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Chapter 1

Notation and assumptions

In this chapter we will state the main definitions and assumptions used throughout this work. We will start by defining the estimator to be considered and introduce all necessary notation for the remaining chapters.

Recall the following definition

$$W_{i:n}^{se} = \frac{m(Z_{i:n}, \hat{\theta}_n)}{n-i+1} \prod_{j=1}^{i-1} \left(1 - \frac{m(Z_{j:n}, \hat{\theta}_n)}{n-j+1}\right)$$

Now we define for $n \geq 2$

$$S_{2,n}^{se} = \sum\sum_{1 \leq i < j \leq n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n}^{se} W_{j:n}^{se}$$

This process will be called semiparametric U-Statistic of degree 2 throughout this thesis. Furthermore define

$$W_{i:n}(q) = \frac{q(Z_{i:n})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right]$$

and

$$S_n(q) = \sum\sum_{1 \leq i < j \leq n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n}(q) W_{j:n}(q)$$

Example 1.1. Let $q(Z_{i:n}) = \delta_{[i:n]}$ for $1 \leq i \leq n$. Then $W_{i:n}(q) = W_{i:n}^{km}$ and therefore

$$S_n(q) = S_{2,n}^{km}$$

Example 1.2. Let $q(t) = m(t, \hat{\theta}_n)$ for $t \in \mathbb{R}^+$. Then $W_{i:n}(q) = W_{i:n}^{se}$ and therefore

$$S_n(q) = S_{2,n}^{se}$$

Moreover define

$$\mathcal{F}_n = \sigma\{Z_{1:n}, \ldots, Z_{n:n}, Z_{n+1}, Z_{n+2}, \ldots\}$$

Throughout this work we will write $S_n := S_n(q)$ and $W_{i:n} := W_{i:n}(q)$. for $1 \leq i \leq n$.

The following assumptions will be needed throughout this work, in order to prove the SLLN for S_n.

(A1) The kernel $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is measurable, non-negative and symmetric in its arguments. In effect $\phi(s,t) = \phi(t,s)$ for all $s, t \in \mathbb{R}_+$.

(A2) The d. f. H is continuous and concentrated on the non-negative real line.

(A3) For $s, t \in \mathbb{R}_+$ the following statement holds true

$$\int_0^s \int_0^t \frac{\phi(s,t)}{m(s,\theta_0)m(t,\theta_0)(1-H(s))^{\epsilon}(1-H(t))^{\epsilon}} F(dt)F(ds) < \infty$$

for some $0 < \epsilon \leq 1$.

(A4) There exists $c_1 < \infty$ s. t. $\sup_x (m \circ H^{-1})'(x) \leq c_1$.

(A5) We have $m \circ H^{-1}(1) = 1$.

We will need the following assumptions about the Censoring Model m and the Maximum Likelihood estimate $\hat{\theta}_n$:

(M1) $\hat{\theta}_n$ is measurable and tends to θ_0

(M2) For any $\epsilon > 0$ there exists a neighborhood $V(\epsilon, \theta_0) \subset \Theta$ of θ_0 s. t. for all $\theta \in V(\epsilon, \theta_0)$

$$\sup_{x \geq 0} |m(x, \theta) - m(x, \theta_0)| < \epsilon$$

Chapter 2

Modifying the Martingale Convergence Theorem

Define $\Pi_n := \{\pi = (\pi_1, \dots, \pi_n) | \pi \text{ is permutation of } \{1, \dots, n\}\}$. Now let for $\pi \in \Pi_n$

$$I_n^{\pi} := \mathbb{I}\{R_n(Z_l) = \pi_l, 1 \le l \le n\} = \prod_{l=1}^{n} \mathbb{I}\{R_n(Z_l) = \pi_l\}$$

Lemma 2.1. *Suppose condition (A3) is satisfied. Then the following statement holds true*

$$\lim_{n\to\infty} \sum\sum_{1\le i<j\le n+1} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n}) W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}} < \infty$$

Proof. Let (A3) be satisfied. Consider the following

$$\begin{aligned}
&\sum\sum_{1\le i<j\le n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n}) W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}} \\
&= \sum\sum_{1\le i<j\le n} \mathbb{E}\Bigg[\phi^2(Z_{i:n}, Z_{j:n}) \frac{q^2(Z_{i:n})}{(n-i+1)^2} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right]^2 \\
&\qquad\qquad \times \frac{q^2(Z_{j:n})}{(n-j+1)^2} \prod_{l=1}^{j-1} \left[1 - \frac{q(Z_{l:n})}{n-l+1}\right]^2\Bigg]^{\frac{1}{2}} \\
&\le \sum\sum_{1\le i<j\le n} \mathbb{E}\Bigg[\phi^2(Z_{i:n}, Z_{j:n}) \frac{q^2(Z_{i:n})}{(n-i+1)^2} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right] \\
&\qquad\qquad \times \frac{q^2(Z_{j:n})}{(n-j+1)^2} \prod_{l=1}^{j-1} \left[1 - \frac{q(Z_{l:n})}{n-l+1}\right]\Bigg]^{\frac{1}{2}} .
\end{aligned} \tag{2.1}$$

Next we will modify the products above. Recall the following definition

$$B_n(s) := \prod_{k=1}^{n} \left[1 + \frac{1 - q(Z_k)}{n - R_{k,n}}\right]^{\mathbb{I}\{Z_k < s\}}$$

and note that for $i = 1, \ldots, n$

$$
\begin{aligned}
B_n(Z_{i:n}) &= \prod_{k=1}^{n} \left[1 + \frac{1 - q(Z_k)}{n - R_{k,n}} \right]^{\mathbb{I}\{Z_k < Z_{i:n}\}} \\
&= \prod_{k=1}^{n} \left[1 + \frac{1 - q(Z_{k:n})}{n - k} \right]^{\mathbb{I}\{Z_{k:n} < Z_{i:n}\}} \\
&= \prod_{k=1}^{i-1} \left[1 + \frac{1 - q(Z_{k:n})}{n - k} \right] .
\end{aligned}
$$

Moreover consider that for $i = 1, \ldots, n$

$$
\begin{aligned}
\frac{1}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1} \right] &= \frac{1}{n-i+1} \prod_{k=1}^{i-1} \left[\frac{n-k+1-q(Z_{k:n})}{n-k+1} \right] \\
&= \frac{1}{n-i+1} \prod_{k=1}^{i-1} \left[\frac{n-k+1-q(Z_{k:n})}{n-k} \cdot \frac{n-k}{n-k+1} \right] \\
&= \frac{1}{n} \prod_{k=1}^{i-1} \left[1 + \frac{1-q(Z_{k:n})}{n-k} \right] \\
&= \frac{B_n(Z_{i:n})}{n} .
\end{aligned}
$$

Now combining the above with (2.1) yields

$$
\begin{aligned}
&\sum\sum_{1 \le i < j \le n} \mathbb{E} \left[\phi^2(Z_{i:n}, Z_{j:n}) W_{i:n}^2 W_{j:n}^2 \right]^{\frac{1}{2}} \\
&\le \sum\sum_{1 \le i < j \le n} \mathbb{E} \left[\phi^2(Z_{i:n}, Z_{j:n}) \frac{q^2(Z_{i:n})}{n(n-i+1)} \frac{q^2(Z_{j:n})}{n(n-j+1)} B_n(Z_{i:n}) B_n(Z_{j:n}) \right]^{\frac{1}{2}} \\
&\le \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\mathbb{E} \left[\phi^2(Z_{i:n}, Z_{j:n}) q^2(Z_{i:n}) q^2(Z_{j:n}) B_n(Z_{i:n}) B_n(Z_{j:n}) \right]^{\frac{1}{2}}}{(n-i+1)^{\frac{1}{2}} (n-j+1)^{\frac{1}{2}}} .
\end{aligned}
$$

Consider that

$$
\begin{aligned}
&\mathbb{E} \left[\phi^2(Z_{i:n}, Z_{j:n}) q^2(Z_{i:n}) q^2(Z_{j:n}) B_n(Z_{i:n}) B_n(Z_{j:n}) \right]^{\frac{1}{2}} \\
&\le \max \left(1, \mathbb{E} \left[\phi^2(Z_{i:n}, Z_{j:n}) q^2(Z_{i:n}) q^2(Z_{j:n}) B_n(Z_{i:n}) B_n(Z_{j:n}) \right]^{\frac{1}{2}} \right)
\end{aligned}
$$

$$\leq \max\left(1, \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n})q^2(Z_{i:n})q^2(Z_{j:n})B_n(Z_{i:n})B_n(Z_{j:n})\right]\right)$$
$$\leq 1 + \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n})q^2(Z_{i:n})q^2(Z_{j:n})B_n(Z_{i:n})B_n(Z_{j:n})\right]$$

Hence we obtain

$$\begin{aligned}
&\sum\sum_{1\leq i<j\leq n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n})W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1 + \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n})q^2(Z_{i:n})q^2(Z_{j:n})B_n(Z_{i:n})B_n(Z_{j:n})\right]}{(n-i+1)^{\frac{1}{2}}(n-j+1)^{\frac{1}{2}}} \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}\left[\frac{\phi^2(Z_{i:n}, Z_{j:n})q^2(Z_{i:n})q^2(Z_{j:n})B_n(Z_{i:n})B_n(Z_{j:n})}{(n-i+1)^{\frac{1}{2}}(n-j+1)^{\frac{1}{2}}}\right] \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{(n-i+1)^{\frac{1}{2}}(n-j+1)^{\frac{1}{2}}} \ .
\end{aligned}$$

