041	0.324	0.520	2.170
RealGARCH	1.454	0.040	0.321	0.537	2.321
RECH-X	**1.416**	**0.037**	**0.317**	**0.501**	**1.933**

Table 5: N225 data: Predictive performance. The numbers in bold indicate the best values.

4.2 The effect of financial-economic indicators on volatility forecast

Scholars have shown that key financial-economic indicators such as economic growth, exchange rates, gold prices, oil prices, interest rates, inflation and money supply, might have an impact on stock market volatility. Furthermore, fluctuations of major stock markets in the world can also affect the stock market of other countries due to the spillover effect. This study focuses on examining how exchange rates, gold prices, oil prices, and the uncertainty of the US stock market influence the volatility of five stock markets: Vietnam (VN100), Japan (N255), France (CAC40), Australia (ASX), and Brazil (BVSP). Other macroeconomic indicators such as GDP can also be considered. We use these four variables as the inputs z_t of the RECH-X model. It is important to note that, as the input variables to the RNN, these exogenous variables should be standardized. More precisely, for each input time series, the in-sample data are standardized to have a zero mean and standard deviation of 1, and the out-of-sample data are standardized accordingly using the mean and standard deviation computed from the in-sample data.

Exchange rates: There is mixed evidence about the effect of exchange rates on the volatility of stock markets. Tursoy et al. (2008) and Abugri (2008) find that the exchange rate does not affect stock market volatility. On the other hand, Adam and Tweneboah (2008) obtain a negative effect while Maysami et al. (2004) support the hypothesis of a positive relationship between exchange rates and stock returns.

Gold: Gold and stocks are two alternative options that investors often consider. Gold is commonly regarded as a safe haven asset that offers financial stability during economic downturns, while stocks present the potential for higher long-term returns but also come with higher risk and uncertainty. Smith (2001) demonstrates bi-directional causality between gold and stock prices using the Granger test. Garefalakis et al. (2011) argue that the volatility of gold prices negatively affects investment returns on the Hong Kong stock market. Akbar et al. (2019) employ a vector autoregressive model to illustrate a reverse effect between gold price and stock price, while Singhal et al. (2019) show that the world gold price has a positive influence on Mexico's stock price. These studies provide valuable insights into the complex relationship between gold and stocks in various markets.

Crude oil: Empirical results on the relationship between oil and stock prices have been shown to be positive or negative depending on the stock markets considered. Singhal et al. (2019) show that oil prices negatively affect stock prices of Mexico, whereas Mokni and Youssef (2019) and Khan et al. (2021) show a positive relationship between crude oil price and stock market.

The uncertainty of the US stock market: The VIX is a popular measure of the US stock market volatility expectations based on S&P500 index options. Due to the interconnectedness of global financial markets, it is generally expected that the volatility of other countries' stock markets will be influenced by the volatility of the US stock market, as indicated by the VIX. The persistence of financial returns volatility further supports the notion that the VIX variable might have a positive effect on the long-term volatility of another nation's stock market, given the high persistence of volatility in financial returns.

We now test the effect of these four exogenous variables on the volatility of the five stock markets mentioned above. As the realized volatilities are not publicly available for those countries, we only compare RECH-X with GARCH-X and GARCH. Table 6 summarizes the in-sample estimation results. Further results on residual analysis can be found in Appendix C. As can be seen, the log marginal likelihood estimates show that the data strongly support the RECH-X model.

Table 7 shows the impact of each individual financial-economic variable. We draw the following conclusions: Firstly, the uncertainty of the US stock market has a positive effect on the stock market volatility of all the five countries, and this effect is significant in the case of CAC40, N225, ASX and BVSP. Among the four exogenous variables, the uncertainty of the US stock market has a positive and the most strong impact, as expected.

Secondly, comparing the coefficients of gold price across the five markets, it can be seen that the volatility in Vietnam, France, Australia, and Brazil markets are quite strongly influenced by the gold price. For France, Australia, and Brazil markets, we observe that gold price has

