Bayesian Learning (732A73) Lab 1

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Assisgnment 1

1. *Daniel Bernoulli*
   Let $y_1, ..., y_n|\theta \sim \text{Bern}(\theta)$, and assume that you have obtained a sample with $s = 8$ successes in $n = 24$ trials. Assume a $\text{Beta}(\alpha_0, \beta_0)$ prior for θ and let $\alpha_0 = \beta_0 = 3$.

   (a) Draw random numbers from the posterior $\theta|y \sim \text{Beta}(\alpha_0 + s, \beta_0 + f)$, where $y = (y_1, \ldots, y_n)$, and verify graphically that the posterior mean and standard deviation converges to the true values as the number of random draws grows large.

   (b) Use simulation (`nDraws = 10000`) to compute the posterior probability $\Pr(\theta > 0.4|y)$ and compare with the exact value [Hint: `pbeta()`].

   (c) Compute the posterior distribution of the log-odds $\phi = \log \frac{\theta}{1-\theta}$ by simulation (`nDraws = 10000`). [Hint: `hist()` and `density()` can be utilized]

(a)

True mean for $Beta(\alpha, \beta)$ is:

$$E[\theta] = \frac{\alpha}{\alpha + \beta}$$

True σ^2 for $Beta(\alpha, \beta)$ is:

$$\sigma^2(\theta) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

The prior function for this problem is:

$$p(\theta) \propto \theta^{\alpha_0 - 1}(1 - \theta^{\beta_0 - 1})$$

The likelihood function is :

$$p(y_1, \ldots, y_n|\theta) \propto \prod_{i=1}^{n} p(y_i|\theta) = \prod_{i=1}^{n} \theta^{y_i}(1-\theta)^{1-y_i} = \theta^s(1-\theta)^f$$

posterior is :

posterior

$$p(\theta|y) \propto Beta(\alpha_0 + s, \beta_0 + f) = Beta(11, 19)$$

In [1]:
```
s = 8
n = 24
f = 16
alpha_0 = beta_0 = 3
```

In [2]:
```
NDraws = 10000
res = matrix(,NDraws,3)
colnames(res) = c("iter","mean","sd")
alpha_n=alpha_0+s
beta_n = beta_0+f
TrueMean = (alpha_n)/(alpha_n+beta_n)
TrueVar = sqrt((alpha_n*beta_n)/((alpha_n+beta_n)^2*(alpha_n+beta_n+1)))
TrueMean
TrueVar
```

0.366666666666667

0.0865507910219387

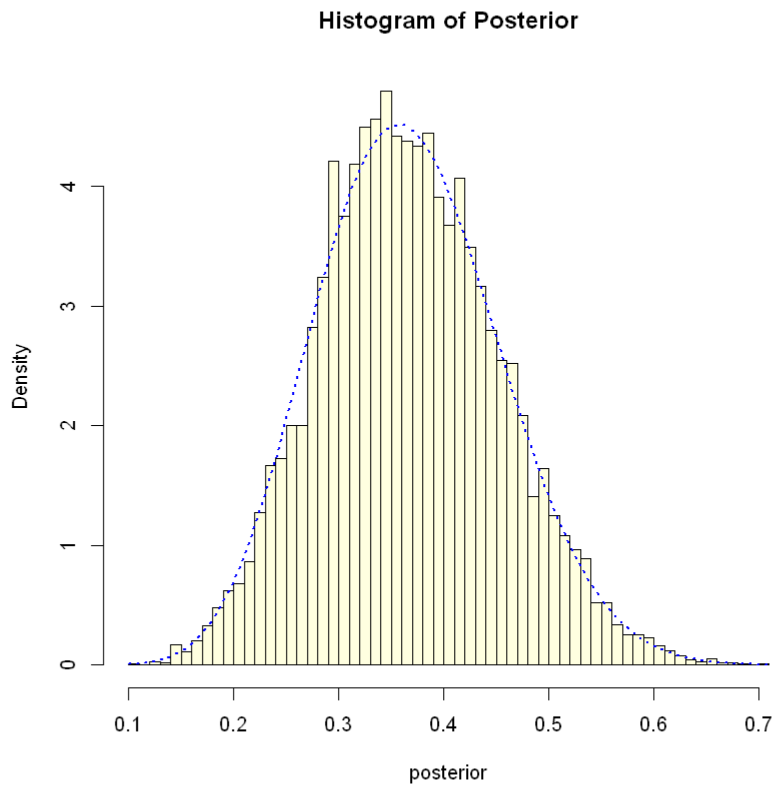
In [3]:
```
PostB <- function(alpha= alpha_n,beta =beta_n ,n=NDraws){
    for(i in 1:n){
        postB = rbeta(i,alpha,beta)
        res[i,] = c(i,mean(postB),sd(postB))

    }

    return(list(post=postB,mean_sd=res))
}
Posterior = PostB()
```

