1 Killing Scaling

Let the Killing vector be proportional to some basis vector such that $\xi = s\widetilde{A}$, where s is specific to the fact that the $l = 1$ modes of L are normalized. Then

$$\frac{3}{16\pi}\oint {}^2R\widetilde{A}\cdot\widetilde{A}\mathrm{d}\Omega = \frac{1}{s} \tag{1}$$

$$\frac{3}{16\pi}\oint {}^2R(s\widetilde{A})\cdot(s\widetilde{A})\mathrm{d}\Omega = \frac{3}{16\pi}\oint {}^2R\xi\cdot\xi\mathrm{d}\Omega = s \tag{2}$$

Similarly, let ζ be proportional to $\widetilde{A}$ such that $\zeta = \alpha\widetilde{A}$, where α represents the fact that the $l = 1$ modes are not necessarily normalized. Then

$$\frac{3}{16\pi}\oint {}^2R(\alpha\widetilde{A})\cdot(\alpha\widetilde{A})\mathrm{d}\Omega = \frac{3}{16\pi}\oint {}^2R\zeta\cdot\zeta\mathrm{d}\Omega = \frac{\alpha^2}{s} \equiv \beta \tag{3}$$

How do we relate ξ to ζ directly using the spherical harmonic functions? We are only interested in the Y_a^m coefficients, so look at the Y_1^m terms:

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \tag{4}$$

$$Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}. \tag{5}$$

We can write the orthogonal directions (x, y, z) in terms of trig functions on a sphere:

$$n^1 = \sin\theta\cos\phi = \hat{x}$$
$$n^2 = \sin\theta\sin\phi = \hat{y}$$
$$n^3 = \cos\theta \qquad = \hat{z}$$

Then in terms of the Y_1^m's, we have

$$n^1 = -\frac{1}{2}\sqrt{\frac{8\pi}{3}}\left(Y_1^1 - Y_1^{-1}\right) \tag{6}$$

$$n^2 = -\frac{1}{2i}\sqrt{\frac{8\pi}{3}}\left(Y_1^1 + Y_1^{-1}\right) \tag{7}$$

$$n^3 = \sqrt{4\pi}3Y_1^0 \tag{8}$$

Note: n^i are real, which implies

$$n^1 = -\frac{1}{2}\sqrt{\frac{8\pi}{3}}\left(Y_1^1 - Y_1^{-1}\right) \tag{9}$$

$$= -\frac{1}{2}\sqrt{\frac{8\pi}{3}}\sin\theta\left(-e^{i\phi} - e^{-i\phi}\right) \tag{10}$$

$$= -\frac{1}{2}\sqrt{\frac{8\pi}{3}}\sin\theta\left(-Y_1^{-1*} + Y_1^{1*}\right) \tag{11}$$

$$n^1 = n^{1*} \tag{12}$$

and similarly for n^2, n^3.

Imagine a scenario in which we rotate the coordinates such that the primary axis z' differs from the original axis z by an amount (γ, δ). [Inserting a graphic might be handy here.] In terms of the old coordinates (θ, ϕ), we can write the expansion in the new coordinates (θ', ϕ') as (see Merzbacher Eq. 16.60):

$$Y_1^0(\theta', \phi') = \sqrt{\frac{4\pi}{3}} \sum_m Y_1^m(\theta, \phi) Y_1^m(\gamma, \delta) \tag{13}$$

Can we make a similar statement about n^3?

