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Chapter 1

Modifying the Martingale Convergence Theorem

1.1 Definitions and Assumptions

We're considering the estimator

$$S_n = \sum\sum_{1\leq i<j\leq n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n} W_{j:n}$$

where

$$W_{i:n} = \frac{q(Z_{i:n})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right]$$

Define $\mathcal{F}_n := \sigma\{Z_{1:n}, \ldots, Z_{n:n}, Z_{n+1}, Z_{n+2}, \ldots\}$. Furthermore we will need the following definitions in order to get into a framework that is more similar to that of (forward) sub-martingales. Define

$$\tilde{S}_n^N := S_{N-n+1},\, \mathcal{F}_n^N := \mathcal{F}_{N-n+1}$$

Let $U_n[a, b]$ denote the number of upcrossings of $\tilde{S}_1^N, \ldots, \tilde{S}_n^N$ and define

$$Y_n^N := \tilde{S}_1^N + \sum_{i=1}^{n-1} \epsilon_i (\tilde{S}_{i+1}^N - \tilde{S}_i^N)$$

with

$$\epsilon_i := \begin{cases} 1 & (\tilde{S}_1^N, \ldots, \tilde{S}_i^N) \in B \\ 0 & \text{o.w.} \end{cases}$$

for some Borel set $B \in \mathcal{B}(\mathbb{R}^i)$. We can show that

$$(b-a)\mathbb{E}[U_n[a,b]] \leq \mathbb{E}[Y_n^N] \leq \mathbb{E}[\tilde{S}_n^N] - \sum_{k=1}^{n-1} \mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N - \tilde{S}_k^N | \mathcal{F}_k^N]]$$

We need to show

$$\begin{aligned}
&\lim_{N\to\infty} (b-a)\mathbb{E}[U_N[a,b]] \\
&\leq \lim_{N\to\infty} \mathbb{E}[Y_N^N] \\
&\leq \lim_{N\to\infty} \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1} \mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N - \tilde{S}_k^N | \mathcal{F}_k^N]] \\
&\leq \lim_{N\to\infty} \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1} \mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N | \mathcal{F}_k^N] - \tilde{S}_k^N] \\
&< \infty
\end{aligned}$$

So the main concern is to show that the sum of increases of $\tilde{S}_k^N$ on the right hand side converges. We will need the following assumptions in order to prove the above:

(A1) $\sup_n \mathbb{E}[S_n] < \infty$

TODO The above assumption originates from the original Upcrossing Theorem. I have to refine the statement for our situation. I need to find conditions on ϕ and q (or m) s. t. the above is satisfied. This is pretty much the same problem as for (A4), which I am still investigating.

(A2) There exists $c_1 \in \mathbb{R}^+$ s. t. $\sup_x (q \circ H^{-1})'(x) \leq c_1$.

(A3) We have $q \circ H^{-1}(1) = 1$.

(A4) There exists $c_3 \in \mathbb{R}^+$ s. t.

$$\sum\sum_{1\leq i<j\leq N-k+1} \mathbb{E}\left[\phi^2(Z_{i:N-k+1}, Z_{j:N-k+1}) W_{i:N-k+1}^2 W_{j:N-k+1}^2\right]^{\frac{1}{2}} \leq c_3$$

TODO This assumption is not final yet. I need to complete the proof of Lemma 1.6 (last section of this document) in order to formulate the assumption properly.

1.2 Generalized Upcrossing Theorem

Theorem 1.1. *Assume that (A1) through (A4) hold. Then we have*

$$\begin{aligned}
&\lim_{N\to\infty}(b-a)\mathbb{E}[U_N[a,b]] \\
&\leq \lim_{N\to\infty}\mathbb{E}[Y_N^N] \\
&\leq \lim_{N\to\infty}\mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1}\mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N|\mathcal{F}_k^N] - \tilde{S}_k^N] \\
&< \infty
\end{aligned}$$

We will first establish all necessary lemmas and then continue with the proof of Theorem 1.1 at the end of this section. The following lemma establishes a representation for the conditional expectation under the sum above, that is similar to Dikta (2000).

Lemma 1.2. *Define*

$$Q_{ij}^{n+1} := \begin{cases} Q_i^{n+1} & j \leq n \\ Q_i^{n+1} - \frac{(n+1)\pi_i\pi_n(1-q(Z_{n:n+1}))}{(n-i+1)(2-q(Z_{n:n+1}))} & j = n+1 \end{cases}$$

with

$$Q_i^{n+1} := (n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2 + \frac{\pi_i\pi_{i+1}}{n-i+1}\right\}$$

and

$$\pi_i^{n+1} := \prod_{k=1}^{i-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})} \right]$$

Then

$$\mathbb{E}[S_n|\mathcal{F}_{n+1}] = \sum\sum_{1\le i<j\le n+1} \phi(Z_{i:n+1}, Z_{j:n+1}) W_{i:n+1} W_{j:n+1} Q_{i,j}^{n+1}$$

Proof. This lemma has been proven in my thesis. We already checked the calculations. □

We will need the following result on the increases of the Q_i^{n+1}'s later in the proof of Theorem 1.1.

Lemma 1.3. *Let Q_i^{n+1} be defined as above. Then*

$$Q_{i+1}^{n+1} - Q_i^{n+1} = \frac{\tilde{\pi}_i^2 (n-i+2)^2}{n+1} \left\{ \frac{(q_i - q_{i+1})(n-i)(n-i+1) - q_{i+1}(1-q_i)(n-i+1-q_i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \right\}$$

where $q_i := q(Z_{i:n+1})$ and

$$\tilde{\pi}_i := \pi_i \frac{n+1}{n-i+2}$$

Note that $\tilde{\pi}_i \le 1$ for all $i \le n+1$.

