Chapter 1

Modifying the Martingale Convergence Theorem

We're considering the estimator

$$S_n = \sum\sum_{1 \le i < j \le n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n} W_{j:n}$$

where

$$W_{i:n} = \frac{q(Z_{i:n})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1} \right]$$

Define $\mathcal{F}_n := \sigma\{Z_{1:n}, \dots, Z_{n:n}, Z_{n+1}, Z_{n+2}, \dots\}$. Furthermore we will need the following definitions in order to get into a (forward) martingale framework. Define

$$\tilde{S}_n^N := S_{N-n+1},\ \mathcal{F}_n^N := \mathcal{F}_{N-n+1}$$

We assume $\sup_n \mathbb{E}[S_n] < \infty$.

Let $U_n[a,b]$ denote the number of upcrossings of $\tilde{S}_1^N, \dots, \tilde{S}_n^N$ and define

$$Y_n^N := \tilde{S}_1^N + \sum_{i=1}^{n-1} \epsilon_i (\tilde{S}_{i+1}^N - \tilde{S}_i^N)$$

with

$$\epsilon_i := \begin{cases} 1 & (\tilde{S}_1^N, \dots, \tilde{S}_i^N) \in B \\ 0 & \text{o.w.} \end{cases}$$

for some Borel set $B \in \mathcal{B}(\mathbb{R}^i)$.

We can show that

$$(b-a)\mathbb{E}[U_n[a,b]] \le \mathbb{E}[Y_n^N] \le \mathbb{E}[\tilde{S}_n^N] - \sum_{k=1}^{n-1} \mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N - \tilde{S}_k^N | \mathcal{F}_k^N]]$$

Note that we need to show

$$\begin{aligned} &\lim_{N\to\infty} (b-a)\mathbb{E}[U_N[a,b]] \\ &\le \lim_{N\to\infty} \mathbb{E}[Y_N^N] \\ &\le \lim_{N\to\infty} \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1} \mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N - \tilde{S}_k^N | \mathcal{F}_k^N]] \qquad (1.1) \\ &< \infty \end{aligned}$$

So the main concern is to show that the sum of increases of $\tilde{S}_k^N$ on the right hand side converges. In order to do that, we need the following result.

Lemma 1.1. *Define*

$$Q_{ij}^{n+1} := \begin{cases} Q_i^{n+1} & j \le n \\ Q_i^{n+1} - \frac{(n+1)\pi_i\pi_n(1-q(Z_{n:n+1}))}{(n-i+1)(2-q(Z_{n:n+1}))} & j = n+1 \end{cases}$$

with

$$Q_i^{n+1} := (n+1)\left\{ \sum_{r=1}^{i-1} \left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})} \right]^2 + \frac{\pi_i\pi_{i+1}}{n-i+1} \right\}$$

and

$$\pi_i := \prod_{k=1}^{i-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})} \right]$$

Then

$$\mathbb{E}[S_n|\mathcal{F}_{n+1}] = \sum\sum_{1\le i<j\le n} \phi(Z_{i:n+1}, Z_{j:n+1}) W_{i:n+1} W_{j:n+1} Q_{i,j}^{n+1}$$

Note that we directly get the following from the lemma above.

$$\begin{aligned}\mathbb{E}[\tilde{S}_{k+1}^N|\tilde{\mathcal{F}}_k^N] &= \mathbb{E}[S_{N-k}|\mathcal{F}_{N-k+1}] \\ &= \sum\sum_{1\le i<j\le N-k+1} \phi(Z_{i:N-k+1}, Z_{j:N-k+1})W_{i:N-k+1}W_{j:N-k+1}Q_{i,j}^{N-k+1}\end{aligned}$$

For the sake of notation let's denote $Z_{(i)} := Z_{i:N-k+1}$ and $Z_{(i)} := Z_{i:N-k+1}$. Now we get

