1 Inner Product Notes

IP1 is the equation

$$\frac{1}{2}\oint {}^2R(D^k v_i)(D_k v_j)\mathrm{d}\Omega = \frac{2}{3}4\pi\delta_{ij}. \tag{1}$$

IP2 is the equation

$$\frac{1}{2}\oint {}^2R(D^k v_i)(D_k v_j)\mathrm{dA} = \frac{2}{3}\pi A\delta_{ij}. \tag{2}$$

IP4 is the equation

$$\frac{1}{2}\oint (D^k v_i)(D_k v_j)\mathrm{d}\Omega = \frac{2}{3}4\pi\delta_{ij}. \tag{3}$$

IP5 is the equation

$$\frac{1}{2}\oint (D^k v_i)(D_k v_j)\mathrm{dA} = \frac{2}{3}\pi A\delta_{ij}. \tag{4}$$

In general,

$$
\begin{align}
{}^2R &= \frac{2}{r^2\psi^4}(1 - 2\ {}_s\nabla^2\ln\psi) \tag{5}\\
S_{ij} &= \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \tag{6}\\
h_{ij} &= r^2\psi^4 S_{ij}. \tag{7}
\end{align}
$$

For a general surface, the LHS of [IP1, IP2] become

$$
\begin{align}
\frac{1}{2}\oint {}^2R(D^k v_i)(D_k v_j)\ \mathrm{d}[\Omega, A] &= \frac{1}{2}\oint \frac{2}{r^2\psi^4}(1 - 2\ {}_s\nabla^2\ln\psi)(h^{kl}\partial_l v_i)(\partial_k v_j)\ \mathrm{d}[\Omega, A] \tag{8}\\
&= \oint \frac{1}{r^2\psi^4}(1 - 2\ {}_s\nabla^2\ln\psi)\frac{1}{r^2\psi^4}S^{kl}(\partial_l v_i)(\partial_k v_j)\ \mathrm{d}[\Omega, A] \tag{9}\\
&= \oint \frac{1}{r^4\psi^8}(1 - 2\ {}_s\nabla^2\ln\psi)S^{kl}(\partial_l v_i)(\partial_k v_j)\ \mathrm{d}[\Omega, A] \tag{10}
\end{align}
$$

In SpherePack, a real scalar function $f(\theta,\phi)$ is represented as

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}{}'\, P_l^m(\cos\theta)\{a_{lm}\cos(m\phi) - b_{lm}\sin(m\phi)\}, \tag{11}$$

where $\sum'$ indicates that the $m = 0$ term is to be multiplied by 1/2. As a shorthand, define

$$
\begin{align}
a_{lm}^f &\equiv P_l^m(\cos\theta)a_{lm}\cos(m\phi) \tag{12}\\
b_{lm}^f &\equiv P_l^m(\cos\theta)b_{lm}\sin(m\phi) \tag{13}
\end{align}
$$

for a given expansion of the function f.
Define the following notation for expansion using only the $l = 0$ terms:

$$
\begin{align}
\langle r^2\psi^4\rangle &\equiv \sum_{l=0}^{0}\sum_{m=0}^{l}{}'\, P_l^m(\cos\theta)\{a_{lm}\cos(m\phi) - b_{lm}\sin(m\phi)\} \tag{14}\\
&= \frac{1}{2}a_{00}^{r^2\psi^4} \tag{15}
\end{align}
$$

1

2 Sphere

For a sphere, the $l = 1$ terms of L are non-zero, and the $l = 0$ terms of $r^2\psi^4$ are non-zero. We see that

$$r^2\psi^4 = \sum_{l=0}^{0}\sum_{m=0}^{0}{}'\{a_{lm}^{r^2\psi^4} - b_{lm}^{r^2\psi^4}\} = \frac{1}{2}a_{00}^{r^2\psi^4} \tag{16}$$

$$L_i = \sum_{l=1}^{1}\sum_{m=i}{}'\{a_{lm}^{L} - b_{lm}^{L}\} = a_{1i}^{L} - b_{1i}^{L} \tag{17}$$

Then v can be expressed as

$$v_i \equiv \sum_{l=0}^{\infty}\sum_{m=0}^{l}{}'\{a_{lm}^{v} - b_{lm}^{v}\} \tag{18}$$

$$_s\nabla^2 v_i = \sum_{l=0}^{\infty}\sum_{m=0}^{l}{}' l(l+1)\{a_{lm}^{v} - b_{lm}^{v}\} \tag{19}$$

$$_s\nabla^2 v_i = -2r^2\psi^4 L_i = -a_{00}^{r^2\psi^4}\{a_{1i}^{L} - b_{1i}^{L}\} \tag{20}$$

$$\implies v_i = -\frac{1}{2}a_{00}^{r^2\psi^4}\{a_{1i}^{L} - b_{1i}^{L}\}. \tag{21}$$

