THE COMPLETE MBA COURSEWORK SERIES

100 BUSINESS MATH QUESTIONS AND ANSWERS

A Tale of Two Brothers

Hicham and Mohamed Ibnalkadi

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"MY LORD, INCREASE MY KNOWLEDGE" Noble Quran

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Preface

Motivation

This book is designed as a part of “The Complete MBA Coursework Series”, established to equip the professionals and students with the eminent capabilities and hone their skillset. The motivation behind this series is the need to establish a thorough and complete MBA coursework, following the core and elective courses of prestigious institutions like Wharton and Harvard’s Business Schools. With this self-motivated study of the MBA curriculum, students and professionals can tailor their MBA according to their interests and need. Thus, embarking on this self-study MBA coursework rather than a traditional, costly, and lengthy MBA degree program is worth the time.

MBA degree programs are very costly, although the skill boost associated is worth acquiring. Thus, as a part of the MBA coursework series, the following book helps the students learn the basic terminologies and techniques associated with Busines Math. This book does not attempt to provide a self-contained discussion of mathematics. Instead is a decent introduction to Busines Math courses used in top MBA programs. Further, the introductory is stemmed from our professional experience. Finally, we want to thank Hina Aslam and Sara Aslam for providing us with many suggestions and valuable feedback on this book.

Prerequisites

To make the book as accessible as possible, the reader should have the basic working knowledge of linear algebra, calculus, and basic statistics. However, prior exposure to undergraduate Linear Algebra is also not required. The emphasis throughout the book is on the understanding of basic terminologies and valid application of techniques onto the real data and problems.

Final few words

Thank you for buying this book, and feedback is highly appreciated for enhancing future versions. I hope that all readers gain something useful from this book and boost their knowledge of econometrics that the author aimed at while writing it.

Although this book has been thoroughly checked and proofread, typos, errors, inconsistencies in notation, and instances where I have got it wrong are bound to sneak in. Any readers spotting such errors or addressing certain questions or comments are kindly requested to contact our customer service through this email (introductory.books@gmail.com) before writing any review online. Finally, feel free to return the book and ask for a refund if unsatisfied.

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Hicham and Mohamed Ibnalkadi
Introductory.books@gmail.com

Q1. How would you define statistics to a layperson?

Answer: Statistics is the field of Mathematics that analyzes and makes sense of numerical data. It can also be defined as a set of tools and methods for numerical data analysis.

Its goal is to help extract information and meaning from numerical data sets, like representations, predictions, estimations, behaviors, tendencies, patterns, probabilities and so on.

Q2. What is a population data set?

Answer: In Statistics, a population data set is the set of all members that are being studied, which usually have at least one common characteristic. It is the set of all members which produce the data of a particular study or research.

Let us clarify with some examples:

a. For a study on the grades of the students of a given school, the population data set is the set which contains all students of the school.
b. For a study on the mean velocity of a road, the population data set is the set of all cars that travel through that road.
c. For an election poll, the population data set is the set of all citizens who are able to vote on that election.

Sometimes it is not feasible to research and collect data from every member of a population data set because of its size or distances to be traveled. Then, a study can be done on a representative subset of that population, which is called a sample data set.

Q3. What are data frequency tables?

Answer: Data frequency tables are tables that contain the values of a particular variable being studied and the number of times that it occurs within the members of a population (its frequency).

Each entry of the table is a row that contains two columns, one that represents the value, and another one that represents the frequency (i.e. how many times that value appears) with which that value occurs in the population.

Usually the values on a frequency table are arranged according to their ascending order.

Q4. How do you define location statistics?

Answer: Location statistics is the branch of statistics that stablishes ways of determining typical values for a statistical population, i.e. values around which a certain statistical distribution varies, according to its mathematical characteristics. Those typical values are called locations and the most usual ones are the mean, the median and the mode.

Q5. What is the mean or average?

Answer: It is measure of location, i.e. a value that represents in general all members of a studied population. It is calculated by summing up all the values of the members of a population data set and dividing the result by the number of members of that population data set.

Considering n values $x_1, x_2, x_3, \ldots, x_n$, the average value μ is calculated as:

$$\mu = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

Calculating the mean value or average value is equivalent to finding the value of each member of a data set, if all of them had the same value and still added up to the same total. It is a value around which all members of the population orbit, some above it and some under it, in a way such that the sum of all the differences from each member to the mean value add up to zero.

Let us clarify with some examples:

a. For a study on the grades of the students of a given school, the average student grade is calculated by the sum of the grades of all students divided by the amount of students in that school.
b. For a study on the ages of customers, the average age is calculated dividing the sum of all customers' ages by the number of customers.

Q6. What is a weighted mean?

Answer: It is a measure of location, i.e. a value that represents in general all members of a studied population. It is a variation of an arithmetic mean or average. While in an arithmetic mean all values contribute equally to the result, in a weighted mean, each value contributes more or less, i.e. each value has a different weight.

The weighted mean is calculated by summing up the products of each value by its weight and then dividing the result by the sum of all weights.

Considering n values $x_1, x_2, x_3, \ldots, x_n$ whose weights are $w_1, w_2, w_3, \ldots, w_n$, the weighted mean $\bar{x}_w$ is calculated as:

$$\bar{x}_w = \frac{w_1 x_1 + w_2 x_2 + w_3 x_3 + \ldots + w_n x_n}{w_1 + w_2 + w_3 + \ldots + w_n}$$

It can also be thought of as a different way of calculating an arithmetic mean when values appear more than once in the distribution, i.e. values with frequencies greater than one. In that case, the arithmetic mean is calculated as the weighted mean of the values, where their weights are their frequencies, because it is equivalent to summing up all values of the distribution and then dividing the result by the number of values.

Considering n values $x_1, x_2, x_3, \ldots, x_n$ whose frequencies are $f_1, f_2, f_3, \ldots, f_n$, the average value μ is calculated as:

$$\mu = \frac{f_1 x_1 + f_2 x_2 + f_3 x_3 + \ldots + f_n x_n}{f_1 + f_2 + f_3 + \ldots + f_n}$$

By its own definition, the weighted mean has applications specially where values have different weights to the desired result, for example, if a company wants to know where it should place a distribution center or a facility, it could be interesting to place it closer to the locations where its products are consumed the most, in order to reduce transportation costs and maximize the profits.

Q7. What is a median?

Answer: It is a measure of location, i.e. a value that represents in general all members of a studied population. To find the median of a distribution, it is necessary to write all of its values in ascending order and then finding the element that lies exactly in the central position.

Being more specific, let us consider the following set of n numbers, written in ascending order: $x_1,\ x_2,\ x_3,\ \ldots\ ,\ x_n$, and its respective median $\tilde{x}$.

If the n is an odd number, then the value of the median of the set is the value of its central element:

$$\tilde{x} = x_{n/2\ +\ 1}$$

If the n is an even number, then the value of the median of the set is the mean value of its two central elements:

$$\tilde{x} = \frac{x_{n/2} + x_{n/2+1}}{2}$$

Some examples:

a. If there are 21 students in a class, the median height of the group is the height of the 11th student in line if we lined all of them according to the ascending order of their heights.
b. If there are 22 students in a class, the median height of the group is the average value between the heights of the 11th and the 12th student in line if we lined all of them according to the ascending order of their heights.

The median is a value around which all members of the population orbit, some above it and some under it, but there is no further numeric relations between them, it is just a central element.

Q8. What is meant by “mode”?

Answer: It is a measure of location, i.e. a value that represents in general all members of a studied population. The mode of a population is the values that occurs the most on it, i.e. the most common value of that population.

Considering that we have the following values in a population: 1, 1, 2, 3, 3, 3, 5, 5, 5, 5, the mode m_0 would be 5, because that is the value that occurs the most on the population (4 times), i.e. the value of greatest frequency ($f = 4$).

Note that it is possible to have more than one mode, as two values may have the highest frequency of the distribution.

Q9. Define quartiles, quintiles, deciles, and percentiles.

Answer: They are values that delimit groups that represent ranges of classification of the members of a data set.

Quartiles divide a data set into four categories according the numeric values it contains. There are three quartiles:

- If a member of the data set is greater than the first quartile, it means that its value is greater than a quarter of the members of that distribution, therefore it is located in its upper three quarters.
- If a member of the data set is greater than the second quartile, it means that its value is greater than two quarters of the members of

that distribution, therefore it is located in its upper two quarters.

- If a member of the data set is greater than the third quartile, it means that its value is greater than three quarters of the members of that distribution, therefore it is located in its upper quarter.

There are three values for quartiles, which divide the distribution into four groups. The same logic apply to all other values: quintiles are four values that divide a distribution into five groups, deciles are nine values that divide a distribution into ten groups and percentiles are ninety-nine values that divide the distribution into a hundred groups.

A good example of the usage of percentiles is test scoring. All test-takers' scores can be arranged into a hundred groups according to their grades and the percentile results define each taker's performance according to his/her score in relation to all of the other test-takers. If a student scored at the 75th percentile, it means that he/she got a score greater than 75% of all the takers.

One famous example of the usage of percentiles is Mensa High IQ Society, which only admits people who score in the 98th percentile of an IQ test. It corresponds to the 2% of people with highest IQ in the world.

Q10. Define standard deviation.

Answer: Standard deviation is a measure of the variability of a data set. It is a way to estimate the mean distance from each value to the average of the data set.

It is represented by Greek letter σ (sigma) and it is calculated by the following formula:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \mu)^2}{n}}$$

Where x_i is each of the values of the data set, μ is the mean value of the data set and n is the number of elements in the data set.

The greater the standard deviation is for a data set, the further its values are far from (below and above) the average value, i.e. the more they vary, and vice versa.

Q11. What is the variance of a data set?

Answer: Variance is a measure of the variability of a data set. It is a way to estimate the distance from each value to the average of the data set. It is equal to the standard deviation of a data set squared, being, therefore, represented by σ^2 (squared Greek letter sigma) and it is calculated by the following formula:

$$\sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \mu)^2}{n}$$

Where x_i is each of the values of the data set, μ is the mean value of the data set and n is the number of elements in the data set.

The greater the variance is for a data set, the further its values are far from (below and above) the average value, i.e. the more they vary, and vice versa.

Q12. What is unconditional probability?

Answer: Unconditional probability, also known as marginal probability is the probability of occurrence of an event when it is not affected by any other previous events, which means that the event is independent. For example, a coin toss always has a 50% chance of landing with heads facing up and a 50% change of landing with tails facing up, regardless of how many coin tosses have been performed before. That is because each coin toss is independent of the previous ones, i.e. it has an unconditional probability of occurrence.

Q13. What is conditional probability?

Answer: Conditional probability is the probability of occurrence of an event that is affected by one or more previous events, which means that the event is somehow dependent of them, being more or less likely to happen, given that other events have or have not happened.

Q14. What is the multiplication rule of probability?

Answer: The probability of two or more events happening is equal to the multiplication of the probabilities for each individual event. Let us say that the probabilities of n events are: $p_1, p_2, p_3, \ldots, p_n$. The probability take place p is, then:

$$p = p_1 \times p_2 \times p_3 \times \ldots \times p_n$$

Q15. What is a normal distribution?

Answer: Normal distribution, also known as Gaussian distribution, is a type of continuous probability distribution that precisely describes so many natural phenomena, that it can be considered the standard one. Its probability density is defined by the following formula:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x - \mu}{\sigma}\right)^2}$$

Where μ is the mean value of the distribution and σ is the standard deviation of the distribution.

There are infinite possible normal distributions according to the values of the parameters μ and σ . If the probability distribution of a variable X is normal with mean value μ and standard deviation σ , it is considered a normal distribution and it is represented as:

$$X \sim N(\mu, \sigma^2)$$

Q16. What is a standard normal curve?

Answer: It is a normal distribution, of a standardized variable Z with $\mu = 0$ and $\sigma = 1$, i.e. $Z \sim N(0, 1^2)$ and, therefore $Z \sim N(0, 1)$. The process of converting a variable $X \sim N(\mu, \sigma^2)$ into a normalized variable Z is called normalization and it is performed by applying the following formula:

$$Z = \frac{X - \mu}{\sigma}$$

The standard normal curves are used in order to make it easier to perform calculations involving normal distributions through the usage standardized tables of values.

Q17. What are some of the other continuous probability distributions?

Answer: In addition to the Normal Distribution, there are many other types of probability distributions like: Uniform Distribution, Bernoulli Distribution, Binomial Distribution, Poisson Distribution, Exponential Distribution, and so on.

Q18. What is a binomial probability distribution?

Answer: It is the probability distribution for an unconditional experiment (whose outcome is completely independent of previous outcomes) with only two possible outcomes (therefore the name binomial = two values) that is performed a given number of times. It describes the probability of having a certain outcome another given number of times.

If an experiment has two outcomes with probabilities of happening p and $1 - p$, then the probability $P(x)$ of having event with probability p occurring x times on a series of n experiments is described by the binomial distribution, whose formula is:

$$P(x) = \binom{n}{x} p^x (1 - p)^{n - x} = \frac{n!}{(n - x)!x!} p^x (1 - p)^{n - x}$$

Q19. What is correlation and what does it do?

Answer: Correlation is a measure of the degree of relation between two variables in a data set. It measures to which degree two variables are linearly dependent of one another. For example, if there is a strong correlation between two variables, then an increase or decrease in one of them certainly leads to a proportional increase or decrease in the other one and vice-versa. If there is a weak correlation between two variables, then an increase or decrease in one of them is less likely to lead to a proportional increase or decrease in the other one and vice-versa. Finally, if there is no correlation between two variables, then an increase or decrease in one of them will not lead to any noticeable change in the other one.

It is very useful tool when looking for causes and effects and when stablishing data profiles. For example, in a commerce business, if there is a strong correlation between the age of the customers and the amount they spend, it may help to target market actions to better serve and reach those who are most profitable.

Q20. What is the formula for calculating the correlation coefficient?

Answer: For a population, the correlation coefficient between variables x and y is represented by ρ_{xy} and its formula is:

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

Where σ_x is the standard deviation of variable x , σ_y is the standard deviation of variable y and σ_{xy} is the covariance between x and y , given by the following formulas, respectively:

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \mu_x)^2}{n}}$$

$$\sigma_y = \sqrt{\frac{\sum_{i=1}^{n} (y_i - \mu_y)^2}{n}}$$

$$\sigma_{xy} = \frac{\sum_{i=1}^{n} (x_i - \mu_x)(y_i - \mu_y)}{n}$$

Where μ_x is the mean value of variable x , μ_y is the mean value of variable y and n is the size of the population.

Q21. How would you calculate a correlation coefficient in Excel?

Answer: Correlation can be calculated using the function $\mathrm{CORREL(X,Y)}$. The arguments X and Y are two arrays of same size, i.e. two areas of same size in number of cells. The function returns (i.e. outputs) the corresponding correlation coefficient for the population ρ_{xy} according to the following formula:

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

Where σ_x is the standard deviation of variable x , σ_y is the standard deviation of variable y and σ_{xy} is the covariance between x and y .

Q22. What is meant by the term “linear regression”?

Answer: Linear regression is the process of finding a straight line equation which represents the relationship between two variables as closely as possible. It consists of treating the relationship between two variables as linear to find a line equation, which makes it possible make estimations about the pairs of values that are not present in the data set.

There are a couple of techniques to estimate the parameters of the line equation of a linear regression, the most common one being the Least Squares Estimation.

Q23. How would you execute regression on Excel?

Answer: Performing a linear regression means finding a straight line equation which describes the relationship between two variables x and y , writing y as a function of x , that is $y = f(x) = a + bx$, where a and b are constant values whose values must be estimated.

The most common method for estimating parameters a and b is the Least Squares Estimation, which can be performed very easily in Excel by using the functions $\mathrm{SLOPE(X,Y)}$ and $\mathrm{INTERCEPT(X,Y)}$.

The $\mathrm{SLOPE(X,Y)}$ function returns the estimated value for b . The arguments X and Y are two arrays of same size (i.e. two areas with the same number of cells) which contain the values for variables x and y , respectively. The returned value for b is calculated according to the following formula.

$$b = \frac{\sum_{i=1}^{n}(x_i - \mu_x)(y_i - \mu_y)}{\sum_{i=1}^{n}(x_i - \mu_x)^2}$$

The $\mathrm{INTERCEPT}(X,Y)$ function returns the estimated value for a . The arguments X and Y are two arrays of same size (i.e. two areas with the same number of cells) which contain the values for variables x and y , respectively. The returned value for a is calculated according to the following formula.

$$a = \mu_y - b\mu_x$$

For both formulas above, μ_x is the mean value of variable x and μ_y is the mean value of variable y .

