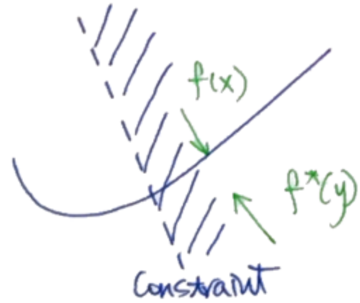
291A Convex Optimization 10/17/2017 P1

- Conjugate Function

  Given $f: R^n \to R$, $f^*: R^n \to R$

  $$f^*(y) = \sup_{x \in \text{dom} f} y^T x - f(x)$$

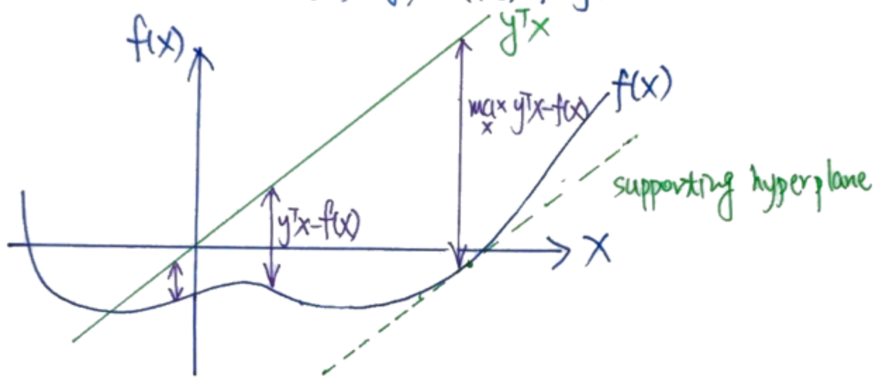
  * constraint: $f^*(y)$ has to be bounded (finite)



- $f^*(y)$ is always convex $\Rightarrow$ pointwise maximum

  <proof>

  $$f^*(\theta y_1 + (1-\theta) y_2) = \sup_x \{[\theta y_1 + (1-\theta) y_2]^T x - f(x)\}$$
  $$\leq \sup_x \{\theta y_1^T x - \theta f(x)\} + \sup_x \{(1-\theta) y_2^T x - (1-\theta) f(x)\}$$
  $$= \theta \cdot f^*(y_1) + (1-\theta) \cdot f^*(y_2)$$

- 

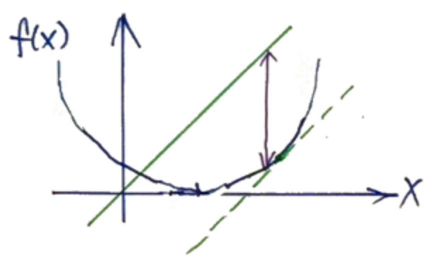
  * Given y, the supporting hyperplane of epi f can be expressed as $y^T x - f^*(y)$.
  By definition, $f(x) \geq y^T x - f^*(y)$

- Example 1:

  Given $f(x) = (x-1)^2$, find $f^*(y)\big|_{y=1}$

  $\Rightarrow f'(x) = 2x - 2 = 1 \Rightarrow x = \frac{3}{2}$

  $\Rightarrow f^*(y)\big|_{y=1} = 1 \cdot \frac{3}{2} - \left(\frac{3}{2} - 1\right)^2 = \frac{3}{2} - \frac{1}{4} = \frac{5}{4}$



- Example 2:

  Given $f(x) = (x-1)^2$, find $f^*(y)\big|_{y=2}$

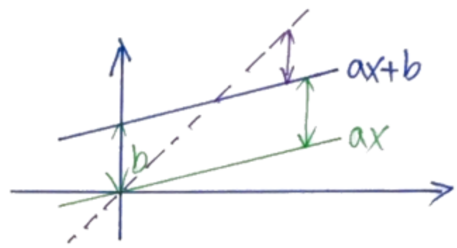
  $\Rightarrow f'(x) = 2x - 2 = 2 \Rightarrow x = 2$

  $\Rightarrow f^*(y)\big|_{y=2} = 2 \cdot 2 - (2-1)^2 = 4 - 1 = 3$

∘ Example:

Let $f(x) = ax+b$, $x \in R$

$$f^*(y) = \sup_x yx - f(x)$$
$$= \sup_x yx - ax - b$$
$$= \sup_x (y-a)x - b$$



If $y>a$, $\sup \to \infty$
$y<a$, $\sup \to \infty$

Thus, $y=a$, $f^*(y) = -b$

We can conclude that $y=a$, $f^*(y)=b$ supporting hyperplane $yx - f^*(y) = ax+b$

∘ Example:

$f(x) = -\log x$, $x \in R^n_{++}$

$f^*(y) = \sup_x yx - f(x) = \sup_x yx + \log x$

(1) If $y \geq 0$, $f^*(y) \to \infty$

(2) If $y<0$, let $g(x) = yx + \log x$, $g'(x) = y + \frac{1}{x} = 0 \Rightarrow x = -\frac{1}{y}$

Thus, $f^*(y) = y \cdot (-\frac{1}{y}) + \log(-\frac{1}{y}) = -1 - \log(-y)$, when $\text{dom } f^* = -R_{++}$

∘ Example:

$f(x) = e^x$, $x \in R$

$f^*(y) = \sup_x yx - e^x$

(1) If $y<0$, $f^*(y) \to \infty$

(2) If $y>0$, let $g(x) = yx - e^x$, $g'(x) = y - e^x = 0 \Rightarrow x = \log y$

(3) If $y=0$, $f^*(y) = 0$

Thus, $f^*(y) = y \log y - y$, when $\text{dom } f^* = R_+$

∘ Example:

$f(x) = x \log x$, $x \in R_+$

$f^*(y) = \sup_x yx - x\log x$,

let $g(x) = yx - x\log x$, $g'(x) = y - \log x - 1 = 0 \Rightarrow x = e^{y-1}$

Thus, $f^*(y) = e^{y-1}$, when $\text{dom } f^* = R_+$

- Example:

$$f(x) = \frac{1}{2} x^T Q x, \quad x \in R^n, \quad Q \in S^n_{++}$$

$$f^*(y) = \sup_x \, y^T x - \frac{1}{2} x^T Q x$$

let $g(x) = y^T x - \frac{1}{2} x^T Q x$

$$g'(x) = y - Qx = 0, \quad x = Q^{-1} y$$

Thus, $f^*(y) = y^T Q^{-1} y - \frac{1}{2} (Q^{-1} y)^T Q (Q^{-1} y) = \frac{1}{2} y^T Q^{-1} y$

- Conjugate's conjugate is the same as original one, if original function is $\left[\begin{array}{l} \text{convex} \\ \text{closed} \end{array} \right.$

- Conjugate Function:

$$f^*(y) = \sup_x \, y^T x - f(x)$$

① $f(x) + f^*(y) \geq y^T x$

② $f^{**} = f$ if f is convex and closed

③ If f is convex and differentiable, dom $f = R^n$,

$$\max_x \, y^T x - f(x) \Rightarrow y = \nabla f(x)$$

$\Rightarrow \nabla f(x)^T x - f(x)$ is the supporting hyperplane