Contents

Chapter 1

Notation and assumptions

In this chapter we will state the main definitions and assumptions used throughout this work. We will start by defining the estimator to be considered and introduce all necessary notation for the remaining chapters.

Recall the following definitions for $n \geq 2$

$$W_{i:n}^{se} = \frac{m(Z_{i:n}, \hat{\theta}_n)}{n-i+1} \prod_{j=1}^{i-1} \left(1 - \frac{m(Z_{j:n}, \hat{\theta}_n)}{n-j+1}\right)$$

and

$$S_{2,n}^{se} = \sum\sum_{1 \leq i < j \leq n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n}^{se} W_{j:n}^{se}$$

This process will be called semiparametric U-Statistic of degree 2 throughout this thesis. Furthermore define

$$W_{i:n}(q) = \frac{q(Z_{i:n})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right]$$

and

$$S_n(q) = \sum\sum_{1 \leq i < j \leq n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n}(q) W_{j:n}(q)$$

for some measurable function q s. t. $q(t) \in [0, 1]$ for all $t \in \mathbb{R}_+$. Next define

$$\mathcal{F}_n = \sigma\{Z_{1:n}, \ldots, Z_{n:n}, Z_{n+1}, Z_{n+2}, \ldots\}$$

The following quantities will be needed in section ??. Define for $n \geq 2$ and $s < t$

$$\begin{aligned}
B_n(s,q) &:= \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n-R_{k,n}}\right]^{\mathbb{I}\{Z_k<s\}} \\
C_n(s,q) &:= \sum_{i=1}^{n+1} \left[\frac{1-q(s)}{n-i+2}\right] \mathbb{I}\{Z_{i-1:n} < s \leq Z_{i:n}\} \\
D_n(s,t,q) &:= \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n-R_{k,n}+2}\right]^{2\mathbb{I}\{Z_k<s\}} \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n-R_{k,n}+1}\right]^{\mathbb{I}\{s<Z_k<t\}} \\
\Delta_n(s,t,q) &:= \mathbb{E}\left[D_n(s,t,q)\right] \\
\bar{\Delta}_n(s,t,q) &:= \mathbb{E}\left[C_n(s,q)D_n(s,t,q)\right] \ .
\end{aligned}$$

We will write $B_n(s) \equiv B_n(s,q)$, $C_n(s) \equiv C_n(s,q)$, $D_n(s,t) \equiv D_n(s,t,q)$, $\Delta_n(s,t) \equiv \Delta_n(s,t,q)$ and $\bar{\Delta}_n(s,t) \equiv \bar{\Delta}_n(s,t,q)$. Next define

$$\bar{S}_n(q) := \sum\sum_{1\leq i<j\leq n} \phi(Z_{i:n}, Z_{j:n}) \bar{W}_{i:n}(q) \bar{W}_{j:n}(q)$$

where

$$\bar{W}_{i:n}(q) := \prod_{k=1}^{n} \left(1 - \frac{q(Z_{k:n})}{n-k+1}\right) \ .$$

Moreover let

$$\begin{aligned}
S(q) := \frac{1}{2}\int_0^\infty \int_0^\infty \phi(s,t)q(s)q(t) \exp\left(\int_0^s \frac{1-q(x)}{1-H(x)} H(dx)\right) \\
\times \exp\left(\int_0^t \frac{1-q(x)}{1-H(x)} H(dx)\right) H(ds)H(dt)
\end{aligned}$$

and

$$\begin{aligned}
\bar{S}(q) := \frac{1}{2}\int_0^\infty \int_0^\infty \phi(s,t) \exp\left(\int_0^s \frac{1-q(x)}{1-H(x)} H(dx)\right) \\
\times \exp\left(\int_0^t \frac{1-q(x)}{1-H(x)} H(dx)\right) H(ds)H(dt) \ .
\end{aligned}$$

We will write $S_n \equiv S_n(q)$, $W_{i:n} \equiv W_{i:n}(q)$, $S \equiv S(q)$ and $\bar{S} \equiv \bar{S}(q)$ throughout this thesis.

The following assumptions will be needed throughout this work, in order to prove the SLLN for S_n.

(A1) The kernel $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is measurable, non-negative and symmetric in its arguments. In effect $\phi(s,t) = \phi(t,s)$ for all $s, t \in \mathbb{R}_+$.

(A2) H is continuous and concentrated on the non-negative real line.

(A3) The following statement holds true

$$\int_0^{\tau_H} \int_0^{\tau_H} \frac{\phi(s,t)}{m(s,\theta_0)m(t,\theta_0)(1-H(s))^\epsilon(1-H(t))^\epsilon} F(dt)F(ds) < \infty$$

for some $0 < \epsilon \leq 1$.

