Parallel Bayesian inference for high dimensional dynamic factor copulas

Hoang Nguyen
M. Concepción Ausín and Pedro Galeano

Universidad Carlos III de Madrid

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- Vine copula GARCH models are commonly used.
  - But ... Most of the vine copulas are static (except Almeida, Czado, and Manner 2015)
- Alternatively, Factor copulas proposed by Krupskii and Joe 2013 have several advantages. The observed variables are conditionally independent given one or more latent variables (see Oh and Patton 2016, Creal and Tsay 2015).

Motivation and research questions (cont.)

- We propose new models of time varying dependence in high dimensional time series and use the Bayesian approach to estimate the different factor copula models.
  - The dynamic factor loadings are modelled as generalized autoregressive score (GAS) processes.
  - The high dimensional problem is handled with a factor structure which also allows the inference strategy to run in a parallel setting.
  - Using group hyperbolic skew Student copula (HSST), we obtain different tail behaviour and asymmetric dependence among financial time series.
  - The model is extendible in which deleting or adding more variables does not change the form of model results.

AGENDA

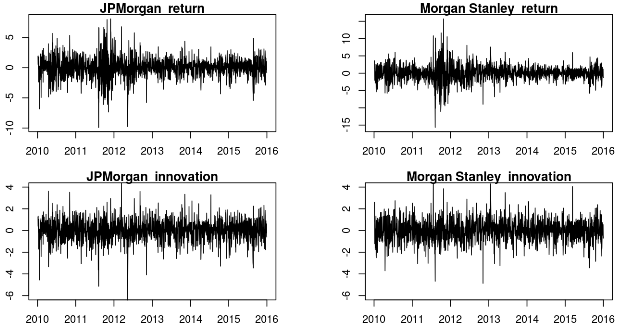
An example of GARCH copulas

Let $r_t = (r_{1t}, \ldots, r_{dt})$ be a d-dimensional financial return time series at time t where $t = 1, \ldots, T$. Each marginal returns i are filtered out the conditional mean and conditional variance using $AR(k) - GARCH(p, q)$ model:

$$\begin{aligned} r_{it} &= c_i + \phi_{i1} r_{i,t-1} + \ldots + \phi_{ik} r_{i,t-k} + a_{it} \\ a_{it} &= \sigma_{it}\eta_{it} \text{ where } \eta_{it} \sim F_i(.|\vartheta_i) \\ \sigma_{it}^2 &= \omega_i + \alpha_{i1} a_{i,t-1}^2 + \ldots + \alpha_{ip} a_{i,t-p}^2 + \beta_{i1}\sigma_{i,t-1}^2 + \ldots + \beta_{iq}\sigma_{i,t-q}^2 \end{aligned}$$

The standardized innovations η_{it} are taken from the filter.
Note that other models such as EGARCH, IGARCH, GJR-GARCH, Stochastic volatility could also be applied.

An example of GARCH copulas (cont.)



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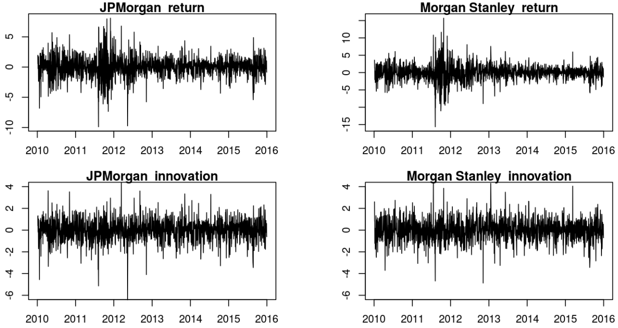

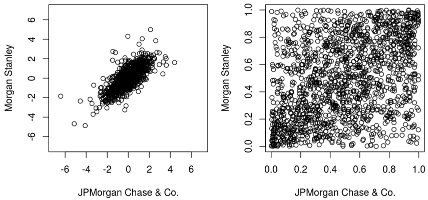

An example of GARCH copulas (cont.)

Copula data $u_{it} = F_i(\eta_{it}|\vartheta_i)$ is known to be an uniform U(0,1) distribution for $i = 1, ..., d$. Thus, the joint cdf of the vector $U_t = (u_{1t}, ..., u_{dt})$, is given by a copula cdf

$$F(u_{1t}, ..., u_{dt}|\theta_t) = C(u_{1t}, ..., u_{dt}|\theta_t)$$

The computation for the likelihood $C(u_{1t}, ..., u_{dt}|\theta_t)$ is often expensive (see Krupskii and Joe 2013).