Proof. I proved this lemma in my thesis. □

Lemma 1.4. *Let (A2) be satisfied. Then the following statements hold true for $k \le n-1$*

(i) We have

$$\mathbb{E}[|q(Z_{k:n}) - q(Z_{k+1:n})|] \le \frac{c_1}{n+1} \tag{1.1}$$

(ii) Furthermore assume that (A3) holds. Then

$$\mathbb{E}[1 - q(Z_{k:n})] \le \frac{c_1(n-k+1)}{n+1} \tag{1.2}$$

Proof. Let $q_H := q \circ H^{-1}$ and consider that we can write

$$q(H^{-1}(x)) = q(H^{-1}(x_0)) + q_H'(\hat{x})(x - x_0) \tag{1.3}$$

using Taylor expansion for some $\hat{x}$ in between x and x_0. Therefore we have

$$q(H^{-1}(x)) - q(H^{-1}(x_0)) = q_H'(\hat{x})(x - x_0)$$

and hence

$$|q(H^{-1}(x)) - q(H^{-1}(x_0))| = |q_H'(\hat{x})| \cdot |x - x_0| \tag{1.4}$$

Now let $U_1, \dots, U_n$ be i. i. d. $Uni[0,1]$ and set $x = U_{k:n}$ and $x_0 = U_{k+1:n}$. Consider the left hand side of the expression above. We have

$$\begin{aligned}\mathbb{E}[|q(H^{-1}(x)) - q(H^{-1}(x_0))|] &= \mathbb{E}[|q(H^{-1}(U_{k:n})) - q(H^{-1}(U_{k+1:n}))|] \\ &= \mathbb{E}[|q(Z_{k:n}) - q(Z_{k+1:n})|]\end{aligned}$$

Thus we get from (1.4)

$$\begin{aligned}\mathbb{E}[|q(Z_{k:n}) - q(Z_{k+1:n})|] &= \mathbb{E}[|q'(H^{-1}(\hat{x}))| \cdot |U_{k:n} - U_{k+1:n}|] \\ &= \mathbb{E}[|q'(H^{-1}(\hat{x}))| \cdot (U_{k+1:n} - U_{k:n})]\end{aligned}$$

where $\hat{x} \in [U_{k:n}, U_{k+1:n}]$. From assumption (A2) directly follows that

$$\left|q'(H^{-1}(x))\right| \leq c_1$$

for all $x \in [0,1]$. Hence we have

$$\mathbb{E}[|q(Z_{k:n}) - q(Z_{k+1:n})|] = c_1 \mathbb{E}[U_{k+1:n} - U_{k:n}]$$

From Shorack and Wellner (2009) (p. 271), we get

$$\mathbb{E}[U_{k+1:n} - U_{k:n}] = \frac{1}{n+1} \tag{1.5}$$

Therefore we may conclude

$$\begin{aligned} \mathbb{E}[|q(Z_{k:n}) - q(Z_{k+1:n})|] &\leq c_1 \mathbb{E}[U_{k+1:n} - U_{k:n}] \\ &= \frac{c_1}{n+1} \end{aligned} \tag{1.6}$$

Thus proving part (i). We will now continue with the proof of part (ii). Consider

$$\begin{aligned} 1 - q(Z_{k:n}) &= 1 - q(Z_{n:n}) + \sum_{l=k}^{n-1} (q(Z_{l+1:n}) - q(Z_{l:n})) \\ &\leq 1 - q(Z_{n:n}) + \sum_{l=k}^{n-1} |q(Z_{l+1:n}) - q(Z_{l:n})| \end{aligned}$$

Taking expectations on each side yields

$$1 - \mathbb{E}[q(Z_{k:n})] \leq 1 - \mathbb{E}[q(Z_{n:n})] + \sum_{l=k}^{n-1} \mathbb{E}[|q(Z_{l+1:n}) - q(Z_{l:n})|]$$

Now we apply inequality (1.6) to the expectation under the sum to get

$$1 - \mathbb{E}[q(Z_{k:n})] \leq 1 - \mathbb{E}[q(Z_{n:n})] + \frac{c_1(n-k)}{n+1} \tag{1.7}$$

Recall the Taylor expansion from above

$$q(H^{-1}(x)) = q(H^{-1}(x_0)) + q'_H(\hat{x})(x - x_0)$$

Setting $x = 1$ and $x_0 = U_{n:n}$ and taking expectations on both sides yields

$$\mathbb{E}[q(H^{-1}(1))] = \mathbb{E}[q(Z_{n:n})] + \mathbb{E}[q'_H(\hat{x}_n)(1 - U_{n:n})]$$

Now we get from assumption (A2) that

$$\begin{aligned}\mathbb{E}[q(Z_{n:n})] &= \mathbb{E}[q(H^{-1}(1))] - \mathbb{E}[q'_H(\hat{x}_n)(1 - U_{n:n})] \\ &\geq \mathbb{E}[q(H^{-1}(1))] - c_1\mathbb{E}[1 - U_{n:n}]\end{aligned}$$

Using Shorack and Wellner (2009) again, we obtain

$$\mathbb{E}[q(Z_{n:n})] = \mathbb{E}[q(H^{-1}(1))] - \frac{c_1}{n+1}$$

TODO Should I prove $\mathbb{E}[1 - U_{n:n}] = \frac{1}{n+1}$? Applying (A3) yields

$$\mathbb{E}[q(Z_{n:n})] \geq 1 - \frac{c_1}{n+1}$$

By combining the above with (1.7) we get

$$1 - \mathbb{E}[q(Z_{k:n})] \leq 1 - 1 + \frac{c_1}{n+1} + \frac{c_1(n-k)}{n+1} = \frac{c_1(n-k+1)}{n+1}$$

This concludes the proof of part (ii). □

The following lemma contains some upper bounds that will be needed later in the proof of Theorem 1.1.

Lemma 1.5. *For $n \geq 2$ the following statements hold true*

(i)

$$\sum_{k=1}^{n-1} \frac{1}{k} \leq \ln(n-1) + 1 \tag{1.8}$$

(ii)

$$\frac{\ln(n-1)+1}{(n+1)^{\frac{1}{3}}} \leq 3 \tag{1.9}$$

Proof. We will start with the proof of part (i). Consider

$$
\begin{aligned}
\sum_{k=1}^{n-1} \frac{1}{k} &\leq \ln(n-1)+1 \\
\Leftrightarrow \quad \sum_{k=1}^{n-1} \frac{1}{k} - 1 &\leq \ln(n-1) \\
\Leftrightarrow \quad \sum_{k=2}^{n-1} \frac{1}{k} &\leq \ln(n-1) \\
\Leftrightarrow \quad \prod_{k=2}^{n-1} \exp\left(\frac{1}{k}\right) &\leq n-1 \qquad (1.10)
\end{aligned}
$$

Now we will continue by induction. For $n = 2$ inequality (1.10) is obviously satisfied, as the product is empty in this case. Now assume that (1.10) holds for any n, then it should hold for $n+1$. It remains to show that

$$\prod_{k=2}^{n} \exp\left(\frac{1}{k}\right) \leq n$$

Consider

$$
\begin{aligned}
\prod_{k=2}^{n} \exp\left(\frac{1}{k}\right) &= \exp\left(\frac{1}{n}\right) \prod_{k=2}^{n-1} \exp\left(\frac{1}{k}\right) \\
&\leq \exp\left(\frac{1}{n}\right)(n-1)
\end{aligned}
$$

