Question 1 Find an antiderivative of $\frac{1}{2x+4}$ with respect to x.

Question 1 $\frac{1}{2}\log\left(|2x+4|\right)+1$

We arbitrarily choose our constant of integration to be 1. It can be any real number, including zero.

Question 2 equation 1 is: $kx-2y=k+2$
equation 2 is: $2x+y(k-5)=2$
a. Find the values of k for which there are no solutions.

b. Find the values of k for which there is a unique solution.

Question 2
a. For the lines to be parallel, the ratios of coefficients between x and y must be the same:
$\frac{k}{2}=\frac{-2}{k-5}$
$k^2-5k+4=0$
$k=1,4$

b. $k\neq 1,4$

Question 3 The function

$$f(x) = \begin{cases} \frac{2x}{7} - \frac{18}{35} & \text{for}\, x \in [0,5] \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function for the continuous random variable X.

a. Find $Pr(X \geq \frac{5}{2})$.

b. Find the value of a such that $Pr(X \leq a) = -\frac{9}{20}$.

Question 3

a. $= \displaystyle\int_{\frac{5}{2}}^{5} \frac{2x}{7} - \frac{18}{35}\, dx$

$= \left[\frac{x^2}{7} - \frac{18x}{35}\right]_{\frac{5}{2}}^{5}$

$= \frac{39}{28}$

b. $Pr(X \leq a) = \displaystyle\int_0^a \frac{2x}{7} - \frac{18}{35}\, dx$

$= \left[\frac{x^2}{7} - \frac{18x}{35}\right]_0^a$

$= \frac{a^2}{7} - \frac{18a}{35} - 0 = -\frac{9}{20}$

$\frac{a^2}{7} - \frac{18a}{35} = -\frac{9}{20}$

$a = \frac{3}{2}, \frac{21}{10}$

but $a \in [0,5]$, so $a = \frac{3}{2}, \frac{21}{10}$

Question 4 The function

$$f(x) = \begin{cases} k\,(x-4)\,(x+3) & \text{for } x \in [-3,4] \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function for the continuous random variable X.

a. Show that $k = -\frac{6}{343}$.

b. Find $Pr(X \leq \frac{1}{2})$.

Question 4

a. $\int_{-3}^{4} k\,(x-4)\,(x+3)\ dx = 1$

$\left[\frac{kx^3}{3} - \frac{kx^2}{2} - 12kx\right]_{-3}^{4} = -\frac{104k}{3} - \frac{45k}{2} = -\frac{343k}{6} = 1$

$\therefore k = -\frac{6}{343}$

b. $= \int_{-3}^{\frac{1}{2}} k\,(x-4)\,(x+3)\ dx$

$= \left[\frac{kx^3}{3} - \frac{kx^2}{2} - 12kx\right]_{-3}^{\frac{1}{2}}$

$= -\frac{343k}{12}$

Question 5 Solve $e^{2x}+3e^x-4$ for x.

Question 5 Let $X=e^x$

$\therefore e^{2x}+3e^x-4=X^2+3X-4$

$=(X-1)(X+4)$

$X=-4,1$

but $X\geq 0$, so $X=1$

$e^x=1$

$x=0$

Question 6 Let X be a normally distributed random variable with mean 85 and variance 25 and let Z be the random variable with the standard normal distribution.
a. If $Pr(75 \leq X \leq 85) = a$, find $Pr(X \geq 95)$ in terms of a.

b. Find $Pr(X \geq 85)$.

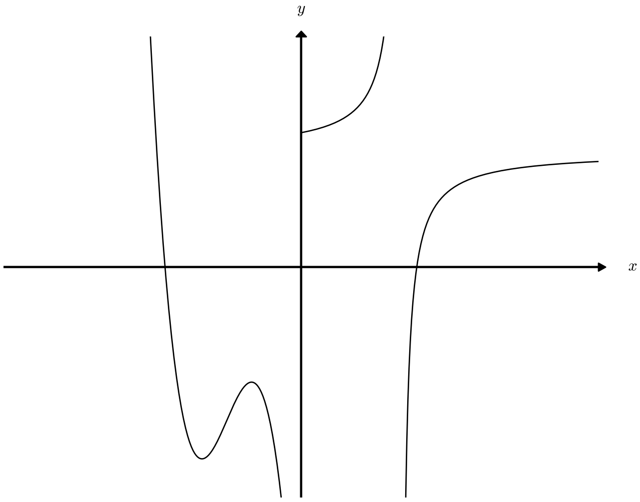
c. Find c such that $Pr(X \geq c) = Pr(Z \geq \frac{2}{5})$.

