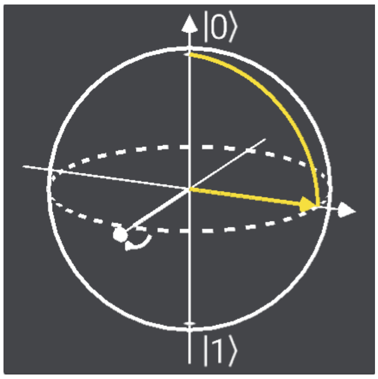
QUESTION 1

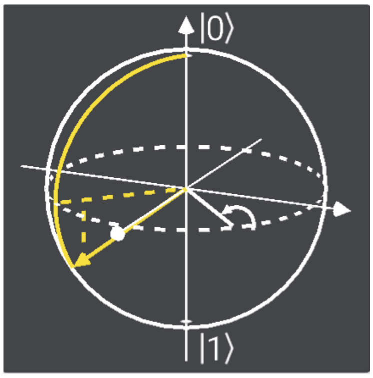


a. $H|0\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$

b. Rotation of angle $\theta = -\frac{\pi}{2}$ about the axis $\tilde{\boldsymbol{n}} = (1, 0, 0)$

c. $|0\rangle \rightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}i|1\rangle, \theta_B = \frac{\pi}{2}, \phi_B = \frac{\pi}{2}$

d. Rotation of angle $\theta = \frac{2\pi}{3}$ about the axis $\tilde{\boldsymbol{n}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$



e. $|0\rangle \rightarrow \frac{1}{\sqrt{2}}|0\rangle + \left(-\frac{\sqrt{3}}{2}e^{-i\frac{\pi}{4}}\right)|1\rangle$

$= \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}\left(-\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)|1\rangle$

$= \frac{1}{2}\Big|0\rangle + \left(-\frac{\sqrt{3}}{2\sqrt{2}} + i\frac{\sqrt{3}}{2\sqrt{2}}\right)\Big|1\rangle, \theta_B = \frac{2\pi}{3}, \phi_B = -\frac{\pi}{4}$

f. Rotation of angle $\theta = \frac{2\pi}{3}$ about the axis $\tilde{\boldsymbol{n}} = \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$

QUESTION 2

a. $H^{\otimes n}|x\rangle = \frac{1}{\sqrt{N}}\sum_{z=0}^{N-1}(-1)^{x \bullet z}|z\rangle$, where $N = 2^n$

We can ignore the $\sum_{z=0}^{N-1}(-1)^{x \bullet z}$ part as -1 will always come in a pair since we are using an even number for n and thus would always result in a positive product if we are finding the product of the first m amplitudes and m is an even number.

$\therefore a_0 \times a_1 \times a_2 \times a_3 = \left(\frac{1}{\sqrt{N}}\right)^4 = \frac{1}{N^2}$

b. Following the above logic where we ignore the latter part of the equation,

Product $= \left(\frac{1}{\sqrt{N}}\right)^{2^n} = \frac{1}{N^{2^{n-1}}}$

QUESTION 3

a. $H: a_0|0\rangle + a_1|1\rangle \xrightarrow{H} \frac{a_0+a_1}{\sqrt{2}}|0\rangle + \frac{a_0-a_1}{\sqrt{2}}|1\rangle$

$T: a_0|0\rangle + a_1|1\rangle \xrightarrow{T} a_0|0\rangle + e^{i\frac{\pi}{4}}a_1|1\rangle$

$T^\dagger: a_0|0\rangle + a_1|1\rangle \xrightarrow{T^\dagger} a_0|0\rangle + e^{-i\frac{\pi}{4}}a_1|1\rangle$

b. $|110\rangle \xrightarrow{H_3} \frac{1}{\sqrt{2}}|110\rangle + \frac{1}{\sqrt{2}}|111\rangle$

$\xrightarrow{CNOT_{2,3}} \frac{1}{\sqrt{2}}|110\rangle + \frac{1}{\sqrt{2}}|111\rangle$