It is well known that for any $x > 1$

$$\exp(x) < \frac{1}{1-x}$$

and hence

$$\exp\left(\frac{1}{n}\right) < \frac{1}{1-\frac{1}{n}} = \frac{n}{n-1}$$

Thus we get

$$\prod_{k=2}^{n} \exp\left(\frac{1}{k}\right) < \frac{n}{n-1}(n-1) = n$$

This concludes the proof of part (i). We will continue with the proof of part (ii). Note that (1.9) is equivalent to showing

$$\ln(n-1) + 1 \leq 3(n+1)^{\frac{1}{3}}$$

Since $\ln(n-1) \leq \ln(n+1)$ it remains to show

$$\ln(n+1) + 1 \leq 3(n+1)^{\frac{1}{3}} \tag{1.11}$$

It is easy to check that inequality (1.11) holds for $n = 2$. Now consider that

$$\frac{d}{dn}(\ln(n+1) + 1) = \frac{1}{n+1}$$

and

$$\frac{d}{dn}3(n+1)^{\frac{1}{3}} = \frac{1}{(n+1)^{\frac{2}{3}}}$$

Now for $n \geq 2$ we have

$$\frac{\frac{1}{n+1}}{\frac{1}{(n+1)^{\frac{2}{3}}}} = \frac{(n+1)^{\frac{2}{3}}}{n+1} = \frac{1}{(n+1)^{\frac{1}{3}}} < 1$$

and hence

$$\frac{d}{dn}(\ln(n+1) + 1) \leq \frac{d}{dn}3(n+1)^{\frac{1}{3}} \tag{1.12}$$

for all $n \geq 2$. Now the result in (ii) follows directly from (1.11) and (1.12) . $\square$

Now we established everything we need in order to proceed with the proof of Theorem 1.1. Recall that we need to show

$$\lim_{N\to\infty}(b-a)\mathbb{E}[U_N[a,b]] < \infty$$

Proof of Theorem 1. Let (A1) through (A4) be satisfied. Recall the following inequality (proven in my thesis). We have for $n \leq N$

$$(b-a)\mathbb{E}[U_n[a,b]] \leq \mathbb{E}[Y_n^N] \leq \mathbb{E}[\tilde{S}_n^N] - \sum_{k=1}^{n-1}\mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N|\mathcal{F}_k^N] - \tilde{S}_k^N]$$

Now consider that we get the following from Lemma 1.2

$$\begin{aligned}\mathbb{E}[\tilde{S}_{k+1}^N|\tilde{\mathcal{F}}_k^N] &= \mathbb{E}[S_{N-k}|\mathcal{F}_{N-k+1}] \\ &= \sum\sum_{1\leq i<j\leq N-k+1}\phi(Z_{i:N-k+1}, Z_{j:N-k+1})W_{i:N-k+1}W_{j:N-k+1}Q_{i,j}^{N-k+1}\end{aligned}$$

Therefore we have

$$\begin{aligned}\mathbb{E}[Y_N^N] &\leq \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1}\mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N|\mathcal{F}_k^N] - \tilde{S}_k^N] \\ &= \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1}\mathbb{E}\left[(1-\epsilon_k)\sum\sum_{1\leq i<j\leq N-k+1}\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right. \\ &\qquad \left.\times W_{i:N-k+1}W_{j:N-k+1}(Q_{i,j}^{N-k+1}-1)\right] \\ &= \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1}\sum\sum_{1\leq i<j\leq N-k+1}\mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right. \\ &\qquad \left.\times W_{i:N-k+1}W_{j:N-k+1}(Q_{i,j}^{N-k+1}-1)\right] \\ &\leq \mathbb{E}[\tilde{S}_N^N] + \left|\sum_{k=1}^{N-1}\sum\sum_{1\leq i<j\leq N-k+1}\mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right.\right. \\ &\qquad \left.\left.\times W_{i:N-k+1}W_{j:N-k+1}(Q_{i,j}^{N-k+1}-1)\right]\right|\end{aligned}$$

$$\le \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} \big|\mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1}, Z_{j:N-k+1}) \right.$$
$$\left. \times\ W_{i:N-k+1}W_{j:N-k+1}(Q_{i,j}^{N-k+1}-1)\right]\big|$$

Now using Jensen's inequality yields

$$\mathbb{E}[Y_N^N] \le \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1}, Z_{j:N-k+1}) \right.$$
$$\left. \times\ W_{i:N-k+1}W_{j:N-k+1}\big|(Q_{i,j}^{N-k+1}-1)\big|\right]$$
$$\le \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[\phi(Z_{i:N-k+1}, Z_{j:N-k+1}) \right.$$
$$\left. \times\ W_{i:N-k+1}W_{j:N-k+1}\big|(Q_{i,j}^{N-k+1}-1)\big|\right]$$

The latter inequality above holds, because $1-\epsilon_k \le 1$ for all $k \le N-1$. By applying the Cauchy-Schwarz inequality on the expectation above, we obtain

$$\mathbb{E}[Y_N^N] \le \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[\phi^2(Z_{i:N-k+1}, Z_{j:N-k+1})W_{i:N-k+1}^2 W_{j:N-k+1}^2\right]^{\frac{1}{2}}$$
$$\times\, \mathbb{E}\left[(Q_{i,j}^{N-k+1}-1)^2\right]^{\frac{1}{2}} \tag{1.13}$$

We will now proceed to find an upper bound for $\mathbb{E}\left[(Q_{i,j}^{N-k+1}-1)^2\right]^{\frac{1}{2}}$. For the purpose of simpler notation we set $n := n(k,N) = N-k$. The inequality above can now be written as

$$\mathbb{E}[Y_N^N] \le \mathbb{E}[S_1] + \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le n+1} \mathbb{E}\left[\phi^2(Z_{i:n+1}, Z_{j:n+1})W_{i:n+1}^2 W_{j:n+1}^2\right]^{\frac{1}{2}}$$
$$\times\, \mathbb{E}\left[(Q_{i,j}^{n+1}-1)^2\right]^{\frac{1}{2}}$$