AN EXAMPLE OF GARCH COPULAS (CONT.)

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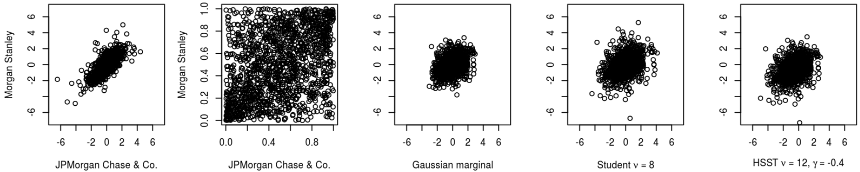
Alternatively, Creal and Tsay 2015, Oh and Patton 2016 choose to work with transformation of copula data into real domain



Gaussian one factor copulas

Assume that $u_t = \{u_{1t}, \ldots, u_{dt}\} \sim C^{Gauss}(.|R_t)$
If we define $x_{it} = \Phi^{-1}(u_{it})$, then,
$x_t = \{x_{1t}, \ldots, x_{dt}\} \sim MultiNorm(\cdot \mid \mu = 0, R_t)$

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Assume that the variables x_{it} are conditionally independent given one standard Normal latent variable z_t and they are bivariate Gaussian with correlation ρ_{it}.

$$x_{it} = \rho_{it} z_t + \sqrt{1 - \rho_{it}^2}\epsilon_{it}$$

where $\epsilon_{it} \sim iidN(0, 1)$ is the individual shock and $z_t \sim iidN(0, 1)$ is one state latent variable. x_t follow a multivariate normal one factor.

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where $\epsilon_{it} \sim iidN(0,1)$ is the individual shock and $z_t \sim iidN(0,1)$ is one state latent variable. x_t follow a multivariate normal one factor. Then, the correlation matrix $R_t = \rho_t \rho_t' + diag(1 - \rho_t^2)$ where $\rho_t = (\rho_{1t}, \ldots, \rho_{dt})$.

Gaussian one factor copulas (cont.)

The dynamic Gaussian one factor copulas are constructed by allowing $\rho_t = (\rho_{1t}, \dots, \rho_{dt})$ to vary across time as the generalized autoregressive score (GAS, see Creal, Koopman, and Lucas 2013) of a modified logistic transformation of $f_t = (f_{1t}, \dots, f_{dt})$

$$
\begin{aligned}
\rho_{it} &= \frac{1 - \exp(-f_{it})}{1 + \exp(-f_{it})} \\
f_{i,t+1} &= (1-b) f_{i0} + a s_{it} + b f_{it} \\
s_{it} &= \frac{\partial \log p(u_t | z_t, f_t, \mathcal{F}_t, \theta)}{\partial f_{it}} \\
s_{it} &= \frac{1}{2} x_{it} z_t + \frac{1}{2} \rho_{it} - \rho_{it} \frac{x_{it}^2 + z_t^2 - 2\rho_{it} x_{it} z_t}{2(1 - \rho_{it}^2)}
\end{aligned}
\tag{1}
$$

Student one factor copulas

Assume that $u_t = \{u_{1t}, \ldots, u_{dt}\} \sim C^{St}(\cdot \mid R_t, \nu)$
If we define $x_{it} = F_{St}^{-1}(u_{it} \mid \nu)$ then,
$x_t = \{x_{1t}, \ldots, x_{dt}\} \sim MultiSt(\cdot \mid \mu = 0, R_t, \nu)$

Student one factor copulas

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If we define $x_{it} = F^{-1}_{St}(u_{it} \mid \nu)$ then,
$x_t = \{x_{1t}, \dots, x_{dt}\} \sim MultiSt(\cdot \mid \mu = 0, R_t, \nu)$
Assuming that for all assets i,

$$x_{it} = \sqrt{\zeta_t}(\rho_{it} z_t + \sqrt{1-\rho_{it}^2}\epsilon_{it})$$

where $\zeta_t \sim Inv - Gamma(\frac{\nu}{2}, \frac{\nu}{2})$ is the mixing variable who shares the same value for asset i. And further, if we define $\dot{x}_{it} = \frac{x_{it}}{\sqrt{\zeta_t}}$, then
$\dot{x}_t = \{\dot{x}_{1t}, \dots, \dot{x}_{dt}\} \sim MultiNorm(\cdot \mid \mu = 0, R_t)$ and substitute in the dynamic equation (1),

$$s_{it} = \frac{\partial \log p(u_t | z_t, \zeta_t, f_t, \mathcal{F}_t, \theta)}{\partial f_{it}} = \frac{1}{2}\dot{x}_{it} z_t + \frac{1}{2}\rho_{it} - \rho_{it}\frac{\dot{x}_{it}^2 + z_t^2 - 2\rho_{it}\dot{x}_{it} z_t}{2(1-\rho_{it}^2)}$$

