- There are two ways to express a set:
  - Implicit (Quantification)
  - Explicit (Enumeration)

- A convex set can be written as:
  - $\{x \mid Ax \le b\}$ … Quantification
  - $\{\sum_{i=1}^{k} \theta_i u_i \mid \sum \theta_i = 1,\ \theta_i \ge 0,\ \forall i = 1 \dots k\}$ … Enumeration

- Example:

Quantification

$$\begin{aligned} x_1 + x_2 &\le 2 \\ x_1 &\le 1 \\ -x_1 &\le 1 \\ -x_2 &\le 1 \end{aligned}$$

Enumeration

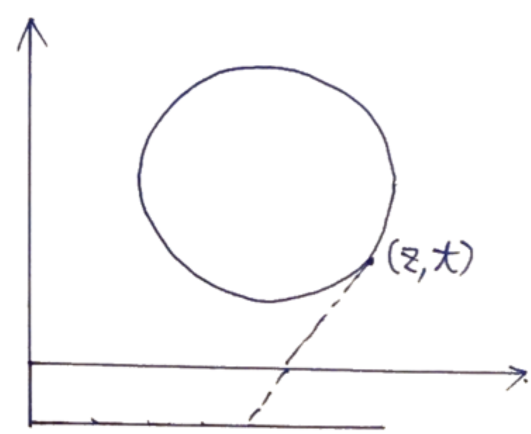
$$S = \{\theta_1 u_1 + \theta_2 u_2 + \theta_3 u_3 + \theta_4 u_4,\ \theta_1 + \theta_2 + \theta_3 + \theta_4 = 1,\ \theta_i \ge 0,\ i = 1, \dots 4\}$$

- Operation that preserves convex set
  1. Intersection of convex sets → convex set
  2. Perspective of convex set → convex set

     mapping from high-d to low-d

<fact> Given $\begin{bmatrix} z_1 \\ \vdots \\ z_t \end{bmatrix} \in$ convex set S, then

$$\left\{ \begin{bmatrix} \frac{z_1}{t} \\ \vdots \\ \frac{z_t}{t} \end{bmatrix} \;\middle|\; \begin{bmatrix} z \\ t \end{bmatrix} \in S \right\} \text{ is convex}$$



<proof>

$(z_1, t_1), (z_2, t_2) \in S \Rightarrow \left(\frac{z_1}{t_1}, \frac{z_2}{t_2}\right)$

if $\alpha(z_1, t_1) + \beta(z_2, t_2) \in S$

$= (\alpha z_1 + \beta z_2,\ \alpha t_1 + \beta t_2),\ \forall \alpha + \beta = 1,\ \alpha, \beta \ge 0$

$$\Rightarrow \frac{\alpha z_1 + \beta z_2}{\alpha t_1 + \beta t_2} = \frac{\alpha z_1}{\alpha t_1 + \beta t_2} + \frac{\beta z_2}{\alpha t_1 + \beta t_2} = \underbrace{\frac{\alpha t_1}{\alpha t_1 + \beta t_2}}_{\alpha'} \cdot \frac{z_1}{t_1} + \underbrace{\frac{\beta t_2}{\alpha t_1 + \beta t_2}}_{\beta'} \cdot \frac{z_2}{t_2}$$