(A4) $m(z, \theta_0)$ is increasing in z.

We will need the following assumptions about the Censoring Model m and the Maximum Likelihood estimate $\hat{\theta}_n$:

(M1) $\hat{\theta}_n$ is measurable and tends to θ_0

(M2) For any $\epsilon > 0$ there exists a neighborhood $V(\epsilon, \theta_0) \subset \Theta$ of θ_0 s.t. for all $\theta \in V(\epsilon, \theta_0)$

$$\sup_{x \geq 0} |m(x,\theta) - m(x,\theta_0)| < \epsilon$$

Chapter 2

Uniform Almost Sure Convergence

Recall the following quantities

$$H^1(x) = \int_0^x m(z,\theta_0) H(dz)$$

and

$$H_n^1(x) = \int_0^x m(z,\theta_0) H_n(dz) = \frac{1}{n}\sum_{i=1}^n \mathbb{I}\{Z_{i:n} \le x\} m(Z_{i:n},\theta_0) \ ,$$

c. f. Dikta (1998), Lemma 3.12.

2.1 New results

The following lemma contains an integration by parts result, which will be useful in order to prove Lemma 2.2.

Lemma 2.1. *For any* $0 \le s < t \le T$ *we have*

$$\begin{aligned}
&\int_s^{t-} \frac{1}{1-H(z)} H_n(dz) - \int_s^t \frac{1}{1-H(z)} H(dz) \\
&= \frac{H_n(t)-H(t)}{1-H(t)} - \frac{H_n(s-)-H(s)}{1-H(s)} - \int_s^t \frac{H_n(z-)-H(z)}{(1-H(z))^2} H(dz) - \gamma_n(t) \qquad (2.1)
\end{aligned}$$

and

$$\begin{aligned}
&\int_s^{t-} \frac{1}{1-H(z)} H_n^1(dz) - \int_s^t \frac{1}{1-H(z)} H^1(dz) \\
&= \frac{H_n^1(t)-H^1(t)}{1-H(t)} - \frac{H_n^1(s-)-H^1(s)}{1-H(s)} - \int_s^t \frac{H_n^1(z-)-H^1(z)}{(1-H(z))^2} H(dz) - \gamma_n^1(t) \qquad (2.2)
\end{aligned}$$

where

$$\gamma_n(t) = \frac{H_n(t) - H_n(t-)}{1 - H(t)} \qquad \textit{and} \qquad \gamma_n^1(t) = \frac{H_n^1(t) - H_n^1(t-)}{1 - H(t)} \ .$$

Proof. First consider that we can write

$$\int_s^t \frac{1}{1 - H(z)} H_n(dz) = \int_s^{t-} \frac{1}{1 - H(z)} H_n(dz) + \gamma_n(s) \ .$$

Thus we have

$$\begin{aligned}
\int_s^{t-} \frac{1}{1 - H(z)} H_n(dz) &= \int_s^t \frac{1}{1 - H(z)} H_n(dz) - \gamma_n(s) \\
&= \int_s^t \left(\frac{1}{1 - H(z)} - 1 \right) H_n(dz) + \int_s^t 1 H_n(dz) - \gamma_n(s) \\
&= \int_s^t \frac{H(z)}{1 - H(z)} H_n(dz) + H_n(t) - H_n(s-) - \gamma_n(s)
\end{aligned}$$

since we have

$$\int_s^t 1 H_n(dz) = \int_0^t 1 H_n(dz) - \int_0^{s-} 1 H_n(dz) = H_n(t) - H_n(s-) \ .$$

We will now use a version of integration by parts (see Cohn (2013), p. 164) to show

$$\begin{aligned}
&\int_s^t \frac{H(z)}{1 - H(z)} H_n(dz) + H_n(t) - H_n(s-) \\
&= \frac{H_n(t)}{1 - H(t)} - \frac{H_n(s-)}{1 - H(s)} - \int_s^t \frac{H_n(z)}{(1 - H(z))^2} H(dz)
\end{aligned}$$

First let's define $\tilde{G}(x) := H_n(x)$ and

$$\tilde{F}(x) := \frac{H(x)}{1 - H(x)}$$

Moreover denote $\mu_{\tilde{F}}$ and $\mu_{\tilde{G}}$ the measures induced by $\tilde{F}$ and $\tilde{G}$ respectively. Note that we have

$$\mu_{\tilde{F}}(]s, t]) = \tilde{F}(t) - \tilde{F}(s) \tag{2.3}$$

Next consider that we can write

$$\tilde{F}(x) = \int_0^x \frac{1}{(1-H(z))^2} H(dz)$$

since we have

$$\begin{aligned} \int_0^x \frac{1}{(1-H(z))^2} H(dz) &= \int_0^{H(x)} \frac{1}{(1-u)^2} du \\ &= \int_0^{H(x)} \frac{1}{(1-u)^2} du \\ &= \frac{1}{1-H(x)} - 1 \\ &= \frac{H(x)}{1-H(x)} \ . \end{aligned}$$