$\xrightarrow{{T^\dagger}_3} \frac{1}{\sqrt{2}}|110\rangle + \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}|111\rangle$

$$
\begin{aligned}
&\xrightarrow{CNOT_{1,3}} && \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}|110\rangle + \frac{1}{\sqrt{2}}|111\rangle \\
&\xrightarrow{T_3} && \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}|110\rangle + \frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}|111\rangle \\
&\xrightarrow{CNOT_{2,3}} && \frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}|110\rangle + \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}|111\rangle \\
&\xrightarrow{T^{\dagger}{}_3} && \frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}|110\rangle + \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{2}}|111\rangle \\
&\xrightarrow{CNOT_{1,3}} && \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{2}}|110\rangle + \frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}|111\rangle \\
&\xrightarrow{T_2, T_3} && \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}|110\rangle + \frac{1}{\sqrt{2}}e^{i\frac{3\pi}{4}}|111\Big\rangle \\
&\xrightarrow{CNOT_{1,2}, H_3} && \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}|101\rangle \\
&\xrightarrow{T_1, T^{\dagger}{}_2} && |101\rangle \\
&\xrightarrow{CNOT_{1,2}} && |111\rangle
\end{aligned}
$$

c. A Toffoli gates is a lot more computationally expensive than one- or two-qubit gates. This is because one-qubit gates can be easily represented as a rotation along a particular axis and two-qubit gates can be easily represented as a matrix multiplication.

QUESTION 4

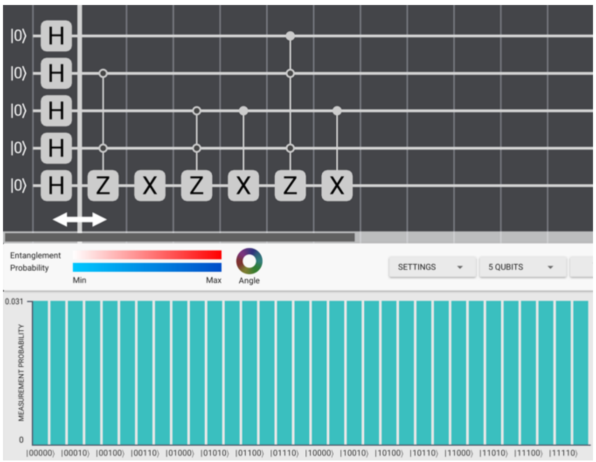
a. https://qui.science.unimelb.edu.au/circuits/60789b6d97f05a005fd20e83

b. https://qui.science.unimelb.edu.au/circuits/60789ba7e69cef001d7bfa0d

QUESTION 5

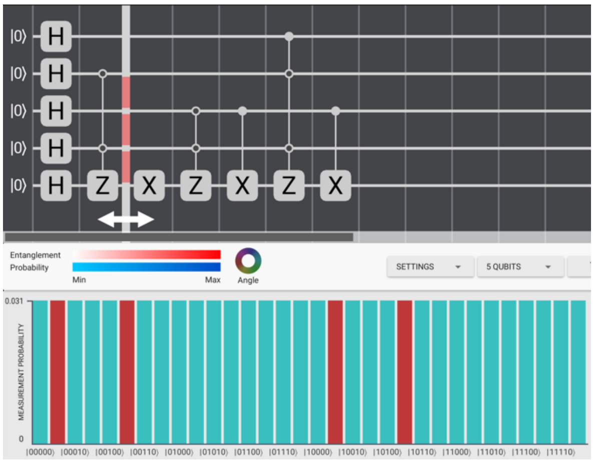
a. https://qui.science.unimelb.edu.au/circuits/60787fb7eb73ad0028424b29

Though not in the implemented oracle, here I added 5 Hadamard gates to allow better visualisation to make it easier to follow and understand the steps taken in creating the oracle. In a way, my implementation could be divided into two parts: marking wanted states (and a few others) and unmarking the unwanted states.



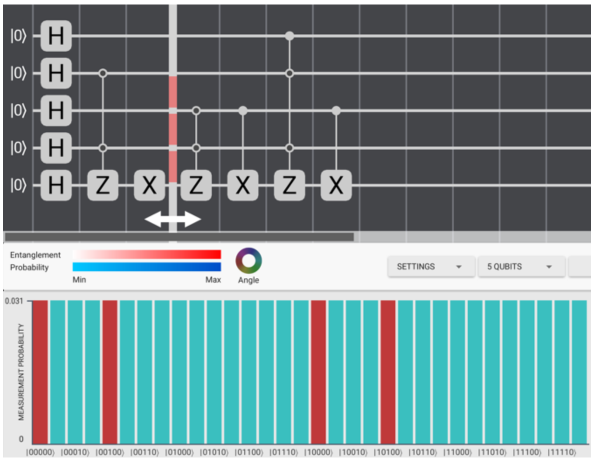
In my first part, I first applied a Z-gate with two control on 0 on the 2nd and 4th qubits. This applied a phase to the following states:

$|00001\rangle = |1\rangle$
$|00101\rangle = |5\rangle$
$|10001\rangle = |17\rangle$
$|10101\rangle = |21\rangle$



