Investment Hw1

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PROBLEM 1

By symmetric of $\tilde{r}_i, i = 1...n$, we know that

$$E\left[\tilde{r}_j \,\middle|\, \frac{1}{n}\sum_{i=1}^{n}\tilde{r}_i\right] = E\left[\tilde{r}_k \,\middle|\, \frac{1}{n}\sum_{i=1}^{n}\tilde{r}_i\right] \ \forall j,k = 1...n$$

which means that $E\left[\tilde{r}_j \,\middle|\, \frac{1}{n}\sum_{i=1}^{n}\tilde{r}_i\right]$ is independent of j.
Then we can derive

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}\tilde{r}_i &= E\left[\frac{1}{n}\sum_{j=1}^{n}\tilde{r}_j \,\middle|\, \frac{1}{n}\sum_{i=1}^{n}\tilde{r}_i\right] \\
&= \frac{1}{n}\sum_{i=1}^{n} E\left[\tilde{r}_j \,\middle|\, \frac{1}{n}\sum_{i=1}^{n}\tilde{r}_i\right] \\
&= E\left[\tilde{r}_j \,\middle|\, \frac{1}{n}\sum_{i=1}^{n}\tilde{r}_i\right]
\end{aligned}$$

Based on Theorem 7 (iii), we should introduce a random variable $\tilde{z}$, such that $\tilde{z} = \mathbf{w}'\tilde{\mathbf{r}} - (\mathbf{w}^*)'\tilde{\mathbf{r}}$ and $1 + \mathbf{w}'\tilde{\mathbf{r}} \sim (\mathbf{w}^*)'\tilde{\mathbf{r}}$, which means $\forall k. E[\tilde{z}|(\mathbf{w}^*)'\tilde{\mathbf{r}} = k] = 0$. It can be derived as following

$$\begin{aligned}
E[\tilde{z}|(\mathbf{w}^*)'\tilde{\mathbf{r}} = k] &= E[\mathbf{w}'\tilde{\mathbf{r}} - (\mathbf{w}^*)'\tilde{\mathbf{r}}|(\mathbf{w}^*)'\tilde{\mathbf{r}}] \\
&= E\left[\sum_{j=1}^{n} w_j\tilde{r}_j - \frac{1}{n}\sum_{j=1}^{n}\tilde{r}_j \,\middle|\, \frac{1}{n}\sum_{j=1}^{n}\tilde{r}_j\right] \\
&= E\left[\sum_{j=1}^{n} w_j\tilde{r}_j \,\middle|\, \frac{1}{n}\sum_{j=1}^{n}\tilde{r}_j\right] - \frac{1}{n}\sum_{j=1}^{n}\tilde{r}_j \\
&= \sum_{j=1}^{n} w_j E\left[\tilde{r}_j \,\middle|\, \frac{1}{n}\sum_{j=1}^{n}\tilde{r}_j\right] - \frac{1}{n}\sum_{j=1}^{n}\tilde{r}_j = 0
\end{aligned}$$

Problem 2

Based on risk premium, we can get

$$E[u(W_0+\tilde{x})] = u(W_0-\rho_u)$$
$$E[v(W_0+\tilde{x})] = v(W_0-\rho_v)$$

According to Theorem 7, if U is more risk averse than V, then we can get an f, such that $f: f' > 0 > f'', u(\bullet) = f(v(\bullet))$. Thus,

$$f(E[v(W_0+\tilde{x})]) = f(v(W_0-\rho_v))$$

By Jensen Inequality, we know that

$$f(E[v(W_0+\tilde{x})]) \geq E[f(v(W_0+\tilde{x}))]$$

Conclude from above sentences, and apply Theorem 7 again, we can get

$$\begin{aligned} u(W_0-\rho_u) &= f(v(W_0-\rho_v)) \\ &= f(E[v(W_0+\tilde{x})]) \\ &\geq E[f(v(W_0+\tilde{x}))] \\ &= E[u(W_0+\tilde{x})] \end{aligned}$$

Problem 3

Subproblem (i)

By definition

$$\tilde{w} = \sum_{j=1}^{n} a_j \tilde{R}_j + \left(W_0 - \sum_{j=1}^{n} a_j\right) R_f = \sum_{j=1}^{n} a_j(\tilde{R}_j - R_f) + W_0 Rf$$

