Investment Hw2

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PROBLEM 1

It is hard to find out the extreme value with constrains in original problem:

$$\max_{x_1^j,x_2^j,x_3^j} E\left[\log\left(\sum_{i=1}^{3} x_i^j \tilde{z}_i\right)\right],\ j \in [1,3]$$

where

$$x_1^j + px_2^j + qx_3^j \le t,\ (j,t) \in [(1,1),(2,p),(3,q)]$$
$$x_i^1 + x_i^2 + x_i^3 = 1,\ i \in [1,3]$$

Hence, we apply Lagrange Duality to solve this problem and get our new objective function:

$$\min_{v} \sup_{x_1^j,x_2^j,x_3^j} \left(E\left[\log\left(\sum_{i=1}^{3} x_i^j \tilde{z}_i\right)\right] - v(x_1^j + px_2^j + qx_3^j - t)\right),\ (j,t) \in [(1,1),(2,p),(3,q)]$$

After that, we can find out the optimal solution for each j must satified following three equations:

$$E\left[\frac{\tilde{z}_1}{\left(\sum_{i=1}^{3} x_i^j \tilde{z}_i\right)}\right] = \lambda,\ E\left[\frac{\tilde{z}_2}{\left(\sum_{i=1}^{3} x_i^j \tilde{z}_i\right)}\right] = p\lambda,\ E\left[\frac{\tilde{z}_3}{\left(\sum_{i=1}^{3} x_i^j \tilde{z}_i\right)}\right] = q\lambda$$

and thus

$$p = \frac{E\left[\frac{\tilde{z}_2}{\left(\sum_{i=1}^{3} x_i^j \tilde{z}_i\right)}\right]}{E\left[\frac{\tilde{z}_1}{\left(\sum_{i=1}^{3} x_i^j \tilde{z}_i\right)}\right]},\ q = \frac{E\left[\frac{\tilde{z}_3}{\left(\sum_{i=1}^{3} x_i^j \tilde{z}_i\right)}\right]}{E\left[\frac{\tilde{z}_1}{\left(\sum_{i=1}^{3} x_i^j \tilde{z}_i\right)}\right]}$$

Besides, from homework 1 problem 3, we know that W_0 and $\frac{a_j^*}{W_0}$ are independent in logarithm utility function. To be more specific, we can utilize the property that $x_1^j : x_2^j : x_3^j$ has the same ratio for

$j \in [1,3]$. In other words, we can eliminate x_i^j both numerator and denominator from p^*, q^* and get

$$p = \frac{E\left[\frac{\tilde{z}_2}{\sum_{i=1}^{3}\tilde{z}_i}\right]}{E\left[\frac{\tilde{z}_1}{\sum_{i=1}^{3}\tilde{z}_i}\right]},\ q = \frac{E\left[\frac{\tilde{z}_3}{\sum_{i=1}^{3}\tilde{z}_i}\right]}{E\left[\frac{\tilde{z}_1}{\sum_{i=1}^{3}\tilde{z}_i}\right]}$$

Problem 2

Subproblem (i)

From the EU Theory, we know that investor i will have the maximized utility by solving

$$\begin{aligned}\max_{D_i \in \mathscr{R}} E\left[-e^{-\rho_i[D_i\tilde{x}+(1-D_i)P(1+r_f)]}\right] &= e^{-\rho_i[(1-D_i)P(1+r_f)]} + E\left[-e^{-\rho_i D_i \tilde{x}}\right] \\ &= e^{-\rho_i[(1-D_i)P(1+r_f)]}\left(-e^{-\rho_i D_i\tilde{x}+\frac{1}{2}\rho_i^2 D_i\sigma^2}\right)\end{aligned}$$