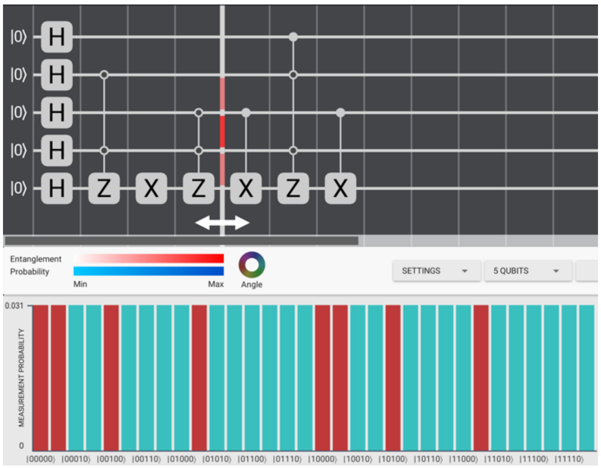
I then applied an X-gate on the 5th qubit. This wouldn't have any effect on most of the states, but would mean that the states with the phases set above would have the last qubit flipped to unmark the above states and mark the following:

$|00000\rangle = |0\rangle$
$|00100\rangle = |4\rangle$
$|10000\rangle = |16\rangle$
$|10100\rangle = |20\rangle$



Lastly, I applied another Z-gate with two controls on 0 on the 3rd and 4th qubits. This applied a phase to the following states:

$|00001\rangle = |1\rangle$
$|01001\rangle = |9\rangle$
$|10001\rangle = |17\rangle$
$|11001\rangle = |25\rangle$



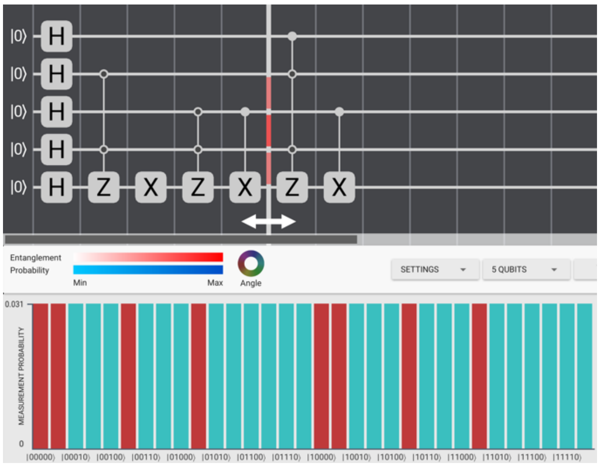
After this first part, I have marked a total of 8 states: $|0\rangle, |1\rangle, |4\rangle, |9\rangle, |16\rangle, |17\rangle, |20\rangle, |25\rangle$. This includes all the 6 square numbers. The next part focuses on unmarking $|17\rangle, |20\rangle$.

In this part, I first applied an X-gate with a control on 0 on the 3rd qubit to flip the following states:

$|00100\rangle = |4\rangle$
$|10100\rangle = |20\rangle$

This then applies phase to:

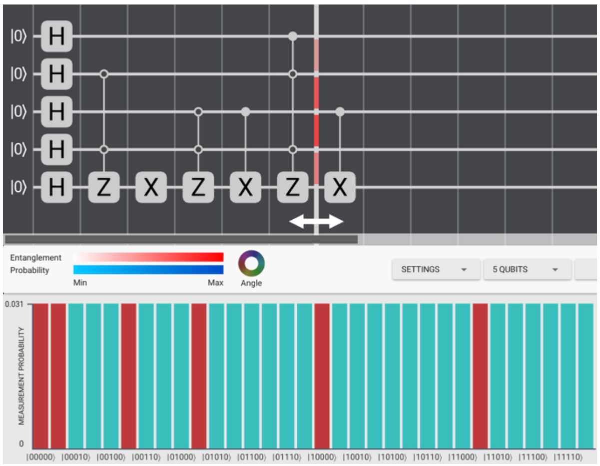
$|00101\rangle = |5\rangle$
$|10101\rangle = |21\rangle$



With the above, the unwanted states are:

$|10001\rangle = |17\rangle$
$|10101\rangle = |21\rangle$

I then applied a Z-gate on the 5th qubit with controls on 0 on the 2nd and 4th qubit and 1 on the 1st qubit as this combination is unique to these two states out of all that is already marked. This unmarks the above two states.



