Eigenvalues and eigenvectors.

1. Consider the matrix

$$A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

Show that A doesn't have eigenvectors when considered in $\mathrm{Mat}_{n\times n}(\mathbb{R})$. Show that A is diagonalizable when considered in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ and find the eigenvectors of A.

2. Give the eigenvalues of $\mathrm{lin}(\mathrm{Pr}_{H,\mathbf{v}})$, $\mathrm{lin}(\mathrm{Ref}_{H,\mathbf{v}})$. What can you say about the eigenvectors?

3. Find the eigenvalues and eigenvectors of the following matrices in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$:

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix},\quad \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix},\quad \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}.$$

4. Let $\phi:\mathbb{R}^3\to\mathbb{R}^3$ be the linear map

$$\phi(x,y,z)=(x+y-z,y+z,2x).$$

Find the matrix $\mathrm{M}_{\boldsymbol{b},\boldsymbol{b}}(\phi)$ where

$$\boldsymbol{b}=\{(1,1,0),(-1,0,1),(1,1,1)\}.$$

5. Calculate the eigenvalues and their algebraic and geometric multiplicities for the following matrices in $\mathrm{Mat}_{3\times 3}(\mathbb{R})$, and deduce whether or not they are diagonalizable:

$$\begin{bmatrix}-6 & 2 & -5\\ -4 & 4 & -2\\ 10 & -3 & 8\end{bmatrix},\quad \begin{bmatrix}1 & 0 & 0\\ 0 & -3 & -15\\ 0 & 2 & 8\end{bmatrix}.$$

Rotations.

6. Show that an isometry is bijective.

7. Determine the matrix form of a rotation with angle 45° having the same center of rotation as the rotation

$$f(\mathbf{x})=\frac{1}{\sqrt{13}}\begin{bmatrix}2 & -3\\ 3 & 2\end{bmatrix}\mathbf{x}+\begin{bmatrix}1\\ -2\end{bmatrix}.$$

8. Determine the cosine of the angle of the rotation f given in the previous exercise and find the inverse rotation, f^{-1}.

9. Let T be the isometry obtained by applying a rotation of angle $-\frac{\pi}{3}$ around the origin after a transation with vector $(-2,5)$. Determine the inverse transformation, T^{-1}.

10. Find the eigenvectors for each of the following symmetric matrices:

$$A=\begin{bmatrix}73 & 36\\ 36 & 52\end{bmatrix},\quad B=\begin{bmatrix}-94 & 180\\ 180 & 263\end{bmatrix}\quad\text{and}\quad C=\begin{bmatrix}128 & 240\\ 240 & 450\end{bmatrix}.$$

11. Determine the sum-of-angles formulas for sine and cosine using rotation matrices.

12. Verify that the matrices

$$A = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{11}\begin{bmatrix} -9 & -2 & 6 \\ 6 & -6 & 7 \\ 2 & 9 & 6 \end{bmatrix}$$

belong to $SO(3)$. Moreover, determine the axis of rotation and the rotation angle.

1. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Show that A doesn't have eigenvectors when considered in $\mathrm{Mat}_{n\times n}(\mathbb{R})$. Show that A is diagonalizable when considered in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ and find the eigenvectors of A.

- $P_A = \det(A - T \cdot I_n) = \det \begin{vmatrix} -T & 1 \\ -1 & -T \end{vmatrix} = T^2 + 1$
- P_A does not have roots in $\mathbb{R} \Rightarrow A$ is not diagonalizable
- P_A has roots in $\mathbb{C}$ so A viewed in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ is diagonalizable

$$T^2+1=0 \qquad \lambda_1 = i \qquad \lambda_2 = -i$$

- to find the eigenvectors we solve the system

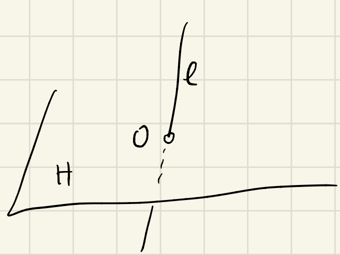
$$A \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = i \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Leftrightarrow \begin{cases} v_2 = i v_1 \\ -v_1 = i v_2 \end{cases}$$ the solutions are $\begin{bmatrix} v_1 \\ i v_1 \end{bmatrix}$

$$\Leftrightarrow V_i(A) = \left\langle \begin{bmatrix} 1 \\ i \end{bmatrix} \right\rangle$$

$$B \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -i \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Leftrightarrow \begin{cases} v_2 = -i v_1 \\ -v_1 = -i v_2 \end{cases}$$ the solutions are $\begin{bmatrix} v_1 \\ -i v_1 \end{bmatrix}$

$$\Leftrightarrow V_{-i}(B) = \left\langle \begin{bmatrix} 1 \\ -i \end{bmatrix} \right\rangle$$

2. Give the eigenvalues of lin($Pr_{H,\mathbf{v}}$), lin($Ref_{H,\mathbf{v}}$). What can you say about the eigenvectors?



- let $\phi = \text{lin}(Pr_{H,v})$ then $\phi(\overrightarrow{AB}) = \overrightarrow{Pr_{H,v}(A)\, Pr_{H,v}(B)}$
- let W be the vector subspace of vectors represented in H

  let $w = \{w_2, \dots, w_n\}$ be a basis of W
- let l be a line with direction vector v
- let $\{O\} = l \cap H$ then $v = \overrightarrow{OP_1}$ $\quad w_i = \overrightarrow{OP_i}$ $\quad i = 2, \dots, n$

$$\Rightarrow \quad \phi(\overrightarrow{OP_1}) = \overrightarrow{Pr_{H,v}(O)\, Pr_{H,v}(P_1)} = \overrightarrow{OO} = 0$$

$$\phi(\overrightarrow{OP_i}) = \overrightarrow{Pr_{H,v}(O)\, Pr_{H,v}(P_i)} = \overrightarrow{OP_i} = \overrightarrow{OP_i} \quad \forall i = 2, \dots, n$$

- Since $v \nparallel H$, $b = \{v, w_2, \dots, w_n\}$ is a basis of V and

$$[\phi]_b = \begin{bmatrix} 0 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \; 1 \end{bmatrix} = \begin{bmatrix} 0 & 0_{1,n-1} \\ 0_{n-1,1} & I_{n-1} \end{bmatrix}$$

- ϕ has two eigenvalues: 0 and 1

$$V_0(\phi) = \langle v \rangle \qquad V_1(\phi) = W$$

- let $\phi = \text{lin}\, Ref_{H,v}$ with W, w_i, P_i, O as above $\phi(\overrightarrow{OP_i}) = \overrightarrow{OP_i}$ $i = 2, n$

and $\phi(\overrightarrow{OP_1}) = \overrightarrow{Ref_{H,v}(O)\, Ref_{H,v}(P_1)} = \overrightarrow{OP'} = -\overrightarrow{OP}$ since O is midpoint of $[PP']$

$\Rightarrow$ $[\phi]_b = \begin{bmatrix} -1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & \ddots \; 1 \end{bmatrix}$, spectrum of ϕ is $\{1, -1\}$, $V_{-1}(\phi) = \langle v \rangle$ $\; V_1(\phi) = W$

3. Find the eigenvalues and eigenvectors of the following matrices in $\mathrm{Mat}_{2\times 2}(\mathbb{R})$:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ the matrix is already diagonal

- so the eigenvalues are 1 and -1
- $V_1(A) = \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$ $\quad V_{-1}(A) = \langle \begin{bmatrix} 0 \\ -1 \end{bmatrix} \rangle$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad P_A = \det(A - T\cdot I_2) = \det\begin{bmatrix} 1-T & 1 \\ 0 & 1-T \end{bmatrix} = (1-T)^2$

- so A has only one eigenvalue $\lambda = 1$

$A \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 1 \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \iff \begin{cases} v_1 + v_2 = v_1 \\ v_2 = v_2 \end{cases}$ the solutions are $\begin{bmatrix} v_1 \\ 0 \end{bmatrix}$

$\Rightarrow V_1(A) = \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$

notice that $\underbrace{\dim V_1(A)}_{\text{geometric multiplicity}} \lneq \underbrace{h_A(1)}_{\text{algebraic multiplicity}}$

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\quad P_A = \det(A - T\cdot I_2) = \det\begin{bmatrix} 1-T & 0 \\ 1 & 1-T \end{bmatrix} = (1-T)^2$

- so A has only one eigenvalue $\lambda = 1$

$A \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 1 \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \iff \begin{cases} v_1 = v_1 \\ v_1 + v_2 = v_2 \end{cases}$ the solutions are $\begin{bmatrix} 0 \\ v_2 \end{bmatrix}$

$\Rightarrow V_1(A) = \langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle$

notice that $\underbrace{\dim V_1(A)}_{\text{geometric multiplicity}} \lneq \underbrace{h_A(1)}_{\text{algebraic multiplicity}}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad P_A = \det(A - T \cdot I_2) = \det \begin{bmatrix} 1-T & 1 \\ 1 & 1-T \end{bmatrix} = (1-T)^2 - 1$

$$= T^2 - 2T = T(T-2)$$

- so A has two eigenvalues $\lambda_1 = 0$ and $\lambda = 2$

$A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \iff \begin{cases} v_1 + v_2 = 0 \\ v_1 + v_2 = 0 \end{cases}$ the solutions are $\begin{bmatrix} v_1 \\ -v_1 \end{bmatrix}$

$\Rightarrow V_0(A) = \left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle$

$A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \iff \begin{cases} v_1 + v_2 = 2v_1 \\ v_1 + v_2 = 2v_2 \end{cases}$ the solutions are $\begin{bmatrix} v_1 \\ v_1 \end{bmatrix}$

$\Rightarrow V_2(A) = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$

4. Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map

$$\phi(x, y, z) = (x + y - z, y + z, 2x).$$

Find the matrix $M_{\boldsymbol{b},\boldsymbol{b}}(\phi)$ where

$$\boldsymbol{b} = \{(1,1,0), (-1,0,1), (1,1,1)\}.$$

Let e be the canonical basis of $\mathbb{R}^3$ $\quad e = \left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right)$

then

$$[\phi_e(x, y, z)]_e = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$[\phi]_b = M_{e,b}^{-1} [\phi]_e M_{e,b}$$

$$M_{e,b} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow M_{e,b}^{-1} = \frac{1}{\det M_{e,b}} \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & 2 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\det M_{e,b} = 0 + 1 + 0 - 0 - (-1) - 1$$

$$\Rightarrow [\phi]_b = \begin{bmatrix} -1 & 2 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} -1 & 2 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -2 & 1 \\ 1 & 1 & 2 \\ 2 & -2 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 6 & 1 \\ -1 & 3 & 1 \\ 3 & -5 & 1 \end{bmatrix}$$

5. Calculate the eigenvalues and their algebraic and geometric multiplicities for the following matrices in $\mathrm{Mat}_{3\times 3}(\mathbb{R})$, and deduce whether or not they are diagonalizable:

$$\begin{bmatrix} -6 & 2 & -5 \\ -4 & 4 & -2 \\ 10 & -3 & 8 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -15 \\ 0 & 2 & 8 \end{bmatrix}.$$

$$P_A = \det(A - T\cdot I_3) = \det\begin{bmatrix} -6-T & 2 & -5 \\ -4 & 4-T & -2 \\ 10 & -3 & 8-T \end{bmatrix} =$$

$$= (-6-T)\begin{vmatrix} 4-T & -2 \\ -3 & 8-T \end{vmatrix} - 2\begin{vmatrix} -4 & -2 \\ 10 & 8-T \end{vmatrix} - 5\begin{vmatrix} -4 & 4-T \\ 10 & -3 \end{vmatrix}$$

$$= -(6+T)\left(32-12T+T^2-6\right) - 2\left(4T-32+20\right) - 5\left(12-40+10T\right)$$

$$= \dots = -T^3 + 6T^2 - 12T + 8$$

$$T^3 - 3\cdot 2T^2 + 3\cdot 2^2 T + 2^3 = 0$$

$$\Leftrightarrow \quad (T-2)^3 = 0 \quad \Rightarrow \quad 2 \text{ is the only eigenvalue of } A$$

To find the eigenvectors we solve

$$A\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2\begin{bmatrix} x \\ y \\ z \end{bmatrix} \Leftrightarrow \begin{bmatrix} -8 & 2 & -5 \\ -4 & 2 & -2 \\ 10 & -3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} -8 & 2 & -5 \\ -4 & 2 & -2 \\ 10 & -3 & 6 \end{bmatrix} \underset{L_3 + L_1 + \frac{1}{2}L_2}{\sim} \begin{bmatrix} -8 & 2 & -5 \\ -4 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

has rank 2 $\Rightarrow$ the solution space has dimension 1

$$\Rightarrow \dim V_2(A) = 1 \underset{\neq}{<} h_2(A) = 3$$

- $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -15 \\ 0 & 2 & 8 \end{bmatrix}$ We see that 1 is an eigenvalue and that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector for 1.

$$P_B = \det \begin{bmatrix} 1-T & 0 & 0 \\ 0 & -3-T & -15 \\ 0 & 2 & 8-T \end{bmatrix} = (1-T)\left(\underbrace{(3+T)(T-8)+30}_{T^2+5T-24+30}\right) = (1-T)(T+2)(T+3)$$

$\Rightarrow$ the spectrum of B is $\{1, -2, -3\}$

B is diagonalizable by prop. 7.9

6. Show that an isometry is bijective.

$\downarrow$

map which preserves distances $f : \mathbb{E}^n \to \mathbb{E}^n$

$d(P,Q) = d(f(P), f(Q))$

injectivity: $f(P) = f(Q) \overset{?}{\Rightarrow} P = Q$

if $f(P) = f(Q) \Rightarrow d(f(P), f(Q)) = 0$

$\Rightarrow d(P,Q) = 0 \Rightarrow P = 0$

surjectivity: isometries are affine maps $f \in AGL(\mathbb{E}^n)$

$\Rightarrow f(x) = Ax + b$ w.r.t some coordinate system

- it is the composition of two maps $f_1(x) = Ax$ and $f_2(x) = x + b$
- f_2 is bijective $\Rightarrow$ $\begin{cases} f \text{ is surjective} \Leftrightarrow f_1(x) \text{ is surj.} \\ f \text{ is injective} \Leftrightarrow f_1(x) \text{ is injective} \end{cases}$
- by the first part f_1 is injective, it is an injective linear map $\mathbb{R}^n \to \mathbb{R}^n$ $\Rightarrow$ it is also surj.