Another option for executing linear regression on Excel is to use the $\mathrm{LINEST}(X,Y)$ function. The arguments X and Y are two arrays of same size (i.e. two areas with the same number of cells) which contain the values for variables x and y , respectively. The function returns an array whose first position corresponds to the value of parameter b and the second position corresponds to the value of parameter a .

Other arguments can also be passed to the $\mathrm{LINEST}(X,Y)$ function, so that it returns a more complete array with all relevant statistical characteristics of the linear regression of the data set, but it requires some knowledge of array functions to retrieve the relevant data.

Q24. What is understood by “neural network”?

Answer: A neural network is a network circuit composed of elements called neurons, which can be either biological of artificial, also called as nodes. A neural network composed of biological neurons is called a biological neural network and a neural network composed of artificial neurons is called artificial neural network.

In the context of data science and technology, neural networks are artificial neural networks implemented in computation systems in order to simulate the behavior of a nervous system of a biological living being. That kind of neural network is implemented in order to give computational systems the ability to learn skills based on inputs and expected outputs, through the use of techniques such as machine learning, without the usage of specific programming.

Q25. An investment has a monthly interest rate of 2.5%. Calculate the total interest for periods of one, two and three years considering: **(a)** Simple interest; **(b)** Compound interest.

Answer: The total interest I_{period} for the period P_{months} (in months), at a monthly interest rate $i_{monthly}$, considering simple interest is:

$$I_{period} = P_{months} \times i_{monthly}$$

And, considering compound interest:

$$I_{period} = (1 + i_{monthly})^{P_{months}} - 1$$

Now we can calculate the interests for 1 year (12 months), 2 years (24 months) and 3 years (36 months) as follows.

(a) For a period of one year:

$$I_{1year} = P_{months} \times i_{monthly} = (12)(0.025) = 0.3 = 30\%$$

For a period of two years

$$I_{2year} = P_{months} \times i_{monthly} = (24)(0.025) = 0.6 = 60\%$$

For a period of three years

$$I_{3year} = P_{months} \times i_{monthly} = (36)(0.025) = 0.9 = 90\%$$

(b) For a period of one year:

$$I_{1year} = (1 + i_{monthly})^{P_{months}} - 1 = (1.025)^{12} - 1 \cong 0.3449 \cong 34.49\%$$

For a period of two years

$$I_{2year} = (1 + i_{monthly})^{P_{months}} - 1 = (1.025)^{24} - 1 \cong 0.8087 \cong 80.87\%$$

For a period of three years

$$I_{3year} = (1 + i_{monthly})^{P_{months}} - 1 = (1.025)^{36} - 1 \cong 1.4325 \cong 143.25\%$$

Q26. How much time do we need to keep \$100,000.00 invested at a monthly interest rate of 1.2% in order to have a net profit of \$50,000.00? When liquidating the investment there will be a tax payment of 20% over the gross profit. Consider compound interests.

Answer: The gross profit P_{gross} is the desired net profit P_{net} minus the tax payment of 20%:

$$P_{gross} - 20\% = P_{net} \Rightarrow P_{gross} \times 0.8 = P_{net} \Rightarrow P_{gross} = \frac{P_{net}}{0.8} = \frac{50{,}000}{0.8}$$

$$P_{gross} = 62{,}500.00$$

Thus, the total value P_f we expect for the future is, considering the present value $P_p = 100{,}00.00$:

$$P_f = P_{gross} + P_p = 100{,}000.00 + 62{,}500.00 = 162{,}500.00$$

That value is the future value of a compound interest investment at an interest (monthly) rate, $i = 1.2\% = 0.012$. Therefore:

$$P_f = P_p \times (1 + i)^n \Rightarrow (1 + i)^n = \frac{P_f}{P_p} \Rightarrow (1.012)^n = \frac{162{,}500}{100{,}000} \Rightarrow$$

$$\Rightarrow (1.012)^n = 1.625 \Rightarrow n = \log_{1.012}(1.625)$$

Changing the logarithm to base 10:

$$n = \frac{\log(1.625)}{\log(1.012)} \cong \frac{0.21085}{0.0051805} \cong 40.7007$$

Because n must be an integer number, we round it up to $n = 41$. Therefore, the minimum amount of time that we need to keep the money invested is 41 months.

Q27. What must be the monthly payment value for the complete amortization of a loan of $35,000.00 with an interest rate of 2% per month in a period of 5 years? How much will be paid in total? Consider compound amortization.

Answer: The monthly payment p_m is given by the following formula, considering compound amortization.

$$p_m = \frac{P \times i \times (1 + i)^n}{(1 + i)^n - 1}$$

Where $P = 35{,}000.00$ is the principal of the loan, $i = 2\% = 0.02$ is the monthly interest rate and $n = 5 \text{ years} = 60 \text{ months}$ is the number of periods for the amortization. Plugging in the values:

$$p_m = \frac{(35,000)(0.02)(1.02)^n}{(1.02)^n - 1} \cong \frac{2,296.722}{2.28103}$$

$$\therefore p_m \cong \$1,006.88$$

Once the monthly payment has been determined, we can simply multiply it by the amount of periods to get the total paid T as follows:

$$T = n \times p_m \cong (60)(1,006.88) \cong \$60,412.80$$

Therefore, the monthly payment will be $\$1,006.88$ and the total amount paid will be $\$60,412.80$.

Q28. A loan of $60,000.00 with monthly payments of $1,000.00 at an interest rate of 1.25% per month, with compound amortization and interests, had a delay of 18 months on its first payment. Answer to following questions:

- **(a)** How many months would it take to pay off the loan had the delay not happened?
- **(b)** What is the principal of the loan after the first 18 months?
- **(c)** How many months will it take to pay off the loan counting from the first payment?

Answer: The monthly interest rate is $i = 1.25\% = 0.0125$. Thus, $1 + i = 1.0125$ in the calculations below:

- **(a)** The monthly payment p_m is given by the following formula, considering compound amortization.

$$p_m = \frac{P \times i \times (1 + i)^n}{(1 + i)^n - 1}$$

Where $P = 60,000.00$ is the principal of the loan and $p_m = 1,000.00$ is the monthly payment. Plugging in the values:

$$1,000 = \frac{(60,000)(0.0125)(1.0125)^n}{(1.0125)^n - 1} \Rightarrow (1.0125)^n - 1 = (60)(0.0125)(1.0125)^n \Rightarrow$$

$$\Rightarrow (1.0125)^n - 1 = (0.75)(1.0125)^n \Rightarrow -1 = (0.75 - 1)(1.0125)^n \Rightarrow -1 = (-0.25)(1.0125)^n \Rightarrow$$

$$\Rightarrow (1.0125)^n = \frac{1}{0.25} \Rightarrow (1.0125)^n = 4 \Rightarrow \log(1.0125)^n = \log(4) \Rightarrow n \times \log(1.0125) = \log(4) \Rightarrow$$

$$\Rightarrow n = \frac{\log(4)}{\log(1.0125)} \Rightarrow n \cong \frac{0.60206}{0.005395} \cong 111,595$$

$$\therefore n \cong 112$$

Because n must be an integer number, we round it up to $n = 112$ months.

(b) Because of the payment delay, the interests during the first $n = 18$ months compound over the present principal $P_p = 60,000.00$. The future principal P_f is given by the following formula, considering compound interests:

$$P_f = P_p \times (1 + i)^n$$

$$P_f = (60,000)(1.0125)^{18}$$

$$\therefore P_f = \$75,035.34$$

Therefore, the principal of the loan after the first 18 months, increases to $\$75,035.34$.

(c) The monthly payment p_m is still given by the following formula, considering compound amortization.

$$p_m = \frac{P \times i \times (1 + i)^n}{(1 + i)^n - 1}$$

But now we consider $P = 75,035.34$ as the principal of the loan and the rest remains the same. Plugging in the values:

$$1,000 = \frac{(75,035.34)(0.0125)(1.0125)^n}{(1.0125)^n - 1} \Rightarrow (1.0125)^n - 1 = (75.03534)(0.0125)(1.0125)^n \Rightarrow$$

$$\Rightarrow (1.0125)^n - 1 = (0.937933)(1.0125)^n \Rightarrow$$

$$\Rightarrow -1 = (0.937933 - 1)(1.0125)^n \Rightarrow -1 = (-0.062067)(1.0125)^n \Rightarrow$$

$$\Rightarrow (1.0125)^n = \frac{1}{0.062067} \Rightarrow (1.0125)^n = 16.11163 \Rightarrow$$

$$\Rightarrow \log(1.0125)^n = \log(16.11163) \Rightarrow n \times \log(1.0125) = \log(16.11163) \Rightarrow$$

$$\Rightarrow n = \frac{\log(16.11163)}{\log(1.0125)} \Rightarrow n \cong \frac{1.20714}{0.005395} \cong 223.75$$

$$\therefore n \cong 224$$

Because n must be an integer number, we round it up to $n = 224$ months.

Q29. A monthly capitalized (with compound interests) investment of $30,000.00 is intended to have a profit of $12,000.00 in a period of one year. What is the monthly interest rate?

Answer: The present value is $P_p = 30,000$, the future value is, considering a profit of $12,000$, is $P_f = 30,000 + 12,000 = 42,000$ and the period of investment is $n = 1 \text{ year} = 12 \text{ months}$. Considering compound interests capitalization:

$$P_f = P_p \times (1 + i)^n$$

$$42,000 = 30,000 \times (1 + i)^{12} \Rightarrow (1 + i)^{12} = \frac{42,000}{30,000} \Rightarrow (1 + i)^{12} = 1.4 \Rightarrow$$

$$\Rightarrow 1 + i = 1.4^{\frac{1}{12}} \Rightarrow i \cong 1,0284 - 1 \Rightarrow i \cong 0,0284$$

$$\therefore i \cong 2.84\%$$

Therefore, monthly interest rate is 2.84%.

Q30. A net profit of 60% is required for a compound interests investment. The investment is monthly capitalized at a rate of 3.2% per month. In the end of the whole period there is a payment of 20% of taxes over the gross profit. What is the minimum amount of time that the money must be kept in the investment?

Answer: The gross profit P_{gross} is the desired net profit P_{net} minus the tax payment of 20%:

$$P_{gross} - 20\% = P_{net} \Rightarrow P_{gross} \times 0.8 = P_{net} \Rightarrow P_{gross} = \frac{P_{net}}{0.8} = \frac{60\%}{0.8}$$

$$P_{gross} = 75\%$$

Thus the total value we expect for the future P_f is, considering the present value P_p:

$P_f = P_{gross} + P_p \Rightarrow P_f = P_p + 75\% \Rightarrow P_f = P_p \times 1.75 \Rightarrow$

$$\Rightarrow \frac{P_f}{P_p} = 1.75$$

Considering a compound interest investment at an interest (monthly) rate $i = 3.2\% = 0.032$:

$$P_f = P_p \times (1 + i)^n \Rightarrow (1 + i)^n = \frac{P_f}{P_p} \Rightarrow (1.032)^n = 1.75 \Rightarrow$$

$$\Rightarrow (1.032)^n = 1.75 \Rightarrow n = \log_{1.032}(1.75)$$

Changing the logarithm to base 10:

$$n = \frac{\log(1.75)}{\log(1.032)} \cong \frac{0.243038}{0.0136797} \cong 17.76$$

Because n must be an integer number, we round it up to $n = 18$. Therefore, the minimum amount of time that we need to keep the money invested is 18 months or 1.5 year.

Q31. A business person invested a certain amount of money on a compound interest investment which is capitalized at a fixed rate per month and, after a two year period, the gross profit was 100%. Then, the investor decided to live from the income of that investment, without reducing the principal, with a monthly wage of $8,790.67. Considering that no taxes nor fees apply on the profit of the investment and that the monthly interest rate remains the same during the whole period, how much was initially invested?

Answer: A profit of 100% on the first two years (which is 24 months) mean that the principal doubled during that amount of time due to compound interests. Therefore we can calculate the monthly interest rate:

$$(1 + i)^n = \frac{P_f}{P_p} \Rightarrow (1 + i)^{24} = 2 \Rightarrow i = 2^{\frac{1}{24}} - 1 \cong 1.0293 - 1 \cong 0.293$$

$$\therefore i \cong 2.93\%$$

To be able to live from the investment without reducing the principal, the monthly wage must be equal to the monthly revenue of the investment, which means that the wage \$8,790.67 is 2.93% of the principal P. Therefore, P can be calculated as:

$$P = \frac{8,790.67}{0.0293} \cong 300,000.00$$

But that is the principal after a two year period, in which the money doubled. Thus the initial principal P_i is half of that amount:

$$P_i = \frac{P}{2} = \frac{300,000}{2} = 150,000$$

Therefore, the initial investment was approximately $\$150,000.00$.

Q32. What is the main difference between simple interest and compound interest?

Answer: Simple interest is calculated only on the principal and compound interest is calculated on the principal plus all the amount of interest so far, which means interest on interest. Another way of stating that is: with simple interest, the return or debt grows linearly while with compound interest, it grows exponentially. Another important difference is that simple interests are not very used while most of the investments and loans are calculated considering compound interests.

Q33. Define Return Over Investment (ROI) or Rate of Return (ROR).

Answer: Return Over Investment (ROI) or Rate of Return (ROR) is a financial metric used to evaluate how much an investment is expected to grow, in percent, over time. That means that it is the difference between the future value and the invested value divided by the invested value, i.e. it is calculated as:

$$(\text{ROI}) = \frac{(\text{Future Value}) \text{ - } (\text{Present Value})}{(\text{Present Value})} \times 100\%$$

Q34. Define Internal Return Rate (IRR).

Answer: Internal Return Rate (IRR) a financial metric used to evaluate and compare the returns of an investment with other available investments and/or depreciation rates. It is the discount rate applied to the cash flows in each period so that the sum of all of them equal to zero. The main idea is to have a metric to compare two or more investments in a way that the one of higher IRR value will be the one that pays off first. Being n the amount of periods with cash flow, C_i

the value of cash flow in period i and r the IRR:

$$\sum_{i=0}^{n} \frac{C_i}{(1+r)^i} = 0$$

Calculating the IRR by the formula above can be tricky, so Excel and other calculation tools are often used for that purpose.

Q35. Consider an investment at an interest rate i per capitalization period, with a total of n periods. How would you calculate the present value P_v as a function of the future value F_v ? How would you calculate F_v as a function of P_v ?

Answer: To calculate F_v as a function of P_v :

$F_v = P_v \times (1+i)^n$

To calculate P_v as a function of F_v :

$$P_v = \frac{F_v}{(1+i)^n}$$

Q36. How would you define gain and loss?

Answer: Gain is how much money is made by a business and loss is how much money is lost by a business. When any financial activity gets more money than it spends, there is a gain which corresponds to the difference between the money that comes in and the money that goes out. When any financial activity gets less money than it spends, there is a loss which corresponds to the difference between the money that goes out and the money that comes in.

Q37. How would you define depreciation of an asset?

Answer: Depreciation is a way to take in account how much an asset loses its value over time. For example, a machine has an expected useful life and as it ages and its components wear off, it loses its original value. The difference between the original value of the asset and its final value, after its period of usage, is its depreciation.

There are several different depreciation regimes and uses, but all of them have the same goal of taking into account how much an asset loses its value or require expenses over its life spam.

Q38. How would you define markup and markdown?

Answer: Markup is the amount added to the cost of a good so that its selling price covers the expenses of the business and profit. Markdown is the amount subtracted from the selling price of a good in order to keep it competitive in the market, i.e. attractive from the customers' standpoint, to promote sells or to correct errors in the original costs.

Q39. When simultaneously rolling three six sided dice (each side of each die contains a number from 1 to 6), what is the probability of getting three consecutive numbers?

