Investment Hw4

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Problem 1

Subproblem (i)

From the textbook, we know that

$$p_z(1,E) = \frac{\pi^*(w_1|E) \times p_z(2,w_1) + \pi^*(w_2|E) \times p_z(2,w_2)}{1+r_f(1,E)}$$

Since the price of asset 2 at date 2 on status w_i is $p_2(1,E) \times (1+r_f(1,E))$, we can derive

$$\begin{aligned} p_z^*(1,E) = \frac{p_z(1,E)}{\frac{p_2(1,E)}{1+r_f(1,E)}} &= \pi^*(w_1|E) \times \frac{p_z(2,w_1)}{p_2(2,w_1)} + \pi^*(w_2|E) \times \frac{p_z(2,w_2)}{p_2(2,w_2)} \\ &= \pi^*(w_1|E) \times p_z^*(2,w_1) + \pi^*(w_2|E) \times p_z^*(2,w_2) \end{aligned}$$

Similarly, we can obtain

$$p_z^*(1,E^c) = \pi^*(w_3|E^c) \times p_z^*(2,w_3) + \pi^*(w_4|E^c) \times p_z^*(2,w_4)$$

and

$$p_z^*(0,\Omega) = \pi^*(E) \times p_z^*(1,E) + \pi^*(E^c) \times p_z^*(1,E^c)$$

Subproblem (ii)

Simply write down all the figures, and list the equations.

$$\begin{aligned} \hat{p}_1(1,E) &= \frac{1.1}{\frac{5}{6}} = \frac{1}{1} * \frac{Q(w_1)}{Q(w_1)+Q(w_2)} + \frac{1.48}{1} * \frac{Q(w_2)}{Q(w_1)+Q(w_2)} \\ \hat{p}_2(1,E) &= \frac{1.1}{\frac{5}{6}} = \frac{1.32}{1} * \frac{Q(w_1)}{Q(w_1)+Q(w_2)} + \frac{1.32}{1} * \frac{Q(w_2)}{Q(w_1)+Q(w_2)} \\ \hat{p}_1(1,E^c) &= \frac{2.2}{1} = \frac{3.3}{1} * \frac{Q(w_3)}{Q(w_3)+Q(w_4)} + \frac{1.1}{1} * \frac{Q(w_4)}{Q(w_3)+Q(w_4)} \\ \hat{p}_2(1,E^c) &= \frac{1.1}{1} = \frac{1.1}{1} * \frac{Q(w_3)}{Q(w_3)+Q(w_4)} + \frac{1.1}{1} * \frac{Q(w_4)}{Q(w_3)+Q(w_4)} \\ \hat{p}_1(0) &= \frac{\frac{7}{4}}{\frac{115}{132}} = \hat{p}_1(1,E) \times (Q(w_1)+Q(w_2)) + \hat{p}_1(1,E^c) \times (Q(w_3)+Q(w_4)) \\ \hat{p}_2(0) &= \frac{1}{\frac{115}{132}} = \hat{p}_2(1,E) \times (Q(w_1)+Q(w_2)) + \hat{p}_2(2,E^c) \times (Q(w_3)+Q(w_4)) \end{aligned}$$

After simplification, we can get:

$$
\begin{aligned}
Q(w_1) &= \frac{5}{23} \times \frac{1}{3} \\
Q(w_2) &= \frac{5}{23} \times \frac{2}{3} \\
Q(w_3) &= \frac{18}{23} \times \frac{1}{2} \\
Q(w_4) &= \frac{18}{23} \times \frac{1}{2}
\end{aligned}
$$

Subproblem (iii)

Based on the calculation in course, we know that

$$
\begin{aligned}
\hat{p_1}(0) = G(0) &= \frac{p_1(0)}{B(0)} \\
\hat{p_1}(1,E) = G(1,E) &= \frac{p_1(1,E)}{B(1,E)} \\
\hat{p_1}(1,E^c) = G(1,E^c) &= \frac{p_1(1,E^c)}{B(1,E^c)}
\end{aligned}
$$

Subproblem (iv)

Very similar to the subproblem ii, what we need to do is to write down all the equations. Simply write down all the figures, and list the equations.

