Contents

Chapter 1

Modifying the Martingale Convergence Theorem

1.1 Definitions and Assumptions

We're considering the estimator

$$S_n = \sum\sum_{1\le i<j\le n} \phi(Z_{i:n}, Z_{j:n}) W_{i:n} W_{j:n}$$

where

$$W_{i:n} = \frac{q(Z_{i:n})}{n-i+1} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right]$$

Define $\mathcal{F}_n := \sigma\{Z_{1:n}, \ldots, Z_{n:n}, Z_{n+1}, Z_{n+2}, \ldots\}$. Furthermore we will need the following definitions in order to get into a framework that is more similar to that of (forward) sub-martingales. Define

$$\tilde{S}_n^N := S_{N-n+1},\ \mathcal{F}_n^N := \mathcal{F}_{N-n+1}$$

Let $U_n[a, b]$ denote the number of upcrossings of $\tilde{S}_1^N, \ldots, \tilde{S}_n^N$ and define

$$Y_n^N := \tilde{S}_1^N + \sum_{i=1}^{n-1} \epsilon_i (\tilde{S}_{i+1}^N - \tilde{S}_i^N)$$

with

$$\epsilon_i := \begin{cases} 1 & (\tilde{S}_1^N, \ldots, \tilde{S}_i^N) \in B \\ 0 & \text{o.w.} \end{cases}$$

for some Borel set $B \in \mathcal{B}(\mathbb{R}^i)$. We can show that

$$(b-a)\mathbb{E}[U_n[a,b]] \leq \mathbb{E}[Y_n^N] \leq \mathbb{E}[\tilde{S}_n^N] - \sum_{k=1}^{n-1} \mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N - \tilde{S}_k^N | \mathcal{F}_k^N]]$$

We need to show

$$\begin{aligned}
&\lim_{N\to\infty} (b-a)\mathbb{E}[U_N[a,b]] \\
&\leq \lim_{N\to\infty} \mathbb{E}[Y_N^N] \\
&\leq \lim_{N\to\infty} \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1} \mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N - \tilde{S}_k^N | \mathcal{F}_k^N]] \\
&\leq \lim_{N\to\infty} \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1} \mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N | \mathcal{F}_k^N] - \tilde{S}_k^N] \\
&< \infty
\end{aligned}$$

So the main concern is to show that the sum of increases of $\tilde{S}_k^N$ on the right hand side converges. We will need the following assumptions in order to prove the above:

(A1) The following holds

$$\int_0^\infty \int_0^\infty \phi^2(s,t) H(ds) H(dt) < \infty$$

(A2) There exists $c_1 \in \mathbb{R}^+$ s. t. $\sup_x (q \circ H^{-1})'(x) \leq c_1$.

(A3) We have $q \circ H^{-1}(1) = 1$.

1.2 Generalized Upcrossing Theorem

Theorem 1.1. *Assume that (A1) through (A3) hold. Then we have*

$$
\begin{aligned}
&\lim_{N\to\infty}(b-a)\mathbb{E}[U_N[a,b]] \\
&\leq \lim_{N\to\infty}\mathbb{E}[Y_N^N] \\
&\leq \lim_{N\to\infty}\mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1}\mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N|\mathcal{F}_k^N] - \tilde{S}_k^N] \\
&< \infty
\end{aligned}
$$

We will first establish all necessary lemmas and then continue with the proof of Theorem 1.1 at the end of this section. The following lemma establishes a representation for the conditional expectation under the sum above, that is similar to Dikta (2000).

Lemma 1.2. *Define*

$$
Q_{ij}^{n+1} := \begin{cases} Q_i^{n+1} & j \leq n \\ Q_i^{n+1} - \frac{(n+1)\pi_i\pi_n(1-q(Z_{n:n+1}))}{(n-i+1)(2-q(Z_{n:n+1}))} & j = n+1 \end{cases}
$$

with

$$
Q_i^{n+1} := (n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2 + \frac{\pi_i\pi_{i+1}}{n-i+1}\right\}
$$

and

$$
\pi_i := \prod_{k=1}^{i-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]
$$

Then

$$
\mathbb{E}[S_n|\mathcal{F}_{n+1}] = \sum\sum_{1\leq i<j\leq n+1}\phi(Z_{i:n+1}, Z_{j:n+1})W_{i:n+1}W_{j:n+1}Q_{i,j}^{n+1}
$$

Proof. This lemma has been proven in my thesis. We already checked the calculations. □

We will need the following result on the increases of the Q_i^{n+1}'s later in the proof of Theorem 1.1.

Lemma 1.3. *Let Q_i^{n+1} be defined as above. Then*

$$Q_{i+1}^{n+1}-Q_i^{n+1} = \frac{\tilde{\pi}_i^2(n-i+2)^2}{n+1}\left\{\frac{(q_i-q_{i+1})(n-i)(n-i+1)-q_{i+1}(1-q_i)(n-i+1-q_i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})}\right\}$$

where $q_i := q(Z_{i:n+1})$ and

$$\tilde{\pi}_i := \pi_i\frac{n+1}{n-i+2}$$

Note that $\tilde{\pi}_i \leq 1$ for all $i \leq n+1$.

Proof. I proved this lemma in my thesis. □

Lemma 1.4. *Let (A2) be satisfied. Then the following statements hold true for $k \leq n-1$*

(i) We have

$$\mathbb{E}[|q(Z_{k:n})-q(Z_{k+1:n})|] \leq \frac{c_1}{n+1} \tag{1.1}$$

(ii) Furthermore assume that (A3) holds. Then

$$\mathbb{E}[1-q(Z_{k:n})] \leq \frac{c_1(n-k+1)}{n+1} \tag{1.2}$$

Proof. Let $q_H := q \circ H^{-1}$ and consider that we can write

$$q(H^{-1}(x)) = q(H^{-1}(x_0)) + q_H'(\hat{x})(x-x_0) \tag{1.3}$$

using Taylor expansion for some $\hat{x}$ in between x and x_0. Therefore we have

$$q(H^{-1}(x)) - q(H^{-1}(x_0)) = q'_H(\hat{x})(x - x_0)$$

and hence

$$|q(H^{-1}(x)) - q(H^{-1}(x_0))| = |q'_H(\hat{x})| \cdot |x - x_0| \tag{1.4}$$

Now let $U_1, \ldots, U_n$ be i. i. d. $Uni[0,1]$ and set $x = U_{k:n}$ and $x_0 = U_{k+1:n}$ to get

$$\mathbb{E}[|q(H^{-1}(U_{k:n})) - q(H^{-1}(U_{k+1:n}))|] = \mathbb{E}[|q(Z_{k:n}) - q(Z_{k+1:n})|]$$

Thus we get from (1.4)

$$\mathbb{E}[|q(Z_{k:n}) - q(Z_{k+1:n})|] = E[|q'_H(\hat{x})| \cdot (U_{k+1:n} - U_{k:n})]$$

where $\hat{x} \in [U_{k:n}, U_{k+1:n}]$. From assumption (A2) directly follows that $|q'_H(x)| \leq c_1$ for all $x \in [0,1]$. Hence we have

$$\mathbb{E}[|q(Z_{k:n}) - q(Z_{k+1:n})|] = c_1 \mathbb{E}[U_{k+1:n} - U_{k:n}]$$

According to Shorack and Wellner (2009) (p. 271), we have

$$\mathbb{E}[U_{k+1:n} - U_{k:n}] = \frac{1}{n+1} \tag{1.5}$$

Therefore we may conclude

$$\begin{aligned}\mathbb{E}[|q(Z_{k:n}) - q(Z_{k+1:n})|] &\leq c_1 \mathbb{E}[U_{k+1:n} - U_{k:n}] \\ &= \frac{c_1}{n+1} \qquad (1.6)\end{aligned}$$

This completes the proof part (i). We will now continue with the proof of part (ii).

Consider

$$
\begin{aligned}
1 - q(Z_{k:n}) &= 1 - q(Z_{n:n}) + \sum_{l=k}^{n-1} (q(Z_{l+1:n}) - q(Z_{l:n})) \\
&\leq 1 - q(Z_{n:n}) + \sum_{l=k}^{n-1} |q(Z_{l+1:n}) - q(Z_{l:n})|
\end{aligned}
$$

Taking expectations on each side yields

$$
1 - \mathbb{E}[q(Z_{k:n})] \leq 1 - \mathbb{E}[q(Z_{n:n})] + \sum_{l=k}^{n-1} \mathbb{E}[|q(Z_{l+1:n}) - q(Z_{l:n})|]
$$

Now we apply inequality (1.6) to the expectation under the sum to get

$$
1 - \mathbb{E}[q(Z_{k:n})] \leq 1 - \mathbb{E}[q(Z_{n:n})] + \frac{c_1(n-k)}{n+1} \tag{1.7}
$$

Recall the Taylor expansion from above

$$
q(H^{-1}(x)) = q(H^{-1}(x_0)) + q'_H(\hat{x})(x - x_0)
$$

Setting $x = 1$ and $x_0 = U_{n:n}$ and taking expectations on both sides yields

$$
\mathbb{E}[q(H^{-1}(1))] = \mathbb{E}[q(Z_{n:n})] + \mathbb{E}[q'_H(\hat{x})(1 - U_{n:n})]
$$

where $\hat{x} \in [U_{n:n}, 1]$ Now we get from assumption (A2) that

$$
\begin{aligned}
\mathbb{E}[q(Z_{n:n})] &= \mathbb{E}[q(H^{-1}(1))] - \mathbb{E}[q'_H(\hat{x})(1 - U_{n:n})] \\
&\geq \mathbb{E}[q(H^{-1}(1))] - c_1\mathbb{E}[1 - U_{n:n}]
\end{aligned}
$$

Using Shorack and Wellner (2009) (p. 271) again, we obtain

$$\mathbb{E}[q(Z_{n:n})] = \mathbb{E}[q(H^{-1}(1))] - \frac{c_1}{n+1}$$

Applying (A3) yields

$$\mathbb{E}[q(Z_{n:n})] \geq 1 - \frac{c_1}{n+1}$$

By combining the above with (1.7) we get

$$1 - \mathbb{E}[q(Z_{k:n})] \leq 1 - 1 + \frac{c_1}{n+1} + \frac{c_1(n-k)}{n+1} = \frac{c_1(n-k+1)}{n+1}$$

This concludes the proof of part (ii). $\square$

The following lemma contains some upper bounds that will be needed later in the proof of Theorem 1.1.

