14.12.2021.

Seminar 11. Hogmogi Ana-Maria Cristina

51 a)

$$I = \int_0^{\infty} \frac{\operatorname{arctg} x}{1+x^2}\, dx = \lim_{t\to\infty}\left(\int_0^t \frac{\operatorname{arctg} x}{1+x^2}\, dx\right) = \lim_{t\to\infty}\left(\int_0^t \operatorname{arctg} x \cdot (\operatorname{arctg} x)'\, dx\right)$$

$$u = \operatorname{arctg} x \Rightarrow du = \frac{1}{1+x^2}\, dx \Rightarrow u = \operatorname{arctg} 0 = 0,\ u = \operatorname{arctg} t$$

$$I = \lim_{t\to\infty}\left(\int_0^{\operatorname{arctg} t} u\, du\right) = \lim_{t\to\infty}\left(\frac{u^2}{2}\Big|_0^{\operatorname{arctg} t}\right) = \lim_{t\to\infty}\left(\frac{\operatorname{arctg}^2 u}{2} - 0\right) =$$

$$= \frac{\left(\frac{\pi}{2}\right)^2}{2} = \frac{\pi^2}{8}$$

b)

$$I = \int_2^{\infty} \frac{x-1}{x^2+x+1}\, dx = \lim_{t\to\infty}\left(\int_2^t \frac{x-1}{x^2+x+1}\, dx\right) = \lim_{t\to\infty}\left(\frac{1}{2}\int_2^t \frac{2x-2}{x^2+x+1}\, dx\right) =$$

$$= \lim_{t\to\infty}\left(\frac{1}{2}\int_2^t \frac{2x-2+1-1}{x^2+x+1}\, dx\right) = \lim_{t\to\infty}\left(\frac{1}{2}\int_2^t \frac{(x^2+x+1)'}{x^2+x+1}\, dx - \frac{3}{2}\int_2^t \frac{1}{x^2+x+1}\, dx\right) =$$

$$= \lim_{t\to\infty}\left(\frac{1}{2}\ln|x^2+x+1|\Big|_2^t - \frac{3}{2}\int_2^t \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}\, dx\right) =$$

$$= \lim_{t\to\infty}\left(\frac{1}{2}\ln|t^2+t+1| - \frac{1}{2}\ln 7 - \frac{3}{2}\int_2^t \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\, dx\right)$$

$$u = x + \frac{1}{2} \Rightarrow du = dx \Rightarrow u = 2 + \frac{1}{2} = \frac{5}{2},\ u = t + \frac{1}{2} = \frac{2t+1}{2}$$

$$I = \lim_{t\to\infty}\left(\frac{1}{2}\ln\frac{t^2+t+1}{7} - \frac{3}{2}\int_{5/2}^{(2t+1)/2} \frac{1}{u^2+\frac{3}{4}}\, du\right) =$$

$$= \lim_{t\to\infty}\left(\frac{1}{2}\ln\frac{t^2+t+1}{7}\right) - \lim_{t\to\infty}\left(\frac{3}{2}\frac{1}{\sqrt{\frac{3}{4}}}\cdot \operatorname{arctg}\frac{u}{\sqrt{\frac{3}{4}}}\Big|_{5/2}^{(2t+1)/2}\right) =$$

$$= \infty - \lim_{t\to\infty}\left(\frac{3}{2}\cdot\frac{2}{\sqrt{3}}\left(\operatorname{arctg}\frac{2\,\frac{2t+1}{2}}{\sqrt{3}} - \operatorname{arctg}\frac{2\cdot\frac{5}{2}}{\sqrt{3}}\right)\right) =$$

$$= \infty - \lim_{t \to \infty} \left(\frac{3}{\sqrt{3}} \left(\operatorname{arctg} \frac{2t+1}{\sqrt{3}} - \operatorname{arctg} \frac{5}{\sqrt{3}} \right) \right) =$$

$$= \infty - \lim_{t \to \infty} \left(\frac{\cancel{3}\sqrt{3}}{\cancel{3}} \left(\operatorname{arctg} \frac{2\sqrt{3}t+1}{3} - \operatorname{arctg} \frac{5\sqrt{3}}{3} \right) \right) =$$

$$= \infty - \lim_{t \to \infty} \left(\sqrt{3} \operatorname{arctg} \frac{2\sqrt{3}t+\sqrt{3}}{3} - \sqrt{3} \operatorname{arctg} \frac{5\sqrt{3}}{3} \right) =$$

$$= \infty - \sqrt{3} \lim_{t \to \infty} \left(\operatorname{arctg} \frac{2\sqrt{3}t+\sqrt{3}}{3} \right) - \sqrt{3} \operatorname{arctg} \frac{5\sqrt{3}}{3} =$$

$$= \infty - \sqrt{3} \operatorname{arctg} \left(\lim_{t \to \infty} \frac{2\sqrt{3}t+\sqrt{3}}{3} \right) = \infty - \sqrt{3} \operatorname{arctg}(\infty) =$$

$$= \infty - \frac{\pi\sqrt{3}}{2} = \infty$$

c) $I = \int_0^1 (\ln x)^2 \, dx = \int_0^1 x' (\ln x)^2 \, dx = x \ln^2 x \Big|_0^1 - 2 \int_0^1 \cancel{x} \frac{1}{\cancel{x}} \ln x \, dx =$

$$= x \ln^2 x \Big|_0^1 - 2 \int_0^1 \ln x \, dx = x \ln^2 x \Big|_0^1 - 2 \int_0^1 x' \ln x \, dx =$$

$$= x \ln^2 x \Big|_0^1 - 2x \ln x \Big|_0^1 + 2 \int_0^1 \cancel{x} \frac{1}{\cancel{x}} \, dx =$$

$$= x \ln^2 x \Big|_0^1 - 2x \ln x \Big|_0^1 + 2 \int_0^1 dx =$$

$$= x \ln^2 x \Big|_0^1 - 2x \ln x \Big|_0^1 + 2x \Big|_0^1 =$$

$$= \lim_{t \to 0} \left(x \ln^2 x \Big|_t^1 - 2x \ln x \Big|_t^1 + 2x \Big|_t^1 \right) =$$

$$= \lim_{t \to 0} \left(\underbrace{\ln^2 1}_{0} - t \ln^2 t - 2 \underbrace{\ln 1}_{0} + 2t \ln t + 2 - 2t \right) =$$

$$= \lim_{t \to 0} \left(-t \ln^2 t + 2t \ln t - 2t \right) + 2$$

$$2\lim_{t\to 0}(t\ln t) = 2\lim_{t\to 0}\frac{\ln t}{\frac{1}{t}} \underset{l'H}{\overset{\frac{-\infty}{\infty}}{=}} 2\lim_{t\to 0}\frac{\frac{1}{t}}{-\frac{1}{t^2}} = 2\lim_{t\to 0}(-t) = 0$$

$$\lim_{t\to 0}(t\ln^2 t) = \lim_{t\to 0}\frac{\ln^2 t}{\frac{1}{t}} \underset{l'H}{\overset{\frac{-\infty}{\infty}}{=}} \lim_{t\to 0}\frac{\frac{1}{t}\ln t}{-\frac{1}{t^2}} = \lim_{t\to 0}(-t\ln t) = 0$$

$$\Rightarrow I = \lim_{t\to 0}(-t\ln^2 t) + 2\lim_{t\to 0}(t\ln t) - 2\lim_{t\to 0} t + 2 =$$

$$= 0 + 0 - 0 + 2 = 2$$