Answer: For each die there are six possible values that can repeat, therefore the amount of possible results R is:

$$R = 6^3$$

If each side can contain a number from 1 to 6, there are four possibilities of getting 3 consecutive numbers: 1-2-3, 2-3-4, 3-4-5 and 4-5-6. And each of those combinations may occur in 3 different dice, i.e. 3! Permutations. Thus, the amount of results in which there are 3 consecutive numbers R_C is:

$$R_C = 4 \times 3! = 24$$

Then, the probability of getting 3 consecutive numbers p_C is:

$$p_C = \frac{R_C}{R} = \frac{24}{6^3}$$

$$\therefore p_C = \frac{1}{9} \cong 11{,}11\%$$

Q40. On a prize draw bet, a sum is payed for someone who scores five out of fifty possible numbers (the order is not important). If a hundred thousand people made each a different bet, what is the probability that nobody will win? (In order to make calculations simpler, consider $49 \times 47 \times 46 = 100{,}000$)

Answer: The question states that 5 numbers are drawn out of 50. The amount of possible result combinations R is:

$$R = C(50,5) = \binom{50}{5} = \frac{50!}{45!5!} = \frac{50 \times 49 \times 48 \times 47 \times 46}{5 \times 4 \times 3 \times 2}$$

Considering $49 \times 47 \times 46 = 100{,}000$, according to the statement of the question:

$$R = \frac{50 \times 48 \times 100{,}000}{5 \times 4 \times 3 \times 2} = 20 \times 100{,}000$$

Because 100,000 people made a different bet each, there are 100,000 winning combinations, therefore, the probability of someone winning p_W is:

$$p_W = \frac{100{,}000}{20 \times 100{,}000} = \frac{1}{20} = 5\%$$

The probability of nobody winning p_L is complementary to p_W. Thus:

$$p_L = 1 - p_W = 1 - \frac{1}{20}$$

$$\therefore p_L = \frac{19}{20} = 95\%$$

Therefore there is a probability of 95% that nobody will win that draw.

Q41. Inside a ballot box there are 4 black chips, 3 white chips and 2 blue chips. After randomly taking three ball out of the box without reposition, what is the probability that we took: **(a)** One chip of each color? **(b)** Three chips of same color? **(c)** Both blue chips?

Answer: First we calculate all takings possibilities T, which is the number of possible combinations of all nine chips taken three at a time (we are not considering the order relevant):

$$T = C(9,3) = \binom{9}{3} = \frac{9!}{6!3!} = \frac{9 \times 8 \times 7}{3 \times 2} = 84$$

(c) The amount of possibilities where there is one chip of each color $C_{3colors}$ is the amount of combinations of 1 black chip out of 4, 1 white chip out of 3 and 1 blue chip out of 2, without considering the order as relevant. Thus:

$$C_{3colors} = 4 \times 3 \times 2 = 24$$

Therefore, the probability of taking one chip of each color $p_{3colors}$ is:

$$p_{3colors} = \frac{C_{3colors}}{T} = \frac{24}{84}$$

$$\therefore p_{3colors} = \frac{2}{7}$$

(d) The amount of possibilities where there are 3 chips of same color C_{1color} is the amount of combinations of 3 out of the 4 black chip plus one possibility which contains three white chips. Thus:

$$C_{1color} = C(4,3) + 1 = \frac{4!}{1!3!} + 1 = 4 + 1 = 5$$

Therefore, the probability of taking three chips of same color p_{1color} is:

$$p_{1color} = \frac{C_{1color}}{T} = \frac{5}{84}$$

$$\therefore p_{1color} = \frac{5}{84}$$

(e) The combinations where there are 2 blue chips C_{2blue} is all combinations where both blue chips appear and one of the seven remaining chips. Thus:

$$C_{2blue} = 7$$

Therefore, the probability of taking three chips and two of them being blue p_{2blue} is:

$$p_{2blue} = \frac{C_{2blue}}{T} = \frac{7}{84}$$

$$\therefore p_{2blue} = \frac{1}{12}$$

Q42. In study on the smoking habits of a city's population, it was stablished that 72% of the population are non-smokers. If 60% of the population is male and 25% of the females smoke, what is the probability of randomly picking a smoker from the male population?

Answer: If 72% of the population are smokers then $100\% - 72\% = 28\%$ of the population smoke. If 60% of the population is male then $100\% - 60\% = 40\%$ of the population is female. As 25% of the female smoke, it means that $(0.25)(0.40) = 0.1 = 10\%$ of the population are woman who smoke.

Because 28% of the population smoke and 10% of the population are woman who smoke, $28\% - 10\% = 18\%$ are men who smoke. Thus the probability of picking a smoker from the male population is:

$$p(S|M) = \frac{p(M \cap S)}{p(M)} = \frac{18\%}{60\%} = \frac{0.18}{0.6} = 0.3$$

$$\therefore p(S|M) = 30\%$$

Therefore, the probability of picking a smoker from the male population is 30%.

Q43. The probabilities of occurrence of three independent events A, B and C are respectively 80%, 30% and 10%. What is the probability that all three events occur simultaneously?

Answer: According to the multiplication rule, the probability simultaneous occurrence of three independent events is the multiplication of their probabilities. Thus, the probability of all three events occurring at the same time $p(A \cap B \cap C)$ is:

$$p(A \cap B \cap C) = p(A) \times p(B) \times p(C)$$

Where $p(A)$, $p(B$), and $p(C)$ are the probabilities of events A, B and C, whose values are given:

$p(A \cap B \cap C) = (0.80)(0.30)(0.10) = (0.24)(0.1) = 0.024$

$\therefore p(A \cap B \cap C) = 2.4\%$

Therefore, the probability of all three events occurring at the same time is 2.4% .

Q44. The probability of occurrence of an event A is 70%. Determine the probability of occurrence of an event B, whose probability of occurrence is 50% when A occurs and 90% when A does not occur.

Answer: The probability of occurrence of B is $p(B)$. Because B can occur with or without the occurrence of A, its probability of occurrence can be expressed as:

$p(B) = p(B \cap A) + p(B \cap \overline{A})$

Where $p(B \cap A)$ is the probability of A and B occurring and $p(B \cap \overline{A})$ is the probability of A not occurring and B occurring. But we know that for any events X and Y, the probability of X given Y $p(X|Y)$ is:

$$p(X|Y) = \frac{p(X \cap Y)}{p(Y)} \Rightarrow p(X \cap Y) = p(X|Y)p(Y)$$

Thus, we can express $p(B)$ as:

$p(B) = p(B|A) \times p(A) + p(B|\overline{A}) \times p(\overline{A})$

Knowing that $p(\overline{A}) = 1 - p(A)$:

$p(B) = p(B|A) \times p(A) + p(B|\overline{A}) \times [1 - p(A)]$

The probabilities $p(A) = 70\% = 0.7$; $p(B|A) = 50\% = 0.5$ and $p(B|\overline{A}) = 90\% = 0.9$ can now be replaced in the expression:

$p(B) = (0.7)(0.5) + (1 - 0.7)(0.9) = 0.35 + (0.3)(0.9) = 0.35 + 0.27 = 0.62$

$$\therefore p(B) = 62\%$$

Therefore, the probability of occurrence of B is 62% .

Q45. In a game of darts, the probability of hitting the dartboard is 90%. Of each twenty-five darts that hit the dartboard, one hits the bullseye. When a player throws a dart, what is the probability that he will not hit the bullseye?

Answer: If the probability of hitting the dartboard is $p_{\text{dartboard}} = 90\%$ then the probability of not hitting the dartboard is

$\overline{p}_{\text{dartboard}} = 100\% - 90\% = 10\%$. If 1 in each 25 darts that hit the dartboard goes for the bullseye, then the probability of hitting the bullseye given that the dartboard was hit $p_{\text{bulseye|dartboard}}$ is:

$$p_{\text{bulseye|dartboard}} = \frac{1}{25} = 0.04 = 4\%$$

Thus the probability of not hitting the bullseye given that the dartboard was hit $\overline{p}_{\text{bulseye|dartboard}}$ is:

$$\overline{p}_{\text{bulseye|dartboard}} = 1 - p_{\text{bulseye|dartboard}} = 1 - 0.04 = 0.96 = 96\%$$

Because $\overline{p}_{\text{bulseye|dartboard}}$ is a conditional probability, the following relation is valid:

$$\overline{p}_{\text{bulseye|dartboard}} = \frac{p_{\text{ dartboard and not bulseye}}}{p_{\text{ dartboard}}}$$

Therefore:

$$0.96 = \frac{p_{\text{ dartboard and not bulseye}}}{0.90} \Rightarrow p_{\text{ dartboard and not bulseye}} = (0.90)(0.96) = 0.864 = 86.4$$

The probability of not hitting the bullseye is:

$$\overline{p}_{\text{bulseye}} = \overline{p}_{\text{dartboard}} + p_{\text{ dartboard and not bulseye}}$$

$$\overline{p}_{\text{bulseye}} = 0.1 + 0.864 = 0.964$$

$$\therefore \overline{p}_{\text{bulseye}} = 96.4\%$$

Therefore, the probability of not hitting the bullseye is 96.4% .

Q46. What is the factorial of a number?

Answer: The factorial of a number is an integer operator which is defined as the multiplication of all numbers from one to a given number n . It is represented by the symbol “!”.

Being integer number n , its factorial is represented by $n!$ and it is calculated as:

$$n! = \prod_{i = 1}^{n} i = n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1$$

For example:

$$3! = 3 \times 2 \times 1$$

$$4! = 3 \times 2 \times 1$$

$$5! = 3 \times 2 \times 1$$

$$6! = 3 \times 2 \times 1$$

Q47. How would you define a permutation?

Answer: Given an ordered set with a given number of elements, any other ordered set with the same amount of elements, but with a different order, is considered a permutation of the original set. It can also be considered as the act of producing such new different ordered set.

Another way of defining a permutation is a subset of a global set of elements in which the order is relevant.

For example, the letters A, B and C can be arranged in 6 different ways: ABC, ACB, BAC, BCA, CAB, CBA. Each of those different combinations is a permutation of the others. We can also say that there are six possible three letter permutations for the set of letters A, B and C.

Let us say now that we had letters A, B, C and D and we want to know how many three letter ordered subsets, i.e. how many three letter permutations we could form. In that case the answer would be 24: ABC, ABD, ACB, ACD, ADB, ADC, BAC, BAD, BCA, BCD, BDA, BDC, CAB, CAD, CBA, CBD, CDA, CDB, DAB, DAC, DBA, DBC, DCA, DCB.

Q48. How would you define a combination?

Answer: Given a set with a given number n of elements, any subset of that global set with m elements is considered a combination of m elements from the n original elements, or a combination of n elements taken m at a time, without repetition.

Another way of defining a combination is a subset of a global set of elements in which the order is not relevant.

Let us say that we had letters A, B, C and D and we want to know how many three letter subsets, i.e. how many three letter combinations we could form. In that case the answer would be 4: ABC, ACD, ABD, BCD. Each of those different combinations is a combination of three elements of the original set of letters.

Q49. How much is the sum of the elements of the *n* th row of Pascal's Triangle?

Answer: The sum of the elements of the *n* th row of Pascal's Triangle is equal to 2^n .

Q50. A coach has 12 players available and he must assemble a team with seven players. How many different teams can he assemble?

Answer: The answer is the number of possible combinations of seven elements out of twelve, which is given by the following formula:

$$C(12,\ 7) = \binom{12}{7} = \frac{12!}{(12 - 7)!7!}$$

Performing the calculations:

$$C(12,7) = \frac{12!}{(12 - 7)!7!} = \frac{12!}{5!7!} = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7!}{5 \times 4 \times 3 \times 2 \times 1 \times 7!} = \frac{12 \times 11 \times 10 \times 9 \times 8}{10 \times 12} = 11 \times 72$$

$$\therefore C(12,\ 7) = 792$$

Therefore, the coach could form 792 different teams.

Q51. Jack has five books on his shelf. How many different ways could he arrange his book in?

Answer: The answer is the possible permutations of five elements without repetition out of a total of five elements, which is given by the following formula:

$$P(5,5) = 5!$$

Performing the calculations:

$$P(5,5) = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Therefore, Jack could arrange his books in 120 different ways on his shelf.

Q52. In a relay racing event each team has four runners, according to a specific order. If a coach can choose from a list of seven different runners, how many different teams can he assemble for the competition?

Answer: The answer is the possible permutations of four elements without repetition out of a total of seven elements, which is given by the following formula:

$$P(7,4) = \frac{7!}{(7-4)!} = \frac{7!}{3!}$$

Performing the calculations:

$$P(7,4) = \frac{7!}{3!} = \frac{7 \times 6 \times 5 \times 4 \times 3!}{3!} = 42 \times 20 = 840$$

Therefore, the coach could form 840 different teams.

Q53. In a soccer team there are always 11 players, being 1 goal keeper and 10 field players. The field players can be of three positions: defense, midfield and attack. A soccer coach has the following players available: 3 goalkeepers, 5 defense players, 8 midfield players and 6 attack players. How many different teams can he assemble if he decides to play with: **(a)** 1 goalkeeper, 3 defense players, 5 midfield players and 2 attack players; **(b)** 1 goalkeeper, 4 defense players, 4 midfield players and 2 attack players; **(c)** 1 goalkeeper, 4 defense players, 5 midfield players and 1 attack player;

Answer: The answer is the multiplication of possible combinations for each position, given the amount of players available:

$$C = C_{\text{goalkeeper}} \times C_{\text{defense}} \times C_{\text{midfield}} \times C_{\text{attack}}$$

(a) Determining the amount of players available and the number of combinations for each position:

$$C = C_{\text{goalkeeper}} \times C_{\text{defense}} \times C_{\text{midfield}} \times C_{\text{attack}} =$$
$$= C(3,1) \times C(5,3) \times C(8,5) \times C(6,2)$$

Performing the calculations:

$$C = C(3,1) \times C(5,3) \times C(8,5) \times C(6,2) =$$
$$= \binom{3}{1} \times \binom{5}{3} \times \binom{8}{5} \times \binom{6}{2} = \frac{3!}{2!1!} \times \frac{5!}{2!3!} \times \frac{8!}{3!5!} \times \frac{6!}{4!2!}$$

$$\therefore C = 25{,}200$$

Therefore, the coach could form $25{,}200$ different teams.

(b) Determining the amount of players available and the number of combinations for each position:

$$C = C_{\text{goalkeeper}} \times C_{\text{defense}} \times C_{\text{midfield}} \times C_{\text{attack}} =$$

$= C(3,1) \times C(5,4) \times C(8,4) \times C(6,2)$

Performing the calculations:

$C = C(3,1) \times C(5,4) \times C(8,4) \times C(6,2) =$

$$= \binom{3}{1} \times \binom{5}{4} \times \binom{8}{4} \times \binom{6}{2} = \frac{3!}{2!1!} \times \frac{5!}{1!4!} \times \frac{8!}{4!4!} \times \frac{6!}{4!2!}$$

$$\therefore C = 9{,}450$$

Therefore, the coach could form $9{,}450$ different teams.

(c) Determining the amount of players available and the number of combinations for each position:

$C = C_{\text{goalkeeper}} \times C_{\text{defense}} \times C_{\text{midfield}} \times C_{\text{attack}} =$

$= C(3,1) \times C(5,4) \times C(8,5) \times C(6,1)$

Performing the calculations:

$C = C(3,1) \times C(5,3) \times C(8,5) \times C(6,2) =$

$$= \binom{3}{1} \times \binom{5}{4} \times \binom{8}{5} \times \binom{6}{1} = \frac{3!}{2!1!} \times \frac{5!}{1!4!} \times \frac{8!}{3!5!} \times \frac{6!}{5!1!}$$

$$\therefore C = 5{,}040$$

Therefore, the coach could form $5{,}040$ different teams.

Q54. Consider a common deck of cards composed of four suits (hearts, spades, clubs and diamonds) of thirteen cards of different values (ace, one, two, three, four, five, six, seven, eight, nine, ten, jack, queen and king), adding up to a total of 52 cards. If we randomly pick three cards, how many different combinations of cards can we get?

Answer: The answer is the amount of combinations of 52 elements picked 3 at each time:

$$C(52,3) = \binom{52}{3} = \frac{52!}{49!3!} = \frac{52 \times 51 \times 50}{3 \times 2} = 22{,}100 \text{ combinations}$$

Q55. Consider a common deck of cards composed of four suits (hearts, spades, clubs and diamonds) of thirteen cards of different values (ace, one, two, three, four, five, six, seven, eight, nine, ten, jack, queen and king), adding up to a total of 52 cards. Suppose that we randomly pick four cards. How many times the amount of possible combinations with all cards of the same suit is greater than the amount of possible combinations of with all cards of the same value?