$$\begin{aligned}\mathbb{E}[Y_N^N] &\le \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1} \mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N - \tilde{S}_k^N|\mathcal{F}_k^N]] \\ &= \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1} \mathbb{E}\left[(1-\epsilon_k)\sum\sum_{1\le i<j\le N-k+1} \phi(Z_{(i)}, Z_{(j)})W_{(i)}W_{(j)}(Q_{i,j}^{N-k+1}-1)\right] \\ &= \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[(1-\epsilon_k)\phi(Z_{(i)}, Z_{(j)})W_{(i)}W_{(j)}(Q_{i,j}^{N-k+1}-1)\right] \\ &\le \mathbb{E}[\tilde{S}_N^N] + \left|\sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[(1-\epsilon_k)\phi(Z_{(i)}, Z_{(j)})W_{(i)}W_{(j)}(Q_{i,j}^{N-k+1}-1)\right]\right| \\ &\le \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} \left|\mathbb{E}\left[(1-\epsilon_k)\phi(Z_{(i)}, Z_{(j)})W_{(i)}W_{(j)}(Q_{i,j}^{N-k+1}-1)\right]\right| \\ &\overset{Jensen}{\le} \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[(1-\epsilon_k)\phi(Z_{(i)}, Z_{(j)})W_{(i)}W_{(j)}\left|(Q_{i,j}^{N-k+1}-1)\right|\right] \\ &\le \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[\phi(Z_{(i)}, Z_{(j)})W_{(i)}W_{(j)}\left|(Q_{i,j}^{N-k+1}-1)\right|\right] \\ &\overset{C.S.}{\le} \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[\phi^2(Z_{(i)}, Z_{(j)})W_{(i)}^2W_{(j)}^2\right]^{\frac{1}{2}} \mathbb{E}\left[(Q_{i,j}^{N-k+1}-1)^2\right]^{\frac{1}{2}}\end{aligned} \tag{1.2}$$

Now consider

$$Q_i^{n+1} - 1 = Q_1^{n+1} + \sum_{k=1}^{i-1}(Q_{k+1}^{n+1} - Q_k^{n+1}) - 1$$

$$= \sum_{k=1}^{i-1} Q_{k+1}^{n+1} - Q_k^{n+1}$$

Hence

$$(Q_i^{n+1} - 1)^2 = \sum_{k=1}^{i-1} \sum_{l=1}^{i-1} (Q_{k+1}^{n+1} - Q_k^{n+1})(Q_{l+1}^{n+1} - Q_l^{n+1}) \tag{1.3}$$

Lemma 1.2. *Let Q_i^{n+1} be defined as above. Then*

$$Q_{i+1}^{n+1} - Q_i^{n+1} = \frac{\tilde{\pi}_i^2 (n-i+2)^2}{n+1} \left\{ \frac{(q_i - q_{i+1})(n-i)(n-i+1) - q_{i+1}(1-q_i)(n-i+1-q_i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \right\}$$

where $q_i := q(Z_{i:n+1})$ and

$$\tilde{\pi}_i := \pi_i \frac{n+1}{n-i+2}$$

Note that $\tilde{\pi}_i \leq 1$ for all $i \leq n+1$.

Consider the following

$$\begin{aligned}
&Q_{i+1}^{n+1} - Q_i^{n+1} \\
&\leq \frac{\tilde{\pi}_i^2 (n-i+2)^2}{n+1} \left\{ \frac{(q_i - q_{i+1})(n-i)(n-i+1)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \right\} \\
&\leq \left| \frac{\tilde{\pi}_i^2 (n-i+2)^2}{n+1} \left\{ \frac{(q_i - q_{i+1})(n-i)(n-i+1)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \right\} \right| \\
&\leq \sum_{k=1}^{i-1} \frac{\tilde{\pi}_i^2 (n-i+2)^2}{n+1} \left\{ \frac{|q_i - q_{i+1}|(n-i)(n-i+1)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \right\} \\
&\leq \sum_{k=1}^{i-1} \frac{(n-i+2)^2}{n+1} \left\{ \frac{|q_i - q_{i+1}|(n-i)(n-i+1)}{(n-i)(n-i+1)(n-i+1)^2(n-i)} \right\} \\
&= \sum_{k=1}^{i-1} \frac{(n-i+2)^2}{n+1} \left\{ \frac{|q_i - q_{i+1}|(n-i)(n-i+1)}{(n-i)^2(n-i+1)^3} \right\} \\
&\leq \frac{4}{n+1} \sum_{k=1}^{i-1} \frac{|q_i - q_{i+1}|(n-i)(n-i+1)}{(n-i)^2(n-i+1)}
\end{aligned}$$

The latter inequality above holds since

$$\frac{n-i+2}{n-i+1} = 1 + \frac{1}{n-i+1} \leq 2$$

Thus we have

$$
\begin{aligned}
&(Q_{k+1}^{n+1} - Q_k^{n+1})(Q_{l+1}^{n+1} - Q_l^{n+1}) \\
&\quad \leq \frac{16}{(n+1)^2} \frac{|q_k - q_{k+1}|(n-k)(n-k+1)}{(n-k)^2(n-k+1)} \\
&\qquad\qquad \times \frac{|q_l - q_{l+1}|(n-l)(n-l+1)}{(n-l)^2(n-l+1)} \\
&\quad = \frac{16|q_k - q_{k+1}||q_l - q_{l+1}|(n-k)(n-k+1)(n-l)(n-l+1)}{(n+1)^2(n-k)^2(n-k+1)(n-l)^2(n-l+1)} \\
&\quad = \frac{16|q_k - q_{k+1}||q_l - q_{l+1}|}{(n+1)^2(n-k)(n-l)}
\end{aligned}
$$