Next consider that we have

$$\begin{aligned}
\sum_{j=1}^{n} \frac{1}{(n-j+1)^{\frac{1}{2}}} &= \sum_{j=1}^{n} \frac{1}{j^{\frac{1}{2}}} \\
&= 1 + \sum_{j=2}^{n} \int_{j-1}^{j} \frac{1}{\sqrt{j}} dx \\
&\leq 1 + \sum_{j=2}^{n} \int_{j-1}^{j} \frac{1}{\sqrt{x}} dx \\
&\leq 2\sqrt{n} \qquad (2.2)
\end{aligned}$$

for all $n \geq 1$. Therefore we get

$$\begin{aligned}
&\sum\sum_{1\leq i<j\leq n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n})W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}\left[\frac{\phi^2(Z_{i:n}, Z_{j:n})q^2(Z_{i:n})q^2(Z_{j:n})B_n(Z_{i:n})B_n(Z_{j:n})}{(n-i+1)^{\frac{1}{2}}(n-j+1)^{\frac{1}{2}}}\right] + 4 \ .
\end{aligned}$$

Let's now consider the double sum above. We have

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\frac{\phi^2(Z_{i:n},Z_{j:n})q^2(Z_{i:n})q^2(Z_{j:n})B_n(Z_{i:n})B_n(Z_{j:n})}{(n-i+1)^{\frac{1}{2}}(n-j+1)^{\frac{1}{2}}}\right]\\
&=\frac{1}{n}\mathbb{E}\left[\sum_{\pi\in\Pi_n} I_n^\pi\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\phi^2(Z_{i:n},Z_{j:n})q^2(Z_{i:n})q^2(Z_{j:n})B_n(Z_{i:n})B_n(Z_{j:n})}{(n-i+1)^{\frac{1}{2}}(n-j+1)^{\frac{1}{2}}}\right]\\
&=\frac{1}{n}\mathbb{E}\left[\sum_{\pi\in\Pi_n} I_n^\pi\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\phi^2(Z_i,Z_j)q^2(Z_i)q^2(Z_j)B_n(Z_i)B_n(Z_j)}{(n-\pi_i+1)^{\frac{1}{2}}(n-\pi_j+1)^{\frac{1}{2}}}\right]\\
&=\frac{1}{n}\sum_{\pi\in\Pi_n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\mathbb{E}\left[I_n^\pi\phi^2(Z_i,Z_j)q^2(Z_i)q^2(Z_j)B_n(Z_i)B_n(Z_j)\right]}{(n-\pi_i+1)^{\frac{1}{2}}(n-\pi_j+1)^{\frac{1}{2}}}
\end{aligned}
$$

According to Lemma 2.3, we have

$$
\begin{aligned}
&\mathbb{E}\left[I_n^\pi\phi^2(Z_i,Z_j)q^2(Z_i)q^2(Z_j)B_n(Z_i)B_n(Z_j)\right]\\
&=\mathbb{E}\left[I_n^\pi\phi^2(Z_1,Z_2)q^2(Z_1)q^2(Z_2)B_n(Z_1)B_n(Z_2)\right]\ .
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\frac{\phi^2(Z_{i:n},Z_{j:n})q^2(Z_{i:n})q^2(Z_{j:n})B_n(Z_{i:n})B_n(Z_{j:n})}{(n-i+1)^{\frac{1}{2}}(n-j+1)^{\frac{1}{2}}}\right]\\
&=\frac{1}{n}\sum_{\pi\in\Pi_n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\mathbb{E}\left[I_n^\pi\phi^2(Z_1,Z_2)q^2(Z_1)q^2(Z_2)B_n(Z_1)B_n(Z_2)\right]}{(n-\pi_i+1)^{\frac{1}{2}}(n-\pi_j+1)^{\frac{1}{2}}}\\
&=\frac{1}{n}\sum_{\pi\in\Pi_n}\mathbb{E}\left[I_n^\pi\phi^2(Z_1,Z_2)q^2(Z_1)q^2(Z_2)B_n(Z_1)B_n(Z_2)\right]\\
&\qquad+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{(n-\pi_i+1)^{\frac{1}{2}}(n-\pi_j+1)^{\frac{1}{2}}}\\
&\le 4\cdot\sum_{\pi\in\Pi_n}\mathbb{E}\left[I_n^\pi\phi^2(Z_1,Z_2)q^2(Z_1)q^2(Z_2)B_n(Z_1)B_n(Z_2)\right]\\
&=4\cdot\mathbb{E}\left[\phi^2(Z_1,Z_2)q^2(Z_1)q^2(Z_2)B_n(Z_1)B_n(Z_2)\right]
\end{aligned}
$$

Since q and ϕ are Borel-measurable, we can apply Lemma 2.4 to obtain

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\frac{\phi^2(Z_{i:n},Z_{j:n})q^2(Z_{i:n})q^2(Z_{j:n})B_n(Z_{i:n})B_n(Z_{j:n})}{(n-i+1)^{\frac{1}{2}}(n-j+1)^{\frac{1}{2}}}\right]$$
$$\leq 8\cdot\mathbb{E}\left[\phi^2(Z_1,Z_2)q^2(Z_1)q^2(Z_2)(\Delta_{n-2}(Z_1,Z_2)+\bar{\Delta}_{n-2}(Z_1,Z_2))\right]\ .$$

Note that $0\leq C_n(s)\leq 1$ for all $n\geq 1$ and $s\in\mathbb{R}_+$. Thus

$$\bar{\Delta}_n(s,t)=\mathbb{E}[C_n(s)D_n(s,t)]\leq\Delta_n(s,t)$$

for all $n\geq 1$ and $s<t$. Therefore we get

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\frac{\phi^2(Z_{i:n},Z_{j:n})q^2(Z_{i:n})q^2(Z_{j:n})B_n(Z_{i:n})B_n(Z_{j:n})}{(n-i+1)^{\frac{1}{2}}(n-j+1)^{\frac{1}{2}}}\right]$$
$$\leq 16\cdot\mathbb{E}\left[\phi^2(Z_1,Z_2)q^2(Z_1)q^2(Z_2)\Delta_{n-2}(Z_1,Z_2)\right]\ .$$

By virtue of Lemma 2.8, we have

$$\Delta_n(s,t)=\mathbb{E}[D_n(s,t)]=\mathbb{E}[D_n(s,t)|\mathcal{F}_\infty]\nearrow D(s,t)\ .$$

But this implies in particular that $\mathbb{E}[D_n(s,t)]\leq D(s,t)$ for all $n\geq 1$. Hence

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\frac{\phi^2(Z_{i:n},Z_{j:n})q^2(Z_{i:n})q^2(Z_{j:n})B_n(Z_{i:n})B_n(Z_{j:n})}{(n-i+1)^{\frac{1}{2}}(n-j+1)^{\frac{1}{2}}}\right]$$
$$\leq 16\cdot\mathbb{E}\left[\phi^2(Z_1,Z_2)q^2(Z_1)q^2(Z_2)D(Z_1,Z_2)\right]\ .$$

by the Monotone Convergence Theorem. Next consider that for each $s<t$ s. t. $H(t)<1$

$$D(s,t)=\exp\left(2\int_0^s\frac{1-q(z)}{1-H(z)}H(dz)+\int_s^t\frac{1-q(z)}{1-H(z)}H(dz)\right)$$

$$
\begin{aligned}
&\leq \exp\left(2\int_0^s \frac{1}{1-H(z)} H(dz) + \int_s^t \frac{1}{1-H(z)} H(dz)\right) \\
&= \exp\left(\int_0^s \frac{1}{1-H(z)} H(dz) + \int_0^t \frac{1}{1-H(z)} H(dz)\right) \\
&= \exp\left(-\ln(1-H(s)) - \ln(1-H(t))\right) \\
&= \frac{1}{(1-H(s))(1-H(t))} \,.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\frac{\phi^2(Z_{i:n}, Z_{j:n})q^2(Z_{i:n})q^2(Z_{j:n})B_n(Z_{i:n})B_n(Z_{j:n})}{(n-i+1)^{\frac{1}{2}}(n-j+1)^{\frac{1}{2}}}\right] \\
&\leq 16\cdot\mathbb{E}\left[\frac{\phi^2(Z_1, Z_2)}{(1-H(Z_1))(1-H(Z_2))}\right] \\
&\leq 16\cdot\left\{\int_0^{Z_1}\int_0^{Z_2}\frac{\phi^2(s,t)}{(1-H(s))(1-H(t))}H(ds)H(dt)\right\} \,.
\end{aligned}
$$

Now taking into consideration the Radon-Nikodym derivatives (c. f. Dikta (2000), page 8)

$$
\frac{H^1(dt)}{H(dt)} = m(t,\theta_0) \text{ and } \frac{H^1(dt)}{F(dt)} = 1 - G(t),
$$

yields

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\frac{\phi^2(Z_{i:n}, Z_{j:n})q^2(Z_{i:n})q^2(Z_{j:n})B_n(Z_{i:n})B_n(Z_{j:n})}{(n-i+1)^{\frac{1}{2}}(n-j+1)^{\frac{1}{2}}}\right] \\
&\leq 16\cdot\left\{\int_0^{Z_1}\int_0^{Z_2}\frac{\phi^2(s,t)}{m(s,\theta_0)m(t,\theta_0)(1-H(s))(1-H(t))}H^1(ds)H^1(dt)\right\}^{\frac{1}{2}} \\
&= 16\cdot\left\{\int_0^{Z_1}\int_0^{Z_2}\frac{\phi^2(s,t)}{m(s,\theta_0)m(t,\theta_0)(1-F(s))(1-F(t))}F(ds)F(dt)\right\}^{\frac{1}{2}}
\end{aligned}
$$

since $1 - H(x) = (1 - F(x))(1 - G(x))$ for all $x \in \mathbb{R}_+$. But now the integral above is finite under **TODO** adjust condition (A3).