Subproblem (ii)

First calculate the first order derivative of $u(\bullet)$

$$u(x) = \frac{x^{1-\rho}}{1-\rho}, u'(x) = x^{-\rho}$$

Then we can get

$$\begin{aligned} E[u'(\tilde{w})(\tilde{R}_j - R_f)] &= E\left[\left(W_0 R_f + \sum_{j=1}^{n} a_j(\tilde{R}_j - R_f)\right)^{-\rho} (\tilde{R}_j - R_f)\right] \\ &= W_0^{-\rho} E\left[\left(R_f + \sum_{j=1}^{n} \frac{a_j}{W_0}(\tilde{R}_j - R_f)\right)^{-\rho} (\tilde{R}_j - R_f)\right] \end{aligned}$$

where $\frac{a_j}{W_0}$ is independent of W_0, and also $1 - \sum_{j=1}^{n} a_j$.

Subproblem (iii)

Similar to previous problem, calculate the first order differential of $u(\bullet)$

$$u(x) = log(x), u'(x) = \frac{1}{x}$$

Then we can get

$$\begin{aligned}
E[u'(\tilde{w})(\tilde{R}_j - R_f)] &= E\left[\frac{1}{W_0 R_f + \sum\limits_{j=1}^{n} a_j(\tilde{R}_j - R_f)}(\tilde{R}_j - R_f)\right] \\
&= \frac{1}{W_0} E\left[\left(R_f + \sum_{j=1}^{n} \frac{a_j}{W_0}(\tilde{R}_j - R_f)\right)(\tilde{R}_j - R_f)\right]
\end{aligned}$$

where $\frac{a_j}{W_0}$ is independent of W_0, and also $1 - \sum\limits_{j=1}^{n} a_j$.

Subproblem (iv)

Similar to previous problem, calculate the first order differential of $u(\bullet)$

$$u(x) = -e^{-\rho x}, u'(x) = \rho e^{-\rho x}$$

Then we can get

$$\begin{aligned}
E[u'(\tilde{w})(\tilde{R}_j - R_f)] &= E\left[\rho e^{-\rho\left(W_0 R_f + \sum\limits_{j=1}^{n} a_j(\tilde{R}_j - R_f)\right)}(\tilde{R}_j - R_f)\right] \\
&= \rho e^{-\rho(W_0 R_f)} E\left[e^{-\rho\left(\sum\limits_{j=1}^{n} a_j(\tilde{R}_j - R_f)\right)}(\tilde{R}_j - R_f)\right]
\end{aligned}$$

where a_j is independent of W_0, which also means that $\frac{a_j}{\sum\limits_{j=1}^{n} a_j}$ is independent of W_0. In other words, when W_0 varies, only porfolio of asset 0 will be affected.

Problem 4

Subproblem (i)

$$W_1 = a(1 + r_1) + (W_0 - a) \Rightarrow 1140 = 1.2a \Rightarrow 1.2a + (1000 - a) \Rightarrow 0.2a = 140 \Rightarrow a = 700$$

Subproblem (ii)

$$W_2 = b(1 + r_2) + (W_1 - b) \Rightarrow 1294 = \frac{61}{50}b \Rightarrow b + (1140 - b) \Rightarrow \frac{11}{50}b = 154 \Rightarrow b = 700$$

Problem 5

Subproblem (a)

Simply substitute the value required by problems, we can get

$$
\begin{aligned}
p = 0, u(w) = u(w-\pi) \Rightarrow \pi = 0 \Rightarrow \pi(0,x) = 0 \\
p = 1, u(w-x) = u(w-\pi) \Rightarrow \pi = x \Rightarrow \pi(1,x) = 0 \\
x = 0, (1-p)u(w) + pu(w) = u(w) == \Rightarrow \pi = 0 \Rightarrow \pi(0,x) = 0
\end{aligned}
$$

That is, we proved $\pi(0,x) = \pi(1,x) = \pi(p,0) = 0$.

