SOLUTIONS FOR RUNGE PHENOMENON

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Abstract. This is an article that introduces and proposes several mitigating methods for Runge phenomenon. While making guesses of functions with discrete data points, one can use polynomial interpolation. However, there were some problems using this method. Proposed higher degree polynomial are oscillating at the end of the given interval. This problem grows errors of the proposed polynomial. In this article, we are going to introduce several solutions for erasing errors for given high order polynomials.

AMS subject classifications. 65G30, 65G40, 65G99

Key words. Runge Phenomenon, polynomial Interpolation

1. Introduction. The Runge phenomenon which was discovered by Carl David Tolme Runge[2] is an oscillation where polynomial interpolation with high degree do not converge at the edge of the given interval. In order to observe Runge phenomenon, we introduce the following problem settings. First, suppose the function $f(x)$ and discrete nodes x_i proposed by [1].

$$f(x) = \frac{1}{1+25x^2} \tag{1.1}$$

$$x_i = \frac{2i}{n} - 1 \quad (i = 0, 1, \cdots, n) \tag{1.2}$$

The above function $f(x)$ is a transformed function of [1] to deal with the interval $[-1, 1]$. We are going to get n^{th} lagrange interpolation polynomial $p_n(x)$ using total $(n+1)$ discrete data nodes of x_i and $f(x_i)$. The Figure 1 is lagrange interpola-

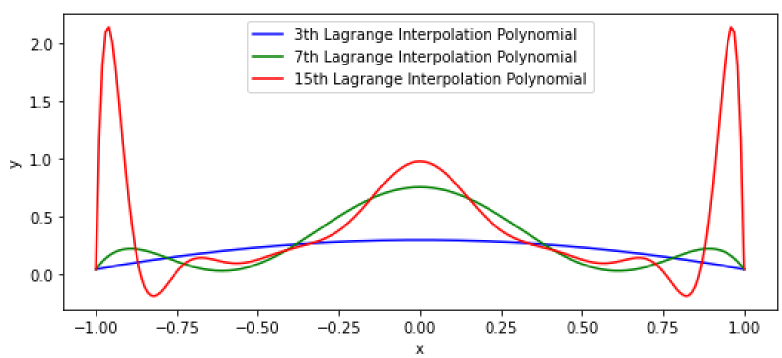

Fig. 1: Runge Phenomenon with different degree of Lagrange interpolation polynomials

tion polynomials with $p_3(x)$, $p_7(x)$ $p_{15}(x)$ each. We can observe that as the degree grows, polynomials diverge and oscillate at the edge of the interval. Then, why this phenomenon exists? We can suggest major two reasons of this phenomenon.

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Table 1: Maximum value of n^{th} order derivatives of $f(x)$

n	$\max\limits_{-1\leq x\leq 1} f^{(n)}(x)$
3	5.8357×10^{2}
4	1.5000×10^{4}
5	3.1393×10^{5}
6	5.9714×10^{6}
7	3.4307×10^{8}
8	1.5750×10^{10}
9	6.3331×10^{11}
10	2.3377×10^{13}
11	1.7722×10^{15}
12	1.1694×10^{17}
13	7.0001×10^{18}
14	3.9009×10^{20}
15	3.7110×10^{22}

1.1. Major Two Reasons of Runge Phenomenon. First, the the high order derivative becoming larger so fast when n increases. This Table 1 shows the maximal value of high order derivatives with given $f(x)$. Given $f(x)$ has rapid change at the center and this fact goes along with rapid change of its high order derivatives at the center. On the other hand, at the edge of the given interval, $f(x)$ shows gradually slow change and this does not match with its high order rapidly changing derivatives. Therefore, when we try to take high degree Lagrange interpolation method, we can observe oscillation at the end of the interval.

