1 Killing Scaling

Let the Killing vector be proportional to some basis vector such that $\xi = s\widetilde{A}$, where s is specific to the fact that the $l = 1$ modes of L are normalized. Then

$$\begin{aligned} \frac{3}{16\pi}\oint {}^2R\widetilde{A}\cdot\widetilde{A}\mathrm{d}\Omega &= \frac{1}{s} \qquad (1)\\ \frac{3}{16\pi}\oint {}^2R(s\widetilde{A})\cdot(s\widetilde{A})\mathrm{d}\Omega &= \frac{3}{16\pi}\oint {}^2R\xi\cdot\xi\mathrm{d}\Omega = s \qquad (2)\end{aligned}$$

Similarly, let ζ be proportional to $\widetilde{A}$ such that $\zeta = \alpha\widetilde{A}$, where α represents the fact that the $l = 1$ modes are not necessarily normalized. Then

$$\frac{3}{16\pi}\oint {}^2R(\alpha\widetilde{A})\cdot(\alpha\widetilde{A})\mathrm{d}\Omega = \frac{3}{16\pi}\oint {}^2R\zeta\cdot\zeta\mathrm{d}\Omega = \frac{\alpha^2}{s} \equiv \beta \qquad (3)$$

How do we relate ξ to ζ directly using the spherical harmonic functions? We are only interested in the Y_a^m coefficients, so look at the Y_1^m terms:

$$\begin{aligned} Y_1^0 &= \sqrt{\frac{3}{4\pi}}\cos\theta \qquad (4)\\ Y_1^{\pm1} &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}. \qquad (5)\end{aligned}$$

We can write the orthogonal directions (x, y, z) in terms of trig functions on a sphere:

$$\begin{aligned} n_1 &= \sin\theta\cos\phi = x \qquad (6)\\ n_2 &= \sin\theta\sin\phi = y \qquad (7)\\ n_3 &= \cos\theta \qquad = z \qquad (8)\end{aligned}$$

Then in terms of the Y_1^m's, we have

$$\begin{aligned} n_1 &= -\frac{1}{2}\sqrt{\frac{8\pi}{3}}\left(Y_1^1 - Y_1^{-1}\right)\\ n_2 &= -\frac{1}{2i}\sqrt{\frac{8\pi}{3}}\left(Y_1^1 + Y_1^{-1}\right) \qquad (9)\\ n_3 &= \sqrt{4\pi}3Y_1^0\end{aligned}$$

Note: n_i are real, which implies

$$\begin{aligned} n_1 &= -\frac{1}{2}\sqrt{\frac{8\pi}{3}}\left(Y_1^1 - Y_1^{-1}\right) \qquad (10)\\ &= -\frac{1}{2}\sqrt{\frac{8\pi}{3}}\sin\theta\left(-e^{i\phi} - e^{-i\phi}\right) \qquad (11)\\ &= -\frac{1}{2}\sqrt{\frac{8\pi}{3}}\sin\theta\left(-Y_1^{-1*} + Y_1^{1*}\right) \qquad (12)\\ n_1 &= n_{1*} \qquad (13)\end{aligned}$$

and similarly for n_2, n_3.

Imagine a scenario in which we rotate the coordinates such that the primary axis z' differs from the original axis z by an amount (γ, δ). [Inserting a graphic might be handy here.] In terms of the old coordinates (θ, ϕ), we can write the expansion in the new coordinates (θ', ϕ') as (see Merzbacher Eq. 16.60):

$$Y_1^0(\theta', \phi') = \sqrt{\frac{4\pi}{3}} \sum_m Y_1^m(\theta, \phi) Y_1^m(\gamma, \delta) \tag{14}$$

Can we make a similar statement about n_3?

