Large sample properties of U-Statistics under semiparametric Random Censorship

by

Jan Hoft

A Dissertation Submitted in

Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

in

Mathematics

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Graduate School Approval Date

Abstract

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Jan Hoft

The University of Wisconsin–Milwaukee, 2018
Under the Supervision of Professor Gerhard Dikta and Professor Jugal Ghorai

About this document

This is a **draft version** of my thesis. So far I was able to establish the SLLN for the the semiparametric U-Statistics of degree 2. The mathematics in here should be correct. But I still want to expand the introduction and I need to write the conclusion and the abstract. Also I want to include a simulation about the semiparametric U-Statistics of degree 2. Moreover I am not quite sure about the section titles yet, so I might change those.

How to read this work

- I put

  Comment comment

  statements to mark problematic spots in this thesis and to share my thoughts about those.

- All **TODO** todo statements are to mark parts of the text for myself, which I need to change later.

TODO Summarize your paper here, including the basic methods used in the study. A signature line for your advisor will be included at the end of the abstract. NOTE:

The abstract can have multiple pages, but is restricted to 400 words in length!

Major Professor Date

Contents

List of notes

Chapter 1

Introduction

Assume that $X_1, ..., X_n$ are independent and identically distributed (i. i. d.) random variables (r. v.) on $\mathbb{R}$ which are defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Denote their common probability distribution function (d. f.) by F. For some $k \geq 1$ let $\phi : \mathbb{R}^k \longrightarrow \mathbb{R}$ be a symmetric Borel-measurable function. Define

$$\theta_F = \int ... \int \phi \prod_{j=1}^{k} dF. \tag{1.1}$$

Examples of this kind of parameters include the expected value, variance and any higher moments of the X's. One approach to estimate those integrals is given by the so called U-Statistics. To obtain this estimator we need to replace the true d. f. F by the empirical d. f. F_n which is defined by

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{X_i \leq t\}}.$$

Now plugging F_n into (1.1) yields

$$\int ... \int \phi \prod_{j=1}^{k} dF_n = \frac{1}{n^k} \sum_{i_1=1}^{n} ... \sum_{i_k=1}^{n} \phi(X_{i_1}, ..., X_{i_k})$$

The expression on the right hand side in the equation above is known as V-statistic. It includes repeated observations. An unbiased estimate of θ_F, based on distinct observations only, can be expressed as

$$U_{kn}(\phi) = \binom{n}{k}^{-1} \sum_{[n,k]} \phi(X_{i_1}, ..., X_{i_k}) \ , \tag{1.2}$$

where the sum iterates over all sets $\{i_1, ..., i_k\}$ s. t. $1 \leq i_1 < i_2 < ... < i_n \leq n$. We call (1.2) U-Statistics of order k. In Lee (1990) it was shown, that the U-Statistics is the unbiased minimum variance estimator for (1.1). Observe that for $k = 2$, equation (1.2) simplifies to

$$U_{2n}(\phi) = \frac{2}{n(n-1)} \sum\sum_{1 \leq i < j \leq n} \phi(X_i, X_j)$$

and we have

$$\mathbb{E}(U_{2n}(\phi)) = \int\int \phi dF dF$$

One of the major problems in lifetime analysis is to handle incomplete observations. The incompleteness is often caused by censoring. In this thesis we are concerned with right censored data. A framework to model this kind of data is provided by the Random Censorship Model (RCM). Here we observe data of the form $(Z_i, \delta_i), i = 1, ..., n$ where the Z_i are the observed sample values, which might include censoring and the δ_i indicate whether the corresponding Z_i was censored or not. Here the sequence $(Z_i, \delta_i), i = 1, ..., n$ is assumed to be independent and identically distributed (i. i. d.). Furthermore we can write for $i = 1, ..., n$

$$Z_i = min(X_i, Y_i) \text{ and } \delta_i = I_{X_i \leq Y_i}$$

where X_i is the true lifetime and Y_i is the so called censoring time. The sequences X_i and Y_i are also i. i. d.and they are assumed to be independent of each other. Throughout this work the probability distribution functions (d. f.) of X, Y and Z will be notated F, G and H respectively. We assume that those d. f.'s are continuous and concentrated on $R \cap [0, \infty]$.

Within this framework we want to study the large sample properties of estimators of $\int\int \phi dFdF$. In particular we will be concerned with the asymptotic properties of the U-statistic defined above based on our observations (Z_i, δ_i). To do so, we need new estimates for our d. f. F which are based on our observations (Z_i, δ_i) rather than the X's. If there can not be any further assumptions made about the censorship, except for the RCM itself, then the commonly used estimator of F is the well known product limit estimator by Kaplan and Meier (1958). It is defined by

$$1 - F_n^{km}(t) = \prod_{i:Z_i \le t} \left(\frac{n - R_{i,n}}{n - R_{i,n} + 1} \right)^{\delta_i}$$

where $R_{i,n}$ denotes the rank of Z_i. If we now consider ordered observations, we get

$$1 - F_n^{km}(t) = \prod_{i=1}^{n} \left(1 - \frac{\delta_{[i:n]}}{n - i + 1} \right)^{\mathbb{1}_{\{Z_{i:n} \le t\}}}$$

where $Z_{1:n} \le ... \le Z_{n:n}$ and $\delta_{[i:n]}$ denotes the concomitant of the i-th order statistics. That means $\delta_{[i:n]} = \delta_j$ whenever $Z_{i:n} = Z_j$.

Let's go back to our integral (1.1) and consider the case $k = 1$. The integral then becomes

$$\int \phi dF \tag{1.3}$$

Replacing the true F in (1.3) by F_n^{km} yields

$$S_{1,n}^{km}(\phi) := \int_0^\infty \phi dF_n^{km} = \sum_{i=1}^{n} \phi(Z_{i:n}) W_{i:n}^{km}$$

where $W_{i:n}^{km}$ denotes the weight placed on $Z_{i:n}$ by F_n^{km}. That is

$$\begin{aligned} W_{i:n}^{km} &= F_n^{km}(Z_{i:n}) - F_n^{km}(Z_{i-1:n}) \\ &= \tfrac{\delta_{[i:n]}}{n-i+1} \prod_{j=1}^{i-1} \left(\tfrac{n-j}{n-j+1} \right)^{\delta_{[j:n]}} \end{aligned}$$

It is easy to see, that the Kaplan-Meier estimator only puts mass at uncensored

Z-values, i. e.

$$W_{i:n}^{km} = \begin{cases} 0 & \text{if } \delta_{[i:n]} = 0 \\ \frac{1}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{\delta_{[k:n]}}{n-k+1}\right] > 0 & \text{if } \delta_{[i:n]} = 1 \end{cases} .$$

The strong law of large numbers (SLLN) for $S_{1,n}^{km}(\phi)$ has been established by Stute and Wang (1993). Let's now consider the case $k = 2$. Define

$$S_{2,n}^{km}(\phi) = \sum\sum_{1 \le i < j \le n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n}^{km} W_{j:n}^{km}.$$

The above estimator will be called Kaplan-Meier U-Statistics of degree 2. Moreover we can define the normalized version of $S_{2,n}^{km}(\phi)$ as

$$U_{2,n}^{km}(\phi) = \frac{S_{2,n}^{km}(\phi)}{S_{2,n}^{km}(1)} = \frac{\sum\sum_{1 \le i < j \le n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n}^{km} W_{j:n}^{km}}{\sum\sum_{1 \le i < j \le n} W_{i:n}^{km} W_{j:n}^{km}} .$$

The normalizing factor $(S_{2,n}^{km}(1))^{-1}$ was introduced by Bose and Sen (1999), since the following holds true for uncensored data:

$$\frac{W_{i:n}^{km} W_{j:n}^{km}}{\sum\sum_{1 \le u < v \le n} W_{u:n}^{km} W_{v:n}^{km}} = \binom{n}{2}^{-1}.$$

The strong law of large numbers for $U_{2,n}^{km}$ has been established in Bose and Sen (1999). Asymptotic distributions of this estimator have been derived in Bose and Sen (2002).

For the semiparametric Random Censorship Model (SRCM) we make, besides the assumptions of the RCM, the further assumption that

$$m(z) = \mathbb{P}(\delta = 1 | Z = z) = \mathbb{E}(\delta | Z = z)$$

belongs to some parametric family, i. e.

$$m(z) = m(z, \theta_0)$$

where $\theta_0 = (\theta_{0,1}, ..., \theta_{0,p}) \in \Theta \subset \mathbb{R}^p$.

TODO Discuss Dikta (1998), Dikta, 2014, Asymptotic Optimality

Now the semiparametric estimator is defined by

$$1 - F_n^{se}(t) = \prod_{i:Z_i \leq t} \left(1 - \frac{m(Z_i, \hat{\theta}_n)}{n - R_i + 1}\right)$$

as it was introduced Dikta (2000). Here $\hat{\theta}_n$ denotes the Maximum Likelihood Estimate (MLE) of θ_0. That is, $\hat{\theta}_n$ is the maximizer of

$$L_n(\theta) = \prod_{i=1}^{n} m(Z_i, \theta)^{\delta_i} (1 - m(Z_i, \theta))^{1-\delta_i}.$$

Now again by replacing the true d. f. F by F_n^{se} in the integral (1.3) we obtain the semiparametric version of $S_{1,n}^{km}$, namely

$$S_{1,n}^{se}(\phi) = \int_0^\infty \phi dF_n^{se} = \sum_{i=1}^{n} \phi(Z_{i:n}) W_{i:n}^{se}$$

where

$$W_{i:n}^{se} = \frac{m(Z_{i:n}, \hat{\theta}_n)}{n - i + 1} \prod_{j=1}^{i-1} \left(1 - \frac{m(Z_{j:n}, \hat{\theta}_n)}{n - j + 1}\right)$$

is the mass that F_n^{se} assigns to $Z_{i:n}$. $W_{i:n}^{se}$ will be called i-th semiparametric weight throughout this document. The SLLN and the CLT for the semiparametric U-Statistic $S_{1,n}^{se}$ have been established in Dikta (2000) and Dikta et al. (2005) respectively.

The goal of this thesis will be to study the strong law of large numbers for the semiparametric U-Statistic of degree 2, which will be defined in the next chapter. The main statement of this thesis is contained in Theorem 4.16 at the end of Chapter 4.

TODO Examples for different Kernels ϕ.

Chapter 2

Notation and assumptions

In this chapter we will state the main definitions and assumptions used throughout this work. We will start by defining the estimator to be considered and introduce all necessary notation for the remaining chapters.

Recall the following definition

$$W_{i:n}^{se} = \frac{m(Z_{i:n}, \hat{\theta}_n)}{n-i+1} \prod_{j=1}^{i-1} \left(1 - \frac{m(Z_{j:n}, \hat{\theta}_n)}{n-j+1} \right)$$

Now we define for $n \geq 2$

$$S_{2,n}^{se} = \sum\sum_{1 \leq i < j \leq n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n}^{se} W_{j:n}^{se}$$

This process will be called semiparametric U-Statistic of degree 2 throughout this thesis. Furthermore define

$$W_{i:n}(q) = \frac{q(Z_{i:n})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1} \right]$$

and

$$S_n(q) = \sum\sum_{1 \leq i < j \leq n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n}(q) W_{j:n}(q)$$

for some measurable function q s. t. $q(t) \in [0,1]$ for all $t \in \mathbb{R}_+$.

Example 2.1. Let $q(Z_{i:n}) = \delta_{[i:n]}$ for $1 \leq i \leq n$. Then $W_{i:n}(q) = W_{i:n}^{km}$ and therefore

$$S_n(q) = S_{2,n}^{km}$$

Example 2.2. Let $q(t) = m(t, \hat{\theta}_n)$ for $t \in \mathbb{R}^+$. Then $W_{i:n}(q) = W_{i:n}^{se}$ and therefore

$$S_n(q) = S_{2,n}^{se}$$

Moreover define

$$\mathcal{F}_n = \sigma\{Z_{1:n}, \ldots, Z_{n:n}, Z_{n+1}, Z_{n+2}, \ldots\}$$

Throughout this work we will write $S_n := S_n(q)$ and $W_{i:n} := W_{i:n}(q)$. for $1 \leq i \leq n$.

The following assumptions will be needed throughout this work, in order to prove the SLLN for S_n.

(A1) The kernel $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is measurable, non-negative and symmetric in its arguments. In effect $\phi(s,t) = \phi(t,s)$ for all $s, t \in \mathbb{R}_+$.

(A2) H is continuous and concentrated on the non-negative real line.

(A3) For $s, t \in \mathbb{R}_+$ the following statement holds true

$$\int_0^s \int_0^t \frac{\phi(s,t)}{m(s,\theta_0)m(t,\theta_0)(1-H(s))^{\epsilon}(1-H(t))^{\epsilon}} F(dt)F(ds) < \infty$$

for some $0 < \epsilon \leq 1$.

(A4) $m(z, \theta_0)$ is increasing in z.

We will need the following assumptions about the Censoring Model m and the Maximum Likelihood estimate $\hat{\theta}_n$:

(M1) $\hat{\theta}_n$ is measurable and tends to θ_0

(M2) For any $\epsilon > 0$ there exists a neighborhood $V(\epsilon, \theta_0) \subset \Theta$ of θ_0 s. t. for all

$\theta \in V(\epsilon, \theta_0)$

$$\sup_{x \geq 0} |m(x,\theta) - m(x,\theta_0)| < \epsilon$$

Chapter 3

Existence of the almost sure limit

Within this chapter we will establish basic properties of $\mathbb{E}[S_n|\mathcal{F}_n]$. In Section 3.1 a representation is derived for $\mathbb{E}[S_n|\mathcal{F}_{n+1}]$, which is similar to the result established in Bose and Sen (1999), Lemma 1. Later on in this section we will derive properties of the process above based on this representation. In Stute and Wang (1993) the proof of existence of the limit of the considered estimator was based on a reverse supermartingale argument. Later in Dikta (1998) and Bose and Sen (1999) used the same type of argument for their the estimators they considered. We will not be able to establish the reverse supermartingale property for $S_{2,n}^{se}$ in general. But we will be able to state a condition on q, s. t. $S_n(q)$ is indeed a supermartingale. This will be discussed in more detail within Section 3.2. In Section **??** we will show how this implies the almost sure existence by the same argument as in Stute and Wang (1993).

3.1 Basic results about $\mathbb{E}[S_n|\mathcal{F}_{n+1}]$

We will first derive an explicit representation for $\mathbb{E}[S_n|\mathcal{F}_{n+1}]$, which is similar to the one established in the proof of Bose and Sen (1999), Lemma 1.

Lemma 3.1. *Define for* $1 \leq i < j \leq n$

$$Q_{ij}^{n+1} = \begin{cases} Q_i^{n+1} & j \leq n \\ Q_i^{n+1} - \frac{(n+1)\pi_i \pi_n (1-q(Z_{n:n+1}))}{(n-i+1)(2-q(Z_{n:n+1}))} & j = n+1 \end{cases}$$

where

$$Q_i^{n+1} = (n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2 + \frac{\pi_i\pi_{i+1}}{n-i+1}\right\} \tag{3.1}$$

and

$$\pi_i = \prod_{k=1}^{i-1}\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})} \ .$$

Then we have

$$\mathbb{E}[S_n|\mathcal{F}_{n+1}] = \sum\sum_{1\le i<j\le n+1} \phi(Z_{i:n+1}, Z_{j:n+1})W_{i:n+1}W_{j:n+1}Q_{ij}^{n+1} \ .$$

Proof. We will need the following result for the proof of lemma 3.1. Let

$$A_i = \pi_i + \sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]$$

for $1 \le i \le n$ with π_i as defined above. Note that $\pi_1 = 1$, since the product is empty and hence taken as 1. Therefore we have $A_1 = \pi_1 = 1$. Now for any $1 \le i \le n-1$

$$\begin{aligned}
A_{i+1} &= \pi_{i+1} + \sum_{r=1}^{i}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right] \\
&= \pi_i\left[\frac{n-i+1-q(Z_{i:n+1})}{n-i+2-q(Z_{i:n+1})}\right] + \sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right] + \left[\frac{\pi_i}{n-i+2-q(Z_{i:n+1})}\right] \\
&= \pi_i + \sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right] \\
&= A_i \ .
\end{aligned}$$

And therefore

$$1 = A_1 = A_2 = \cdots = A_{n-1} = A_n \ . \tag{3.2}$$

Now let's establish lemma 3.1. Let F_n^q denote the measure that assigns mass to

$Z_{1:n}, \dots, Z_{n:n}$, then

$$
\begin{aligned}
\mathbb{E}[S_n|\mathcal{F}_{n+1}] &= \mathbb{E}[\sum\sum_{1\le i<j\le n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n} W_{j:n} |\mathcal{F}_{n+1}] \\
&= \mathbb{E}[\sum\sum_{1\le i<j\le n+1} \phi(Z_{i:n+1}, Z_{j:n+1}) F_n^q\{Z_{i:n+1}\} F_n^q\{Z_{j:n+1}\} |\mathcal{F}_{n+1}] \\
&= \sum\sum_{1\le i<j\le n+1} \phi(Z_{i:n+1}, Z_{j:n+1}) \mathbb{E}[F_n^q\{Z_{i:n+1}\} F_n^q\{Z_{j:n+1}\} |\mathcal{F}_{n+1}] \ .
\end{aligned}
$$

Consider for $1 \le i < j \le n$

$$
\begin{aligned}
&\mathbb{E}[F_n^q\{Z_{i:n+1}\} F_n^q\{Z_{j:n+1}\} |\mathcal{F}_{n+1}] \\
&= \mathbb{E}\left[\sum_{r=1}^{n+1} F_n^q\{Z_{i:n+1}\} F_n^q\{Z_{j:n+1}\} I_{\{Z_{n+1}=Z_{r:n+1}\}} |\mathcal{F}_{n+1}\right] \ .
\end{aligned}
$$

Define the set $A_{rn} := \{Z_{n+1} = Z_{r:n+1}\}$. Note that on A_{rn} we have for $1 \le l \le n+1$

$$
Z_{l:n+1} = \begin{cases} Z_{l:n} & l < r \\ Z_{l-1:n} & l > r \end{cases} \tag{3.3}
$$

and therefore

$$
F_n^q\{Z_{l:n+1}\} = \begin{cases} W_{l:n} & l < r \\ 0 & l = r \\ W_{l-1:n} & l > r \end{cases} \ . \tag{3.4}
$$

Now we have

$$
\begin{aligned}
&\sum_{r=1}^{n+1} F_n^q\{Z_{i:n+1}\} F_n^q\{Z_{j:n+1}\} I_{\{Z_{n+1}=Z_{r:n+1}\}} \\
&= \sum_{r=1}^{n+1} F_n^q\{Z_{i:n+1}\} F_n^q\{Z_{j:n+1}\} I_{A_{rn}} \\
&= \sum_{r=1}^{i-1} W_{i-1:n} W_{j-1:n} I_{A_{rn}} + \sum_{r=i+1}^{j-1} W_{i:n} W_{j-1:n} I_{A_{rn}} + \sum_{r=j+1}^{n+1} W_{i:n} W_{j:n} I_{A_{rn}}
\end{aligned}
$$

$$=: T_1 + T_2 + T_3 \ . \tag{3.5}$$

Let's now consider each of the sums T_1, T_2, and T_3 in the above equation individually. First consider T_1. We have

$$\begin{aligned}
T_1 &= \sum_{r=1}^{i-1} \frac{q(Z_{i-1:n})}{n-i+2} \prod_{k=1}^{i-2} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right] \\
&\quad \times \frac{q(Z_{j-1:n})}{n-j+2} \prod_{k=1}^{j-2} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right] I_{A_{rn}} \\
&= \sum_{r=1}^{i-1} \frac{q(Z_{i:n+1})}{n-i+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1}\right] \prod_{k=r}^{i-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] \\
&\quad \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1}\right] \prod_{k=r}^{j-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] I_{A_{rn}}
\end{aligned}$$

using (3.3). We will now continue to find an expression for T_1 in terms of $W_{i:n+1}$ and $W_{j:n+1}$. We have

$$\begin{aligned}
T_1 &= \sum_{r=1}^{i-1} \frac{q(Z_{i:n+1})}{n-i+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1}\right] \prod_{k=r}^{i-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] \\
&\quad \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1}\right] \prod_{k=r}^{j-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] I_{A_{rn}} \\
&= \sum_{r=1}^{i-1} \frac{q(Z_{i:n+1})}{n-i+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right] \prod_{k=r}^{i-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] \\
&\quad \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right] \prod_{k=r}^{j-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] I_{A_{rn}} \\
&\quad \times \left[\frac{\prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1}\right]}{\prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right]}\right]^2 \\
&= \sum_{r=1}^{i-1} \frac{q(Z_{i:n+1})}{n-i+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right] \prod_{k=r}^{i-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] \\
&\quad \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right] \prod_{k=r}^{j-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] I_{A_{rn}}
\end{aligned}$$

$$\times \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]^2 \prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right]^2 \ .$$