$$
\begin{aligned}
\tilde{p_1}(1,E) &= \frac{1.1}{1.1} = \frac{1}{1} * \frac{R(w_1)}{R(w_1)+R(w_2)} + \frac{1.48}{1.48} * \frac{R(w_2)}{R(w_1)+R(w_2)} \\
\tilde{p_2}(1,E) &= \frac{1.1}{1.1} = \frac{1.32}{1} * \frac{R(w_1)}{R(w_1)+R(w_2)} + \frac{1.32}{1.48} * \frac{R(w_2)}{R(w_1)+R(w_2)} \\
\tilde{p_1}(1,E^c) &= \frac{2.2}{2.2} = \frac{3.3}{3.3} * \frac{R(w_3)}{R(w_3)+R(w_4)} + \frac{1.1}{1.1} * \frac{R(w_4)}{R(w_3)+R(w_4)} \\
\tilde{p_2}(1,E^c) &= \frac{1.1}{2.2} = \frac{1.1}{3.3} * \frac{R(w_3)}{R(w_3)+R(w_4)} + \frac{1.1}{1.1} * \frac{R(w_4)}{R(w_3)+R(w_4)} \\
\tilde{p_1}(0) &= \frac{\frac{7}{4}}{\frac{115}{132}} = \tilde{p_1}(1,E) \times (R(w_1)+R(w_2)) + \tilde{p_1}(1,E^c) \times (R(w_3)+R(w_4)) \\
\tilde{p_2}(0) &= \frac{1}{\frac{115}{132}} = \tilde{p_2}(1,E) \times (R(w_1)+R(w_2)) + \tilde{p_2}(2,E^c) \times (R(w_3)+R(w_4))
\end{aligned}
$$

After simplification, we can get:

$$
\begin{aligned}
R(w_1) &= \frac{1}{7} \times \frac{25}{99} \\
R(w_2) &= \frac{1}{7} \times \frac{74}{99} \\
R(w_3) &= \frac{6}{7} \times \frac{3}{4} \\
R(w_4) &= \frac{6}{7} \times \frac{1}{4}
\end{aligned}
$$

Problem 2

We skip the repeated procedure of finding the martingale probability and r_f, then we can obtain

$$p_z(1,E) = \frac{215028 * \frac{1}{3} + 107514 * \frac{2}{3}}{1+0.2} = 119460$$
$$p_z(1,E^c) = \frac{39820 * \frac{1}{2} + 39820 * \frac{1}{2}}{1+0.0} = 39820$$
$$p_z(0,\Omega) = \frac{119460 * \frac{1}{4} + 39820 * \frac{3}{4}}{1+0.1} = 54300$$

It is obvious that we can generate riskless profit $55 \times 1000 - 54300 = 700$ by shorting that asset.

Problem 3

We can imagine firm A as the debt issuer, and firm B as the debt receiver. Thus, date 2 price of bonds are as follows:

$$A = \begin{bmatrix} \min(250000, 160000) \\ \min(180000, 160000) \\ \min(140000, 160000) \\ \min(100000, 160000) \end{bmatrix} = \begin{bmatrix} 160000 \\ 160000 \\ 140000 \\ 100000 \end{bmatrix}$$

$$B = \begin{bmatrix} \max(160000 - 120000, 0) \\ \max(160000 - 120000, 0) \\ \max(140000 - 120000, 0) \\ \max(100000 - 120000, 0) \end{bmatrix} = \begin{bmatrix} 40000 \\ 40000 \\ 20000 \\ 0 \end{bmatrix}$$

Then, with similar procedure as problem 3, we can obtain date 0 price of firm B bond

$$p_z(1,E) = \frac{40000 * \frac{1}{3} + 40000 * \frac{2}{3}}{1+0.2} = \frac{40000}{1.2}$$
$$p_z(1,E^c) = \frac{20000 * \frac{1}{2} + 0 * \frac{1}{2}}{1+0.0} = 10000$$
$$p_z(0,\Omega) = \frac{\frac{40000}{1.2} * \frac{1}{4} + 10000 * \frac{3}{4}}{1+0.1} \approx 14394$$

Thus, the expected return rate can be calculated as $\frac{\frac{1}{4}(40000+40000+20000+0)}{14394} - 1 \approx 0.736842$

Problem 4

Subproblem (i)

We can derive the result from previous calculated data

$$G(0) = \frac{p_1(0)}{B(0)} = \frac{\frac{7}{4}}{\frac{115}{132}} = \frac{231}{115}$$
$$G(1,E) = \frac{p_1(1,E)}{B(1,E)} = \frac{1.1}{\frac{5}{6}} = 1.32$$
$$G(1,E^c) = \frac{p_1(1,E^c)}{B(1,E^c)} = \frac{2.2}{1} = 2.2$$

Subproblem (ii)

