Topic 6: Fixed Effects, Difference-in-differences, and Factor Models

ECON 5783 — University of Arkansas

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Estimating the effect of AP classes

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For example, the average teacher quality Z might be higher in schools that offer AP classes

- Taking AP classes is confounded with attending different quality schools

Omitted Variable Bias

Let $s(i)$ denote the school that student i attended and assume we see multiple students from each school

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Say true causal model determining the probability of completing a college degree is given by

$$y_i = D_i \tau + Z_{s(i)} \gamma + u_i$$

- $Z_{s(i)}$ is the quality of the teachers at student i's school
- Assume $\mathbb{E}\left[u_i \mid D_i, Z_{s(i)}\right] = 0$, i.e. there are no other variables correlated with D_i that impact y_i (for the sake of illustration)

Omitted Variable Bias

From our omitted variables bias formula, regressing y_i on D_i (but not $Z_{s(i)}$) would yield

$$\tau_{\text{OLS}} = \tau + \gamma \frac{\text{Cov}\left(D_i, Z_{s(i)}\right)}{\text{Var}\left(D_i\right)}$$

The estimated effect of taking AP classes is biased (up) by the fact that students who take AP classes typically have higher-quality teachers

School indicator variables

The simplest solution would be to measure $Z_{s(i)}$ for each student and control for it using the Selection on Observables tools

- But this variable might not be in our data or might be hard to measure accurately

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Instead, consider running the regression of y_i on D_i and a set of indicators for each school $\mathbb{1}[s(i) = k]$ for $k = 1, \ldots, K$:

$$y_i = D_i \tau + \sum_{k=1}^{K} \mathbb{1}[s(i) = k] \alpha_k + u_i$$

- The schools being labeled $1, \ldots, K$ only makes notation easier

School indicator variables

$$y_i = D_i \tau + \sum_{k=1}^{K} \mathbb{1}[s(i) = k]\alpha_k + u_i$$

The set of school indicator variables are often referred to as school 'fixed effects'

- This sum is a bit of a pain to write, so people will often short-hand this model to $y_i = D_i \tau + \alpha_{s(i)} + u_i$ or even just α_s leaving the $s(i)$ implicit.

School indicator variables

To show you what this method does, consider the infeasible regression of y_i on D_i, $Z_{s(i)}$, and indicators for each school

$$y_i = D_i \tau + Z_{s(i)} \gamma + \sum_{k=1}^{K} \mathbb{1}[s(i) = k] \alpha_k + u_i$$

We know from the Frisch-Waugh-Lovell theorem that we can 'residualize' off the school indicator variables and that leaves:

$$\tilde{y}_i = \tilde{D}_i \tau + \tilde{Z}_{s(i)} \gamma + v_i$$

Residualizing $Z_{s(i)}$ on school indicators

What does the residuals of $Z_{s(i)}$ look like? Well it's the regression of

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The values of $Z_{s(i)}$ only varies at the school level, so this model will fit the data *perfectly* (set $\alpha_k = Z_k$)

- Consequently, $\tilde{Z}_{s(i)} = 0$ for all observations!

School indicator variables

Therefore, we have that

$$\begin{aligned}\tilde{y}_i &= \tilde{D}_i \tau + \underbrace{\tilde{Z}_{s(i)}}_{=0} \gamma + v_i \\ &= \tilde{D}_i \tau + v_i\end{aligned}$$

So the infeasible regression is, in some sense, actually *feasible*

- We only need to residualize y_i and D_i by the school fixed-effects and the impact of $Z_{s(i)}$ is removed all together

‘Fixed Effects’ and controlling for *unobservables*

This logic extends almost immediately to the case where you have *many* factors that vary at the school-level. Say you have $Z_{1,s(i)}$, $Z_{2,s(i)}$, and $Z_{3,s(i)}$

The FWL theorem logic before is the same

- Residualizing each $Z_{\ell,s(i)}$ separately on school indicator variables will remove them all!

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The school fixed-effects absorb *all* the school-level confounders

- The identifying assumption is therefore that there is no *student-level* confounders

‘Fixed Effects’ and controlling for *unobservables*

This is why ‘fixed effects’ are viewed powerfuly in economics and are so commonly used

- They control for many different confounders (that vary at the level of the fixed-effect)
- E.g. any omitted-variables bias story that occurs at the school level is removed by school fixed-effects

What ‘variation’ remains?

So, clearly fixed effects are a powerful tool; but the question is what ‘variation’ remains in $\tilde{D}_i$?

