Fair die rolled 3 times

1) E_i = event that total $= i$

$$P(E_3) = P(\{(1,1,1)\}) = \frac{1}{|S|}$$

$$|S| = 6^3$$

$$\Rightarrow P(E_3) = \frac{1}{6^3} = \frac{1}{216} \approx 4.63 \times 10^{-3}$$

$$P(E_4) = P(\{(1,1,2), (1,2,1), (2,1,1)\})$$

$$= \frac{3}{6^3} = \frac{1}{72} \approx 1.39 \times 10^{-2}$$

$$E_5 = \{(1,1,3), (1,3,1), (3,1,1), (1,2,2), (2,1,2), (2,2,1)\}$$

$$P(E_5) = \frac{6}{6^3} = \frac{1}{36} \approx 2.78 \times 10^{-2}$$

3) (2)

E = event all 12 rolls are ≤ 4

Under Ho (the null hypothesis)
= dice are fair

$$P(E) = \left(\frac{4}{6}\right)^{12} \approx 7.7 \times 10^{-3}$$

Since $P(E) < 0.01$, we reject Ho.

4) Let F_i = event ith ~~pa~~ roll of pair has sum ≤ 6

Let F = event that every pair of dice has sum ≤ 6

Under Ho: dice are fair

$$P(F) = P\left(\bigcap_{i=1}^{4} F_i\right)$$

$$= \prod_{i=1}^{4} P(F_i)$$

$F_i = \{(1,1), (1,2), (1,3), (1,4), (1,5)$ →

(2,1), (2,2), (2,3), (2,4)
(3,1), (3,2), (3,3),
(4,1), (4,2),
(5,1)}

$|F_i| = 15$

Let S_i = sample space for ith roll of pair of dice

Then $|S_i| = 6^2 = 36$

$$P(F_i) = \frac{|F_i|}{|S_i|} = \frac{15}{36} = \frac{5}{12} \approx 0.417$$

$$P(F) \cong (0.417)^4 \approx 3.01 \times 10^{-2}$$

We cannot reject the hypothesis (H_0) that the die are fair because $P(F) = 3.01 \times 10^{-2} > 0.01$

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5) Let G = event every pair has sum ≤ 6 for 6 rolls

$$P(G) \cong (0.417)^6 \approx 5.3 \times 10^{-3}$$

Since $P(G) < 0.01$, we reject the null hypothesis (that the dice are fair) at the $p < 0.01$ level.

6) Let H = event that result is heads
F = event that coin is fair

$$P(H) = P(H|F)P(F) + P(H|\bar{F})P(\bar{F})$$

$$= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) + (1)\left(\frac{1}{3}\right)$$

1 two-head coin out of 3 total

2 fair out of 3 total

H & T are equally likely for fair coin

$$= \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right) = \frac{2}{3}$$

7) Let H_i = heads on flip i (5)

$$P(H_2 | H_1) = \frac{P(H_1 \cap H_2)}{P(H_1)}$$

$P(H_1) = 2/3$ from prob. 6

$$P(H_1 \cap H_2) = P(H_1 \cap H_2 | F)P(F) + P(H_1 \cap H_2 | \bar{F})P(\bar{F})$$

conditionally s.i. given the coin is fair

$$P(H_1 \cap H_2) = P(H_1|F)P(H_2|F)\cancel{P(F)}\left(\tfrac{2}{3}\right) + (1)\left(\tfrac{1}{3}\right)$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right) + \left(\tfrac{1}{3}\right) = \tfrac{1}{2}$$

$$P(H_2 | H_1) = \frac{\tfrac{1}{2}}{\tfrac{2}{3}} = \frac{3}{4}$$

As we expect $P(H_2|H_1) > P(H_1)$ because every time we observe a heads outcome, the more likely it is that the magician is using the two-headed coin

8) $$P(H_3|H_1 \cap H_2) = \frac{P(H_1 \cap H_2 \cap H_3)}{P(H_1 \cap H_2)}$$

$P(H_1 \cap H_2) = \frac{1}{2}$ from intermediate result in prob. 7

$$P(H_1 \cap H_2 \cap H_3) = P(H_1 \cap H_2 \cap H_3 | F)\, \cancel{P(F)}\left(\tfrac{2}{3}\right) + \underbrace{\cancel{P(H_1 \cap H_2 \cap H_3 | \bar{F})}}_{(1)}\ \underbrace{\cancel{P(\bar{F})}}_{(\frac{1}{3})}$$

$$= \left(\frac{1}{2}\right)^3\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right) = \frac{5}{12}$$

$$P(H_3 | H_1 \cap H_2) = \frac{\frac{5}{12}}{\frac{1}{2}} = \frac{10}{12} = \frac{5}{6}$$

Again, this should match your intuition. $P(H_3 | H_1 \cap H_2) > P(H_2 | H_1)$
$> P(H_1)$