Data	Model	w	α	β	ν	β_1	Mar.llh
VN100	GARCH	0.033	0.078	0.83	5.53		−1308.5
		(0.013)	(0.019)	(0.041)	(0.818)		(0.094)
	GARCH-X	0.083	0.092	0.585	5.547		-1322.2
		(0.041)	(0.025)	(0.141)	(0.843)		(0.785)
	RECH-X	0.162	0.064	0.708	6.156	0.169	**-1303.5**
		(0.284)	(0.022)	(0.085)	(1.007)	(0.073)	(0.36)
CAC40	GARCH	0.022	0.089	0.841	5.199		−1413.1
		(0.012)	(0.023)	(0.041)	(0.830)		(0.083)
	GARCH-X	0.032	0.117	0.762	5.342		-1431.0
		(0.019)	(0.030)	(0.065)	(0.820)		(0.289)
	RECH-X	0.055	0.045	0.616	5.705	0.313	**-1388.7**
		(0.243)	(0.020)	(0.116)	(0.808)	(0.092)	(0.27)
N225	GARCH	0.052	0.099	0.773	4.034		−1538.2
		(0.021)	(0.024)	(0.050)	(0.523)		(0.528)
	GARCH-X	0.031	0.105	0.743	4.087		-1524.6
		(0.019)	(0.028)	(0.063)	(0.542)		(1.156)
	RECH-X	0.245	0.029	0.583	4.962	0.406	**-1483.5**
		(0.207)	(0.018)	(0.095)	(0.712)	(0.107)	(0.403)
ASX	GARCH	0.778	0.234	0.238	7.53		−1786.3
		(0.128)	(0.045)	(0.076)	(1.378)		(0.107)
	GARCH-X	0.508	0.221	0.195	7.807		-1785.4
		(0.117)	(0.045)	(0.062)	(1.458)		(0.321)
	RECH-X	0.272	0.199	0.216	7.663	0.294	**-1781.9**
		(0.298)	(0.047)	(0.072)	(1.322)	(0.153)	(0.246)
BVSP	GARCH	0.108	0.051	0.861	7.056		−1770.6
		(0.076)	(0.017)	(0.063)	(1.282)		(0.064)
	GARCH-X	0.544	0.021	0.170	7.615		-1766.2
		(0.181)	(0.025)	(0.019)	(0.149)		(0.179)
	RECH-X	0.281	0.038	0.668	7.108	0.209	**-1765.3**
		(0.292)	(0.018)	(0.164)	(1.211)	(0.088)	(0.39)

Table 6: Data from January 2015 to April 2023: Posterior means, with the posterior standard deviations in brackets, of the main model parameters. The last column shows the log marginal likelihood, the number in bold indicates the best value.

a negative effect, which is consistent with the finding of Garefalakis et al. (2011), while in the other two markets, volatility is positively correlated with the gold price, similar to Akbar et al. (2019), Singhal et al. (2019). The volatility of the Japanese stock market is less influenced by the gold price.

Thirdly, consistent with the previous studies, the oil price has a mixed effect on the volatility. In France, Japan, Australia and Brazil, the volatility tends to decrease when the oil price increases as studied by Singhal et al. (2019). Particularly, in Australia rising oil prices can cause a significant reduction in volatility in the stock market. Only in Vietnam, oil price and volatility have a positive relationship similar to the results of Mokni and Youssef (2019), Khan et al. (2021), with a relatively small influence.

Lastly, with the foreign exchange rate, we find that it has a positive influence and has a great impact on the volatility of the Brazilian stock market while the remaining 4 countries have a relatively low influence. In Vietnam and Australia, the correlation is opposite as the conclusion of Adam and Tweneboah (2008), the remaining countries are positively correlated as the conclusion of Maysami et al. (2004).

Data	**OIL**	**GOLD**	**VIX**	**EXR**
VN100	0.086	0.237	0.190	-0.058
	(0.121)	(0.133)	(0.115)	(0.08)
CAC40	-0.115	-0.135	0.336	0.037
	(0.075)	(0.079)	(0.137)	(0.066)
N225	-0.042	0.004	0.569	0.014
	(0.059)	(0.046)	(0.160)	(0.067)
ASX	-0.351	-0.374	0.451	-0.014
	(0.247)	(0.241)	(0.273)	(0.244)
BVSP	-0.096	-0.408	0.319	0.302
	(0.148)	(0.227)	(0.162)	(0.164)

Table 7: Posterior mean and standard deviation of the estimated coefficients corresponding to the covariates in the RECH-X model

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We now test the combined influence of the financial-economic indicators on volatility forecasting. Table 8 shows that, with the inclusion of these four financial-economic indicators, the

RECH-X model delivers a better volatility forecast than the GARCH and GARCH-X model.

Data	Model	PPS	QS
VN100	GARCH	1.59	0.056
	GARCH-X	1.62	**0.054**
	RECH-X	**1.59**	0.055
CAC40	GARCH	1.541	0.051
	GARCH-X	1.57	0.051
	RECH-X	**1.490**	**0.045**
N225	GARCH	1.622	0.047
	GARCH-X	1.67	0.054
	RECH-X	**1.604**	**0.046**
ASX	GARCH	2.046	0.059
	GARCH-X	2.09	0.065
	RECH-X	**2.040**	**0.059**
BVSP	GARCH	1.781	0.058
	GARCH-X	1.801	0.061
	RECH-X	**1.767**	**0.057**

Table 8: Predictive performance. The numbers in bold indicate the best values.

