Run Generation Revisited:
What Goes Up May or May Not Come Down

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Abstract

In this paper, we revisit the classic problem of run generation. Run generation is the first phase of external-memory sorting, where the objective is to scan through the data, reorder elements using a small buffer of size M, and output ***runs*** (contiguously sorted chunks of elements) that are as long as possible.

We develop algorithms for minimizing the total number of runs (or equivalently, maximizing the average run length) when the runs are allowed to be sorted or reverse sorted. We study the problem in the online setting, both with and without resource augmentation, and in the offline setting.

- We analyze alternating-up-down replacement selection (runs alternate between sorted and reverse sorted), which was studied by Knuth as far back as 1963. We show that this simple policy is asymptotically optimal. Specifically, we show that alternating-up-down replacement selection is 2-competitive and no deterministic online algorithm can perform better.

- We give online algorithms having smaller competitive ratios with resource augmentation. Specifically, we exhibit a deterministic algorithm that, when given a buffer of size $4M$, is able to match or beat any optimal algorithm having a buffer of size M. Furthermore, we present a randomized online algorithm which is $7/4$-competitive when given a buffer twice that of the optimal.

- We demonstrate that performance can also be improved with a small amount of foresight. We give an algorithm, which is $3/2$-competitive, with foreknowledge of the next $3M$ elements of the input stream. For the extreme case where all future elements are known, we design a PTAS for computing the optimal strategy a run generation algorithm must follow.

- We present algorithms tailored for "nearly sorted" inputs which are guaranteed to have optimal solutions with sufficiently long runs.

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1 Introduction

External-memory sorting algorithms are tailored for data sets too large to fit in main memory. Generally, these algorithms begin their sort by bringing chunks of data into main memory, sorting within memory, and writing back out to disk in sorted sequences, called ***runs*** [15, 19, 26, 34].

We revisit the classic problem of how to maximize the length of these runs, the ***run-generation problem***. The run-generation problem has been studied in its various guises for over 50 years [14, 17–19, 25, 30, 31, 34].

The most well-known external-memory sorting algorithm is multi-way merge sort [1, 8, 15, 22, 28, 29, 40, 42, 44]. The multi-way merge sort is formalized in the ***disk-access machine***[1] (***DAM***) model of Aggarwal and Vitter [1]. If M is the size of RAM and data is transferred between main memory and disk in blocks of size B, then an M/B-way merge sort has a complexity of $O\big((N/B)\log_{M/B}(N/B)\big)$ I/Os, where N is the number of elements to be sorted. This is the best possible [1].

A top-down description of multi-way merge sort follows. Divide the input into M/B subproblems, recursively sort each subproblem, and merge them together in one final scan through the input. The base case is reached when each subproblem has size $O(M)$, and therefore fits into RAM.

A bottom-up description of the algorithm starts with the base case, which is the run-generation phase. Naïvely, we can always generate runs of length M: ingest M elements into memory, sort them, write them to disk, and then repeat.

The point of run generation is to produce runs *longer* than M. After all, with typical values of N and M, we rarely need more than one or two passes over the data after the initial run-generation phase. Longer runs can mean fewer passes over the data or less memory consumption during the merge phase of the sort. Because there are few scans to begin with, even if we only do one fewer scan, the cost of a merge sort is decreased by a significant percentage. Run generation has further advantages in databases even when a full sort is not required [22, 24].

Replacement Selection. The classic algorithm for run generation is called ***replacement selection*** [20, 26, 28]. We describe replacement selection below by assuming that the elements can be read into memory and written to disk *one at a time*.

To create an increasing run starting from an initially full internal memory, proceed as follows:

1. From main memory, select the smallest element[2] at least as large as every element in the current run.
2. If no such element exists, then the run ends; select the smallest element in the buffer.
3. ***Eject*** that element, and ***ingest*** the next element, so that the memory stays full.

Replacement selection can deal with input elements one at a time, even though the DAM model transfers input between RAM and disk B elements at a time. To see why, consider two additional blocks in memory, an "input block," which stores elements recently read from disk, and an "output block," which stores elements that have already been placed in a run and will be written back to disk. To ingest, take an element from the input block, and to eject an element, put the element in the output block. When the input block becomes empty, fill it from disk and when the output block fills up, flush it to disk. Similar to previous work, in this paper, we ignore these two blocks.

Properties of Replacement Selection. It has been known for decades that when the input appears in random order, then the expected length of a run is actually $2M$, not M [18, 19, 25]. In [26], Knuth gives memorable intuition about this result, conceptualizing the buffer as a snowplow traveling along a circular track.

Replacement selection performs particularly well on nearly sorted data (for many intuitive notions of "nearly"), and the runs generated are much larger than M. For example, when each element in the input appears at a distance at most M from its actual rank, replacement selection produces a single run.

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[1]The external-memory model, also called the I/O model, applies to any two levels of the memory hierarchy.

[2]Observe that data structures such as in-memory heaps can be used to identify the smallest elements in memory. However, from the perspective of minimizing I/Os, this does not matter—computation is free in the DAM model.

On the other hand, replacement selection performs poorly on reverse-sorted data. It produces runs of length M, which is the worst possible.

Up-Down Replacement Selection. From the perspective of the sorting algorithm, it matters little, or not at all, whether the initially generated runs are sorted or reverse sorted.

This observation has motivated researchers to think about run generation when the replacement-selection algorithm has a choice about whether to generate an ***up run*** or a ***down run***, each time a new run begins.

Knuth [25] analyzes the performance of replacement selection that alternates deterministically between generating up runs and down runs. He shows that for randomly generated data, this alternative policy performs *worse*, generating runs of expected length $3M/2$, instead of $2M$.

Martinez-Palau et al. [34] revive this idea in an experimental study. Their two-way-replacement-selection algorithms heuristically choose between whether the run generation should go up or down. Their experiments find that two-way replacement selection (1) is slightly worse than replacement selection for random input (in accordance with Knuth [25]) and (2) produces significantly longer runs on inputs that have mixed up-down runs and reverse-sorted inputs.

Our Contributions. The results in our present paper complement these earlier results. In contrast to Knuth's negative result for random inputs [25], we show that strict up-down alternation is best possible for worst-case inputs. Moreover, we give better competitive ratios with resource augmentation and lookahead, which helps explain why heuristically choosing between up and down runs based on what is currently in memory may lead to better solutions. Resource augmentation is a standard tool used in competitive analysis [9, 11–13, 38, 39] to empower an online algorithm when comparing against an omniscient and all-powerful optimal algorithm.

Up-down run generation boils down to figuring out, each time a run ends, whether the next run should be an up run or a down run. The objective is to minimize the number of runs output.[3] We establish the following:

1. *Analysis of alternating-up-down replacement selection.* We revisit (online) alternating-up-down replacement selection, which was earlier analyzed by Knuth [25]. We prove that alternating-up-down replacement selection is 2-competitive and asymptotically optimal for deterministic algorithms. To put this result in context, it is known that up-only replacement selection is a constant factor better than up-down replacement selection for random inputs, but can be an unbounded factor worse than optimal for arbitrary inputs.

2. *Resource augmentation with extra buffer.* We analyze the effect of augmenting the buffer available to an online algorithm on its performance. We show that with a constant factor larger buffer, it is possible to perform better than twice optimal. Specifically, we exhibit a deterministic algorithm that, when given a buffer of size $4M$, matches or beats any optimal algorithm having a buffer of size M. We also design a randomized online algorithm which is 7/4-competitive using a $2M$-size buffer.

3. *Resource augmentation with extra visibility.* We show that performance factors can also be improved, without augmenting the buffer, if an algorithm has limited foreknowledge of the input. In particular, we propose a deterministic algorithm which attains a competitive ratio of 3/2, using its regular buffer of size M, with a *lookahead* of $3M$ incoming elements of the input (at each step).

4. *Better bounds for nearly sorted data.* We give algorithms that perform well on inputs that have some inherent sortedness. We show that the greedy offline algorithm is optimal for inputs on which the optimal runs are at least $5M$ elements long. We also give a 3/2-competitive algorithm with $2M$-size buffer when the optimal runs are at least $3M$ long. These results are reminiscent of previous literature studying sorting on inputs with "bounded disorder" [10] and adaptive sorting algorithms [16, 33, 41].

5. *PTAS for the offline problem.* We give a polynomial-time approximation scheme for the offline run-generation problem. Specifically, our offline polynomial-time approximation algorithm guarantees a $(1 + \varepsilon)$-approximation to the optimal solution. We first give an algorithm with the running time of $O(2^{1/\varepsilon} N \log N)$ and then improve the running time to $O\left(\left(\frac{1+\sqrt{5}}{2}\right)^{1/\varepsilon} N \log N\right)$.

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[3]Note that for a given input, minimizing the number of runs output is equivalent to maximizing the average length of runs output.

Paper Outline. The paper is organized as follows. In Section 2, we formalize the up-down run generation problem and provide necessary notation. Section 3 contains important structural properties of run generation and key lemmas used in analyzing our algorithms. Analysis of alternating-up-down replacement selection and online lower bounds are in Section 4. Algorithms with resource augmentation, along with properties of the greedy algorithm, are presented in Section 5. The offline version of the problem is studied in Section 6. Improvements on well-sorted input are presented in Section 7. Section 8 summarizes related work and we conclude with open problems in Section 9. Due to space constraints, we defer some proofs to the appendix (Appendix A).

2 Up-Down Run Generation

In this section, we formalize the up-down run generation problem and introduce notation.

2.1 Problem Definition

An instance of the up-down run generation problem is a stream I of N elements. The elements of I are presented to the algorithm one by one, in order. They can be stored in the memory of size M available to the algorithm, which we henceforth refer to as the ***buffer***. Each element occupies one slot of the buffer. In general, the model allows duplicate elements, although some results, particularly in Section 5 and Section 7, do require uniqueness.

We say that an algorithm A ***reads*** an element of I when A transfers the element from the input sequence to the buffer. We say that an algorithm A ***writes*** an element when A ejects the element from its buffer and appends it to the ***output sequence*** S.

