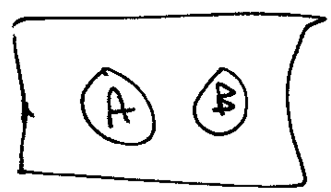
EEL 4930/Stats HW 3 Solutions

1.

(a) either $A$ or $B$ occurs if $A$ and $B$ are mutually exclusive

$$P(A \cup B) = P(A) + P(B)$$
$$= 0.7$$

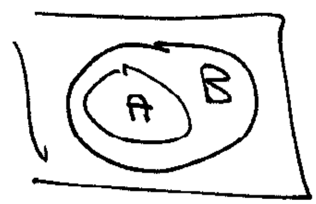



(b) either $A$ or $B$ occurs if $A$ and $B$ are statistically independent

$$P(A \cup B) = P(A) + P(B) - P(A)P(B)$$
$$= 0.7 - 0.12 = 0.58$$
$$P(A \cup B) = 1 - P(\bar{A} \cap \bar{B}) = 1 - (0.7)(0.6) = 0.58$$

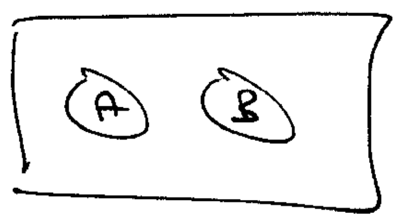
(c) either $A$ or $B$ occurs if $A$ is a subset of $B$

$$P(A \cup B) = P(B) = 0.4$$




(d) $A$ occurs but $B$ does not occur if $A$ and $B$ are mutually exclusive

$$P(A \cap \bar{B}) = P(A)$$
$$= 0.3$$

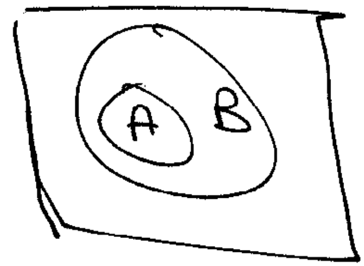



(e) $A$ occurs but $B$ does not occur if $A$ and $B$ are statistically independent

$$P(A \cap \bar{B}) = P(A)P(\bar{B}) = (0.3)(0.6) = 0.18$$

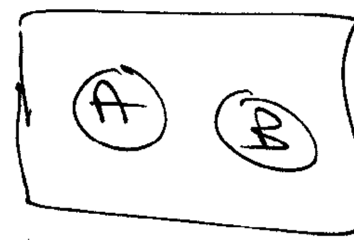
(f) $A$ occurs but $B$ does not occur if $A$ is a subset of $B$

$$P(A \cap \bar{B}) = P(\emptyset) = 0$$




(g) both $A$ and $B$ occur if $A$ and $B$ are mutually exclusive

$$P(A \cap B) = P(\emptyset) = 0$$

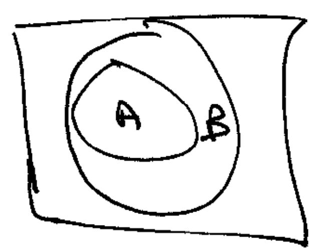



(h) both $A$ and $B$ occur if $A$ and $B$ are statistically independent

$$P(A \cap B) = P(A)P(B) = (0.3)(0.4) = 0.12$$

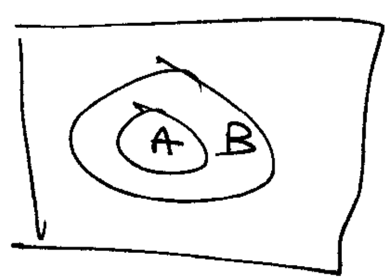
(i) both $A$ and $B$ occur if $A$ is a subset of $B$

$$P(A \cap B) = P(A) = 0.3$$




(j) $B$ occurs but $A$ does not occur if $A$ is a subset of $B$

$$P(B \cap \bar{A}) = P(B) - P(A)$$
$$= 0.4 - 0.3 = 0.1$$



2. See the separate Jupyter Notebook solution

3. Problem 61 from Leon-Garcia.

   This is a Bayes' theorem problem. Let D be the event that a chip is defective. We are given $P(D|A) = 0.001$, $P(D|B) = 0.005$, and $P(D|C) = 0.01$, and we are asked to find $P(A|D)$ and $P(C|D)$.

