Large sample properties of U-Statistics under semiparametric Random Censorship

by

Jan Hoft

A Dissertation Submitted in

Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

in

Mathematics

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Graduate School Approval Date

Abstract

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The University of Wisconsin–Milwaukee, 2018
Under the Supervision of Professor Gerhard Dikta and Professor Jugal Ghorai

About this document

This is a **draft version** of my thesis. So far I was able to establish the SLLN for the the semiparametric U-Statistics of degree 2. The mathematics in here should be correct. But I still want to expand the introduction and I need to write the conclusion and the abstract. Also I want to include a simulation about the semiparametric U-Statistics of degree 2. Moreover I am not quite sure about the section titles yet, so I might change those.

How to read this work

- I put

  Comment comment

  statements to mark problematic spots in this thesis and to share my thoughts about those.

- All **TODO** todo statements are to mark parts of the text for myself, which I need to change later.

TODO Summarize your paper here, including the basic methods used in the study. A signature line for your advisor will be included at the end of the abstract. NOTE:

The abstract can have multiple pages, but is restricted to 400 words in length!

Major Professor Date

Contents

Chapter 1

Introduction

Assume that $X_1, ..., X_n$ are independent and identically distributed (i. i. d.) random variables (r. v.) on $\mathbb{R}$ which are defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Denote their common probability distribution function (d. f.) by F. For some $k \geq 1$ let $\phi : \mathbb{R}^k \longrightarrow \mathbb{R}$ be a symmetric Borel-measurable function. Define

$$\theta_F = \int ... \int \phi \prod_{j=1}^{k} dF. \tag{1.1}$$

Examples of this kind of parameters include the expected value, variance and any higher moments of the X's. One approach to estimate those integrals is given by the so called U-Statistics. To obtain this estimator we need to replace the true d. f. F by the empirical d. f. F_n which is defined by

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{X_i \leq t\}}.$$

Now plugging F_n into (1.1) yields

$$\int ... \int \phi \prod_{j=1}^{k} dF_n = \frac{1}{n^k} \sum_{i_1=1}^{n} ... \sum_{i_k=1}^{n} \phi(X_{i_1}, ..., X_{i_k})$$

The expression on the right hand side in the equation above is known as V-statistic. It includes repeated observations. An unbiased estimate of θ_F based on distinct observations only can be expressed as

$$U_{kn}(\phi) = \binom{n}{k}^{-1} \sum_{[n,k]} \phi(X_{i_1}, ..., X_{i_k}) \tag{1.2}$$

where the sum iterates over all sets $\{i_1, ..., i_k\}$ s. t. $1 \leq i_1 < i_2 < ... < i_n \leq n$. We call (1.2) U-Statistics of order k. In Lee (1990) it was shown, that that the U-Statistics is the unbiased minimum variance estimator for (1.1). Observe that for $k = 2$, equation (1.2) simplifies to

$$U_{2n}(\phi) = \frac{2}{n(n-1)} \sum\sum_{1 \leq i < j \leq n} \phi(X_i, X_j)$$

and we have

$$\mathbb{E}(U_{2n}(\phi)) = \int\int \phi dF dF$$

One of the major problems in lifetime analysis is to handle incomplete observations. This incompleteness is often caused by censoring. Here we are looking at right censored data. A framework to model this kind of data is provided by the Random Censorship Model (RCM). Here we observe data of the form $(Z_i, \delta_i), i = 1, ..., n$ where the Z_i are the observed sample values, which might include censoring and the δ_i indicate whether the corresponding Z_i was censored or not. Here the sequence $(Z_i, \delta_i), i = 1, ..., n$ is assumed to be independent and identically distributed (i. i. d.). Furthermore we can write for $i = 1, ..., n$

$$Z_i = min(X_i, Y_i) \text{ and } \delta_i = I_{X_i \leq Y_i}$$

where X_i is the true lifetime and Y_i is the so called censoring time. The sequences X_i and Y_i are also i. i. d.and they are assumed to be independent of each other. Throughout this work the probability distribution functions (d. f.) of X, Y and Z will be notated F, G and H respectively. We assume that those d. f.'s are continuous and concentrated on $R \cap [0, \infty]$.

Within this framework we want to study the large sample properties of estimators of $\int\int \phi dF dF$. In particular we will be concerned with the asymptotic properties of the U-statistic defined above based on our observations (Z_i, δ_i). To do so, we need new estimates for our d. f. F which are based on our observations (Z_i, δ_i) rather than the X's. If there can not be any further assumptions made about the censorship, except for the RCM itself, then the commonly used estimator of F is the well known product limit estimator by Kaplan and Meier (1958). It is defined by

$$1 - F_n^{km}(t) = \prod_{i:Z_i \le t} \left(\frac{n - R_{i,n}}{n - R_{i,n} + 1} \right)^{\delta_i}$$

where $R_{i,n}$ denotes the rank of Z_i. If we now consider ordered observations, we get

$$1 - F_n^{km}(t) = \prod_{i=1}^{n} \left(1 - \frac{\delta_{[i:n]}}{n - i + 1} \right)^{\mathbb{1}_{\{Z_{i:n} \le t\}}}$$

where $Z_{1:n} \le ... \le Z_{n:n}$ and $\delta_{[i:n]}$ denotes the concomitant of the i-th order statistics. That means $\delta_{[i:n]} = \delta_j$ whenever $Z_{i:n} = Z_j$.

Let's go back to our integral (1.1) and consider the case $k = 1$. The integral then becomes

$$\int \phi dF \tag{1.3}$$

Replacing the true F in (1.3) by F_n^{km} yields

$$S_{1,n}^{km}(\phi) := \int_0^\infty \phi dF_n^{km} = \sum_{i=1}^{n} \phi(Z_{i:n}) W_{i:n}^{km}$$

where $W_{i:n}^{km}$ denotes the weight placed on $Z_{i:n}$ by F_n^{km}. That is

$$\begin{aligned} W_{i:n}^{km} &= F_n^{km}(Z_{i+1:n}) - F_n^{km}(Z_{i:n}) \\ &= \tfrac{\delta_{[i:n]}}{n-i+1} \textstyle\prod_{j=1}^{i-1} \left(\tfrac{n-j}{n-j+1} \right)^{\delta_{[j:n]}} \end{aligned}$$

It is easy to see, that the Kaplan-Meier estimator only puts mass at uncensored

Z-values. Consider

$$W_{i:n}^{km} = \begin{cases} 0 & \text{if } \delta_{[i:n]} = 0 \\ \frac{1}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{\delta_{[k:n]}}{n-k+1}\right] > 0 & \text{if } \delta_{[i:n]} = 1 \end{cases}$$

The strong law of large numbers (SLLN) for $S_{1,n}^{km}(\phi)$ has been established by Stute and Wang (1993). Let's now consider the case $k = 2$. Define

$$S_{2,n}^{km}(\phi) = \sum\sum_{1 \le i < j \le n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n}^{km} W_{j:n}^{km}.$$

The above estimator will be called Kaplan-Meier U-Statistics of degree 2. Moreover we can define the normalized version of $S_{2,n}^{km}(\phi)$ as

$$U_{2,n}^{km}(\phi) = \frac{S_{2,n}^{km}(\phi)}{S_{2,n}^{km}(1)} = \frac{\sum\sum_{1 \le i < j \le n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n}^{km} W_{j:n}^{km}}{\sum\sum_{1 \le i < j \le n} W_{i:n}^{km} W_{j:n}^{km}}$$

The normalizing factor $(S_{2,n}^{km}(1))^{-1}$ was introduced by Bose and Sen (1999), since the following holds true for uncensored data:

$$\frac{W_{i:n}^{km} W_{j:n}^{km}}{\sum\sum_{1 \le u < v \le n} W_{u:n}^{km} W_{v:n}^{km}} = \binom{n}{2}^{-1}.$$

The strong law of large numbers for $U_{2,n}^{km}$ has been established in Bose and Sen (1999). Asymptotic distributions of this estimator have been derived in Bose and Sen (2002).

For the semiparametric Random Censorship Model (SRCM) we make, besides the assumptions of the RCM, the further assumption that

$$m(z) = \mathbb{P}(\delta = 1 | Z = z) = \mathbb{E}(\delta | Z = z)$$

belongs to some parametric family, i. e.

$$m(z) = m(z, \theta_0)$$

where $\theta_0 = (\theta_{0,1}, ..., \theta_{0,p}) \in \Theta \subset \mathbb{R}^p$. Now the semiparametric estimator is defined by

$$1 - F_n^{se}(t) = \prod_{i: Z_i \le t} \left(1 - \frac{m(Z_i, \hat{\theta}_n)}{n - R_i + 1}\right)$$

as it was introduced Dikta (2000). Here $\hat{\theta}_n$ denotes the Maximum Likelihood Estimate (MLE) of θ_0. That is, $\hat{\theta}_n$ is the maximizer of

$$L_n(\theta) = \prod_{i=1}^{n} m(Z_i, \theta)^{\delta_i} (1 - m(Z_i, \theta))^{1-\delta_i}.$$

Now again by replacing the true d. f. F by F_n^{se} in the integral (1.3) we obtain the semiparametric version of $S_{1,n}^{km}$, namely

$$S_{1,n}^{se}(\phi) = \int_0^\infty \phi \, dF_n^{se} = \sum_{i=1}^{n} \phi(Z_{i:n}) W_{i:n}^{se}$$

where

$$W_{i:n}^{se} = \frac{m(Z_{i:n}, \hat{\theta}_n)}{n - i + 1} \prod_{j=1}^{i-1} \left(1 - \frac{m(Z_{j:n}, \hat{\theta}_n)}{n - j + 1}\right)$$

is the mass that F_n^{se} assigns to $Z_{i:n}$. $W_{i:n}^{se}$ will be called i-th semiparametric weight throughout this document. The SLLN and the CLT for the semiparametric U-Statistic $S_{1,n}^{se}$ have been established in Dikta (2000) and Dikta et al. (2005) respectively.

TODO Discuss Dikta (1998)

The goal of this thesis will be to study the asymptotic behavior of the semiparametric U-Statistic of degree 2, which will be defined in the next chapter. In particular, we are interested in the strong law of large numbers for $S^{se}_{2,n}$. The main statement of this thesis is contained in Theorem 5.7 at the end of Chapter 5.

TODO Examples for different Kernels ϕ.

Chapter 2

Notation and assumptions

In this chapter we will state the main definitions and assumptions used throughout this work. We will start by defining the estimator to be considered and introduce all necessary notation for the remaining chapters.

Recall the following definition

$$W_{i:n}^{se} = \frac{m(Z_{i:n}, \hat{\theta}_n)}{n-i+1} \prod_{j=1}^{i-1} \left(1 - \frac{m(Z_{j:n}, \hat{\theta}_n)}{n-j+1}\right)$$

Now we define for $n \geq 2$

$$S_{2,n}^{se} = \sum\sum_{1 \leq i < j \leq n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n}^{se} W_{j:n}^{se}$$

This process will be called semiparametric U-Statistic of degree 2 throughout this thesis. Furthermore define

$$W_{i:n}(q) = \frac{q(Z_{i:n})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right]$$

and

$$S_n(q) = \sum\sum_{1 \leq i < j \leq n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n}(q) W_{j:n}(q)$$

Example 2.1. Let $q(Z_{i:n}) = \delta_{[i:n]}$ for $1 \leq i \leq n$. Then $W_{i:n}(q) = W_{i:n}^{km}$ and therefore

$$S_n(q) = S_{2,n}^{km}$$

Example 2.2. Let $q(t) = m(t, \hat{\theta}_n)$ for $t \in \mathbb{R}^+$. Then $W_{i:n}(q) = W_{i:n}^{se}$ and therefore

$$S_n(q) = S_{2,n}^{se}$$

Moreover define

$$\mathcal{F}_n = \sigma\{Z_{1:n}, \ldots, Z_{n:n}, Z_{n+1}, Z_{n+2}, \ldots\}$$

Throughout this work we will write $S_n := S_n(q)$ and $W_{i:n} := W_{i:n}(q)$. for $1 \le i \le n$.

The following assumptions will be needed throughout this work, in order to prove the SLLN for S_n.

(A1) The kernel $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is measurable, non-negative and symmetric in its arguments. In effect $\phi(s,t) = \phi(t,s)$ for all $s, t \in \mathbb{R}_+$.

(A2) The d. f. H is continuous and concentrated on the non-negative real line.

(A3) For $s, t \in \mathbb{R}_+$ the following statement holds true

$$\int_0^s \int_0^t \frac{\phi(s,t)}{m(s,\theta_0)m(t,\theta_0)(1-H(s))^\epsilon(1-H(t))^\epsilon k} F(ds)F(dt) < \infty.$$

for some $0 < \epsilon \le 1$.

(A4) There exists $c_1 < \infty$ s. t. $\sup_x (m \circ H^{-1})'(x) \le c_1$.

(A5) We have $m \circ H^{-1}(1) = 1$.

We will need the following assumptions about the Censoring Model m and the Maximum Likelihood estimate $\hat{\theta}_n$:

(M1) $\hat{\theta}_n$ is measurable and tends to θ_0

(M2) For any $\epsilon > 0$ there exists a neighborhood $V(\epsilon, \theta_0) \subset \Theta$ of θ_0 s. t. for all $\theta \in V(\epsilon, \theta_0)$

$$\sup_{x \geq 0} |m(x, \theta) - m(x, \theta_0)| < \epsilon$$

Chapter 3

Basic results

Within this chapter we will establish basic properties of $\mathbb{E}[S_n]$. In Section 3.1 a representation is derived for $\mathbb{E}[S_n|\mathcal{F}_{n+1}]$, which is similar to the result established in Bose and Sen (1999), Lemma 1. Later on in this section we will derive properties of the process above based on this representation. In Bose and Sen (1999) and Dikta (1998), the proof of existence of the limit of the considered estimator was based on a reverse supermartingale argument. We will not be able to establish the reverse supermartingale property for $S_{2,n}^{se}$. This will be discussed in more detail within Section 3.2. Consequently, we will derive a generalized version of Doob's Upcrossing Theorem within Chapter 4 in order to show that the limit of $S_{2,n}^{se}$ exists with probability one.

3.1 Basic results about $\mathbb{E}[S_n|\mathcal{F}_{n+1}]$

We will first derive an explicit representation for $\mathbb{E}[S_n|\mathcal{F}_{n+1}]$, which is similar to the one established in the proof of Bose and Sen (1999), Lemma 1.

Lemma 3.1. *Define for* $1 \leq i < j \leq n$

$$Q_{ij}^{n+1} = \begin{cases} Q_i^{n+1} & j \leq n \\ Q_i^{n+1} - \frac{(n+1)\pi_i\pi_n(1-q(Z_{n:n+1}))}{(n-i+1)(2-q(Z_{n:n+1}))} & j = n+1 \end{cases}$$

where

$$Q_i^{n+1} = (n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2 + \frac{\pi_i\pi_{i+1}}{n-i+1}\right\} \tag{3.1}$$

and

$$\pi_i = \prod_{k=1}^{i-1} \frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}$$

Then we have

$$\mathbb{E}[S_n|\mathcal{F}_{n+1}] = \sum\sum_{1\leq i<j\leq n+1} \phi(Z_{i:n+1}, Z_{j:n+1}) W_{i:n+1} W_{j:n+1} Q_{ij}^{n+1}$$

Proof. We will need the following result for the proof of lemma 3.1. Let

$$A_i = \pi_i + \sum_{r=1}^{i-1} \left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]$$

for $1 \leq i \leq n$ with π_i as defined above. Note that $\pi_1 = 1$, since the product is empty and hence taken as 1. Therefore we have $A_1 = \pi_1 = 1$. Now for any $1 \leq i \leq n-1$

$$\begin{aligned}
A_{i+1} &= \pi_{i+1} + \sum_{r=1}^{i} \left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right] \\
&= \pi_i \left[\frac{n-i+1-q(Z_{i:n+1})}{n-i+2-q(Z_{i:n+1})}\right] + \sum_{r=1}^{i-1} \left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right] + \left[\frac{\pi_i}{n-i+2-q(Z_{i:n+1})}\right] \\
&= \pi_i + \sum_{r=1}^{i-1} \left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right] \\
&= A_i
\end{aligned}$$

And therefore

$$1 = A_1 = A_2 = \cdots = A_{n-1} = A_n \tag{3.2}$$

Now let's establish lemma 3.1. Let F_n^q denote the measure that assigns mass to $Z_{1:n}, \ldots, Z_{n:n}$, then

$$\begin{aligned}
\mathbb{E}[S_n|\mathcal{F}_{n+1}] &= \mathbb{E}[\sum\sum_{1\leq i<j\leq n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n} W_{j:n} |\mathcal{F}_{n+1}] \\
&= \mathbb{E}[\sum\sum_{1\leq i<j\leq n+1} \phi(Z_{i:n+1}, Z_{j:n+1}) F_n^q\{Z_{i:n+1}\} F_n^q\{Z_{j:n+1}\} |\mathcal{F}_{n+1}]
\end{aligned}$$

$$= \sum\sum_{1\le i<j\le n+1} \phi(Z_{i:n+1}, Z_{j:n+1})\mathbb{E}[F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}|\mathcal{F}_{n+1}]$$

Consider for $1 \le i < j \le n$

$$\begin{aligned}
&\mathbb{E}[F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}|\mathcal{F}_{n+1}] \\
&= \mathbb{E}\left[\sum_{r=1}^{n+1} F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}I_{\{Z_{n+1}=Z_{r:n+1}\}}|\mathcal{F}_{n+1}\right]
\end{aligned}$$

Define the set $A_{rn} := \{Z_{n+1} = Z_{r:n+1}\}$. Note that on A_{rn} we have for $1 \le l \le n+1$

$$Z_{l:n+1} = \begin{cases} Z_{l:n} & l < r \\ Z_{l-1:n} & l > r \end{cases} \tag{3.3}$$

and therefore

$$F_n^q\{Z_{l:n+1}\} = \begin{cases} W_{l:n} & l < r \\ 0 & l = r \\ W_{l-1:n} & l > r \end{cases} \tag{3.4}$$

Now we have

$$\begin{aligned}
&\sum_{r=1}^{n+1} F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}I_{\{Z_{n+1}=Z_{r:n+1}\}} \\
&= \sum_{r=1}^{n+1} F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}I_{A_{rn}} \\
&= \sum_{r=1}^{i-1} W_{i-1:n}W_{j-1:n}I_{A_{rn}} + \sum_{r=i+1}^{j-1} W_{i:n}W_{j-1:n}I_{A_{rn}} + \sum_{r=j+1}^{n+1} W_{i:n}W_{j:n}I_{A_{rn}} \\
&=: T_1 + T_2 + T_3 \qquad (3.5)
\end{aligned}$$

Let's now consider each of the sums T_1, T_2, and T_3 in the above equation individually.

First consider T_1. We have

$$\begin{aligned}
T_1 &= \sum_{r=1}^{i-1} \frac{q(Z_{i-1:n})}{n-i+2} \prod_{k=1}^{i-2} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right] \\
&\quad \times \frac{q(Z_{j-1:n})}{n-j+2} \prod_{k=1}^{j-2} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right] I_{A_{rn}} \\
&= \sum_{r=1}^{i-1} \frac{q(Z_{i:n+1})}{n-i+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1}\right] \prod_{k=r}^{i-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] \\
&\quad \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1}\right] \prod_{k=r}^{j-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] I_{A_{rn}}
\end{aligned}$$

using (3.3). We will now continue to find an expression for T_1 in terms of $W_{i:n+1}$ and $W_{j:n+1}$. We have

$$\begin{aligned}
T_1 &= \sum_{r=1}^{i-1} \frac{q(Z_{i:n+1})}{n-i+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1}\right] \prod_{k=r}^{i-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] \\
&\quad \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1}\right] \prod_{k=r}^{j-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] I_{A_{rn}} \\
&= \sum_{r=1}^{i-1} \frac{q(Z_{i:n+1})}{n-i+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right] \prod_{k=r}^{i-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] \\
&\quad \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right] \prod_{k=r}^{j-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] I_{A_{rn}} \\
&\quad \times \left[\frac{\prod_{k=1}^{r-1}\left[1 - \frac{q(Z_{k:n+1})}{n-k+1}\right]}{\prod_{k=1}^{r-1}\left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right]}\right]^2 \\
&= \sum_{r=1}^{i-1} \frac{q(Z_{i:n+1})}{n-i+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right] \prod_{k=r}^{i-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] \\
&\quad \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right] \prod_{k=r}^{j-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1}\right] I_{A_{rn}} \\
&\quad \times \prod_{k=1}^{r-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]^2 \prod_{k=1}^{r-1} \left[\frac{n-k+2}{n-k+1}\right]^2
\end{aligned}$$

Now using index transformation on the products $\prod_{k=r}^{i-2}[\dots]$ and $\prod_{k=r}^{j-2}[\dots]$ yields

$$
\begin{aligned}
T_1 = & \sum_{r=1}^{i-1} \frac{q(Z_{i:n+1})}{n-i+2} \prod_{k=1}^{r-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] \prod_{k=r+1}^{i-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] \\
& \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] \prod_{k=r+1}^{j-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] I_{A_{rn}} \\
& \times \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]^2 \prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right]^2 \\
= & \sum_{r=1}^{i-1} \frac{q(Z_{i:n+1})}{n-i+2} \prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] \left[1-\frac{q(Z_{r:n+1})}{n-r+2}\right]^{-1} \\
& \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{j-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right] \left[1-\frac{q(Z_{r:n+1})}{n-r+2}\right]^{-1} I_{A_{rn}} \\
& \times \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]^2 \prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right]^2 \\
= & W_{i:n+1} W_{j:n+1} \sum_{r=1}^{i-1} \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]^2 \prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right]^2 \\
& \times \left[\frac{n-r+2}{n-r+2-q(Z_{r:n+1})}\right]^2 I_{A_{rn}}
\end{aligned}
$$

Note that

$$
\begin{aligned}
\prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right] &= \frac{n+1}{n} \cdot \frac{n}{n-1} \cdots \frac{n-r+4}{n-r+3} \cdot \frac{n-r+3}{n-r+2} \\
&= \frac{n+1}{n-r+2} \qquad (3.6)
\end{aligned}
$$

and recall the following definition

$$
\pi_r = \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]
$$

Now we finally get

$$
\begin{aligned}
T_1 &= W_{i:n+1} W_{j:n+1} \sum_{r=1}^{i-1} \prod_{k=1}^{r-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})} \right]^2 \\
&\quad \times \left[\frac{n+1}{n-r+2} \right]^2 \left[\frac{n-r+2}{n-r+2-q(Z_{r:n+1})} \right]^2 I_{A_{rn}} \\
&= W_{i:n+1} W_{j:n+1} \sum_{r=1}^{i-1} \pi_r^2 \left[\frac{n+1}{n-r+2-q(Z_{r:n+1})} \right]^2 I_{A_{rn}}
\end{aligned}
$$

Now let's consider T_2. We will, again, firstly express T_2 completely in terms of the ordered Z values w. r. t. order $n+1$ using (3.3). Consider

$$
\begin{aligned}
T_2 &= \sum_{r=i+1}^{j-1} \frac{q(Z_{i:n})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1} \right] \\
&\quad \times \frac{q(Z_{j-1:n})}{n-j+2} \prod_{k=1}^{j-2} \left[1 - \frac{q(Z_{k:n})}{n-k+1} \right] I_{A_{rn}} \\
&= \sum_{r=i+1}^{j-1} \frac{q(Z_{i:n+1})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1} \right] \\
&\quad \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1} \right] \prod_{k=r}^{j-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1} \right] I_{A_{rn}}
\end{aligned}
$$

Now let's find a representation of T_2 which relies on $W_{i:n+1}$ and $W_{j:n+1}$ only. Consider

$$
\begin{aligned}
T_2 &= \sum_{r=i+1}^{j-1} \left[\frac{n-i+2}{n-i+1} \right] \left[\frac{q(Z_{i:n+1})}{n-i+2} \right] \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2} \right] \\
&\quad \times \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2} \right] \prod_{k=r}^{j-2} \left[1 - \frac{q(Z_{k+1:n+1})}{n-k+1} \right] I_{A_{rn}} \\
&\quad \times \prod_{k=1}^{i-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})} \right] \prod_{k=1}^{i-1} \left[\frac{n-k+2}{n-k+1} \right] \\
&\quad \times \prod_{k=1}^{r-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})} \right] \prod_{k=1}^{r-1} \left[\frac{n-k+2}{n-k+1} \right]
\end{aligned}
$$

$$= \left[\frac{n-i+2}{n-i+1}\right]\left[\frac{q(Z_{i:n+1})}{n-i+2}\right]\prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]$$
$$\times \prod_{k=1}^{i-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{i-1}\left[\frac{n-k+2}{n-k+1}\right]$$
$$\times \sum_{r=i+1}^{j-1}\frac{q(Z_{j:n+1})}{n-j+2}\prod_{k=1}^{r-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]\prod_{k=r}^{j-2}\left[1-\frac{q(Z_{k+1:n+1})}{n-k+1}\right]I_{A_{rn}}$$
$$\times \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right]$$