Now using index transformation on the products $\prod_{k=r}^{i-2}[\ldots]$ and $\prod_{k=r}^{j-2}[\ldots]$ yields

$$\begin{aligned}
T_1 &= \sum_{r=1}^{i-1} \frac{q(Z_{i:n+1})}{n-i+2} \prod_{k=1}^{r-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] \prod_{k=r+1}^{i-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] \\
&\quad \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] \prod_{k=r+1}^{j-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] I_{A_{rn}} \\
&\quad \times \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]^2 \prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right]^2 \\
&= \sum_{r=1}^{i-1} \frac{q(Z_{i:n+1})}{n-i+2} \prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] \left[1-\frac{q(Z_{r:n+1})}{n-r+2}\right]^{-1} \\
&\quad \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{j-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] \left[1-\frac{q(Z_{r:n+1})}{n-r+2}\right]^{-1} I_{A_{rn}} \\
&\quad \times \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]^2 \prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right]^2 \\
&= W_{i:n+1}W_{j:n+1} \sum_{r=1}^{i-1} \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]^2 \prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right]^2 \\
&\quad \times \left[\frac{n-r+2}{n-r+2-q(Z_{r:n+1})}\right]^2 I_{A_{rn}} \ .
\end{aligned}$$

Note that

$$\begin{aligned}
\prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right] &= \frac{n+1}{n}\cdot\frac{n}{n-1}\cdots\frac{n-r+4}{n-r+3}\cdot\frac{n-r+3}{n-r+2} \\
&= \frac{n+1}{n-r+2} \ . \qquad (3.6)
\end{aligned}$$

and recall the following definition

$$\pi_r = \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right] \ .$$

Now we finally get

$$
\begin{aligned}
T_1 &= W_{i:n+1} W_{j:n+1} \sum_{r=1}^{i-1} \prod_{k=1}^{r-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})} \right]^2 \\
&\quad \times \left[\frac{n+1}{n-r+2} \right]^2 \left[\frac{n-r+2}{n-r+2-q(Z_{r:n+1})} \right]^2 I_{A_{rn}} \\
&= W_{i:n+1} W_{j:n+1} \sum_{r=1}^{i-1} \pi_r^2 \left[\frac{n+1}{n-r+2-q(Z_{r:n+1})} \right]^2 I_{A_{rn}} \ .
\end{aligned}
$$

Now let's consider T_2. We will, again, firstly express T_2 completely in terms of the ordered Z values w. r. t. order $n+1$ using (3.3). Consider

$$
\begin{aligned}
T_2 &= \sum_{r=i+1}^{j-1} \frac{q(Z_{i:n})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1} \right] \\
&\quad \times \frac{q(Z_{j-1:n})}{n-j+2} \prod_{k=1}^{j-2} \left[1 - \frac{q(Z_{k:n})}{n-k+1} \right] I_{A_{rn}} \\
&= \sum_{r=i+1}^{j-1} \frac{q(Z_{i:n+1})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1} \right] \\
&\quad \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1} \right] \prod_{k=r}^{j-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1} \right] I_{A_{rn}} \ .
\end{aligned}
$$

Now let's find a representation of T_2 which relies on $W_{i:n+1}$ and $W_{j:n+1}$ only. Consider

$$
\begin{aligned}
T_2 &= \sum_{r=i+1}^{j-1} \left[\frac{n-i+2}{n-i+1} \right] \left[\frac{q(Z_{i:n+1})}{n-i+2} \right] \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2} \right] \\
&\quad \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2} \right] \prod_{k=r}^{j-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1} \right] I_{A_{rn}} \\
&\quad \times \prod_{k=1}^{i-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})} \right] \prod_{k=1}^{i-1} \left[\frac{n-k+2}{n-k+1} \right] \\
&\quad \times \prod_{k=1}^{r-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})} \right] \prod_{k=1}^{r-1} \left[\frac{n-k+2}{n-k+1} \right] \\
&= \left[\frac{n-i+2}{n-i+1} \right] \left[\frac{q(Z_{i:n+1})}{n-i+2} \right] \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2} \right]
\end{aligned}
$$

$$
\begin{aligned}
&\times \prod_{k=1}^{i-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{i-1}\left[\frac{n-k+2}{n-k+1}\right]\\
&\times \sum_{r=i+1}^{j-1}\frac{q(Z_{j:n+1})}{n-j+2}\prod_{k=1}^{r-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]\prod_{k=r}^{j-2}\left[1-\frac{q(Z_{k+1:n+1})}{n-k+1}\right]I_{A_{rn}}\\
&\times \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right]\ .
\end{aligned}
$$

Now using (3.6) on $\prod_{k=1}^{i-1}[\ldots]$ yields

$$
\begin{aligned}
T_2 &= \left[\frac{n+1}{n-i+1}\right]\left[\frac{q(Z_{i:n+1})}{n-i+2}\right]\prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]\\
&\quad\times \prod_{k=1}^{i-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\\
&\quad\times \sum_{r=i+1}^{j-1}\frac{q(Z_{j:n+1})}{n-j+2}\prod_{k=1}^{r-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]\prod_{k=r}^{j-2}\left[1-\frac{q(Z_{k+1:n+1})}{n-k+1}\right]I_{A_{rn}}\\
&\quad\times \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right]\\
&= \left[\frac{n+1}{n-i+1}\right]W_{i:n+1}\pi_i\\
&\quad\times \sum_{r=i+1}^{j-1}\frac{q(Z_{j:n+1})}{n-j+2}\prod_{k=1}^{r-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]\prod_{k=r}^{j-2}\left[1-\frac{q(Z_{k+1:n+1})}{n-k+1}\right]I_{A_{rn}}\\
&\quad\times \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right]\ .
\end{aligned}
$$

Again doing an index transformation on $\prod_{k=r}^{j-2}[\ldots]$ yields

$$
\begin{aligned}
&= \left[\frac{n+1}{n-i+1}\right]W_{i:n+1}\pi_i\\
&\quad\times \sum_{r=i+1}^{j-1}\frac{q(Z_{j:n+1})}{n-j+2}\prod_{k=1}^{r-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]\prod_{k=r+1}^{j-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]I_{A_{rn}}\\
&\quad\times \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right]I_{A_{rn}}\\
&= W_{i:n+1}\pi_i\frac{n+1}{n-i+1}\sum_{r=i+1}^{j-1}\frac{q(Z_{j:n+1})}{n-j+2}\prod_{k=1}^{j-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]\left[1-\frac{q(Z_{r:n+1})}{n-r+2}\right]^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&\times \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right] I_{A_{rn}} \\
= {} & W_{i:n+1}W_{j:n+1}\pi_i \frac{n+1}{n-i+1} \\
&\times \sum_{r=i+1}^{j-1}\prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right] \\
&\times \frac{n-r+2}{n-r+2-q(Z_{r:n+1})} I_{A_{rn}} \ .
\end{aligned}
$$

Now applying (3.6) to the latter product yields

$$
T_2 = W_{i:n+1}W_{j:n+1}\pi_i \frac{n+1}{n-i+1}\sum_{r=i+1}^{j-1}\pi_r \frac{n+1}{n-r+2-q(Z_{r:n+1})} I_{A_{rn}} \ .
$$

We will now proceed similarly for T_3. Consider

$$
T_3 = \sum_{r=j+1}^{n+1} W_{i:n}W_{j:n}\mathbb{1}_{\{A_{rn}\}} \ .
$$

Note that for $j = n+1$ the sum above is empty and hence zero. Now consider for $j \leq n$

$$
\begin{aligned}
T_3 = {} & \sum_{r=j+1}^{n+1} \frac{q(Z_{i:n})}{n-i+1}\prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n})}{n-k+1}\right] \\
&\times \frac{q(Z_{j:n})}{n-j+1}\prod_{k=1}^{j-1}\left[1-\frac{q(Z_{k:n})}{n-k+1}\right]\mathbb{1}_{\{A_{rn}\}} \\
= {} & \sum_{r=j+1}^{n+1} \frac{q(Z_{i:n+1})}{n-i+1}\prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+1}\right] \\
&\times \frac{q(Z_{j:n+1})}{n-j+1}\prod_{k=1}^{j-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+1}\right]\mathbb{1}_{\{A_{rn}\}} \\
= {} & \sum_{r=j+1}^{n+1} \frac{n-i+2}{n-i+1}\frac{q(Z_{i:n+1})}{n-i+2}\prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] \\
&\times \frac{n-j+2}{n-j+1}\frac{q(Z_{j:n+1})}{n-j+2}\prod_{k=1}^{j-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\times \prod_{k=1}^{i-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})} \right] \prod_{k=1}^{i-1} \left[\frac{n-k+2}{n-k+1} \right] \\
&\times \prod_{k=1}^{j-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})} \right] \prod_{k=1}^{j-1} \left[\frac{n-k+2}{n-k+1} \right] \mathbb{1}_{\{A_{rn}\}} \\
= &\sum_{r=j+1}^{n+1} \frac{n-i+2}{n-i+1} \frac{n-j+2}{n-j+1} \pi_i \pi_j W_{i:n+1} W_{j:n+1} \\
&\times \prod_{k=1}^{i-1} \left[\frac{n-k+2}{n-k+1} \right] \prod_{k=1}^{j-1} \left[\frac{n-k+2}{n-k+1} \right] \mathbb{1}_{\{A_{rn}\}} .
\end{aligned}
$$

Again, by (3.6), we have

$$
T_3 = \sum_{r=j+1}^{n+1} \frac{(n+1)^2 \pi_i \pi_j}{(n-i+1)(n-j+1)} W_{i:n+1} W_{j:n+1} \mathbb{1}_{\{A_{rn}\}} .
$$

Therefore

$$
T_3 = \begin{cases} W_{i:n+1} W_{j:n+1} \pi_i \pi_j \left[\frac{(n+1)^2}{(n-i+1)(n-j+1)} \right] \sum_{r=j+1}^{n+1} \mathbb{1}_{\{A_{rn}\}} & j \leq n \\ 0 & j = n+1 \end{cases}
$$

for $1 \leq i < j \leq n$. Now using these expressions for T_1, T_2 and T_3 in equation (3.5) together with the fact that

$$
\mathbb{E}[I_{A_{rn}} | \mathcal{F}_{n+1}] = \frac{1}{n+1}
$$

yields

$$
\begin{aligned}
&\mathbb{E}[F_n^q\{Z_{i:n+1}\} F_n^q\{Z_{j:n+1}\} | \mathcal{F}_{n+1}] \\
&= \mathbb{E}[T_1 + T_2 + T_3 | \mathcal{F}_{n+1}] \\
&= W_{i:n+1} W_{j:n+1} \times \left\{ \sum_{r=1}^{i-1} \pi_r^2 \left[\frac{n+1}{n-r+2-q(Z_{r:n+1})} \right]^2 \mathbb{E}[I_{A_{rn}} | \mathcal{F}_{n+1}] \right.
\end{aligned}
$$

$$+\sum_{r=i+1}^{j-1}\pi_i\pi_r\left[\frac{n+1}{n-i+1}\right]\left[\frac{n+1}{n-r+2-q(Z_{r:n+1})}\right]\mathbb{E}[I_{A_{rn}}|\mathcal{F}_{n+1}]$$
$$\left.+\pi_i\pi_j\frac{(n+1)^2}{(n-i+1)(n-j+1)}[1-I_{\{j=n+1\}}]\sum_{i=j+1}^{n+1}\mathbb{E}[I_{A_{rn}}|\mathcal{F}_{n+1}]\right\}$$
$$=W_{i:n+1}W_{j:n+1}\left[\frac{1}{n+1}\right]\times\left\{\sum_{r=1}^{i-1}\pi_r^2\left[\frac{n+1}{n-r+2-q(Z_{r:n+1})}\right]^2\right.$$
$$+\sum_{r=i+1}^{j-1}\pi_i\pi_r\left[\frac{n+1}{n-i+1}\right]\left[\frac{n+1}{n-r+2-q(Z_{r:n+1})}\right]$$
$$\left.+\pi_i\pi_j\frac{(n+1)^2}{n-i+1}[1-I_{\{j=n+1\}}]\right\}\ .$$

Next consider that we have

$$\mathbb{E}[F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}|\mathcal{F}_{n+1}]$$
$$=W_{i:n+1}W_{j:n+1}(n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2\right.$$
$$\left.+\frac{\pi_i}{n-i+1}\left[\sum_{r=i+1}^{j-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]+\pi_j\right]\right\}\ .$$

for $1\le i<j\le n$. Now applying (3.2) yields

$$\mathbb{E}[F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}|\mathcal{F}_{n+1}]$$
$$=W_{i:n+1}W_{j:n+1}(n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2\right.$$
$$\left.+\frac{\pi_i}{n-i+1}(A_j-A_{i+1}+\pi_{i+1})\right\}$$
$$=W_{i:n+1}W_{j:n+1}(n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2\right.$$
$$\left.+\frac{\pi_i\pi_{i+1}}{n-i+1}\right\}$$
$$=W_{i:n+1}W_{j:n+1}Q_i^{n+1}\ .$$

It remains to consider the case $j = n+1$. We have

$$
\begin{aligned}
&\mathbb{E}[F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}|\mathcal{F}_{n+1}] \\
&= W_{i:n+1}W_{n+1:n+1}(n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2\right. \\
&\qquad \left. + \frac{\pi_i}{n-i+1}\sum_{r=i+1}^{n}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]\right\} \\
&= W_{i:n+1}W_{n+1:n+1}(n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2\right. \\
&\qquad \left. + \frac{\pi_i}{n-i+1}\left[\sum_{r=1}^{n}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right] - \sum_{r=1}^{i}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]\right]\right\} \\
&= W_{i:n+1}W_{n+1:n+1}(n+1)\left\{\frac{Q_i^{n+1}}{n+1} - \frac{\pi_i\pi_{i+1}}{n-i+1}\right. \\
&\qquad \left. + \frac{\pi_i}{n-i+1}\left[\sum_{r=1}^{n}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right] - \sum_{r=1}^{i}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]\right]\right\} .
\end{aligned}
$$

Now using (3.2) again yields

$$
\begin{aligned}
&\mathbb{E}[F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}|\mathcal{F}_{n+1}] \\
&= W_{i:n+1}W_{n+1:n+1}(n+1)\left\{\frac{Q_i^{n+1}}{n+1} - \frac{\pi_i\pi_{i+1}}{n-i+1}\right. \\
&\qquad \left. +\frac{\pi_i}{n-i+1}\left[A_{n+1} - \pi_{n+1} - (A_{i+1} - \pi_{i+1})\right]\right\} \\
&= W_{i:n+1}W_{n+1:n+1}(n+1)\left\{\frac{Q_i^{n+1}}{n+1} - \frac{\pi_i\pi_{i+1}}{n-i+1}\right. \\
&\qquad \left. +\frac{\pi_i}{n-i+1}\left[\pi_{i+1} - \pi_{n+1}\right]\right\} .
\end{aligned}
$$

Note that for $1 \le i < n$ we have

$$
\pi_{i+1} = \frac{\pi_i(1 - q(Z_{i:n+1}))}{2 - q(Z_{i:n+1})} .
$$

Thus we obtain

$$
\begin{aligned}
&\mathbb{E}[F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}|\mathcal{F}_{n+1}] \\
&= W_{i:n+1}W_{n+1:n+1}(n+1)\left\{\frac{Q_i^{n+1}}{n+1} - \frac{\pi_i\pi_{i+1}}{n-i+1}\right. \\
&\qquad\qquad \left. + \frac{\pi_i}{n-i+1}\left[\pi_{i+1} - \frac{\pi_n(1-q(Z_{n:n+1}))}{2-q(z_{n:n+1})}\right]\right\} \\
&= W_{i:n+1}W_{n+1:n+1}(n+1)\left\{\frac{Q_i^{n+1}}{n+1} - \frac{\pi_i\pi_n(1-q(Z_{n:n+1}))}{(n-i+1)(2-q(Z_{n:n+1}))}\right\} \\
&= W_{i:n+1}W_{n+1:n+1}\left\{Q_i^{n+1} - \frac{\pi_i\pi_n(n+1)(1-q(Z_{n:n+1}))}{(n-i+1)(2-q(Z_{n:n+1}))}\right\} \ .
\end{aligned}
$$

□

The following lemma contains a result on the increases of Q_i^{n+1} w. r. t. i. It is especially useful, since we can express Q_i^{n+1}, since

$$
Q_i^{n+1} = Q_1^{n+1} + \sum_{k=1}^{n} Q_{k+1}^{n+1} - Q_k^{n+1} \ ,
$$

which will be used in Lemma 4.10.