Every time an element is written, its slot in the buffer becomes free. Unless stated otherwise, the next element from the input takes up the freed slot. Thus the buffer is always full, except when the end of the input is reached and there are fewer than M unwritten elements.[4]

An algorithm can decide which element to eject from its buffer based on (a) the current contents of the buffer and (b) the last element written. The algorithm may also use $o(M)$ additional words to maintain its internal state (for example, it can store the direction of the current run). However, the algorithm cannot arbitrarily access S or I—it can only append elements to S, and access the next in-order element of I. We say the algorithm is at ***time step*** t if it has written exactly t elements.

A ***run*** is a sequence of sorted or reverse-sorted elements. The cost of the algorithm is the smallest number of runs we can use to partition its output. Specifically, the number of runs in an output S, denoted $R(S)$, is the smallest number of mutually disjoint sequences $S_1, S_2, \ldots, S_{R(S)}$ such that each S_i is a run and $S = S_1 \circ \cdots \circ S_{R(S)}$ where $\circ$ indicates concatenation.

We let $\text{OPT}(I)$ be the minimum number of runs of any possible output sequence on input I, i.e., the number of runs generated by the optimal offline algorithm. If I is clear from context, we denote this as OPT. Our goal is to give algorithms that perform well compared to OPT for every I. We say that an online algorithm is β-***competitive*** if on any input, its output S satisfies $R(S) \le \beta\text{OPT}$.

At any time step, an algorithm's ***unwritten-element sequence*** is comprised of the contents of the buffer, concatenated with the remaining (not yet ingested) input elements. For the purpose of this definition, we assume that the elements in the buffer are stored in their arrival order (their order in the input sequence I).

Time step t is a ***decision point*** or ***decision time step*** for an algorithm A if $t = 0$ or if A finished writing a run at t. At a decision point, A needs to decide whether the next run will be increasing or decreasing.

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[4]Reading in the next element of the input when there is a free slot in the buffer never hurts the performance of any algorithm. However, we allow the algorithm in the proof of Lemma 16 to maintain free slots in the buffer to simplify the analysis.

2.2 Notation

We employ the following notation. We use $(x \nearrow y)$ to denote the increasing sequence $x, x+1, x+2, \ldots, y$ and $(x \searrow y)$ to denote the decreasing sequence $x, x-1, x-2 \ldots, y$. We use $\circ$ to denote concatenation: if $A = a_1, a_2, \ldots, a_k$ and $B = b_1, b_2, \ldots, b_\ell$ then $A \circ B = a_1, a_2, \ldots a_k, b_1, b_2, \ldots, b_\ell$.

Let $A = a_1, a_2, \ldots, a_k$. We use $A \oplus x$ to denote the sequence $a_1 + x, a_2 + x, \ldots a_k + x$. Similarly, we use $A \otimes x$ to denote the sequence $a_1 x, a_2 x, \ldots, a_k x$.

Let A, B be sequences. We say A **covers** B if for all $e \in B, e \in A$. A **subsequence** of a sequence $A = a_1, \ldots, a_k$ is a sequence $B = a_{n_1}, a_{n_2}, \ldots, a_{n_\ell}$ where $1 \leq n_1 < n_2 < \ldots < n_\ell \leq k$.

3 Structural Properties

In this section, we identify structural properties of the problem and the tools used in the analysis of our algorithms, which will be important in the rest of the paper.

3.1 Maximal Runs

We show that in run generation, it is never a good idea to end a run early, and never a good idea to "skip over" an element (keeping it in buffer instead of writing it out as part of the current run).

To begin, we show that adding elements to an input sequence never decreases the number of runs. Note that if S' is a subsequence of S, then $R(S') \leq R(S)$ by definition.

Lemma 1. *Consider two input streams I and I'. If I' is a subsequence of I, then* $\mathrm{OPT}(I') \leq \mathrm{OPT}(I)$.

Proof. Let A be an algorithm with input stream I and output S. Suppose that A produces the optimal number of runs on I, that is $R(S) = \mathrm{OPT}(I)$. Consider an algorithm A' on I'. Algorithm A' performs the same operations as A, but when it reaches an element that is not in I' (but is in I), it executes a no-op. These no-ops mean that the buffer of A' may not be completely full, since elements that A has in buffer do not exist in the buffer of A'. Let S' be the output of A'; S' is a subsequence of S.

Then $\mathrm{OPT}(I') \leq R(S') \leq R(S) = \mathrm{OPT}(I)$. $\qquad\square$

A **maximal increasing run** is a run generated using the following rules (a **maximal decreasing run** is defined similarly):
1. Start with the smallest element in the buffer and always write the smallest element that is larger than the last element written.
2. End the run only when no element in the buffer can continue the run, i.e., all elements in buffer are smaller than the last element written.

Lemma 2. *At any decision time step, a maximal increasing (decreasing) run r covers every other (non-maximal) increasing (decreasing) run r'.*

A **proper algorithm** is an algorithm that always writes maximal runs. We say an output is proper if it is generated by a proper algorithm. We show that there always exists an optimal proper algorithm.

Theorem 3. *For any input I, there exists a proper algorithm A with output S such that $R(S) = \mathrm{OPT}(I)$.*

Proof. We prove this by induction on the number of runs. If there is only one run, it must be maximal. Assume that all inputs I_t with $\mathrm{OPT}(I_t) = t$ have a maximal proper algorithm. Consider an input I_{t+1} with $\mathrm{OPT}(I_{t+1}) = t + 1$. Assume that an optimal algorithm on I_{t+1} is A_O, and it is not proper; we will construct a proper A with the same number of runs. The first run A writes is maximal and has the same direction of

the first run that A_O writes; the first run A_O writes may or may not be maximal. Then A is left with an unwritten-element sequence I_A and A_O is left with I_O. Note that $\text{OPT}(I_O) = t$ by definition.

By Lemma 2, I_O is a subsequence of I_A. Then by Lemma 1, $\text{OPT}(I_A) \leq \text{OPT}(I_O)$. Then by the inductive hypothesis, I_A has an optimal proper algorithm. Thus A is a proper algorithm generating the optimal number of runs. $\square$

In conclusion, we have established that it always makes sense for an algorithm to write maximal runs. Furthermore, we use the following property of proper algorithms throughout the rest of the paper.

Property 4. *Any proper algorithm satisfies the following two properties:*
1. *At each decision point, the elements of the buffer must have arrived while the previous run was being written.*
2. *A new element can not be included in the current run if the element just written out is larger (smaller) and the current run is increasing (respectively, decreasing).*

3.2 Analysis Toolbox

We now present observations and lemmas that play an integral role in analysing the algorithms presented in the rest of the paper.

Observation 5. *Consider algorithms A_1 and A_2 on input I. Suppose that at time step t_1 algorithm A_1 has written out all the elements that algorithm A_2 already wrote out by some previous time step t_2. Then, the unwritten-element sequence of algorithm A_1 at time step t_1 forms a subsequence of the unwritten-element sequence of algorithm A_2 at time step t_2.*

Lemma 6. *Consider a proper algorithm A. At some decision time step, A can write k runs $p_1 \circ \cdots \circ p_k$ or ℓ runs $q_1 \circ \cdots \circ q_\ell$ such that $|p_1 \circ \cdots \circ p_k| \geq |q_1 \circ \cdots \circ q_\ell|$. Then $p_1 \circ \cdots \circ p_k \circ p_{k+1}$, where p_{k+1} is either an up or down run, covers $q_1 \circ \cdots \circ q_\ell$.*

Therefore, the unwritten-element sequence after A writes p_{k+1} (if A writes $p_1 \circ \cdots \circ p_{k+1}$) is a subsequence of the unwritten-element sequence after A writes q_ℓ (if A writes $q_1 \circ \cdots \circ q_\ell$).

Proof. Since $|p_1 \circ \cdots \circ p_k| \geq |q_1 \circ \cdots \circ q_\ell|$, the set of elements that are in $q_1 \circ \cdots \circ q_\ell$ but not in $p_1 \circ \cdots \circ p_k$ have to be in the buffer when p_k ends. By 4, p_{k+1} will write all such elements. $\square$

The next theorem serves as a template for analyzing the algorithms in this paper. It helps us restrict our attention to comparing the output of our algorithm against that of the optimal in small *partitions*. We show that if in every partition i, an algorithm writes x_i runs that cover the first y_i runs of an optimal output (on the current unwritten-element sequence), and $x_i/y_i \leq \beta$, then the algorithm outputs no more than βOPT runs.

Theorem 7. *Let A be an algorithm with output S. Partition S into k contiguous subsequences $S_1, S_2 \ldots S_k$. Let x_i be the number of runs in S_i. For $1 < i \leq k$, let I_i be the unwritten-element sequence after A outputs S_{i-1}; let $I_1 = I$ and $I_{k+1} = \emptyset$. Let $\alpha, \beta \geq 1$. For each I_i, let S_i' be the output of an optimal algorithm on I_i.*

If for all $i \leq k$, S_i covers the first y_i runs of S_i', and $x_i/y_i \leq \beta$, then $R(S) \leq \beta\text{OPT}$. Similarly, if for all $i \leq k$, S_i covers the first y_i runs of S_i', and $\mathbb{E}[x_i]/y_i \leq \alpha$, then $\mathbb{E}[R(S)] \leq \alpha\text{OPT}$.