   Well, $P(A|D) = P(A \cap D)/P(D) = P(D|A)P(A)/P(D)$. We don't know $P(D)$, but we find it from the law of total probability.

$$\begin{aligned} P(D) &= P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C) \\ &= \frac{0.001}{3} + \frac{0.005}{3} + \frac{0.01}{3} \\ &= 5.33 \times 10^{-3}. \end{aligned}$$

   Thus,

$$\begin{aligned} P(A|D) &\approx \frac{(0.001)(0.333)}{(5.33 \times 10^{-3})} \\ &= 0.0625, \end{aligned}$$

   and

$$\begin{aligned} P(C|D) &\approx \frac{(0.01)(0.333)}{(5.33 \times 10^{-3})} \\ &= 0.625. \end{aligned}$$

4.

$C =$ event man has cancer
$E =$ " PSA level elevated

a) $$P(C|E) = \frac{P(E|C)P(C)}{P(E)}$$

$$= \frac{P(E|C)P(C)}{P(E|C)P(C) + P(E|\bar{C})P(\bar{C})}$$

$$= \frac{(0.268)(0.7)}{(0.268)(0.7) + (0.135)(0.3)}$$

$$= 0.822$$

b) $$P(C|\bar{E}) = \frac{P(\bar{E}|C)P(C)}{P(\bar{E})}$$

$$= \frac{P(\bar{E}|C)P(C)}{P(\bar{E}|C)P(C) + P(\bar{E}|\bar{C})P(\bar{E})}$$

$$= \frac{(0.732)(0.7)}{(0.732)(0.7) + (0.865)(0.3)}$$

$$= 0.6638$$

(Note that the a priori estimate $P(C)$ was 0.7, so the negative test did not significantly alter the prob. that he has cancer.)

c) $P(C|E)$ for $P(C) = 0.3$

$$P(C|E) = \frac{(0.268)(0.3)}{(0.268)(0.3) + (0.135)(0.7)}$$

$$\approx 0.46$$

d) $$P(C|E) = \frac{(0.732)(0.3)}{(0.732)(0.3) + (0.865)(0.7)}$$

$$\approx 0.266$$

5. Communication system problem from *Random Signal Analysis in Engineering Systems* by John J. Komo

   This is another problem that is solved using Bayes' rule. We need to find $P(A_i|B_j)$ and we know $P(B_j|A_i)$ and $P(A_i)$. I provide the solution for $B_0$ received here:

$$P(A_0|B_0) = \frac{P(B_0|A_0)P(A_0)}{P(B_0)}.$$

   So we need to find $P(B_0)$ from the total probability law:

$$\begin{aligned} P(B_0) &= P(B_0|A_0)P(A_0) + P(B_0|A_1)P(A_1) \\ &= (0.5)(0.6) + (0.1)(0.4) \\ &= 0.34. \end{aligned}$$

   Similarly, $P(B_1) = 0.27$, and $P(B_2) = 0.39$.

   Then $P(A_0|B_0) = (0.5)(0.6)/(0.34) \approx 0.882$, and $P(A_1|B_0) = 1 - P(A_0|B_0) \approx 0.118$. Thus the best decision for the receiver given that $B_0$ is received is to choose $A_0$ because that will be the correct decision with probability 0.882. The probability of error given that $B_0$ is received is then 0.118.

   Similarly, given that $B_1$ is received, the conditional probability that $A_0$ was sent is $\approx 0.556$ and the conditional probabiltiy that $A_1$ was sent is $\approx 0.444$. The best decision rule is to decide that $A_0$ was sent, and the conditional probability of error given that $B_1$ was received is 0.444.

   Given that $B_2$ is received, $P(A_0|B_2) \approx 0.385$, and $P(A_1|B_2) \approx 0.615$. The best decision rule is to decide that $A_1$ was transmitted, and the conditional probability of error is 0.385.

   The overall probability of error $P(E)$ can also be found using the law of total probability, $P(E) = P(E|B_0)P(B_0) + P(E|B_1)P(B_1) + P(E|B_2)P(B_2) = 0.310$.