Answer: First we calculate the amount of combinations with four cards of same value. There are four suits and each suit has thirteen different values. So we have thirteen ways of picking one card of each suit:

$$C_{\text{value}} = 13$$

Then we calculate the amount of combinations with four cards of same suit. There are four suits and each suit has thirteen different values. So we have four ways of picking four cards out of thirteen possible values:

$$C_{\text{suit}} = 4 \times C(13,4) = 4 \times \binom{13}{4} = 4 \times \frac{13!}{9!4!} = \frac{4 \times 13 \times 12 \times 11 \times 10}{4 \times 3 \times 2} = 2{,}860$$

Now we calculate the ratio of both calculated values:

$$\frac{C_{\text{suit}}}{C_{\text{value}}} = \frac{2860}{13} = 220$$

Therefore there are 220 times more combinations with all four cards of same suit than there are combinations with all four cards of same value.

Q56. How many anagrams are there for the words: **(a)** SEVEN, **(b)** SEVENTY, **(c)** SEVENTEEN?

Answer:

(a) There are five letters, with two E’s, in the word SEVEN. Therefore, the amount of anagrams is:

$$\frac{5!}{2!} = 5 \times 4 \times 3 = 60$$

(b) There are seven letters, with two E’s, in the word SEVENTY. Therefore, the amount of anagrams is:

$$\frac{7!}{2!} = 7 \times 6 \times 5 \times 4 \times 3 = 2{,}520$$

(c) There are nine letters, with four E’s and two N’s, in the word SEVENTEEN. Therefore, the amount of anagrams is:

$$\frac{9!}{4!2!} = \frac{9 \times 8 \times 7 \times 6 \times 5}{2} = 7{,}560$$

Q57. What is a symmetric matrix?

Answer: A symmetric matrix is a square matrix of order n,

$$A_{n \times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$ in which $a_{ij} = a_{ji}$. That is equivalent to say that it is a matrix in which every element in row i and column j is equal to the element in row j and column i .

A symmetric matrix can also be defined as a square matrix that is equal to its transpose.

Examples:

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1/3 & 1/4 \\ 1/3 & 4 & 1/2 \\ 1/4 & 1/2 & 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 18 & 0 & 1 & 14 \\ 0 & -3 & -1 & 9 \\ 1 & -1 & 7.21 & 8.13 \\ 14 & 9 & 8.13 & 4 \end{bmatrix}$$

Q58. What is a diagonal matrix? How is the determinant calculated for a diagonal matrix?

Answer: A diagonal matrix is a square matrix in which all elements have value 0

(ZERO) except for those in its main diagonal. That is equivalent to say that for each element a_{ij} of matrix *A* , if $i = j$ then $a_{ij} = 0$. A diagonal matrix is a special case of a symmetric matrix.

If we were to represent a diagonal matrix as a square, then all elements that do not rest on the diagonal that goes from the top-left corner to the bottom right corner must be 0 (ZERO).

The determinant of a diagonal matrix calculated by multiplying the elements of its main diagonal.

Examples:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \boxed{\begin{matrix} 1 & 0 \\ 0 & -2 \end{matrix}} \qquad \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = (1)(-2) = -2$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \qquad \boxed{\begin{matrix} -1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 0.5 \end{matrix}}$$

$$\begin{vmatrix} -1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 0.5 \end{vmatrix} = (-1)(-6)(0.5) = 3$$

Q59. How is the determinant calculated for a 3×3 matrix?

Answer: The determinant of a 3×3 matrix is calculated by the rule of Sarrus, which is described as follows.

Given matrix $A_{3 \times 3}$,

$$A_{3 \times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Its determinant is represented by:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

We repeat columns 1 and 2 to its right side, like so:

$$\left|\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}\right|\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

Now we perform the diagonal products top down. The left-to-right ones have a positive value and the right-to-left ones have negative values, as schematically shown below:

Therefore, the algebraic expression for the determinant is:

$$\det(A) = + a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Q60. How is the determinant calculated for a 2×2 matrix?

Answer: The determinant of a 2×2 matrix is calculated by the difference between the product of the elements of its main diagonal and the product of the elements of its secondary diagonal.

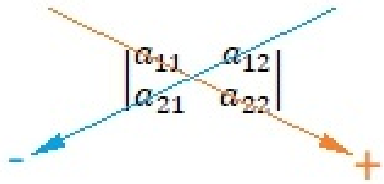
Given matrix $A_{2 \times 2}$,

$$A_{2 \times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Its determinant is represented by:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

To calculate it, we perform the products of both diagonals in the top- down direction, so that the left-to-right diagonal (the main diagonal) has as positive value and the right-to-left diagonal (the secondary diagonal) has a negative value, as schematically shown below:



Therefore, the algebraic expression for the determinant is:

$\det(A) = + a_{11}a_{22} - a_{12}a_{21}$

Q61. What does the matrix notation $A_{m \times n}$ represent?

Answer: It represents a matrix named *A* which is composed of *m* rows and *n* columns.

Q62. Calculate the determinant of the following matrix.

$$A = \begin{bmatrix} 1 & 2 \\ 0.5 & 3 \end{bmatrix}$$

Answer: The determinant of A is represented by $\det(A)$ and it is calculated by the difference between the product of the elements of the main diagonal and the product of the elements of the secondary diagonal, as follows.

$\det(A) = (1 \times 3) - (2 \times 0.5) = 3 - 1 = 2$

Q63. Calculate the determinant of the following matrix.

$$B = \begin{bmatrix} 1 & 3 & -2 \\ 1 & 2 & 1 \\ 4 & 0.5 & -1 \end{bmatrix}$$

Answer: The determinant of B is represented by $\det(B)$ and it is calculated by the Rule of Sarrus, as follows:

$$\det(B) = + 1 \times 2 \times (-1) + 3 \times 1 \times 4 + (-2) \times 1 \times 0.5 - (-2) \times 2 \times 4 - 1 \times 1 \times 0.5 - 3 \times 1 \times (-1)$$

$\det(B) = + (-2) + (12) + (-1) - (-16) - (0.5) - (-3)$

$\det(B) = - 2 + 12 - 1 + 16 - 0.5 + 3 = 27.5$

Q64. Determine the value of x .

$$\begin{vmatrix} x & 1 & 2 \\ -1 & x & 3 \\ 2 & 1 & 1 \end{vmatrix} = -5$$

Answer: The notation above indicates that the determinant of the matrix $\begin{bmatrix} x & 1 & 2 \\ -1 & x & 3 \\ 2 & 1 & 1 \end{bmatrix}$ is -5. Therefore, applying the Rule of Sarrus, we will have:

$$\begin{vmatrix} x & 1 & 2 \\ -1 & x & 3 \\ 2 & 1 & 1 \end{vmatrix} = +x \times x \times 1 + 1 \times 3 \times 2 + (-1) \times 1 \times 2 - 2 \times x \times 2 - 1 \times (-1) \times 1 \times - x \times 3 \times 1 = x^2 + 6 - 2 - 4x - 3x + 1 = x^2 - 7x + 5$$

Thus, we must solve the following equation:

$$x^2 - 7x + 5 = -5$$

Rearranging the equation:

$$x^2 - 7x + 10 = 0$$

Factorizing the left side of the equation:

$$(x - 5) \times (x - 2) = 0$$

The solutions for this equation are $x_1 = 2$ and $x_2 = 5$.

Q65. Determine matrix B such that $B = A^{-1}$, where $A = \begin{bmatrix} -2 & 3 & -20 \\ 1 & 2 & 1 \\ 0 & 1 & -4 \end{bmatrix}$.

Answer: Matrix B is the inverse of matrix A, represented by A^{-1}.

$$A^{-1} = \frac{1}{\det(A)} \times \mathrm{cof}^t(A)$$

Where $\det(A)$ is the determinant of matrix A and $\operatorname{cof}^t(A)$ is the transposed of the matrix of the cofactors of matrix A. The determinant of matrix A is calculated by the Rule of Sarrus as follows:

$$\det(A) = \begin{vmatrix} -2 & 3 & -20 \\ 1 & 2 & 1 \\ 0 & 1 & -4 \end{vmatrix} = 16 + 0 - 20 - 0 + 2 + 12 = 30 - 20 = 10$$

The matrix of the cofactors of matrix A is calculated is the following matrix:

$$\operatorname{cof}(A) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

In which each entry c_{mn} is calculated by the following formula:

$$c_{mn} = (-1)^{m+n} \times \det(A_{mn})$$

Where A_{mn} is the resulting matrix when we remove row m and column n from matrix A. Calculating each value for c_{mn}, we have:

$$c_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix} = (-1)^{1+1}(-8 - 1) = -9$$

$$c_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 0 & -4 \end{vmatrix} = (-1)^{1+2}(-4 - 0) = 4$$

$$c_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = (-1)^{1+3}(1 - 0) = 1$$

$$c_{21} = (-1)^{2+1}\begin{vmatrix} 3 & -20 \\ 1 & -4 \end{vmatrix} = (-1)^{2+1}(-12 + 20) = -8$$

$$c_{22} = (-1)^{2+2}\begin{vmatrix} -2 & -20 \\ 0 & -4 \end{vmatrix} = (-1)^{2+2}(8 - 0) = 8$$

$$c_{23} = (-1)^{2+3}\begin{vmatrix} -2 & 3 \\ 0 & 1 \end{vmatrix} = (-1)^{2+3}(-2 - 0) = 2$$

$$c_{31} = (-1)^{3+1}\begin{vmatrix} 3 & -20 \\ 2 & 1 \end{vmatrix} = (-1)^{3+1}(3 + 40) = 43$$

$$c_{32} = (-1)^{3+2}\begin{vmatrix} -2 & -20 \\ 1 & 1 \end{vmatrix} = (-1)^{3+2}(-2 + 20) = -18$$

$$c_{33} = (-1)^{3+3}\begin{vmatrix} -2 & 3 \\ 1 & 2 \end{vmatrix} = (-1)^{3+3}(-4 - 3) = -7$$

Then, we can assemble matrix $\text{cof}(A)$ as below:

$$\text{cof}(A) = \begin{bmatrix} -9 & 4 & 1 \\ -8 & 8 & 2 \\ 43 & -18 & -7 \end{bmatrix}$$

Transposing that matrix:

$$\text{cof}^t(A) = \begin{bmatrix} -9 & -8 & 43 \\ 4 & 8 & -18 \\ 1 & 2 & -7 \end{bmatrix}$$

Now we can calculate matrix B :

$$B = A^{-1} = \frac{1}{\det(A)} \times \text{cof}^t(A) = 0.1 \times \begin{bmatrix} -9 & -8 & 43 \\ 4 & 8 & -18 \\ 1 & 2 & -7 \end{bmatrix}$$

$$\therefore B = \begin{bmatrix} -0.9 & -0.8 & 4.3 \\ 0.4 & 0.8 & -1.8 \\ 0.1 & 0.2 & -0.7 \end{bmatrix}$$

Q66. If $A = \begin{bmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix}$, then calculate $(A^{-1} \times B)^t$.

Answer: First, we need to calculate the inverse of matrix A , represented by A^{-1} .

$$A^{-1} = \frac{1}{\det(A)} \times \text{cof}^t(A)$$

Where $\det(A)$ is the determinant of matrix A and $\text{cof}^t(A)$ is the transposed of the matrix of the cofactors of matrix A . The determinant of matrix A is calculated the differences of the products of the diagonals as follows:

$$\det(A) = \begin{vmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{vmatrix} = 1 \times 0 - \left(-\frac{1}{2}\right) \times \frac{1}{2} = \frac{1}{4}$$

The matrix of the cofactors of matrix A is calculated is the following matrix:

$$\text{cof}(A) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

In which each entry c_{mn} is calculated by the following formula:

$c_{mn} = (-1)^{m+n} \times \det(A_{mn})$

Where A_{mn} is the resulting matrix when we remove row m and column n from matrix A. Calculating each value for c_{mn}, we have:

$$c_{11} = (-1)^{1+1}(0) = 0$$

$$c_{12} = (-1)^{1+2}\left(-\frac{1}{2}\right) = \frac{1}{2}$$

$$c_{21} = (-1)^{2+1}\left(\frac{1}{2}\right) = -\frac{1}{2}$$

$$c_{22} = (-1)^{2+2}(1) = 1$$

Then, we can assemble matrix $\operatorname{cof}(A)$ as below:

$$\operatorname{cof}(A) = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

Transposing that matrix:

$$\operatorname{cof}^t(A) = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

Now we can calculate A^{-1}:

$$A^{-1} = \frac{1}{\det(A)} \times \operatorname{cof}^t(A) = 4 \times \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 4 \end{bmatrix}$$

The next step is to multiply that matrix by matrix B:

$$A^{-1} \times B = \begin{bmatrix} 0 & -2 \\ 2 & 4 \end{bmatrix} \times \begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -2 \\ 12 & 10 \end{bmatrix}$$

Finally, transposing the result:

$(A^{-1} \times B)^t = \begin{bmatrix} -6 & -2 \\ 12 & 10 \end{bmatrix}^t = \begin{bmatrix} -6 & 12 \\ -2 & 10 \end{bmatrix}$

Q67. Calculate the eigenvalues of matrix $A = \begin{bmatrix} 3 & 3 \\ -2 & -4 \end{bmatrix}$.

Answer: The eigenvalues are the real values for λ, so that:

$$A\vec{v} = \lambda I_2 \vec{v}$$

Where $\vec{v}$ is a vector and I_2 is the identity matrix of order 2. For that condition to be met, it is necessary that:

$$\det(A - \lambda I_2) = 0$$

Calculating the matrix $A - \lambda I_2$:

$A - \lambda I_2 = \begin{bmatrix} 3 & 3 \\ -2 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -2 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} (3-\lambda) & 3 \\ -2 & (-4-\lambda) \end{bmatrix} = 0$

Calculating the determinant of $A - \lambda I_2$:

$\det(A - \lambda I_2) = \begin{vmatrix} (3-\lambda) & 3 \\ -2 & (-4-\lambda) \end{vmatrix} = (3-\lambda)(-4-\lambda) - (-2)(3)$

Plugging in that expression into $\det(A - \lambda I_2) = 0$:

$(3-\lambda)(-4-\lambda) - (-2)(3) = 0$

$(-12 - 3\lambda + 4\lambda + \lambda^2) - (-6) = 0$

$\lambda^2 + \lambda - 12 + 6 = 0$

$$\lambda^2 + \lambda - 6 = 0$$

Applying the second degree equation formula:

$\lambda = \dfrac{-1 \pm \sqrt{1^2 - 4 \times 1 \times (-6)}}{2 \times 1}$

$$\lambda = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2}$$

Therefore the eigenvalues matrix A are:

$$\lambda_1 = \frac{-1+5}{2} = \frac{4}{2} = 2$$

$$\lambda_2 = \frac{-1-5}{2} = \frac{-6}{2} = -3$$

Q68. If $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0.5 \\ 0 & 0.5 \\ 1 & 2 \end{bmatrix}$, then calculate: **(a)** $(A \times B)$; **(b)** $(A^t \times B^t)$.

Answer:

(a)

$$A \times B = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} -1 & 0.5 \\ 0 & 0.5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -2+0+1 & 1-0.5+2 \\ -3+0+3 & 1.5+1+6 \end{bmatrix} = \begin{bmatrix} -1 & 2.5 \\ 0 & 8.5 \end{bmatrix}$$

(b)

$$A^t \times B^t = \begin{bmatrix} -1 & 0 & 1 \\ 0.5 & 0.5 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ -1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -2+0+1 & -3+0+3 \\ 1-0.5+2 & 1.5+1+6 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2.5 & 8.5 \end{bmatrix}$$

Therefore, the following property (which could be used to calculate item **b**) is demonstrated:

$$\therefore A^t \times B^t = (A \times B)^t$$

Q69. What is a homogeneous linear system?

Answer: It is a linear system in which all constant terms (i.e. the terms that do not multiply any variable) are equal to zero. A homogeneous linear systems always has at least one solution in which all of its variables are equal to ZERO. That solution is called a trivial solution.