Lemma 3.2. *Let Q_i^{n+1} be defined as in Lemma 3.1 for $1 \le i \le n$. Moreover define*

$$
\tilde{\pi}_i := \prod_{k=1}^{i-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{i-1}\left[\frac{n-k+2}{n-k+1}\right] \ .
$$

Then we have

$$
\begin{aligned}
Q_{i+1}^{n+1} - Q_i^{n+1} &= \frac{(q_i - q_{i+1})(n-i)(n-i+1) - q_{i+1}(1-q_i)(n-i+1-q_i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\
&\quad \times \frac{\tilde{\pi}_i(n-i+2)^2}{n+1} \ .
\end{aligned}
$$

Proof. For the sake of simplicity we will write $q_i \equiv q(Z_{i:n+1})$ during this proof. From equation (3.1) we get

$$\begin{aligned}
\frac{Q_{i+1}^{n+1} - Q_i^{n+1}}{n+1} &= \left\{ \sum_{r=1}^{i} \left[\frac{\pi_r}{n-r+2-q_r} \right]^2 + \frac{\pi_{i+1}\pi_{i+2}}{n-i} \right\} \\
&\quad - \left\{ \sum_{r=1}^{i-1} \left[\frac{\pi_r}{n-r+2-q_r} \right]^2 + \frac{\pi_i \pi_{i+1}}{n-i+1} \right\} \\
&= \frac{\pi_i^2}{(n-i+2-q_i)^2} + \frac{\pi_{i+1}\pi_{i+2}}{n-i} - \frac{\pi_i\pi_{i+1}}{n-i+1} \\
&= \frac{\pi_i^2}{(n-i+2-q_i)^2} + \frac{\pi_i^2 (n-i+1-q_i)^2 (n-i-q_{i+1})}{(n-i)(n-i+2-q_i)^2 (n-i+1-q_{i+1})} \\
&\quad - \frac{\pi_i^2 (n-i+1-q_i)}{(n-i+1)(n-i+2-q_i)} \\
&= \pi_i^2 \left\{ \frac{1}{(n-i+2-q_i)^2} + \frac{(n-i+1-q_i)^2 (n-i-q_{i+1})}{(n-i)(n-i+2-q_i)^2 (n-i+1-q_{i+1})} \right. \\
&\qquad \left. - \frac{n-i+1-q_i}{(n-i+1)(n-i+2-q_i)} \right\} \\
&=: \pi_i^2 \left\{ a(n,i) + b(n,i) - c(n,i) \right\} \ .
\end{aligned} \tag{3.7}$$

Now consider

$$\begin{aligned}
& b(n,i) - c(n,i) \\
&= (n-i+1-q_i) \left[\frac{(n-i+1-q_i)(n-i-q_{i+1})}{(n-i)(n-i+2-q_i)^2 (n-i+1-q_{i+1})} \right. \\
&\qquad \left. - \frac{1}{(n-i+1)(n-i+2-q_i)} \right] \\
&= (n-i+1-q_i) \left[\frac{(n-i+1-q_i)(n-i-q_{i+1})(n-i+1)}{(n-i)(n-i+1)(n-i+2-q_i)^2 (n-i+1-q_{i+1})} \right. \\
&\qquad \left. - \frac{(n-i+2-q_i)(n-i+1-q_{i+1})(n-i)}{(n-i)(n-i+1)(n-i+2-q_i)^2 (n-i+1-q_{i+1})} \right] \ .
\end{aligned} \tag{3.8}$$

Next we will simplify the difference of the numerators above. We have

$$\begin{aligned}
& (n-i+1-q_i)(n-i-q_{i+1})(n-i+1) \\
& - (n-i+2-q_i)(n-i+1-q_{i+1})(n-i) \\
&= (n-i+1-q_i)(n-i)(n-i+1) - q_{i+1}(n-i+1-q_i)(n-i+1)
\end{aligned}$$

$$
\begin{aligned}
&- (n-i+2-q_i)(n-i+1-q_{i+1})(n-i) \\
&= (n-i+1-q_i)(n-i)(n-i+1) - q_{i+1}(n-i+1-q_i)(n-i+1) \\
&- (n-i+1-q_i)(n-i+1-q_{i+1})(n-i) - (n-i+1-q_{i+1})(n-i) \\
&= (n-i+1-q_i)(n-i)(n-i+1) - q_{i+1}(n-i+1-q_i)(n-i+1) \\
&- (n-i+1-q_i)(n-i+1)(n-i) + q_{i+1}(n-i+1-q_i)(n-i) \\
&- (n-i+1-q_{i+1})(n-i) \\
&= -q_{i+1}(n-i+1-q_i) - (n-i+1-q_{i+1})(n-i)\ .
\end{aligned}
$$

Hence we get, according to (3.8)

$$
\begin{aligned}
&b(n,i) - c(n,i) \\
&= -(n-i+1-q_i)\left[\frac{q_{i+1}(n-i+1-q_i) + (n-i+1-q_{i+1})(n-i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})}\right]\ .
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&a(n,i) + b(n,i) - c(n,i) \\
&= \frac{1}{(n-i+2-q_i)^2} \\
&\quad - \frac{q_{i+1}(n-i+1-q_i)^2 + (n-i+1-q_i)(n-i+1-q_{i+1})(n-i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\
&= \frac{(n-i)(n-i+1)(n-i+1-q_{i+1})}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\
&\quad - \frac{q_{i+1}(n-i+1-q_i)^2 + (n-i+1-q_i)(n-i+1-q_{i+1})(n-i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})}\ .
\end{aligned}
$$

Consider again the numerator of the latter expression. We have

$$
\begin{aligned}
&= (n-i)(n-i+1)(n-i+1-q_{i+1}) - q_{i+1}(n-i+1-q_i)^2 \\
&- (n-i)(n-i+1-q_i)(n-i+1-q_{i+1}) \\
&= q_i(n-i)(n-i+1-q_{i+1}) - q_{i+1}(n-i+1-q_i)^2
\end{aligned}
$$

$$
\begin{aligned}
&= q_i(n-i)^2 + q_i(1-q_{i+1})(n-i) - q_{i+1}(n-i)^2 \\
&\quad - 2q_{i+1}(1-q_i)(n-i) - q_{i+1}(1-q_i)^2 \\
&= (q_i - q_{i+1})(n-i)^2 + q_i(n-i) - q_i q_{i+1}(n-i) \\
&\quad - 2q_{i+1}(n-i) + 2q_i q_{i+1}(n-i) - q_{i+1}(1-q_i)^2 \\
&= (q_i - q_{i+1})(n-i)^2 + (q_i + q_i q_{i+1} - 2q_{i+1})(n-i) - q_{i+1}(1-q_i)^2 \ .
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
&a(n,i) + b(n,i) - c(n,i) \\
&= \frac{(q_i - q_{i+1})(n-i)^2 + (q_i + q_i q_{i+1} - 2q_{i+1})(n-i) - q_{i+1}(1-q_i)^2}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\
&= \frac{(q_i - q_{i+1})(n-i)^2 + [(q_i - q_{i+1}) - q_{i+1}(1-q_i))(n-i) - q_{i+1}(1-q_i)^2}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\
&= \frac{(q_i - q_{i+1})(n-i)(n-i+1) - q_{i+1}(1-q_i)(n-i+1-q_i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \ . \qquad (3.9)
\end{aligned}
$$

Finally note that

$$
\begin{aligned}
\tilde{\pi}_i &= \frac{n+1}{n-i+2} \prod_{k=1}^{i-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})} \right] \\
&= \pi_i \cdot \frac{n+1}{n-i+2} \qquad (3.10)
\end{aligned}
$$

with π_i as defined in Lemma 3.1. Now the statement of the lemma follows directly by combining (3.7), (3.9) and (3.10) □

3.2 S_n is not a reverse supermartingale in general

As discussed in Chapter 1, the Strong Law of Large Numbers for Kaplan-Meier U-Statistics of degree 2 was established by Bose and Sen (1999). Recall the definition of the estimator they considered:

$$S_n^{km} = \sum\sum_{1 \le i < j \le n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n}^{km} W_{j:n}^{km}$$

with

$$W_{i:n}^{km} = \frac{\delta_{[i:n]}}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{\delta_{[k:n]}}{n-k+1}\right] .$$

The proof of existence of the limit $S = \lim_{n\to\infty} S_n^{km}$ was here essentially based upon a supermartingale argument together with Neveu (1975), proposition V-3-11. In Lemma 1 of Bose and Sen (1999) a representation for $\mathbb{E}[S_n^{km}|\mathcal{F}_{n+1}]$ was derived, which is similar to our lemma 3.1. It was shown that for $1 \le i < j \le n$

$$\mathbb{E}[S_n^{km}|\mathcal{F}_{n+1}] = \sum\sum_{1 \le i < j \le n+1} \phi(Z_{i:n+1}, Z_{j:n+1}) W_{i:n+1}^{km} W_{j:n+1}^{km} Q_{ij}^{km}$$

where

$$Q_{ij}^{km} = \begin{cases} Q_i^{km} & \text{if } j \le n \\ Q_i^{km} - \pi_i \pi_n (1 - \delta_{[n:n+1]})) \frac{n-i+2}{(n+1)(n-i+1)} & \text{if } j = n+1 \end{cases}$$

and

$$Q_i^{km} = \frac{1}{n+1} \left\{ \sum_{r=1}^{i-1} \pi_r^2 \left[\frac{n-r+2}{n-r+1}\right]^{2\delta_{[r:n+1]}} \right. \\ \left. + \pi_i^2 (n-i+2) \left[\frac{(n-i)(n-i+2)}{(n-i+1)^2}\right]^{\delta_{[i:n+1]}} \right\} .$$

Then Bose and Sen (1999) show that $Q_{ij}^{km} \le 1$ for $1 \le i < j \le n$, in order to establish the reverse time supermartingale property for $(S_n^{km}, \mathcal{F}_n)$. However their prove relies on the fact that

$$W_{i:n}^{km} = \frac{\delta_{[i:n]}}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{\delta_{[k:n]}}{n-k+1}\right]$$

$$= \frac{\delta_{[i:n]}}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{1}{n-k+1}\right]^{\delta_{[k:n]}} .$$

But the corresponding statement is not true for $W_{i:n}$, since we have in general that

$$\begin{aligned} W_{i:n} &= \frac{q(Z_{i:n})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right] \\ &\neq \frac{q(Z_{i:n})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{1}{n-k+1}\right]^{q(Z_{k:n})} . \end{aligned}$$

In Dikta (2000), the following estimator was considered

$$S_n^{se}(q) = \sum_{i=1}^{n} \phi(Z_{i:n}) W_{i:n}^{se} .$$

The proof of existence of the limit $S^{se} = \lim_{n\to\infty} S_n^{se}$ shows a similar structure, as the one in Bose and Sen (1999). In Lemma 2.1 of Dikta (2000), it was shown that $\mathbb{E}[\mu_n\{Z_{1:n+1}\}|\mathcal{F}_{n+1}] = W_{1:n}^{se}$ and for $2 \leq i \leq n$

$$\mathbb{E}[\mu_n\{Z_{i:n+1}\}|\mathcal{F}_{n+1}] = W_{i:n}^{se} Q_i^{se} ,$$

where μ_n is the measure assigning mass $W_{i:n}$ to $Z_{i:n}$ and

$$Q_i^{se} = \pi_i + \sum_{k=1}^{i-1} \frac{\pi_k}{n-k+2-q(Z_{k:n+2})} .$$

Here π_i is defined as in Lemma 3.1. Furthermore it was shown that $Q_i^{se} = Q_{i+1}^{se} = 1$ for all $2 \leq i \leq n$, which, among other arguments, implies the reverse supermartingale property for S_n^{se}. So far we have seen, that we are not able to establish the supermartingale property for S_n without further restrictions. The following Lemma will establish the supermartingale property for S_n under an additional assumption on q.

Lemma 3.3. *Let q be monotone increasing. Then $S_n(q)$ is a non-negative reverse* *supermartingale.*

Proof. First note that

$$Q_1^{n+1} = (n+1)\frac{\pi_1\pi_2}{n} = \frac{(n+1)(n-q_k)}{n(n+1-q_k)} = \frac{n(n+1)-q_k(n+1)}{n(n+1)-q_k n} \leq 1 \tag{3.11}$$

Now recall that we have

$$\begin{aligned} Q_{i+1}^{n+1} - Q_i^{n+1} = & \frac{(q_i - q_{i+1})(n-i)(n-i+1) - q_{i+1}(1-q_i)(n-i+1-q_i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\ & \times \frac{\tilde{\pi}_i(n-i+2)^2}{n+1} \ . \end{aligned}$$

according to Lemma 3.2. Next consider that $q_i - q_{i+1} < 0$ and $q_{i+1}(1-q_i) \geq q_{i+1} - q_i > 0$, since q is monotone increasing. Therefore we obtain

$$Q_{i+1}^{n+1} - Q_i^{n+1} < 0 \tag{3.12}$$

Consider that we can write Q_i^{n+1} as

$$Q_i^{n+1} = Q_1^{n+1} + \sum_{k=1}^{n} \left(Q_{k+1}^{n+1} - Q_i^{n+1}\right)$$

Applying inequalities (3.11) and (3.12) to the latter equation yields $Q_i^{n+1} \leq 1$ for all $i \leq n+1$. Recall from Lemma 3.1 that

$$Q_{ij}^{n+1} = \begin{cases} Q_i^{n+1} & j \leq n \\ Q_i^{n+1} - \frac{(n+1)\pi_i\pi_n(1-q(Z_{n:n+1}))}{(n-i+1)(2-q(Z_{n:n+1}))} & j = n+1 \end{cases}$$

Thus $Q_{ij}^{n+1} \leq Q_i^{n+1} \leq 1$ for all $1 \leq i < j \leq n+1$. Now the latter together with Lemma 3.1 imply the statement of the Lemma . □

The assumption, that q is monotone increasing, is transfered to the censoring model m by (A4). This poses a restriction on the models m which can be used. This will be discussed in more detail in Section 4.3.

3.3 Existence of the Limit $S_\infty(q)$

As we have seen in the preceding section, $(S_n, D_n)_{n\geq 2}$ is not necessarily a supermartingale. However, we can show that if q is monotone increasing, then $S_n(q)$ is indeed a supermartingale, as we have seen in Lemma 3.3. We will now show how this implies the almost sure existence of S_∞, by a standard argument. The following result will be needed to prove the almost sure existence of S in Theorem 3.5.

Lemma 3.4. *Let $\mathcal{F}_\infty = \bigcap_{n\geq 2} \mathcal{F}_n$. Then we have for each $A \in \mathcal{F}_\infty$ that $\mathbb{P}(A) \in \{0, 1\}$.*

Proof. Denote $\tilde{Z} := (Z_1, Z_2, \dots) \in \mathbb{R}^\infty$ and let $1 \leq n < \infty$ fixed but arbitrary. We will use the Hewitt-Savage zero-one law to prove the statement of this lemma. Let π be a map

$$\begin{aligned} \pi : (\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty)) &\longrightarrow (\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty)) \\ (Z_1, Z_2, \dots, Z_n, Z_{n+1}, \dots) &\longmapsto (Z_{\tilde{\pi}(1)}, Z_{\tilde{\pi}(2)}, \dots, Z_{\tilde{\pi}(n)}, Z_{n+1}, \dots) \ . \end{aligned}$$

where $\tilde{\pi}$ is some permutation of $\{1, \dots, n\}$. Denote Π_n the set of all $n!$ of such maps. We need to show that for all $A \in \mathcal{F}_\infty$ and for all $\pi_0 \in \Pi$ there exists $B \in \mathcal{B}(\mathbb{R}^\infty)$ s. t.

$$A = \{\omega | \tilde{Z}(\omega) \in B\} = \{\omega | \pi_0(\tilde{Z}(\omega)) \in B\} \ . \tag{3.13}$$

Let $A \in \mathcal{F}_\infty$, then $A \in \mathcal{F}_n$ for all $n \in \mathbb{N}$. Since the map $(Z_{1:n}, \dots, Z_{n:n}, Z_{n+1}, Z_{n+2}, \dots)$ is measurable, there must exist $\tilde{B} \in \mathcal{B}(\mathbb{R}^\infty)$ such that

$$A = \{\omega | (Z_{1:n}(\omega), \dots, Z_{n:n}(\omega), Z_{n+1}(\omega), Z_{n+2}(\omega), \dots) \in \tilde{B}\} \ .$$

Note that each of the maps $\pi \in \Pi_n$ is measurable. Hence we can write A as

$$
\begin{aligned}
A &= \bigcup_{\pi \in \Pi_n} \left\{ \omega | \pi(\tilde{Z}) \in \tilde{B} \right\} \\
&= \bigcup_{\pi \in \Pi_n} \left\{ \omega | \tilde{Z} \in \pi^{-1}(\tilde{B}) \right\} \\
&= \left\{ \omega | \tilde{Z} \in \bigcup_{\pi \in \Pi_n} \pi^{-1}(\tilde{B}) \right\} \\
&= \left\{ \omega | \tilde{Z} \in B \right\} \ ,
\end{aligned}
$$

with

$$
B := \bigcup_{\pi \in \Pi_n} \pi^{-1}(\tilde{B}) \ .
$$

Clearly $B \in \mathcal{B}(\mathbb{R}^\infty)$, since it is expressed as a countable union of sets in $\mathcal{B}(\mathbb{R}^\infty)$. Moreover note that

$$
\bigcup_{\pi \in \Pi_n} \pi^{-1}(\tilde{B}) = \bigcup_{\pi \in \Pi_n} (\pi_0 \circ \pi)^{-1}(\tilde{B}) \ ,
$$

since the union is iterating over all $\pi \in \Pi_n$. Thus we can write

$$
\begin{aligned}
A &= \left\{ \omega | \tilde{Z} \in \bigcup_{\pi \in \Pi_n} (\pi_0 \circ \pi)^{-1}(\tilde{B}) \right\} \\
&= \bigcup_{\pi \in \Pi_n} \left\{ \omega | \tilde{Z} \in (\pi_0 \circ \pi)^{-1}(\tilde{B}) \right\} \\
&= \bigcup_{\pi \in \Pi_n} \left\{ \omega | \pi_0(\tilde{Z}) \in \pi^{-1}(\tilde{B}) \right\} \\
&= \left\{ \omega | \pi_0(\tilde{Z}) \in B \right\} \ .
\end{aligned}
$$

Whence establishing (3.13). □

Theorem 3.5. *Let q be monotone increasing. Then $S_\infty = \lim_{n\to\infty} S_n(q)$ exists almost surely and*

$$
\lim_{n\to\infty} S_n = \lim_{n\to\infty} \mathbb{E}[S_n] = S
$$

Proof. According to Lemma 3.3 $(S_n, D_n)_{n\geq 2}$ is a non-negative supermartingale. Hence S_n converges almost surely to a limit S, according to Neveu (1975), Lemma V-3-11. Furthermore we have $\mathbb{E}[S_n|F_\infty] \nearrow S$. Now the latter and Lemma 3.4 imply

$$\lim_{n\to\infty} \mathbb{E}[S_n] = \lim_{n\to\infty} \mathbb{E}[S_n|F_\infty] = S = \lim_{n\to\infty} S_n$$

□

Chapter 4

Identifying the limit

In the previous chapter we established the existence of the limit

$$\lim_{n\to\infty} S_n = S_\infty \ .$$

We will now continue to identify the limit $S(m(\cdot, \hat{\theta}_n))$ throughout this chapter. The interdependence structure of the proofs within this chapter in figure 4.1 below.

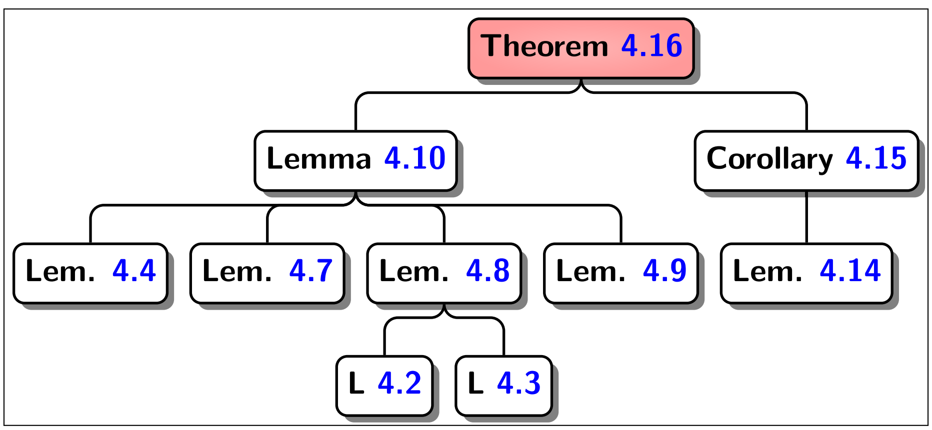

Figure 4.1: Interdependence Structure of the lemmas and theorems within this chapter.

4.1 The reverse supermartingale D_n

Let's first define the following quantities for $n \geq 1$ and $s < t$:

$$B_n(s) := \prod_{k=1}^{n} \left[1 + \frac{1 - q(Z_k)}{n - R_{k,n}}\right]^{\mathbb{1}_{\{Z_k < s\}}}$$

$$
\begin{aligned}
C_n(s) &:= \sum_{i=1}^{n+1} \left[\frac{1-q(s)}{n-i+2}\right] \mathbb{1}_{\{Z_{i-1:n} < s \leq Z_{i:n}\}} \\
D_n(s,t) &:= \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n-R_{k,n}+2}\right]^{2\mathbb{1}_{\{Z_k<s\}}} \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n-R_{k,n}+1}\right]^{\mathbb{1}_{\{s<Z_k<t\}}} \\
\Delta_n(s,t) &:= \mathbb{E}\left[D_n(s,t)\right] \\
\bar{\Delta}_n(s,t) &:= \mathbb{E}\left[C_n(s) D_n(s,t)\right] \ .
\end{aligned}
$$

Here $Z_{0:n} := 0$ and $Z_{n+1:n} := \infty$.

During this section, we will derive a representation of $\mathbb{E}[S_n]$ which involves the process D_n. This will be done in Lemma 4.2 and Lemma 4.3. We will then show that $\{D_n, \mathcal{F}_n\}$ is a reverse supermartingale in Lemma 4.6 and identify the limit of D_n in Lemma 4.4. Moreover Lemmas 4.2, 4.3 and 4.7 will play a central role in identifying the limit S in Chapter 4.