Proof. Consider I_i', the unwritten element sequence at the end of the first y runs of S_{i-1}' (we let $I_1' = I$). We show that $\text{OPT}(I_i) \leq \text{OPT} - \sum_{j=1}^{i-1} y_i$ for all $1 \leq i \leq k$ using induction. Note that $\text{OPT}(I_1) = \text{OPT}$ (the base case). Induction hypothesis: assume $\text{OPT}(I_i) \leq \text{OPT} - \sum_{j=1}^{i-1} y_i$. Since S_{i+1} covers the first y runs of S_{i+1}', by 5, I_{i+1} is a subsequence of I_{i+1}'. Then by Lemma 1, $\text{OPT}(I_{i+1}) \leq \text{OPT}(I_{i+1}')$. By definition, for $i > 1$,

$$\text{OPT}(I_{i+1}') = \text{OPT}(I_i) - y_i \leq \text{OPT} - \sum_{j=1}^{i} y_i \ .$$

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Therefore, $\text{OPT}(I_{i+1}) \le \text{OPT} - \sum_{j=1}^{i} y_i$. When $i = k$, we have $\text{OPT}(I_{k+1}) \le \text{OPT} - \sum_{j=1}^{k} y_i$. But since I_{k+1} contains no elements, $\text{OPT}(I_{k+1}) = 0$, and we have $\sum_{j=1}^{k} y_i \le \text{OPT}$. Since $R(S) = \sum_{j=1}^{k} x_i$, and $\sum_{i=1}^{k} x_i \le \beta \sum_{i=1}^{k} y_i$, we have the following:

$$R(S) = \frac{\sum_{i=1}^{k} x_i}{\text{OPT}} \cdot \text{OPT} \le \frac{\sum_{i=1}^{k} x_i}{\sum_{i=1}^{k} y_i} \cdot \text{OPT} \le \beta \text{OPT}.$$

We also have the same in expectation, that is,

$$\mathbb{E}[R(S)] = \mathbb{E}[\sum_{i=1}^{n} x_i] \le \alpha \sum_{i=1}^{n} y_i \le \alpha \cdot \text{OPT}.$$

$\square$

4 Up-Down Replacement Selection

We begin by analyzing the *alternating up-down replacement selection*, which deterministically alternates between writing (maximal) up and down runs. Knuth [25] showed that when the input elements arrive in a random order (all permutations of the input are equally likely), alternating-up-down replacement selection performs worse than standard replacement selection (all up runs). Specifically, he showed that the expected length of runs generated by up-down-replacement selection is $1.5M$ on random input, compared to the expected length of $2M$ of replacement selection.

In this section, we show that for deterministic online algorithms, alternating-up-down replacement selection is, in fact, asymptotically optimal for *any* input. It generates at most twice the optimal number of runs in the worst case. This is the best possible—no deterministic algorithm can have a better competitive ratio.

4.1 Alternating-Up-Down Replacement Selection is 2-competitive

We begin by giving a structural lemma, analyzing identical runs on two inputs in which one input is a subsequence of the other.

Lemma 8. *Consider two inputs I_1 and I_2, where I_2 is a subsequence of I_1. Let S_1 and S_2 be proper outputs of I_1 and I_2 such that:*
1. *S_1 and S_2 have initial runs r_1 and r_2 respectively,*
2. *r_1 and r_2 have the same direction*

Let the unwritten-element sequence after r_1 and r_2 be I_1' and I_2' respectively. Then I_2' is a subsequence of I_1'.

Proof. Assume that r_1 and r_2 are up runs (a similar analysis works for down runs). Let r_2' be a run that is a subsequence of r_1, consisting of all elements of r_1 that are also in I_2. Then r_2' can be produced by an algorithm A' that mirrors the algorithm A that generates r_1. When A reads or writes an element in I_2, A' reads or writes that element; when A reads or writes an element not in I_2, A' does nothing. Since r_2 is maximal, it covers r_2' by Lemma 2. $\square$

Theorem 9. *Alternating up-down replacement selection is 2-competitive.*

Proof. We show that we can apply Theorem 7 to this algorithm with $\beta = 2$.

In any partition that is not the last one of the output, the alternating algorithm writes a maximal up run r_u and then writes a maximal down run r_d. We must show that $r_u \circ r_d$ covers any run r_O written by a proper optimal algorithm on I_r, the unwritten element sequence at the beginning of the partition.

If r_O is an up run, then $r_O = r_u$ and thus is covered by $r_u \circ r_d$. If r_O is a down run, consider I', the unwritten-element sequence after r_u is written; I' is a subsequence of I_r. By Lemma 8 (with $I_1 = I_r$ and $I_2 = I'$), $r_u \circ r_d$ covers r_O.

In the last partition, the algorithm can write at most two runs while any optimal output must contain at least one run. Hence $x_i / y_i \leq 2$ in all partitions as required. □

4.2 Lower Bounds on Online Algorithms for Up-Down Run Generation

Now, we show that no deterministic online algorithm can hope to perform better than alternating-up-down replacement selection. Then, we partially answer the question of whether randomization helps overcome this impossibility result. Specifically, we show that no randomized algorithm can achieve a competitive ratio better than $3/2$. We provide the main ideas of the proofs here and defer the details to Appendix A.

Theorem 10. *Let A be any online deterministic algorithm with output S_I on input I. Then there are arbitrarily long I such that $R(S_I) \geq 2\text{OPT}(I)$.*

Proof Sketch. Given any M elements in the buffer, every time A commits to a run direction (up/down), the adversary sets the incoming elements such that they do not help the current run. Thus, A is forced to have runs of length at most M while OPT (since it has knowledge of the future) can do better. □

We also give a lower bound for randomized algorithms using similar ideas; however, in this case we do not have a matching upper bound. We use Yao's minimax principle to prove this bound. That is, we generate a randomized input and show that any deterministic algorithm cannot perform better than $3/2$ times OPT on that input against an oblivious adversary.

Theorem 11. *Let A be any online, randomized algorithm. Then there are arbitrarily long input sequences such that $\mathbb{E}[R(S_I)] \geq (3/2)\text{OPT}(I)$.*

5 Run Generation with Resource Augmentation

In this section, we use resource augmentation to circumvent the impossibility result on the performance of deterministic online algorithms. We consider two kinds of augmentation:
- *Extra Buffer:* The algorithm's buffer is actually a constant factor larger, that is, it can use its large buffer to read elements from the input, rearrange them, and write to the output.
- *Extra Visibility:* The algorithm's buffer is restricted to be of size M but it has prescience—the algorithm can *see* some elements in the immediate future (say, the next $3M$ elements), without the ability to write them early.

We present algorithms that, under the above conditions, achieve a competitive ratio better than 2 when compared against an optimal offline algorithm with a buffer of size M.

Resource augmentation is a common tool used in competitive analysis [9, 11–13, 38, 39]. It gives the online algorithm power to make better decisions and exclude worst case inputs, allowing us to compare the performance, more realistically, against an all-powerful offline optimal algorithm.

The results in this section require the elements of the input to be unique. Duplicate elements can nullify the extra ability to see or write future (non-repeated) elements which is provided by visibility and buffer-augmentation respectively. For example, consider the input,

$$I = (99, 101, \underbrace{100, \ldots, 100}_{cM-2 \text{ times}}, \ldots).$$

On input I, any algorithm with cM-size buffer or visibility is as powerless as the one without any augmentation.

Note that the assumption of distinct elements in run generation is not new. Knuth's analysis of the average run lengths [25] also requires uniqueness.

We begin by analyzing the **_greedy algorithm_** for run generation. Greedy is a proper algorithm which looks into the future at each decision point, determines the length of the next up and down run and writes the longer run.

Greedy is not an online algorithm. However, it is central to our resource augmentation results. The idea of resource augmentation, in part, is that the algorithm can use the extra buffer or visibility to determine, at each decision point, which direction (up or down) leads to the longer next run.

We next look at some guarantees on the length of a run chosen by greedy (or the *greedy run*) and also on the run that is not chosen by greedy (or the *non-greedy run*).

5.1 Greedy is Good but not Great

We first show that greedy is not optimal. The following example demonstrates that greedy can be a factor of $3/2$ away from optimal.

Example 12. *Consider the input $I = I_1 \circ (I_1 \oplus 10M) \circ (I_1 \oplus 20M) \circ \cdots \circ (I_1 \oplus 10cM)$, where*

$$I_1 = (4M + 4 \nearrow 5M + 3) \circ (M + 2) \circ (5M + 4 \nearrow 6M + 3)$$
$$\circ (2M + 1 \nearrow 3M - 1) \circ (4M + 3 \searrow 3M + 4) \circ (2M \searrow M + 3) \circ (M + 1 \searrow 1).$$

On input I above, writing down runs repeatedly produces $2c$ runs; two for each $I \oplus i10M$. On the other hand, the output of greedy is $S_1 \circ (S_1 \oplus 10M) \circ \cdots \circ (S_1 \oplus c10M)$, where $S_1 = (4M + 4 \nearrow 6M + 3) \circ (M + 2) \circ (2M + 1 \nearrow 3M - 1) \circ (3M + 4 \nearrow 4M + 3) \circ (2M \searrow M + 3) \circ (M + 1 \searrow 1)$ which contains $3c$ runs.

Next, we show that all the runs written by the greedy algorithm (except the last two) are guaranteed to have length at least $5M/4$. In contrast, up-down replacement selection can have have runs of length M in the worst case.

Theorem 13. *Each greedy run, except the last two runs, has length at least $M + \lceil \lfloor M/2 \rfloor / 2 \rceil$.*

We now bound how far into the future an algorithm must see to be able to determine which direction greedy would pick at a particular decision point. Intuitively, an algorithm should never have to choose between a very long up run and a very long down run. We formalize this idea about the non-greedy run not being too long in the following lemma.

Lemma 14. *Given an input I with no duplicate elements. Let the two possible initial increasing and decreasing runs be r_1 and r_2. Then $|r_1| < 3M$ or $|r_2| < 3M$.*

The next example shows that the above bound is tight.

Example 15. *Consider the input $I = I_1 \circ I_2 \circ I_3$, where*
$$I_1 = (1 \nearrow (M - 1)) \otimes M, \quad I_2 = (M^2 \searrow M^2 - M + 1)$$
$$I_3 = (M - 1 \searrow 1) \circ (M^2 + 2 \nearrow M^2 + M + 1).$$
Then,
$$r_1 = ((1 \nearrow (M - 1)) \otimes M) \circ (M^2 - M + 1 \nearrow M^2 + M + 1)$$
$$r_2 = (M^2 \searrow M^2 - M + 1) \circ ((M - 1 \searrow 1) \otimes M) \circ (M - 1 \searrow 1).$$
Thus, we have $|r_1| = 3M$ and $|r_2| = 3M - 1$.

The following lemma sheds some light on the choices made by an optimal algorithm with respect to that of greedy. It says, roughly, that if at any decision point, an optimal algorithm chooses to write the non-greedy run, and then writes the next run in the opposite direction, it performs no better than an optimal algorithm which chooses the greedy run in the first place.

