Determining $D4\sigma$ Beam Width Using Python CV2 Moments

$D4\sigma$ is the ISO standard for measuring beam width. It is defined as 4 times the standard deviation of the intensity profile with respect to the centroid (mean) for both x and y directions of a beam profile.

2σ of a Gaussian profile means at that radius from the centroid, the intensity falls off to 5% of the peak value. $D4\sigma$ is simply twice the 2σ value for opposing radii, meaning it is a measure of beam *diameter*.

We define $D4\sigma$ statistically using a the (not normalized) distribution for intensity $I(x,y)$:

$$D4\sigma_x = 4\sigma_x = 4\sqrt{\frac{\iint I(x,y)(x-\bar{x})^2 dxdy}{\iint I(x,y)dxdy}}.$$

Now the polynomial in the numerator integral can be expanded:

$$\iint I(x,y)(x-\bar{x})^2 dxdy = \iint I(x,y)x^2 dxdy - 2\bar{x}\iint I(x,y)xdxdy + \bar{x}^2\iint I(x,y)dxdy.$$

Writing this in terms of image moments:

$$\iint I(x,y)(x-\bar{x})^2 dxdy = M_{20} - 2\bar{x}M_{10} + \bar{x}^2 M_{00}.$$

$$D4\sigma_x = 4\sqrt{\frac{M_{20} - 2\bar{x}M_{10} + \bar{x}^2 M_{00}}{M_{00}}}.$$

Note that the centroid is given by:

$$\bar{x} = \frac{\iint I(x,y)xdxdy}{\iint I(x,y)dxdy} = \frac{M_{10}}{M_{00}}.$$

Thus,

$$\frac{M_{20} - 2\bar{x}M_{10} + \bar{x}^2 M_{00}}{M_{00}} = \frac{M_{20}}{M_{00}} - 2\bar{x}^2 + \bar{x}^2 = \frac{M_{20}}{M_{00}} - \bar{x}^2,$$

$$D4\sigma_x = 4\sqrt{\frac{M_{20}}{M_{00}} - \bar{x}^2}.$$

$$D4\sigma_y = 4\sqrt{\frac{M_{02}}{M_{00}} - \bar{y}^2}$$

See:

https://en.wikipedia.org/wiki/Beam_diameter

https://en.wikipedia.org/wiki/Image_moment

https://en.wikipedia.org/wiki/Variance