5 Conclusion

This paper proposes a deep learning enhanced volatility model RECH-X that allows for the incorporation of high-frequency data and financial-economic information as exogenous covariates. Bayesian inference and prediction is performed using Sequential Monte Carlo samplers. We find that the RECH-X model using a realized measure as the covariate improves performance compared to GARCH, GARCH-X and RealGARCH. Incorporating financial-economic indicators into the input of RECH-X leads to some interesting findings, and a significant improvement in volatility modeling and forecasting. It is possible to extend the RECH-X model into a mix-frequency context similar to what has been considered by Engle et al. (2013), Asgharian et al. (2013), Conrad and Kleen (2020) and Virbickaite et al. (2023); this research is in progress.

Appendix

A Forecast evaluation

In order to assess the predictive performance, we use five predictive scores for both point and density forecasts. The first is the partial predictive score (PPS),

$$\text{PPS} := -\frac{1}{T_{\text{test}}}\sum_{t=T+1}^{T+T_{\text{test}}} \log\ p(y_t|y_{1:t-1}),$$

where T_{test} is the number of observations in the test data. The second is quantile score defined as (Taylor, 2019)

$$\text{QS} := \frac{1}{T_{\text{test}}}\sum_{t=T+1}^{T+T_{\text{test}}} (\alpha - I_{y_t \le q_{t,\alpha}})(y_t - q_{t,\alpha}),$$

where $q_{t,\alpha}$ is the α-VaR forecast of y_t, conditional on $y_{1:t-1}$. The smaller the quantile score, the better the Value-at-Risk forecast. The next three predictive scores compare the model-based volatility forecast $\widehat{v}_t^2$ with the realized volatility in the test data (Hansen and Lunde, 2005):

$$\begin{aligned}
\text{MSE} &= \frac{1}{T_{\text{test}}}\sum_{t=T+1}^{T+T_{\text{test}}} (\text{RV}_t - \widehat{v}_t)^2,\\
\text{MAE} &= \frac{1}{T_{\text{test}}}\sum_{t=T+1}^{T+T_{\text{test}}} |\text{RV}_t - \widehat{v}_t)|,\\
\text{R}^2\text{LOG} &= \frac{1}{T_{\text{test}}}\sum_{t=T+1}^{T+T_{\text{test}}} \left[\log(\text{RV}_t^2\widehat{v}_t^{-2})\right]^2,
\end{aligned}$$

where $\widehat{v}_t^2 = \nu\widehat{\sigma}_t^2/(\nu-2)$ with $\widehat{\sigma}_t^2$ a forecast of σ_t^2 using the volatility model with Student's t innovations.

B Predictive performance of the RECH-X model with realize volatility

This section assesses the predictive performance of the three models on a wide range of index returns. We opt to not present the in-sample estimation results, except the log marginal likelihood estimates listed in the last column of Table 10. The RECH-X model has the highest log marginal likelihood estimates in all cases except for the BVSP data. The results in Table 10 show that the RECH-X model consistently has an impressive predictive performance across all the datasets. The RealGARCH model is, in general, better than GARCH.

Symbol	Stock Name	Start Date	End Date	Size
AEX	Amsterdam Exchange Index	2012-03-20	2020-01-24	2001
AORD	Australia All Ordinaries Index	2012-02-28	2020-01-24	2001
BFX	Brussels Stock Exchange Index	2012-03-20	2020-01-24	2001
BVSP	Sao Paulo Stock Exchange	2011-12-13	2020-01-24	2001
DJI	Dow Jones Industrial Average	2012-02-02	2020-01-24	2001
FCHI	Euronext Paris CAC 40 Index	2012-03-22	2020-01-24	2001
FTSE	Financial Times Stock Exchange	2012-02-23	2020-01-24	2001
DAX	Frankfurt Stock Exchange	2012-02-21	2020-01-24	2001
N225	Nikkei Stock Average 225	2011-11-16	2020-01-24	2001
SPX	The Standard and Poor's 500	2012-02-06	2020-01-24	2001

