# CSE 291-A: Homework 1

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## Exercises from textbook chapter 2

### 2.9 Voronoi sets and polyhedral decomposition.

Let $x_0, \cdots, x_K \in \mathbf{R}^n$. Consider the set of points that are closer (in Euclidean norm) to $x_0$ than the other $x_i$, i.e.,

$$V = \{x \in \mathbf{R}^n \mid \|x - x_0\|_2 \leq \|x - x_i\|_2, i = 1, \cdots, K\}.$$

$V$ is called the Voronoi region around $x_0$ with respect to $x_1, \ldots, x_K$.

**(a)**

Show that $V$ is a polyhedron. Express $V$ in the form $V = \{x | Ax \leq b\}$.

*Sol.* For every $i$, we can have the following derivation.

$$\begin{aligned}
\|x - x_0\|_2 &\leq \|x - x_i\|_2 \\
(x - x_0)^2 &\leq (x - x_i)^2 \\
x^\intercal x - 2x^\intercal x_0 + x_0^\intercal x_0 - x^\intercal x + 2x^\intercal x_i - x_i^\intercal x_i &\leq 0 \\
2x^\intercal \cdot (x_i - x_0) &\leq (x_i^\intercal x_i - x_0^\intercal x_0)
\end{aligned}$$

Thus, $A \in R^{K \times n}$ and $b \in R^{K \times 1}$, where

$$\begin{aligned}
A_i &= 2(x_i - x_0) \\
b_i &= x_i^\intercal x_i - x_0^\intercal x_0.
\end{aligned}$$

□

**(b)**

Conversely, given a polyhedron $P$ with nonempty interior, show how to find $x_0, \cdots, x_K$ so that the polyhedron is the Voronoi region of $x_0$ with respect to $x_0, \cdots, x_K$.

*Sol.* According to the definition of Voronoi, one can pick up a point $x_0$ arbitrary inside the polyhedron and mirrored $x_i$ outside the polyhedron with hyperplane $\{x | A_i x = b_i\}$.

We can easily calculate the distance from $x_0$ to hyperplane $\{x | A_i x = b_i\}$ is $\frac{b_i - A_i x_0}{\|A_i\|_2}$. Thus $x_i$ can be expressed by fixed point plus twice the distance times normal vector, which is $x_0 + 2 \cdot \frac{b_i - A_i x_0}{\|A_i\|_2} \cdot A_i$.

□

**(c)**

We can also consider the sets

$$V = \{x \in \mathbf{R}^n | \, \|x - x_i\|_2 \leq \|x - x_i\|_2 \, , i \neq k\}$$

The set $V_k$ consists of points in $\mathbf{R}^n$ for which the closest point in the set $\{x_0, \cdots, x_K\}$ is $x_k$.

The sets $V_0, \cdots, V_K$ give a polyhedral decomposition of $\mathbf{R}^n$. More precisely, the sets $V_k$ are polyhedra, $\cup_{k=0}^{K} V_k = \mathbf{R}^n$, and $\textbf{int} V_i \cap \textbf{int} V_j = \phi$ for $i \neq j$, i.e., $V_i$ and $V_j$ intersect at most along a boundary.

Suppose that $P_1, \cdots, P_m$ are polyhedra such that $\cup_{i=1}^{m} P_i = \mathbf{R}^n$, and $\textbf{int} P_i \cap \textbf{int} P_j = \phi$ for $i \neq j$. Can this polyhedral decomposition of $\mathbf{R}^n$ be described as the Voronoi regions generated by an appropriate set of points?

*Sol.* Fig. 1 shown one of the counter examples to this problem. Assume there are two hyperplanes $HP_0$ and $HP_1$ separating the whole hyperspace into four subspaces, i.e. $HS_0$, $HS_1$, $HS_2$, and $HS_3$. If there are two points $P_0$ and $P_1$, such that $P_0$ in $HS_0$ and $P_1$ in $HS_1$ respectively, mirrored points $\bar{P}_0$ and $\bar{P}_1$ are going to located in $HS_2$, which makes it impossible to express feasible point $P_2$ in subspace $HS_2$.