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Lemma 16. *At any decision time step consider two possible next maximal runs r_1 and r_2. If $|r_1| \geq |r_2|$, then one of the following is the prefix of an optimal output on the unwritten-element sequence:*

1. *$r_1 \circ r_3$ where r_3 is a maximal run after r_1 and it can be either up or down.*
2. *$r_2 \circ r_4$ where r_4 is maximal run after r_2 with the same direction of r_2.*

5.2 Online Algorithms with Resource Augmentation

We now present several online algorithms which use resource augmentation (buffer or visibility) to determine an up-down replacement selection strategy, beating the competitive ratio of 2. For a concise summary of results, see Figure 1.

Matching OPT using $4M$-size Buffer. We present an algorithm with $4M$-size buffer that writes no more runs than an optimal algorithm with an M-size buffer. Later on, we prove that $(4M - 2)$-size is necessary even to be $3/2$-competitive; thus this augmentation result is optimal up to a constant.

Consider the following deterministic algorithm with a $4M$-size buffer. The algorithm reads elements until its buffer is full. It then uses the contents of its buffer to determine, for an algorithm with buffer size M, if the maximal up run or the maximal down run would be longer. If the maximal up run is longer, the algorithm uses its full buffer (of size $4M$) to write a maximal up run; otherwise it writes a maximal down run. The algorithm stops when there is no element left to write.

Theorem 17. *Let A be the algorithm with a $4M$-size buffer described above. On any input I, A never writes more runs than an optimal algorithm with buffer size M.*

Proof Sketch. At each decision point, A determines the direction that a greedy algorithm on the same unwritten element sequence, but with a buffer of size M, would have picked. It is able to do so using its $4M$-size buffer because, by Lemma 14, we know the length of the non-greedy run is bounded by $3M$. Note that it does not need to write any elements during this step. In each partition, A writes a maximal run r in the greedy direction and thus covers the greedy run by Lemma 2. Furthermore, r covers the non-greedy run as well since all of the elements of this run must already be in A's initial buffer and hence get written out. An optimal algorithm (with M-size buffer), on the unwritten-element-sequence, has to choose between the greedy and the non-greedy run. Since A covers both choices of the optimal in one run, by Theorem 7, it is able to match or beat OPT. □

A natural question is whether resource augmentation boosts performance automatically, without using the run-simulation technique. However, the following example shows that our 2-competitive algorithm, even when allowed to have $4M$-size buffer, may still be as bad when using M-size buffer.

Example 18. *Consider the input, $(8M \searrow 1) \circ (16M \searrow 8M + 1) \circ \cdots \circ (8cM \searrow 8(c-1)M + 1)$. The alternating algorithm from Section 4.1 which alternates maximal up and maximal down runs will write $2c$ runs given a $4M$-size buffer. In contrast, the optimal number of runs with an M-size buffer has c runs.*

$3/2$-competitive using $4M$-visibility. When we say that an algorithm has X-visibility ($X \geq M$) or $(X - M)$-lookahead, it means that the algorithm has knowledge of the next X elements of its unwritten element sequence, and can use this knowledge when deciding what to write.

However, only the usual M-size buffer is used for reading and writing. Furthermore, the algorithm must continue to read elements into its buffer sequentially from I, even if it sees elements further down the stream it would like to read or rearrange instead.

We present a deterministic algorithm which uses $4M$-visibility to achieve a competitive ratio of $3/2$. At each decision point, similar to the algorithm in Theorem 17, we can use $3M$-lookahead to determine the direction leading to the longer (greedy) run. However, unlike Theorem 17 we cannot use a large buffer to

Buffer size	Lookahead	Competitive ratio	Comments
M	-	2	Deterministic
$2M$	-	1.75	Randomized
M	$3M$	1.5	Deterministic
$4M$	-	1	Deterministic

Figure 1: Summary of online algorithms on run generation on any input

write future elements. Instead, we do the following—write a maximal greedy run, followed by two additional maximal runs in the same direction and opposite direction respectively.

We show that, at each decision point, the above algorithm is able to cover two runs of optimal (on the unwritten-element-sequence) using three runs. Lemma 16 and Lemma 6 are key in this analysis (see Appendix A for details). Thus, we have the following.

Theorem 19. *Let* OPT *be the optimal number of runs on input I given an M-size buffer, where I has no duplicate elements. Then there exists an online algorithm A with an M-size buffer and $4M$-visibility such that A always outputs S satisfying $R(S) \leq (3/2)$OPT.*

7/4-competitive using $2M$-size buffer. We have seen that it is possible to achieve a competitive ratio of $3/2$ using a standard M-size buffer as long as the algorithm is able to determine the direction leading to the longer (greedy) run (see Theorem 19). Now we only have a $2M$-size buffer. The algorithm will pick a direction randomly, and write a maximal run in that direction using its regular M buffer. It use the additional M-size buffer to simulate a run in the opposite direction (and thus figure out which one is longer).

With probability $1/2$, the algorithm is lucky and picks the greedy direction. In this case, we can cover the first two runs of optimal (on the unwritten-element sequence) with three runs as in Theorem 19. With probability $1/2$, the algorithm picks the wrong direction and we spend four (alternating) runs to cover two runs of optimal. Thus, in expectation we achieve a competitive ratio of $1/2(3/2) + 1/2(4/2) = 7/4$.

Theorem 20. *Let* OPT *be the optimal number of runs on input I given an M-size buffer, where I has no duplicate elements. Then there exists an online algorithm A with a $2M$-size buffer such that A always outputs S satisfying $\mathbb{E}[R(S)] \leq (7/4)$OPT and $R(S) \leq 2$OPT.*

5.3 Lower Bound for Resource Augmentation

We show that with less than $(4M - 2)$-augmentation, no deterministic online algorithm can be $3/2$-competitive on all inputs. Thus, an algorithm with $(4M - 2)$-size buffer cannot be optimal, so Theorem 17 is nearly tight. Similarly, Theorem 19 is nearly tight, since $4M - 2$-size buffer implies $4M - 2$-visibility.

Theorem 21. *With buffer size less than $(4M - 2)$, for any deterministic online algorithms A, there exists an input I such that if S is the output of A on I, then $R(S) \geq (3/2)$OPT.*

6 Offline Algorithms for Run Generation

We give offline algorithms for run generation. The offline problem is the following—given the entire input, compute (using a standard polynomial computation time algorithm) the optimal strategy which when executed by a run generation algorithm (with a buffer of size M) produces the minimum possible number of runs.

For any ε, we provide an offline polynomial time approximation algorithm that gives a $(1 + \varepsilon)$-approximation to the optimal solution. This is called a polynomial-time approximation scheme, or PTAS. The

running time of our first attempt is $O(2^{1/\varepsilon} N \log N)$. We then improve the running time to $O(\varphi^{1/\varepsilon} N \log N)$ where $\varphi = (1 + \sqrt{5})/2 \approx 1.618$ is the well-known golden ratio.

Simple PTAS. Our first attempt breaks the output into sequences with a small number of runs, and uses brute force to find which set of runs writes the most elements. We show that for any ε, we can achieve a $1 + \varepsilon$ approximation in polynomial time using this strategy.

Theorem 22. *There exists an offline algorithm A that always writes an S satisfying $R(S) \leq (1 + \varepsilon) \cdot \mathrm{OPT}$. The running time of A is $O(2^{1/\varepsilon} N \log N)$.*

Improved PTAS. We reduce the running time by bounding the number of choices we need to consider in a brute-force search. We do this using Lemma 16.

At each decision point, an algorithm chooses between starting an increasing run r_1 and a decreasing run r_2. If $|r_1| \geq |r_2|$, then by Lemma 16, we are able to discard r_2 followed by an increasing run.

Let F_d be the number of run sequences we need to consider if d runs remain to be written (for example, naïve PTAS has $F_d = 2^d$). First, the algorithm must handle all run sequences beginning with r_1; this is the same as an instance of F_{d-1}. Then the algorithm handles all run sequences beginning with r_2 followed by a decreasing run; this is an instance of F_{d-2}. Thus $F_d = F_{d-1} + F_{d-2}$; by examination, $F_1 = 1$ and $F_2 = 2$. This is the Fibonacci sequence, which gives us the $\varphi^{1/\varepsilon}$ factor in the running time.

Theorem 23. *There exists an offline algorithm A that writes S such that $R(S) \leq (1+\varepsilon) \cdot \mathrm{OPT}$. The running time of A is $O(\varphi^{1/\varepsilon} N \log N)$ where φ is the golden ratio $(1 + \sqrt{5})/2$.*

7 Run Generation on Nearly Sorted Input

This section presents results proving that up-down replacement selection performs better when the input has inherent sortedness (or "bounded-disorder" [34]). Replacement selection produces longer runs on nearly sorted data. In particular, if every input element is M away from its target position, then a single run is produced. Similarly, we give algorithms which perform well on inputs, where the optimal runs are also long.

In particular, we say that an input is *c-**nearly-sorted*** if there exists a proper optimal algorithm whose outputs consists of runs of length at least cM.

3/2-competitive using $2M$-size Buffer. We provide a randomized online algorithm that, on inputs which are 3-nearly-sorted, achieves a competitive ratio of $3/2$, while using an augmented-buffer of size $2M$.

A sketch of the algorithm follows. At each decision point, the algorithm picks a run direction at random. It starts a maximal run in that direction, but uses its extra M-buffer to simulate the run in the opposite direction. By Lemma 14, the algorithm can tell if it picked the same run as greedy (with M-buffer), similar to Theorem 20. If the algorithm got lucky and picked the greedy run, it repeats the process.

If the algorithm picked the non-greedy run, it uses some careful bookkeeping to write elements and simulate the run in the opposite direction. In doing so, the algorithm winds up at the same point in the input it would have reached, had it written the greedy run in the first place, but with an additional cost of one run.

Theorem 24. *There exists a randomized online algorithm A using M space in addition to its buffer such that, on any 3-nearly-sorted input I that has no duplicates, A is a $3/2$-approximation in expectation. Furthermore, A is at worst a 2-approximation regardless of its random choices.*

Exact Offline Algorithm on Nearly Sorted Input. We show that the greedy (offline) algorithm is a linear time optimal algorithm on inputs which are 5-nearly-sorted. We first prove the following lemma.