Question 6
a. $Pr(X \geq 95) = Pr(X \leq 75) = Pr(X \leq 85) - Pr(75 \leq X \leq 85)$
$= (\frac{1}{2}) - (a) = -a + \frac{1}{2}$

b. $\frac{1}{2}$

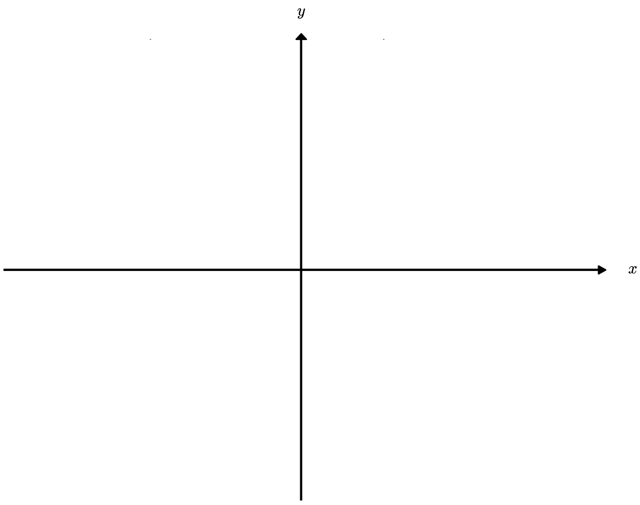
c. $Pr(Z \geq \frac{2}{5}) = Pr(X \geq \frac{2}{5} \times 5 + 85) = Pr(X \geq 87)$
$\therefore c = 87$

Question 7 The graph of the function f is shown, where $f = \begin{cases} -4x^3 - 18x^2 - 24x - 13 & \text{for } x < 0 \\ 3 - \frac{1}{x-2} & \text{for } x \geq 0 \end{cases}$



a. The stationary points of the function are labelled with their coordinates. Write down the domain of the derivative function f'.

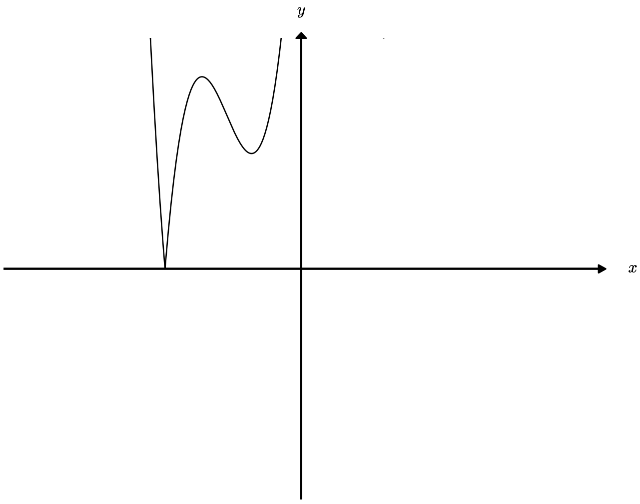
b. By referring to the graph of f, sketch the graph of the function with rule $y = \left|-4x^3 - 18x^2 - 24x - 13\right|$, for $[-6, 0)$. Label stationary points with their coordinates (do not attempt to find x-axis intercepts).



Question 7

a. $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$

b.



Question 8 Evaluate $\int_{\frac{5\pi}{6}}^{\pi} \sin(x) + 1 \; dx$.