For example, consider the following linear system, where $x_1, x_2, x_3, \ldots, x_n$ are the variables, $a_{11}, a_{12}, a_{13}, \ldots, a_{nn}$ are the variables' coefficients and

$b_1, b_2, b_3, \ldots, b_n$ are the constant terms, as shown below:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots a_{1n}x_n + b_1 = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots a_{2n}x_n + b_2 = 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots a_{3n}x_n + b_1 = 0 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots a_{nn}x_n + b_n = 0 \end{cases}$$

The system above will be homogeneous if $b_1 = b_2, = b_3 = \ldots = b_n = 0$ and is trivial solution is $x_1 = x_2 = x_3 = \ldots = x_n = 0$.

Q70. What are the three types of linear systems, according to their number of solutions?

Answer: A linear system can be of three types:

- Inconsistent: if the linear system does not admit any solution at all. It is impossible to find variable values that satisfy the conditions of the system.
- Consistent and Independent: if the linear system admit exactly one solution other than the trivial one, in the case of homogeneous systems. It is possible to find variable values that satisfy the conditions of the system and there is exactly one set of variables that satisfies the system.
- Consistent and Dependent: if the linear system admit infinite solutions. It is possible to find variable values that satisfy the conditions of the system and there are infinite sets of variables that satisfy the system.

Q71. How many equations does a linear system with n variables need to have in order to admit one single solution, other than an eventual trivial solution?

Answer: For a linear system to be consistent and independent, which means it admits only one solution, the number of equations needs to be equal to the number of variables. Therefore the linear system must consist of n equations. This condition is necessary, but it is not enough to ensure that the system will have exactly one solution.

Q72. Solve the following system for x and y.

$$\begin{cases} x + 2y = 9 \\ 2x - 3y = -10 \end{cases}$$

Answer:

(a) Using the method of substitution.

Let us consider equations (I) and (II), like below

$$\begin{cases} x + 2y = 9 \text{ (I)} \\ 2x - 3y = -10 \text{ (II)} \end{cases}$$

Obtaining equation (III) by isolating x in (I):

$$x = 9 - 2y \text{ (III)}$$

We can now plug the value of x from (III) in (II):

$$2x - 3y = -10$$

$2(9 - 2y) - 3y = -10$

$18 - 4y - 3y = -10$

$$-7y = -28$$

$$y = 4$$

Now we can plug the value of y found back in (III) to find the value of x :

$x = 9 - 2y = 9 - 2 \times 4 = 9 - 8$

$$x = 1$$

Thus the solution set is:

$$S = \{(1,4)\}$$

(b) Using the method of sum.

Let us consider equations (I) and (II), like below

$$\begin{cases} x + 2y = 9 \text{ (I)} \\ 2x - 3y = -10 \text{ (II)} \end{cases}$$

Multiplying both sides of equations of (I) by -2 :

$$\begin{cases} (x + 2y) \times (-2) = 9 \times (-2) \\ 2x - 3y = -10 \end{cases}$$

$$\begin{cases} -2x - 4y = -18 \\ 2x - 3y = -10 \end{cases}$$

Summing both equations, we can get rid of the terms $-2x$ and $2x$ and then isolate y :

$$\begin{cases} -2x - 4y = -18 \\ 2x - 3y = -10 \end{cases} +$$

$$2x - 2x - 4y - 3y = -18 - 10$$

Solving the resulting equation:

$$2x - 2x - 4y - 3y = -18 - 10$$

$$-7y = -28$$

$$y = 4$$

Now we can plug the value of y found back in (I) to find the value of x :

$$x + 2y = 9$$

$$x + 2 \times 4 = 9$$

$$x + 8 = 9$$

$$x = 1$$

Thus the solution set is:

$$S = \{(1,4)\}$$

Q73. Solve the following system for x , y and z .

$$\begin{cases} x + 2y - z = 3 \\ 2x + 3y - 2z = 3 \\ x + y - 3z = -4 \end{cases}$$

Answer:

(a) Using the method of Gaussian elimination.

Representing the system as a matrix multiplication:

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -2 \\ 1 & 1 & -3 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$$

Subtracting 2 times row 1 from row 2 and subtracting row 1 from row 3:

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & -2 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -7 \end{bmatrix}$$

Adding 2 times row 2 to row 1 and subtracting row 2 from row 3:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -4 \end{bmatrix}$$

Multiplying row 2 by -1 and dividing row 3 by -2:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 2 \end{bmatrix}$$

Adding row 3 to row 1:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$

Performing the matrix multiplication:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$

Comparing the resulting matrices:

$$\begin{cases} x = -1 \\ y = 3 \\ z = 2 \end{cases}$$

Thus the solution set is:

$$S = \{(-1,3,2)\}$$

(b) Using Cramer’s rule.

Representing the system as a matrix multiplication:

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -2 \\ 1 & 1 & -3 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$$

Let us call the coefficient’s matrix M, then:

$$M = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -2 \\ 1 & 1 & -3 \end{bmatrix}$$

Let us call the matrices used for calculating x, y and z: M_x, M_y and M_z respectively. Each of them is obtained by replacing the column of the respective variable by the vector of the independent terms, so:

$$M_x = \begin{bmatrix} 3 & 2 & -1 \\ 3 & 3 & -2 \\ -4 & 1 & -3 \end{bmatrix}$$

$$M_y = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 3 & -2 \\ 1 & -4 & -3 \end{bmatrix}$$

$$M_z = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \\ 1 & 1 & -4 \end{bmatrix}$$

Now we calculate the determinants of M , M_x , M_y and M_z .

$$\det(M) = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & -2 \\ 1 & 1 & -3 \end{vmatrix} = -9 - 4 - 2 + 3 + 2 + 12 = 2$$

$$\det(M_x) = \begin{vmatrix} 3 & 2 & -1 \\ 3 & 3 & -2 \\ -4 & 1 & -3 \end{vmatrix} = -27 + 16 - 3 - 12 + 18 + 6 = -2$$

$$\det(M_y) = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 3 & -2 \\ 1 & -4 & -3 \end{vmatrix} = -9 - 6 + 8 + 3 - 8 + 18 = 6$$

$$\det(M_z) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \\ 1 & 1 & -4 \end{vmatrix} = -12 + 6 + 6 - 9 - 3 + 16 = 4$$

According to Crammer's rule, we can calculate x, y and z as:

$$x = \frac{\det(M_x)}{\det(M)}$$

$$y = \frac{\det(M_y)}{\det(M)}$$

$$z = \frac{\det(M_z)}{\det(M)}$$

Plugging in the calculated values for the determinants:

$$x = \frac{\det(M_x)}{\det(M)} = \frac{-2}{2} = -1$$

$$y = \frac{\det(M_y)}{\det(M)} = \frac{6}{2} = 3$$

$$z = \frac{\det(M_z)}{\det(M)} = \frac{4}{2} = 2$$

$$\begin{cases} x = -1 \\ y = 3 \\ z = 2 \end{cases}$$

Thus the solution set is:

$$S = \{(-1,3,2)\}$$

Q74. Knowing that $m \in R$ and that $\left(\frac{1}{2}, -\frac{1}{2}, 3\right)$ is a solution to the following system, what is the value of m ?

$$\begin{cases} 2x - 2y - mz = -1 \\ x + my + z = 3 \\ -3x + y + z = 1 \end{cases}$$

Answer:

Representing the system as a matrix multiplication:

$$\begin{bmatrix} 2 & -2 & -m \\ 1 & m & 1 \\ -3 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

Replacing x, y and z by the known solution:

$$\begin{bmatrix} 2 & -2 & -m \\ 1 & m & 1 \\ -3 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

Performing the matrix multiplication:

$$\begin{bmatrix} (2)\left(\frac{1}{2}\right) + (-2)\left(-\frac{1}{2}\right) + (-m)(3) \\ (1)\left(\frac{1}{2}\right) + (m)\left(-\frac{1}{2}\right) + (1)(3) \\ (-3)\left(\frac{1}{2}\right) + (1)\left(-\frac{1}{2}\right) + (1)(3) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} (1 + 1 - 3m) \\ \left(\frac{1}{2} - \frac{m}{2} + 3\right) \\ \left(-\frac{3}{2} - \frac{1}{2} + 3\right) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} (2 - 3m) \\ \left(\frac{1}{2} - \frac{m}{2} + 3\right) \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

Comparing all the elements in the respective positions:

$$\begin{cases} 2 - 3m = -1 \\ \frac{1}{2} - \frac{m}{2} + 3 = 3 \\ 1 = 1 \end{cases}$$

Solving for m in the first two equations:

$$\begin{cases} -3m = -1 - 2 \\ -\frac{m}{2} = -\frac{1}{2} \\ 1 = 1 \end{cases}$$

$$\begin{cases} -3m = -3 \\ -m = -1 \\ 1 = 1 \end{cases}$$

$$\begin{cases} m = 1 \\ m = 1 \\ 1 = 1 \end{cases}$$

Because both values found for m are equal to one another and because the third equation is valid, then the system is possible. Therefore $m = 1$ is the answer.

Q75. For the following system, we know that $x = 1$ .What are the values of y, z and t ?

$$\begin{cases} x + y + z + t = 1 \\ x + 2y + z + 2t = 0 \\ 2x + y - z + 2t = 2 \end{cases}$$

Answer:

Plugging $x = 1$ into all equations of the system and reorganizing:

$$\begin{cases} 1 + y + z + t = 1 \\ 1 + 2y + z + 2t = 0 \\ 2 \times 1 + y - z + 2t = 2 \end{cases}$$

$$\begin{cases} y + z + t = 0 \\ 2y + z + 2t = -1 \\ y - z + 2t = 0 \end{cases}$$

Now we just need to solve the system above. For the sake of this resolution, we will be using Cramer’s rule, as follows:

Writing the matrix of the coefficients M, and finding its determinant:

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \Rightarrow \det(M) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & -1 & 2 \end{vmatrix} = 2 + 2 - 2 - 1 - 4 + 2 = -1$$

Let us call the matrices used for calculating y , z and t : M_y , M_z and M_t respectively. Each of them is obtained by replacing the column of the respective variable by the vector of the independent terms. Below we show the resulting matrices and their determinants:

$$M_y = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \Rightarrow \det(M_y) = \begin{vmatrix} 0 & 1 & 1 \\ -1 & 1 & 2 \\ 0 & -1 & 2 \end{vmatrix} = 1 + 2 = 3$$

$$M_z = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & 2 \end{bmatrix} \Rightarrow \det(M_z) = \begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & 2 \end{vmatrix} = -2 + 1 = -1$$

$$M_t = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \Rightarrow \det(M_t) = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = -1 - 1 = -2$$

According to Crammer’s rule, we can calculate y, z and t as:

$$y = \frac{\det(M_y)}{\det(M)} = \frac{3}{-1} = -3$$

$$z = \frac{\det(M_z)}{\det(M)} = \frac{-1}{-1} = 1$$

$$t = \frac{\det(M_t)}{\det(M)} = \frac{-2}{-1} = 2$$

Q76. How many nontrivial solutions are there for the following system?

$$\begin{cases} -x + 2y + z = 0 \\ 3x + y - 2z = 0 \\ x + 5y = 0 \end{cases}$$

Answer:

The coefficients’ matrix is:

$$\begin{bmatrix} -1 & 2 & 1 \\ 3 & 1 & -2 \\ 1 & 5 & 0 \end{bmatrix}$$

Because all independent terms are equal to ZERO, there is at least one trivial solution (0,0,0). To know if there are nontrivial solutions and how many, we need to perform the Gaussian elimination as follows.

Adding 3 times row 1 to row 2 and adding row 1 to row 3:

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 7 & 1 \\ 0 & 7 & 1 \end{bmatrix}$$

Subtracting row 2 from row 3:

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 7 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Because there is a row of zeroes, it will not be possible to perform the Gaussian elimination until we obtain a diagonal matrix and its determinant will always be zero. Therefore, there are infinite many solutions to the system.

Let us exemplify the reason by continuing the Gaussian elimination. Subtracting row 2 from row 1.

$$\begin{bmatrix} -1 & -5 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If we now perform the matrix multiplication in the following equation:

$$\begin{bmatrix} -1 & -5 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -x - 5y \\ 7y + z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Comparing the resulting matrices:

$$\begin{cases} -x - 5y = 0 \\ 7y + z = 0 \end{cases}$$

We have a system with 2 equations and 3 variables, and it is not possible to solve such system, we can only find a family of solutions, for example by finding x and z as functions of y : $\begin{cases} x = -5y \\ z = -7y \end{cases}$

So, a generic solution to the system would be:

$(-5y, y, -7y) = y(-5, 1, -7)$

We can replace y by any real number and we will have a valid solution. Because there are infinite real numbers, there are also infinite possible solutions for the system.

Q77. John is three times as old as Phillip. Three years from now John will be only twice as old as Phillip. How old are John and Phillip today?

Answer:

Considering x the age of John today and y the age of Phillip today. We can stablish the following equation for their ages today:

$$x = 3y \quad \text{(I)}$$

Three years in the future their ages will be three units greater than today's. So, we can stablish following equation for their future ages:

$$x + 3 = 2(y + 3) \quad \text{(II)}$$

Plugging x from I into II, and solving for y :

$3y + 3 = 2(y + 3)$

$$3y + 3 = 2y + 6$$

$$y = 3$$

Plugging the value of y in I:

$x = 3y = 3 \times 3 = 9$

Therefore, John is nine years old and Phillip is three years old (in three years they will be twelve and six years old respectively).

Q78. Knowing that $m \in R$, what must be the value(s) of m for the following system to have just one single possible solution?

$$\begin{cases} x + 4y + z = 3 \\ x + 3y + 3z = 2 \\ x + 5y + (1 - m)z = 4 \end{cases}$$

Answer:

The coefficients' matrix is:

$$M = \begin{bmatrix} 1 & 4 & 1 \\ 1 & 3 & 3 \\ 1 & 5 & (1 - m) \end{bmatrix}$$

The condition for the system to be possible and determinate (i.e. only one single solution possible) is that its determinant must be different of ZERO:

$$\det(M) \neq 0$$

Now we only need to calculate $\det(M)$, and solve the resulting equation for m as follows:

$$\det(M) \neq 0 \Rightarrow \begin{vmatrix} 1 & 4 & 1 \\ 1 & 3 & 3 \\ 1 & 5 & (1 - m) \end{vmatrix} \neq 0$$

$$3(1 - m) + 12 + 5 - 3 - 15 - 4(1 - m) \neq 0$$

$$3 - 3m - 4 + 4m - 1 \neq 0 \Rightarrow m - 2 \neq 0$$

$$\therefore m \neq 2$$

It means that the system has one single solution for all real numbers different of 2.

Q79. Calculate the quotient and the remainder of the division of $(2x^5 - 3x^4 - 4x^3 + 2x^2 + 3x - 1)$ by $(1 - x^2)$.

Answer: We start by arranging both polynomials according to the descending degrees of their terms.

$P(x) = 2x^5 - 3x^4 - 4x^3 + 2x^2 + 3x - 1$

$$D(x) = - x^2 + 1$$

Now we place the polynomials according to the division algorithm arrangement, so that we can find the quotient polynomial $Q(x)$ and the remainder polynomial $R(x)$, as follows:

$P(x)$	$D(x)$
$R(x)$	$Q(x)$

Replacing $P(x)$ and $D(x)$ by their terms.

$2x^5 - 3x^4 - 4x^3 + 2x^2 + 3x - 1$	$- x^2 + 1$

Now we proceed with the term by term division, starting by the ones of greater degree, i.e. the left most ones. The first division is $2x^5$ by $- x^2$ which results in $- 2x^3$. We note that result as the first term of the quotient.

$2x^5 - 3x^4 - 4x^3 + 2x^2 + 3x - 1$	$- x^2 + 1$
	$- 2x^3$

Now we multiply the newest term of the quotient by the divider and subtract the result from the dividend.