Lemma 25. *If a proper algorithm produces runs of length at least $5M$ on a given input with no duplicates, then it is optimal.*

Thus, we get our required linear time exact offline algorithm.

Theorem 26. *The greedy offline algorithm, i.e., picking the longer run at each decision point, is optimal on a 5-nearly-sorted input that contain no duplicates. The running time of the algorithm is $O(N)$.*

8 Additional Related Work

Replacement Selection. The classic algorithm for run generation is replacement selection [20]. While replacement selection considers only up runs, Knuth [25] analyzed alternating up-down replacement selection in 1963. He showed that for uniformly random input, alternating up-down replacement selection produces runs of expected length $3M/2$, compared to $2M$ of the standard replacement selection [18, 19, 26].

Recently, Martinez-Palau et al. [34] introduce *Two-way replacement selection* (2WRS), reviving the idea of up-down replacement selection. The 2WRS algorithm maintains two heaps in memory for up and down runs and heuristically decides in which heap each element must be placed. Their simulations show that 2WRS performs significantly better on inputs with mixed up-down, alternating up-down, and random sequences.

Replacement selection with a fixed-sized *reservoir* appears in [17, 40]. Larson [28] introduced batched replacement selection, a cache-conscious replacement selection which works for variable-length records. Koltsidas, Müller, and Viglas [27] study replacement selection for sorting hierarchical data.

Improvements for the merge phase of external sorting have been considered in [10, 15, 37, 43, 44], but this is beyond the scope of this paper.

Reordering Buffer Management. Run generation problem is reminiscent of the *buffer reordering problem* (also known as the *sorting buffer problem*), introduced by Räcke et al. [36]. It consists of a sequence of n elements that arrive over time, each having a certain color. A buffer, that can store up to k elements, is used to rearrange them. When the buffer becomes full, an element must be output. A cost is incurred every time an element is output that has a color different from the previous element in the output sequence. The goal is to design a scheduling strategy for the order in which elements must be output, so as to minimize the total number of color changes. The buffer reordering problem models a number of important problems in manufacturing processes and network routing and has been extensively studied, both in the online and offline case [3–7, 9, 12, 23]. The offline version of the buffer reordering problem is NP hard [9], while the complexity of our problem remains unresolved.

Patience Sort and Longest Increasing Subsequence. An old sorting technique used to sort decks of playing cards, *Patience Sort* [33] has two phases—the creation of sorted *piles* or runs, and the merging of these runs. The elements arrive one at a time and each one can be added to an existing run or starts a new run of its own. Unlike this paper, a legal run only consists of elements decreasing in value, and patience sort can form any number of parallel runs. The goal is to minimize the number of runs. The greedy strategy of placing an element to the left-most legal run is optimal. Moreover, the minimum number of such runs is the length of the longest increasing subsequence of the input [2]. Patience sort has been studied in the streaming model [21].

Similar to Replacement Selection, Patience Sort is able to leverage partially sorted input data. Chandramouli and Goldstein [10] present improvements to patience sort, and combine it with replacement selection to achieve practical speed up.

Adaptive Sorting Algorithms. Python's inbuilt sorting algorithm, *Timsort* [41] works by finding contiguous runs of increasing or decreasing value during the run generation phase. External memory sorting for well-ordered or "partially sorted" data has been studied by Liu et al. [32]. They minimize the I/O cost of run generation phase by finding "naturally occurring runs". See [16] for a survey on adaptive sorting algorithms.

9 Conclusion and Open Problems

In this paper, we present an in-depth analysis of algorithms for run generation. We establish that considering both up and down runs can substantially reduce the number of runs in an external sort. The notion of up-down replacement selection has received relatively little attention since Knuth's negative result [25], until its promise was acknowledged by the experimental work of Martinez-Palau et al. [34].

The results in our paper complement the findings of Knuth [25] and Martinez-Palau et al. [34]. In particular, strict up-down alternation being the best possible strategy explains why heuristics for up-down run-generation can lead to better performance in some cases. Moreover, our constant-factor competitive ratios with resource augmentation and lookahead may guide followup heuristics and practical speed-ups.

We conclude with open problems.

Can randomization help circumvent the lower bound of 2 on the competitive ratio of online algorithms (without resource augmentation)? We know that no randomized online algorithm can have a competitive ratio better than $3/2$, but there is still a gap. What is the performance of the greedy offline algorithm compared to optimal? We show that greedy can as bad as $3/2$ times optimal. Is there a matching upper bound? Can we design a polynomial, exact, algorithm for the offline run-generation problem? We find it intriguing that our attempts at an exact dynamic program requires maintaining too many buffer states to run in polynomial time.

10 Acknowledgments

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A   Appendix: Omitted proofs

Proof of Lemma 2. Without loss of generality, assume r and r' are increasing runs. Consider any time step t when elements from both r and r' are being written. Let B_t and B'_t be the buffer of r and r' at time step t respectively; let C_t be the set of elements in B_t that are eventually written to r, and C'_t be the set of elements in B'_t that are eventually written to r'. We prove inductively that $C'_t \subseteq C_t$. This implies that r covers r'. The base case is true as r and r' start with the same buffer. Since we have $C'_t \subseteq C_t$, we must show that (a) the element z written to r is not in C'_{t+1} and (b) the element e read into C'_{t+1} must also be in C_{t+1}.

Consider the elements z' and z written by r' and r, respectively, at time $t+1$. We must have $z' \geq z$; thus either $z = z'$, or z is never written to r' (either way it is not in C'_{t+1}).

Since e is in C'_{t+1}, it is eventually written by r'; thus $e \geq z'$. Thus $e \geq z$, but that means e is eventually written by r. Since e was just read it is in B_{t+1}; thus $e \in C_{t+1}$. □

Observation A. *If A has just written an element e, and is writing a down (up) run, then A cannot write any element larger (smaller) than e in the same run. Similarly, if A has just written e, then A cannot write both an element larger than e and an element smaller than e in the same run.*

Proof of Theorem 10. Let I_1, the first M elements of the input, be $I_1 = 1, 2, \ldots, M$. We divide the rest of the input into segments of size M. Let the $(t+1)$st such segment be I_{t+1}. Then,

$$I_{t+1} = (1 + tM \nearrow M + tM) \text{ or } (-(1 + tM) \searrow -(M + tM)) .$$

Call this a positive segment and a negative segment respectively. At time $M(t-1)+1$ we decide whether I_{t+1} is a positive or a negative segment based on A.

Specifically, we choose I_{t+1} using either the direction of the run A is writing, or the value of the most recent element written. If A is writing a down run, I_{t+1} is a positive segment; if A is writing an up run, I_{t+1} is a negative segment. It may be that A has only written one element of a run (so A could turn this into either an up run or a down run). If this element was the smallest element in the buffer of A, I_{t+1} is a negative segment. Otherwise, I_{t+1} is a positive segment.

First we show that A must write at least one new run for each I_t; thus $R(S) \geq t$. At least one run is required for A to write I_1, so for the remainder of the proof we assume $t > 1$. Consider time $M(t-2)+1$, when I_t begins. We assume that I_t is a positive segment—a mirroring argument works when I_t is a negative segment. Furthermore, note that the elements of I_t are the largest in the instance so far.

There are two cases: A is currently writing a down run, or the initial element of a new run.

Case 1: Algorithm A is currently writing a down run. Then the elements of I_t must be larger than any element in A's down run. Thus A must use another run to write the elements of I_t by Observation A.

Case 2: Algorithm A is writing the initial element of a new run. By construction, the element written is not the smallest element in A's buffer, but is smaller than all elements in I_t. Then A must spend one run to write the smallest element in its buffer, and another to write I_t. Thus, I_t causes A to write a run in addition to its current run by Observation A.

On the other hand, an offline algorithm can write I_{2i} and I_{2i+1} in one run. Assume that I_{2i} is a positive segment—a mirroring argument works when I_{2i} is a negative segment. If I_{2i+1} is positive, both can be written using an up run. If I_{2i+1} is negative, both can be written using a down run. Thus OPT is no more than $\lceil t/2 \rceil$. □

Proof of Theorem 11. Our lower bound uses the same basic principles as Theorem 10. We first show the lower bound with some repeated elements, then show how to perturb the elements to avoid repetitions. We generate a randomized input and show that any deterministic algorithm cannot perform better than $3/2$ times OPT on that input. The theorem is then proven by Yao's minimax principle. Yao's minimax principle states

that the best expected performance of a randomized algorithm is at least as large as the expected performance of the best *deterministic* algorithm over a (known) distribution of inputs. (See, e.g., [35] for details.)

As in Theorem 10, we divide the input into segments of size $4M$. Call these segments I_t for $t = 1, 2, \ldots, \lfloor N/4M \rfloor$. Note that this input is randomized: for each t, we pick one of two inputs, each with probability $1/2$. We choose either

$$
\begin{array}{ll}
I_t^1 = (1 \nearrow M) & I_t^1 = (1 \nearrow M) \\
I_t^2 = (2M \searrow M+1) & I_t^2 = (-2M \nearrow -M-1) \\
I_t^3 = (3M \searrow 2M+1) \quad \text{or} \quad & I_t^3 = (-3M \nearrow -2M-1) \\
I_t^4 = (4M \searrow 3M+1) & I_t^4 = (-4M \nearrow -3M-1). \\
\text{(a positive segment)} & \text{(a negative segment)}
\end{array}
$$

Let $I_t = I_t^1 \circ I_t^2 \circ I_t^3 \circ I_t^4$. A positive or negative segment is chosen randomly for each I_t with probability $1/2$.

The optimal algorithm spends no more than one run per I_t, using an up run for a positive segment or a down run for a negative segment.

We show that any deterministic algorithm requires at least one new run to write I_{t-1}^4 and I_t^1 for $t > 1$. Further analysis shows that with probability $1/2$, any deterministic algorithm requires at least one run to write the remainder of I_t^1, I_t^2, and I_t^3. Note that one run is also required to write I_1^1; summing, this gives a total expected cost of $(3/2)\text{OPT}$.

Consider a segment I_t; $t > 1$. Once all of I_{t-1}^4 has been read into its buffer, at least one element of I_{t-1}^3 has been written. Once all of I_t^1 has been read into its buffer, at least one element of I_{t-1}^4 has been written. Finally, once all of I_t^2 has been read into its buffer, at least one element x of I_t^1 has been written. Applying Observation A, at least one new run is required to write these three elements.