Question 8 $\int_{\frac{5\pi}{6}}^{\pi} \sin(x) + 1 \; dx = [x - \cos(x)]_{\frac{5\pi}{6}}^{\pi}$

$= 1 + \pi - (\frac{\sqrt{3}}{2} + \frac{5\pi}{6})$

$= -\frac{\sqrt{3}}{2} + \frac{\pi}{6} + 1$

Question 9 Every Tuesday John goes to see a movie. he always goes to one of two nearby cinemas - the Pollos or the Altiplano.
If he goes to the Pollos one Tuesday, the probability that he goes to the Altiplano the next week is 0.10. If he goes to the Altiplano one Tuesday, then the probability that he goes to the Pollos the next Tuesday is 0.65.
On any given Tuesday the cinema he goes to depends only on the cinema he went to on the previous Tuesday.
If he goes to the Pollos one Tuesday, what is the probability that he goes to the Pollos on exactly 2 of the next 3 Tuesdays?

Question 9 Let X represent the number of times John goes to Pollos over the 3 Tuesdays
$Pr(X=2) = Pr(X_0 = \text{Pollos}) \times Pr(X_1 = \text{Pollos}) \times Pr(X_2 = \text{Altiplano}) + Pr(X_0 = \text{Pollos}) \times Pr(X_1 = \text{Altiplano}) \times Pr(X_2 = \text{Pollos})$
$= 0.90 \times 0.10 + 0.10 \times 0.65$
$= 0.1550$

Question 10 When Emma drives to work each morning they pass a number of intersections with traffic lights. The number X of traffic lights that are green when Emma is driving to work is a random variable with probability distribution given by

x	0	1	2	3	4
Pr(X = x_i)	0.3	0.3	0.1	0.2	0.1

a. Find Var(X), the variance of X.

b. What is the probability that the number of green traffic lights Emma gets is the same on each of 2 consecutive days?

c. Find $Pr(X >= 2 | X >= 1)$.

d. Emma receives green traffic lights on 2 consecutive days. What is the probability that Emma receives a total of 2 green traffic lights over these 2 days?

Question 10

a. $Var(X) = E(X^2) - (E(X))^2$

$= [0^2 \times \frac{3}{10} + 1^2 \times \frac{3}{10} + 2^2 \times \frac{1}{10} + 3^2 \times \frac{1}{5} + 4^2 \times \frac{1}{10}] - [0 \times \frac{3}{10} + 1 \times \frac{3}{10} + 2 \times \frac{1}{10} + 3 \times \frac{1}{5} + 4 \times \frac{1}{10}]^2$

$= \frac{41}{10} - \frac{3}{2}^2 = \frac{37}{20}$

b. $Pr(X$ is the same number 2 days in a row$) = Pr(X=0)^2 + Pr(X=1)^2 + Pr(X=2)^2 + Pr(X=3)^2 + Pr(X=4)^2$
$= (\frac{3}{10})^2 + (\frac{3}{10})^2 + (\frac{1}{10})^2 + (\frac{1}{5})^2 + (\frac{1}{10})^2 = \frac{6}{25}$

c. $Pr(X >= 2 \,|\, X >= 1) = \frac{Pr(X>=2 \cap X>=1)}{Pr(X>=1)}$
$= \frac{\frac{2}{5}}{\frac{7}{10}} = \frac{4}{7}$

d. $Pr(X_1 + X_2 = 2) = Pr(ball_1 = 0 \cap ball_2 = 2) + Pr(ball_1 = 1 \cap ball_2 = 1) + Pr(ball_1 = 2 \cap ball_2 = 0)$
$= \frac{3}{100} + \frac{9}{100} + \frac{3}{100}$
$= \frac{3}{20}$

Question 11 State the amplitude and period of the function $f, R \rightarrow R, f(x) = 3\sin(2\pi(x+1)) - 2$

Question 11 The amplitude is 3
The period is $\frac{1}{2}$

Question 12 Solve the equation $2\log(2x+2) - \log(3x+3) = \log(-x+2)$ for x.

