Seminar 8. EX. 37 c)

$$\iint_A (\sin x + \sin y)\, dx\, dy, \quad \text{where } A = [0, \pi/2] \times [0, \pi/4]$$

We denote $I = \iint_A (\sin x + \sin y)\, dx\, dy$

$$I = \int_0^{\pi/2} \left(\int_0^{\pi/4} (\sin x + \sin y)\, dy \right) dx$$

$$\int_0^{\pi/4} (\sin x + \sin y)\, dy = \int_0^{\pi/4} \sin x\, dy + \int_0^{\pi/4} \sin y\, dy \overset{\substack{x \text{ const} \\ y \text{ var.}}}{=\!=} \sin x \cdot y \Big|_0^{\pi/4} - \cos y \Big|_0^{\pi/4} =$$

$$= \sin x \left(\frac{\pi}{4} - 0 \right) - \left(\cos \frac{\pi}{4} - \cos 0 \right) = \frac{\pi}{4} \sin x - \left(\frac{\sqrt{2}}{2} - 1 \right) =$$

$$= \frac{\pi}{4} \sin x - \frac{\sqrt{2}}{2} + 1$$

$$I = \int_0^{\pi/2} \left(\frac{\pi}{4} \sin x + 1 - \frac{\sqrt{2}}{2} \right) dx = \frac{\pi}{4} \int_0^{\pi/2} \sin x\, dx + \int_0^{\pi/2} 1\, dx - \int_0^{\pi/2} \frac{\sqrt{2}}{2}\, dx =$$

$$= \frac{\pi}{4} (-\cos x) \Big|_0^{\pi/2} + x \Big|_0^{\pi/2} - \frac{\sqrt{2}}{2} x \Big|_0^{\pi/2} = \frac{\pi}{4} \left(-\cos \frac{\pi}{2} + \cos 0 \right) + \frac{\pi}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\pi}{2} =$$

$$= \frac{\pi}{4} (0 + 1) + \frac{\pi}{2} \left({}^{2)}1 - \frac{\sqrt{2}}{2} \right) = \frac{\pi}{4} + \frac{\pi}{2} \cdot \frac{2 - \sqrt{2}}{2} = \frac{\pi}{4} \left(1 + 2 - \sqrt{2} \right) = \frac{\pi}{4} (3 - \sqrt{2})$$

$$\Longrightarrow \iint_A (\sin x + \sin y)\, dx\, dy = \frac{\pi}{4} (3 - \sqrt{2})$$

where $A = [0, \pi/2] \times [0, \pi/4]$

Seminar 8. Ex. 37 d)

$$\int\ldots\int_A e^{x_1+\ldots+x_m}\,dx_1\ldots dx_m \text{ , where } A=[0,1]\times\ldots\times[0,1]$$

$$\int\ldots\int_A e^{x_1+\ldots+x_m}\,dx_1\ldots dx_m = \int\ldots\int_A e^{x_1}\cdot\ldots\cdot e^{x_m}\,dx_1\ldots dx_m$$

We denote $I^m = \int\ldots\int_A e^{x_1+\ldots+x_m}\,dx_1\ldots dx_m$

$$\int_0^1 e^{x_1}\cdot\ldots\cdot e^{x_m}\,dx_1 \;\; \frac{e^{x_1}\text{ var}}{e^{x_2}\ldots e^{x_m}\text{ const}} \;\; e^{x_2}\cdot\ldots\cdot e^{x_m}\cdot e^{x_1}\Big|_0^1 =$$

$$= e^{x_2}\ldots e^{x_m}(e^1-e^0) = e^{x_2}\ldots e^{x_m}(e-1)$$

$$\int_0^1 e^{x_2}\ldots e^{x_m}(e-1)\,dx_2 \;\; \frac{e^{x_2}\text{ var}}{e^{x_3}\ldots e^{x_m}(e-1)\text{ const}} \;\; e^{x_3}\ldots e^{x_m}(e-1)\cdot e^{x_2}\Big|_0^1 =$$

$$= e^{x_3}\ldots e^{m}(e-1)(e-1) = e^{x_3}\ldots e^{x_m}(e-1)^2$$

$$\Rightarrow P(m):\ I^m = \int\ldots\int_A e^{x_1+\ldots+x_m}\,dx_1\ldots dx_m =$$

$$= \left(\int_0^1 e^x\,dx\right)^m \text{ , where } A=[0,1]\times\ldots\times[0,1]$$

$m\geq 2,\ m\in\mathbb{N}$

1. First stage

$$P(2):\ \iint_A e^{x_1+x_2}\,dx_1\,dx_2 = \int_0^1\left(\int_0^1 e^{x_1}\cdot e^{x_2}\,dx_1\right)dx_2 =$$

where $A=[0,1]\times[0,1]$

$$= \int_0^1\left(e^{x_1}\Big|_0^1\cdot e^{x_2}\right)dx_2 = \int_0^1 (e^1-e^0)\,e^{x_2}\,dx_2 =$$

$$= \int_0^1 (e-1)\,e^{x_2}\,dx_2 = e^{x_2}\Big|_0^1\cdot(e-1) =$$

$$= (e-1)(e-1) = (e-1)^2 \quad\Rightarrow \text{TRUE}$$

2. Second stage

We assume that there is a true P(k) and we proof P(k+1) where $k \le m$

$$P(k): \int\dots\int_A e^{x_1+\dots+x_k} d_1 \dots dx_k = \left(\int_0^1 e^x dx\right)^k \quad \text{„T"}$$

where $A = [0,1] \times \dots \times [0,1]$

$$P(k+1): \int\dots\int\int_A e^{x_1+\dots+x_k+x_{k+1}} d_1 \dots dx_{k+1} = \left(\int_0^1 e^x dx\right)^{k+1}$$

where $A = [0,1] \times \dots \times [0,1] \times [0,1]$

$$\int\dots\int_A e^{x_1} \cdot \dots \cdot e^{x_k} \cdot e^{x_{k+1}} d_1 \dots dx_k \, dx_{k+1} =$$

$$\overset{P(k)}{=} \int_0^1 \left(\int_0^1 e^x dx\right)^k e^{x_{k+1}} dx_{k+1} = \int_0^1 (e-1)^k e^{x_{k+1}} dx_{k+1} =$$

$$= e^{x_{k+1}} \Big|_0^1 \cdot (e-1)^k = (e-1)(e-1)^k = (e-1)^{k+1} \Rightarrow$$

$\Rightarrow$ P(k+1) it is true $\Rightarrow$ P(m) it is true

$$\Rightarrow I^m = \left(\int_0^1 e^x dx\right)^m = \left(e^x \Big|_0^1\right)^m = \left(e^1 - e^0\right)^m = (e-1)^m$$

$$\Rightarrow \int\dots\int_A e^{x_1+\dots+x_m} dx_1 \dots dx_m = (e-1)^m$$

where $A = [0,1] \times \dots \times [0,1]$

Hognogi Ana-Maria
Cristina
Group 913