7. Determine the matrix form of a rotation with angle 45° having the same center of rotation as the rotation

$$f(\mathbf{x}) = \frac{1}{\sqrt{13}}\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

- that f is a rotation, follows from the classification of isometries in dim 2
- the center of the rotation f is the point satisfying the equation

$$f(x) = x$$

$$\Leftrightarrow \frac{1}{\sqrt{13}}\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} \left(\frac{2}{\sqrt{13}} - 1\right)x_1 - \frac{3}{\sqrt{13}}x_2 = -1 \\ \frac{3}{\sqrt{13}}x_1 + \left(\frac{2}{\sqrt{13}} - 1\right)x_2 = 2 \end{cases}$$

$$\begin{vmatrix} -1 & -\frac{3}{\sqrt{13}} \\ 2 & \frac{2}{\sqrt{13}} - 1 \end{vmatrix} = -\frac{2}{\sqrt{13}} + 1 + \frac{6}{\sqrt{13}} = \frac{4}{\sqrt{13}} + 1 = \frac{4 + \sqrt{13}}{\sqrt{13}}$$

$$\begin{vmatrix} \frac{2}{\sqrt{13}} - 1 & -1 \\ \frac{3}{\sqrt{13}} & 2 \end{vmatrix} = \frac{4}{\sqrt{13}} - 2 + \frac{3}{\sqrt{13}} = \frac{7 - 2\sqrt{13}}{\sqrt{13}}$$

$$\begin{vmatrix} \frac{2}{\sqrt{13}} - 1 & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} - 1 \end{vmatrix} = \frac{1}{13}\begin{vmatrix} 2 - \sqrt{13} & -3 \\ 3 & 2 - \sqrt{13} \end{vmatrix} = \frac{1}{13}\left[4 - 4\sqrt{13} + 13 + 9\right] = \frac{1}{13}(26 - 4\sqrt{13})$$

$$= \frac{2\sqrt{13} - 4}{\sqrt{13}} = 2\left(\frac{\sqrt{13} - 2}{\sqrt{13}}\right)$$

$$\Rightarrow x_1 = \frac{4 + \sqrt{13}}{2(\sqrt{13} - 2)} = \frac{1}{2}\frac{(4 + \sqrt{13})(\sqrt{13} + 2)}{(13 - 4)} = \frac{1}{18}\left(4\sqrt{13} + 8 + 13 + 2\sqrt{13}\right) = \frac{1}{18}\left(21 + 6\sqrt{13}\right)$$

$$\Rightarrow x_1 = \frac{1}{6}\left(7 + 2\sqrt{13}\right)$$

and $$x_2 = \frac{7 - 2\sqrt{13}}{2(\sqrt{13} - 2)} = \frac{1}{18}(7 - 2\sqrt{13})(\sqrt{13} + 2) = \frac{1}{18}\left(7\sqrt{13} + 14 - 26 - 4\sqrt{13}\right) = \frac{1}{18}\left(-12 + 3\sqrt{13}\right)$$

$$\Rightarrow x_2 = \frac{1}{6}\left(-4 + \sqrt{13}\right)$$

$\Rightarrow$ the center of the rotation f is $C_f = \begin{bmatrix} \frac{7 + 2\sqrt{13}}{6} \\ \frac{-4 + \sqrt{13}}{6} \end{bmatrix}$

A rotation of angle 45° around C_f is obtained by
1. translating C_f in the origin
2. rotating with angle 45° around the origin
3. translating back.

$$\begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \underbrace{\underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \Bigg(\underbrace{\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{7+2\sqrt{13}}{6} \\ \frac{-4+\sqrt{13}}{6} \end{bmatrix}}_{1} \Bigg)}_{2} + \begin{bmatrix} \frac{7+2\sqrt{13}}{6} \\ \frac{-4+\sqrt{13}}{6} \end{bmatrix}}_{3}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \frac{1}{6\sqrt{2}} \begin{bmatrix} 11+\sqrt{13} \\ 3+3\sqrt{13} \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 7+2\sqrt{13} \\ -4+\sqrt{13} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{6\sqrt{2}} \begin{bmatrix} -11-\sqrt{13}+7\sqrt{2}+2\sqrt{26} \\ -3-3\sqrt{13}-4\sqrt{2}+\sqrt{26} \end{bmatrix}$$

$$f(\mathbf{x}) = \frac{1}{\sqrt{13}} \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

8. Determine the cosine of the angle of the rotation f given in the previous exercise and find the inverse rotation, f^{-1}.

denote it by θ

then $\cos\theta = \frac{2}{\sqrt{13}}$

$f(x) = Ax + b$ if $f^{-1}(x) = \tilde{A}x + \tilde{b}$ then $x = f^{-1} \circ f(x) = \tilde{A}(Ax+b) + \tilde{b}$

$= \tilde{A}Ax + \tilde{A}b + \tilde{b}$

$\Rightarrow \tilde{A} = A^{-1}$ and $\tilde{b} = -\tilde{A}b$

$$\left(\frac{1}{\sqrt{13}} \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \right)^{-1} = \frac{1}{\frac{4+9}{13}} \begin{bmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{bmatrix}^T = \frac{1}{\sqrt{13}} \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$$

we expected this since $A \in O(2)$
so $\det A = 1$

we expected this
since $A \in O(2) \Rightarrow A^{-1} = A^T$

$$\tilde{b} = -\frac{1}{\sqrt{13}}\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{13}}\begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$\Rightarrow f^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{13}}\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{\sqrt{13}}\begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

Rem $\quad$ $\operatorname{tr} A = \operatorname{tr} A^T = \operatorname{tr} A^{-1}$ for $A \in O(n)$

$$\Rightarrow \cos\theta_A = \cos\theta_{A^{-1}}$$

obviously $\theta_{A^{-1}} = -\theta_A$ but with the cosine-trace formula we don't see this

9. Let T be the isometry obtained by applying a rotation of angle $-\frac{\pi}{3}$ around the origin after a transation with vector $(-2, 5)$. Determine the inverse transformation, T^{-1}.

$$T(x) = Rot_{-\frac{\pi}{3}} \circ T_{(-2,5)}(x)$$

$$\Rightarrow T^{-1}(x) = \left(Rot_{-\frac{\pi}{3}} \circ T_{(-2,5)}\right)^{-1}(x) = T^{-1}_{(-2,5)} \circ Rot^{-1}_{-\frac{\pi}{3}}(x)$$

$$= T_{(2,-5)} \circ Rot_{\frac{\pi}{3}}(x)$$

$$= \begin{bmatrix} 1/2 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1/2 \end{bmatrix} x + \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

10. Find the eigenvectors for each of the following symmetric matrices:

$$A = \begin{bmatrix} 73 & 36 \\ 36 & 52 \end{bmatrix}, \quad B = \begin{bmatrix} -94 & 180 \\ 180 & 263 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 128 & 240 \\ 240 & 450 \end{bmatrix}.$$

A:

$$\begin{vmatrix} 73-T & 36 \\ 36 & 52-T \end{vmatrix} = (73-T)(52-T) - 6^4 = 4\cdot 13\cdot 73 - 125T + T^2 - 6^4$$

$$= T^2 - 125T + 4\underbrace{(949 - \underbrace{2^2\cdot 3^4}_{324})}_{625}$$

$$= T^2 - 5^3 T + 4\cdot 5^4$$

$$\Delta = 5^6 - 2^4\cdot 5^4$$
$$= 5^4(25-16)$$

$$\Rightarrow T_{1,2} = \frac{5^3 \pm 5^2 3}{2}$$

$$= \frac{25(5\pm 3)}{2} \begin{cases} 100 \\ 25 \end{cases}$$

B: 338, −169

C: 578, 0

11. Determine the sum-of-angles formulas for sine and cosine using rotation matrices.

$$\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}\begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} = \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & \cdots \\ \sin\theta_1\cos\theta_1 + \sin\theta_2\cos\theta_1 & \cdots \end{bmatrix} = \begin{bmatrix} \cos(\theta_1+\theta_2) & \cdots \\ \sin(\theta_1+\theta_2) & \cdots \end{bmatrix}$$

12. Verify that the matrices

$$A = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{11}\begin{bmatrix} -9 & -2 & 6 \\ 6 & -6 & 7 \\ 2 & 9 & 6 \end{bmatrix}$$

belong to $SO(3)$. Moreover, determine the axis of rotation and the rotation angle.

- $A \in SO(3) \iff AA^t = I_3$ & $\det A = 1$

$$A^T = \frac{1}{3}\begin{bmatrix} -1 & -2 & -2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{bmatrix}$$

$$A \cdot A^T = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} -1 & -2 & -2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

On checks that $\det A = 1$ so, yeah, $A \in SO(3)$

- the axis of rotation is the line passing through the origin in the direction of the eigenvectors for the eigenvalue 1

this is obtained by solving $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \iff \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\iff \begin{bmatrix} -4 & 2 & -2 \\ -2 & -5 & -1 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 & -2 \\ -2 & -5 & -1 \\ -2 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 1 \\ -2 & -5 & -1 \\ -2 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 1 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow y = 0 \Big\} \Rightarrow 2x + z = 0 \Rightarrow z = -2x$$

$\Rightarrow$ eigenspace for $\lambda = 1$ is $V_1 = \{(t, 0, -2t) : t \in \mathbb{R}\}$ ← this is a line passing through the origin, it is the rotation axis

the eigenvectors are the non-zero vectors in V_1

- the angle of rotation θ is determined by

$$\operatorname{tr}(A) = 1 + 2\cos\theta \iff \cos\theta = \frac{\operatorname{tr}A - 1}{2} = \frac{-\frac{1}{3} - 1}{2} = -\frac{2}{3}$$

so $\quad \theta = \arccos\left(-\frac{2}{3}\right)$

the calculation for B is similar.

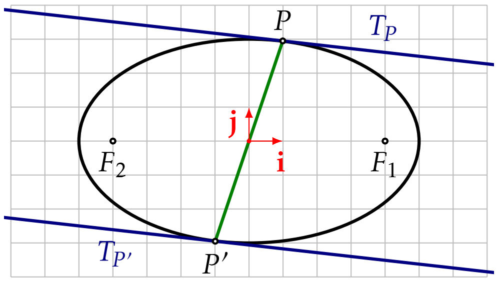
1. Determine the foci (focal points) of the Ellipse $9x^2 + 25y^2 - 225 = 0$

2. Determine the intersection of the line $\ell : x + 2y - 7 = 0$ and the ellipse $\mathcal{E} : x^2 + 3y^2 - 25 = 0$.

3. Determine the position of the line $\ell : 2x + y - 10 = 0$ relative to the ellipse $\mathcal{E} : \frac{x^2}{9} + \frac{y^2}{4} - 1 = 0$.

4. Determine an equation of a line which is orthogonal to $\ell : 2x - 2y - 13 = 0$ and tangent to the ellipse $\mathcal{E} : x^2 + 4y^2 - 20 = 0$.

5. A *diameter* of an ellipse is the line segment determined by the intersection points of the ellipse with a line passing through the center of the ellipse. Show that the tangent lines to an ellipse at the endpoints of a diameter are parallel.



6. Using the gradient, prove the reflective properties of an ellipse.

7. Determine the common tangents to the ellipses

$$\frac{x^2}{45} + \frac{y^2}{9} = 1 \quad \text{and} \quad \frac{x^2}{9} + \frac{y^2}{18} = 1.$$

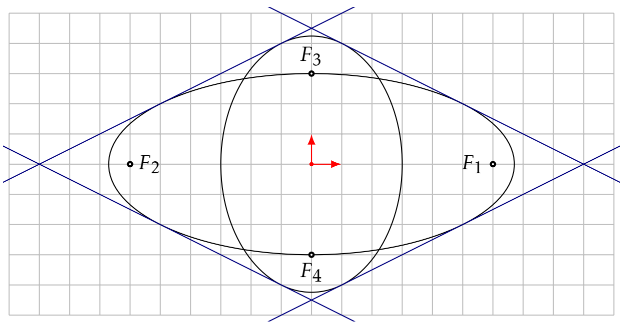


Figure 0.1: Gemeinsame Tangenten.

8. Consider the ellipse $\mathcal{E} : \frac{x^2}{4} + y^2 - 1$ with focal points F_1 and F_2. Determine the points M, situated on the ellipse for which

1. the angle $\measuredangle F_1MF_2$ is right;
2. the angle $\measuredangle F_1MF_2$ is θ;
3. the angle $\measuredangle F_1MF_2$ is maximal.

9. Consider the ellipse $\mathcal{E} : x^2 + 4y^2 = 25$. Find the chords on the ellipse which have the point $A(7/2, 7/4)$ as midpoint.

10. Consider the ellipse $\mathcal{E} : \frac{x^2}{25} + \frac{y^2}{9} = 1$. Determine the geometric locus of the midpoints of the chords on the ellipse which are parallel to the line $\ell : x + 2y = 1$.

11. Find the equation of the circle:

1. passing through $A(3,1)$ and $B(-1,3)$ and having the center on the line $\ell : 3x - y - 2 = 0$;
2. passing through $A(1,1)$, $B(1,-1)$ and $C(2,0)$;
3. tangent to both $\ell_1 : 2x+y-5 = 0$ and $\ell_2 : 2x+y+15 = 0$ if the tangency point with ℓ_1 is $M(3,-1)$.

1. Determine the foci (focal points) of the Ellipse $9x^2 + 25y^2 - 225 = 0$ $(*)$

$(*) \Leftrightarrow \quad \dfrac{x^2}{\frac{15^2}{3^2}} + \dfrac{y^2}{\frac{15^2}{5^2}} = 1 \Leftrightarrow \quad \dfrac{x^2}{5^2} + \dfrac{y^2}{3^2} = 1$ so $a=5$ and $b=3$

$b^2 = a^2 - c^2$ or $c^2 = a^2 - b^2$ so $c^2 = 16$

$\Rightarrow$ the focal points are $F_1(4,0)$ and $F_2(-4,0)$

2. Determine the intersection of the line $\ell : x + 2y - 7 = 0$ and the ellipse $\mathcal{E} : x^2 + 3y^2 - 25 = 0$.

$\ell \cap \mathcal{E} : \begin{cases} x^2 + 3y^2 - 25 = 0 \\ x + 2y - 7 = 0 \Leftrightarrow x = 7 - 2y \end{cases} \Leftrightarrow \begin{cases} (7-2y)^2 + 3y^2 - 25 = 0 & (*) \\ x = 7 - 2y \end{cases}$

$(*) \Leftrightarrow \quad 49 - 28y + 4y^2 + 3y^2 - 25 = 0$

$7y^2 - 28y + 24 = 0$

$\Delta = 4^2 \cdot 7^2 - 4 \cdot 7 \cdot 24 = 4^2 \cdot 7(7-6) \Rightarrow y_{1,2} = \dfrac{4 \cdot 7 \pm 4 \cdot \sqrt{7}}{2 \cdot 7} = 2 \pm \dfrac{2}{\sqrt{7}}$

$\Rightarrow$ the two intersection points are $A(7-2y_1, y_1)$ and $B(7-2y_2, y_2)$

So $A\left(3 - \frac{4}{\sqrt{7}}, 2 + \frac{2}{\sqrt{7}}\right)$ and $B\left(3 + \frac{4}{\sqrt{7}}, 2 - \frac{2}{\sqrt{7}}\right)$

3. Determine the position of the line $\ell : 2x + y - 10 = 0$ relative to the ellipse $\mathcal{E} : \frac{x^2}{9} + \frac{y^2}{4} - 1 = 0$.

$\ell \cap \mathcal{E} : \begin{cases} 2x + y - 10 = 0 \Rightarrow y = 10 - 2x \\ \frac{x^2}{9} + \frac{y^2}{4} - 1 = 0 \Rightarrow 4x^2 + 9y^2 - 36 = 0 \end{cases} \Leftrightarrow \begin{cases} y = 10 - 2x \\ 4x^2 + 9(10-2x)^2 - 36 = 0 & (*) \end{cases}$

$(*) \Leftrightarrow 4x^2 + 9(100 - 40x + 4x^2) - 36 = 0$

$4x^2 + 900 - 360x + 36x^2 - 36 = 0$

$40x^2 - 360x + 864 = 0 \Leftrightarrow 5x^2 - 45x + 2^2 \cdot 3^3 = 0$

$\begin{array}{r|l} 360 & 4 \cdot 9 \\ 10 & 2 \cdot 5 \end{array}$

$$\Delta = 3^4 5^2 - 2^2\cdot 5\cdot 2^2\cdot 3^3 = 3^3\cdot 5\left(3\cdot 5 - 2^4\right) < 0$$

$\Rightarrow$ l does not intersect $\mathcal{E}$

5. A *diameter* of an ellipse is the line segment determined by the intersection points of the ellipse with a line passing through the center of the ellipse. Show that the tangent lines to an ellipse at the endpoints of a diameter are parallel.