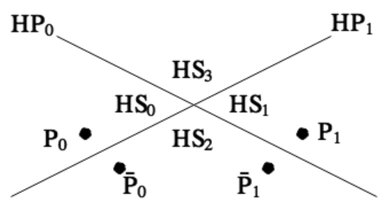


Figure 1: Counter Example for 2.9 (c)

$\square$

**2.28 Positive semidefinite cone for $n = 1, 2, 3$.**

Give an explicit description of the positive semidefinite cone $\mathbf{S}_+^n$, in terms of the matrix coefficients and ordinary inequalities, for $n = 1, 2, 3$. To describe a general element of $\mathbf{S}^n$, for $n = 1, 2, 3$, use the notation

$$x_1, \qquad \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}, \qquad \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix}$$

*Sol.* For $n = 1, 2, 3$, I derive all requirements individually.

For $n = 1$, the requirement is straightforward: $x_1 \geq 0$.

For $n = 2$, it can be derived that for all $a, b$

$$\begin{aligned} [a \quad b] \cdot \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} &\geq 0 \\ [ax_1 + bx_2 \quad ax_2 + bx_3] \cdot \begin{bmatrix} a \\ b \end{bmatrix} &\geq 0 \\ a^2 x_1 + 2ab x_2 + b^2 x_3 &\geq 0 \\ (a+b)^2 x_2 + a^2(x_1 - x_2) + b^2(x_3 - x_2) &\geq 0. \end{aligned}$$

Thus, the requirements are

$$\begin{aligned} x_2 &\geq 0 \\ x_1 - x_2 &\geq 0 \\ x_3 - x_2 &\geq 0. \end{aligned}$$

For $n = 3$, it can be derived that for all $a, b, c$

$$
\begin{aligned}
\begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} &\geq 0 \\
\begin{bmatrix} ax_1 + bx_2 + cx_3 & ax_2 + bx_4 + cx_5 & ax_3 + bx_5 + cx_6 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} &\geq 0 \\
a^2x_1 + b^2x_4 + c^2x_6 + 2abx_2 + 2acx_3 + 2bcx_5 &\geq 0 \\
(a+b)^2x_2 + (a+c)^2x_3 + (b+c)^2x_5 + a^2(x_1 - x_2 - x_3) + b^2(x_3 - x_2 - x_5) + c^2(x_6 - x_3 - x_5) &\geq 0.
\end{aligned}
$$

Thus, the requirements are

$$
\begin{aligned}
x_2 &\geq 0 \\
x_3 &\geq 0 \\
x_5 &\geq 0 \\
x_1 - x_2 - x_3 &\geq 0 \\
x_4 - x_2 - x_5 &\geq 0 \\
x_6 - x_3 - x_5 &\geq 0
\end{aligned}
$$

□

**2.31 Properties of dual cones.**

Let $K^*$ be the dual cone of a convex cone $K$, as defined in (2.19). Prove the following.

**(a)**

$K^*$ is indeed a convex cone.

*Sol.* According to the dual cone definition, $K^*$ can be written as $\{y | x^\intercal y \geq 0, x \in K\}$. Assume there are two points $y_1, y_2 \in K^*$, I need to prove $\theta_1 y_1 + \theta_2 y_2$ is still in $K^*$, where $\theta_1, \theta_2 \geq 0$.

For $x \in K$, I have $x^\intercal y_1 \geq 0$ and $x^\intercal y_2 \geq 0$. Thus,

$$
\begin{aligned}
x^\intercal(\theta_1 y_1 + \theta_2 y_2) &= \theta_1 x^\intercal y_1 + \theta_2 x^\intercal y_2 \\
&\geq 0 \cdot 0 + 0 \cdot 0 = 0
\end{aligned}
$$

□

**(b)**

$K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$.

*Sol.* Assume there is a point $y'$ that $y' \in K_2^*$ and $y' \notin K_1^*$. According to dual cone definition, $K_2^* = \{y | y^\intercal x \geq 0, x \in K_2\}$, which implies that

$$
\begin{aligned}
&y'^\intercal x \geq 0, x \in K_2 \\
&y'^\intercal x \geq 0, x \in K_1 \qquad \because K_1 \subseteq K_2.
\end{aligned}
$$

With the above formula, I know that $y' \in K_1^*$. However, this contradicts our assumption in the beginning. Thus, $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$. □

**(c)**

$K^*$ is closed.

