Homework 1

September 6, 2016

Except for your response to Question 1, your solutions must be submitted through Canvas as a single PDF document as an attachment in response to the homework assignment. No paper documents will be accepted. Multiple files, JPEGs, Word documents, etc. will not be accepted. Please submit your work as a single, non-password protected PDF document.

The following homework set has two types of problems. Those labeled "SS" are self-study problems that do not need to be turned in. However, these are problems that I suggest you complete to ensure that you understand all the material in the class. The problems not labeled SS should be turned for a grade. In general, the SS problems are easier than the graded problems, so they can be used to practice and build expertise before doing the graded problems.

1. Send an email to jshea@ece.ufl.edu with subject "EEE 5544 Student Info" that includes:
   (a) your full name
   (b) the name you would like to be called
   (c) your home country
   (d) your undergraduate university
   (e) a recent picture of you

Sample Spaces, Events, and Sets

SS-1. A box contains three marbles, one red, one green, one blue.

(a) Consider an experiment that consists of taking 1 marble from the box, then replacing it in the box and drawing a second marble from the box. Describe the sample space.

(b) Repeat when the second marble is drawn without replacing the first marble.

SS-2. Problem 2.5 from Leon-Garcia (see problems at end of this PDF)

SS-3. Problem 2.7 from Leon-Garcia. Experiment E_6 is "A block of information is transmitted repeatedly over a noisy channel until an error-free block arrives at the receiver. Count the number of transmissions required."

SS-4. Two dice are thrown. Let E be the event that the sum of the dice is odd; let F be the event that at least one of the dice lands on 1; and let G be the event that the sum is 5. Describe the events:

(a) $E \cap F$

(b) $E \cup F$

(c) $F \cap G$

(d) $E \cap \overline{F}$

(e) $E \cap F \cap G$

2. A hospital administrator codes incoming patients suffering from gunshot wounds according to whether they have insurance (coding 1 if they do and 0 if they do not) and according to their condition, which is rated as either good (g), fair (f), or serious (s). Consider an experiment that consists of coding such a patient.

(a) Give the sample space for the experiment.

(b) Let A be the event that the patient is in serious condition. Specify the outcomes in A.

(c) Let B be the event that the patient is uninsured. Specify the outcomes in B.

(d) Give all the outcomes in the event $\overline{B} \cup A$.

SS-5. **IMPORTANT: If you are planning to take Stochastic Methods for Engineering II, you should answer this question:**

For any sequence of events $E_1, E_2, \ldots$, define a new sequence $F_1, F_2, \ldots$ of *mutually exclusive* events such that for all $n \geq 1$,

$$\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i \tag{1}$$

holds for ALL n.

Hint: Start with $n = 1$. Then F_1 should be obvious. Then what is F_2? What is a general formula for any F_i?

SS-6.

Let E, F, and G be three events. Find expressions for the events so that of E, F, and G:

(a) only E occurs;
(b) both E and G but not F occur;
(c) at least one of the events occurs;
(d) at least two of the events occur;
(e) all three occur;
(f) none of the events occurs;
(g) at most one of them occurs;
(h) at most two of them occur;
(i) exactly two of them occur;
(j) at most three of them occur.

Combinatorics

Use combinatorics to solve the following problems:

3. Calculate the probabilities of winning the first five prizes (the Grand Prize through the \$100 prizes) of the Powerball lottery (see `http://www.powerball.com/powerball/pb_prizes.asp`) and verify if the posted probabilities (under the erroneous title "Odds") are correct.

SS-7. Two cards are randomly drawn from an ordinary playing deck (only the 52 suited cards, no jokers). What is the probability that they form a blackjack? That is, what is the probability that one of the cards is an ace and the other one is either a ten, a jack, a queen, or a king? *(Answer: 128/2652)*

SS-8. Poker dice is played by simultaneously rolling five dice. Show that

(a) $P[\text{no two alike}] = 0.0926$
(b) $P[\text{one pair}] = 0.4630$
(c) $P[\text{two pair}] = 0.2315$
(d) $P[\text{three alike}] = 0.1543$
(e) $P[\text{full house}] = 0.0386$
(f) $P[\text{four alike}] = 0.0193$
(g) $P[\text{five alike}] = 0.0.0008$

Note that a full house is three of one number plus a pair of another number.

4. In a group of 100 items, a number of items are tested, and the group is rejected if too many of the tested items are found defective.