With the European call option on asset 1 on date 2, we know

$$\begin{bmatrix} \max(p_1(w_1)-2.2,0) \\ \max(p_1(w_2)-2.2,0) \\ \max(p_1(w_3)-2.2,0) \\ \max(p_1(w_4)-2.2,0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.1 \\ 0 \end{bmatrix}$$

After calculation, we get

$$\begin{aligned} c(1,E) &= \frac{0\times\frac{1}{3}+0\times\frac{2}{3}}{1+0.2} = 0 \\ c(1,E^c) &= \frac{1.1\times\frac{1}{2}+0\times\frac{1}{2}}{1+0.0} = 0.55 \\ c(0) &= \frac{0\times\frac{1}{4}+0.55\times\frac{3}{4}}{1+0.1} = 0.375 \end{aligned}$$

Subproblem (iii)

With the European put option on asset 1 at date 2, we know

$$\begin{bmatrix} \max(2.2-p_1(w_1),0) \\ \max(2.2-p_1(w_2),0) \\ \max(2.2-p_1(w_3),0) \\ \max(2.2-p_1(w_4),0) \end{bmatrix} = \begin{bmatrix} 1.2 \\ 0.72 \\ 0 \\ 1.1 \end{bmatrix}$$

After calculation, we get

$$\begin{aligned} P(1,E) &= \frac{1.2\times\frac{1}{3}+0.72\times\frac{2}{3}}{1+0.2} = \frac{11}{15} \\ P(1,E^c) &= \frac{0\times\frac{1}{2}+1.1\times\frac{1}{2}}{1+0.0} = 0.55 \\ P(0) &= \frac{\frac{11}{15}\times\frac{1}{4}+0.55\times\frac{3}{4}}{1+0.1} = \frac{13}{24} \end{aligned}$$

Subproblem (iv)

It is easy to find out that it is bad to execute the American call at date 1,

$$\begin{aligned} C(1,E) &= \max(p_1(1,E)-2.2,0) = 0 \le c(1,E) = 0 \\ C(1,E^c) &= \max(p_1(1,E^c)-2.2,0) = 0 \le c(1,E^c) = 0.55 \end{aligned}$$

It is also easy to find out that it is not worthwhile to execute it at date 0

$$C(0) = \max(p_1(0)-2.2,0) = 0 \le c(0) = 0.375$$

Subproblem (v)

With the European call at date 1, we can derive

$$\begin{bmatrix} \max(c(1,E)-\frac{1}{2},0) \\ \max(c(1,E^c)-\frac{1}{2},0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0.05 \end{bmatrix}$$

After calculation, we can get

$$\frac{0\times\frac{1}{4}+0.05\times\frac{3}{4}}{1+0.1} = \frac{3}{88}$$

Subproblem (vi)

With the lookback option at date 2, we can derive

$$\begin{bmatrix} \max(\max(p_1(0),p_1(1,E),p_1(2,w_1))-1.4,0) \\ \max(\max(p_1(0),p_1(1,E),p_1(2,w_2))-1.4,0) \\ \max(\max(p_1(0),p_1(1,E^c),p_1(2,w_3))-1.4,0) \\ \max(\max(p_1(0),p_1(1,E^c),p_1(2,w_4))-1.4,0) \end{bmatrix} = \begin{bmatrix} 0.35 \\ 0.35 \\ 1.9 \\ 0.8 \end{bmatrix}$$

After calculation, we can get

$$\begin{aligned} l(1,E) &= \frac{0.35\times\frac{1}{3}+0.35\times\frac{2}{3}}{1+0.2} = \frac{7}{24} \\ l(1,E^c) &= \frac{1.9\times\frac{1}{2}+0.8\times\frac{1}{2}}{1+0.0} = 1.35 \\ l(0) &= \frac{\frac{7}{24}\times\frac{1}{4}+1.35\times\frac{3}{4}}{1+0.1} = \frac{521}{528} \end{aligned}$$

Subproblem (vii)

With the lookback option at date 2, we can derive

$$\begin{bmatrix} \max(p_1(2,w_1)-\frac{1}{6}p_1(0)-\frac{1}{3}p_1(1,E)-\frac{1}{2}p_1(2,w_1)) \\ \max(p_1(2,w_2)-\frac{1}{6}p_1(0)-\frac{1}{3}p_1(1,E)-\frac{1}{2}p_1(2,w_2)) \\ \max(p_1(2,w_3)-\frac{1}{6}p_1(0)-\frac{1}{3}p_1(1,E^c)-\frac{1}{2}p_1(2,w_3)) \\ \max(p_1(2,w_4)-\frac{1}{6}p_1(0)-\frac{1}{3}p_1(1,E^c)-\frac{1}{2}p_1(2,w_4)) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{49}{600} \\ \frac{5}{8} \\ 0 \end{bmatrix}$$