The lemma below contains a basic result needed to prove Lemma 4.3.

Lemma 4.1. *Let $i \neq j$. Then the conditional expectation*

$$\mathbb{E}[B_n(s)B_n(t)|Z_i = s, Z_j = t]$$

is independent of i, j and hence

$$\mathbb{E}[B_n(s)B_n(t)|Z_i = s, Z_j = t] = \mathbb{E}[B_n(s)B_n(t)|Z_1 = s, Z_2 = t]$$

holds almost surely.

Proof. For the sake of notational simplicity denote for $s < t$ $s_k^n := \mathbb{1}_{\{Z_{k:n}<s\}}$ and $t_k^n := \mathbb{1}_{\{s \leq Z_{k:n} < t\}}$. Note that $i \neq j$ implies $s \neq t$, since the $(Z_i)_{i \leq n}$ are pairwise distinct. Now consider on $\{s < t\}$

$$
\begin{aligned}
&\mathbb{E}\left[B_n(s)B_n(t)|Z_i = s, Z_j = t\right] \\
&= \mathbb{E}\left[\prod_{k=1}^{n}\left(1+\frac{1-q(Z_{k:n})}{n-k}\right)^{2s_k^n+t_k^n} |Z_i = s, Z_j = t\right] \\
&= \mathbb{E}\left[\sum_{k_1=1}^{n-1}\sum_{k_2=2}^{n} \mathbb{1}_{\{Z_{k_1:n}=s\}}\mathbb{1}_{\{Z_{k_2:n}=t\}}\left(1+\frac{1-q(s)}{n-k_1}\right)\right. \\
&\qquad \times \prod_{k=1}^{k_1-1}\left(1+\frac{1-q(Z_{k:n})}{n-k}\right)^{2s_k^n+t_k^n} \\
&\qquad \times \prod_{k=k_1+1}^{k_2-1}\left(1+\frac{1-q(Z_{k:n})}{n-k}\right)^{2s_k^n+t_k^n} \\
&\qquad \left.\times \prod_{k=k_2+1}^{n}\left(1+\frac{1-q(Z_{k:n})}{n-k}\right)^{2s_k^n+t_k^n} |Z_i = s, Z_j = t\right]
\end{aligned}
$$

since $s_{k_1}^n = 0$, $t_{k_1}^n = 1$, $s_{k_2}^n = 0$ and $t_{k_2}^n = 0$. Moreover we have

$$
\begin{cases}
s_k^n = 1 \text{ and } t_k^n = 0 & \text{if } k < k_1 \\
s_k^n = 0 \text{ and } t_k^n = 1 & \text{if } k_1 < k < k_2 \\
s_k^n = 0 \text{ and } t_k^n = 0 & \text{if } k_2 < k
\end{cases} .
$$

Therefore we obtain

$$
\begin{aligned}
&\mathbb{E}\left[B_n(s)B_n(t)|Z_i = s, Z_j = t\right] \\
&= \mathbb{E}\left[\sum_{k_1=1}^{n-1}\sum_{k_2=2}^{n} \mathbb{1}_{\{Z_{k_1:n}=s\}}\mathbb{1}_{\{Z_{k_2:n}=t\}}\left(1+\frac{1-q(s)}{n-k_1}\right)\right. \\
&\qquad \times \prod_{k=1}^{k_1-1}\left(1+\frac{1-q(Z_{k:n})}{n-k}\right)^{2s_k^n} \\
&\qquad \left.\times \prod_{k=k_1+1}^{k_2-1}\left(1+\frac{1-q(Z_{k:n})}{n-k}\right)^{t_k^n} |Z_i = s, Z_j = t\right] .
\end{aligned}
$$

Next we need to introduce some more notation. For $1 \leq i, j \leq n$ and $n \geq 2$, let $\{Z_{k:n-2}\}_{k\le n-2}$ denote the ordered Z-values among $Z_1,\ldots,Z_n$ with Z_i and Z_j removed from the sample. Note that

$$Z_{k:n} = \begin{cases} Z_{k:n-2} & k < k_1 \\ Z_{k-1:n-2} & k_1 < k < k_2 \end{cases} . \tag{4.1}$$

Thus we have

$$\begin{aligned}
&\mathbb{E}\left[B_n(s)B_n(t)|Z_i = s, Z_j = t\right] \\
&= \mathbb{E}\Bigg[\sum_{k_1=1}^{n}\sum_{k_2=1}^{n} \mathbb{1}_{\{Z_{k_1-1:n-2}<s\le Z_{k_1:n-2}\}}\mathbb{1}_{\{Z_{k_2-2:n-2}<t\le Z_{k_2-1:n-2}\}} \\
&\qquad \times\left(1+\frac{1-q(s)}{n-k_1}\right)\prod_{k=1}^{k_1-1}\left(1+\frac{1-q(Z_{k:n-2})}{n-k}\right)^{2s_k^{n-2}} \\
&\qquad \times\prod_{k=k_1+1}^{k_2-1}\left(1+\frac{1-q(Z_{k-1:n-2})}{n-k}\right)^{t_{k-1}^{n-2}} |Z_i = s, Z_j = t\Bigg] \\
&= \mathbb{E}\Bigg[\sum_{k_1=1}^{n}\sum_{k_2=1}^{n} \mathbb{1}_{\{Z_{k_1-1:n-2}<s\le Z_{k_1:n-2}\}}\mathbb{1}_{\{Z_{k_2-2:n-2}<t\le Z_{k_2-1:n-2}\}} \\
&\qquad \times\left(1+\frac{1-q(s)}{n-k_1}\right)\prod_{k=1}^{k_1-1}\left(1+\frac{1-q(Z_{k:n-2})}{n-k}\right)^{2s_k^{n-2}} \\
&\qquad \times\prod_{k=k_1}^{k_2-2}\left(1+\frac{1-q(Z_{k:n-2})}{n-k-1}\right)^{t_k^{n-2}}\Bigg] \\
&= \mathbb{E}\Bigg[\sum_{k_1=1}^{n} \mathbb{1}_{\{Z_{k_1-1:n-2}<s\le Z_{k_1:n-2}\}}\left(1+\frac{1-q(s)}{n-k_1}\right) \\
&\qquad \times\prod_{k=1}^{n-2}\left(1+\frac{1-q(Z_{k:n-2})}{n-k}\right)^{2s_k^{n-2}} \\
&\qquad \times\prod_{k=k_1}^{n-2}\left(1+\frac{1-q(Z_{k:n-2})}{n-k-1}\right)^{t_k^{n-2}}\Bigg]
\end{aligned}$$

which is independent of i, j. Next consider the case $t < s$. Define $\tilde{t}_k^n := \mathbb{1}_{\{Z_{k:n}<t\}}$

and $\tilde{s}_k^n := \mathbb{1}_{\{t \leq Z_{k:n} < s\}}$. Using similar arguments we can show that in this case

$$\begin{aligned}
&\mathbb{E}\left[B_n(s)B_n(t)|Z_i = s, Z_j = t\right] \\
&= \mathbb{E}\left[\sum_{k_1=1}^{n} \mathbb{1}_{\{Z_{k_1-1:n-2} < t \leq Z_{k_1:n-2}\}} \left(1 + \frac{1-q(t)}{n-k_1}\right)\right. \\
&\qquad \times \prod_{k=1}^{n-2} \left(1 + \frac{1-q(Z_{k:n-2})}{n-k}\right)^{2\tilde{t}_k^{n-2}} \\
&\qquad \left. \times \prod_{k=k_1}^{n-2} \left(1 + \frac{1-q(Z_{k:n-2})}{n-k-1}\right)^{\tilde{s}_k^{n-2}} \right]
\end{aligned}$$

which is independent of i, j as well. Thus we have on $\{s \neq t\}$ that $\mathbb{E}\left[B_n(s)B_n(t)|Z_i = s, Z_j = t\right]$ is independent of i, j and hence

$$\mathbb{E}\left[B_n(s)B_n(t)|Z_i = s, Z_j = t\right] = \mathbb{E}\left[B_n(s)B_n(t)|Z_1 = s, Z_2 = t\right] \ .$$

$\square$

Lemma 4.2. *Let $\tilde{\phi} : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ be a Borel-measurable function. Then we have for any $n \geq 2$*

$$\begin{aligned}
&\mathbb{E}[\tilde{\phi}(Z_i, Z_j)B_n(Z_i)B_n(Z_j)] \\
&= \mathbb{E}[\tilde{\phi}(Z_1, Z_2)B_n(Z_1)B_n(Z_2)] \ .
\end{aligned}$$

Proof. Consider that $\{Z_i = Z_j\}$ is a measure zero set, since H is continuous. Therefore the following holds for $1 \leq i, j \leq n$

$$\begin{aligned}
&\mathbb{E}\left[\tilde{\phi}(Z_i, Z_j)\mathbb{E}\left[B_n(Z_i)B_n(Z_j)|Z_i, Z_j\right]\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\{Z_i \neq Z_j\}}\tilde{\phi}(Z_i, Z_j)\mathbb{E}\left[B_n(Z_i)B_n(Z_j)|Z_i, Z_j\right]\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\{i \neq j\}}\tilde{\phi}(Z_i, Z_j)\mathbb{E}\left[B_n(Z_i)B_n(Z_j)|Z_i, Z_j\right]\right] \\
&= \int_0^\infty \int_0^\infty \mathbb{1}_{\{i \neq j\}}\tilde{\phi}(s, t)\mathbb{E}\left[B_n(s)B_n(t)|Z_i = s, Z_j = t\right] H(ds)H(dt) \ . \qquad (4.2)
\end{aligned}$$

According to Lemma 4.1 we have for $1 \le i \neq j \le n$

$$\mathbb{E}[B_n(s)B_n(t)|Z_i = s, Z_j = t] = \mathbb{E}[B_n(s)B_n(t)|Z_1 = s, Z_2 = t]$$

Therefore we obtain, according to (4.2), that

$$\begin{aligned}\mathbb{E}\left[\tilde{\phi}(Z_i, Z_j)B_n(Z_i)B_n(Z_j)\right] &= \mathbb{E}\left[\tilde{\phi}(Z_i, Z_j)\mathbb{E}\left[B_n(Z_i)B_n(Z_j)|Z_i, Z_j\right]\right] \\ &= \mathbb{E}\left[\tilde{\phi}(Z_1, Z_2)B_n(Z_1)B_n(Z_2)\right] \ .\end{aligned}$$

□

Lemma 4.3. *Let $\tilde{\phi} : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ be a Borel-measurable function. Then we have for any $s < t$ and $n \ge 2$*

$$\begin{aligned}&\mathbb{E}[\tilde{\phi}(Z_1, Z_2)B_n(Z_1)B_n(Z_2)] \\ &= \mathbb{E}[2\tilde{\phi}(Z_1, Z_2)\{\Delta_{n-2}(Z_1, Z_2) + \bar{\Delta}_{n-2}(Z_1, Z_2)\}\mathbb{1}_{\{Z_1 < Z_2\}}] \ .\end{aligned}$$

Proof. Note that w.l.o.g. we can assume that the $(Z_i)_{i \le n}$ are pairwise distinct, since H is continuous. Consider the following

$$\begin{aligned}B_n(Z_1)B_n(Z_2) &= \prod_{k=1}^{n}\left[1 + \frac{1 - q(Z_k)}{n - R_{k,n}}\right]^{\mathbb{1}_{\{Z_k < Z_1\}} + \mathbb{1}_{\{Z_k < Z_2\}}} \\ &= \left[1 + \frac{1 - q(Z_1)}{n - R_{1,n}}\right]^{\mathbb{1}_{\{Z_1 < Z_2\}}} \left[1 + \frac{1 - q(Z_2)}{n - R_{2,n}}\right]^{\mathbb{1}_{\{Z_2 < Z_1\}}} \\ &\quad \times \prod_{k=3}^{n}\left[1 + \frac{1 - q(Z_k)}{n - R_{k,n}}\right]^{\mathbb{1}_{\{Z_k < Z_1\}} + \mathbb{1}_{\{Z_k < Z_2\}}} \\ &= \mathbb{1}_{\{Z_1 < Z_2\}}\left[1 + \frac{1 - q(Z_1)}{n - R_{1,n}}\right] \\ &\qquad \times \prod_{k=1}^{n-2}\left[1 + \frac{1 - q(Z_{k+2})}{n - R_{k+2,n}}\right]^{\mathbb{1}_{\{Z_{k+2} < Z_1\}} + \mathbb{1}_{\{Z_{k+2} < Z_2\}}} \\ &\quad + \mathbb{1}_{\{Z_1 > Z_2\}}\left[1 + \frac{1 - q(Z_2)}{n - R_{2,n}}\right]\end{aligned}$$

$$\times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n - R_{k+2,n}}\right]^{\mathbb{1}_{\{Z_{k+2}<Z_1\}}+\mathbb{1}_{\{Z_{k+2}<Z_2\}}}$$
$$+ \mathbb{1}_{\{Z_1=Z_2\}} \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n - R_{k+2,n}}\right]^{2\mathbb{1}_{\{Z_{k+2}<Z_1\}}} . \tag{4.3}$$

On $\{Z_1 < Z_2\}$ we have

$$\prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n - R_{k+2,n}}\right]^{\mathbb{1}_{\{Z_{k+2}<Z_2\}}} = \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n - \tilde{R}_{k,n-2}}\right]^{\mathbb{1}_{\{Z_{k+2}<Z_1\}}}$$
$$\times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n - \tilde{R}_{k,n-2} - 1}\right]^{\mathbb{1}_{\{Z_1<Z_{k+2}<Z_2\}}}$$

where $\tilde{R}_{k,n-2}$ denotes the rank of the Z_k, $k = 3, \ldots, n$ among themselves. The above holds since

$$R_{k+2,n} = \begin{cases} \tilde{R}_{k,n-2} & \text{if } Z_{k+2} < Z_1 \\ \tilde{R}_{k,n-2} + 1 & \text{if } Z_1 < Z_{k+2} < Z_2 \end{cases}$$

for $k = 1, \ldots, n-2$. Therefore (4.3) yields

$$B_n(Z_1)B_n(Z_2) = \mathbb{1}_{\{Z_1<Z_2\}} \left[1 + \frac{1-q(Z_1)}{n - R_{1,n}}\right]$$
$$\times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n - \tilde{R}_{k,n-2}}\right]^{2\mathbb{1}_{\{Z_{k+2}<Z_1\}}}$$
$$\times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n - \tilde{R}_{k,n-2} - 1}\right]^{\mathbb{1}_{\{Z_1<Z_{k+2}<Z_2\}}}$$
$$+ \mathbb{1}_{\{Z_2<Z_1\}} \left[1 + \frac{1-q(Z_2)}{n - R_{2,n}}\right]$$
$$\times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n - \tilde{R}_{k,n-2}}\right]^{2\mathbb{1}_{\{Z_{k+2}<Z_2\}}}$$
$$\times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n - \tilde{R}_{k,n-2} - 1}\right]^{\mathbb{1}_{\{Z_2<Z_{k+2}<Z_1\}}}$$
$$+ \mathbb{1}_{\{Z_1=Z_2\}} \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n - \tilde{R}_{k,n-2}}\right]^{2\mathbb{1}_{\{Z_{k+2}<Z_1\}}} . \tag{4.4}$$

Now let's denote $Z_{k:n-2}$ the ordered Z-values among $Z_3, \ldots, Z_n$ for $k = 1, \ldots, n-2$. Consider that we can write

$$\left[1 + \frac{1-q(Z_1)}{n-R_{1,n}}\right] = \sum_{i=1}^{n-1} \left[1 + \frac{1-q(s)}{n-i}\right] \mathbb{1}_{\{Z_{i-1:n-2} < Z_1 \leq Z_{i:n-2}\}} \ .$$

Recall that we set $Z_{0:n} = 0$ and $Z_{n-1:n-2} = \infty$. Now note that $Z_{k:n-2}$ is independent of Z_1 and Z_2 for $k = 1, \ldots, n-2$. Therefore we obtain the following, by conditioning (4.4) on Z_1, Z_2:

$$\begin{aligned}
&\mathbb{E}[B_n(Z_1)B_n(Z_2)|Z_1 = s, Z_2 = t] \\
&= \mathbb{1}_{\{s<t\}}\mathbb{E}\left[\left(\sum_{i=1}^{n-1}\left[1+\frac{1-q(s)}{n-i}\right]\mathbb{1}_{\{Z_{i-1:n-2}<s\leq Z_{i:n-2}\}}\right)\right. \\
&\qquad\qquad \times \prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{1}_{\{Z_{k:n-2}<s\}}} \\
&\qquad\qquad \left.\times \prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k-1}\right]^{\mathbb{1}_{\{s<Z_{k:n-2}<t\}}}\right] \\
&\quad + \mathbb{1}_{\{t<s\}}\mathbb{E}\left[\left(\sum_{i=1}^{n-1}\left[1+\frac{1-q(t)}{n-i}\right]\mathbb{1}_{\{Z_{i-1:n-2}<t\leq Z_{i:n-2}\}}\right)\right. \\
&\qquad\qquad \times \prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{1}_{\{Z_{k:n-2}<t\}}} \\
&\qquad\qquad \left.\times \prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k-1}\right]^{\mathbb{1}_{\{t<Z_{k:n-2}<s\}}}\right] \\
&\quad + \mathbb{1}_{\{s=t\}}\mathbb{E}\left[\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{1}_{\{Z_{k:n-2}<s\}}}\right] \\
&= \alpha(s,t) + \alpha(t,s) + \beta(s,t)
\end{aligned}$$

where

$$\alpha(s,t) := \mathbb{1}_{\{s<t\}}\mathbb{E}\left[\left(\sum_{i=1}^{n-1}\left[1+\frac{1-q(s)}{n-i}\right]\mathbb{1}_{\{Z_{i-1:n-2}<s\leq Z_{i:n-2}\}}\right)\right.$$

$$\times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{1}_{\{Z_{k:n-2}<s\}}}$$
$$\left.\times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k:n-2})}{n-k-1}\right]^{\mathbb{1}_{\{s<Z_{k:n-2}<t\}}}\right]$$

and

$$\beta(s,t) := \mathbb{1}_{\{s=t\}}\mathbb{E}\left[\prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{1}_{\{Z_{k:n-2}<s\}}}\right] .$$

Consider that we have

$$\mathbb{E}[\alpha(Z_1, Z_2)] = \mathbb{E}[\alpha(Z_2, Z_1)] \ ,$$

because Z_1 and Z_2 are i. i. d. and α is symmetric in its arguments. Moreover

$$\mathbb{E}[\beta(Z_1, Z_2)] = 0$$

since H is continuous. Therefore we get

$$\begin{aligned}
&\mathbb{E}[\tilde{\phi}(Z_1, Z_2)B_n(Z_1)B_n(Z_2)] \\
&= \mathbb{E}[\tilde{\phi}(Z_1, Z_2)(\alpha(Z_1, Z_2) + \alpha(Z_2, Z_1) + \beta(Z_1, Z_2))] \\
&= \mathbb{E}[2\tilde{\phi}(Z_1, Z_2)\alpha(Z_1, Z_2)] \ .
\end{aligned} \tag{4.5}$$

under (A1). Next consider that

$$\begin{aligned}
\alpha(s,t) &= \mathbb{1}_{\{s<t\}}\mathbb{E}\left[(1 + C_{n-2}(s))D_{n-2}(s,t)\right] \\
&= \mathbb{1}_{\{s<t\}}(\Delta_{n-2}(s,t) + \bar{\Delta}_{n-2}(s,t)) \ .
\end{aligned}$$

