1 Inner Product Notes

IP1 is the equation

$$\frac{1}{2}\oint {}^2R(D^k v_i)(D_k v_j)\mathrm{d}\Omega = \frac{2}{3}4\pi\delta_{ij}. \tag{1}$$

IP2 is the equation

$$\frac{1}{2}\oint {}^2R(D^k v_i)(D_k v_j)\mathrm{dA} = \frac{2}{3}\pi A\delta_{ij}. \tag{2}$$

IP4 is the equation

$$\frac{1}{2}\oint (D^k v_i)(D_k v_j)\mathrm{d}\Omega = \frac{2}{3}4\pi\delta_{ij}. \tag{3}$$

IP5 is the equation

$$\frac{1}{2}\oint (D^k v_i)(D_k v_j)\mathrm{dA} = \frac{2}{3}\pi A\delta_{ij}. \tag{4}$$

(Note that $\mathrm{dA} = r^2\psi^4\mathrm{d}\Omega$.) In general,

$$\begin{aligned}
{}^2R &= \frac{2}{r^2\psi^4}(1 - 2\ {}_s\nabla^2\ln\psi) && (5)\\
S_{ij} &\equiv \begin{pmatrix}1 & 0\\ 0 & \sin^2\theta\end{pmatrix} && (6)\\
h_{ij} &= r^2\psi^4 S_{ij}. && (7)
\end{aligned}$$

For a general surface, the LHS of [IP1, IP2] become

$$\begin{aligned}
\frac{1}{2}\oint {}^2R(D^k v_i)(D_k v_j)\ \mathrm{d}[\Omega,\mathrm{A}] &= \frac{1}{2}\oint \frac{2}{r^2\psi^4}(1 - 2\ {}_s\nabla^2\ln\psi)(h^{kl}\partial_l v_i)(\partial_k v_j)\ \mathrm{d}[\Omega,\mathrm{A}] && (8)\\
&= \oint \frac{1}{r^2\psi^4}(1 - 2\ {}_s\nabla^2\ln\psi)\frac{1}{r^2\psi^4}S^{kl}(\partial_l v_i)(\partial_k v_j)\ \mathrm{d}[\Omega,\mathrm{A}] && (9)\\
&= \oint \frac{1}{r^4\psi^8}(1 - 2\ {}_s\nabla^2\ln\psi)S^{kl}(\partial_l v_i)(\partial_k v_j)\ \mathrm{d}[\Omega,\mathrm{A}] && (10)
\end{aligned}$$

In SpherePack, a real scalar function $f(\theta,\phi)$ is represented as

$$f(\theta,\phi) = \sum_{l=0}^{\infty}{\sum_{m=0}^{l}}' P_l^m(\cos\theta)\{a_{lm}\cos(m\phi) - b_{lm}\sin(m\phi)\}, \tag{11}$$

where $\sum'$ indicates that the $m = 0$ term is to be multiplied by $1/2$. As a shorthand, define

$$\begin{aligned}
a_{lm}^f &\equiv P_l^m(\cos\theta)a_{lm}\cos(m\phi) && (12)\\
b_{lm}^f &\equiv P_l^m(\cos\theta)b_{lm}\sin(m\phi) && (13)
\end{aligned}$$

for a given expansion of the function f.
Define the following notation for expansion using only the $l = 0$ terms:

$$\begin{aligned}
\langle r^2\psi^4\rangle &\equiv \sum_{l=0}^{0}{\sum_{m=0}^{l}}' P_l^m(\cos\theta)\{a_{lm}\cos(m\phi) - b_{lm}\sin(m\phi)\} && (14)\\
&= \frac{1}{2}a_{00}^{r^2\psi^4} && (15)
\end{aligned}$$

2 Sphere

For a sphere, the $l = 1$ terms of L are non-zero, and the $l = 0$ terms of $r^2\psi^4$ are non-zero. We see that

$$r^2\psi^4 = \sum_{l=0}^{0}{\sum_{m=0}^{0}}' \{a_{lm}^{r^2\psi^4} - b_{lm}^{r^2\psi^4}\} = \frac{1}{2}a_{00}^{r^2\psi^4} \tag{16}$$

$$L_i = \sum_{l=1}^{1}{\sum_{m=i}}' \{a_{lm}^{L} - b_{lm}^{L}\} = a_{1i}^{L} - b_{1i}^{L} \tag{17}$$

Then v can be expressed as

$$v_i \equiv \sum_{l=0}^{\infty}{\sum_{m=0}^{l}}' \{a_{lm}^{v} - b_{lm}^{v}\} \tag{18}$$

$${}_s\nabla^2 v_i = -\sum_{l=0}^{\infty}{\sum_{m=0}^{l}}' l(l+1)\{a_{lm}^{v} - b_{lm}^{v}\} \tag{19}$$

$${}_s\nabla^2 v_i = -2r^2\psi^4 L_i = -a_{00}^{r^2\psi^4}\{a_{1i}^{L} - b_{1i}^{L}\} \tag{20}$$

$$\Longrightarrow v_i = -\frac{1}{2}a_{00}^{r^2\psi^4}\{a_{1i}^{L} - b_{1i}^{L}\}. \tag{21}$$

