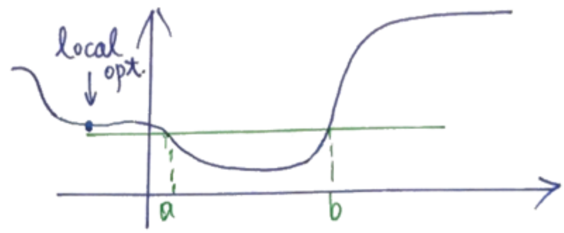
291A Convex Optimization 10/26/17

- Quasiconvex Functions $f: R^n \to R$

sublevel set

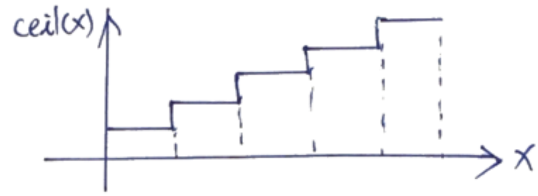
$$S_t = \{x \mid x \in \mathrm{dom}\, f,\ f(x) \le t\}$$



ex: $\log x$ is concave but quasiconvex (in fact, it is quasilinear)

ex: Ceiling function

$$\mathrm{ceil}(x) = \inf\{z \in Z \mid z > x\}$$

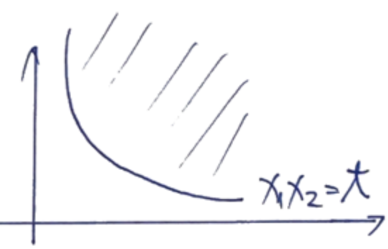


ex: $f(x_1, x_2) = x_1 x_2$. $\nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\Rightarrow$ it is neither convex nor concave

$$S_t = \{x \mid x \in R_+^2,\ x_1 x_2 \le t\}$$

$\Rightarrow$ not concvex $\Rightarrow f(x_1, x_2)$ is not quasiconvex



$$S_t = \{x \mid x \in R_+^2,\ x_1 x_2 \ge t\}$$

$\Rightarrow$ Convex $\Rightarrow f(x_1, x_2)$ is quasiconcave

ex: $f(x) = \dfrac{a^T x + b}{c^T x + d}$, for $c^T x + d > 0$

$$\left[S_t = \{x \mid c^T x + d > 0 \text{ and } \frac{a^T x + b}{c^T x + d} \underset{\ge}{\le} t\} = \{x \mid c^T x + d > 0 \text{ and } (a^T - t c^T) \cdot x + b - t d^T \underset{\ge}{\le} 0\} \right.$$

$\Rightarrow$ quasilinear

- Quasiconvex Opt.

$\min f_0(x)$, s.t. $f_i(x) \le 0$ and $Ax = b$

- Alg. Bisection Method for Quasiconvex Opt.

Given $l \le p^* \le u$, $\epsilon > 0$

$f_0(x) \le t \Leftrightarrow \phi_t(x) \le 0$

Repeat:

1) $t = (l+r)/2$, 2) solve find x s.t. $\phi_t(x) \le 0$, $f(x) < 0$, $Ax = b$

3) if solution is feasible, $l = t$, 4) keep in loop till $u - l < \varepsilon$.

- Duality
  - Primal

$$\min f_0(x), \text{ s.t. } \quad f_i(x) \le 0, \quad h_i(x) = 0$$

$$\begin{cases} D_0 : \text{dom } f_0 \\ D_{f_i} : \text{dom } f_i \rightarrow D = D_0 \cap D_{f_i} \cap D_{h_i} \\ D_{h_i} : \text{dom } h_i \end{cases}$$

  - Lagrangian $L: R^n \times R^m \times R^p \rightarrow R$

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i \cdot f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x), \text{ where}$$

$$\begin{cases} \lambda_i \text{ and } \nu_i \text{ are Lagrange multiplier (shadow price)} \\ \lambda_i \in R_+ \\ \nu_i \in R \end{cases}$$

  - Lagrange Dual Function:

$$g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu) \text{ , the problem becomes } \max_{\lambda, \nu} g(\lambda, \nu)$$

  - Properties of $g(\lambda, \nu)$

(1) $g(\lambda, \nu)$ is concave

$$-g(\lambda, \nu) = \sup_{x \in D} -L(x, \lambda, \nu) = \sup_{x \in D} -f_0(x) - \sum_{i=1}^{m} \lambda_i f_i(x) - \sum_{i=1}^{p} \nu_i h_i(x)$$

(2) $g(\lambda, \nu) \le p^*$, p^* is an optimal value

Proof: for any feasible $\tilde{x}$ and $\lambda \ge 0$

$$\underbrace{f_0(\tilde{x})}_{\text{primal}} \ge L(\tilde{x}, \lambda, \nu) \ge \underbrace{g(\lambda, \nu)}_{\text{dual}}$$