- Primal Problem

  $\min f_0(x)$, $x \in R^n$

  subject to

  $f_i(x) \le 0$, $i = 1 \dots m$

  $h_i(x) = 0$, $i = 1 \dots p$

  - Domain: $\bigcap_{i=0}^{m} f_i \bigcap_{i=0}^{p} h_i$

    $\mathcal{D}$

  - Feasibile range: x that satisfies all the constraints

  - Lagrange:

    $$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

    subject to

    $$\lambda_i \ge 0, \quad \forall i$$

  - Lagrange dual function:

    $$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu)$$

- Dual Problem

  $$\max_{\lambda, \nu} g(\lambda, \nu) \quad \text{s.t. } \lambda_i \ge 0$$

  - Property 1: $g(\lambda, \nu)$ is concave

  - Property 2: $f_0(x) \ge L(x, \lambda, \nu) \ge g(\lambda, \nu)$, for x in feasible range

    $\Rightarrow p^* \ge g(\lambda, \nu)$

- Examples:

  - Example 1:

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$$\min_x \ c^T x, \ \text{s.t.} \quad Ax \le b, \quad x \ge 0$$

<primal>

$$\Downarrow$$

$$\max_{\lambda_1, \lambda_2} \ -\lambda_1^T b \ \text{s.t.} \quad A^T\lambda_1 + C - \lambda_2 = 0, \quad \lambda_i \ge 0$$

<dual>

$$\Longrightarrow \quad \max_\lambda \ -\lambda^T b \ \text{s.t.} \quad A^T\lambda + C \ge 0$$

<simplified dual>

* Lagrange: $L(x,\lambda,\nu) = C^T x + \lambda_1^T(Ax-b) + \lambda_2^T(-x)$

$$= -\lambda_1^T b + (A^T\lambda_1 - \lambda_2 + C^T)x$$

* Dual function:

$$g(\lambda,\nu) = \inf_x L(x,\lambda,\nu) = \inf_x \left[-\lambda_1^T b + (A^T\lambda_1 - \lambda_2 + C^T)x\right]$$

$$\Rightarrow g(\lambda,\nu) = \begin{cases} -\lambda_1^T b & A^T\lambda_1 - \lambda_2 + C^T = 0 \\ -\infty & \text{otherwise} \end{cases}$$

  - Example 2:

$$\min_{x_1, x_2} [-1, -1] \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ \text{s.t.} \quad \begin{bmatrix} 1 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad x_1, x_2 \ge 0$$

<primal>

$$\Longrightarrow \quad \max_{\lambda_1, \lambda_2} \ \lambda^T \cdot \begin{bmatrix} -3 \\ -2 \end{bmatrix} \ \text{s.t.} \quad \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} \ge 0$$

<dual>

  - Example 3:

$$\min_x \ x^T x, \ x \in \mathbb{R}^n \quad \text{s.t. } Ax = b$$

<primal>

$$\Downarrow$$

$$\max_\nu \ -\frac{1}{4}\nu^T A A^T \nu - b^T \nu$$

<dual>

* Lagrange: $L(x,\nu) = x^T x + \nu^T(Ax - b)$

$$= x^T x + \nu^T A x - \nu^T b$$

* Dual function:

$$g(\nu) = \inf_x L(x,\nu) = \inf_x x^T x + \nu^T A x - \nu^T b$$

$$\frac{\partial L(x,\nu)}{\partial x} = 2x + A^T\nu = 0 \Rightarrow x = \frac{-A^T\nu}{2}$$

$$\frac{\partial g(\nu)}{\partial \nu} = -\frac{1}{2}AA^T\nu - b = 0 \qquad \Rightarrow g(\nu) = -\frac{1}{4}\nu^T A A^T \nu - b^T\nu$$

$\Rightarrow$ maximum ? $\nu = 2(AA^T)^{-1} \cdot b$

- Example 4:

$$\min x^T W x, \quad x \in \mathbb{R}^n,$$
$$\text{s.t. } x_i^2 = 1, \quad w_{ij} \in \mathbb{R}$$

* Lagrange: $L(x, \nu) = x^T W x + \sum_{i=1}^{n} \nu_i (x_i^2 - 1)$

$$= x^T (W + \operatorname{diag}(\nu)) x - \mathbf{1}^T \nu$$

* Dual Function:

$$g(\nu) = \inf_x L(x, \nu) = \inf_x x^T (W + \operatorname{diag}(\nu)) x - \mathbf{1}^T \cdot \nu$$

$$= \begin{cases} -\mathbf{1}^T \nu & W + \operatorname{diag}(\nu) \succeq 0 \\ -\infty & \text{otherwise} \end{cases}$$

$$\max_\nu g(\nu) = -\mathbf{1}^T \nu,$$
$$\text{s.t. } W + \operatorname{diag}(\nu) \succeq 0$$

$$\Rightarrow \nu = -\lambda_{\min}(W) \cdot \mathbf{1} \Rightarrow p^* \geq d^* = -\mathbf{1}^T \nu = n \cdot \lambda_{\min}(W)$$