GP Compatible Breguet Range Formulations

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1 Introduction

The Breguet range equation is used to calculate aircraft fuel burn during cruise. It is possible to develop multiple sets of GP compatible constraints that model the Breguet range equation. However, when the objective is to minimize total fuel burn, there is a non-obvious upwards pressure on aircraft weight. In one GP formulation, this pressure causes aircraft weight to be pushed to infinity.

2 Derivation

During cruise, the rate of change of an aircraft's gross weight is equal to the mass flow rate of fuel out of the aircraft. Equation 1 is the differential equation modeling this process. W is the aircraft gross weight at any instant in time.

$$\frac{-dW}{dt} = g\dot{m}_f \tag{1}$$

In equation 2, the fuel mass flow rate is written in terms of thrust specific fuel consumption ($TSFC$). T is net engine thrust.

$$g\dot{m}_f = T(TSFC) \tag{2}$$

Substituting equation 2 into equation 1 and multiplying though by dt yields equation 3. [1]

$$dW = -T(TSFC)dt \tag{3}$$

From this point, two different forms of the Breguet range equation can be derived.

2.1 Breguet Range Derivation 1

Dividing equation 3 by gross weight (W) and integrating yields equation 5, one form of the Breguet range equation.

$$\frac{dW}{W} = \frac{-T(TSFC)}{W}dt \tag{4}$$

$$ln\left(\frac{W_{start}}{W_{end}}\right) = \frac{T(TSFC)}{W}t \tag{5}$$

Assuming steady level flight during cruise (equation 9), thrust equals drag (D). Therefore, equation 5 is equivalent to equation 6.

$$ln\left(\frac{W_{start}}{W_{end}}\right) = \frac{D(TSFC)}{W}t \tag{6}$$

2.2 Breguet Range Derivation 2 [1]

Noting TSFC is the inverse of specific impulse, I_{sp}, equation 3 can be rewritten as equation 8.

$$TSFC = \frac{1}{I_{sp}} \tag{7}$$

$$dW = -\frac{T}{I_{sp}}dt \tag{8}$$

Rewriting thrust with the steady level flight relations (equations 9 and 10) yields equation 11.

$$T = D, \quad L = W \tag{9}$$

$$W = D\left(\frac{L}{D}\right) = T\left(\frac{L}{D}\right) \tag{10}$$

$$dW = -\frac{W}{(L/D)I_{sp}}dt \tag{11}$$

Dividing through by gross weight produces equation 12, an easily integrated differential equation.

$$\frac{dW}{W} = -\frac{dt}{(L/D)I_{sp}} \tag{12}$$

$$ln\left(\frac{W_i}{W_f}\right) = -\frac{t}{(L/D)I_{sp}} \tag{13}$$

$$ln\left(\frac{W_i}{W_f}\right)(L/D)I_{sp} = t \tag{14}$$

Integration of equation 12 gives equation 13, which is rewritten as equation 14. After multiplying both sides of equation 14 by V, and noting $V(t)$ is range, the Breguet range equation can be written as either equation 15 or 16.

$$ln\left(\frac{W_i}{W_f}\right)(L/D)I_{sp}V = Range \tag{15}$$

$$ln\left(\frac{W_i}{W_f}\right)(L/D)V(1/TSFC) = Range \tag{16}$$

3 GP Modeling of the Breguet Range Equation

3.1 Removal of the Natural Logarithm

The natural logarithm in equations 5, 15, and 16 is not GP compatible. After rewriting the weight ratio a Taylor expansion used to remove the natural logarithm. First, expand the weight ratio, as done in equation 17.

$$\frac{W_{start}}{W_{end}} = \frac{W_{end} + W_{fuel}}{W_{end}} = 1 + \frac{W_{fuel}}{W_{end}} \tag{17}$$

The natural log term is now set equal to a dummy variable, z_{bre}.

$$ln\left(\frac{W_{start}}{W_{end}}\right) = ln\left(1 + \frac{W_{fuel}}{W_{end}}\right) = z_{bre} \tag{18}$$