With utility function and its strictly increasing property, we can modify our problem to solve

$$\max_{D_i \in \mathscr{R}} (1-D_i)P(1+r_f) + D_i\mu - \frac{1}{2}\rho_i D_i^2\sigma^2$$

and it is easy to find out $D_i^*(P) = \frac{-P(1+r_f)+u}{\rho_i\sigma^2}$.
Now replacing P with P^* in D_i then we can get

$$\sum_i^n D_i(P^*) = n\,;\ p^* = \frac{-\bar{\rho}+\mu}{1+r_f}$$

where

$$\bar{\rho} = \frac{n}{\sum_i^n \frac{1}{\rho_i}}\sigma^2$$

Subproblem (ii)

To begin with, we can find out that $D_i(P^*) = \frac{\bar{\rho}}{\rho_i}$. We also know that if the demand from investor i larger than 1, he would like to borrow in the data-0 equilibrium. Thus, if $D_1(P^*) > \ldots > D_k(P^*) \geq 1 \geq D_{k+1}(P^*) > \ldots > D_n(P^*)$, then investor 1 to investor k are borrowing in the data-0 equilibrium.

Problem 3

Subproblem (i)

The problem can be modeled as

$$\max_{\theta_A^1,\theta_A^2}\left(\frac{1}{2}\ln(100\theta_A^1 + 0\theta_A^2 + 20) + \frac{1}{2}\ln(200\theta_A^1 + 50\theta_A^2 + 20)\right)$$

where

$$\theta_A^1 p_1 + \theta_A^2 p_2 \leq 0.8 p_1$$

We can easily prove that the optimal θ_A^1 and $theta_A^2$ appear only when the budget is consumed completely. Then, we apply Lagrange Multiplier into this problem and get:

$$\max_{\theta_A^1,\theta_A^2}\left(\frac{1}{2}\ln(100\theta_A^1+0\theta_A^2+20)+\frac{1}{2}\ln(200\theta_A^1+50\theta_A^2+20)-\lambda(\theta_A^1 p_1+\theta_A^2 p_2-0.8p_1)\right)$$

where

$$\theta_A^1 p_1+\theta_A^2 p_2-0.8p_1=0$$

With the first order derivatives, we now are going to solve three equations:

$$\begin{aligned}\lambda p_1 &= \frac{1}{2}\frac{100}{100\theta_A^1+20}+\frac{1}{2}\frac{200}{200\theta_A^1+50\theta_A^2+20}\\ \lambda p_2 &= \frac{1}{2}\frac{50}{200\theta_A^1+50\theta_A^2+20}\\ 0 &= \theta_A^1 p_1+\theta_A^2 p_2-0.8p_1\end{aligned}$$

After several steps of simplifications with first and second equations, we get:

$$\begin{aligned}\theta_A^1 &= \frac{1}{100}\left(\frac{50}{(p_1-4p_2)\lambda}-20\right)\\ \theta_A^2 &= \frac{1}{50}\left(\frac{25}{p_2\lambda}-\frac{100}{(p_1-4p_2)\lambda}+20\right)\end{aligned}$$

Substitute it θ_A^1 and θ_A^2 into third equation will obtain

$$\lambda=\frac{1}{p_1-0.4p_2}$$

Then, we finally know

$$\begin{aligned}\theta_A^1 &= \frac{3p_1+6p_2}{10p_1-40p_2}\\ \theta_A^2 &= \frac{5p_1-2p_2}{10p_2}-\frac{10p_1-4p_2}{5p_1-20p_2}+\frac{2}{5}\end{aligned}$$

Similar to the previous steps, we can solve Lagrange Multiplier that

$$\max_{\theta_B^1,\theta_B^2}\left(\frac{1}{2}\ln(100\theta_B^1+0\theta_B^2+100)+\frac{1}{2}\ln(200\theta_B^1+50\theta_B^2+0)-\lambda(\theta_B^1 p_1+\theta_B^2 p_2-0.2p_1)\right)$$

where

$$\theta_B^1 p_1+\theta_B^2 p_2-0.2p_1=0$$

then get

$$\begin{aligned}\lambda &= \frac{1}{1.2p_1+4p_2}\\ \theta_B^1 &= \frac{-2p_1+10p_2}{5p_1-20p_2}\\ \theta_B^2 &= \frac{3p_1-10p_2}{5p_2}-\frac{12p_1-40p_2}{5p_1-20p_2}+4\end{aligned}$$