Now combining the above with (2.3) yields

$$\mu_{\tilde{F}}(]s,t]) = \tilde{F}(t) - \tilde{F}(s) = \int_s^t \frac{1}{(1-H(z))^2} H(dz) \ .$$

Therefore the Radon Nikodym derivative of $\mu_{\tilde{F}}$ w. r. t. H is given by

$$\frac{\mu_{\tilde{F}}(dx)}{H(dx)} = \frac{1}{(1-H(x))^2} \ . \tag{2.4}$$

Note that $\tilde{F}$ and $\tilde{G}$ are bounded, right-continuous and vanish at $-\infty$. Thus we can apply Cohn (2013), p. 164, to obtain

$$\int_s^t \tilde{F}(z)\mu_{\tilde{G}}(dz) = \tilde{F}(t)\tilde{G}(t) - \tilde{F}(s-)\tilde{G}(s-) - \int_s^t \tilde{G}(z-)\mu_{\tilde{F}}(dz) \ .$$

Now we get by (2.4) and by definition of $\tilde{F}$ and $\tilde{G}$ that

$$\begin{aligned} \int_0^s \frac{H(z)}{1-H(z)} H_n(dz) &= \frac{H_n(t)H(t)}{1-H(t)} - \frac{H_n(s-)H(s)}{1-H(s)} - \int_s^t H_n(z-)\mu_{\tilde{F}}(dz) \\ &= \frac{H_n(t)H(t)}{1-H(t)} - \frac{H_n(s-)H(s)}{1-H(s)} - \int_s^t \frac{H_n(z-)}{(1-H(z))^2} H(dz) \ . \end{aligned}$$

Therefore we have

$$\begin{aligned}\int_s^{t-} \frac{1}{1-H(z)} H_n(dz) &= \int_s^t \frac{H(z)}{1-H(z)} H_n(dz) + H_n(t) - H_n(s-) - \gamma_n(s) \\ &= \frac{H_n(t)H(t)}{1-H(t)} - \frac{H_n(s-)H(s)}{1-H(s)} - \int_0^s \frac{H_n(z-)}{(1-H(z))^2} H(dz) \\ &\quad + H_n(t) - H_n(s-) - \gamma_n(s) \\ &= \frac{H_n(t)}{1-H(t)} - \frac{H_n(s-)}{1-H(s)} - \int_0^s \frac{H_n(z-)}{(1-H(z))^2} H(dz) \\ &\quad - \gamma_n(s) \ . \end{aligned} \tag{2.5}$$

The latter equality holds, since

$$\frac{H_n(t)H(t)}{1-H(t)} + H_n(t) = \frac{H_n(t)}{1-H(t)}$$

and

$$\frac{H_n(s-)H(s)}{1-H(s)} + H_n(s-) = \frac{H_n(s-)}{1-H(s)} \ .$$

Now consider the following

$$\int_s^t \frac{1}{1-H(z)} H(dz) = \int_s^t \frac{H(z)}{1-H(z)} H(dz) + H(t) - H(s)$$

Define $\bar{G}(x) := H(x)$ and note that $\bar{G}(x)$ is bounded, right-continuous and vanishes at $-\infty$. Therefore applying Cohn (2013), p. 164, to $\tilde{F}$ and $\bar{G}$ yields

$$\int_s^t \frac{H(z)}{1-H(z)} H(dz) = \frac{H^2(t)}{1-H(t)} - \frac{H^2(s)}{1-H(s)} - \int_s^t \frac{H(z)}{(1-H(z))^2} H(dz) \ .$$

Hence we have

$$\int_s^t \frac{1}{1-H(z)} H(dz) = \frac{H^2(t)}{1-H(t)} - \frac{H^2(s)}{1-H(s)} - \int_s^t \frac{H(z)}{(1-H(z))^2} H(dz)$$

$$
\begin{aligned}
&+ H(t) - H(s) \\
&= \frac{H(t)}{1-H(t)} - \frac{H(s)}{1-H(s)} - \int_s^t \frac{H(z)}{(1-H(z))^2} H(dz) \; . \qquad (2.6)
\end{aligned}
$$

Combining (2.5) and (2.6) yields

$$
\begin{aligned}
&\int_s^{t-} \frac{1}{1-H(z)} H_n(dz) - \int_s^t \frac{1}{1-H(z)} H(dz) \\
&= \frac{H_n(t) - H(t)}{1-H(t)} - \frac{H_n(s-) - H(s)}{1-H(s)} - \int_s^t \frac{H_n(z-) - H(z)}{1-H(z)} H(dz) - \gamma_n(t) \; .
\end{aligned}
$$

