Lab Nr. 10, Numerical Calculus

Numerical Integration I

Newton-Cotes Formulas; Adaptive Quadratures

Matlab functions

- *int*: computes definite and indefinite integrals symbolically
- *integral*: computes definite integrals numerically

1. Implement the composite rectangle, trapezoidal and Simpson's formulas.

2. Implement adaptive quadratures based on the three rules above.

Applications

1. Approximate $\ln 2$ with 3 correct decimals, applying the repeated rectangle, trapezoidal and Simpson's rules to the integral

$$I = \int_1^2 \frac{dx}{x},$$

with the appropriate number of subintervals. Compute the errors of the approximations.

2. Recall from Calculus that for a function $f : [a, b] \to \mathbb{R}_+$, the *area* under the curve $y = f(x)$ is given by the integral

$$A(f) = \int_a^b f(x)\, dx.$$

a) Plot the graph of the function

$$f(x) = \frac{xe^{-x}}{x^2+1},\ x \in [0, 1].$$

b) Approximate the area under f, using all three composite rules mentioned above, for $n = 2, 4, 8, 16, \dots, 256$. Compute the errors and the ratios of errors. Are the rates of convergence consistent with the theoretical ones?

3. The **error function** (also known as the **Gauss error function**), defined as

$$\operatorname{erf}(x) = \frac{2}{\pi} \int_0^x e^{-t^2}\, dt,\ x \geq 0,$$

is an important function in mathematics, with applications in expressing solutions of the *heat equation*, in computing the cumulative distribution function (CDF) of the Normal distribution, and many more. Use adaptive trapezoidal quadratures to tabulate its values for $x = 0.1, 0.2, \dots, 1$. Compare the results with those given by Matlab functions *integral* and *erf*.

4. Recall from Calculus that the *length* of the curve represented by a differentiable function $y = f(x)$ on an interval $[a, b]$, is given by the integral

$$l(f) = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

a) Plot the graph of the function $f : [0, 1] \to \mathbb{R}$, $f(x) = \sin(\pi x)$.

b) Use adaptive Simpson quadratures to approximate the length of the curve.

Optional

5. The theoretical rates of convergence for the rectangle, trapezoidal and Simpson's rules assume a certain degree of smoothness of the integrand ($f \in C^2[a, b]$ for the first two and $f \in C^4[a, b]$ for the last one). These formulas *will not* perform as well for functions f that do not satisfy those conditions, in which case, other procedures might be necessary for numerically integrating them. Here is such an example.

a) Apply Simpson's rule to the integral

$$I = \int_0^1 \sin(\sqrt[3]{x})\, dx,$$

with $n = 2, 4, 8, \ldots, 256$. Compute the errors and their ratios. What can be noticed?

b) Transform I by using a suitable change of variables that will restore the higher speed of convergence of Simpson's rule and repeat part **a)**.