Table 9: Description of the stock index data

Data	Model	PPS	QS	MSE	MAE	R2LOG	Mar.llh
DJI	GARCH	1.249	0.034	0.132	0.282	0.877	−1498.1
	RealGARCH	1.232	0.037	0.096	0.235	0.684	−1547.3
	RECH-X	**1.191**	**0.032**	**0.095**	**0.226**	**0.567**	**-1446.5**
AORD	GARCH	1.022	0.029	0.093	0.259	1.098	−1639.5
	RealGARCH	1.033	0.028	0.097	0.274	1.196	−1636.3
	RECH-X	**1.010**	**0.027**	**0.086**	**0.240**	**0.963**	**-1618.8**
FTSE	GARCH	1.195	0.029	0.094	0.232	0.656	−1743.7
	RealGARCH	1.181	0.028	**0.076**	0.205	0.567	−1738.7
	RECH-X	**1.166**	**0.027**	0.077	**0.201**	**0.500**	**-1707.0**
BVSP	GARCH	1.655	0.040	0.302	0.510	1.365	−2636.1
	RealGARCH	1.642	**0.039**	0.286	0.491	1.242	**-2613.6**
	RECH-X	**1.639**	**0.039**	**0.278**	**0.484**	**1.207**	−2620.0
BFX	GARCH	1.256	0.030	0.091	0.254	0.663	−1919.1
	RealGARCH	1.278	0.030	0.098	0.280	0.805	−1913.6
	RECH-X	**1.222**	**0.028**	**0.089**	**0.239**	**0.608**	**-1878.4**
AEX	GARCH	1.212	0.034	0.108	0.279	0.898	−1944.4
	RealGARCH	1.243	0.032	0.124	0.322	1.134	−1942.5
	RECH-X	**1.142**	**0.027**	**0.090**	**0.239**	**0.656**	**-1890.6**
FCHI	GARCH	1.276	0.034	0.133	0.312	0.977	−2157.0
	RealGARCH	1.272	0.033	0.117	0.306	0.982	−2131.4
	RECH-X	**1.204**	**0.029**	**0.105**	**0.259**	**0.674**	**-2100.1**
DAX	GARCH	1.387	0.033	0.148	0.335	0.939	−2178.4
	RealGARCH	1.359	0.032	**0.104**	0.287	0.772	−2146.3
	RECH-X	**1.334**	**0.031**	0.113	**0.281**	**0.674**	**-2131.9**

Table 10: Predictive performance of GARCH, RealGARCH and RECH-X across a range of index returns. The numbers in bold indicate the best values.

C Predictive performance of the RECH-X model with financial-economic variables

VN100

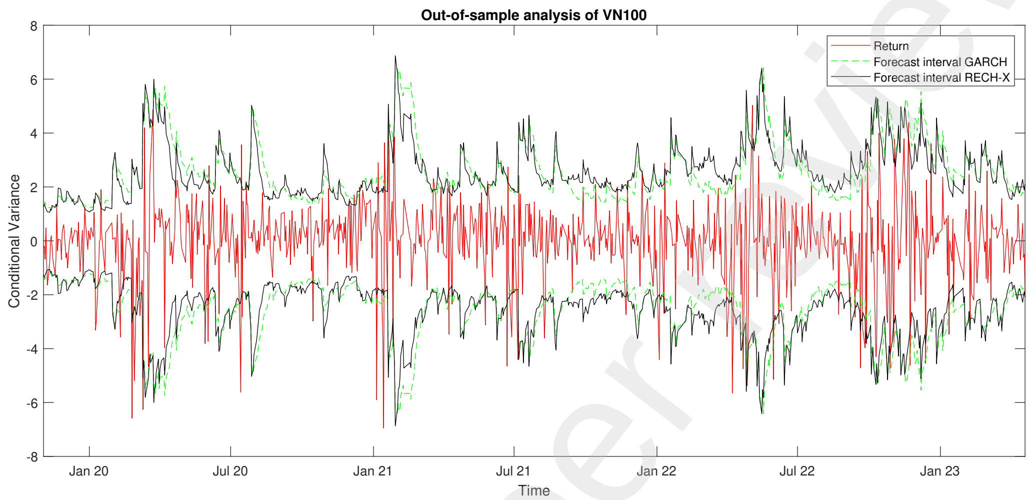


Figure 3: VN100: plots of return indices in the testing data together with 95% one-step-ahead forecast intervals from GARCH (dashed green)and RECH-X (solid black).