Lemma 1.5. *For $n \geq 2$ the following statements hold true*

(i)

$$\sum_{k=1}^{n-1} \frac{1}{k} \leq \ln(n-1) + 1 \tag{1.8}$$

(ii)

$$\frac{\ln(n-1)+1}{(n+1)^{\frac{1}{3}}} \leq 3 \tag{1.9}$$

Proof. We will start with the proof of part (i). Consider

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{1}{k} &\leq \ln(n-1) + 1 \\ \Leftrightarrow \quad \sum_{k=1}^{n-1} \frac{1}{k} - 1 &\leq \ln(n-1) \end{aligned}$$

$$\Leftrightarrow \quad \sum_{k=2}^{n-1} \frac{1}{k} \quad \leq \quad \ln(n-1) \tag{1.10}$$

Moreover we have

$$\begin{aligned}
\sum_{k=2}^{n-1} \frac{1}{k} &= \sum_{k=2}^{n-1} \int_{k-1}^{k} \frac{1}{k} dx \\
&\leq \sum_{k=2}^{n-1} \int_{k-1}^{k} \frac{1}{x} dx \\
&\leq \sum_{k=2}^{n-1} \ln(k) - \ln(k-1) \\
&\leq \ln(n-1) - \ln(1) \\
&= \ln(n-1)
\end{aligned}$$

Thus proving part (i). We will continue with the proof of part (ii). Note that (1.9) is equivalent to showing

$$\ln(n-1) + 1 \leq 3(n+1)^{\frac{1}{3}}$$

Since $\ln(n-1) \leq \ln(n+1)$, this will be implied by the following

$$\ln(n+1) + 1 \leq 3(n+1)^{\frac{1}{3}} \tag{1.11}$$

It is easy to check that inequality (1.11) holds for $n = 2$. Now consider that

$$\frac{d}{dn}(\ln(n+1) + 1) = \frac{1}{n+1}$$

and

$$\frac{d}{dn} 3(n+1)^{\frac{1}{3}} = \frac{1}{(n+1)^{\frac{2}{3}}}$$

to get

$$\frac{d}{dn}(\ln(n+1) + 1) \leq \frac{d}{dn} 3(n+1)^{\frac{1}{3}} \tag{1.12}$$

for all $n \geq 2$. Now the result in (ii) follows directly from (1.11) and (1.12) . $\square$

Now we established everything we need in order to proceed with the proof of Theorem 1.1. Recall that we need to show

$$\lim_{N\to\infty}(b-a)\mathbb{E}[U_N[a,b]] < \infty$$

Proof of Theorem 1. Let (A1) through (A3) be satisfied. Recall the following inequality (proven in my thesis). We have for $n \leq N$

$$(b-a)\mathbb{E}[U_n[a,b]] \leq \mathbb{E}[Y_n^N] \leq \mathbb{E}[\tilde{S}_n^N] - \sum_{k=1}^{n-1}\mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N|\mathcal{F}_k^N] - \tilde{S}_k^N]$$

Moreover we get from Lemma 1.2

$$\begin{aligned}\mathbb{E}[\tilde{S}_{k+1}^N|\tilde{\mathcal{F}}_k^N] &= \mathbb{E}[S_{N-k}|\mathcal{F}_{N-k+1}] \\ &= \sum\sum_{1\leq i<j\leq N-k+1}\phi(Z_{i:N-k+1}, Z_{j:N-k+1})W_{i:N-k+1}W_{j:N-k+1}Q_{i,j}^{N-k+1}\end{aligned}$$

Therefore we get

$$\begin{aligned}\mathbb{E}[Y_N^N] &\leq \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1}\mathbb{E}[(1-\epsilon_k)\mathbb{E}[\tilde{S}_{k+1}^N|\mathcal{F}_k^N] - \tilde{S}_k^N] \\ &= \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1}\mathbb{E}\Bigg[(1-\epsilon_k)\sum\sum_{1\leq i<j\leq N-k+1}\phi(Z_{i:N-k+1}, Z_{j:N-k+1}) \\ &\qquad\qquad \times W_{i:N-k+1}W_{j:N-k+1}(Q_{i,j}^{N-k+1}-1)\Bigg] \\ &= \mathbb{E}[\tilde{S}_N^N] - \sum_{k=1}^{N-1}\sum\sum_{1\leq i<j\leq N-k+1}\mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right. \\ &\qquad\qquad \left.\times W_{i:N-k+1}W_{j:N-k+1}(Q_{i,j}^{N-k+1}-1)\right] \\ &\leq \mathbb{E}[\tilde{S}_N^N] + \left|\sum_{k=1}^{N-1}\sum\sum_{1\leq i<j\leq N-k+1}\mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1}, Z_{j:N-k+1})\right.\right.\end{aligned}$$

$$
\begin{aligned}
&\qquad \times\, W_{i:N-k+1}W_{j:N-k+1}(Q_{i,j}^{N-k+1}-1)\big]\Bigg| \\
&\le \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1}\sum\sum_{1\le i<j\le N-k+1} \big|\mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1},Z_{j:N-k+1})\right. \\
&\qquad \times\, W_{i:N-k+1}W_{j:N-k+1}(Q_{i,j}^{N-k+1}-1)\big]\big|
\end{aligned}
$$

Now using Jensen's inequality yields

$$
\begin{aligned}
\mathbb{E}[Y_N^N] &\le \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1}\sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[(1-\epsilon_k)\phi(Z_{i:N-k+1},Z_{j:N-k+1})\right. \\
&\qquad \times\, W_{i:N-k+1}W_{j:N-k+1}\big|(Q_{i,j}^{N-k+1}-1)\big|\big] \\
&\le \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1}\sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[\phi(Z_{i:N-k+1},Z_{j:N-k+1})\right. \\
&\qquad \times\, W_{i:N-k+1}W_{j:N-k+1}\big|(Q_{i,j}^{N-k+1}-1)\big|\big]
\end{aligned}
$$

The latter inequality above holds, because $1-\epsilon_k \le 1$ for all $k \le N-1$. By applying the Cauchy-Schwarz inequality on the expectation above, we obtain

$$
\begin{aligned}
\mathbb{E}[Y_N^N] \le \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1}\sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[\phi^2(Z_{i:N-k+1},Z_{j:N-k+1})W_{i:N-k+1}^2W_{j:N-k+1}^2\right]^{\frac{1}{2}} \\
\times\, \mathbb{E}\left[(Q_{i,j}^{N-k+1}-1)^2\right]^{\frac{1}{2}} \qquad (1.13)
\end{aligned}
$$

We will now proceed to find an upper bound for $\mathbb{E}\left[(Q_{i,j}^{N-k+1}-1)^2\right]^{\frac{1}{2}}$. For the purpose of simpler notation we set $n := n(k,N) = N-k$. The inequality above can now be written as

$$
\begin{aligned}
\mathbb{E}[Y_N^N] \le \mathbb{E}[S_1] + \sum_{k=1}^{N-1}\sum\sum_{1\le i<j\le n+1} \mathbb{E}\left[\phi^2(Z_{i:n+1},Z_{j:n+1})W_{i:n+1}^2W_{j:n+1}^2\right]^{\frac{1}{2}} \\
\times\, \mathbb{E}\left[(Q_{i,j}^{n+1}-1)^2\right]^{\frac{1}{2}}
\end{aligned}
$$

Note k_1 and k_2 below do **not** correspond to k above in any way. Consider

$$Q_i^{n+1} - 1 = Q_1^{n+1} + \sum_{k_1=1}^{i-1}(Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}) - 1 \tag{1.14}$$

and recall the following definition

$$Q_i^{n+1} := (n+1)\left\{\sum_{r=1}^{i-1}\left[\frac{\pi_r}{n-r+2-q(Z_{r:n+1})}\right]^2 + \frac{\pi_i\pi_{i+1}}{n-i+1}\right\}$$

where

$$\pi_i := \prod_{k=1}^{i-1}\left[\frac{n-k+1-q(Z_{k:n+1})}{n-k+2-q(Z_{k:n+1})}\right]$$

We have $\pi_1 = 1$, since the product above is empty for $i = 1$ and

$$\pi_2 = \frac{n - q(Z_{1:n+1})}{n+1-q(Z_{1:n+1})}$$

Thus we get

$$\begin{aligned}
Q_1^{n+1} - 1 &= (n+1)\frac{\pi_1\pi_2}{n} - 1 \\
&= \frac{(n+1)(n-q(Z_{1:n+1}))}{n(n+1-q(Z_{1:n+1}))} - 1 \\
&= \frac{n(n+1-q(Z_{1:n+1})) - q(Z_{1:n+1})}{n(n+1-q(Z_{1:n+1}))} - 1 \\
&= 1 - \frac{q(Z_{1:n+1})}{n(n+1-q(Z_{1:n+1}))} - 1 \\
&= -\frac{q(Z_{1:n+1})}{n(n+1-q(Z_{1:n+1}))}
\end{aligned}$$

Therefore we get from (1.14)

$$Q_i^{n+1} - 1 = \sum_{k_1=1}^{i-1}(Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}) - \frac{q(Z_{1:n+1})}{n(n+1-q(Z_{1:n+1}))}$$

Moreover we have

$$
\begin{aligned}
(Q_i^{n+1}-1)^2 &= \sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}(Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1})(Q_{k_2+1}^{n+1}-Q_{k_2}^{n+1}) \\
&\quad - \frac{2q(Z_{1:n+1})}{n(n+1-q(Z_{1:n+1}))}\sum_{k=1}^{i-1}(Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}) \\
&\quad + \frac{q^2(Z_{1:n+1})}{n^2(n+1-q(Z_{1:n+1}))^2} \\
&\le \sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}|\cdot|Q_{k_2+1}^{n+1}-Q_{k_2}^{n+1}| \\
&\quad + \frac{2q(Z_{1:n+1})}{n(n+1-q(Z_{1:n+1}))}\sum_{k_1=1}^{i-1}|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}| \\
&\quad + \frac{q^2(Z_{1:n+1})}{n^2(n+1-q(Z_{1:n+1}))^2} \\
&\le \sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}|\cdot|Q_{k_2+1}^{n+1}-Q_{k_2}^{n+1}| \\
&\quad + \frac{2}{n^2}\sum_{k_1=1}^{i-1}|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}| + \frac{1}{n^4}
\end{aligned} \tag{1.15}
$$