The endpoints of a diameter are symmetric relative to the origin

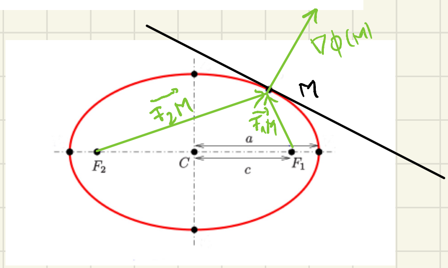
If $M(x_0, y_0)$ is one endpoint then $M(-x_0, -y_0)$ is the other endpoint.

now $T_M \mathcal{E}_{a,b}: \quad \dfrac{x x_0}{a^2} + \dfrac{y y_0}{b^2} = 1$

and $T_M \mathcal{E}_{a,b}: \quad \dfrac{x(-x_0)}{a^2} + \dfrac{y(-y_0)}{b^2} = 1 \iff \dfrac{x x_0}{a^2} + \dfrac{y y_0}{b^2} = -1$

both lines admit $\left(\dfrac{x_0}{a^2}, \dfrac{y_0}{b^2}\right)$ as normal vectors so they are parallel.

6. Using the gradient, prove the reflective properties of an ellipse.



$$\mathcal{E}_{a,b}: \quad d(M, F_1) + d(M, F_2) - 2a = 0$$

$$\iff \underbrace{\sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} - 2a}_{\phi(x,y)} = 0$$

$$\nabla\phi(x,y) = \left(\frac{\partial\phi}{\partial x}(x,y), \frac{\partial\phi}{\partial y}(x,y)\right)$$

$$\frac{\partial\phi}{\partial x}(x,y) = \frac{\partial}{\partial x}\left(\sqrt{(x-c)^2+y^2} + \sqrt{(x-c)^2+y^2} - 2a\right) = \frac{x-c}{\sqrt{(x-c)^2+y^2}} + \frac{x+c}{\sqrt{(x+c)^2+y^2}}$$

$$\frac{\partial\phi}{\partial x}(x,y) = \frac{y}{\sqrt{(x-c)^2+y^2}} + \frac{y}{\sqrt{(x+c)^2+y^2}} =$$

$$\Rightarrow \nabla\phi(M) = \left(\frac{x_M - c}{\|\overrightarrow{F_1M}\|} + \frac{x+c}{\|\overrightarrow{F_2M}\|}, \frac{y}{\|\overrightarrow{F_1M}\|} + \frac{y}{\|\overrightarrow{F_2M}\|}\right)$$

$$= \frac{1}{\|\overrightarrow{F_1M}\|}(x_M - c, y) + \frac{1}{\|\overrightarrow{F_2M}\|}(x+c, y)$$

$$= \frac{\overrightarrow{F_1M}}{\|\overrightarrow{F_1M}\|} + \frac{\overrightarrow{F_2M}}{\|\overrightarrow{F_2M}\|}$$

$\Rightarrow \nabla\phi(M)$ is a direction vector for the angle $\widehat{F_1MF_2}$ (*)

Reflecting in $\mathcal{E}$ at point $M \Leftrightarrow$ reflecting in $T_M\mathcal{E}$

$\Leftrightarrow$ incoming angle $\equiv$ outgoing angle $\Leftrightarrow$ (*)

7. Determine the common tangents to the ellipses

$$\frac{x^2}{45} + \frac{y^2}{9} = 1 \quad \text{and} \quad \frac{x^2}{9} + \frac{y^2}{18} = 1.$$

Consider the equation of tangent lines if the slope is given

$$y = kx \pm \sqrt{a^2k^2 + b^2}$$

For the first ellipse we have

$$y = kx \pm \sqrt{45k^2 + 9}$$

Such a line is tangent to the second ellipse if the intersection point with the second ellipse is a double intersection point

$\Leftrightarrow$ discriminant is zero for the equation

$$\frac{x^2}{9} + \frac{\left(kx \pm \sqrt{45k^2+9}\right)^2}{18} = 1$$

$$(2+k^2)\,x^2 \pm 2\sqrt{45k^2+9}\,kx + 45k^2 - 9 = 0$$

$$\ldots \quad \Delta = -72\,(2k-1)(2k+1)$$

so the two slopes are $k=\pm\frac{1}{2}$ and the four tangent lines are

$$\pm x+2y\pm 9=0$$

Method II the tangent lines for the two ellipses are

$$y=kx\pm\sqrt{45k^2+9} \quad \text{and} \quad y=kx\pm\sqrt{9k^2+18}$$

they are equal if $\sqrt{45k^2+9}=\sqrt{9k^2+18}$

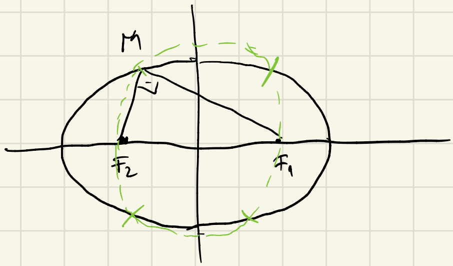
$$45k^2+9=9k^2+18$$

$$36k^2=9$$

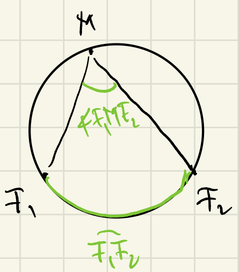
$$k^2=\frac{1}{4} \iff k=\pm\frac{1}{2}$$

8. Consider the ellipse $\mathcal{E}:\frac{x^2}{4}+y^2-1=0$ with focal points F_1 and F_2. Determine the points M, situated on the ellipse for which

1. the angle $\measuredangle F_1MF_2$ is right;
2. the angle $\measuredangle F_1MF_2$ is θ;
3. the angle $\measuredangle F_1MF_2$ is maximal.



1. recall that for a circle passing through M, F_1, F_2 the measure of $\measuredangle F_1MF_2$ is twice the measure of the arc $\widehat{F_1F_2}$



So $\measuredangle F_1MF_2$ is a right angle if and only if F_1F_2 is a diameter

$\Rightarrow$ The points in 1. are the intersection of $\mathcal{E}$ with the circle centered at O and of radius c

2. $\cos\theta = \cos \widehat{F_1 M F_2} = \dfrac{\overrightarrow{MF_1}\cdot\overrightarrow{MF_2}}{\|\overrightarrow{MF_1}\|\cdot\|\overrightarrow{MF_2}\|}$

$\|\overrightarrow{MF_1}\| = a - \frac{c}{a}x_M$ $\quad \|\overrightarrow{MF_2}\| = a + \frac{c}{a}x_M$ (see lecture)

$a=2 \quad c = \sqrt{4-1} = \sqrt{3}$

$\overrightarrow{MF_1} = (c - x_M, y_M) \quad \overrightarrow{MF_2} = (-c - x_M, y_M)$

$\Rightarrow \overrightarrow{MF_1}\cdot\overrightarrow{MF_2} = x_M^2 - c^2 + y_M^2 = x_M^2 - 3 + 1 - \frac{x_M^2}{4} = \frac{3}{4}x_M^2 - 2$

$\Rightarrow \cos(\theta) = \dfrac{\frac{3}{4}x_M^2 - 2}{4 - \frac{3}{4}x_M^2}$

$\Rightarrow \theta = \arccos \dfrac{\frac{3}{4}x_M^2 - 2}{4 - \frac{3}{4}x_M^2}$

$^{12}/_{4} \quad - \quad \frac{6}{4}$

3. Consider $f(x) = \dfrac{\frac{3}{4}x - 2}{4 - \frac{3}{4}x}$ $\quad f'(x) = \dfrac{\frac{3}{4}x\left(4 - \frac{3}{4}x\right) + \frac{3}{4}x\left(\frac{3}{4}x - 2\right)}{\left(4 - \frac{3}{4}x\right)^2} = \dfrac{\frac{3}{2}x}{\left(4 - \frac{3}{4}x\right)^2}$

$\Rightarrow x = 0$ local minima/maxima

$f''(x) = \dfrac{\frac{3}{2}\left(4 - \frac{3}{4}x\right)^2 - 2\left(4 - \frac{3}{4}x\right)\cdot -\frac{3}{4}\cdot\frac{3}{2}x}{\left(4 - \frac{3}{4}x\right)^4} = \underbrace{\frac{3}{2}\cdot\frac{\overbrace{\left(4 - \frac{3}{4}x\right)}^{>0}}{\left(4 - \frac{3}{4}x\right)^4}}_{>0}\Big[\underbrace{4 - \frac{3}{4}x + \frac{3}{2}x}_{>0}\Big]$

$\Rightarrow x = 0$ local minima for $\cos(\theta)$

$\Rightarrow M(0,1)$ is the point with $\measuredangle F_1MF_2$ maximal

this is visible in the picture

9. Consider the ellipse $\mathcal{E} : x^2 + 4y^2 = 25$. Find the chords on the ellipse which have the point $A(7/2, 7/4)$ as midpoint.

consider the right hand side of the ellipse:

$$\left(x = \sqrt{25-4y^2},\ y\right)$$

The midpoint of two such points is

$$\left(\tfrac{1}{2}\left(\sqrt{25-4y_1^2}+\sqrt{25-4y_2^2}\right),\ \tfrac{1}{2}(y_1+y_2)\right) = \left(\tfrac{7}{2}, \tfrac{7}{4}\right)$$

so $\quad y_1 + y_2 = \frac{7}{2} \Rightarrow y_2 = \frac{7}{2} - y_1 \quad$ and

$$\sqrt{25-4y_1^2} + \sqrt{25-4\left(\tfrac{7}{2}-y_1\right)^2} = 7$$

$$\sqrt{25-(7-2y_1)^2} = 7 - \sqrt{25-4y_1^2} \quad \Big| (\,)^2$$

$$\cancel{25} - 49 + 28y_1 - \cancel{4y_1^2} = 49 - 14\sqrt{25-4y_1^2} + \cancel{25} - \cancel{4y_1^2} \quad \Big| :7$$

$$-7 \quad 4 \qquad\qquad 7 \quad -2$$

$$4y_1 - 14 = -2\sqrt{25-4y_1^2}$$

$$2y_1 - 7 = \sqrt{25-4y_1^2} \quad \Big| (\,)^2$$

$$4y_1^2 - 28y_1 + 49 = 25 - 4y_1^2$$

$$8y_1^2 - 28y_1 + 24 = 0 \quad \Big| :4$$

$$2y_1^2 - 7y_1 + 6 = 0$$

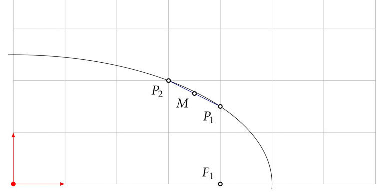
$$\Delta = 49 - 48 \qquad y_1 \begin{cases} \frac{7-1}{4} = \frac{3}{2} \\ \frac{7+1}{4} = 2 \end{cases}$$

$y \in [-b, b] = \left[-\frac{5}{2}, \frac{5}{2}\right]$

so both values are ok

$\Rightarrow$ we obtained two points $\quad x = \sqrt{25-4y^2} \begin{cases} 4 & M\left(4, \frac{3}{2}\right) \\ 3 & N(3, 2) \end{cases}$

$\Rightarrow$ the chord is the line segment $[MN]$



10. Consider the ellipse $\mathcal{E} : \frac{x^2}{25} + \frac{y^2}{9} = 1$. Determine the geometric locus of the midpoints of the chords on the ellipse which are parallel to the line $\ell : x + 2y = 1$.

lines parallel to ℓ have an eq. of the form

$$\ell_m : x + 2y = m \quad \text{for } m \in \mathbb{R}$$

let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be the intersection points of ℓ with $\mathcal{E}$

the midpoint of the chord P_1P_2 is

$$M_m\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \underset{x = m-2y}{=} \left(\frac{m-2y_1+m-2y_2}{2}, \frac{y_1+y_2}{2}\right)$$

so $M_m\left(m - y_1 - y_2, \frac{y_1+y_2}{2}\right)$

So the geometric locus that we look for is the set of points M_m which exist, these correspond to the values m for which ℓ_m intersects

the intersection of ℓ_m with $\mathcal{E}$ are obtained with

$$\frac{(m-2y)^2}{25} - \frac{y^2}{9} = 1$$

view this as an equation in y and impose the condition $\Delta \geq 0$ (at least 1 sol) to get the values for m

11. Find the equation of the circle:

1. passing through $A(3,1)$ and $B(-1,3)$ and having the center on the line $\ell : 3x - y - 2 = 0$;
2. passing through $A(1,1)$, $B(1,-1)$ and $C(2,0)$;
3. tangent to both $\ell_1 : 2x + y - 5 = 0$ and $\ell_2 : 2x + y + 15 = 0$ if the tangency point with ℓ_1 is $M(3,-1)$.

1) A generic pt. on ℓ is $P(x, 3x-2)$

$d(P,A)^2 = (3-x)^2 + (3-3x)^2 = 18 - 24x + 10x^2$

$d(P,B)^2 = \dots = 26 - 28x + 10x^2$

since $r^2 = d(P,A)^2 = d(P,B)^2$

$\Rightarrow 18 - 24x + 10x^2 = 26 - 28x + 10x^2$

$\Rightarrow x = 2$

$\Rightarrow$ center of circle is $I(2,4)$

and radius is $r = d(I,A) = \sqrt{1^2+3^2} = \sqrt{10}$

So the circle is $\mathcal{C}: (x-2)^2 + (y-4)^2 = 10$

2.) $\mathcal{C} \ni A(1,1), B(1,-1), C(2,0)$

$\mathcal{C}: (x-a)^2 + (y-b)^2 = r^2$

$A \in \mathcal{C} \Rightarrow (1-a)^2 + (1-b)^2 = r^2$

$1 - 2a + a^2 + 1 - 2b + b^2 = r^2$ (1)

$B \in \mathcal{C} \Rightarrow (1-a)^2 + (1+b)^2 = r^2$

$1 - 2a + a^2 + 1 + 2b + b^2 = r^2$ (2)

$C \in \mathcal{C} \Rightarrow (2-a)^2 + b^2 = r^2$

$4 - 4a + a^2 + b^2 = r^2$ (3)

so, a, b, r have to satisfy

$$(*)\quad \begin{cases} 1-2a+a^2+1-2b+b^2=r^2 & (1) \\ 1-2a+a^2+1+2b+b^2=r^2 & (2) \\ 4-4a+a^2+b^2=r^2 & (3) \end{cases}$$

"(1) − (2)": $\quad -4b=0 \Rightarrow \boxed{b=0}$

$$\Rightarrow (*) \overset{b=0}{\Longleftrightarrow} \begin{cases} 2-2a+a^2=r^2 \\ 4-4a+a^2=r^2 \end{cases} \quad -$$

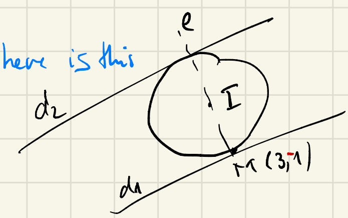
$$-2+2a=0 \Rightarrow \boxed{a=1} \Rightarrow r^2=1$$

so $\mathscr{C}: (x-1)^2+y^2=1.$

3.) the normal vectors of the two lines are the same

so in fact $d_1 \parallel d_2$

so in fact the picture here is this



a line $\ell \perp d_1$, $\ell \ni M$ is

$$\ell: \begin{cases} x=3+2t \\ y=-1+t \end{cases} \quad \text{and } \ell \cap d_2: \quad 2(3+2t)+(-1+t)+15=0$$

$6+4t$

$5t+20=0 \Rightarrow t=-4$

$\Rightarrow \ell \cap d_2 = N(-5,-5)$

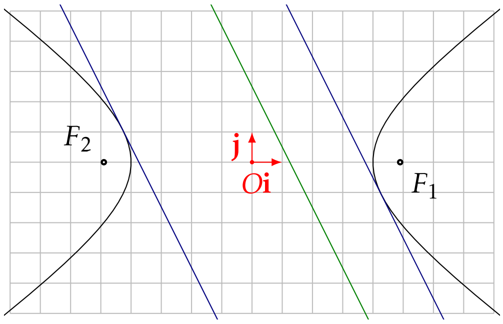
$\Rightarrow$ center I of $\mathscr{C}$ is midpoint of MN, i.e. $I(-1,-3)$

and radius is $d(I,M)=d(I,N)=\sqrt{20}$

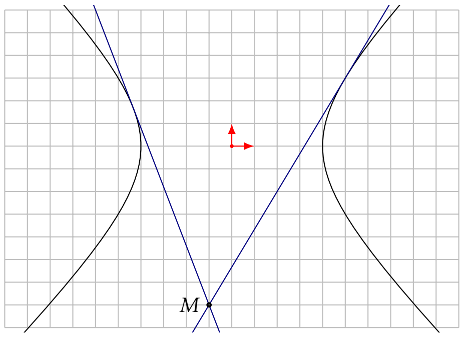
$\Rightarrow \mathscr{C}: (x+1)^2+(y+3)^2=20$

1. Determine the intersection points between the line $\ell : 2x - y - 10 = 0$ and the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$.

