

Chapter 1

Modifying the Martingale Convergence Theorem

We're considering the estimator

$$S_n = \sum_{1 \leq i < j \leq n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n} W_{j:n}$$

where

$$W_{i:n} = \frac{q(Z_{i:n})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1} \right]$$

Define $\mathcal{F}_n := \sigma\{Z_{1:n}, \dots, Z_{n:n}, Z_{n+1}, Z_{n+2}, \dots\}$. Furthermore we will need the following definitions in order to get into a (forward) martingale framework. Define

$$\tilde{S}_n^N := S_{N-n+1}, \mathcal{F}_n^N := \mathcal{F}_{N-n+1}$$

We assume $\sup_n \mathbb{E}[S_n] < \infty$.

Let $U_n[a, b]$ denote the number of upcrossings of $\tilde{S}_1^N, \dots, \tilde{S}_n^N$ and define

$$Y_n^N := \tilde{S}_1^N + \sum_{i=1}^{n-1} \epsilon_i (\tilde{S}_{i+1}^N - \tilde{S}_i^N)$$

with

$$\epsilon_i := \begin{cases} 1 & (\tilde{S}_1^N, \dots, \tilde{S}_i^N) \in B \\ 0 & \text{o.w.} \end{cases}$$

for some Borel set $B \in \mathcal{B}(\mathbb{R}^i)$.

We can show that

$$(b-a)\mathbb{E}[U_n[a, b]] \leq \mathbb{E}[Y_n^N] \leq \mathbb{E}[\tilde{S}_n^N] - \sum_{k=1}^{n-1} \mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N - \tilde{S}_k^N | \mathcal{F}_k^N]]$$

Note that we need to show

$$\begin{aligned} & \lim_{N \rightarrow \infty} (b-a)\mathbb{E}[U_N[a, b]] \\ & \leq \lim_{N \rightarrow \infty} \mathbb{E}[Y_N^N] \\ & \leq \lim_{N \rightarrow \infty} \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1} \mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N - \tilde{S}_k^N | \mathcal{F}_k^N]] \\ & < \infty \end{aligned} \tag{1.1}$$

So the main concern is to show that the sum of increases of \tilde{S}_k^N on the right hand side converges. In order to do that, we need the following result.

Lemma 1.1. *Define*

$$Q_{ij}^{n+1} := \begin{cases} Q_i^{n+1} & j \leq n \\ Q_i^{n+1} - \frac{(n+1)\pi_i\pi_n(1-q(Z_{n:n+1}))}{(n-i+1)(2-q(Z_{n:n+1}))} & j = n+1 \end{cases}$$

with

$$Q_i^{n+1} := (n+1) \left\{ \sum_{r=1}^{i-1} \left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})} \right]^2 + \frac{\pi_i\pi_{i+1}}{n-i+1} \right\}$$

and

$$\pi_i := \prod_{k=1}^{i-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})} \right]$$

Then

$$\mathbb{E}[S_n | \mathcal{F}_{n+1}] = \sum_{1 \leq i < j \leq n} \phi(Z_{i:n+1}, Z_{j:n+1}) W_{i:n+1} W_{j:n+1} Q_{i,j}^{n+1}$$

Note that we directly get the following from the lemma above.

$$\begin{aligned}\mathbb{E}[\tilde{S}_{k+1}^N | \tilde{\mathcal{F}}_k^N] &= \mathbb{E}[S_{N-k} | \mathcal{F}_{N-k+1}] \\ &= \sum_{1 \leq i < j \leq N-k+1} \phi(Z_{i:N-k+1}, Z_{j:N-k+1}) W_{i:N-k+1} W_{j:N-k+1} Q_{i,j}^{N-k+1}\end{aligned}$$