Hyperbolic skew Student one factor copulas

Assuming that $u_t = \{u_{1t}, \ldots, u_{dt}\} \sim C^{HS}(\cdot \mid R_t, \nu, \gamma)$
If we define $x_{it} = F_{HS}^{-1}(u_{it} \mid \nu, \gamma)$ then,
$x_t = \{x_{1t}, \ldots, x_{dt}\} \sim MultiHS(\cdot \mid \mu = 0, R_t, \nu, \gamma)$

HYPERBOLIC SKEW STUDENT ONE FACTOR COPULAS

Assuming that $u_t = \{u_{1t}, \ldots, u_{dt}\} \sim C^{HS}(\cdot \mid R_t, \nu, \gamma)$
If we define $x_{it} = F_{HS}^{-1}(u_{it} \mid \nu, \gamma)$ then,
$x_t = \{x_{1t}, \ldots, x_{dt}\} \sim MultiHS(\cdot \mid \mu = 0, R_t, \nu, \gamma)$
Assuming that for all assets i,

$$x_{it} = \gamma\zeta_t + \sqrt{\zeta_t}(\rho_{it} z_t + \sqrt{1 - \rho_{it}^2 \epsilon_{it}})$$

where γ accounts for the asymmetric distribution in each individual assets. And further, if we define $\tilde{x}_{it} = \frac{x_{it} - \gamma\zeta_t}{\sqrt{\zeta_t}}$, then $\tilde{x}_t = \{\tilde{x}_{1t}, \ldots, \tilde{x}_{dt}\} \sim MultiNorm(\cdot \mid \mu = 0, R_t)$, the dynamic equation (1) remains

$$s_{it} = \frac{\partial \log p(u_{it} | z_t, \zeta_t, \gamma, f_t, \mathcal{F}_t, \theta)}{\partial f_{it}} = \frac{1}{2}\tilde{x}_{it} z_t + \frac{1}{2}\rho_{it} - \rho_{it}\frac{\tilde{x}_{it}^2 + z_t^2 - 2\rho_{it}\tilde{x}_{it} z_t}{2(1 - \rho_{it}^2)}$$

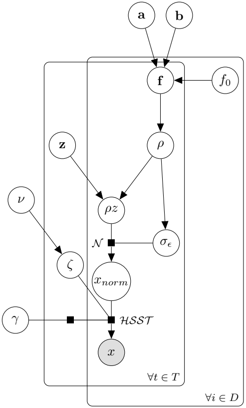
Group hyperbolic skew Student one factor copulas

The fact that using a few parameters to control for the comovement in extreme events is really restrictive. The strategy (suggested by Creal and Tsay 2015) is to split the assets into G smaller groups that can share the same characteristics.

$$\begin{aligned} x_{it} &= \gamma_g \zeta_{gt} + \sqrt{\zeta_{gt}}(\rho_{it} z_t + \sqrt{1-\rho_{it}^2}\epsilon_{it}) \\ \rho_{it} &= \frac{1-\exp(-f_{it})}{1+\exp(-f_{it})} \\ f_{i,t+1} &= (1-b_g) f_{i0} + a_g s_{it} + b_g f_{it} \end{aligned} \tag{2}$$

where $(a_g, b_g, \nu_g, \gamma_g)$ accounts for the dynamic and asymmetric dependence between individual assets belong to the same group. The pseudo data are obtained by $x_{it} = F_{HS}^{-1}(u_{it}|\nu_g, \gamma_g)$.

