Question 1 Find an antiderivative of $-\sin(x-3)$ with respect to x.

Question 1 $\cos(x-3)+1$

We arbitrarily choose our constant of integration to be 1. It can be any real number, including zero.

Question 2 equation 1 is: $ky+6x=4$
equation 2 is: $x(k-3)+3y=k-3$
a. Find the values of k for which there are no solutions.

b. Find the values of k for which there is a unique solution.

Question 2
a. For the lines to be parallel, the ratios of coefficients between x and y must be the same:
$\frac{6}{k-3}=\frac{k}{3}$
$-k^2+3k+18=0$
$k=-3,6$

b. $k\neq -3,6$

Question 3 The function

$$f(x) = \begin{cases} \frac{2x}{11} - \frac{4}{55} & \text{for } x \in [-1, 4] \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function for the continuous random variable X.

a. Find $Pr(X \leq \frac{3}{2})$.

b. Find the value of a such that $Pr(X \geq a) = \frac{31}{55}$.

Question 3

a. $= \displaystyle\int_{-1}^{\frac{3}{2}} \frac{2x}{11} - \frac{4}{55}\, dx$

$= \left[\frac{x^2}{11} - \frac{4x}{55}\right]_{-1}^{\frac{3}{2}}$

$= -\frac{3}{44}$

b. $Pr(X \geq a) = \displaystyle\int_{a}^{4} \frac{2x}{11} - \frac{4}{55}\, dx$

$= \left[\frac{x^2}{11} - \frac{4x}{55}\right]_{a}^{4}$

$= \frac{64}{55} - (\frac{a^2}{11} - \frac{4a}{55}) = \frac{31}{55}$

$-\frac{a^2}{11} + \frac{4a}{55} = -\frac{3}{5}$

$a = -\frac{11}{5}, 3$

but $a \in [-1, 4]$, so $a = 3$

Question 4 The function

$$f(x) = \begin{cases} k\cos\left(\frac{\pi x}{2}\right) & \text{for } k \in [1,2] \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function for the continuous random variable X.

a. Show that $k = -\frac{\pi}{2}$.

b. Find $Pr(X \geq \frac{5}{4})$.

Question 4

a. $\displaystyle\int_1^2 k\cos\left(\frac{\pi x}{2}\right)\,dx = 1$

$\left[\frac{2k}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_1^2 = 0 - \frac{2k}{\pi} = -\frac{2k}{\pi} = 1$

$\therefore k = -\frac{\pi}{2}$

b. $= \displaystyle\int_{\frac{5}{4}}^2 k\cos\left(\frac{\pi x}{2}\right)\,dx$

$= \left[\frac{2k}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_{\frac{5}{4}}^2$

$= -\frac{2k}{\pi}\sqrt{\frac{\sqrt{2}}{4} + \frac{1}{2}}$

Question 5 Solve $e^{2x} - 16$ for x.

Question 5 Let $X = e^x$
$\therefore e^{2x} - 16 = X^2 - 16$
$= (X-4)(X+4)$
$X = -4, 4$
but $X \geq 0$, so $X = 4$
$e^x = 4$
$x = \log(4)$

Question 6 Let X be a normally distributed random variable with mean 20 and variance 81 and let Z be the random variable with the standard normal distribution.

a. If $Pr(16 \leq X \leq 20) = a$, find $Pr(X \geq 24)$ in terms of a.

b. Find $Pr(X \geq 20)$.

c. Find c such that $Pr(X \leq c) = Pr(Z \geq -\frac{1}{9})$.