- I.e. what remains after regressing D_i on the set of school indicators

Our regression is $D_i = \sum_{k=1}^{K} \mathbb{1}[s(i) = k]\alpha_k + e_i$

- Predict whether you took an AP class based on the school you go to

What variation remains?

$$D_i = \sum_{k=1}^{K} \mathbb{1}[s(i) = k]\alpha_k + e_i$$

Since the set of school indicators are mutually exclusive and exhaustive, we know that $\hat{\alpha}_k$ estimates the proportion of kids in school k that take AP classes:

- $\hat{\alpha}_k = \mathbb{E}[D_i \mid s(i) = k]$

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- $\hat{\alpha}_k = \mathbb{E}[D_i \mid s(i) = k]$

Then the variable $\tilde{D}_i$ equals $D_i - \hat{\alpha}_{s(i)}$

- This takes one of two values within a school: $1 - \hat{\alpha}_{s(i)}$ or $-\hat{\alpha}_{s(i)}$

The intuition is that we are still comparing people with larger or smaller values of D_i, but we are removing bad variation from y_i

Within-school variation

Say you have a relatively small sample of students per school

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$\implies$ Schools' with no variation in D_i do not contribute to the estimate

Example of Fixed Effects usage

This semester, Sarah Cordes from Temple University presented her work estimating the returns to high-quality schooling on student outcomes among low-income families in New York City

She uses the fact that families are placed randomly within public-housing complexes and are therefore sent to different NYC public schools (of varying quality)

Example of Fixed Effects usage

Cordes et. al. ran a regression of educational gains (y_i) on school quality (q_i) and a set of indicator variables for the public-housing complex a family is assigned to:

$$y_i = q_i \tau + \alpha_c + u_i$$

- α_c denotes complex fixed-effects

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- α_c denotes complex fixed-effects

$\tilde{q}_i = q_i - \mathbb{E}[q_i \mid c = c_i]$ is the within-complex variation in school-quality

- Some families are located near better schools than others within the complex

“Within” estimator

The fixed-effect estimator is sometimes referred to as the ‘within’-estimator.

For example, here is how an economist typically talks about the previous example

- ‘Our estimator compares two kids *within the same public-housing complex*, one assigned to a better than expected quality and the other to a lower than expected quality school’

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- ‘Our estimator compares two kids *within the same public-housing complex*, one assigned to a better than expected quality and the other to a lower than expected quality school’

Note this feels very similar to matching; look within people with the same X_i and argue that treatment is randomly assigned within that group

“Within” estimator

‘Our estimator compares two kids within the same public-housing complex, one assigned to a better than expected quality and the other to a lower than expected quality school’

I do not love this language since it implies (to me) that you are running separate regressions for each public-housing complex (you are not!)

“Within” estimator

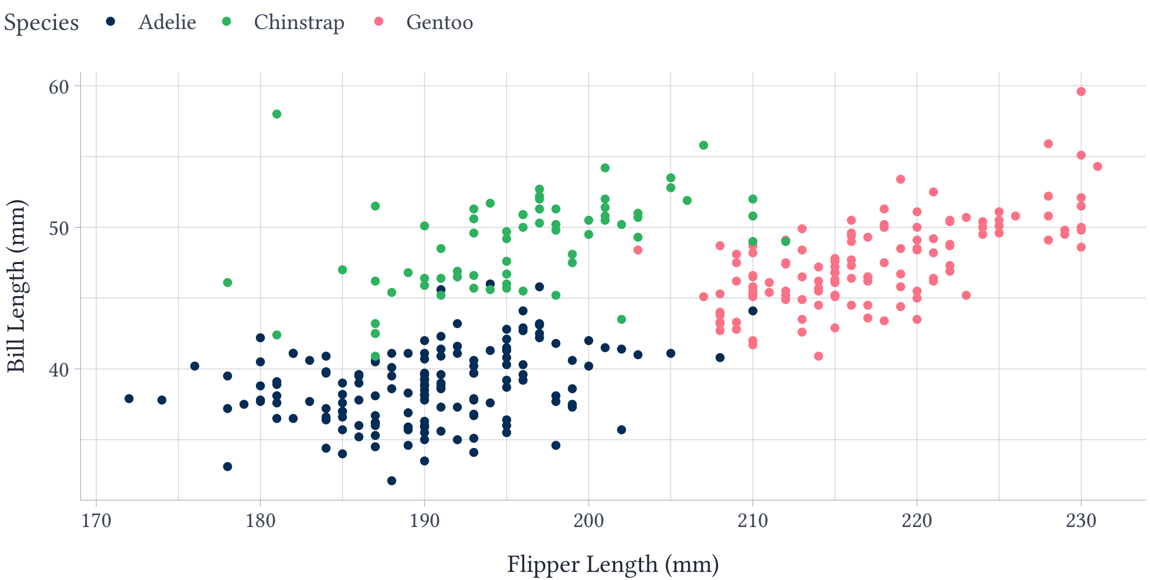
What fixed effects are actually doing:

