A Computational Model for Inviscid Flow over a Finite Wing

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1 Aerodynamic Relations

This section provides the necessary aerodynamic relations implemented by the vortex lattice solver.

1.1 Computing the Vortex Circulation Strength

We may write the downwash tensor as follows, where $s = b/N$ is the width of each vortex panel.

$$k_{ij} = \frac{1}{4\pi s}[\frac{-1}{j-i+\frac{1}{2}} + \frac{1}{j-i-\frac{1}{2}}]$$

From the downwash tensor, we may compute the downwash on section i as follows. Γ_j is the circulation strength of horseshoe vortex j.

$$w_i = k_{ij}\Gamma_j$$

The normalwash may be computed as follows, defining β to be the angle the panel forms with the horizontal.

$$n_i = w_i \cos\beta_i$$

The effective angle of attack may be computed from the freestream velocity and the normalwash as follows.

$$\alpha_{e,i} = \alpha_i + \frac{n_i}{u_\infty}$$

The sectional coefficient of lift may be computed as follows.

$$C_i' = 2\pi\alpha_{e,i}$$

Using the Kutta-Joukowski theorem, we may determine a linear system of equations to solve for the distributed vorticity across the wing. Here, c is the mean aerodynamic chord.

$$L_i' = \frac{1}{2}\rho_\infty u_\infty^2 C_i' c$$

$$\Gamma_i = u_\infty \pi \alpha_{e,i} c$$

We substitute all equations to determine an expression for the circulation tensor q_{ij}.

$$\Gamma_i = u_\infty \pi c(\alpha_i + \frac{k_{ij}\Gamma_j \cos\beta_i}{u_\infty})$$

$$\frac{\Gamma_i}{u_\infty \pi c} = \alpha_i + \frac{k_{ij}\Gamma_j \cos\beta_i}{u_\infty}$$

$$\alpha_i = \frac{\Gamma_i}{u_\infty \pi c} - \frac{k_{ij}\Gamma_j \cos\beta_i}{u_\infty}$$

In order to find the circulation tensor q_{ij}, we must take advantage of the index-replacing property of the Kronecker delta δ_{ij}. By this property, $\Gamma_i = \delta_{ij}\Gamma_j$.

$$\alpha_i = \frac{\delta_{ij}\Gamma_j}{u_\infty \pi c} - \frac{k_{ij}\Gamma_j \cos\beta_i}{u_\infty}$$

$$\alpha_i = (\frac{\delta_{ij}}{u_\infty \pi c} - \frac{\cos\beta_i}{4\pi s u_\infty}[\frac{-1}{j-i+\frac{1}{2}} + \frac{1}{j-i-\frac{1}{2}}])\Gamma_j$$

We next define the non-dimensional panel width $\xi_i = s/c = b/Nc$ in order to non-dimensionalize the circulation tensor.

$$\alpha_i = \frac{1}{u_\infty c}(\frac{\delta_{ij}}{\pi} + \frac{\cos\beta_i}{4\pi\xi_i}[\frac{1}{j-i+\frac{1}{2}} + \frac{-1}{j-i-\frac{1}{2}}])\Gamma_j$$

The non-dimensional circulation tensor q_{ij} is therefore expressed as follows.

$$\boxed{q_{ij} = \frac{\delta_{ij}}{\pi} + \frac{\cos\beta_i}{4\pi\xi_i}[\frac{2}{2j-2i+1} + \frac{-2}{2j-2i-1}]}$$

The circulation tensor has the following property.

$$\alpha_i = q_{ij}\frac{\Gamma_j}{c u_\infty}$$

We may rewrite this relation using the non-dimensionalized circulation strength γ_i.

$$\alpha_i = q_{ij}\gamma_i$$

Using an input geometry file, the computational model can quickly assemble and invert the circulation tensor in order to compute the distributed vortex circulation strengths from provided angle of attack distribution.

1.2 Computing the Coefficients of Lift and Drag

Using the Kutta-Joukowski theorem, we can compute the sectional lift and induced drag forces on each segment.

$$L_i' = \gamma_i \cos\beta_i \rho_\infty c u_\infty^2$$

$$D_i' = \gamma_i \cos\beta_i \rho_\infty c u_\infty^2 \cdot \left(-\frac{n_i}{u_\infty}\right)$$