Thus equation (2.1) from the statement of the lemma has been established. Next define $\tilde{G}^1(x) := H_n^1(x)$ and apply Cohn (2013), p. 164, to $\tilde{F}$ and $\tilde{G}^1$ to obtain

$$
\int_s^t \frac{H(z)}{1-H(z)} H_n^1(dz) = \frac{H_n^1(t)H(t)}{1-H(t)} - \frac{H_n^1(s-)H(s)}{1-H(s)} - \int_s^t \frac{H_n^1(z)}{(1-H(z))^2} H(dz) \qquad (2.7)
$$

Next define $\bar{G}^1(x) := H^1(x)$ and apply Cohn (2013), p. 164, to $\tilde{F}$ and $\bar{G}^1$ to obtain

$$
\int_s^t \frac{H(z)}{1-H(z)} H^1(dz) = \frac{H^1(t)H(t)}{1-H(t)} - \frac{H^1(s-)H(s)}{1-H(s)} - \int_s^t \frac{H^1(z)}{(1-H(z))^2} H(dz) \qquad (2.8)
$$

Finally consider the following

$$
\begin{aligned}
&\int_s^{t-} \frac{1}{1-H(z)} H_n^1(dz) - \int_s^t \frac{1}{1-H(z)} H^1(dz) \\
&= \int_s^t \frac{1}{1-H(z)} H_n^1(dz) - \int_s^t \frac{1}{1-H(z)} H^1(dz) - \gamma_n^1(t) \\
&= \int_s^t \frac{H(z)}{1-H(z)} H_n^1(dz) + H_n^1(t) - H_n^1(s-) \\
&\quad - \int_s^t \frac{1}{1-H(z)} H(dz) + H^1(t) - H^1(s-) - \gamma_n^1(t) \; .
\end{aligned}
$$

Now combining the above with equations (2.7) and (2.8) yields the second part of the lemma. $\square$

The lemma below contains a statement about uniform convergence of processes considered in the proof of Lemma 2.4. It will be used to establish Corollary 2.3.

Lemma 2.2. *The following holds for any $T < \tau_H$.*

$$\sup_{0\le s<t\le T}\left|\int_s^{t-}\frac{1-m(z,\theta_0)}{1-H(z)}H_n(dz)-\int_s^t\frac{1-m(z,\theta_0)}{1-H(z)}H(dz)\right|\to 0$$

almost surely as $n\to\infty$.

Proof. First consider the following

$$\begin{aligned}
&\sup_{0\le s<t\le T}\left|\int_s^{t-}\frac{1-m(z,\theta_0)}{1-H(z)}H_n(dz)-\int_s^t\frac{1-m(z,\theta_0)}{1-H(z)}H(dz)\right|\\
&=\sup_{0\le s<t\le T}\left|\int_s^{t-}\frac{1}{1-H(z)}H_n(dz)-\int_s^{t-}\frac{1}{1-H(z)}H(dz)\right.\\
&\qquad\left.+\int_s^{t-}\frac{m(z,\theta_0)}{1-H(z)}H(dz)-\int_s^{t-}\frac{m(z,\theta_0)}{1-H(z)}H_n(dz)\right|\\
&=\sup_{0\le s<t\le T}\left|\int_s^{t-}\frac{1}{1-H(z)}H_n(dz)-\int_s^{t-}\frac{1}{1-H(z)}H(dz)\right.\\
&\qquad\left.+\int_s^{t-}\frac{1}{1-H(z)}H^1(dz)-\int_s^{t-}\frac{1}{1-H(z)}H_n^1(dz)\right|\\
&\le\sup_{0\le s<t\le T}\left|\int_s^{t-}\frac{1}{1-H(z)}H_n(dz)-\int_s^{t-}\frac{1}{1-H(z)}H(dz)\right|\\
&\qquad+\sup_{0\le s<t\le T}\left|\int_s^{t-}\frac{1}{1-H(z)}H^1(dz)-\int_s^{t-}\frac{1}{1-H(z)}H_n^1(dz)\right|\,.\,. \qquad (2.9)
\end{aligned}$$

