

# Topic 5: Instrumental Variables

*ECON 5783 – University of Arkansas*

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## **Introducing Instrumental Variables**

**What makes a good instrument?**

**Two-stage least squares estimator**

**IV with Heterogeneous Effects**

# Instrumental Variables

Instrumental Variables are one of the oldest 'causal' identification strategies

- Their roots go back to the early 1900s where economics was primarily the study of particular markets
- Demand and supply curves were a new phenomenon and economists wanted to try and estimate them

# Supply and Demand Curves

Consider a simple supply and demand curve setup

$$\text{quantity}_d = \alpha_d + \text{price} \gamma_d + u_d \quad (1)$$

$$\text{quantity}_s = \alpha_s + \text{price} \gamma_s + u_s \quad (2)$$

Here, this is the theoretical relationship between prices and quantity *at the same point in time* (potential outcomes)

- Equilibrium is determined by the price such that  $\text{quantity}_d = \text{quantity}_s$

Note that 'market conditions' (e.g. preferences, production technologies, etc.) are implicitly embedded in the coefficients

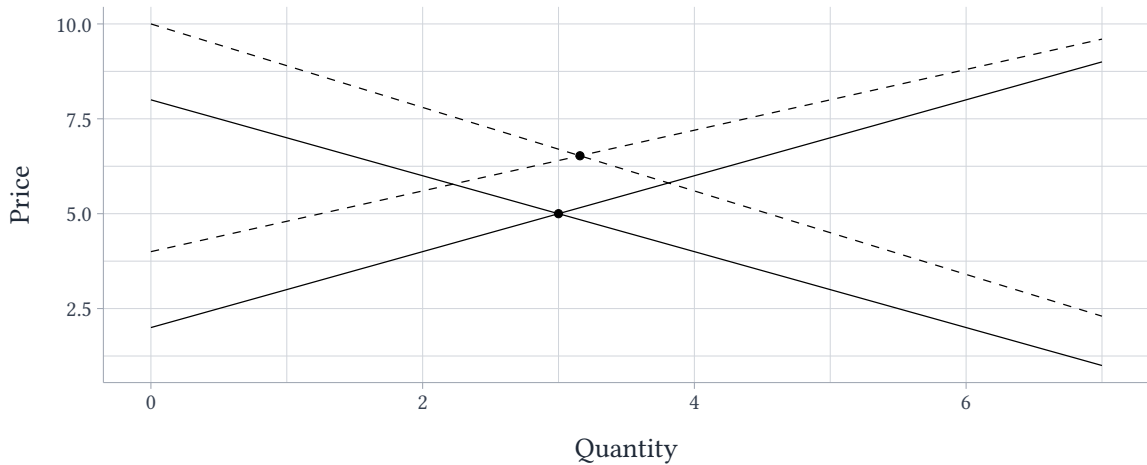
# Observed market outcomes

What we observe is a set of markets  $t$  with their corresponding price and quantities  $(p_t, q_t)$

The problem here is that market-shocks could move both the demand curve and the supply curve

→ The data is not tracing 'along' the demand curve or the supply curve

— Market 1    - - Market 2



# Tracing out the demand curve

Phillip Wright wrote a book in 1928 about 'animal and vegetable oils'

- he was trying to argue that recent tariffs had negatively impacted the market

In this book, there is a now famous "Appendix B" proposing the first IV estimator

- There's debate who wrote this chapter: father or son.
- It was likely the son; see Stock and Trebbi, 2003