Now consider

$$
\begin{aligned}
&\mathbb{E}[(Q_{k+1}^{n+1} - Q_k^{n+1})(Q_{l+1}^{n+1} - Q_l^{n+1})] \\
&\quad = \frac{16\mathbb{E}[|q_k - q_{k+1}||q_l - q_{l+1}|]}{(n+1)^2(n-k)(n-l)} \\
&\quad \leq \frac{16\mathbb{E}[|q_k - q_{k+1}|]}{(n+1)^2(n-k)(n-l)}
\end{aligned}
$$

Here the latter inequality holds, since $|q_l - q_{l+1}| \leq 1$.

TODO Let's assume for the moment that

$$
\mathbb{E}[|q_k - q_{k+1}|] \overset{?}{\leq} \frac{c_1}{n+1}
$$

Then we get

$$
\mathbb{E}[(Q_{k+1}^{n+1} - Q_k^{n+1})(Q_{l+1}^{n+1} - Q_l^{n+1})] \leq \frac{16c_1}{(n+1)^3(n-k)(n-l)}
$$

Now combining the latter with (1.3) yields

$$
\mathbb{E}[(Q_i^{n+1} - 1)^2] \leq \sum_{k=1}^{i-1} \sum_{l=1}^{i-1} \frac{16c_1}{(n+1)^3(n-k)(n-l)}
$$

$$= \frac{16c_1}{(n+1)^3} \sum_{k=1}^{i-1} \frac{1}{(n-k)} \sum_{l=1}^{i-1} \frac{1}{(n-l)}$$
$$= \frac{16c_1}{(n+1)^3} \sum_{k=n-i+1}^{n-1} \frac{1}{k} \sum_{l=n-i+1}^{n-1} \frac{1}{l}$$

TODO Note that (I believe) I can show

$$\sum_{k=1}^{n-1} \frac{1}{k} \leq \ln(n-1)+1$$

and

$$\frac{\ln(n-1)+1}{(n+1)^{\frac{1}{3}}} \leq 3$$

(Everything of the form $(n+1)^{\frac{1}{2}-\epsilon}$ with $\epsilon > 0$ will work for the denominator.)

Therefore

$$\mathbb{E}[(Q_i^{n+1}-1)^2] \leq \frac{16c_1}{(n+1)^3}(\ln(n-1)+1)^2$$

and hence

$$\begin{aligned}
\mathbb{E}[(Q_i^{n+1}-1)^2]^{\frac{1}{2}} &\leq \frac{4\sqrt{c_1}(\ln(n-1)+1)}{(n+1)^{\frac{3}{2}}} \\
&= \frac{4\sqrt{c_1}}{(n+1)^{\frac{7}{6}}} \cdot \frac{(\ln(n-1)+1)}{(n+1)^{\frac{1}{3}}} \\
&\leq \frac{12\sqrt{c_1}}{(n+1)^{\frac{7}{6}}}
\end{aligned}$$

Now combining the above with (1.2) yields

$$\begin{aligned}
\mathbb{E}[Y_N^N] &\leq \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\leq i<j\leq N-k+1} \mathbb{E}\left[\phi^2(Z_{(i)}, Z_{(j)})W_{(i)}^2 W_{(j)}^2\right]^{\frac{1}{2}} \mathbb{E}\left[(Q_{i,j}^{N-k+1}-1)^2\right]^{\frac{1}{2}} \\
&\leq \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \mathbb{E}\left[\sum\sum_{1\leq i<j\leq N-k+1} \phi^2(Z_{(i)}, Z_{(j)})W_{(i)}^2 W_{(j)}^2\right]^{\frac{1}{2}} \times \frac{12\sqrt{c_1}}{(N-k+1)^{\frac{7}{6}}}
\end{aligned}$$

Thus it remains to show that

$$\mathbb{E}\left[\sum\sum_{1\leq i<j\leq N-k+1} \phi^2(Z_{(i)}, Z_{(j)}) W_{(i)}^2 W_{(j)}^2\right]^{\frac{1}{2}} \leq c_2$$

is constant. Then we would have

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}[U_N[a,b]] &\leq \lim_{N\to\infty} \mathbb{E}[Y_N^N] \\
&\leq \lim_{N\to\infty} \mathbb{E}[\tilde{S}_N^N] + c_2 \sum_{k=1}^{N-1} \frac{\sqrt{64}}{(N-k+1)^{\frac{7}{6}}} \\
&\leq \sup_N \mathbb{E}[\tilde{S}_N^N] + \lim_{N\to\infty} c_2 \sum_{k=1}^{N-1} \frac{\sqrt{64}}{(N-k+1)^{\frac{7}{6}}} \\
&< \infty
\end{aligned}$$

And therefore we may finally conclude that $S = \lim_{n\to\infty} S_n$ exists.

$\boxed{\textbf{TODO}}$:

1. Show that
$$\mathbb{E}[|q_k - q_{k+1}|] \overset{?}{\leq} \frac{c_1}{n+1}$$
using Taylor expansion. DONE.

2. Show
$$\mathbb{E}\left[\sum\sum_{1\leq i<j\leq N-k+1} \phi^2(Z_{(i)}, Z_{(j)}) W_{(i)}^2 W_{(j)}^2\right]^{\frac{1}{2}} \leq const.$$
using comparison with $\tilde{S}_n^N$ for some $n \leq N$. DONE.

3. Show (or find reference for) the following
$$\sum_{k=1}^{n-1} \frac{1}{k} \leq \ln(n-1) + 1$$

and

$$\frac{\ln(n-1)+1}{(n+1)^{\frac{1}{3}}} \leq 3$$

DONE.

1.1 q-Spacings

We need to show that

$$\mathbb{E}[q(Z_{k:n}) - q(Z_{k+1:n})] \leq \frac{c_1}{n+1}$$

where $Z_1, \ldots, Z_n \sim H$.

We know (David) that for $U_1, \ldots, U_n \sim \text{UNI}[0,1]$

$$\mathbb{E}[U_{k+1:n} - U_{k:n}] = \frac{1}{n+1}$$

We can write

$$q(H^{-1}(x)) = q(H^{-1}(x_0)) + q'(\hat{x})(x - x_0)$$

by Taylor's expansion for some $\hat{x}$ in a neighborhood of x_0. Therefore we have

$$q(H^{-1}(x)) - q(H^{-1}(x_0)) = q'(\hat{x})(x - x_0)$$

and hence

$$|q(H^{-1}(x)) - q(H^{-1}(x_0))| = |q'(\hat{x})| \cdot |x - x_0|$$

Now consider that for $x = U_{k:n}$ and $x_0 = U_{k+1:n}$ we have

$$|q(H^{-1}(x)) - q(H^{-1}(x_0))| = |q(Z_{k:n}) - q(Z_{k+1:n})|$$

Thus we get

$$
\begin{aligned}
|q(Z_{k:n}) - q(Z_{k+1:n})| &= |q'(\hat{x})| \cdot |U_{k:n} - U_{k+1:n}| \\
&= |q'(\hat{x})| \cdot (U_{k+1:n} - U_{k:n})
\end{aligned}
$$

Assume

$$q'(x) \leq c_1$$

for all $x \in \mathbb{R}+$.

Then we get

$$|q(Z_{k:n}) - q(Z_{k+1:n})| = c_1(U_{k+1:n} - U_{k:n})$$

Now by taking expectations on both sides we may conclude

$$
\begin{aligned}
\mathbb{E}[|q(Z_{k:n}) - q(Z_{k+1:n})|] &\leq c_1\mathbb{E}[U_{k+1:n} - U_{k:n}] \\
&= \frac{c_1}{n+1}
\end{aligned}
$$

1.2 The Expectation above

We need

$$\mathbb{E}\left[\sum\sum_{1\leq i<j\leq N-k+1} \phi^2(Z_{(i)}, Z_{(j)})W_{(i)}^2 W_{(j)}^2\right]^{\frac{1}{2}} \leq const.$$

Consider that

$$
\begin{aligned}
(\tilde{S}_k^N)^2 = &\left(\sum\sum_{1\leq i<j\leq N-k+1} \phi(Z_{(i)}, Z_{(j)})W_{(i)}W_{(j)}\right) \\
&\times \left(\sum\sum_{1\leq i<j\leq N-k+1} \phi(Z_{(i)}, Z_{(j)})W_{(i)}W_{(j)}\right) \\
= &\left(\sum\sum_{1\leq i<j\leq N-k+1} \phi^2(Z_{(i)}, Z_{(j)})W_{(i)}^2 W_{(j)}^2\right)
\end{aligned}
$$

$$
\begin{aligned}
& + \left(\sum\sum_{1\le i<j\le N-k+1} \phi(Z_{(i)}, Z_{(j)}) W_{(i)} W_{(j)} \right) \\
& \times \left(\sum\sum_{1\le l<m\le N-k+1} \phi(Z_{(l)}, Z_{(m)}) W_{(l)} W_{(m)} \right) \mathbb{1}_{\{(i,j)\neq(l,m)\}} \\
\geq & \sum\sum_{1\le i<j\le N-k+1} \phi^2(Z_{(i)}, Z_{(j)}) W_{(i)}^2 W_{(j)}^2
\end{aligned}
$$