After calculation, we can get

$$\begin{aligned} A(1,E) &= \frac{0\times\frac{1}{3}+\frac{49}{600}\times\frac{2}{3}}{1+0.2} = \frac{49}{1080} \\ A(1,E^c) &= \frac{\frac{5}{8}\times\frac{1}{2}+0\times\frac{1}{2}}{1+0.0} = \frac{5}{16} \\ A(0) &= \frac{\frac{49}{1080}\times\frac{1}{4}+\frac{5}{16}\times\frac{3}{4}}{1+0.1} \approx 0.223380 \end{aligned}$$

Subproblem (viii)

It is easy to find out that it is good to execute American put option at date 1 for asset 1, but not for asset 2.

$$P^*(1,E) = \max(2.2 - p_1(1,E), 0) = 1.1 \geq P(1,E) = \frac{11}{15}$$
$$P^*(1,E^c) = \max(2.2 - p_1(1,E^c), 0) = 0 \leq P(1,E^c) = 0.55$$

Then, we can find out that it is bad to execute it at date 0.

$$P^*(0) = \max(2.2 - p_1(0), 0) = 0.45 \leq \frac{1.1 \times \frac{1}{4} + 0.55 \times \frac{3}{4}}{1 + 0.1} = 0.525$$

Problem 5

Subproblem (i)

First we can observe that asset 1 and asset2 at any status are linearly with one another. This property fulfills the definition of dynamically complete.
Furthermore, we can tell that the figure of this subproblem is just the same as our previous problems. Thus, the martingale probabilities and interest rate is just the same as previous ones.

Subproblem (ii)

Write down all the equations as follows.

$$\begin{aligned}
0 &= 1 \times \theta_1(2,E) + 1.32 \times \theta_2(2,E) \\
0 &= 1.48 \times \theta_1(2,E) + 1.32 \times \theta_2(2,E) \\
1.1 &= 3.3 \times \theta_1(2,E^c) + 1.1 \times \theta_2(2,E^c) \\
0 &= 1.1 \times \theta_1(2,E^c) + 1.1 \times \theta_2(2,E^c)
\end{aligned}$$

We also know that

$$p_1(1,E) \times \theta_1(2,E) + p_2(1,E) \times \theta_2(2,E) = \big(p_1(1,E) + x_1(1,E)\big) \times \theta_1(1) + \big(p_2(1,E) + x_2(1,E)\big) \times \theta_2(1)$$
$$p_1(1,E^c) \times \theta_1(2,E^c) + p_2(1,E^c) \times \theta_2(2,E^c) = \big(p_1(1,E^c) + x_1(1,E^c)\big) \times \theta_1(1) + \big(p_2(1,E^c) + x_2(1,E^c)\big) \times \theta_2(1)$$

After solving the equations, we can get

$$\begin{aligned}
\theta_1(2,E) &= \frac{3}{4} \\
\theta_2(2,E) &= -\frac{9}{8} \\
\theta_1(2,E^c) &= 0 \\
\theta_2(2,E^c) &= 0 \\
\theta_1(1) &= \frac{1}{2} \\
\theta_2(1) &= -\frac{1}{2}
\end{aligned}$$

Subproblem (iii)

First, we can observe when on status E that executing American call is the same as European call, since $p_1(1,E) = 1.1 < 2.2$ and $p_1(1,E) + x_1(1,E) = \frac{3.3}{2} < 2.2$.
Similarly, we know that executing it on status E^c is still not a good idea. Though $p_1(1,E^c) = 2.2 = 2.2$ and $p_1(1,E^c) + x_1(1,E^c) = \frac{14.3}{6} > 2.2$, remain the option can actually get $\frac{1.1}{2}$.
Finally, when we look into date 0, we find out that this American call is just the same as European call, because $p_1(0) = 2 < 2.2$. Thus, the detailed calculation is just the same as those in subproblem ii.

Subproblem (iv)

Very similar to the subproblem iii, however, the decision on status E^c will be changed because $p_1(1,E^c) + x_1(1,E^c) - 2.2 = \frac{3.3}{2} > \frac{1.1}{2}$. Then we can derive the value if keeping the option to date 1 as follows.

$$\frac{0 \times \frac{1}{4} + \frac{3.3}{2} \times \frac{3}{4}}{1 + 0.1} = \frac{9}{8}$$

which is better than the previous conlcusion 0.8.