Subproblem (b)

It is easy to find out that

$$\pi(p,x) = \pi - px = w - u^{-1}\big((1-p)u(w) + pu(w-x)\big) - px$$

First, it is obvious that π and px are differentiable anywhere. Besides, since $u' > 0 > u''$, it can be shown that u^{-1} is differentiable, and that is to say, u^{-1} is continuous anywhere. It is also known that the combination of continuous functions is still contiuous. We proved that $\pi(\bullet,\bullet)$ is continuous.

Subproblem (c)

Just calculate the second order derivative of p as following

$$
\begin{aligned}
\frac{\partial \pi}{\partial p} &= -\frac{\partial u^{-1}}{\partial p}\Bigg((1-p)u(w) + pu(w-x)\Bigg)(-u(w) + u(w-x)) - x \\
\frac{\partial^2 \pi}{\partial p^2} &= -\frac{\partial u^{-1}}{\partial p}\Bigg((1-p)u(w) + pu(w-x)\Bigg)(-u(w) + u(w-x))^2
\end{aligned}
$$

Since u^{-1} is a convex function with $\frac{\partial u^{-1}}{\partial p} > 0$, $\frac{\partial u^{-1}}{\partial p}(\bullet) > 0$, and thus $\frac{\partial^2 \pi}{\partial p^2} < 0$.

Subproblem (d)

Calculatet the first order derivative of x as following

$$
\begin{aligned}
\frac{\partial \pi}{\partial x} &= -\frac{\partial u^{-1}}{\partial x}\Bigg((1-p)u(w) + pu(w-x)\Bigg)p\frac{\partial u}{\partial x}\Bigg(w-x\Bigg)(-1) - p \\
&= p(\frac{\partial u^{-1}}{\partial x}\Bigg((1-p)u(w) + pu(w-x)\Bigg)\frac{\partial u}{\partial x}\Bigg(w-x\Bigg) - 1) \\
&\geq p(\frac{\partial u^{-1}}{\partial x}\Bigg((1-p)u(w) + pu(w-x)\Bigg)\frac{\partial u}{\partial x}\Bigg(w-\pi\Bigg) - 1) \\
&= p(1-1) = 0
\end{aligned}
$$

Subproblem (e)

First define a function $f : f(x) = px + k_1$ as the proposed insurance policy price. Mr. D wants to purchase an insured policy if $\pi(p,x) = \pi - px \geq f(x) - px$.

$$
\begin{aligned}
\pi &\geq px + k_1 \\
\pi - px &\geq k_1 \\
\pi(p,x) &\geq k_1
\end{aligned}
$$

(e-1)

If $\exists x^*$, we have $\pi(p,x^*) = k_1$. Besides, since $\pi(p,x)$ is an increasing function, we can prove that

$$\forall x.\ x > x^*, \text{s.t. } \pi(p,x) > \pi(p,x^*) = k_1$$

(e-2)

If $\exists \underline{p}$ and $\bar{p}$, we have $\pi(\underline{p},x) = \pi(x,\bar{p}) = k_1$. Furthermore, since $\pi(p,x)$ is a concave function, we know that

$$\forall p = a\underline{p} + (1-a)\bar{p}, a \in [0,1], \text{s.t. } \pi(p,x) \geq a\pi(\underline{p},x) + (1-a)\pi(\bar{p},x) = k_1$$

Subproblem (f)

No implementation.

Subproblem (g)

For simplicity, we first define three notations: U_{z_1}, U_{z_2}, U_n as following.

$$\begin{aligned}
U_{z_1} &= \frac{\partial u}{\partial z_1}\left(W - x_1 + z_1 - k_1 - \sum_{j=1}^{n} p_j z_j\right) \\
U_{z_2} &= \frac{\partial u}{\partial z_2}\left(W - x_2 + z_2 - k_1 - \sum_{j=1}^{n} p_j z_j\right) \\
U_n &= \frac{\partial u}{\partial z_2}\left(W - k_1 - \sum_{j=1}^{n} p_j z_j\right)
\end{aligned}$$

Then, we calculate the derivatives of z_1 and z_2 individually.