For the sake of notation let's denote $Z_{(i)} := Z_{i:N-k+1}$ and $Z_{(j)} := Z_{j:N-k+1}$. Now we get

$$\begin{aligned}\mathbb{E}[Y_N^N] &\leq \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1} \mathbb{E}[(1 - \epsilon_k) \mathbb{E}[\tilde{S}_{k+1}^N - \tilde{S}_k^N | \tilde{\mathcal{F}}_k^N]] \\ &= \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1} \mathbb{E} \left[(1 - \epsilon_k) \sum_{1 \leq i < j \leq N-k+1} \phi(Z_{(i)}, Z_{(j)}) W_{(i)} W_{(j)} (Q_{i,j}^{N-k+1} - 1) \right] \\ &= \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1} \sum_{1 \leq i < j \leq N-k+1} \mathbb{E} [(1 - \epsilon_k) \phi(Z_{(i)}, Z_{(j)}) W_{(i)} W_{(j)} (Q_{i,j}^{N-k+1} - 1)] \\ &\leq \mathbb{E}[\tilde{S}_N^N] + \left| \sum_{k=1}^{N-1} \sum_{1 \leq i < j \leq N-k+1} \mathbb{E} [(1 - \epsilon_k) \phi(Z_{(i)}, Z_{(j)}) W_{(i)} W_{(j)} (Q_{i,j}^{N-k+1} - 1)] \right| \\ &\leq \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum_{1 \leq i < j \leq N-k+1} |\mathbb{E} [(1 - \epsilon_k) \phi(Z_{(i)}, Z_{(j)}) W_{(i)} W_{(j)} (Q_{i,j}^{N-k+1} - 1)]| \\ &\stackrel{Jensen}{\leq} \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum_{1 \leq i < j \leq N-k+1} \mathbb{E} [(1 - \epsilon_k) \phi(Z_{(i)}, Z_{(j)}) W_{(i)} W_{(j)} | (Q_{i,j}^{N-k+1} - 1)|] \\ &\leq \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum_{1 \leq i < j \leq N-k+1} \mathbb{E} [\phi(Z_{(i)}, Z_{(j)}) W_{(i)} W_{(j)} | (Q_{i,j}^{N-k+1} - 1)|] \\ &\stackrel{C.S.}{\leq} \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum_{1 \leq i < j \leq N-k+1} \mathbb{E} [\phi^2(Z_{(i)}, Z_{(j)}) W_{(i)}^2 W_{(j)}^2]^{\frac{1}{2}} \mathbb{E} [(Q_{i,j}^{N-k+1} - 1)^2]^{\frac{1}{2}}\end{aligned}\tag{1.2}$$

Now consider

$$Q_i^{n+1} - 1 = Q_1^{n+1} + \sum_{k=1}^{i-1} (Q_{k+1}^{n+1} - Q_k^{n+1}) - 1$$

$$= \sum_{k=1}^{i-1} Q_{k+1}^{n+1} - Q_k^{n+1}$$

Hence

$$(Q_i^{n+1} - 1)^2 = \sum_{k=1}^{i-1} \sum_{l=1}^{i-1} (Q_{k+1}^{n+1} - Q_k^{n+1})(Q_{l+1}^{n+1} - Q_l^{n+1}) \quad (1.3)$$

Lemma 1.2. *Let Q_i^{n+1} be defined as above. Then*

$$Q_{i+1}^{n+1} - Q_i^{n+1} = \frac{\tilde{\pi}_i^2 (n-i+2)^2}{n+1} \left\{ \frac{(q_i - q_{i+1})(n-i)(n-i+1) - q_{i+1}(1-q_i)(n-i+1-q_i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \right\}$$

where $q_i := q(Z_{i:n+1})$ and

$$\tilde{\pi}_i := \pi_i \frac{n+1}{n-i+2}$$

Note that $\tilde{\pi}_i \leq 1$ for all $i \leq n+1$.