Now using (3.6) on $\prod_{k=1}^{i-1}[\ldots]$ yields

$$= \left[\frac{n+1}{n-i+1}\right]\left[\frac{q(Z_{i:n+1})}{n-i+2}\right]\prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]$$
$$\times \prod_{k=1}^{i-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]$$
$$\times \sum_{r=i+1}^{j-1}\frac{q(Z_{j:n+1})}{n-j+2}\prod_{k=1}^{r-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]\prod_{k=r}^{j-2}\left[1-\frac{q(Z_{k+1:n+1})}{n-k+1}\right]I_{A_{rn}}$$
$$\times \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right]$$
$$= \left[\frac{n+1}{n-i+1}\right]W_{i:n+1}\pi_i$$
$$\times \sum_{r=i+1}^{j-1}\frac{q(Z_{j:n+1})}{n-j+2}\prod_{k=1}^{r-1}\left[1-\frac{q(Z_{k:n+1})}{n-k+2}\right]\prod_{k=r}^{j-2}\left[1-\frac{q(Z_{k+1:n+1})}{n-k+1}\right]I_{A_{rn}}$$
$$\times \prod_{k=1}^{r-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{r-1}\left[\frac{n-k+2}{n-k+1}\right]$$

Again doing an index transformation on $\prod_{k=r}^{j-2}[\ldots]$ yields

$$= \left[\frac{n+1}{n-i+1}\right]W_{i:n+1}\pi_i$$

$$
\begin{aligned}
&\times \sum_{r=i+1}^{j-1} \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{r-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right] \prod_{k=r+1}^{j-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right] I_{A_{rn}} \\
&\times \prod_{k=1}^{r-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right] \prod_{k=1}^{r-1} \left[\frac{n-k+2}{n-k+1}\right] I_{A_{rn}} \\
= {} & W_{i:n+1}\pi_i \frac{n+1}{n-i+1} \sum_{r=i+1}^{j-1} \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{j-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right] \left[1 - \frac{q(Z_{r:n+1})}{n-r+2}\right]^{-1} \\
&\times \prod_{k=1}^{r-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right] \prod_{k=1}^{r-1} \left[\frac{n-k+2}{n-k+1}\right] I_{A_{rn}} \\
= {} & W_{i:n+1} W_{j:n+1} \pi_i \frac{n+1}{n-i+1} \\
&\times \sum_{r=i+1}^{j-1} \prod_{k=1}^{r-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right] \prod_{k=1}^{r-1} \left[\frac{n-k+2}{n-k+1}\right] \\
&\times \frac{n-r+2}{n-r+2-q(Z_{r:n+1})} I_{A_{rn}}
\end{aligned}
$$

Now applying (3.6) to the latter product yields

$$
T_2 = W_{i:n+1} W_{j:n+1} \pi_i \frac{n+1}{n-i+1} \sum_{r=i+1}^{j-1} \pi_r \frac{n+1}{n-r+2-q(Z_{r:n+1})} I_{A_{rn}}
$$

We will now proceed similarly for T_3. Consider

$$
T_3 = \sum_{r=j+1}^{n+1} W_{i:n} W_{j:n} \mathbb{1}_{\{A_{rn}\}}
$$

Note that for $j = n+1$ the sum above is empty and hence zero. Now consider for $j \leq n$

$$
\begin{aligned}
T_3 = \sum_{r=j+1}^{n+1} & \frac{q(Z_{i:n})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right] \\
& \times \frac{q(Z_{j:n})}{n-j+1} \prod_{k=1}^{j-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right] \mathbb{1}_{\{A_{rn}\}}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{r=j+1}^{n+1} \frac{q(Z_{i:n+1})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1}\right] \\
&\quad \times \frac{q(Z_{j:n+1})}{n-j+1} \prod_{k=1}^{j-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+1}\right] \mathbb{1}_{\{A_{rn}\}} \\
&= \sum_{r=j+1}^{n+1} \frac{n-i+2}{n-i+1} \frac{q(Z_{i:n+1})}{n-i+2} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right] \\
&\quad \times \frac{n-j+2}{n-j+1} \frac{q(Z_{j:n+1})}{n-j+2} \prod_{k=1}^{j-1} \left[1 - \frac{q(Z_{k:n+1})}{n-k+2}\right] \\
&\quad \times \prod_{k=1}^{i-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right] \prod_{k=1}^{i-1} \left[\frac{n-k+2}{n-k+1}\right] \\
&\quad \times \prod_{k=1}^{j-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right] \prod_{k=1}^{j-1} \left[\frac{n-k+2}{n-k+1}\right] \mathbb{1}_{\{A_{rn}\}} \\
&= \sum_{r=j+1}^{n+1} \frac{n-i+2}{n-i+1} \frac{n-j+2}{n-j+1} \pi_i \pi_j W_{i:n+1} W_{j:n+1} \\
&\quad \times \prod_{k=1}^{i-1} \left[\frac{n-k+2}{n-k+1}\right] \prod_{k=1}^{j-1} \left[\frac{n-k+2}{n-k+1}\right] \mathbb{1}_{\{A_{rn}\}}
\end{aligned}
$$

Again, by (3.6), we have

$$
T_3 = \sum_{r=j+1}^{n+1} \frac{(n+1)^2 \pi_i \pi_j}{(n-i+1)(n-j+1)} W_{i:n+1} W_{j:n+1} \mathbb{1}_{\{A_{rn}\}}
$$

Therefore

$$
T_3 = \begin{cases} W_{i:n+1} W_{j:n+1} \pi_i \pi_j \left[\frac{(n+1)^2}{(n-i+1)(n-j+1)}\right] \sum_{i=j+1}^{n+1} \mathbb{1}_{\{A_{rn}\}} & j \le n \\ 0 & j = n+1 \end{cases}
$$

for $1 \le i < j \le n$. Now using these expressions for T_1, T_2 and T_3 in equation (3.5) together with the fact that

$$\mathbb{E}[I_{A_{rn}}|\mathcal{F}_{n+1}] = \frac{1}{n+1}$$

yields

$$\begin{aligned}
&\mathbb{E}[F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}|\mathcal{F}_{n+1}] \\
&= \mathbb{E}[T_1 + T_2 + T_3|\mathcal{F}_{n+1}] \\
&= W_{i:n+1}W_{j:n+1} \times \left\{ \sum_{r=1}^{i-1} \pi_r^2 \left[\frac{n+1}{n-r+2-q(Z_{r:n+1})}\right]^2 \mathbb{E}[I_{A_{rn}}|\mathcal{F}_{n+1}] \right. \\
&\qquad + \sum_{r=i+1}^{j-1} \pi_i\pi_r \left[\frac{n+1}{n-i+1}\right] \left[\frac{n+1}{n-r+2-q(Z_{r:n+1})}\right] \mathbb{E}[I_{A_{rn}}|\mathcal{F}_{n+1}] \\
&\qquad \left. +\pi_i\pi_j \frac{(n+1)^2}{(n-i+1)(n-j+1)}[1 - I_{\{j=n+1\}}] \sum_{i=j+1}^{n+1} \mathbb{E}[I_{A_{rn}}|\mathcal{F}_{n+1}] \right\} \\
&= W_{i:n+1}W_{j:n+1} \left[\frac{1}{n+1}\right] \times \left\{ \sum_{r=1}^{i-1} \pi_r^2 \left[\frac{n+1}{n-r+2-q(Z_{r:n+1})}\right]^2 \right. \\
&\qquad + \sum_{r=i+1}^{j-1} \pi_i\pi_r \left[\frac{n+1}{n-i+1}\right] \left[\frac{n+1}{n-r+2-q(Z_{r:n+1})}\right] \\
&\qquad \left. +\pi_i\pi_j \frac{(n+1)^2}{n-i+1}[1 - I_{\{j=n+1\}}] \right\}
\end{aligned}$$

Next consider that we have

$$\begin{aligned}
&\mathbb{E}[F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}|\mathcal{F}_{n+1}] \\
&= W_{i:n+1}W_{j:n+1}(n+1) \left\{ \sum_{r=1}^{i-1} \left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2 \right. \\
&\qquad \left. + \frac{\pi_i}{n-i+1} \left[\sum_{r=i+1}^{j-1} \left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right] + \pi_j \right] \right\}
\end{aligned}$$

for $1 \le i < j \le n$. Now applying (3.2) yields

$$
\begin{aligned}
&= W_{i:n+1}W_{j:n+1}(n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2\right.\\
&\quad\left.+\frac{\pi_i}{n-i+1}(A_j - A_{i+1}+\pi_{i+1})\right\}\\
&= W_{i:n+1}W_{j:n+1}(n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2\right.\\
&\quad\left.+\frac{\pi_i\pi_{i+1}}{n-i+1}\right\}\\
&= W_{i:n+1}W_{j:n+1}Q_i^{n+1}
\end{aligned}
$$

It remains to consider the case $j = n+1$. We have

$$
\begin{aligned}
&\mathbb{E}[F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}|\mathcal{F}_{n+1}]\\
&= W_{i:n+1}W_{n+1:n+1}(n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2\right.\\
&\quad\left.+\frac{\pi_i}{n-i+1}\sum_{r=i+1}^{n}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]\right\}\\
&= W_{i:n+1}W_{n+1:n+1}(n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2\right.\\
&\quad\left.+\frac{\pi_i}{n-i+1}\left[\sum_{r=1}^{n}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]-\sum_{r=1}^{i}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]\right]\right\}\\
&= W_{i:n+1}W_{n+1:n+1}(n+1)\left\{\frac{Q_i^{n+1}}{n+1}-\frac{\pi_i\pi_{i+1}}{n-i+1}\right.\\
&\quad\left.+\frac{\pi_i}{n-i+1}\left[\sum_{r=1}^{n}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]-\sum_{r=1}^{i}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]\right]\right\}
\end{aligned}
$$

Now using (3.2) again yields

$$
\begin{aligned}
&= W_{i:n+1}W_{n+1:n+1}(n+1)\left\{\frac{Q_i^{n+1}}{n+1}-\frac{\pi_i\pi_{i+1}}{n-i+1}\right.\\
&\quad\left.+\frac{\pi_i}{n-i+1}\left[A_{n+1}-\pi_{n+1}-(A_{i+1}-\pi_{i+1})\right]\right\}
\end{aligned}
$$

$$= W_{i:n+1}W_{n+1:n+1}(n+1)\left\{\frac{Q_i^{n+1}}{n+1} - \frac{\pi_i\pi_{i+1}}{n-i+1} \right.$$
$$\left. + \frac{\pi_i}{n-i+1}\left[\pi_{i+1} - \pi_{n+1}\right]\right\}$$

Note that for $1 \leq i \leq n$ we have

$$\pi_{i+1} = \frac{\pi_i(1 - q(Z_{i:n+1}))}{2 - q(Z_{i:n+1})}$$

Thus we obtain

$$\begin{aligned}
&\mathbb{E}[F_n^q\{Z_{i:n+1}\}F_n^q\{Z_{j:n+1}\}|\mathcal{F}_{n+1}] \\
&= W_{i:n+1}W_{n+1:n+1}(n+1)\left\{\frac{Q_i^{n+1}}{n+1} - \frac{\pi_i\pi_{i+1}}{n-i+1} \right. \\
&\qquad\qquad \left. + \frac{\pi_i}{n-i+1}\left[\pi_{i+1} - \frac{\pi_n(1-q(Z_{n:n+1}))}{2-q(z_{n:n+1})}\right]\right\} \\
&= W_{i:n+1}W_{n+1:n+1}(n+1)\left\{\frac{Q_i^{n+1}}{n+1} - \frac{\pi_i\pi_n(1-q(Z_{n:n+1}))}{(n-i+1)(2-q(Z_{n:n+1}))}\right\} \\
&= W_{i:n+1}W_{n+1:n+1}\left\{Q_i^{n+1} - \frac{\pi_i\pi_n(n+1)(1-q(Z_{n:n+1}))}{(n-i+1)(2-q(Z_{n:n+1}))}\right\}
\end{aligned}$$

□

The following result on the increases of Q_i^{n+1} will be useful for the proof of Lemma 4.15.

Lemma 3.2. *Let Q_i^{n+1} be defined as in lemma 3.1 for $1 \leq i \leq n$. Moreover define*

$$\tilde{\pi}_i := \prod_{k=1}^{i-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]\prod_{k=1}^{i-1}\left[\frac{n-k+2}{n-k+1}\right]$$

Then we have

$$Q_{i+1}^{n+1} - Q_i^{n+1} = \frac{(q_i - q_{i+1})(n-i)(n-i+1) - q_{i+1}(1-q_i)(n-i+1-q_i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})}$$
$$\times \frac{\tilde{\pi}_i(n-i+2)^2}{n+1}$$

Proof. For the sake of simplicity we will write $q_i \equiv q(Z_{i:n+1})$ during this proof. From equation (3.1) we get

$$\begin{aligned}
\frac{Q_{i+1}^{n+1} - Q_i^{n+1}}{n+1} &= \left\{ \sum_{r=1}^{i} \left[\frac{\pi_r}{n-r+2-q_r} \right]^2 + \frac{\pi_{i+1}\pi_{i+2}}{n-i} \right\} \\
&- \left\{ \sum_{r=1}^{i-1} \left[\frac{\pi_r}{n-r+2-q_r} \right]^2 + \frac{\pi_i \pi_{i+1}}{n-i+1} \right\} \\
&= \frac{\pi_i^2}{(n-i+2-q_i)^2} + \frac{\pi_{i+1}\pi_{i+2}}{n-i} - \frac{\pi_i \pi_{i+1}}{n-i+1} \\
&= \frac{\pi_i^2}{(n-i+2-q_i)^2} + \frac{\pi_i^2 (n-i+1-q_i)^2 (n-i-q_{i+1})}{(n-i)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\
&- \frac{\pi_i^2 (n-i+1-q_i)}{(n-i+1)(n-i+2-q_i)} \\
&= \pi_i^2 \left\{ \frac{1}{(n-i+2-q_i)^2} + \frac{(n-i+1-q_i)^2(n-i-q_{i+1})}{(n-i)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \right. \\
&\qquad \left. - \frac{n-i+1-q_i}{(n-i+1)(n-i+2-q_i)} \right\} \\
&=: \pi_i^2 \left\{ a(n,i) + b(n,i) - c(n,i) \right\} \qquad (3.7)
\end{aligned}$$

Now consider

$$\begin{aligned}
&b(n,i) - c(n,i) \\
&= (n-i+1-q_i) \left[\frac{(n-i+1-q_i)(n-i-q_{i+1})}{(n-i)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \right. \\
&\qquad \left. - \frac{1}{(n-i+1)(n-i+2-q_i)} \right] \\
&= (n-i+1-q_i) \left[\frac{(n-i+1-q_i)(n-i-q_{i+1})(n-i+1)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \right. \\
&\qquad \left. - \frac{(n-i+2-q_i)(n-i+1-q_{i+1})(n-i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \right] \qquad (3.8)
\end{aligned}$$

Next we will simplify the difference of the numerators above. We have

$$(n-i+1-q_i)(n-i-q_{i+1})(n-i+1)$$

$$
\begin{aligned}
&- (n-i+2-q_i)(n-i+1-q_{i+1})(n-i) \\
&= (n-i+1-q_i)(n-i)(n-i+1) - q_{i+1}(n-i+1-q_i)(n-i+1) \\
&\quad - (n-i+2-q_i)(n-i+1-q_{i+1})(n-i) \\
&= (n-i+1-q_i)(n-i)(n-i+1) - q_{i+1}(n-i+1-q_i)(n-i+1) \\
&\quad - (n-i+1-q_i)(n-i+1-q_{i+1})(n-i) - (n-i+1-q_{i+1})(n-i) \\
&= (n-i+1-q_i)(n-i)(n-i+1) - q_{i+1}(n-i+1-q_i)(n-i+1) \\
&\quad - (n-i+1-q_i)(n-i+1)(n-i) + q_{i+1}(n-i+1-q_i)(n-i) \\
&\quad - (n-i+1-q_{i+1})(n-i) \\
&= -q_{i+1}(n-i+1-q_i) - (n-i+1-q_{i+1})(n-i)
\end{aligned}
$$

Hence we get, according to (3.8)

$$
\begin{aligned}
&b(n,i) - c(n,i) \\
&= -(n-i+1-q_i)\left[\frac{q_{i+1}(n-i+1-q_i) + (n-i+1-q_{i+1})(n-i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})}\right]
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&a(n,i) + b(n,i) - c(n,i) \\
&= \frac{1}{(n-i+2-q_i)^2} \\
&\quad - \frac{q_{i+1}(n-i+1-q_i)^2 + (n-i+1-q_i)(n-i+1-q_{i+1})(n-i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\
&= \frac{(n-i)(n-i+1)(n-i+1-q_{i+1})}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\
&\quad - \frac{q_{i+1}(n-i+1-q_i)^2 + (n-i+1-q_i)(n-i+1-q_{i+1})(n-i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})}
\end{aligned}
$$

Consider again the numerator of the latter expression. We have

$$
= (n-i)(n-i+1)(n-i+1-q_{i+1}) - q_{i+1}(n-i+1-q_i)^2
$$

$$
\begin{aligned}
&- (n-i)(n-i+1-q_i)(n-i+1-q_{i+1}) \\
&= q_i(n-i)(n-i+1-q_{i+1}) - q_{i+1}(n-i+1-q_i)^2 \\
&= q_i(n-i)^2 + q_i(1-q_{i+1})(n-i) - q_{i+1}(n-i)^2 \\
&\quad - 2q_{i+1}(1-q_i)(n-i) - q_{i+1}(1-q_i)^2 \\
&= (q_i - q_{i+1})(n-i)^2 + q_i(n-i) - q_iq_{i+1}(n-i) \\
&\quad - 2q_{i+1}(n-i) + 2q_iq_{i+1}(n-i) - q_{i+1}(1-q_i)^2 \\
&= (q_i - q_{i+1})(n-i)^2 + (q_i + q_iq_{i+1} - 2q_{i+1})(n-i) - q_{i+1}(1-q_i)^2
\end{aligned}
$$

Note that $q_i - q_{i+1} \geq -1$ and $q_i + q_iq_{i+1} - 2q_{i+1} \geq -2$, since $0 \leq q_i \leq 1$ for $1 \leq i \leq n$. Similarly we have $q_{i+1}(1-q_i)^2 \leq 1$. Thus we get

$$
\begin{aligned}
&a(n,i) + b(n,i) - c(n,i) \\
&= \frac{(q_i - q_{i+1})(n-i)^2 + (q_i + q_iq_{i+1} - 2q_{i+1})(n-i) - q_{i+1}(1-q_i)^2}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\
&= \frac{(q_i - q_{i+1})(n-i)^2 + [(q_i - q_{i+1}) - q_{i+1}(1-q_i))(n-i) - q_{i+1}(1-q_i)^2}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\
&= \frac{(q_i - q_{i+1})(n-i)(n-i+1) - q_{i+1}(1-q_i)(n-i+1-q_i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \qquad (3.9)
\end{aligned}
$$

Finally note that

$$
\begin{aligned}
\tilde{\pi}_i &= \frac{n+1}{n-i+2} \prod_{k=1}^{i-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})} \right] \\
&= \pi_i \cdot \frac{n+1}{n-i+2} \qquad (3.10)
\end{aligned}
$$

with π_i as defined in Lemma 3.1. Now the statement of the lemma follows directly from (3.7), (3.9) and (3.10) $\square$

3.2 S_n is not necessarily a reverse supermartingale

As discussed in Chapter 1, the Strong Law of Large Numbers for Kaplan-Meier U-Statistics of degree 2 was established by Bose and Sen (1999). Recall definition of said estimator:

$$S_n^{km} = \sum\sum_{1\le i<j\le n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n}^{km} W_{j:n}^{km}$$

with

$$W_{i:n}^{km} = \frac{\delta_{[i:n]}}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{\delta_{[k:n]}}{n-k+1}\right]$$

The prove of existence of the limit $S = \lim_{n\to\infty} S_n^{km}$ was here essentially based on a supermartingale argument together with Neveu (1975), proposition 5-3-11. In Lemma 1 of Bose and Sen (1999) a representation for $\mathbb{E}[S_n^{km}|\mathcal{F}_{n+1}]$ was derived, which is similar to our lemma 3.1. It was shown that for $1 \le i < j \le n$

$$\mathbb{E}[S_n^{km}|\mathcal{F}_{n+1}] = \sum\sum_{1\le i<j\le n+1} \phi(Z_{i:n+1}, Z_{j:n+1}) W_{i:n+1}^{km} W_{j:n+1}^{km} Q_{ij}^{km}$$

where

$$Q_{ij}^{km} = \begin{cases} Q_i^{km} & \text{if } j \le n \\ Q_i^{km} - \pi_i \pi_n (1 - \delta_{[n:n+1]}) \frac{n-i+2}{(n+1)(n-i+1)} & \text{if } j = n+1 \end{cases}$$

and

$$Q_i^{km} = \frac{1}{n+1} \left\{ \sum_{r=1}^{i-1} \pi_r^2 \left[\frac{n-r+2}{n-r+1}\right]^{2\delta_{[r:n+1]}} \right. \\ \left. + \pi_i^2 (n-i+2) \left[\frac{(n-i)(n-i+2)}{(n-i+1)^2}\right]^{\delta_{[i:n+1]}} \right\}$$

Then Bose and Sen (1999) show that $Q_{ij}^{km} \le 1$ for $1 \le i < j \le n$, in order to establish the reverse time supermartingale property for $(S_n^{km}, \mathcal{F}_n)$. However their prove relies on the fact that

$$\begin{aligned} W_{i:n}^{km} &= \frac{\delta_{[i:n]}}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{\delta_{[k:n]}}{n-k+1} \right] \\ &= \frac{\delta_{[i:n]}}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{1}{n-k+1} \right]^{\delta_{[k:n]}} \end{aligned}$$

But the corresponding statement is not true for $W_{i:n}$, since we have in general that

$$\begin{aligned} W_{i:n} &= \frac{q(Z_{i:n})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1} \right] \\ &\neq \frac{q(Z_{i:n})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{1}{n-k+1} \right]^{q(Z_{k:n})} \end{aligned}$$

In Dikta (2000), the following estimator was considered

$$S_n^{se}(q) = \sum_{i=1}^{n} \phi(Z_{i:n}) W_{i:n}^{se}$$

The proof of existence of the limit $S^{se} = \lim_{n\to\infty} S_n^{se}$ shows a similar structure, as the one in Bose and Sen (1999). In Lemma 2.1 of Dikta (2000), it was shown that $\mathbb{E}[\mu_n\{Z_{1:n+1}\}|\mathcal{F}_{n+1}] = W_{1:n}^{se}$ and for $2 \leq i \leq n$

$$\mathbb{E}[\mu_n\{Z_{i:n+1}\}|\mathcal{F}_{n+1}] = W_{i:n}^{se} Q_i^{se}$$

where μ_n is the measure assigning mass $W_{i:n}$ to $Z_{i:n}$ and

$$Q_i^{se} = \pi_i + \sum_{k=1}^{i-1} \frac{\pi_k}{n-k+2-q(Z_{k:n+2})}$$

Here π_i is defined as in Lemma 3.1. Furthermore it was shown that $Q_i^{se} = Q_{i+1}^{se} = 1$ for all $2 \leq i \leq n$, which, among other arguments, implies the reverse supermartingale property for S_n^{se}.

Within the framework of this thesis, we were neither able to show $Q_i = Q_{i+1} = 1$, nor to show that $Q_i \leq 1$ for all $1 \leq i \leq n$. Therefore it is not clear, if $\{S^{se}_{2,n}, \mathcal{F}_{n+1}\}$ is a reverse supermartingale. However, in Lemma 3.2 we derived a representation for the increases $Q_{i+1} - Q_i$, which we will use in Chapter 4 in order to generalize Doob's Upcrossing Theorem to our framework. Afterwards we will derive conditions under which the expected number of upcrossings is finite in Chapter 4. This will then imply the almost sure existence of the limit of $S^{se}_{2,n}$.