Note k_1 and k_2 below do **not** correspond to k above in any way. Consider

$$Q_i^{n+1} - 1 = Q_1^{n+1} + \sum_{k_1=1}^{i-1} (Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}) - 1 \tag{1.14}$$

and recall the following definition

$$Q_i^{n+1} := (n+1) \left\{ \sum_{r=1}^{i-1} \left[\frac{\pi_r}{n - r + 2 - q(Z_{r:n+1})} \right]^2 + \frac{\pi_i \pi_{i+1}}{n - i + 1} \right\}$$

where

$$\pi_i := \prod_{k=1}^{i-1} \left[\frac{n - k + 1 - q(Z_{k:n+1})}{n - k + 2 - q(Z_{k:n+1})} \right]$$

We have $\pi_1 = 1$, since the product above is empty for $i = 1$ and

$$\pi_2 = \frac{n - q(Z_{1:n+1})}{n + 1 - q(Z_{1:n+1})}$$

Thus we get

$$\begin{aligned}
Q_1^{n+1} - 1 &= (n+1) \frac{\pi_1 \pi_2}{n} - 1 \\
&= \frac{(n+1)(n - q(Z_{1:n+1}))}{n(n + 1 - q(Z_{1:n+1}))} - 1 \\
&= \frac{n(n + 1 - q(Z_{1:n+1})) - q(Z_{1:n+1})}{n(n + 1 - q(Z_{1:n+1}))} - 1 \\
&= 1 - \frac{q(Z_{1:n+1})}{n(n + 1 - q(Z_{1:n+1}))} - 1 \\
&= -\frac{q(Z_{1:n+1})}{n(n + 1 - q(Z_{1:n+1}))}
\end{aligned}$$

Therefore we get from (1.14)

$$Q_i^{n+1} - 1 = \sum_{k_1=1}^{i-1} (Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}) - \frac{q(Z_{1:n+1})}{n(n + 1 - q(Z_{1:n+1}))}$$

Moreover we have

$$
\begin{aligned}
(Q_i^{n+1}-1)^2 &= \sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}(Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1})(Q_{k_2+1}^{n+1}-Q_{k_2}^{n+1}) \\
&\quad - \frac{2q(Z_{1:n+1})}{n(n+1-q(Z_{1:n+1}))}\sum_{k=1}^{i-1}(Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}) \\
&\quad + \frac{q^2(Z_{1:n+1})}{n^2(n+1-q(Z_{1:n+1}))^2} \\
&\le \sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}|\cdot|Q_{k_2+1}^{n+1}-Q_{k_2}^{n+1}| \\
&\quad + \frac{2q(Z_{1:n+1})}{n(n+1-q(Z_{1:n+1}))}\sum_{k_1=1}^{i-1}|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}| \\
&\quad + \frac{q^2(Z_{1:n+1})}{n^2(n+1-q(Z_{1:n+1}))^2} \\
&\le \sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}|\cdot|Q_{k_2+1}^{n+1}-Q_{k_2}^{n+1}| \\
&\quad + \frac{2}{n^2}\sum_{k_1=1}^{i-1}|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}| + \frac{1}{n^4}
\end{aligned} \tag{1.15}
$$

We get from Lemma 1.3 that

$$
\begin{aligned}
&|Q_{i+1}^{n+1}-Q_i^{n+1}| \\
&= \frac{\tilde{\pi}_i^2(n-i+2)^2}{n+1}\cdot\left|\frac{(q_i-q_{i+1})(n-i)(n-i+1)-q_{i+1}(1-q_i)(n-i+1-q_i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})}\right| \\
&\le \frac{\tilde{\pi}_i^2(n-i+2)^2}{n+1}\cdot\frac{|q_i-q_{i+1}|(n-i)(n-i+1)+q_{i+1}(1-q_i)(n-i+1-q_i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\
&\le \frac{(n-i+2)^2}{n+1}\left\{\frac{|q_i-q_{i+1}|(n-i)(n-i+1)+q_{i+1}(1-q_i)(n-i+1)}{(n-i)(n-i+1)(n-i+1)^2(n-i)}\right\} \\
&= \frac{(n-i+2)^2}{n+1}\left\{\frac{|q_i-q_{i+1}|(n-i)+q_{i+1}(1-q_i)}{(n-i)^2(n-i+1)^2}\right\} \\
&\le \frac{4|q_i-q_{i+1}|}{(n+1)(n-i)}+\frac{4(1-q_i)}{(n+1)(n-i)^2}
\end{aligned} \tag{1.16}
$$

The latter inequality above holds since

$$\frac{n-i+2}{n-i+1} = 1 + \frac{1}{n-i+1} \leq 2$$

and $q_{i+1} \leq 1$. Thus we have

$$\begin{aligned}
&|Q^{n+1}_{k_1+1} - Q^{n+1}_{k_1}| \cdot |Q^{n+1}_{k_2+1} - Q^{n+1}_{k_2}| \\
&\leq \left[\frac{4|q_{k_1} - q_{k_1+1}|}{(n+1)(n-k_1)} + \frac{4(1-q_{k_1})}{(n+1)(n-k_1)^2}\right] \\
&\quad \times \left[\frac{4|q_{k_2} - q_{k_2+1}|}{(n+1)(n-k_2)} + \frac{4(1-q_{k_2})}{(n+1)(n-k_2)^2}\right] \\
&= \frac{16|q_{k_1} - q_{k_1+1}||q_{k_2} - q_{k_2+1}|}{(n+1)^2(n-k_1)(n-k_2)} + \frac{16|q_{k_1} - q_{k_1+1}|(1-q_{k_2})}{(n+1)^2(n-k_1)(n-k_2)^2} \\
&\quad + \frac{16(1-q_{k_1})|q_{k_2} - q_{k_2+1}|}{(n+1)^2(n-k_1)^2(n-k_2)} + \frac{16(1-q_{k_1})(1-q_{k_2})}{(n+1)^2(n-k_1)^2(n-k_2)^2} \\
&\leq \frac{16|q_{k_1} - q_{k_1+1}|}{(n+1)^2(n-k_1)(n-k_2)} + \frac{16|q_{k_1} - q_{k_1+1}|}{(n+1)^2(n-k_1)(n-k_2)^2} \\
&\quad + \frac{16|q_{k_2} - q_{k_2+1}|}{(n+1)^2(n-k_1)^2(n-k_2)} + \frac{16(1-q_{k_1})}{(n+1)^2(n-k_1)^2(n-k_2)^2}
\end{aligned}$$

Here the latter inequality holds, since we have $|q_k - q_{k+1}| \leq 1$ and $1 - q_k \leq 1$ for all $k \leq n-1$.

