Differential equations

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Table of contents

Introduction

Differential equation

- Equations of the form $\dot{x} = f(x, t)$
- x is the state of the system (an "array of floats")
- f is the function that computes the derivative of x with respect to time
- General purpose toolbox for dealing with them

Introduction

How does this apply to us?

- x is the state of all our rigid bodies
- f gives us the derivative of the state: velocity, acceleration, *angular velocity*, torque

Introduction

x	$f(x)$
position	velocity
velocity/linear momentum	acceleration/force
rotation	*angular velocity* $(\omega \star R)$
angular velocity/angular momentum	angular acceleration/torque

Numerical solutions

Numerical solutions

- We cannot just compute the state at time t
- There is no closed form for it, unless the problem is really simple [a]
- We look for an *approximate* solution

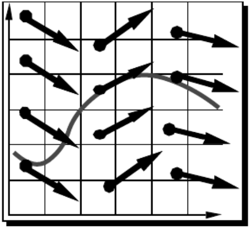
[a] Yeah, you wish :)

Numerical solutions

Field of derivatives

- We know how to compute f
- This means that we can compute the *field of slopes* of x

Slope field



The derivative function

$$\dot{x} = f(x,t)$$

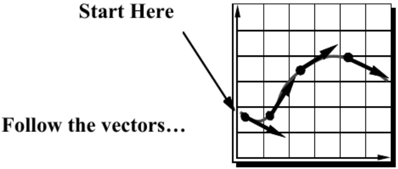
forms a vector field.

Numerical solutions

Field of derivatives

- Start at x_0
- Follow the slope
- Discrete steps

Following the slope

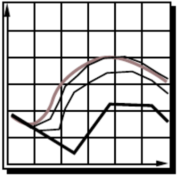


Numerical solutions

Euler's method

- Big, discrete steps along the slope: $x(t+h) = x(t) + h\dot{x}(t)$
- Actually works correctly and *may be acceptable* in some cases
- If it is acceptable, it's simple to code and fast to run
- Otherwise it requires a very small time-step to compensate (more computation!)

Euler's method



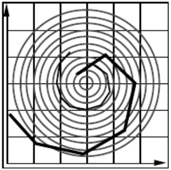
- **Simplest numerical solution method**
- **Discrete time steps**
- **Bigger steps, bigger errors.**

Numerical solutions

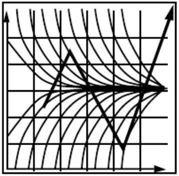
Euler's method

- Why may Euler's method not work?
- Euler's method is
  - Inaccurate, that is it may jump from one trajectory to another
  - Not stable, that is it may increase the energy of the state where it should decrease

Euler's method



Inaccuracy:
Error turns x(t) from a circle into the spiral of your choice.



Instability: off to Neptune!

Numerical solutions

Taylor series

- Any function can be expressed as the infinite sum of its derivatives
- $x(t+h) = x(t) + h\dot{x}(t) + \frac{h^2}{2}\ddot{x}(t) + \cdots + \frac{h^n}{n!}\frac{d^n}{dt^n}x(t) + \dots$

Numerical solutions

Taylor series

- Any function can be expressed as the infinite sum of its derivatives
- $x(t+h) = x(t) + h\dot{x}(t) + \frac{h^2}{2}\ddot{x}(t) + \cdots + \frac{h^n}{n!}\frac{d^n}{dt^n}x(t) + \ldots$
- We can *truncate* this summation to *approximate* the original function
- All differential equations solvers are based on this technique

Numerical solutions

Deriving Euler's method

- $x(t+h) = x(t) + h\dot{x}(t) + \frac{h^2}{2}\ddot{x}(t) + \cdots + \frac{h^n}{n!}\frac{d^n}{dt^n}x(t) + \ldots$
- $x(t+h) \approx x(t) + h\dot{x}(t)$

Numerical solutions

Deriving the midpoint method

- $x(t+h) \approx x(t) + h\dot{x}(t) + \frac{h^2}{2}\ddot{x}(t)$

Numerical solutions

Deriving the midpoint method

- $x(t+h) \approx x(t) + h\dot{x}(t) + \frac{h^2}{2}\ddot{x}(t)$
- We must now compute $\ddot{x}(t)$
- $\ddot{x} = \frac{d}{dt}\dot{x} = \frac{d}{dt}f(x(t)) = f'(x(t))\dot{x} = f'(x(t))f(x(t))$