2. Determine the tangents to the hyperbola $\mathcal{H} : \frac{x^2}{16} - \frac{y^2}{8} - 1 = 0$ which are parallel to the line $\ell : 4x + 2y - 5 = 0$.



3. Determine the tangents to the hyperbola $\mathcal{H} : x^2 - y^2 = 16$ which contain the point $M(-1,7)$.



4. Find the area of the triangle determined by the asymptotes of the hyperbola $\mathcal{H} : \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the line $\ell : 9x + 2y - 24 = 0$.

5. Find an equation for the tangent lines to:

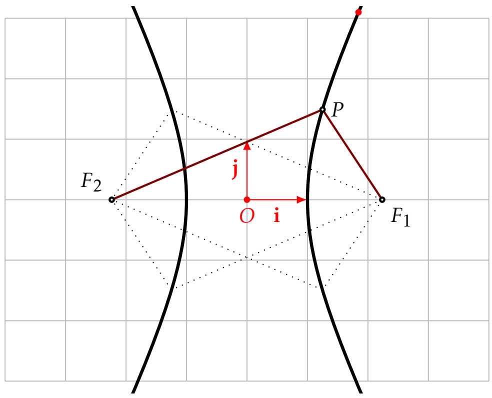
1. the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$, orthogonal to the line $\ell : 4x + 3y - 7 = 0$;
2. the parabola $\mathcal{P} : y^2 - 8x = 0$, parallel to $\ell : 2x + 2y - 3 = 0$.

6. Find an equation for the tangent lines to:

1. the hyperbola $\mathcal{H} : \frac{x^2}{3} - \frac{y^2}{5} - 1 = 0$, passing through $P(1, -5)$;
2. the paraola $\mathcal{P} : y^2 - 36x = 0$, passing through $P(2, 9)$.

7. Consider the hyperbola $\mathcal{H} : x^2 - \frac{y^2}{4} - 1 = 0$ with focal points F_1 and F_2. Find the points M situated on the hyperbola such that

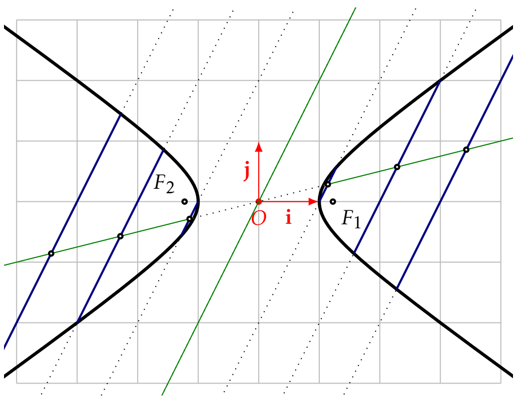
1. The angle $\measuredangle F_1 M F_2$ is right;
2. The angle $\measuredangle F_1 M F_2$ is $60°$;
3. The angle $\measuredangle F_1 M F_2$ is θ.



8. Consider the tangents to the parabola $\mathcal{P} : y^2 - 10x = 0$ passing through the point $P(-3, 12)$. Calculate the distance from the point P to the chord of the parabola which is formed by the two contact points.

9. Using the gradient, prove the reflective properties of the hyperbola and of the parabola.

10. Consider the hyperbola $\mathcal{H} : x^2 - 2y^2 = 1$. Determine the geometric locus described by the midpoints of the chords of $\mathcal{H}$ which are parallel to the line $2x - y = 0$.

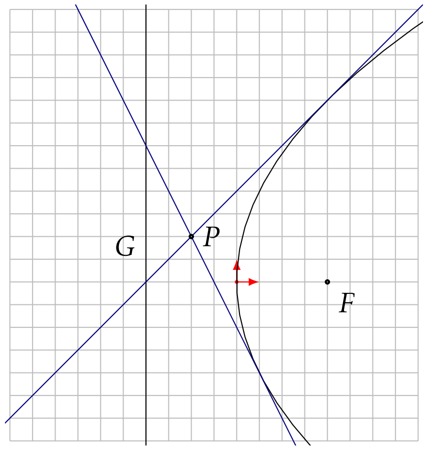


11. For which value k is the line $y = kx + 2$ tangent to the parabola $\mathcal{P} : y^2 = 4x$?

12. Consider the parabola $\mathcal{P} : y^2 = 16x$. Determine the tangents to $\mathcal{P}$ which are

1. parallel to the line $\ell : 3x - 2y + 30 = 0$;
2. perpendicular to the line $\ell : 4x + 2y + 7 = 0$.

13. Determine the tangents to the parabola $\mathcal{P} : y^2 = 16x$ which contain the point $P(-2, 2)$.



1. Determine the intersection points between the line $\ell : 2x - y - 10 = 0$ and the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$.

$$\ell \cap \mathcal{H} : \begin{cases} \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0 \\ 2x - y - 10 = 0 \end{cases} \iff \begin{cases} \frac{x^2}{20} - \frac{(2x-10)^2}{5} - 1 = 0 \\ y = 2x - 10 \end{cases}$$

$$\frac{x^2}{20} - \frac{4(x-5)^2}{5} - 1 = 0 \quad \Big| \cdot 20$$

$$x^2 - 16(x^2 - 10x + 25) - 20 = 0$$

$$-15x^2 + 160x - 420 = 0 \quad \Big| : 5$$

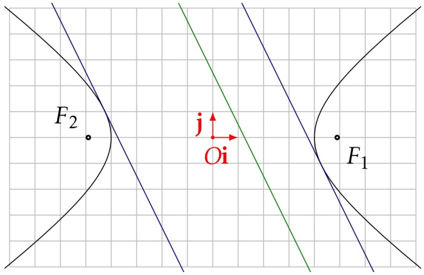
$$-3x^2 + 32x - 84 = 0$$

$$\Delta = \underbrace{32^2}_{2^{10}} - \underbrace{4 \cdot 3 \cdot 84}_{2^2 \cdot 3 \cdot 2^2 \cdot 3 \cdot 7} = 2^4(\underbrace{2^6}_{64} - \underbrace{3^2 \cdot 7}_{63}) = 2^4 \Rightarrow x_{1,2} = \frac{-32 \pm 4}{-6} = \begin{cases} \frac{-36}{-6} = 6 \\ \frac{-28}{-6} = \frac{14}{3} \end{cases}$$

$2x - 10$

So we have two intersection points $P_1(6, 2)$ and $P_2\left(\frac{14}{3}, -\frac{2}{3}\right)$.

2. Determine the tangents to the hyperbola $\mathcal{H} : \frac{x^2}{16} - \frac{y^2}{8} - 1 = 0$ which are parallel to the line ℓ : $4x + 2y - 5 = 0$.



The tangents to $\mathcal{H}$ of slope k are

$$\ell_k : y = kx \pm \sqrt{a^2k^2 - b^2} \qquad \text{if} \quad k \in \left(-\infty, -\tfrac{b}{a}\right) \cup \left(\tfrac{b}{a}, \infty\right)$$

In our case $a = 4$ and $b = 2\sqrt{2}$ and $k = -2 < -\frac{2\sqrt{2}}{4}$

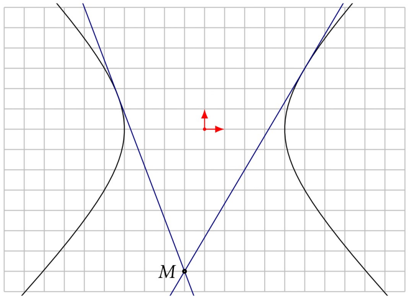
So $\sqrt{a^2k^2 - b^2} = \sqrt{16 \cdot 4 - 8} = 2\sqrt{14}$

and the two tangent lines are

$$\ell_1 : y = -2x + 2\sqrt{14} \qquad \text{and} \qquad \ell_2 : y = -2x - 2\sqrt{14}$$

Remark: when we deduce the equation of a tangent line for a given slope k, it is clear that $\sqrt{a^2k^2 - b^2}$ has to be a real number in order for the equation to describe a line in the plane $\mathbb{E}^2$

3. Determine the tangents to the hyperbola $\mathcal{H}: x^2 - y^2 = 16$ which contain the point $M(-1,7)$.



We search for the possible tangents passing through M in the form

$$\ell: \; y = kx \pm \sqrt{a^2k^2 - b^2} \quad \text{with } k \in \left(-\infty, -\tfrac{b}{a}\right) \cup \left(\tfrac{b}{a}, \infty\right)$$

since any tangent line to $\mathcal{H}$ has such an equation, in particular those who pass through M

We know that $M(-1,7) \in \ell$ and that $a = b = 4$

$$7 = -k \pm \sqrt{16k^2 - 16}$$

$$\Rightarrow \quad 7 + k = \pm\sqrt{16k^2 - 16} \quad \Big| \, (\,)^2$$

$$\Rightarrow \quad 49 + 14k + k^2 = 16k^2 - 16 \quad \left(\text{since } k^2 \geq \frac{b^2}{a^2}\right)$$

$$\Rightarrow \quad 15k^2 - 14k - 65 = 0$$

$$\Delta = 14^2 + 4 \cdot \underbrace{15 \cdot 65}_{3 \cdot 5^2 \cdot 13} = 4\left(49 + 975\right) = 4 \cdot (\underbrace{1024}_{2^{10}}) = 2^{12}$$

$$\Rightarrow \quad k_{1,2} = \frac{14 \pm 64}{30} \quad \begin{cases} -\frac{50}{30} = -\frac{5}{3} \\[1ex] \frac{78}{30} = \frac{13}{5} \end{cases}$$

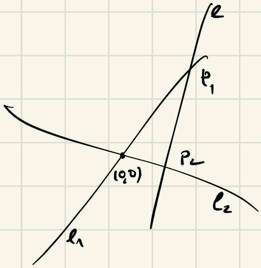
4. Find the area of the triangle determined by the asymptotes of the hyperbola $\mathcal{H} : \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the line $\ell : 9x + 2y - 24 = 0$.

$a = 2 \qquad b = 3$

$\Rightarrow$ the asymptotes of $\mathcal{H}$ are $\quad \ell_1 : y = \frac{3}{2}x \quad$ and $\quad \ell_2 : y = \frac{-3}{2}x$

$\ell \cap \ell_1 : \quad 9x + 2\,\frac{3}{2}x - 24 = 0 \qquad x = \frac{24}{11} \Rightarrow y = \frac{36}{11}$

$\ell \cap \ell_2 : \quad 9x + 2\,\frac{-3}{2}x - 24 = 0 \qquad x = 4 \Rightarrow y = -6$



area of triangle $= \frac{1}{2}\left| \begin{vmatrix} 0 & 0 & 1 \\ \frac{24}{11} & \frac{36}{11} & 1 \\ 4 & -6 & 1 \end{vmatrix} \right| = \frac{144}{11}$

5. Find an equation for the tangent lines to:

1. the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$, orthogonal to the line $\ell : 4x + 3y - 7 = 0$;
2. the parabola $\mathcal{P} : y^2 - 8x = 0$, parallel to $\ell : 2x + 2y - 3 = 0$.

1.) A tangent line to $\mathcal{H}$ has an equation of the form

$$\ell_k : y = kx \pm \sqrt{a^2k^2 - b^2}$$

Since $\ell_k \perp \ell : y = -\frac{4}{3}x + \frac{7}{3}$ $\Rightarrow$ the slope k of the tangent is $\frac{3}{4}$ $(*)$

$$\Rightarrow \quad \ell_k : \quad y = \frac{3}{4}x \pm \sqrt{20 \cdot \frac{9}{16} - 5} = \frac{3}{4}x \pm \frac{5}{2}$$

so the two tangents are $\ell_1 : y = \frac{3}{4}x + \frac{5}{2}$ and $\ell_2 : y = \frac{3}{4}x - \frac{5}{2}$

2.) We consider the tangent lines to $\mathcal{P}$ in the form

$$\ell_k : y = kx + \frac{p}{2k}$$

Since $\ell_k \parallel \ell : y = -x + \frac{3}{2}$ $\Rightarrow$ the slope k of the tangent is 1

$p = 4$

So, the tangent is $y = x + 2$

6. Find an equation for the tangent lines to:

1. the hyperbola $\mathcal{H}: \frac{x^2}{3} - \frac{y^2}{5} - 1 = 0$, passing through $P(1,-5)$;
2. the paraola $\mathcal{P}: y^2 - 36x = 0$, passing through $P(2,9)$.

Method I

$$\ell_k: \quad y = kx \pm \sqrt{3k^2-5}$$

$$P(1,-5) \in \ell_k \Rightarrow -5 = k \pm \sqrt{3k^2-5}$$

$$\Rightarrow -5-k = \sqrt{3k^2-5} \quad \Big| ( )^2$$

$$\Rightarrow 25 + 10k + k^2 = 3k^2 - 5$$

$$\Rightarrow 2k^2 - 10k - 30 = 0$$

$$\Rightarrow k^2 - 5k - 15 = 0 \qquad \Delta = 25 + 60 \qquad k_{1,2} = \frac{10 \pm \sqrt{85}}{4}$$

So the possible tangent lines are $y = \frac{10 \pm \sqrt{85}}{4} x \pm \sqrt{3\left(\frac{10 \pm \sqrt{85}}{4}\right)^2 - 5}$

we have four lines here

now we can check which of these four equations are satisfied by the coordinates of P

Method II we look for the tangent lines in the form

$$\ell_{(x_0,y_0)}: \frac{x x_0}{3} - \frac{y y_0}{5} = 1$$

since $P \in \ell_{(x_0,y_0)}$ we have $\frac{x_0}{3} + y_0 = 1 \Rightarrow y_0 = 1 - \frac{x_0}{3}$

since $(x_0,y_0) \in \mathcal{H}$: $\frac{x_0^2}{3} - \frac{\left(1-\frac{x_0}{3}\right)^2}{5} - 1 = 0$

.... $x_{1,2} = -\frac{3}{14} \pm \frac{3\sqrt{85}}{14}$

so we obtain the points

$(x_0,y_0) = \left(-\frac{3}{14} + \frac{3\sqrt{85}}{14}, \frac{15}{14} - \frac{\sqrt{85}}{14}\right)$ and $(x_0,y_0) = \left(-\frac{3}{14} - \frac{3\sqrt{85}}{14}, \frac{15}{14} + \frac{\sqrt{85}}{14}\right)$

and the corresponding lines in these two points.