Chapter 4

Existence of the limit

As we have seen during the previous chapter, we were not able to establish that $(S_n, \mathcal{F}_n)_{n\geq 1}$ is a reverse time supermartingale, and hence we can not establish the almost sure existence of the limit of S_n in the same way as e. g. in Dikta (2000) and Bose and Sen (1999). The purpose of this chapter is to generalize Doob's Upcrossing Theorem for our estimator S_n in order to show that $S = \lim_{n\to\infty} S_n$ exists $\mathbb{P}$-almost surely.

Figure 4.1 below shows the interdependence structure of the proofs within this chapter.

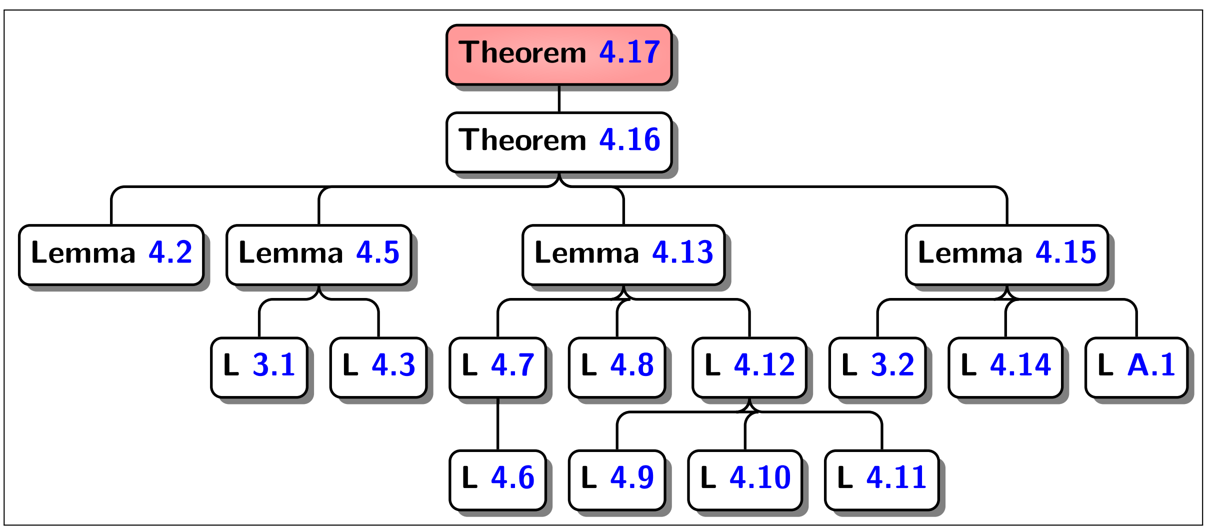

Figure 4.1: Interdependence Structure of the lemmas and theorems within this chapter.

The following assumptions on q will be needed in Section 4.3.

(Q1) There exists $c_1 < \infty$ s. t. $\sup_x (q \circ H^{-1})'(x) \leq c_1$.

(Q2) We have $q \circ H^{-1}(1) = 1$.

4.1 Modifying Doob's Upcrossing Theorem

During this section we will generalize Doob's Upcrossing Theorem to our framework. To get closer to the situation of Doob's Upcrossing Theorem, we define the following quantities. Let $N < \infty$ and define for $1 \leq n \leq N$

$$\tilde{S}_n^N := S_{N-n+1},\ \tilde{\mathcal{F}}_n^N := \mathcal{F}_{N-n+1} \quad \text{and} \quad \tilde{\xi}_n^N := \xi_{N-n+1}$$

Note that $\{\tilde{\mathcal{F}}_n^N\}_{1\leq n\leq N}$ is now an increasing σ-field in n. Below we will define everything needed, in order to generalize Doob's Upcrossing Theorem.

Definition 4.1. Let $N \geq 2$. For $1 \leq n \leq N$ and $a, b \in \mathbb{R}$ with $a < b$, let

$$\begin{aligned}
T_0 &:= 0 \\
T_1 &:= \begin{cases} \min\{1 \leq n \leq N | \tilde{S}_n^N \leq a\} & \text{if } \{1 \leq n \leq N | \tilde{S}_n^N \leq a\} \neq \emptyset \\ N & \text{if } \{1 \leq n \leq N | \tilde{S}_n^N \leq a\} = \emptyset \end{cases} \\
T_2 &:= \begin{cases} \min\{T_1 \leq n \leq N | \tilde{S}_n^N \geq b\} & \text{if } \{T_1 \leq n \leq N | \tilde{S}_n^N \leq a\} \neq \emptyset \\ N & \text{if } \{T_1 \leq n \leq N | \tilde{S}_n^N \geq b\} = \emptyset \end{cases} \\
&\vdots \quad \vdots \quad \vdots \\
T_{2m-1} &:= \begin{cases} \min\{T_{2m-2} \leq n \leq N | \tilde{S}_n^N \leq a\} & \text{if } \{T_{2m-2} \leq n \leq N | \tilde{S}_n^N \leq a\} \neq \emptyset \\ N & \text{if } \{T_{2m-2} \leq n \leq N | \tilde{S}_n^N \leq a\} = \emptyset \end{cases}
\end{aligned}$$

$$T_{2m} := \begin{cases} \min\{T_{2m-1} \leq n \leq N | \tilde{S}_n^N \geq b\} & \text{if } \{T_{2m-1} \leq n \leq N | \tilde{S}_n^N \leq a\} \neq \emptyset \\ N & \text{if } \{T_{2m-1} \leq n \leq N | \tilde{S}_n^N \geq b\} = \emptyset \end{cases}$$

Now we can define the number of upcrossings of $[a, b]$ by $\tilde{S}_1^N, ..., \tilde{S}_n^N$ as follows:

$$U_n^N[a, b] := \begin{cases} \max\{1 \leq m \leq N | T_{2m} < N\} & \text{if } \{1 \leq m \leq N | T_{2m} < N\} \neq \emptyset \\ 0 & \text{if } \{1 \leq m \leq N | T_{2m} < N\} = \emptyset \end{cases}$$

Furthermore let for $1 \leq k \leq n - 1$

$$\epsilon_k := \begin{cases} 0 & \text{if } k < T_1 \\ 1 & \text{if } T_1 \leq k < T_2 \\ 0 & \text{if } T_2 \leq k < T_3 \\ 1 & \text{if } T_3 \leq k < T_4 \\ \dots & \text{if } \dots \end{cases}$$

and define

$$Y_n^N := \tilde{S}_1^N + \sum_{k=1}^{n-1} \epsilon_k (\tilde{S}_{k+1}^N - \tilde{S}_k^N)$$

for $1 \leq n \leq N$.

The following lemma shows that the expected number of upcrossings of $[a, b]$ is bounded above by the expected value of Y_n^N. Thus afterwards it will remain to show that the limit $\mathbb{E}[Y_N^N]$ has a finite limit, which will be done in Section 4.2.

Lemma 4.2. *For $1 \leq n \leq N$ we have*

$$\mathbb{E}[U_n^N[a, b]] \leq \frac{\mathbb{E}[Y_n^N]}{b - a}$$

Proof. Consider for $1 \le n \le N$ and $N \ge 2$

$$
\begin{aligned}
Y_n^N &= \tilde{S}_1^N + \sum_{k=1}^{n-1} \epsilon_k (\tilde{S}_{k+1}^N - \tilde{S}_k^N) \\
&= \tilde{S}_1^N + \sum_{k=1}^{n} (\tilde{S}_{T_{2k}}^N - \tilde{S}_{T_{2k-1}}^N) \\
&\ge \sum_{k=1}^{n} (\tilde{S}_{T_{2k}}^N - \tilde{S}_{T_{2k-1}}^N)
\end{aligned}
$$

by definition of ϵ_k. The latter inequality above holds, since $\tilde{S}_1^N \ge 0$. Note that by definition of $T_1, T_2, \ldots$ we have

$$
\sum_{k=1}^{n} (\tilde{S}_{T_{2k}}^N - \tilde{S}_{T_{2k-1}}^N) \ge (b-a) U_n^N[a,b]
$$

From here the assertion follows directly. □

The following lemma provides a representation for the expectation of the process Y_N^N.

Lemma 4.3. *For $1 \le n \le N$ let*

$$
Y_n^N := \tilde{S}_1^N + \sum_{k=1}^{n-1} \epsilon_k (\tilde{S}_{k+1}^N - \tilde{S}_k^N)
$$

with

$$
\epsilon_k := \begin{cases} 1 & (\tilde{S}_1^N, \ldots, \tilde{S}_k^N) \in B_k \\ 0 & \textit{otherwise} \end{cases}
$$

for $k = 1, \ldots, n-1$. Here B_k is an arbitrary set in $\mathfrak{B}(\mathbb{R}^k)$. Then we have

$$
\mathbb{E}[Y_n^N] = \mathbb{E}[\tilde{S}_n^N] - \sum_{k=1}^{n-1} \mathbb{E}\left[(1-\epsilon_k)\left(\mathbb{E}[\tilde{S}_{k+1}^N | \tilde{\mathcal{F}}_k^N] - \tilde{S}_k^N\right)\right] \tag{4.1}
$$

Proof. Consider for $1 \leq n \leq N$ and $N \geq 2$

$$
\begin{aligned}
&\tilde{S}_{n+1}^N - Y_{n+1}^N \\
&= (1-\epsilon_1)(\tilde{S}_2^N - \tilde{S}_1^N) + (1-\epsilon_2)(\tilde{S}_3^N - \tilde{S}_2^N) + ... + (1-\epsilon_k)(\tilde{S}_{n+1}^N - \tilde{S}_n^N) \\
&= (\tilde{S}_n^N - Y_n^N) + (1-\epsilon_n)(\tilde{S}_{n+1}^N - \tilde{S}_n^N)
\end{aligned}
$$

Conditioning on $\tilde{\mathcal{F}}_n^N$ on both sides yields

$$
\mathbb{E}[\tilde{S}_{n+1}^N - Y_{n+1}^N | \tilde{\mathcal{F}}_n^N] = \tilde{S}_n^N - Y_n^N + (1-\epsilon_n)\left(\mathbb{E}[(\tilde{S}_{n+1}^N)|\tilde{\mathcal{F}}_n^N] - \tilde{S}_n^N\right)
$$

Now taking expectations on both sides yields

$$
\mathbb{E}[\tilde{S}_{n+1}^N - Y_{n+1}^N] \geq \mathbb{E}[\tilde{S}_n^N - Y_n^N] + \mathbb{E}\left[(1-\epsilon_n)\left(\mathbb{E}[\tilde{S}_{n+1}^N|\tilde{\mathcal{F}}_n^N] - \tilde{S}_n^N\right)\right]
$$

Note that

$$
\begin{aligned}
\mathbb{E}[\tilde{S}_2^N - Y_2^N] &= \mathbb{E}[\tilde{S}_1^N - Y_1^N] + \mathbb{E}\left[(1-\epsilon_1)\left(\mathbb{E}[\tilde{S}_2^N|\tilde{\mathcal{F}}_1^N] - \tilde{S}_1^N\right)\right] \\
&= \mathbb{E}\left[(1-\epsilon_1)\left(\mathbb{E}[\tilde{S}_2^N|\tilde{\mathcal{F}}_1^N] - \tilde{S}_1^N\right)\right]
\end{aligned}
$$

since $Y_1^N = \tilde{S}_1^N$. Moreover we have

$$
\begin{aligned}
\mathbb{E}[\tilde{S}_3^N - Y_3^N] &= \mathbb{E}[\tilde{S}_2^N - Y_2^N] + \mathbb{E}\left[(1-\epsilon_2)\left(\mathbb{E}[\tilde{S}_3^N|\tilde{\mathcal{F}}_2^N] - \tilde{S}_2^N\right)\right] \\
&= \mathbb{E}\left[(1-\epsilon_1)\left(\mathbb{E}[\tilde{S}_2^N|\tilde{\mathcal{F}}_1^N] - \tilde{S}_1^N\right)\right] \\
&\quad + \mathbb{E}\left[(1-\epsilon_2)\left(\mathbb{E}[\tilde{S}_3^N|\tilde{\mathcal{F}}_2^N] - \tilde{S}_2^N\right)\right] \\
&\cdots
\end{aligned}
$$

$$\mathbb{E}[\tilde{S}_n^N - Y_n^N] = \sum_{k=1}^{n-1} \mathbb{E}\left[(1-\epsilon_k)\left(\mathbb{E}[\tilde{S}_{k+1}^N|\tilde{\mathcal{F}}_k^N] - \tilde{S}_k^N\right)\right]$$

Hence we get

$$\mathbb{E}[Y_n^N] = \mathbb{E}[\tilde{S}_n^N] - \sum_{k=1}^{n-1} \mathbb{E}\left[(1-\epsilon_k)\left(\mathbb{E}[\tilde{S}_{k+1}^N|\tilde{\mathcal{F}}_k^N] - \tilde{S}_k^N\right)\right]$$

□

Remark 4.4. Note that we have $Y_1^N = \tilde{S}_1^N$, as the sum in the definition above is in this case empty and hence treated as zero. Moreover note that we have $Y_{n+1}^N = \tilde{S}_{n+1}^N$ if $\epsilon_k = 1$ for all $1 \leq k \leq n$.

The Lemma below establishes an upper bound for $\mathbb{E}[Y_N^N]$.

Lemma 4.5. *We have for $N \geq 2$*

$$\mathbb{E}[Y_N^N] \leq \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \alpha_{N-k+1} \tag{4.2}$$

where

$$\alpha_{N-k+1} := \sum\sum_{1\leq i<j\leq N-k+1} \mathbb{E}\left[\phi^2(Z_{i:N-k+1}, Z_{j:N-k+1}) W_{i:N-k+1}^2 W_{j:N-k+1}^2\right]^{\frac{1}{2}} \times \mathbb{E}\left[(Q_{i,j}^{N-k+1} - 1)^2\right]^{\frac{1}{2}}$$

Proof. Combining Lemmas 4.3 and 4.2 yields the following for $n \leq N$

$$(b-a)\mathbb{E}[U_n[a,b]] \leq \mathbb{E}[Y_n^N] \leq \mathbb{E}[\tilde{S}_n^N] - \sum_{k=1}^{n-1} \mathbb{E}[(1-\epsilon_k)\left(\mathbb{E}[\tilde{S}_{k+1}^N|\mathcal{F}_k^N] - \tilde{S}_k^N\right)]$$

Moreover we get from Lemma 3.1

$$\mathbb{E}[\tilde{S}_{k+1}^N|\tilde{\mathcal{F}}_k^N] = \mathbb{E}[S_{N-k}|\mathcal{F}_{N-k+1}]$$

$$= \sum\sum_{1\le i<j\le N-k+1} \phi(Z_{i:N-k+1}, Z_{j:N-k+1}) W_{i:N-k+1} W_{j:N-k+1} Q_{i,j}^{N-k+1}$$

Therefore we obtain

$$\begin{aligned}
\mathbb{E}[Y_N^N] &\le \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1} \mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N | \mathcal{F}_k^N] - \tilde{S}_k^N] \\
&= \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right. \\
&\qquad\qquad \left.\times\, W_{i:N-k+1} W_{j:N-k+1}(Q_{i,j}^{N-k+1} - 1)\right] \\
&\le \mathbb{E}[\tilde{S}_N^N] + \left| \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right.\right. \\
&\qquad\qquad \left.\left.\times\, W_{i:N-k+1} W_{j:N-k+1}(Q_{i,j}^{N-k+1} - 1)\right] \right| \\
&\le \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} \left|\mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right.\right. \\
&\qquad\qquad \left.\left.\times\, W_{i:N-k+1} W_{j:N-k+1}(Q_{i,j}^{N-k+1} - 1)\right]\right|
\end{aligned}$$

Now using Jensen's inequality yields

$$\begin{aligned}
\mathbb{E}[Y_N^N] &\le \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right. \\
&\qquad\qquad \left.\times\, W_{i:N-k+1} W_{j:N-k+1} \cdot |(Q_{i,j}^{N-k+1} - 1)|\right] \\
&\le \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right. \\
&\qquad\qquad \left.\times\, W_{i:N-k+1} W_{j:N-k+1} \cdot |(Q_{i,j}^{N-k+1} - 1)|\right]
\end{aligned}$$

The latter inequality above holds, because $1 - \epsilon_k \le 1$ for all $k \le N-1$. By applying the Cauchy-Schwarz inequality on the expectation above, we obtain

$$\mathbb{E}[Y_N^N] \leq \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\leq i<j\leq N-k+1} \mathbb{E}\left[\phi^2(Z_{i:N-k+1}, Z_{j:N-k+1}) W_{i:N-k+1}^2 W_{j:N-k+1}^2\right]^{\frac{1}{2}} \times \mathbb{E}\left[(Q_{i,j}^{N-k+1} - 1)^2\right]^{\frac{1}{2}} \tag{4.3}$$

□

4.2 The reverse supermartingale D_n

Let's first define the following quantities for $n \geq 1$ and $s < t$:

$$B_n(s) := \prod_{k=1}^{n} \left[1 + \frac{1 - q(Z_k)}{n - R_{k,n}}\right]^{\mathbb{1}_{\{Z_k < s\}}}$$

$$C_n(s) := \sum_{i=1}^{n+1} \left[\frac{1 - q(s)}{n - i + 2}\right] \mathbb{1}_{\{Z_{i-1:n} < s \leq Z_{i:n}\}}$$

$$D_n(s,t) := \prod_{k=1}^{n} \left[1 + \frac{1 - q(Z_k)}{n - R_{k,n} + 2}\right]^{2\mathbb{1}_{\{Z_k < s\}}} \prod_{k=1}^{n} \left[1 + \frac{1 - q(Z_k)}{n - R_{k,n} + 1}\right]^{\mathbb{1}_{\{s < Z_k < t\}}}$$

$$\Delta_n(s,t) := \mathbb{E}\left[D_n(s,t)\right]$$

$$\bar{\Delta}_n(s,t) := \mathbb{E}\left[C_n(s) D_n(s,t)\right]$$

Here $Z_{0:n} := -\infty$ and $Z_{n+1:n} := \infty$.

During this section, we will derive a representation of $\mathbb{E}[S_n]$ which involves the process D_n. This will be done in Lemma 4.7 and Lemma 4.8. We will then show that $\{D_n, \mathcal{F}_n\}$ is a reverse supermartingale in Lemma 4.10 and identify the limit of D_n in Lemma 4.9. Those results will lead to Lemma 4.12, which will be central to develop certain bounds in Section 4.3, and hence to show the almost sure existence of the limit S. Moreover Lemmas 4.7, 4.8 and 4.12 will play a central role in identifying the limit S in Chapter 5.

The lemma below contains a basic result needed to prove Lemma 4.8.

Lemma 4.6. *Let* $Z_{1:n-2}, \ldots, Z_{n-2:n-2}$ *denote the ordered* Z*-values among* $Z_1, \ldots, Z_{i-1}, Z_{i+1}, \ldots, Z_{j-1}, Z_{j+1}, \ldots, Z_n$ *for* $1 \leq i, j \leq n$ *and* $n \geq 2$. *Moreover let* $Z_i = s$ *and* $Z_j = t$. *Then we have*

$$\begin{aligned} B_n(s)B_n(t) = \prod_{k=1}^{n-2}\left(1 + \frac{1-q(Z_{k:n-2})}{n-k}\right)^{2\mathbb{1}_{\{Z_{k:n-2}<s\}}} \\ + \prod_{k=1}^{n-2}\left(1 + \frac{1-q(Z_{k:n-2})}{n-k}\right)^{\mathbb{1}_{\{s<Z_{k:n-2}<t\}}} \end{aligned}$$

Proof. Consider the following for $s < t$

$$\begin{aligned} B_n(s)B_n(t) = \prod_{k=1}^{n}\left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{2\mathbb{1}_{\{Z_{k:n}<s\}}} \\ \times \prod_{k=1}^{n}\left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{\mathbb{1}_{\{s<Z_{k:n}<t\}}} \end{aligned} \tag{4.4}$$

Now let's consider the first product above. We have

$$\begin{aligned} \prod_{k=1}^{n}\left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{2\mathbb{1}_{\{Z_{k:n}<s\}}} &= \prod_{k=1}^{n}\left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{2\mathbb{1}_{\{Z_{k:n}<s\}}} \\ &= \sum_{k_1=1}^{n}\sum_{k_2=1}^{n} \mathbb{1}_{\{Z_{k_1:n}=Z_i\}}\mathbb{1}_{\{Z_{k_2:n}=Z_j\}} \\ &\quad \times \prod_{k=1}^{k_1-1}\left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{2\mathbb{1}_{\{Z_{k:n}<s\}}} \\ &\quad \times \prod_{k=k_1+1}^{k_2-1}\left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{2\mathbb{1}_{\{Z_{k:n}<s\}}} \\ &\quad \times \prod_{k=k_2+1}^{n}\left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{2\mathbb{1}_{\{Z_{k:n}<s\}}} \end{aligned}$$

$$= \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \mathbb{1}_{\{Z_{k_1:n}=Z_i\}} \mathbb{1}_{\{Z_{k_2:n}=Z_j\}}$$
$$\times \prod_{k=1}^{k_1-1} \left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{2\mathbb{1}_{\{Z_{k:n}<s\}}}$$

since $Z_{k:n} < s$ if and only if $k < k_1$. Recall that $Z_i = s$ and $Z_j = t$. Thus we obtain for $s < t$ that $Z_{1:n} \leq s \leq Z_{n-1:n}$ and $Z_{2:n} \leq t \leq Z_{n:n}$, or in other words $1 \leq k_1 \leq n-1$ and $2 \leq k_1 \leq n$. Hence

$$\prod_{k=1}^{n} \left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{2\mathbb{1}_{\{Z_{k:n}<s\}}}$$
$$= \sum_{k_1=1}^{n-1} \sum_{k_2=2}^{n} \mathbb{1}_{\{Z_{k_1:n}=Z_i\}} \mathbb{1}_{\{Z_{k_2:n}=Z_j\}}$$
$$\times \prod_{k=1}^{k_1-2} \left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{2\mathbb{1}_{\{Z_{k:n}<s\}}}$$
$$= \sum_{k_1=1}^{n-1} \sum_{k_2=2}^{n} \mathbb{1}_{\{Z_{k_1:n}=Z_i\}} \mathbb{1}_{\{Z_{k_2:n}=Z_j\}}$$
$$\times \prod_{k=1}^{n-2} \left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{2\mathbb{1}_{\{Z_{k:n}<s\}}}$$

Moreover we have

$$Z_{k:n} = \begin{cases} Z_{k:n-2} & Z_{k:n} < s \\ Z_{k-1:n-2} & s < Z_{k:n} < t \end{cases}. \tag{4.5}$$

Therefore

$$\prod_{k=1}^{n} \left(1 + \frac{1-q(Z_{k:n})}{n-k}\right)^{2\mathbb{1}_{\{Z_{k:n}<s\}}}$$
$$= \sum_{k_1=1}^{n-1} \sum_{k_2=2}^{n} \mathbb{1}_{\{Z_{k_1-1:n-2}<Z_i\leq Z_{k_1:n-2}\}} \mathbb{1}_{\{Z_{k_2-2:n-2}<Z_j\leq Z_{k_2-1:n-2}\}}$$
$$\times \prod_{k=1}^{n-2} \left(1 + \frac{1-q(Z_{k:n-2})}{n-k}\right)^{2\mathbb{1}_{\{Z_{k:n-2}<s\}}}$$

$$= \prod_{k=1}^{n-2}\left(1+\frac{1-q(Z_{k:n-2})}{n-k}\right)^{2\mathbb{1}_{\{Z_{k:n-2}<s\}}}$$