Thus

$$
\mathbb{E}\left[\sum\sum_{1\le i<j\le N-k+1} \phi^2(Z_{(i)}, Z_{(j)}) W_{(i)}^2 W_{(j)}^2 \right]^{\frac{1}{2}} \leq \mathbb{E}[(\tilde{S}_k^N)^2]^{\frac{1}{2}} \leq \left(\sup_{k,N} \mathbb{E}[(\tilde{S}_k^N)^2] \right)^{\frac{1}{2}}
$$

I think this leads to the following additional assumptions on S_n:

- Assume $(S_n)^2$ is integrable for all $n \geq 2$.
- Assume $\sup_n \mathbb{E}[(S_n)^2] < \infty$.

1.3 Some limits

First we want to show that for $n \geq 2$

$$
\sum_{k=1}^{n-1} \frac{1}{k} \leq \ln(n-1) + 1 \tag{1.4}
$$

Proof. Consider

$$
\begin{aligned}
& \sum_{k=1}^{n-1} \frac{1}{k} \quad \leq \quad \ln(n-1) + 1 \\
\Leftrightarrow \quad & \sum_{k=1}^{n-1} \frac{1}{k} - 1 \quad \leq \quad \ln(n-1) \\
\Leftrightarrow \quad & \sum_{k=2}^{n-1} \frac{1}{k} \quad \leq \quad \ln(n-1)
\end{aligned}
$$

$$\Leftrightarrow \quad \prod_{k=2}^{n-1} \exp\left(\frac{1}{k}\right) \quad \leq \quad n-1 \tag{1.5}$$

Now we will continue by induction. For $n = 2$ inequality (1.5) is obviously satisfied. Now assume that (1.5) holds for any n, then it should hold for $n+1$. It remains to show that

$$\prod_{k=2}^{n} \exp\left(\frac{1}{k}\right) \leq n$$

Consider

$$\begin{aligned}\prod_{k=2}^{n} \exp\left(\frac{1}{k}\right) &= \exp\left(\frac{1}{n}\right) \prod_{k=2}^{n-1} \exp\left(\frac{1}{k}\right) \\ &\leq \exp\left(\frac{1}{n}\right)(n-1)\end{aligned}$$

It is well known that for any $x > 1$

$$\exp(x) < \frac{1}{1-x}$$

and hence

$$\exp\left(\frac{1}{n}\right) < \frac{1}{1-\frac{1}{n}} = \frac{n}{n-1}$$

Thus we get

$$\prod_{k=2}^{n} \exp\left(\frac{1}{k}\right) < \frac{n}{n-1}(n-1) = n$$

$\square$

Next we will show that for $n \geq 2$

$$\frac{\ln(n-1)+1}{(n+1)^{\frac{1}{3}}} \leq 3$$

which is equivalent to showing

$$\ln(n-1)+1 \leq 3(n+1)^{\frac{1}{3}} \tag{1.6}$$

Proof. Since $\ln(n-1) \leq \ln(n+1)$ it remains to show

$$\ln(n+1)+1 \leq 3(n+1)^{\frac{1}{3}} \tag{1.7}$$

It is easy to check that inequality (1.7) holds for $n=2$. Now consider that

$$\frac{d}{dn}(\ln(n+1)+1) = \frac{1}{n+1}$$

and

$$\frac{d}{dn}3(n+1)^{\frac{1}{3}} = \frac{1}{(n+1)^{\frac{2}{3}}}$$

Now for $n \geq 2$ we have

$$\frac{\frac{1}{n+1}}{\frac{1}{(n+1)^{\frac{2}{3}}}} = \frac{(n+1)^{\frac{2}{3}}}{n+1} = \frac{1}{(n+1)^{\frac{1}{3}}} < 1$$

and hence

$$\frac{d}{dn}(\ln(n+1)+1) \leq \frac{d}{dn}3(n+1)^{\frac{1}{3}} \tag{1.8}$$

But now the fact that (1.7) holds for $n=2$ and (1.8) holds for $n \geq 2$ implies that (1.7) holds for all $n \geq 2$. □