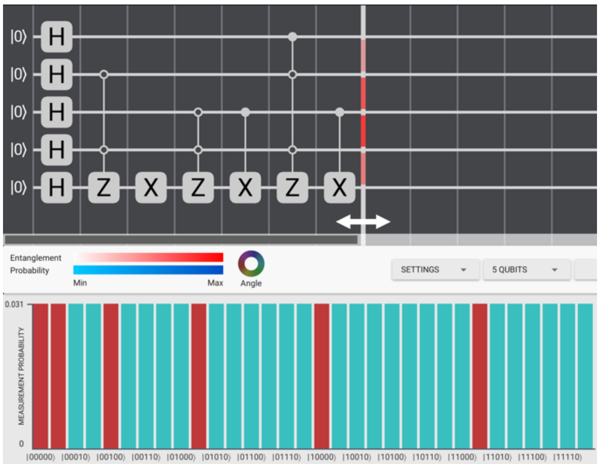
Lastly, I then apply the X-gate with a control on 0 on the 3rd qubit again to flip the state:

$|00101\rangle = |5\rangle$

This results in it flipping back to being unmarked and marking:

$|00100\rangle = |4\rangle$

Now, all that remains as marked at the states for the square numbers.



a. https://qui.science.unimelb.edu.au/circuits/60789be5eb73ad0028424b2a

$$\text{Mean} = \left(\frac{26}{\sqrt{32}} - \frac{6}{\sqrt{32}}\right) \times \frac{1}{\sqrt{32}} = \frac{5}{8\sqrt{32}}$$

Inverse the probability of 1 of the marked states about the mean:

$$\text{Difference} = \left|-\frac{1}{\sqrt{32}} - \frac{5}{8\sqrt{32}}\right| = \frac{13}{8\sqrt{32}}$$

$\text{Amplitude}(marked\ state) = \frac{5}{8\sqrt{32}} + \frac{13}{8\sqrt{32}} = \frac{9}{4\sqrt{32}}$

$P(success) = \left(\frac{9}{4\sqrt{32}}\right)^2 \times 6 = \frac{486}{512} \approx 0.949$

b.

Iteration, j	$(2j+1)\theta$ rad	α_j	$\lvert\alpha_j\rvert^2$
0	0.4478	0.4330	0.1875
1	1.3434	0.9743	0.9492
2	2.239	0.7848	0.6160

Ans: 1 time. More iterations and we greatly reduce the probability of success.

c. There are 1024 square numbers strictly less than 2^{20}, including 0.

$\sin\theta = \frac{\sqrt{1024}}{\sqrt{2^{20}}} = \frac{1}{32}$

θ is very small. Hence $\theta \approx \frac{1}{32}$

$n \approx \frac{\pi \times 32}{4} \approx 25$ iterations

QUESTION 6

a. $QS + RS + RT - QT = (R+Q)S + (R-Q)T$

$R + Q = 0$ or ± 2. Hence, $(R+Q)S = 0$ or ± 2

$R - Q = 0$ or ± 2. Hence, $(R-Q)T = 0$ or ± 2

$R - Q$ and $R + Q$ cannot both be 0 and ± 2. Hence at least one will be 0 and the other will be ± 2

$\therefore E(QS + RS + RT - QT) \leq 2$

Upper bound of expected value $= 2$

b. Eigenvalues:

Q: ± 1

R: ± 1

S: $\frac{1}{\sqrt{2}}$

T: $\frac{1}{\sqrt{2}}$

Max $\langle QS + RS + RT - QT \rangle = 2\sqrt{2}$

c. $\phi = \frac{k\pi}{4}$, where $k \in \mathbb{Z}$

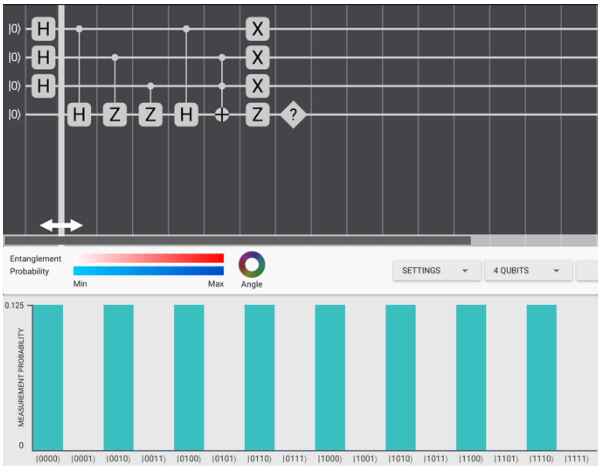
The maximum value of $2\sqrt{2}$ violates the Bell's inequality where with CHSH: $QS + RS + RT - QT = 2\sqrt{2}$

But Bell's inequality states that it must be ≤ 2.