Recall that

$$\begin{aligned}
(Q^{n+1}_i - 1)^2 \leq &\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1} \left|Q^{n+1}_{k_1+1} - Q^{n+1}_{k_1}\right|\left|Q^{n+1}_{k_2+1} - Q^{n+1}_{k_2}\right| \\
&+ \frac{2}{n^2}\sum_{k_1=1}^{i-1}\left|Q^{n+1}_{k_1+1} - Q^{n+1}_{k_1}\right| + \frac{1}{n^4}
\end{aligned}$$

Taking expectations on each side yields

$$\mathbb{E}[(Q^{n+1}_i - 1)^2] \leq \sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1} \mathbb{E}\left[\left|Q^{n+1}_{k_1+1} - Q^{n+1}_{k_1}\right|\left|Q^{n+1}_{k_2+1} - Q^{n+1}_{k_2}\right|\right]$$

$$+\frac{2}{n^2}\sum_{k_1=1}^{i-1}\mathbb{E}[|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}|]+\frac{1}{n^4} \tag{1.17}$$

Consider the expectation under the double sum above. We have

$$\begin{aligned}
&\mathbb{E}\left[\left|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}\right|\left|Q_{k_2+1}^{n+1}-Q_{k_2}^{n+1}\right|\right]\\
&\leq \frac{16\mathbb{E}\left[|q_{k_1}-q_{k_1+1}|\right]}{(n+1)^2(n-k_1)(n-k_2)}+\frac{16\mathbb{E}\left[|q_{k_1}-q_{k_1+1}|\right]}{(n+1)^2(n-k_1)(n-k_2)^2}\\
&\quad+\frac{16\mathbb{E}\left[|q_{k_2}-q_{k_2+1}|\right]}{(n+1)^2(n-k_1)^2(n-k_2)}+\frac{16\mathbb{E}\left[(1-q_{k_1})\right]}{(n+1)^2(n-k_1)^2(n-k_2)^2}
\end{aligned} \tag{1.18}$$

We will now use Lemma 1.4 to establish an upper bound for the expectation above. Combining (1.1) and (1.2) above with (1.18) yields

$$\begin{aligned}
&\mathbb{E}[\left|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}\right|\left|Q_{k_2+1}^{n+1}-Q_{k_2}^{n+1}\right|]\\
&\leq \frac{16c_1}{(n+1)^3(n-k_1)(n-k_2)}+\frac{16c_1}{(n+1)^3(n-k_1)(n-k_2)^2}\\
&\quad+\frac{16c_1}{(n+1)^3(n-k_1)^2(n-k_2)}+\frac{16c_1(n-k_1)+16}{(n+1)^3(n-k_1)^2(n-k_2)^2}
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\mathbb{E}[\left|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}\right|\left|Q_{k_2+1}^{n+1}-Q_{k_2}^{n+1}\right|]\\
&\leq \sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\frac{16c_1}{(n+1)^3(n-k_1)(n-k_2)}+\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\frac{16c_1}{(n+1)^3(n-k_1)(n-k_2)^2}\\
&\quad+\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\frac{16c_1}{(n+1)^3(n-k_1)^2(n-k_2)}+\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\frac{16c_1(n-k_1)}{(n+1)^3(n-k_1)^2(n-k_2)^2}\\
&\quad+\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\frac{16}{(n+1)^3(n-k_1)^2(n-k_2)^2}\\
&=\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\frac{16c_1}{(n+1)^3(n-k_1)(n-k_2)}+\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\frac{32c_1}{(n+1)^3(n-k_1)(n-k_2)^2}\\
&\quad+\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\frac{16c_1}{(n+1)^3(n-k_1)^2(n-k_2)}+\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\frac{16}{(n+1)^3(n-k_1)^2(n-k_2)^2}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{16c_1}{(n+1)^3}\sum_{k_1=1}^{i-1}\frac{1}{(n-k_1)}\sum_{k_2=1}^{i-1}\frac{1}{(n-k_2)} + \frac{32c_1}{(n+1)^3}\sum_{k_1=1}^{i-1}\frac{1}{n-k_1}\sum_{k_2=1}^{i-1}\frac{1}{(n-k_2)^2} \\
&\quad + \frac{16c_1}{(n+1)^3}\sum_{k_1=1}^{i-1}\frac{1}{(n-k_1)^2}\sum_{k_2=1}^{i-1}\frac{1}{n-k_2} + \frac{16}{(n+1)^3}\sum_{k_1=1}^{i-1}\frac{1}{(n-k_1)^2}\sum_{k_2=1}^{i-1}\frac{1}{(n-k_2)^2} \\
&\leq \frac{16c_1}{(n+1)^3}\sum_{k_1=n-i+1}^{n-1}\frac{1}{k_1}\sum_{k_2=n-i+1}^{n-1}\frac{1}{k_2} + \frac{32c_1}{(n+1)^3}\sum_{k_1=n-i+1}^{n-1}\frac{1}{k_1}\sum_{k_2=n-i+1}^{n-1}\frac{1}{k_2^2} \\
&\quad + \frac{16c_1}{(n+1)^3}\sum_{k_1=n-i+1}^{n-1}\frac{1}{k_1^2}\sum_{k_2=n-i+1}^{n-1}\frac{1}{k_2} + \frac{16}{(n+1)^3}\sum_{k_1=n-i+1}^{n-1}\frac{1}{k_1^2}\sum_{k_2=1}^{i-1}\frac{1}{k_2^2} \qquad (1.19)
\end{aligned}$$

Now using (1.8) and (1.9) from Lemma 1.5 on inequality (1.19) yields

$$\begin{aligned}
&\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\mathbb{E}[|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}||Q_{k_2+1}^{n+1}-Q_{k_2}^{n+1}|] \\
&\leq \frac{16c_1}{(n+1)^3}(\ln(n-1)+1)^2 + \frac{64c_1}{(n+1)^3}(\ln(n-1)+1) \\
&\qquad + \frac{32c_1}{(n+1)^3}(\ln(n-1)+1) + \frac{64}{(n+1)^3} \\
&\leq \frac{144c_1}{(n+1)^{\frac{7}{3}}} + \frac{288c_1}{(n+1)^{\frac{8}{3}}} + \frac{64}{(n+1)^3} \\
&\leq \frac{432c_1+64}{(n+1)^{\frac{7}{3}}} \qquad (1.20)
\end{aligned}$$

