EEL 4930 Lecture 7

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  - the relative frequencies converge to the probabilities

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- Consider a random experiment that has K possible outcomes, $K < \infty$
- Let $N_k(n) =$ the number of times the outcome is k and let the relative frequency of outcome k be

$$r_k(n) = \frac{N_k(n)}{n}.$$

- If the relative frequencies converge, i.e. $\lim_{n\to\infty} r_k(n)$ is a constant for each k, then:

> For experiments with statistical regularity,
>
> $$\lim_{n\to\infty} r_k(n) = p_k$$
>
> is called the *probability of outcome* k.

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Dividing by n yields

$$0 \leq \frac{N_k(n)}{n} = r_k(n) \leq 1, \quad \forall k = 1, \ldots, K \tag{1}$$

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Again, dividing by n yields

$$\sum_{k=1}^{K} r_k(n) = 1. \tag{2}$$

Consider the event E that an even number occurs.

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What can we say about the number of times E is observed in n trials?

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What have we assumed in developing this equation?

2, 4, and 6 are outcomes & cannot occur at the same time (mutually exclusive)

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- **General property:** If A and B are 2 events that cannot occur simultaneously, and C is the event that either A or B occurs, then

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f_C(n) = f_A(n) + f_B(n). \tag{3}
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1. It is not clear when and in what sense the limit exists.
2. It is not possible to perform an experiment an infinite number of times, so the probabilities can never be known exactly.
3. We cannot use this definition if the experiment cannot be repeated.

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However, any such model should

1. be useful for solving real problems
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3. agree with our intuition (*where appropriate!*)

Probability Spaces

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- We have already defined the **sample space** Ω and **event class** $\mathscr{F}$

- We need to specify P:

> The *probability measure*, denoted by P is a numerically-valued set function that maps all members of $\mathscr{F}$ onto $\mathbb{R}$.

Axioms of Probability

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AXIOMS OF PROBABILITY

Axioms: We specify a minimal set of rules that P must obey:

I. $\forall E \in \mathscr{F}, P(E) \geq 0$

II. $P(\Omega) = 1$

III. $\forall E, F \in \mathscr{F}, P(E \cup F) = P(E) + P(F)$ if $\underbrace{E \cap F = \emptyset}_{\text{m. e.}}$

III. $\forall E, F \in \mathscr{F}, P(E \cup F) = P(E) + P(F)$ if $E \cap F = \emptyset$

III'. If $A_1, A_2, \ldots$ is a sequence of event such that $A_i \cap A_j = \emptyset \; \forall i \neq j$, then

$$P\left[\bigcup_{k=1}^{\infty} A_k\right] = \sum_{k=1}^{\infty} P\left[A_k\right].$$

- We can use Axioms I-III to deal with finite sample spaces. However, Axiom III' is required instead of Axiom III for infinite sample spaces

- Axiom III is a special case of Axiom III'

- The unions and summations in Axiom III' are over *countable* index sets only

Corollaries

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Let $A \in \mathscr{F}$ and $B \in \mathscr{F}$. Then the following properties of P can be derived from the axioms and the mathematical structure of $\mathscr{F}$:

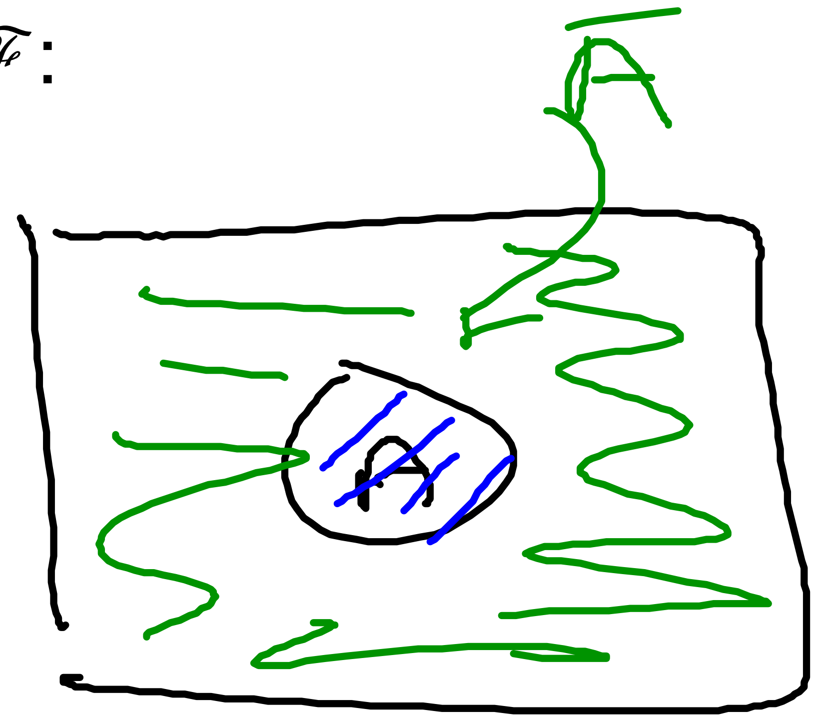
1. $P(A^c) = 1 - P(A)$

$\bar{A}$

$P(S) = 1$

$P(A \cup A^c) = 1$

$P(A) + P(A^c) = 1$



$A \cap A^c = \emptyset$

$A \cup A^c = S$

Corollaries

Let $A \in \mathscr{F}$ and $B \in \mathscr{F}$. Then the following properties of P can be derived from the axioms and the mathematical structure of $\mathscr{F}$:

1. $P(A^c) = 1 - P(A)$

2. $P(A) \leq 1$

$$P(A) = 1 - \underbrace{P(\bar{A})}_{\geq 0}$$

$$\Rightarrow P(A) \leq 1$$

2. $P(A) \leq 1$

3. $P(\emptyset) = 0$

$P(\emptyset) = 1 - P(S) = 1 - 1 = 0$

$\emptyset = \overline{S}$

2. $P(A) \leq 1$

3. $P(\emptyset) = 0$

4. If $A_1, A_2, \ldots, A_n$ are pairwise mutually exclusive, then

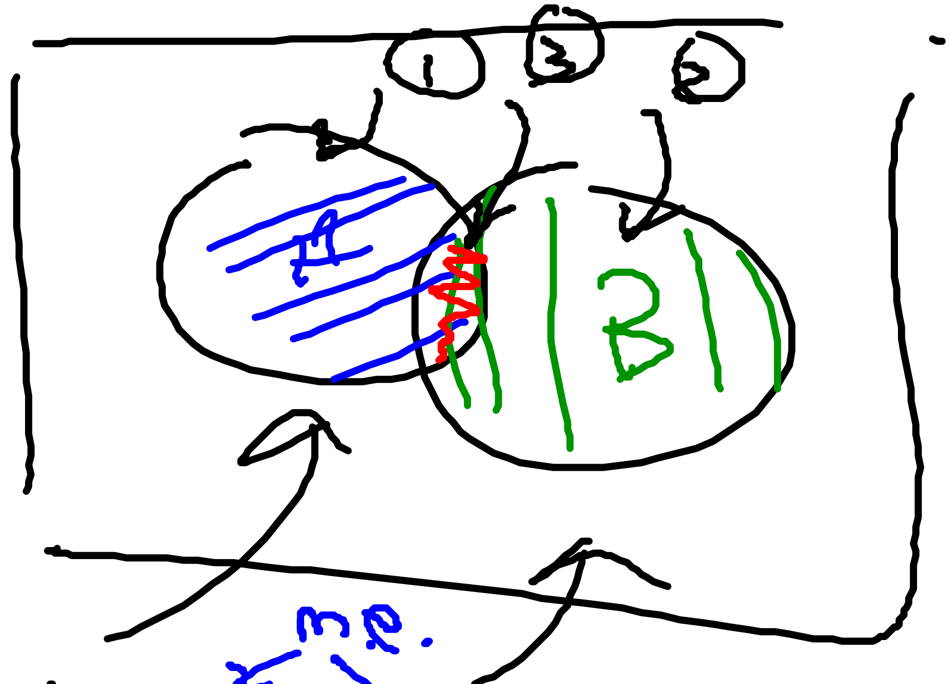
$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_k)$$

Proof is by induction

5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= P(A \cap \bar{B}) + P(B \cap \bar{A}) + P(A \cap B)$

$= [P(A) - P(A \cap B)] + [P(B) - P(A \cap B)] + P(A \cap B)$



m.e.

$A = (A \cap B) \cup (A \cap \bar{B})$

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$B = (B \cap A) \cup (B \cap \bar{A})$

$P(A) = P(A \cap B) + P(A \cap \bar{B})$

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5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

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Proof:

EX **Take 2:** A fair six-sided die is rolled twice. What is the probability of observing either a 1 or a 2 on the top face on either roll?

Let E_i = 1 or 2 on roll i

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= \frac{1}{3} + \frac{1}{3} - \frac{4}{36}$$

(1,1) (1,2)
(2,1) (2,2)

$$= \frac{2}{6} - \frac{1}{9} = 5/9$$

6.

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} P(A_j) - \sum_{j<k} P(A_j \cap A_k) + \cdots$$
$$+(-1)^{(n+1)} P(A_1 \cap A_2 \cap \cdots \cap A_n)$$

Add all single events, subtract off all intersections of pairs of events, add in all intersections of 3 events, ...

6.

$$
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Proof is by induction.

7. If $A \subset B$, then $P(A) \leq P(B)$.

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Proof: