EEL 4930 Lecture 8

1 Introduction to Conditional Probability

(See the Jupyter notebook)

Another example

EX **EX Defective computers in a lab**

A computer lab contains

- two computer from manufacturer A, one of which is defective
- three computers from manufacturer B, two of which are defective

A user sits down at a computer at random. Let the properties of the computer he sits down at be denoted by a two letter code, where the first letter is the manufacturer and the second letter is D for a defective computer and N for a non-defective computer. (We add a subscript to differentiate computers with the same two-letter code.)

$$S = \{AD, AN, BD_1, BD_2, BN\}$$

Let

- E_A be the event that the selected computer is from manufacturer A
- E_B be the event that the selected computer is from manufacturer B
- E_D be the event that the selected computer is defective

Find

$P(E_A) =$ ________	$P(E_B) =$ ________	$P(E_D) =$ ________

- Now, suppose that I select a computer and tell you its manufacturer. Does that influence the probability that the computer is defective?

- Ex: Suppose I tell you the computer is from manufacturer A. Then what is the prob. that it is defective?

We denote this prob. as $P(E_D|E_A)$
(means: *the conditional probability of event E_D given that event E_A occured*)

$$S = \{AD, AN, BD_1, BD_2, BN\}$$

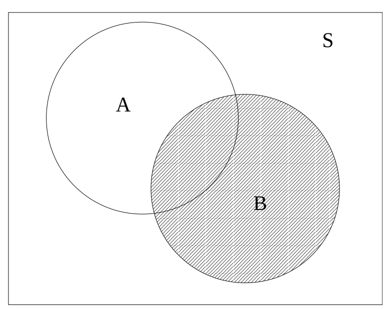
Find

- $P(E_D|E_B) =$________
- $P(E_A|E_D) =$________
- $P(E_B|E_D) =$________

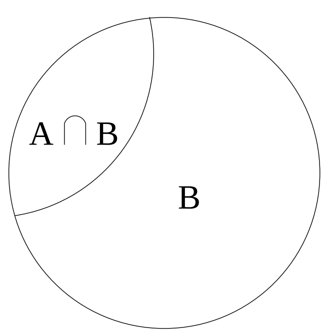
We need a systematic way of determining probabilities given additional information about the experiment outcome.

2 Formally Definining Conditional Probabilty

Consider the Venn diagram:



If we condition on B having occurred, then we can form the new Venn diagram:



This diagram suggests that if $A \cap B = \emptyset$ then if B occurs, A could not have occurred.

Similarly if $B \subset A$, then if B occurs, the diagram suggests that A must have occurred.
A definition of conditional probability that agrees with these and other observations is:

> For $A \in \mathcal{F}$, $B \in \mathcal{F}$, the *conditional probability* of event A *given* that event B occurred is
>
> $$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ for } P(B) > 0.$$

Claim: If $P(B) > 0$, the conditional probability $P(|B)$ satisfies the axioms on the original sample space
$(S, \mathcal{F}, P(\cdot|B))$

Check the axioms:

1.

$$P(S|B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

2. Given $A \in \mathcal{F}$,

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

and $P(A \cap B) \geq 0$, $P(B) \geq 0$

$$\Rightarrow P(A|B) \geq 0$$

3. If $A \subset \mathcal{F}$, $C \subset \mathcal{F}$, and $A \cap C = \emptyset$,

$$\begin{aligned} P(A \cup C|B) &= \frac{P[(A \cup C) \cap B]}{P[B]} \\ &= \frac{P[(A \cap B) \cup (C \cap B)]}{P[B]}. \end{aligned}$$

Note that $A \cap C = \emptyset \Rightarrow (A \cap B) \cap (C \cap B) = \emptyset$, so

$$\begin{aligned} P(A \cup C|B) &= \frac{P[A \cap B]}{P[B]} + \frac{P[C \cap B]}{P[B]} \\ &= P(A|B) + P(C|B) \end{aligned}$$

EX **Check prev. example:** $S = \{AD, AN, BD, BD, BN\}$

$P(E_D|E_A) =$

$P(E_D|E_B) =$

$P(E_A|E_D) =$

$P(E_B|E_D) =$

Relating Conditional and Unconditional Probs

Which of the following statements are true?

(a) $P(A|B) \geq P(A)$

(b) $P(A|B) \leq P(A)$

(c) Not necessarily (a) or (b)

2.1 Conditional Probability for Discrete Sample Spaces with Equal Probabilities

Conditional probability, independence, and mutually exclusive events for discrete sample spaces with equal probabilities:

EX **Take 2:** XOR of Two Independent Binary Values

Flip a fair coin with sides labeled '0' and '1' two times. Let E_i denote a '1' on the top face on flip i. Let F denote the event that the XOR of the values observed on the top faces on the two flips is '1'.

3 Using Conditional Probability to Decompose Events: Chain Rules, Partitions, and Total Probability

3.1 Chain Rules

Chain rule for expanding intersections

$$\begin{aligned}
\text{Note that } P(A|B) &= \frac{P(A \cap B)}{P(B)} \\
\Rightarrow P(A \cap B) &= P(A|B)P(B) \qquad (1)
\end{aligned}$$

$$\begin{aligned}
\text{and } P(B|A) &= \frac{P(A \cap B)}{P(A)} \\
\Rightarrow P(A \cap B) &= P(B|A)P(A) \qquad (2)
\end{aligned}$$

- Eqns. (1) and (2) are chain rules for expanding the probability of the intersection of two events
- The chain rule can be easily generalized to more than two events

  Ex: Intersection of 3 events

$$\begin{aligned}
P(A \cap B \cap C) &= \frac{P(A \cap B \cap C)}{P(B \cap C)} \cdot \frac{P(B \cap C)}{P(C)} \cdot P(C) \\
&= P(A|B \cap C)P(B|C)P(C)
\end{aligned}$$

3.2 Statistical Independence

- In the last example, we had several probabilities of the form $P(A|B) = P(A)$
- In this case, we say that A is *statistically independent (s.i.)* of B, since the probabilities of A are not affected by knowledge of A having occurred
- By the chain rule, $P(A \cap B) = P(A|B)P(B)$. So if $P(A) > 0$, $P(B > 0)$, and $P(A|B) = P(A)$, then

$$\begin{aligned}
P(B|A) &= \frac{P(A \cap B)}{P(A)} \\
&= \frac{P(A|B)P(B)}{P(A)} \\
&= \frac{P(A)P(B)}{P(A)} \\
&= P(B)
\end{aligned}$$

- So, if A is statistically independent of B, then B is statistically independent of A
- Thus, we can write $P(A \cap B) = P(A)P(B)$, and we use this for our definition of statistical independence because it works even when one of $P(A) = 0$ or $P(B) = 0$

Events A and B are *statistically independent (s.i.)* if and only if (iff)

$$P(A \cap B) = P(A)P(B).$$

- Events that arise from completely separate random phenomena are statistically independent.

EX

Take 3: A fair six-sided die is rolled twice. What is the probability of observing a 1 or a 2 on the top face on either roll of the die?