Now we show that with probability $1/2$, an additional run is required to write I_t^2. Let x be the first element written by I_t^1 (thus, the cost of writing x itself was handled in the above case—we show when an additional run is required). Note that the algorithm must choose an x before it sees any element of I_t^2 (so it does now know if I_t is positive or negative).

Let $x \neq 1$ and I_t be a positive segment. By Observation A, an additional run is required to write both 1 and any element of I_t^2. If 1 is not written, all of I_t^2 cannot be stored in the buffer—but then, I_t^2 and I_t^3 cannot be written using one run. Similarly, let $x = 1$ and I_t be a negative segment. By Observation A, an additional run is required to write both M and any element of I_t^2; otherwise I_t^2 and I_t^3 require an additional run to be written.

Thus any deterministic algorithm cannot perform better than a $3/2$-approximation. Applying Yao's minimax principle proves the theorem.

Now we perturb the input to avoid duplicate elements. We multiply each element by $\lfloor N/4M \rfloor$, and add t to each element of I_t. In other words, we use a new segment

$$
I_t' = (I_t \otimes \lfloor N/4M \rfloor) \oplus t.
$$

Our arguments above only depended on the relative ordering of the elements, which is preserved by this perturbation. For example, assume I_t and I_{t-1} are both positive segments. Then all elements of I_t^1 are less than all elements of I_{t-1}^4, and all elements of I_t^2 are greater than all elements of I_t^1. $\qquad \square$

Proof of Theorem 13. We will build S constructively. At each time t where greedy chooses an up or down run, we show that one of its choices leads to a run of length at least $5M/4$. Since the greedy algorithm always picks the longer run at each decision time step, the run with length less than $5M/4$ can never be part of its output.

Consider any time step t where $t < N - 5M/4$. If t is larger than this value, the final run will have length $n - t$. Let the contents of buffer at time t be $\sigma_1, \ldots, \sigma_M$. Let I' be the sequence of $\lfloor M/2 \rfloor$ elements of I arriving after t.

Consider the run starting at σ_M and continuing downwards; call this down run r_1. Let r_2 be the up run starting at σ_1 and continuing upwards. Any element of I' less than $\sigma_{\lceil M/2 \rceil}$ will be (eventually) written out to r_1; any that are greater than $\sigma_{\lfloor M/2 \rfloor}$ will be written out to r_2. Every number must fall into one of these categories, so there must be at least $\lceil \lfloor M/2 \rfloor /2 \rceil$ numbers added to the larger run. Each run at least includes the elements already in the buffer, so the larger run has length at least $M + \lceil \lfloor M/2 \rfloor /2 \rceil$.

The last $5M/4$ elements can be handled by greedy in at most 2 runs (since each trivially has length at least M). Thus the above applies to all but the last two runs of greedy. $\square$

Proof of Lemma 14. Let S_1 be an output that writes r_1 initially and S_2 be an output that writes r_2 initially. Without loss of generality, suppose that r_1 is increasing and r_2 is decreasing. Let $r_1 = r_1(1), r_1(2), \ldots, r_1(k)$ and $r_2 = r_2(1), r_2(2), \ldots, r_2(\ell)$. The idea of the proof is to split these runs into two phases (a) elements of r_1 are smaller than the corresponding elements of r_2 and (b) when the elements of r_1 are greater than or equal to those of r_2. During each of these two phases, we use the fact that incoming elements written by S_1 have to be in the buffer of S_2 (and vice versa) to bound their length. We assume that both runs write exactly one element for each element they read in; this cannot affect the length of the runs.

Let B_0 be the original buffer, i.e., the first M elements of I. Let i be the transition point between the two phases mentioned above; in other words, $r_1(i+1) \geq r_2(i+1)$ but $r_1(i) < r_2(i)$.

Divide r_1 into s_1 and t_1, where s_1 is the first i elements of r_1, and t_1 is the remainder of r_1. We further divide s_1 into s_1^B, the elements of s_1 that are in B_0, and s_1^N, the elements of s_1 that are not in B_0. Let t_1^B be the elements of t_1 that are in B_0. Let f_1 be the set of elements in r_1 that are read in after x_{i+1} is written. Let u_1 be the set of elements not in r_1 that are read in before x_{i+1} is written. We define the corresponding sets for r_2 as well: $s_2^B, s_2^N, t_2^B, f_2, r_2$, and u_2.

We can bound the size of several of these sets by M. Note that s_1 cannot have more than M elements, since all must be stored in the buffer while s_2 is being written. Thus $|s_1^N| + |s_1^B| \leq M$. Similarly, $|s_2^N| + |s_2^B| \leq M$. We must also have $|u_1| \leq M$ and $|u_2| \leq M$. Finally, consider $s_2^N \cup t_1^B$. Any element in s_2^N must be read before time step i. Since s_1 is disjoint from s_2 (by definition of i), all elements of s_2^N must be in S_1's buffer at time step i. All elements of t_1^B must also be in S_1's buffer at time step i, so $|s_2^N| + |t_1^B| \leq M$.

Starting from time step $i+1$, any new element e that is read in cannot be in both r_1 and r_2. This means that all elements of f_1 must be in the buffer of S_2 until r_2 ends, and all elements of f_2 must be in the buffer of S_1 until r_1 ends. On the other hand, all elements of u_2 must eventually be a part of r_1, and similarly for u_1 and r_2.

To begin, we show a weaker version of the lemma for runs of length $4M$. We have $|r_1| \leq (|s_1^N| + |s_1^B|) + (|u_2|) + (|s_2^N| + |t_1^B|) + |f_1| \leq 3M + |f_1|$. Then if $|r_1| \geq 4M$, then $|f_1| \geq M$. Since all elements of f_1 must be stored in the buffer of S_2 until r_2 ends, r_2 must end when the Mth element of f_1 is read in. Then we must have $|r_2| < |r_1|$.

We have $|f_1| \geq M - |u_2|$; otherwise, $|r_1| \leq (|s_1^N| + |s_1^B|) + (|s_2^N| + |t_1^B|) + (|u_2| + |f_1|) < 3M$. Consider the first $M - |u_2| - 1$ elements read in after i that are eventually written to r_1 (this is a prefix of f_1), call them f_1'. Since $|f_1| \geq M - |u_2|$, there must be another element $e \in f_1$ that is read after all elements of f_1'. Note $e \in r_1$. Let t be the time when e arrives.

At t, the buffer of S_2 must contain all elements of u_2, as well as all elements of f_1' and e. The buffer of S_2 is then full of elements that cannot be written in r_2. Hence, S_2 is forced to start a new run at time $t < |r_1|$, so $|r_2| < |r_1|$. Then we must have $|f_2| + |u_1| < M$, none of the elements in these sets are in r_1, and must be stored in S_1's buffer until r_1 ends (which is after r_2 ends). Finally, we have,

$$|r_2| \leq (|s_2^N| + |s_2^B|) + (|s_1^N| + |t_2^B|) + (|u_1| + |f_2|) < 3M \ ,$$

as required.

$\square$

Proof of Theorem 17. Algorithm A will simulate the maximal up run, r_1, and maximal down run, r_2, to see which is longer, but it does not actually need to write any elements during this simulation. By Lemma 14, if we find that one run has length at least $3M$, it must be the longer run.

We now describe exactly how to simulate a run r using $4M$ space without writing any elements. Algorithm A simulates the run one step at a time. We describe the actions and buffer of an algorithm with M-size buffer writing r as the ***simulated algorithm***. Without loss of generality, assume r is an up run.

Assume that all elements are stored in the buffer in the order they arrive. Thus, after t elements have been written, the first $M + t$ elements of the buffer are exactly the elements the simulated algorithm has read from the input up to time t. Of these elements, M must be in the buffer of the simulated algorithm, while the other t will have been written to r; however, A does not explicitly keep track of which elements are in the buffer.

The algorithm A keeps track of ℓ, the last element written to r, because at each t, all of the first $M + t$ elements larger than ℓ are: (a) in the simulated algorithm's buffer at time t and (b) will be written to r at a later point. Thus, once no item in the first $M + t$ elements is larger than ℓ, r must end. At each time step, the smallest element larger than ℓ is written to r.

Specifically, at time step t, A finds the smallest element e in the first $M + t$ elements of the buffer that is larger than ℓ. This is the next element of r. Thus in the next time step, A updates $\ell \leftarrow e$, and repeats. If no such e can be found, no element in $M + t$ (and thus no element in the buffer) can continue the run, so the run ends at time t.

The last time A can update ℓ is when the simulated algorithm has seen all elements in the buffer; in other words, $t = 3M$. By Lemma 14, this is sufficient to determine which run is longer.

The algorithm now knows which run is longer; without loss of generality, assume $|r_1| \geq |r_2|$. Then the algorithm writes a maximal run r using its $4M$-size buffer in the direction of r_1. Run r is guaranteed to contain all elements of r_1 by Lemma 2. Since r_2 has length less than $3M$ by Lemma 14, all of its elements must already be in the $4M$-size buffer. Thus they are written during r because a maximal run always writes its buffer contents. The first initial run of a proper optimal algorithm on the unwritten-element sequence has to be either r_1 or r_2. Since r covers both r_1 and r_2, by Theorem 7 with $\beta = 1$, A never writes more runs than an optimal algorithm with an M-size buffer. $\qquad\square$

Lemma A. *Consider two algorithms A_1 and A_2 that have the same remaining input I when they both start writing a new maximal run, called r_1 and r_2. Let their buffers at this point be B_1 and B_2 (that may not be full) respectively, and assume $\max(B_1 \setminus B_2) \leq \min(B_2 \setminus B_1)$. If r_1 and r_2 are increasing then all elements in I written to r_2 are also written to r_1. Similarly, if r_1 and r_2 are decreasing, all elements in I written to r_1 are also written to r_2.*

Proof. It suffices to prove the first case where r_1 and r_2 are both increasing as the other case can be proven similarly. After $r_1(t)$ and $r_2(t)$ were written, let $C_1(t)$ and $C_2(t)$ be the set of elements in the buffers of A_1 and A_2 that will be written in r_1 and r_2 at some point in the future, i.e., the set of elements that are at least as large as $r_1(t)$ or $r_2(t)$ respectively.