Therefore can finally conclude

$$\lim_{n\to\infty}\sum\sum_{1\leq i<j\leq n}\mathbb{E}\left[\phi^2(Z_{i:n},Z_{j:n})W_{i:n}^2W_{j:n}^2\right]^{\frac{1}{2}}<\infty\ .$$

□

2.1 Modified Lemmas from my thesis

The following is an adapted version of Lemma 4.6 of my thesis.

Lemma 2.2. *Let $s\neq t$. Then the conditional expectation*

$$\mathbb{E}[I_n^\pi B_n(s)B_n(t)|Z_i=s,Z_j=t]$$

is independent of i,j and hence

$$\mathbb{E}[I_n^\pi B_n(s)B_n(t)|Z_i=s,Z_j=t]=\mathbb{E}[I_n^\pi B_n(s)B_n(t)|Z_1=s,Z_2=t]$$

holds almost surely.

Proof. First let $s<t$. For the sake of notational simplicity denote $s_k^n:=\mathbb{I}\{Z_{k:n}<s\}$ and $t_k^n:=\mathbb{I}\{s\leq Z_{k:n}<t\}$ and consider

$$\begin{aligned}
&\mathbb{E}\left[I_n^\pi B_n(s)B_n(t)|Z_i=s,Z_j=t\right]\\
&=\mathbb{E}\left[I_n^\pi\prod_{k=1}^{n}\left(1+\frac{1-q(Z_{k:n})}{n-k}\right)^{2s_k^n+t_k^n}|Z_i=s,Z_j=t\right]\\
&=\mathbb{E}\left[\sum_{\alpha=1}^{n-1}\sum_{\beta=2}^{n}\mathbb{I}\{Z_{\alpha:n}=Z_i\}\mathbb{I}\{Z_{\beta:n}=Z_j\}I_n^\pi\left(1+\frac{1-q(Z_i)}{n-\alpha}\right)\right.\\
&\qquad\times\prod_{k=1}^{\alpha-1}\left(1+\frac{1-q(Z_{k:n})}{n-k}\right)^{2s_k^n+t_k^n}
\end{aligned}$$

$$\times \prod_{k=\alpha+1}^{\beta-1} \left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{2s_k^n + t_k^n}$$
$$\left. \times \prod_{k=\beta+1}^{n} \left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{2s_k^n + t_k^n} |Z_i = s, Z_j = t \right]$$

since $s_\alpha^n = 0$, $t_\alpha^n = 1$, $s_\beta^n = 0$ and $t_\beta^n = 0$. Moreover we have

$$\begin{cases} s_k^n = 1 \text{ and } t_k^n = 0 & \text{if } k < \alpha \\ s_k^n = 0 \text{ and } t_k^n = 1 & \text{if } \alpha < k < \beta \\ s_k^n = 0 \text{ and } t_k^n = 0 & \text{if } \beta < k \end{cases} .$$

Therefore we obtain

$$\begin{aligned} &\mathbb{E}\left[I_n^\pi B_n(s) B_n(t) | Z_i = s, Z_j = t\right] \\ &= \mathbb{E}\left[\sum_{\alpha=1}^{n-1} \sum_{\beta=2}^{n} \mathbb{I}\{Z_{\alpha:n} = s\} \mathbb{I}\{Z_{\beta:n} = t\} I_n^\pi \left(1 + \frac{1-q(Z_i)}{n-\alpha}\right)\right. \\ &\qquad \times \prod_{k=1}^{\alpha-1} \left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{2s_k^n} \\ &\qquad \left. \times \prod_{k=\alpha+1}^{\beta-1} \left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{t_k^n} |Z_i = s, Z_j = t \right] . \end{aligned} \tag{2.3}$$

Next define

$$\tilde{I}_n^\pi(\alpha, \beta) := \prod_{l=1}^{i-1} \mathbb{I}\{Z_{\pi_l:n} = Z_l\} \prod_{l=i+1}^{j-1} \mathbb{I}\{Z_{\pi_l:n} = Z_l\} \prod_{l=j+1}^{n} \mathbb{I}\{Z_{\pi_l:n} = Z_l\}$$

and note that

$$\sum_{\alpha=1}^{n-1} \sum_{\beta=2}^{n} \mathbb{I}\{Z_{\alpha:n} = Z_i\} \mathbb{I}\{Z_{\beta:n} = Z_j\} I_n^\pi = \sum_{\alpha=1}^{n-1} \sum_{\beta=2}^{n} \mathbb{I}\{Z_{\alpha:n} = Z_i\} \mathbb{I}\{Z_{\beta:n} = Z_j\} \tilde{I}_n^\pi(\alpha, \beta) ,$$

since $\mathbb{I}\{Z_{\alpha:n} = Z_i\} \mathbb{I}\{Z_{\pi_i:n} = Z_i\} = 1$ and $\mathbb{I}\{Z_{\beta:n} = Z_j\} \mathbb{I}\{Z_{\pi_j:n} = Z_j\} = 1$ if and only if $\pi_i = \alpha$ and $\pi_j = \beta$ respectively. Now we can rewrite (2.3) as

$$\begin{aligned}
&\mathbb{E}\left[I_n^\pi B_n(s)B_n(t)|Z_i = s, Z_j = t\right] \\
&= \mathbb{E}\left[\sum_{\alpha=1}^{n-1}\sum_{\beta=2}^{n} \mathbb{I}\{Z_{\alpha:n} = Z_i\}\mathbb{I}\{Z_{\beta:n} = Z_j\}\tilde{I}_n^\pi(\alpha,\beta)\left(1 + \frac{1-q(Z_i)}{n-\alpha}\right)\right. \\
&\qquad \times \prod_{k=1}^{\alpha-1}\left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{2s_k^n} \\
&\qquad \left. \times \prod_{k=\alpha+1}^{\beta-1}\left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{t_k^n} |Z_i = s, Z_j = t\right] \\
&= \mathbb{E}\left[\sum_{\alpha=1}^{n-1}\sum_{\beta=2}^{n} \mathbb{I}\{Z_{\alpha:n} = s\}\mathbb{I}\{Z_{\beta:n} = t\}\tilde{I}_n^\pi(\alpha,\beta)\left(1 + \frac{1-q(s)}{n-\alpha}\right)\right. \\
&\qquad \times \prod_{k=1}^{\alpha-1}\left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{2s_k^n} \\
&\qquad \left. \times \prod_{k=\alpha+1}^{\beta-1}\left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{t_k^n} |Z_i = s, Z_j = t\right] .
\end{aligned}$$

Next we need to introduce some more notation. For $1 \leq i, j \leq n$ and $n \geq 2$, let $\{Z_{k:n-2}\}_{k\leq n-2}$ denote the ordered Z-values among $Z_1, \ldots, Z_n$ with Z_i and Z_j removed from the sample. Note that

$$Z_{k:n} = \begin{cases} Z_{k:n-2} & k < \alpha \\ Z_{k-1:n-2} & \alpha < k < \beta \\ Z_{k-2:n-2} & k > \beta \end{cases} . \tag{2.4}$$