2.1 Unit Sphere

For a unit sphere,

$$r = 1 \tag{22}$$

$$\psi = 1 \tag{23}$$

$$v_i = -(1)L_i = -\{a_{1i}^{L} - b_{1i}^{L}\}. \tag{24}$$

The relations IP1 and IP2 become

$$\oint \frac{1}{r^4\psi^8}(1 - 2_s\nabla^2\ln\psi)S^{kl}(\partial_l v_i)(\partial_k v_j)\,\mathrm{d}[\Omega, A] = \oint S^{kl}(\partial_l L_i)(\partial_k L_j)\,\mathrm{d}[\Omega, A] \tag{25}$$

2.2 Arbitrary Sphere

For an arbitrary sphere,

$$r = \text{constant} \tag{26}$$

$$\psi = \text{constant} \tag{27}$$

$$v_i = -r^2\psi^4 L_i = -\frac{1}{2}a_{00}^{r^2\psi^4}\{a_{1i}^{L} - b_{1i}^{L}\}. \tag{28}$$

The relations IP1 and IP2 become

$$\oint \frac{1}{r^4\psi^8}(1 - 2_s\nabla^2\ln\psi)S^{kl}(\partial_l v_i)(\partial_k v_j)\,\mathrm{d}[\Omega, A]$$

$$= \oint \frac{1}{r^4\psi^8}S^{kl}(\partial_l r^2\psi^4 L_i)(\partial_k r^2\psi^4 L_j)\,\mathrm{d}[\Omega, A] \tag{29}$$

$$= \oint S^{kl}(\partial_l L_i)(\partial_k L_j)\,\mathrm{d}[\Omega, A] \tag{30}$$

2

3   Arbitrary Surface

For an arbitrary surface, $l \neq 1$ terms may contribute to the expansions of L and $r^2\psi^4$. The function L is represented as

$$
\begin{aligned}
L_i &= \sum_{l=0}^{\infty}\sum_{m=0}^{l}{}' P_l^m \{a_{lm}\cos(m\phi) - b_{lm}\sin(m\phi)\} && (31)\\
&= \{a_{1i}^L - b_{1i}^L\} + \frac{1}{2}a_{00}^L + \sum_{l=2}^{\infty}\sum_{m=0}^{l}{}' \{a_{lm}^L - b_{lm}^L\}. && (32)
\end{aligned}
$$

Similarly, for the conformal factor,

$$
r^2\psi^4 = \frac{1}{2}a_{00}^{r^2\psi^4} + \sum_{l=1}^{\infty}\sum_{m=0}^{l}{}' \{a_{lm}^{r^2\psi^4} - b_{lm}^{r^2\psi^4}\} \tag{33}
$$

Then the v_i terms become

$$
\begin{aligned}
v_i &= -r^2\psi^4 L_i && (34)\\
&= -\frac{1}{2}a_{00}^{r^2\psi^4}\{a_{1i}^L - b_{1i}^L\} \\
&\quad -\frac{1}{2}a_{00}^{r^2\psi^4}\left[\frac{1}{2}a_{00}^L + \sum_{l=2}^{\infty}\sum_{m=0}^{l}{}' \{a_{lm}^L - b_{lm}^L\}\right] \\
&\quad -\frac{1}{2}a_{00}^L \sum_{l=1}^{\infty}\sum_{m=0}^{l}{}' \{a_{lm}^{r^2\psi^4} - b_{lm}^{r^2\psi^4}\} && (35)\\
&\quad -\{a_{1i}^L - b_{1i}^L\}\sum_{l=1}^{\infty}\sum_{m=0}^{l}{}' \{a_{lm}^{r^2\psi^4} - b_{lm}^{r^2\psi^4}\} \\
&\quad -\sum_{l=1}^{\infty}\sum_{m=0}^{l}{}' \{a_{lm}^{r^2\psi^4} - b_{lm}^{r^2\psi^4}\}\sum_{l=2}^{\infty}\sum_{m=0}^{l}{}' \{a_{lm}^L - b_{lm}^L\}
\end{aligned}
$$