Applying Lemma 2.1 equation (2.1) to the first term above yields

$$\begin{aligned}
&\sup_{0\le s<t\le T}\left|\int_s^{t-}\frac{1}{1-H(z)}H_n(dz)-\int_s^{t-}\frac{1}{1-H(z)}H(dz)\right|\\
&=\sup_{0\le s<t\le T}\left|\frac{H_n(t)-H(t)}{1-H(t)}-\frac{H_n(s-)-H(s)}{1-H(s)}\right.\\
&\qquad\left.-\int_s^t\frac{H_n(z-)-H(z)}{(1-H(z))^2}H(dz)-\frac{H_n(t-)-H_n(t)}{1-H(t)}\right|\\
&\le\sup_{0\le s<t\le T}\left|\frac{H_n(t)-H(t)}{1-H(t)}\right|+\sup_{0\le s<t\le T}\left|\frac{H_n(s-)-H(s)}{1-H(s)}\right|\\
&\qquad+\sup_{0\le s<t\le T}\left|\int_s^t\frac{H_n(z-)-H(z)}{(1-H(z))^2}H(dz)\right|+\sup_{0\le s<t\le T}\left|\frac{H_n(t-)-H_n(t)}{1-H(t)}\right|\,.
\end{aligned}$$

Next consider that we have

$$\sup_{0\le s<t\le T}\left|\frac{H_n(t)-H(t)}{1-H(t)}\right| \le \frac{\sup_{x\le T}|H_n(x)-H(x)|}{1-H(T)}$$

and

$$\sup_{0\le s<t\le T}\left|\frac{H_n(s-)-H(s)}{1-H(s)}\right| \le \frac{\sup_{x\le T}|H_n(x)-H(x)|+\frac{1}{n}}{1-H(T)} \; .$$

Furthermore consider

$$\begin{aligned}\sup_{0\le s<t\le T}\left|\int_s^t \frac{H_n(z-)-H(z)}{(1-H(z))^2}H(dz)\right| &\le \sup_{0\le s<t\le T}\left|\int_0^t \frac{H_n(z-)-H(z)}{(1-H(z))^2}H(dz)\right| \\ &\quad + \sup_{0\le s<t\le T}\left|\int_0^s \frac{H_n(z-)-H(z)}{(1-H(z))^2}H(dz)\right| \\ &\le 2\cdot\frac{\sup_{x\le T}|H_n(x)-H(x)|+\frac{1}{n}}{(1-H(T))^2} \; ,\end{aligned}$$

since we have for $t\le T$

$$\left|\int_0^t \frac{H_n(z-)-H(z)}{(1-H(z))^2}H(dz)\right| \le \int_0^t \frac{|H_n(z-)-H(z)|}{(1-H(T))^2}H(dz) \le \frac{\sup_{x\le T}|H_n(x)-H(x)|+\frac{1}{n}}{(1-H(T))^2}$$

using Jensen's inequality. Moreover note that $H_n(s-)-H_n(s)\le n^{-1}$ for any $0\le s\le T$ and hence

$$\sup_{0\le s<t\le T}\left|\frac{H_n(s-)-H_n(s)}{1-H(s)}\right| \le \frac{1}{n(1-H(T))} \; .$$

Therefore we obtain

$$\begin{aligned}&\sup_{0\le s<t\le T}\left|\int_s^{t-}\frac{1}{1-H(z)}H_n(dz)-\int_s^{t-}\frac{1}{1-H(z)}H(dz)\right| \\ &\le \frac{\sup_{x\le T}|H_n(x)-H(x)|}{1-H(T)}+\frac{\sup_{x\le T}|H_n(x)-H(x)|+\frac{1}{n}}{1-H(T)}\end{aligned}$$

$$+ 2 \cdot \frac{\sup\limits_{x \le T} |H_n(x) - H(x)| + \frac{1}{n}}{(1 - H(T))^2} + \frac{1}{n(1 - H(T))}$$
$$\to 0$$

almost surely as $n \to \infty$ by the Glivenko-Cantelli Theorem and since $H(T) < 1$. Now let's consider the latter term in (2.9). Applying Lemma 2.1 equation (2.2) yields

$$\begin{aligned}
&\sup_{0 \le s < t \le T} \left| \int_s^{t-} \frac{1}{1 - H(z)} H_n^1(dz) - \int_s^{t-} \frac{1}{1 - H(z)} H^1(dz) \right| \\
&= \sup_{0 \le s < t \le T} \left| \frac{H_n^1(t) - H^1(t)}{1 - H(t)} - \frac{H_n^1(s-) - H^1(s)}{1 - H(s)} \right. \\
&\qquad\qquad \left. - \int_s^t \frac{H_n^1(z-) - H^1(z)}{(1 - H(z))^2} H(dz) - \frac{H_n^1(t-) - H_n^1(t)}{1 - H(t)} \right| \\
&\le \sup_{0 \le s < t \le T} \left| \frac{H_n^1(t) - H^1(t)}{1 - H(t)} \right| + \sup_{0 \le s < t \le T} \left| \frac{H_n^1(s-) - H^1(s)}{1 - H(s)} \right| \\
&\quad + \sup_{0 \le s < t \le T} \left| \int_s^t \frac{H_n^1(z-) - H^1(z)}{(1 - H(z))^2} H(dz) \right| + \sup_{0 \le s < t \le T} \left| \frac{H_n^1(t-) - H_n^1(t)}{1 - H(t)} \right| \\
&\le \frac{\sup\limits_{x \le T} |H_n^1(x) - H^1(x)|}{1 - H(T)} + \frac{\sup\limits_{x \le T} |H_n^1(x) - H^1(x)| + \frac{1}{n}}{1 - H(T)} \\
&\quad + 2 \cdot \frac{\sup\limits_{x \le T} |H_n^1(x) - H^1(x)| + \frac{1}{n}}{(1 - H(T))^2} + \frac{1}{n(1 - H(T))} \\
&\to 0
\end{aligned}$$