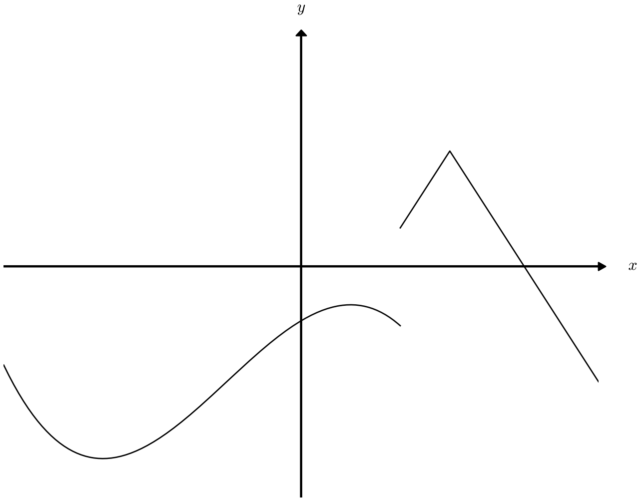
Question 6

a. $Pr(X \geq 24) = Pr(X \leq 16) = Pr(X \leq 20) - Pr(16 \leq X \leq 20)$
$= (\frac{1}{2}) - (a) = -a + \frac{1}{2}$

b. $\frac{1}{2}$

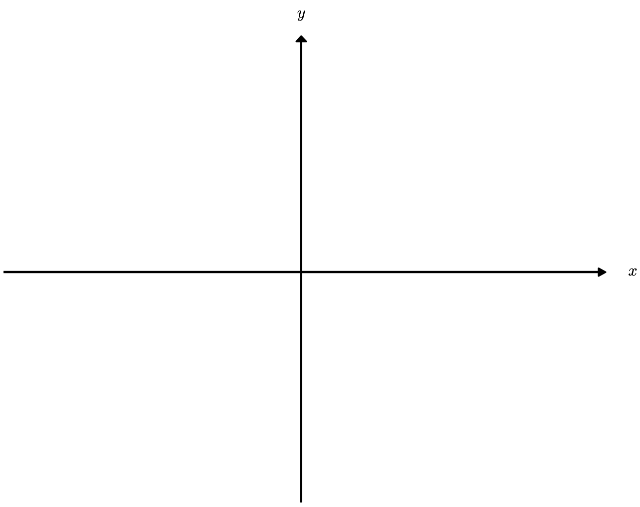
c. $Pr(Z \geq -\frac{1}{9}) = Pr(X \geq -\frac{1}{9} \times 9 + 20) = Pr(X \geq 19)$
$= Pr(Z \leq -19)$
$\therefore c = -19$

Question 7 The graph of the function f is shown, where $f = \begin{cases} -\frac{8x^3}{125} - \frac{36x^2}{125} + \frac{96x}{125} - \frac{177}{125} & \text{for } x < 2 \\ -2\,|x-3| + 3 & \text{for } x \geq 2 \end{cases}$



a. The stationary points of the function are labelled with their coordinates. Write down the domain of the derivative function f'.

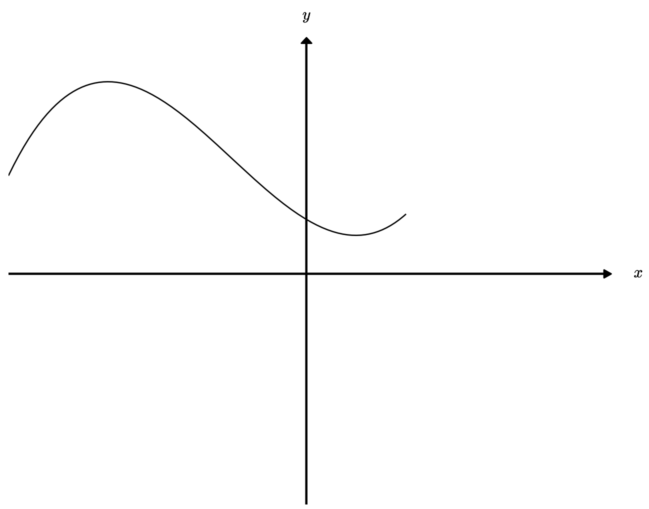
b. By referring to the graph of f, sketch the graph of the function with rule $y = \left| -\frac{8x^3}{125} - \frac{36x^2}{125} + \frac{96x}{125} - \frac{177}{125} \right|$, for $[-6, 2)$. Label stationary points with their coordinates (do not attempt to find x-axis intercepts).



Question 7

a. $(-\infty, 2) \cup (2, 3) \cup (3, \infty)$

b.



Question 8 Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \sin(x) - 2\, dx$.