(a) Consider the case where two items are tested and the group is rejected if either of the tested items is defective. Find the probabilities of **accepting** a group for groups with 5 defective items and for groups with 10 defective items. **Use combinatorics.**

(b) Consider the case where three items are tested and the group is rejected if more than one item is found to be defective. Find the probabilities of **accepting** a group for groups with 5 defective items and for groups with 10 defective items. **Use combinatorics.**

5. An urn contains 3 red and 7 black balls. Players A and B withdraw balls from the urn consecutively until a red ball is selected. Find the probability that A selects the red ball. (A draws the first ball, then B, and so on. There is no replacement of the balls drawn.) **Use combinatorics.** *Hint:* Imagine that all balls are withdrawn.

Axioms of Probability and Corrolaries

6. Problem 2.29 from Leon-Garcia

SS-9. Problem 2.30 from Leon-Garcia

7. Problem 2.38 from Leon-Garcia

Independent Events

SS-10. If you had to construct a mathematical model for events E and F, as described in parts (a) through (e), would you assume that they were independent events? Explain your reasoning.

(a) E is the event that a businesswoman has blue eyes, and F is the event that her secretary has blue eyes. event that he has a web site.

(b) E is the event that a professor owns a car, and F is the event that the professor has a web site for his course.

(c) E is the event that a man is under 6 feet tall, and F is the event that he weighs over 200 pounds.

(d) E is the event that a woman lives in the United States, and F is the event that she lives in the western hemisphere.

(e) E is the event that it will rain today, and F is the event that it will rain tomorrow.

SS-11. In a class there are 4 freshman boys, 6 freshman girls, and 6 sophomore boys. How many sophomore girls must be present if gender and class are to be independent when a student is selected at random? Answer: 9

8. A system has m controllers and n actuators. The system has redundancy built in so that it will be operational if at least one controller and two actuators have not failed. All controllers and actuators fail independently.

   Part I: If the probability that a controller fails is $p_c = 0.25$, and the probability that an actuator fails is $p_a = 0.1$, find the probability that the system will be operational for:

   (a) 1 controller, 2 actuators

   (b) 2 controllers, 2 actuators

   (c) 2 controllers, 3 actuators

   (d) 2 controllers, 4 actuators

   (e) 3 controllers, 4 actuators

   Part II: Now evaluate the benefit of redundancy in a different way. If $p_c = p_a$, find the value of p_c such that the system of part (a) is operational with the same probability as the system in part (e).

   Bonus: Solve Part I by writing a Python function that returns the answer to the general problem in Part I and provide your answers in a Jupyter notebook.

9. Suppose I want to roll a fair die M times.

   (a) If I want the probability of seeing at least one outcome of 1 or 2 on any of the rolls to be at least 0.95, how large should M be?

   (b) If I want the probability of seeing at least one outcome of 1 or 2 on any of the rolls to be at least 0.999, how large should M be?

   (c) If I want the probability of seeing at least one outcome of 1 on any of the rolls to be at least 0.95, how large should M be?

   (d) If I want the probability of seeing at least one outcome of 1 on any of the rolls to be at least 0.999, how large should M be?

7. W. Feller, *An Introduction to Probability Theory and Its Applications*, 3d ed., Wiley, New York, 1968.
8. A. N. Kolmogorov and S. V. Fomin, *Introductory Real Analysis*, Dover Publications, New York, 1970.
9. P. J. G. Long, "Introduction to Octave," University of Cambridge, September 2005, available online.
10. A. M. Law and W. D. Kelton, *Simulation Modeling and Analysis*, McGraw-Hill, New York, 2000.

PROBLEMS

Section 2.1: Specifying Random Experiments

2.1. The (loose) minute hand in a clock is spun hard and the hour at which the hand comes to rest is noted.

(a) What is the sample space?

(b) Find the sets corresponding to the events: A = "hand is in first 4 hours"; B = "hand is between 2nd and 8th hours inclusive"; and D = "hand is in an odd hour."

(c) Find the events: $A \cap B \cap D$, $A^c \cap B$, $A \cup (B \cap D^c)$, $(A \cup B) \cap D^c$.

2.2. A die is tossed twice and the number of dots facing up in each toss is counted and noted in the order of occurrence.

(a) Find the sample space.

(b) Find the set A corresponding to the event "number of dots in first toss is not less than number of dots in second toss."

(c) Find the set B corresponding to the event "number of dots in first toss is 6."

(d) Does A imply B or does B imply A?

(e) Find $A \cap B^c$ and describe this event in words.

(f) Let C correspond to the event "number of dots in dice differs by 2." Find $A \cap C$.

2.3. Two dice are tossed and the magnitude of the difference in the number of dots facing up in the two dice is noted.

(a) Find the sample space.

(b) Find the set A corresponding to the event "magnitude of difference is 3."

(c) Express each of the elementary events in this experiment as the union of elementary events from Problem 2.2.