The latter equality holds, since

$$\sum_{i=1}^{n-1} \left[1 + \frac{1-q(s)}{n-i}\right] \mathbb{1}_{\{Z_{i-1:n-2}<s\leq Z_{i:n-2}\}}$$

$$= \sum_{i=1}^{n-1} \mathbb{1}_{\{Z_{i-1:n-2} < s \le Z_{i:n-2}\}} + \sum_{i=1}^{n-1} \left[\frac{1-q(s)}{n-i}\right] \mathbb{1}_{\{Z_{i-1:n-2} < s \le Z_{i:n-2}\}}$$

$$= 1 + C_{n-2}(s) \ .$$

Now the statement of the lemma follows directly from (4.5). □

The next lemma identifies the almost sure limit of D_n for $n \to \infty$. Define for $s < t$

$$D(s,t) := \exp\left(2\int_0^s \frac{1-q(z)}{1-H(z)} H(dz) + \int_s^t \frac{1-q(z)}{1-H(z)} H(dz)\right)$$

Lemma 4.4. *For any $s < t$ s. t. $H(t) < 1$, we have*

$$\lim_{n\to\infty} D_n(s,t) = D(s,t) \ .$$

Proof. First recall the following definition

$$D_n(s,t) := \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n - R_{k,n} + 2}\right]^{2\mathbb{1}_{\{Z_k < s\}}} \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n - R_{k,n} + 1}\right]^{\mathbb{1}_{\{s < Z_k < t\}}} \quad .$$

Next let

$$x_k := \frac{1-q(Z_k)}{n(1-H_n(Z_k) + 2/n)}$$
$$y_k := \frac{1-q(Z_k)}{n(1-H_n(Z_k) + 1/n)}$$
$$s_k := \mathbb{1}_{\{Z_k < s\}}$$
$$t_k := \mathbb{1}_{\{s < Z_k < t\}}$$

for $s < t$ and $k = 1, \ldots, n$. Now consider

$$D_n(s,t) = \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n(1-H_n(Z_k) + 2/n)} \mathbb{1}_{\{Z_k < s\}}\right]^2$$

$$\begin{aligned}
&\times \prod_{k=1}^{n}\left[1+\frac{1-q(Z_k)}{n(1-H_n(Z_k)+1/n)}\mathbb{1}_{\{s<Z_k<t\}}\right]\\
&=\prod_{k=1}^{n}\left[1+x_ks_k\right]^2\prod_{k=1}^{n}\left[1+y_kt_k\right]\\
&=\exp\left(2\sum_{k=1}^{n}\ln\left[1+x_ks_k\right]+\sum_{k=1}^{n}\ln\left[1+y_kt_k\right]\right) .
\end{aligned}$$

Note that $0 \leq x_ks_k \leq 1$ and $0 \leq y_kt_k \leq 1$. Consider that the following inequality holds

$$-\frac{x^2}{2} \leq \ln(1+x) - x \leq 0$$

for any $x \geq 0$ (cf. Stute and Wang (1993), p. 1603). This implies

$$-\frac{1}{2}\sum_{k=1}^{n}x_k^2s_k \leq \sum_{k=1}^{n}\ln(1+x_ks_k) - \sum_{k=1}^{n}x_ks_k \leq 0 .$$

But now

$$\begin{aligned}
\sum_{k=1}^{n}x_k^2s_k &= \frac{1}{n^2}\sum_{k=1}^{n}\left(\frac{1-q(Z_k)}{1-H_n(Z_k)+\frac{2}{n}}\right)^2\mathbb{1}_{\{Z_k<s\}}\\
&\leq \frac{1}{n^2}\sum_{k=1}^{n}\left(\frac{1}{1-H_n(s)+\frac{1}{n}}\right)^2\\
&= \frac{1}{n(1-H_n(s)+n^{-1})^2} \longrightarrow 0
\end{aligned}$$

almost surely as $n \to \infty$, since $H(s) < H(t) < 1$ (c. f. Stute and Wang (1993), p. 1603). Therefore we have

$$|\sum_{k=1}^{n}\ln(1+x_ks_k) - \sum_{k=1}^{n}x_ks_k| \longrightarrow 0$$

with probability 1 as $n \to \infty$. Similarly we obtain

$$|\sum_{k=1}^{n}\ln(1+y_kt_k) - \sum_{k=1}^{n}y_kt_k| \longrightarrow 0$$

with probability 1 as $n \to \infty$. Hence

$$\lim_{n\to\infty} D_n(s) = \lim_{n\to\infty} \exp\left(2\sum_{k=1}^{n} x_k s_k + \sum_{k=1}^{n} y_k t_k\right) .$$

Now consider

$$\begin{aligned}
\sum_{k=1}^{n} x_k s_k &= \frac{1}{n}\sum_{k=1}^{n} \frac{1-q(Z_k)}{1-H_n(Z_k)+\frac{2}{n}} \mathbb{1}_{\{Z_k<s\}} \\
&= \int_0^{s-} \frac{1-q(z)}{1-H_n(z)+\frac{2}{n}} H_n(dz) \\
&= \int_0^{s-} \frac{1-q(z)}{1-H(z)} H_n(dz) + \int_0^{s-} \frac{1-q(z)}{1-H_n(z)+\frac{2}{n}} - \frac{1-q(z)}{1-H(z)} H_n(dz) \\
&= \int_0^{s-} \frac{1-q(z)}{1-H(z)} H_n(dz) + \int_0^{s-} \frac{(1-q(z))(H_n(z)-H(z)-\frac{2}{n})}{(1-H_n(z)+\frac{2}{n})(1-H(z))} H_n(dz) .
\end{aligned} \tag{4.6}$$

Note that the second term on the right hand side of the latter equation above tends to zero for $n \to \infty$, because

$$\begin{aligned}
&\left|\int_0^{s-} \frac{(1-q(z))(H_n(z)-H(z)-\frac{2}{n})}{(1-H_n(z)+\frac{2}{n})(1-H(z))} H_n(dz)\right| \\
&\leq \frac{\sup_z |H_n(z)-H(z)-\frac{2}{n}|}{1-H(s)} \int_0^{s-} \frac{1}{1-H_n(z)} H_n(dz) \longrightarrow 0
\end{aligned}$$

almost surely as $n \to \infty$, by the Glivenko-Cantelli Theorem and since $H(s) < 1$. Moreover we have

$$\int_0^{s-} \frac{1-q(z)}{1-H(z)} H_n(dz) \longrightarrow \int_0^{s} \frac{1-q(z)}{1-H(z)} H(dz)$$

by the SLLN. Therefore we obtain

$$\lim_{n\to\infty} \sum_{k=1}^{n} x_k s_k = \int_0^{s} \frac{1-q(z)}{1-H(z)} H(dz) .$$

By the same arguments, we can show that

$$\lim_{n\to\infty}\sum_{k=1}^{n} y_k t_k = \int_s^t \frac{1-q(z)}{1-H(z)} H(dz) \ .$$

Thus we finally conclude

$$\lim_{n\to\infty} D_n(s,t) = \exp\left(2\int_0^s \frac{1-q(z)}{1-H(z)} H(dz) + \int_s^t \frac{1-q(z)}{1-H(z)} H(dz)\right)$$

almost surely. $\square$

Remark 4.5. The measure zero set $\{\omega | D_n(s,t;\omega) \to D(s,t) \text{ as } n\to\infty\}$ is independent of s and t.

Lemma 4.6. $\{D_n, \mathcal{F}_n\}_{n\geq 1}$ *is a non-negative reverse supermartingale.*

Proof. Consider that for $s<t$ and $n\geq 1$

$$\begin{aligned}
\mathbb{E}[D_n(s,t)|\mathcal{F}_{n+1}] &= \mathbb{E}\left[\prod_{k=1}^{n}\left(1+\frac{1-q(Z_{k:n})}{n-k+2}\right)^{2\mathbb{1}_{\{Z_{k:n}<s\}}}\right.\\
&\qquad \left.\times\prod_{k=1}^{n}\left(1+\frac{1-q(Z_{k:n})}{n-k+1}\right)^{\mathbb{1}_{\{s<Z_{k:n}<t\}}} |\mathcal{F}_{n+1}\right]\\
&= \sum_{i=1}^{n+1}\mathbb{E}\left[\mathbb{1}_{\{Z_{n+1}=Z_{i:n+1}\}}\prod_{k=1}^{n}\ldots|\mathcal{F}_{n+1}\right]\\
&= \sum_{i=1}^{n+1}\mathbb{E}\left[\mathbb{1}_{\{Z_{n+1}=Z_{i:n+1}\}}\prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+2}\right)^{2\mathbb{1}_{\{Z_{k:n+1}<s\}}}\right.\\
&\qquad \times\prod_{k=i}^{n}\left(1+\frac{1-q(Z_{k+1:n+1})}{n-k+2}\right)^{2\mathbb{1}_{\{Z_{k+1:n+1}<s\}}}\\
&\qquad \times\prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+1}\right)^{\mathbb{1}_{\{s<Z_{k:n+1}<t\}}}\\
&\qquad \left.\times\prod_{k=i}^{n}\left(1+\frac{1-q(Z_{k+1:n+1})}{n-k+1}\right)^{\mathbb{1}_{\{s<Z_{k+1:n+1}<t\}}} |\mathcal{F}_{n+1}\right]\\
&= \sum_{i=1}^{n+1}\mathbb{E}\left[\mathbb{1}_{\{Z_{n+1}=Z_{i:n+1}\}}\prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+2}\right)^{2\mathbb{1}_{\{Z_{k:n+1}<s\}}}\right.
\end{aligned}$$

$$\times \prod_{k=i+1}^{n+1} \left(1 + \frac{1-q(Z_{k:n+1})}{n-k+3}\right)^{2\mathbb{1}_{\{Z_{k:n+1}<s\}}}$$
$$\times \prod_{k=1}^{i-1} \left(1 + \frac{1-q(Z_{k:n+1})}{n-k+1}\right)^{\mathbb{1}_{\{s<Z_{k:n+1}<t\}}}$$
$$\times \prod_{k=i+1}^{n+1} \left(1 + \frac{1-q(Z_{k:n+1})}{n-k+2}\right)^{\mathbb{1}_{\{s<Z_{k:n+1}<t\}}} |\mathcal{F}_{n+1}\Bigg] \ .$$

Now each product within the conditional expectation is measurable w. r. t. $\mathcal{F}_{n+1}$. Moreover we have for $i = 1, \ldots, n$

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{\{Z_{n+1}=Z_{i:n+1}\}}|\mathcal{F}_{n+1}] &= \mathbb{P}(Z_{n+1} = Z_{i:n+1}|\mathcal{F}_{n+1}) \\ &= \mathbb{P}(R_{n+1,n+1} = i) \\ &= \frac{1}{n+1} \ .\end{aligned}$$

Thus we obtain

$$\begin{aligned}\mathbb{E}[D_n(s,t)|\mathcal{F}_{n+1}] = \frac{1}{n+1} \sum_{i=1}^{n+1} \prod_{k=1}^{i-1} &\left(1 + \frac{1-q(Z_{k:n+1})}{n-k+2}\right)^{2\mathbb{1}_{\{Z_{k:n+1}<s\}}} \\ &\times \left(1 + \frac{1-q(Z_{k:n+1})}{n-k+1}\right)^{\mathbb{1}_{\{s<Z_{k:n+1}<t\}}} \\ \times \prod_{k=i+1}^{n+1} &\left(1 + \frac{1-q(Z_{k:n+1})}{n-k+3}\right)^{2\mathbb{1}_{\{Z_{k:n+1}<s\}}} \\ &\times \left(1 + \frac{1-q(Z_{k:n+1})}{n-k+2}\right)^{\mathbb{1}_{\{s<Z_{k:n+1}<t\}}} \ .\end{aligned} \tag{4.7}$$

We will now proceed by induction on n. First let

$$x_k := 1 - q(Z_{k:2}),\ s_k := \mathbb{1}_{\{Z_{k:2}<s\}} \text{ and } t_k := \mathbb{1}_{\{s<Z_{k:2}<t\}}$$

for $k = 1, 2$. Note that that x_k and y_k are different, compared to the corresponding definitions in lemma 4.4, as they involves the ordered Z-values here. Next consider

$$\begin{aligned}\mathbb{E}[D_1(s,t)|\mathcal{F}_2] &= \frac{1}{2}\left[\left(1+\frac{1-q(Z_{2:2})}{2}\right)^{2\mathbb{1}_{\{Z_{2:2}<s\}}} \times (1+(1-q(Z_{2:2})))^{\mathbb{1}_{\{s<Z_{2:2}<t\}}}\right.\\ &\qquad \left. + \left(1+\frac{1-q(Z_{1:2})}{2}\right)^{2\mathbb{1}_{\{Z_{1:2}<s\}}} \times (1+(1-q(Z_{1:2})))^{\mathbb{1}_{\{s<Z_{1:2}<t\}}}\right]\\ &= \frac{1}{2}\left[\left(1+\frac{x_2}{2}s_2\right)^2 \times (1+x_2t_2) + \left(1+\frac{x_1}{2}s_1\right)^2 \times (1+x_1t_1)\right] .\end{aligned}$$

Moreover we have

$$\begin{aligned}D_2(s,t) &= \prod_{k=1}^{2}\left[1+\frac{1-q(Z_{k:2})}{4-k}\right]^{2\mathbb{1}_{\{Z_{k:2}<s\}}} \prod_{k=1}^{2}\left[1+\frac{1-q(Z_{k:2})}{3-k}\right]^{\mathbb{1}_{\{s<Z_{k:2}<t\}}}\\ &= \left[1+\frac{x_1}{3}s_1\right]^2 \times \left[1+\frac{x_1}{2}t_1\right] \times \left[1+\frac{x_2}{2}s_2\right]^2 \times [1+x_2t_2]\\ &= \left[1+\frac{x_1}{2}t_1 + \left(\frac{x_1^2}{9}+\frac{2}{3}x_1\right)s_1\right] \times \left[1+x_2t_2+\left(\frac{x_2^2}{4}+x_2\right)s_2\right] .\end{aligned}$$

Therefore we obtain

$$\mathbb{E}[D_1(s,t)|\mathcal{F}_2] - D_2(s,t) \le \frac{x_1^2}{72} - \frac{x_1}{6} \le 0 .$$

since $0 \le x_1 \le 1$. Thus $\mathbb{E}[D_1(s,t)|\mathcal{F}_2] \le D_2(s,t)$ for any $s<t$, as needed. Now assume that

$$\mathbb{E}[D_n(s,t)|\mathcal{F}_{n+1}] \le D_{n+1}(s,t)$$

holds for any $n \ge 1$. Note that the latter is equivalent to assuming

$$\begin{aligned}&\frac{1}{n+1}\sum_{i=1}^{n+1}\prod_{k=1}^{i-1}\left(1+\frac{1-q(y_k)}{n-k+2}\right)^{2\mathbb{1}_{\{y_k<s\}}}\left(1+\frac{1-q(y_k)}{n-k+1}\right)^{\mathbb{1}_{\{s<y_k<t\}}}\\ &\qquad \times \prod_{k=i+1}^{n+1}\left(1+\frac{1-q(y_k)}{n-k+3}\right)^{2\mathbb{1}_{\{y_k<s\}}}\left(1+\frac{1-q(y_k)}{n-k+2}\right)^{\mathbb{1}_{\{s<y_k<t\}}}\\ &\le \prod_{k=1}^{n+1}\left(1+\frac{1-q(y_k)}{n-k+3}\right)^{2\mathbb{1}_{\{y_k<s\}}}\prod_{k=1}^{n+1}\left(1+\frac{1-q(y_k)}{n-k+2}\right)^{\mathbb{1}_{\{s<y_k<t\}}} \qquad (4.8)\end{aligned}$$

holds for arbitrary $y_k \geq 0$. Next define for $s < t$ and $n \geq 1$

$$A_{n+2}(s,t) := \prod_{k=2}^{n+2} \left[1 + \frac{1 - q(Z_{k:n+2})}{n-k+4}\right]^{2\mathbb{1}_{\{Z_{k:n+2}<s\}}} \times \left[1 + \frac{1 - q(Z_{k:n+2})}{n-k+3}\right]^{\mathbb{1}_{\{s<Z_{k:n+2}<t\}}} .$$

Now consider that we get from (4.7)

$$\begin{aligned}
&\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}] \\
&= \frac{1}{n+2} \sum_{i=1}^{n+2} \prod_{k=1}^{i-1} \left(1 + \frac{1 - q(Z_{k:n+2})}{n-k+3}\right)^{2\mathbb{1}_{\{Z_{k:n+2}<s\}}} \left(1 + \frac{1 - q(Z_{k:n+2})}{n-k+2}\right)^{\mathbb{1}_{\{s<Z_{k:n+2}<t\}}} \\
&\qquad \times \prod_{k=i+1}^{n+2} \left(1 + \frac{1 - q(Z_{k:n+2})}{n-k+4}\right)^{2\mathbb{1}_{\{Z_{k:n+2}<s\}}} \left(1 + \frac{1 - q(Z_{k:n+2})}{n-k+3}\right)^{\mathbb{1}_{\{s<Z_{k:n+2}<t\}}} \\
&= \frac{A_{n+2}}{n+2} + \frac{1}{n+2} \sum_{i=2}^{n+2} \prod_{k=1}^{i-1} \cdots \times \prod_{k=i+1}^{n+2} \cdots \\
&= \frac{A_{n+2}}{n+2} + \frac{1}{n+2} \sum_{i=1}^{n+1} \prod_{k=1}^{i} \cdots \times \prod_{k=i+2}^{n+2} \cdots \\
&= \frac{A_{n+2}}{n+2} + \frac{1}{n+2} \left(1 + \frac{1 - q(Z_{1:n+2})}{n+2}\right)^{2\mathbb{1}_{\{Z_{1:n+2}<s\}}} \left(1 + \frac{1 - q(Z_{1:n+2})}{n+1}\right)^{\mathbb{1}_{\{s<Z_{1:n+2}<t\}}} \\
&\qquad \times \sum_{i=1}^{n+1} \prod_{k=1}^{i-1} \left(1 + \frac{1 - q(Z_{k+1:n+2})}{n-k+2}\right)^{2\mathbb{1}_{\{Z_{k+1:n+2}<s\}}} \\
&\qquad\qquad \times \left(1 + \frac{1 - q(Z_{k+1:n+2})}{n-k+1}\right)^{\mathbb{1}_{\{s<Z_{k+1:n+2}<t\}}} \\
&\qquad \times \prod_{k=i+1}^{n+1} \left(1 + \frac{1 - q(Z_{k+1:n+2})}{n-k+3}\right)^{2\mathbb{1}_{\{Z_{k+1:n+2}<s\}}} \\
&\qquad\qquad \times \left(1 + \frac{1 - q(Z_{k+1:n+2})}{n-k+2}\right)^{\mathbb{1}_{\{s<Z_{k+1:n+2}<t\}}} .
\end{aligned}$$

Using (4.8) on the right hand side of the equation above yields

$$\begin{aligned}
&\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}] \\
&\leq \frac{A_{n+2}}{n+2} + \frac{n+1}{n+2} \left(1 + \frac{1 - q(Z_{1:n+2})}{n+2}\right)^{2\mathbb{1}_{\{Z_{1:n+2}<s\}}} \left(1 + \frac{1 - q(Z_{1:n+2})}{n+1}\right)^{\mathbb{1}_{\{s<Z_{1:n+2}<t\}}} \\
&\qquad \times \prod_{k=1}^{n+1} \left(1 + \frac{1 - q(Z_{k+1:n+2})}{n-k+3}\right)^{2\mathbb{1}_{\{Z_{k+1:n+2}<s\}}}
\end{aligned}$$

$$\times\left(1+\frac{1-q(Z_{k+1:n+2})}{n-k+2}\right)^{\mathbb{1}_{\{s<Z_{k+1:n+2}<t\}}}$$

$$= A_{n+2}\left[\frac{1}{n+2}+\frac{n+1}{n+2}\left(1+\frac{1-q(Z_{1:n+2})}{n+2}\right)^{2\mathbb{1}_{\{Z_{1:n+2}<s\}}}\right.$$
$$\left.\times\left(1+\frac{1-q(Z_{1:n+2})}{n+1}\right)^{\mathbb{1}_{\{s<Z_{1:n+2}<t\}}}\right] .$$