In [4]:
```
Posterior$mean_sd[1,3]=0
head(Posterior$mean_sd)
```

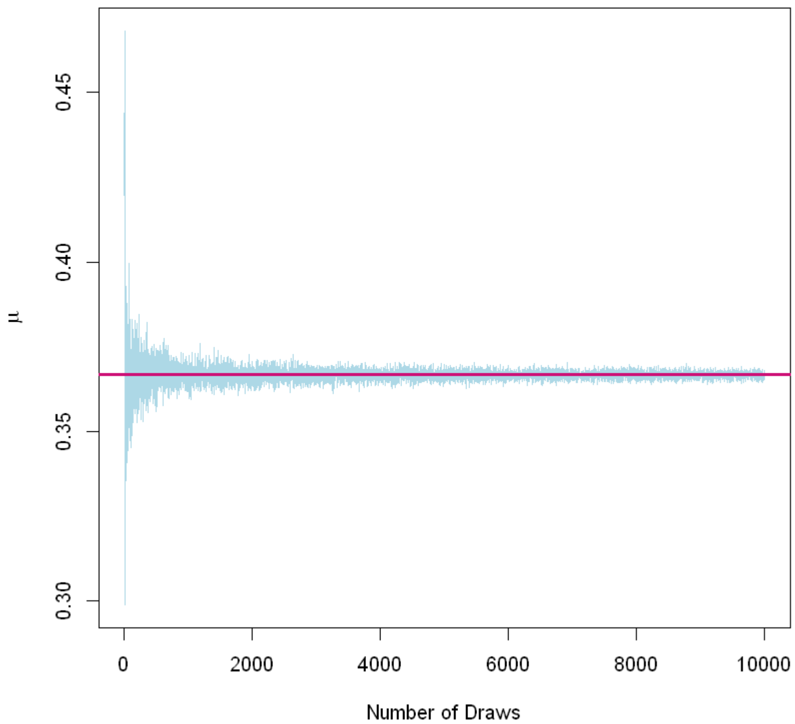
A matrix: 6 × 3 of type dbl

iter	mean	sd
1	0.4197914	0.00000000
2	0.4681419	0.02404432
3	0.2988524	0.02031929
4	0.3461667	0.08010881
5	0.3839380	0.05407235
6	0.3565289	0.07064133

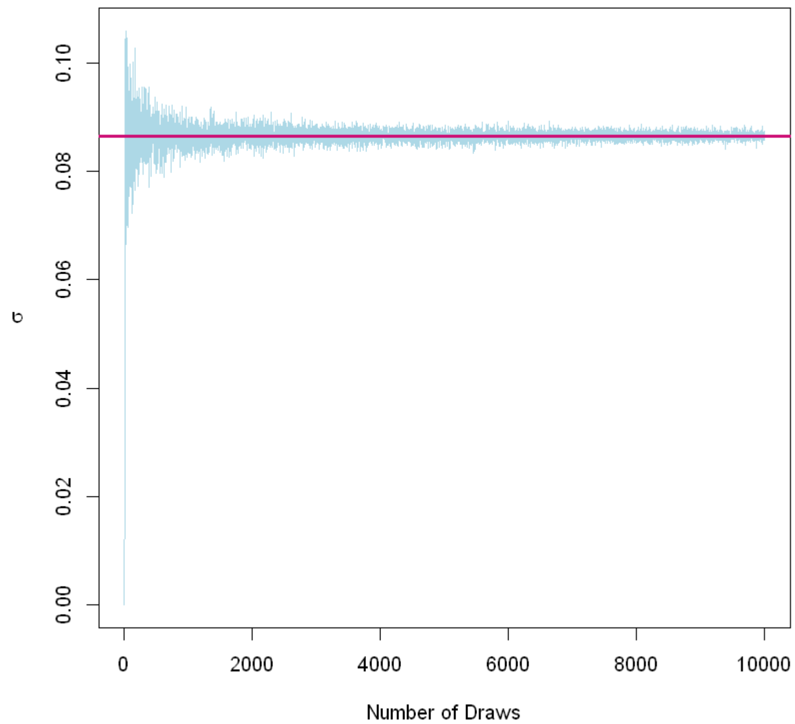
In [5]:
```
x <-seq( from =-5, to =10 , by =0.001)
hist(Posterior$post,col='lightyellow',freq = FALSE,breaks=50,xlab='posterior',mai
curve(dbeta(x,alpha_n,beta_n),add=TRUE,col='blue', lwd =2,lty=3)
```