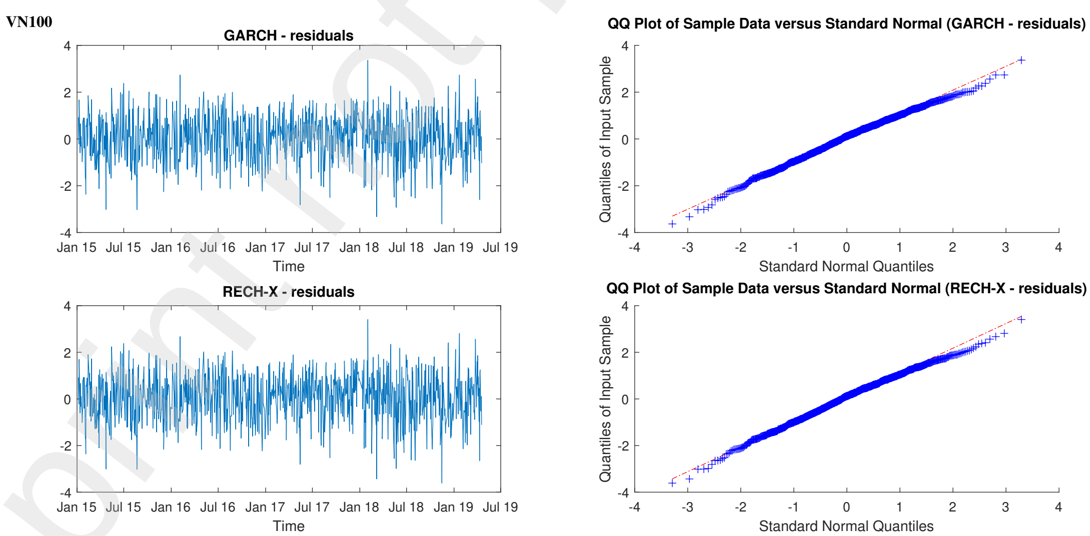


Figure 4: VN100: plots of sample data versus Standard Normal

CAC40

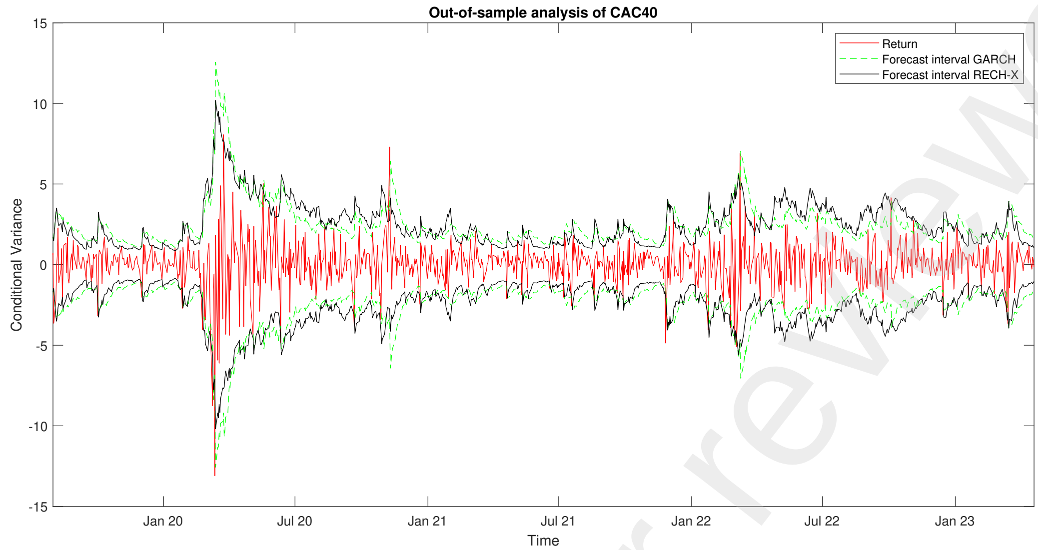


Figure 5: CAC40: plots of return indices in the testing data together with 95% one-step-ahead forecast intervals from GARCH (dashed green)and RECH-X (solid black).