Second, the points are equally distributed. Figure 2 shows $f(x)$ and used 16 data points at 15^{th} Lagrange interpolation polynomials. Although we have different

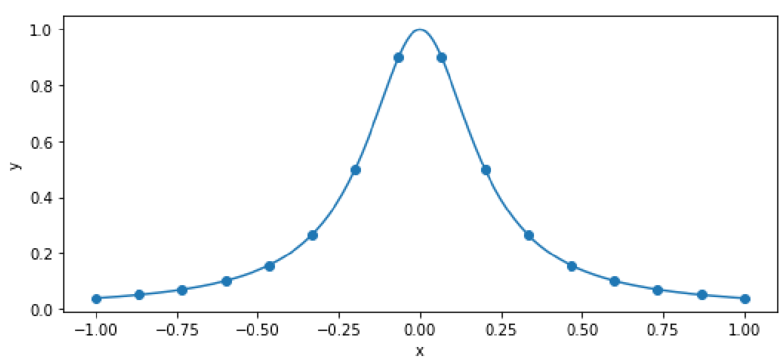


Fig. 2: $f(x) = \frac{1}{1+25x^2}$ and 16 nodes for Lagrange interpolation

inclinations near 0.0 and 1.0, we give the same number of nodes to interpolation polynomial. These equal spaced nodes are not enough to represent the different inclinations of given function $f(x)$.

Due to the two reasons above, high order interpolation polynomials oscillate at the edge of the interval, now it is popular as Runge phenomenon. In this paper, we are going to introduce several solutions to deal with Runge phenomenon. We wrote the proof that upper bound for error goes to infinity at Appendix A.

2. Methods. We introduce three main techniques in our experiments: Chebyshev Nodes, Cubic Spline Curves, Bernstein Polynomial.

2.1. Chebyshev Node: giving denser nodes at the edge of the interval. In order to deal with equally distributed nodes problem, we suggest using Chebyshev node instead. Chebyshev node refers to the points projecting the equally divided arc points of a semicircle to x-axis.

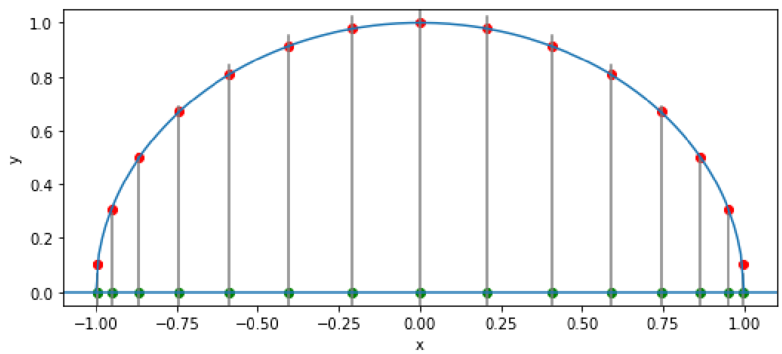


Fig. 3: Chebyshev nodes

$$x_k = \cos\frac{(2k-1)\pi}{2n} \quad (k = 1, \cdots, n) \tag{2.1}$$

Figure 3 shows 15 chebyshev nodes at $[-1, 1]$ using (2.1) with $n = 15$. These nodes are distributed much denser at the edge of the given interval. As we need more points nearby the edge part, this proposed chebyshev nodes alleviate oscillation. Also, due to same reason, other methods making more points at the edge of given interval can powerfully work to manage runge phenomenon.

2.2. Cubic Spline Curves: having more piecewise polynomials than growing the degree of polynomial. To avoid making general high order polynomial, we can set more piecewise-polynomials. The most well-known piecewise-polynomial approximation is cubic spline interpolation.

Definition 2.1 (Cubic Spline). *Given a function f defined on $[a, b]$ and a set of nodes $a = x_0 < x_1 < \cdots < x_n = b$, a cubic spline interpolant S for f is a function that satisfies the following conditions:*

(a) $S(x)$ is a cubic polynomial, denoted $S_j(x)$, on the subinterval $[x_j, x_{j+1}]$ for each $j = 0, 1, \cdots, n-1$;
(b) $S_j(x_j) = f(x_j)$ and $S_j(x_{j+1}) = f(x_{j+1})$ for each $j = 0, 1, \cdots, n-1$;
(c) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ for each $j = 0, 1, \cdots, n-2$;
(d) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for each $j = 0, 1, \cdots, n-2$;
(e) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for each $j = 0, 1, \cdots, n-2$;
(f) One of the following sets of boundary conditions is sastisfiend:
  (i) $S''(x_0) = S''(x_n) = 0$ (natural boundary);
  (ii) $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$ (clamped boundary);

Using above Definition 2.1, we can set a number of pieces on the given interval and cubic polynomials. As we take these conditions about derivatives of first and second order to make interpolation polynomial $p_n(x)$, this new $p_n(x)$ has a constraint which does not need to touch high order derivatives of $f(x)$. Therefore, we can prevent Runge phenomenon.

