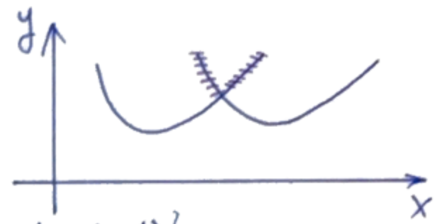
- Operations that preserve the convexity
  - non-negative multiples : $\alpha f,\ \alpha \geq 0$
  - Summation : $f_1 + f_2$
  - Composition of affine function: $f(Ax+b)$
  - point-wise maximum : $f(x) = \max_i \{f_i(x)\}$, where $f_i(x)$ is convex

    ex: $g(x) = \max_{y \in A} f(x,y)$ is convex

    * there is no constraint on A



<proof>

$$f(\theta x + (1-\theta)\cdot y) = \max\{f_1(\theta x + (1-\theta)y),\ f_2(\theta x + (1-\theta)\cdot y)\}$$

$$\leq \max\{\theta \cdot f_1(x) + (1-\theta)\cdot f_1(y),\ \theta \cdot f_2(x) + (1-\theta)\cdot f_2(y)\}$$

$$\leq \max\{\theta \cdot f_1(x),\ \theta f_2(x)\} + \max\{(1-\theta) f_1(x),\ (1-\theta) f_2(x)\}$$

$$\leq \theta \cdot f(x) + (1-\theta)\cdot f(y)$$

- partial minimization :

  If $f(x,y)$ and C are convex, $g(x) = \min_{y \in C} f(x,y)$ is convex

  * C must be convex

<proof>

$$\theta \cdot g(x_1) + (1-\theta)\cdot g(x_2) = \theta \cdot f(x_1,y_1) + (1-\theta) f(x_2,y_2),$$

(where $y_1 \in \{y \mid \min_{y \in C} g(x_1,y)\}$ and $y_2 \in \{y \mid \min_{y\in C} g(x_2,y)\}$)

$$\overset{f(x,y)\text{ is convex}}{\geq} f(\theta x_1 + (1-\theta)x_2,\ \theta y_1 + (1-\theta)y_2)$$

$$\overset{C\text{ is convex}}{\geq} \min_{y \in C} f(\theta x_1 + (1-\theta)x_2,\ y)$$

$$= f(\theta x_1 + (1-\theta)x_2)$$

ex: Given $f(x,y) = [x^T\ y^T]\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, $x \in R^n, y \in R^m$, $A \in S_+^n, C \in S_+^m$, $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \in S_+^{n+m}$

Let $g(x) = \min_y f(x,y) = x^T(A - BC^{\dagger}B^T)x$, where $C^{\dagger}$ is pseudo inverse (C^{-1})

(1) $f(x,y)$ is convex

(2) C is convex

(3) from (1) + (2), $g(x)$ is convex, i.e. $A - BC^{\dagger}B^T \geq 0$

- Composition

  $g: R^n \to R$ and $h: R \to R$

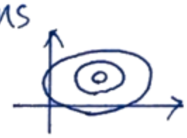
  $f(x) = h(g(x))$

  $f'(x) = h'(g(x)) \cdot g'(x)$

  $f''(x) = h''(g(x)) \cdot [g'(x)]^2 + h'(g(x)) \cdot g''(x)$

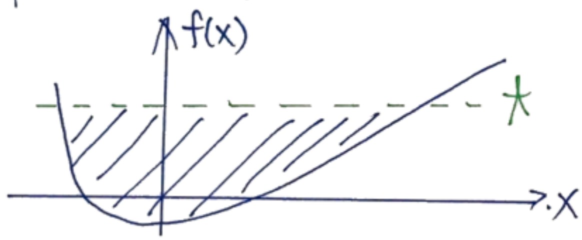
  $*$ f is convex if $\begin{cases} g \text{ is convex, } h \text{ is convex, non-decreasing } (h' \geq 0) \\ g \text{ is concave, } h \text{ is convex, non-increasing } (h' \leq 0) \end{cases}$

  $*$ f is concave if $\begin{cases} g \text{ is convex, } h \text{ is concave, non-increasing } (h' \leq 0) \\ g \text{ is concave, } h \text{ is concave, non-decreasing } (h' \geq 0) \end{cases}$

- Different Views of Functions
  - Equi-potential map 
  - Graph of f 

  - Epigraph:

    $\text{epi } f = \{(x, t) \mid x \in \text{dom } f,\ f(x) \leq t\}$



$\Rightarrow$ f is convex iff epi f is convex