BAYESIAN APPROACH



Algorithm 1: BUG illustration - HSST copulas

Prior: $a \sim U(-1, 1)$;
$b \sim U(0, 1)$;
$f_0 \sim U(-5, 5)$;
$\nu \sim 4 + G(2, 2.5)$;
$\gamma \sim N(0, 1)$;
$z_t \sim N(0, 1)$;
$\zeta_g \sim IG(\nu_g/2, \nu_g/2)$;

Data transformation

$$x_{it} = F^{-1}(u_{it}|\nu_g, \gamma_g)$$

$$\tilde{x}_{it} = \frac{x_{it} - \gamma_g \zeta_{gt}}{\sqrt{\zeta_{gt}}}$$

Parameters transformation

$$s_{it} = \frac{1}{2}\tilde{x}_{it} z_t + \frac{1}{2}\rho_{it} - \rho_{it}\frac{\tilde{x}_{it}^2 + z_t^2 - 2\rho_{it}\tilde{x}_{it} z_t}{2(1-\rho_{it}^2)}$$

$$f_{i,t+1} = f_{0i} + a_g s_{it} + b_g f_{it}$$

$$\rho_{i,t+1} = \frac{1-\exp(-f_{i,t+1})}{1+\exp(-f_{i,t+1})};$$

Conditional distribution

$$\tilde{x}_{it} \sim N(\rho_{it} z_t, \sqrt{1-\rho_{it}^2});$$

https://github.com/hoanguc3m/FactorCopula

ESTIMATION RESULTS I

We illustrate an empirical example using $d = 150$ stock returns divided into G = 14 groups from 01/01/2010 to 31/12/2015 of the companies listed in S&P 500 index. The daily data contain $T = 1509$ observation days. We use AR(1)-GARCH(1,1) - HSST innovation to marginalize each stock returns.

Table : Estimation results for alternative copula models

	Gaussian block equi (1)	Gaussian 1G (2)	Gaussian dynamic (3)	Student-t block equi (4)	Student-t 1G (5)	Student-t dynamic (6)	Skew Student block equi (7)	Skew Student 1G (8)	Skew Student dynamic (9)
AIC	-139990	-140848	-141211	-168123	-153480	-168957	-168709	-153690	**-169362**
BIC	-139766	-140039	-140264	-167825	-152666	-167935	-168336	-152871	**-168266**
DIC	-139968	-140851	-141214	-168145	-153500	-169037	-169092	-153772	**-170008**
# params	42	152	178	56	153	192	70	154	206
a	[0.022,0.144]	0.039	[0.023,0.140]	[0.017,0.078]	0.026	[0.023,0.135]	[0.017,0.062]	0.025	[0.023,0.132]
b	[0.793,0.998]	0.982	[0.611,0.996]	[0.901,0.999]	0.993	[0.585,0.997]	[0.924,0.999]	0.993	[0.597,0.997]
ν				[7.734,20.802]	35.606	[7.766,21.253]	[8.242,35.468]	34.234	[8.121,35.857]
γ							[-1.678,-0.058]	-0.378	[-1.689,-0.036]
f_0	[1.303,1.767]	[0.935,2.439]	[0.913,2.476]	[1.366,1.834]	[0.953,2.521]	[0.933,2.520]	[1.361,1.825]	[0.949,2.519]	[0.907,2.518]

Estimation results II

Table : Estimation results for alternative copula models

	Gaussian block equi (1)	Gaussian 1G (2)	Gaussian dynamic (3)
AIC	-139,990	-140,848	-141,211
BIC	-139,766	-140,039	-140,264
DIC	-139,968	-140,851	-141,214
# params	42	152	178
a	[0.022,0.144]	0.039	[0.023,0.140]
b	[0.793,0.998]	0.982	[0.611,0.996]
ν			
γ			
f_0	[1.303,1.767]	[0.935,2.439]	[0.913,2.476]

ESTIMATION RESULTS

Table : Estimation results for alternative copula models

	Student-t block equi (4)	Student-t 1G (5)	Student-t dynamic (6)
AIC	-168,123	-153,480	-168,957
BIC	-167,825	-152,666	-167,935
DIC	-168,145	-153,500	-169,037
# params	56	153	192
a	[0.017,0.078]	0.026	[0.023,0.135]
b	[0.901,0.999]	0.993	[0.585,0.997]
ν	[7.734,20.802]	35.606	[7.766,21.253]
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Estimation results