Question 12 $\log\left(\frac{(2x+2)^2}{3x+3}\right) = \log(-x+2)$

$\therefore \frac{(2x+2)^2}{3x+3} = -x+2$

$4x^2+8x+4 = -3x^2+3x+6$

$7x^2+5x-2=0$

$x=-1, \frac{2}{7}$

but $x=-1$ is an invalid solution since a log can only accept positive values

$\therefore x = \frac{2}{7}$

Question 13 The discrete random variable X has the probability distribution

x	-1	0	1	2
Pr(X = x_i)	$\frac{1}{4}\left(8k^2+1\right)$	$\frac{k}{4}$	$\frac{k}{2}(2k+1)$	$\frac{k}{2}$

Find the value of k.

Question 13 $E(X) = \sum_{i=1}^{n} x_i \times Pr(X = x_i) = 3k^2 + \frac{5k}{4} + \frac{1}{4} = 1$

$3k^2 + \frac{5k}{4} - \frac{3}{4} = 0$

$24k^2 + 10k - 6 = 0$

$k = \dfrac{-10 \pm \sqrt{10^2 - 4 \times 24 \times (-6)}}{2 \times 24} = \frac{1}{3}, -\frac{3}{4}$

but $k > 0$ so k = $\frac{1}{3}$

Question 14 The transformation $T : R^2 \to R^2$ is defined by

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

The image of the curve $y = 2e^{-x+2} - 2$ under the transformation T has equation y = $ae^{bx+c} + d$. Find the values of a, b, c, d.

Question 14 $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x-4 \\ 3y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x'+4 \\ \frac{y'}{3} \end{bmatrix}$

$\therefore y = 2e^{-x+2} - 2$ transforms to $\frac{y'}{3} = 2e^{-x'+4+2} - 2$

The transformed equation is $y = 6e^{-x-2} - 6$

$\therefore a = 6, b = -1, c = -2, d = -6$

Question 15 6 identical balls are numbered 1, 2, 3, 4, 5 and 6 and put into a box. A ball is randomly drawn from the box, and not returned to the box. A second balls is then randomly drawn from the box.

a. Find the probability that the values of the balls are 3 and 1, respectively.

b. What is the probability that the sum of the numbers on the 2 balls is 8?

c. Given that the sum of the numbers on the 2 balls is 9, what is the probability that the second is numbered 5?

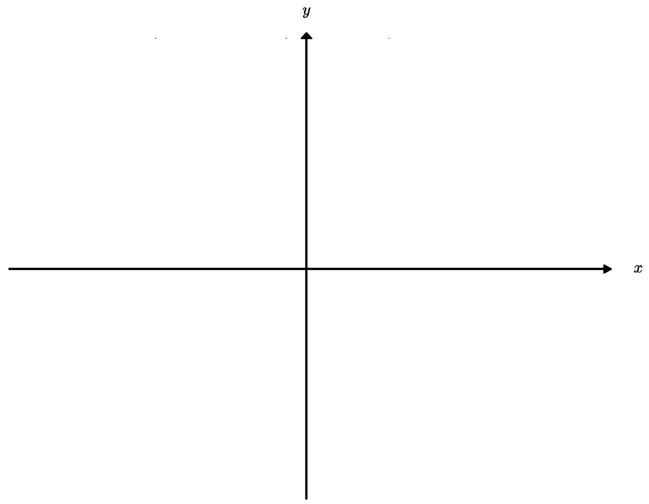
Question 15

a. $Pr(ball_1 = 3 \cap ball_2 = 1) = \frac{1}{6} \times \frac{1}{5} = \frac{1}{30}$

b. $Pr(\text{sum} = 8) = 2! \times Pr(ball_1 = 2 \cap ball_2 = 6) + 2! \times Pr(ball_1 = 3 \cap ball_2 = 5)$
$= 2 \times \frac{1}{30} + 2 \times \frac{1}{30}$
$= \frac{2}{15}$

c. $Pr(\text{sum} = 9) = 2! \times Pr(ball_1 = 3 \cap ball_2 = 6) + 2! \times Pr(ball_1 = 4 \cap ball_2 = 5)$
$= 2 \times \frac{1}{30} + 2 \times \frac{1}{30}$
$= \frac{2}{15}$
$Pr(ball_2 = 5 \cap \text{sum} = 9) = Pr(ball_1 = 4 \cap ball_2 = 5)$
$= \frac{1}{30}$
$Pr(ball_2 = 5 | sum = 9) = \frac{Pr(ball_2 = 5 \cap sum = 9)}{Pr(sum = 9)} = \frac{\frac{1}{30}}{\frac{2}{15}} = \frac{1}{4}$

Question 16 Sketch the graph of $f: [-4,5] \to R, f(x) = 3 + \frac{1}{x}$. Label the axes intercepts and endpoints with their coordinates.