Moreover let $Z_{0:n-2} := 0$ and $Z_{n-1:n-2} = \infty$. Furthermore consider $\pi \in \Pi_n$ as a mapping

$$\pi : (1, \ldots, n) \longrightarrow (\pi_1, \ldots, \pi_n)$$

Note that π is a permutation of $\{1, \ldots, n\}$ and hence bijective. We will denote its inverse as π^{-1}. Here $\pi^{-1}(i) = k$ whenever $\pi_k = i$. Now we can rewrite $\tilde{I}_n^\pi(\alpha, \beta)$ as follows

$$
\begin{aligned}
\tilde{I}_n^{\pi}(\alpha,\beta) &= \prod_{l=1}^{i-1} \mathbb{I}\{Z_{\pi_l:n} = Z_l\} \prod_{l=i+1}^{j-1} \mathbb{I}\{Z_{\pi_l:n} = Z_l\} \prod_{l=j+1}^{n} \mathbb{I}\{Z_{\pi_l:n} = Z_l\} \\
&= \prod_{l=1}^{\alpha-1} \mathbb{I}\{Z_{l:n} = Z_{\pi_l^{-1}}\} \prod_{l=\alpha+1}^{\beta-1} \mathbb{I}\{Z_{l:n} = Z_{\pi_l^{-1}}\} \prod_{l=\beta+1}^{n} \mathbb{I}\{Z_{l:n} = Z_{\pi_l^{-1}}\} \\
&= \prod_{l=1}^{\alpha-1} \mathbb{I}\{Z_{l:n-2} = Z_{\pi_l^{-1}}\} \prod_{l=\alpha+1}^{\beta-1} \mathbb{I}\{Z_{l-1:n-2} = Z_{\pi_l^{-1}}\} \prod_{l=\beta+1}^{n} \mathbb{I}\{Z_{l-2:n-2} = Z_{\pi_l^{-1}}\}
\end{aligned}
$$

by equation (2.4). Now $\tilde{I}_n^{\pi}(\alpha,\beta)$ is independent of Z_i and Z_j, since $\pi_\alpha^{-1} = i$ and $\pi_\beta^{-1} = j$. Moreover we have (c. f. Bose and Sen (1999), page 193)

$$
\begin{aligned}
&\mathbb{E}\left[I_n^{\pi} B_n(s) B_n(t) | Z_i = s, Z_j = t\right] \\
&= \mathbb{E}\Bigg[\sum_{\alpha=1}^{n-1}\sum_{\beta=2}^{n} \mathbb{I}\{Z_{\alpha-1:n-2} < s \le Z_{\alpha:n-2}\}\mathbb{I}\{Z_{\beta-2:n-2} < t \le Z_{\beta-1:n-2}\}\tilde{I}_n^{\pi}(\alpha,\beta) \\
&\qquad \times \left(1 + \frac{1-q(s)}{n-\alpha}\right) \prod_{k=1}^{\alpha-1}\left(1 + \frac{1-q(Z_{k:n-2})}{n-k}\right)^{2s_k^{n-2}} \\
&\qquad \times \prod_{k=\alpha+1}^{\beta-1}\left(1 + \frac{1-q(Z_{k-1:n-2})}{n-k}\right)^{t_{k-1}^{n-2}} | Z_i = s, Z_j = t\Bigg] \\
&= \mathbb{E}\Bigg[\sum_{\alpha=1}^{n-1}\sum_{\beta=2}^{n} \mathbb{I}\{Z_{\alpha-1:n-2} < s \le Z_{\alpha:n-2}\}\mathbb{I}\{Z_{\beta-2:n-2} < t \le Z_{\beta-1:n-2}\}\tilde{I}_n^{\pi}(\alpha,\beta) \\
&\qquad \times \left(1 + \frac{1-q(s)}{n-\alpha}\right) \prod_{k=1}^{\alpha-1}\left(1 + \frac{1-q(Z_{k:n-2})}{n-k}\right)^{2s_k^{n-2}} \\
&\qquad \times \prod_{k=\alpha}^{\beta-2}\left(1 + \frac{1-q(Z_{k:n-2})}{n-k-1}\right)^{t_k^{n-2}}\Bigg] \\
&= \mathbb{E}\Bigg[\sum_{\alpha=1}^{n-1} \mathbb{I}\{Z_{\alpha-1:n-2} < s \le Z_{\alpha:n-2}\}\tilde{I}_n^{\pi}(\alpha,\beta)\left(1 + \frac{1-q(s)}{n-\alpha}\right) \\
&\qquad \times \prod_{k=1}^{n-2}\left(1 + \frac{1-q(Z_{k:n-2})}{n-k}\right)^{2s_k^{n-2}} \\
&\qquad \times \prod_{k=1}^{n-2}\left(1 + \frac{1-q(Z_{k:n-2})}{n-k-1}\right)^{t_k^{n-2}}\Bigg]
\end{aligned}
$$

which is independent of i, j. Next consider the case $t < s$. Define $\tilde{t}_k^n := \mathbb{I}\{Z_{k:n} < t\}$ and $\tilde{s}_k^n := \mathbb{I}\{t \le Z_{k:n} < s\}$. Using similar arguments we can show that in this case

$$\begin{aligned}
&\mathbb{E}\left[I_n^\pi B_n(s)B_n(t)|Z_i = s, Z_j = t\right] \\
&= \mathbb{E}\left[\sum_{\alpha=1}^{n-1} \mathbb{I}\{Z_{\alpha-1:n-2} < t \le Z_{\alpha:n-2}\}\tilde{I}_n^\pi(\alpha,\beta)\left(1 + \frac{1-q(t)}{n-\alpha}\right)\right. \\
&\qquad \times \prod_{k=1}^{n-2}\left(1 + \frac{1-q(Z_{k:n-2})}{n-k}\right)^{2\tilde{t}_k^{n-2}} \\
&\qquad \left. \times \prod_{k=\alpha}^{n-2}\left(1 + \frac{1-q(Z_{k:n-2})}{n-k-1}\right)^{\tilde{s}_k^{n-2}}\right]
\end{aligned}$$

which is independent of i, j as well. Thus we have on $\{s \neq t\}$ that $\mathbb{E}\left[B_n(s)B_n(t)|Z_i = s, Z_j = t\right]$ is independent of i, j and hence

$$\mathbb{E}\left[I_n^\pi B_n(s)B_n(t)|Z_i = s, Z_j = t\right] = \mathbb{E}\left[I_n^\pi B_n(s)B_n(t)|Z_1 = s, Z_2 = t\right] \ .$$

$\square$

The following is an adapted version of Lemma 4.7 of my thesis.

Lemma 2.3. *Let $\tilde{\phi} : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ be a Borel-measurable function. Then we have for any $n \geq 2$*

$$\begin{aligned}
&\mathbb{E}[I_n^\pi \tilde{\phi}(Z_i, Z_j)B_n(Z_i)B_n(Z_j)] \\
&= \mathbb{E}[I_n^\pi \tilde{\phi}(Z_1, Z_2)B_n(Z_1)B_n(Z_2)] \ .
\end{aligned}$$

Proof. Consider that $\{Z_i = Z_j\}$ is a measure zero set, since H is continuous. Therefore the following holds for $1 \le i, j \le n$

$$\begin{aligned}
&\mathbb{E}[I_n^\pi \tilde{\phi}(Z_i, Z_j)B_n(Z_i)B_n(Z_j)] \\
&= \mathbb{E}\left[\mathbb{I}\{Z_i \neq Z_j\}I_n^\pi \tilde{\phi}(Z_i, Z_j)B_n(Z_i)B_n(Z_j)\right]
\end{aligned}$$

$$= \mathbb{E}\left[\mathbb{I}\{Z_i \neq Z_j\}\tilde{\phi}(Z_i, Z_j)\mathbb{E}\left[I_n^\pi B_n(Z_i)B_n(Z_j)|Z_i, Z_j\right]\right]$$
$$= \int_0^\infty \int_0^\infty \mathbb{I}\{s \neq t\}\tilde{\phi}(s,t)\mathbb{E}\left[I_n^\pi B_n(s)B_n(t)|Z_i = s, Z_j = t\right] H(ds)H(dt)$$

Applying Lemma 2.2 we obtain for $1 \leq i, j \leq n$

$$\mathbb{I}\{s \neq t\}\mathbb{E}[I_n^\pi B_n(s)B_n(t)|Z_i = s, Z_j = t]$$
$$= \mathbb{I}\{s \neq t\}\mathbb{E}[I_n^\pi B_n(s)B_n(t)|Z_1 = s, Z_2 = t]$$

Therefore we conclude

$$\begin{aligned}\mathbb{E}\left[I_n^\pi\tilde{\phi}(Z_i, Z_j)B_n(Z_i)B_n(Z_j)\right] &= \mathbb{E}\left[\tilde{\phi}(Z_i, Z_j)\mathbb{E}\left[I_n^\pi B_n(Z_i)B_n(Z_j)|Z_i, Z_j\right]\right] \\ &= \mathbb{E}\left[I_n^\pi\tilde{\phi}(Z_1, Z_2)B_n(Z_1)B_n(Z_2)\right] \ .\end{aligned}$$

□

2.2 Lemmas from my thesis (unchanged)

Lemma 2.4. *Let $\tilde{\phi} : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ be a Borel-measurable function. Then we have for any $s < t$ and $n \geq 2$*

$$\mathbb{E}[\tilde{\phi}(Z_1, Z_2)B_n(Z_1)B_n(Z_2)]$$
$$= \mathbb{E}[2\tilde{\phi}(Z_1, Z_2)\{\Delta_{n-2}(Z_1, Z_2) + \bar{\Delta}_{n-2}(Z_1, Z_2)\}\mathbb{I}\{Z_1 < Z_2\}] \ .$$