2.3. Bernstein polynomial: using well-known converging method. We suggest using converging non-interpolating method which has more computational cost.

$$b_{(v,n)}(x) = \binom{n}{v} x^v (1-x)^{n-v} \tag{2.2}$$

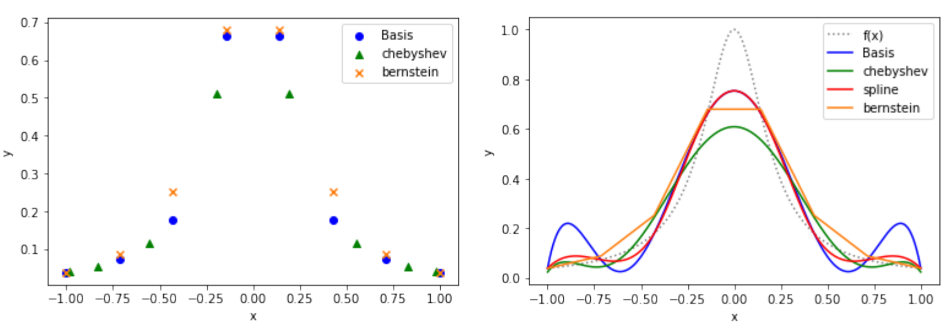
$$B_n(f)(x) = \sum_{v=0}^{n} f\left(\frac{v}{n}\right) b_{(v,n)}(x) \tag{2.3}$$

The convergence of this method is proved by Theorem B.1. Although we have to compute the sum of $b_{(i,n)}(x)$ and this is not an interpolating method, we can get non-oscillating polynomial.

3. Experimental results. To compare the level of oscillating between above solutions, we made several experimental settings. With the convenience of this comparison, we call polynomials by lagrange interpolation with equal distribution as *Basis*, polynomials by lagrange interpolation with non-equal distribution using chebyshev nodes as *Chebyshev*, polynomials by cubic spline interpolation as *Spline* and polynomials by bernstein polynomial as *Bernstein*. For representing original answer $f(x)$, it is expressed as gray dotted line for all following experiments.

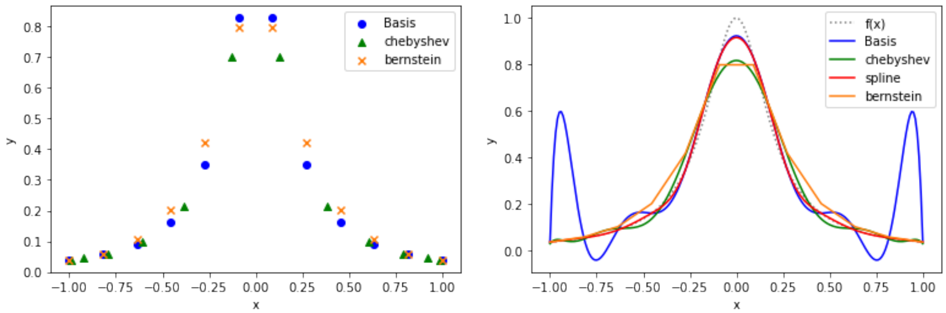
3.1. $n = 7$**.** Figure 4a represents nodes which were made with equal space (basis), non-equal space (Chebyshev) and used node at Bernstein polynomial. Figure 4b shows $f(x)$, basis, Chebyshev, cubic spline, Bernstein polynomials each. Although there is not much runge phenomenon even for basis, the other polynomials also stick to the given $f(x)$. However, the error at the center is significantly found.