We will now proceed with the second sum in (1.17). We get from (1.16)

$$\mathbb{E}[|Q_{i+1}^{n+1}-Q_i^{n+1}|] \leq \frac{4\mathbb{E}[|q_i-q_{i+1}|]}{(n+1)(n-i)} + \frac{4\mathbb{E}[1-q_i]}{(n+1)(n-i)^2}$$

Therefore we obtain

$$\frac{2}{n^2}\sum_{k_1=1}^{i-1}\mathbb{E}[|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}|] \leq \frac{8}{n^2(n+1)^2}\sum_{k_1=1}^{i-1}\frac{\mathbb{E}[|q_{k_1}-q_{k_1+1}|]}{n-k_1} + \frac{\mathbb{E}[1-q_{k_1}]}{(n-k_1)^2}$$

Again using (1.1) and (1.2) reveals

$$\frac{2}{n^2}\sum_{k_1=1}^{i-1}\mathbb{E}[|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}|] \leq \frac{8}{n^2(n+1)^2}\left\{\sum_{k_1=1}^{i-1}\frac{c_1}{(n-k_1)} + \sum_{k_1=1}^{i-1}\frac{c_1(n-k_1+1)}{(n-k_1)^2}\right\}$$

$$= \frac{8}{n^2(n+1)^2}\left\{2\sum_{k_1=1}^{i-1}\frac{c_1}{(n-k_1)} + \sum_{k_1=1}^{i-1}\frac{c_1}{(n-k_1)^2}\right\}$$
$$= \frac{8}{n^2(n+1)^2}\left\{2\cdot\sum_{k_1=n-i+1}^{n-1}\frac{c_1}{k_1} + \sum_{k_1=n-i+1}^{n-1}\frac{c_1}{k_1^2}\right\}$$

By using (1.8) and (1.9) again we obtain

$$\begin{aligned}
\frac{2}{n^2}\sum_{k_1=1}^{i-1}\mathbb{E}[|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}|] &\le \frac{8\cdot\{2c_1(\ln(n-1)+1)+2c_1\}}{n^2(n+1)^2} \\
&= \frac{16c_1(\ln(n-1)+1)}{n^2(n+1)^2} + \frac{16c_1}{n^2(n+1)^2} \\
&\le \frac{48c_1}{n^2(n+1)^{\frac{5}{3}}} + \frac{16c_1}{n^2(n+1)^2} \\
&\le \frac{64c_1}{n^2(n+1)^{\frac{5}{3}}} \qquad (1.21)
\end{aligned}$$

Again recall the following fact

$$\begin{aligned}
\mathbb{E}[(Q_i^{n+1}-1)^2] = &\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\mathbb{E}\big[\big|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}\big|\big|Q_{k_2+1}^{n+1} - Q_{k_2}^{n+1}\big|\big] \\
&+ \frac{2}{n^2}\sum_{k_1=1}^{i-1}\mathbb{E}[|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}|] + \frac{1}{n^4}
\end{aligned}$$

Combining the above with (1.20) and (1.21) yields

$$\begin{aligned}
\mathbb{E}[(Q_i^{n+1}-1)^2] &\le \frac{432c_1+64}{(n+1)^{\frac{7}{3}}} + \frac{64c_1}{n^2(n+1)^{\frac{5}{3}}} + \frac{1}{n^4} \\
&\le \frac{432c_1+64}{n^{\frac{7}{3}}} + \frac{64c_1}{n^{\frac{11}{3}}} + \frac{1}{n^4} \\
&\le \frac{1}{n^{\frac{7}{3}}}\left[432c_1 + 64 + \frac{64c_1}{n^{\frac{4}{3}}} + \frac{1}{n^{\frac{5}{3}}}\right] \\
&\le \frac{496c_1+65}{n^{\frac{7}{3}}} \\
&= \frac{c_2}{n^{\frac{7}{3}}}
\end{aligned}$$

with $c_2 := 496c_1 + 65$. Therefore

$$\mathbb{E}[(Q_i^{n+1} - 1)^2]^{\frac{1}{2}} \leq \frac{\sqrt{c_2}}{n^{\frac{7}{6}}}$$

Recall that we set $n = N - k$. Thus we can write

$$\mathbb{E}[(Q_i^{N-k+1} - 1)^2]^{\frac{1}{2}} \leq \frac{\sqrt{c_2}}{(N-k)^{\frac{7}{6}}}$$

Now combining the latter with (1.13) yields

$$\begin{aligned}
\mathbb{E}[Y_N^N] &\leq \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1 \leq i < j \leq N-k+1} \mathbb{E}\left[\phi^2(Z_{i:N-k+1}, Z_{j:N-k+1}) W_{i:N-k+1}^2 W_{j:N-k+1}^2\right]^{\frac{1}{2}} \\
&\qquad \times \mathbb{E}\left[(Q_{i,j}^{N-k+1} - 1)^2\right]^{\frac{1}{2}} \\
&\leq \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1 \leq i < j \leq N-k+1} \mathbb{E}\left[\phi^2(Z_{i:N-k+1}, Z_{j:N-k+1}) W_{i:N-k+1}^2 W_{j:N-k+1}^2\right]^{\frac{1}{2}} \\
&\qquad \times \frac{\sqrt{c_2}}{(N-k)^{\frac{7}{6}}}
\end{aligned}$$

Thus it remains to show that

$$\sum\sum_{1 \leq i < j \leq N-k+1} \mathbb{E}\left[\phi^2(Z_{(i)}, Z_{(j)}) W_{(i)}^2 W_{(j)}^2\right]^{\frac{1}{2}} \leq c_3$$

is constant. Then we would have

$$\begin{aligned}
\lim_{N \to \infty} \mathbb{E}[U_N[a,b]] &\leq \lim_{N \to \infty} \mathbb{E}[Y_N^N] \\
&\leq \lim_{N \to \infty} \left\{ \mathbb{E}[\tilde{S}_N^N] + c_3 \sum_{k=1}^{N-1} \frac{\sqrt{c_2}}{(N-k)^{\frac{7}{6}}} \right\} \\
&\leq \sup_N \mathbb{E}[\tilde{S}_N^N] + \sqrt{c_2} c_3 \left\{ \lim_{N \to \infty} \sum_{k=1}^{N-1} \frac{1}{(N-k)^{\frac{7}{6}}} \right\} \\
&< \infty
\end{aligned}$$

And therefore we may finally conclude that $S = \lim_{n\to\infty} S_n$ exists. Note that there is more argumentation about the relationship between $U_N[a,b]$ and $\lim\limits_{n\to\infty} S_n$ in my thesis.

