Investment Hw5

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PROBLEM 1

SUBPROBLEM (i)

To solve this problem, we need to first find out the value of zero-coupon bonds. We know

$$\begin{aligned} 1050 \times B(0,1) &= 945 \\ 40 \times B(0,1) + 1040 \times B(0,2) &= 868 \\ 50 \times B(0,1) + 50 \times B(0,2) + 1050 \times B(0,3) &= 715 \end{aligned}$$

After simplification, we can obtain

$$\begin{aligned} B(0,1) &= 0.9 \\ B(0,2) &= 0.8 \\ B(0,3) &= 0.6 \end{aligned}$$

To fulfill lending 500000 at date 2, and gain repaid at date3, we need to solve

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1050 & 40 & 50 \\ 0 & 1040 & 50 \\ 0 & 0 & 1050 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -500000 \\ 500000 \times \frac{B(0,2)}{B(0,3)} \end{bmatrix}$$

After matrix multification, we can get

$$\begin{aligned} a_1 &= -\frac{185000}{17199} \\ a_2 &= -\frac{837500}{1638} \\ a_3 &= -\frac{40000}{63} \end{aligned}$$

Subproblem (ii)

This can be rewritten into an simple optimization problem that

$$\max \log(c_2) + \log(c_3)$$

where

$$c_2 B(0,2) + c_3 B(0,3) = 100000$$

It is easy to know that the optimal solution happens when $c_2 B(0,2) + c_3 B(0,3)$. Then we know

$$\begin{aligned} c_2 &= 62500 \\ c_3 &= \frac{250000}{3} \end{aligned}$$

Subproblem (iii)

What we have to do first is to construct the asset 4 by other three assets in complete market. Since our main purpose is to just find out $d_4 = 1$, we can gain d_1, d_2, d_3 by solving

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 1050 & 40 & 50 \\ 0 & 1040 & 50 \\ 0 & 0 & 1050 \end{bmatrix}^{-1} \begin{bmatrix} -30 \\ -30 \\ -1030 \end{bmatrix}$$

Then, we get

$$\begin{aligned} d_1 &= \frac{100}{5733} \\ d_2 &= \frac{5}{273} \\ d_3 &= -\frac{103}{105} \end{aligned}$$

Problem 2

We need to find out the maximum possible portfolio ($\underline{P}$) that creates date t cash flow smaller than C_t; also the minimum portfolio ($\overline{P}$) that creates date t cash flow larger than C_t.
We solve the former part with optimization problem as follows.

$$\max_{a,b} 669 w_a + 715 w_b$$

where

$$\begin{aligned} 30 w_a + 50 w_b &\le 200 \\ 30 w_a + 50 w_b &\le 300 \\ 1030 w_a + 1050 w_b &\le 100 \end{aligned}$$

After simplification, we can obtain

$$\begin{aligned} w_a^* &= -\frac{205}{20} \\ w_b^* &= \frac{203}{20} \end{aligned}$$

Then, $\underline{P} = 669(-\frac{205}{20}) + 715 * \frac{203}{20} = 400$

We then solve the latter part with similar procedure

$$\min_{a,b} 669w_a + 715w_b$$

where

$$\begin{aligned} 30w_a + 50w_b &\geq 200 \\ 30w_a + 50w_b &\geq 300 \\ 1030w_a + 1050w_b &\geq 100 \end{aligned}$$

After simplification again, we can obtain

$$\begin{aligned} w_a^* &= -\frac{155}{10} \\ w_b^* &= \frac{153}{10} \end{aligned}$$

Then, $\overline{P} = 669(-\frac{155}{10}) + 715 * \frac{153}{10} = 570$

Problem 3

Subproblem (i)

With the information provided in question, we know that one with these three assets will either execute the European call option or European put option on date 2. And the cash flow generated are $K = 28$. Buying one unit asset for p_1, c, P respectively at date 0 will generate 28 cash flow in date 2. In other words, buying $20 + \frac{25}{8} + \frac{45}{24}$ copies of $B(0,2)$ can generate 28 cash flow. Then, $B(0,2) = \frac{75}{112}$.

Subproblem (ii)

Simliar to what we have done in last homework, by solving equations

$$\begin{aligned} 20 = p_1(0) &= \frac{24 \times \pi^*(E) + 24 \times (1 - \pi^*(E))}{1 + r_f(0)} \\ \frac{25}{8} = p_1(0) &= \frac{3 \times \pi^*(E) + \frac{9}{2} \times (1 - \pi^*(E))}{1 + r_f(0)} \\ \frac{45}{24} = p_1(0) &= \frac{3 \times \pi^*(E) + \frac{3}{2} \times (1 - \pi^*(E))}{1 + r_f(0)} \end{aligned}$$

we can get $r_f(0) = 0.2$, and $\pi^* = 0.5$

After that, we can derive $B(0) = \frac{z(w_1) \times 0.5 + z(w_2) \times 0.5}{1+0.2} = \frac{5}{6}$

Subproblem (iii)

To consume 1120000 on date 2, one needs to spend $1120000 \times \frac{B(0,2)}{B(0,1)} = 900000$ on date 1. In this way, we know that there will 100000 left in date 1, which can be equally seen as $\frac{100000}{1+0.2} = \frac{500000}{6}$ on date 0.

Subproblem (iv)

It is easy to find out that $1 + y = \frac{B(0,1)}{B(0,2)} = \frac{122}{90}$. Thus, $y = \frac{20}{90}$.

Subproblem (v)

Since the riskless portfolio created by combination of three assets should share exactly the same rate of return on date t to date $t+1$, we can derive

$$\frac{1}{1+r_f(1,E)} = \frac{1}{K}(24-3+3) = \frac{6}{7}$$
$$\frac{1}{1+r_f(1,E^c)} = \frac{1}{K}(24-\frac{9}{2}+\frac{3}{2}) = \frac{3}{4}$$

and

$$r_f(1,E) = \frac{1}{6}$$
$$r_f(1,E^c) = \frac{1}{3}$$

Then, we know

$$z_1 = 1000000 \times (1+0.2) \times (1+\frac{1}{6}) = 1400000$$
$$z_4 = 1000000 \times (1+0.2) \times (1+\frac{1}{3}) = 1600000$$

Subproblem (vi)

It is also similar to what we have done in last homework. By the definition we get

$$G(1,E) = \frac{24}{\frac{6}{7}} = 28$$
$$G(1,E^c) = \frac{24}{\frac{3}{4}} = 32$$

and also $G(0) = G(1,E) + G(1,E^c) = 60$.

Problem 4

From the statement we know that

$$G(1,E) = \frac{20}{B(1,E)} = H(1,E) = 25$$
$$G(1,E^c) = \frac{18}{B(1,E)} = H(1,E^c) = 20$$

then we can get

$$B(1,E) = 0.8$$
$$B(1,E^c) = 0.9$$

We also know that

$$H(0,\Omega) = \frac{45}{2} = 25 \times \pi^*(E) + 20 \times \left(1-\pi^*(E)\right)$$

and thus, $\pi^*(E) = 0.5$.
After finding out the martingale probability, we can now find out $r_f(0)$ by concerning asset 1

$$19 = \frac{20 \times 0.5 + 18 \times 0.5}{1+r_f(0)}$$

then we get $r_f = 0$.
Then, we can derive $B(0,\Omega) = \frac{0.8\times0.5+0.9\times0.5}{1+0} = 0.85$
Finally, we can calculate the answer with $\frac{19\times17000}{0.85} = 380000$.