Now let's consider the second product in (4.4). We have

$$\begin{aligned}
&\prod_{k=1}^{n}\left(1+\frac{1-q(Z_{k:n})}{n-k}\right)^{\mathbb{1}_{\{s<Z_{k:n}<t\}}} \\
&= \sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\mathbb{1}_{\{Z_{k_1:n}=Z_i\}}\mathbb{1}_{\{Z_{k_2:n}=Z_j\}}\prod_{k=1}^{k_1-1}\left(1+\frac{1-q(Z_{k:n})}{n-k}\right)^{\mathbb{1}_{\{s<Z_{k:n}<t\}}} \\
&\quad \times \prod_{k=k_1+1}^{k_2-1}\left(1+\frac{1-q(Z_{k:n})}{n-k}\right)^{\mathbb{1}_{\{s<Z_{k:n}<t\}}} \times \prod_{k=k_2+1}^{n}\left(1+\frac{1-q(Z_{k:n})}{n-k}\right)^{\mathbb{1}_{\{s<Z_{k:n}<t\}}} \\
&= \sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\mathbb{1}_{\{Z_{k_1:n}=Z_i\}}\mathbb{1}_{\{Z_{k_2:n}=Z_j\}}\prod_{k=k_1+1}^{k_2-1}\left(1+\frac{1-q(Z_{k:n})}{n-k}\right)^{\mathbb{1}_{\{s<Z_{k:n}<t\}}}
\end{aligned}$$

since $s < Z_{k:n} < t$ if and only if $k_1 < k < k_2$. Using again the fact that $1 \le k_1 \le n-1$ and $2 \le k_1 \le n$, we obtain

$$\begin{aligned}
\prod_{k=1}^{n}\left(1+\frac{1-q(Z_{k:n})}{n-k}\right)^{\mathbb{1}_{\{s<Z_{k:n}<t\}}} &= \sum_{k_1=1}^{n-1}\sum_{k_2=2}^{n}\mathbb{1}_{\{Z_{k_1:n}=Z_i\}}\mathbb{1}_{\{Z_{k_2:n}=Z_j\}} \\
&\quad \times \prod_{k=2}^{n-1}\left(1+\frac{1-q(Z_{k:n})}{n-k}\right)^{\mathbb{1}_{\{s<Z_{k:n}<t\}}}
\end{aligned}$$

Now using (4.5) again yields

$$\begin{aligned}
\prod_{k=1}^{n}\left(1+\frac{1-q(Z_{k:n})}{n-k}\right)^{\mathbb{1}_{\{s<Z_{k:n}<t\}}} &= \sum_{k_1=1}^{n-1}\sum_{k_2=2}^{n}\mathbb{1}_{\{Z_{k_1:n}=Z_i\}}\mathbb{1}_{\{Z_{k_2:n}=Z_j\}} \\
&\quad \times \prod_{k=2}^{n-1}\left(1+\frac{1-q(Z_{k-1:n-2})}{n-k}\right)^{\mathbb{1}_{\{s<Z_{k-1:n-2}<t\}}} \\
&= \prod_{k=1}^{n-2}\left(1+\frac{1-q(Z_{k:n-2})}{n-k-1}\right)^{\mathbb{1}_{\{s<Z_{k:n-2}<t\}}}
\end{aligned}$$

With similar arguments we can show the same result on $\{s \geq t\}$. $\square$

Lemma 4.7. *Let $\tilde{\phi} : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ be a Borel-measurable function. Then we have for any $s < t$ and $n \geq 2$*

$$\begin{aligned} &\mathbb{E}[\tilde{\phi}(Z_i, Z_j) B_n(Z_i) B_n(Z_j)] \\ &= \mathbb{E}[\tilde{\phi}(Z_1, Z_2) B_n(Z_1) B_n(Z_2)] \end{aligned}$$

Proof. Consider that $\{Z_i = Z_j\}$ is a measure zero set, since H is continuous. Therefore the following holds for $1 \leq i, j \leq n$

$$\begin{aligned} &\mathbb{E}\left[\tilde{\phi}(Z_i, Z_j)\mathbb{E}\left[B_n(Z_i)B_n(Z_j)|Z_i, Z_j\right]\right] \\ &= \mathbb{E}\left[\mathbb{1}_{\{Z_i<Z_j\}}\tilde{\phi}(Z_i, Z_j)\mathbb{E}\left[B_n(Z_i)B_n(Z_j)|Z_i, Z_j\right]\right] \\ &\quad + \mathbb{E}\left[\mathbb{1}_{\{Z_i>Z_j\}}\tilde{\phi}(Z_i, Z_j)\mathbb{E}\left[B_n(Z_i)B_n(Z_j)|Z_i, Z_j\right]\right] \\ &= \int_0^\infty \int_0^\infty \mathbb{1}_{\{s<t\}}\tilde{\phi}(s,t)\mathbb{E}\left[B_n(s)B_n(t)|Z_i = s, Z_j = t\right] H(ds)H(dt) \\ &\quad + \int_0^\infty \int_0^\infty \mathbb{1}_{\{s>t\}}\tilde{\phi}(s,t)\mathbb{E}\left[B_n(s)B_n(t)|Z_i = s, Z_j = t\right] H(ds)H(dt) \\ &=: \int_0^\infty \int_0^\infty \tilde{\phi}(s,t)I_1(s,t)H(ds)H(dt) + \int_0^\infty \int_0^\infty \tilde{\phi}(s,t)I_2(s,t)H(ds)H(dt) \end{aligned} \tag{4.6}$$

Now let's consider I_1 above. Using Lemma 4.6 we obtain for $1 \leq i, j \leq n$

$$\begin{aligned} I_1(s,t) &= \mathbb{1}_{\{s<t\}}\mathbb{E}[B_n(s)B_n(t)|Z_i = s, Z_j = t] \\ &= \mathbb{1}_{\{s<t\}}\mathbb{E}\left[\prod_{k=1}^{n-2}\left(1 + \frac{1-q(Z_{k:n-2})}{n-k}\right)^{2\mathbb{1}_{\{Z_{k:n-2}<s\}}} |Z_i = s, Z_j = t\right] \\ &\quad \times \mathbb{1}_{\{s<t\}}\mathbb{E}\left[\prod_{k=1}^{n-2}\left(1 + \frac{1-q(Z_{k:n-2})}{n-k-1}\right)^{\mathbb{1}_{\{s\leq Z_{k:n-2}<t\}}} |Z_i = s, Z_j = t\right] \\ &= \mathbb{1}_{\{s<t\}}\mathbb{E}\left[\prod_{k=1}^{n-2}\left(1 + \frac{1-q(Z_{k:n-2})}{n-k}\right)^{2\mathbb{1}_{\{Z_{k:n-2}<s\}}}\right] \\ &\quad \times \mathbb{1}_{\{s<t\}}\mathbb{E}\left[\prod_{k=1}^{n-2}\left(1 + \frac{1-q(Z_{k:n-2})}{n-k-1}\right)^{\mathbb{1}_{\{s\leq Z_{k:n-2}<t\}}}\right] \end{aligned}$$

which is independent of i, j. Similarly we obtain

$$\begin{aligned} I_2(s,t) &= \mathbb{1}_{\{s>t\}}\mathbb{E}[B_n(s)B_n(t)|Z_i = s, Z_j = t] \\ &= \mathbb{1}_{\{s>t\}}\mathbb{E}\left[\prod_{k=1}^{n-2}\left(1+\frac{1-q(Z_{k:n-2})}{n-k}\right)^{2\mathbb{1}_{\{Z_{k:n-2}<s\}}}\right] \\ &\quad \times \mathbb{1}_{\{s<t\}}\mathbb{E}\left[\prod_{k=1}^{n-2}\left(1+\frac{1-q(Z_{k:n-2})}{n-k-1}\right)^{\mathbb{1}_{\{s\le Z_{k:n-2}<t\}}}\right] \end{aligned}$$

which is independent of i, j as well. Therefore we get, according to (4.6), that $\mathbb{E}\left[\tilde{\phi}(Z_i, Z_j)\mathbb{E}\left[B_n(Z_i)B_n(Z_j)|Z_i, Z_j\right]\right]$ is independent of i, j. Hence

$$\begin{aligned} \mathbb{E}\left[\tilde{\phi}(Z_i, Z_j)B_n(Z_i)B_n(Z_j)\right] &= \mathbb{E}\left[\tilde{\phi}(Z_i, Z_j)\mathbb{E}\left[B_n(Z_i)B_n(Z_j)|Z_i, Z_j\right]\right] \\ &= \mathbb{E}\left[\tilde{\phi}(Z_1, Z_2)B_n(Z_1)B_n(Z_2)\right] \end{aligned}$$

$\square$

Lemma 4.8. *Let $\tilde{\phi} : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ be a Borel-measurable function. Then we have for any $s < t$ and $n \geq 2$*

$$\begin{aligned} &\mathbb{E}[\tilde{\phi}(Z_1, Z_2)B_n(Z_1)B_n(Z_2)] \\ &= \mathbb{E}[2\tilde{\phi}(Z_1, Z_2)\{\Delta_{n-2}(Z_1, Z_2) + \bar{\Delta}_{n-2}(Z_1, Z_2)\}\mathbb{1}_{\{Z_1<Z_2\}}] \end{aligned}$$

Proof. Consider the following

$$\begin{aligned} B_n(Z_1)B_n(Z_2) &= \prod_{k=1}^{n}\left[1+\frac{1-q(Z_k)}{n-R_{k,n}}\right]^{\mathbb{1}_{\{Z_k<Z_1\}}+\mathbb{1}_{\{Z_k<Z_2\}}} \\ &= \left[1+\frac{1-q(Z_1)}{n-R_{1,n}}\right]^{\mathbb{1}_{\{Z_1<Z_2\}}}\left[1+\frac{1-q(Z_2)}{n-R_{2,n}}\right]^{\mathbb{1}_{\{Z_2<Z_1\}}} \\ &\quad \times \prod_{k=3}^{n}\left[1+\frac{1-q(Z_k)}{n-R_{k,n}}\right]^{\mathbb{1}_{\{Z_k<Z_1\}}+\mathbb{1}_{\{Z_k<Z_2\}}} \\ &= \mathbb{1}_{\{Z_1<Z_2\}}\left[1+\frac{1-q(Z_1)}{n-R_{1,n}}\right] \end{aligned}$$

$$\times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n-R_{k+2,n}}\right]^{\mathbb{1}_{\{Z_{k+2}<Z_1\}}+\mathbb{1}_{\{Z_{k+2}<Z_2\}}}$$
$$+ \mathbb{1}_{\{Z_1>Z_2\}} \left[1 + \frac{1-q(Z_2)}{n-R_{2,n}}\right]$$
$$\times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n-R_{k+2,n}}\right]^{\mathbb{1}_{\{Z_{k+2}<Z_1\}}+\mathbb{1}_{\{Z_{k+2}<Z_2\}}}$$
$$+ \mathbb{1}_{\{Z_1=Z_2\}} \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n-R_{k+2,n}}\right]^{2\mathbb{1}_{\{Z_{k+2}<Z_1\}}} \tag{4.7}$$

On $\{Z_1 < Z_2\}$ we have

$$\prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n-R_{k+2,n}}\right]^{\mathbb{1}_{\{Z_{k+2}<Z_2\}}} = \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}}\right]^{\mathbb{1}_{\{Z_{k+2}<Z_1\}}}$$
$$\times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}-1}\right]^{\mathbb{1}_{\{Z_1<Z_{k+2}<Z_2\}}}$$

where $\tilde{R}_{k,n-2}$ denotes the rank of the Z_k, $k = 3, \dots, n$ among themselves. The above holds since

$$R_{k+2,n} = \begin{cases} \tilde{R}_{k,n-2} & \text{if } Z_{k+2} < Z_1 \\ \tilde{R}_{k,n-2} + 1 & \text{if } Z_1 < Z_{k+2} < Z_2 \end{cases}$$

for $k = 1, \dots, n-2$. Therefore (4.7) yields

$$B_n(Z_1)B_n(Z_2) = \mathbb{1}_{\{Z_1<Z_2\}} \left[1 + \frac{1-q(Z_1)}{n-R_{1,n}}\right]$$
$$\times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}}\right]^{2\mathbb{1}_{\{Z_{k+2}<Z_1\}}}$$
$$\times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}-1}\right]^{\mathbb{1}_{\{Z_1<Z_{k+2}<Z_2\}}}$$
$$+ \mathbb{1}_{\{Z_2<Z_1\}} \left[1 + \frac{1-q(Z_2)}{n-R_{2,n}}\right]$$
$$\times \prod_{k=1}^{n-2} \left[1 + \frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}}\right]^{2\mathbb{1}_{\{Z_{k+2}<Z_2\}}}$$

$$
\begin{aligned}
&\times \prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}-1}\right]^{\mathbb{1}_{\{Z_2<Z_{k+2}<Z_1\}}} \\
&+\mathbb{1}_{\{Z_1=Z_2\}}\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}}\right]^{2\mathbb{1}_{\{Z_{k+2}<Z_1\}}}
\end{aligned} \tag{4.8}
$$

Now let's denote $Z_{k:n-2}$ the ordered Z-values among $Z_3,\ldots,Z_n$ for $k=1,\ldots,n-2$. Consider that we can write

$$
\left[1+\frac{1-q(Z_1)}{n-R_{1,n}}\right]=\sum_{i=1}^{n-1}\left[1+\frac{1-q(s)}{n-i}\right]\mathbb{1}_{\{Z_{i-1:n-2}<Z_1\leq Z_{i:n-2}\}}
$$

Note that $Z_{k:n-2}$ is independent of Z_1 and Z_2 for $k=1,\ldots,n-2$. Therefore we obtain the following, by conditioning (4.8) on Z_1, Z_2:

$$
\begin{aligned}
&\mathbb{E}[B_n(Z_1)B_n(Z_2)|Z_1=s,Z_2=t] \\
&=\mathbb{1}_{\{s<t\}}\mathbb{E}\Bigg[\left(\sum_{i=1}^{n-1}\left[1+\frac{1-q(s)}{n-i}\right]\mathbb{1}_{\{Z_{i-1:n-2}<s\leq Z_{i:n-2}\}}\right) \\
&\qquad\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{1}_{\{Z_{k:n-2}<s\}}} \\
&\qquad\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k-1}\right]^{\mathbb{1}_{\{s<Z_{k:n-2}<t\}}}\Bigg] \\
&\quad+\mathbb{1}_{\{t<s\}}\mathbb{E}\Bigg[\left(\sum_{i=1}^{n-1}\left[1+\frac{1-q(t)}{n-i}\right]\mathbb{1}_{\{Z_{i-1:n-2}<t\leq Z_{i:n-2}\}}\right) \\
&\qquad\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{1}_{\{Z_{k:n-2}<t\}}} \\
&\qquad\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k-1}\right]^{\mathbb{1}_{\{t<Z_{k:n-2}<s\}}}\Bigg] \\
&\quad+\mathbb{1}_{\{s=t\}}\mathbb{E}\left[\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{1}_{\{Z_{k:n-2}<s\}}}\right] \\
&=\alpha(s,t)+\alpha(t,s)+\beta(s,t)
\end{aligned}
$$

where

$$\alpha(s,t) := \mathbb{1}_{\{s<t\}}\mathbb{E}\left[\left(\sum_{i=1}^{n-1}\left[1+\frac{1-q(s)}{n-i}\right]\mathbb{1}_{\{Z_{i-1:n-2}<s\le Z_{i:n-2}\}}\right)\right.$$
$$\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{1}_{\{Z_{k:n-2}<s\}}}$$
$$\left.\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k-1}\right]^{\mathbb{1}_{\{s<Z_{k:n-2}<t\}}}\right]$$

and

$$\beta(s,t) := \mathbb{1}_{\{s=t\}}\mathbb{E}\left[\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{1}_{\{Z_{k:n-2}<s\}}}\right]$$

Consider that we have

$$\mathbb{E}[\alpha(Z_1,Z_2)] = \mathbb{E}[\alpha(Z_2,Z_1)]$$

under (A1), because Z_1 and Z_2 are i. i. d. and

$$\mathbb{E}[\beta(Z_1,Z_2)] = 0$$

since H is continuous. Therefore we get

$$\begin{aligned}
&\mathbb{E}[\tilde{\phi}(Z_1,Z_2)B_n(Z_1)B_n(Z_2)]\\
&=\mathbb{E}[\tilde{\phi}(Z_1,Z_2)(\alpha(Z_1,Z_2)+\alpha(Z_2,Z_1)+\beta(Z_1,Z_2))]\\
&=\mathbb{E}[2\tilde{\phi}(Z_1,Z_2)\alpha(Z_1,Z_2)] \qquad (4.9)
\end{aligned}$$

Next consider that

$$\begin{aligned}
\alpha(s,t) &= \mathbb{1}_{\{s<t\}}\mathbb{E}\left[(1+C_n(s))D_{n-2}(s,t)\right]\\
&= \mathbb{1}_{\{s<t\}}(\Delta_{n-2}(s,t)+\bar{\Delta}_{n-2}(s,t))
\end{aligned}$$

The latter equality holds, since

$$\begin{aligned}
&\sum_{i=1}^{n-1}\left[1+\frac{1-q(s)}{n-i}\right]\mathbb{1}_{\{Z_{i-1:n-2}<s\le Z_{i:n-2}\}}\\
&=\sum_{i=1}^{n-1}\mathbb{1}_{\{Z_{i-1:n-2}<s\le Z_{i:n-2}\}}+\sum_{i=1}^{n-1}\left[\frac{1-q(s)}{n-i}\right]\mathbb{1}_{\{Z_{i-1:n-2}<s\le Z_{i:n-2}\}}\\
&=1+C_n(s)
\end{aligned}$$

Now the statement of the lemma follows directly from (4.9). $\square$

The next lemma identifies the $\mathbb{P}$-almost sure limit of D_n for $n\to\infty$. Define for $s<t$

$$D(s,t):=\exp\left(2\int_0^s\frac{1-q(z)}{1-H(z)}H(dz)+\int_s^t\frac{1-q(z)}{1-H(z)}H(dz)\right)$$

Lemma 4.9. *For any $s<t$ s. t. $H(t)<1$, we have*

$$\lim_{n\to\infty}D_n(s,t)=D(s,t)$$

Proof. First define for $s<t$ and $k=1,\dots,n$

$$\begin{aligned}
x_k&:=\frac{1-q(Z_k)}{n(1-H_n(Z_k)+2/n)}\\
y_k&:=\frac{1-q(Z_k)}{n(1-H_n(Z_k)+1/n)}\\
s_k&:=\mathbb{1}_{\{Z_k<s\}}\\
t_k&:=\mathbb{1}_{\{s<Z_k<t\}}
\end{aligned}$$

Next consider

$$\begin{aligned}
D_n(s,t)=&\prod_{k=1}^{n}\left[1+\frac{1-q(Z_k)}{n(1-H_n(Z_k)+2/n)}\mathbb{1}_{\{Z_k<s\}}\right]^2\\
&\times\prod_{k=1}^{n}\left[1+\frac{1-q(Z_k)}{n(1-H_n(Z_k)+1/n)}\mathbb{1}_{\{s<Z_k<t\}}\right]
\end{aligned}$$

$$= \prod_{k=1}^{n} [1 + x_k s_k]^2 \prod_{k=1}^{n} [1 + y_k t_k]$$
$$= \exp\left(2 \sum_{k=1}^{n} \ln [1 + x_k s_k] + \sum_{k=1}^{n} \ln [1 + y_k t_k] \right)$$

Note that $0 \leq x_k s_k \leq 1$ and $0 \leq y_k t_k \leq 1$. Consider that the following inequality holds

$$-\frac{x^2}{2} \leq \ln(1 + x) - x \leq 0$$

for any $x \geq 0$ (cf. Stute and Wang (1993), p. 1603). This implies

$$-\frac{1}{2} \sum_{k=1}^{n} x_k^2 s_k \leq \sum_{k=1}^{n} \ln(1 + x_k s_k) - \sum_{k=1}^{n} x_k s_k \leq 0$$

But now

$$\begin{aligned} \sum_{k=1}^{n} x_k^2 s_k &= \frac{1}{n^2} \sum_{k=1}^{n} \left(\frac{1 - q(Z_k)}{1 - H_n(Z_k) + \frac{2}{n}} \right)^2 \mathbb{1}_{\{Z_k < s\}} \\ &\leq \frac{1}{n^2} \sum_{k=1}^{n} \left(\frac{1}{1 - H_n(s) + \frac{1}{n}} \right)^2 \\ &= \frac{1}{n(1 - H_n(s) + n^{-1})^2} \longrightarrow 0 \end{aligned}$$

$\mathbb{P}$-almost surely as $n \to \infty$, since $H(s) < H(t) < 1$ (c. f. Stute and Wang (1993), p. 1603). Therefore we have

$$|\sum_{k=1}^{n} \ln(1 + x_k s_k) - \sum_{k=1}^{n} x_k s_k| \longrightarrow 0$$

with probability 1 as $n \to \infty$. Similarly we obtain

$$|\sum_{k=1}^{n} \ln(1 + y_k t_k) - \sum_{k=1}^{n} y_k t_k| \longrightarrow 0$$

with probability 1 as $n \to \infty$. Hence

$$\lim_{n\to\infty} D_n(s) = \lim_{n\to\infty} \exp\left(2\sum_{k=1}^{n} x_k s_k + \sum_{k=1}^{n} y_k t_k\right)$$

Now consider

$$\begin{aligned}
\sum_{i=1}^{n} x_k s_k &= \frac{1}{n}\sum_{k=1}^{n} \frac{1-q(Z_k)}{1-H_n(Z_k)+\frac{2}{n}} \mathbb{1}_{\{Z_k<s\}} \\
&= \int_0^{s-} \frac{1-q(z)}{1-H_n(z)+\frac{2}{n}} H_n(dz) \\
&= \int_0^{s-} \frac{1-q(z)}{1-H(z)} H_n(dz) + \int_0^{s-} \frac{1-q(z)}{1-H_n(z)+\frac{2}{n}} - \frac{1-q(z)}{1-H(z)} H_n(dz) \\
&= \int_0^{s-} \frac{1-q(z)}{1-H(z)} H_n(dz) + \int_0^{s-} \frac{(1-q(z))(H_n(z)-H(z)-\frac{2}{n})}{(1-H_n(z)+\frac{2}{n})(1-H(z))} H_n(dz)
\end{aligned} \tag{4.10}$$

Note that the second term on the right hand side of the latter equation above tends to zero for $n \to \infty$, because

$$\begin{aligned}
&\int_0^{s-} \frac{(1-q(z))(H_n(z)-H(z)-\frac{2}{n})}{(1-H_n(z)+\frac{2}{n})(1-H(z))} H_n(dz) \\
\le\ & \frac{\sup_z |H_n(z)-H(z)-\frac{2}{n}|}{1-H(s)} \int_0^{s-} \frac{1}{1-H_n(z)} H_n(dz) \longrightarrow 0
\end{aligned}$$

$\mathbb{P}$-almost surely as $n \to \infty$, by the Glivenko-Cantelli Theorem and since $H(s) < 1$. Moreover we have

$$\int_0^{s-} \frac{1-q(z)}{1-H(z)} H_n(dz) \longrightarrow \int_0^{s} \frac{1-q(z)}{1-H(z)} H(dz)$$

by the SLLN. Therefore we obtain

$$\lim_{n\to\infty} \sum_{i=1}^{n} x_k s_k = \int_0^{s} \frac{1-q(z)}{1-H(z)} H(dz)$$

By the same arguments, we can show that

$$\lim_{n\to\infty}\sum_{i=1}^{n} y_k t_k = \int_s^t \frac{1-q(z)}{1-H(z)} H(dz)$$

Thus we finally conclude

$$\lim_{n\to\infty} D_n(s,t) = \exp\left(2\int_0^s \frac{1-q(z)}{1-H(z)} H(dz) + \int_s^t \frac{1-q(z)}{1-H(z)} H(dz)\right)$$

$\mathbb{P}$-almost surely. $\square$

Lemma 4.10. $\{D_n, \mathcal{F}_n\}_{n\geq 1}$ *is a non-negative reverse supermartingale.*