$\square$

1.3 The missing bound

It remains to show that

$$\sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[\phi^2(Z_{i:N-k+1}, Z_{j:N-k+1})W^2_{i:N-k+1}W^2_{j:N-k+1}\right]^{\frac{1}{2}} \le c_3$$

for some constant $c_3 \in R^+$. For the sake of simplicity we will set $n = N - k$ again. The following lemma contains the result needed to prove Theorem 1.1.

Lemma 1.6. *There exists constant $c_3 \in R^+$ s. t.*

$$\sum\sum_{1\le i<j\le n+1} \mathbb{E}\left[\phi^2(Z_{i:n+1}, Z_{j:n+1})W^2_{i:n+1}W^2_{j:n+1}\right]^{\frac{1}{2}} \le c_3$$

Proof. Consider

$$\begin{aligned}
&\sum\sum_{1\le i<j\le n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n})W^2_{i:n}W^2_{j:n}\right]^{\frac{1}{2}} \\
&= \sum\sum_{1\le i<j\le n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n})\frac{q^2(Z_{i:n})}{n-i+1}\prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n})}{n-k+1}\right]^2\right. \\
&\qquad\qquad \left.\times\ \frac{q^2(Z_{j:n})}{n-j+1}\prod_{l=1}^{j-1}\left[1-\frac{q(Z_{l:n})}{n-l+1}\right]^2\right]^{\frac{1}{2}} \\
&= \frac{1}{n^2}\sum\sum_{1\le i<j\le n} \mathbb{E}\Bigg[\phi^2(Z_{i:n}, Z_{j:n})q^2(Z_{i:n})q^2(Z_{j:n})
\end{aligned}$$

$$\times \prod_{k=1}^{i-1}\left[1+\frac{1-q(Z_{k:n})}{n-k}\right]^2 \prod_{l=1}^{j-1}\left[1+\frac{1-q(Z_{l:n})}{n-l}\right]^2\Bigg]^{\frac{1}{2}} \tag{1.22}$$

The latter equality holds since

$$\begin{aligned}
\frac{q^2(Z_{i:n})}{n-i+1}\prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n})}{n-k+1}\right]^2 &= \frac{q^2(Z_{i:n})}{n-i+1}\prod_{k=1}^{i-1}\left[\frac{n-k+1-q(Z_{k:n})}{n-k+1}\right]^2 \\
&= \frac{q^2(Z_{i:n})}{n-i+1}\prod_{k=1}^{i-1}\left[\frac{n-k+1-q(Z_{k:n})}{n-k}\cdot\frac{n-k}{n-k+1}\right]^2 \\
&= \frac{q^2(Z_{i:n})}{n^2}\prod_{k=1}^{i-1}\left[\frac{n-k+1-q(Z_{k:n})}{n-k}\right]^2 \\
&= \frac{q^2(Z_{i:n})}{n^2}\prod_{k=1}^{i-1}\left[1+\frac{1-q(Z_{k:n})}{n-k}\right]^2
\end{aligned}$$

Moreover we have for all $i \leq n$ that

$$\prod_{k=1}^{i-1}\left[1+\frac{1-q(Z_{k:n})}{n-k}\right] = \prod_{k=1}^{n}\left[1+\frac{1-q(Z_k)}{n-R_{k,n}}\right]^{\mathbb{1}_{\{Z_k<Z_{i:n}\}}}$$

Hence the right hand side of (1.22) turns into

$$\begin{aligned}
&\frac{1}{n^2}\sum\sum_{1\leq i<j\leq n}\mathbb{E}\Bigg[\phi^2(Z_{i:n},Z_{j:n})q^2(Z_{i:n})\prod_{k=1}^{i-1}\left[1+\frac{1-q(Z_{k:n})}{n-k}\right]^2 \\
&\qquad\qquad \times\, q^2(Z_{j:n})\prod_{l=1}^{j-1}\left[1+\frac{1-q(Z_{l:n})}{n-l}\right]^2\Bigg]^{\frac{1}{2}} \\
&= \frac{1}{n^2}\sum\sum_{1\leq i<j\leq n}\mathbb{E}\Bigg[\phi^2(Z_{i:n},Z_{j:n})q^2(Z_{i:n})\prod_{k=1}^{n}\left[1+\frac{1-q(Z_k)}{n-R_{k,n}}\right]^{2\mathbb{1}_{\{Z_k<Z_{i:n}\}}} \\
&\qquad\qquad \times\, q^2(Z_{j:n})\prod_{l=1}^{n}\left[1+\frac{1-q(Z_l)}{n-R_{l,n}}\right]^{2\mathbb{1}_{\{Z_l<Z_{j:n}\}}}\Bigg]^{\frac{1}{2}} \\
&\stackrel{?}{=} \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\Bigg[\phi^2(Z_i,Z_j)\mathbb{1}_{\{Z_i<Z_j\}}q^2(Z_i)\prod_{k=1}^{n}\left[1+\frac{1-q(Z_k)}{n-R_{k,n}}\right]^{2\mathbb{1}_{\{Z_k<Z_i\}}}
\end{aligned}$$

$$\times\ q^2(Z_j)\prod_{l=1}^{n}\left[1+\frac{1-q(Z_l)}{n-R_{l,n}}\right]^{2\mathbb{1}_{\{Z_l<Z_j\}}}\Bigg]^{\frac{1}{2}}$$

$$\leq \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\Bigg[\phi^2(Z_i,Z_j)q^2(Z_i)\prod_{k=1}^{n}\left[1+\frac{1-q(Z_k)}{n-R_{k,n}}\right]^{2\mathbb{1}_{\{Z_k<Z_i\}}}$$

$$\times\ q^2(Z_j)\prod_{l=1}^{n}\left[1+\frac{1-q(Z_l)}{n-R_{l,n}}\right]^{2\mathbb{1}_{\{Z_l<Z_j\}}}\Bigg]^{\frac{1}{2}}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\Bigg[\phi^2(Z_1,Z_2)q^2(Z_1)\prod_{k=1}^{n}\left[1+\frac{1-q(Z_k)}{n-R_{k,n}}\right]^{2\mathbb{1}_{\{Z_k<Z_1\}}}$$

$$\times\ q^2(Z_2)\prod_{l=1}^{n}\left[1+\frac{1-q(Z_l)}{n-R_{l,n}}\right]^{2\mathbb{1}_{\{Z_l<Z_2\}}}\Bigg]^{\frac{1}{2}}$$

$$= \mathbb{E}\Bigg[\phi^2(Z_1,Z_2)q^2(Z_1)\prod_{k=1}^{n}\left[1+\frac{1-q(Z_k)}{n-R_{k,n}}\right]^{2\mathbb{1}_{\{Z_k<Z_1\}}}$$

$$\times\ q^2(Z_2)\prod_{l=1}^{n}\left[1+\frac{1-q(Z_l)}{n-R_{l,n}}\right]^{2\mathbb{1}_{\{Z_l<Z_2\}}}\Bigg]^{\frac{1}{2}}$$