Remember that we set $q_i := q(Z_{i:n+1})$. We get from Lemma 1.3 that

$$
\begin{aligned}
&|Q_{i+1}^{n+1}-Q_i^{n+1}| \\
&= \frac{\tilde{\pi}_i^2(n-i+2)^2}{n+1}\cdot\left|\frac{(q_i-q_{i+1})(n-i)(n-i+1)-q_{i+1}(1-q_i)(n-i+1-q_i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})}\right| \\
&\le \frac{\tilde{\pi}_i^2(n-i+2)^2}{n+1}\cdot\frac{|q_i-q_{i+1}|(n-i)(n-i+1)+q_{i+1}(1-q_i)(n-i+1-q_i)}{(n-i)(n-i+1)(n-i+2-q_i)^2(n-i+1-q_{i+1})} \\
&\le \frac{(n-i+2)^2}{n+1}\left\{\frac{|q_i-q_{i+1}|(n-i)(n-i+1)+q_{i+1}(1-q_i)(n-i+1)}{(n-i)(n-i+1)(n-i+1)^2(n-i)}\right\} \\
&= \frac{(n-i+2)^2}{n+1}\left\{\frac{|q_i-q_{i+1}|(n-i)+q_{i+1}(1-q_i)}{(n-i)^2(n-i+1)^2}\right\} \\
&\le \frac{4|q_i-q_{i+1}|}{(n+1)(n-i)}+\frac{4(1-q_i)}{(n+1)(n-i)^2}
\end{aligned} \tag{1.16}
$$

The latter inequality above holds since

$$\frac{n-i+2}{n-i+1} = 1 + \frac{1}{n-i+1} \leq 2$$

and $q_{i+1} \leq 1$. Thus we have

$$\begin{aligned}
&|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}| \cdot |Q_{k_2+1}^{n+1} - Q_{k_2}^{n+1}| \\
&\leq \left[\frac{4|q_{k_1} - q_{k_1+1}|}{(n+1)(n-k_1)} + \frac{4(1-q_{k_1})}{(n+1)(n-k_1)^2}\right] \\
&\quad \times \left[\frac{4|q_{k_2} - q_{k_2+1}|}{(n+1)(n-k_2)} + \frac{4(1-q_{k_2})}{(n+1)(n-k_2)^2}\right] \\
&= \frac{16|q_{k_1} - q_{k_1+1}||q_{k_2} - q_{k_2+1}|}{(n+1)^2(n-k_1)(n-k_2)} + \frac{16|q_{k_1} - q_{k_1+1}|(1-q_{k_2})}{(n+1)^2(n-k_1)(n-k_2)^2} \\
&\quad + \frac{16(1-q_{k_1})|q_{k_2} - q_{k_2+1}|}{(n+1)^2(n-k_1)^2(n-k_2)} + \frac{16(1-q_{k_1})(1-q_{k_2})}{(n+1)^2(n-k_1)^2(n-k_2)^2} \\
&\leq \frac{16|q_{k_1} - q_{k_1+1}|}{(n+1)^2(n-k_1)(n-k_2)} + \frac{16|q_{k_1} - q_{k_1+1}|}{(n+1)^2(n-k_1)(n-k_2)^2} \\
&\quad + \frac{16|q_{k_2} - q_{k_2+1}|}{(n+1)^2(n-k_1)^2(n-k_2)} + \frac{16(1-q_{k_1})}{(n+1)^2(n-k_1)^2(n-k_2)^2}
\end{aligned}$$

Here the latter inequality holds, since we have $|q_k - q_{k+1}| \leq 1$ and $1 - q_k \leq 1$ for all $k \leq n-1$.

Recall that

$$\begin{aligned}
(Q_i^{n+1} - 1)^2 \leq & \sum_{k_1=1}^{i-1} \sum_{k_2=1}^{i-1} |Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}||Q_{k_2+1}^{n+1} - Q_{k_2}^{n+1}| \\
& + \frac{2}{n^2} \sum_{k_1=1}^{i-1} |Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}| + \frac{1}{n^4}
\end{aligned}$$

Taking expectations on each side yields

$$\mathbb{E}[(Q_i^{n+1} - 1)^2] \leq \sum_{k_1=1}^{i-1} \sum_{k_2=1}^{i-1} \mathbb{E}\big[|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}||Q_{k_2+1}^{n+1} - Q_{k_2}^{n+1}|\big]$$

$$+\frac{2}{n^2}\sum_{k_1=1}^{i-1}\mathbb{E}[|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}|]+\frac{1}{n^4} \tag{1.17}$$

Consider the expectation under the double sum above. We have

$$\begin{aligned}
&\mathbb{E}\left[\left|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}\right|\left|Q_{k_2+1}^{n+1}-Q_{k_2}^{n+1}\right|\right] \\
&\le \frac{16\mathbb{E}\left[|q_{k_1}-q_{k_1+1}|\right]}{(n+1)^2(n-k_1)(n-k_2)}+\frac{16\mathbb{E}\left[|q_{k_1}-q_{k_1+1}|\right]}{(n+1)^2(n-k_1)(n-k_2)^2} \\
&\quad +\frac{16\mathbb{E}\left[|q_{k_2}-q_{k_2+1}|\right]}{(n+1)^2(n-k_1)^2(n-k_2)}+\frac{16\mathbb{E}\left[(1-q_{k_1})\right]}{(n+1)^2(n-k_1)^2(n-k_2)^2}
\end{aligned} \tag{1.18}$$

We will now use Lemma 1.4 to establish an upper bound for the expectation above. Combining (1.1) and (1.2) above with (1.18) yields

$$\begin{aligned}
&\mathbb{E}[\left|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}\right|\left|Q_{k_2+1}^{n+1}-Q_{k_2}^{n+1}\right|] \\
&\le \frac{16c_1}{(n+1)^3(n-k_1)(n-k_2)}+\frac{16c_1}{(n+1)^3(n-k_1)(n-k_2)^2} \\
&\quad +\frac{16c_1}{(n+1)^3(n-k_1)^2(n-k_2)}+\frac{16c_1(n-k_1)+16c_1}{(n+1)^3(n-k_1)^2(n-k_2)^2}
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\mathbb{E}[\left|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}\right|\left|Q_{k_2+1}^{n+1}-Q_{k_2}^{n+1}\right|] \\
&\le \sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\frac{16c_1}{(n+1)^3(n-k_1)(n-k_2)}+\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\frac{16c_1}{(n+1)^3(n-k_1)(n-k_2)^2} \\
&\quad +\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\frac{16c_1}{(n+1)^3(n-k_1)^2(n-k_2)}+\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\frac{16c_1(n-k_1)}{(n+1)^3(n-k_1)^2(n-k_2)^2} \\
&\quad +\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\frac{16c_1}{(n+1)^3(n-k_1)^2(n-k_2)^2} \\
&= \sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\frac{16c_1}{(n+1)^3(n-k_1)(n-k_2)}+\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\frac{32c_1}{(n+1)^3(n-k_1)(n-k_2)^2} \\
&\quad +\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\frac{16c_1}{(n+1)^3(n-k_1)^2(n-k_2)}+\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\frac{16c_1}{(n+1)^3(n-k_1)^2(n-k_2)^2}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{16c_1}{(n+1)^3}\sum_{k_1=1}^{i-1}\frac{1}{(n-k_1)}\sum_{k_2=1}^{i-1}\frac{1}{(n-k_2)}+\frac{32c_1}{(n+1)^3}\sum_{k_1=1}^{i-1}\frac{1}{n-k_1}\sum_{k_2=1}^{i-1}\frac{1}{(n-k_2)^2}\\
&\quad+\frac{16c_1}{(n+1)^3}\sum_{k_1=1}^{i-1}\frac{1}{(n-k_1)^2}\sum_{k_2=1}^{i-1}\frac{1}{n-k_2}+\frac{16c_1}{(n+1)^3}\sum_{k_1=1}^{i-1}\frac{1}{(n-k_1)^2}\sum_{k_2=1}^{i-1}\frac{1}{(n-k_2)^2}\\
&\leq \frac{16c_1}{(n+1)^3}\sum_{k_1=n-i+1}^{n-1}\frac{1}{k_1}\sum_{k_2=n-i+1}^{n-1}\frac{1}{k_2}+\frac{32c_1}{(n+1)^3}\sum_{k_1=n-i+1}^{n-1}\frac{1}{k_1}\sum_{k_2=n-i+1}^{n-1}\frac{1}{k_2^2}\\
&\quad+\frac{16c_1}{(n+1)^3}\sum_{k_1=n-i+1}^{n-1}\frac{1}{k_1^2}\sum_{k_2=n-i+1}^{n-1}\frac{1}{k_2}+\frac{16c_1}{(n+1)^3}\sum_{k_1=n-i+1}^{n-1}\frac{1}{k_1^2}\sum_{k_2=n-i+1}^{n-1}\frac{1}{k_2^2} \qquad (1.19)
\end{aligned}
$$

Now using (1.8) and (1.9) from Lemma 1.5 on inequality (1.19) yields

$$
\begin{aligned}
&\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\mathbb{E}[|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}||Q_{k_2+1}^{n+1}-Q_{k_2}^{n+1}|]\\
&\leq \frac{16c_1}{(n+1)^3}(\ln(n-1)+1)^2+\frac{64c_1}{(n+1)^3}(\ln(n-1)+1)\\
&\quad+\frac{32c_1}{(n+1)^3}(\ln(n-1)+1)+\frac{64c_1}{(n+1)^3}\\
&\leq \frac{144c_1}{(n+1)^{\frac{7}{3}}}+\frac{288c_1}{(n+1)^{\frac{8}{3}}}+\frac{64c_1}{(n+1)^3}\\
&\leq \frac{496c_1}{(n+1)^{\frac{7}{3}}} \qquad (1.20)
\end{aligned}
$$

We will now proceed with the second sum in (1.17). We get from (1.16)

$$\mathbb{E}[|Q_{i+1}^{n+1}-Q_i^{n+1}|]\leq\frac{4\mathbb{E}[|q_i-q_{i+1}|]}{(n+1)(n-i)}+\frac{4\mathbb{E}[1-q_i]}{(n+1)(n-i)^2}$$

Therefore we obtain

$$\frac{2}{n^2}\sum_{k_1=1}^{i-1}\mathbb{E}[|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}|]\leq\frac{8}{n^2(n+1)}\sum_{k_1=1}^{i-1}\frac{\mathbb{E}[|q_{k_1}-q_{k_1+1}|]}{n-k_1}+\frac{\mathbb{E}[1-q_{k_1}]}{(n-k_1)^2}$$

Again using (1.1) and (1.2) reveals

$$\frac{2}{n^2}\sum_{k_1=1}^{i-1}\mathbb{E}[|Q_{k_1+1}^{n+1}-Q_{k_1}^{n+1}|]\leq\frac{8}{n^2(n+1)^2}\left\{\sum_{k_1=1}^{i-1}\frac{c_1}{(n-k_1)}+\sum_{k_1=1}^{i-1}\frac{c_1(n-k_1+1)}{(n-k_1)^2}\right\}$$

$$= \frac{8}{n^2(n+1)^2}\left\{2\sum_{k_1=1}^{i-1}\frac{c_1}{(n-k_1)} + \sum_{k_1=1}^{i-1}\frac{c_1}{(n-k_1)^2}\right\}$$
$$= \frac{8}{n^2(n+1)^2}\left\{2\cdot\sum_{k_1=n-i+1}^{n-1}\frac{c_1}{k_1} + \sum_{k_1=n-i+1}^{n-1}\frac{c_1}{k_1^2}\right\}$$