Table : Estimation results for alternative copula models

	HSST block equi (7)	HSST 1G (8)	HSST dynamic (9)
AIC	-168,709	-153,690	**-169,362**
BIC	-168,336	-152,871	**-168,266**
DIC	-169,092	-153,772	**-170,008**
# params	70	154	206
a	[0.017,0.062]	0.025	[0.023,0.132]
b	[0.924,0.999]	0.993	[0.597,0.997]
ν	[8.242,35.468]	34.234	[8.121,35.857]
γ	[-1.678,-0.058]	-0.378	[-1.689,-0.036]
f_0	[1.361,1.825]	[0.949,2.519]	[0.907,2.518]

Empirical estimations

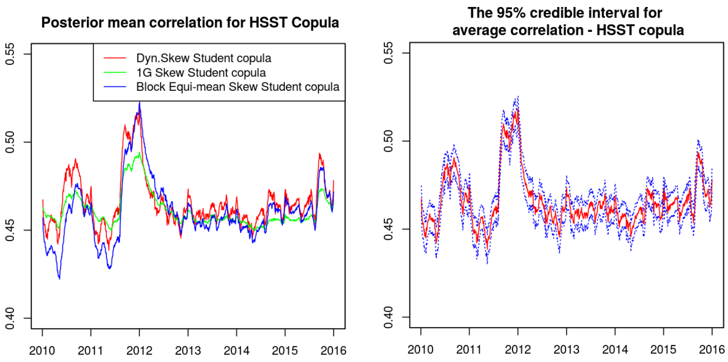

Figure : The distribution of posterior correlation among factor copula models

EMPIRICAL ESTIMATIONS

Table : Posterior estimation of grouped Students-t copula

	Oil	Food & Bev.	Pharma.	Plastics	Paper	Steel	Home App.
a	0.035	0.045	0.079	0.037	0.043	0.032	0.041
	(0.006)	(0.009)	(0.031)	(0.007)	(0.006)	(0.003)	(0.009)
b	0.996	0.989	0.902	0.988	0.986	0.995	0.960
	(0.002)	(0.006)	(0.146)	(0.005)	(0.004)	(0.001)	(0.017)
ν	12.562	13.103	7.766	14.271	14.916	17.733	15.883
	(1.215)	(1.207)	(0.596)	(1.041)	(0.906)	(1.005)	(0.804)
f_0	[1.28,1.83]	[1.10,1.99]	[1.26,1.41]	[1.21,1.97]	[1.13,1.90]	[1.33,2.03]	[1.40,2.50]
	(0.177)	(0.119)	(0.070)	(0.096)	(0.093)	(0.134)	(0.060)
# firms	5	6	5	8	12	13	14
$\lambda_L = \lambda_U$	0.031	0.022	0.072	0.026	0.019	0.019	0.038
	Electronics	Transportation	Telecom.	Retail	Insurance	Finance	Services
a	0.135	0.042	0.061	0.023	0.052	0.051	0.034
	(0.017)	(0.007)	(0.014)	(0.003)	(0.007)	(0.008)	(0.004)
b	0.585	0.978	0.973	0.997	0.972	0.984	0.991
	(0.064)	(0.007)	(0.014)	(0.002)	(0.008)	(0.006)	(0.002)
ν	17.344	9.287	8.134	21.253	9.792	9.483	20.681
	(1.106)	(0.388)	(0.470)	(1.963)	(0.375)	(0.356)	(1.013)
f_0	[1.39,2.35]	[1.00,2.12]	[0.93,2.10]	[1.10,1.87]	[1.30,2.48]	[1.45,2.52]	[1.31,2.24]
	(0.049)	(0.075)	(0.086)	(0.153)	(0.074)	(0.099)	(0.097)
# firms	12	12	7	8	14	17	17
$\lambda_L = \lambda_U$	0.026	0.086	0.087	0.007	0.106	0.099	0.011