In [6]:
```
plot(x=1:NDraws,y=Posterior$mean_sd[,2],col='lightblue',lwd =1,type='l',xlab="Num
abline(h=TrueMean,col='deeppink3',lwd=3)
```



In [7]:
```
plot(x=1:NDraws,y=Posterior$mean_sd[,3],col='lightblue',lwd =1,type='l',xlab="Nur
abline(h=TrueVar,col='deeppink3',lwd=3)
```



As we can see from the two plots above,for while the number of draws increases, both μ and σ converge to the true values.

(b)

we can see that our estimation of $Pr(\theta > 0.4|y)$ is very close to true theoratical value.

In [8]:
```
exact_prb =1- pbeta(0.4,alpha_n,beta_n)
pos_prob = length(Posterior$post[Posterior$post > 0.4])/length(Posterior$post)
prb = data.frame(exact_prb,pos_prob)
colnames(prb) = c('excact_prob',"simulated_value")
prb
```

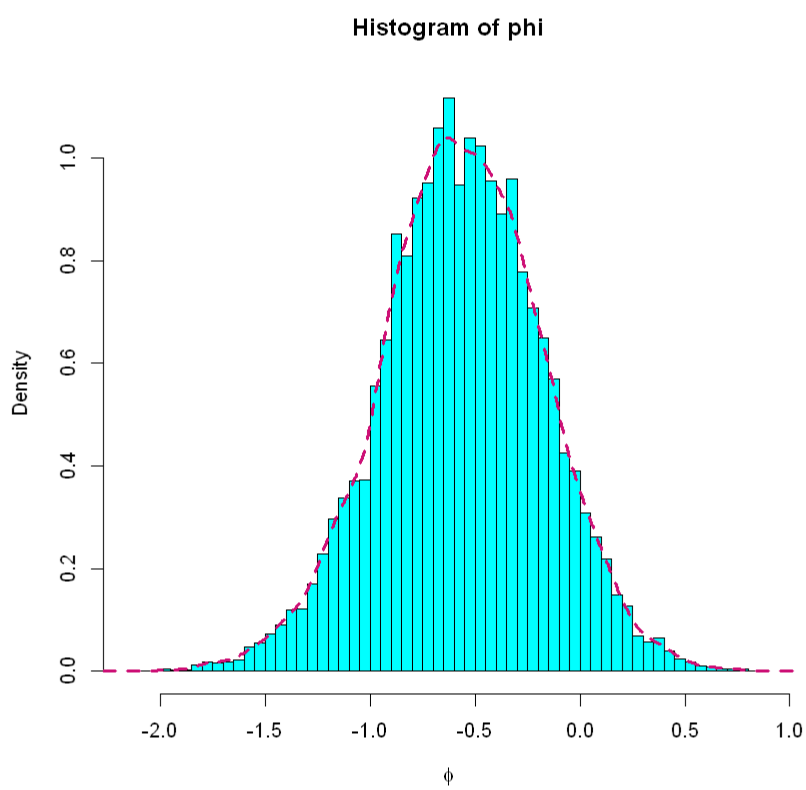
A data.frame: 1 × 2

excact_prob	simulated_value
<dbl>	<dbl>
0.3426654	0.3425

(C)

Histogram and kernel density of of the data simulated from the posterior distribution of the $\phi = log(\theta\,(1-\theta))$ with 10000 draws.