Proof. Consider the following

$$\begin{aligned}B_n(Z_1)B_n(Z_2) &= \prod_{k=1}^n \left[1 + \frac{1 - q(Z_k)}{n - R_{k,n}}\right]^{\mathbb{I}\{Z_k < Z_1\} + \mathbb{I}\{Z_k < Z_2\}} \\ &= \left[1 + \frac{1 - q(Z_1)}{n - R_{1,n}}\right]^{\mathbb{I}\{Z_1 < Z_2\}} \left[1 + \frac{1 - q(Z_2)}{n - R_{2,n}}\right]^{\mathbb{I}\{Z_2 < Z_1\}}\end{aligned}$$

$$
\begin{aligned}
&\times \prod_{k=3}^{n} \left[1 + \frac{1-q(Z_k)}{n-R_{k,n}}\right]^{\mathbb{I}\{Z_k<Z_1\}+\mathbb{I}\{Z_k<Z_2\}} \\
&= \mathbb{I}\{Z_1 < Z_2\} \left[1 + \frac{1-q(Z_1)}{n-R_{1,n}}\right] \\
&\qquad \times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n-R_{k+2,n}}\right]^{\mathbb{I}\{Z_{k+2}<Z_1\}+\mathbb{I}\{Z_{k+2}<Z_2\}} \\
&\quad + \mathbb{I}\{Z_1 > Z_2\} \left[1 + \frac{1-q(Z_2)}{n-R_{2,n}}\right] \\
&\qquad \times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n-R_{k+2,n}}\right]^{\mathbb{I}\{Z_{k+2}<Z_1\}+\mathbb{I}\{Z_{k+2}<Z_2\}} \\
&\quad + \mathbb{I}\{Z_1 = Z_2\} \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n-R_{k+2,n}}\right]^{2\mathbb{I}\{Z_{k+2}<Z_1\}} .
\end{aligned}
\tag{2.5}
$$

On $\{Z_1 < Z_2\}$ we have

$$
\begin{aligned}
\prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n-R_{k+2,n}}\right]^{\mathbb{I}\{Z_{k+2}<Z_2\}} &= \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}}\right]^{\mathbb{I}\{Z_{k+2}<Z_1\}} \\
&\quad \times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}-1}\right]^{\mathbb{I}\{Z_1<Z_{k+2}<Z_2\}}
\end{aligned}
$$

where $\tilde{R}_{k,n-2}$ denotes the rank of the Z_k, $k = 3, \ldots, n$ among themselves. The above holds since

$$
R_{k+2,n} = \begin{cases} \tilde{R}_{k,n-2} & \text{if } Z_{k+2} < Z_1 \\ \tilde{R}_{k,n-2} + 1 & \text{if } Z_1 < Z_{k+2} < Z_2 \end{cases}
$$

for $k = 1, \ldots, n-2$. Therefore (2.5) yields

$$
\begin{aligned}
B_n(Z_1)B_n(Z_2) = \mathbb{I}\{Z_1 < Z_2\} &\left[1 + \frac{1-q(Z_1)}{n-R_{1,n}}\right] \\
&\times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}}\right]^{2\mathbb{I}\{Z_{k+2}<Z_1\}} \\
&\times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}-1}\right]^{\mathbb{I}\{Z_1<Z_{k+2}<Z_2\}}
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbb{I}\{Z_2 < Z_1\}\left[1+\frac{1-q(Z_2)}{n-R_{2,n}}\right] \\
&\quad\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}}\right]^{2\mathbb{I}\{Z_{k+2}<Z_2\}} \\
&\quad\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}-1}\right]^{\mathbb{I}\{Z_2<Z_{k+2}<Z_1\}} \\
&+\mathbb{I}\{Z_1 = Z_2\}\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}}\right]^{2\mathbb{I}\{Z_{k+2}<Z_1\}} .
\end{aligned}
\tag{2.6}
$$

Now let's denote $Z_{k:n-2}$ the ordered Z-values among $Z_3, \ldots, Z_n$ for $k = 1, \ldots, n-2$. Consider that we can write

$$
\left[1+\frac{1-q(Z_1)}{n-R_{1,n}}\right] = \sum_{i=1}^{n-1}\left[1+\frac{1-q(s)}{n-i}\right]\mathbb{I}\{Z_{i-1:n-2} < Z_1 \le Z_{i:n-2}\} .
$$

Note that $Z_{k:n-2}$ is independent of Z_1 and Z_2 for $k = 1, \ldots, n-2$. Therefore we obtain the following, by conditioning (2.6) on Z_1, Z_2:

$$
\begin{aligned}
&\mathbb{E}[B_n(Z_1)B_n(Z_2)|Z_1 = s, Z_2 = t] \\
&= \mathbb{I}\{s<t\}\mathbb{E}\left[\left(\sum_{i=1}^{n-1}\left[1+\frac{1-q(s)}{n-i}\right]\mathbb{I}\{Z_{i-1:n-2} < s \le Z_{i:n-2}\}\right)\right. \\
&\qquad\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{I}\{Z_{k:n-2}<s\}} \\
&\qquad\left.\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k-1}\right]^{\mathbb{I}\{s<Z_{k:n-2}<t\}}\right] \\
&\quad+\mathbb{I}\{t<s\}\mathbb{E}\left[\left(\sum_{i=1}^{n-1}\left[1+\frac{1-q(t)}{n-i}\right]\mathbb{I}\{Z_{i-1:n-2} < t \le Z_{i:n-2}\}\right)\right. \\
&\qquad\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{I}\{Z_{k:n-2}<t\}} \\
&\qquad\left.\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k-1}\right]^{\mathbb{I}\{t<Z_{k:n-2}<s\}}\right] \\
&\quad+\mathbb{I}\{s=t\}\mathbb{E}\left[\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{I}\{Z_{k:n-2}<s\}}\right]
\end{aligned}
$$

$$= \alpha(s,t) + \alpha(t,s) + \beta(s,t)$$

where

$$\alpha(s,t) := \mathbb{I}\{s < t\}\mathbb{E}\left[\left(\sum_{i=1}^{n-1}\left[1 + \frac{1-q(s)}{n-i}\right]\mathbb{I}\{Z_{i-1:n-2} < s \le Z_{i:n-2}\}\right) \times \prod_{k=1}^{n-2}\left[1 + \frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{I}\{Z_{k:n-2}<s\}} \times \prod_{k=1}^{n-2}\left[1 + \frac{1-q(Z_{k:n-2})}{n-k-1}\right]^{\mathbb{I}\{s<Z_{k:n-2}<t\}}\right]$$

and

$$\beta(s,t) := \mathbb{I}\{s = t\}\mathbb{E}\left[\prod_{k=1}^{n-2}\left[1 + \frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{I}\{Z_{k:n-2}<s\}}\right] .$$

Consider that we have

$$\mathbb{E}[\alpha(Z_1, Z_2)] = \mathbb{E}[\alpha(Z_2, Z_1)]$$

under (A1), because Z_1 and Z_2 are i. i. d. and

$$\mathbb{E}[\beta(Z_1, Z_2)] = 0$$

since H is continuous. Therefore we get

$$\begin{aligned}
&\mathbb{E}[\tilde{\phi}(Z_1, Z_2)B_n(Z_1)B_n(Z_2)] \\
&= \mathbb{E}[\tilde{\phi}(Z_1, Z_2)(\alpha(Z_1, Z_2) + \alpha(Z_2, Z_1) + \beta(Z_1, Z_2))] \\
&= \mathbb{E}[2\tilde{\phi}(Z_1, Z_2)\alpha(Z_1, Z_2)] \ . \qquad (2.7)
\end{aligned}$$

Next consider that

$$\alpha(s,t) = \mathbb{I}\{s < t\}\mathbb{E}\left[(1 + C_n(s))D_{n-2}(s,t)\right]$$

$$= \mathbb{I}\{s < t\}(\Delta_{n-2}(s,t) + \bar{\Delta}_{n-2}(s,t)) \ .$$

The latter equality holds, since

$$\begin{aligned}
&\sum_{i=1}^{n-1} \left[1 + \frac{1-q(s)}{n-i}\right] \mathbb{I}\{Z_{i-1:n-2} < s \le Z_{i:n-2}\} \\
&= \sum_{i=1}^{n-1} \mathbb{I}\{Z_{i-1:n-2} < s \le Z_{i:n-2}\} + \sum_{i=1}^{n-1} \left[\frac{1-q(s)}{n-i}\right] \mathbb{I}\{Z_{i-1:n-2} < s \le Z_{i:n-2}\} \\
&= 1 + C_n(s) \ .
\end{aligned}$$

Now the statement of the lemma follows directly from (2.7). □

The next lemma identifies the almost sure limit of D_n for $n \to \infty$. Define for $s < t$

$$D(s,t) := \exp\left(2\int_0^s \frac{1-q(z)}{1-H(z)} H(dz) + \int_s^t \frac{1-q(z)}{1-H(z)} H(dz)\right)$$

Lemma 2.5. *For any $s < t$ s. t. $H(t) < 1$, we have*

$$\lim_{n\to\infty} D_n(s,t) = D(s,t) \ .$$

Proof. First recall the following definition

$$D_n(s,t) := \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n-R_{k,n}+2}\right]^{2\mathbb{I}\{Z_k<s\}} \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n-R_{k,n}+1}\right]^{\mathbb{I}\{s<Z_k<t\}} \ .$$