For the moment, let

$$x_1 := 1-q(Z_{1:n+2}),\ s_1 := \mathbb{1}_{\{Z_{1:n+2}<s\}} \text{ and } t_1 := \mathbb{1}_{\{s<Z_{1:n+2}<t\}}$$

Now we can rewrite the above as

$$\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}] \leq A_{n+2}\left[\frac{1}{n+2}+\frac{n+1}{n+2}\left(1+\frac{x_1s_1}{n+2}\right)^2\left(1+\frac{x_1t_1}{n+1}\right)\right] . \quad (4.9)$$

Next consider

$$\begin{aligned}\left(1+\frac{x_1t_1}{n+1}\right) &= \left(1+\frac{x_1t_1}{n+2}-\frac{1}{n+2}\right)\left(1+\frac{1}{n+1}\right)\\ &= \left(1+\frac{x_1t_1}{n+2}\right)+\frac{1}{n+1}\left(1+\frac{x_1t_1}{(n+2)}\right)-\frac{1}{n+1}\\ &= \left(1+\frac{x_1t_1}{n+2}\right)+\frac{x_1t_1}{(n+1)(n+2)} .\end{aligned}$$

Thus we get

$$\begin{aligned}&\frac{n+1}{n+2}\left(1+\frac{x_1s_1}{n+2}\right)^2\left(1+\frac{x_1t_1}{n+1}\right)\\ &=\frac{n+1}{n+2}\left(1+\frac{x_1s_1}{n+2}\right)^2\left(1+\frac{x_1t_1}{n+2}\right)+\left(1+\frac{x_1s_1}{n+2}\right)^2\frac{x_1t_1}{(n+2)^2} .\end{aligned}$$

But now

$$\left(1+\frac{x_1s_1}{n+2}\right)^2\frac{x_1t_1}{(n+2)^2}=\left(1+2\frac{x_1s_1}{n+2}+\frac{x_1^2s_1}{(n+2)^2}\right)\frac{x_1t_1}{(n+2)^2}$$

$$= \frac{x_1 t_1}{(n+2)^2}$$

since $s_1 \cdot t_1 = 0$ for all $s < t$. Hence we can rewrite the term in brackets in (4.9) as

$$\begin{aligned}
&\frac{1}{n+2} + \frac{n+1}{n+2}\left(1 + \frac{x_1 s_1}{n+2}\right)^2 \left(1 + \frac{x_1 t_1}{n+1}\right) \\
&= \frac{1}{n+2} + \frac{x_1 t_1}{(n+2)^2} + \frac{n+1}{n+2}\left(1 + \frac{x_1 s_1}{n+2}\right)^2 \left(1 + \frac{x_1 t_1}{n+2}\right) \\
&= \frac{1}{n+2}\left(1 + \frac{x_1 t_1}{n+2}\right) + \frac{n+1}{n+2}\left(1 + \frac{x_1 s_1}{n+2}\right)^2 \left(1 + \frac{x_1 t_1}{n+2}\right) \\
&= \left[\frac{1}{n+2} + \frac{n+1}{n+2}\left(1 + \frac{x_1}{n+2}\right)^{2s_1}\right] \left(1 + \frac{x_1}{n+2}\right)^{t_1} \\
&\leq \left(1 + \frac{x_1}{n+3}\right)^{2s_1} \left(1 + \frac{x_1}{n+2}\right)^{t_1} \ .
\end{aligned}$$

The latter inequality above holds, since

$$\left[\frac{1}{n+2} + \frac{n+1}{n+2}\left(1 + \frac{x}{n+2}\right)^2\right] \leq \left(1 + \frac{x}{n+3}\right)^2$$

for any $0 \leq x \leq 1$. (c. f. Bose and Sen (1999), page 197). Therefore we can rewrite (4.9) as

$$\begin{aligned}
\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}] \leq A_{n+2} &\left(1 + \frac{1 - q(Z_{1:n+2})}{n+3}\right)^{2\mathbb{1}_{\{Z_{1:n+2} < s\}}} \\
&\times \left(1 + \frac{1 - q(Z_{1:n+2})}{n+2}\right)^{\mathbb{1}_{\{s < Z_{1:n+2} < t\}}} \\
= D_{n+2}(s,t) \ .
\end{aligned}$$

This concludes the proof. $\square$

Lemma 4.7. *Let $s < t$ s. t. $H(t) < 1$. Then $\Delta_n(s,t) \nearrow D(s,t)$.*

Proof. Consider that we have for $n \geq 2$

$$\Delta_n(s,t) = \mathbb{E}[D_n(s,t)] = \mathbb{E}[D_n(s,t)|\mathcal{F}_\infty]$$

by definition of $\Delta_n(s,t)$ and Lemma 3.4. Next note that we have $D_n(s,t) \to D(s,t)$ almost surely, according to Lemma 4.4. Moreover we get from Lemma 4.6, that $\{D_n, \mathcal{F}_n\}_{n\geq 1}$ is a reverse supermartingale. Now this together with Proposition 5-3-11 of Neveu (1975) yields

$$\mathbb{E}[D_n(s,t)|\mathcal{F}_\infty] \nearrow D(s,t) \ .$$

This proves the lemma. □

We will now proceed to find an explicit representation for $\mathbb{E}[S_n]$ in terms of the reverse supermartingale D_n (see Section **??**) to identify the limit $S = S(q)$. Consider the following lemma.

Lemma 4.8. *For continuous $H(\cdot)$, we have*

$$\mathbb{E}[S_n(q)] = \frac{n-1}{n}\mathbb{E}[\phi(Z_1,Z_2)q(Z_1)q(Z_2)\{\Delta_{n-2}(Z_1,Z_2) + \bar{\Delta}_{n-2}(Z_1,Z_2)\}\mathbb{1}_{\{Z_1<Z_2\}}] \ .$$

Proof. The proof of the lemma above is similar to the proof of Lemma **??**. Consider

$$\begin{aligned}\mathbb{E}[S_n(q)] &= \sum\sum_{1\leq i<j\leq n}\mathbb{E}\left[\phi(Z_{i:n},Z_{j:n})\frac{q(Z_{i:n})}{n-i+1)}\prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n})}{n-k+1}\right]\right.\\ &\qquad\left.\times\frac{q(Z_{j:n})}{n-j+1}\prod_{l=1}^{j-1}\left[1-\frac{q(Z_{l:n})}{n-l+1}\right]\right]\\ &= \frac{1}{n^2}\sum\sum_{1\leq i<j\leq n}\mathbb{E}\left[\phi(Z_{i:n},Z_{j:n})q(Z_{i:n})\prod_{k=1}^{i-1}\left[1+\frac{1-q(Z_{k:n})}{n-k+1}\right]\right.\end{aligned}$$

$$
\begin{aligned}
&\times\, q(Z_{j:n}) \prod_{l=1}^{j-1}\left[1+\frac{1-q(Z_{l:n})}{n-l+1}\right]\Bigg] \\
&= \frac{1}{n^2}\sum\sum_{1\le i<j\le n} \mathbb{E}\left[\phi(Z_{i:n}, Z_{j:n})q(Z_{i:n})q(Z_{j:n})B_n(Z_{i:n})B_n(Z_{j:n})\right] \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{1}_{\{R_{i,n}<R_{j,n}\}}\mathbb{E}\left[\phi(Z_i, Z_j)q(Z_i)q(Z_j)B_n(Z_i)B_n(Z_j)\right] \ . \qquad (4.10)
\end{aligned}
$$

According to Lemma 4.2 we obtain

$$
\mathbb{E}[S_n(q)] = \frac{n-1}{2n}\mathbb{E}\left[\phi(Z_1, Z_2)q(Z_1)q(Z_2)B_n(Z_1)B_n(Z_2)\right] \ .
$$

Now, since ϕ and q are measurable, we can apply Lemma 4.3 to obtain the result. □

The following result is necessary for the proof of Lemma 4.10.

Lemma 4.9. *For continuous $H(\cdot)$ and $t < s$, we have $C_n(t) \to 0$ as $n \to \infty$ w. p. 1, and $C_n(t) \in [0, 1]$ for all $n \ge 1$ and $t \ge 0$.*

Proof. It is easy to see that $0 \le C_n(t) \le 1$ for any $t \ge 0$ and $n \ge 2$, since $0 \le q(t) \le 1$ and $\mathbb{1}_{\{Z_{i-1:n}<t\le Z_{i:n}\}} = 1$ for exactly one $i \in \{1, \ldots, n\}$. Let's now consider

$$
\begin{aligned}
C_n(t) &= \sum_{i=1}^{n+1}\frac{1-q(t)}{n-i+2}[\mathbb{1}_{\{Z_{i-1:n}<t\}} - \mathbb{1}_{\{Z_{i:n}<t\}}] \\
&= \sum_{i=1}^{n+1}\frac{1-q(t)}{n-i+2}\mathbb{1}_{\{Z_{i-1:n}<t\}} - \sum_{i=1}^{n+1}\frac{1-q(t)}{n-i+2}\mathbb{1}_{\{Z_{i:n}<t\}} \\
&= \sum_{i=0}^{n}\frac{1-q(t)}{n-i+1}\mathbb{1}_{\{Z_{i:n}<t\}} - \sum_{i=1}^{n}\frac{1-q(t)}{n-i+2}\mathbb{1}_{\{Z_{i:n}<t\}} \\
&= \sum_{i=1}^{n}\frac{1-q(t)}{n-i+1}\mathbb{1}_{\{Z_{i:n}<t\}} + \frac{(1-q(t))}{n+1} - \sum_{i=1}^{n}\frac{1-q(t)}{n-i+2}\mathbb{1}_{\{Z_{i:n}<t\}} \\
&= (1-q(t))\left\{\frac{1}{n+1} + \sum_{i=1}^{n}\left[\frac{1}{n-i+1} - \frac{1}{n-i+2}\right]\mathbb{1}_{\{Z_{i:n}<t\}}\right\} \\
&= (1-q(t))\sum_{i=1}^{n}\left[\frac{1}{n-nH_n(Z_{i:n})+1}\frac{1}{n-nH_n(Z_{i:n})+2}\right]\mathbb{1}_{\{Z_{i:n}<t\}} \\
&\quad + \frac{1-q(t)}{n+1}
\end{aligned}
$$

$$= (1 - q(t)) \int_0^t \left[\frac{1}{1 - H_n(x) + \frac{1}{n}} - \frac{1}{1 - H_n(x) + \frac{2}{n}} \right] H_n(dx) \\ + \frac{1 - q(t)}{n+1} \ . \tag{4.11}$$

In Lemma 4.4 we have seen that

$$\int_0^t \frac{1}{1 - H_n(x) + \frac{2}{n}} H_n(dx) \to \int_0^t \frac{1}{1 - H(x)} H(dx) \ .$$

By the same arguments we obtain

$$\int_0^t \frac{1}{1 - H_n(x) + \frac{1}{n}} H_n(dx) \to \int_0^t \frac{1}{1 - H(x)} H(dx) \ .$$

Therefore the right hand side of (4.11) converges to zero. □

We will now identify the almost sure limits of $S_n(q)$ and $\bar{S}_n(q)$ in Lemma 4.10. Define for $n \geq 2$

$$\bar{S}_n(q) := \sum\sum_{1 \leq i < j \leq n} \phi(Z_{i:n}, Z_{j:n}) \bar{W}_{i:n}(q) \bar{W}_{j:n}(q)$$

where

$$\bar{W}_{i:n}(q) := \prod_{k=1}^{n} \left(1 - \frac{q(Z_{k:n})}{n - k + 1} \right) \ .$$

Moreover let

$$S(q) := \frac{1}{2} \int_0^\infty \int_0^\infty \phi(s,t) q(s) q(t) \exp\left(\int_0^s \frac{1 - q(x)}{1 - H(x)} H(dx) \right) \\ \times \exp\left(\int_0^t \frac{1 - q(x)}{1 - H(x)} H(dx) \right) H(ds) H(dt)$$

and

$$\bar{S}(q) := \frac{1}{2} \int_0^\infty \int_0^\infty \phi(s,t) \exp\left(\int_0^s \frac{1 - q(x)}{1 - H(x)} H(dx) \right)$$

$$\times \exp\left(\int_0^t \frac{1-q(x)}{1-H(x)} H(dx)\right) H(ds)H(dt) \; .$$

Lemma 4.10. *Let H be continuous and q be increasing. Then the following statements hold*

$$\lim_{n\to\infty} S_n(q) = S(q)$$

and

$$\lim_{n\to\infty} \bar{S}_n(q) = \bar{S}(q)$$

with probability one, if the limits above exist.

Proof. Suppose H is continuous and q is increasing. First consider that S exists almost surely and we have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \mathbb{E}[S_n] = S$$

according to Theorem 3.5. Next consider

$$\begin{aligned}
\mathbb{E}[S_n(q)] &= \frac{n-1}{n}\mathbb{E}[\phi(Z_1,Z_2)q(Z_1)q(Z_2)\{\Delta_{n-2}(Z_1,Z_2)+\bar{\Delta}_{n-2}(Z_1,Z_2)\}\mathbb{1}_{\{Z_1<Z_2\}}] \\
&= \frac{n-1}{n}\mathbb{E}[\phi(Z_1,Z_2)q(Z_1)q(Z_2)\Delta_{n-2}(Z_1,Z_2)\mathbb{1}_{\{Z_1<Z_2\}}] \\
&\quad + \frac{n-1}{n}\mathbb{E}[\phi(Z_1,Z_2)q(Z_1)q(Z_2)\bar{\Delta}_{n-2}(Z_1,Z_2)\mathbb{1}_{\{Z_1<Z_2\}}]
\end{aligned}$$

by Lemma 4.8. Note that we get from Lemma 4.4 that $D_n(s,t) \to D(s,t)$ w. p. 1, and according to Lemma 4.9 $C_n(s) \to 0$ w. p. 1. Thus $\bar{\Delta}_n(s,t) \to 0$ as $n \to \infty$ for each $s < t$. Moreover, the fact that $0 \le C_n(s) \le 1$ and Lemma 4.7 imply that $\bar{\Delta}_n(s,t) \le \Delta_n(s,t) \le D(s,t)$ for all $n \ge 2$. Now note that $D(s,t)$ is integrable, since on $\{Z_1 < Z_2\}$ we have

$$\mathbb{E}[D(Z_1,Z_2)] = \mathbb{E}\left[\int_0^s \frac{1-q(x)}{1-H(x)}H(dx) + \int_0^t \frac{1-q(x)}{1-H(x)}H(dx)\right]$$

$$
\begin{aligned}
&\le \mathbb{E}\left[\int_0^{Z_{n:n}} \frac{1}{1-H(x)} H(dx) + \int_0^{Z_{n:n}} \frac{1}{1-H(x)} H(dx)\right] \\
&\le \mathbb{E}\left[-2\ln(1-H(Z_{n:n}))\right] \\
&< \infty \ .
\end{aligned}
$$

Therefore applying the Dominated Convergence Theorem yields

$$
\lim_{n\to\infty} \mathbb{E}[2\phi(Z_1,Z_2)q(Z_1)q(Z_2)\bar{\Delta}_{n-2}(Z_1,Z_2)\mathbb{1}_{\{Z_1<Z_2\}}] = 0 \ .
$$

Furthermore, according to Lemma 4.7, we have $\Delta_n(s,t) \nearrow D(s,t)$ for $s < t$ and $H(t) < 1$. Hence we have

$$
\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}[\phi(Z_1,Z_2)q(Z_1)q(Z_2)\Delta_{n-2}(Z_1,Z_2)\mathbb{1}_{\{Z_1<Z_2\}}] \\
&= \mathbb{E}[\phi(Z_1,Z_2)q(Z_1)q(Z_2)D(Z_1,Z_2)\mathbb{1}_{\{Z_1<Z_2\}}] \ .
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
\lim_{n\to\infty} \mathbb{E}[S_n(q)] &= \mathbb{E}[\phi(Z_1,Z_2)q(Z_1)q(Z_2)D(Z_1,Z_2)\mathbb{1}_{\{Z_1<Z_2\}}] \\
&= \int\int_{\{Z_1<Z_2\}} \phi(s,t)q(s)\exp\left(\int_0^s \frac{1-q(z)}{1-H(z)} H(dz)\right) \\
&\qquad\qquad \times q(t)\exp\left(\int_0^t \frac{1-q(z)}{1-H(z)} H(dz)\right) H(ds)H(dt) \\
&= \frac{1}{2}\int_0^\infty \int_0^\infty \phi(s,t)q(s)\exp\left(\int_0^s \frac{1-q(z)}{1-H(z)} H(dz)\right) \\
&\qquad\qquad \times q(t)\exp\left(\int_0^t \frac{1-q(z)}{1-H(z)} H(dz)\right) H(ds)H(dt)
\end{aligned}
$$

almost surely, since $\phi(s,t)q(s)q(t)D(s,t)$ is symmetric by (A1), and Z_1 and Z_2 are i. i. d.. This concludes the argument for S_n. By similar arguments, we obtain $\bar{S}_n \to \bar{S}$ w. p. 1. □

4.2 On the censoring model m

TODO Relationship of model m and hazard rates. However, we can show that common choices for m (c. f. TODO cite) will work within this framework, though some under restrictions. Consider the following examples:

Example 4.11. Let $\theta = (\theta_1, \theta_2)$. For $\theta_1 > 0$ and $\theta_2 < 0$ let

$$m(t, \theta) := \frac{\theta_1}{\theta_1 + t_2^\theta} \ .$$

Clearly $m(t, \theta)$ is increasing in t. TODO More Weibull framework. Conclude that $\theta_2 > 0$ is restriction.

Example 4.12. For $\theta > 0$ let

$$m(t, \theta) := \frac{1}{1 + + \exp(-\theta t)} \ .$$

The logistic model works without restrictions, since $m(t, \theta)$ is increasing whenever $\theta > 0$.

Example 4.13. For $\theta > 0$ let

$$m(t, \theta) := \frac{1}{1 - \exp(-\exp(\theta t))} \ .$$

The complementary log-log model is increasing if $\theta > 0$.