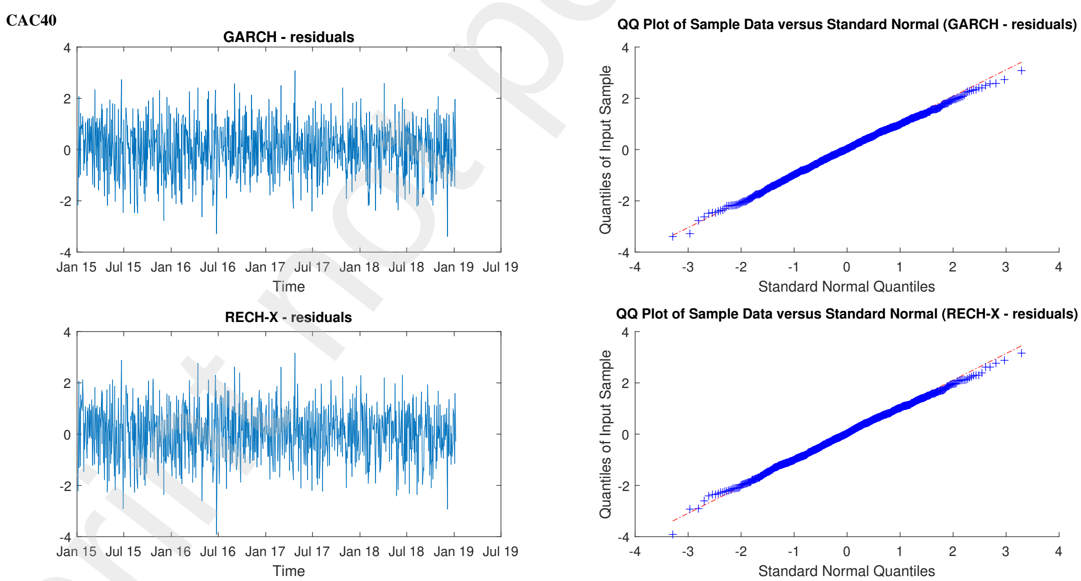


Figure 6: CAC40: plots of sample data versus Standard Normal

N225

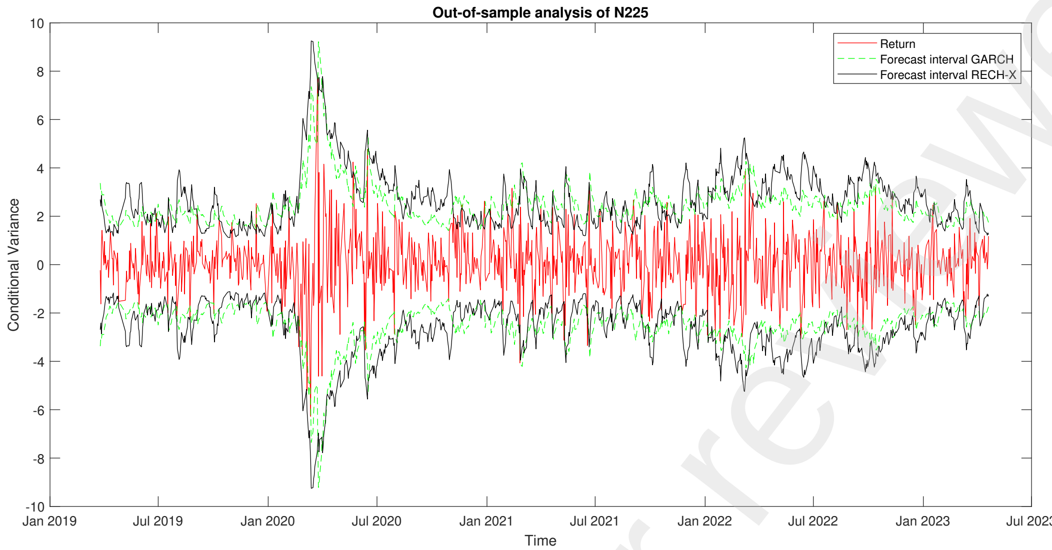


Figure 7: N225: plots of return indices in the testing data together with 95% one-step-ahead forecast intervals from GARCH (dashed green)and RECH-X (solid black).