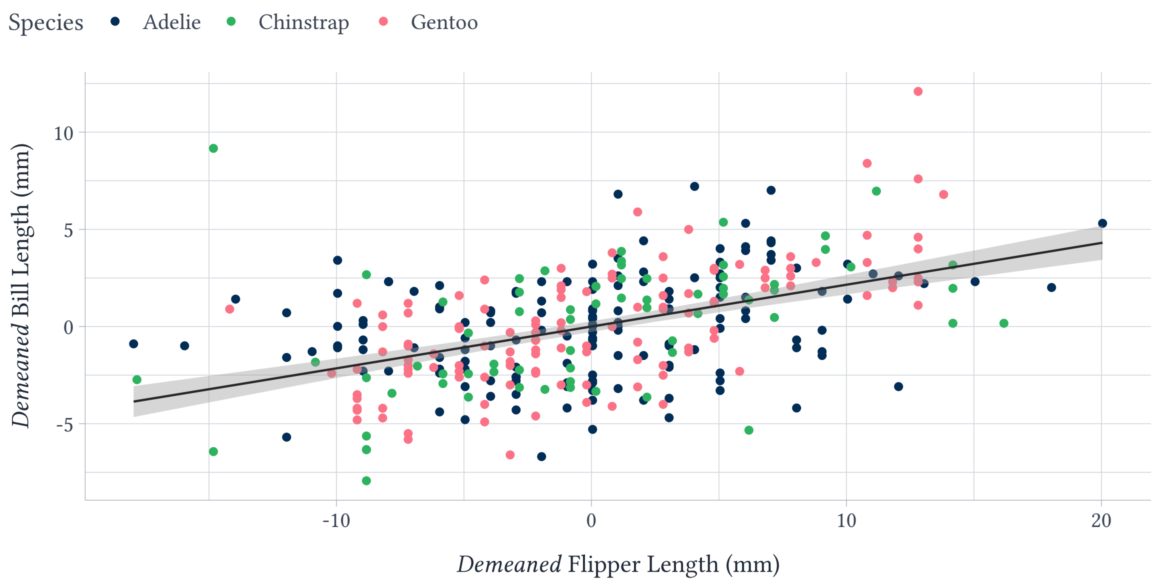
- Within each complex, take deviations between observed q_i and the complex’s average q_i: $\tilde{q}_i$
- Across complexes, regress $\tilde{y}_i$ on $\tilde{q}_i$
  - $\rightarrow$ OLS pools across complexes, putting more weight on public-housing complexes with more variation in $\tilde{q}_i$

Penguins Example

I'll show you an example to try and help make this clearer. I have a dataset with penguin's flipper length and their bill length

- I will regress bill length on flipper length while controlling for species fixed effects (there are 3 different species in this dataset)


Species
Adelie
Chinstrap
Gentoo
60
50
40
Bill Length (mm)
170
180
190
200
210
220
230
Flipper Length (mm)


Species
Adelie
Chinstrap
Gentoo
10
5
0
-5
-10
0
10
20
Demeaned Bill Length (mm)
Demeaned Flipper Length (mm)

“Within” estimator

Am I being a bit pedantic? Kind of...

“Within” estimator

Am I being a bit pedantic? Kind of...; but it can sometimes matter

Remember in our selection on observables section we discussed the weird weighting of units’ treatment effects that can occur with a linear regression:

- In our case, we put more weight on appartment complexes that have more variation in $\tilde{q}_i$ (by least-squares algebra)

“Within” estimator

Am I being a bit pedantic? Kind of...; but it can sometimes matter

Remember in our selection on observables section we discussed the weird weighting of units’ treatment effects that can occur with a linear regression:

- In our case, we put more weight on appartment complexes that have more variation in $\tilde{q}_i$ (by least-squares algebra)
  - $\rightarrow$ In this case, treatment effects probably don’t change much across apartment complex (minimal heterogeneity), so this weighting is probably fine

This will turn out to matter when we discuss difference-in-differences!