It is easy to prove by induction that the invariant

$$\max(C_1(t) \setminus C_2(t)) \leq \min(C_2(t) \setminus C_1(t))$$

always holds. We note that this invariant implies $r_1(t + 1) \leq r_2(t + 1)$. Therefore, if this invariant is true for all $t < \min\{|r_1|, |r_2|\}$, any incoming element $e \in I$ satisfies $e \in r_2 \Rightarrow e \in r_1$ as required. We now prove the invariant:

The base case is true since $C_1(0) = B_1, C_2(0) = B_2$. Suppose the invariant holds for t, then $r_1(t + 1) \leq r_2(t + 1)$ and a new element e is read in.

Case 1: if $e \in C_2(t + 1) \Rightarrow e \geq r_2(t + 1) \geq r_1(t + 1) \Rightarrow e \in C_1(t + 1)$.

Case 2: if $e \in C_1(t + 1)$ and $e \notin C_2(t + 1)$, then $e < r_2(t) = \min(C_2(t)) \leq \min(C_2(t + 1))$.

Hence, the invariant holds for $t + 1$.

$\square$

Observation B. *On an input I, let $r_1 \circ \ldots \circ r_k$ be the first k runs of an optimal output. If $r'_1 \circ \ldots \circ r'_k$ be k runs that cover $r_1 \circ \ldots \circ r_k$. Then, $r'_1 \circ \ldots \circ r'_k$ are also the first k runs of an optimal output.*

Proof of Lemma 16. Without loss of generality, assume r_1 and r_2 are initial maximal increasing and decreasing runs respectively and $|r_1| \geq |r_2|$. Suppose $r_2 \circ r_3$, where r_3 is an increasing, is prefix of an optimal output $S_{\text{OPT}}(I)$.

Consider the case $|r_1| = |r_2|$. Let their buffers at the end of these two runs be B_1, B_2 and let j be the smallest index such that $r_1(j + 1) > r_2(j + 1)$. Consider any new element $e \in I$ that is read in before $r_1(j + 1)$ is written. Obviously, we have:

$$e \in B_1 \Rightarrow e \leq r_1(j),$$
$$e \in B_2 \Rightarrow e \geq r_2(j).$$

Consider any new element e which is read in after $r_1(j + 1), r_2(j + 1)$ were written. It is easy to see the followings:

$$e \in B_1 \setminus B_2 \Rightarrow e \leq r_1(j + 1),$$
$$e \in B_2 \setminus B_1 \Rightarrow e \geq r_2(j + 1).$$

Therefore, $\max(B_1 \setminus B_2) \leq \max(r_1(j), r_2(j + 1)) \leq \min(r_2(j), r_1(j + 1)) \leq \min(B_2 \setminus B_1)$. The situation can be visualized in Figure 2 as follows. If an incoming element cannot be written in the current run, it lies below or above (depending on whether the run is increasing or decreasing) the last element written. The regions are marked with their associated sets described above.
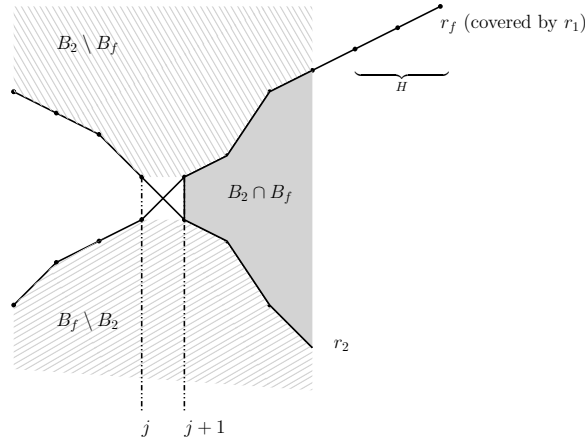

Figure 2: Visualizing the buffer states.

Consider $r_1 \circ r_4$, where r_4 is a maximal increasing run. Every elements in r_2 will be written in either r_1 or r_4 by Lemma 6. If $e \in r_3$, then we consider the cases where $e \in B_2$ or $e \in r_3 \setminus B_2$. If $e \in B_2$, then e is either in r_1 or in B_1 which means e is either in r_1 or r_4. If $e \in r_3 \setminus B_2$, $e \in r_4$ by Lemma A using the fact that $\max(B_1 \setminus B_2) \leq \min(B_2 \setminus B_1)$. Thus, $r_1 \circ r_4$ covers $r_2 \circ r_3$. As a result, r_1 is also a prefix of an optimal output $S_{\text{OPT}}(I)'$ by Observation B.

If $|r_1| > |r_2|$ and $|r_2| = k$. Then the simplest argument goes as follows. Instead of arguing based on r_1 directly, we consider r_f that is increasing but may not be maximal. Consider an algorithm A_f that writes $r_f(1) = r_1(1), \ldots, r_f(k) = r_1(k)$. Then, without reading any new element in, it finishes its first

run r_f by writing out all elements in its buffer that are larger than $r_1(k)$ (the set of these elements is H in Figure 2). After this extra step, let the buffer be B_f. Use the exact same argument as above, we have that $\max(B_f \setminus B_2) \leq \max(r_f(j), r_2(j+1)) \leq \min(r_2(j), r_f(j+1)) \leq \min(B_2 \setminus B_f)$. Using the same argument as in the first case, we have that r_f followed by a maximal increasing run will cover r_2 followed by a maximal increasing run. Hence, r_f is a prefix of an optimal output. Since r_1 covers r_f, it is also a prefix of an optimal output by Observation B . $\qquad\square$

Proof of Theorem 19. In any partition that is not the last one, let I_r be the unwritten-element sequence and let r_1, r_2 be the two possible maximal initial runs where r_1 is increasing and r_2 is decreasing. Without loss of generality, suppose $|r_1| \geq |r_2|$. We use the simulation technique of Theorem 17 to determine which run is longer.

The algorithm writes $r_1 \circ r_3 \circ r_4$ where r_3 and r_4 are maximal runs that have the same and opposite directions as r_1 respectively. The algorithm stops when there is no element left to write. We break our analysis into cases based on what runs are in an optimal output. In each case, we show that Theorem 7 proves a competitive ratio of $\beta = 3/2$.

If r_2 is a prefix of a proper optimal output $S_{\mathrm{OPT}}(I_r)$, let r_5 be the maximal run after r_2 in $S_{\mathrm{OPT}}(I_r)$. After writing r_3, the algorithm already writes out all elements in r_2 by Lemma 6. Let the unwritten-element sequence after writing r_3 be I_3 and let the unwritten-element sequence after writing r_2 be I_2. By Lemma 6, I_3 is a subsequence of I_2. According to Lemma 16, r_5 has to be decreasing in order to possibly have fewer runs than writing r_1 initially. Hence, applying Lemma 8 to r_4 and r_5, we know that at the end of r_4, the algorithm has written all elements of r_2 and r_5. Thus, $r_1 \circ r_3 \circ r_4$ covers $r_2 \circ r_5$.

If $r_1 \circ r_3$ is a prefix of $S_{\mathrm{OPT}}(I_R)$, then we are done as $r_1 \circ r_3 \circ r_4$ trivially covers $r_1 \circ r_3$.

If r_1 is a prefix of $S_{\mathrm{OPT}}(I_R)$ but $r_1 \circ r_3$ is not a prefix of $S_{\mathrm{OPT}}(I_R)$. Then, let r_3' be the opposite maximal run to r_3, i.e., r_1, r_3' are the first two runs of $S_{\mathrm{OPT}}(I_R)$. We have I_3 is a subsequence of I_1. Hence, applying Lemma 8 to r_3' on input I_1 and r_4 on input I_3, we have that at the end of r_4, the algorithm has written out all elements in $r_1 \circ r_3'$. Thus, $r_1 \circ r_3 \circ r_4$ covers $r_1 \circ r_3'$.

In the last partition, since A outputs at most 3 runs, it can only achieve a ratio worse than $3/2$ if the optimal algorithm wrote out a single run. But then that run is longer, and A would choose it. Therefore, we have $R(S) \leq (3/2) \cdot \mathrm{OPT}$. $\qquad\square$

Proof of Theorem 20. In each partition, let the unwritten-element sequence be I_r and the optimal proper output of I_r be $S_{\mathrm{OPT}}(I_r)$. The algorithm randomly picks the direction of the next run and writes a maximal runs in that direction using M-size buffer. It uses the extra M buffer slots to simulate the buffer state of the maximal run in the other direction to check if the run it chose is at least as long as the other run. If the algorithm picked the run that is at least as long as the other run, it then writes a maximal run in the same direction followed by another maximal run in the opposite direction. The algorithm stops when there is no more element to write. In the proof of Theorem 19, we showed that these three runs will cover the first two runs of $S_{\mathrm{OPT}}(I_r)$.

If the algorithm picked the shorter run, then it writes three more maximal runs with alternating directions. We know that the first two runs with alternating directions cover the first run of $S_{\mathrm{OPT}}(I_r)$ as argued in the proof of Theorem 9; hence, the next two runs with alternating directions cover the second run of $S_{\mathrm{OPT}}(I_r)$.

In the last partition, if $\mathrm{OPT}(I_r) = 2$, the analysis is the same. If $\mathrm{OPT}(I_r) = 1$, then the optimal output must be the longer maximal run. The algorithm, if picked the shorter run, then will cover the longer run when it writes the next maximal run in the opposite direction as showed in the proof of Theorem 9. Therefore, we have $\mathbb{E}[x_i]/y_i \leq 1/2.(4/2) + 1/2.(3/2) = 7/4$.

Applying Theorem 7 with $\alpha = 1.75$ and $\beta = 2$, we have: $\mathbb{E}[R(S)] \leq (7/4)\mathrm{OPT}$ and $R(S) = 2\mathrm{OPT}$. $\qquad\square$

Proof of Theorem 21. Suppose an algorithm has $(4M - 3)$-size buffer. Consider the input $I_1 \circ e \circ I_2$ where $I_1 = (1 \nearrow M - 1) \circ (2M - 1 \searrow M) \circ (3M \nearrow 4M - 2) \circ (-M \searrow -2M + 2)$.