In [9]:
```
phi = log(Posterior$post/(1- Posterior$post))
hist(phi,breaks=100,probability = TRUE,col='cyan',xlab=expression(phi))
lines(density(phi),col='deeppink3',lty=2,lwd=3)
```



Assignment 2

2. *Log-normal distribution and the Gini coefficient.*
Assume that you have asked 10 randomly selected persons about their monthly income (in thousands Swedish Krona) and obtained the following ten observations: 38, 20, 49, 58, 31, 70, 18, 56, 25 and 78. A common model for non-negative continuous variables is the log-normal distribution. The log-normal distribution $\log \mathcal{N}(\mu, \sigma^2)$ has density function

$$p(y|\mu, \sigma^2) = \frac{1}{y \cdot \sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(\log y - \mu)^2\right],$$

where $y > 0$, $\mu > 0$ and $\sigma^2 > 0$. The log-normal distribution is related to the normal distribution as follows: if $y \sim \log \mathcal{N}(\mu, \sigma^2)$ then $\log y \sim \mathcal{N}(\mu, \sigma^2)$. Let $y_1, ..., y_n|\mu, \sigma^2 \overset{iid}{\sim} \log \mathcal{N}(\mu, \sigma^2)$, where $\mu = 3.8$ is assumed to be known but σ^2 is unknown with non-informative prior $p(\sigma^2) \propto 1/\sigma^2$. The posterior for σ^2 is the $Inv - \chi^2(n, \tau^2)$ distribution, where

$$\tau^2 = \frac{\sum_{i=1}^{n}(\log y_i - \mu)^2}{n}.$$

1- Simulate 10, 000 draws from the posterior of σ2 (assuming μ = 3.8) and com- pare it with the theoretical Inv − χ2(n, τ 2) posterior distribution.

In [10]:
```
m = 10000 #sample size
mu = 3.8
observ = c(38,20, 49, 58, 31, 70, 18, 56, 25,78)
n = length(observ)
```

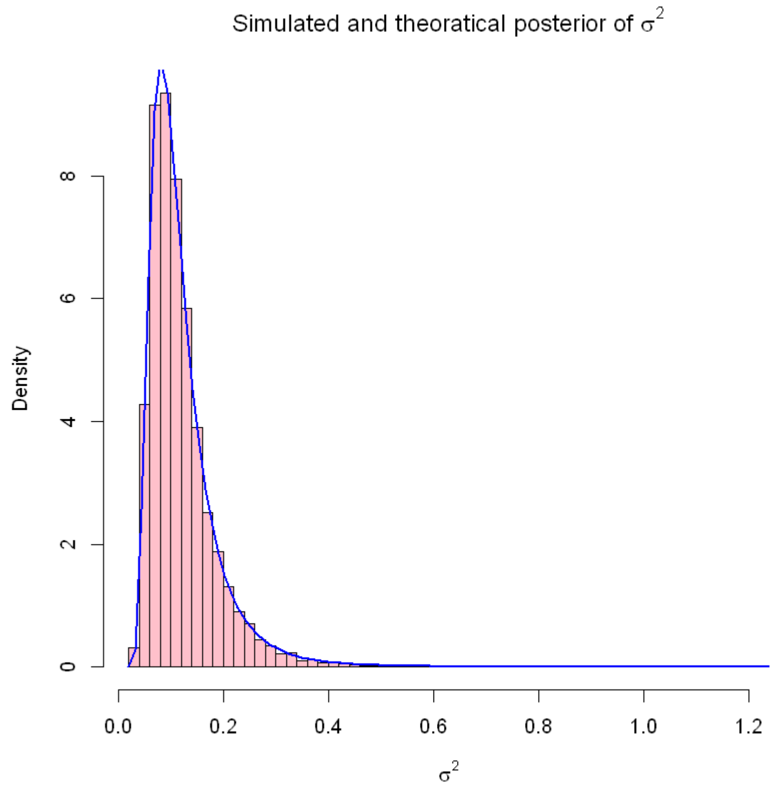
In [11]:
```
tau2 = function(data,mu){
    sum((log(data) -mu)^2 )/n
}
```