By using (1.8) and (1.9) again we obtain

$$\begin{aligned}
\frac{2}{n^2}\sum_{k_1=1}^{i-1}\mathbb{E}[|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}|] &\leq \frac{8\cdot\{2c_1(\ln(n-1)+1)+2c_1\}}{n^2(n+1)^2} \\
&= \frac{16c_1(\ln(n-1)+1)}{n^2(n+1)^2} + \frac{16c_1}{n^2(n+1)^2} \\
&\leq \frac{48c_1}{n^2(n+1)^{\frac{5}{3}}} + \frac{16c_1}{n^2(n+1)^2} \\
&\leq \frac{64c_1}{n^2(n+1)^{\frac{5}{3}}} \qquad (1.21)
\end{aligned}$$

Again recall the following fact

$$\begin{aligned}
\mathbb{E}[(Q_i^{n+1}-1)^2] = &\sum_{k_1=1}^{i-1}\sum_{k_2=1}^{i-1}\mathbb{E}[\left|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}\right|\left|Q_{k_2+1}^{n+1} - Q_{k_2}^{n+1}\right|] \\
&+ \frac{2}{n^2}\sum_{k_1=1}^{i-1}\mathbb{E}[|Q_{k_1+1}^{n+1} - Q_{k_1}^{n+1}|] + \frac{1}{n^4}
\end{aligned}$$

Combining the above with (1.20) and (1.21) yields

$$\begin{aligned}
\mathbb{E}[(Q_i^{n+1}-1)^2] &\leq \frac{496c_1}{(n+1)^{\frac{7}{3}}} + \frac{64c_1}{n^2(n+1)^{\frac{5}{3}}} + \frac{1}{n^4} \\
&\leq \frac{496c_1}{n^{\frac{7}{3}}} + \frac{64c_1}{n^{\frac{11}{3}}} + \frac{1}{n^4} \\
&\leq \frac{1}{n^{\frac{7}{3}}}\left[496c_1 + \frac{64c_1}{n^{\frac{4}{3}}} + \frac{1}{n^{\frac{5}{3}}}\right] \\
&\leq \frac{560c_1+1}{n^{\frac{7}{3}}} \\
&= \frac{c_2}{n^{\frac{7}{3}}}
\end{aligned}$$

with $c_2 := 560c_1 + 1$. Therefore

$$\mathbb{E}[(Q_i^{n+1} - 1)^2]^{\frac{1}{2}} \leq \frac{\sqrt{c_2}}{n^{\frac{7}{6}}}$$

Recall that we set $n = N - k$. Thus we can write

$$\mathbb{E}[(Q_i^{N-k+1} - 1)^2]^{\frac{1}{2}} \leq \frac{\sqrt{c_2}}{(N-k)^{\frac{7}{6}}}$$

Now combining the latter with (1.13) yields

$$\begin{aligned}
\mathbb{E}[Y_N^N] \leq \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\leq i<j\leq N-k+1} & \mathbb{E}\left[\phi^2(Z_{i:N-k+1}, Z_{j:N-k+1}) W_{i:N-k+1}^2 W_{j:N-k+1}^2\right]^{\frac{1}{2}} \\
& \times \mathbb{E}\left[(Q_{i,j}^{N-k+1} - 1)^2\right]^{\frac{1}{2}} \\
\leq \mathbb{E}[\tilde{S}_N^N] + \sum_{k=1}^{N-1} \sum\sum_{1\leq i<j\leq N-k+1} & \mathbb{E}\left[\phi^2(Z_{i:N-k+1}, Z_{j:N-k+1}) W_{i:N-k+1}^2 W_{j:N-k+1}^2\right]^{\frac{1}{2}} \\
& \times \frac{\sqrt{c_2}}{(N-k)^{\frac{7}{6}}}
\end{aligned}$$

Thus it remains to show that

$$\sum\sum_{1\leq i<j\leq N-k+1} \mathbb{E}\left[\phi^2(Z_{(i)}, Z_{(j)}) W_{(i)}^2 W_{(j)}^2\right]^{\frac{1}{2}} \leq c_3 < \infty$$

is bounded above. Then we would have

$$\begin{aligned}
\lim_{N\to\infty} \mathbb{E}[U_N[a,b]] &\leq \lim_{N\to\infty} \mathbb{E}[Y_N^N] \\
&\leq \lim_{N\to\infty} \left\{ \mathbb{E}[\tilde{S}_N^N] + c_3 \sum_{k=1}^{N-1} \frac{\sqrt{c_2}}{(N-k)^{\frac{7}{6}}} \right\} \\
&\leq \sup_N \mathbb{E}[\tilde{S}_N^N] + \sqrt{c_2} c_3 \left\{ \lim_{N\to\infty} \sum_{k=1}^{N-1} \frac{1}{(N-k)^{\frac{7}{6}}} \right\} \\
&< \infty
\end{aligned}$$

And therefore we may finally conclude that $S = \lim_{n\to\infty} S_n$ exists. Note that there is more argumentation about the relationship between $U_N[a,b]$ and $\lim\limits_{n\to\infty} S_n$ in my thesis.

□

1.3 The missing bound

It remains to show that

$$\sum\sum_{1\le i<j\le N-k+1} \mathbb{E}\left[\phi^2(Z_{i:N-k+1}, Z_{j:N-k+1}) W^2_{i:N-k+1} W^2_{j:N-k+1}\right]^{\frac{1}{2}}$$

is bounded above. For the sake of simplicity we will set $n = N - k + 1$ again. The following lemma contains the result needed to prove Theorem 1.1.

Lemma 1.6. *Suppose (A1) holds. Then there exists $c_3 < \infty$ s. t.*

$$\sum\sum_{1\le i<j\le n+1} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n}) W^2_{i:n} W^2_{j:n}\right]^{\frac{1}{2}} \le c_3$$

The prove of the lemma above will be given at the end of this section. In the following I will establish a few lemmas, that will be needed to prove lemma 1.6 above. Define the following quantities for $n \ge 1$ and $s < t$:

$$B_n(s) := \prod_{k=1}^{n}\left[1 + \frac{1-q(Z_k)}{n - R_{k,n}}\right]^{\mathbb{I}\{Z_k<s\}}$$

$$C_n(s) := \sum_{i=1}^{n+1}\left[\frac{1-q(s)}{n-i+2}\right]\mathbb{I}\{Z_{i-1:n} < s \le Z_{i:n}\}$$

$$D_n(s,t) := \prod_{k=1}^{n}\left[1 + \frac{1-q(Z_k)}{n - R_{k,n} + 2}\right]^{2\mathbb{I}\{Z_k<s\}} \prod_{k=1}^{n}\left[1 + \frac{1-q(Z_k)}{n - R_{k,n} + 1}\right]^{\mathbb{I}\{s<Z_k<t\}}$$

$$\Delta_n(s,t) := \mathbb{E}\left[D_n(s,t)\right]$$

$$\bar{\Delta}_n(s,t) := \mathbb{E}\left[C_n(s)D_n(s,t)\right]$$

Here $Z_{0:n} := -\infty$ and $Z_{n+1:n} := \infty$.

Lemma 1.7. *Let $\tilde{\phi} : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ be a Borel-measurable function. Then we have for any $s < t$ and $n \geq 1$*

$$\begin{aligned}
&\mathbb{E}[\tilde{\phi}(Z_1, Z_2)B_n(Z_1)B_n(Z_2)] \\
&= \mathbb{E}[2\tilde{\phi}(Z_1, Z_2)\{\Delta_{n-2}(Z_1, Z_2) + \bar{\Delta}_{n-2}(Z_1, Z_2)\}\mathbb{I}\{Z_1 < Z_2\}]
\end{aligned}$$

Proof. Consider the following

$$\begin{aligned}
B_n(Z_1)B_n(Z_2) &= \prod_{k=1}^{n}\left[1 + \frac{1-q(Z_k)}{n - R_{k,n}}\right]^{\mathbb{I}\{Z_k<Z_1\}+\mathbb{I}\{Z_k<Z_2\}} \\
&= \left[1 + \frac{1-q(Z_1)}{n - R_{1,n}}\right]^{\mathbb{I}\{Z_1<Z_2\}} \left[1 + \frac{1-q(Z_2)}{n - R_{2,n}}\right]^{\mathbb{I}\{Z_2<Z_1\}} \\
&\quad \times \prod_{k=3}^{n}\left[1 + \frac{1-q(Z_k)}{n - R_{k,n}}\right]^{\mathbb{I}\{Z_k<Z_1\}+\mathbb{I}\{Z_k<Z_2\}} \\
&= \mathbb{I}\{Z_1 < Z_2\}\left[1 + \frac{1-q(Z_1)}{n - R_{1,n}}\right] \\
&\qquad \times \prod_{k=1}^{n-2}\left[1 + \frac{1-q(Z_{k+2})}{n - R_{k+2,n}}\right]^{\mathbb{I}\{Z_{k+2}<Z_1\}+\mathbb{I}\{Z_{k+2}<Z_2\}} \\
&\quad + \mathbb{I}\{Z_1 > Z_2\}\left[1 + \frac{1-q(Z_2)}{n - R_{2,n}}\right] \\
&\qquad \times \prod_{k=1}^{n-2}\left[1 + \frac{1-q(Z_{k+2})}{n - R_{k+2,n}}\right]^{\mathbb{I}\{Z_{k+2}<Z_1\}+\mathbb{I}\{Z_{k+2}<Z_2\}} \\
&\quad + \mathbb{I}\{Z_1 = Z_2\}\prod_{k=1}^{n-2}\left[1 + \frac{1-q(Z_{k+2})}{n - R_{k+2,n}}\right]^{2\mathbb{I}\{Z_{k+2}<Z_1\}} \qquad (1.22)
\end{aligned}$$

On $\{Z_1 < Z_2\}$ we have

$$\begin{aligned}\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k+2})}{n-R_{k+2,n}}\right]^{\mathbb{I}\{Z_{k+2}<Z_2\}} &= \prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}}\right]^{\mathbb{I}\{Z_{k+2}<Z_1\}}\\ &\quad\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}-1}\right]^{\mathbb{I}\{Z_1<Z_{k+2}<Z_2\}}\end{aligned}$$

where $\tilde{R}_{k,n-2}$ denotes the rank of the Z_k, $k = 3, \ldots, n$ among themselves. The above holds since

$$R_{k+2,n} = \begin{cases}\tilde{R}_{k,n-2} & \text{if } Z_{k+2} < Z_1\\ \tilde{R}_{k,n-2}+1 & \text{if } Z_1 < Z_{k+2} < Z_2\end{cases}$$

for $k = 1, \ldots, n-2$. Therefore (1.22) yields

$$\begin{aligned}B_n(Z_1)B_n(Z_2) = \mathbb{I}\{Z_1 < Z_2\}&\left[1+\frac{1-q(Z_1)}{n-R_{1,n}}\right]\\ &\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}}\right]^{2\mathbb{I}\{Z_{k+2}<Z_1\}}\\ &\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}-1}\right]^{\mathbb{I}\{Z_1<Z_{k+2}<Z_2\}}\\ +\mathbb{I}\{Z_2 < Z_1\}&\left[1+\frac{1-q(Z_2)}{n-R_{2,n}}\right]\\ &\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}}\right]^{2\mathbb{I}\{Z_{k+2}<Z_2\}}\\ &\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}-1}\right]^{\mathbb{I}\{Z_2<Z_{k+2}<Z_1\}}\\ +\mathbb{I}\{Z_1 = Z_2\}&\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k+2})}{n-\tilde{R}_{k,n-2}}\right]^{2\mathbb{I}\{Z_{k+2}<Z_1\}}\end{aligned}\qquad(1.23)$$

Now let's denote for $k = 1, \ldots, n-2$, $Z_{k:n-2}$ the ordered Z-values among $Z_3, \ldots, Z_n$.