4.3 Calculating the limit

In order to identify the limit of $S_{2,n}^{se} = S_n(m(\cdot, \hat{\theta}_n))$ we need the statement of Corollary 4.15, which is based upon the following lemma. Define for any $\epsilon > 0$ let

$$M_{1,\epsilon}(x) := \max(0, m(x, \theta_0) - \epsilon)) \text{ and } M_{2,\epsilon}(x) := \min(1, m(x, \theta_0) + \epsilon))$$

Lemma 4.14. *Suppose (M1) and (M2) hold. Then the following statements hold for each* $0 < \epsilon \leq 1$ *and* n *large enough*

(i) $M_{1,\epsilon}(x) \leq m(x, \hat{\theta}_n) \leq M_{2,\epsilon}(x)$

(ii) $M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 4\epsilon \leq m(x, \hat{\theta}_n)m(y, \hat{\theta}_n) \leq M_{1,\epsilon}(x)M_{1,\epsilon}(y) + 4\epsilon.$

Proof. First we will introduce some notation. We will write $m_n(x) := m(x, \theta_n)$ and $m(x) := m(x, \theta_0)$. Let's start with part *(i)*. Suppose $M_{1,\epsilon}(x) = 0$, then the condition above is trivially satisfied since $m_n(x) \geq 0$. Now suppose $M_{1,\epsilon}(x) = m(x) - \epsilon$.

$$\begin{aligned} m_n(x) &= (m_n(x) - m(x)) + m(x) \\ &\geq m(x) - \left| m_n(x) - m(x) \right| . \end{aligned}$$

Now using assumption (M1), we have for n large enough that for some $\epsilon > 0$ $\theta_n \in V(\epsilon, \theta_0)$. Now we get, according to (M2), that

$$\sup_{x \geq 0} |m_n(x) - m(x)| < \epsilon$$

Therefore we obtain $m_n(x) \geq m(x) - \epsilon = M_{1,\epsilon}(x)$. Let's now consider $M_{2,\epsilon}$. The case $M_{2,\epsilon} = 1$ is trivial again, since $m_n(x) \leq 1$. Now suppose $M_{2,\epsilon} = m(x) + \epsilon$. Then we obtain, for n large enough

$$\begin{aligned} m_n(x) &= (m_n(x) - m(x)) + m(x) \\ &\leq m(x) + |m_n(x) - m(x)| \\ &\leq m(x) + \epsilon \\ &= M_{2,\epsilon}(x) \ . \end{aligned}$$

This concludes the proof of part *(i)*. Now note that, according to (M1) and (M2), the following holds for n large enough and $\epsilon > 0$

$$\begin{aligned} m_n(x) &= (m_n(x) - m(x)) + m(x) \\ &\leq |m_n(x) - m(x)| + m(x) \\ &\leq m(x) + \epsilon \ . \end{aligned} \tag{4.12}$$

Moreover consider that

$$\begin{aligned} m_n(x)m_n(y) &= (m_n(x) - m(x))(m_n(y) - m(y)) \\ &\qquad + m(x)m_n(y) + m_n(x)m(y) - m(x)m(y) \\ &\leq \epsilon^2 + m(x)m_n(y) + m_n(x)m(y) - m(x)m(y) \ . \end{aligned}$$

Using on the right hand side of the latter inequality (4.12) yields

$$\begin{aligned} m_n(x)m_n(y) &\leq \epsilon^2 + m(x)(m(y) + \epsilon) + (m(x) + \epsilon)m(y) - m(x)m(y) \\ &= m(x)m(y) + \epsilon(m(x) + m(y)) + \epsilon^2 \ . \end{aligned} \tag{4.13}$$

Now suppose $M_{1,\epsilon}(x) = 0$ and $M_{1,\epsilon}(y) = 0$ for $x, y \in \mathbb{R}_+$. Then $m(x) \leq \epsilon$ and $m(y) \leq \epsilon$. Hence, using (4.13) yields

$$m_n(x)m_n(y) \leq 4\epsilon^2 \ .$$

Next suppose $M_{1,\epsilon}(x) = 0$ and $M_{1,\epsilon}(y) = m(y) - \epsilon$. Using (4.13) again, we obtain

$$\begin{aligned} m_n(x)m_n(y) &\leq m(x)m(y) + \epsilon(m(x) + m(y)) + \epsilon^2 \\ &\leq \epsilon + \epsilon(1 + \epsilon) + \epsilon^2 \\ &= 2\epsilon(1 + \epsilon) \ , \end{aligned}$$

since $m(x) \leq \epsilon$ and $m(y) \leq 1$. By similar calculations, we obtain the exact same result for the case $M_{1,\epsilon}(x) = m(x) - \epsilon$ and $M_{1,\epsilon}(y) = 0$. Now suppose $M_{1,\epsilon}(x) = m(x) - \epsilon$ and $M_{1,\epsilon}(y) = m(y) - \epsilon$ and note that

$$\begin{aligned} M_{1,\epsilon}(x)M_{1,\epsilon}(y) &= (m(x) - \epsilon)(m(y) - \epsilon) \\ &= m(x)m(y) - \epsilon(m(x) + m(y)) + \epsilon^2 \ . \end{aligned}$$

Now (4.13) implies

$$\begin{aligned} m_n(x)m_n(y) &\leq m(x)m(y) + \epsilon(m(x) + m(y)) + \epsilon^2 \\ &= M_{1,\epsilon}(x)M_{1,\epsilon}(y) + 2\epsilon(m(x) + m(y)) \\ &\leq M_{1,\epsilon}(x)M_{1,\epsilon}(y) + 4\epsilon \ . \end{aligned}$$

Thus we have for $0 \leq \epsilon \leq 1$ that

$$m_n(x)m_n(y) \leq M_{1,\epsilon}(x)M_{1,\epsilon}(y) + 4\epsilon$$

as claimed in the statement of this lemma. It remains to show that $M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 4\epsilon \leq m_n(x)m_n(y)$. By calculations similar to those, that lead to (4.12) and (4.13) we obtain

$$m_n(x) \geq m(x) - \epsilon$$

and

$$m_n(x)m_n(y) \geq m(x)m(y) - \epsilon(m(x) + m(y)) - \epsilon^2 \ . \tag{4.14}$$

Now we will continue and look at $M_{2,\epsilon}$ case by case. Suppose $M_{2,\epsilon}(x) = 1$ and $M_{2,\epsilon}(y) = 1$. This is equivalent to $m(x) \geq 1 - \epsilon$ and $m(y) \geq 1 - \epsilon$. Therefore (4.14) implies

$$\begin{aligned} m_n(x)m_n(y) &\geq (1-\epsilon)^2 - 2\epsilon - \epsilon^2 \\ &= 1 - 4\epsilon \\ &= M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 4\epsilon \ . \end{aligned}$$

Next consider the case $M_{2,\epsilon}(x) = 1$ and $M_{2,\epsilon}(y) = m(y) + \epsilon$. Then we have $m(x) \geq 1 - \epsilon$ and $m(y) \leq 1 - \epsilon$. Moreover we have $M_{2,\epsilon}(x)M_{2,\epsilon}(y) = m(y) + \epsilon$. Hence we obtain the following, according to (4.14)

$$\begin{aligned} m_n(x)m_n(y) &\geq (1-\epsilon)m(y) - \epsilon((1 + (1-\epsilon)) - \epsilon^2 \\ &= m(y) - \epsilon m(y) - 2\epsilon \\ &\geq m(y) - \epsilon(1-\epsilon) - 2\epsilon \\ &\geq m(y) - 3\epsilon \\ &= M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 4\epsilon \ . \end{aligned}$$

By similar calculations we obtain the same result, if $M_{2,\epsilon}(x) = m(x) + \epsilon$ and $M_{2,\epsilon}(y) = 1$. Finally consider the case $M_{2,\epsilon}(x) = m(x) + \epsilon$ and $M_{2,\epsilon}(y) = m(y) + \epsilon$. Then we have $m(x) \geq 1 - \epsilon$ and $m(y) \geq 1 - \epsilon$. Furthermore we have

$$\begin{aligned} M_{2,\epsilon}(x)M_{2,\epsilon}(y) &= (m(x) + \epsilon)(m(y) + \epsilon) \\ &= m(x)m(y) + \epsilon(m(x) + m(y)) + \epsilon^2 \ . \end{aligned}$$

Therefore, using (4.14) again, yields

$$\begin{aligned} m_n(x)m_n(y) &\geq m(x)m(y) - \epsilon(m(x) + m(y)) - \epsilon^2 \\ &= M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 2\epsilon(m(x) + m(y)) - 2\epsilon^2 \end{aligned}$$

$$
\begin{aligned}
&= M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 4\epsilon(1-\epsilon) - 2\epsilon^2 \\
&\geq M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 4\epsilon \ .
\end{aligned}
$$

This concludes the proof. □

Corollary 4.15. *Suppose (M1) and (M2) hold and H is continuous. Then we have for each $0 < \epsilon \leq 1$ and n large enough*

$$
S_n(M_{2,\epsilon}) - 4\epsilon \bar{S}_n(M_{2,\epsilon}) \leq S_n(m(\cdot, \hat{\theta}_n)) \leq S_n(M_{1,\epsilon}) + 4\epsilon \bar{S}_n(M_{1,\epsilon}).
$$

Proof. Consider that we have the following for any $n \geq 1$

$$
\begin{aligned}
S_n(M_{2,\epsilon}) - 4\epsilon \bar{S}_n(M_{2,\epsilon}) = &\sum\sum_{1\leq i<j\leq n} \phi(Z_{i:n}, Z_{j:n})(M_{2,\epsilon}(Z_{i:n})M_{2,\epsilon}(Z_{j:n}) - 4\epsilon) \\
&\times \prod_{k=1}^{i-1}\left[1 - \frac{M_{2,\epsilon}(Z_{k:n})}{n-k+1}\right]\prod_{k=1}^{j-1}\left[1 - \frac{M_{2,\epsilon}(Z_{k:n})}{n-k+1}\right] \ .
\end{aligned}
$$

But according to Lemma 4.14 we have

$$
m(x, \hat{\theta}_n) \leq M_{2,\epsilon}(x) \text{ and } M_{2,\epsilon}(x)M_{2,\epsilon}(y) \leq m(x, \hat{\theta}_n)m(y, \hat{\theta}_n)
$$

for all $x, y \in \mathbb{R}_+$. Hence we obtain

$$
\begin{aligned}
S_n(M_{2,\epsilon}) - 4\epsilon \bar{S}_n(M_{2,\epsilon}) \leq &\sum\sum_{1\leq i<j\leq n} \phi(Z_{i:n}, Z_{j:n}) m(Z_{i:n}, \hat{\theta}_n) m(Z_{j:n}, \hat{\theta}_n) \\
&\times \prod_{k=1}^{i-1}\left[1 - \frac{m(Z_{k:n}, \hat{\theta}_n)}{n-k+1}\right]\prod_{k=1}^{j-1}\left[1 - \frac{m(Z_{k:n}, \hat{\theta}_n)}{n-k+1}\right] \\
= &\ S_n(m(\cdot, \hat{\theta}_n)).
\end{aligned}
$$

Similarly we obtain

$$S_n(M_{1,\epsilon}) + 4\epsilon\bar{S}_n(M_{1,\epsilon}) \geq S_n(m(\cdot,\hat{\theta}_n)).$$

□

Now we are in a position, to identify $S = \lim_{n\to\infty} S_{2,n}^{se}$. The following theorem gives the main statement of this thesis.

Theorem 4.16. *Suppose (A1) through (A4), (M1) and (M2) hold. Then we have*

$$\lim_{n\to\infty} S_n(m(\cdot,\hat{\theta}_n)) = \frac{1}{2}\int_0^{\tau_H}\int_0^{\tau_H} \phi(s,t)F(ds)F(dt)\ .$$

Proof. Consider that we have

$$S_n(M_{2,\epsilon}) - 4\epsilon\bar{S}_n(M_{2,\epsilon}) \leq S_n(m(\cdot,\hat{\theta}_n)) \leq S_n(M_{1,\epsilon}) + 4\epsilon\bar{S}_n(M_{1,\epsilon})$$

by Corollary 4.15 under (M1) and (M2). Next take note of the Radon-Nikodym derivatives (c. f.Dikta (2000), page 8)

$$m(s,\theta_0) = \frac{H^1(ds)}{H(ds)} \text{ and } (1-G(s)) = \frac{H^1(ds)}{F(ds)}\ .$$

Moreover consider that we have

$$\int_0^s \frac{1-m(x,\theta_0)}{1-H(x)} = -\ln(1-G(s))$$

and

$$\int_0^s \frac{\epsilon}{1-H(x)} = -\ln((1-H(s))^\epsilon)$$

according to Dikta (2000). Consider that we have

$$\begin{aligned} M_{1,\epsilon}(x) &= \mathbb{1}_{\{m(x,\theta_0)>\epsilon\}}(m(x,\theta_0)-\epsilon) \\ &\leq m(x,\theta_0)-\epsilon\ . \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \bar{S}(M_{1,\epsilon}) &\leq \frac{1}{2}\int_0^\infty\int_0^\infty \phi(s,t)\exp\left(\int_0^s \frac{1-m(x,\theta_0)}{1-H(x)}+\frac{\epsilon}{1-H(x)}H(dx)\right) \\ &\qquad \times\exp\left(\int_0^t \frac{1-m(x,\theta_0)}{1-H(x)}+\frac{\epsilon}{1-H(x)}H(dx)\right)H(ds)H(dt) \\ &= \frac{1}{2}\int_0^\infty\int_0^\infty \frac{\phi(s,t)}{(1-G(s))(1-G(t))(1-H(s))^\epsilon(1-H(t))^\epsilon}H(ds)H(dt) \\ &= \frac{1}{2}\int_0^\infty\int_0^\infty \frac{\phi(s,t)}{m(s,\theta_0)m(t,\theta_0)(1-H(s))^\epsilon(1-H(t))^\epsilon}F(ds)F(dt)\ . \end{aligned}$$

But by condition (A3), the integral above is finite. Moreover $M_{1,\epsilon}(x)$ is increasing in x, since m is increasing under (A4). Therefore $S(M_{1,\epsilon})$ exists almost surely under (A1) through (A4), by Theorem 3.5. Hence we have that for each $0<\epsilon\leq 1$ we have $S_n(M_{1,\epsilon})+4\epsilon\bar{S}_n(M_{1,\epsilon})\to S(M_{1,\epsilon})+4\epsilon\bar{S}(M_{1,\epsilon})$ w. p. 1 as $n\to\infty$, according to Lemma 4.10 . Next consider that

$$\begin{aligned} S(M_{1,\epsilon})+4\epsilon\bar{S}(M_{1,\epsilon}) \leq \frac{1}{2}\int_0^\infty\int_0^\infty & \frac{\phi(s,t)}{(1-H(s))^\epsilon(1-H(t))^\epsilon} \\ & \times\frac{m(s,\theta_0)m(t,\theta_0)+4\epsilon}{(1-G(s))(1-G(t))}H(ds)H(dt)\ . \end{aligned}$$

By similar arguments we can show that $S_n(M_{2,\epsilon})-4\epsilon\bar{S}_n(M_{2,\epsilon})\to S(M_{2,\epsilon})-4\epsilon\bar{S}(M_{2,\epsilon})$ w. p. 1 as $n\to\infty$ and

$$\begin{aligned} S(M_{2,\epsilon})-4\epsilon\bar{S}(M_{2,\epsilon}) \geq \frac{1}{2}\int_0^\infty\int_0^\infty & \phi(s,t)(1-H(s))^\epsilon(1-H(t))^\epsilon \\ & \times\frac{m(s,\theta_0)m(t,\theta_0)-4\epsilon}{(1-G(s))(1-G(t))}H(ds)H(dt)\ . \end{aligned}$$

Let's summarize the above. So far we have shown, that for $0 < \epsilon \leq 1$ small enough

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty \int_0^\infty \phi(s,t)(1-H(s))^\epsilon(1-H(t))^\epsilon \\
&\qquad\qquad \times \frac{m(s,\theta_0)m(t,\theta_0)-4\epsilon}{(1-G(s))(1-G(t))} H(ds)H(dt) \\
&\leq \liminf_{n\to\infty} \int_0^\infty \phi F_n^{se} F_n^{se} \\
&\leq \limsup_{n\to\infty} \int_0^\infty \phi F_n^{se} F_n^{se} \\
&\leq \frac{1}{2}\int_0^\infty \int_0^\infty \frac{\phi(s,t)}{(1-H(s))^\epsilon(1-H(t))^\epsilon} \\
&\qquad\qquad \times \frac{m(s,\theta_0)m(t,\theta_0)+4\epsilon}{(1-G(s))(1-G(t))} H(ds)H(dt) \ .
\end{aligned}$$

Finally let $\epsilon \searrow 0$ and apply the Monotone Convergence Theorem to obtain that the upper and lower bound converge both to the same limit. In effect

$$\begin{aligned}
&\lim_{\epsilon\searrow 0} \frac{1}{2}\int_0^\infty \int_0^\infty \frac{\phi(s,t)(1-H(s))^\epsilon(1-H(t))^\epsilon}{(1-G(s))(1-G(t))} H(ds)H(dt) \\
&= \frac{1}{2}\int_0^\infty \int_0^\infty \frac{\phi(s,t)m(s,\theta_0)m(t,\theta_0)}{(1-G(s))(1-G(t))} H(ds)H(dt) \\
&= \frac{1}{2}\int_0^\infty \int_0^\infty \phi(s,t)F(ds)F(dt) \\
&= \lim_{\epsilon\searrow 0} \frac{1}{2}\int_0^\infty \int_0^\infty \frac{\phi(s,t)}{(1-G(s))(1-G(t))} \\
&\qquad\qquad \times \frac{1}{(1-H(s))^\epsilon(1-H(t))^\epsilon} H(ds)H(dt) \ .
\end{aligned}$$

Hereby the proof of Theorem 4.16 is concluded. □

Remark 4.17. Note that according to Theorem 4.16

$$S_n(1) = \sum_{1\leq i<j\leq n} W_{i:n}W_{j:n} \to \frac{1}{2}\int_0^{\tau_H}\int_0^{\tau_H} F(ds)F(dt) = \frac{1}{2}F^2(\tau_H) \ .$$

Therefore we have

$$\lim_{n\to\infty} \frac{S_n(\phi)}{S_n(1)} = F^{-2}(\tau_H) \int_0^{\tau_H} \int_0^{\tau_H} \phi(s,t) F(ds) F(dt) \ .$$

which is the normalized version of S_n.

Chapter 5

Simulations

TODO under construction..

Chapter 6

Discussion

- Summarize the SLLN for semiparametric U-Statistics
- Discussion of main assumptions and examples of kernels ϕ and censoring models m
- Future (or perhaps in this thesis): Transfer that property to Prof. Dikta's new estimator using stochastic equivalence
- Future: CLT based on Bose and Sen (2002)

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Appendix: Supplementary Results

Lemma A.1. *For $n \geq 2$ the following statements hold true*

(i)

$$\sum_{k=1}^{n-1} \frac{1}{k} \leq \ln(n-1) + 1 \tag{A1}$$

(ii)

$$\frac{\ln(n-1) + 1}{(n+1)^{\frac{1}{3}}} \leq 3 \tag{A2}$$

Proof. We will start with the proof of part (i). Consider

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{1}{k} &\leq \ln(n-1) + 1 \\ \Leftrightarrow \quad \sum_{k=1}^{n-1} \frac{1}{k} - 1 &\leq \ln(n-1) \\ \Leftrightarrow \quad \sum_{k=2}^{n-1} \frac{1}{k} &\leq \ln(n-1) \end{aligned} \tag{A3}$$

Moreover we have

$$\begin{aligned} \sum_{k=2}^{n-1} \frac{1}{k} &= \sum_{k=2}^{n-1} \int_{k-1}^{k} \frac{1}{k} dx \\ &\leq \sum_{k=2}^{n-1} \int_{k-1}^{k} \frac{1}{x} dx \\ &\leq \sum_{k=2}^{n-1} \ln(k) - \ln(k-1) \\ &\leq \ln(n-1) - \ln(1) \\ &= \ln(n-1) \end{aligned}$$

Thus proving part (i). We will continue with the proof of part (ii). Note that (A2) is equivalent to showing

$$\ln(n-1) + 1 \leq 3(n+1)^{\frac{1}{3}}$$

Since $\ln(n-1) \leq \ln(n+1)$, this will be implied by the following

$$\ln(n+1) + 1 \leq 3(n+1)^{\frac{1}{3}} \tag{A4}$$

It is easy to check that inequality (A4) holds for $n = 2$. Now consider that

$$\frac{d}{dn}(\ln(n+1) + 1) = \frac{1}{n+1}$$

and

$$\frac{d}{dn} 3(n+1)^{\frac{1}{3}} = \frac{1}{(n+1)^{\frac{2}{3}}}$$

to get

$$\frac{d}{dn}(\ln(n+1) + 1) \leq \frac{d}{dn} 3(n+1)^{\frac{1}{3}} \tag{A5}$$

for all $n \geq 2$. Now the result in (ii) follows directly from (A4) and (A5) . $\square$

Appendix: Thoughts on finding weaker assumptions

In Section 3.2, we were only able to show that $S_n(q)$ is a reverse supermartingale under the assumption that q is monotone increasing. To establish the almost sure existence of limits of supermartingale processes, one considers the number of up-crossings of an interval $[a, b]$ by the process. This was done in the famous Upcrossing Theorem by Doob **TODO** cite. During this section we will generalize Doob's Upcrossing Theorem to our framework in order to explore ways to establish weaker assumptions. To get closer to the situation of Doob's Upcrossing Theorem, we define the following quantities. Let $N < \infty$ and define for $1 \leq n \leq N$

$$\tilde{S}_n^N := S_{N-n+1},\ \tilde{\mathcal{F}}_n^N := \mathcal{F}_{N-n+1} \quad \text{and} \quad \tilde{\xi}_n^N := \xi_{N-n+1}\ .$$

Note that $\{\tilde{\mathcal{F}}_n^N\}_{1 \leq n \leq N}$ is now an increasing σ-field in n. Below we will define everything needed, in order to generalize Doob's Upcrossing Theorem.

