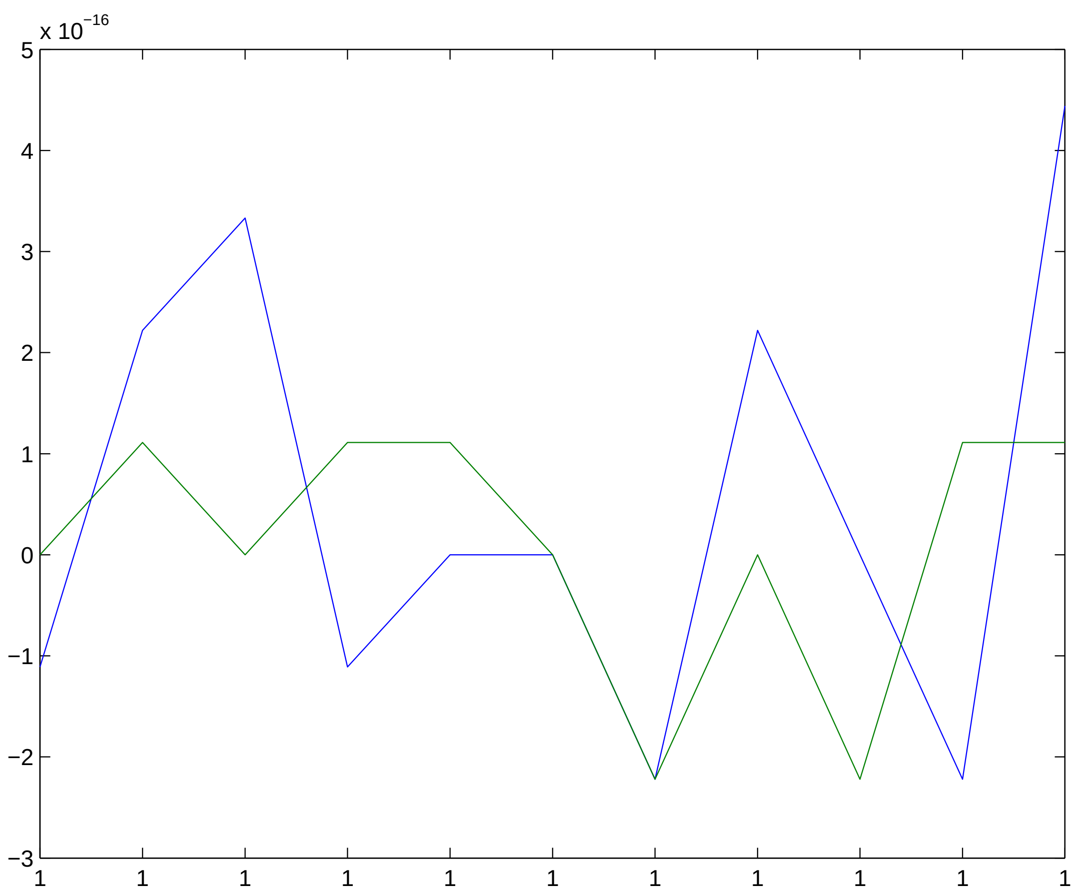
ERRORS and LIMITS to ACCURACY

Example : compute $1 - 3x + 3x^2 - x^3$ near 1 in DP FP.

```
for x = 1+10^(-6)*[-1:.2:1]
 disp([ x  1-3*x+3*x^2-x^3   1+x*(-3+x*(3-x)) ])
end
0.999999  -1.1102230246e-16  0
0.9999992  2.2204460492e-16  1.1102230246e-16
0.9999994  3.3306690738e-16  0
0.9999996 -1.1102230246e-16  1.1102230246e-16
0.9999998  0                 1.1102230246e-16
1          0                 0
1.0000002 -2.2204460492e-16 -2.2204460492e-16
1.0000004  2.2204460492e-16  0
1.0000006  0                -2.2204460492e-16
1.0000008 -2.2204460492e-16  1.1102230246e-16
1.000001   4.4408920985e-16  1.1102230246e-16
```



LIMITS to ACCURACY CONT.