2.4. A binary communication system transmits a signal X that is either a $+2$ voltage signal or a -2 voltage signal. A malicious channel reduces the magnitude of the received signal by the number of heads it counts in two tosses of a coin. Let Y be the resulting signal.

(a) Find the sample space.

(b) Find the set of outcomes corresponding to the event "transmitted signal was definitely $+2$."

(c) Describe in words the event corresponding to the outcome $Y = 0$.

2.5. A desk drawer contains six pens, four of which are dry.

(a) The pens are selected at random one by one until a good pen is found. The sequence of test results is noted. What is the sample space?

(b) Suppose that only the number, and not the sequence, of pens tested in part a is noted. Specify the sample space.

(c) Suppose that the pens are selected one by one and tested until both good pens have been identified, and the sequence of test results is noted. What is the sample space?

(d) Specify the sample space in part c if only the number of pens tested is noted.

2.6. Three friends (Al, Bob, and Chris) put their names in a hat and each draws a name from the hat. (Assume Al picks first, then Bob, then Chris.)

(a) Find the sample space.

(b) Find the sets A, B, and C that correspond to the events "Al draws his name," "Bob draws his name," and "Chris draws his name."

(c) Find the set corresponding to the event, "no one draws his own name."

(d) Find the set corresponding to the event, "everyone draws his own name."

(e) Find the set corresponding to the event, "one or more draws his own name."

2.7. Let M be the number of message transmissions in Experiment E_6.

(a) What is the set A corresponding to the event "M is even"?

(b) What is the set B corresponding to the event "M is a multiple of 3"?

(c) What is the set C corresponding to the event "6 or fewer transmissions are required"?

(d) Find the sets $A \cap B$, $A - B$, $A \cap B \cap C$ and describe the corresponding events in words.

2.8. A number U is selected at random from the unit interval. Let the events A and B be: A = "U differs from 1/2 by more than 1/4" and B = "$1 - U$ is less than 1/2." Find the events $A \cap B$, $A^c \cap B$, $A \cup B$.

2.9. The sample space of an experiment is the real line. Let the events A and B correspond to the following subsets of the real line: $A = (-\infty, r]$ and $B = (-\infty, s]$, where $r \le s$. Find an expression for the event $C = (r, s]$ in terms of A and B. Show that $B = A \cup C$ and $A \cap C = \varnothing$.

2.10. Use Venn diagrams to verify the set identities given in Eqs. (2.2) and (2.3). You will need to use different colors or different shadings to denote the various regions clearly.

2.11. Show that:

(a) If event A implies B, and B implies C, then A implies C.

(b) If event A implies B, then B^c implies A^c.

2.12. Show that if $A \cup B = A$ and $A \cap B = A$ then $A = B$.

2.13. Let A and B be events. Find an expression for the event "exactly one of the events A and B occurs." Draw a Venn diagram for this event.

2.14. Let A, B, and C be events. Find expressions for the following events:

(a) Exactly one of the three events occurs.

(b) Exactly two of the events occur.

(c) One or more of the events occur.

(d) Two or more of the events occur.

(e) None of the events occur.

2.15. Figure P2.1 shows three systems of three components, C_1, C_2, and C_3. Figure P2.1(a) is a "series" system in which the system is functioning only if all three components are functioning. Figure 2.1(b) is a "parallel" system in which the system is functioning as long as at least one of the three components is functioning. Figure 2.1(c) is a "two-out-of-three"

2.28. A hexadecimal character consists of a group of three bits. Let A_i be the event "ith bit in a character is a 1."

(a) Find the probabilities for the following events: A_1, $A_1 \cap A_3$, $A_1 \cap A_2 \cap A_3$ and $A_1 \cup A_2 \cup A_3$. Assume that the values of bits are determined by tosses of a fair coin.

(b) Repeat part a if the coin is biased.

2.29. Let M be the number of message transmissions in Problem 2.7. Find the probabilities of the events $A, B, C, C^c, A \cap B, A - B, A \cap B \cap C$. Assume the probability of successful transmission is 1/2.

2.30. Use Corollary 7 to prove the following:

(a) $P[A \cup B \cup C] \leq P[A] + P[B] + P[C]$.

(b) $P\left[\bigcup_{k=1}^{n} A_k\right] \leq \sum_{k=1}^{n} P[A_k]$.

(c) $P\left[\bigcap_{k=1}^{n} A_k\right] \geq 1 - \sum_{k=1}^{n} P[A_k^c]$.

The second expression is called the **union bound**.

2.31. Let p be the probability that a single character appears incorrectly in this book. Use the union bound for the probability of there being any errors in a page with n characters.

2.32. A die is tossed and the number of dots facing up is noted.

(a) Find the probability of the elementary events if faces with an even number of dots are twice as likely to come up as faces with an odd number.