Consider that we can write

$$\left[1+\frac{1-q(Z_1)}{n-R_{1,n}}\right]=\sum_{i=1}^{n-1}\left[1+\frac{1-q(s)}{n-i}\right]\mathbb{I}\{Z_{i-1:n-2}<Z_1\leq Z_{i:n-2}\}$$

Therefore we obtain the following, by conditioning (1.23) on Z_1, Z_2:

$$\begin{aligned}
&\mathbb{E}[B_n(Z_1)B_n(Z_2)|Z_1=s,Z_2=t]\\
&=\mathbb{I}\{s<t\}\mathbb{E}\left[\left(\sum_{i=1}^{n-1}\left[1+\frac{1-q(s)}{n-i}\right]\mathbb{I}\{Z_{i-1:n-2}<s\leq Z_{i:n-2}\}\right)\right.\\
&\qquad\qquad\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{I}\{Z_{k:n-2}<s\}}\\
&\qquad\qquad\left.\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k-1}\right]^{\mathbb{I}\{s<Z_{k:n-2}<t\}}\right]\\
&\quad+\mathbb{I}\{t<s\}\mathbb{E}\left[\left(\sum_{i=1}^{n-1}\left[1+\frac{1-q(t)}{n-i}\right]\mathbb{I}\{Z_{i-1:n-2}<t\leq Z_{i:n-2}\}\right)\right.\\
&\qquad\qquad\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{I}\{Z_{k:n-2}<t\}}\\
&\qquad\qquad\left.\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k-1}\right]^{\mathbb{I}\{t<Z_{k:n-2}<s\}}\right]\\
&\quad+\mathbb{I}\{s=t\}\mathbb{E}\left[\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{I}\{Z_{k:n-2}<s\}}\right]\\
&=\alpha(s,t)+\alpha(t,s)+\beta(s,t)
\end{aligned}$$

where

$$\begin{aligned}
\alpha(s,t):=\mathbb{I}\{s<t\}\mathbb{E}&\left[\left(\sum_{i=1}^{n-1}\left[1+\frac{1-q(s)}{n-i}\right]\mathbb{I}\{Z_{i-1:n-2}<s\leq Z_{i:n-2}\}\right)\right.\\
&\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{I}\{Z_{k:n-2}<s\}}\\
&\left.\times\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k-1}\right]^{\mathbb{I}\{s<Z_{k:n-2}<t\}}\right]
\end{aligned}$$

and

$$\beta(s,t) := \mathbb{I}\{s=t\}\mathbb{E}\left[\prod_{k=1}^{n-2}\left[1+\frac{1-q(Z_{k:n-2})}{n-k}\right]^{2\mathbb{I}\{Z_{k:n-2}<s\}}\right]$$

Consider that we have

$$\mathbb{E}[\alpha(Z_1,Z_2)] = \mathbb{E}[\alpha(Z_2,Z_1)]$$

because Z_1 and Z_2 are i. i. d., and

$$\mathbb{E}[\beta(Z_1,Z_2)] = 0$$

since H is continuous. Therefore we get

$$\begin{aligned} &\mathbb{E}[\tilde{\phi}(Z_1,Z_2)B_n(Z_1)B_n(Z_2)] \\ &= \mathbb{E}[\tilde{\phi}(Z_1,Z_2)(\alpha(Z_1,Z_2)+\alpha(Z_2,Z_1)+\beta(Z_1,Z_2))] \\ &= \mathbb{E}[2\tilde{\phi}(Z_1,Z_2)\alpha(Z_1,Z_2)] \end{aligned} \tag{1.24}$$

Next consider that

$$\begin{aligned} \alpha(s,t) &= \mathbb{I}\{s<t\}\mathbb{E}\left[(1+C_n(s))D_{n-2}(s,t)\right] \\ &= \mathbb{I}\{s<t\}(\Delta_{n-2}(s,t)+\bar{\Delta}_{n-2}(s,t)) \end{aligned}$$

The latter equality holds, since

$$\begin{aligned} &\sum_{i=1}^{n-1}\left[1+\frac{1-q(s)}{n-i}\right]\mathbb{I}\{Z_{i-1:n-2}<s\le Z_{i:n-2}\} \\ &= \sum_{i=1}^{n-1}\mathbb{I}\{Z_{i-1:n-2}<s\le Z_{i:n-2}\} + \sum_{i=1}^{n-1}\left[\frac{1-q(s)}{n-i}\right]\mathbb{I}\{Z_{i-1:n-2}<s\le Z_{i:n-2}\} \\ &= 1+C_n(s) \end{aligned}$$

Now the statement of the lemma follows directly from (1.24). □

The next lemma identifies the $\mathbb{P}$-almost sure limit of D_n for $n \to \infty$. Define for $s < t$

$$D(s,t) := \exp\left(2\int_0^s \frac{1-q(z)}{1-H(z)}H(dz) + \int_s^t \frac{1-q(z)}{1-H(z)}H(dz)\right)$$

Lemma 1.8. *For any $s < t$ s. t. $H(t) < 1$, we have*

$$\lim_{n\to\infty} D_n(s,t) = D(s,t)$$

Proof. First let for $s < t$ and $k = 1, \dots, n$

$$x_k := \frac{1-q(Z_k)}{n(1-H_n(Z_k)+2/n)}$$
$$y_k := \frac{1-q(Z_k)}{n(1-H_n(Z_k)+1/n)}$$
$$s_k := \mathbb{I}\{Z_k < s\}$$
$$t_k := \mathbb{I}\{s < Z_k < t\}$$

Next consider

$$\begin{aligned}
D_n(s,t) &= \prod_{k=1}^n \left[1 + \frac{1-q(Z_k)}{n(1-H_n(Z_k)+2/n)}\mathbb{I}\{Z_k < s\}\right]^2 \\
&\quad \times \prod_{k=1}^n \left[1 + \frac{1-q(Z_k)}{n(1-H_n(Z_k)+1/n)}\mathbb{I}\{s < Z_k < t\}\right] \\
&= \prod_{k=1}^n [1 + x_k s_k]^2 \prod_{k=1}^n [1 + y_k t_k] \\
&= \exp\left(2\sum_{k=1}^n \ln[1 + x_k s_k] + \sum_{k=1}^n \ln[1 + y_k t_k]\right)
\end{aligned}$$

Note that $0 \leq x_k s_k \leq 1$ and $0 \leq y_k t_k \leq 1$. Consider that the following inequality holds

$$-\frac{x^2}{2} \leq \ln(1+x) - x \leq 0$$

for any $x \geq 0$ (cf. Stute and Wang (1993), p. 1603). This implies

$$-\frac{1}{2}\sum_{k=1}^{n} x_k^2 s_k \leq \sum_{k=1}^{n} \ln(1+x_k s_k) - \sum_{k=1}^{n} x_k s_k \leq 0$$

But now

$$\begin{aligned}
\sum_{k=1}^{n} x_k^2 s_k &= \frac{1}{n^2}\sum_{k=1}^{n}\left(\frac{1-q(Z_k)}{1-H_n(Z_k)+\frac{2}{n}}\right)^2 \mathbb{I}\{Z_k < s\} \\
&\leq \frac{1}{n^2}\sum_{k=1}^{n}\left(\frac{1}{1-H_n(s)+\frac{1}{n}}\right)^2 \\
&= \frac{1}{n(1-H_n(s)+n^{-1})^2} \longrightarrow 0
\end{aligned}$$

$\mathbb{P}$-almost surely as $n \to \infty$, since $H(s) < H(t) < 1$. Therefore we have

$$\left|\sum_{k=1}^{n} \ln(1+x_k s_k) - \sum_{k=1}^{n} x_k s_k\right| \longrightarrow 0$$

with probability 1 as $n \to \infty$. Similarly we obtain

$$\left|\sum_{k=1}^{n} \ln(1+y_k t_k) - \sum_{k=1}^{n} y_k t_k\right| \longrightarrow 0$$

with probability 1 as $n \to \infty$. Hence

$$\lim_{n\to\infty} D_n(s) = \lim_{n\to\infty} \exp\left(2\sum_{k=1}^{n} x_k s_k + \sum_{k=1}^{n} y_k t_k\right)$$

Now consider

$$
\begin{aligned}
\sum_{i=1}^{n} x_k s_k &= \frac{1}{n} \sum_{k=1}^{n} \frac{1-q(Z_k)}{1-H_n(Z_k)+\frac{2}{n}} \mathbb{I}\{Z_k < s\} \\
&= \int_0^{s-} \frac{1-q(z)}{1-H_n(z)+\frac{2}{n}} H_n(dz) \\
&\leq \int_0^{s-} \frac{1-q(z)}{1-H_n(z)} H_n(dz) \\
&= \int_0^{s-} \frac{1-q(z)}{1-H(z)} H_n(dz) + \int_0^{s-} \frac{1-q(z)}{1-H_n(z)} - \frac{1-q(z)}{1-H(z)} H_n(dz) \\
&= \int_0^{s-} \frac{1-q(z)}{1-H(z)} H_n(dz) + \int_0^{s-} \frac{(1-q(z))(H(z)-H_n(z))}{(1-H_n(z))(1-H(z))} H_n(dz) \quad (1.25)
\end{aligned}
$$