Numerical solutions

Deriving the midpoint method

- $\ddot{x} = f'(x(t))f(x(t))$
- We now approximate f with Euler's method
- $f(x_0 + \Delta x) = f(x_0) + \Delta x f'(x_0)$

Numerical solutions

Deriving the midpoint method

- We (arbitrarily) choose $\Delta x = \frac{h}{2} f(x_0)$ (which is an acceptable solution, check units of measure!)
- Substitute the approximation of f
- $f(x_0 + \frac{h}{2} f(x_0)) = f(x_0) + \frac{h}{2} f(x_0) f'(x_0)$
- We multiply both sides by h:
- $h(f(x_0 + \frac{h}{2} f(x_0)) - f(x_0)) = \underbrace{\frac{h^2}{2} f(x_0) f'(x_0)}_{\frac{h^2}{2}\ddot{x}}$

Numerical solutions

Deriving the midpoint method

- $x(t+h) \approx x(t) + h\dot{x}(t) + \frac{h^2}{2}\ddot{x}(t)$
- $\ddot{x} = f'(x(t))f(x(t))$
- $h(f(x_0 + \frac{h}{2}f(x_0)) - f(x_0)) = \frac{h^2}{2}f(x_0)f'(x_0) = \frac{h^2}{2}\ddot{x}$

Numerical solutions

Deriving the midpoint method

- $x(t+h) \approx x(t) + h\dot{x}(t) + \frac{h^2}{2}\ddot{x}(t)$
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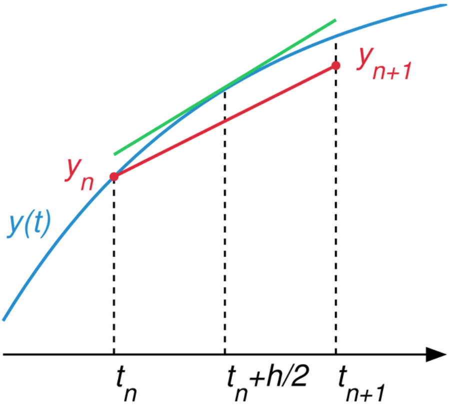
- $x(t+h) \approx x(t) + h\dot{x}(t) + h(f(x_0 + \frac{h}{2}f(x_0)) - f(x_0))$

Numerical solutions

Deriving the midpoint method

- $x(t+h) \approx x(t) + h\dot{x}(t) + h(f(x_0 + \frac{h}{2}f(x_0)) - f(x_0))$
- We sample the derivative at the midpoint of the step, hence the name

RK2



Numerical solutions

About the midpoint method

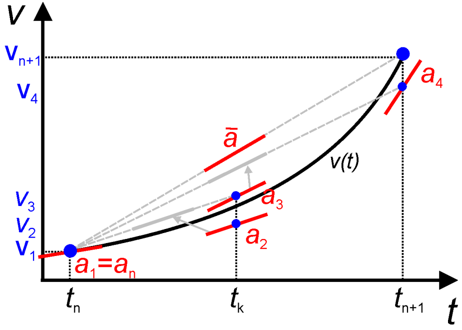
- $x(t+h) \approx x(t) + h\dot{x}(t) + h(f(x_0 + \frac{h}{2}f(x_0)) - f(x_0))$
- We compute f twice; in general though, the higher number of evaluations of f in advanced methods is more than offset by the far smaller time step needed

Numerical solutions

RK4

- The "best" numerical method though is RK4
- $k_1 = hf(x_0)$, $k_2 = hf(x_0 + \frac{k_1}{2})$, $k_3 = hf(x_0 + \frac{k_2}{2})$, $k_4 = hf(x_0 + \frac{k_3}{2})$
- $x(t+h) = x_0 + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4$

RK4



Numerical solutions

To the stars and beyond?

- Why not use RK5, RK6, or even more?
- Is it not more precise after all?

Numerical solutions

To the stars and beyond?

- Why not use RK5, RK6, or even more?
- Is it not more precise after all?
- Suspense :)

Numerical solutions

To the stars and beyond?

- Why not use RK5, RK6, or even more?
- Is it not more precise after all?
- Suspense :)
- NO!
- We can approximate the higher order derivatives with the first derivative only so far
- After a while we are just not inserting any new information
- Very high order methods would only work if we could reliably compute third or fourth order derivatives
- Otherwise we insert spurious oscillations and new errors in the system

That's it

Thank you!