Next let

$$\begin{aligned}
x_k &:= \frac{1-q(Z_k)}{n(1-H_n(Z_k)+2/n)} \\
y_k &:= \frac{1-q(Z_k)}{n(1-H_n(Z_k)+1/n)} \\
s_k &:= \mathbb{I}\{Z_k < s\} \\
t_k &:= \mathbb{I}\{s < Z_k < t\}
\end{aligned}$$

for $s < t$ and $k = 1, \ldots, n$. Now consider

$$\begin{aligned}
D_n(s,t) &= \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n(1-H_n(Z_k)+2/n)} \mathbb{I}\{Z_k < s\} \right]^2 \\
&\quad \times \prod_{k=1}^{n} \left[1 + \frac{1-q(Z_k)}{n(1-H_n(Z_k)+1/n)} \mathbb{I}\{s < Z_k < t\} \right] \\
&= \prod_{k=1}^{n} [1 + x_k s_k]^2 \prod_{k=1}^{n} [1 + y_k t_k] \\
&= \exp\left(2 \sum_{k=1}^{n} \ln [1 + x_k s_k] + \sum_{k=1}^{n} \ln [1 + y_k t_k] \right) .
\end{aligned}$$

Note that $0 \le x_k s_k \le 1$ and $0 \le y_k t_k \le 1$. Consider that the following inequality holds

$$-\frac{x^2}{2} \le \ln(1+x) - x \le 0$$

for any $x \ge 0$ (cf. Stute and Wang (1993), p. 1603). This implies

$$-\frac{1}{2} \sum_{k=1}^{n} x_k^2 s_k \le \sum_{k=1}^{n} \ln(1 + x_k s_k) - \sum_{k=1}^{n} x_k s_k \le 0 .$$

But now

$$\begin{aligned}
\sum_{k=1}^{n} x_k^2 s_k &= \frac{1}{n^2} \sum_{k=1}^{n} \left(\frac{1-q(Z_k)}{1-H_n(Z_k)+\frac{2}{n}} \right)^2 \mathbb{I}\{Z_k < s\} \\
&\le \frac{1}{n^2} \sum_{k=1}^{n} \left(\frac{1}{1-H_n(s)+\frac{1}{n}} \right)^2 \\
&= \frac{1}{n(1-H_n(s)+n^{-1})^2} \longrightarrow 0
\end{aligned}$$

almost surely as $n \to \infty$, since $H(s) < H(t) < 1$ (c.f. Stute and Wang (1993), p. 1603). Therefore we have

$$\left| \sum_{k=1}^{n} \ln(1 + x_k s_k) - \sum_{k=1}^{n} x_k s_k \right| \longrightarrow 0$$

with probability 1 as $n \to \infty$. Similarly we obtain

$$\left| \sum_{k=1}^{n} \ln(1 + y_k t_k) - \sum_{k=1}^{n} y_k t_k \right| \longrightarrow 0$$

with probability 1 as $n \to \infty$. Hence

$$\lim_{n\to\infty} D_n(s) = \lim_{n\to\infty} \exp\left(2\sum_{k=1}^{n} x_k s_k + \sum_{k=1}^{n} y_k t_k \right) .$$

Now consider

$$\begin{aligned}
\sum_{k=1}^{n} x_k s_k &= \frac{1}{n} \sum_{k=1}^{n} \frac{1 - q(Z_k)}{1 - H_n(Z_k) + \frac{2}{n}} \mathbb{I}\{Z_k < s\} \\
&= \int_0^{s-} \frac{1 - q(z)}{1 - H_n(z) + \frac{2}{n}} H_n(dz) \\
&= \int_0^{s-} \frac{1 - q(z)}{1 - H(z)} H_n(dz) + \int_0^{s-} \frac{1 - q(z)}{1 - H_n(z) + \frac{2}{n}} - \frac{1 - q(z)}{1 - H(z)} H_n(dz) \\
&= \int_0^{s-} \frac{1 - q(z)}{1 - H(z)} H_n(dz) + \int_0^{s-} \frac{(1 - q(z))(H_n(z) - H(z) - \frac{2}{n})}{(1 - H_n(z) + \frac{2}{n})(1 - H(z))} H_n(dz) .
\end{aligned} \tag{2.8}$$

Note that the second term on the right hand side of the latter equation above tends to zero for $n \to \infty$, because

$$\begin{aligned}
&\int_0^{s-} \frac{(1 - q(z))(H_n(z) - H(z) - \frac{2}{n})}{(1 - H_n(z) + \frac{2}{n})(1 - H(z))} H_n(dz) \\
&\leq \frac{\sup_z |H_n(z) - H(z) - \frac{2}{n}|}{1 - H(s)} \int_0^{s-} \frac{1}{1 - H_n(z)} H_n(dz) \longrightarrow 0
\end{aligned}$$

almost surely as $n \to \infty$, by the Glivenko-Cantelli Theorem and since $H(s) < 1$. Moreover we have

$$\int_0^{s-} \frac{1 - q(z)}{1 - H(z)} H_n(dz) \longrightarrow \int_0^{s} \frac{1 - q(z)}{1 - H(z)} H(dz)$$

by the SLLN. Therefore we obtain

$$\lim_{n\to\infty}\sum_{k=1}^{n} x_k s_k = \int_0^s \frac{1-q(z)}{1-H(z)} H(dz) \ .$$

By the same arguments, we can show that

$$\lim_{n\to\infty}\sum_{k=1}^{n} y_k t_k = \int_s^t \frac{1-q(z)}{1-H(z)} H(dz) \ .$$

Thus we finally conclude

$$\lim_{n\to\infty} D_n(s,t) = \exp\left(2\int_0^s \frac{1-q(z)}{1-H(z)} H(dz) + \int_s^t \frac{1-q(z)}{1-H(z)} H(dz)\right)$$

almost surely. $\square$

Lemma 2.6. $\{D_n, \mathcal{F}_n\}_{n\geq 1}$ *is a non-negative reverse supermartingale.*

Proof. Consider that for $s<t$ and $n\geq 1$

$$\begin{aligned}
\mathbb{E}[D_n(s,t)|\mathcal{F}_{n+1}] &= \mathbb{E}\left[\prod_{k=1}^{n}\left(1+\frac{1-q(Z_{k:n})}{n-k+2}\right)^{2\mathbb{I}\{Z_{k:n}<s\}}\right.\\
&\qquad \left.\times\prod_{k=1}^{n}\left(1+\frac{1-q(Z_{k:n})}{n-k+1}\right)^{\mathbb{I}\{s<Z_{k:n}<t\}} |\mathcal{F}_{n+1}\right]\\
&= \sum_{i=1}^{n+1}\mathbb{E}\left[\mathbb{I}\{Z_{n+1}=Z_{i:n+1}\}\prod_{k=1}^{n}\ldots|\mathcal{F}_{n+1}\right]\\
&= \sum_{i=1}^{n+1}\mathbb{E}\left[\mathbb{I}\{Z_{n+1}=Z_{i:n+1}\}\prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+2}\right)^{2\mathbb{I}\{Z_{k:n+1}<s\}}\right.\\
&\qquad \times\prod_{k=i}^{n}\left(1+\frac{1-q(Z_{k+1:n+1})}{n-k+2}\right)^{2\mathbb{I}\{Z_{k+1:n+1}<s\}}\\
&\qquad \times\prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+1}\right)^{\mathbb{I}\{s<Z_{k:n+1}<t\}}\\
&\qquad \left.\times\prod_{k=i}^{n}\left(1+\frac{1-q(Z_{k+1:n+1})}{n-k+1}\right)^{\mathbb{I}\{s<Z_{k+1:n+1}<t\}} |\mathcal{F}_{n+1}\right]
\end{aligned}$$

$$
\begin{aligned}
= \sum_{i=1}^{n+1} \mathbb{E} \Bigg[& \mathbb{I}\{Z_{n+1} = Z_{i:n+1}\} \prod_{k=1}^{i-1} \left(1 + \frac{1 - q(Z_{k:n+1})}{n-k+2}\right)^{2\mathbb{I}\{Z_{k:n+1}<s\}} \\
& \times \prod_{k=i+1}^{n+1} \left(1 + \frac{1 - q(Z_{k:n+1})}{n-k+3}\right)^{2\mathbb{I}\{Z_{k:n+1}<s\}} \\
& \times \prod_{k=1}^{i-1} \left(1 + \frac{1 - q(Z_{k:n+1})}{n-k+1}\right)^{\mathbb{I}\{s<Z_{k:n+1}<t\}} \\
& \times \prod_{k=i+1}^{n+1} \left(1 + \frac{1 - q(Z_{k:n+1})}{n-k+2}\right)^{\mathbb{I}\{s<Z_{k:n+1}<t\}} |\mathcal{F}_{n+1} \Bigg] .
\end{aligned}
$$

Now each product within the conditional expectation is measurable w. r. t. $\mathcal{F}_{n+1}$. Moreover we have for $i = 1, \ldots, n$

$$
\begin{aligned}
\mathbb{E}[\mathbb{I}\{Z_{n+1} = Z_{i:n+1}\}|\mathcal{F}_n + 1] &= \mathbb{P}(Z_{n+1} = Z_{i:n+1}|\mathcal{F}_{n+1}) \\
&= \mathbb{P}(R_{n+1,n+1} = i) \\
&= \frac{1}{n+1} .
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
\mathbb{E}[D_n(s,t)|\mathcal{F}_{n+1}] = \frac{1}{n+1} \sum_{i=1}^{n+1} \prod_{k=1}^{i-1} & \left(1 + \frac{1 - q(Z_{k:n+1})}{n-k+2}\right)^{2\mathbb{I}\{Z_{k:n+1}<s\}} \\
& \times \left(1 + \frac{1 - q(Z_{k:n+1})}{n-k+1}\right)^{\mathbb{I}\{s<Z_{k:n+1}<t\}} \\
\times \prod_{k=i+1}^{n+1} & \left(1 + \frac{1 - q(Z_{k:n+1})}{n-k+3}\right)^{2\mathbb{I}\{Z_{k:n+1}<s\}} \\
& \times \left(1 + \frac{1 - q(Z_{k:n+1})}{n-k+2}\right)^{\mathbb{I}\{s<Z_{k:n+1}<t\}} .
\end{aligned} \tag{2.9}
$$