Consider the following

$$\begin{aligned} & Q_{i+1}^{n+1} - Q_i^{n+1} \\ & \leq \frac{\tilde{\pi}_i^2 (n-i+2)^2}{n+1} \left\{ \frac{(q_i - q_{i+1})(n-i)(n-i+1)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \right\} \\ & \leq \left| \frac{\tilde{\pi}_i^2 (n-i+2)^2}{n+1} \left\{ \frac{(q_i - q_{i+1})(n-i)(n-i+1)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \right\} \right| \\ & \leq \sum_{k=1}^{i-1} \frac{\tilde{\pi}_i^2 (n-i+2)^2}{n+1} \left\{ \frac{|q_i - q_{i+1}|(n-i)(n-i+1)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \right\} \\ & \leq \sum_{k=1}^{i-1} \frac{(n-i+2)^2}{n+1} \left\{ \frac{|q_i - q_{i+1}|(n-i)(n-i+1)}{(n-i)(n-i+1)(n-i+1)^2(n-i)} \right\} \\ & = \sum_{k=1}^{i-1} \frac{(n-i+2)^2}{n+1} \left\{ \frac{|q_i - q_{i+1}|(n-i)(n-i+1)}{(n-i)^2(n-i+1)^3} \right\} \\ & \leq \frac{4}{n+1} \sum_{k=1}^{i-1} \frac{|q_i - q_{i+1}|(n-i)(n-i+1)}{(n-i)^2(n-i+1)} \end{aligned}$$

The latter inequality above holds since

$$\frac{n-i+2}{n-i+1} = 1 + \frac{1}{n-i+1} \leq 2$$

Thus we have

$$\begin{aligned}
 & (Q_{k+1}^{n+1} - Q_k^{n+1})(Q_{l+1}^{n+1} - Q_l^{n+1}) \\
 & \leq \frac{16}{(n+1)^2} \frac{|q_k - q_{k+1}|(n-k)(n-k+1)}{(n-k)^2(n-k+1)} \\
 & \quad \times \frac{|q_l - q_{l+1}|(n-l)(n-l+1)}{(n-l)^2(n-l+1)} \\
 & = \frac{16|q_k - q_{k+1}||q_l - q_{l+1}|(n-k)(n-k+1)(n-l)(n-l+1)}{(n+1)^2(n-k)^2(n-k+1)(n-l)^2(n-l+1)} \\
 & = \frac{16|q_k - q_{k+1}||q_l - q_{l+1}|}{(n+1)^2(n-k)(n-l)}
 \end{aligned}$$

Now consider

$$\begin{aligned}
 & \mathbb{E}[(Q_{k+1}^{n+1} - Q_k^{n+1})(Q_{l+1}^{n+1} - Q_l^{n+1})] \\
 & = \frac{16\mathbb{E}[|q_k - q_{k+1}||q_l - q_{l+1}|]}{(n+1)^2(n-k)(n-l)} \\
 & \leq \frac{16\mathbb{E}[|q_k - q_{k+1}|]}{(n+1)^2(n-k)(n-l)}
 \end{aligned}$$

Here the latter inequality holds, since $|q_l - q_{l+1}| \leq 1$.

TODO Let's assume for the moment that

$$\mathbb{E}[|q_k - q_{k+1}|] \stackrel{?}{\leq} \frac{1}{n+1}$$

Then we get

$$\mathbb{E}[(Q_{k+1}^{n+1} - Q_k^{n+1})(Q_{l+1}^{n+1} - Q_l^{n+1})] \leq \frac{16}{(n+1)^3(n-k)(n-l)}$$

Now combining the latter with (1.3) yields

$$\mathbb{E}[(Q_i^{n+1} - 1)^2] \leq \sum_{k=1}^{i-1} \sum_{l=1}^{i-1} \frac{16}{(n+1)^3(n-k)(n-l)}$$

$$\begin{aligned}
 &= \frac{16}{(n+1)^3} \sum_{k=1}^{i-1} \frac{1}{(n-k)} \sum_{l=1}^{i-1} \frac{1}{(n-l)} \\
 &= \frac{16}{(n+1)^3} \sum_{k=n-i+1}^{n-1} \frac{1}{k} \sum_{l=n-i+1}^{n-1} \frac{1}{l}
 \end{aligned}$$

TODO Note that (I believe) I can show

$$\sum_{k=1}^{n-1} \frac{1}{k} \leq \ln(n-1) + 1$$

and

$$\frac{\ln(n-1) + 1}{(n+1)^{\frac{1}{3}}} \leq 2$$

(Everything of the form $(n+1)^{\frac{1}{2}-\epsilon}$ with $\epsilon > 0$ will work for the denominator.)