almost surely as $n \to \infty$ by the Glivenko Cantelli Theorem and since $H(T) < 1$. □

The following Corollary is important for the proof of Theorem **??**.

Corollary 2.3. *The measure zero sets* $\{\omega | C_n(s, m; \omega) \nrightarrow C(s, m)$ *as* $n \to \infty\}$ *and* $\{\omega | D_n(s, t, m; \omega) \nrightarrow D(s, t, m)$ *as* $n \to \infty\}$ *are independent of* s *and* t.

Proof. In Lemma 2.4 we have seen that $D_n(s, t, q)$ converges almost surely to $D(s, t, q)$ by Glivenko Cantelli and the SLLN. In order to show the statement of the corollary we need to show that this convergence is uniform in s and t. Let $q \equiv m(\cdot, \theta_0)$ and recall from the proof of Lemma 2.4 that we have

$$\begin{aligned}&\left|\int_0^{s-} \frac{(1-q(z))(H_n(z)-H(z)-\frac{2}{n})}{(1-H_n(z)+\frac{2}{n})(1-H(z))} H_n(dz)\right| \\ &\leq \frac{\sup_{z\leq T}\left|H_n(z)-H(z)-\frac{2}{n}\right|}{1-H(T)} \int_0^{T-} \frac{1}{1-H_n(z)} H_n(dz) \longrightarrow 0\end{aligned}$$

almost surely as $n \to \infty$. Note that the right hand side above converges to zero independent of s and t. Next recall that

$$\int_0^{s-} \frac{1-q(z)}{1-H(z)} H_n(dz) \longrightarrow \int_0^{s} \frac{1-q(z)}{1-H(z)} H(dz) \tag{2.10}$$

by the SLLN. Note that this means pointwise convergence. But according to Lemma 2.2 we also have

$$\sup_{0\leq s\leq T}\left|\int_0^{s-} \frac{1-m(z,\theta_0)}{1-H(z)} H_n(dz) - \int_0^{s} \frac{1-m(z,\theta_0)}{1-H(z)} H(dz)\right| \to 0$$

almost surely as $n \to \infty$. Thus we can show that the convergence in (2.10) is indeed uniform in s and t. For the last part of the proof, we need

$$\sup_{0\leq s<t\leq T}\left|\int_s^{t-} \frac{1-m(z,\theta_0)}{1-H(z)} H_n(dz) - \int_s^{t} \frac{1-m(z,\theta_0)}{1-H(z)} H(dz)\right| \to 0$$

almost surely as $n \to \infty$, which is provided by Lemma 2.2 as well. Hence $D_n(s,t,m) \to D(s,t,m)$ almost surely, uniformly in s and t as $n \to \infty$. By similar arguments we get that $C_n(s,m) \to C(s,m)$ almost surely, uniformly in s and t as $n \to \infty$, considering the proof of Lemma 2.5. $\square$

2.2 Unchanged from my thesis

Recall the following definition from the beginning of this chapter:

$$D_n(s,t) := \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n - R_{k,n} + 2}\right]^{2\mathbb{I}\{Z_k < s\}} \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n - R_{k,n} + 1}\right]^{\mathbb{I}\{s < Z_k < t\}}$$

The next lemma identifies the almost sure limit of D_n for $n \to \infty$. Define for $s < t$

$$D(s,t) := \exp\left(2\int_0^s \frac{1-q(z)}{1-H(z)} H(dz) + \int_s^t \frac{1-q(z)}{1-H(z)} H(dz)\right)$$

Lemma 2.4. *For any $s < t \leq T$ s. t. $H(T) < 1$, we have*

$$\lim_{n\to\infty} D_n(s,t) = D(s,t)\ .$$

Proof. First recall the following definition

$$D_n(s,t) := \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n - R_{k,n} + 2}\right]^{2\mathbb{I}\{Z_k < s\}} \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n - R_{k,n} + 1}\right]^{\mathbb{I}\{s < Z_k < t\}}\ .$$