## Identifying movement along one curve

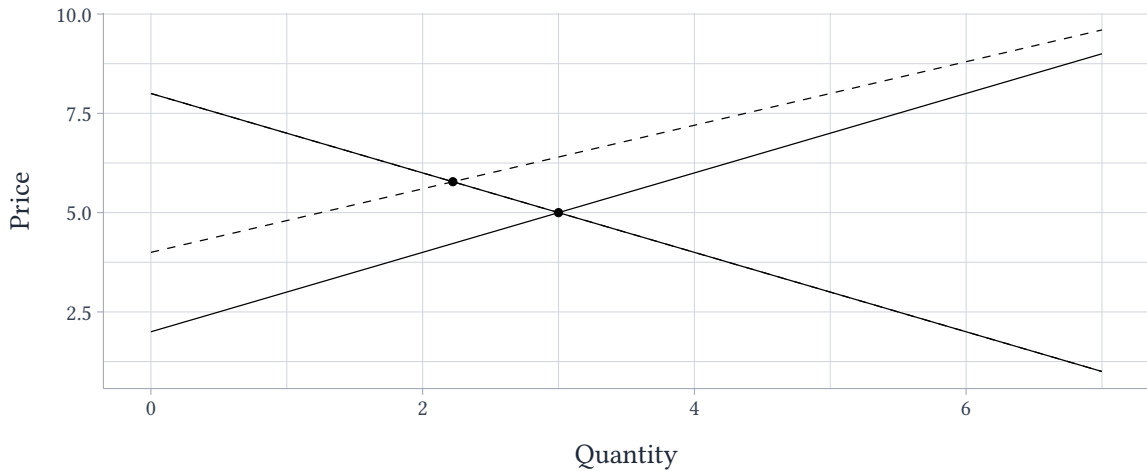
As we saw in the previous graph, we can not take two points  $(p_1, q_1)$  and  $(p_2, q_2)$  and know if these fall on the same demand curve or the same supply curve

Say somehow we *know* that one of the curves did not change from market 1 to market 2 (say from one day to the next)

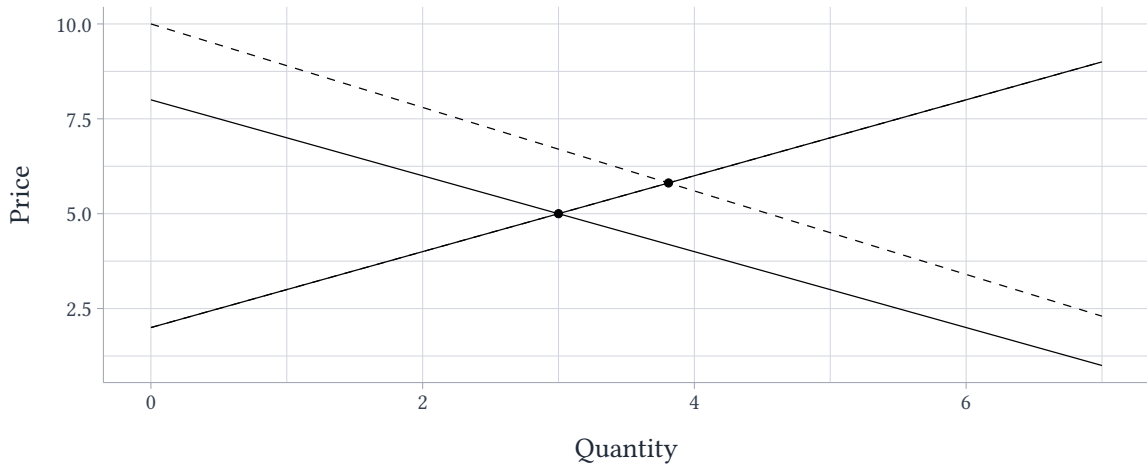
- Then our two points both must fall on the curve that did not shift (equilibrium conditions)



— Market 1    - - Market 2



— Market 1    - - Market 2



## “Demand Shifters” and “Supply Shifters”

Wright proposed to use a variable that shifts only one of the curves and not the other, what we now call instruments, to estimate the demand/supply curve:

- E.g. a demand shifter being the change in the price of a substitute good
- E.g. a supply shifter being a change in rain from one year to the next

## “Demand Shifters” and “Supply Shifters”

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- E.g. a demand shifter being the change in the price of a substitute good
- E.g. a supply shifter being a change in rain from one year to the next

Since these ‘shifters’ only affect one curve and not the other, then we can leverage these to estimate the other curve (e.g. demand shifter to estimate supply)

# Fulton Fish Market

Angrist, Graddy, and Imbens (2000) give a great example of this.

Graddy, then a graduate student, woke up every morning before sunrise and traveled from New Jersey to the Fulton Fish Market in downtown Manhattan

- Recorded data on the price and quantity sold of fish
- Also recorded each day's ocean-weather, an important supply curve shifter



# Fulton Fish Market IV Estimator

Let  $Z_t$  denote the weather observed in market  $t$  (e.g.  $Z_t$  is the wind speeds on the ocean).