Question 8 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \sin(x) - 2\, dx = [-2x - \cos(x)]_{-\frac{\pi}{2}}^{\frac{\pi}{6}}$

$= -\frac{\pi}{3} - \frac{\sqrt{3}}{2} - \pi$

$= -\frac{4\pi}{3} - \frac{\sqrt{3}}{2}$

Question 9 Every Thursday Addison goes to see a movie. she always goes to one of two nearby cinemas - the Pollos or the Altiplano.
If she goes to the Pollos one Thursday, the probability that she goes to the Altiplano the next week is 0.20. If she goes to the Altiplano one Thursday, then the probability that she goes to the Pollos the next Thursday is 0.15.
On any given Thursday the cinema she goes to depends only on the cinema she went to on the previous Thursday.
If she goes to the Pollos one Thursday, what is the probability that she goes to the Pollos on exactly 2 of the next 3 Thursdays?

Question 9 Let X represent the number of times Addison goes to Pollos over the 3 Thursdays
$Pr(X=2) = Pr(X_0 = \text{Pollos}) \times Pr(X_1 = \text{Pollos}) \times Pr(X_2 = \text{Altiplano}) + Pr(X_0 = \text{Pollos}) \times Pr(X_1 = \text{Altiplano}) \times Pr(X_2 = \text{Pollos})$
$= 0.80 \times 0.20 + 0.20 \times 0.15$
$= 0.1900$

Question 10 When Ella drives to work each morning they pass a number of intersections with traffic lights. The number X of traffic lights that are green when Ella is driving to work is a random variable with probability distribution given by

x	0	1	2	3	4
Pr(X = x_i)	0.1	0.3	0.1	0.4	0.1

a. Find the mode of X.

b. What is the probability that the number of green traffic lights Ella gets is the same on each of 3 consecutive days?

c. Find $Pr(X >= 3 | X >= 1)$.

d. Ella receives green traffic lights on 2 consecutive days. What is the probability that Ella receives a total of 3 green traffic lights over these 2 days?

Question 10

a. 3

b. $Pr(X$ is the same number 3 days in a row$) = Pr(X = 0)^3 + Pr(X = 1)^3 + Pr(X = 2)^3 + Pr(X = 3)^3 + Pr(X = 4)^3$
$= (\frac{1}{10})^3 + (\frac{3}{10})^3 + (\frac{1}{10})^3 + (\frac{2}{5})^3 + (\frac{1}{10})^3 = \frac{47}{500}$

c. $Pr(X >= 3 | X >= 1) = \frac{Pr(X>=3 \cap X>=1)}{Pr(X>=1)}$
$= \frac{\frac{1}{2}}{\frac{9}{10}} = \frac{5}{9}$

d. $Pr(X_1 + X_2 = 3) = Pr(ball_1 = 0 \cap ball_2 = 3) + Pr(ball_1 = 1 \cap ball_2 = 2) + Pr(ball_1 = 2 \cap ball_2 = 1) + Pr(ball_1 = 3 \cap ball_2 = 0)$
$= \frac{1}{25} + \frac{3}{100} + \frac{3}{100} + \frac{1}{25}$
$= \frac{7}{50}$

Question 11 State the range and period of the function $f, R \rightarrow R, f(x) = \sin(\pi(x-3)) - 1$

Question 11 The range is $[-2, 0]$
The period is 1

Question 12 Solve the equation $\log(2x-5) - 2\log(2x+2) = -\log(3x+3)$ for x.

Question 12 $\log\left(\frac{2x-5}{(2x+2)^2}\right) = \log\left(\frac{1}{3x+3}\right)$

$\therefore \frac{2x-5}{(2x+2)^2} = \frac{1}{3x+3}$

$6x^2 - 9x - 15 = 4x^2 + 8x + 4$

$2x^2 - 17x - 19 = 0$

$x = -1, \frac{19}{2}$

but $x = -1$ is an invalid solution since a log can only accept positive values

$\therefore x = \frac{19}{2}$

Question 13 The discrete random variable X has the probability distribution

x	-2	-1	0	1
Pr(X = x_i)	$\frac{3k}{2}$	$\frac{1}{6}$	$2k^2$	k^2

Find the value of k.