2.1 Unit Sphere

For a unit sphere,

$$r = 1 \tag{22}$$

$$\psi = 1 \tag{23}$$

$$v_i = -(1)L_i = -\{a_{1i}^{L} - b_{1i}^{L}\}. \tag{24}$$

The relations IP1 and IP2 become

$$\oint \frac{1}{r^4\psi^8}(1 - 2{}_s\nabla^2 \ln\psi) S^{kl}(\partial_l v_i)(\partial_k v_j)\ \mathrm{d}[\Omega, \mathrm{A}] = \oint S^{kl}(\partial_l L_i)(\partial_k L_j)\ \mathrm{d}[\Omega, \mathrm{A}] \tag{25}$$

2.2 Arbitrary Sphere

For an arbitrary sphere,

$$r = \text{constant} \tag{26}$$

$$\psi = \text{constant} \tag{27}$$

$$v_i = -r^2\psi^4 L_i = -\frac{1}{2}a_{00}^{r^2\psi^4}\{a_{1i}^{L} - b_{1i}^{L}\}. \tag{28}$$

The relations IP1 and IP2 become

$$\oint \frac{1}{r^4\psi^8}(1 - 2{}_s\nabla^2 \ln\psi) S^{kl}(\partial_l v_i)(\partial_k v_j)\ \mathrm{d}[\Omega, \mathrm{A}]$$

$$= \oint \frac{1}{r^4\psi^8} S^{kl}(\partial_l r^2\psi^4 L_i)(\partial_k r^2\psi^4 L_j)\ \mathrm{d}[\Omega, \mathrm{A}] \tag{29}$$

$$= \oint S^{kl}(\partial_l L_i)(\partial_k L_j)\ \mathrm{d}[\Omega, \mathrm{A}] \tag{30}$$

3 Arbitrary Surface

3.1 With Ricci term

For an arbitrary surface, $l \neq 1$ terms may contribute to the expansions of L and $r^2\psi^4$. The function L is represented as

$$
\begin{aligned}
L_i &= \sum_{l=0}^{\infty} {\sum_{m=0}^{l}}' P_l^m \{a_{lm}\cos(m\phi) - b_{lm}\sin(m\phi)\} && (31) \\
&= \{a^L_{1i} - b^L_{1i}\} + \frac{1}{2}a^L_{00} + \sum_{l=2}^{\infty} {\sum_{m=0}^{l}}' \{a^L_{lm} - b^L_{lm}\}. && (32)
\end{aligned}
$$

Similarly, for the conformal factor,

$$
r^2\psi^4 = \frac{1}{2}a^{r^2\psi^4}_{00} + \sum_{l=1}^{\infty} {\sum_{m=0}^{l}}' \{a^{r^2\psi^4}_{lm} - b^{r^2\psi^4}_{lm}\} \qquad (33)
$$

Then the v_i terms become

$$
\begin{aligned}
v_i &= -r^2\psi^4 L_i && (34) \\
&= -\frac{1}{2}a^{r^2\psi^4}_{00}\{a^L_{1i} - b^L_{1i}\} \\
&\quad -\frac{1}{2}a^{r^2\psi^4}_{00}\left[\frac{1}{2}a^L_{00} + \sum_{l=2}^{\infty} {\sum_{m=0}^{l}}' \{a^L_{lm} - b^L_{lm}\}\right] \\
&\quad -\frac{1}{2}a^L_{00}\sum_{l=1}^{\infty} {\sum_{m=0}^{l}}' \{a^{r^2\psi^4}_{lm} - b^{r^2\psi^4}_{lm}\} && (35) \\
&\quad -\{a^L_{1i} - b^L_{1i}\}\sum_{l=1}^{\infty} {\sum_{m=0}^{l}}' \{a^{r^2\psi^4}_{lm} - b^{r^2\psi^4}_{lm}\} \\
&\quad -\sum_{l=1}^{\infty} {\sum_{m=0}^{l}}' \{a^{r^2\psi^4}_{lm} - b^{r^2\psi^4}_{lm}\}\sum_{l=2}^{\infty} {\sum_{m=0}^{l}}' \{a^L_{lm} - b^L_{lm}\}
\end{aligned}
$$