(a) $n = 7$ nodes (b) $n = 7$ polynomials



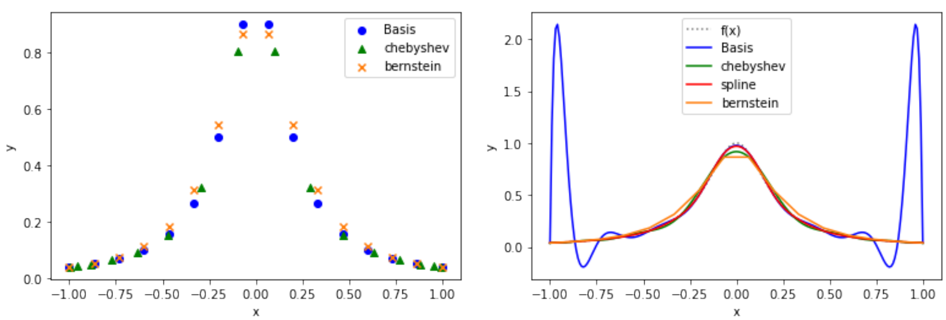
3.2. $n = 11$**.** Figure 5a represents nodes which were made with equal space (basis), non-equal space (Chebyshev) and used node at Bernstein polynomial. Figure 5b shows $f(x)$, basis, Chebyshev, cubic spline, Bernstein polynomials each. The oscillation is much bigger than Figure 4b, the error at the center gets smaller. We also find the desity for chebyshev node at the edge of interval increased than Figure 4a.

(a) $n = 11$ nodes (b) $n = 11$ polynomials



3.3. $n = 15$**.** Figure 6a represents nodes which were made with equal space (basis), non-equal space (Chebyshev) and used node at Bernstein polynomial. Figure 6b shows $f(x)$, basis, Chebyshev, cubic spline, Bernstein polynomials each. While Basis shows tremendous oscillation at the edge of the interval, all the other polynomials approximate the given $f(x)$ without diverging at the edge.

(a) $n = 15$ nodes (b) $n = 15$ polynomials



4. Discussions. As the degree gets higher, the overall approximation precision gets better at all four comparisons except for basis near 1.0 ad -1.0. Also we made more higher degree experiments, the general tendency such as approximations becoming more well-fitted and oscillation getting bigger maintained. Therefore, we did not append more experiments.

5. Conclusions. The experimental results from our experiments show that all three solutions have meaningful effect to mitigating runge phenomenon. Also, when the degree goes up a certain level, there is no meaning to compare between Chevbyshev, cubic spline and Bernstein. They all approximate well to given $f(x)$. One direction for future work is suggesting another way to get denser at the edge of the interval. For Chebyshev it uses a circle for making denser nodes, we can use ellipse or another geometric shape instead.

Also, we open the source code at here.

Appendix A. Mathematical perspective for Runge phenomenon. First, let error term between $f(x)$ and polynomial.

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\Pi_{i=0}^{n}(x - x_i), \quad \xi \in (-1, 1)$$

Therefore,

$$\max_{-1\leq x\leq 1} |f(x) - p_n(x)| \leq \max_{-1\leq x\leq 1} \frac{|f^{n+1}(\xi)|}{(n+1)!} \max_{-1\leq x\leq 1} \Pi_{i=0}^{n}(x - x_i)$$

Second, let $q_n(x)$ such as

$$q_n(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$$

Also let $Q_n(x)$ for maximum value of $q_n(x)$.

$$Q_n(x) = \max_{-1\leq x\leq 1} |q_n(x)|$$

As we have x_i for

$$x_i = \frac{2i}{n} - 1 \quad (i = 0, 1, \cdots, n)$$

We can set following inequality.

$$Q_n(x) \leq n!\left(\frac{2}{n}\right)^{n+1}$$

Third, suppose we have bound for high order derivative of f.

$$\max_{-1\leq x\leq 1} |f^{(n+1)}(\xi)| \leq M_{n+1}$$

Thus,

$$\max_{-1\leq x\leq 1} |f(x) - p_n(x)| \leq M_{n+1}\frac{\left(\frac{2}{n}\right)^{n+1}}{(n+1)}$$

As for our given $f(x) = \frac{1}{(1+25x^2)}$ has growing $f^{(n)}(x)$ as $n \to \infty$, these upper bound for error term goes to infinity.

Appendix B. The Weierstrass Approximation Theorem.

THEOREM B.1 (The Weierstrass Approximation Theorem). *Given $f \in C[a, b]$ and $\epsilon > 0$, there is a polynomial p such that $\|f - p\|_\infty < \epsilon$. Hence, there is a sequence of polynomials (p_n) such that $p_n \rightrightarrows f$ on $[a, b]$.*

Acknowledgments. We would like to thank professor Jae-Hun Jung and his dedication for Intro. to Numerical Analysis lecture for proposing this significant interest of numerical analysis.

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