9. $\mathcal{P}: y^2 - 36x = 0 \qquad P(2,9)$

$\ell_k: \quad y = kx + \frac{p}{2k} \ni P(2,9)$

$9 = 2k + \frac{18}{2k} \quad \Rightarrow \quad 9k = 2k^2 + 9$

$2k^2 - 9k + 9 = 0$

$\Delta = 81 - 8\cdot 9 = 9 \qquad k_{1,2} = \frac{9 \pm 3}{4} \begin{cases} 3 \\ \frac{3}{2} \end{cases}$

$k = 3: \qquad y = 3x + 3 \ni P(2,9) \qquad \leftarrow (\ast\ast)$

$k = \frac{3}{2} \qquad y = \frac{3}{2}x + \frac{9}{\frac{3}{2}} \iff y = \frac{3}{2}x + 6 \ni P(2,9) \qquad \leftarrow (\ast)$

Method II $\qquad \ell_{(x_0,y_0)}: \quad yy_0 = 18(x + x_0) \ni P(2,9)$

$y_0 9 = 36 + 18x_0 \quad |:9$

$y_0 = 4 + 2x_0$

$(x_0,y_0) \in \mathcal{P} \qquad (4+2x_0)^2 = 36x_0$

$(2+x_0)^2 = 9x_0$

$4 + 4x_0 + x_0^2 = 9x_0$

$x_0^2 - 5x_0 + 4 = 0 \qquad \Delta = 25 - 16 = 9 \qquad x_{1,2} = \frac{5 \pm 3}{2} \begin{cases} 4 \\ 1 \end{cases}$

So $(x_0, y_0) = (4, 12)$ or $(x_0, y_0) = (1, 6)$

$\downarrow$ $\qquad\qquad$ $\downarrow$

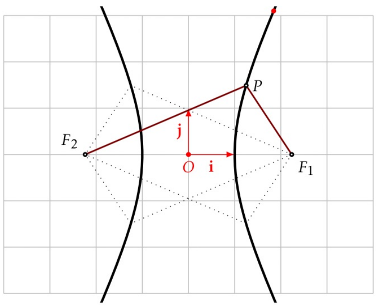
$12y = 18(x+4)$ $\qquad\qquad$ $6y = 18(x+1)$

$2y = 3x + 12$ $\qquad\qquad$ $y = 3x + x$

this is (*) $\qquad\qquad$ this is (**)

7. Consider the hyperbola $\mathcal{H}: x^2 - \frac{y^2}{4} - 1 = 0$ with focal points F_1 and F_2. Find the points M situated on the hyperbola such that

1. The angle $\measuredangle F_1 M F_2$ is right;
2. The angle $\measuredangle F_1 M F_2$ is $60°$;
3. The angle $\measuredangle F_1 M F_2$ is θ.



1. the points are the intersection of $\mathcal{H}$ with a circle centered in the origin and passing through the focal points F_1 and F_2 (see similar problem in previous problem set.)

2. One could use a geometric argument as in 1:
   - consider the equilateral triangle F_1F_2A with A above the x-axis
   - intersect $\mathcal{H}$ with the circumcenter of this triangle
   - the intersection points above the x-axis are two of the points we need
   - the other two are obtain with A below the x-axis.

3. With an algebraic calculation we can give a solution for all θ

We know that for a point $P \in \mathcal{H}$ we have

$$\cos\theta = \cos \sphericalangle F_1 P F_2 = \frac{\overrightarrow{PF_1} \cdot \overrightarrow{PF_2}}{\|\overrightarrow{PF_1}\| \cdot \|\overrightarrow{PF_2}\|}$$

so, ... we can calculate

$P(x_0, y_0) \quad F_1(c, 0) \quad F_2(-c, 0)$

$\Rightarrow \quad \overrightarrow{PF_1}\left(c - x_0, y_0\right), \quad \overrightarrow{PF_2}\left(-c - x_0, y_0\right)$

$$\frac{y^2}{b^2} = \frac{x^2}{a} - 1$$

$$\|\overrightarrow{PF_1}\| = \sqrt{(c - x_0)^2 + y_0^2} = \dots = |c x_0 - a| = c x_0 - a$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\|\overrightarrow{PF_2}\| = \sqrt{(-c - x_0)^2 + y_0^2} = \dots = |c x_0 + a| = c x_0 + a$$

$$\Rightarrow \frac{\overrightarrow{PF_1} \cdot \overrightarrow{PF_2}}{\|\overrightarrow{PF_1}\| \cdot \|\overrightarrow{PF_2}\|} = \frac{(c - x_0)(-c - x_0) + y_0^2}{(c x_0 - a)(c x_0 + a)} = \frac{-c^2 + x_0^2 + y_0^2}{c^2 x_0^2 - a^2} = \frac{x_0^2 - c^2 + b^2\left(\frac{x_0^2}{a^2} - 1\right)}{c^2 x_0^2 - a^2}$$

$a = 1 \quad b = 2 \quad \Rightarrow \quad c = \sqrt{a^2 + b^2} = \sqrt{5}$

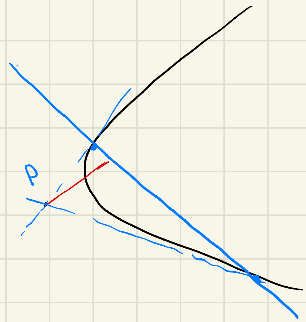
so in our case $\quad \cos\theta = \dfrac{x_0^2 - 5 + 4(x_0^2 - 1)}{5 x_0^2 - 1} = \dfrac{5x_0^2 - 9}{5x_0^2 - 1}$

So in the previous case $\theta = 60°$

$$\frac{1}{2} = \cos\theta = \frac{5x_0^2 - 9}{5x_0^2 - 1} \iff x_0 = \pm\sqrt{\frac{17}{5}}$$

So the four point are $M\left(\pm\sqrt{\frac{17}{5}}, \pm 4\sqrt{\frac{3}{5}}\right)$

8. Consider the tangents to the parabola $\mathcal{P}: y^2 - 10x = 0$ passing through the point $P(-3,12)$. Calculate the distance from the point P to the chord of the parabola which is formed by the two contact points.



We need the two contact points so it makes sense to work with a tangent line in the form $yy_0 = p(x+x_0)$

since $P(-3,12)$ lies on such a line, we have

$$12y_0 = 5(x_0 - 3) \quad \Rightarrow \quad x_0 = \frac{12}{5}y_0 + 3$$

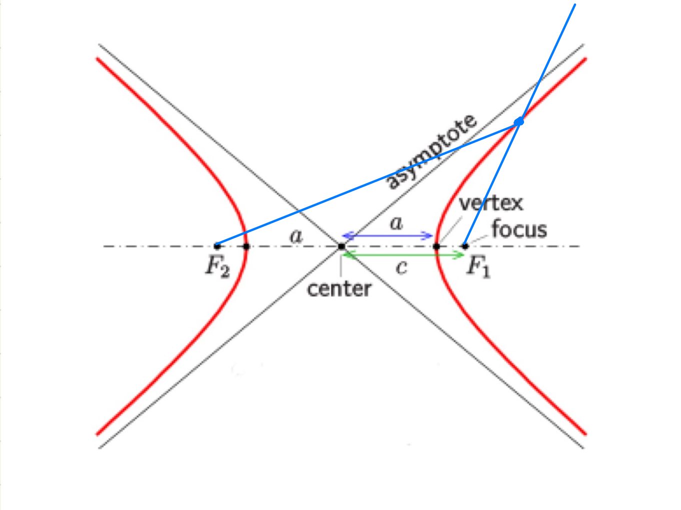
$$(x_0,y_0) \in \mathcal{P} \Rightarrow \quad y_0^2 - 24y_0 - 30 = 0 \qquad \Delta = 24^2 + 4\cdot 30 = 4\ 174$$

$$\Rightarrow y_0 \in \{12 - \sqrt{174}\ ,\ 12 + \sqrt{174}\}$$

$$\Rightarrow (x_0, y_0) \in \{P_1(\ldots;\ldots)\ ,\ P_2(\ldots;\ldots)\}$$

So we can write down the line P_1P_2 and calculate the distance from P to this line.

9. Using the gradient, prove the reflective properties of the hyperbola and of the parabola.



Hyperbola

a ray starting in F_2 is reflected in the curve away from F_1 on a line passing through the contact point and F_1

$$\mathcal{H}_{a,b}: \ d(M, F_1) - d(M, F_2) \pm 2a = 0$$

$$\Leftrightarrow \underbrace{\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} \pm 2a}_{\phi(x,y)} = 0$$

$$\nabla \phi(x,y) = \left(\frac{\partial \phi}{\partial x}(x,y), \frac{\partial \phi}{\partial y}(x,y) \right)$$

$$\frac{\partial \phi}{\partial x}(x,y) = \frac{\partial}{\partial x}\left(\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} \pm 2a \right) = \frac{x-c}{\sqrt{(x-c)^2+y^2}} - \frac{x+c}{\sqrt{(x+c)^2+y^2}}$$

$$\frac{\partial \phi}{\partial x}(x,y) = \frac{y}{\sqrt{(x-c)^2+y^2}} - \frac{y}{\sqrt{(x+c)^2+y^2}}$$

$$\Rightarrow \nabla \phi(M) = \left(\frac{x_M - c}{\|\overrightarrow{F_1M}\|} - \frac{x+c}{\|\overrightarrow{F_2M}\|}, \frac{y}{\|\overrightarrow{F_1M}\|} - \frac{y}{\|\overrightarrow{F_2M}\|} \right)$$

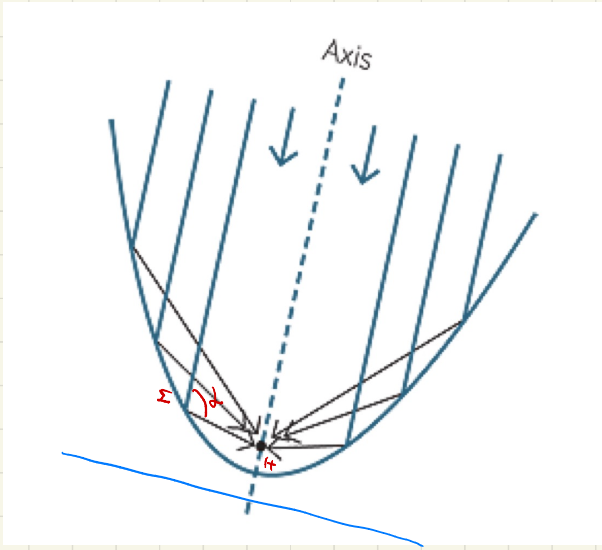
$$= \frac{1}{\|\overrightarrow{F_1M}\|}(x_M - c, y) - \frac{1}{\|\overrightarrow{F_2M}\|}(x+c, y)$$

$$= \frac{\overrightarrow{F_1M}}{\|\overrightarrow{F_1M}\|} - \frac{\overrightarrow{F_2M}}{\|\overrightarrow{F_2M}\|}$$

$\Rightarrow$ $\nabla\phi(M)$ is a direction vector for the exterior angle bisector of $\widehat{F_1MF_2}$ (*)

Parabola

rays parallel to the axis of the parabola which hit the curve get reflected in the focal point



$$\mathcal{P}_p : \quad d(M, F) - d(M, \text{directrix}) = 0$$

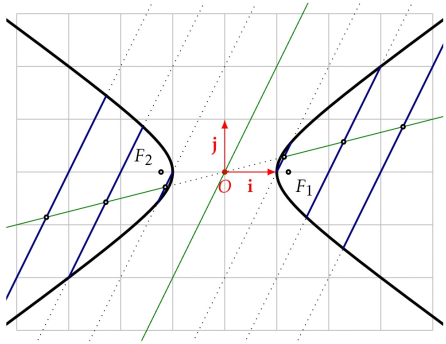
$$d(M,F) \downarrow F\left(\frac{p}{2}, 0\right) \qquad d(M, \text{directrix}) \downarrow x = -\frac{p}{2}$$

$$\Leftrightarrow \quad \underbrace{\sqrt{(x_M - c)^2 + y_M^2} - \left(x + \frac{p}{2}\right)}_{\phi(x_M, y_M)} = 0$$

$$\nabla\phi(x,y) = \left(\frac{x-c}{\sqrt{(x-c)^2+y^2}} - 1, \frac{y}{\sqrt{(x-c)^2+y^2}}\right) = \frac{1}{\|\overrightarrow{FM}\|}\overrightarrow{FM} - (1,0)$$

$\Rightarrow$ $\nabla\phi(M)$ is a dir. vector for the angle α

10. Consider the hyperbola $\mathcal{H} : x^2 - 2y^2 = 1$. Determine the geometric locus described by the midpoints of the chords of $\mathcal{H}$ which are parallel to the line $2x - y = 0$.



A line parallel to the given line has an equation of the form

$$\ell_m : y = 2x + m$$

Then $\ell_m \cap \mathcal{H}$: $\begin{cases} x^2 - 2(2x+m)^2 = 1 \\ y = 2x + m \end{cases}$ $\iff x^2 - 8x^2 - 8xm - 2m^2 - 1 = 0$

$$-7x^2 - 8mx - 2m^2 - 1 = 0$$

$$\Delta = 64m^2 - 4 \cdot 7(2m^2+1)$$
$$= 4(16m^2 - 14m^2 - 7)$$
$$= 4(2m^2 - 7)$$

So ℓ_m intersects $\mathcal{H}$ if $2m^2 - 7 > 0 \iff m^2 > \frac{7}{2} \iff m \in (-\infty, -\frac{7}{2}) \cup (\frac{7}{2}, \infty)$

For such m, the two intersection points are

$$P_1\left(x_1 = \frac{8m + 2\sqrt{2m^2-7}}{-14}, 2x_1 + m\right) \text{ and } P_2\left(x_2 = \frac{8m - \sqrt{\Delta}}{-14}, 2x_2 + m\right)$$

The mid point of the segment $[P_1P_2]$ is

$$M\left(\frac{x_1+x_2}{2}, 2\frac{x_1+x_2}{2} + m\right) = M\left(-\frac{4}{7}m, -\frac{1}{7}m\right)$$

So, the geometric locus is the set of points $\left\{\left(-\frac{4}{7}m, -\frac{1}{7}m\right) : m \in (-\infty, -\frac{7}{2}) \cup (\frac{7}{2}, \infty)\right\}$

11. For which value k is the line $\underbrace{y = kx + 2}_{\ell_k}$ tangent to the parabola $\underbrace{\mathcal{P} : y^2 = 4x}_{\mathcal{P}_2}$?

$$\ell_k \cap \mathcal{P}_2 : \begin{cases} y^2 = 4x \\ y = kx+2 \end{cases} \iff \begin{cases} (kx+2)^2 = 4x \\ y = kx+2 \end{cases}$$

$$k^2x^2 + 4kx + 4 = 4x$$

$$k^2x^2 + 4(k-1)x + 4$$

$$\Delta = 4^2(k-1)^2 - 4\cdot 4\cdot k^2$$
$$= 4^2(k^2 - 2k - 1 - k^2)$$
$$= 4^2(-2k-1)$$

ℓ_k is tangent to $\mathcal{P}_2$ if $\Delta = 0 \iff k = -\frac{1}{2}$

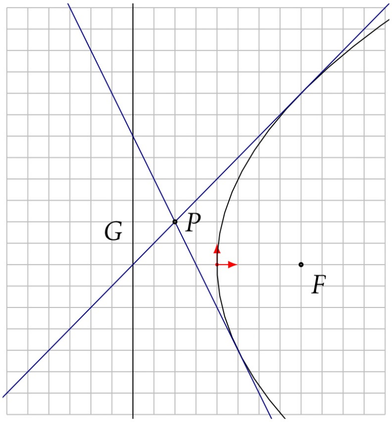
12. Consider the parabola $\mathcal{P} : y^2 = 16x$. Determine the tangents to $\mathcal{P}$ which are

1. parallel to the line $\ell : 3x - 2y + 30 = 0$;
2. perpendicular to the line $\ell : 4x + 2y + 7 = 0$.

use the form $y = kx + \frac{p}{2k}$ for a tangent

similar to exer. 5

13. Determine the tangents to the parabola $\mathcal{P} : y^2 = 16x$ which contain the point $P(-2, 2)$.



use the form $y = kx + \frac{p}{2k}$ for a tangent

similar to exer. 6

1. For each of the following, find the matrix $M \in SO(2)$ which diagonalizes the given symmetric matrix:

1. $\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$
2. $\begin{bmatrix} 5 & -13 \\ -15 & 5 \end{bmatrix}$
3. $\begin{bmatrix} 7 & -2 \\ -2 & 5/3 \end{bmatrix}$

2. For each of the symmetric matrices A in the previous exercise write down a quadratic equation with associated matrix A.

3. For each of the following equations write down the associated matrix.

1. $-x^2 + xy - y^2 = 0$,
2. $6xy + x - y = 0$.