Question 13 $E(X) = \sum_{i=1}^{n} x_i \times Pr(X = x_i) = 3k^2 + \frac{3k}{2} + \frac{1}{6} = 1$

$3k^2 + \frac{3k}{2} - \frac{5}{6} = 0$

$18k^2 + 9k - 5 = 0$

$k = \dfrac{-9 \pm \sqrt{9^2 - 4 \times 18 \times (-5)}}{2 \times 18} = \frac{1}{3}, -\frac{5}{6}$

but $k > 0$ so k = $\frac{1}{3}$

Question 14 The transformation $T : R^2 \to R^2$ is defined by

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

The image of the curve $y = e^{2x} - 5$ under the transformation T has equation y = $ae^{bx+c} + d$. Find the values of a, b, c, d.

Question 14 $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x-3 \\ 2y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x'+3 \\ \frac{y'}{2} \end{bmatrix}$

$\therefore y = e^{2x} - 5$ transforms to $\frac{y'}{2} = e^{2(x'+3)} - 5$

The transformed equation is $y = 2e^{2x+6} - 10$

$\therefore a = 2, b = 2, c = 6, d = -10$

Question 15 5 identical balls are numbered 1, 2, 3, 4 and 5 and put into a box. A ball is randomly drawn from the box, and not returned to the box. A second and third balls are then randomly drawn from the box.

a. Find the probability that the values of the balls are 3, 1 and 2, respectively.

b. What is the probability that the sum of the numbers on the 3 balls is 7?

c. Given that the sum of the numbers on the 3 balls is 11, what is the probability that the third is numbered 2?

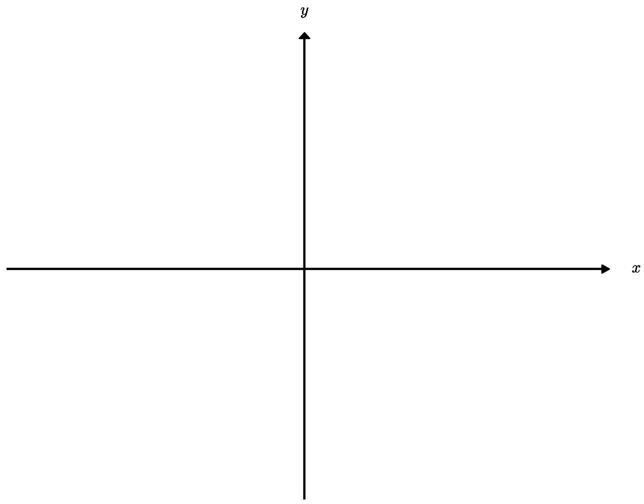
Question 15

a. $Pr(ball_1 = 3 \cap ball_2 = 1 \cap ball_3 = 2) = \frac{1}{5} \times \frac{1}{4} \times \frac{1}{3} = \frac{1}{60}$

b. $Pr(\text{sum} = 7) = 3! \times Pr(ball_1 = 1 \cap ball_2 = 2 \cap ball_3 = 4)$
$= 6 \times \frac{1}{60}$
$= \frac{1}{10}$

c. $Pr(\text{sum} = 11) = 3! \times Pr(ball_1 = 2 \cap ball_2 = 4 \cap ball_3 = 5)$
$= 6 \times \frac{1}{60}$
$= \frac{1}{10}$
$Pr(ball_3 = 2 \cap \text{sum} = 11) = Pr(ball_1 = 4 \cap ball_2 = 5 \cap ball_3 = 2) + Pr(ball_1 = 5 \cap ball_2 = 4 \cap ball_3 = 2)$
$= \frac{1}{30}$
$Pr(ball_3 = 2|sum = 11) = \frac{Pr(ball_3=2 \cap sum=11)}{Pr(sum=11)} = \frac{\frac{1}{30}}{\frac{1}{10}} = \frac{1}{3}$

Question 16 Sketch the graph of $f : [-4, 4] \to R, f(x) = -\frac{2}{x-1}$. Label the axes intercepts and endpoints with their coordinates.