Note that the second term on the right hand side of the latter equation above tends to zero for $n \to \infty$, because

$$
\begin{aligned}
&\int_0^{s-} \frac{(1-q(z))(H(z)-H_n(z)}{(1-H_n(z))(1-H(z))} H_n(dz) \\
&\leq \frac{\sup_z |H(z)-H_n(z)|}{1-H(s)} \int_0^{s-} \frac{1}{1-H_n(z)} H_n(dz) \longrightarrow 0
\end{aligned}
$$

$\mathbb{P}$-almost surely as $n \to \infty$, by the Glivenko-Cantelli Theorem and since $H(s) < 1$. Now consider the first term in (1.25). We have

$$
\int_0^{s-} \frac{1-q(z)}{1-H(z)} H_n(dz) \longrightarrow \int_0^{s} \frac{1-q(z)}{1-H(z)} H(dz)
$$

by the SLLN. Therefore we obtain

$$
\lim_{n\to\infty} \sum_{i=1}^{n} x_k s_k = \int_0^{s} \frac{1-q(z)}{1-H(z)} H(dz)
$$

By the same arguments, we can show that

$$
\lim_{n\to\infty} \sum_{i=1}^{n} y_k t_k = \int_s^{t} \frac{1-q(z)}{1-H(z)} H(dz)
$$

Thus we finally conclude

$$\lim_{n\to\infty} D_n(s,t) = \exp\left(2\int_0^s \frac{1-q(z)}{1-H(z)}H(dz) + \int_s^t \frac{1-q(z)}{1-H(z)}H(dz)\right)$$

$\mathbb{P}$-almost surely. □

Lemma 1.9. $\{D_n, \mathcal{F}_n\}_{n\geq 1}$ *is a non-negative reverse supermartingale.*

Proof. Consider that for $s<t$ and $n \geq 1$

$$\begin{aligned}
\mathbb{E}[D_n(s,t)|\mathcal{F}_{n+1}] &= \mathbb{E}\left[\prod_{k=1}^{n}\left(1+\frac{1-q(Z_{k:n})}{n-k+2}\right)^{2\mathbb{I}\{Z_{k:n}<s\}}\right. \\
&\qquad \left.\times \prod_{k=1}^{n}\left(1+\frac{1-q(Z_{k:n})}{n-k+1}\right)^{\mathbb{I}\{s<Z_{k:n}<t\}} |\mathcal{F}_{n+1}\right] \\
&= \sum_{i=1}^{n+1}\mathbb{E}\left[\mathbb{I}\{Z_{n+1}=Z_{i:n+1}\}\dots|\mathcal{F}_{n+1}\right] \\
&= \sum_{i=1}^{n+1}\mathbb{E}\left[\mathbb{I}\{Z_{n+1}=Z_{i:n+1}\}\prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+2}\right)^{2\mathbb{I}\{Z_{k:n+1}<s\}}\right. \\
&\qquad \times \prod_{k=i}^{n}\left(1+\frac{1-q(Z_{k+1:n+1})}{n-k+2}\right)^{2\mathbb{I}\{Z_{k+1:n+1}<s\}} \\
&\qquad \times \prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+1}\right)^{\mathbb{I}\{s<Z_{k:n+1}<t\}} \\
&\qquad \left.\times \prod_{k=i}^{n}\left(1+\frac{1-q(Z_{k+1:n+1})}{n-k+1}\right)^{\mathbb{I}\{s<Z_{k+1:n+1}<t\}} |\mathcal{F}_{n+1}\right] \\
&= \sum_{i=1}^{n+1}\mathbb{E}\left[\mathbb{I}\{Z_{n+1}=Z_{i:n+1}\}\prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+2}\right)^{2\mathbb{I}\{Z_{k:n+1}<s\}}\right. \\
&\qquad \times \prod_{k=i+1}^{n+1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+3}\right)^{2\mathbb{I}\{Z_{k:n+1}<s\}} \\
&\qquad \times \prod_{k=1}^{i-1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+1}\right)^{\mathbb{I}\{s<Z_{k:n+1}<t\}} \\
&\qquad \left.\times \prod_{k=i+1}^{n+1}\left(1+\frac{1-q(Z_{k:n+1})}{n-k+2}\right)^{\mathbb{I}\{s<Z_{k:n+1}<t\}} |\mathcal{F}_{n+1}\right]
\end{aligned}$$

Now each product within the conditional expectation is measurable w. r. t. $\mathcal{F}_{n+1}$.

Moreover we have for $i = 1, \ldots, n$

$$\begin{aligned}
\mathbb{E}[\mathbb{I}\{Z_{n+1} = Z_{i:n+1}\}|\mathcal{F}_n + 1] &= \mathbb{P}(Z_{n+1} = Z_{i:n+1}|\mathcal{F}_{n+1}) \\
&= \mathbb{P}(R_{n+1,n+1} = i) \\
&= \frac{1}{n+1}
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\mathbb{E}[D_n(s,t)|\mathcal{F}_{n+1}] = \frac{1}{n+1} \sum_{i=1}^{n+1} \prod_{k=1}^{i-1} &\left(1 + \frac{1 - q(Z_{k:n+1})}{n-k+2}\right)^{2\mathbb{I}\{Z_{k:n+1} < s\}} \\
&\times \left(1 + \frac{1 - q(Z_{k:n+1})}{n-k+1}\right)^{\mathbb{I}\{s < Z_{k:n+1} < t\}} \\
\times \prod_{k=i+1}^{n+1} &\left(1 + \frac{1 - q(Z_{k:n+1})}{n-k+3}\right)^{2\mathbb{I}\{Z_{k:n+1} < s\}} \\
&\times \left(1 + \frac{1 - q(Z_{k:n+1})}{n-k+2}\right)^{\mathbb{I}\{s < Z_{k:n+1} < t\}}
\end{aligned} \tag{1.26}$$

We will now proceed by induction on n. First let

$$x_k := 1 - q(Z_{k:2}),\ s_k := \mathbb{I}\{Z_{k:2} < s\} \text{ and } t_k := \mathbb{I}\{s < Z_{k:2} < t\}$$

for $k = 1, 2$. Next consider

$$\begin{aligned}
\mathbb{E}[D_1(s,t)|\mathcal{F}_2] &= \frac{1}{2}\Bigg[\left(1 + \frac{1 - q(Z_{2:2})}{2}\right)^{2\mathbb{I}\{Z_{2:2} < s\}} \times (1 + (1 - q(Z_{2:2})))^{\mathbb{I}\{s < Z_{2:2} < t\}} \\
&\quad + \left(1 + \frac{1 - q(Z_{1:2})}{2}\right)^{2\mathbb{I}\{Z_{1:2} < s\}} \times (1 + (1 - q(Z_{1:2})))^{\mathbb{I}\{s < Z_{1:2} < t\}}\Bigg] \\
&= \frac{1}{2}\left[\left(1 + \frac{x_2}{2} s_2\right)^2 \times (1 + x_2 t_2) + \left(1 + \frac{x_1}{2} s_1\right)^2 \times (1 + x_1 t_1)\right]
\end{aligned}$$

Moreover we have

$$
\begin{aligned}
D_2(s,t) &= \left[1+\frac{x_1}{3}s_1\right]^2 \times \left[1+\frac{x_1}{2}t_1\right] \times \left[1+\frac{x_2}{2}s_2\right]^2 \times [1+x_2t_2] \\
&= \left[1+\frac{x_1}{2}t_1+\left(\frac{x_1^2}{9}+\frac{2}{3}x_1\right)s_1\right] \times \left[1+x_2t_2+\left(\frac{x_2^2}{4}+x_2\right)s_2\right]
\end{aligned}
$$

Therefore we obtain

$$
\mathbb{E}[D_1(s,t)|\mathcal{F}_2] - D_2(s,t) \le \frac{x_1^2}{72} - \frac{x_1}{6} \le 0
$$

since $0 \le x_1 \le 1$. Thus $\mathbb{E}[D_1(s,t)|\mathcal{F}_2] \le D_2(s,t)$ for any $s < t$, as needed. Now assume that

$$
\mathbb{E}[D_n(s,t)|\mathcal{F}_{n+1}] \le D_{n+1}(s,t)
$$

holds for any $n \ge 1$. Note that the latter is equivalent to assuming

$$
\begin{aligned}
&\frac{1}{n+1}\sum_{i=1}^{n+1}\prod_{k=1}^{i-1}\left(1+\frac{1-q(y_k)}{n-k+2}\right)^{2\mathbb{I}\{y_k<s\}}\left(1+\frac{1-q(y_k)}{n-k+1}\right)^{\mathbb{I}\{s<y_k<t\}} \\
&\qquad\times\prod_{k=i+1}^{n+1}\left(1+\frac{1-q(y_k)}{n-k+3}\right)^{2\mathbb{I}\{y_k<s\}}\left(1+\frac{1-q(y_k)}{n-k+2}\right)^{\mathbb{I}\{s<y_k<t\}} \\
&\le \prod_{k=1}^{n+1}\left(1+\frac{1-q(y_k)}{n-k+3}\right)^{2\mathbb{I}\{y_k<s\}}\prod_{k=1}^{n+1}\left(1+\frac{1-q(y_k)}{n-k+2}\right)^{\mathbb{I}\{s<y_k<t\}}
\end{aligned}
\tag{1.27}
$$

holds for arbitrary $y_k \ge 0$. Next define for $s < t$ and $n \ge 1$

$$
A_{n+2}(s,t) := \prod_{k=2}^{n+2}\left[1+\frac{1-q(Z_{k:n+2})}{n-k+4}\right]^{2\mathbb{I}\{Z_{k:n+2}<s\}} \times \left[1+\frac{1-q(Z_{k:n+2})}{n-k+3}\right]^{\mathbb{I}\{s<Z_{k:n+2}<t\}}
$$

Now consider that we get from (1.26)

