Your Name:

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1. (24 points) Let $f_X(x)$ be the density for a random variable X, where

$$f_X(x) = \begin{cases} a\sqrt{x}, & 0 \le x < 1 \\ 0, & \text{otherwise,} \end{cases}$$

and a is a constant.

(a) Find the numeric value of a for $f_X(x)$ to be a valid density function.

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

$$\int_0^1 a\, x^{1/2}\, dx = 1$$

$$a \left. \frac{x^{3/2}}{3/2} \right|_0^1 = 1$$

$$\frac{2a}{3}\left[1 - 0\right] = 1$$

$$\boxed{a = \frac{3}{2}}$$

(b) Find the expected value (mean) of X.

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \int_0^1 x \, \frac{3}{2} x^{1/2}\, dx$$

$$= \frac{3}{2} \frac{x^{5/2}}{5/2} \Big|_0^1$$

$$= \frac{3}{5}\left[1 - 0\right] = \boxed{3/5}$$

(c) Find the variance of X.

$$\mathrm{Var}[X] = E[X^2] - E[X]^2$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$= \int_0^1 x^2 \frac{3}{2} x^{1/2}\,dx$$

$$= \frac{3}{2} \left.\frac{x^{7/2}}{7/2}\right|_0^1$$

$$= \frac{3}{7}\left[1 - 0\right] = \frac{3}{7}$$

$$\mathrm{Var}[X] = \frac{3}{7} - \left(\frac{3}{5}\right)^2 \approx 0.0686$$

correction

(d) Find $E[2\sqrt{\cancel{X}}+1]$

$$E\left[2\sqrt{\cancel{X}}+1\right]$$

$$= 2E\left[\sqrt{X}\right]+1 \quad \text{by linearity}$$

$$E\left[\sqrt{X}\right] = \int_{-\infty}^{\infty} \sqrt{\cancel{x}}\, f_X(x)\, dx$$

$$= \int_{\cancel{-\infty}\ 0}^{\cancel{\infty}\ 1} x^{1/2}\ \frac{3}{2} x^{1/2}\, dx$$

$$= \frac{3}{2}\frac{x^2}{2}\Big|_0^1$$

$$= \frac{3}{4}$$

$$E\left[2\sqrt{X}+1\right] = 2\left(\frac{3}{4}\right)+1$$

$$= \frac{3}{2}+1 = \boxed{\frac{5}{2}}$$

2. (20 points)

Part I. A professor makes an average of 2 errors in the problem descriptions of his final exams.

(a) What is the probability that a given final exam has no errors?

Rate given, no specific max # $\Rightarrow$ Poisson

$X \sim \text{Poisson}(\alpha = 2)$

$$P[X=0] = e^{-\alpha} \frac{\cancel{\alpha^0}}{\cancel{0!}} = e^{-2} \approx 0.135$$

(b) What is the probability that the exam will have 3 or more errors?

$$P[X \geq 3] = 1 - P[X<3]$$

$$= 1 - P[X \leq 2]$$

$$= 1 - P[X=0] - P[X=1] - P[X=2]$$

$$= 1 - e^{-\alpha}\left[1 + \frac{\alpha^1}{1!} + \frac{\alpha^2}{2!}\right]$$

$$= 1 - e^{-2}\left[1 + \frac{2}{1} + \frac{4}{2}\right]$$

$$= 1 - e^{-2}[5]$$

$$\boxed{\approx 0.323}$$

Part II. Over many years, it is observed that a professor makes an average of 100 mistakes during lectures per semester.

(c) Give an **approximation** for the probability that the professor makes more than 125 mistakes during a semester.

Again, Poisson

X, $\alpha = 100$

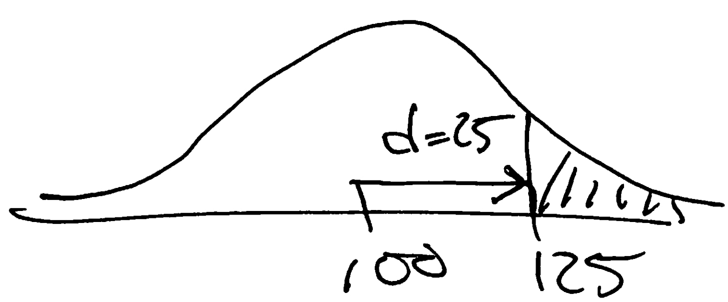
$P[X > 125]$

$X \approx \text{Gaussian}(\mu, \sigma^2)$

$\mu = E[X] = \alpha = 100$

$\sigma^2 = \text{Var}[X] = \alpha = 100$

(mean & variance are equal for Poisson)



$P[X > 125] \approx Q\left(\frac{d}{\sigma}\right) = Q\left(\frac{25}{10}\right)$

$= Q(2.5) \approx 6.21 \times 10^{-3}$

(from table)