EEE 5544 Lectures 30

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Conditioning with Random Variables

- There are multiple ways that random variables can depend on events or on other random variables

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Conditioning with Random Variables

- There are multiple ways that random variables can depend on events or on other random variables
- One of the most straight-forward is when the distribution of the random variable depends on some event

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- For example, in a binary communication system, the received signal is a noisy version of the transmitted signal. In the presence of thermal noise, the received signal can be modeled as $X = s_i + N$, where:

  – $s_i \in \{-1, 1\}$ depends on which signal is transmitted (0 or 1), and
  – N is a Gaussian random variable with mean 0 and variance σ^2, which determines the signal-to-noise ratio

- Thus, the received signal has a conditional distribution, depending on which signal is transmitted:

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- (Here only the mean changes and not the variance, but this is an accurate model of what happens in most binary communication systems.)

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- We can use this model to calculate error probabilities as a function of the noise level (σ^2) given a decision rule

- However, we cannot use this model to find an optimal MAP decision rule because the model only gives us $f_X(x|0 \text{ Tx})$ and $f_X(x|1 \text{ Tx})$. These are likelihoods

- To find the *a posteriori* probabilities, we need to introduce a new form of conditional probability:

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- Suppose we want to evaluate the probability of an event A given that $X=x$, where X is a continuous random variable.

- Then, if we use the definition of conditional probability

$$P(A|X=x)=\frac{P(A\cap X=x)}{P(X=x)}=\frac{0}{0},$$

so the previous definition of conditional prob. will not work.

$$P(A|X = x)$$

$$P(A|X = x) \ = \ \lim_{\Delta x \to 0} P(A|x < X \leq x + \Delta x)$$

$$
\begin{aligned}
P(A|X=x) &= \lim_{\Delta x \to 0} P(A|x < X \leq x + \Delta x) \\
&= \lim_{\Delta x \to 0} \frac{F_X(x+\Delta x|A) - F_X(x|A)}{F_X(x+\Delta x) - F_X(x)} P(A)
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&= \frac{f_X(x|A)}{f_X(x)} P(A),
\end{aligned}
$$

if $f_X(x|A)$ and $f_X(x)$ exist, and $f_X(x) \neq 0$.

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(Continuous Version of Law of Total Probability)

Point Conditioning Form of Baye’s Rule:

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If $\{A_i,\ i = 0, 1, \ldots, n-1\}$ form a partition of S, then

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If $\{A_i,\ i = 0, 1, \ldots, n-1\}$ form a partition of S, then

$$P(A_i | X = x) \ = \ \frac{f_X(x|A_i)P(A_i)}{\sum_{i=0}^{n-1} f_X(x|A_i)P(A_i)}$$

EX MAP Detection in a Binary Communication System

Show how point conditioning applies to finding the MAP decision rule for the binary communication system above. Use Python to show the MAP decision regions. Find the ML decision rule.

Given $X = x$ is observed, what was most likely Tx (0 or 1)?

MAP Rule w/ pt. cond.

$$P(0\text{ Tx} \mid X = x) \underset{1}{\overset{0}{\gtrless}} P(1\text{ Tx} \mid X = x)$$