The natural logarithm can be removed through exponentiation (equation 19). The fuel weight ratio is now equated to the Taylor expansion of $e^{z_{bre}} - 1$, as shown in equation 20.

$$\frac{W_{fuel}}{W_{end}} = e^{z_{bre}} - 1 \tag{19}$$

$$z_{bre} + \frac{z_{bre}^2}{2} + \frac{z_{bre}^3}{6} = \frac{W_{fuel}}{W_{end}} \tag{20}$$

Substituting equation 18 into equations 5 and 15 allows the Breguet range equation to be modeled with either equations 20 and 21 or equations 20 and 22.

$$z_{bre} = \frac{T(TSFC)}{W}t \tag{21}$$

$$z_{bre}(L/D)V(I_{sp}) = Range \tag{22}$$

3.2 Weight Buildup

Taking the aircraft's end weight as known (equivalent to assuming we know the aircraft's zero fuel weight) and assuming the only change in aircraft weight is due to fuel burn, the aircraft's start weight is simply its end weight plus the weight of fuel it burns.

$$W_{start} = W_{end} + W_{fuel} \tag{23}$$

3.3 Drag Modeling

It is assumed thrust equals drag during cruise (steady level flight). A drag model is required if using equation 5 or 6. Similarly, a drag model is required to accurately estimate L/D in equation 15 or 16. Any drag model will suffice. A good starting place is the parabolic drag model given by equation 24. For the purposes of this report, C_{D_0} and K are assumed constant. However, they can be modeled if a higher fidelity model is desired.

$$D = .5\rho V^2(C_{D_0} + KC_L^2) \tag{24}$$

Equation 24 requires some sort of lift model. Assuming steady level flight, lift is equal to weight. Thus, $W = .5\rho S(V^2)C_L$. This introduces the question of what weight to use when determining C_L. Using the segment start or end weight introduces inaccuracy, and there is no GP compatible constraint for determining the arithmetic average of the segment start and end weight. However, it is easy to write a GP compatible constraint for the geometric average of the segment start and end weights, as is done in equation 25. This is the recommended procedure.

$$W_{avg} = \sqrt{W_{start}W_{end}} \tag{25}$$

Using the geometric mean of segment weights has been shown to increase model stability.

3.4 Time Constraint

Assuming the aircraft's cruise speed is a constant, or determined by another model, time is constrained by the range relation, presented below.

$$Range = V(t) \tag{26}$$

3.5 Fuel Burn Modeling

To accurately model fuel burn, an engine model is required. If this is desired, a GP compatible Breguet range model could be linked to the Hoburg Research Group's turbofan model. However, for simplicity, this report assumes a fixed value for TSFC.

3.6 Relaxation of Equalities

The final step in developing a set of GP constraints that model the Breguet range equation is to relax equalities between monomial and posynomial quantities. Note that equations 25 and 26 are of the form monomial equals monomial and are therefore valid GP equality constraints.

From this point on it, is assumed the objective is to minimize total fuel burn (W_{fuel}). This determines the pressures on variables.

3.6.1 Relaxation of Drag Equality

To decrease fuel burn, it is beneficial to minimize drag (equivalent to minimzing thrust in steady level flight). Thus, equation 24 can be rewritten as 27.

$$D \geq .5\rho V^2(C_{D_0} + KC_L^2) \tag{27}$$

3.6.2 Relaxation of Equalities for Breguet Range Derivation 1

The downward pressure on fuel weight allows the equality in equation 20 to be replaced with a less than or equal to sign. Equation 28 is now a valid GP constraint.

$$z_{bre} + \frac{z_{bre}^2}{2} + \frac{z_{bre}^3}{6} \leq \frac{W_{fuel}}{W_{end}} \tag{28}$$

$$z_{bre} + \frac{z_{bre}^2}{2} + \frac{z_{bre}^3}{6} \geq \frac{W_{fuel}}{W_{end}} \qquad z_{bre} \geq \frac{TSFC\ (t)\ D}{W_{avg}}$$
$$D \geq 0.5(S)rho(V^2)(C_{D_0} + K(C_L^2)) \qquad Range = t(V)$$
$$W_{avg} = 0.5(S)rho(C_L)V^2 \qquad W_{avg} = \sqrt{W_{start}W_{end}}$$
$$W_{start} \geq W_{fuel} + W_{end}$$