$2x^5 - 3x^4 - 4x^3 + 2x^2 + 3x - 1$	$- x^2 + 1$
$- 2x^5 + 0x^4 + 2x^3$	$- 2x^3$
$- 3x^4 - 2x^3$	

We can then bring the next factor down, which is $2x^2$ and proceed with the division of the next term, first division is $- 3x^4$ by $- x^2$ and so on. The final result is below:

$2x^5 - 3x^4 - 4x^3 + 2x^2 + 3x - 1$	$- x^2 + 1$
$- 2x^5 + 0x^4 + 2x^3$	$- 2x^3 + 3x^2 + 2x + 1$

	$-3x^4 - 2x^3 + 2x^2$					
	$+3x^4 + 0x^3 - 3x^2$					
		$-2x^3 - x^2 + 3x$				
		$2x^3 - 0x^2 - 2x$				
			$-x^2 + x - 1$			
			$x^2 + 0x - 1$			
					$x - 2$	

Therefore, the quotient polynomial $Q(x)$ and the remainder polynomial $R(x)$ are:

$Q(x) = -2x^3 + 3x^2 + 2x + 1$

$$R(x) = x - 2$$

Q80. Decompose the fraction below into its partial fractions:

$$Q(x) = \frac{4x^2 - 18x + 24}{x^3 - 6x^2 + 8x}$$

Answer: First we need to factorize the denominator. Because all of its terms are multiples of x, we can put x in evidence:

$$Q(x) = \frac{4x^2 - 18x + 24}{x(x^2 - 6x + 8)}$$

Now we need to check whether it is possible to factorize $(x^2 - 6x + 8)$.

If $(x^2 - 6x + 8) = (x - a)(x - b)$, then $x^2 - 6x + 8 = x^2 - (a + b)x + ab.$ Comparing the terms of both expressions, we need to find a and b so that a $a + b = 6$ and $a \times b = 8$. Those values can be found as $a = 2$ and $b = 4$. Therefore:

$$Q(x) = \frac{4x^2 - 18x + 24}{x(x - 2)(x - 4)}$$

The factors of the denominator are the denominators of the partial fractions, whose numerators A , B and C must be found, so that:

$$Q(x) = \frac{A}{x} + \frac{B}{x - 2} + \frac{C}{x - 4}$$

The LCM (least common multiple) of the denominators is the original denominator of $Q(x)$, so that:

$$Q(x) = \frac{A}{x} + \frac{B}{x - 2} + \frac{C}{x - 4} = \frac{A(x - 2)(x - 4) + Bx(x - 4) + Cx(x - 2)}{x(x - 2)(x - 4)}$$

Performing the distributive multiplications on the resulting numerator we have

$$Q(x) = \frac{A(x^2 - 6x + 8) + B(x^2 - 4x) + C(x^2 - 2x)}{x(x - 2)(x - 4)} = \frac{Ax^2 - 6Ax + 8A + Bx^2 - 4Bx + Cx^2 - 2Cx}{x(x - 2)(x - 4)}$$

Grouping the terms of same degree on in the numerator:

$$Q(x) = \frac{(A + B + C)x^2 - (6A + 4B + 2C)x + 8A}{x(x - 2)(x - 4)}$$

Comparing that expression with the expression $Q(x) = \frac{4x^2 - 18x + 24}{x(x - 2)(x - 4)}$, we find that:

$$\frac{4x^2 - 18x + 24}{x(x - 2)(x - 4)} = \frac{(A + B + C)x^2 - (6A + 4B + 2C)x + 8A}{x(x - 2)(x - 4)}$$

Comparing the terms of same degree in the numerators of both sides of that equation, we find the following system:

$$\begin{cases} A + B + C = 4 \\ 6A + 4B + 2C = 18 \\ 8A = 24 \end{cases}$$

Isolating A in the third equation:

$$8A = 24 \Rightarrow A = 3$$

Replacing the value of A in the two remaining equations, results in the following system:

$$\begin{cases} 3 + B + C = 4 \\ 18 + 4B + 2C = 18 \end{cases}$$

Leaving only multiples of B and C on the left side of both equations:

$$\begin{cases} B + C = 1 \\ 4B + 2C = 0 \end{cases}$$

Dividing both sides of the second equation by two:

$$\begin{cases} B + C = 1 \\ 2B + C = 0 \end{cases}$$

Isolating C in the third equation:

$2B + C = 0 \Rightarrow C = - 2B$

Plugging in the value of C in the first equation:

$B + C = 1 \Rightarrow B - 2B = 1 \Rightarrow - B = 1$

$$\therefore B = - 1$$

Plugging in the value of B in $C = - 2B$:

$C = - 2B \Rightarrow C = (- 2)(- 1)$

$$\therefore C = 2$$

Therefore the resulting decomposition of $Q(x)$ into partial fractions is:

$Q(x) = \frac{3}{x} - \frac{1}{x - 2} + \frac{2}{x - 4}$

Or:

$\frac{4x^2 - 18x + 24}{x^3 - 6x^2 + 8x} = \frac{3}{x} - \frac{1}{x - 2} + \frac{2}{x - 4}$

Q81. Knowing that $\log_b x = 3$, $\log_b y = 2$, $\log_b z = m$ and $b^n = a$, calculate the value of $\log_a\left(\frac{xy}{\sqrt[m]{z}}\right)^n$.

Answer: In order to make the substitutions easier we will turn $b^n = a$ into its logarithmical form:

$b^n = a \Rightarrow \log_b a = n$

To calculate the value of the given expression we need to perform a change of base in the logarithm, because all values we have are given in base b :

$$\log_a\left(\frac{xy}{\sqrt[m]{z}}\right)^n = \frac{\log_b\left(\frac{xy}{\sqrt[m]{z}}\right)^n}{\log_b a}$$

But previously we found that $\log_b a = n$, therefore we can plug in that expression so that:

$$\log_a\left(\frac{xy}{\sqrt[m]{z}}\right)^n = \frac{\log_b\left(\frac{xy}{\sqrt[m]{z}}\right)^n}{n}$$

Considering the property of the logarithm of a power, we can remove n from the exponent and multiply it by the logarithm on the numerator, which will cancel with the n on the denominator as follows:

$$\log_a\left(\frac{xy}{\sqrt[m]{z}}\right)^n = \frac{\log_b\left(\frac{xy}{\sqrt[m]{z}}\right)^n}{n} = \frac{n \times \log_b\left(\frac{xy}{\sqrt[m]{z}}\right)}{n} = \log_b\left(\frac{xy}{\sqrt[m]{z}}\right)$$

The logarithm of a multiplication is the sum of the logarithms of the numbers being multiplied and the logarithm of a division is the difference between the logarithm of the dividend and the logarithm of the divisor, thus:

$$\log_a\left(\frac{xy}{\sqrt[m]{z}}\right)^n = \log_b\left(\frac{xy}{\sqrt[m]{z}}\right) = \log_b x + \log_b y - \log_b \sqrt[m]{z}$$

We know that the logarithm of a root is the logarithm of the radicand divided by the degree of the root, then:

$$\log_a\left(\frac{xy}{\sqrt[m]{z}}\right)^n = \log_b x + \log_b y - \log_b \sqrt[m]{z} = \log_b x + \log_b y - \frac{\log_b z}{m}$$

All of the logarithms that appear on that last expression were given in the statement of the problem. So, all we need to do now is to plug in their values:

$$\log_a\left(\frac{xy}{\sqrt[m]{z}}\right)^n = \log_b x + \log_b y - \frac{\log_b z}{m} = 3 + 2 - \frac{m}{m} = 3 + 2 - 1$$

$$\therefore \log_a\left(\frac{xy}{\sqrt[m]{z}}\right)^n = 4$$

Q82. In an arithmetic progression, the common difference between two terms is twice the first term. If the sum of the first nine terms of that progression is 27, determine the formula for the n^{th} term of the arithmetic progression.

Answer: The general formula for the n^{th} term of an arithmetic progression is below, where a and d are the first term and the common difference, respectively:

$$T_n = a + d(n - 1)$$

According to the statement of the exercise, $d = 2a$. Plugging in the value of d in the previous formula:

$$T_n = a + d(n - 1) = a + 2a(n - 1) = a + 2an - 2a = 2an - a$$

$$\therefore T_n = a(2n - 1)$$

The formula for the sum of the first n terms of an arithmetic progression is:

$$S_n = (a + T_n)\frac{n}{2}$$

Substituting the value of T_n in the formula of the sum, we have:

$$S_n = (a + T_n)\frac{n}{2} = [a + a(2n - 1)]\frac{n}{2} = [1 + 2n - 1]\frac{na}{2} = (2n)\frac{na}{2}$$

$$S_n = n^2a \Rightarrow a = \frac{S_n}{n^2}$$

According to the problem, the values of the sum of the first 9 elements is 27:

$$a = \frac{S_n}{n^2} = \frac{27}{9^2} = \frac{1}{3}$$

But we know that $d = 2a$, so:

$$d = 2a = 2 \times \frac{1}{3} = \frac{2}{3}$$

Substituting a and d in the formula for the n^th^ term:

$$T_n = a + d(n - 1)$$

$$\therefore T_n = \frac{1}{3} + \frac{2}{3}(n - 1)$$

Q83. The sum of the first two terms of a geometric progression is equal to $\frac{80}{9}$ and the sum of infinite terms of that geometric progression tends to value 9. Knowing that the common ratio of the progression is positive, what is the formula for nth term?

Answer: The general formula for the nth term of a geometric progression is below, where a and r are the first term and the common ratio, respectively:

$$T_n = ar^{n - 1}$$

The formula for the sum of the first n elements of a geometric progression is:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

When n tends to infinity:

$$S_\infty = \frac{a}{1 - r}$$

According to the problem, $S_\infty = 9$, so:

$$S_\infty = \frac{a}{1 - r} = 9 \Rightarrow \frac{a}{r - 1} = -9$$

We can now plug in the value for this last expression in the expression for S_n as follows:

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a}{r - 1}(r^n - 1) = -9(r^n - 1) \Rightarrow$$

$$\Rightarrow S_n = -9r^n + 9$$

According to the statement of the problem $S_n = \frac{80}{9}$, thus:

$$S_n = -9r^n + 9 \Rightarrow S_2 = -9r^2 + 9 \Rightarrow \frac{80}{9} = -9r^2 + 9$$

$$\frac{80}{9} - 9 = -9r^2 \Rightarrow -r^2 = \frac{80}{81} - 1 \Rightarrow -r^2 = \frac{80}{81} - \frac{81}{81} \Rightarrow -r^2 = -\frac{1}{81} \Rightarrow$$

$$\Rightarrow r^2 = \frac{1}{81} \Rightarrow r = \pm\sqrt{\frac{1}{81}}$$

$$\therefore r = \pm\frac{1}{9}$$

According to the statement of the problem, the common ratio must be a positive number, so we will only consider the positive value for r. Substituting the value of r in the expression for S_∞:

$$S_\infty = \frac{a}{1 - r} = 9 \Rightarrow \frac{a}{1 - \frac{1}{9}} = 9 \Rightarrow a = 9\left(1 - \frac{1}{9}\right) \Rightarrow a = 9 - 1$$

$$\therefore a = 8$$

Substituting a and r in the formula for the n^{th} term:

$$T_n = ar^{n - 1}$$

$$\therefore T_n = 8\left(\frac{1}{9}\right)^{n - 1}$$

Q84. Solve the following equation for x .

$$8 \times 2^{x^2} = \left(\frac{8}{2^{5x}}\right)^{- 1}$$

Answer: Applying the negative exponent on the right hand side of the equation:

$$8 \times 2^{x^2} = \left(\frac{8}{2^{5x}}\right)^{- 1} \Rightarrow 8 \times 2^{x^2} = \frac{2^{5x}}{8}$$

Multiplying both sides of the equation by 8:

$$8 \times 2^{x^2} = \frac{2^{5x}}{8} \Rightarrow 64 \times 2^{x^2} = 2^{5x}$$

Expressing 64 as a power of 2:

$$64 \times 2^{x^2} = 2^{5x} \Rightarrow 2^6 \times 2^{x^2} = 2^{5x}$$

Performing the multiplication on the left hand-side of the equation:

$$2^6 \times 2^{x^2} = 2^{5x} \Rightarrow 2^{x^2 + 6} = 2^{5x}$$

For these powers of same base to be equal, the exponents must be equal, therefore:

$$x^2 + 6 = 5x \Rightarrow x^2 - 5x + 6 = 0$$

Applying the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)} = \frac{5 \pm \sqrt{25 - 24}}{2} = \frac{5 \pm 1}{2}$$

$$x = \frac{5 \pm 1}{2} \Rightarrow \begin{cases} x_1 = \frac{5+1}{2} = 3 \\ x_2 = \frac{5-1}{2} = 2 \end{cases}$$

Therefore, the solution set for the equation is:

$$S = \{2, 3\}$$

Q85. The return profit y of an investment, as a function of the invested value x, both in millions of dollars, is $y(x) = -x^2 + 5x - 4$. What value must be invested and why? What will be the profit in that case?

Answer: The profit function is a quadratic function with a negative coefficient of its highest degree term. It means that the function is concave downwards, which, in its turn, means that it has a maximum value, whose coordinates (x_V, y_V) are given by the vertex of the parabola it describes. That maximum value y_V is the maximum profit we can obtain, by investing x_V.

The coordinates of the vertex of a parabola $y(x) = ax^2 + bx + c$ are calculated by the following formulas:

$$x_V = \frac{-b}{2a}$$

$$y_V = \frac{-(b^2 - 4ac)}{4a}$$

Replacing the coefficients of the function given:

$$x_v = \frac{-5}{2(-1)} = \frac{5}{2} = 2.5$$

$$y_v = \frac{-(b^2 - 4ac)}{4a} = \frac{-[5^2 - 4(-1)(-4)]}{4(-1)} = \frac{25 - 16}{4} = \frac{9}{4} = 2.25$$

Therefore, the value that must be invested is $2.5 million for a profit of $2.25 million, because that is the maximum profit we can obtain from the investment. Any other invested value, less than or greater than $2.5 million, will have a return profit of less than $2.25 million.

Q86. A runner took twenty laps on a track during a training session. The lap times were registered according to the following table. What was the mean lap time of the runner? What was the standard deviation for the lap times?

Lap	Time (s)	Lap	Time (s)	Lap	Time (s)	Lap	Time (s)
1	123.5	6	123	11	123.5	16	123.4
2	122	7	123.1	12	122.5	17	124.5
3	122.2	8	123.6	13	124	18	123.5
4	121.8	9	122.1	14	124.2	19	123
5	122.5	10	123.3	15	124.4	20	121.9

Answer: Considering n the amount of laps taken (which is 20), t_i the time in seconds for lap i, we can rewrite the lap table and sum up all the lap values as follows:

i	t_i
1	123.5
2	122
3	122.2
4	121.8
5	122.5
6	123
7	123.1
8	123.6

9	122.1
10	123.3
11	123.5
12	122.5
13	124
14	124.2
15	124.4
16	123.4
17	124.5
18	123.5
19	123
20	121.9
Σ [sum]	**2462**

The mean time for the distribution μ_t is calculated using the following formula:

$$\mu_t = \frac{\sum_{i=1}^{n} t_i}{n} \Rightarrow \mu_t = \frac{2462}{20}$$

$$\therefore \mu_t = 123.1 \text{ s}$$

In order to calculate the standard deviation for the distribution, we need to compute $(t_i - \mu_t)^2$ for each lap i . Rewriting the table and performing the calculations (note that $\sum_{i=1}^{n} t_i - \mu_t$ must always be zero):

i	t_i	$t_i - \mu_t$	$(t_i - \mu_t)^2$
1	123.5	0.4	0.16
2	122	-1.1	1.21
3	122.2	-0.9	0.81

4	121.8	-1.3	1.69
5	122.5	-0.6	0.36
6	123	-0.1	0.01
7	123.1	0	0
8	123.6	0.5	0.25
9	122.1	-1	1
10	123.3	0.2	0.04
11	123.5	0.4	0.16
12	122.5	-0.6	0.36
13	124	0.9	0.81
14	124.2	1.1	1.21
15	124.4	1.3	1.69
16	123.4	0.3	0.09
17	124.5	1.4	1.96
18	123.5	0.4	0.16
19	123	-0.1	0.01
20	121.9	-1.2	1.44
Σ	**2462**	**0**	**13.42**

The standard deviation for the distribution σ_t is calculated using the following formula:

$$\sigma_t = \sqrt{\frac{\sum_{i=1}^{n}(t_i - \mu_t)^2}{n}} \Rightarrow \sigma_t = \sqrt{\frac{13.42}{20}} = \sqrt{0.671}$$

$$\therefore \sigma_t \cong 0.8191 \text{ s}$$

Q87. Find the mean value (μ) and the median ($\tilde{x}$) for the following frequency distribution:

Frequency (	2	1	2	4

f)				
Value (x)	3.6	4	4.2	5

Answer: The mean value for the frequency distribution is the weighted mean of the values of the distribution weighted by the respective frequencies, where each entry i contains a value x_i and a respective frequency f_i :

$$\mu = \frac{\sum_{i=1}^{n} (f_i \times x_i)}{\sum_{i=1}^{n} f_i}$$

Rewriting the table and making the calculations:

i	f_i	x_i	$f_i \times x_i$
1	2	3.6	7.2
2	1	4	4
3	2	4.2	8.4
4	4	5	20
Σ	**9**	**-**	**39.6**

Considering the values calculated above, the mean value can be calculated using the previous formula.

$$\mu = \frac{\sum_{i=1}^{n} (f_i \times x_i)}{\sum_{i=1}^{n} f_i} = \frac{39.6}{9}$$

$$\therefore \mu = 4.4$$

In order to calculate the median value, we need to reorganize the table to view all the population in ascending order, where each entry i contains a value x_i :

i	1	2	3	4	5	6	7	8	9
x_i	3.6	3.6	4	4.2	4.2	5	5	5	5

Because the population contains an odd amount of values (9), the median value will be the central element of our table (highlighted in orange):

$$\tilde{x} = x_{n/2 + 1} = x_5 = 4.2$$

Q88. A company has a goal of having a mean annual revenue of at least \$2,000,000.00 in the period from 2018 to 2020. The revenue in 2018 was \$1,800,000.00 and the revenue in 2019 was \$2,400,000.00. What must be the revenue in 2020 for the goal to be achieved?