Empirical estimations

Table : Posterior estimation of Grouped hyperbolic skew Students-t copula

	Oil	Food & Bev.	Pharma.	Plastics	Paper	Steel	Home App.
a	0.034	0.045	0.049	0.037	0.045	0.033	0.038
	(0.006)	(0.009)	(0.019)	(0.007)	(0.006)	(0.003)	(0.012)
b	0.996	0.991	0.976	0.989	0.985	0.995	0.957
	(0.002)	(0.005)	(0.023)	(0.005)	(0.004)	(0.001)	(0.036)
ν	12.971	14.069	35.857	15.368	15.248	17.982	17.012
	(1.389)	(1.554)	(5.343)	(1.504)	(1.062)	(1.104)	(1.036)
γ	-0.109	-0.237	-1.689	-0.202	-0.146	-0.173	-0.242
	(0.075)	(0.075)	(0.163)	(0.073)	(0.044)	(0.048)	(0.043)
f_0	[1.27,1.84]	[1.09,1.95]	[1.27,1.43]	[1.21,1.97]	[1.12,1.89]	[1.33,2.02]	[1.40,2.48]
	(0.174)	(0.127)	(0.099)	(0.098)	(0.092)	(0.132)	(0.062)
# firms	5	6	5	8	12	13	14
λ_L	0.047	0.056	0.423	0.051	0.035	0.039	0.086
λ_U	0.016	0.005	0.000	0.008	0.008	0.008	0.010
	Electronics	Transportation	Telecom.	Retail	Insurance	Finance	Services
a	0.132	0.042	0.065	0.023	0.054	0.052	0.034
	(0.017)	(0.007)	(0.017)	(0.004)	(0.007)	(0.009)	(0.003)
b	0.597	0.974	0.969	0.997	0.971	0.982	0.991
	(0.060)	(0.008)	(0.019)	(0.002)	(0.007)	(0.007)	(0.002)
ν	17.560	10.220	8.121	21.196	9.845	9.405	21.333
	(1.133)	(1.028)	(0.515)	(1.885)	(0.402)	(0.345)	(1.151)
γ	-0.195	-0.214	-0.075	-0.208	-0.109	-0.036	-0.318
	(0.043)	(0.089)	(0.035)	(0.055)	(0.037)	(0.029)	(0.040)
f_0	[1.38,2.33]	[0.99,2.10]	[0.91,2.08]	[1.09,1.85]	[1.29,2.46]	[1.44,2.52]	[1.30,2.23]
	(0.049)	(0.070)	(0.082)	(0.154)	(0.074)	(0.091)	(0.097)
# firms	12	12	7	8	14	17	17
λ_L	0.057	0.200	0.130	0.016	0.176	0.119	0.035
λ_U	0.010	0.022	0.054	0.003	0.060	0.082	0.002

EMPIRICAL ESTIMATIONS

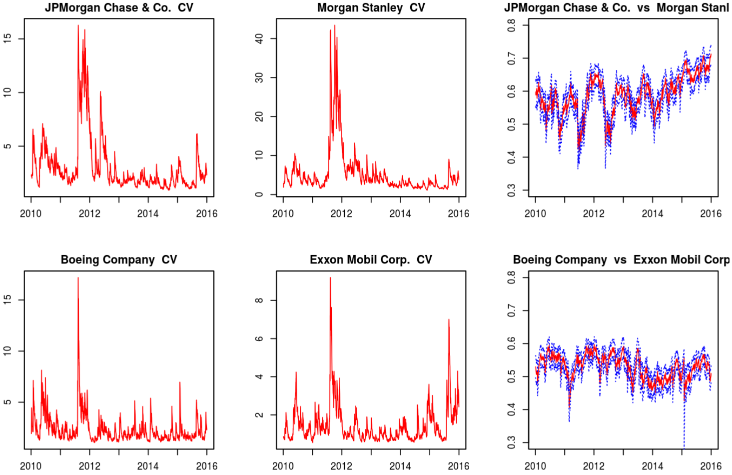

Figure : Posterior correlation among each time series

Main contribution

We employ the Bayesian approach to estimate the different specifications of the factor copula models.

- Condition on the latent factors, time series become independent which allows the algorithm to run in a parallel setting.
- Factor loadings could be modelled as generalized autoregressive score (GAS) processes imposing a dynamic dependence structure in their densities.
- Using hyperbolic skew Student copulas, we obtain different types of tail and asymmetric dependence.
- Furthermore, the model is extendible. The number of parameters scales linearly with the dimension in parametric models.

Discussion and Extension

- The more complex copula functions are build based on the distribution of ζ_{gt}.
- Computational expensive on $x_{it} = F^{-1}(u_{it}|\theta)$.
- Dynamic multi-skewness factor copulas.
- Understanding about the tail behaviour of the HSST copula.
- Taking into account more factors.

Watch out,

The devil is in the tail

Thank you
and have a good time!

References I

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Krupskii, Pavel and Harry Joe (2013). "Factor copula models for multivariate data". In: *Journal of Multivariate Analysis*.

