EEL 5544 Midterm Examination Number 1

October 7, 2009

The time for this test is 2 hours. This is a closed book test, but you are allowed one formula sheet. The formula sheet cannot contain any examples. You should **write your name on the formula sheet** and turn it in with your exam. You may use a calculator on this test. You must show your work to receive credit for a problem. Note that some problems are worth more points than other problems, and the problems are not necessarily sorted in order of difficulty or point value. You must sign the honor statement at the end of the test in order to receive any credit.

Solution

1. (18 points) Let $(S, \mathcal{F}, P)$ be a probability space. Suppose A, B, and C are events (i.e., are members of $\mathcal{F}$) with non-zero probability. Consider the following equations. If the equation is true for all A, B, and C, write "TRUE". If the equation is not true for any A, B, C, write "FALSE". If the equation is true given some restrictions on A, B, and C, then specify the restrictions. You do not have to prove your answer.

(a)

$$P(A \cup B) = P(A) + P(B)$$

TRUE if A & B are mutually exclusive.

(b)

$$P(A \cap B) = P(A)P(B)$$

TRUE if A & B are statistically independent

(c)

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

TRUE for any A & B.

(d)

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

TRUE for any A & B.

(e)

$$P(B|A) = P(B)$$

TRUE if A & B are statistically independent

(f)

$$P(B|A) = 1$$

TRUE if $A \subseteq B$.

2. (24 points) Suppose there are 100 students in a class.

   Part I. In this part, suppose that there are 10 students in the class who did their undergraduate work at UF. Suppose the professor randomly partitions the class into 10 groups of equal size.

   (a) What is the probability that the first group created has exactly two students who did their undergraduate work at UF?

$$\frac{\binom{10}{2}\binom{90}{8}}{\binom{100}{10}} = 0.2015$$

   (b) What is the probability that the first group created has two or more students who did their undergraduate work at UF?

$$1 - \frac{\binom{90}{10}+\binom{10}{1}\binom{90}{9}}{\binom{100}{10}} = 0.2615$$

Part II. In this part, suppose that the probability that a student in the class did their undergraduate at UF is 0.1, and the events that students did their undergraduate at UF are independent for different students. Suppose the professor randomly partitions the class into 10 groups of equal size.

(c) What is the probability that the first group created has exactly two students who did their undergraduate work at UF?

$$\binom{10}{2}(0.1)^2(0.9)^8 = 0.1937$$

(d) What is the probability that the first group created has two or more students who did their undergraduate work at UF?

$$1 - (0.9)^{10} - \binom{10}{1}(0.1)^1(0.9)^9 = 0.2639$$

Part III. In this part, suppose again that there are 10 students in the class who did their undergraduate work at UF. Suppose the professor randomly partitions the class into 2 groups of equal size.

(e) What is the probability that all of the students who did their undergraduate work at UF end up in one group?

$$\frac{2\cdot\binom{10}{10}\binom{90}{40}}{\binom{100}{50}} = 0.0012$$

(f) Suppose the professor wants to create 2 groups, but that he does not want either group to contain 7 or more students who did their undergraduate work at UF. The professor partitions the group at random, but discards the partition if it doesn't satisfy that condition. What is the probability that the professor has to try 3 or more partitions?

Let p = probability of acceptable partitions.

then p = probability of having 4, 5 or 6 people in group 1.

$$p = \frac{\binom{10}{4}\binom{90}{44}+\binom{10}{5}\binom{90}{45}+\binom{10}{6}\binom{90}{44}}{\binom{100}{50}} = 0.6822$$

Then Prob (3 or more partitions) $= (1-p)^2 = 0.1010$

3. (18 Points) Consider the function

$$f_X(x) = \begin{cases} cx^2, & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find c for $f_X(x)$ to be a valid density function.

$$\int_{-1}^{1} f(x)\,dx = 1$$

$$\int_{-1}^{1} cx^2\,dx = 1$$

$$\frac{c}{3}x^3 \Big|_{-1}^{1} = 1$$

$$\therefore\ c = \frac{3}{2}$$

(b) Let X be a random variable with density function $f_X(x)$. Find the distribution function of X.

Let $-1 \leq x \leq 1$

$$\int_{-1}^{x} f(x)\,dx = \int_{-1}^{x} \frac{3}{2}x^2\,dx$$

$$= \frac{1}{2}x^3 \Big|_{-1}^{x}$$

$$= \frac{1}{2}(x^3+1)$$

thus, $F_X(x) = \begin{cases} 0, & x < -1 \\ \frac{1}{2}(x^3+1), & -1 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$

(c) Find $P[(X+0.5)^2 > 1]$.