*Sol.* According to the conclusion from (a), $K^*$ is actually a convex cone, which is closed based on textbook. □

**(d)**

The interior of $K^*$ is given by $\mathbf{int}K^* = \{y|y^\intercal x > 0 \text{ for all } x \in \mathbf{cl}K\}$.

*Sol.* Based on definition, $y^\intercal x > 0$ for all $x \in K$. Therefore, for all sufficient small $u$, I have $(y+u)^\intercal x > 0$. Thus, I can conclude that $y \in \mathbf{int}K^*$. $\square$

**(e)**

If $K$ has nonempty interior then $K^*$ is pointed.

*Sol.* According to the definition of point set, if $K^*$ is not pointed, there exists non-zero $y$ such that $y \in K^*$ and $-y \in K^*$. This can infer both $y^\intercal x \geq 0$ and $-y^\intercal x \geq 0$ for all $x \in K$, and thus, $y^\intercal x = 0$ for all $x \in K$. To fulfill this criteria, $x$ must be a single point at origin, which has empty interior. $\square$

**(f)**

$K^{**}$ is the closure of $K$. (Hence if $K$ is closed, $K^{**} = K$.)

*Sol.* The dual cone of a dual cone $K^*$ can still be imagined as an intersection of all homogeneous halfspaces with non-zero normal vector $y$, which will contains convex cone $K$. Since $K^{**}$ is the smallest set that contains $K$, $K^{**}$ is the closure of $K$ based on the definition of "closure". $\square$

**(g)**

If the closure of $K$ is pointed then $K^*$ has nonempty interior.

*Sol.* If $K^*$ has empty interior, there must exist non-zero $x$ such that $x^\intercal y = 0$ for all $y \in K^*$. In other words, both $x$ and $-x$ exist in $K^{**}$. Based on our conclusion from (f), since $K^{**}$ is the closure of $K$, non-pointed $K^{**}$ indicates non-pointed $K$, which violates our assumption. $\square$

**2.32**

Find the dual cone of $\{Ax|x \geq 0\}$,where $A \in \mathbf{R}^{m\times n}$.

*Sol.* Based on the definition in class, $\{x|Ax \geq 0\}$ and $\{A^\intercal v|v \geq 0\}$ are mutual dual cone with one another. In other words, the dual cone for $\{Ax|x \geq 0\}$ can be expressed as $\{x|A^\intercal x \geq 0\}$. $\square$

## Assignments

### [Polyhedron Example]

Given a sphere in the first octant of $\mathbb{R}^3$. The sphere is centered at $(r, r, r)$ and its radius is equal to $r$. Find a polyhedron, $\{x|Ax \leq b\}$, where matrix $A \in \mathbb{R}^{4\times 3}$, vector $x \in \mathbb{R}^3$, that contains the sphere with a minimum volume.

*Sol.* First of all, from the dimension of $A$, I know that there will be only four equations as boundaries. In other words, the sphere will be bounded within the intersection of four hyperplanes, or said, a tetrahedron. On top of that, since the tetrahedron who covers the sphere must have the minimum volume, the tetrahedron needs to be a regular one.

It is obvious that there are infinite solutions for this regular tetrahedron. I am going to choose the one with the following four vertices: $A(0, 0, 0)$, $B(\sqrt{6}r, 0, 0)$, $C(\frac{\sqrt{6}}{2}r, \frac{3\sqrt{2}}{2}r, 0)$, and $D(r, r, 4r)$. Thus, I can easily know the normal vector for four triangles: $\triangle ABC$, $\triangle ABD$, $\triangle ACD$, and $\triangle BCD$.