Estimating fixed effects in `R`

One final note, to estimate fixed effects in `R`, we will continue to use the `fixest` package:

```
feols(y ~ x | school, data = df, vcov = "hc1")
```

- Put the fixed effects after `|`
- Could do `i(school)`, but will be much slower and clog up your output with extra coefficients

A note on estimation of fixed-effects

Thinking about estimation of fixed-effects, our set of indicator variables can be written as:

$$\begin{pmatrix} \mathbb{1}[s(i) = 1] & \mathbb{1}[s(i) = 2] & \dots & \mathbb{1}[s(i) = S] \end{pmatrix}$$

- This is an $S \times N$ matrix; in larger datasets, your computer would run out of memory creating this matrix

But this matris is very sparse (mostly 0s) and we know what residualizing does:

- In the case of one fixed-effect, we know we can just demean the variables within i
- In the case of multiple fixed-effects, we do something like multiple demeaning

A note on estimation of fixed-effects

Faster methods are implementedin Stata (the O.G.) by `reghdfe` and in R by `fixest`

- It manually demeans y and $\boldsymbol{X}$ and then runs the much simpler $\tilde{y}$ on $\tilde{\boldsymbol{X}}$

In the case of large datasets with many fixed effects, this can take estimation from many hours to a few seconds. Short: use `fixest/reghdfe`

- I find this stuff really interesting, but ymmv

Fixed Effects in Panel Data

Panel Data

Now, we are at the point in the course where we will consider panel data

- We observe a set of individuals $i \in \{1, \dots, N\}$ over a set of time periods $t \in \{1, \dots, T\}$

A balanced panel observers each individual in all T time periods ($N \times T$ total)

- Otherwise, we call this an unbalanced panel

Advantages of Panel Data

In the absence of a clear ‘quasi-experimental’ method in cross-section, people often turn to panel data

Panel data helps us estimate effects by allowing us to remove some key sources of confounding

- Observing a person before they enter into treatment might help us better understand their $Y_{it}(0)$

Caffeine and Productivity

Say you observe a panel dataset of workers (i) on different workdays (t). You want to know if your company should provide free coffee for the workers by evaluating the impact of caffeine (d_{it}) on productivity (y_{it})

The first problem is that worker's that drink coffee probably look different than those that do not

- People with higher (average) d_{it} might differ in other characteristics

Caffeine and Productivity

Say worker's producivity is determined by

$$y_{it} = p_i + \tau d_{it} + \varepsilon_{it}$$

- Here, p_i is a worker's underlying productivity that is time-invariant
- τ is the true treatment effect
- ε_{it} are shocks to productivity on a given day

Unit “Fixed Effects”

$$y_{it} = p_i + \tau d_{it} + \varepsilon_{it}$$

The p_i term is our ‘fixed effect’

- A person-specific effect that is *fixed* over days t
- When we estimate this model, this is just a set of N indicator variables (e.g. a indicator for being person 1, an indicator for being person 2, ...)

Unit “Fixed Effects”

After residualizing out the estimated fixed effects

$$\tilde{y}_{it} = \tau \tilde{d}_{it} + \tilde{\varepsilon}_{it}$$

From least-squares mechanics, we have $\tilde{d}_{it} = d_{it} - \bar{d}_i$, where $\bar{d}_i$ is the (sample) average caffeine intake for a person

- $\tilde{d}_{it}$ represents the average caffeine intake relative to the worker’s average intake

Identifying assumption

$$(y_{it} - \bar{y}_i) = \tau \left(d_{it} - \bar{d}_i\right) + \tilde{\varepsilon}_{it}$$

Our regression compares workers on days where they have above (their) average caffeine intake to days where they have below (their) average

- Our ‘ideal experiment’ is that workers *randomly* have more or less caffeine than their average

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Our 'ideal experiment' is that workers randomly have more or less caffeine than their average

Do we think this is a plausible assumption?

- Maybe on days when the worker is feeling very tired, they drink an extra cup of coffee
- Them being tired might have a direct effect on their productivity (showing up in $\tilde{\varepsilon}_{it}$)

‘Fixed Effects’ vs. ‘Fixed Characteristics’

A lot of people confuse ‘fixed effects’ with ‘fixed characteristics’

- E.g. your people skills might be a ‘fixed characteristic’
- But over time the labor market returns to people skill might change

So even though your people skills might be fixed, it does not have a ‘fixed *effect*’ on the outcome

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So even though your people skills might be fixed, it does not have a ‘fixed *effect*’ on the outcome

This can cause a bias in our treatment effect

- E.g. If people with high people skills select into treatment when the returns to people skills is going up, that will contaminate treatment effect