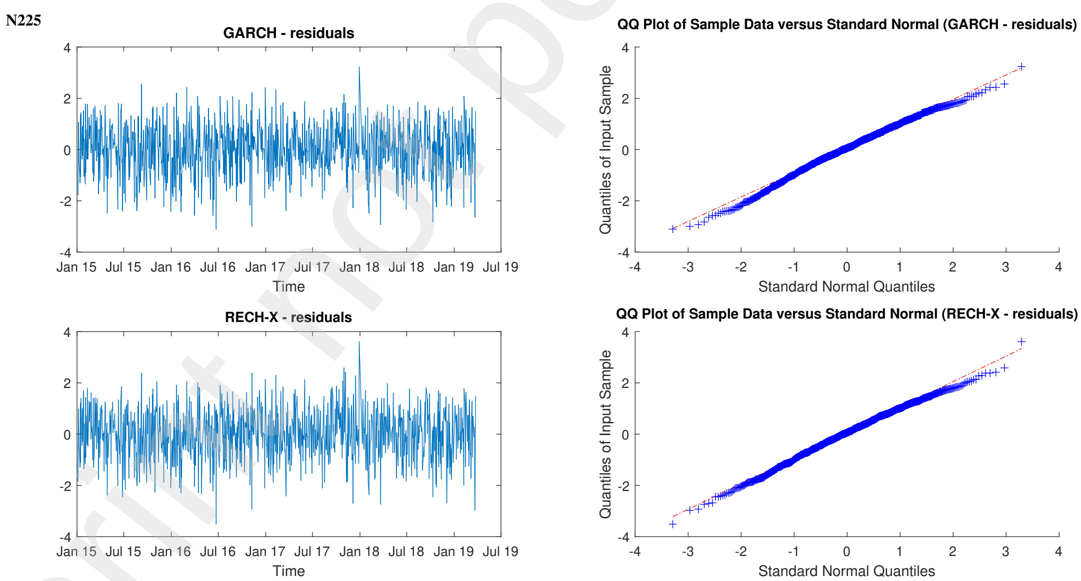


Figure 8: N225: plots of sample data versus Standard Normal

ASX

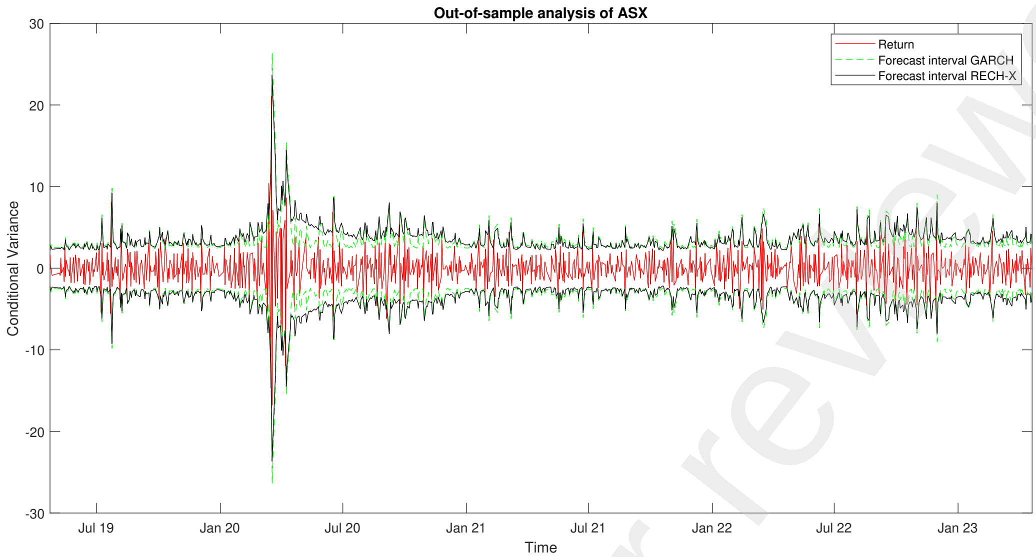


Figure 9: ASX: plots of return indices in the testing data together with 95% one-step-ahead forecast intervals from GARCH (dashed green)and RECH-X (solid black).