Answer: The goal stablished can be represented as the mean revenue in the 3 year period being greater than or equal to \$2 million:

$$\mu \geq 2M$$

The mean value is calculated by the arithmetic mean of the annual revenues, where x_1 , x_2 and x_3 are the revenues in 2018, 2019 and 2020 respectively:

$$\mu = \frac{x_1 + x_2 + x_3}{3} = \frac{1.8M + 2.4M + x_3}{3} = \frac{4.2M + x_3}{3}$$

Plugging the value of μ found in the second equation into the first inequation:

$$\frac{4.2M + x_3}{3} \geq 2M \Rightarrow 4.2M + x_3 \geq 3 \times 2M \Rightarrow x_3 \geq 6M - 4.2M$$

$$\therefore x_3 \geq 1.8M$$

That means that the revenue in 2020 must be greater than or equal to $1,800,000.00.

Q89. A company has a goal of having a mean annual revenue of $1.5 million for a seven year period. The mean annual revenue was $2.3 million in the first three years of the period and $0.2 million in the next two years after that. What must be the minimum mean annual revenue in the rest of the period for the goal to be achieved?

Answer: The goal stablished can be represented as the mean revenue in the 7 year period being greater than or equal to $1.5 million:

$$\mu \geq 1.5M$$

The mean value is calculated by the weighted mean of the mean annual revenues, where x_1, x_2 and x_3 are the revenues in the first three years, in the two following years and in the final two years respectively. Each mean annual revenue is weighted by its amount of years:

$$\mu = \frac{3x_1 + 2x_2 + 2x_3}{7} = \frac{(3)(2.3M) + (2)(0.2M) + 2x_3}{7} = \frac{7.3M + 2x_3}{7}$$

Plugging the value of μ found in the second equation into the first inequation:

$$\frac{7.3M + 2x_3}{7} \geq 1.5M \Rightarrow 7.3M + 2x_3 \geq (7)(1.5M) \Rightarrow 2x_3 \geq 10.5M - 7.3M \Rightarrow 2x_3 \geq 3.2M$$

$$\therefore x_3 \geq 1.6M$$

That means that minimum mean annual revenue in the final three years must be $1,600,000.00.

Q90. In a frequency distribution, value x has a frequency of 2, value y has a frequency of 3 and value z has a frequency of 5. The distribution has mean value of 47.2, median value of 50 and mode value of 54. Find the values of x, y and z, knowing that $x < y < z$.

Answer: The mode (m_0) for a distribution is the value with highest frequency. Analyzing the frequencies given, we can see that the value with highest frequency is z, with a frequency of 5, therefore:

$$m_0 = z = 54$$

Because $x < y < z$, when we sort the values of the population, we get the following table. Notice that there are two central elements (highlighted in orange) because the amount of elements in the population is even.

i	1	2	3	4	5	6	7	8	9	10
x_i	x	x	y	y	y	z	z	z	z	z

The median ($\tilde{x}$) for an even population is the arithmetic mean of its two central elements and its value is 50. Therefore:

$$\tilde{x} = \frac{x_{n/2} + x_{n/2+1}}{2} = \frac{x_5 + x_6}{2} = \frac{y + z}{2} = 50 \Rightarrow$$

$$\Rightarrow y + z = 100 \Rightarrow$$

$$\Rightarrow y + 54 = 100 \Rightarrow$$

$$\Rightarrow y = 100 - 54$$

$$\therefore y = 46$$

Finally, the mean value is calculated by the weighted mean of the values of the distribution, weighted by their respective frequencies and its value is 47.2. Therefore:

$$\mu = \frac{2x_1 + 3y + 5z}{10} = \frac{2x + (3)(46) + (5)(54)}{10} = \frac{2x + 408}{10} = 47.2 \Rightarrow$$

$$\Rightarrow 2x + 408 = 472 \Rightarrow$$

$$\Rightarrow 2x = 472 - 408 \Rightarrow$$

$$\Rightarrow 2x = 64$$

$$\therefore x = 32$$

We can then conclude that the values of x, y and z are 32, 46 and 54 respectively.

Q91. In a binary options investment, there are two possible outcomes: gain or loss. The probability of gaining is approximately 40%. What is the probability of gaining in four investments in a series of five?

Answer: An event with two possible outcomes is has a binomial distribution whose probability is determined by the following formula:

$$P(x) = \frac{n!}{(n - x)!x!}p^x(1 - p)^{n - x}$$

Where $x = 4$ is the number of gaining investments, $n = 5$ is the total number of investments and $p = 40\% = 0.4$ is the probability of gaining in each investment. Making the substitutions and performing the calculations:

$$P(4) = \frac{5!}{(5 - 4)!4!}(0.4)^4(0.6)^{5 - 4} = (5)(0.4)^4(0.6)^1 = (3)(0.0256) = 0.0768$$

$$P(4) = 7.68\%$$

Therefore the probability of gaining in four investments in a series of five is 7.68%.

Q92. The test scores of a group of 25,000 students follow a normal distribution with a mean value of 50 and a standard deviation of 16. Consider the table below for the area of the standard normal curve of probability density with respect to z. How many students are expected to have a score of 90 or above?

z	0.5	1	1.5	2	2.5	3
Area from 0 to z	0.1915	0.3413	0.4332	0.4772	0.4938	0.4987

Answer: First, we need to normalize the desired value of $x = 90$ to z, so that we can work with the given table. That is made by using the following formula, with mean value $\mu = 50$ and standard deviation $\sigma = 16$:

$$z = \frac{x - \mu}{\sigma} = \frac{90 - 50}{16} = \frac{40}{16} = 2.5$$

The probability p of finding a value greater than or equal to $z = 2.5$ corresponds to the area under the standard normal curve in the region from $z = 2.5$ to positive infinity. In a standard normal curve, the area of the region from 0 to positive infinity is 0.5 and we can read the area of the region from 0 to $z = 2.5$, which is 0.4938. Therefore, the probability p can be calculated by subtracting those values:

$$p = 0.5 - 0.4938 = 0.0062$$

The amount of people n that corresponds to that probability, considering a population of 25,000 students is:

$$n = 0.0062 \times 25{,}000 = 155$$

Therefore the amount of students which are expected to have a score of 90 or above is $155.$

Q93. In a factory's manufacturing process, the size of a product may vary around a mean value of 16 inches, with standard deviation of 2 inches, according to a normal distribution. The acceptable range for selling the product in the market, with a profit of $0.80 per unit, is from 14 inches to 20 inches and all products whose size is outside of that range are sold as an input material for other industry, with a profit of $0.05 per unit. Knowing that the factory produces 40,000 units of that product per month, what is the total monthly profit of the factory? Consider the table below for the area of the standard normal curve of probability density with respect to z .

z	0.5	1	1.5	2	2.5	3
Area from 0 to z	0.1915	0.3413	0.4332	0.4772	0.4938	0.4987

Answer: First, we need to normalize the boundary values of $x_1 = 14$ and $x_2 = 20$ to z , so that we can work with the given table. That is made by using the following formula, with mean value $\mu = 16$ and standard deviation $\sigma = 2$:

$$z = \frac{x - \mu}{\sigma}$$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{14 - 16}{2} = \frac{-2}{2} = -1$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{20 - 16}{2} = \frac{4}{2} = 2$$

The normal distribution density probability curve is symmetrical around $x = 0$. Thus, the area under the curve between $z_1 = -1$ and $z = 0$ is equal to the area delimited by $z = 0$ and $z = 1$, which is the probability p_1 of finding an element whose value is between $z_1 = -1$ and $z = 0$. Reading p_1 from the table:

$$p_1 = 0.3413$$

Now we can do the same and read the probability p_2 of finding a value between $z = 0$ and $z_1 = 2$ directly from the table. It corresponds to the are from $z = 0$ and $z = 2$:

$$p_2 = 0.4772$$

The probability p of finding a value greater than or equal to $z_1 = -1$ and less than or equal to $z_2 = 2$ corresponds to the area under the standard normal curve in the region from $z_1 = -1$ to $z_2 = 2$.. Therefore, the probability p can be calculated by adding the previously found probabilities p_1 and p_2:

$$p = p_1 + p_2 = 0.3413 + 0.4772 = 0.8185$$

It means that 81.85% of the factory's production can be sold in the market. The rest will be sold to another industry with a lower gain. The corresponding percentage is:

$$1 - p = 1 - 0.8185 = 0.1815 = 18.15\%$$

In order to calculate the monthly gain G_m, we simply multiply the monthly production by the sum of each percentage multiplied by the respective per unit gain:

$$G_m = 40{,}000 \times (0.8185 \times 0.80 + 0.1815 \times 0.05) = (32741 \times 0.80) + (7260 \times 0.05) = 26192 + 363$$

$$\therefore G_m = \$26{,}555.00$$

Therefore, the monthly profit of the factory is $\$26{,}555.00$.

Q94. A mobile app profits with in-app purchases. It is known that 15% of the users spend $1.00 a day, 5% of the users spend $5.00 a day and 80% of the users do not spend any money with in-app purchases. If the app's creators spend $0.10 in average for each user, what is expected daily profit per user?

Answer: The solution consists in finding the expected value E_v of the profit of the distribution, which is given by the following formula:

$$E_v = \sum_{i=1}^{n} (p_i \times x_i)$$

Where there are n possible outcomes, with each outcome value x_i having a probability p_i within the distribution. The probabilities are given as the percentages of users in the statement of the question. The values of x_i are the profits:

- 5% of the users spend $5.00 and they cost $0.10:

$$x_1 = 5 - 0.1 = 4.9$$

$$p_1 = 5\% = 0.05$$

- 15% of the users spend $1.00 and they cost $0.10:

$$x_2 = 1 - 0.1 = 0.9$$

$$p_2 = 15\% = 0.15$$

- 80% of the users spend $0.00 and they cost $0.10:

$x_3 = 0 - 0.1 = - 0.1$

$$p_3 = 80\% = 0.8$$

Plugging in the calculated values into the expected value formula:

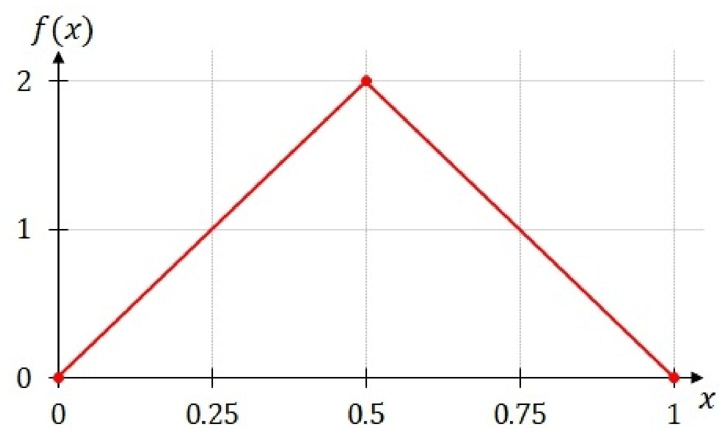
$$E_v = \sum_{i=1}^{n} (p_i \times x_i) = p_1 \times x_1 + p_2 \times x_2 + p_3 \times x_3 = (0.05)(4.9) + (0.15)(0.9) + (0.8)(-0.1) =$$

$$= 0.245 + 0.135 - 0.08 = 0.38 - 0.08 = 0.3$$

$$\therefore E_v = 0.30$$

Therefore, the expected daily profit is $\$0.30$ per user.

Q95. The probability density function $f(x)$ is plotted in the chart below as a function of its value x. For a given process, the relevant values are in the interval from $x = 0.1$ to $x = 0.9$. What is the percentage of this population which is not relevant?



Answer: The answer A is the area under the curve from $x = 0$ to $x = 0.1$ plus the area from $x = 0.9$ to $x = 1$. Because of the symmetry of $f(x)$, these two areas are equal and thus, it is only necessary to calculate one of them and then multiply it by two.

The curve from $x = 0$ to $x = 2$ is a straight line, which means that all of its values are in direct proportion. Therefore, reading $f(0.5) = 2$ from the chart, we can calculate the value of $f(0.1)$ as:

$$\frac{f(0.1)}{0.1} = \frac{f(0.5)}{0.5} \Rightarrow \frac{f(0.1)}{0.1} = \frac{2}{0.5} \Rightarrow f(0.1) = \frac{2}{5}$$

$$\therefore f(0.1) = 0.4$$

The area A_1 under the curve from $x = 0$ to $x = 0.1$ is the area of a triangle whose base is $b = 0.1$ and whose height is $h = f(0.1) = 0.4$. Thus:

$$A_1 = \frac{b \times h}{2} = \frac{0.1 \times 0.4}{2} = 0.02$$

Now, all we need to do is multiply that area by 2 to obtain the answer:

$$A = 2A_1 = 2 \times 0.02 = 0.04$$

$$\therefore A = 4\%$$

Therefore, 4% of the population is not relevant.

Q96. A statistic research on the customers of an e-commerce company assessed three variables: the amount spent by the customer x , the age of the customer y and the monthly income of the customer z . The standard deviation values for each variable are $\sigma_x = 50$, $\sigma_y = 20$ and $\sigma_z = 30$. The covariance between variables x and y is $\sigma_{xy} =- 800$ and the covariance between variables x and z is $\sigma_{xz} = 1290$. The company intends to perform a marketing campaign targeting those who spend the most on their platform, with that in mind: **(a)** Calculate the correlation coefficient ρ_{xy} between variables x and y and the correlation coefficient ρ_{xz} between variables x and z . **(b)** Based on the results obtained in item (a), define the characteristics of age and monthly income of the audience of the marketing campaign.

Answer: We need to calculate the correlation coefficients first to see if age and monthly income have any relation with the amount spent by the user. Then, with those results, we can determine whether those who spend more are older or younger and whether they have higher or lower monthly incomes. The correlation between variables x and y is calculated by:

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

(a) Calculating the correlation coefficients:

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y} = \frac{(-800)}{(50)(20)} = \frac{-800}{1000} \Rightarrow \rho_{xy} = -0.8$$

$$\rho_{xz} = \frac{\sigma_{xz}}{\sigma_x \sigma_z} = \frac{(1290)}{(50)(30)} = \frac{1290}{1500} \Rightarrow \rho_{xy} = 0.86$$

(b) Both calculated correlation coefficients have an absolute value which is closer to one than it is to zero. Therefore, there is a strong relationship between the amount spent by the customers and their age as well as their monthly income and both of them can be considered as relevant. Now we need to determine what the characteristics are for those who spend more, as follows:

- $\rho_{xy} = -0.8 < 0$ indicates a negative correlation, which means that x increases as y decreases and vice-versa. **Thus, those who spend more are the younger ones** .
- $\rho_{xz} = 0.86 > 0$ indicates a positive correlation, which means that x increases as z increases and vice-versa. **Thus, those who spend more are those with higher monthly incomes.**

Therefore, the marketing campaign must target the younger customers of higher monthly income.