From $\theta_A^1+\theta_B^1=1$, we can derive

$$\begin{aligned}\frac{3p_1+6p_2}{10p_1-40p_2}+\frac{-2p_1+10p_2}{5p_1-20p_2}&=1\\ p_1&=6p_2\end{aligned}$$

Since $\phi_1+\phi_2=1$, and we know $\frac{p_2}{p_1}=\frac{0\phi_1+50\phi_2}{200\phi_1+100\phi_2}=\frac{1}{6}$, we can derive $\phi_1=\phi_2=0.5,\ p_1=150,\ p_2=25.$

Subproblem (ii)

Since the state price will not change, we can get the result with

$$\max(120-100,0)\phi_1+\max(100-200,0)\phi_2=10$$

Subproblem (iii)

It is obvious that the new investors' utility functions' first derivatives on day-0 and day-1 are $\frac{1}{e_0}$, $\frac{1}{e_1}$, respectively. Then we can apply the pricing formula from EU theory:

$$\begin{aligned}\frac{1}{1+r_f}&=E\left[\frac{u_1^{'}(e_1)}{u_0^{'}(e_0))}\right]\\&=\frac{\frac{1}{200}}{\frac{1}{180}}=0.9\\r_f&=\frac{1}{9}\end{aligned}$$

Problem 4

Subproblem (i)

The first order derivative of expected utility function with respect to a_i is

$$E\left[U^{'}\left(f(a_i^*)+\tilde{\epsilon_i}+(\mathbf{z}-\mathbf{p})^T\mathbf{x}_i^*\right)f^{'}(a_i^*)\right]-C^{'}(a_i^*)=0$$

We can then utilize the first order of derivative of $U(w_i)$: $U^{'}(w_i)=1-bw_i$ to get

$$\begin{aligned}[1-bf(a_i^*)-b(\mathbf{z}-\mathbf{p})^T\mathbf{x}_i^*]f^{'}(a_i^*)-C^{'}(a_i^*)=0\\1-b(\mathbf{z}-\mathbf{p})^T\mathbf{x}_i^*=\frac{C^{'}(a_i^*}{f^{'}(a_i^*)}+bf(a_i^*)=H(a_i^*)\end{aligned}$$

It is obvious that when there is no opening market, we can simply make $\mathbf{x}_i^*$ be all zero. This can be inferred that the optimal a_i^* is equal for all i, and $H(a_i^*)=1$. Besides that, since $H(\bullet)$ is an increasing, and convex function, we can derive

$$H\left(\frac{1}{N}\sum_{i=1}^N a_i^*\right)\leq\frac{1}{N}\sum_{i=1}^N H(a_i^*)=\frac{1}{N}\sum_{i=1}^N[1-b(\mathbf{z}-\mathbf{p})^T\mathbf{x}_i^*]=1-\frac{b}{N}(\mathbf{z}-\mathbf{p})^T\left(\sum_{i=1}^N\mathbf{x}_i^*\right)=1$$

It can also be shown that each element of aggregate productivity, said $f(a)-C(a)$ is an increasing, and concave function. This can help prove

$$\begin{aligned}\sum_{i=1}^N\left(f(a^0)-C(a^0)\right)&=N\left(f(a^0)-C(a^0)\right)\\&\geq N\left(f(\frac{1}{N}\sum_{i=1}^N a_i^*)-C(\frac{1}{N}\sum_{i=1}^N a_i^*)\right)\\&\geq\sum_{i=1}^N\left(f(a_i^*)-C(a_i^*)\right)\end{aligned}$$

Subproblem (ii)

It is easy to derive that $H(a)=ba+ca=(b+c)a$, which meets the former presume of $H(\bullet)$, i.e. $H(\bullet)$ is convex in a. In other words, the expected productivity will be reduced if opening the markets.