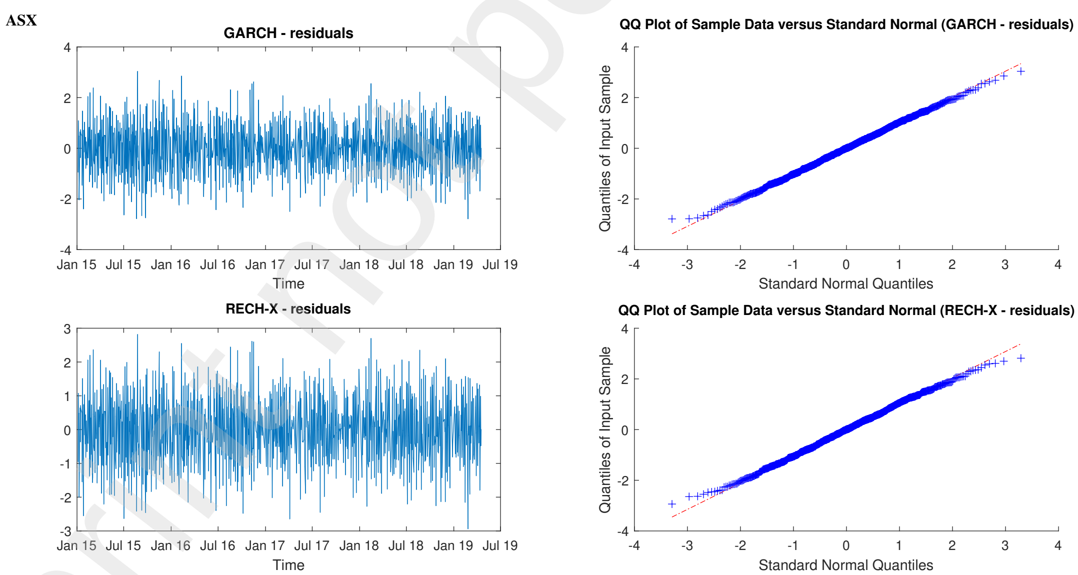


Figure 10: ASX: plots of sample data versus Standard Normal

BVSP

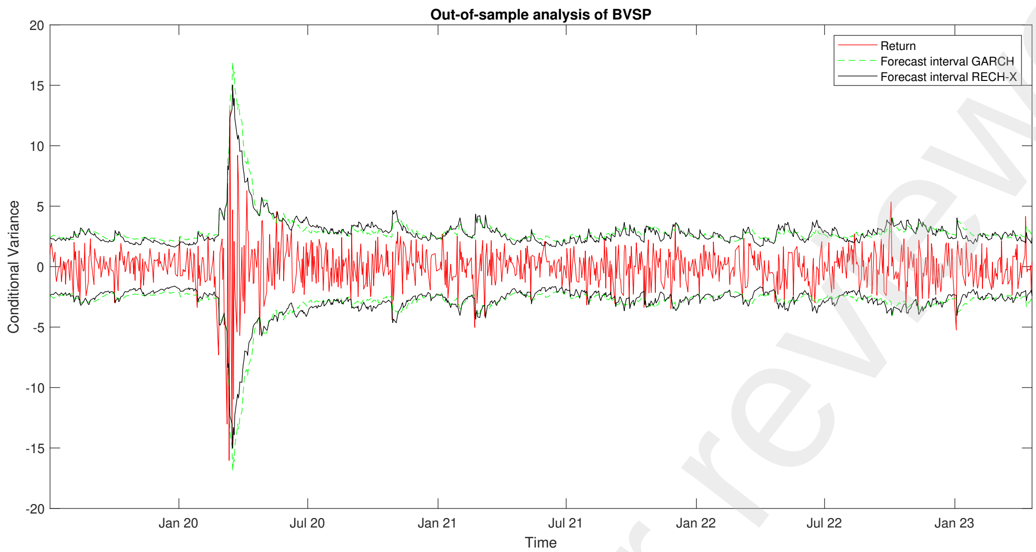
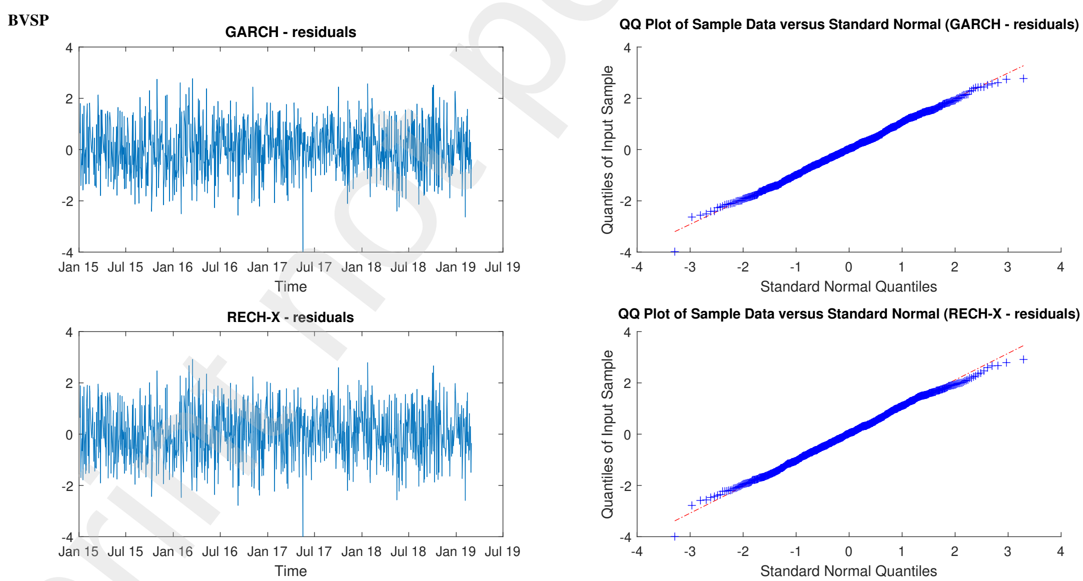


Figure 11: BVSP: plots of return indices in the testing data together with 95% one-step-ahead forecast intervals from GARCH (dashed green)and RECH-X (solid black).



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