MATH 4650: Review for Exam #1

First, before itemizing all you need to know, let us start by the good part. Here what you do not need to know. We skipped these sections so you do not need to know them.

1. Section 2.5.3 – Convergence of iterative methods
2. Section 2.6.3 – Conjugate Gradient method
3. Section 2.6.4 – Preconditioning
4. Section 2.7.2 – Broyden's method

We went over all this but you do not need to know this for the exam #1.

1. Section 0.1 – Evaluating Polynomials (we will review this starting next week anyhow)
2. Section 1.3.2 – The Wilkinson polynomial
3. Section 1.3.3 – Sensitivity of Root Finding
4. In Section 1.5.1, you need to know Secant's method but you do not need to know False Position, Muller's, nor IQI.
5. Section 1.5.2 – Brent's Method

Here what you need to know for the exam #1.

1. Chapter 0
   (a) Convert between binary and decimal and vice versa. Integer part and fractional/decimal part.
   (b) Floating point numbers in IEEE-754 standard: mantissa+exponent+bit sign, hidden bit, machine epsilon (ε_{mach}).
   (c) Avoid loss of precision when subtracting two numbers which are close from each other (whenever possible). E.g. for the quadratic formula.
   (d) Mean Value Theorem.
   (e) Intermediate Value Theorem.
   (f) Taylor's Theorem with remainder
      - Taylor polynomial of degree k at x_0 with Taylor remainder:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(k)}}{k!}(x_0)(x - x_0)^k + \frac{f^{(k+1)}(c)}{(k+1)!}(x - x_0)^{k+1}.$$

   (g) Able to use Taylor's theorem to compute an approximation of a function at a given point. Able to predict the accuracy of the approximation. (A typical exercise would be HW#1.6.)
2. Chapter 1
   (a) Bisection Method
      i. bracketing a root: find interval $[a, b]$ such that $f(a)f(b) < 0$
      ii. compute approximation x_a to root r of $f(x) = 0$ using bisection method on $[a, b]$
      iii. error at nth step: $|x_a - r| \leq \dfrac{b-a}{2^{n+1}}$
      iv. remember that function f needs to be continuous on interval
   (b) Fixed Point Iteration
      i. Assume that g is continuously differentiable, that $g(r) = r$, and that $S = |g'(r)| < 1$. Then FPI converges linearly with rate S to the fixed point r for initial guess sufficiently close to r.

   ii. find $g(x)$ such that $f(x) = g(x) - x$ with $|g'(r)| < 1$ for a given $f(x)$.
   iii. error convergence: $e_{n+1} \approx Se_n$ where $S = |g'(r)|$.

(c) Forward and Backward Error, and Error Magnification Factor in root finding
   i. forward error (error in output): $|r - x_a|$
   ii. backward error (error in input): $|f(r) - f(x_a)| = |f(x_a)|$ since $f(r) = 0$.
   iii. error magnification factor: relative forward error divided by relative backward error.

(d) Newton's Method
   i. Definition of *root of multiplicty m*.
   ii. Definition of *quadratic convergence*.
   iii. Being able to derive Newton's method from a picture
   iv. Let $f(x)$ be twice continuously differentiable at r and $f(r) = 0$. If $f'(r) \neq 0$, then

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

   is locally and quadratically convergent to r.
   v. error convergence: $e_{n+1} \approx Me_n^2$ (quadratic) where $M = \left|\frac{f''(r)}{2f'(r)}\right|$
   vi. for multiple root: $m+1$ times continuously differentiable function f on $[a, b]$ with multiplicity m root at r. Locally linear convergent with $e_{n+1} \approx Se_n$ where $S = (m-1)/m$.

(e) Secant method
   i. Being able to derive the Secant method from a picture
   ii. Assuming that the Secant method converges to r and $f'(r) \neq 0$, then

$$e_{i+1} = \left|\frac{f''(r)}{2f'(r)}\right| e_i e_{i-1}$$

   and

$$e_{i+1} = \left|\frac{f''(r)}{2f'(r)}\right|^{\alpha-1} e_i^{\alpha}$$

   where $\alpha = (1 + \sqrt{5})/2$
   iii. convergence to simple roots is superlinear

3. Chapter 2

(a) infinity norm and matrix norm

(b) Forward and Backward Error, and Error Magnification Factor
   i. forward error (error in output): $\|x - x_c\|_\infty$
   ii. backward error (error in input): $\|b - Ax_c\|_\infty$
   iii. error magnification factor: relative forward error divided by relative backward error.
   iv. condition number: $\kappa = \|A\|_\infty \|A^{-1}\|_\infty$
   v. condition number is the maximum possible error magnification factor for solving $Ax = b$ over all right-hand sides b and all possible errors of a given amplitude

(c) Direct method: $A = LU$ and $PA = LU$ factorization
   i. solve the system $Ax = b$, up through 4×4 using $PA = LU$ with back substitution
   ii. approximate number of operations is $2n^3/3$
   iii. permutation matrices and elementary matrices

(d) Iterative methods
   i. strictly diagonally dominant

ii. Jacobi: $x_{k+1} = D^{-1}(b - (L + U)x_k)$

iii. Gauss-Seidel: $x_{k+1} = D^{-1}(b - Ux_k - Lx_{k+1})$ or $x_{k+1} = (D + L)^{-1}(b - Ux_k)$

(e) Symmetric Positive Definite matrices

i. symmetric: $A^T = A$

ii. symmetric positive definite: $A^T = A$ and $x^T Ax > 0$ for all $x \neq 0$

iii. symmetric positive definite if and only if symmetric and all eigenvalues of A are positive.

iv. show a matrix is symmetric positive definite (or not)

(f) Cholesky Factorization

i. If A is symmetric positive definite, then $A = R^T R$ where R is upper triangular

ii. A is SPD if and only if the Cholesky factorization of A exists

iii. compute the Cholesky factorization, R, for A of size 3×3

iv. approximate number of operations is $n^3/3$

(g) Nonlinear systems of equations, $F(x) = 0$

i. Jacobian matrix, $DF(x)$

ii. Linear approximation, $L(x)$, using Taylor's expansion

iii. Multivariate Newton's Method

$$x_{k+1} = x_k - (DF(x_k))^{-1}F(x)$$