$$\begin{aligned}
n_3(\theta', \phi') &= \sqrt{\frac{4\pi}{3}} Y_1^0(\theta', \phi') && (15)\\
&= \sqrt{\frac{4\pi}{3}}\sqrt{\frac{4\pi}{3}} \sum_m Y_1^m(\theta, \phi) Y_1^m(\gamma, \delta) && (16)\\
&= \frac{4\pi}{3}\left(\frac{3}{8\pi}\sin\theta e^{i\phi}\sin\gamma e^{-i\delta} + \frac{3}{4\pi}\cos\theta\cos\gamma + \frac{3}{8\pi}\sin\theta e^{-i\phi}\sin\gamma e^{i\delta}\right) && (17)\\
&= \cos\theta\cos\gamma + \frac{4\pi}{3}\frac{3}{8\pi}\sin\theta\sin\gamma\left(e^{i\phi}e^{-i\delta} + e^{-i\phi}e^{i\delta} + 0\right) && (18)\\
&= n_3(\theta, \phi) n_3(\gamma, \delta) && (19)\\
&\quad + \frac{2\pi}{3}\frac{3}{8\pi}\sin\theta\sin\gamma\left(2e^{i\phi}e^{-i\delta} + 2e^{-i\phi}e^{i\delta} + e^{i\phi}e^{i\delta} - e^{i\phi}e^{i\delta} + e^{-i\phi}e^{-i\delta} - e^{-i\phi}e^{-i\delta}\right)\\
&= n_3(\theta, \phi) n_3(\gamma, \delta) && (20)\\
&\quad + \frac{2\pi}{3}\frac{3}{8\pi}\sin\theta\sin\gamma\left(e^{i\phi}e^{i\delta} + e^{i\phi}e^{-i\delta} + e^{-i\phi}e^{i\delta} + e^{-i\phi}e^{-i\delta}\right)\\
&\quad + \frac{2\pi}{3}\frac{3}{8\pi}\sin\theta\sin\gamma\left(-e^{i\phi}e^{i\delta} + e^{i\phi}e^{-i\delta} + e^{-i\phi}e^{i\delta} - e^{-i\phi}e^{-i\delta}\right)\\
&= n_3(\theta, \phi) n_3(\gamma, \delta) && (21)\\
&\quad + \frac{2\pi}{3}\left(-Y_1^1(\theta, \phi)Y_1^{-1*}(\gamma, \delta) + Y_1^1(\theta, \phi)Y_1^{1*}(\gamma, \delta) + Y_1^{-1}(\theta, \phi)Y_1^{-1*}(\gamma, \delta) - Y_1^{-1}(\theta, \phi)Y_1^{1*}(\gamma, \delta)\right)\\
&\quad + \frac{2\pi}{3}\left(Y_1^1(\theta, \phi)Y_1^{-1*}(\gamma, \delta) + Y_1^1(\theta, \phi)Y_1^{1*}(\gamma, \delta) + Y_1^{-1}(\theta, \phi)Y_1^{-1*}(\gamma, \delta) + Y_1^{-1}(\theta, \phi)Y_1^{1*}(\gamma, \delta)\right)\\
&= n_3(\theta, \phi) n_3(\gamma, \delta) && (22)\\
&\quad + \frac{2\pi}{3}\left(Y_1^1(\theta, \phi) - Y_1^{-1}(\theta, \phi)\right)\left(Y_1^{1*}(\gamma, \delta) - Y_1^{-1*}(\gamma, \delta)\right)\\
&\quad + \frac{2\pi}{3}\left(Y_1^1(\theta, \phi) + Y_1^{-1}(\theta, \phi)\right)\left(Y_1^{1*}(\gamma, \delta) + Y_1^{-1*}(\gamma, \delta)\right)\\
&= n_3(\theta, \phi) n_3(\gamma, \delta) + n_1(\theta, \phi) n_{1*}(\gamma, \delta) + n_2(\theta, \phi) n_{2*}(\gamma, \delta)\\
n_3(\theta', \phi') &= \sum_m n_m(\theta, \phi) n_{m*}(\gamma, \delta) = \sum_m n_m(\theta, \phi) n_m(\gamma, \delta) && (23)
\end{aligned}$$

When γ, δ are chosen carefully such that the z' axis is now a symmetry axis, then the $m \neq 0$ modes vanish in the $Y_1^m(\theta', \phi')$ expansion.

Let $\vec{A}$ be an arbitrary vector which has a spherical harmonic decomposition

$$\vec{A} = a_1^0 Y_1^0 + a_1^1 Y_1^1 + a_1^{-1} Y_1^{-1}. \tag{24}$$

When we take advantage of the orthogonality of the (non-normalized) n_i from Eq. 9, we can write the dot product of $\vec{A}$ with itself as

$$\begin{aligned}
\vec{A} \cdot \vec{A} &= \left[(a_1^1 Y_1^1 + a_1^{-1} Y_1^{-1})(a_1^1 Y_1^1 + a_1^{-1} Y_1^{-1})\right] + i^2 \left[(a_1^1 Y_1^1 - a_1^{-1} Y_1^{-1})(a_1^1 Y_1^1 - a_1^{-1} Y_1^{-1})\right] \qquad (25) \\
&\quad + \left[(a_1^0 Y_1^0)(a_1^0 Y_1^0)\right] \\
&= \left[(a_1^1 Y_1^1)^2 + 2a_1^1 a_1^{-1} Y_1^1 Y_1^{-1} + (a_1^{-1} Y_1^{-1})^2 - (a_1^1 Y_1^1)^2 + 2a_1^1 a_1^{-1} Y_1^1 Y_1^{-1} - (a_1^{-1} Y_1^{-1})^2\right] \qquad (26) \\
&\quad + (a_1^0 Y_1^0)^2 \\
&= 4a_1^1 a_1^{-1} + (a_1^0 Y_1^0)^2 \qquad (27) \\
&= (a_1^0 Y_1^0)^2 \qquad (28)
\end{aligned}$$

The cross-terms vanish because the spherical harmonics are orthogonal in l and m.

From Eq. 9, we can say that

$$\begin{aligned}
b_1 &= (a_1^1 - a_1^1) \qquad (29) \\
b_2 &= (a_1^1 + a_1^1) \qquad (30) \\
b_3 &= a_1^0. \qquad (31)
\end{aligned}$$