$$P\left[(X+0.5)^2 > 1\right]$$

$$= P\left[(X+0.5)>1 \cup (X+0.5)<-1\right]$$

$$= P\left[(X>0.5) \cup (X<-1.5)\right]$$

$$= 1 - P[X \leq 0.5] + P[X \leq -1.5)]$$

$$= 1 - \frac{1}{2}(x^3+1)\Big|_{x=0.5} + 0$$

$$= 1 - \frac{1}{2}\left(\frac{1}{8}+1\right)$$

$$= \frac{7}{16}$$

$$= 0.4375$$

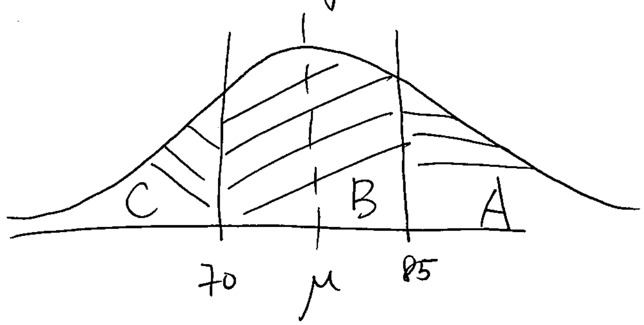
4. (20 Points) A professor is designing an exam. Let A be the event that a student gets an A on the exam, etc. The professor wishes to design the exam so that $P(A) = 0.4$, $P(B) = 0.5$, and $P(C) = 0.1$. Let X be the grade, where X is a Gaussian random variable with parameter μ and σ^2.

   If

$$A = \{X \geq 85\}$$
$$B = \{70 \leq X < 85\}$$
$$C = \{X < 70\},$$

   find the values of μ and σ^2 that can achieve the professor's target values for $P(A)$, $P(B)$, and $P(C)$.

Gaussian distribution as follows.



$$P(A) = Q\left(\frac{85-\mu}{\sigma}\right) = 0.4 \Rightarrow \frac{85-\mu}{\sigma} = 0.25$$

$$P(C) = Q\left(\frac{\mu-70}{\sigma}\right) = 0.1 \Rightarrow \frac{\mu-70}{\sigma} = 1.28$$

hence $\mu = 82.5491$

$\sigma = 9.8040$

$\sigma^2 = 96.1184$

Additional room for problem 4

5. (XX points) According to a recent study, the probability of contracting the H1N1 flu virus from contact with a contaminated surface is 0.31, the probability of contracting H1N1 from inhaling airborne particles laden with the H1N1 virus are 0.17, and the probability of contracting H1N1 from contact with cough sprays from an infected person are 0.52. Suppose that if you do not come into contact with any infected surface, airborne particles, or cough sprays containing the H1N1 virus, then you will not contract it.

   Suppose that in a day, the probability of coming into contact with an infected surface is 0.01, the probability of coming into contact with cough sprays from an infected person is 0.005.

   (a) Let p be the probability of coming into contact with airborne particles laden with the H1N1 virus. Find the minimum value of p such that airborne particles is the MAP source of a H1N1 flu infection.

Let S = event of coming into contact with an infected surface

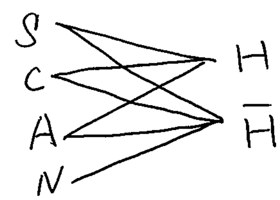
C = event of coming into contact with cough sprays from an infected person

A = event of coming into contact with airborne particles

N = event of none of the above

H = event of H1N1 flu infection.

Then the above describes a channel with <u>four</u> inputs and <u>two</u> outputs as follows



where $p(S) = 0.01$, $p(C) = 0.005$, $p(A) = p$, $p(N) = 0.985 - p$

$p(H|S) = 0.31$, $p(H|C) = 0.52$, $p(H|A) = 0.17$, $p(H|N) = 0$

We need to compare $p(S|H)$, $p(C|H)$ and $p(A|H)$

$$p(S|H) = \frac{p(H|S)\,p(S)}{p(H)} = \frac{0.01 \times 0.31}{p(H)}$$

$$p(C|H) = \frac{p(H|C)\,p(C)}{p(H)} = \frac{0.005 \times 0.52}{p(H)}$$

$$p(A|H) = \frac{p(H|A)\,p(A)}{p(H)} = \frac{0.17\,p}{p(H)}$$

hence $0.17\,p \geq 3.1 \times 10^{-3}$

$p \geq 0.0182$

(b) Suppose $p = 0.2$. If someone contracts the H1N1 flu, what is the probability that it was contracted from surface particles?

$$P(S|H) = \frac{P(H|S)P(S)}{P(H)}$$

$$= \frac{P(H|S)P(S)}{P(H|S)P(S) + P(H|C)P(C) + P(H|A)P(A)}$$

$$= \frac{0.01 \times 0.31}{0.01 \times 0.31 + 0.005 \times 0.52 + 0.02 \times 0.17}$$

$$= 0.3407$$

(c) Suppose $p = 0.02$. Given that you do NOT get the H1N1 flu from a day's activities, what is the probability that you were exposed to it indirectly (through airborne particles or surface contact)?

$$P(S|\bar{H}) + P(A|\bar{H})$$

$$= \frac{P(\bar{H}|S)P(S) + P(\bar{H}|A)P(A)}{P(\bar{H})}$$

$$= \frac{[1 - P(H|S)]P(S) + [1 - P(H|A)]P(A)}{1 - P(H)}$$

$$= \frac{(1 - 0.31) \times 0.01 + (1 - 0.17) \times 0.02}{1 - 9.1 \times 10^{-3}}$$

$$= 0.0237.$$