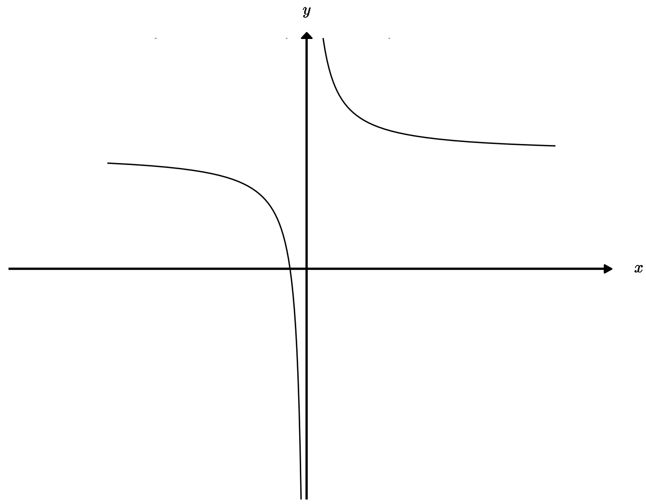
a. Find the coordinates of the image of the point $\left(4,\ \frac{13}{4}\right)$ under a translation of 1 in the positive direction of the x-axis, followed by a dilation of factor 2 from the y-axis.

b. Find the equation of the image of the graph of f under a translation of 1 in the positive direction of the x-axis, followed by a dilation of factor 1 from the y-axis.

Question 16

a. $(x,\ y) \to (x+1,\ y) \to (2x+2,\ y)$

$\therefore \left(4,\ \frac{13}{4}\right) \to \left(1,\ \frac{13}{4}\right)$



b. Let our new $(x,\quad y)$ coordinates be $(x',\quad y')$.
$(x,\quad y) \to (x+1,\quad y) \to (x+1,\quad \frac{y}{2})$
Hence, $\begin{bmatrix} x+1 \\ \frac{y}{2} \end{bmatrix} \to (x',\quad y')$ and $(x,\quad y) \to (x'-1,\quad 2y')$
Now we apply the mapping to the equation:
$2y' = 3 + \frac{1}{x'-1}$
Hence our mapped equation is $y = \frac{3}{2} + \frac{1}{2x-2}$.

Question 17 The area of the region bounded by the y-axis, the x-axis, the curve $y = 2e^{-2x-1} + 5$ and the line x = C, where C is a negative real constant, is $-\frac{1}{e} + e^{-\frac{1}{3}} + \frac{5}{3}$. Find C.

Question 17 A = $\int_{C}^{0} 2e^{-2x-1} + 5\ dx$

$= \left[5x - e^{-2x-1}\right]_{C}^{0}$

$= -\frac{1}{e} - \left(5C - e^{-2C-1}\right) = -\frac{1}{e} + e^{-\frac{1}{3}} + \frac{5}{3}$

$C = -\frac{1}{3}$

Question 18 Let $f : \left(-\infty, \frac{1}{2}\right) \to R$, where $f(x) = -3\log(-2x+1) - 2$. Find f^{-1}, the inverse function of f.

Question 18 $d_{f^{-1}} = r_f = (-\infty, \infty)$
$f^{-1}(x) = -\frac{1}{2}e^{-\frac{x}{3}-\frac{2}{3}} + \frac{1}{2}$

Question 19

a. Let $f(x) = \left(2x^2 + 3x\right)^4$. Find $f'(x)$.

b. If $f(x) = x\cos(2x)$, find $f'(\frac{\pi}{2})$.