Proof. Consider that for $s < t$ and $n \geq 1$

$$\begin{aligned}
\mathbb{E}[D_n(s,t)|\mathcal{F}_{n+1}] &= \mathbb{E}\left[\prod_{k=1}^{n}\left(1+\frac{1-q(Z_{k:n})}{n-k+2}\right)^{2\mathbb{1}_{\{Z_{k:n}<s\}}}\right. \\
&\qquad \left.\times \prod_{k=1}^{n}\left(1+\frac{1-q(Z_{k:n})}{n-k+1}\right)^{\mathbb{1}_{\{s<Z_{k:n}<t\}}} |\mathcal{F}_{n+1}\right] \\
&= \sum_{i=1}^{n+1}\mathbb{E}\left[\mathbb{1}_{\{Z_{n+1}=Z_{i:n+1}\}}\prod_{k=1}^{n}\ldots|\mathcal{F}_{n+1}\right] \\
&= \sum_{i=1}^{n+1}\mathbb{E}\left[\mathbb{1}_{\{Z_{n+1}=Z_{i:n+1}\}}\prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+2}\right)^{2\mathbb{1}_{\{Z_{k:n+1}<s\}}}\right. \\
&\qquad \times \prod_{k=i}^{n}\left(1+\frac{1-q(Z_{k+1:n+1})}{n-k+2}\right)^{2\mathbb{1}_{\{Z_{k+1:n+1}<s\}}} \\
&\qquad \times \prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+1}\right)^{\mathbb{1}_{\{s<Z_{k:n+1}<t\}}} \\
&\qquad \left.\times \prod_{k=i}^{n}\left(1+\frac{1-q(Z_{k+1:n+1})}{n-k+1}\right)^{\mathbb{1}_{\{s<Z_{k+1:n+1}<t\}}} |\mathcal{F}_{n+1}\right] \\
&= \sum_{i=1}^{n+1}\mathbb{E}\left[\mathbb{1}_{\{Z_{n+1}=Z_{i:n+1}\}}\prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+2}\right)^{2\mathbb{1}_{\{Z_{k:n+1}<s\}}}\right. \\
&\qquad \times \prod_{k=i+1}^{n+1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+3}\right)^{2\mathbb{1}_{\{Z_{k:n+1}<s\}}}
\end{aligned}$$

$$\times \prod_{k=1}^{i-1} \left(1 + \frac{1-q(Z_{k:n+1})}{n-k+1}\right)^{\mathbb{1}_{\{s<Z_{k:n+1}<t\}}}$$
$$\times \prod_{k=i+1}^{n+1} \left(1 + \frac{1-q(Z_{k:n+1})}{n-k+2}\right)^{\mathbb{1}_{\{s<Z_{k:n+1}<t\}}} |\mathcal{F}_{n+1}\Bigg]$$

Now each product within the conditional expectation is measurable w. r. t. $\mathcal{F}_{n+1}$. Moreover we have for $i = 1, \ldots, n$

$$\begin{aligned}
\mathbb{E}[\mathbb{1}_{\{Z_{n+1}=Z_{i:n+1}\}}|\mathcal{F}_n + 1] &= \mathbb{P}(Z_{n+1} = Z_{i:n+1}|\mathcal{F}_{n+1}) \\
&= \mathbb{P}(R_{n+1,n+1} = i) \\
&= \frac{1}{n+1}
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\mathbb{E}[D_n(s,t)|\mathcal{F}_{n+1}] = \frac{1}{n+1} \sum_{i=1}^{n+1} \prod_{k=1}^{i-1} &\left(1 + \frac{1-q(Z_{k:n+1})}{n-k+2}\right)^{2\mathbb{1}_{\{Z_{k:n+1}<s\}}} \\
&\times \left(1 + \frac{1-q(Z_{k:n+1})}{n-k+1}\right)^{\mathbb{1}_{\{s<Z_{k:n+1}<t\}}} \\
&\times \prod_{k=i+1}^{n+1} \left(1 + \frac{1-q(Z_{k:n+1})}{n-k+3}\right)^{2\mathbb{1}_{\{Z_{k:n+1}<s\}}} \\
&\times \left(1 + \frac{1-q(Z_{k:n+1})}{n-k+2}\right)^{\mathbb{1}_{\{s<Z_{k:n+1}<t\}}}
\end{aligned} \tag{4.11}$$

We will now proceed by induction on n. First let

$$x_k := 1 - q(Z_{k:2}),\ s_k := \mathbb{1}_{\{Z_{k:2}<s\}} \text{ and } t_k := \mathbb{1}_{\{s<Z_{k:2}<t\}}$$

for $k = 1, 2$. Next consider

$$\begin{aligned}
\mathbb{E}[D_1(s,t)|\mathcal{F}_2] = \frac{1}{2} \Bigg[&\left(1 + \frac{1-q(Z_{2:2})}{2}\right)^{2\mathbb{1}_{\{Z_{2:2}<s\}}} \times (1 + (1 - q(Z_{2:2})))^{\mathbb{1}_{\{s<Z_{2:2}<t\}}} \\
&+ \left(1 + \frac{1-q(Z_{1:2})}{2}\right)^{2\mathbb{1}_{\{Z_{1:2}<s\}}} \times (1 + (1 - q(Z_{1:2})))^{\mathbb{1}_{\{s<Z_{1:2}<t\}}} \Bigg]
\end{aligned}$$

$$= \frac{1}{2}\left[\left(1+\frac{x_2}{2}s_2\right)^2 \times (1+x_2t_2) + \left(1+\frac{x_1}{2}s_1\right)^2 \times (1+x_1t_1)\right]$$

Moreover we have

$$\begin{aligned} D_2(s,t) &= \left[1+\frac{x_1}{3}s_1\right]^2 \times \left[1+\frac{x_1}{2}t_1\right] \times \left[1+\frac{x_2}{2}s_2\right]^2 \times [1+x_2t_2] \\ &= \left[1+\frac{x_1}{2}t_1 + \left(\frac{x_1^2}{9}+\frac{2}{3}x_1\right)s_1\right] \times \left[1+x_2t_2+\left(\frac{x_2^2}{4}+x_2\right)s_2\right] \end{aligned}$$

Therefore we obtain

$$\mathbb{E}[D_1(s,t)|\mathcal{F}_2] - D_2(s,t) \le \frac{x_1^2}{72} - \frac{x_1}{6} \le 0$$

since $0 \le x_1 \le 1$. Thus $\mathbb{E}[D_1(s,t)|\mathcal{F}_2] \le D_2(s,t)$ for any $s < t$, as needed. Now assume that

$$\mathbb{E}[D_n(s,t)|\mathcal{F}_{n+1}] \le D_{n+1}(s,t)$$

holds for any $n \ge 1$. Note that the latter is equivalent to assuming

$$\begin{aligned} &\frac{1}{n+1}\sum_{i=1}^{n+1}\prod_{k=1}^{i-1}\left(1+\frac{1-q(y_k)}{n-k+2}\right)^{2\mathbb{1}_{\{y_k<s\}}}\left(1+\frac{1-q(y_k)}{n-k+1}\right)^{\mathbb{1}_{\{s<y_k<t\}}} \\ &\qquad \times \prod_{k=i+1}^{n+1}\left(1+\frac{1-q(y_k)}{n-k+3}\right)^{2\mathbb{1}_{\{y_k<s\}}}\left(1+\frac{1-q(y_k)}{n-k+2}\right)^{\mathbb{1}_{\{s<y_k<t\}}} \\ &\le \prod_{k=1}^{n+1}\left(1+\frac{1-q(y_k)}{n-k+3}\right)^{2\mathbb{1}_{\{y_k<s\}}}\prod_{k=1}^{n+1}\left(1+\frac{1-q(y_k)}{n-k+2}\right)^{\mathbb{1}_{\{s<y_k<t\}}} \end{aligned} \tag{4.12}$$

holds for arbitrary $y_k \ge 0$. Next define for $s < t$ and $n \ge 1$

$$A_{n+2}(s,t) := \prod_{k=2}^{n+2}\left[1+\frac{1-q(Z_{k:n+2})}{n-k+4}\right]^{2\mathbb{1}_{\{Z_{k:n+2}<s\}}} \times \left[1+\frac{1-q(Z_{k:n+2})}{n-k+3}\right]^{\mathbb{1}_{\{s<Z_{k:n+2}<t\}}}$$

Now consider that we get from (4.11)

$$
\begin{aligned}
&\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}]\\
&= \frac{1}{n+2}\sum_{i=1}^{n+2}\prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k:n+2})}{n-k+3}\right)^{2\mathbb{1}_{\{Z_{k:n+2}<s\}}}\left(1+\frac{1-q(Z_{k:n+2})}{n-k+2}\right)^{\mathbb{1}_{\{s<Z_{k:n+2}<t\}}}\\
&\qquad\qquad \times \prod_{k=i+1}^{n+2}\left(1+\frac{1-q(Z_{k:n+2})}{n-k+4}\right)^{2\mathbb{1}_{\{Z_{k:n+2}<s\}}}\left(1+\frac{1-q(Z_{k:n+2})}{n-k+3}\right)^{\mathbb{1}_{\{s<Z_{k:n+2}<t\}}}\\
&= \frac{A_{n+2}}{n+2}+\frac{1}{n+2}\sum_{i=2}^{n+2}\prod_{k=1}^{i-1}\cdots\times\prod_{k=i+1}^{n+2}\cdots\\
&= \frac{A_{n+2}}{n+2}+\frac{1}{n+2}\sum_{i=1}^{n+1}\prod_{k=1}^{i}\cdots\times\prod_{k=i+2}^{n+2}\cdots\\
&= \frac{A_{n+2}}{n+2}+\frac{1}{n+2}\left(1+\frac{1-q(Z_{1:n+2})}{n+2}\right)^{2\mathbb{1}_{\{Z_{1:n+2}<s\}}}\left(1+\frac{1-q(Z_{1:n+2})}{n+1}\right)^{\mathbb{1}_{\{s<Z_{1:n+2}<t\}}}\\
&\qquad\qquad \times\sum_{i=1}^{n+1}\prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k+1:n+2})}{n-k+2}\right)^{2\mathbb{1}_{\{Z_{k+1:n+2}<s\}}}\\
&\qquad\qquad\qquad \times\left(1+\frac{1-q(Z_{k+1:n+2})}{n-k+1}\right)^{\mathbb{1}_{\{s<Z_{k+1:n+2}<t\}}}\\
&\qquad\qquad \times\prod_{k=i+1}^{n+1}\left(1+\frac{1-q(Z_{k+1:n+2})}{n-k+3}\right)^{2\mathbb{1}_{\{Z_{k+1:n+2}<s\}}}\\
&\qquad\qquad\qquad \times\left(1+\frac{1-q(Z_{k+1:n+2})}{n-k+2}\right)^{\mathbb{1}_{\{s<Z_{k+1:n+2}<t\}}}
\end{aligned}
$$

Using (4.12) on the right hand side of the equation above yields

$$
\begin{aligned}
&\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}]\\
&\leq \frac{A_{n+2}}{n+2}+\frac{n+1}{n+2}\left(1+\frac{1-q(Z_{1:n+2})}{n+2}\right)^{2\mathbb{1}_{\{Z_{1:n+2}<s\}}}\left(1+\frac{1-q(Z_{1:n+2})}{n+1}\right)^{\mathbb{1}_{\{s<Z_{1:n+2}<t\}}}\\
&\qquad\qquad \times\prod_{k=1}^{n+1}\left(1+\frac{1-q(Z_{k+1:n+2})}{n-k+3}\right)^{2\mathbb{1}_{\{Z_{k+1:n+2}<s\}}}\\
&\qquad\qquad\qquad \times\left(1+\frac{1-q(Z_{k+1:n+2})}{n-k+2}\right)^{\mathbb{1}_{\{s<Z_{k+1:n+2}<t\}}}\\
&= A_{n+2}\left[\frac{1}{n+2}+\frac{n+1}{n+2}\left(1+\frac{1-q(Z_{1:n+2})}{n+2}\right)^{2\mathbb{1}_{\{Z_{1:n+2}<s\}}}\right.
\end{aligned}
$$

$$\times \left(1 + \frac{1 - q(Z_{1:n+2})}{n+1}\right)^{\mathbb{1}_{\{s<Z_{1:n+2}<t\}}}\Bigg]$$

For the moment, let

$$x_1 := 1 - q(Z_{1:n+2}),\ s_1 := \mathbb{1}_{\{Z_{1:n+2}<s\}} \text{ and } t_1 := \mathbb{1}_{\{s<Z_{1:n+2}<t\}}$$

Now we can rewrite the above as

$$\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}] \leq A_{n+2}\left[\frac{1}{n+2} + \frac{n+1}{n+2}\left(1 + \frac{x_1 s_1}{n+2}\right)^2\left(1 + \frac{x_1 t_1}{n+1}\right)\right] \quad (4.13)$$

Next consider

$$\begin{aligned}
\left(1 + \frac{x_1 t_1}{n+1}\right) &= \left(1 + \frac{x_1 t_1}{n+2} - \frac{1}{n+2}\right)\left(1 + \frac{1}{n+1}\right) \\
&= \left(1 + \frac{x_1 t_1}{n+2}\right) + \frac{1}{n+1}\left(1 + \frac{x_1 t_1}{(n+2)}\right) - \frac{1}{n+1} \\
&= \left(1 + \frac{x_1 t_1}{n+2}\right) + \frac{x_1 t_1}{(n+1)(n+2)}
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\frac{n+1}{n+2}\left(1 + \frac{x_1 s_1}{n+2}\right)^2\left(1 + \frac{x_1 t_1}{n+1}\right) \\
&= \frac{n+1}{n+2}\left(1 + \frac{x_1 s_1}{n+2}\right)^2\left(1 + \frac{x_1 t_1}{n+2}\right) + \left(1 + \frac{x_1 s_1}{n+2}\right)^2 \frac{x_1 t_1}{(n+2)^2}
\end{aligned}$$

But now

$$\begin{aligned}
\left(1 + \frac{x_1 s_1}{n+2}\right)^2 \frac{x_1 t_1}{(n+2)^2} &= \left(1 + 2\frac{x_1 s_1}{n+2} + \frac{x_1^2 s_1}{(n+2)^2}\right)\frac{x_1 t_1}{(n+2)^2} \\
&= \frac{x_1 t_1}{(n+2)^2}
\end{aligned}$$

since $s_1 \cdot t_1 = 0$ for all $s < t$. Hence we can rewrite the term in brackets in (4.13) as

$$\begin{aligned}
&\frac{1}{n+2} + \frac{n+1}{n+2}\left(1 + \frac{x_1 s_1}{n+2}\right)^2 \left(1 + \frac{x_1 t_1}{n+1}\right) \\
&= \frac{1}{n+2} + \frac{x_1 t_1}{(n+2)^2} + \frac{n+1}{n+2}\left(1 + \frac{x_1 s_1}{n+2}\right)^2 \left(1 + \frac{x_1 t_1}{n+2}\right) \\
&= \frac{1}{n+2}\left(1 + \frac{x_1 t_1}{n+2}\right) + \frac{n+1}{n+2}\left(1 + \frac{x_1 s_1}{n+2}\right)^2 \left(1 + \frac{x_1 t_1}{n+2}\right) \\
&= \left[\frac{1}{n+2} + \frac{n+1}{n+2}\left(1 + \frac{x_1}{n+2}\right)^{2s_1}\right]\left(1 + \frac{x_1}{n+2}\right)^{t_1} \\
&\leq \left(1 + \frac{x_1}{n+3}\right)^{2s_1}\left(1 + \frac{x_1}{n+2}\right)^{t_1}
\end{aligned}$$

The latter inequality above holds, since

$$\left[\frac{1}{n+2} + \frac{n+1}{n+2}\left(1 + \frac{x}{n+2}\right)^2\right] \leq \left(1 + \frac{x}{n+3}\right)^2$$

for any $0 \leq x \leq 1$. (c. f. Bose and Sen (1999), page 197). Therefore we can rewrite (4.13) as

$$\begin{aligned}
\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}] \leq A_{n+2} &\left(1 + \frac{1 - q(Z_{1:n+2})}{n+3}\right)^{2\mathbb{1}_{\{Z_{1:n+2} < s\}}} \\
&\times \left(1 + \frac{1 - q(Z_{1:n+2})}{n+2}\right)^{\mathbb{1}_{\{s < Z_{1:n+2} < t\}}} \\
= D_{n+2}(s,t)&
\end{aligned}$$

This concludes the proof. $\square$

Lemma 4.11. *Let $\mathcal{F}_\infty = \bigcap_{n \geq 2} \mathcal{F}_n$. Then we have for each $A \in \mathcal{F}_\infty$ that $\mathbb{P}(A) \in \{0, 1\}$.*

Proof. Define

$$\Pi_n := \{\pi | \pi \text{ is permutation of } 1, \ldots, n\}$$

and

$$\Pi := \bigcup_{n\in\mathbb{N}} \Pi_n$$

We will use the Hewitt-Savage zero-one law xin order to show the statement of the lemma. Thus we need to show that for all $A \in \mathcal{F}_\infty$ and all $\pi \in \Pi$ there exists $B \in \mathcal{B}^*_{\mathbb{N}}$ s. t.

$$A = \{\omega | (Z_i(\omega))_{i\in\mathbb{N}} \in B\} = \{\omega | (Z_{\pi(i)}(\omega))_{i\in\mathbb{N}} \in B\} \tag{4.14}$$

Let $A \in \mathcal{F}_\infty$, then $A \in \mathcal{F}_n$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ fixed but arbitrary and $A \in \mathcal{F}_n$. Since the map $(Z_{1:n}, \dots, Z_{n:n}, Z_{n+1}, Z_{n+2}, \dots)$ is measurable, there must exist $\tilde{B} \in \mathcal{B}^*_{\mathbb{N}}$ such that

$$A = (Z_{1:n}, \dots, Z_{n:n}, Z_{n+1}, Z_{n+2}, \dots)^{-1}(\tilde{B}) \tag{4.15}$$

For fixed $\omega \in \Omega$ define the map

$$T : (\mathbb{R}^{\mathbb{N}}, \mathcal{B}^*_{\mathbb{N}}) \ni (Z_i(\omega))_{i\in\mathbb{N}} \longrightarrow T((Z_i(\omega))_{i\in\mathbb{N}}) \in (\mathbb{R}^{\mathbb{N}}, \mathcal{B}^*_{\mathbb{N}})$$

with

$$T((Z_i(\omega))_{i\in\mathbb{N}}) := (Z_{1:n}, \dots, Z_{n:n}, Z_{n+1}, Z_{n+2}, \dots)(\omega)$$

Note that for any $\pi \in \Pi_n$ we have

$$\begin{aligned} T((Z_i(\omega))_{i\in\mathbb{N}}) &= T((Z_{\pi(i)}(\omega))_{i\in\mathbb{N}}) \\ &= (Z_{1:n}, \dots, Z_{n:n}, Z_{n+1}, Z_{n+2}, \dots)(\omega) \end{aligned} \tag{4.16}$$

Hence on the one hand, we get from (4.15)

$$\begin{aligned} A &= (T((Z_i)_{i\in\mathbb{N}}))^{-1}(\tilde{B}) \\ &= ((Z_i)_{i\in\mathbb{N}})^{-1}(T^{-1}(\tilde{B})) \end{aligned}$$

$$= \{\omega | (Z_i(\omega))_{i\in\mathbb{N}} \in B\}$$

where $B = T^{-1}(\tilde{B})$. On the other hand we get from (4.16) and again by (4.15) that

$$\begin{aligned} A &= (T((Z_{\pi(i)})_{i\in\mathbb{N}}))^{-1}(\tilde{B}) \\ &= ((Z_{\pi(i)})_{i\in\mathbb{N}})^{-1}(T^{-1}(\tilde{B})) \\ &= \{\omega | (Z_{\pi(i)}(\omega))_{i\in\mathbb{N}} \in B\} \end{aligned}$$

Now since $n \in \mathbb{N}$ was chosen arbitrarily, the above statement holds for all $n \in \mathbb{N}$ and hence for all $\pi \in \Pi$. Whence establishing (4.14). □

Lemma 4.12. *For any $s < t$ s. t. $H(t) < 1$ the following statement holds true*

$$\Delta_n(s,t) = \mathbb{E}[D_n(s,t)] = \mathbb{E}[D_n(s,t)|\mathcal{F}_\infty] \nearrow D(s,t)$$

Proof. Consider that we have for $n \geq 2$

$$\Delta_n(s,t) = \mathbb{E}[D_n(s,t)] = \mathbb{E}[D_n(s,t)|\mathcal{F}_\infty]$$

by definition of $\Delta_n(s,t)$ and Lemma 4.11. Next note that we have $D_n(s,t) \to D(s,t)$ $\mathbb{P}$-almost surely, according to Lemma 4.9. Moreover we get from Lemma 4.10, that $\{D_n, \mathcal{F}_n\}_{n\geq 1}$ is a reverse supermartingale. Now this together with Proposition 5-3-11 of Neveu (1975) yields

$$\mathbb{E}[D_n(s,t)|\mathcal{F}_\infty] \nearrow D(s,t)$$

This proves the lemma. □

4.3 Upper bound for α_n

In Lemma 4.8 the previous section we derived a representation of S_n, which contains a reverse supermartingale D_n. Moreover we derived different properties of D_n. We will now use the results of the previous section, to show that

$$\lim_{n\to\infty} \sum\sum_{1\leq i<j\leq n+1} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n}) W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}}$$

is finite. Moreover we will establish an upper bound for $\mathbb{E}[(Q_i^{n+1}-1)^2]$. Those results will be used in Section 4.4 to prove the almost sure existence of the limit of S_n as $n \to \infty$.