$$\begin{aligned}
\frac{\partial f}{\partial z_1} &= p_1 U_{z_1}(1-p_1) + p_2 U_{z_2}(-p_1) + (1-p_1-p_2)U_n(-p_1) \\
&= p_1\big((U_{z_1} - U_n) - p_1(U_{z_1} - U_n) - p_2(U_{z_2} - U_n)\big) \\
&= p_1\big((1-p_1-p_2)(U_{z_1} - U_n)\big)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial z_2} &= p_1 U_{z_2}(-p_2) + p_2 U_{z_2}(1-p_2) + (1-p_1-p_2)U_n(-p_2) \\
&= p_2\big((U_{z_2} - U_n) - p_1(U_{z_2} - U_n) - p_2(U_{z_2} - U_n)\big) \\
&= p_2\big((1-p_1-p_2)(U_{z_2} - U_n)\big)
\end{aligned}$$

If neither p_1 nor p_2 is 0, then we can remove p_1 and p_2 from equation when solving the extremum and get

$$\begin{aligned}
(1-p_1-p_2)(U_{z_1} - U_n) = 0 \\
(1-p_1-p_2)(U_{z_2} - U_n) = 0
\end{aligned}$$

By simple subtraction, we can find that

$$U_{z_1} = U_{z_2} = U_n = 0$$

and thus, extremum happens when $z_1 = x_1$ and $z_2 = x_2$.
However, based on the requirement of question, $0`p_1 + p_2`1$, it means that either p_1 or p_2 can be 0. Take $p_1 = 0$ and $p_2 \neq 0$ as an example, we can derive only $U_{z_2} = U_n$ and thus z_1 can be any arbitrary number. We conclude that the question statement is wrong.

Subproblem (h)

The process is very similar to the last subproblem, but we think that there may have been some errors in the previous question.

Subproblem (i)

Skipped with the same reason.

Problem 6

Subproblem (i)

We can first find out that $\tilde{\epsilon_1} \sim \mathcal{N}(\eta_1, 1)$, $\tilde{\epsilon_2} \sim \mathcal{N}(\eta_2, 1)$, $\tilde{m} \sim \mathcal{N}(\mu_1, 1)$, $\tilde{e} \sim \mathcal{N}(0,1)$, then calculate expectation and variance of W.

$$\begin{aligned}
\tilde{W} &= W_0 + a_1\tilde{r_1} + a_2\tilde{r_2} \\
E[\tilde{W}] &= W_0 + a_1(\beta_1\mu + \eta_1) + a_2(\beta_2\mu + \eta_2) \\
V[\tilde{W}] &= a_1^2(\beta_1^2 + 1) + a_2^2(\beta_2^2 + 1) + 2a_1a_2\beta_1\beta_2
\end{aligned}$$

After that, we can write down the objective function that we need to maximize.

$$f(a_1, a_2) = W_0 + a_1(\beta_1\mu + \eta_1) + a_2(\beta_2\mu + \eta_2) - \frac{A}{2}\left(a_1^2(\beta_1^2 + 1) + a_2^2(\beta_2^2 + 1) + 2a_1a_2\beta_1\beta_2\right)$$

The derivatives of a_1 and a_2 are as following

$$\begin{aligned}
\frac{\partial f}{\partial a_1} &= \beta_1\mu + \eta_1 - \frac{A}{2}\left(2a_1(\beta_1^2 + 1) + 2a_2\beta_1\beta_2\right) \\
\frac{\partial f}{\partial a_2} &= \beta_2\mu + \eta_2 - \frac{A}{2}\left(2a_2(\beta_2^2 + 1) + 2a_1\beta_1\beta_2\right)
\end{aligned}$$

To begin with, simplify a_1 with first equation being to 0, and get

$$a_1 = \frac{\frac{\beta_1\mu+\eta_1}{A} - a_2\beta_1\beta_2}{\beta_1^2 + 1}$$

Substitute a_1 wtih second equation being 0

$$\begin{aligned}
0 &= \beta_2\mu + \eta_2 - \frac{A}{2}\left(2a_2(\beta_2^2 + 1) + 2\frac{\frac{\beta_1+\eta_1}{A} - a_2\beta_1\beta_2}{\beta_1^2 + 1}\beta_1\beta_2\right) \\
&= (\beta_1^2 + 1)(\beta_2\mu + \eta_2) - a_2A(\beta_1^2 + 1)(\beta_2^2 + 1) - (\beta_1 + \eta_1)\beta_1\beta_2 + a_2\beta_1^2\beta_2^2A \\
&= -a_2A(\beta_1^2 + \beta_2^2 + 1) + \beta_1^2\beta_2\mu + \beta_2\mu + (\beta_1^2 + 1)\eta_2 - \beta_1\mu\beta_1\beta_2 - \eta_1\beta_1\beta_2 \\
&= -a_2A(\beta_1^2 + \beta_2^2 + 1) + \beta_2\mu + (\beta_1^2 + 1)\eta_2 - \eta_1\beta_1\beta_2
\end{aligned}$$