We will now proceed by induction on n. First let

$$x_k := 1 - q(Z_{k:2}),\ s_k := \mathbb{I}\{Z_{k:2} < s\} \text{ and } t_k := \mathbb{I}\{s < Z_{k:2} < t\}$$

for $k = 1, 2$. Note that that x_k and y_k are different, compared to the corresponding definitions in lemma 2.5, as they involves the ordered Z-values here. Next consider

$$\begin{aligned}\mathbb{E}[D_1(s,t)|\mathcal{F}_2] &= \frac{1}{2}\left[\left(1+\frac{1-q(Z_{2:2})}{2}\right)^{2\mathbb{I}\{Z_{2:2}<s\}} \times (1+(1-q(Z_{2:2})))^{\mathbb{I}\{s<Z_{2:2}<t\}}\right.\\ &\qquad\left.+\left(1+\frac{1-q(Z_{1:2})}{2}\right)^{2\mathbb{I}\{Z_{1:2}<s\}} \times (1+(1-q(Z_{1:2})))^{\mathbb{I}\{s<Z_{1:2}<t\}}\right]\\ &= \frac{1}{2}\left[\left(1+\frac{x_2}{2}s_2\right)^2 \times (1+x_2t_2) + \left(1+\frac{x_1}{2}s_1\right)^2 \times (1+x_1t_1)\right] .\end{aligned}$$

Moreover we have

$$\begin{aligned}D_2(s,t) &= \prod_{k=1}^{2}\left[1+\frac{1-q(Z_{k:2})}{4-k}\right]^{2\mathbb{I}\{Z_{k:2}<s\}} \prod_{k=1}^{2}\left[1+\frac{1-q(Z_{k:2})}{3-k}\right]^{\mathbb{I}\{s<Z_{k:2}<t\}}\\ &= \left[1+\frac{x_1}{3}s_1\right]^2 \times \left[1+\frac{x_1}{2}t_1\right] \times \left[1+\frac{x_2}{2}s_2\right]^2 \times [1+x_2t_2]\\ &= \left[1+\frac{x_1}{2}t_1+\left(\frac{x_1^2}{9}+\frac{2}{3}x_1\right)s_1\right] \times \left[1+x_2t_2+\left(\frac{x_2^2}{4}+x_2\right)s_2\right] .\end{aligned}$$

Therefore we obtain

$$\mathbb{E}[D_1(s,t)|\mathcal{F}_2] - D_2(s,t) \leq \frac{x_1^2}{72} - \frac{x_1}{6} \leq 0 .$$

since $0 \leq x_1 \leq 1$. Thus $\mathbb{E}[D_1(s,t)|\mathcal{F}_2] \leq D_2(s,t)$ for any $s < t$, as needed. Now assume that

$$\mathbb{E}[D_n(s,t)|\mathcal{F}_{n+1}] \leq D_{n+1}(s,t)$$

holds for any $n \geq 1$. Note that the latter is equivalent to assuming

$$\begin{aligned}&\frac{1}{n+1}\sum_{i=1}^{n+1}\prod_{k=1}^{i-1}\left(1+\frac{1-q(y_k)}{n-k+2}\right)^{2\mathbb{I}\{y_k<s\}}\left(1+\frac{1-q(y_k)}{n-k+1}\right)^{\mathbb{I}\{s<y_k<t\}}\\ &\qquad\times\prod_{k=i+1}^{n+1}\left(1+\frac{1-q(y_k)}{n-k+3}\right)^{2\mathbb{I}\{y_k<s\}}\left(1+\frac{1-q(y_k)}{n-k+2}\right)^{\mathbb{I}\{s<y_k<t\}}\\ &\leq \prod_{k=1}^{n+1}\left(1+\frac{1-q(y_k)}{n-k+3}\right)^{2\mathbb{I}\{y_k<s\}}\prod_{k=1}^{n+1}\left(1+\frac{1-q(y_k)}{n-k+2}\right)^{\mathbb{I}\{s<y_k<t\}} \qquad (2.10)\end{aligned}$$

holds for arbitrary $y_k \geq 0$. Next define for $s < t$ and $n \geq 1$

$$A_{n+2}(s,t) := \prod_{k=2}^{n+2}\left[1+\frac{1-q(Z_{k:n+2})}{n-k+4}\right]^{2\mathbb{I}\{Z_{k:n+2}<s\}} \times \left[1+\frac{1-q(Z_{k:n+2})}{n-k+3}\right]^{\mathbb{I}\{s<Z_{k:n+2}<t\}} .$$

Now consider that we get from (2.9)

$$\begin{aligned}
&\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}] \\
&= \frac{1}{n+2}\sum_{i=1}^{n+2}\prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k:n+2})}{n-k+3}\right)^{2\mathbb{I}\{Z_{k:n+2}<s\}}\left(1+\frac{1-q(Z_{k:n+2})}{n-k+2}\right)^{\mathbb{I}\{s<Z_{k:n+2}<t\}} \\
&\qquad\qquad \times \prod_{k=i+1}^{n+2}\left(1+\frac{1-q(Z_{k:n+2})}{n-k+4}\right)^{2\mathbb{I}\{Z_{k:n+2}<s\}}\left(1+\frac{1-q(Z_{k:n+2})}{n-k+3}\right)^{\mathbb{I}\{s<Z_{k:n+2}<t\}} \\
&= \frac{A_{n+2}}{n+2}+\frac{1}{n+2}\sum_{i=2}^{n+2}\prod_{k=1}^{i-1}\cdots\times\prod_{k=i+1}^{n+2}\cdots \\
&= \frac{A_{n+2}}{n+2}+\frac{1}{n+2}\sum_{i=1}^{n+1}\prod_{k=1}^{i}\cdots\times\prod_{k=i+2}^{n+2}\cdots \\
&= \frac{A_{n+2}}{n+2}+\frac{1}{n+2}\left(1+\frac{1-q(Z_{1:n+2})}{n+2}\right)^{2\mathbb{I}\{Z_{1:n+2}<s\}}\left(1+\frac{1-q(Z_{1:n+2})}{n+1}\right)^{\mathbb{I}\{s<Z_{1:n+2}<t\}} \\
&\qquad\qquad \times \sum_{i=1}^{n+1}\prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k+1:n+2})}{n-k+2}\right)^{2\mathbb{I}\{Z_{k+1:n+2}<s\}} \\
&\qquad\qquad\qquad \times \left(1+\frac{1-q(Z_{k+1:n+2})}{n-k+1}\right)^{\mathbb{I}\{s<Z_{k+1:n+2}<t\}} \\
&\qquad\qquad\qquad \times \prod_{k=i+1}^{n+1}\left(1+\frac{1-q(Z_{k+1:n+2})}{n-k+3}\right)^{2\mathbb{I}\{Z_{k+1:n+2}<s\}} \\
&\qquad\qquad\qquad \times \left(1+\frac{1-q(Z_{k+1:n+2})}{n-k+2}\right)^{\mathbb{I}\{s<Z_{k+1:n+2}<t\}} .
\end{aligned}$$

Using (2.10) on the right hand side of the equation above yields

$$\begin{aligned}
&\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}] \\
&\leq \frac{A_{n+2}}{n+2}+\frac{n+1}{n+2}\left(1+\frac{1-q(Z_{1:n+2})}{n+2}\right)^{2\mathbb{I}\{Z_{1:n+2}<s\}}\left(1+\frac{1-q(Z_{1:n+2})}{n+1}\right)^{\mathbb{I}\{s<Z_{1:n+2}<t\}}
\end{aligned}$$

$$\times \prod_{k=1}^{n+1} \left(1 + \frac{1 - q(Z_{k+1:n+2})}{n-k+3}\right)^{2\mathbb{I}\{Z_{k+1:n+2}<s\}}$$
$$\times \left(1 + \frac{1 - q(Z_{k+1:n+2})}{n-k+2}\right)^{\mathbb{I}\{s<Z_{k+1:n+2}<t\}}$$
$$= A_{n+2}\left[\frac{1}{n+2} + \frac{n+1}{n+2}\left(1 + \frac{1-q(Z_{1:n+2})}{n+2}\right)^{2\mathbb{I}\{Z_{1:n+2}<s\}} \times \left(1 + \frac{1-q(Z_{1:n+2})}{n+1}\right)^{\mathbb{I}\{s<Z_{1:n+2}<t\}}\right].$$