In [12]:
```
#install.packages('invgamma')
library(invgamma)
tau2(observ,mu )
```

0.261043665681099

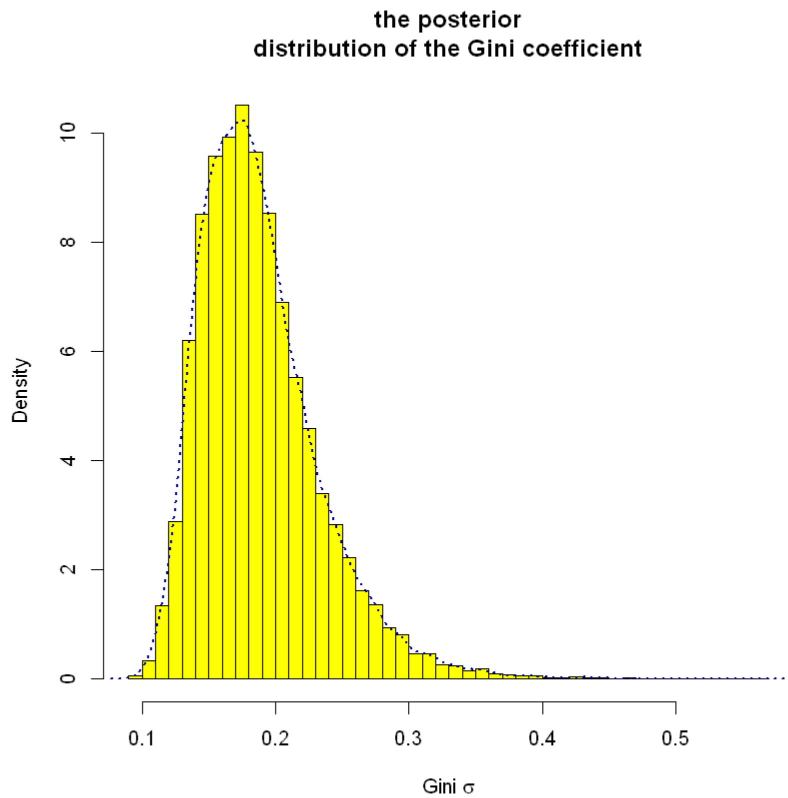
In [13]:
```
post_sigma2 <- function(m){
    set.seed(12345)
    rinvchisq(n = m,df = n,ncp = tau2(observ,mu))
}
```

In [14]:
```
x <-seq( from =0, to =10 , by =0.001)
hist(post_sigma2(m),probability = TRUE,col='pink',breaks = 50,main=expression(pas
    ,xlab = expression(paste(sigma ^2)))
curve(dinvchisq(x ,df = n,ncp = tau2(observ,mu)),add=TRUE,col='blue',lwd=2)
```



(b) The most common measure of income inequality is the Gini coefficient, G, where $0 \leq G \leq 1$. $G = 0$ means a completely equal income distribution, whereas $G = 1$ means complete income inequality (see Wikipedia for more information about the Gini coefficient). It can be shown that $G = 2\Phi\left(\sigma/\sqrt{2}\right) - 1$ when incomes follow a $\log \mathcal{N}(\mu, \sigma^2)$ distribution. $\Phi(z)$ is the cumulative distribution function (CDF) for the standard normal distribution with mean zero and unit variance. Use the posterior draws in a) to compute the posterior distribution of the Gini coefficient G for the current data set.

In [15]:
```
sigma2 = post_sigma2(m)
g_sigma = sqrt(sigma2)/sqrt(2)
gpdf =2* pnorm(q = g_sigma,mean = 0,sd = 1)-1
hist(gpdf,probability = TRUE,col='yellow',breaks=50,main='the posterior
distribution of the Gini coefficient',xlab=expression(paste('Gini ', sigma)))
lines(density(gpdf),col='darkblue',lty=3,lwd=2)
```



(c) Use the posterior draws from b) to compute a 90% equal tail credible interval for G. A 90% equal tail interval (a, b) cuts off 5% percent of the posterior probability mass to the left of a, and 5% to the right of b. Also, do a kernel density estimate of the posterior of G using the `density` function in R with default settings, and use that kernel density estimate to compute a 90% Highest Posterior Density Interval (HPDI) for G. Compare the two intervals.