(b) Repeat parts b and c of Problem 2.21.

2.33. Consider Problem 2.1 where the minute hand in a clock is spun. Suppose that we now note the *minute* at which the hand comes to rest.

(a) Suppose that the minute hand is very loose so the hand is equally likely to come to rest anywhere in the clock. What are the probabilities of the elementary events?

(b) Now suppose that the minute hand is somewhat sticky and so the hand is 1/2 as likely to land in the second minute than in the first, 1/3 as likely to land in the third minute as in the first, and so on. What are the probabilities of the elementary events?

(c) Now suppose that the minute hand is very sticky and so the hand is 1/2 as likely to land in the second minute than in the first, 1/2 as likely to land in the third minute as in the second, and so on. What are the probabilities of the elementary events?

(d) Compare the probabilities that the hand lands in the last minute in parts a, b, and c.

2.34. A number x is selected at random in the interval $[-1, 2]$. Let the events $A = \{x < 0\}$, $B = \{|x - 0.5| < 0.5\}$, and $C = \{x > 0.75\}$.

(a) Find the probabilities of A, B, $A \cap B$, and $A \cap C$.

(b) Find the probabilities of $A \cup B$, $A \cup C$, and $A \cup B \cup C$, first, by directly evaluating the sets and then their probabilities, and second, by using the appropriate axioms or corollaries.

2.35. A number x is selected at random in the interval $[-1, 2]$. Numbers from the subinterval $[0, 2]$ occur half as frequently as those from $[-1, 0)$.

(a) Find the probability assignment for an interval completely within $[-1, 0)$; completely within $[0, 2]$; and partly in each of the above intervals.

(b) Repeat Problem 2.34 with this probability assignment.

2.36. The lifetime of a device behaves according to the probability law $P[(t, \infty)] = 1/t$ for $t > 1$. Let A be the event "lifetime is greater than 4," and B the event "lifetime is greater than 8."

(a) Find the probability of $A \cap B$, and $A \cup B$.

(b) Find the probability of the event "lifetime is greater than 6 but less than or equal to 12."

2.37. Consider an experiment for which the sample space is the real line. A probability law assigns probabilities to subsets of the form $(-\infty, r]$.

(a) Show that we must have $P[(-\infty, r]] \le P[(-\infty, s]]$ when $r < s$.

(b) Find an expression for $P[(r, s]]$ in terms of $P[(-\infty, r]]$ and $P[(-\infty, s]]$

(c) Find an expression for $P[(s, \infty)]$.

2.38. Two numbers (x, y) are selected at random from the interval $[0, 1]$.

(a) Find the probability that the pair of numbers are inside the unit circle.

(b) Find the probability that $y > 2x$.

*Section 2.3: Computing Probabilities Using Counting Methods

2.39. The combination to a lock is given by three numbers from the set $\{0, 1, \dots, 59\}$. Find the number of combinations possible.

2.40. How many seven-digit telephone numbers are possible if the first number is not allowed to be 0 or 1?

2.41. A pair of dice is tossed, a coin is flipped twice, and a card is selected at random from a deck of 52 distinct cards. Find the number of possible outcomes.

2.42. A lock has two buttons: a "0" button and a "1" button. To open a door you need to push the buttons according to a preset 8-bit sequence. How many sequences are there? Suppose you press an arbitrary 8-bit sequence; what is the probability that the door opens? If the first try does not succeed in opening the door, you try another number; what is the probability of success?

2.43. A Web site requires that users create a password with the following specifications:

- Length of 8 to 10 characters
- Includes at least one special character {!, @, #, $, %, ^, &, *, (,), +, =, {, }, |, <, >, \, _, -, [,], /, ?}
- No spaces
- May contain numbers (0–9), lower and upper case letters (a–z, A–Z)
- Is case-sensitive.

How many passwords are there? How long would it take to try all passwords if a password can be tested in 1 microsecond?

2.44. A multiple choice test has 10 questions with 3 choices each. How many ways are there to answer the test? What is the probability that two papers have the same answers?

2.45. A student has five different t-shirts and three pairs of jeans ("brand new," "broken in," and "perfect").

(a) How many days can the student dress without repeating the combination of jeans and t-shirt?

(b) How many days can the student dress without repeating the combination of jeans and t-shirt and without wearing the same t-shirt on two consecutive days?

2.46. Ordering a "deluxe" pizza means you have four choices from 15 available toppings. How many combinations are possible if toppings can be repeated? If they cannot be repeated? Assume that the order in which the toppings are selected does not matter.

2.47. A lecture room has 60 seats. In how many ways can 45 students occupy the seats in the room?