Next let

$$\begin{aligned} x_k &:= \frac{1-q(Z_k)}{n(1-H_n(Z_k)+2/n)} \\ y_k &:= \frac{1-q(Z_k)}{n(1-H_n(Z_k)+1/n)} \\ s_k &:= \mathbb{I}\{Z_k < s\} \\ t_k &:= \mathbb{I}\{s < Z_k < t\} \end{aligned}$$

for $s < t$ and $k = 1, \ldots, n$. Now consider

$$D_n(s,t) = \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n(1-H_n(Z_k)+2/n)} \mathbb{I}\{Z_k < s\}\right]^2$$

$$
\begin{aligned}
&\times \prod_{k=1}^{n}\left[1+\frac{1-q(Z_k)}{n(1-H_n(Z_k)+1/n)}\mathbb{I}\{s<Z_k<t\}\right] \\
&= \prod_{k=1}^{n}\left[1+x_k s_k\right]^2 \prod_{k=1}^{n}\left[1+y_k t_k\right] \\
&= \exp\left(2\sum_{k=1}^{n}\ln\left[1+x_k s_k\right]+\sum_{k=1}^{n}\ln\left[1+y_k t_k\right]\right) .
\end{aligned}
$$

Note that $0 \leq x_k s_k \leq 1$ and $0 \leq y_k t_k \leq 1$. Consider that the following inequality holds

$$
-\frac{x^2}{2} \leq \ln(1+x) - x \leq 0
$$

for any $x \geq 0$ (cf. Stute and Wang (1993), p. 1603). This implies

$$
-\frac{1}{2}\sum_{k=1}^{n} x_k^2 s_k \leq \sum_{k=1}^{n}\ln(1+x_k s_k) - \sum_{k=1}^{n} x_k s_k \leq 0 .
$$

But now

$$
\begin{aligned}
\sum_{k=1}^{n} x_k^2 s_k &= \frac{1}{n^2}\sum_{k=1}^{n}\left(\frac{1-q(Z_k)}{1-H_n(Z_k)+\frac{2}{n}}\right)^2 \mathbb{I}\{Z_k<s\} \\
&\leq \frac{1}{n^2}\sum_{k=1}^{n}\left(\frac{1}{1-H_n(s)+\frac{1}{n}}\right)^2 \\
&= \frac{1}{n(1-H_n(s)+n^{-1})^2} \longrightarrow 0
\end{aligned}
$$

almost surely as $n \to \infty$, since $H(s) < H(t) < 1$ (c. f. Stute and Wang (1993), p. 1603). Therefore we have

$$
\left|\sum_{k=1}^{n}\ln(1+x_k s_k) - \sum_{k=1}^{n} x_k s_k\right| \longrightarrow 0
$$

with probability 1 as $n \to \infty$. Similarly we obtain

$$
\left|\sum_{k=1}^{n}\ln(1+y_k t_k) - \sum_{k=1}^{n} y_k t_k\right| \longrightarrow 0
$$

with probability 1 as $n \to \infty$. Hence

$$\lim_{n\to\infty} D_n(s,t) = \lim_{n\to\infty} \exp\left(2\sum_{k=1}^{n} x_k s_k + \sum_{k=1}^{n} y_k t_k\right) .$$

Now consider

$$\begin{aligned}
\sum_{k=1}^{n} x_k s_k &= \frac{1}{n}\sum_{k=1}^{n} \frac{1-q(Z_k)}{1-H_n(Z_k)+\frac{2}{n}}\mathbb{I}\{Z_k < s\} \\
&= \int_0^{s-} \frac{1-q(z)}{1-H_n(z)+\frac{2}{n}} H_n(dz) \\
&= \int_0^{s-} \frac{1-q(z)}{1-H(z)} H_n(dz) + \int_0^{s-} \frac{1-q(z)}{1-H_n(z)+\frac{2}{n}} - \frac{1-q(z)}{1-H(z)} H_n(dz) \\
&= \int_0^{s-} \frac{1-q(z)}{1-H(z)} H_n(dz) + \int_0^{s-} \frac{(1-q(z))(H_n(z)-H(z)-\frac{2}{n})}{(1-H_n(z)+\frac{2}{n})(1-H(z))} H_n(dz) .
\end{aligned} \tag{2.11}$$