In [16]:
```
alpha_conf = 0.1
q_lower = quantile(gpdf,alpha_conf/2)
q_upper = quantile(gpdf,1-alpha_conf/2)
c(q_lower, q_upper)
true_mean = mean(gpdf)
```

5%: 0.130774883443954 **95%:** 0.271906220898121

In [17]:
```
true_mean
```

0.188106773563226

HPD: shortest possible interval that under the posterior has the siginificance probability (ie. 0.9)

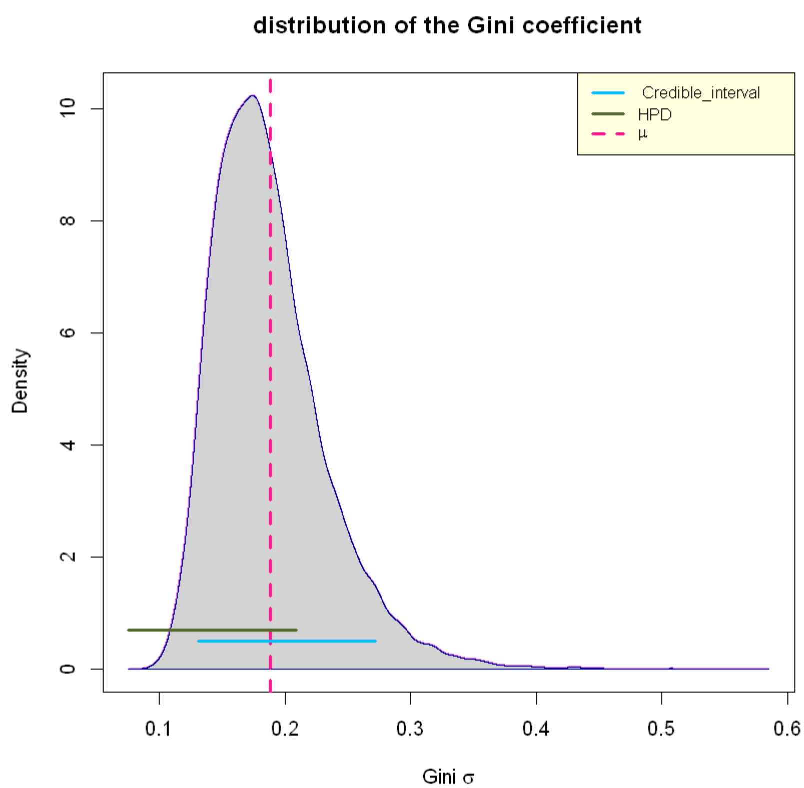
In [18]:
```
KerDenG=density(gpdf)
xx = KerDenG$x
yy = sort(KerDenG$y,decreasing = TRUE,index.return=TRUE)
```

we will use the cumulative sum of kernel density values to find the shortest interval in data that contains 0.9 of the data. the reason we can use cumulative to find HPD interval is because our data is unimodal.

In [19]:
```
CumSum = cumsum(yy$x)
HPD_D = CumSum[length(CumSum)]*0.9
temp = which(CumSum<HPD_D)
HPD_interval = range(xx[which(CumSum<HPD_D)])
```

In [20]:
```
df = data.frame('HPD_interval'=HPD_interval,'credible_interval'=c(q_lower, q_uppe
```