, and we get $a_2^* = \frac{\beta_2\mu + (\beta_1^2+1)\eta_2 - \eta_1\beta_1\beta_2}{A(\beta_1^2+\beta_2^2+1)}$. Similarly, $a_1^* = \frac{\beta_1\mu + (\beta_2^2+1)\eta_1 - \eta_2\beta_1\beta_2}{A(\beta_1^2+\beta_2^2+1)}$.
If $\eta_1 = \eta_2 = 0$, $a_1^* = \frac{\beta_1\mu}{A(\beta_1^2+\beta_2^2+1)}$, $a_2^* = \frac{\beta_2\mu}{A(\beta_1^2+\beta_2^2+1)}$. Since the denominator is always positive, it is obvious that a_i^* have the same sign as β_i.

Subproblem (ii)

If $\beta_1 = \beta_2 = 0$, we can easily find out that $a_1^* = \frac{\eta_1}{A}$, $a_2^* = \frac{\eta_2}{A}$

Subproblem (iii)

We can obtain the conditional mean of $\tilde{\epsilon_1}$ given $\tilde{s}$ is

$$E[\tilde{\epsilon_1}|\tilde{s}] = \eta_1 + \frac{cov[\tilde{\epsilon_1}, \tilde{s}]}{V[\tilde{s}]}(\tilde{s} - E[\tilde{s}]) = \frac{\eta_1 + \tilde{s}}{2}$$

, and also the variance of it

$$V[\tilde{\epsilon_1}|\tilde{s}] = V[\tilde{\epsilon_1}] - V[E[\tilde{\epsilon_1}|\tilde{s}]] = 1 - V\left[\frac{\eta_1 + \tilde{s}}{2}\right] = \frac{1}{2}$$

After that, we can calculate the $\tilde{W}$ given $\tilde{s}$, and obtain

$$E[\tilde{W}|\tilde{s}] = W_0 + a_1(\beta_1\mu + \frac{\eta_1 + \tilde{s}}{2}) + a_2(\beta_2\mu + \eta_2)$$
$$V[\tilde{W}|\tilde{s}] = a_1^2(\beta_1^2 + \frac{1}{2}) + a_2^2(\beta_2^2 + 1) + 2a_1a_2\beta_1\beta_2$$

That is to say the new objective function becomes

$$f(a_1, a_2; \tilde{s}) = W_0 + a_1(\beta_1\mu + \frac{\eta_1 + \tilde{s}}{2}) + a_2(\beta_2\mu + \eta_2) - \frac{A}{2}\left(a_1^2(\beta_1^2 + \frac{1}{2}) + a_2^2(\beta_2^2 + 1) + 2a_1a_2\beta_1\beta_2\right)$$

, ans thus we can obtain a new extremum of a_i^*

$$a_1^* = \frac{\beta_1\mu + (\beta_2^2 + 1)(\frac{\eta_1+\tilde{s}}{2}) - \eta_2\beta_1\beta_2}{A(\beta_1^2 + \frac{\beta_2^2}{2} + \frac{1}{2})}$$
$$a_2^* = \frac{\beta_2\frac{\mu}{2} + (\beta_1^2 + \frac{1}{2})\eta_2 - (\frac{\eta_1+\tilde{s}}{2})\beta_1\beta_2}{A(\beta_1^2 + \frac{\beta_2^2}{2} + \frac{1}{2})}$$

First, since $(\beta_2^2 + 1) > 0$ and $A(\beta_1^2 + \frac{\beta_2^2}{2} + \frac{1}{2}) > 0$, the increase of $\tilde{s}$ will definitely raise a_1^*.
Second, if $\tilde{r_1}$ and $\tilde{r_2}$ are negatively correlated, then $\beta_1\beta_2 < 0$. We can conclude that the increase of $\tilde{s}$ will also raise a_2^*.