$$\begin{aligned}
n^3(\theta', \phi') &= \sqrt{\frac{4\pi}{3}} Y_1^0(\theta', \phi') && (14)\\
&= \sqrt{\frac{4\pi}{3}} \sqrt{\frac{4\pi}{3}} \sum_m Y_1^m(\theta, \phi) Y_1^m(\gamma, \delta) && (15)\\
&= \frac{4\pi}{3} \left(\frac{3}{8\pi} \sin\theta e^{i\phi} \sin\gamma e^{-i\delta} + \frac{3}{4\pi} \cos\theta \cos\gamma + \frac{3}{8\pi} \sin\theta e^{-i\phi} \sin\gamma e^{i\delta} \right) && (16)\\
&= \cos\theta\cos\gamma + \frac{4\pi}{3}\frac{3}{8\pi} \sin\theta \sin\gamma \left(e^{i\phi}e^{-i\delta} + e^{-i\phi}e^{i\delta} + 0 \right) && (17)\\
&= n^3(\theta, \phi) n^3(\gamma, \delta) && (18)\\
&\quad + \frac{2\pi}{3}\frac{3}{8\pi} \sin\theta\sin\gamma \left(2e^{i\phi}e^{-i\delta} + 2e^{-i\phi}e^{i\delta} + e^{i\phi}e^{i\delta} - e^{i\phi}e^{i\delta} + e^{-i\phi}e^{-i\delta} - e^{-i\phi}e^{-i\delta} \right) \\
&= n^3(\theta, \phi) n^3(\gamma, \delta) && (19)\\
&\quad + \frac{2\pi}{3}\frac{3}{8\pi} \sin\theta\sin\gamma \left(e^{i\phi}e^{i\delta} + e^{i\phi}e^{-i\delta} + e^{-i\phi}e^{i\delta} + e^{-i\phi}e^{-i\delta} \right) \\
&\quad + \frac{2\pi}{3}\frac{3}{8\pi} \sin\theta\sin\gamma \left(-e^{i\phi}e^{i\delta} + e^{i\phi}e^{-i\delta} + e^{-i\phi}e^{i\delta} - e^{-i\phi}e^{-i\delta} \right) \\
&= n^3(\theta, \phi) n^3(\gamma, \delta) && (20)\\
&\quad + \frac{2\pi}{3} \left(-Y_1^1(\theta,\phi) Y_1^{-1*}(\gamma,\delta) + Y_1^1(\theta,\phi) Y_1^{1*}(\gamma,\delta) + Y_1^{-1}(\theta,\phi) Y_1^{-1*}(\gamma,\delta) - Y_1^{-1}(\theta,\phi) Y_1^{1*}(\gamma,\delta) \right) \\
&\quad + \frac{2\pi}{3} \left(Y_1^1(\theta,\phi) Y_1^{-1*}(\gamma,\delta) + Y_1^1(\theta,\phi) Y_1^{1*}(\gamma,\delta) + Y_1^{-1}(\theta,\phi) Y_1^{-1*}(\gamma,\delta) + Y_1^{-1}(\theta,\phi) Y_1^{1*}(\gamma,\delta) \right) \\
&= n^3(\theta, \phi) n^3(\gamma, \delta) && (21)\\
&\quad + \frac{2\pi}{3} \left(Y_1^1(\theta,\phi) - Y_1^{-1}(\theta,\phi) \right) \left(Y_1^{1*}(\gamma,\delta) - Y_1^{-1*}(\gamma,\delta) \right) \\
&\quad + \frac{2\pi}{3} \left(Y_1^1(\theta,\phi) + Y_1^{-1}(\theta,\phi) \right) \left(Y_1^{1*}(\gamma,\delta) + Y_1^{-1*}(\gamma,\delta) \right) \\
&= n^3(\theta,\phi) n^3(\gamma,\delta) + n^1(\theta,\phi) n^{1*}(\gamma,\delta) + n^2(\theta,\phi) n^{2*}(\gamma,\delta) \\
n^3(\theta', \phi') &= \sum_m n^m(\theta,\phi) n^{m*}(\gamma,\delta) = \sum_m n^m(\theta,\phi) n^m(\gamma,\delta) && (22)
\end{aligned}$$

When γ, δ are chosen carefully such that the z' axis is now a symmetry axis, then the $m \neq 0$ modes vanish in the $Y_1^m(\theta', \phi')$ expansion.