In [21]:
```
y_temp=density(gpdf,n=10000)
plot(x=y_temp$x,y=y_temp$y,type = 'l', lwd = 2, col = 'violet',main= 'distributic
     ,xlab=expression(paste('Gini ', sigma)),ylab='Density')
polygon(x=y_temp$x,y=y_temp$y, col = 'lightgrey',border = 'darkblue')
abline(v = true_mean, col="deeppink", lwd=3, lty=2)
#abline(v=q_upper)
#abline(v=q_lower)
#abline(v=HPD_interval[1],col='blue')
#abline(v=HPD_interval[2],col='blue')

segments(x0 =q_lower,y0 =0.5,x1 =q_upper,y1 =  0.5,col='deepskyblue1' ,lwd=3)
segments(x0=HPD_interval[1],y0=0.7,x1=HPD_interval[2],y=0.7,col='darkolivegreen',

legend("topright",
  legend = c(" Credible_interval", "HPD",expression(mu)),
  col = c('deepskyblue1','darkolivegreen','deeppink'), lty=c(1,1,2), cex=0.8,lwd=
```



Assignment 3

3. *Bayesian inference for the concentration parameter in the von Mises distribution.* This exercise is concerned with directional data. The point is to show you that the posterior distribution for somewhat weird models can be obtained by plotting it over a grid of values. The data points are observed wind directions at a given location on ten different days. The data are recorded in degrees:

$$(40, 303, 326, 285, 296, 314, 20, 308, 299, 296),$$

where North is located at zero degrees (see Figure 1 on the next page, where the angles are measured clockwise). To fit with Wikipedias description of probability distributions for circular data we convert the data into radians $-\pi \leq y \leq \pi$. The 10 observations in radians are

$$(-2.44, 2.14, 2.54, 1.83, 2.02, 2.33, -2.79, 2.23, 2.07, 2.02).$$

Assume that these data points are independent observations following the von Mises distribution

$$p(y|\mu,\kappa) = \frac{\exp\left[\kappa \cdot \cos(y-\mu)\right]}{2\pi I_0(\kappa)}, \quad -\pi \leq y \leq \pi,$$

where $I_0(\kappa)$ is the modified Bessel function of the first kind of order zero [see `?besselI` in R]. The parameter μ ($-\pi \leq \mu \leq \pi$) is the mean direction and $\kappa > 0$ is called the concentration parameter. Large κ gives a small variance around μ, and vice versa. Assume that μ is known to be 2.39. Let $\kappa \sim \text{Exponential}(\lambda = 1)$ a priori, where λ is the rate parameter of the exponential distribution (so that the mean is $1/\lambda$).

(a) Plot the posterior distribution of κ for the wind direction data over a fine grid of κ values.

(b) Find the (approximate) posterior mode of κ from the information in a).

(a) Posterior :

$$p(\kappa \mid y_1, y_2, \ldots, y_n) \propto p(y_1, y_2, \ldots, y_n \mid \kappa) \cdot p(\kappa)$$

$$p(\kappa \mid y_1, y_2, \ldots, y_n) \propto \left[\frac{1}{I_0(\kappa)}\right]^n \cdot \exp\left[\sum_{i=1}^{n} \kappa \cdot cos(y_i - \mu) - \lambda\kappa\right]$$

$$p(\kappa \mid y_1, y_2, \ldots, y_n) \propto \left[\frac{1}{I_0(\kappa)}\right]^n \cdot \exp\left[\sum_{i=1}^{n} \kappa \cdot cos(y_i - 2.39) - \kappa\right]$$

In [22]:
```
y_data =c(-2.44, 2.14, 2.54, 1.83, 2.02, 2.33, -2.79, 2.23, 2.07, 2.02)
```

In [23]:
```
n =length(y_data)
mu = 2.39
n
```

10

In [24]:
```
# a function to compute prior for k (exponential with lambda = 1)
prior_k <- function(k){
    dexp(x = k,rate = 1)
}
```