Q97. Two variables A and B of a population data set are studied and their values are shown in the table below. Considering $\sqrt{20} = 4.4$, calculate the correlation coefficient ρ_{AB} between those variables and classify their correlation according to the following criteria:

- If $-1 \leq \rho < -0.7$, the correlation is classified as STRONG AND

NEGATIVE;

- If $-0.7 \le \rho < -0.3$, the correlation is classified as MODERATE AND NEGATIVE;
- If $-0.3 \le \rho < 0$, the correlation is classified as WEAK AND NEGATIVE;
- If $\rho = 0$, the correlation is classified as NON-EXISTENT;
- If $0 < \rho \le 0.3$, the correlation is classified as WEAK AND POSITIVE;
- If $0.3 < \rho \le 0.7$, the correlation is classified as MODERATE AND POSITIVE;
- If $0.7 < \rho \le 1$, the correlation is classified as STRONG AND POSITIVE;

A	2	4	6	8	10
B	3	10	11	4	7

Answer: The correlation between two variable A and B is calculated by:

$$\rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \sigma_B}$$

Where σ_{AB} is the covariance between A and B and σ_A and σ_B are the standard deviation values for variables A and B respectively. These three values are calculated by the following formulas:

$$\sigma_{AB} = \frac{\sum_{i=1}^{n} (A_i - \mu_A)(B_i - \mu_B)}{n}$$

$$\sigma_A = \sqrt{\frac{\sum_{i=1}^{n} (A_i - \mu_A)^2}{n}}$$

$$\sigma_B = \sqrt{\frac{\sum_{i=1}^{n} (B_i - \mu_B)^2}{n}}$$

Plugging in these equations into the equation for ρ_{AB}, we can find an expression that will make calculations easier:

$$\rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \sigma_B} = \frac{\frac{\sum_{i=1}^{n} (A_i - \mu_A)(B_i - \mu_B)}{n}}{\sqrt{\frac{\sum_{i=1}^{n} (A_i - \mu_A)^2}{n}} \sqrt{\frac{\sum_{i=1}^{n} (B_i - \mu_B)^2}{n}}} = \frac{\frac{\sum_{i=1}^{n} (A_i - \mu_A)(B_i - \mu_B)}{n}}{\sqrt{\frac{\sum_{i=1}^{n} (A_i - \mu_A)^2}{n} \times \frac{\sum_{i=1}^{n} (B_i - \mu_B)^2}{n}}} =$$

$$= \frac{\frac{\sum_{i=1}^{n} (A_i - \mu_A)(B_i - \mu_B)}{n}}{\sqrt{\frac{\sum_{i=1}^{n} (A_i - \mu_A)^2 \sum_{i=1}^{n} (B_i - \mu_B)^2}{n^2}}} = \frac{\frac{\sum_{i=1}^{n} (A_i - \mu_A)(B_i - \mu_B)}{n}}{\frac{\sqrt{\sum_{i=1}^{n} (A_i - \mu_A)^2 \sum_{i=1}^{n} (B_i - \mu_B)^2}}{n}}$$

$$\therefore \rho_{AB} = \frac{\sum_{i=1}^{n} (A_i - \mu_A)(B_i - \mu_B)}{\sqrt{\sum_{i=1}^{n} (A_i - \mu_A)^2 \sum_{i=1}^{n} (B_i - \mu_B)^2}}$$

Where μ_A is the mean value of variable A, μ_B is the mean value of variable B and n is the size of the population (which is 5, the amount of value rows we have for each variable).

We can now rewrite the table for variables A and B, considering i a sequential number for each value of A and B and performing the sums of the rows:

i	A_i	B_i
1	2	3

2	4	10
3	6	11
4	8	4
5	10	7
Σ	**30**	**35**

Calculating the mean values for both variables:

$$\mu_A = \frac{\sum_{i=1}^{n} A_i}{n} = \frac{30}{5} = 6$$

$$\mu_B = \frac{\sum_{i=1}^{n} B_i}{n} = \frac{35}{5} = 7$$

Now we need to compute $(A_i - \mu_A)(B_i - \mu_B)$, $(A_i - \mu_A)^2$ and $(B_i - \mu_B)^2$ for each value of A and B . Adding the respective columns to the table and performing the calculations:

i	A_i	B_i	$A_i - \mu_A$	$B_i - \mu_B$	$(A_i - \mu_A)(B_i - \mu_B)$	$(A_i - \mu_A)^2$	$(B_i - \mu_B)^2$
1	2	3	-4	-4	16	16	16
2	4	10	-2	3	-6	4	9
3	6	11	0	4	0	0	16
4	8	4	2	-3	-6	4	9
5	10	7	4	0	0	16	0
Σ	**30**	**35**	**-**	**-**	**4**	**40**	**50**

Plugging in the values calculated above into the formula for ρ_{AB} :

$$\rho_{AB} = \frac{\sum_{i=1}^{n}(A_i - \mu_A)(B_i - \mu_B)}{\sqrt{\sum_{i=1}^{n}(A_i - \mu_A)^2 \sum_{i=1}^{n}(B_i - \mu_B)^2}} = \frac{4}{\sqrt{40 \times 50}} = \frac{4}{\sqrt{20 \times 100}} = \frac{4}{\sqrt{20}\sqrt{100}} = \frac{4}{10\sqrt{20}}$$

Considering $\sqrt{20} = 4.4$:

$$\rho_{AB} = \frac{4}{10\sqrt{20}} = \frac{4}{10 \times 4.4} = \frac{4}{44} = \frac{1}{11} \cong 0.091$$

Because $0 < 0.091 \leq 0.3$, the correlation is classified as WEAK AND POSITIVE.

Q98. The profit and the respective invested amount for a particular investment are shown in the table below in thousands of Dollars. Estimate the profit for an invested amount of $3,000.00 by performing a linear regression using the Least Squares Estimation.

Invested Amount	1	2	4	5
Profit	0.6	3.2	5.8	9.6

Answer: Considering the invested amount as variable x and the profit as variable y , the goal of the regression is to find an equation of the following type:

$$y = f(x) = a + bx$$

Using the Least Squares Estimation, the coefficients above are calculated as:

$$b = \frac{\sum_{i=1}^{n}(x_i - \mu_x)(y_i - \mu_y)}{\sum_{i=1}^{n}(x_i - \mu_x)^2}$$

$$a = \mu_y - b\mu_x$$

Where i is an index value for each value of x and y , $n = 4$ is the amount of columns in the data table, μ_x is the mean value of variable x and μ_y is the mean value of variable y . In order to calculate those values we will reorganize the given table and perform the calculations.

i	x_i	y_i
1	1	0.6
2	2	3.2
3	4	5.8
4	5	9.6
Σ	**12**	**19.2**

Calculating the mean values for both variables:

$$\mu_x = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{12}{4} = 3$$

$$\mu_y = \frac{\sum_{i=1}^{n} y_i}{n} = \frac{19.2}{4} = 4.8$$

Now we need to compute $(x_i - \mu_x)(y_i - \mu_y)$ and $(x_i - \mu_x)^2$ for each value of x and y . Adding the respective columns to the table and performing the calculations:

i	x	y	$x_i - \mu_x$	$y_i - \mu_y$	$(x_i - \mu_x)(y_i - \mu_y)$	$(x_i - \mu_x)^2$
1	1	0.6	-2	-4.2	8.4	4
2	2	3.2	-1	-1.6	1.6	1
3	4	5.8	1	1	1	1
4	5	9.6	2	4.8	9.6	4
				Σ	**20.6**	**10**

Calculating the coefficients a and b :

$$b = \frac{20.6}{10} = 2.06$$

$a = 4.8 - (2.06)(3) =- 1.38$

Therefore, we have the following expression:

$$y = a + bx$$

$y =- 1.38 + 2.06x$

Plugging in $x = 3$, we can calculate y:

$y =- 1.38 + 2.06x =- 1.38 + (2.06)(3) = 6.18 - 1.38 = 4.8$

Therefore, can estimate that the gain for an investment of $3,000.00 will be $4,800.00.

Q99. The revenue of the sales of a product as a function of the cost of production of that same product was the subject of a statistic study of a population of several different products. The data of the study show that the average revenue is $21,000, the average production cost is $2500 with a standard deviation of $300 and the covariance between revenue and production cost is $540,000. Using the Least Squares Estimation calculate a linear relationship between revenue and production cost and then estimate the revenue for a production cost of $20,000.

Answer: Considering production cost as variable x and revenue as variable y, the goal of the regression is to find an equation of the following type:

$y = f(x) = a + bx$

Using the Least Squares Estimation, the coefficients above are calculated as:

$$b = \frac{\sum_{i=1}^{n} (x_i - \mu_x)(y_i - \mu_y)}{\sum_{i=1}^{n} (x_i - \mu_x)^2}$$

$$a = \mu_y - b\mu_x$$

Where n is the size of the population (unknown), μ_x is the mean value of variable x and μ_y is the mean value of variable y . Both, μ_x and μ_y are known and, therefore, we just need to find a way to calculate b in order to calculate a . To do that, we need to find a relationship between b , the standard deviation of x (σ_x) and the covariance between x and y (σ_{xy}) whose values are known. The expressions for σ_x and σ_{xy} are:

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \mu_x)^2}{n}}$$

$$\sigma_{xy} = \frac{\sum_{i=1}^{n} (x_i - \mu_x)(y_i - \mu_y)}{n}$$

Isolating the summations in both expressions:

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \mu_x)^2}{n}} \Rightarrow \sum_{i=1}^{n} (x_i - \mu_x)^2 = n\sigma_x^{2}$$

$$\sigma_{xy} = \frac{\sum_{i=1}^{n} (x_i - \mu_x)(y_i - \mu_y)}{n} \Rightarrow \sum_{i=1}^{n} (x_i - \mu_x)(y_i - \mu_y) = n\sigma_{xy}$$

Substituting the expressions for those summations in the expression for b :

$$b = \frac{\sum_{i=1}^{n} (x_i - \mu_x)(y_i - \mu_y)}{\sum_{i=1}^{n} (x_i - \mu_x)^2} = \frac{n\sigma_{xy}}{n\sigma_x^{2}} \Rightarrow b = \frac{\sigma_{xy}}{\sigma_x^{2}}$$

Now we can calculate b by plugging in the values for σ_{xy} and σ_x :

$$b = \frac{540{,}000}{300^2} = \frac{540{,}000}{90{,}000} = 6$$

Then we calculate a , by plugging in the values for b , μ_x and μ_y :

$$a = 21{,}000 - 6 \times 2{,}500 = 21{,}000 - 15{,}000 = 6{,}000$$

Thus, the equation for the linear regression is:

$$y = a + bx$$

$$y = 6{,}000 + 6x$$

We can now estimate the value of y for $x = 20{,}000$:

$$y = 6{,}000 + 6x = 6{,}000 + 6 \times 20{,}000 = 6{,}000 + 120{,}000$$

$$\therefore y = 126{,}000$$

Therefore, the estimated revenue for a production cost of \$20,000 is \$126,000.

Q100. Three variables A , B and C of a population data set are subject of a statistic study, which showed that $\sigma_A = 25$, $\sigma_B = 20$, $\sigma_C = 24$, $\sigma_{AB} =- 300$ And $\sigma_{AC} = 540$. Calculate the correlation coefficients ρ_{AB} between A and B and ρ_{AC} between A and C and classify both correlations according to the following criteria:

- If $-1 \leq \rho <- 0.7$, the correlation is classified as STRONG AND NEGATIVE;
- If $-0.7 \leq \rho <- 0.3$, the correlation is classified as MODERATE AND NEGATIVE;
- If $-0.3 \leq \rho < 0$, the correlation is classified as WEAK AND NEGATIVE;
- If $\rho = 0$, the correlation is classified as NON-EXISTENT;
- If $0 < \rho \leq 0.3$, the correlation is classified as WEAK AND POSITIVE;
- If $0.3 < \rho \leq 0.7$, the correlation is classified as MODERATE

AND POSITIVE;

- If $0.7 < \rho \leq 1$, the correlation is classified as STRONG AND POSITIVE;

Answer: The correlation coefficient between variables A and B is calculated by:

$$\rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \sigma_B}$$

Plugging in the values given in the statement of the question:

$$\rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \sigma_B} = \frac{-300}{25 \times 20} = \frac{-300}{500} = \frac{-3}{5}$$

$$\rho_{AB} = -0.6$$

Because $-0.7 \leq -0.6 < -0.3$, the correlation between A and B is classified as MODERATE AND NEGATIVE.

The correlation coefficient between variables A and C is calculated by:

$$\rho_{AC} = \frac{\sigma_{AC}}{\sigma_A \sigma_C}$$

Plugging in the values given in the statement of the question:

$$\rho_{AC} = \frac{\sigma_{AC}}{\sigma_A \sigma_C} = \frac{540}{25 \times 24} = \frac{540}{600} = \frac{9}{10}$$

$$\rho_{AC} = 0.9$$

Because $0.7 < 0.9 \leq 1$, the correlation is classified as STRONG AND POSITIVE.

Q101. Given the following matrices $A = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 4 \\ 2 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 10 & -2 \\ 8 & 4 \end{bmatrix}$ and $X = \frac{1}{2}(C^t - A \times B)$, calculate det(X).

Answer: First we calculate the product $A \times B$:

$$A \times B = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 4 & 4 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{4}\right)(4) + \left(\frac{1}{2}\right)(2) & \left(\frac{1}{4}\right)(4) + \left(\frac{1}{2}\right)(6) \\ (-1)(4) + (2)(2) & (-1)(4) + (2)(6) \end{bmatrix} = \begin{bmatrix} 1 + 1 & 1 + 3 \\ -4 + 4 & -4 + 12 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} 2 & 4 \\ 0 & 8 \end{bmatrix}$$

Then, we calculate the transposed of matrix C:

$$C^t = \begin{bmatrix} 10 & 8 \\ -2 & 4 \end{bmatrix}$$

Now we can calculate matrix X:

$$X = \frac{1}{2}(C^t - A \times B) = \frac{1}{2}\left(\begin{bmatrix} 10 & 8 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 0 & 8 \end{bmatrix}\right) = \frac{1}{2} \times \begin{bmatrix} 10-2 & 8-4 \\ -2-0 & 4-8 \end{bmatrix} = \frac{1}{2} \times \begin{bmatrix} 8 & 4 \\ -2 & -4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \times 8 & \frac{1}{2} \times 4 \\ \frac{1}{2} \times (-2) & \frac{1}{2} \times (-4) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ -1 & -2 \end{bmatrix}$$

$$X = \begin{bmatrix} 4 & 2 \\ -1 & -2 \end{bmatrix}$$

Finally, we can calculate the determinant of matrix X:

$$\det(X) = \begin{vmatrix} 4 & 2 \\ -1 & -2 \end{vmatrix} = (4)(-2) - (2)(-1) = -8 + 2$$

$$\therefore \det(X) = -6$$

Q102. What is a row matrix?

Answer: A row matrix is a matrix that contains one single row. That is equivalent to say that, given a matrix $A_{m \times n}$, if $m = 1$, then *A* is a row matrix.

Q103. What is a column matrix?

Answer: A column matrix is a matrix that contains one single column. That is equivalent to say that, given a matrix $A_{m \times n}$, if $n = 1$, then *A* is a column matrix.

Q104. What is a square matrix?

Answer: A square matrix is a matrix that contains the same amount of rows and columns. That is equivalent to say that, given a matrix $A_{m \times n}$, if $m = n$, then *A* is a square matrix.

Q105. What is the condition necessary for the following matrix multiplication to be possible? What will the dimensions of C be?

$C = A_{m \times n} \times B_{p \times q}$

Answer: The required condition is that $n = p$, i.e. from left to right, the number of columns of the first matrix must be equal to the number of rows of the second matrix. Then C will be m rows height and q rows wide, which is noted as $C_{m \times q}$.