$$
\begin{aligned}
\triangle ABC &= \overrightarrow{AB} \times \overrightarrow{AC} \\
&= [0 \cdot 0 - 0 \cdot \frac{3\sqrt{2}}{2}r, 0 \cdot \frac{\sqrt{6}}{2}r - \sqrt{6}r \cdot 0, \sqrt{6}r \cdot \frac{3\sqrt{2}}{2}r - 0 \cdot \frac{\sqrt{6}}{2}r] \\
&= [0, 0, 3\sqrt{3}r^2] \\
&\propto [0, 0, -1] \\
\triangle ABD &= \overrightarrow{AB} \times \overrightarrow{AD} \\
&= [0 \cdot 4r - 0 \cdot r, 0 \cdot r - \sqrt{6}r \cdot 4r, \sqrt{6}r \cdot r - 0 \cdot r] \\
&= [0, -4\sqrt{6}r^2, \sqrt{6}r^2] \\
&\propto [0, -4, 1] \\
\triangle ACD &= \overrightarrow{AC} \times \overrightarrow{AD} \\
&= [\frac{3\sqrt{2}}{2}r \cdot 4r - 0 \cdot r, 0 \cdot r - \frac{\sqrt{6}}{2}r \cdot 4r, \frac{\sqrt{6}}{2}r \cdot r - \frac{3\sqrt{2}}{2}r \cdot r] \\
&= [6\sqrt{2}r^2, -2\sqrt{6}r^2, \frac{\sqrt{6}}{2}r^2 - \frac{3\sqrt{2}}{2}r^2] \\
&\propto [-4\sqrt{3}, 4, \sqrt{3} - 1] \\
\triangle BCD &= \overrightarrow{BC} \times \overrightarrow{BD} \\
&= [\frac{3\sqrt{2}}{2}r \cdot 4r - 0 \cdot r, 0 \cdot (r - \sqrt{6}r) + \frac{\sqrt{6}}{2}r \cdot 4r, -\frac{\sqrt{6}}{2}r \cdot r - \frac{3\sqrt{2}}{2}r \cdot (r - \sqrt{6}r)] \\
&= [6\sqrt{2}r^2, 2\sqrt{6}r^2, 3\sqrt{3}r^2 - \frac{3\sqrt{2}}{2}r^2 - \frac{\sqrt{6}}{2}r^2] \\
&\propto [-4\sqrt{3}, -4, \sqrt{3} + 1 - 3\sqrt{2}]
\end{aligned}
$$

Thus, I can derive $A$ and $b$ after putting proper sign symbol on the normal vectors, which is

$$
A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -4 & 1 \\ -4\sqrt{3} & 4 & \sqrt{3} - 1 \\ -4\sqrt{3} & -4 & \sqrt{3} + 1 - 3\sqrt{2} \end{bmatrix}
$$

$$
b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -12\sqrt{2}r \end{bmatrix}
$$

□

**[Convex space Conversion]**

$$
\begin{cases} 2x + y \leq 6 \\ -2x - 3y \leq -6 \\ -x + 2y \leq 4 \end{cases}
$$

Find the matrix $V$ such that $\{V\theta | \sum \theta_i = 1, \theta_i \geq 0\}$, has the equivalent solution set as above.

*Sol.* After plotting three lines on a 2D plane, the original formula expresses a closed triangle, whose three vertices are $(3, 0)$, $(0, 2)$, and $(1.6, 2.8)$.

If we fix $(3, 0)$, the triangle can be expressed as $(3, 0) + \theta_1(-3, 2) + \theta_2(-1.4, 2.8)$, where $\theta_1 + \theta_2 = 1$ and $\theta_1, \theta_2 \leq 0$. To fit previous formula into the target $\{V\theta | \sum \theta_i = 1, \theta_i \geq 0\}$, I propose to formulate

$\theta$ as $[\theta_1, \theta_2, \theta_3]^\intercal$, where $\theta_3 = 1 - \theta_2 - \theta_3$. Thus, $V$ can be derived as

$$\begin{bmatrix} 0 & 1.6 & 3 \\ 2 & 2.8 & 0 \end{bmatrix}.$$

To verify my answer, I have

$$\begin{aligned}\begin{bmatrix} 0 & 1.6 & 3 \\ 2 & 2.8 & 0 \end{bmatrix} \cdot \begin{bmatrix} \theta_1 \\ \theta_2 \\ 1 - \theta_1 - \theta_2 \end{bmatrix} &= \begin{bmatrix} 3 - 3\theta_1 - 1.4\theta_2 \\ 2\theta_1 + 2.8\theta_2 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \theta_1 \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \theta_2 \begin{bmatrix} -1.4 \\ 2.8 \end{bmatrix},\end{aligned}$$

which is exactly the same as my goal. $\square$