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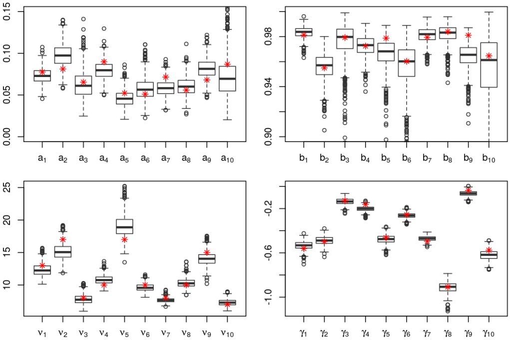
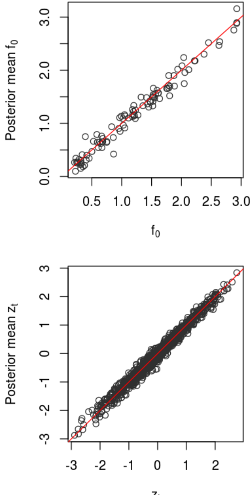
Data simulation

We simulate $d = 100$, $G = 10$, $T = 1000$, iterations $= 10.000$

It took 13 minutes, 35 minutes, 45 minutes for Gaussian, Student, Hyperbolic student copulas on Intel i7-4770 PC (4 cores - 8 threads - 3.4GHz).





TAIL DEPENDENCE FOR HSST COPULA

Consider the bivariate HSST copula, we derive the tail dependence of pseudo observable x_1 and x_2 from the Equation (2) as

$$\begin{aligned} x_1 &= \gamma_g \zeta_g + \sqrt{\zeta_g}(\rho_1 z + \sqrt{1-\rho_1^2}\epsilon_1) \\ x_2 &= \gamma_g \zeta_g + \sqrt{\zeta_g}(\rho_2 z + \sqrt{1-\rho_2^2}\epsilon_2) \end{aligned} \tag{3}$$

$$\begin{aligned} &C_{HSST}(u, u|\nu_g, \gamma_g) = P(x_1 \leq F_{HS}^{-1}(u), x_2 \leq F_{HS}^{-1}(u)) \\ &= \mathbb{E}\left(P\left(\epsilon_1 \leq \frac{F_{HS}^{-1}(u) - \gamma_g\zeta_g}{\sqrt{\zeta_g}\sqrt{1-\rho_1^2}} - \frac{\rho_1 z}{\sqrt{1-\rho_1^2}}, \epsilon_2 \leq \frac{F_{HS}^{-1}(u) - \gamma_g\zeta_g}{\sqrt{\zeta_g}\sqrt{1-\rho_2^2}} - \frac{\rho_2 z}{\sqrt{1-\rho_2^2}}\right)\right) \\ &= \mathbb{E}(\Phi(p_1).\Phi(p_2)) \end{aligned} \tag{4}$$

Then, we obtain $C_{HS}(u, u|\nu_g, \gamma_g)$ as the numerical integral over $z \sim N(0, 1)$ and $\zeta_g \sim IG(\nu_g/2, \nu_g/2)$ As taking the equation to the limit, the tail dependence are

$$\lambda_L = \lim_{u\to 0} \frac{C(u, u)}{u} \; ; \lambda_U = 2 + \lim_{u\to 0} \frac{C(1-u, 1-u) - 1}{u}$$

Model selection

The statistics of model selection are calculated based on the average of log likelihood. We take the average of the log likelihood after MCMC iterations at the posterior mean of the interested parameters as the integral over the nuisance parameter space. The set parameters of interest is $\theta_{int} = \{a, b, f_0, \nu, \gamma\}$ and the nuisance parameters are $\theta_{nui} = \{z, \zeta\}$.

$$\begin{aligned} AIC &= -2\mathbb{E}_{\theta_{nui}}(\texttt{log}(p(u|\bar{\theta}_{int}, f, \mathcal{F}))) + 2p \\ BIC &= -2\mathbb{E}_{\theta_{nui}}(\texttt{log}(p(u|\bar{\theta}_{int}, f, \mathcal{F}))) + p\texttt{log}T \\ DIC &= -4\mathbb{E}_{\theta}[\texttt{log}p(u|\theta, f, \mathcal{F})|u] + 2E_{\theta_{nui}}(\texttt{log}(p(u|\bar{\theta}_{int}, f, \mathcal{F}))) \end{aligned}$$

where p is the number of parameters of interest.