Therefore

$$\begin{aligned}
 \mathbb{E}[(Q_i^{n+1} - 1)^2]^{\frac{1}{2}} &\leq \frac{\sqrt{16}(\ln(n-1) + 1)}{(n+1)^{\frac{3}{2}}} \\
 &= \frac{\sqrt{16}}{(n+1)^{\frac{7}{6}}} \cdot \frac{(\ln(n-1) + 1)}{(n+1)^{\frac{1}{3}}} \\
 &\leq \frac{\sqrt{64}}{(n+1)^{\frac{7}{6}}}
 \end{aligned}$$

Now combining the above with (1.2) yields

$$\begin{aligned}
 \mathbb{E}[Y_N^N] &\leq \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum_{1 \leq i < j \leq N-k+1} \mathbb{E}[\phi^2(Z_{(i)}, Z_{(j)}) W_{(i)}^2 W_{(j)}^2]^{\frac{1}{2}} \mathbb{E}[(Q_{i,j}^{N-k+1} - 1)^2]^{\frac{1}{2}} \\
 &\leq \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \mathbb{E} \left[\sum_{1 \leq i < j \leq N-k+1} \phi^2(Z_{(i)}, Z_{(j)}) W_{(i)}^2 W_{(j)}^2 \right]^{\frac{1}{2}} \times \frac{\sqrt{64}}{(N-k+1)^{\frac{7}{6}}}
 \end{aligned}$$

Thus it remains to show that

$$\mathbb{E} \left[\sum_{1 \leq i < j \leq N-k+1} \phi^2(Z_{(i)}, Z_{(j)}) W_{(i)}^2 W_{(j)}^2 \right]^{\frac{1}{2}} \leq \text{const.}$$

Then we would have

$$\begin{aligned}
 \lim_{N \rightarrow \infty} \mathbb{E}[U_N[a, b]] &\leq \lim_{N \rightarrow \infty} \mathbb{E}[Y_N^N] \\
 &\leq \lim_{N \rightarrow \infty} \mathbb{E}[\tilde{S}_N^N] + c \sum_{k=1}^{N-1} \frac{\sqrt{64}}{(N-k+1)^{\frac{7}{6}}} \\
 &\leq \sup_N \mathbb{E}[\tilde{S}_N^N] + \lim_{N \rightarrow \infty} c \sum_{k=1}^{N-1} \frac{\sqrt{64}}{(N-k+1)^{\frac{7}{6}}} \\
 &< \infty
 \end{aligned}$$

And therefore we may finally conclude that $S = \lim_{n \rightarrow \infty} S_n$ exists.

TODO :

1. Show that

$$\mathbb{E}[|q_k - q_{k+1}|] \stackrel{?}{\leq} \frac{1}{n+1}$$

using Taylor expansion.

2. Show

$$\mathbb{E} \left[\sum_{1 \leq i < j \leq N-k+1} \phi^2(Z_{(i)}, Z_{(j)}) W_{(i)}^2 W_{(j)}^2 \right]^{\frac{1}{2}} \leq \text{const.}$$

using comparison with \tilde{S}_n^N for some $n \leq N$.

3. Show (or find reference for) the following

$$\sum_{k=1}^{n-1} \frac{1}{k} \leq \ln(n-1) + 1$$

and

$$\frac{\ln(n-1) + 1}{(n+1)^{\frac{1}{3}}} \leq 2$$