Question 19

a. Let $f(x) = \left(2x^2 + 3x\right)^4 = u^4, u = 2x^2 + 3x$

$f'(x) = \frac{dy}{du} \times \frac{du}{dx} = 4u^3 \times u'$

$f'(x) = 4x^3\,(2x+3)^3\,(4x+3)$

b. $f'(x) = -2x\sin(2x) + \cos(2x)$

$f'(\frac{\pi}{2}) = -2x\sin(2x) + \cos(2x)$

$\sin\left(2\frac{\pi}{2}\right) = 0$, $\cos\left(2\frac{\pi}{2}\right) = -1$

$f'(\frac{\pi}{2}) = -1$

Question 20 Solve the equation $\tan(-x) = \frac{\sqrt{3}}{3}$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Question 20 $-x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

The base solution for $\tan(x) = \frac{\sqrt{3}}{3}$ is $\frac{\pi}{6}$

$\therefore -x = \frac{\pi}{6}$

$= \frac{\pi}{6}$

$-x = \frac{\pi}{6} + 1$

$x = -1 - \frac{\pi}{6}$

Question 21 A rocket is flying vertically towards outer-space at a constant rate, forming a right-angled with an observer standing 400km away from the position where the rocket launched. The rocket's altitude is is increasing at a rate of 80km per minute. Find the rate of change of the angle of the observer's view between the launch position and the rocket's current location when the angle is $\frac{\pi}{3}$ radians. Give an exact answer, with units of radians per minute.

Question 21 $\frac{dh}{dt} = 80$

$\frac{dh}{d\theta} = 400\tan^2(\theta) + 400$

$\frac{d\theta}{dh} = \frac{1}{\frac{dh}{d\theta}} = \frac{1}{400\tan^2(\theta)+400}$

$\frac{d\theta}{dt} = \frac{d\theta}{dh} \times \frac{dh}{dt}$

$\therefore \frac{d\theta}{dt} = \frac{80}{400\tan^2(\theta)+400}$

$\frac{d\theta}{dt}\left(\frac{\pi}{3}\right) = \frac{1}{20}$radians/minute

Question 22 Find p given that $\int_{-1}^{0} \frac{1}{2x-2}\,dx = \frac{1}{2}\log(p)$

Question 22 $\left[\frac{1}{2}\log(|2x-2|)\right]_{-1}^{0}$
$= \frac{1}{2}\log(2) - \frac{1}{2}\log(4)$
$= \frac{1}{2}\log\left(\frac{1}{2}\right)$
$\therefore p = \frac{1}{2}.$

Question 23

a. Use the relationship $f(x+h) \approx f(x) + hf'(x)$ for a small positive value of h, to find an approximate value for 2.08^2.

b. Explain why this approximate value is less than than the exact value for 2.08^2.

Question 23

a. $y = x^2, y' = 2x$

$y(2) = 4$

$y'(2) = 4$

$\therefore y(2.08) \approx 0.08 \times 4 + 4 = 4.32$

b. The function is convex at $x = 2$ since the second derivative $\frac{d^2x}{dy^2}(2) = 2$ is greater than than 0. Therefore, the gradient of x^2 increases as x moves from 2 to 2.08 and as a result, linear approximation underestimates the true value of 2.08^2

Question 24 Let $f(x) = x\cos(x)$.

a. Find $f'(x)$.

b. Use the result of part a. to find the value of $\int_{\frac{4\pi}{3}}^{\frac{5\pi}{3}} x\sin(x)\,dx$.

Question 24

a. $f'(x) = -x\sin(x) + \cos(x)$

b. $x\sin(x) = -f'(x) + \cos(x)$

$\int_{\frac{4\pi}{3}}^{\frac{5\pi}{3}} x\sin(x)\,dx = \int_{\frac{4\pi}{3}}^{\frac{5\pi}{3}} -f'(x) + \cos(x)\,dx$

$= [-f(x) + \sin(x)]_{\frac{4\pi}{3}}^{\frac{5\pi}{3}}$

$= [-x\cos(x) + \sin(x)]_{\frac{4\pi}{3}}^{\frac{5\pi}{3}}$

$= -\frac{5\pi}{6} - \frac{\sqrt{3}}{2} - (-\frac{\sqrt{3}}{2} + \frac{2\pi}{3})$

$= -\frac{3\pi}{2}$