4. Bring the equations from the previous exercise in canonical form.

5. Decide if the following equations describe an ellipse, a hyperbola or a parabola.

1. $x^2 - 4xy + 2y^2 = 1$,
2. $x^2 + 4xy + 2y^2 = 2$,
3. $x^2 + 4xy + 4y^2 = 3$.

6. Using the classification of quadrics, decide what surfaces are described by the following equations.

1. $x^2 + 2y^2 + z^2 + xy + yz + zx = 1$,
2. $xy + yz + zx = 1$,
3. $x^2 + xy + yz + zx = 1$,
4. $xy + yz + zx = 0$.

7. Consider the rotation R_{90° of $\mathbb{E}^2$ around the origin and the translation $T_{\mathbf{v}}$ of $\mathbb{E}^2$ with vector $\mathbf{v}(1,0)$.

1. Give the algebraic form of the isometries R_{90°, $T_{\mathbf{v}}$ and $T_{\mathbf{v}} \circ R_{90^\circ}$.
2. Determine the equations of the hyperbola $\mathcal{H} : \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the parabola $\mathcal{P} : y^2 - 8x = 0$ after transforming them with R_{90° and with $T_{\mathbf{v}} \circ R_{90^\circ}$ respectively.

8. Let $\boldsymbol{e}$ and $\boldsymbol{f}$ be two orthonormal bases of a $\mathbb{V}^n$. Show that $\mathrm{M}_{\boldsymbol{e},\boldsymbol{f}}$ is orthogonal, i.e. that $\mathrm{M}_{\boldsymbol{e},\boldsymbol{f}} \in O(n)$.

9. Let $\boldsymbol{e} = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ be an orthonormal basis of $\mathbb{V}^n$. If π is a permutation of $\{1, \dots, n\}$ let $\pi(\boldsymbol{e}) = (\mathbf{e}_{\pi(1)}, \dots, \mathbf{e}_{\pi(n)})$. Show that $\mathrm{M}_{\boldsymbol{e},\pi(\boldsymbol{e})} \in O(n)$.

1. For each of the following, find the matrix $M \in SO(2)$ which diagonalizes the given symmetric matrix:

1. $\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} = A$

the matrix M is the base change matrix $M_{e',e}$ where e is the basis with respect to which the matrix A is given and e' is an orthonormal basis of eigenvectors

$$\det(\lambda I_2 - A) = \begin{vmatrix} \lambda-6 & -2 \\ -2 & \lambda-9 \end{vmatrix} = (\lambda-6)(\lambda-9) - 4 = \lambda^2 - 15\lambda + 54 - 4 = 0$$

$$\lambda^2 - 15\lambda + 50 = 0 \iff (\lambda-10)(\lambda-5) = 0$$

so, the eigenvalues are 5 and 10

$\lambda = 5$ $\quad A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} x \\ y \end{bmatrix} \iff (5 \cdot I - A)\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$$\iff \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \sim \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix} \quad \text{so } -x - 2y = 0$$

$$\Rightarrow x = -2y$$

the eigenspace for the eigenvalue $\lambda = 5$ is

$$\left\{ \begin{bmatrix} -2t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \left\langle \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\rangle$$

$\lambda = 10$ $\quad (10 I_2 - A)\begin{bmatrix} x \\ y \end{bmatrix} = 0 \iff \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{so } -2x + y = 0 \Rightarrow y = 2x$$

the eigenspace for the eigenvalue $\lambda = 10$ is

$$\left\{ \begin{bmatrix} t \\ 2t \end{bmatrix} : t \in \mathbb{R} \right\} = \left\langle \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\rangle$$

$\Rightarrow$ an orthonormal basis of eigenvectors is $\left\{ \frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix} \right\}$

so M could be $M_{e,e'} = \frac{1}{\sqrt{5}}\begin{bmatrix}-2 & 1\\ 1 & 2\end{bmatrix}$ $\quad M\cdot M^t (= M^2) = \frac{1}{5}\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix} = I_2$

$\det M = \frac{1}{5}\cdot(-5) = -1$

$\Downarrow$

we can replace $\frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\end{bmatrix}$ by $\frac{1}{\sqrt{5}}\begin{bmatrix}2\\-1\end{bmatrix}$

then $M_{e,e'} = \frac{1}{\sqrt{5}}\begin{bmatrix}2 & 1\\ -1 & 2\end{bmatrix}$ has $M\cdot M^t = \frac{1}{5}\begin{bmatrix}2 & 1\\ -1 & 2\end{bmatrix}\cdot\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix} = I_2$

and $\det M = \frac{1}{5}5 = 1$

$\Rightarrow M \in SO(2)$

Moreover $\quad M^{-1}AM = M^tAM = \frac{1}{\sqrt{5}}\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}\cdot\begin{bmatrix}6 & 2\\ 2 & 9\end{bmatrix}\cdot\frac{1}{\sqrt{5}}\begin{bmatrix}2 & 1\\ -1 & 2\end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}\begin{bmatrix}10 & 10\\ -5 & 20\end{bmatrix} = \frac{1}{5}\begin{bmatrix}25 & 0\\ 0 & 50\end{bmatrix} = \begin{bmatrix}5 & 0\\ 0 & 10\end{bmatrix}$

2. For each of the symmetric matrices A in the previous exercise write down a quadratic equation with associated matrix A.

1. $\begin{bmatrix}6 & 2\\ 2 & 9\end{bmatrix}$ $\quad 6x^2 + 9y^2 + 4xy = 0$

3. For each of the following equations write down the associated matrix.

1. $-x^2 + xy - y^2 = 0$,
2. $6xv + x - v = 0$.

1. $A = \begin{bmatrix}-1 & 1/2\\ 1/2 & -1\end{bmatrix}$

4. Bring the equations from the previous exercise in canonical form.

as in exer 1, see also lecture notes

5. Decide if the following equations describe an ellipse, a hyperbola or a parabola.

1. $x^2 - 4xy + 2y^2 = 1$,
2. $x^2 + 4xy + 2y^2 = 2$,
3. $x^2 + 4xy + 4y^2 = 3$.

1. the associated matrix is $\begin{pmatrix} 1 & -2 \\ -2 & 2 \end{pmatrix}$

the characteristic polynomial is $\begin{vmatrix} 1-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 4$

$= \lambda^2 - 3\lambda - 2$

$\Delta = 9 + 8$

$\lambda_{1,2} = \frac{3 \pm \sqrt{17}}{2}$

$\Rightarrow$ signature is $(1,1)$

in this case we have a hyperbola

6. Using the classification of quadrics, decide what surfaces are described by the following equations.

1. $x^2 + 2y^2 + z^2 + xy + yz + zx = 1$,

$\hookrightarrow$ has matrix $\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 2 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix} \rightarrow$ has char. poly $-x^3 + 4x^2 - \frac{17}{4}x + \frac{5}{4}$

$= -\frac{1}{4}(x-1)(2x-5)(2x-1)$

$\Rightarrow$ has signature $(3,0)$

in this case the equation describes an ellipsoid (possibly imaginary)

7. Consider the rotation $R_{90°}$ of $\mathbb{E}^2$ around the origin and the translation $T_{\mathbf{v}}$ of $\mathbb{E}^2$ with vector $\mathbf{v}(1,0)$.

1. Give the algebraic form of the isometries $R_{90°}$, $T_{\mathbf{v}}$ and $T_{\mathbf{v}} \circ R_{90°}$.
2. Determine the equations of the hyperbola $\mathcal{H} : \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the parabola $\mathcal{P} : y^2 - 8x = 0$ after transforming them with $R_{90°}$ and with $T_{\mathbf{v}} \circ R_{90°}$ respectively.

$$R_{90°} = \begin{bmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$T_v \circ R_{90°} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathcal{P}: y^2 - 8x = 0 \iff [x \ y] \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + [-8 \ 0] \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

replace this with $\begin{bmatrix} x' \\ y' \end{bmatrix} = Rot_{90°} \begin{bmatrix} x \\ y \end{bmatrix}$

and $\begin{bmatrix} x' \\ y' \end{bmatrix} = T_v \circ Rot \begin{bmatrix} x \\ y \end{bmatrix}$

in order to get the equations.

8. Let $\boldsymbol{e}$ and $\boldsymbol{f}$ be two orthonormal bases of a $\mathbb{V}^n$. Show that $M_{e,f}$ is orthogonal, i.e. that $M_{e,f} \in O(n)$.

$$M_{e,f} = \begin{bmatrix} \cdots & f_i & \cdots \end{bmatrix}$$

↳ components of f_i w.r.t e

$$M_{e,f}^t M_{e,f} = \begin{bmatrix} \vdots \\ - f_i - \\ \vdots \end{bmatrix} \begin{bmatrix} \cdots & f_j & \cdots \end{bmatrix} = \begin{bmatrix} \cdots & f_i \cdot f_j & \cdots \end{bmatrix}$$

scalar product $f_i \cdot f_j$

$$\Rightarrow M_{ef}^t M_{ef} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

1. Determine the intersection of the ellipsoid

$$\mathcal{E}_{4,2\sqrt{3},2} : \frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} - 1 = 0 \quad \text{with the line} \quad \ell = \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix} + \left\langle \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} \right\rangle.$$

Write down the equations of the tangent planes in the intersection points.

2. Determine the intersection of the ellipsoid

$$\mathcal{E}_{2,3,4} : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

with planes parallel to the coordinate planes. Treat the various cases separately.

3. Determine the intersection of the ellipsoid

$$\mathcal{E}_{2,\sqrt{3},3} : \frac{x^2}{4} + \frac{y^2}{3} + \frac{z^2}{9} = 1 \quad \text{with the line} \quad \ell : x = y = z.$$

Write down the equations of the tangent planes in the intersection points.

4. Determine the tangent planes to the ellipsoid

$$\mathcal{E}_{2,3,2\sqrt{2}} : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{8} = 1$$

which are parallel to the plane $\pi : 3x - 2y + 5z + 1 = 0$.

5. Use the classification of quadrics to determine what surfaces are described by the following equations

1. $xz + xy + yz = 1$
2. $x^2 - 2xz - y^2 - z^2 = 1$
3. $xz + xy + yz = -1$
4. $5x^2 + 3y^2 + xz = 1$

6. Determine the points P of the ellipsoid

$$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

for which the tangent space $T_P\mathcal{E}$ intersects the coordinate axis in congruent segments.

7. Show that the line

$$\begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix} + \left\langle \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\rangle \quad \text{is tangent to the quadric} \quad \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} - 1 = 0$$

and determine the tangency point.

8. Prove that the intersection of a quadric in $\mathbb{E}^3$ with a plane is either the empty set or a point or a line or two lines or an ellipse or a hyperbola or a parabola.

9. Prove that the intersection of an ellipsoid with a plane is either the empty set or a point or an ellipse.

10. Show that the ellipsoid $\mathcal{E}_{a,b,b}$ is the locus of points for which the sum of the distances to two given points is constant. Such a surface is called *ellipsoid of revolution*.

11. Use a parametrization of an ellipse and a rotation matrix to deduce a parametrization of an ellipsoid of revolution.

12. For the surface $\mathcal{S}$ with parametrization

$$\mathcal{S}: \begin{cases} x = 4\cos(s)\cos(t) \\ y = 4\sin(s)\cos(t) \\ z = 2\sin(t) \end{cases} \qquad s \in [0, 2\pi[\quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}[$$

- Give an equation of $\mathcal{S}$.
- Find the parameters of the point $P(3, \sqrt{3}, 1)$.
- Calculate a parametrization of the tangent plane $T_P\mathcal{S}$ using partial derivatives.
- Give an equation of $T_P\mathcal{S}$.

13. Prove that the intersection of an elliptic cone with a plane is either a point or a line or an ellipse or a hyperbola or a parabola.

1. Determine the intersection of the ellipsoid

$$\mathcal{E}_{4,2\sqrt{3},2}: \frac{x^2}{16}+\frac{y^2}{12}+\frac{z^2}{4}-1=0 \quad \text{with the line} \quad \ell = \begin{bmatrix}4\\-6\\-2\end{bmatrix}+\left\langle\begin{bmatrix}2\\-3\\-2\end{bmatrix}\right\rangle.$$

$\mathcal{E} \cap \ell =$ points on ℓ satisfying the eq. of the ellipsoid

$$\frac{(4+2t)^2}{16}+\frac{(-6-3t)^2}{12}+\frac{(-2-2t)^2}{4}-1=0$$

$$\Leftrightarrow \quad (2+t)^2+(1+t)^2-1=0$$

$$\Leftrightarrow \quad (t+1)(t+2)=0$$

the parameters $t=-1$ and $t=-2$ correspond to points on ℓ which satisfy the eq. of $\mathcal{E}$

$t=-1$ corresponds to $P_1(2,-3,0)$

$t=-2$ — " — $P_2(0,0,2)$

The tangent planes in these points are

$$T_{P_1}\mathcal{E}: \frac{2x}{16}-\frac{3y}{12}+\frac{0\cdot z}{4}-1=0 \quad \Leftrightarrow \quad x-2y-8=0$$

$$T_{P_2}\mathcal{E}: \frac{0x}{16}+\frac{0\cdot y}{12}+\frac{2\cdot z}{4}-1=0 \quad \Leftrightarrow \quad z=2$$

2. Determine the intersection of the ellipsoid

$$\mathcal{E}_{2,3,4} : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

with planes parallel to the coordinate planes. Treat the various cases separately.

Intersection with planes $\parallel$ Oxy: $\mathcal{E} \cap z=h$ for some $h \in \mathbb{R}$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 - \frac{h^2}{16}$$

$$\iff \frac{x^2}{4\left(1-\frac{h^2}{16}\right)} + \frac{y^2}{9\left(1-\frac{h^2}{16}\right)} = 1 \quad (\text{in the plane } z=h)$$

which gives

- the empty set if $1 < \frac{h^2}{16} \iff h \in \left]-\infty,-4\right[\cup \left]4,\infty\right[$
- the point $(0,0,4)$ if $h=4$
- the point $(0,0,-4)$ if $h=-4$
- an ellipse if $h \in (-4,4)$
  with semi-major axis $\sqrt{9\left(1-\frac{h^2}{16}\right)}$
  and semi-minor axis $\sqrt{4\left(1-\frac{h^2}{16}\right)}$

The other cases are treated similarly.

3. Determine the intersection of the ellipsoid

$$\mathcal{E}_{2,\sqrt{3},3} : \frac{x^2}{4} + \frac{y^2}{3} + \frac{z^2}{9} = 1 \quad \text{with the line} \quad \ell : x = y = z.$$

Write down the equations of the tangent planes in the intersection points.

$$\mathcal{E} \cap \ell : \begin{cases} \frac{x^2}{4} + \frac{y^2}{3} + \frac{z^2}{9} = 1 \\ y = x \\ z = x \end{cases}$$

$$\Rightarrow \frac{x^2}{4} + \frac{x^2}{3} + \frac{x^2}{9} = 1$$

$$\Rightarrow 25x^2 = 4 \cdot 9$$

$$\Rightarrow x = \pm \frac{6}{5}$$

$\Rightarrow$ we obtain two points: $P_1\left(\frac{6}{5}, \frac{6}{5}, \frac{6}{5}\right)$ and

$P_2\left(-\frac{6}{5}, \frac{-6}{5}, -\frac{6}{5}\right)$

- The tangent planes in these points are

$$T_{P_1}\mathcal{E} : \frac{\frac{6}{5}x}{4} + \frac{\frac{6}{5}y}{3} + \frac{\frac{6}{5}z}{9} = 1$$

$$\Leftrightarrow \frac{x}{4} + \frac{y}{3} + \frac{z}{9} = \frac{5}{6}$$

and $T_{P_2}\mathcal{E} : \frac{x}{4} + \frac{y}{3} + \frac{z}{9} = -\frac{5}{6}$

4. Determine the tangent planes to the ellipsoid

$$\mathcal{E}_{2,3,2\sqrt{2}} : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{8} = 1$$

which are parallel to the plane $\pi : 3x - 2y + 5z + 1 = 0$.