Lemma 4.13. *Suppose condition (A3) is satisfied. Then the following statement holds true*

$$\lim_{n\to\infty} \sum\sum_{1\leq i<j\leq n+1} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n}) W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}} < \infty$$

Proof. Let (A3) be satisfied. Consider the following

$$\begin{aligned}
&\sum\sum_{1\leq i<j\leq n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n}) W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}} \\
&= \sum\sum_{1\leq i<j\leq n} \mathbb{E}\Bigg[\phi^2(Z_{i:n}, Z_{j:n}) \frac{q^2(Z_{i:n})}{(n-i+1)^2} \prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n})}{n-k+1}\right]^2 \\
&\qquad\qquad \times \frac{q^2(Z_{j:n})}{(n-j+1)^2} \prod_{l=1}^{j-1}\left[1-\frac{q(Z_{l:n})}{n-l+1}\right]^2\Bigg]^{\frac{1}{2}} \\
&\leq \sum\sum_{1\leq i<j\leq n} \mathbb{E}\Bigg[\phi^2(Z_{i:n}, Z_{j:n}) \frac{q^2(Z_{i:n})}{(n-i+1)^2} \prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n})}{n-k+1}\right] \\
&\qquad\qquad \times \frac{q^2(Z_{j:n})}{(n-j+1)^2} \prod_{l=1}^{j-1}\left[1-\frac{q(Z_{l:n})}{n-l+1}\right]\Bigg]^{\frac{1}{2}}
\end{aligned} \tag{4.17}$$

Next we will modify the products above. Recall the following definition

$$B_n(s) := \prod_{k=1}^{n}\left[1+\frac{1-q(Z_k)}{n-R_{k,n}}\right]^{\mathbb{1}\{Z_k<s\}}$$

and note that for $i=1,\ldots,n$

$$\begin{aligned}B_n(Z_{i:n}) &= \prod_{k=1}^{n}\left[1+\frac{1-q(Z_k)}{n-R_{k,n}}\right]^{\mathbb{1}\{Z_k<Z_{i:n}\}}\\ &= \prod_{k=1}^{n}\left[1+\frac{1-q(Z_{k:n})}{n-k}\right]^{\mathbb{1}\{Z_{k:n}<Z_{i:n}\}}\\ &= \prod_{k=1}^{i-1}\left[1+\frac{1-q(Z_{k:n})}{n-k}\right]\end{aligned}$$

Moreover consider that for $i=1,\ldots,n$

$$\begin{aligned}\frac{1}{n-i+1}\prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n})}{n-k+1}\right] &= \frac{1}{n-i+1}\prod_{k=1}^{i-1}\left[\frac{n-k+1-q(Z_{k:n})}{n-k+1}\right]\\ &= \frac{1}{n-i+1}\prod_{k=1}^{i-1}\left[\frac{n-k+1-q(Z_{k:n})}{n-k}\cdot\frac{n-k}{n-k+1}\right]\\ &= \frac{1}{n}\prod_{k=1}^{i-1}\left[1+\frac{1-q(Z_{k:n})}{n-k}\right]\\ &= \frac{B_n(Z_{i:n})}{n}\end{aligned}$$

Now combining the above with (4.17) yields

$$\begin{aligned}&\sum\sum_{1\le i<j\le n}\mathbb{E}\left[\phi^2(Z_{i:n},Z_{j:n})W_{i:n}^2W_{j:n}^2\right]^{\frac{1}{2}}\\ &\le \sum\sum_{1\le i<j\le n}\mathbb{E}\left[\phi^2(Z_{i:n},Z_{j:n})\frac{q^2(Z_{i:n})}{n(n-i+1)}\frac{q^2(Z_{j:n})}{n(n-j+1)}B_n(Z_{i:n})B_n(Z_{j:n})\right]^{\frac{1}{2}}\\ &\le \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{(n-i+1)^{\frac{1}{2}}(n-j+1)^{\frac{1}{2}}}\end{aligned}$$

$$
\begin{aligned}
&\times \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n})q^2(Z_{i:n})q^2(Z_{j:n})B_n(Z_{i:n})B_n(Z_{j:n})\right]^{\frac{1}{2}} \\
= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} &\frac{1}{(n-R_{i,n}+1)^{\frac{1}{2}}(n-R_{j,n}+1)^{\frac{1}{2}}} \\
&\times \mathbb{E}\left[\phi^2(Z_i, Z_j)q^2(Z_i)q^2(Z_j)B_n(Z_i)B_n(Z_j)\right]^{\frac{1}{2}}
\end{aligned}
$$

According to Lemma 4.7, we have

$$
\begin{aligned}
&\mathbb{E}\left[\phi^2(Z_i, Z_j)q^2(Z_i)q^2(Z_j)B_n(Z_i)B_n(Z_j)\right] \\
&= \mathbb{E}\left[\phi^2(Z_1, Z_2)q^2(Z_1)q^2(Z_2)B_n(Z_1)B_n(Z_2)\right]
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
&\sum\sum_{1\le i<j\le n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n})W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}} \\
&\le \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{(n-R_{i,n}+1)^{\frac{1}{2}}(n-R_{j,n}+1)^{\frac{1}{2}}} \\
&\qquad \times \mathbb{E}\left[\phi^2(Z_1, Z_2)q^2(Z_1)q^2(Z_2)B_n(Z_1)B_n(Z_2)\right]^{\frac{1}{2}}
\end{aligned}
$$

Next consider that we have

$$
\begin{aligned}
\sum_{j=1}^{n} \frac{1}{(n-R_{j,n}+1)^{\frac{1}{2}}} &= \sum_{j=1}^{n} \frac{1}{j^{\frac{1}{2}}} \\
&= 1 + \sum_{j=2}^{n} \int_{j-1}^{j} \frac{1}{\sqrt{j}} dx \\
&\le 1 + \sum_{j=2}^{n} \int_{j-1}^{j} \frac{1}{\sqrt{x}} dx \\
&\le 2\sqrt{n}
\end{aligned}
$$

for all $n \geq 1$. We therefore obtain

$$\sum\sum_{1 \leq i < j \leq n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n}) W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}}$$
$$\leq 4 \cdot \mathbb{E}\left[\phi^2(Z_1, Z_2) q^2(Z_1) q^2(Z_2) B_n(Z_1) B_n(Z_2)\right]^{\frac{1}{2}} \tag{4.18}$$

Since q and ϕ are Borel-measurable, we can apply Lemma 4.8 to obtain

$$\sum\sum_{1 \leq i < j \leq n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n}) W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}}$$
$$\leq 8 \cdot \mathbb{E}\left[\phi^2(Z_1, Z_2) q^2(Z_1) q^2(Z_2)(\Delta_{n-2}(Z_1, Z_2) + \bar{\Delta}_{n-2}(Z_1, Z_2))\right]^{\frac{1}{2}}$$

Note that $0 \leq C_n(s) \leq 1$ for all $n \geq 1$ and $s \in \mathbb{R}_+$. Thus

$$\bar{\Delta}_n(s,t) = \mathbb{E}[C_n(s) D_n(s,t)] \leq \Delta_n(s,t)$$

for all $n \geq 1$ and $s < t$. Therefore we get

$$\sum\sum_{1 \leq i < j \leq n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n}) W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}}$$
$$\leq 16 \cdot \mathbb{E}\left[\phi^2(Z_1, Z_2) q^2(Z_1) q^2(Z_2) \Delta_{n-2}(Z_1, Z_2)\right]^{\frac{1}{2}}$$

By virtue of Lemma 4.12, we have

$$\Delta_n(s,t) = \mathbb{E}[D_n(s,t)] = \mathbb{E}[D_n(s,t) | \mathcal{F}_\infty] \nearrow D(s,t)$$

But this implies in particular that $\mathbb{E}[D_n(s,t)] \leq D(s,t)$ for all $n \geq 1$. Hence

$$\sum\sum_{1 \leq i < j \leq n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n}) W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}}$$
$$\leq 16 \cdot \mathbb{E}\left[\phi^2(Z_1, Z_2) q^2(Z_1) q^2(Z_2) D(Z_1, Z_2)\right]^{\frac{1}{2}}$$

Next consider that for each $s < t$ s. t. $H(t) < 1$

$$\begin{aligned} D(s,t) &= \exp\left(2\int_0^s \frac{1-q(z)}{1-H(z)}H(dz) + \int_s^t \frac{1-q(z)}{1-H(z)}H(dz)\right) \\ &\le \exp\left(2\int_0^s \frac{1}{1-H(z)}H(dz) + \int_s^t \frac{1}{1-H(z)}H(dz)\right) \\ &= \exp\left(\int_0^s \frac{1}{1-H(z)}H(dz) + \int_0^t \frac{1}{1-H(z)}H(dz)\right) \\ &= \exp\left(-\ln(1-H(s)) - \ln(1-H(t))\right) \\ &= \frac{1}{(1-H(s))(1-H(t))} \end{aligned}$$

Therefore we have

$$\begin{aligned} &\sum\sum_{1\le i<j\le n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n})W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}} \\ &\le 16 \cdot \mathbb{E}\left[\frac{\phi^2(Z_1, Z_2)}{(1-H(Z_1))(1-H(Z_2))}\right]^{\frac{1}{2}} \\ &\le 16 \cdot \left\{\int_0^{Z_1}\int_0^{Z_2} \frac{\phi^2(s,t)}{(1-H(s))(1-H(t))}H(ds)H(dt)\right\}^{\frac{1}{2}} \end{aligned}$$

Now taking into consideration the Radon-Nikodym derivatives (c. f. Dikta (2000), page 8)

$$\frac{H^1(dt)}{H(dt)} = m(t,\theta_0) \text{ and } \frac{H^1(dt)}{F(dt)} = 1 - G(t),$$

yields

$$\begin{aligned} &\sum\sum_{1\le i<j\le n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n})W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}} \\ &\le 16 \cdot \left\{\int_0^{Z_1}\int_0^{Z_2} \frac{\phi^2(s,t)}{m(s,\theta_0)m(t,\theta_0)(1-H(s))(1-H(t))}H^1(ds)H^1(dt)\right\}^{\frac{1}{2}} \\ &= 16 \cdot \left\{\int_0^{Z_1}\int_0^{Z_2} \frac{\phi^2(s,t)}{m(s,\theta_0)m(t,\theta_0)(1-F(s))(1-F(t))}F(ds)F(dt)\right\}^{\frac{1}{2}} \end{aligned}$$

since $1 - H(x) = (1 - F(x))(1 - G(x))$ for all $x \in \mathbb{R}_+$. But now the integral above is finite under (A3), since we have for $0 \leq \epsilon \leq 1$

$$(1 - H(x))^\epsilon = (1 - F(x))^\epsilon (1 - G(x))^\epsilon \leq (1 - F(x))^\epsilon \leq 1 - F(x) \ .$$

Therefore we finally obtain

$$\lim_{n\to\infty} \sum\sum_{1\leq i<j\leq n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n}) W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}} < \infty \ .$$

□

The following results about q in combination with order statistics are necessary for the proof of Lemma 4.15 below.

Lemma 4.14. *Let (Q1) be satisfied. Then the following statements hold true for* $k \leq n - 1$

(i) We have

$$\mathbb{E}[|q(Z_{k:n}) - q(Z_{k+1:n})|] \leq \frac{c_1}{n+1} \tag{4.19}$$

(ii) Furthermore assume that (Q2) holds. Then

$$\mathbb{E}[1 - q(Z_{k:n})] \leq \frac{c_1(n - k + 1)}{n+1} \tag{4.20}$$

Proof. Let $q_H := q \circ H^{-1}$ and consider that we can write

$$q(H^{-1}(x)) = q(H^{-1}(x_0)) + q_H'(\hat{x})(x - x_0) \tag{4.21}$$

using Taylor expansion for some $\hat{x}$ in between x and x_0. Therefore we have

$$q(H^{-1}(x)) - q(H^{-1}(x_0)) = q_H'(\hat{x})(x - x_0)$$

and hence

$$|q(H^{-1}(x)) - q(H^{-1}(x_0))| = |q'_H(\hat{x})| \cdot |x - x_0| \tag{4.22}$$

Now let $U_1, \ldots, U_n$ be i. i. d. $Uni[0,1]$ and set $x = U_{k:n}$ and $x_0 = U_{k+1:n}$ to get

$$\mathbb{E}[|q(H^{-1}(U_{k:n})) - q(H^{-1}(U_{k+1:n}))|] = \mathbb{E}[|q(Z_{k:n}) - q(Z_{k+1:n})|]$$

Thus we get from (4.22)

$$\mathbb{E}[|q(Z_{k:n}) - q(Z_{k+1:n})|] = E[|q'_H(\hat{x})| \cdot (U_{k+1:n} - U_{k:n})]$$

where $\hat{x} \in [U_{k:n}, U_{k+1:n}]$. Assumption (Q1) directly implies that $|q'_H(x)| \leq c_1$ for all $x \in [0,1]$. Hence we have

$$\mathbb{E}[|q(Z_{k:n}) - q(Z_{k+1:n})|] = c_1 \mathbb{E}[U_{k+1:n} - U_{k:n}]$$

According to Shorack and Wellner (2009) (p. 271), we have

$$\mathbb{E}[U_{k+1:n} - U_{k:n}] = \frac{1}{n+1} \tag{4.23}$$

Therefore we may conclude

$$\begin{aligned} \mathbb{E}[|q(Z_{k:n}) - q(Z_{k+1:n})|] &\leq c_1 \mathbb{E}[U_{k+1:n} - U_{k:n}] \\ &= \frac{c_1}{n+1} \end{aligned} \tag{4.24}$$

This completes the proof part (i). We will now continue with the proof of part (ii). Consider

$$1 - q(Z_{k:n}) = 1 - q(Z_{n:n}) + \sum_{l=k}^{n-1} (q(Z_{l+1:n}) - q(Z_{l:n}))$$

$$\leq 1 - q(Z_{n:n}) + \sum_{l=k}^{n-1} |q(Z_{l+1:n}) - q(Z_{l:n})|$$

Taking expectations on each side yields

$$1 - \mathbb{E}[q(Z_{k:n})] \leq 1 - \mathbb{E}[q(Z_{n:n})] + \sum_{l=k}^{n-1} \mathbb{E}[|q(Z_{l+1:n}) - q(Z_{l:n})|]$$

Now we apply inequality (4.24) to the expectation under the sum to get

$$1 - \mathbb{E}[q(Z_{k:n})] \leq 1 - \mathbb{E}[q(Z_{n:n})] + \frac{c_1(n-k)}{n+1} \tag{4.25}$$

Recall the Taylor expansion from above

$$q(H^{-1}(x)) = q(H^{-1}(x_0)) + q'_H(\hat{x})(x - x_0)$$

Setting $x = 1$ and $x_0 = U_{n:n}$ and taking expectations on both sides yields

$$\mathbb{E}[q(H^{-1}(1))] = \mathbb{E}[q(Z_{n:n})] + \mathbb{E}[q'_H(\hat{x})(1 - U_{n:n})]$$

where $\hat{x} \in [U_{n:n}, 1]$ Now we get from assumption (A**??**) that

$$\begin{aligned}
\mathbb{E}[q(Z_{n:n})] &= \mathbb{E}[q(H^{-1}(1))] - \mathbb{E}[q'_H(\hat{x})(1 - U_{n:n})] \\
&\geq \mathbb{E}[q(H^{-1}(1))] - c_1\mathbb{E}[1 - U_{n:n}]
\end{aligned}$$

Using Shorack and Wellner (2009) (p. 271) again, we obtain

$$\mathbb{E}[q(Z_{n:n})] = \mathbb{E}[q(H^{-1}(1))] - \frac{c_1}{n+1}$$

Applying (Q2) yields

$$\mathbb{E}[q(Z_{n:n})] \geq 1 - \frac{c_1}{n+1}$$

By combining the above with (4.25) we get

$$1 - \mathbb{E}[q(Z_{k:n})] \leq 1 - 1 + \frac{c_1}{n+1} + \frac{c_1(n-k)}{n+1} = \frac{c_1(n-k+1)}{n+1}$$

This concludes the proof of part (ii). $\square$

The lemma below identifies an upper bound for the second term in (4.34).

Lemma 4.15. *Let (Q1) and (Q2) be satisfied. Then the following holds for $n \geq 2$ and $1 \leq i \leq n+1$*

$$\mathbb{E}[(Q_i^{n+1} - 1)^2] \leq \frac{c_2}{n^{\frac{7}{3}}}$$

where $c_2 := 560c_1 + 1$ and c_1 as defined in (Q1).

Proof. Consider for $n \geq 2$ and $1 \leq i \leq n+1$

$$Q_i^{n+1} - 1 = Q_1^{n+1} + \sum_{k_1=1}^{i-1} (Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}) - 1 \tag{4.26}$$

and recall the following definition

$$Q_i^{n+1} := (n+1) \left\{ \sum_{r=1}^{i-1} \left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})} \right]^2 + \frac{\pi_i \pi_{i+1}}{n-i+1} \right\}$$

where

$$\pi_i := \prod_{k=1}^{i-1} \left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})} \right]$$

We have $\pi_1 = 1$, since the product above is empty for $i = 1$ and

$$\pi_2 = \frac{n - q(Z_{1:n+1})}{n+1-q(Z_{1:n+1})}$$

Thus we get

$$\begin{aligned}
Q_1^{n+1} - 1 &= (n+1)\frac{\pi_1\pi_2}{n} - 1 \\
&= \frac{(n+1)(n - q(Z_{1:n+1}))}{n(n+1-q(Z_{1:n+1}))} - 1 \\
&= \frac{n(n+1-q(Z_{1:n+1})) - q(Z_{1:n+1})}{n(n+1-q(Z_{1:n+1}))} - 1 \\
&= 1 - \frac{q(Z_{1:n+1})}{n(n+1-q(Z_{1:n+1}))} - 1 \\
&= -\frac{q(Z_{1:n+1})}{n(n+1-q(Z_{1:n+1}))}
\end{aligned}$$

Therefore we get from (4.26)

$$Q_i^{n+1} - 1 = \sum_{k_1=1}^{i-1}(Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}) - \frac{q(Z_{1:n+1})}{n(n+1-q(Z_{1:n+1}))}$$

Moreover we have

$$\begin{aligned}
(Q_i^{n+1} - 1)^2 = &\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}(Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1})(Q_{k_2+1}^{n+1} - Q_{k_2}^{n+1}) \\
&\quad - \frac{2q(Z_{1:n+1})}{n(n+1-q(Z_{1:n+1}))}\sum_{k=1}^{i-1}(Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}) \\
&\quad + \frac{q^2(Z_{1:n+1})}{n^2(n+1-q(Z_{1:n+1}))^2} \\
\leq &\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}| \cdot |Q_{k_2+1}^{n+1} - Q_{k_2}^{n+1}| \\
&\quad + \frac{2q(Z_{1:n+1})}{n(n+1-q(Z_{1:n+1}))}\sum_{k_1=1}^{i-1}|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}| \\
&\quad + \frac{q^2(Z_{1:n+1})}{n^2(n+1-q(Z_{1:n+1}))^2} \\
\leq &\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}| \cdot |Q_{k_2+1}^{n+1} - Q_{k_2}^{n+1}|
\end{aligned}$$

$$+\frac{2}{n^2}\sum_{k_1=1}^{i-1}|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}|+\frac{1}{n^4} \tag{4.27}$$

Remember that we set $q_i := q(Z_{i:n+1})$. According to Lemma 3.2 we have

$$\begin{aligned}
&|Q_{i+1}^{n+1}-Q_i^{n+1}| \\
&= \frac{\tilde{\pi}_i^2(n-i+2)^2}{n+1}\cdot\left|\frac{(q_i-q_{i+1})(n-i)(n-i+1)-q_{i+1}(1-q_i)(n-i+1-q_i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})}\right| \\
&\le \frac{\tilde{\pi}_i^2(n-i+2)^2}{n+1}\cdot\frac{|q_i-q_{i+1}|(n-i)(n-i+1)+q_{i+1}(1-q_i)(n-i+1-q_i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\
&\le \frac{(n-i+2)^2}{n+1}\left\{\frac{|q_i-q_{i+1}|(n-i)(n-i+1)+q_{i+1}(1-q_i)(n-i+1)}{(n-i)(n-i+1)(n-i+1)^2(n-i)}\right\} \\
&= \frac{(n-i+2)^2}{n+1}\left\{\frac{|q_i-q_{i+1}|(n-i)+q_{i+1}(1-q_i)}{(n-i)^2(n-i+1)^2}\right\} \\
&\le \frac{4|q_i-q_{i+1}|}{(n+1)(n-i)}+\frac{4(1-q_i)}{(n+1)(n-i)^2}
\end{aligned} \tag{4.28}$$

The latter inequality above holds since

$$\frac{n-i+2}{n-i+1}=1+\frac{1}{n-i+1}\le 2$$

and $q_{i+1}\le 1$. Thus we have

$$\begin{aligned}
&|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}|\cdot|Q_{k_2+1}^{n+1}-Q_{k_2}^{n+1}| \\
&\le \left[\frac{4|q_{k_1}-q_{k_1+1}|}{(n+1)(n-k_1)}+\frac{4(1-q_{k_1})}{(n+1)(n-k_1)^2}\right] \\
&\quad\times\left[\frac{4|q_{k_2}-q_{k_2+1}|}{(n+1)(n-k_2)}+\frac{4(1-q_{k_2})}{(n+1)(n-k_2)^2}\right] \\
&= \frac{16|q_{k_1}-q_{k_1+1}||q_{k_2}-q_{k_2+1}|}{(n+1)^2(n-k_1)(n-k_2)}+\frac{16|q_{k_1}-q_{k_1+1}|(1-q_{k_2})}{(n+1)^2(n-k_1)(n-k_2)^2} \\
&\quad+\frac{16(1-q_{k_1})|q_{k_2}-q_{k_2+1}|}{(n+1)^2(n-k_1)^2(n-k_2)}+\frac{16(1-q_{k_1})(1-q_{k_2})}{(n+1)^2(n-k_1)^2(n-k_2)^2} \\
&\le \frac{16|q_{k_1}-q_{k_1+1}|}{(n+1)^2(n-k_1)(n-k_2)}+\frac{16|q_{k_1}-q_{k_1+1}|}{(n+1)^2(n-k_1)(n-k_2)^2} \\
&\quad+\frac{16|q_{k_2}-q_{k_2+1}|}{(n+1)^2(n-k_1)^2(n-k_2)}+\frac{16(1-q_{k_1})}{(n+1)^2(n-k_1)^2(n-k_2)^2}
\end{aligned}$$

Here the latter inequality holds, since we have $|q_k - q_{k+1}| \leq 1$ and $1 - q_k \leq 1$ for all $k \leq n-1$.

Next recall that

$$(Q_i^{n+1} - 1)^2 \leq \sum_{k_1=1}^{i-1} \sum_{k_2=1}^{i-1} |Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}||Q_{k_2+1}^{n+1} - Q_{k_2}^{n+1}| \\ + \frac{2}{n^2} \sum_{k_1=1}^{i-1} |Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}| + \frac{1}{n^4}$$

Taking expectations on each side yields

$$\mathbb{E}[(Q_i^{n+1} - 1)^2] \leq \sum_{k_1=1}^{i-1} \sum_{k_2=1}^{i-1} \mathbb{E}[|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}||Q_{k_2+1}^{n+1} - Q_{k_2}^{n+1}|] \\ + \frac{2}{n^2} \sum_{k_1=1}^{i-1} \mathbb{E}[|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}|] + \frac{1}{n^4} \tag{4.29}$$

We will now regard the two sums above individually. Consider the expectation under the double sum above. We have

$$\mathbb{E}\left[|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}||Q_{k_2+1}^{n+1} - Q_{k_2}^{n+1}|\right] \\ \leq \frac{16\mathbb{E}\left[|q_{k_1} - q_{k_1+1}|\right]}{(n+1)^2(n-k_1)(n-k_2)} + \frac{16\mathbb{E}\left[|q_{k_1} - q_{k_1+1}|\right]}{(n+1)^2(n-k_1)(n-k_2)^2} \\ + \frac{16\mathbb{E}\left[|q_{k_2} - q_{k_2+1}|\right]}{(n+1)^2(n-k_1)^2(n-k_2)} + \frac{16\mathbb{E}\left[(1-q_{k_1})\right]}{(n+1)^2(n-k_1)^2(n-k_2)^2} \tag{4.30}$$

According to Lemma 4.14, we have

$$\mathbb{E}[|q(Z_{k:n}) - q(Z_{k+1:n})|] \leq \frac{c_1}{n+1}$$

and

$$\mathbb{E}[1 - q(Z_{k:n})] \leq \frac{c_1(n-k+1)}{n+1}$$

Therefore

$$
\begin{aligned}
&\mathbb{E}[|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}||Q_{k_2+1}^{n+1} - Q_{k_2}^{n+1}|] \\
&\leq \frac{16c_1}{(n+1)^3(n-k_1)(n-k_2)} + \frac{16c_1}{(n+1)^3(n-k_1)(n-k_2)^2} \\
&\quad + \frac{16c_1}{(n+1)^3(n-k_1)^2(n-k_2)} + \frac{16c_1(n-k_1)+16c_1}{(n+1)^3(n-k_1)^2(n-k_2)^2}
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
&\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1} \mathbb{E}[|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}||Q_{k_2+1}^{n+1} - Q_{k_2}^{n+1}|] \\
&\leq \sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1} \frac{16c_1}{(n+1)^3(n-k_1)(n-k_2)} + \sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1} \frac{16c_1}{(n+1)^3(n-k_1)(n-k_2)^2} \\
&\quad + \sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1} \frac{16c_1}{(n+1)^3(n-k_1)^2(n-k_2)} + \sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1} \frac{16c_1(n-k_1)}{(n+1)^3(n-k_1)^2(n-k_2)^2} \\
&\quad + \sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1} \frac{16c_1}{(n+1)^3(n-k_1)^2(n-k_2)^2} \\
&= \frac{16c_1}{(n+1)^3}\sum_{k_1=1}^{i-1}\frac{1}{(n-k_1)}\sum_{k_2=1}^{i-1}\frac{1}{(n-k_2)} + \frac{32c_1}{(n+1)^3}\sum_{k_1=1}^{i-1}\frac{1}{n-k_1}\sum_{k_2=1}^{i-1}\frac{1}{(n-k_2)^2} \\
&\quad + \frac{16c_1}{(n+1)^3}\sum_{k_1=1}^{i-1}\frac{1}{(n-k_1)^2}\sum_{k_2=1}^{i-1}\frac{1}{n-k_2} + \frac{16c_1}{(n+1)^3}\sum_{k_1=1}^{i-1}\frac{1}{(n-k_1)^2}\sum_{k_2=1}^{i-1}\frac{1}{(n-k_2)^2} \\
&= \frac{16c_1}{(n+1)^3}\sum_{k_1=n-i+1}^{n-1}\frac{1}{k_1}\sum_{k_2=n-i+1}^{n-1}\frac{1}{k_2} + \frac{32c_1}{(n+1)^3}\sum_{k_1=n-i+1}^{n-1}\frac{1}{k_1}\sum_{k_2=n-i+1}^{n-1}\frac{1}{k_2^2} \\
&\quad + \frac{16c_1}{(n+1)^3}\sum_{k_1=n-i+1}^{n-1}\frac{1}{k_1^2}\sum_{k_2=n-i+1}^{n-1}\frac{1}{k_2} + \frac{16c_1}{(n+1)^3}\sum_{k_1=n-i+1}^{n-1}\frac{1}{k_1^2}\sum_{k_2=n-i+1}^{n-1}\frac{1}{k_2^2} \qquad (4.31)
\end{aligned}
$$