If S first writes $-2M + 2$, then let $e = -2M + 1$.

- Case 1: if S writes $e = -2M + 1$ next, then let $I_2 = (0 \searrow -(M - 1))$. Thus, S has to spend at least two runs while an optimal output is one run: $(4M - 2 \searrow 3M) \circ (2M - 1 \searrow -2M + 1)$.
- Case 2: if S writes $-2M + 3$ next, let $I_2 = (-2M \searrow -10M) \circ (2M \nearrow 3M - 1)$. Then S has to spend at least 3 runs while an optimal output has 2 runs: $(4M - 2 \searrow 3M) \circ (2M - 2 \searrow -10M) \circ (2M \nearrow 3M - 1)$.

Similarly, if S first writes $4M - 2$, then let $e = 4M - 1$.

- Case 1: if S writes $e = 4M - 1$ next, then let $I_2 = (2M \nearrow 3M - 1)$.
- Case 2: if S writes $4M - 3$ next, let $I_2 = (4M \nearrow 10M) \circ (0 \searrow -M + 1)$.

If S first writes $e' \notin \{-2M + 2, 4M - 2\}$, then let $e = -2M + 1, I_2 = (0 \searrow -(M - 1))$. Thus, S has to spend at least two runs while an optimal output has the following output with one run: $(4M - 2 \searrow 3M) \circ (2M - 1 \searrow -2M + 1)$. $\qquad \square$

Proof of Theorem 22. We apply Theorem 7 with $x = (\lceil 1/\varepsilon \rceil + 1)$ and $y = \lceil 1/\varepsilon \rceil$. In any partition except the last one, the algorithm chooses the combination of $\lceil 1/\varepsilon \rceil$ maximal runs $r_1 \circ \cdots \circ r_{\lceil 1/\varepsilon \rceil}$ whose output is longest (ties are broken arbitrarily) and writes out one extra run $r_{(\lceil 1/\varepsilon \rceil + 1)}$. By Lemma 6, $r_1 \circ \cdots \circ r_{\lceil 1/\varepsilon \rceil + 1}$ covers the first $\lceil 1/\varepsilon \rceil$ runs of a proper optimal output of the unwritten-element sequence I_r in $(1 + \lceil 1/\varepsilon \rceil)$ runs. In the last partition, the algorithm chooses a combination of runs with the smallest number of runs.

Therefore, we obtain an $\beta = 1 + 1/\lceil 1/\varepsilon \rceil \leq 1 + \varepsilon$ approximation.

There are $2^{\lceil 1/\varepsilon \rceil + 1}$ combinations to consider (each run can be up or down). The length of a run can be calculated in $O(N_i)$ time by simulating it directly. where N_i is the length of the longest output, namely, $|r_1 \circ \cdots \circ r_{\lceil 1/\varepsilon \rceil + 1}|$, Since N_i items are then written out, the total running time is $O(\sum_{i=1}^{t} N_i 2^{\lceil 1/\varepsilon \rceil}) = O(N 2^{1/\varepsilon})$. Searching for the shortest way to write out the remaining elements (once $I_r = \emptyset$) takes $O(N 2^{1/\varepsilon})$ time, which does not affect the running time. $\qquad \square$

Proof of Theorem 23. In each partition, we restrict the search for the combination of $(1/\lceil \varepsilon \rceil)$ consecutive runs that writes the longest sequence as described above. By Lemma 16, if d runs remain to be written out, we must examine one subcase with $d - 1$ runs remaining, and one with $d - 2$ runs remaining. Thus, the number of combinations we need to consider is $F_d = F_{d-1} + F_{d-2}$. Therefore, the running time of this step is $O(F_{\lceil 1/\varepsilon \rceil} N_i \log N_i)$. Thus, we have

$$O(F_{\lceil 1/\varepsilon \rceil} N_i \log N_i) = O(\varphi^{\lceil 1/\varepsilon \rceil} N_i \log N_i).$$

This is because $F_{\lceil 1/\varepsilon \rceil} = (\varphi^{\lceil 1/\varepsilon \rceil} - \psi^{\lceil 1/\varepsilon \rceil})/\sqrt{5} \leq \varphi^{\lceil 1/\varepsilon \rceil}$. $\qquad \square$

Proof of Theorem 24. At each decision time step, A flips a coin to pick a direction for the next run r. It begins writing an up or down run according to the coin flip.

Meanwhile, A uses M additional space to simulate r', the run in the opposite direction. In particular, it simulates the contents of the buffer at each time step, as well as the last element written. Note that A does not need to keep track of the most recent element read when simulating r', as it is always the last element in the buffer.

By Lemma 14, the run with the incorrect direction has length less than $3M$ and the run with the correct direction has length of $3M$ or more. Thus, A can tell if it picked the correct direction. With probability $1/2$, A writes the longer run. Therefore, it knows it made the correct direction and repeats, flipping another coin.

Now consider the case where A picks the wrong direction. When r ends (at time t), r' is continuing. Then A must act *exactly* as if it had written r'. Specifically, we cannot simply cover r' and use an argument akin to Theorem 7, as then the unwritten-element sequence may not be 3-nearly-sorted.

To simulate r', A has two tasks: (a) A must write all elements that were written by r' that were not written by r, and (b) A must "undo" writing any element that was written during r that is not in r', in case

these elements are required to make a subsequent run have length $3M$. Divide the buffer into two halves: B_r is the buffer after writing r, and $B_{r'}$ is the buffer being simulated when r ends.

The first task is to ensure that A writes all elements written by r' that were not written by r in the direction of r'. These elements must be in B_r since they were not written by r; and they must not be in $B_{r'}$ since they were written by r'. Thus, A can simply write out each element in B_r that is not in $B_{r'}$ and continue writing r' from that time step.

The second task is to ensure that all elements written during r that were not written during r' cannot affect future run lengths. These elements must be in $B_{r'}$ but not in B_r. We mark these elements as special ghost elements. We can do this with $O(1)$ additional space by moving them to the front of the buffer and keeping track of how many of them there are. During subsequent runs, these are considered to be a part of A's buffer. However, when A would normally want to write one of these elements out, it instead simply deletes it from its buffer without writing any element. That said, A still counts these deletions towards the size of the run. Note that our buffer never overflows, as A continues to write (or delete) one element per time step.

When this simulation is finished, the contents of A's buffer are exactly what they would have been had it written out r' in the first place—however some are ghost elements, and will be deleted instead of written. Then A repeats, flipping another coin.

For each run in the optimal output, either: A writes that run exactly for cost 1 (with probability $1/2$), or A writes another run, and makes up for its mistake by simulating the correct run exactly, for cost 2 (with probability $1/2$). Thus A has expected cost $(3/2)\text{OPT}$. In the worst case, A guesses incorrectly each time for a total cost of 2OPT.

$\square$

Proof of Lemma 25. Suppose we are at a decision time step. Without loss of generality, assume this time step to be 0. Let the next two possible maximal runs be r_1 and r_2 that are up and down respectively. Without loss of generality, suppose $|r_1| \geq 5M$. Let r_3 be the maximal decreasing run that follows r_2. By Lemma 16, either writing r_1 or writing $r_2 \circ r_3$ is optimal on the unwritten-element sequence. Call the two outputs S_1 and S_2 respectively.

Let $s_1^B, s_1^N, t_1^B, f_1, u_1, s_2^B, s_2^N, t_2^B, f_2, u_2$ be the same sets described in the proof of Lemma 14. Let $r_1 = (x_1, \ldots, x_k), r_2 = (y_1, \ldots, y_\ell), r_3 = (y_{\ell+1}, \ldots, y_q)$. Let the buffers of S_1 and S_2 after time step $t + \ell$ be $B_{1,\ell}, B_{2,\ell}$. Let $j \geq \ell$ be the smallest such that $x_{j+1} \geq y_{j+1}$.

Similar to the proof of Lemma 14, we let $r_{1,2} = \{x_{\ell+1}, \ldots, x_j\}$ and $t_{1,2} = \{x_{j+1}, \ldots, x_k\}$. Let $s_{1,2}^N$ be the set of elements in $r_{1,2}$ but not in $B_{1,\ell}$. Let $s_{1,2}^B$ be the set of elements in $r_{1,2}$ and also in $B_{1,\ell} \setminus r_2$. Let $t_{1,2}^B$ be the set of elements in $t_{1,2}$ and also in $B_{1,\ell} \setminus r_2$. Let $u_{1,2}$ be the set of elements not in r_1 and read in before x_{j+1} is written. Let $f_{1,2}$ be the set of elements in r_1 and read in after x_{j+1} is written.

We define $s_3 = \{y_{\ell+1}, \ldots, y_j\}$ and $t_3 = \{y_{j+1}, \ldots, y_q\}$. Let s_3^N be the set of elements in s_3 but not in $B_{2,\ell}$. Let s_3^B be the set of elements in $s_3 \cap B_{2,\ell}$. Let u_3 be the set of elements that are not in r_3 and read in before y_{j+1} is written. Let f_3 be the set of elements in r_3 and read in after y_{j+1} is written.

Since the buffer of S_2 must keep all elements in $s_{1,2}^N, s_{1,2}^B$ at time step $j+1$, we have $|s_{1,2}^N| + |s_{1,2}^B| \leq M$. We have,

$$
\begin{aligned}
|r_1| &= |r_2| + (|u_3| + |s_3^N|) + (|s_{1,2}^N| + |s_{1,2}^B|) + |f_{1,2}| \\
&\leq (2M + |u_1| + |f_2|) + (M - |u_1| - |f_2|) + M + |f_{1,2}| \\
&= 4M + |f_{1,2}|.
\end{aligned}
$$

Since $|f_{1,2}| \geq M$ because of our assumption $|r_1| \geq 5M$, r_3 has to end before r_1 using the same argument as in the proof of Lemma 14. Since $|r_1| \geq |r_2 \circ r_3|$, r_1 followed by any maximal run will cover $r_2 \circ r_3$. Therefore, r_1 is an optimal prefix of the unwritten-element sequence. At every time step, the maximal run of length $5M$ or more is always a prefix of an optimal output on the unwritten-element sequence as required. $\square$