- for $P(x_0, y_0, z_0) \in \mathcal{E}$ we have

$$T_P\mathcal{E} : \frac{x_0 x}{4} + \frac{y_0 y}{9} + \frac{z_0 z}{8} = 1$$

- $T_P\mathcal{E} \parallel \pi \Rightarrow$ the normal vectors $(3, -2, 5)$ and $\left(\frac{x_0}{4}, \frac{y_0}{9}, \frac{z_0}{8}\right)$

are proportional

$$\frac{3}{\frac{x_0}{4}} = \frac{-2}{\frac{y_0}{9}} = \frac{5}{\frac{z_0}{8}}$$

$$\Leftrightarrow \frac{x_0}{12} = \frac{y_0}{-18} = \frac{z_0}{40}$$

$$\Leftrightarrow \begin{cases} y_0 = -\frac{3}{2} x_0 \\ z_0 = \frac{10}{3} x_0 \end{cases}$$

- with the condition that $P \in \mathcal{E}$ we have

$$\frac{x_0^2}{4} + \frac{\left(-\frac{3}{2}x_0\right)^2}{9} + \frac{\left(\frac{10}{3}x_0\right)^2}{8} = 1 \quad \ldots \Leftrightarrow x_0^2 = \frac{9}{17}$$

so we obtain two points $P_1\left(\frac{3}{\sqrt{17}}, -\frac{9}{\sqrt{17}}, \frac{10}{\sqrt{17}}\right)$ and $P_1\left(-\frac{3}{\sqrt{17}}, \frac{9}{\sqrt{17}}, -\frac{10}{\sqrt{17}}\right)$

$\Rightarrow T_{P_1}\mathcal{E} : \frac{x}{4} - \frac{y}{6} - \frac{5z}{12} = \frac{\sqrt{17}}{3}$ and $T_{P_2}\mathcal{E} : \frac{x}{4} - \frac{y}{6} + \frac{5z}{12} = -\frac{\sqrt{17}}{3}$

5. Use the classification of quadrics to determine what surfaces are described by the following equations

1. $xz + xy + yz = 1$

The matrix associated to this equation is $Q = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

the characteristic polynomial is $\begin{vmatrix} \lambda & 1/2 & 1/2 \\ 1/2 & \lambda & 1/2 \\ 1/2 & 1/2 & \lambda \end{vmatrix} = \lambda^3 + \frac{1}{8} + \frac{1}{8} - \frac{\lambda}{4} - \frac{\lambda}{4} - \frac{\lambda}{4}$

$$-\lambda^3 + \frac{3}{4}\lambda + \frac{1}{4} = 0$$

We notice that $\lambda = 1$ is a root

$$\begin{array}{l|l} -\lambda^3 + \frac{3}{4}\lambda + \frac{1}{4} & \lambda - 1 \\ \hline -\lambda^3 + \lambda^2 & -\lambda^2 - \lambda - \frac{1}{4} \\ \hline \quad -\lambda^2 + \frac{3}{4}\lambda + \frac{1}{4} & \\ \quad -\lambda^2 + \lambda & \\ \hline \quad -\frac{1}{4}\lambda + \frac{1}{4} & \end{array}$$

$$(\lambda - 1)(-\lambda^2 - \lambda - \tfrac{1}{4}) = 0$$

$$\Leftrightarrow -(\lambda+1)(\lambda - \tfrac{1}{2})^2 = 0$$

so the roots are -1 and $\frac{1}{2}$ with $\frac{1}{2}$ having multiplicity 2

$\Rightarrow$ rank $Q = 3$ and the signature of Q is $(2, 1)$

two positive eigenvalues ↳ one negative eigenvalue

$\Rightarrow$ the possibilities for this surface are

either a hyperboloid (of one sheet or of two sheets)

or an elliptic cone

Method 2 we can make linear changes of variables which correspond to affine changes of coordinates in order to see the canonical form.

$$xy + yz + zx = 1$$

replace y by $x+y$ to obtain

$$x^2 + xy + xz + yz + zx = 1$$

$$\Leftrightarrow \quad x^2 + xy + yz + 2zx = 1$$

$$\Leftrightarrow \quad \left(x^2 + xy + \frac{y^2}{4}\right) - \frac{y^2}{4} + yz + 2zx = 1$$

$$\Leftrightarrow \quad \left(x + \frac{y}{2}\right)^2 - \left(\frac{y^2}{4} - yz + z^2\right) + z^2 + 2zx = 1$$

$$\Leftrightarrow \quad \left(x + \frac{y}{2}\right)^2 - \left(\frac{y}{2} - z\right)^2 + z^2 + 2zx = 1$$

Replace $x + \frac{y}{2}$ by x

$$\Leftrightarrow \quad x^2 - \left(\frac{y}{2} - z\right)^2 + z^2 + 2z\left(x - \frac{y}{2}\right) = 1$$

$$\Leftrightarrow \quad x^2 - \left(\frac{y}{2} - z\right)^2 + z^2 + 2zx - zy = 1$$

Replace $\frac{y}{2} - z$ by y

$$\Leftrightarrow \quad x^2 - y^2 + z^2 + 2zx - z(2y + 2z) = 1$$

$$\Leftrightarrow \quad x^2 - y^2 + z^2 + 2zx - 2yz - 2z^2 = 1$$

$$\Leftrightarrow \quad x^2 - y^2 - z^2 + 2zx - 2yz = 1$$

$$\Leftrightarrow \quad (x^2 + 2zx + z^2) - z^2 - (y^2 + 2yz + z^2) + z^2 - z^2 = 1$$

$$\Leftrightarrow \quad (x+z)^2 - (y+z)^2 - z^2 = 1$$

Replace $x+z$ by x and $y+z$ by y to obtain

$$x^2 - y^2 - z^2 = 1 \quad \text{or} \quad y^2 + z^2 - x^2 = -1$$ the eq. of a hyperboloid of two sheets.

6. Determine the points P of the ellipsoid

$$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

for which the tangent space $T_P\mathcal{E}$ intersects the coordinate axis in congruent segments.

Let $P(x_0, y_0, z_0)$ be a point on $\mathcal{E}$

Then $T_p\mathcal{E} : \quad \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1$

and $T_p\mathcal{E} \cap Ox = \begin{bmatrix} a^2/x_0 \\ 0 \\ 0 \end{bmatrix}$

$T_p\mathcal{E} \cap Oy = \begin{bmatrix} 0 \\ b^2/y_0 \\ 0 \end{bmatrix}$

$T_p\mathcal{E} \cap Oz = \begin{bmatrix} 0 \\ 0 \\ c^2/z_0 \end{bmatrix}$

"$T_p\mathcal{E}$ intersects the coordinate axis in congruent segments" means that the distances from the intersection points to the origin are equal

so $\left|\frac{x_0}{a^2}\right| = \left|\frac{y_0}{b^2}\right| = \left|\frac{z_0}{c^2}\right| \Rightarrow x_0 = \pm \frac{a^2}{c^2} z_0$ and $y_0 = \pm \frac{b^2}{c^2} z_0$

since $P \in \mathcal{E}$ we have $\frac{\left(\pm \frac{a^2}{c^2} z_0\right)^2}{a^2} + \frac{\left(\pm \frac{b^2}{c^2} z_0\right)^2}{b^2} + \frac{z_0^2}{c^2} = 1$

$$\Leftrightarrow \quad z_0^2 \left(\frac{a^4}{c^4} + \frac{b^4}{c^4} + \frac{1}{c^2}\right) = 1$$

$$\Leftrightarrow \quad z_0 = \pm \frac{c^2}{\sqrt{a^2 + b^2 + c^2}}$$

so $P = \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{bmatrix} \pm a^2 \\ \pm b^2 \\ \pm c^2 \end{bmatrix}$

we obtain 8 points where the property holds. In fact, if you find one such point in one octant then with the symmetries of $\mathcal{E}$ you obtain such points also in the other 7 octants.

7. Show that the line

$$\begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix} + \left\langle \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\rangle \text{ is tangent to the quadric } \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} - 1 = 0$$

and determine the tangency point.

$$\ell : \begin{cases} x = 2 \\ y = -3-t \\ z = 6+2t \end{cases} \qquad t \in \mathbb{R}$$

ℓ is tangent to $\mathcal{E}$ if it intersects $\mathcal{E}$ in a double point

this happens if

$$\frac{2^2}{4} + \frac{(-3-t)^2}{9} + \frac{(6+2t)^2}{16} - 1 = 0 \qquad (*)$$

has a double solution in t

$$(*) \iff (3+t)^2 = 0$$

so, yes, ℓ is tangent to $\mathcal{E}$.

The point of tangency (where ℓ touches $\mathcal{E}$)

corresponds to $t = -3$, it is $P = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

8. Prove that the intersection of a quadric in $\mathbb{E}^3$ with a plane is either the empty set or a point or a line or two lines or an ellipse or a hyperbola or a parabola.

A quadric is the set of solutions to a quadratic equation given w.r.t a coordinate system $K = (O, i, j, k)$

Let S be the quadric and let π be a plane

Fix a second coordinate system $K' = (O', i', j', k')$ with $i', j' \parallel \pi$ and $O \in \pi$

Changing the coordinate system from K to K' will change the equation of S to some other quadratic equation in three variables

But in this coordinate system K' the intersection with π is obtained by setting the third coordinate equal to zero, since $\pi = Ox'y' : z' = 0$

This gives a quadratic equation in the plane π which, by the classification of quadratic curves is one of the possibilities in the statement.

9. Prove that the intersection of an ellipsoid with a plane is either the empty set or a point or an ellipse.

As in the previous exercise

With the extra observation that an ellipsoid is bounded (you can put it in a box)

$\Rightarrow$ $S \cap \pi$ is also bounded

$\Rightarrow$ the possibilities for $S \cap \pi$ are the quadratic curves (possibly degenerate) which are bounded

10. Show that the ellipsoid $\mathcal{E}_{a,b,b}$ is the locus of points for which the sum of the distances to two given points is constant. Such a surface is called *ellipsoid of revolution*.

- View $\mathcal{E}_{a,b,b}$ as the union of ellipses $\Pi \cap \mathcal{E}_{a,b,b}$ where Π is a plane which contains the x-axis

- All such ellipses have the same semi-major and semi-minor axes

  $\Rightarrow$ they have the same focal points.

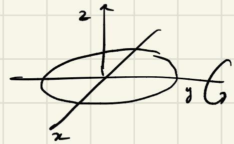
We need to assume $a > b$

$\Rightarrow$ the focal points are on the x-axis

- Since $\mathcal{E}_{a,b,b}$ is the union of such ellipses, all points on $\mathcal{E}_{a,b,b}$ have the required property. (if $a > b$)

11. Use a parametrization of an ellipse and a rotation matrix to deduce a parametrization of an ellipsoid of revolution.

ellipse in Oxy $\begin{cases} x = a\cos\theta \\ y = b\sin\theta \\ z = 0 \end{cases}$



$$\begin{bmatrix} \cos\theta' & 0 & \sin\theta' \\ 0 & 1 & 0 \\ -\sin\theta' & 0 & \cos\theta' \end{bmatrix} \begin{bmatrix} a\cos\theta \\ b\sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} a\cos\theta\cos\theta' \\ b\sin\theta \\ -a\cos\theta\sin\theta' \end{bmatrix}$$

rotation around y-axis — ellipsoid of revolution

12. For the surface $\mathcal{S}$ with parametrization

$$\mathcal{S}: \begin{cases} x = 4\cos(s)\cos(t) \\ y = 4\sin(s)\cos(t) \\ z = 2\sin(t) \end{cases} \qquad s \in [0, 2\pi[\quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}[$$

- Give an equation of $\mathcal{S}$.
- Find the parameters of the point $P(3, \sqrt{3}, 1)$.
- Calculate a parametrization of the tangent plane $T_P\mathcal{S}$ using partial derivatives.
- Give an equation of $T_P\mathcal{S}$.

- $S : \dfrac{x^2}{16} + \dfrac{y^2}{16} + \dfrac{z^2}{4} = 1$

- $P(3, \sqrt{3}, 1)$ $\quad \sin(t) = \frac{1}{2} \Rightarrow t = \frac{\pi}{6}$

$$\Rightarrow \begin{cases} 3 = 4\frac{\sqrt{3}}{2}\cos(s) \\ \sqrt{3} = 4\frac{\sqrt{3}}{2}\sin(s) \end{cases} \Rightarrow \begin{cases} \cos(s) = \frac{\sqrt{3}}{2} \\ \sin(s) = \frac{1}{2} \end{cases} \Rightarrow s = \frac{\pi}{6}$$

so P is obtained with the parameters $(t, s) = (\frac{\pi}{6}, \frac{\pi}{6})$

- $T_pS = p + \langle \frac{\partial\sigma}{\partial t}(p), \frac{\partial\sigma}{\partial s}(p) \rangle$ where $\sigma(t,s) = \begin{bmatrix} 4\cos(s)\cos(t) \\ 4\sin(s)\cos(t) \\ 2\sin(t) \end{bmatrix}$

$$\frac{\partial\sigma}{\partial s}(p) = \begin{bmatrix} -4\sin(s)\cos(t) \\ 4\cos(s)\cos(t) \\ 0 \end{bmatrix}(p) = \begin{bmatrix} -\sqrt{3} \\ 3 \\ 0 \end{bmatrix}$$

$$\frac{\partial r}{\partial t}(p) = \begin{pmatrix} -4\cos(s)\sin(t) \\ -4\sin(s)\sin(t) \\ 2\cos(t) \end{pmatrix}(p) = \begin{pmatrix} -\sqrt{3} \\ -1 \\ \sqrt{3} \end{pmatrix}$$

$$\Rightarrow T_pS : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ \sqrt{3} \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -\sqrt{3} \\ 3 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -\sqrt{3} \\ -1 \\ \sqrt{3} \end{pmatrix}$$

- $T_pS : \frac{3x}{16} + \frac{\sqrt{3}y}{16} + \frac{z}{4} = 1$

13. Prove that the intersection of an elliptic cone with a plane is either a point or a line or an ellipse or a hyperbola or a parabola.

Similar to 8 and 9

1. Determine the intersection of the hyperboloid

$$\mathcal{H}^1_{4,3,1} : \frac{x^2}{16} + \frac{y^2}{9} - \frac{z^2}{1} = 1 \quad \text{with the line} \quad \ell = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} + \langle \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \rangle.$$

Write down the equations of the tangent planes in the intersection points.

2. Determine the tangent plane of the hyperboloid

$$\mathcal{H}^1_{2,3,1} : \frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1$$

in the point $M(2,3,1)$. Show that the tangent plane intersects the surface in two lines.

3. Determine the generators of the hyperboloid

$$\frac{x^2}{36} + \frac{y^2}{9} - \frac{x^2}{4} = 1$$

which are parallel to the plane $x + y + z = 0$.

4. Determine the intersection of the hyperboloid

$$\mathcal{H}^2_{2,1,3} : \frac{x^2}{4} + \frac{y^2}{1} - \frac{z^2}{9} = -1 \quad \text{with the line} \quad \ell = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} + \langle \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \rangle.$$

Write down the equations of the tangent planes in the intersection points.

5. Determine the intersection of the paraboloid

$$\mathcal{P}^h_{2,\frac{1}{2}} : x^2 - 4y^2 = 4z \quad \text{with the line} \quad \ell = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + \langle \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \rangle.$$

Write down the equations of the tangent planes in the intersection points.

6. Determine the tangent plane of

1. the elliptic paraboloid $\frac{x^2}{5} + \frac{y^2}{3} = z$ and of
2. the hyperbolic paraboloid $x^2 - \frac{y^2}{4} = z$

which are parallel to the plane $x - 3y + 2z - 1 = 0$.