TODO Split up and fill in more details. Below will be the part where I need conditional expectation. Define

$$B_n(s) := \prod_{k=1}^{n}\left[1+\frac{1-q(Z_k)}{n-R_{k,n}}\right]^{2\mathbb{1}_{\{Z_k<s\}}}$$

and

$$E_n(s,t) := \mathbb{E}\left[\phi^2(s,t)q(s)q(t)B_n(s)B_n(t)\right]$$

Then

$$E_n(Z_1,Z_2) = \mathbb{E}\Bigg[\phi^2(Z_1,Z_2)q^2(Z_1)\prod_{k=1}^{n}\left[1+\frac{1-q(Z_k)}{n-R_{k,n}}\right]^{2\mathbb{1}_{\{Z_k<Z_1\}}}$$

$$\times\ q^2(Z_2)\prod_{l=1}^{n}\left[1+\frac{1-q(Z_l)}{n-R_{l,n}}\right]^{2\mathbb{1}_{\{Z_l<Z_2\}}}\Bigg]^{\frac{1}{2}}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\phi^2(Z_1,Z_2)q^2(Z_1)\exp\left(2\sum_{k=1}^{n}\mathbb{1}_{\{Z_k<Z_1\}}\ln\left[1+\frac{1-q(Z_k)}{n-R_{k,n}}\right]\right)\right.\\
&\qquad\left.\times\; q^2(Z_2)\exp\left(2\sum_{l=1}^{n}\mathbb{1}_{\{Z_l<Z_2\}}\ln\left[1+\frac{1-q(Z_l)}{n-R_{l,n}}\right]\right)\right]^{\frac{1}{2}}\\
&= \mathbb{E}\left[\phi^2(Z_1,Z_2)q^2(Z_1)\exp\left(2\sum_{k=1}^{n}\mathbb{1}_{\{Z_k<Z_1\}}\ln\left[1+\frac{1-q(Z_k)}{n(1-H_n(Z_k))}\right]\right)\right.\\
&\qquad\left.\times\; q^2(Z_2)\exp\left(2\sum_{l=1}^{n}\mathbb{1}_{\{Z_l<Z_2\}}\ln\left[1+\frac{1-q(Z_l)}{n(1-H_n(Z_l))}\right]\right)\right]^{\frac{1}{2}}
\end{aligned}
\tag{1.23}
$$

Now let

$$x_k = \frac{1-q(Z_k)}{n(1-H_n(Z_k))}$$

and note that

$$\ln(1+x_k) \le x_k$$

Thus the right hand side of (1.23) becomes

$$
\begin{aligned}
&\mathbb{E}\left[\phi^2(Z_1,Z_2)q^2(Z_1)\exp\left(2\sum_{k=1}^{n}\mathbb{1}_{\{Z_k<Z_1\}}\ln\left[1+\frac{1-q(Z_k)}{n(1-H_n(Z_k))}\right]\right)\right.\\
&\qquad\left.\times\; q^2(Z_2)\exp\left(2\sum_{l=1}^{n}\mathbb{1}_{\{Z_l<Z_2\}}\ln\left[1+\frac{1-q(Z_l)}{n(1-H_n(Z_l))}\right]\right)\right]^{\frac{1}{2}}\\
&\le \mathbb{E}\left[\phi^2(Z_1,Z_2)q^2(Z_1)\exp\left(2\sum_{k=1}^{n}\mathbb{1}_{\{Z_k<Z_1\}}\frac{1-q(Z_k)}{n(1-H_n(Z_k))}\right)\right.\\
&\qquad\left.\times\; q^2(Z_2)\exp\left(2\sum_{l=1}^{n}\mathbb{1}_{\{Z_l<Z_2\}}\frac{1-q(Z_l)}{n(1-H_n(Z_l))}\right)\right]^{\frac{1}{2}}\\
&= \mathbb{E}\left[\phi^2(Z_1,Z_2)q^2(Z_1)\exp\left(2\int_0^{Z_1-}\frac{1-q(z)}{n(1-H_n(z))}H_n(dz)\right)\right.\\
&\qquad\left.\times\; q^2(Z_2)\exp\left(2\int_0^{Z_2-}\frac{1-q(z)}{n(1-H_n(z))}H_n(dz)\right)\right]^{\frac{1}{2}}\\
&= \left[\int_0^\infty\int_0^\infty \phi^2(s,t)q^2(s)\exp\left(2\int_0^{s-}\frac{1-q(z)}{n(1-H_n(z))}H_n(dz)\right)\right.
\end{aligned}
$$

$$\times\, q^2(t) \exp\left(2 \int_0^{t-} \frac{1-q(z)}{n(1-H_n(z))} H_n(dz)\right) H(dt)H(ds)\Bigg]^{\frac{1}{2}}$$

where $s, t \leq \inf\{x|H(x) = 1\}$. (?)

TODO Construction work ahead..

- By Glivenko-Cantelli H_n converges uniformly against H w. p. 1
- SLLN on $H_n - H$
- Shorack and Wellner (2009), p. 304f
-

$$\begin{aligned}
\int_0^{s-} \frac{1-q(z)}{n(1-H_n(z))} H_n(dz) &\overset{G.C.}{=} \int_0^{s-} \frac{1-q(z)}{n(1-H(z))} H_n(dz) \\
&= \int_0^{s-} \frac{1-q(z)}{n(1-H(z))} H(dz) \\
&\quad + \int_0^{s-} \frac{1-q(z)}{n(1-H(z))} (H_n - H)(dz) \\
&\xrightarrow{SLLN} \int_0^{s-} \frac{1-q(z)}{n(1-H(z))} H(dz)
\end{aligned}$$

- upper limit c_3 guaranteed?

$\square$

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