Definition A.2. Let $N \geq 2$. For $1 \leq n \leq N$ and $a, b \in \mathbb{R}$ with $a < b$, let

$$\begin{aligned}
T_0 &:= 0 \\
T_1 &:= \begin{cases} \min\{1 \leq n \leq N | \tilde{S}_n^N \leq a\} & \text{if } \{1 \leq n \leq N | \tilde{S}_n^N \leq a\} \neq \emptyset \\ N & \text{if } \{1 \leq n \leq N | \tilde{S}_n^N \leq a\} = \emptyset \end{cases} \\
T_2 &:= \begin{cases} \min\{T_1 \leq n \leq N | \tilde{S}_n^N \geq b\} & \text{if } \{T_1 \leq n \leq N | \tilde{S}_n^N \leq a\} \neq \emptyset \\ N & \text{if } \{T_1 \leq n \leq N | \tilde{S}_n^N \geq b\} = \emptyset \end{cases} \\
&\vdots \quad \vdots \quad \vdots
\end{aligned}$$

$$T_{2m-1} := \begin{cases} \min\{T_{2m-2} \leq n \leq N | \tilde{S}_n^N \leq a\} & \text{if } \{T_{2m-2} \leq n \leq N | \tilde{S}_n^N \leq a\} \neq \emptyset \\ N & \text{if } \{T_{2m-2} \leq n \leq N | \tilde{S}_n^N \leq a\} = \emptyset \end{cases}$$

$$T_{2m} := \begin{cases} \min\{T_{2m-1} \leq n \leq N | \tilde{S}_n^N \geq b\} & \text{if } \{T_{2m-1} \leq n \leq N | \tilde{S}_n^N \leq a\} \neq \emptyset \\ N & \text{if } \{T_{2m-1} \leq n \leq N | \tilde{S}_n^N \geq b\} = \emptyset \end{cases} .$$

Now we can define the number of upcrossings of $[a, b]$ by $\tilde{S}_1^N, ..., \tilde{S}_n^N$ as follows:

$$U_n^N[a, b] := \begin{cases} \max\{1 \leq m \leq N | T_{2m} < N\} & \text{if } \{1 \leq m \leq N | T_{2m} < N\} \neq \emptyset \\ 0 & \text{if } \{1 \leq m \leq N | T_{2m} < N\} = \emptyset \end{cases}$$

Furthermore let for $1 \leq k \leq n-1$

$$\epsilon_k := \begin{cases} 0 & \text{if } k < T_1 \\ 1 & \text{if } T_1 \leq k < T_2 \\ 0 & \text{if } T_2 \leq k < T_3 \\ 1 & \text{if } T_3 \leq k < T_4 \\ \dots & \text{if } \dots \end{cases}$$

and define

$$Y_n^N := \tilde{S}_1^N + \sum_{k=1}^{n-1} \epsilon_k (\tilde{S}_{k+1}^N - \tilde{S}_k^N)$$

for $1 \leq n \leq N$.

Let's now explore how $\lim_{N\to\infty} U_N^N[a, b] < \infty$ implies that S must exist amost surely. Suppose for now, that $\lim_{N\to\infty} U_N^N[a, b] < \infty$ and define the set of all ω for which S_n does not converge as

$$\Lambda := \{\omega | S_n(\omega) \text{ does not converge}\} .$$

Consider that can write

$$\begin{aligned}\Lambda &= \{\omega | \liminf_n S_n(\omega) < \limsup_n S_n(\omega)\} \\ &= \bigcup_{a,b\in\mathbb{Q}} \{\omega | \liminf_n S_n(\omega) < a < b < \limsup_n S_n(\omega)\} \ .\end{aligned}$$

Recall that we have $U_N^N[a,b]$, the number of upcrossings of $[a,b]$ by $\tilde{S}_1^N, \dots, \tilde{S}_N^N$. But this is equal to the number of upcrossings of $[a,b]$ by $S_N, \dots, S_1$. Furthermore recall that

$$U_\infty[a,b] = \lim_{N\to\infty} U_N^N[a,b] \ .$$

Consider that for each $\omega \in \{\omega | \liminf_n S_n(\omega) < a < b < \limsup_n S_n(\omega)\}$ we must have $U_\infty[a,b](\omega) = \infty$. This follows directly from the definitions of lim inf and lim sup. Thus we can write

$$\Lambda = \bigcup_{a,b\in\mathbb{Q}} \{\omega | U_\infty[a,b](\omega) = \infty\} = \bigcup_{a,b\in\mathbb{Q}} \Lambda_{a,b}$$

where $\Lambda_{a,b} := \{\omega | U_\infty[a,b](\omega) = \infty\}$. Consequently we get that

$$\mathbb{E}[\mathbb{1}_{\{\Lambda_{a,b}\}} U_\infty[a,b]] = \begin{cases} \infty & \text{if } \mathbb{P}(\Lambda_{a,b}) > 0 \\ 0 & \text{if } \mathbb{P}(\Lambda_{a,b}) = 0 \end{cases} \ . \tag{A6}$$

Note that $U_N^N[a,b]$ is clearly non-decreasing in N. Now if $\lim_{N\to\infty} \mathbb{E}[U_N^N[a,b]] < \infty$, we can apply the Monotone Convergence Theorem to obtain

$$\lim_{N\to\infty} \mathbb{E}[U_N^N[a,b]] = \mathbb{E}[U_\infty[a,b]] < \infty$$

and hence that

$$\mathbb{E}[\mathbb{1}_{\{\Lambda_{a,b}\}} U_\infty[a,b]] \le \mathbb{E}[U_\infty[a,b]] < \infty \ .$$

Now the latter together with (A6) implies that $\mathbb{P}(\Lambda_{a,b}) = 0$. Therefore we have

$$\mathbb{P}(\Lambda) = \mathbb{P}\left(\bigcup_{a,b\in\mathbb{Q}} \Lambda_{a,b}\right) = \sum_{a,b\in\mathbb{Q}} \mathbb{P}(\Lambda_{a,b}) = 0 \ .$$

The following Lemmas show how Doob's Upcrossing Theorem can be adapted to our framework. We will show that $\mathbb{E}[U_n^N[a,b]]$ is bounded above by $\mathbb{E}[Y_n^N]/(b-a)$.

Lemma A.3. *For $1 \le n \le N$ we have*

$$\mathbb{E}[U_n^N[a,b]] \le \frac{\mathbb{E}[Y_n^N]}{b-a} \ .$$

Proof. Consider for $1 \le n \le N$ and $N \ge 2$

$$\begin{aligned} Y_n^N &= \tilde{S}_1^N + \sum_{k=1}^{n-1} \epsilon_k (\tilde{S}_{k+1}^N - \tilde{S}_k^N) \\ &= \tilde{S}_1^N + \sum_{k=1}^{n} (\tilde{S}_{T_{2k}}^N - \tilde{S}_{T_{2k-1}}^N) \\ &\ge \sum_{k=1}^{n} (\tilde{S}_{T_{2k}}^N - \tilde{S}_{T_{2k-1}}^N) \end{aligned}$$

by definition of ϵ_k. The latter inequality above holds, since $\tilde{S}_1^N \ge 0$. Note that by definition of $T_1, T_2, \ldots$ we have

$$\sum_{k=1}^{n} (\tilde{S}_{T_{2k}}^N - \tilde{S}_{T_{2k-1}}^N) \ge (b-a) U_n^N[a,b] \ .$$

From here the assertion follows directly. $\square$

The following lemma provides a representation for the expectation of the process Y_N^n.

Lemma A.4. *For $1 \leq n \leq N$ let*

$$Y_n^N := \tilde{S}_1^N + \sum_{k=1}^{n-1} \epsilon_k (\tilde{S}_{k+1}^N - \tilde{S}_k^N)$$

with

$$\epsilon_k := \begin{cases} 1 & (\tilde{S}_1^N, \ldots, \tilde{S}_k^N) \in B_k \\ 0 & \textit{otherwise} \end{cases}$$

for $k = 1, \ldots, n-1$. Here B_k is an arbitrary set in $\mathfrak{B}(\mathbb{R}^k)$. Then we have

$$\mathbb{E}[Y_n^N] = \mathbb{E}[\tilde{S}_n^N] - \sum_{k=1}^{n-1} \mathbb{E}\left[(1-\epsilon_k)\left(\mathbb{E}[\tilde{S}_{k+1}^N | \tilde{\mathcal{F}}_k^N] - \tilde{S}_k^N\right)\right] \ . \tag{A7}$$

Proof. Consider for $1 \leq n \leq N$ and $N \geq 2$

$$\begin{aligned}
&\tilde{S}_{n+1}^N - Y_{n+1}^N \\
&= (1-\epsilon_1)(\tilde{S}_2^N - \tilde{S}_1^N) + (1-\epsilon_2)(\tilde{S}_3^N - \tilde{S}_2^N) + ... + (1-\epsilon_k)(\tilde{S}_{n+1}^N - \tilde{S}_n^N) \\
&= (\tilde{S}_n^N - Y_n^N) + (1-\epsilon_n)(\tilde{S}_{n+1}^N - \tilde{S}_n^N) \ .
\end{aligned}$$

Conditioning on $\tilde{\mathcal{F}}_n^N$ on both sides yields

$$\mathbb{E}[\tilde{S}_{n+1}^N - Y_{n+1}^N | \tilde{\mathcal{F}}_n^N] = \tilde{S}_n^N - Y_n^N + (1-\epsilon_n)\left(\mathbb{E}[(\tilde{S}_{n+1}^N) | \tilde{\mathcal{F}}_n^N] - \tilde{S}_n^N\right) \ .$$

Now taking expectations on both sides yields

$$\mathbb{E}[\tilde{S}_{n+1}^N - Y_{n+1}^N] \geq \mathbb{E}[\tilde{S}_n^N - Y_n^N] + \mathbb{E}\left[(1-\epsilon_n)\left(\mathbb{E}[\tilde{S}_{n+1}^N | \tilde{\mathcal{F}}_n^N] - \tilde{S}_n^N\right)\right] \ .$$

Note that

$$\mathbb{E}[\tilde{S}_2^N - Y_2^N] = \mathbb{E}[\tilde{S}_1^N - Y_1^N] + \mathbb{E}\left[(1-\epsilon_1)\left(\mathbb{E}[\tilde{S}_2^N | \tilde{\mathcal{F}}_1^N] - \tilde{S}_1^N\right)\right]$$

$$= \mathbb{E}\left[(1-\epsilon_1)\left(\mathbb{E}[\tilde{S}_2^N|\tilde{\mathcal{F}}_1^N] - \tilde{S}_1^N\right)\right]$$

since $Y_1^N = \tilde{S}_1^N$. Moreover we have

$$\begin{aligned}
\mathbb{E}[\tilde{S}_3^N - Y_3^N] &= \mathbb{E}[\tilde{S}_2^N - Y_2^N] + \mathbb{E}\left[(1-\epsilon_2)\left(\mathbb{E}[\tilde{S}_3^N|\tilde{\mathcal{F}}_2^N] - \tilde{S}_2^N\right)\right] \\
&= \mathbb{E}\left[(1-\epsilon_1)\left(\mathbb{E}[\tilde{S}_2^N|\tilde{\mathcal{F}}_1^N] - \tilde{S}_1^N\right)\right] \\
&\quad + \mathbb{E}\left[(1-\epsilon_2)\left(\mathbb{E}[\tilde{S}_3^N|\tilde{\mathcal{F}}_2^N] - \tilde{S}_2^N\right)\right] \\
&\dots \\
\mathbb{E}[\tilde{S}_n^N - Y_n^N] &= \sum_{k=1}^{n-1} \mathbb{E}\left[(1-\epsilon_k)\left(\mathbb{E}[\tilde{S}_{k+1}^N|\tilde{\mathcal{F}}_k^N] - \tilde{S}_k^N\right)\right] \ .
\end{aligned}$$

Hence we get

$$\mathbb{E}[Y_n^N] = \mathbb{E}[\tilde{S}_n^N] - \sum_{k=1}^{n-1} \mathbb{E}\left[(1-\epsilon_k)\left(\mathbb{E}[\tilde{S}_{k+1}^N|\tilde{\mathcal{F}}_k^N] - \tilde{S}_k^N\right)\right] \ .$$

□

Remark A.5. Note that we have $Y_1^N = \tilde{S}_1^N$, as the sum in the definition above is in this case empty and hence treated as zero. Moreover note that we have $Y_{n+1}^N = \tilde{S}_{n+1}^N$ if $\epsilon_k = 1$ for all $1 \le k \le n$.

The Lemma below establishes an upper bound for $\mathbb{E}[Y_N^N]$ in terms of Q_{ij}^{N-k+1}, as defined in Lemma 3.1.

Lemma A.6. *We have for $N \geq 2$*

$$\mathbb{E}[Y_N^N] \le \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \alpha_{N-k+1} \tag{A8}$$

where

$$\alpha_{N-k+1} := \sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[\phi(Z_{i:N-k+1}, Z_{j:N-k+1}) W_{i:N-k+1} W_{j:N-k+1} (Q_{i,j}^{N-k+1} - 1)\right] \ .$$

Proof. Combining Lemmas A.4 and A.3 yields the following for $n \le N$

$$(b-a)\mathbb{E}[U_n[a,b]] \le \mathbb{E}[Y_n^N] = \mathbb{E}[\tilde{S}_n^N] - \sum_{k=1}^{n-1} \mathbb{E}[(1-\epsilon_k)\left(\mathbb{E}[\tilde{S}_{k+1}^N|\mathcal{F}_k^N] - \tilde{S}_k^N\right)] \ .$$

Moreover we get from Lemma 3.1

$$\begin{aligned}
\mathbb{E}[\tilde{S}_{k+1}^N|\tilde{\mathcal{F}}_k^N] &= \mathbb{E}[S_{N-k}|\mathcal{F}_{N-k+1}] \\
&= \sum\sum_{1\le i<j\le N-k+1} \phi(Z_{i:N-k+1}, Z_{j:N-k+1}) W_{i:N-k+1} W_{j:N-k+1} Q_{i,j}^{N-k+1} \ .
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\mathbb{E}[Y_N^N] &= \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1} \mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N|\mathcal{F}_k^N] - \tilde{S}_k^N] \\
&= \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right. \\
&\qquad\qquad \left.\times\ W_{i:N-k+1} W_{j:N-k+1} (Q_{i,j}^{N-k+1} - 1)\right] \\
&\le \mathbb{E}[\tilde{S}_N^N] + \left| \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right.\right. \\
&\qquad\qquad \left.\left.\times\ W_{i:N-k+1} W_{j:N-k+1} (Q_{i,j}^{N-k+1} - 1)\right]\right| \\
&\le \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} \left|\mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right.\right. \\
&\qquad\qquad \left.\left.\times\ W_{i:N-k+1} W_{j:N-k+1} (Q_{i,j}^{N-k+1} - 1)\right]\right| \ .
\end{aligned}$$

Now using Jensen's inequality yields

$$\begin{aligned}\mathbb{E}[Y_N^N] &\leq \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\leq i<j\leq N-k+1} \mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right. \\ &\qquad\qquad \left. \times\, W_{i:N-k+1}W_{j:N-k+1} \cdot |(Q_{i,j}^{N-k+1} - 1)|\right] \\ &\leq \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\leq i<j\leq N-k+1} \mathbb{E}\left[\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right. \\ &\qquad\qquad \left. \times\, W_{i:N-k+1}W_{j:N-k+1} \cdot |(Q_{i,j}^{N-k+1} - 1)|\right] \ .\end{aligned}$$

The latter inequality above holds, because $1 - \epsilon_k \leq 1$ for all $k \leq N - 1$. $\square$

In addition to the almost sure existence of $S(q)$ we need the following statement

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \mathbb{E}[S_n]$$

in order to identify $S(q)$ in Lemma 4.10. This could be established by the following Lemma.

Lemma A.7. *The following statement holds true:*

$$S_\infty = \lim_{n\to\infty} \mathbb{E}[S_n|\mathcal{F}_\infty] = \lim_{n\to\infty} \mathbb{E}[S_n]$$

almost surely, if the limits above exist.

Proof. Let $a > 0$ and note that, since $S_n \to S$ almost surely as $n \to \infty$, we have

$$\lim_{n\to\infty} \min(S_n, a) = \min(S, a)$$

almost surely, since $\min(\cdot, a)$ is continuous (see van der Vaart (2000), Theorem 2.3). Now $\min(S_n, a)$ is bounded by a. Hence applying the Dominated Convergence

Theorem yields

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}[\min(S_n,a)|\mathcal{F}_\infty] &= \mathbb{E}[\lim_{n\to\infty}\min(S_n,a)|\mathcal{F}_\infty]\\ &= \mathbb{E}[\min(S_\infty,a)|\mathcal{F}_\infty]\ .\end{aligned}$$

Note that S_k is measurable with respect to $\mathcal{F}_n$ whenever $k \geq n$, therefore S_∞ must be $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$. Consequently S_∞ must be F_∞-measurable. Moreover, for $a \in \mathbb{R}$, $\min(\cdot, a)$ is a continuous function. Thus $\min(S_\infty, a)$ is $\mathcal{F}_\infty$-measurable as well. Hence

$$\lim_{n\to\infty}\mathbb{E}[min(S_n,a)|\mathcal{F}_\infty] = \min(S_\infty,a)$$

almost surely. Thus we have

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}[S_n|\mathcal{F}_\infty] &= \lim_{n\to\infty}\lim_{a\to\infty}\mathbb{E}[\min(S_n,a)|\mathcal{F}_\infty]\\ &= \lim_{a\to\infty}\lim_{n\to\infty}\mathbb{E}[\min(S_n,a)|\mathcal{F}_\infty]\\ &= \lim_{a\to\infty}\min(S_\infty,a)\\ &= S_\infty\ . \end{aligned} \tag{A9}$$

almost surely. Moreover we obtain

$$\mathbb{E}[S_n|\mathcal{F}_\infty] = \mathbb{E}[S_n]$$

for all n, by applying Lemma 3.4. Now the latter together with (A9) implies the statement of the lemma. $\square$

Curriculum Vitae

Jan Höft

Stiftherrenstraße 5, 52428 Jülich,

Germany

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email: jan.hoeft86@gmail.com

mobile: +49-176-45933938

Date of Birth: December 27, 1986

Place of Birth: Templin, Brandenburg, Germany

Education:

September 2006 - May 2009	Mathematisch-Technischer Assistent, IHK zu Köln
September 2006 - May 2009	B.Sc. in Scientific Programming at FH-Aachen, University of Applied Sciences **Thesis title:** Rissanalyse Brennstoffzellenmembran
September 2009 - May 2011	M.Sc. in Technomathematik at FH-Aachen, University of Applied Sciences

	Thesis title: U-Statistics under the semiparametric Random Censorship Model
Expected: May 2018	Ph.D. in Mathematics, University of Wisconsin - Milwaukee
	Dissertation title: Large Sample properties of U-Statistics under semiparametric Random Censorship

Teaching & Research Experience:

2006 - 2009	Apprenticeship as Mathematisch-Technischer Assistent at Forschungszentrum Jülich, Instut für Energietechnik - Brennstoffzellen
2010 - May 2011	Teaching Assistant, FH-Aachen, University of Applied Sciences
2011 - 2012	Teaching Assistant, University of Wisconsin - Milwaukee
2012 - 2015	Research Assistant, Atmospheric Sciences, University of Wisconsin - Milwaukee

Awards & Fellowships:

2011 - 2012	Teaching Assistant Fellowship, University of Wisconsin - Milwaukee
2012 - 2015	Research Assistant Fellowship, Atmospheric Sciences, University of Wisconsin - Milwaukee
2011, 2012	Chancellor's Award (4x 1500$), University of Wisconsin, Milwaukee

Major Professor Date