Table 1: Valid Breguet range constraints

Equation 28 provides an upper bound for z_{bre}, therefore, the equality in equation 21 can be replaced with a greater than or equal to sign.

$$z_{bre} \geq \frac{D(TSFC)}{W}t \tag{29}$$

Finally, the equality in equation 23 must be replaced. As the aircraft gets heavier, more lift is required and induced drag increases. This increases the required thrust, and consequently fuel burn (if a real turbofan model is used). There is also a downward pressure on weight due to equation 29. The lighter the aircraft is, the larger, z_{bre} will be for a given TSFC and drag. The larger z_{bre} is, the larger the fuel weight fraction in equation 28 will be. The downward pressure on total weight allows equation 23 to become the constraint given by equation 30.

$$W_{start} \geq W_{end} + W_{fuel} \tag{30}$$

Combining equations 25, 26, 27, 28, 29 and 30 with a constraint on C_L and a TSFC source yields a complete GP compatible Breguet range model. The full set of constraints is presented in table 1.

3.6.3 Relaxation of Equalities for Breguet Range Derivation 2

The drag, weight, and time constraints presented previously are still valid when modeling the second form of the Breguet range equation.

There is an upwards pressure on range (the largest range per unit of fuel is optimal), so equation 22 must become equation 31 in order to bound range.

$$z_{bre}(L/D)V(I_{sp}) \geq Range \tag{31}$$

This causes an upwards pressure on z_{bre} (it is in the numerator of equation 31), so equation 20 must be written as equation 32 below.

$$z_{bre} + \frac{z_{bre}^2}{2} + \frac{z_{bre}^3}{6} \leq \frac{W_{fuel}}{W_{end}} \tag{32}$$

L/D can be written as the following monomial equality constraint.

$$L/D = \frac{W_{avg}}{D} \tag{33}$$

It is tempting to reuse equation 30 as a weight constraint. Physical intuition says that a lighter aircraft will burn less fuel, so there must be a downward pressure on W_{start}. However,

$z_{bre} + \frac{z_{bre}^2}{2} + \frac{z_{bre}^3}{6} \leq \frac{W_{fuel}}{W_{end}}$	$Range \leq \frac{z_{bre}(L/D)V}{TSFC}$
$D \geq 0.5(S)rho(V^2)(C_{D_0} + K(C_L^2))$	$L/D = \frac{W_{avg}}{D}$
$W_{avg} = 0.5(S)rho(C_L)V^2$	$W_{avg} = \sqrt{W_{start}W_{end}}$
$W_{start} \geq W_{fuel} + W_{end}$	$Range = t(V)$

Table 2: Dual infeasible Breguet range constraints

this is not true when using this Breguet range formulation. L/D appears in the numerator of equation 31 leading to an upward pressure on L/D. L/D can be artificially increased by increasing the aircraft's weight, i.e. there is an upward pressure on aircraft weight. Consequently, the constraint given by equation 30 is not valid because W_{start} will grow unbounded. If the constraints given in table 2 are used to model the Breguet range equation, the model will be dual infeasible. If the inequality in equation 30 is flipped, the constraint becomes a signomial. This form of the Breguet range equation is not GP compatible. It is important to note this holds true even if a real turbofan model is used. As aircraft weight increases, induced drag increases. This increases the required thrust and consequently TSFC. However, this increase is small relative to the increase in L/D, so it is still beneficial for the aircraft to be heavy.

References

[1] Z. S. Spakovszky. 16.unified: Thermodynamics and propulsion. http://web.mit.edu/16.unified/www/FALL/thermodynamics/notes/notes.html, 2013.