$\min f_0(x)$, s.t. $Ax=b$

x is optimal iff $\nabla f_0(x) + A^T \nu = 0$, $\exists\, \nu \in R^p$

$\nabla f_0(x)^T (y-x) \geq 0, \forall y \in D$

$\Rightarrow \nabla f_0(x)^T \cdot w = 0, \forall Aw = 0$

$\Rightarrow \nabla f_0(x) \perp N(A)$

$\Rightarrow \nabla f_0(x) = A^T(-\nu), \exists\, \nu \in R^p$

(1) If $\nabla f_0(x) + A^T \nu = 0$

Then for all w, s.t. $Aw=0$, $w^T \nabla f_0(x) = w^T(-A^T\nu) = -(Aw)^T \nu = 0$

(2) If $\nabla f_0(x)$ has component outside the range of A^T,

$\exists\, w$, s.t. $Aw=0$ and $w^T \nabla f_0(x) \neq 0$

Ex 1: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $Null(A) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} d, d \in R$, $R(A^T) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in R^2$

Ex2: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $Null(A) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} d, d \in R$, $R(A^T) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in R^2$

- Linear Fractional Programming

P1 $\min f_0(x) = \dfrac{c^T x + d}{e^T x + f}$, $\mathrm{dom}\, f_0 = \{x \mid e^T x + f > 0\}$, s.t. $Gx \preceq h$, $Ax = b$

P2 Let $y = \dfrac{x}{e^T x + f}$, $z = \dfrac{1}{e^T x + f}$

We have $\min c^T y + dz$, s.t.

$Gy - hz \leq 0$, $Ay - bz = 0$, $e^T y + fz = 1$, $z \geq 0$

$\min f_0(x)$ s.t.

$$f_i(x) \le 1, \quad i=1,\dots,m \qquad\qquad f_i(x) \text{ are posynomials}$$
$$h_i(x) = 1, \quad i=1,\dots,p \qquad\qquad h_i(x) \text{ are monomials}$$
$$x > 0$$

$$f(x) = \sum_{k=1}^{K} C_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}, \quad C_k > 0,\ a_{ik} \in R,\ x \in R^n_{++}$$

$\hookrightarrow$ (Each term is called monomial
$f(x)$ is called posynomial

It is known that

$$\log\left[C_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}\right]$$
$$= \log C_k + a_{1k} \underbrace{\log x_1}_{y_1} + a_{2k} \underbrace{\log x_2}_{y_2} + \cdots + a_{nk} \underbrace{\log x_n}_{y_n}$$
$$\Rightarrow C_k x_1^{a_{1k}} \cdots x_n^{a_{nk}} = e^{a^T y + \log C_k}$$

In other words,

$$f(x) = \log \sum_{k=1}^{K} e^{a^T y + b_k}, \quad \text{where } a^T = [a_{1k} \cdots a_{nk}] \begin{bmatrix} \log x_1 \\ \vdots \\ \log x_n \end{bmatrix} + \log C_k.$$

$f: R^n \to R$, $f(x) > 0$, $\forall x \in \operatorname{dom} f$

(If $\log f$ is convex, f is log-convex
" concave, " log-concave

Ex: $f(x) = a^T x + b$, $(a^T x + b > 0) \Rightarrow$ log-concave

$f(x) = x^a$, $(x \in R_{++}, a \le 0) \Rightarrow$ log-convex

$f(x) = e^{ax}$, log-convex and log-concave

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$ [candidate distribution of Gaussian density] $\Rightarrow$ log-concave

$$f(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} e^{-\frac{1}{2}(x-\bar{x})\Sigma^{-1}(x-\bar{x})} \Rightarrow \text{log-concave}$$

Properties:

$$\nabla^2 \log f(x) = \frac{1}{f(x)} \nabla^2 f(x) - \frac{1}{f(x)^2} \nabla f(x) \cdot \nabla f(x)^T$$

$$\because \nabla \log f(x) = \frac{1}{f(x)} \cdot \nabla f(x)$$

If $\frac{1}{f(x)} \nabla^2 f(x) \geq \frac{1}{f(x)^2} \cdot \nabla f(x) \quad \nabla f(x)^T \Rightarrow$ convex

$\leq \qquad \Rightarrow$ concave

- "Equivalent" Formulation

(1) Eliminate the equality constraints

$$\begin{cases} \min f_0(x), \text{ s.t.} \\ f_i(x) \leq 0 \\ Ax = b \end{cases}$$

Convert $\{x \mid Ax = b\}$ into a set $\{Fz + x_0 \mid x \in R^n\}$

$$\Rightarrow \begin{cases} \min f_0(Fz + x_0), \text{ s.t.} \\ f_i(Fz + x_0) \leq 0 \end{cases}$$

$\Rightarrow$ may destroy the sparsity of the problem

(2) Slack Variables

$$\begin{cases} \min f_0(x) \quad \text{s.t.} \\ f_i(x) \leq 0 \\ Ax = b \end{cases} \Rightarrow \begin{cases} \min t \\ f_0(x) - t \leq 0 \\ f_i(x) \leq 0 \\ Ax = b \end{cases} \Rightarrow \begin{cases} \min f_0(x) \\ f_i(x) + \delta_i = 0 \\ Ax = b \\ \delta_i \geq 0 \end{cases}$$

(3) Matrics

$$|f_i| \leq b \Rightarrow f_i \leq b, \; -f_i \leq b$$