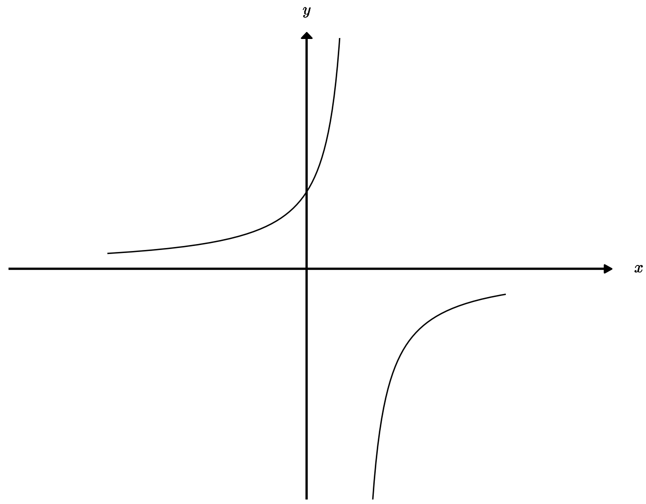
a. Find the coordinates of the image of the point $\left(-3,\ \frac{1}{2}\right)$ under a translation of 5 in the negative direction of the y-axis, followed by a dilation of factor 1 from the y-axis.

b. Find the equation of the image of the graph of f under a translation of 2 in the positive direction of the x-axis, followed by a dilation of factor 3 from the y-axis.

Question 16

a. $(x,\ \ y) \to (x,\ \ y-5) \to \left(x,\ \ \frac{y}{2} - \frac{5}{2}\right)$

$\therefore \left(-3,\ \ \frac{1}{2}\right) \to (-3,\ \ 6)$



b. Let our new $(x, \quad y)$ coordinates be $(x', \quad y')$.
$(x, \quad y) \to (x+2, \quad y) \to (3x+6, \quad y)$
Hence, $\begin{bmatrix} 3x+6 \\ y \end{bmatrix} \to (x', \quad y')$ and $(x, \quad y) \to \left(\frac{x'}{3} - 2, \quad y'\right)$
Now we apply the mapping to the equation:
$y' = -\frac{2}{\frac{x'}{3}-2-1}$
Hence our mapped equation is $y = -\frac{6}{x-9}$.

Question 17 The area of the region bounded by the y-axis, the x-axis, the curve $y = 2e^{-3x-5} - 1$ and the line x = C, where C is a positive real constant, is $-\frac{2}{3e^5} + \frac{2}{3e^6} + \frac{1}{3}$. Find C.

Question 17 Since the curve is below the x-axis, we take the signed area.

$A = -\int_0^C 2e^{-3x-5} - 1\,dx = \int_0^C -2e^{-3x-5} + 1\,dx$

$= \left[x + \frac{2}{3}e^{-3x-5}\right]_0^C$

$= C + \frac{2}{3}e^{-3C-5} - \frac{2}{3e^5} = -\frac{2}{3e^5} + \frac{2}{3e^6} + \frac{1}{3}$

$C = \frac{1}{3}$

Question 18 Let $f:\left(-\infty,\frac{5}{2}\right)\to R$, where $f(x)=2\log\left(-2x+5\right)-3$. Find f^{-1}, the inverse function of f.

Question 18 $d_{f^{-1}}=r_f=(-\infty,\infty)$
$f^{-1}(x)=-\frac{1}{2}e^{\frac{x}{2}+\frac{3}{2}}+\frac{5}{2}$

Question 19

a. Let $f(x) = \left(-2x^2 + x\right)^3$. Find $f'(x)$.

b. If $f(x) = \frac{\sin(x)}{-2x+5}$, find $f'(-\pi)$.

Question 19

a. Let $f(x) = \left(-2x^2 + x\right)^3 = u^3, u = -2x^2 + x$

$f'(x) = \frac{dy}{du} \times \frac{du}{dx} = 3u^2 \times u'$

$f'(x) = -3x^2 (2x-1)^2 (4x-1)$

b. $f'(x) = \frac{1}{(-2x+5)^2} ((-2x+5)\cos(x) + 2\sin(x))$

$f'(-\pi) = \frac{1}{(-2x+5)^2} ((-2x+5)\cos(x) + 2\sin(x))$

$\sin(-\pi) = 0$, $\cos(-\pi) = -1$

$f'(-\pi) = \frac{-2\pi-5}{(5+2\pi)^2}$

Question 20 Solve the equation $3\tan(x)=\sqrt{3}$ for $x\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$.