$\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}]$

$$
\begin{aligned}
&= \frac{1}{n+2} \sum_{i=1}^{n+2} \prod_{k=1}^{i-1} \left(1 + \frac{1-q(Z_{k:n+2})}{n-k+3}\right)^{2\mathbb{I}\{Z_{k:n+2}<s\}} \left(1 + \frac{1-q(Z_{k:n+2})}{n-k+2}\right)^{\mathbb{I}\{s<Z_{k:n+2}<t\}} \\
&\qquad \times \prod_{k=i+1}^{n+2} \left(1 + \frac{1-q(Z_{k:n+2})}{n-k+4}\right)^{2\mathbb{I}\{Z_{k:n+2}<s\}} \left(1 + \frac{1-q(Z_{k:n+2})}{n-k+3}\right)^{\mathbb{I}\{s<Z_{k:n+2}<t\}} \\
&= \frac{A_{n+2}}{n+2} + \frac{1}{n+2} \sum_{i=2}^{n+2} \prod_{k=1}^{i-1} \cdots \times \prod_{k=i+1}^{n+2} \cdots \\
&= \frac{A_{n+2}}{n+2} + \frac{1}{n+2} \sum_{i=1}^{n+1} \prod_{k=1}^{i} \cdots \times \prod_{k=i+2}^{n+2} \cdots \\
&= \frac{A_{n+2}}{n+2} + \frac{1}{n+2} \left(1 + \frac{1-q(Z_{1:n+2})}{n+2}\right)^{2\mathbb{I}\{Z_{1:n+2}<s\}} \left(1 + \frac{1-q(Z_{1:n+2})}{n+1}\right)^{\mathbb{I}\{s<Z_{1:n+2}<t\}} \\
&\qquad \times \sum_{i=1}^{n+1} \prod_{k=1}^{i-1} \left(1 + \frac{1-q(Z_{k+1:n+2})}{n-k+2}\right)^{2\mathbb{I}\{Z_{k+1:n+2}<s\}} \\
&\qquad\qquad \times \left(1 + \frac{1-q(Z_{k+1:n+2})}{n-k+1}\right)^{\mathbb{I}\{s<Z_{k+1:n+2}<t\}} \\
&\qquad\qquad \times \prod_{k=i+1}^{n+1} \left(1 + \frac{1-q(Z_{k+1:n+2})}{n-k+3}\right)^{2\mathbb{I}\{Z_{k+1:n+2}<s\}} \\
&\qquad\qquad \times \left(1 + \frac{1-q(Z_{k+1:n+2})}{n-k+2}\right)^{\mathbb{I}\{s<Z_{k+1:n+2}<t\}}
\end{aligned}
$$

Using (1.27) on the right hand side of the equation above yields

$$
\begin{aligned}
&\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}] \\
&\leq \frac{A_{n+2}}{n+2} + \frac{n+1}{n+2} \left(1 + \frac{1-q(Z_{1:n+2})}{n+2}\right)^{2\mathbb{I}\{Z_{1:n+2}<s\}} \left(1 + \frac{1-q(Z_{1:n+2})}{n+1}\right)^{\mathbb{I}\{s<Z_{1:n+2}<t\}} \\
&\qquad \times \prod_{k=1}^{n+1} \left(1 + \frac{1-q(Z_{k+1:n+2})}{n-k+3}\right)^{2\mathbb{I}\{Z_{k+1:n+2}<s\}} \\
&\qquad\qquad \times \left(1 + \frac{1-q(Z_{k+1:n+2})}{n-k+2}\right)^{\mathbb{I}\{s<Z_{k+1:n+2}<t\}} \\
&= A_{n+2} \left[\frac{1}{n+2} + \frac{n+1}{n+2} \left(1 + \frac{1-q(Z_{1:n+2})}{n+2}\right)^{2\mathbb{I}\{Z_{1:n+2}<s\}} \right. \\
&\qquad\qquad \left. \times \left(1 + \frac{1-q(Z_{1:n+2})}{n+1}\right)^{\mathbb{I}\{s<Z_{1:n+2}<t\}} \right]
\end{aligned}
$$

For the moment, let

$$x_1 := 1 - q(Z_{1:n+2}),\ s_1 := \mathbb{I}\{Z_{1:n+2} < s\} \text{ and } t_1 := \mathbb{I}\{s < Z_{1:n+2} < t\}$$

Now we can rewrite the above as

$$\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}] \leq A_{n+2}\left[\frac{1}{n+2} + \frac{n+1}{n+2}\left(1 + \frac{x_1 s_1}{n+2}\right)^2\left(1 + \frac{x_1 t_1}{n+1}\right)\right] \tag{1.28}$$

Next consider

$$\begin{aligned}
\left(1 + \frac{x_1 t_1}{n+1}\right) &= \left(1 + \frac{x_1 t_1}{n+2} - \frac{1}{n+2}\right)\left(1 + \frac{1}{n+1}\right) \\
&= \left(1 + \frac{x_1 t_1}{n+2}\right) + \frac{1}{n+1}\left(1 + \frac{x_1 t_1}{(n+2)}\right) - \frac{1}{n+1} \\
&= \left(1 + \frac{x_1 t_1}{n+2}\right) + \frac{x_1 t_1}{(n+1)(n+2)}
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\frac{n+1}{n+2}\left(1 + \frac{x_1 s_1}{n+2}\right)^2\left(1 + \frac{x_1 t_1}{n+1}\right) \\
&= \frac{n+1}{n+2}\left(1 + \frac{x_1 s_1}{n+2}\right)^2\left(1 + \frac{x_1 t_1}{n+2}\right) + \left(1 + \frac{x_1 s_1}{n+2}\right)^2 \frac{x_1 t_1}{(n+2)^2}
\end{aligned}$$

But now

$$\begin{aligned}
\left(1 + \frac{x_1 s_1}{n+2}\right)^2 \frac{x_1 t_1}{(n+2)^2} &= \left(1 + 2\frac{x_1 s_1}{n+2} + \frac{x_1^2 s_1}{(n+2)^2}\right)\frac{x_1 t_1}{(n+2)^2} \\
&= \frac{x_1 t_1}{(n+2)^2}
\end{aligned}$$

since $s_1 \cdot t_1 = 0$ for all $s < t$. Hence we can rewrite the term in brackets in (1.28) as

$$\frac{1}{n+2} + \frac{n+1}{n+2}\left(1 + \frac{x_1 s_1}{n+2}\right)^2\left(1 + \frac{x_1 t_1}{n+1}\right)$$

$$
\begin{aligned}
&= \frac{1}{n+2} + \frac{x_1 t_1}{(n+2)^2} + \frac{n+1}{n+2}\left(1 + \frac{x_1 s_1}{n+2}\right)^2 \left(1 + \frac{x_1 t_1}{n+2}\right) \\
&= \frac{1}{n+2}\left(1 + \frac{x_1 t_1}{n+2}\right) + \frac{n+1}{n+2}\left(1 + \frac{x_1 s_1}{n+2}\right)^2 \left(1 + \frac{x_1 t_1}{n+2}\right) \\
&= \left[\frac{1}{n+2} + \frac{n+1}{n+2}\left(1 + \frac{x_1}{n+2}\right)^{2s_1}\right]\left(1 + \frac{x_1}{n+2}\right)^{t_1} \\
&\le \left(1 + \frac{x_1}{n+3}\right)^{2s_1}\left(1 + \frac{x_1}{n+2}\right)^{t_1}
\end{aligned}
$$

The latter inequality above holds, since

$$
\left[\frac{1}{n+2} + \frac{n+1}{n+2}\left(1 + \frac{x}{n+2}\right)^2\right] \le \left(1 + \frac{x}{n+3}\right)^2
$$

for any $0 \le x \le 1$. (**TODO** prove?) Therefore we can rewrite (1.28) as

$$
\begin{aligned}
\mathbb{E}[D_{n+1}(s,t)|\mathcal{F}_{n+2}] \le A_{n+2} &\left(1 + \frac{1 - q(Z_{1:n+2})}{n+3}\right)^{2\mathbb{I}\{Z_{1:n+2} < s\}} \\
&\times \left(1 + \frac{1 - q(Z_{1:n+2})}{n+2}\right)^{\mathbb{I}\{s < Z_{1:n+2} < t\}} \\
&= D_{n+2}(s,t)
\end{aligned}
$$

This concludes the proof. □

Lemma 1.10. *Let* $\mathcal{F}_\infty = \bigcap_{n\ge 2} \mathcal{F}_n$*. Then we have each* $A \in \mathcal{F}_\infty$ *that* $\mathbb{P}(A) \in \{0,1\}$*.*

Proof. Define

$$
\Pi_n := \{\pi | \pi \text{ is permutation of } 1, \ldots, n\}
$$

and

$$
\Pi := \bigcup_{n\in\mathbb{N}} \Pi_n
$$

We will use the Hewitt-Savage zero-one law xin order to show the statement of the lemma. Thus we need to show that for all $A \in \mathcal{F}_\infty$ and all $\pi \in \Pi$ there exists

$B \in \mathcal{B}_{\mathbb{N}}^*$ s. t.

$$A = \{\omega | (Z_i(\omega))_{i\in\mathbb{N}} \in B\} = \{\omega | (Z_{\pi(i)}(\omega))_{i\in\mathbb{N}} \in B\} \tag{1.29}$$

Let $A \in \mathcal{F}_\infty$, then $A \in \mathcal{F}_n$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ fixed but arbitrary and $A \in \mathcal{F}_n$. Then, because of measurability **TODO** wording, there must exist $\tilde{B} \in \mathcal{B}_{\mathbb{N}}^*$ such that

$$A = (Z_{1:n}, \dots, Z_{n:n}, Z_{n+1}, Z_{n+2}, \dots)^{-1}(\tilde{B}) \tag{1.30}$$

For fixed $\omega \in \Omega$ define the map

$$T : (\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{N}}^*) \ni (Z_i(\omega))_{i\in\mathbb{N}} \longrightarrow T((Z_i(\omega))_{i\in\mathbb{N}}) \in (\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{N}}^*)$$

with

$$T((Z_i(\omega))_{i\in\mathbb{N}}) := (Z_{1:n}, \dots, Z_{n:n}, Z_{n+1}, Z_{n+2}, \dots)(\omega)$$

Note that for any $\pi \in \Pi_n$ we have

$$\begin{aligned} T((Z_i(\omega))_{i\in\mathbb{N}}) &= T((Z_{\pi(i)}(\omega))_{i\in\mathbb{N}}) \\ &= (Z_{1:n}, \dots, Z_{n:n}, Z_{n+1}, Z_{n+2}, \dots)(\omega) \end{aligned} \tag{1.31}$$

Hence on the one hand, we get from (1.30)

$$\begin{aligned} A &= (T((Z_i)_{i\in\mathbb{N}}))^{-1}(\tilde{B}) \\ &= ((Z_i)_{i\in\mathbb{N}})^{-1}(T^{-1}(\tilde{B})) \\ &= \{\omega | (Z_i(\omega))_{i\in\mathbb{N}} \in B\} \end{aligned}$$

where $B = T^{-1}(\tilde{B})$. On the other hand we get from (1.31) and again by (1.30) that

$$\begin{aligned} A &= (T((Z_{\pi(i)})_{i\in\mathbb{N}}))^{-1}(\tilde{B}) \\ &= ((Z_{\pi(i)})_{i\in\mathbb{N}})^{-1}(T^{-1}(\tilde{B})) \end{aligned}$$

$$= \{\omega | (Z_{\pi(i)}(\omega))_{i \in \mathbb{N}} \in B\}$$

Now since $n \in \mathbb{N}$ was chosen arbitrarily, the above statement holds for all $n \in \mathbb{N}$ and hence for all $\pi \in \Pi$. Whence establishing (1.29). $\square$