Now using inequalities (A1) and (A2) from Lemma A.1 on inequality (4.31) yields

$$
\begin{aligned}
&\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1} \mathbb{E}[|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}||Q_{k_2+1}^{n+1} - Q_{k_2}^{n+1}|] \\
&\leq \frac{16c_1}{(n+1)^3}(\ln(n-1)+1)^2 + \frac{64c_1}{(n+1)^3}(\ln(n-1)+1)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{32c_1}{(n+1)^3}(\ln(n-1)+1) + \frac{64c_1}{(n+1)^3} \\
&\le \frac{144c_1}{(n+1)^{\frac{7}{3}}} + \frac{288c_1}{(n+1)^{\frac{8}{3}}} + \frac{64c_1}{(n+1)^3} \\
&\le \frac{496c_1}{(n+1)^{\frac{7}{3}}} \qquad (4.32)
\end{aligned}
$$

We will now proceed with the second sum in (4.29). According to (4.28) we have

$$
\mathbb{E}[|Q_{i+1}^{n+1} - Q_i^{n+1}|] \le \frac{4\mathbb{E}[|q_i - q_{i+1}|]}{(n+1)(n-i)} + \frac{4\mathbb{E}[1-q_i]}{(n+1)(n-i)^2}
$$

Therefore we obtain

$$
\frac{2}{n^2}\sum_{k_1=1}^{i-1}\mathbb{E}[|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}|] \le \frac{8}{n^2(n+1)}\sum_{k_1=1}^{i-1}\frac{\mathbb{E}[|q_{k_1} - q_{k_1+1}|]}{n-k_1} + \frac{\mathbb{E}[1-q_{k_1}]}{(n-k_1)^2}
$$

Using (4.19) and (4.20) again reveals

$$
\begin{aligned}
\frac{2}{n^2}\sum_{k_1=1}^{i-1}\mathbb{E}[|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}|] &\le \frac{8}{n^2(n+1)^2}\left\{\sum_{k_1=1}^{i-1}\frac{c_1}{(n-k_1)} + \sum_{k_1=1}^{i-1}\frac{c_1(n-k_1+1)}{(n-k_1)^2}\right\} \\
&= \frac{8}{n^2(n+1)^2}\left\{2\sum_{k_1=1}^{i-1}\frac{c_1}{(n-k_1)} + \sum_{k_1=1}^{i-1}\frac{c_1}{(n-k_1)^2}\right\} \\
&= \frac{8}{n^2(n+1)^2}\left\{2\cdot\sum_{k_1=n-i+1}^{n-1}\frac{c_1}{k_1} + \sum_{k_1=n-i+1}^{n-1}\frac{c_1}{k_1^2}\right\}
\end{aligned}
$$

According to (A1) and (A2) of Lemma A.1 we have

$$
\begin{aligned}
\frac{2}{n^2}\sum_{k_1=1}^{i-1}\mathbb{E}[|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}|] &\le \frac{8\cdot\{2c_1(\ln(n-1)+1)+2c_1\}}{n^2(n+1)^2} \\
&= \frac{16c_1(\ln(n-1)+1)}{n^2(n+1)^2} + \frac{16c_1}{n^2(n+1)^2} \\
&\le \frac{48c_1}{n^2(n+1)^{\frac{5}{3}}} + \frac{16c_1}{n^2(n+1)^2} \\
&\le \frac{64c_1}{n^2(n+1)^{\frac{5}{3}}} \qquad (4.33)
\end{aligned}
$$

Now recall the following fact from (4.29)

$$\mathbb{E}[(Q_i^{n+1}-1)^2] = \sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\mathbb{E}[|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}||Q_{k_2+1}^{n+1}-Q_{k_2}^{n+1}|] + \frac{2}{n^2}\sum_{k_1=1}^{i-1}\mathbb{E}[|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}|] + \frac{1}{n^4}$$

Combining the above with (4.32) and (4.33) yields

$$\begin{aligned}
\mathbb{E}[(Q_i^{n+1}-1)^2] &\leq \frac{496c_1}{(n+1)^{\frac{7}{3}}} + \frac{64c_1}{n^2(n+1)^{\frac{5}{3}}} + \frac{1}{n^4} \\
&\leq \frac{496c_1}{n^{\frac{7}{3}}} + \frac{64c_1}{n^{\frac{11}{3}}} + \frac{1}{n^4} \\
&\leq \frac{1}{n^{\frac{7}{3}}}\left[496c_1 + \frac{64c_1}{n^{\frac{4}{3}}} + \frac{1}{n^{\frac{5}{3}}}\right] \\
&\leq \frac{560c_1+1}{n^{\frac{7}{3}}} \\
&= \frac{c_2}{n^{\frac{7}{3}}}
\end{aligned}$$

This concludes the proof. □

4.4 Finite Limit

During the preceding sections of this chapter, we established everything we need to prove the almost sure existence of the limit $S = \lim_{n\to\infty} S_n$. In this section we will first show, that the number of Upcrossings of $\tilde{S}_1^N, \ldots, \tilde{S}_N^N$ is finite in Lemma 4.16. Afterwards we will show how this implies the almost sure existence of S in Lemma 4.17.

Theorem 4.16. *Assume that (A1) through (A3), (Q1) and (Q2) hold. Then we have*

$$\lim_{N\to\infty} \mathbb{E}[U_N^N[a,b]] < \infty$$

Note: *For $N \geq 2$ we have $\{\tilde{S}_1^N, \ldots, \tilde{S}_N^N\} = \{S_N, \ldots, S_1\}$. Thus $U_N^N[a,b]$ is the number of upcrossings of $S_N, \ldots, S_1$, which coincides with the number of downcrossings of $S_1, \ldots, S_N$ (c.f. Neveu (1975), p. 116).*

Proof. Let (A1) through (A2) be satisfied. According to Lemma 4.5, we have

$$\mathbb{E}[U_N^N[a,b]] \leq \frac{\mathbb{E}[Y_N^N]}{b-a}$$

and furthermore

$$\begin{aligned}\mathbb{E}[Y_N^N] \leq \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1 \leq i < j \leq N-k+1} \mathbb{E}\left[\phi^2(Z_{i:N-k+1}, Z_{j:N-k+1}) W_{i:N-k+1}^2 W_{j:N-k+1}^2\right]^{\frac{1}{2}} \\ \times \mathbb{E}\left[(Q_{i,j}^{N-k+1} - 1)^2\right]^{\frac{1}{2}}\end{aligned} \tag{4.34}$$

First note that $\mathbb{E}[\tilde{S}_N^N] = \mathbb{E}[S_2]$ and that we have

$$\begin{aligned}\mathbb{E}[S_2] &= \int_0^\infty \int_0^\infty \phi(s,t) \frac{q(s)q(t)}{2}\left[1 - \frac{q(s)}{2}\right] H(ds)H(dt) \\ &\leq \frac{1}{2}\int_0^\infty \int_0^\infty \phi(s,t) H(ds)H(dt) \\ &\leq \frac{1}{2}\int_0^\infty \int_0^\infty \frac{\phi(s,t)}{m(s,\theta_0)m(t,\theta_0)(1-H(s))(1-H(t))} H(ds)H(dt) \\ &< \infty\end{aligned}$$

Next consider that Lemma 4.15 identifies an upper bound for the expectation above as

$$\mathbb{E}[(Q_i^{N-k+1} - 1)^2]^{\frac{1}{2}} \leq \frac{\sqrt{c_2}}{(N-k)^{\frac{7}{6}}} \; .$$

Now combining the latter with (4.34) yields

$$\begin{aligned}\mathbb{E}[Y_N^N] \leq \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1 \leq i < j \leq N-k+1} \mathbb{E}\left[\phi^2(Z_{i:N-k+1}, Z_{j:N-k+1}) W_{i:N-k+1}^2 W_{j:N-k+1}^2\right]^{\frac{1}{2}} \\ \times \mathbb{E}\left[(Q_{i,j}^{N-k+1} - 1)^2\right]^{\frac{1}{2}}\end{aligned}$$

$$\leq \left\{\sup_N \mathbb{E}[\tilde{S}_N^N]\right\} + \sum_{k=1}^{N-1} \sum\sum_{1\leq i<j\leq N-k+1} \mathbb{E}\left[\phi^2(Z_{i:N-k+1}, Z_{j:N-k+1}) W_{i:N-k+1}^2 W_{j:N-k+1}^2\right]^{\frac{1}{2}}$$
$$\times \frac{\sqrt{c_2}}{(N-k)^{\frac{7}{6}}}$$

But, according to Lemma 4.13, the limit of the double sum above is finite. Moreover

$$\sum_{k=1}^{N-1} \frac{\sqrt{c_2}}{(N-k)^{\frac{7}{6}}}$$

converges to a finite limit as $N \to \infty$. Furthermore note, that $\sup_N \mathbb{E}[\tilde{S}_N^N]$ is finite under (A3). Hence we have seen that

$$\lim_{N\to\infty} \mathbb{E}[U_N^N[a,b]] \leq \lim_{N\to\infty} \frac{\mathbb{E}[Y_N^N]}{b-a} < \infty$$

□

The following theorem gives the main statement of this chapter.

Theorem 4.17. *Suppose (A1) through (A3), (Q1) and (Q2) are satisfied. Then the limit $S = \lim_{n\to\infty} S_n$ exists $\mathbb{P}$-almost surely.*

Proof. Let's first define the set of all ω for which $S_{2,n}^{se}$ does not converge as

$$\Lambda := \{\omega | S_n(\omega) \text{ does not converge}\}$$

Consider that

$$\begin{aligned}\Lambda &= \{\omega | \liminf_n S_n(\omega) < \limsup_n S_n(\omega)\} \\ &= \bigcup_{a,b\in\mathbb{Q}} \{\omega | \liminf_n S_n(\omega) < a < b < \limsup_n S_n(\omega)\}\end{aligned}$$

Recall that we have $U_N^N[a,b]$, the number of upcrossings of $[a,b]$ by $\tilde{S}_1^N, \ldots, \tilde{S}_N^N$. But this is equal to the number of upcrossings of $[a,b]$ by $S_N, \ldots, S_1$. Furthermore recall that

$$U_\infty[a,b] = \lim_{N\to\infty} U_N^N[a,b]$$

Consider that for each $\omega \in \{\omega | \liminf_n S_n(\omega) < a < b < \limsup_n S_n(\omega)\}$ we must have $U_\infty[a,b](\omega) = \infty$. This follows directly from the definitions of $\liminf$ and $\limsup$. Thus we can write

$$\Lambda = \bigcup_{a,b\in\mathbb{Q}} \{\omega | U_\infty[a,b](\omega) = \infty\} = \bigcup_{a,b\in\mathbb{Q}} \Lambda_{a,b}$$

where $\Lambda_{a,b} := \{\omega | U_\infty[a,b](\omega) = \infty\}$. Consequently we get that

$$\mathbb{E}[\mathbb{1}_{\{\Lambda_{a,b}\}} U_\infty[a,b]] = \begin{cases} \infty & \text{if } \mathbb{P}(\Lambda_{a,b} > 0) \\ 0 & \text{if } \mathbb{P}(\Lambda_{a,b} = 0) \end{cases} \tag{4.35}$$

But, according to Theorem 4.16, we have

$$\lim_{N\to\infty} \mathbb{E}[U_N^N[a,b]] < \infty$$

Clearly $U_N^N[a,b]$ is non-decreasing in N. Hence we get by virtue of the Monotone Convergence Theorem

$$\lim_{N\to\infty} \mathbb{E}[U_N^N[a,b]] = \mathbb{E}[U_\infty[a,b]] < \infty$$

Therefore

$$\mathbb{E}[\mathbb{1}_{\{\Lambda_{a,b}\}} U_\infty[a,b]] \le \mathbb{E}[U_\infty[a,b]] < \infty$$

But this means together with (4.35) that $\mathbb{P}(\Lambda_{a,b}) = 0$. Therefore we have

$$\mathbb{P}(\Lambda) = \mathbb{P}\left(\bigcup_{a,b\in\mathbb{Q}} \Lambda_{a,b}\right) = \sum_{a,b\in\mathbb{Q}} \mathbb{P}(\Lambda_{a,b}) = 0$$

Hereby the proof is concluded. $\square$

Chapter 5

Identifying the limit

In the previous chapter we established the existence of the limit

$$\lim_{n\to\infty} S_n = S_\infty$$

We will now continue to identify the limit $S(m(\cdot, \hat{\theta}_n))$ throughout this chapter. The interdependence structure of the proofs within this chapter in figure 5.1 below.

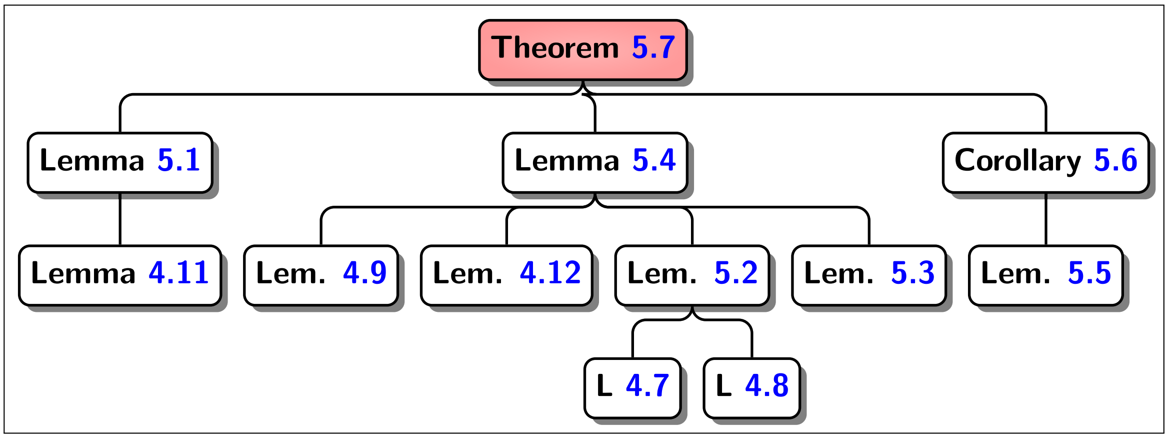

Figure 5.1: Interdependence Structure of the lemmas and theorems within this chapter.

The following lemma is similar to Neveu (1975), Proposition 5-3-11, and it will be essential to identify the limit of S_n.

Lemma 5.1. *The following statement holds true*

$$S_\infty = \lim_{n\to\infty} \mathbb{E}[S_n|\mathcal{F}_\infty] = \lim_{n\to\infty} \mathbb{E}[S_n]$$

if the limits above exist.

Proof. Let $a > 0$ and note that, since $S_n \to S$ almost surely as $n \to \infty$, we have

$$\lim_{n\to\infty} \min(S_n, a) = \min(S, a)$$

almost surely, since $\min(\cdot, a)$ is continuous. TODO reference, Dikta Stochastik script, better: english book Now $\min(S_n, a)$ is bounded by a. Hence applying the Dominated Convergence Theorem yields

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}[\min(S_n, a)|\mathcal{F}_\infty] &= \mathbb{E}[\lim_{n\to\infty} \min(S_n, a)|\mathcal{F}_\infty] \\ &= \mathbb{E}[\min(S_\infty, a)|\mathcal{F}_\infty] \\ &= \min(S_\infty, a) \end{aligned} \tag{5.1}$$

Note that S_k is measurable with respect to $\mathcal{F}_n$ whenever $k \geq n$, therefore S_∞ must be $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$. Consequently S_∞ must be F_∞-measurable. Moreover, for $a \in \mathbb{R}$, $\min(\cdot, a)$ is a continuous function. Thus $\min(S_\infty, a)$ is $\mathcal{F}_\infty$-measurable as well. Hence

$$\lim_{n\to\infty} \mathbb{E}[min(S_n, a)|\mathcal{F}_\infty] = \min(S_\infty, a)$$

almost surely. Thus we have

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}[S_n|\mathcal{F}_\infty] &= \lim_{n\to\infty} \lim_{a\to\infty} \mathbb{E}[\min(S_n, a)|\mathcal{F}_\infty] \\ &= \lim_{a\to\infty} \lim_{n\to\infty} \mathbb{E}[\min(S_n, a)|\mathcal{F}_\infty] \\ &= \lim_{a\to\infty} \min(S_\infty, a) \\ &= S_\infty\end{aligned}$$

$\mathbb{P}$-almost surely. But now, using Lemma 4.11, we obtain

$$\mathbb{E}[S_n|\mathcal{F}_\infty] = \mathbb{E}[S_n]$$

for all n, which implies the statement of the lemma. $\square$

We will now proceed to find an explicit representation for $\mathbb{E}[S_n]$ in terms of the reverse supermartingale D_n (see Section 4.2) to identify the limit $S = S(q)$. Consider the following lemma.

Lemma 5.2. *For continuous $H(\cdot)$, we have*

$$\mathbb{E}[S_n(q)] = \frac{n-1}{n}\mathbb{E}[\phi(Z_1, Z_2)q(Z_1)q(Z_2)\{\Delta_{n-2}(Z_1, Z_2) + \bar{\Delta}_{n-2}(Z_1, Z_2)\}\mathbb{1}_{\{Z_1<Z_2\}}]$$

Proof. The proof of the lemma above is similar to the proof of Lemma 4.13. Consider

$$\begin{aligned}
\mathbb{E}[S_n(q)] &= \sum\sum_{1\le i<j\le n} \mathbb{E}\left[\phi(Z_{i:n}, Z_{j:n})\frac{q(Z_{i:n})}{n-i+1)}\prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n})}{n-k+1}\right]\right. \\
&\qquad\left.\times\ \frac{q(Z_{j:n})}{n-j+1}\prod_{l=1}^{j-1}\left[1-\frac{q(Z_{l:n})}{n-l+1}\right]\right] \\
&= \frac{1}{n^2}\sum\sum_{1\le i<j\le n} \mathbb{E}\left[\phi(Z_{i:n}, Z_{j:n})q(Z_{i:n})\prod_{k=1}^{i-1}\left[1+\frac{1-q(Z_{k:n})}{n-k+1}\right]\right. \\
&\qquad\left.\times\ q(Z_{j:n})\prod_{l=1}^{j-1}\left[1+\frac{1-q(Z_{l:n})}{n-l+1}\right]\right] \\
&= \frac{1}{n^2}\sum\sum_{1\le i<j\le n} \mathbb{E}\left[\phi(Z_{i:n}, Z_{j:n})q(Z_{i:n})q(Z_{j:n})B_n(Z_{i:n})B_n(Z_{j:n})\right] \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{1}_{\{R_{i,n}<R_{j,n}\}}\mathbb{E}\left[\phi(Z_i, Z_j)q(Z_i)q(Z_j)B_n(Z_i)B_n(Z_j)\right] \qquad (5.2)
\end{aligned}$$

According to Lemma 4.7 we obtain

$$\mathbb{E}[S_n(q)] = \frac{n-1}{2n}\mathbb{E}\left[\phi(Z_1, Z_2)q(Z_1)q(Z_2)B_n(Z_1)B_n(Z_2)\right]$$

Now, since ϕ and q are measurable, we can apply Lemma 4.8 to obtain the result. $\square$

The following result is necessary for the proof of Lemma 5.4.

Lemma 5.3. *For continuous $H(\cdot)$ and $t < s$, we have $C_n(t) \to 0$ as $n \to \infty$ w. p. 1, and $C_n(t) \in [0,1]$ for all $n \geq 1$ and $t \geq 0$.*

Proof. It is easy to see that $0 \leq C_n(t) \leq 1$ for any $t \geq 0$ and $n \geq 2$, since $0 \leq q(t) \leq 1$ and $\mathbb{1}_{\{Z_{i-1:n} < t \leq Z_{i:n}\}} = 1$ for exactly one $i \in \{1, \ldots, n\}$. Let's now consider

$$\begin{aligned}
C_n(t) &= \sum_{i=1}^{n+1} \frac{1-q(t)}{n-i+2}[\mathbb{1}_{\{Z_{i-1:n}<t\}} - \mathbb{1}_{\{Z_{i:n}<t\}}] \\
&= \sum_{i=1}^{n+1} \frac{1-q(t)}{n-i+2}\mathbb{1}_{\{Z_{i-1:n}<t\}} - \sum_{i=1}^{n+1} \frac{1-q(t)}{n-i+2}\mathbb{1}_{\{Z_{i:n}<t\}} \\
&= \sum_{i=0}^{n} \frac{1-q(t)}{n-i+1}\mathbb{1}_{\{Z_{i:n}<t\}} - \sum_{i=1}^{n} \frac{1-q(t)}{n-i+2}\mathbb{1}_{\{Z_{i:n}<t\}} \\
&= \sum_{i=1}^{n} \frac{1-q(t)}{n-i+1}\mathbb{1}_{\{Z_{i:n}<t\}} + \frac{(1-q(t))}{n+1} - \sum_{i=1}^{n} \frac{1-q(t)}{n-i+2}\mathbb{1}_{\{Z_{i:n}<t\}} \\
&= (1-q(t))\left\{\frac{1}{n+1} + \sum_{i=1}^{n}\left[\frac{1}{n-i+1} - \frac{1}{n-i+2}\right]\mathbb{1}_{\{Z_{i:n}<t\}}\right\} \\
&= (1-q(t))\sum_{i=1}^{n}\left[\frac{1}{n-nH_n(Z_{i:n})+1}\frac{1}{n-nH_n(Z_{i:n})+2}\right]\mathbb{1}_{\{Z_{i:n}<t\}} \\
&\qquad + \frac{1-q(t)}{n+1} \\
&= (1-q(t))\int_0^t \left[\frac{1}{1-H_n(x)+\frac{1}{n}} - \frac{1}{1-H_n(x)+\frac{2}{n}}\right]H_n(dx) \\
&\qquad + \frac{1-q(t)}{n+1}
\end{aligned} \tag{5.3}$$

In Lemma 4.9 we have seen that

$$\int_0^t \frac{1}{1 - H_n(x) + \frac{2}{n}} H_n(dx) \to \int_0^t \frac{1}{1 - H(x)} H(dx)$$

By the same arguments we obtain

$$\int_0^t \frac{1}{1 - H_n(x) + \frac{1}{n}} H_n(dx) \to \int_0^t \frac{1}{1 - H(x)} H(dx)$$

Therefore the right hand side of (5.3) converges to zero. □

We will now identify the almost sure limits of $S_n(q)$ and $\bar{S}_n(q)$ in Lemma 5.4. Define for $n \geq 2$

$$\bar{S}_n(q) := \sum\sum_{1 \leq i < j \leq n} \phi(Z_{i:n}, Z_{j:n}) \bar{W}_{i:n}(q) \bar{W}_{j:n}(q)$$

where

$$\bar{W}_{i:n}(q) := \prod_{k=1}^{n} \left(1 - \frac{q(Z_{k:n})}{n - k + 1}\right) .$$

Moreover let

$$S(q) := \frac{1}{2} \int_0^\infty \int_0^\infty \phi(s,t) q(s) q(t) \exp\left(\int_0^s \frac{1 - q(x)}{1 - H(x)} H(dx)\right) \times \exp\left(\int_0^t \frac{1 - q(x)}{1 - H(x)} H(dx)\right) H(ds) H(dt)$$

and

$$\bar{S}(q) := \frac{1}{2} \int_0^\infty \int_0^\infty \phi(s,t) \exp\left(\int_0^s \frac{1 - q(x)}{1 - H(x)} H(dx)\right) \times \exp\left(\int_0^t \frac{1 - q(x)}{1 - H(x)} H(dx)\right) H(ds) H(dt)$$

Lemma 5.4. *For continuous H the following statements hold*

$$\lim_{n\to\infty} S_n(q) = S(q)$$

and

$$\lim_{n\to\infty} \bar{S}_n(q) = \bar{S}(q)$$

with probability one, if the limits above exist.