QUESTION 7

a. https://qui.science.unimelb.edu.au/circuits/6078a038f25ca0003e5fdcfc

As with Question 5, I applied the Hadamard gates to make it easy to visualise and follow the steps. The first 3 qubits are for the input while the last (target) qubit will be measured to give 1 (0) if the input has more bits set to 0 (1).



For a 3-bit input to have more bits set to 0 than 1, it must essentially satisfy one or all the following:

- 1st ***AND*** 2nd bit set to 0 ***OR*** [1]
- 1st ***AND*** 3rd bit set to 0 ***OR*** [2]
- 2nd ***AND*** 3rd bit set to 0 [3]

I checked for the first two statements with the following set of gates:

The four steps above can be summarised as:

- (1st ***AND*** 2nd qubits set to 1) ***OR*** (1st ***AND*** 3rd qubits set to 1)

Note the above is "set to 1", not 0, unlike [1] and [2]. This will be dealt with in future steps.

The Hadamard gates and Z-gate (one) together act like a Toffoli gate that acts as an AND condition. This works because all the bits were set to 0 then nothing will happen. If either (not both) the conditions above were satisfied, then it would result in a Hadamard gate controlled on 1 on the 1st qubit, followed by a Z-gate (phase application) controlled on 1 on either the 2nd or 3rd qubits (depending on which of the two conditions was satisfied), and lastly followed by the same Hadamard gate controlled on 1 on the 1st qubit. Overall, this would result in the target qubit being flipped, like an X-gate which can be seen as on the states $|1100\rangle$ and $|1010\rangle$ in the equal superposition from the first step being flipped to $|1101\rangle$ and $|1011\rangle$.

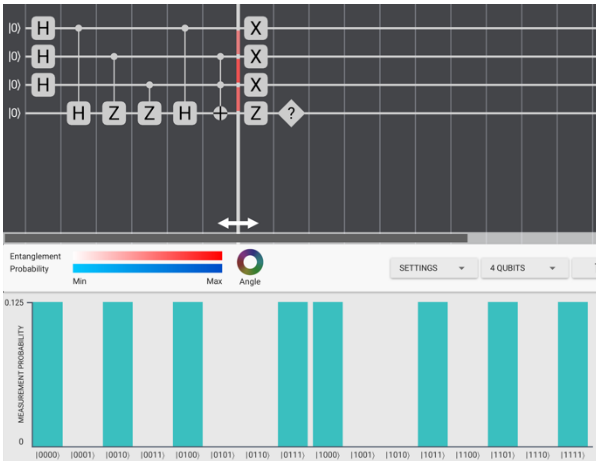
If both the conditions were to be true, it would mean that all 3 qubits are set to 1. This would not have any effect in the above step because satisfying both steps would just result in the equivalent of 2 X-gates being performed on the target qubit, hence having no effect.

Next, I used one Toffoli gate controlled on 1 on the 2nd and 3rd qubits to check for the following condition:

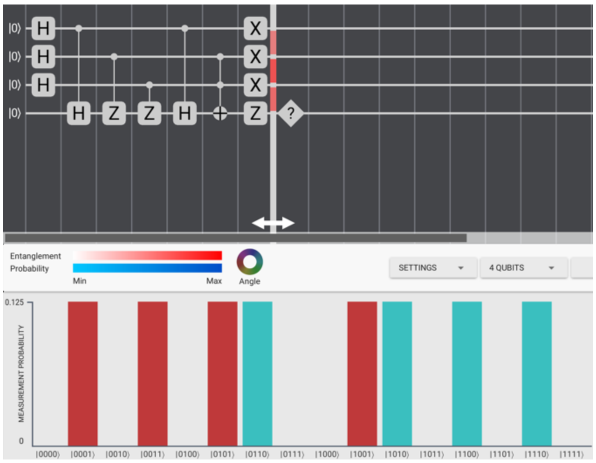
- 2nd ***AND*** 3rd qubit set to 1

Note the "set to 1" again.

This last part flips the states $|1100\rangle$ and $|1110\rangle$ to $|1101\rangle$ and $|1111\rangle$.



Now, all the states with the first 3 qubits having more qubits set to 1 (0) would have the target qubit set to 1 (0). This is the exact opposite of what we want. Hence, all we need to do is flip the first three qubits with X-gates on the 1st, 2nd, and 3rd qubits to flip it so that states with the first 3 qubits having more qubits set to 0 (1) would have the target qubit set to 1 (0).



The last Z-gate is only part of this explanation to apply a phase on these marked states (target qubit set to 1) to make the probabilities in the bottom part of the screenshot red to make it easier to distinguish between possible results. It has no effect on the result that is measured with the measure gate.

b.