Proof of lemma 1.6. Suppose (A1) is satisfied. Consider the following

$$\begin{aligned}
&\sum\sum_{1 \le i < j \le n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n}) W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}} \\
&= \sum\sum_{1 \le i < j \le n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n}) \frac{q^2(Z_{i:n})}{(n-i+1)^2} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right]^2 \right. \\
&\qquad \left. \times \frac{q^2(Z_{j:n})}{(n-j+1)^2} \prod_{l=1}^{j-1} \left[1 - \frac{q(Z_{l:n})}{n-l+1}\right]^2 \right]^{\frac{1}{2}} \\
&\le \sum\sum_{1 \le i < j \le n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n}) \frac{q^2(Z_{i:n})}{(n-i+1)^2} \prod_{k=1}^{i-1} \left[1 - \frac{q(Z_{k:n})}{n-k+1}\right] \right. \\
&\qquad \left. \times \frac{q^2(Z_{j:n})}{(n-j+1)^2} \prod_{l=1}^{j-1} \left[1 - \frac{q(Z_{l:n})}{n-l+1}\right] \right]^{\frac{1}{2}} \qquad (1.32)
\end{aligned}$$

Next we will modify the products above. Recall the following definition

$$B_n(s) := \prod_{k=1}^{n} \left[1 + \frac{1 - q(Z_k)}{n - R_{k,n}}\right]^{\mathbb{I}\{Z_k < s\}}$$

and note that for $i = 1, \ldots, n$

$$\begin{aligned}
B_n(Z_{i:n}) &= \prod_{k=1}^{n} \left[1 + \frac{1 - q(Z_k)}{n - R_{k,n}}\right]^{\mathbb{I}\{Z_k < Z_{i:n}\}} \\
&= \prod_{k=1}^{n} \left[1 + \frac{1 - q(Z_{k:n})}{n - k}\right]^{\mathbb{I}\{Z_{k:n} < Z_{i:n}\}} \\
&= \prod_{k=1}^{i-1} \left[1 + \frac{1 - q(Z_{k:n})}{n - k}\right]
\end{aligned}$$

Moreover consider that for $i = 1, \ldots, n$

$$\begin{aligned}
\frac{1}{n-i+1}\prod_{k=1}^{i-1}\left[1-\frac{q(Z_{k:n})}{n-k+1}\right] &= \frac{1}{n-i+1}\prod_{k=1}^{i-1}\left[\frac{n-k+1-q(Z_{k:n})}{n-k+1}\right] \\
&= \frac{1}{n-i+1}\prod_{k=1}^{i-1}\left[\frac{n-k+1-q(Z_{k:n})}{n-k}\cdot\frac{n-k}{n-k+1}\right] \\
&= \frac{1}{n}\prod_{k=1}^{i-1}\left[1+\frac{1-q(Z_{k:n})}{n-k}\right] \\
&= \frac{B_n(Z_{i:n})}{n}
\end{aligned}$$

Thus we get, according to (1.32)

$$\begin{aligned}
&\sum\sum_{1\le i<j\le n}\mathbb{E}\left[\phi^2(Z_{i:n},Z_{j:n})W_{i:n}^2W_{j:n}^2\right]^{\frac{1}{2}} \\
&\le \sum\sum_{1\le i<j\le n}\mathbb{E}\left[\phi^2(Z_{i:n},Z_{j:n})\frac{q^2(Z_{i:n})}{n(n-i+1)}\frac{q^2(Z_{j:n})}{n(n-j+1)}B_n(Z_{i:n})B_n(Z_{j:n})\right]^{\frac{1}{2}} \\
&\le \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{(n-i+1)^{\frac{1}{2}}(n-j+1)^{\frac{1}{2}}} \\
&\qquad\times\mathbb{E}\left[\phi^2(Z_{i:n},Z_{j:n})q^2(Z_{i:n})q^2(Z_{j:n})B_n(Z_{i:n})B_n(Z_{j:n})\right]^{\frac{1}{2}} \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{(n-R_{i,n}+1)^{\frac{1}{2}}(n-R_{j,n}+1)^{\frac{1}{2}}} \\
&\qquad\times\mathbb{E}\left[\phi^2(Z_i,Z_j)q^2(Z_i)q^2(Z_j)B_n(Z_i)B_n(Z_j)\right]^{\frac{1}{2}}
\end{aligned}$$

Note that for $1 \le i, j \le n$

$$\begin{aligned}
&\mathbb{E}\left[\phi^2(Z_i,Z_j)q^2(Z_i)q^2(Z_j)B_n(Z_i)B_n(Z_j)\right] \\
&= \int_0^\infty\int_0^\infty \phi^2(s,t)q^2(s)q^2(t)B_n(s)B_n(t)H(ds)H(dt) \\
&= \mathbb{E}\left[\phi^2(Z_1,Z_2)q^2(Z_1)q^2(Z_2)B_n(Z_1)B_n(Z_2)\right]
\end{aligned}$$

since $Z_i \sim H$ for $i = 1, \ldots, n$. Hence we get

$$\begin{aligned}
&\sum\sum_{1\le i<j\le n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n}) W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}} \\
&\le \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{(n-R_{i,n}+1)^{\frac{1}{2}}(n-R_{j,n}+1)^{\frac{1}{2}}} \\
&\qquad \times \mathbb{E}\left[\phi^2(Z_1, Z_2) q^2(Z_1) q^2(Z_2) B_n(Z_1) B_n(Z_2)\right]^{\frac{1}{2}}
\end{aligned}$$

Next consider that we have

$$\begin{aligned}
\sum_{j=1}^{n} \frac{1}{(n-R_{j,n}+1)^{\frac{1}{2}}} &= \sum_{j=1}^{n} \frac{1}{j^{\frac{1}{2}}} \\
&= 1 + \sum_{j=2}^{n} \int_{j-1}^{j} \frac{1}{\sqrt{j}} dx \\
&\le 1 + \sum_{j=2}^{n} \int_{j-1}^{j} \frac{1}{\sqrt{x}} dx \\
&= 1 + 2\sum_{j=2}^{n} (\sqrt{j} - \sqrt{j-1}) \\
&= 2\sqrt{n} - 1 \\
&\le 2\sqrt{n}
\end{aligned}$$

for all $n \ge 1$. We therefore obtain

$$\begin{aligned}
&\sum\sum_{1\le i<j\le n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n}) W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}} \\
&\le 4 \cdot \mathbb{E}\left[\phi^2(Z_1, Z_2) q^2(Z_1) q^2(Z_2) B_n(Z_1) B_n(Z_2)\right]^{\frac{1}{2}} \qquad (1.33)
\end{aligned}$$

Since q and ϕ are Borel-measurable, we can apply Lemma 1.7 to obtain

$$\sum\sum_{1\le i<j\le n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n}) W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}}$$

$$\leq 8 \cdot \mathbb{E}\left[\phi^2(Z_1, Z_2)q^2(Z_1)q^2(Z_2)(\Delta_{n-2}(Z_1, Z_2) + \bar{\Delta}_{n-2}(Z_1, Z_2))\right]^{\frac{1}{2}}$$

Note that $0 \leq C_n(s) \leq 1$ for all $n \geq 1$ and $s \in \mathbb{R}_+$. Thus

$$\bar{\Delta}_n(s,t) = \mathbb{E}[C_n(s)D_n(s,t)] \leq \Delta_n(s,t)$$

for all $n \geq 1$ and $s < t$. Therefore we get

$$\begin{aligned}
&\sum\sum_{1\leq i<j\leq n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n})W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}} \\
&\leq 16 \cdot \mathbb{E}\left[\phi^2(Z_1, Z_2)q^2(Z_1)q^2(Z_2)\Delta_{n-2}(Z_1, Z_2)\right]^{\frac{1}{2}}
\end{aligned}$$

According to Lemma 1.8, $D_n(s,t) \to D(s,t)$ $\mathbb{P}$-almost surely. Moreover we get from Lemma 1.9, that $\{D_n, \mathcal{F}_n\}_{n\geq 1}$ is a reverse supermartingale. Now this together with Proposition 5-3-11 of Neveu (1975) and Lemma 1.10 yields

$$\Delta_n(s,t) = \mathbb{E}[D_n(s,t)] = \mathbb{E}[D_n(s,t)|\mathcal{F}_\infty] \nearrow D(s,t)$$

But this implies in particular that $\mathbb{E}[D_n(s,t)] \leq D(s,t)$ for all $n \geq 1$. Hence

$$\begin{aligned}
&\sum\sum_{1\leq i<j\leq n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n})W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}} \\
&\leq 16 \cdot \mathbb{E}\left[\phi^2(Z_1, Z_2)q^2(Z_1)q^2(Z_2)D(Z_1, Z_2)\right]^{\frac{1}{2}}
\end{aligned}$$

Next consider that for $s < t$ s. t. $H(t) < 1$

$$\begin{aligned}
D(s,t) &= \exp\left(2\int_0^s \frac{1-q(z)}{1-H(z)}H(dz) + \int_s^t \frac{1-q(z)}{1-H(z)}H(dz)\right) \\
&\leq \exp\left(2\int_0^s \frac{1}{1-H(z)}H(dz) + \int_s^t \frac{1}{1-H(z)}H(dz)\right) \\
&= \exp\left(\int_0^s \frac{1}{1-H(z)}H(dz) + \int_0^t \frac{1}{1-H(z)}H(dz)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \exp\left(-\ln(1-H(s)) - \ln(1-H(t))\right) \\
&= \frac{1}{(1-H(s))(1-H(t))} < \infty
\end{aligned}
$$

since $H(s) < H(t) < 1$. Thus there must exist $c < \infty$ s. t. $D(s,t) \leq c$ for all $s < t$ and therefore

$$
\begin{aligned}
&\sum\sum_{1\leq i<j\leq n} \mathbb{E}\left[\phi^2(Z_{i:n}, Z_{j:n}) W_{i:n}^2 W_{j:n}^2\right]^{\frac{1}{2}} \\
&\leq 16c \cdot \mathbb{E}\left[\phi^2(Z_1, Z_2)\right]^{\frac{1}{2}}
\end{aligned}
$$

since $0 \leq q(s) \leq 1$ for all $s \in \mathbb{R}_+$. But now, under assumption (A1), the expectation above is finite. $\square$

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