Then [IP1,IP2] become

$$
\begin{aligned}
& \frac{1}{2}\oint \frac{2}{r^2\psi^4}\left(1-2\;{}_s\nabla^2\ln\psi\right)\frac{1}{r^2\psi^4}S^{kl}(\partial_l v_i)(\partial_k v_j)\mathrm{d}[\Omega,\mathrm{A}] && (36)\\
= & \oint \frac{1}{r^4\psi^8}S^{kl}(\partial_l v_i)(\partial_k v_j)\mathrm{d}[\Omega,\mathrm{A}] - \oint \frac{1}{r^4\psi^8}\left(2\;{}_s\nabla^2\ln\psi\right)S^{kl}(\partial_l v_i)(\partial_k v_j)\mathrm{d}[\Omega,\mathrm{A}] && (37)\\
= & \oint \frac{1}{r^4\psi^8}S^{kl}(r^4\psi^8)(\partial_l L_i)(\partial_k L_j)\mathrm{d}[\Omega,\mathrm{A}] + \oint \frac{1}{r^4\psi^8}S^{kl}L_iL_j(\partial_l r^2\psi^4)(\partial_k r^2\psi^4)\mathrm{d}[\Omega,\mathrm{A}] && (38)\\
& - \oint \frac{1}{r^4\psi^8}\left(2\;{}_s\nabla^2\ln\psi\right)S^{kl}(\partial_l v_i)(\partial_k v_j)\mathrm{d}[\Omega,\mathrm{A}] && (39)
\end{aligned}
$$

Does ∂L_{nm} contribute to the integral for $n \neq 1$? Is it correct that we are only looking for $X_{00}, \partial X_{1,m}, \nabla^2 X_{2,m}\ldots$ terms? If so, then:

$$
\begin{aligned}
= & \oint S^{kl}(\partial_l L_{1i})(\partial_k L_{1j})\mathrm{d}[\Omega,\mathrm{A}] + \oint \frac{1}{r^4\psi^8}S^{kl}L_{00}L_{00}\left[\partial_l(r^2\psi^4)_{1i}\right]\left[\partial_k(r^2\psi^4)_{1j}\right]\mathrm{d}[\Omega,\mathrm{A}] && (40)\\
& - \oint \left(2\;{}_s\nabla^2\ln\psi\right)S^{kl}(\partial_l L_i)(\partial_k L_j)\mathrm{d}[\Omega,\mathrm{A}]\\
& - \oint \frac{1}{r^4\psi^8}\left(2\;{}_s\nabla^2\ln\psi\right)S^{kl}L_{00}L_{00}\left[\partial_l(r^2\psi^4)_{1i}\right]\left[\partial_k(r^2\psi^4)_{1j}\right]\mathrm{d}[\Omega,\mathrm{A}]
\end{aligned}
$$

If the above assumption is correct, then we can/must simplify the $\left(2\;{}_s\nabla^2\ln\psi\right)$ term. How do we associate the m-terms of ψ_{2m} with the i-terms of v_i?

$$
\begin{aligned}
= & \oint S^{kl}(\partial_l L_{1i})(\partial_k L_{1j})\mathrm{d}[\Omega,\mathrm{A}] + \oint a_{00}^{1/r^4\psi^8}S^{kl}a^L_{00}a^L_{00}\left[\partial_l(r^2\psi^4)_{1i}\right]\left[\partial_k(r^2\psi^4)_{1j}\right]\mathrm{d}[\Omega,\mathrm{A}] && (41)\\
& - \oint \left(2\;{}_s\nabla^2\ln\psi_{2m}\right)S^{kl}(\partial_l L_i)(\partial_k L_j)\mathrm{d}[\Omega,\mathrm{A}]\\
& - \oint a_{00}^{1/r^4\psi^8}\left(2\;{}_s\nabla^2\ln\psi\right)S^{kl}a^L_{00}a^L_{00}\left[\partial_l(r^2\psi^4)_{1i}\right]\left[\partial_k(r^2\psi^4)_{1j}\right]\mathrm{d}[\Omega,\mathrm{A}]
\end{aligned}
$$

If my assumptions are correct, this is the result. The first term looks like the result for the sphere.

3.2 Without Ricci term

For [IP4,IP5] we get:

$$
\begin{aligned}
& \frac{1}{2}\oint \frac{1}{r^2\psi^4}S^{kl}(\partial_l v_i)(\partial_k v_j)\mathrm{d}[\Omega,\mathrm{A}] && (42)\\
= & \frac{1}{2}\oint \frac{1}{r^2\psi^4}S^{kl}r^4\psi^8(\partial_l L_i)(\partial_k L_j)\mathrm{d}[\Omega,\mathrm{A}] && (43)\\
& +\frac{1}{2}\oint \frac{1}{r^2\psi^4}S^{kl}L_{00}L_{00}\left[\partial_l(r^2\psi^4)_{1i}\right]\left[\partial_k(r^2\psi^4)_{1j}\right]\mathrm{d}[\Omega,\mathrm{A}]\\
= & \frac{1}{2}\oint a_{00}^{r^2\psi^4}S^{kl}(\partial_l L_i)(\partial_k L_j)\mathrm{d}[\Omega,\mathrm{A}] && (44)\\
& +\frac{1}{2}\oint a_{00}^{1/r^2\psi^4}S^{kl}a^L_{00}a^L_{00}\left[\partial_l(r^2\psi^4)_{1i}\right]\left[\partial_k(r^2\psi^4)_{1j}\right]\mathrm{d}[\Omega,\mathrm{A}]
\end{aligned}
$$

The $\mathrm{d}\Omega$ integral of the first term looks like 1/2 IP2 for a sphere.