For the moment, let

$$x_1 := 1 - q(Z_{1:n+2}),\ s_1 := \mathbb{I}\{Z_{1:n+2} < s\} \text{ and } t_1 := \mathbb{I}\{s < Z_{1:n+2} < t\}$$

Now we can rewrite the above as

$$\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}] \leq A_{n+2}\left[\frac{1}{n+2} + \frac{n+1}{n+2}\left(1+\frac{x_1 s_1}{n+2}\right)^2\left(1+\frac{x_1 t_1}{n+1}\right)\right]. \quad (2.11)$$

Next consider

$$\begin{aligned}\left(1+\frac{x_1t_1}{n+1}\right) &= \left(1+\frac{x_1t_1}{n+2}-\frac{1}{n+2}\right)\left(1+\frac{1}{n+1}\right)\\ &= \left(1+\frac{x_1t_1}{n+2}\right)+\frac{1}{n+1}\left(1+\frac{x_1t_1}{(n+2)}\right)-\frac{1}{n+1}\\ &= \left(1+\frac{x_1t_1}{n+2}\right)+\frac{x_1t_1}{(n+1)(n+2)}.\end{aligned}$$

Thus we get

$$\begin{aligned}&\frac{n+1}{n+2}\left(1+\frac{x_1s_1}{n+2}\right)^2\left(1+\frac{x_1t_1}{n+1}\right)\\ &= \frac{n+1}{n+2}\left(1+\frac{x_1s_1}{n+2}\right)^2\left(1+\frac{x_1t_1}{n+2}\right)+\left(1+\frac{x_1s_1}{n+2}\right)^2\frac{x_1t_1}{(n+2)^2}.\end{aligned}$$

But now

$$\begin{aligned}\left(1+\frac{x_1 s_1}{n+2}\right)^2 \frac{x_1 t_1}{(n+2)^2} &= \left(1+2\frac{x_1 s_1}{n+2}+\frac{x_1^2 s_1}{(n+2)^2}\right)\frac{x_1 t_1}{(n+2)^2} \\ &= \frac{x_1 t_1}{(n+2)^2}\end{aligned}$$

since $s_1 \cdot t_1 = 0$ for all $s < t$. Hence we can rewrite the term in brackets in (2.11) as

$$\begin{aligned}&\frac{1}{n+2}+\frac{n+1}{n+2}\left(1+\frac{x_1 s_1}{n+2}\right)^2\left(1+\frac{x_1 t_1}{n+1}\right) \\ &= \frac{1}{n+2}+\frac{x_1 t_1}{(n+2)^2}+\frac{n+1}{n+2}\left(1+\frac{x_1 s_1}{n+2}\right)^2\left(1+\frac{x_1 t_1}{n+2}\right) \\ &= \frac{1}{n+2}\left(1+\frac{x_1 t_1}{n+2}\right)+\frac{n+1}{n+2}\left(1+\frac{x_1 s_1}{n+2}\right)^2\left(1+\frac{x_1 t_1}{n+2}\right) \\ &= \left[\frac{1}{n+2}+\frac{n+1}{n+2}\left(1+\frac{x_1}{n+2}\right)^{2s_1}\right]\left(1+\frac{x_1}{n+2}\right)^{t_1} \\ &\leq \left(1+\frac{x_1}{n+3}\right)^{2s_1}\left(1+\frac{x_1}{n+2}\right)^{t_1} .\end{aligned}$$

The latter inequality above holds, since

$$\left[\frac{1}{n+2}+\frac{n+1}{n+2}\left(1+\frac{x}{n+2}\right)^2\right] \leq \left(1+\frac{x}{n+3}\right)^2$$

for any $0 \leq x \leq 1$. (c. f. Bose and Sen (1999), page 197). Therefore we can rewrite (2.11) as

$$\begin{aligned}\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}] \leq A_{n+2}&\left(1+\frac{1-q(Z_{1:n+2})}{n+3}\right)^{2\mathbb{I}\{Z_{1:n+2}<s\}} \\ &\times\left(1+\frac{1-q(Z_{1:n+2})}{n+2}\right)^{\mathbb{I}\{s<Z_{1:n+2}<t\}} \\ = D_{n+2}(s,t) .&\end{aligned}$$

This concludes the proof. □

Lemma 2.7. *Let* $\mathcal{F}_\infty = \bigcap_{n\geq 2} \mathcal{F}_n$. *Then we have for each* $A \in \mathcal{F}_\infty$ *that* $\mathbb{P}(A) \in \{0,1\}$.

Proof. Denote $\tilde{Z} := (Z_1, Z_2, \dots) \in \mathbb{R}^\infty$ and let $1 \leq n < \infty$ fixed but arbitrary. Moreover define **TODO** phrasing

$$\Pi_n := \{\pi | \pi \text{ is permutation of } 1, \dots, n\} \ ,$$

i. e. for each $\pi \in \Pi_n$ we have

$$\begin{aligned} \pi : (\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty)) &\longrightarrow (\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty)) \\ (Z_1, Z_2, \dots, Z_n, Z_{n+1}, \dots) &\longmapsto (Z_{\pi(1)}, Z_{\pi(2)}, \dots, Z_{\pi(n)}, Z_{n+1}, \dots) \ . \end{aligned}$$

We will now use the Hewitt-Savage zero-one law to prove the statement of this lemma. We need to show that for all $A \in \mathcal{F}_\infty$ and for all $\pi_0 \in \Pi$ there exists $B \in \mathcal{B}(\mathbb{R}^\infty)$ s. t.

$$A = \{\omega | \tilde{Z}(\omega) \in B\} = \{\omega | \pi_0(\tilde{Z}(\omega)) \in B\} \ . \tag{2.12}$$

Let $A \in \mathcal{F}_\infty$, then $A \in \mathcal{F}_n$ for all $n \in \mathbb{N}$. Since the map $(Z_{1:n}, \dots, Z_{n:n}, Z_{n+1}, Z_{n+2}, \dots)$ is measurable, there must exist $\tilde{B} \in \mathcal{B}(\mathbb{R}^\infty)$ such that

$$A = \{\omega | (Z_{1:n}(\omega), \dots, Z_{n:n}(\omega), Z_{n+1}(\omega), Z_{n+2}(\omega), \dots) \in \tilde{B}\} \ .$$

Note that each of the maps $\pi \in \Pi_n$ is measurable. Hence we can write A as

$$\begin{aligned} A &= \bigcup_{\pi \in \Pi_n} \left\{ \omega | \pi(\tilde{Z}) \in \tilde{B} \right\} \\ &= \bigcup_{\pi \in \Pi_n} \left\{ \omega | \tilde{Z} \in \pi^{-1}(\tilde{B}) \right\} \\ &= \left\{ \omega | \tilde{Z} \in \bigcup_{\pi \in \Pi_n} \pi^{-1}(\tilde{B}) \right\} \end{aligned}$$

$$= \left\{ \omega | \tilde{Z} \in B \right\} \ ,$$

with

$$B := \bigcup_{\pi \in \Pi_n} \pi^{-1}(\tilde{B}) \ .$$

Clearly $B \in \mathcal{B}(\mathbb{R}^\infty)$, since it is constructed as countable union of sets in $\mathcal{B}(\mathbb{R}^\infty)$. Moreover note that

$$\bigcup_{\pi \in \Pi_n} \pi^{-1}(\tilde{B}) = \bigcup_{\pi \in \Pi_n} (\pi_0 \circ \pi)^{-1}(\tilde{B}) \ ,$$

since the union is iterating over all $\pi \in \Pi_n$. Thus we can write

$$\begin{aligned} A &= \left\{ \omega | \tilde{Z} \in \bigcup_{\pi \in \Pi_n} (\pi_0 \circ \pi)^{-1}(\tilde{B}) \right\} \\ &= \bigcup_{\pi \in \Pi_n} \left\{ \omega | \tilde{Z} \in (\pi_0 \circ \pi)^{-1}(\tilde{B}) \right\} \\ &= \bigcup_{\pi \in \Pi_n} \left\{ \omega | \pi_0(\tilde{Z}) \in \pi^{-1}(\tilde{B}) \right\} \\ &= \left\{ \omega | \pi_0(\tilde{Z}) \in B \right\} \ . \end{aligned}$$

Whence establishing (2.12). □

Lemma 2.8. *For any $s < t$ s. t. $H(t) < 1$ the following statement holds true*

$$\Delta_n(s,t) = \mathbb{E}[D_n(s,t)] = \mathbb{E}[D_n(s,t) | \mathcal{F}_\infty] \nearrow D(s,t) \ .$$

Proof. Consider that we have for $n \geq 2$

$$\Delta_n(s,t) = \mathbb{E}[D_n(s,t)] = \mathbb{E}[D_n(s,t) | \mathcal{F}_\infty]$$

by definition of $\Delta_n(s,t)$ and Lemma 2.7. Next note that we have $D_n(s,t) \to D(s,t)$

almost surely, according to Lemma 2.5. Moreover we get from Lemma 2.6, that $\{D_n, \mathcal{F}_n\}_{n\geq 1}$ is a reverse supermartingale. Now this together with Proposition 5-3-11 of Neveu (1975) yields

$$\mathbb{E}[D_n(s,t)|\mathcal{F}_\infty] \nearrow D(s,t) \; .$$

This proves the lemma. $\square$

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