Matlab check using zero finding function "fzero"

```
P = @(x) 1-3*x+3*x^2-x^3;
r = fzero( @(x)P(x), 1.5 ); disp([r r-1 P(r)])
  1.00000018358296   1.8358296e-07      0
P = @(x) 1+x*(-3+x*(3-x));
r = fzero( @(x)P(x), 1.5 ); disp([r r-1 P(r)])
  0.999996829235169 -3.1707648e-06      0
```

Forward and Backward Error

- Definitions: if $f(x)$ has root r with $f(r) = 0$, $x_a \approx r$,
  backward error in x_a is $|f(x_a)|$;
  forward error in x_a is $|x_a - r|$.
  Backward error is amount needed to change problem so that x_a is root.
  Forward error is amount needed to change x_a so that x_a is root.
  For prev. example, backward errors are $\approx 10^{-16}$, but the forward errors are $\approx 10^{-6}$.

LIMITS to ACCURACY CONT.

- Multiple roots: a root r has **multiplicity** m if
  $0 = f(r) = f'(r) = \ldots = f^{(m-1)}(r)$, $f^{(m)}(r) \neq 0$.
  If $m > 1$, f has a **multiple root** at r;
  if $m > 0$ is integer: $f(x) = (x-r)^m h(x)$, $h(r) \neq 0$.

- **Sensitivity** of root-finding: trying to solve $f(r) = 0$, we numerically solve $f(r + \Delta r) + \epsilon g(r + \Delta r) = 0$, for small (foward) Δr and (backward) ϵ.
  $f(r + \Delta r) + \epsilon g(r + \Delta r)$
  $\approx f(r) + \Delta r f'(r) + \epsilon g(r) + \epsilon \Delta r g'(r) = 0.$
  If $f(r) = 0$, $f'(r) \neq 0$ (simple root r),

$$\Delta r \approx -\epsilon g(r)/f'(r).$$

  Large forward errors sometimes occur
  even when backward errors are small for simple roots.
  Note: a double root, with $f'(r) = 0$, has
  $f(r + \Delta r) + \epsilon g(r + \Delta r) \approx$
  $f(r) + \Delta r f'(r) + (\Delta r)^2 f''(r)/2 + \epsilon g(r) + \epsilon \Delta r g'(r) = 0.$
  so

$$|\Delta r| \approx \sqrt{2\epsilon g(r)/f''(r)}.$$

  A triple root has $|\Delta r| \approx \sqrt[3]{6\epsilon g(r)/f'''(r)}$.

LIMITS to ACCURACY CONT.

- **Error magnification factor** (EMF)

$$\text{EMF} = \frac{\text{relative forward error}}{\text{relative backward error}}$$
$$= |\frac{\Delta r/r}{\epsilon g(r)/g(r)}| = |\frac{\epsilon g(r)/(rf'(r))}{\epsilon}| = \frac{|g(r)|}{|rf'(r)|}.$$

  Examples:

  a) $P(x) = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6)$
  $= x^6 - 21x^5 + \cdots + 720.$
  If $r = 6,\ \epsilon = 10^{-15}$, $g(x) = x^6$,
  $\Delta r = -10^{-15}(6)^6/120 \approx 4 \times 10^{-13}$, $EMF \approx 400.$

  b) Wilkinson polynomial
  $W(x) = (x-1)(x-2)\cdots(x-20)$
  $= x^{20} - 210x^{19} + \cdots + 20!.$
  If $r = 20,\ \epsilon = 10^{-15}$, $g(x) = x^{20}$,
  $|\Delta r| = 10^{-15}(20)^{20}/19! \approx 9\times 10^{-7}, EMF \approx 900000000.$

  Matlab check

```
W = @(x)prod(x-[1:20]);
r = fzero( @(x) W(x)+x^20/10^15 , 21 ) ;
disp([r r-20 W(r)])
19.999999138002  -8.62e-07 -104857509578.026
```

  c) $P(x) = 1 - 3x + 3x^2 - x^3$, with $r = 1,\ g(r) = 1$,
  $\epsilon = 10^{-15}$, has $|\Delta r| \approx 10^{-5},\ EMF \approx 10^{10}$.

- Conditioning: a problem is **ill-conditioned** if small changes in problem cause large changes in solution, the **condition number** K (or EMF) is $>> 1$.
  A **well-conditioned** problem has $K \approx 1$.
  Multiple root root-finding is ill-conditioned.