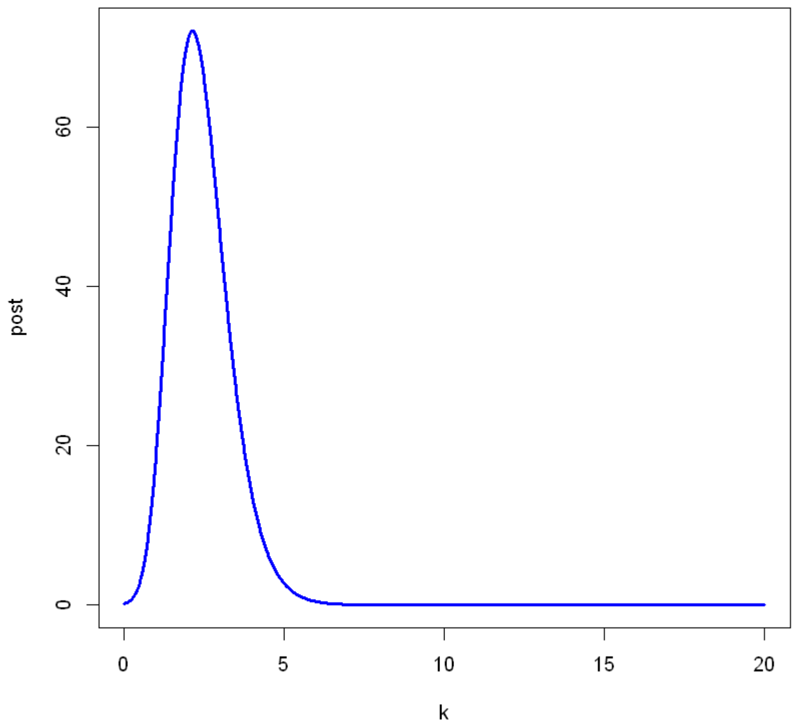
In [25]:
```
likelikood_k <- function(y,mu ,n,k){

    exp(k*sum(cos(y-mu)))/(2*pi*besselI(k,nu = 0)^n)
}
```

In [26]:
```
k= seq(from = 0,to = 20,by = 0.01)
```

In [27]:
```
#
posterior_k <- function(y=y_data,mu=mu ,n=n,k=k){
    likelikood_k(y,mu ,n,k )*prior_k(k)
}
```

In [28]:
```
post=posterior_k(y=y_data,mu=mu ,n=n,k=k)
```

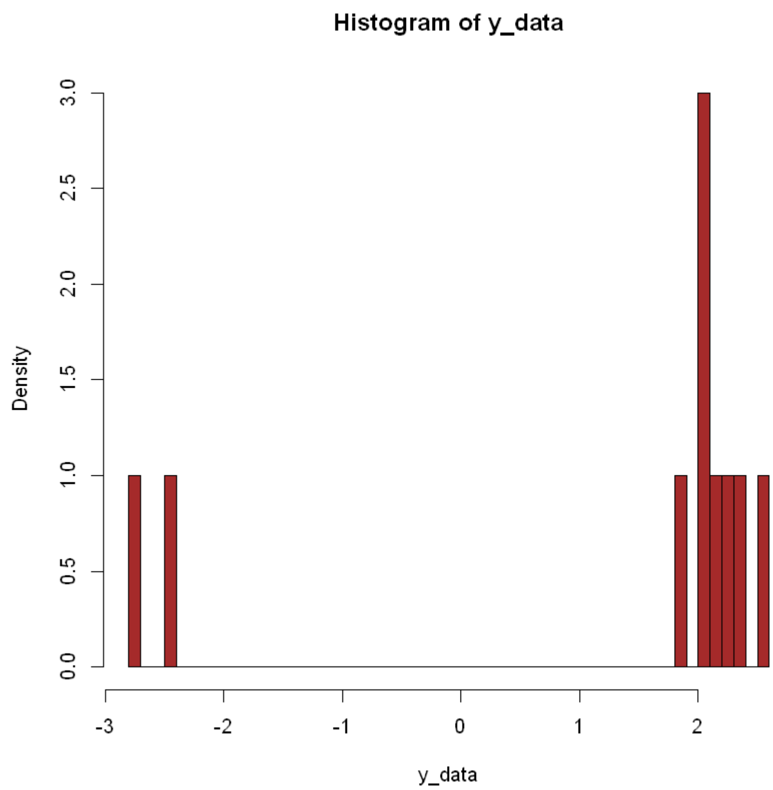
In [29]:
```
plot(y=post,x=k,col='blue',type='l',lwd=3)
```



In [30]:
```
k[which.max(post)]
```

2.12

In [31]:
```
hist(y_data,probability = TRUE,breaks = 50,col='brown')
```



In [32]:
```
y_data[which(y_data<0)]
```

-2.44 · -2.79