Proof. Suppose H is continuous. First consider that we have

$$\begin{aligned}
\mathbb{E}[S_n(q)] &= \frac{n-1}{n}\mathbb{E}[\phi(Z_1,Z_2)q(Z_1)q(Z_2)\{\Delta_{n-2}(Z_1,Z_2)+\bar{\Delta}_{n-2}(Z_1,Z_2)\}\mathbb{1}_{\{Z_1<Z_2\}}] \\
&= \frac{n-1}{n}\mathbb{E}[\phi(Z_1,Z_2)q(Z_1)q(Z_2)\Delta_{n-2}(Z_1,Z_2)\mathbb{1}_{\{Z_1<Z_2\}}] \\
&\quad + \frac{n-1}{n}\mathbb{E}[\phi(Z_1,Z_2)q(Z_1)q(Z_2)\bar{\Delta}_{n-2}(Z_1,Z_2)\mathbb{1}_{\{Z_1<Z_2\}}]
\end{aligned}$$

by Lemma 5.2. Next note that we get from Lemma 4.9 that $D_n(s,t) \to D(s,t)$ w. p. 1, and according to Lemma 5.3 $C_n(s) \to 0$ w. p. 1. Thus $\bar{\Delta}_n(s,t) \to 0$ as $n \to \infty$ for each $s < t$. Moreover, the fact that $0 \le C_n(s) \le 1$ and Lemma 4.12 imply that $\bar{\Delta}_n(s,t) \le \Delta_n(s,t) \le D(s,t)$ for all $n \ge 2$. Now note that $D(s,t)$ is integrable, since on $\{Z_1 < Z_2\}$ we have

$$\begin{aligned}
\mathbb{E}[D(Z_1,Z_2)] &= \mathbb{E}\left[\int_0^s \frac{1-q(x)}{1-H(x)}H(dx) + \int_0^t \frac{1-q(x)}{1-H(x)}H(dx)\right] \\
&\le \mathbb{E}\left[\int_0^{Z_{n:n}} \frac{1}{1-H(x)}H(dx) + \int_0^{Z_{n:n}} \frac{1}{1-H(x)}H(dx)\right] \\
&\le \mathbb{E}\left[-2\ln(1-H(Z_{n:n}))\right] \\
&< \infty
\end{aligned}$$

Therefore applying the Dominated Convergence Theorem yields

$$\lim_{n\to\infty} \mathbb{E}[2\phi(Z_1, Z_2)q(Z_1)q(Z_2)\bar{\Delta}_{n-2}(Z_1, Z_2)\mathbb{1}_{\{Z_1<Z_2\}}] = 0 \ .$$

Furthermore, according to Lemma 4.12, the following holds

$$\Delta_n(s,t) = \mathbb{E}[D_n(s,t)] = \mathbb{E}[D_n(s,t)|\mathcal{F}_\infty] \nearrow D(s,t) \ .$$

Hence we have

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}[\phi(Z_1, Z_2)q(Z_1)q(Z_2)\Delta_{n-2}(Z_1, Z_2)\mathbb{1}_{\{Z_1<Z_2\}}] \\
&= \mathbb{E}[\phi(Z_1, Z_2)q(Z_1)q(Z_2)D(Z_1, Z_2)\mathbb{1}_{\{Z_1<Z_2\}}] \ .
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}[S_n(q)] &= \mathbb{E}[\phi(Z_1, Z_2)q(Z_1)q(Z_2)D(Z_1, Z_2)\mathbb{1}_{\{Z_1<Z_2\}}] \\
&= \int\int_{\{Z_1<Z_2\}} \phi(s,t)q(s)\exp\left(\int_0^s \frac{1-q(z)}{1-H(z)}H(dz)\right) \\
&\qquad \times q(t)\exp\left(\int_0^t \frac{1-q(z)}{1-H(z)}H(dz)\right) H(ds)H(dt) \\
&= \frac{1}{2}\int_0^\infty \int_0^\infty \phi(s,t)q(s)\exp\left(\int_0^s \frac{1-q(z)}{1-H(z)}H(dz)\right) \\
&\qquad \times q(t)\exp\left(\int_0^t \frac{1-q(z)}{1-H(z)}H(dz)\right) H(ds)H(dt)
\end{aligned}$$

$\mathbb{P}$-almost surely, since $\phi(s,t)q(s)q(t)D(s,t)$ is symmetric in s and t by (A1), and Z_1 and Z_2 are i. i. d.. By similar arguments, we obtain $\bar{S}_n \to \bar{S}$ w. p. 1.

□

In order to identify the limit of $S^{se}_{2,n} = S_n(m(\cdot, \hat{\theta}_n))$ we will need the following result in combination with Corollary 5.6. We will need the following definitions. For any $\epsilon > 0$ let

$$M_{1,\epsilon}(x) := \max(0, m(x,\theta_0) - \epsilon)) \text{ and } M_{2,\epsilon}(x) := \min(1, m(x,\theta_0) + \epsilon))$$

Lemma 5.5. *Suppose (M1) and (M2) hold. Then the following statements hold for each $0 < \epsilon \leq 1$ and n large enough*

(i) $M_{1,\epsilon}(x) \leq m(x, \hat{\theta}_n) \leq M_{2,\epsilon}(x)$

(ii) $M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 4\epsilon \leq m(x, \hat{\theta}_n)m(y, \hat{\theta}_n) \leq M_{1,\epsilon}(x)M_{1,\epsilon}(y) + 4\epsilon$

Proof. First we will introduce some notation. We will write $m_n(x) := m(x, \theta_n)$ and $m(x) := m(x, \theta_0)$. Let's start with part *(i)*. Suppose $M_{1,\epsilon}(x) = 0$, then the condition above is trivially satisfied since $m_n(x) \geq 0$. Now suppose $M_{1,\epsilon}(x) = m(x) - \epsilon$.

$$\begin{aligned} m_n(x) &= (m_n(x) - m(x)) + m(x) \\ &\geq m(x) - |m_n(x) - m(x)| \end{aligned}$$

Now using assumption (M1), we have for n large enough that for some $\epsilon > 0$ $\theta_n \in V(\epsilon, \theta_0)$. Now we get, according to (M2), that

$$\sup_{x \geq 0} |m_n(x) - m(x)| < \epsilon$$

Therefore we obtain $m_n(x) \geq m(x) - \epsilon = M_{1,\epsilon}(x)$. Let's now consider $M_{2,\epsilon}$. The case $M_{2,\epsilon} = 1$ is trivial again, since $m_n(x) \leq 1$. Now suppose $M_{2,\epsilon} = m(x) + \epsilon$. Then we obtain, for n large enough

$$\begin{aligned} m_n(x) &= (m_n(x) - m(x)) + m(x) \\ &\leq m(x) + |m_n(x) - m(x)| \\ &\leq m(x) + \epsilon \end{aligned}$$

$$= M_{2,\epsilon}(x)$$

This concludes the proof of part *(i)*. Now note that, according to (M1) and (M2), the following holds for n large enough and $\epsilon > 0$

$$\begin{aligned} m_n(x) &= (m_n(x) - m(x)) + m(x) \\ &\leq |m_n(x) - m(x)| + m(x) \\ &\leq m(x) + \epsilon \end{aligned} \tag{5.4}$$

Moreover consider that

$$\begin{aligned} m_n(x)m_n(y) &= (m_n(x) - m(x))(m_n(y) - m(y)) \\ &\quad + m(x)m_n(y) + m_n(x)m(y) - m(x)m(y) \\ &\leq \epsilon^2 + m(x)m_n(y) + m_n(x)m(y) - m(x)m(y) \end{aligned}$$

Using on the right hand side of the latter inequality (5.4) yields

$$\begin{aligned} m_n(x)m_n(y) &\leq \epsilon^2 + m(x)(m(y) + \epsilon) + (m(x) + \epsilon)m(y) - m(x)m(y) \\ &= m(x)m(y) + \epsilon(m(x) + m(y)) + \epsilon^2 \end{aligned} \tag{5.5}$$

Now suppose $M_{1,\epsilon}(x) = 0$ and $M_{1,\epsilon}(y) = 0$ for $x, y \in \mathbb{R}_+$. Then $m(x) \leq \epsilon$ and $m(y) \leq \epsilon$. Hence, using (5.5) yields

$$m_n(x)m_n(y) \leq 4\epsilon^2$$

Next suppose $M_{1,\epsilon}(x) = 0$ and $M_{1,\epsilon}(y) = m(y) - \epsilon$. Using (5.5) again, we obtain

$$m_n(x)m_n(y) \leq m(x)m(y) + \epsilon(m(x) + m(y)) + \epsilon^2$$

$$\le \epsilon + \epsilon(1+\epsilon) + \epsilon^2$$
$$= 2\epsilon(1+\epsilon)$$

since $m(x) \le \epsilon$ and $m(y) \le 1$. By similar calculations, we obtain the exact same result for the case $M_{1,\epsilon}(x) = m(x) - \epsilon$ and $M_{1,\epsilon}(y) = 0$. Now suppose $M_{1,\epsilon}(x) = m(x) - \epsilon$ and $M_{1,\epsilon}(y) = m(y) - \epsilon$ and note that

$$\begin{aligned} M_{1,\epsilon}(x)M_{1,\epsilon}(y) &= (m(x)-\epsilon)(m(y)-\epsilon) \\ &= m(x)m(y) - \epsilon(m(x)+m(y)) + \epsilon^2 \end{aligned}$$

Now (5.5) implies

$$\begin{aligned} m_n(x)m_n(y) &\le m(x)m(y) + \epsilon(m(x)+m(y)) + \epsilon^2 \\ &= M_{1,\epsilon}(x)M_{1,\epsilon}(y) + 2\epsilon(m(x)+m(y)) \\ &\le M_{1,\epsilon}(x)M_{1,\epsilon}(y) + 4\epsilon \end{aligned}$$

Thus we have for $0 \le \epsilon \le 1$ that

$$m_n(x)m_n(y) \le M_{1,\epsilon}(x)M_{1,\epsilon}(y) + 4\epsilon$$

as claimed in the statement of this lemma. It remains to show that $M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 4\epsilon \le m_n(x)m_n(y)$. By calculations similar to those, that lead to (5.4) and (5.5) we obtain

$$m_n(x) \ge m(x) - \epsilon$$

and

$$m_n(x)m_n(y) \ge m(x)m(y) - \epsilon(m(x)+m(y)) - \epsilon^2 \tag{5.6}$$

Now we will continue and look at $M_{2,\epsilon}$ case by case. Suppose $M_{2,\epsilon}(x) = 1$ and

$M_{2,\epsilon}(y) = 1$. This is equivalent to $m(x) \geq 1 - \epsilon$ and $m(y) \geq 1 - \epsilon$. Therefore (5.6) implies

$$\begin{aligned} m_n(x)m_n(y) &\geq (1-\epsilon)^2 - 2\epsilon - \epsilon^2 \\ &= 1 - 4\epsilon \\ &= M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 4\epsilon \end{aligned}$$

Next consider the case $M_{2,\epsilon}(x) = 1$ and $M_{2,\epsilon}(y) = m(y) + \epsilon$. Then we have $m(x) \geq 1 - \epsilon$ and $m(y) \leq 1 - \epsilon$. Moreover we have $M_{2,\epsilon}(x)M_{2,\epsilon}(y) = m(y) + \epsilon$. Hence we obtain the following, according to (5.6)

$$\begin{aligned} m_n(x)m_n(y) &\geq (1-\epsilon)m(y) - \epsilon((1 + (1-\epsilon)) - \epsilon^2 \\ &= m(y) - \epsilon m(y) - 2\epsilon \\ &\geq m(y) - \epsilon(1-\epsilon) - 2\epsilon \\ &\geq m(y) - 3\epsilon \\ &= M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 4\epsilon \end{aligned}$$

By similar calculations we obtain the same result, if $M_{2,\epsilon}(x) = m(x) + \epsilon$ and $M_{2,\epsilon}(y) = 1$. Finally consider the case $M_{2,\epsilon}(x) = m(x) + \epsilon$ and $M_{2,\epsilon}(y) = m(y) + \epsilon$. Then we have $m(x) \geq 1 - \epsilon$ and $m(y) \geq 1 - \epsilon$. Furthermore we have

$$\begin{aligned} M_{2,\epsilon}(x)M_{2,\epsilon}(y) &= (m(x) + \epsilon)(m(y) + \epsilon) \\ &= m(x)m(y) + \epsilon(m(x) + m(y)) + \epsilon^2 \end{aligned}$$

Therefore, using (5.6) again, yields

$$\begin{aligned} m_n(x)m_n(y) &\geq m(x)m(y) - \epsilon(m(x) + m(y)) - \epsilon^2 \\ &= M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 2\epsilon(m(x) + m(y)) - 2\epsilon^2 \end{aligned}$$

$$= M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 4\epsilon(1-\epsilon) - 2\epsilon^2$$
$$\geq M_{2,\epsilon}(x)M_{2,\epsilon}(y) - 4\epsilon$$

This concludes the proof. □

Corollary 5.6. *Suppose (M1) and (M2) hold and H is continuous. Then we have for each $0 < \epsilon \leq 1$ and n large enough*

$$S_n(M_{2,\epsilon}) - 4\epsilon\bar{S}_n(M_{2,\epsilon}) \leq S_n(m(\cdot,\hat{\theta}_n)) \leq S_n(M_{1,\epsilon}) + 4\epsilon\bar{S}_n(M_{1,\epsilon}).$$

Proof. Consider that we have the following for any $n \geq 1$

$$\begin{aligned} S_n(M_{2,\epsilon}) - 4\epsilon\bar{S}_n(M_{2,\epsilon}) = &\sum\sum_{1\leq i<j\leq n} \phi(Z_{i:n}, Z_{j:n})(M_{2,\epsilon}(Z_{i:n})M_{2,\epsilon}(Z_{j:n}) - 4\epsilon) \\ &\times \prod_{k=1}^{i-1}\left[1 - \frac{M_{2,\epsilon}(Z_{k:n})}{n-k+1}\right]\prod_{k=1}^{j-1}\left[1 - \frac{M_{2,\epsilon}(Z_{k:n})}{n-k+1}\right]. \end{aligned}$$

But according to Lemma 5.5 we have

$$m(x,\hat{\theta}_n) \leq M_{2,\epsilon}(x) \text{ and } M_{2,\epsilon}(x)M_{2,\epsilon}(y) \leq m(x,\hat{\theta}_n)m(y,\hat{\theta}_n).$$

for all $x, y \in \mathbb{R}_+$. Hence we obtain

$$\begin{aligned} S_n(M_{2,\epsilon}) - 4\epsilon\bar{S}_n(M_{2,\epsilon}) \leq &\sum\sum_{1\leq i<j\leq n} \phi(Z_{i:n}, Z_{j:n})m(Z_{i:n},\hat{\theta}_n)m(Z_{j:n},\hat{\theta}_n) \\ &\times \prod_{k=1}^{i-1}\left[1 - \frac{m(Z_{k:n},\hat{\theta}_n)}{n-k+1}\right]\prod_{k=1}^{j-1}\left[1 - \frac{m(Z_{k:n},\hat{\theta}_n)}{n-k+1}\right] \\ = &S_n(m(\cdot,\hat{\theta}_n)). \end{aligned}$$

Similarly we obtain

$$S_n(M_{1,\epsilon}) + 4\epsilon \bar{S}_n(M_{1,\epsilon}) \geq S_n(m(\cdot, \hat{\theta}_n)).$$

□

Now we are in a position, to identify $S = \lim_{n\to\infty} S^{se}_{2,n}$. The following theorem gives the main statement of this thesis.

Theorem 5.7. *Suppose (A1) through (A5), (M1) and (M2) hold. Then we have*

$$\lim_{n\to\infty} S_n(m(\cdot, \hat{\theta}_n)) = \frac{1}{2} \int_0^{\tau_H} \int_0^{\tau_H} \phi(s,t) F(ds) F(dt)$$

Proof. Consider that we have

$$S_n(M_{2,\epsilon}) - 4\epsilon \bar{S}_n(M_{2,\epsilon}) \leq S_n(m(\cdot, \hat{\theta}_n)) \leq S_n(M_{1,\epsilon}) + 4\epsilon \bar{S}_n(M_{1,\epsilon})$$

by Corollary 5.6 under (M1) and (M2). Next take note of the Radon-Nikodym derivatives

$$m(s, \theta_0) = \frac{H^1(ds)}{H(ds)} \text{ and } (1 - G(s)) = \frac{H^1(ds)}{F(ds)}$$

Moreover consider that we have

$$\int_0^s \frac{1 - m(x, \theta_0)}{1 - H(x)} = -\ln(1 - G(s))$$

and

$$\int_0^s \frac{\epsilon}{1 - H(x)} = -\ln((1 - H(s))^\epsilon)$$

according to Dikta (2000), page 8. Consider that we have

$$\begin{aligned} M_{1,\epsilon}(x) &= \mathbb{1}_{\{m(x,\theta_0)>\epsilon\}}(m(x,\theta_0)-\epsilon) \\ &\leq m(x,\theta_0)-\epsilon \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \bar{S}(M_{1,\epsilon}) &\leq \frac{1}{2}\int_0^\infty \int_0^\infty \phi(s,t)\exp\left(\int_0^s \frac{1-m(x,\theta_0)}{1-H(x)} + \frac{\epsilon}{1-H(x)}H(dx)\right) \\ &\qquad\qquad \times \exp\left(\int_0^t \frac{1-m(x,\theta_0)}{1-H(x)} + \frac{\epsilon}{1-H(x)}H(dx)\right) H(ds)H(dt) \\ &= \frac{1}{2}\int_0^\infty \int_0^\infty \frac{\phi(s,t)}{(1-G(s))(1-G(t))(1-H(s))^\epsilon(1-H(t))^\epsilon} H(ds)H(dt) \\ &= \frac{1}{2}\int_0^\infty \int_0^\infty \frac{\phi(s,t)}{m(s,\theta_0)m(t,\theta_0)(1-H(s))^\epsilon(1-H(t))^\epsilon} F(ds)F(dt) \end{aligned}$$

But by condition (A3), the integral above is finite. Moreover we know that $S(M_{1,\epsilon})$ exists almost surely under (A1) through (A5), by Theorem 4.17. Therefore we get by Lemma 5.4 that for each $0 < \epsilon \leq 1$ we have $S_n(M_{1,\epsilon}) + 4\epsilon\bar{S}_n(M_{1,\epsilon}) \to S(M_{1,\epsilon}) + 4\epsilon\bar{S}(M_{1,\epsilon})$ w. p. 1 as $n \to \infty$. Next consider that

$$\begin{aligned} S(M_{1,\epsilon}) + 4\epsilon\bar{S}(M_{1,\epsilon}) \leq \frac{1}{2}\int_0^\infty \int_0^\infty & \frac{\phi(s,t)}{(1-H(s))^\epsilon(1-H(t))^\epsilon} \\ & \times \frac{m(s,\theta_0)m(t,\theta_0)+4\epsilon}{(1-G(s))(1-G(t))} H(ds)H(dt) \end{aligned}$$

By similar arguments we can show that $S_n(M_{2,\epsilon}) - 4\epsilon\bar{S}_n(M_{2,\epsilon}) \to S(M_{2,\epsilon}) - 4\epsilon\bar{S}(M_{2,\epsilon})$ w. p. 1 as $n \to \infty$ and

$$\begin{aligned} S(M_{2,\epsilon}) - 4\epsilon\bar{S}(M_{2,\epsilon}) \geq \frac{1}{2}\int_0^\infty \int_0^\infty & \phi(s,t)(1-H(s))^\epsilon(1-H(t))^\epsilon \\ & \times \frac{m(s,\theta_0)m(t,\theta_0)-4\epsilon}{(1-G(s))(1-G(t))} H(ds)H(dt) \end{aligned}$$

Let's summarize the above. So far we have shown, that for $0 < \epsilon \leq 1$ small enough

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty \int_0^\infty \phi(s,t)(1-H(s))^\epsilon(1-H(t))^\epsilon \\
&\qquad\qquad \times \frac{m(s,\theta_0)m(t,\theta_0)-4\epsilon}{(1-G(s))(1-G(t))}H(ds)H(dt) \\
&\leq \liminf_{n\to\infty}\int_0^\infty \phi F_n^{se}F_n^{se} \\
&\leq \limsup_{n\to\infty}\int_0^\infty \phi F_n^{se}F_n^{se} \\
&\leq \frac{1}{2}\int_0^\infty \int_0^\infty \frac{\phi(s,t)}{(1-H(s))^\epsilon(1-H(t))^\epsilon} \\
&\qquad\qquad \times \frac{m(s,\theta_0)m(t,\theta_0)+4\epsilon}{(1-G(s))(1-G(t))}H(ds)H(dt)
\end{aligned}$$

Finally let $\epsilon \searrow 0$ and apply the Monotone Convergence Theorem to obtain that the upper and lower bound converge both to the same limit. In effect

$$\begin{aligned}
&\lim_{\epsilon\searrow 0}\frac{1}{2}\int_0^\infty \int_0^\infty \frac{\phi(s,t)(1-H(s))^\epsilon(1-H(t))^\epsilon}{(1-G(s))(1-G(t))}H(ds)H(dt) \\
&=\frac{1}{2}\int_0^\infty \int_0^\infty \frac{\phi(s,t)m(s,\theta_0)m(t,\theta_0)}{(1-G(s))(1-G(t))}H(ds)H(dt) \\
&=\frac{1}{2}\int_0^\infty \int_0^\infty \phi(s,t)F(ds)F(dt) \\
&=\lim_{\epsilon\searrow 0}\frac{1}{2}\int_0^\infty \int_0^\infty \frac{\phi(s,t)}{(1-G(s))(1-G(t))} \\
&\qquad\qquad \times \frac{1}{(1-H(s))^\epsilon(1-H(t))^\epsilon}H(ds)H(dt)
\end{aligned}$$

Hereby the proof of Theorem 5.7 is concluded. □

Remark 5.8. Note that according to Theorem 5.7

$$S_n(1) = \sum\sum_{1\leq i<j\leq n} W_{i:n}W_{j:n} \to \frac{1}{2}\int_0^{\tau_H}\int_0^{\tau_H} F(ds)F(dt) = \frac{1}{2}F^2(\tau_H)$$

Therefore we have

$$\lim_{n\to\infty} \frac{S_n(\phi)}{S_n(1)} = F^{-2}(\tau_H) \int_0^{\tau_H} \int_0^{\tau_H} \phi(s,t) F(ds) F(dt)$$

which is the normalized version of S_n.

Chapter 6

Conclusion

TODO Conclusion goes here...

- Summarize the SLLN for semiparametric U-Statistics
- Discussion of main assumptions and examples of kernels ϕ and censoring models m
- Future (or perhaps in this thesis): Transfer that property to Prof. Dikta's new estimator using stochastic equivalence
- Future: CLT based on Bose and Sen (2002)

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Appendix: Supplementary Results

Lemma A.1. *For $n \geq 2$ the following statements hold true*

(i)

$$\sum_{k=1}^{n-1} \frac{1}{k} \leq \ln(n-1) + 1 \tag{A1}$$

(ii)

$$\frac{\ln(n-1)+1}{(n+1)^{\frac{1}{3}}} \leq 3 \tag{A2}$$

Proof. We will start with the proof of part (i). Consider

$$\begin{aligned}
\sum_{k=1}^{n-1} \frac{1}{k} &\leq \ln(n-1) + 1 \\
\Leftrightarrow \sum_{k=1}^{n-1} \frac{1}{k} - 1 &\leq \ln(n-1) \\
\Leftrightarrow \sum_{k=2}^{n-1} \frac{1}{k} &\leq \ln(n-1) \qquad \text{(A3)}
\end{aligned}$$

Moreover we have

$$\begin{aligned}
\sum_{k=2}^{n-1} \frac{1}{k} &= \sum_{k=2}^{n-1} \int_{k-1}^{k} \frac{1}{k} dx \\
&\leq \sum_{k=2}^{n-1} \int_{k-1}^{k} \frac{1}{x} dx \\
&\leq \sum_{k=2}^{n-1} \ln(k) - \ln(k-1) \\
&\leq \ln(n-1) - \ln(1) \\
&= \ln(n-1)
\end{aligned}$$

Thus proving part (i). We will continue with the proof of part (ii). Note that (A2) is equivalent to showing

$$\ln(n-1)+1 \leq 3(n+1)^{\frac{1}{3}}$$

Since $\ln(n-1) \leq \ln(n+1)$, this will be implied by the following

$$\ln(n+1)+1 \leq 3(n+1)^{\frac{1}{3}} \tag{A4}$$

It is easy to check that inequality (A4) holds for $n = 2$. Now consider that

$$\frac{d}{dn}(\ln(n+1)+1) = \frac{1}{n+1}$$

and

$$\frac{d}{dn}3(n+1)^{\frac{1}{3}} = \frac{1}{(n+1)^{\frac{2}{3}}}$$

to get

$$\frac{d}{dn}(\ln(n+1)+1) \leq \frac{d}{dn}3(n+1)^{\frac{1}{3}} \tag{A5}$$

for all $n \geq 2$. Now the result in (ii) follows directly from (A4) and (A5) . $\square$

Curriculum Vitae

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