- Convex Set:

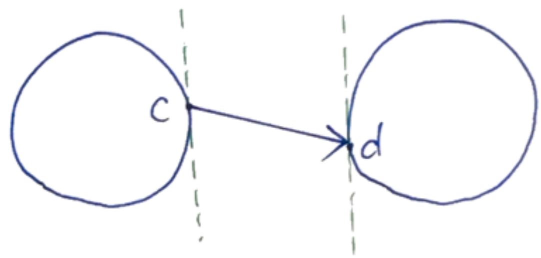
$$a^T x \le b \quad \Rightarrow \quad a^T(x - x_0) \le 0 \quad \Rightarrow \quad \{x \mid Ax \le b,\ cx \le d\}$$

- Hyperplane:

$a = d - c$

$b = \frac{\|d\|_2^2 - \|c\|_2^2}{2}$

$a^T x - b$ or $f(x) = (d-c)^T\left(x - \frac{d+c}{2}\right)$



- <Thm> Given two convex sets C and D, $\exists\, a, b$ s.t. $a^T x \le b,\ \forall x \in C$; $a^T x \ge b,\ \forall x \in D$

<proof> By contradiction.

$\forall\, v \in D$, $a^T v \ge a^T d$ should be true.

(If $\exists\, v \in D$, $a^T v < a^T d$, we can find a point on $\overline{vd}$, i.e. $d + t(v-d)$, is closer to C, for some small $t > 0$)

$$\Rightarrow \frac{\partial \|d + t(v-d) - c\|_2}{\partial t} = 2\cdot\underbrace{(d-c)^T}_{a}\cdot(v-d) + \underbrace{2t(v-d)^T\cdot(v-d)}_{\approx 0}$$

$$\approx 2\cdot(a^T v - a^T d) < 0$$

$\Rightarrow \|d + t(v-d) - c\|_2$ is closer, which contrdicts our assumption that (c,d) is the closest point pair in (C, D)

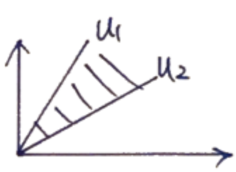
- Supporting Hyperplane:

$$C = \{x \mid a^T x \le a^T x_0\}$$

Def $\tilde{C} = C \setminus$ Boundary of C

$\Rightarrow$ there is a hyperplane that separates $\tilde{C}$ and X_0

- Cone:

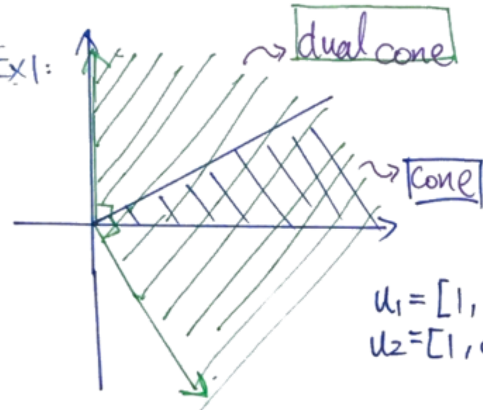


$$C = \left\{\sum_{i=1}^{k} \theta_i u_i \;\middle|\; \theta_i \ge 0\right\}$$

$\Rightarrow \sum_{i=1}^{k} \theta_i$ needs not to be 1

- Dual Cones

Ex1:

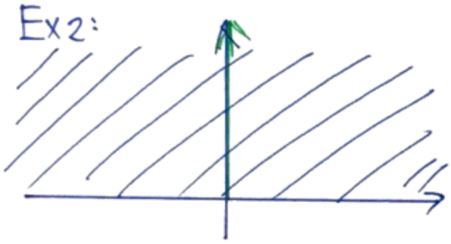


$u_1 = [1,1]$
$u_2 = [1,0]$

$$K^* = \{ y \mid x^T y \ge 0,\ \forall x \in K \}$$

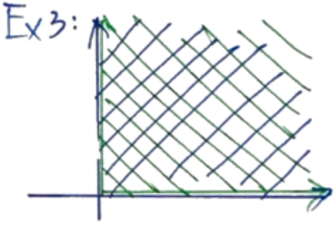
$\Rightarrow$ for each point in Cone K, each point in dual cone K^* has ≤ 90 degrees

Ex2:



$u_1 = [1,0]$
$u_2 = [0,1]$
$u_3 = [-1,0]$

Ex3:



$u_1 = [1,0]$
$u_2 = [0,1]$

- Duality:

$$K = \{ \overset{n\times 1}{x} \mid \overset{m\times n}{A} x \ge 0 \}$$
$$K^* = \{ A^T v \mid \underset{m\times 1}{v} \ge 0 \}$$

if $m \ll n$, dual form might be useful

- Examples:
  1. $K = R_+^n$, $K^* = R_+^n$
  2. $K = S_+^n$, $K^* = S_+^n$
  3. $K = \{(x,t) \mid \|x\|_2 \le t,\ t \ge 0\}$, $K^* = \{(x,t) \mid \|x\|_2 \le t,\ t \ge 0\}$
  4. $K = \{(x,t) \mid \|x\|_p \le t,\ t \ge 0\}$, $K^* = \{(x,t) \mid \|x\|_q \le t,\ t \ge 0\}$; $\boxed{\frac{1}{p} + \frac{1}{q} = 1}$

- Convex Function:

$f: R^n \to R$ is convex if ① dom f is convex, ② $f(\theta x + (1-\theta) y) \le \theta f(x) + (1-\theta) f(y)$, $\forall x, y \in \mathrm{dom} f$, $0 \le \theta \le 1$

  - Example:
    1. $x \in R$
    2. $ax + b$
    3. e^{ax}, for any $a \in R$
    4. x^α on R_{++} for $\alpha \ge 1$ or $\alpha \le 0$
    5. $|x|^p$ on R for $p \ge 1$
    6. $x \log x$, for $x \in R_{++}$