Question 20 $\tan(x)=\frac{\sqrt{3}}{3}$

The base solution for $\tan(x)=\frac{\sqrt{3}}{3}$ is $\frac{\pi}{6}$

$\therefore x=\frac{\pi}{6}+(1\times 2\pi)$

$=\frac{7\pi}{6}$

Question 21 A rocket is flying vertically towards outer-space at a constant rate, forming a right-angled with an observer standing 500km away from the position where the rocket launched. The rocket's altitude is is increasing at a rate of 30km per minute. Find the rate of change of the angle of the observer's view between the launch position and the rocket's current location when the angle is $\frac{\pi}{4}$ radians. Give an exact answer, with units of radians per minute.

Question 21 $\frac{dh}{dt} = 30$

$\frac{dh}{d\theta} = 500\tan^2(\theta) + 500$

$\frac{d\theta}{dh} = \frac{1}{\frac{dh}{d\theta}} = \frac{1}{500\tan^2(\theta)+500}$

$\frac{d\theta}{dt} = \frac{d\theta}{dh} \times \frac{dh}{dt}$

$\therefore \frac{d\theta}{dt} = \frac{30}{500\tan^2(\theta)+500}$

$\frac{d\theta}{dt}(\frac{\pi}{4}) = \frac{3}{100}$radians/minute

Question 22 Find p given that $\int_{-5}^{-4} \frac{1}{-3x-5}\, dx = \frac{1}{3}\log(p)$

Question 22 $\left[-\frac{1}{3}\log(|3x+5|)\right]_{-5}^{-4}$
$= -\frac{1}{3}\log(7) - (-\frac{1}{3}\log(10))$
$= \frac{1}{3}\log\left(\frac{10}{7}\right)$
$\therefore p = \frac{10}{7}$.

Question 23

a. Use the relationship $f(x+h) \approx f(x) + hf'(x)$ for a small positive value of h, to find an approximate value for $\sqrt[3]{7.95}$.

b. Explain why this approximate value is less than than the exact value for $\sqrt[3]{7.95}$.

Question 23

a. $y = \sqrt[3]{x}, y' = \frac{1}{3x^{\frac{2}{3}}}$

$y(8) = 2$

$y'(8) = \frac{1}{12}$

$\therefore y(7.95) \approx -\frac{0.05}{12} + 2 = 1.99583333333333$

b. The function is concave at $x = 8$ since the second derivative $\frac{d^2x}{dy^2}(8) = -\frac{1}{144}$ is less than than 0. Therefore, the gradient of $\sqrt[3]{x}$ decreases as x moves from 8 to 7.95 and as a result, linear approximation underestimates the true value of $\sqrt[3]{7.95}$

Question 24 Let $f(x) = x\sin(x)$.

a. Find $f'(x)$.

b. Use the result of part a. to find the value of $\int_{-\frac{7\pi}{6}}^{\frac{5\pi}{3}} \cos(x)\ dx$.

Question 24

a. $f'(x) = x\cos(x) + \sin(x)$

b. $\cos(x) = \frac{1}{x}\left(f'(x) - \sin(x)\right)$

$$\int_{-\frac{7\pi}{6}}^{\frac{5\pi}{3}} \cos(x)\ dx = \int_{-\frac{7\pi}{6}}^{\frac{5\pi}{3}} \frac{1}{x}\left(f'(x) - \sin(x)\right)\ dx$$

$$= \left[\frac{1}{x}\left(f(x) + \cos(x)\right)\right]_{-\frac{7\pi}{6}}^{\frac{5\pi}{3}}$$

$$= \left[\sin(x)\right]_{-\frac{7\pi}{6}}^{\frac{5\pi}{3}}$$

$$= -\frac{\sqrt{3}}{2} - \frac{1}{2}$$

$$= -\frac{\sqrt{3}}{2} - \frac{1}{2}$$