Note that the second term on the right hand side of the latter equation above tends to zero for $n \to \infty$, because

$$\begin{aligned}
&\left|\int_0^{s-} \frac{(1-q(z))(H_n(z)-H(z)-\frac{2}{n})}{(1-H_n(z)+\frac{2}{n})(1-H(z))} H_n(dz)\right| \\
&\le \frac{\sup_{z\le T}\left|H_n(z)-H(z)-\frac{2}{n}\right|}{1-H(T)} \int_0^{T-} \frac{1}{1-H_n(z)} H_n(dz) \longrightarrow 0
\end{aligned} \tag{2.12}$$

almost surely as $n \to \infty$, by the Glivenko-Cantelli Theorem and since $H(T) < 1$. Moreover we have

$$\int_0^{s-} \frac{1-q(z)}{1-H(z)} H_n(dz) \longrightarrow \int_0^{s} \frac{1-q(z)}{1-H(z)} H(dz)$$

by the SLLN. Therefore we obtain

$$\lim_{n\to\infty} \sum_{k=1}^{n} x_k s_k = \int_0^{s} \frac{1-q(z)}{1-H(z)} H(dz) .$$

By the same arguments, we can show that

$$\lim_{n\to\infty}\sum_{k=1}^{n} y_k t_k = \int_s^t \frac{1-q(z)}{1-H(z)} H(dz) \ .$$

Thus we finally conclude

$$\lim_{n\to\infty} D_n(s,t) = \exp\left(2\int_0^s \frac{1-q(z)}{1-H(z)} H(dz) + \int_s^t \frac{1-q(z)}{1-H(z)} H(dz)\right)$$

almost surely. □

Lemma 2.5. *For continuous H and $t \leq T < \tau_H$, we have $C_n(t) \to 0$ as $n \to \infty$ w. p. 1, and $C_n(t) \in [0,1]$ for all $n \geq 1$ and $t \geq 0$.*

Proof. It is easy to see that $0 \leq C_n(t) \leq 1$ for any $t \geq 0$ and $n \geq 2$, since $0 \leq q(t) \leq 1$ and $\mathbb{I}\{Z_{i-1:n} < t \leq Z_{i:n}\} = 1$ for exactly one $i \in \{1, \ldots, n+1\}$. Let's now consider

$$\begin{aligned}
C_n(t) &= \sum_{i=1}^{n+1} \frac{1-q(t)}{n-i+2} [\mathbb{I}\{Z_{i-1:n} < t\} - \mathbb{I}\{Z_{i:n} < t\}] \\
&= \sum_{i=1}^{n+1} \frac{1-q(t)}{n-i+2} \mathbb{I}\{Z_{i-1:n} < t\} - \sum_{i=1}^{n+1} \frac{1-q(t)}{n-i+2} \mathbb{I}\{Z_{i:n} < t\} \\
&= \sum_{i=0}^{n} \frac{1-q(t)}{n-i+1} \mathbb{I}\{Z_{i:n} < t\} - \sum_{i=1}^{n} \frac{1-q(t)}{n-i+2} \mathbb{I}\{Z_{i:n} < t\} \\
&= \sum_{i=1}^{n} \frac{1-q(t)}{n-i+1} \mathbb{I}\{Z_{i:n} < t\} + \frac{(1-q(t))}{n+1} - \sum_{i=1}^{n} \frac{1-q(t)}{n-i+2} \mathbb{I}\{Z_{i:n} < t\} \\
&= (1-q(t)) \left\{ \frac{1}{n+1} + \sum_{i=1}^{n} \left[\frac{1}{n-i+1} - \frac{1}{n-i+2} \right] \mathbb{I}\{Z_{i:n} < t\} \right\} \\
&= (1-q(t)) \sum_{i=1}^{n} \left[\frac{1}{n-nH_n(Z_{i:n})+1} \frac{1}{n-nH_n(Z_{i:n})+2} \right] \mathbb{I}\{Z_{i:n} < t\} \\
&\quad + \frac{1-q(t)}{n+1} \\
&= (1-q(t)) \int_0^t \left[\frac{1}{1-H_n(x)+\frac{1}{n}} - \frac{1}{1-H_n(x)+\frac{2}{n}} \right] H_n(dx) \\
&\quad + \frac{1-q(t)}{n+1} \ .
\end{aligned} \tag{2.13}$$

In Lemma 2.4 we have seen that

$$\int_0^t \frac{1}{1 - H_n(x) + \frac{2}{n}} H_n(dx) \to \int_0^t \frac{1}{1 - H(x)} H(dx) \ .$$

By the same arguments we obtain

$$\int_0^t \frac{1}{1 - H_n(x) + \frac{1}{n}} H_n(dx) \to \int_0^t \frac{1}{1 - H(x)} H(dx) \ .$$

Therefore the right hand side of (2.13) converges to zero. $\square$

Bibliography

Cohn, D., 2013: *Measure Theory*. Birkhäuser Boston.

Dikta, G., 1998: On semiparametric random censorship models. *Journal of Statistical Planning and Inference*, **66 (2)**, 253–279.

Stute, W. and J.-L. Wang, 1993: The strong law under random censorship. *The Annals of Statistics*, 1591–1607.