7. Determine the plane which contains the line

$$\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \langle \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \rangle \quad \text{and is tangent to the quadric} \quad x^2 + 2y^2 - z^2 + 1 = 0.$$

8. Show that the parabolid $\mathcal{P}^e_{p,p}$ is the locus of points for which the distance from a point equals the distance to a plane. Such a surface is called *elliptic paraboloid of revolution*.

9. Use a parametrization of a parabola and a rotation matrix to deduce a parametrization of an elliptic paraboloid of revolution.

10. For the surface $\mathcal{S}$ with parametrization

$$\mathcal{S}: \begin{cases} x = \sqrt{1+t^2}\cos(s) \\ y = \sqrt{1+t^2}\sin(s) \\ z = 2t \end{cases}$$

- Give the equation of $\mathcal{S}$.
- Find the parameters of the point $P(1,1,2)$.
- Calculate a parametrization of the tangent plane $T_P\mathcal{S}$ using partial derivatives.
- Give the equation of $T_P\mathcal{S}$.

11. For the surface $\mathcal{S}$ with parametrization

$$\mathcal{S}: \begin{cases} x = s \\ y = t \\ z = s^2 - t^2 \end{cases}$$

- Give the equation of $\mathcal{S}$.
- Find the parameters of the point $P(1,1,0)$.
- Calculate a parametrization of the tangent plane $T_P\mathcal{S}$ using partial derivatives.
- Give the equation of $T_P\mathcal{S}$.

12. Determine the generators of the paraboloid

$$4x^2 - 9y^2 = 36z$$

containing the point $P(3\sqrt{2},2,1)$.

13. Determine the generators of the paraboloid

$$\frac{x^2}{16} - \frac{y^2}{4} = z$$

which are parallel to the plane $3x+2y-4z=0$.

14. Which of the following is a hyperboliod?

1. $\mathcal{S}: 2xz + 2xy + 2yz = 1$
2. $\mathcal{S}: 5x^2 + 3y^2 + xz = 1$
3. $\mathcal{S}: 2xy + 2yz + y + z = 2$

1. Determine the intersection of the hyperboloid

$$\mathcal{H}^1_{4,3,1} : \frac{x^2}{16} + \frac{y^2}{9} - \frac{z^2}{1} = 1 \quad \text{with the line} \quad \ell = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} + \left\langle \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \right\rangle.$$

Write down the equations of the tangent planes in the intersection points.

$$\mathcal{H} \cap \ell : \quad \frac{(4+4t)^2}{16} + \frac{(-2)^2}{9} - (1+t)^2 = 0$$

$$\Leftrightarrow \quad \frac{4}{9} = 1$$

so there is no solution, i.e no intersection point

$\Rightarrow$ no tangent planes to write down

2. Determine the tangent plane of the hyperboloid

$$\mathcal{H}^1_{2,3,1} : \frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1$$

in the point $M(2,3,1)$. Show that the tangent plane intersects the surface in two lines.

$$T_M \mathcal{H}' : \frac{x}{2} + \frac{y}{3} - z = 1$$

$$T_M \mathcal{H}' \cap \mathcal{H}' : \begin{cases} \frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1 \\ \frac{x}{2} + \frac{y}{3} - z = 1 \end{cases}$$

so $z = \frac{x}{2} + \frac{y}{3} - 1$

$$\Rightarrow \quad \frac{x^2}{4} + \frac{y^2}{9} - \left(\frac{x}{2} + \frac{y}{3} - 1\right)^2 = 1$$

$$\frac{x^2}{4} + \frac{y^2}{9} + 1 + \frac{xy}{3} - x - \frac{2y}{3}$$

$$\Rightarrow \quad xy - 3x - 2y - 6 = 0$$

$\Rightarrow \quad (x-2)(y-3)=0$

So $T_M \mathcal{H}' \cap \mathcal{H}' : \begin{cases} \frac{x}{2}+\frac{y}{3}-z=1 \\ (x-2)(y-3)=0 \end{cases}$

the solutions to this system is the union of solutions to the two systems

$\ell_1 : \begin{cases} \frac{x}{2}+\frac{y}{3}-z=1 \\ x-2=0 \end{cases}$ and $\ell_2 : \begin{cases} \frac{x}{2}+\frac{y}{3}-z=1 \\ y-3=0 \end{cases}$

i.e. $T_M \mathcal{H}' \cap \mathcal{H}'$ is the union of the two lines ℓ_1 and ℓ_2

3. Determine the generators of the hyperboloid

$$\frac{x^2}{36}+\frac{y^2}{9}-\frac{x^2}{4}=1$$

which are parallel to the plane $x+y+z=0$.

$$\mathcal{H} : \left(\frac{x}{6}-\frac{z}{2}\right)\left(\frac{x}{6}+\frac{z}{2}\right)=\left(1-\frac{y}{3}\right)\left(1+\frac{y}{3}\right)$$

One family of generators is $\quad \ell_\alpha : \begin{cases} \frac{x}{6}-\frac{z}{2}=\alpha\left(1-\frac{y}{3}\right) \\ \alpha\left(\frac{x}{6}+\frac{z}{2}\right)=1+\frac{y}{3} \end{cases} \qquad \alpha\in\mathbb{R}$

$$\Leftrightarrow \begin{cases} x-3z=6\alpha-2\alpha y \\ \alpha x+3\alpha z=6+2y \end{cases}$$

$$\begin{pmatrix} 1 & 2\alpha & -3 & 6\alpha \\ \alpha & -2 & 3\alpha & 6 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2\alpha & -3 & 6\alpha \\ 0 & -2-2\alpha^2 & 6\alpha & 6-6\alpha^2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & \frac{-3(1-\alpha^2)}{1+\alpha^2} & \frac{8\alpha}{1+\alpha^2} \\ 0 & 1 & \frac{-3\alpha}{1+\alpha^2} & -\frac{3(1-\alpha^2)}{1+\alpha^2} \end{pmatrix}$$

you can also calculate a direction vector with the vector product.

$$\Rightarrow \ell_\alpha : \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \frac{1}{1+\alpha^2} \begin{bmatrix} 8\alpha \\ -3(1-\alpha^2) \\ 0 \end{bmatrix} + z \underbrace{\begin{bmatrix} \frac{3(1-\alpha^2)}{1+\alpha^2} \\ \frac{3\alpha}{1+\alpha^2} \\ 1 \end{bmatrix}}_{=v}$$

a direction vector

$$\ell_\alpha \parallel \pi \Leftrightarrow v \parallel \pi$$

$$\Leftrightarrow \frac{3(1-\alpha^2)}{1+\alpha^2} + \frac{3\alpha}{1+\alpha^2} + 1 = 0$$

$$\Leftrightarrow -2\alpha^2 + 3\alpha + 4 = 0$$

$$\Delta = 41 \Rightarrow \alpha_{1,2} = \frac{3 \pm \sqrt{41}}{4}$$

In this family of generators there are two lines parallel to π

$$\ell_{\alpha_1} : \{ \;\cdots \qquad \ell_{\alpha_2} : \{ \;\cdots$$

The second family of generators is $\tilde{\ell}_\alpha$:
$$\begin{cases} \frac{x}{6} - \frac{z}{2} = \alpha\left(1 + \frac{y}{3}\right) \\ \alpha\left(\frac{x}{6} + \frac{z}{2}\right) = 1 - \frac{y}{3} \end{cases}$$

a direction vector for $\tilde{\ell}_\alpha$ is

$$w = \begin{vmatrix} i & j & k \\ \frac{1}{6} & -\frac{\alpha}{3} & -\frac{1}{2} \\ \frac{\alpha}{6} & \frac{1}{3} & \frac{\alpha}{2} \end{vmatrix} = i\left(\frac{-\alpha^2+1}{6}\right) - j\left(\frac{\alpha}{12} + \frac{\alpha}{12}\right) + k\left(\frac{1}{18} + \frac{\alpha^2}{18}\right)$$

$$\tilde{\ell}_\alpha \parallel \pi \Leftrightarrow w \parallel \pi$$

$$\Leftrightarrow \frac{-\alpha^2+1}{6} - \frac{\alpha}{6} + \frac{1+\alpha^2}{18} = 0$$

$$\Leftrightarrow \quad -3\alpha^2+3-3\alpha+1+\alpha^2=0$$

$$\Leftrightarrow \quad -2\alpha^2-3\alpha+4=0$$

$$\Delta = 41 \quad \Rightarrow \alpha_{1,2} = \frac{-3 \pm \sqrt{41}}{4}$$

As expected we obtain two generators also in the family $\tilde{\ell}_\alpha$

$\tilde{\ell}_{\alpha_1} : \{ \ldots$ and $\tilde{\ell}_{\alpha_2} : \{ \ldots$

4. Determine the intersection of the hyperboloid

$$\mathcal{H}^2_{2,1,3} : \frac{x^2}{4} + \frac{y^2}{1} - \frac{z^2}{9} = -1 \quad \text{with the line} \quad \ell = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} + \langle \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \rangle.$$

Write down the equations of the tangent planes in the intersection points.

- $\mathcal{H}^2 \cap \ell$: $\quad \frac{(3+t)^2}{4} + (1+t)^2 - \frac{3^2(2+t)^2}{9} = -1$

$$\Leftrightarrow \quad (t-1)^2 = 0$$

so we obtain a double solution. This means that ℓ intersects $\mathcal{H}^2$ in a double point, i.e ℓ is tangent to $\mathcal{H}^2$

- The intersection point is $p = \begin{pmatrix} 4 \\ 2 \\ 9 \end{pmatrix}$ and the corresponding tangent plane is

$$T_p\mathcal{H}^2 : x+2y-z=-1$$

5. Determine the intersection of the paraboloid

$$\mathcal{P}^h_{2,\frac{1}{2}} : x^2 - 4y^2 = 4z \quad \text{with the line} \quad \ell = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + \langle \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \rangle.$$

Write down the equations of the tangent planes in the intersection points.

• $\mathcal{P}^h \cap \ell$: $(2+2t)^2 - 4t^2 = 4(3-2t)$

$\iff t = \frac{1}{2}$

So we have a single solution which corresponds to a simple point of intersection. The line punctures the surface in the point

$$P = \begin{bmatrix} 3 \\ 1/2 \\ 2 \end{bmatrix}$$

• $\mathcal{P}^h$: $\frac{x^2}{2} - \frac{y^2}{1/2} = 2z \Rightarrow T_P\mathcal{P}^h : \frac{3x}{2} - \frac{\frac{1}{2}y}{1/2} = z + 2$

$\Rightarrow T_P\mathcal{P}^h : \frac{3}{2}x - y - z - 2 = 0$

6. Determine the tangent plane of

1. the elliptic paraboloid $\frac{x^2}{5} + \frac{y^2}{3} = z$ and of
2. the hyperbolic paraboloid $x^2 - \frac{y^2}{4} = z$

which are parallel to the plane $x - 3y + 2z - 1 = 0$.

1) $\mathcal{P}^e : \frac{x^2}{5} + \frac{y^2}{3} = z$ $\qquad \pi : x - 3y + 2z - 1 = 0$

$$p = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \Rightarrow T_p\mathcal{P}^e : \frac{xx_0}{5} + \frac{yy_0}{3} = \frac{z+z_0}{2}$$

$T_p\mathcal{P}^e \parallel \pi \Rightarrow \frac{x_0}{5} = \frac{y_0}{-9} = \frac{-1}{4}$

$\Rightarrow x_0 = -\frac{5}{4}, \quad y_0 = \frac{9}{4}$

since $p \in \mathcal{P}^e$ we also have

$$\frac{\left(\frac{-5}{4}\right)^2}{5} + \frac{\left(\frac{9}{4}\right)^2}{3} = z_0 \Rightarrow z_0 = 2$$

$$\Rightarrow p = \begin{pmatrix} -5/4 \\ 9/4 \\ 2 \end{pmatrix}$$ is the point where $T_p \mathcal{P}^e \parallel \pi$

$$T_p \mathcal{P}^e : -\frac{x}{4} + \frac{3y}{4} = \frac{z+2}{2}$$

$$\Leftrightarrow \quad x - 3y + 2z + 4 = 0$$

2.) $T_p \mathcal{P}^h : \quad x x_0 - \frac{y y_0}{4} = \frac{z + z_0}{2}$

$T_p \mathcal{P}^h \parallel \pi \Rightarrow \quad x_0 = -\frac{1}{4}, \; y_0 = -3$

since $p \in \mathcal{P}^h \Rightarrow \quad z_0 = -\frac{1}{2}$

$\Rightarrow p = \begin{pmatrix} -1/4 \\ -3 \\ -1/2 \end{pmatrix}$ and $T_p \mathcal{P}^h : x - 3y + 2z + 1 = 0$

7. Determine the plane which contains the line

$$\ell : \underset{Q}{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}} + \langle \underset{v}{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}} \rangle \text{ and is tangent to the quadric } \quad x^2 + 2y^2 - z^2 + 1 = 0.$$

for $p = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ $\quad T_p S : \quad x_0 x + 2 y_0 y - z z_0 + 1 = 0$

$n(x_0, 2y_0, -z_0)$ is a normal vector

$\ell \subseteq \pi \Rightarrow n \perp v \Rightarrow n \cdot v = 0 \qquad 2x_0 - 2y_0 = 0 \Rightarrow x_0 = y_0$

$\ell \subseteq \pi \Rightarrow Q \in T_p S \Rightarrow -x_0 + 1 = 0 \quad \Rightarrow x_0 = 1$

$\Rightarrow x_0 = y_0 = 1$

so $T_p S : \quad x + 2y - z z_0 + 1 = 0$

$p \in S \Rightarrow \quad 1 + 2 - z_0^2 + 1 = 0 \quad \Rightarrow z_0 = \pm 2$

$\Rightarrow p = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

8. Show that the parabolid $\mathcal{P}^e_{p,p}$ is the locus of points for which the distance from a point equals the distance to a plane. Such a surface is called *elliptic paraboloid of revolution*.

Similar to problem 10 from last week

9. Use a parametrization of a parabola and a rotation matrix to deduce a parametrization of an elliptic paraboloid of revolution.

Similar to problem 11 from last week

10. For the surface $\mathcal{S}$ with parametrization

$$\mathcal{S}: \begin{cases} x = \sqrt{1+t^2}\cos(s) \\ y = \sqrt{1+t^2}\sin(s) \\ z = 2t \end{cases}$$

- Give the equation of $\mathcal{S}$.
- Find the parameters of the point $P(1,1,2)$.
- Calculate a parametrization of the tangent plane $T_P\mathcal{S}$ using partial derivatives.
- Give the equation of $T_P\mathcal{S}$.

- S is an elliptic paraboloid with equation

$$x^2+y^2=\frac{z}{2}$$

- The parameters of the point $P=\begin{bmatrix}1\\1\\2\end{bmatrix}$ are $t=1$, $s=\frac{\pi}{4}$

- $$T_pS = P + \langle \frac{\partial\sigma}{\partial s}(P), \frac{\partial\sigma}{\partial t}(P)\rangle$$

$$\frac{\partial\sigma}{\partial s}(P) = \begin{bmatrix}-\sqrt{1+t^2}\sin(s)\\ \sqrt{1+t^2}\cos(s)\\ 0\end{bmatrix}(P) = \begin{bmatrix}-1\\1\\0\end{bmatrix}, \qquad \frac{\partial\sigma}{\partial t}(P) = \begin{bmatrix}\frac{t}{\sqrt{1+t^2}}\cos(s)\\ \frac{t}{\sqrt{1+t^2}}\sin(s)\\ 2\end{bmatrix}(P) = \begin{bmatrix}1/2\\1/2\\2\end{bmatrix}$$

$$T_pS: \begin{bmatrix}1\\1\\2\end{bmatrix} + \alpha\begin{bmatrix}-1\\1\\0\end{bmatrix} + \beta\begin{bmatrix}1/2\\1/2\\2\end{bmatrix} \quad \alpha,\beta\in\mathbb{R}$$

- $$T_pS: \frac{xx_0}{a}+\frac{yy_0}{b} = z+z_0 \implies 4x+4y-z-2=0$$