The idea of the IV estimator we will present is to do two things:

1. First, see how weather,  $Z_t$ , impacts the (log) market price,  $\log(p_t)$
2. Second, see how weather,  $Z_t$ , impacts the (log) market quantity,  $\log(q_t)$

Since the demand elasticity is  $\frac{\partial \log(q_t)}{\partial \log(p_t)}$ , we can take the ratio of these two quantities to estimate the demand elasticity

# Fulton Fish Market IV Estimator

Our IV estimator can be written as

$$\hat{\tau}_{IV} = \frac{\text{Cov}(Z_t, q_t)}{\text{Cov}(Z_t, p_t)}$$

- Weather causes a shift in  $q_t$ ,  $\text{Cov}(Z_t, q_t)$ , and a shift in  $p_t$ ,  $\text{Cov}(Z_t, p_t)$

Our IV estimate compares how much weather changes the quantity sold in equilibrium to how much weather changes the price to estimate the demand curve



# Fulton Fish Market IV Estimator

$$\hat{\tau}_{IV} = \frac{\text{Cov}(Z_t, q_t)}{\text{Cov}(Z_t, p_t)}$$

Ideally these shifts come from only the supply curve moving; i.e.  $Z_t$  has no effect on quantity $_t^d$  except from the price

- E.g. this requires fish demand not change with the weather, otherwise both curves would be moving

## **Introducing Instrumental Variables**

**What makes a good instrument?**

**Two-stage least squares estimator**

**IV with Heterogeneous Effects**

# Generalizing the IV Estimator

What are the “essential” ideas of this IV estimator? Say you want to know the causal effect of  $X_i$  on some variable  $y_i$ . We are concerned there are some other variables (in the error term,  $\varepsilon_i$ ) that determine both  $X_i$  and  $y_i$ .

That is:

$$y_i = X_i\beta + \varepsilon_i,$$

where  $\mathbb{E}[\varepsilon_i \mid X_i] \neq 0$

# Generalizing the IV Estimator

$$y_i = X_i\beta + \varepsilon_i \text{ where } \mathbb{E}[\varepsilon_i | X_i] \neq 0$$

Our IV Estimator compares the change in  $y_i$  induced by  $Z_i$  to the change in  $X_i$  induced by  $Z_i$  to estimate the slope parameter:

$$\frac{\Delta y \text{ induced by } Z}{\Delta X \text{ induced by } Z}$$

## IV Requirements

$$y_i = X_i\beta + \varepsilon_i \text{ where } \mathbb{E}[\varepsilon_i | X_i] \neq 0$$

We want an instrument  $Z_i$  that does two things:

1. (Relevancy Condition) The instrument should cause a change in  $X_i$ , so that we could trace out the subsequence impact on  $y_i$

## IV Requirements

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We want an instrument  $Z_i$  that does two things:

1. **(Relevancy Condition)** The instrument should cause a change in  $X_i$ , so that we could trace out the subsequence impact on  $y_i$
2. **(Exclusion Restriction)** The instrument  $Z_i$  should change  $y_i$  *only through* changing  $X_i$

## IV Estimand

$$\hat{\beta}_{\text{IV}} = \frac{\text{Cov}(Z_i, y_i)}{\text{Cov}(Z_i, X_i)}$$

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*Exclusion Restriction*  
= 0

*Relevancy Condition*  
not dividing by 0

1. **Relevancy Condition** requires  $Z_i$  to actually shift  $X_i$
2. **Exclusion Restriction** requires  $Z_i$  to be uncorrelated with other drivers of  $y_i$

## IV Requirements

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1. **Relevancy Condition** requires  $Z_i$  to actually shift  $X_i$
2. **Exclusion Restriction** requires  $Z_i$  to be uncorrelated with other drivers of  $y_i$

The first tends to be the easier of the two to satisfy: we need a 'shifter' of  $X_i$ . The second is the hard part...

## Rainfall IV Example

For example, consider trying to estimate on family income on the years of schooling children receive in the developing world. Papers have used rainfall as an instrument for income

- The idea is that rainfall is 'random' year over year and so that creates 'good variation' in family income

## Rainfall IV Example

In essence, the IV estimator will compare families that had a good rainfall year (and hence more income than normal) to families with a bad rainfall year (and hence less money than expected)

- If high-rainfall families go to school at higher rates, the IV estimator attributes this to higher income

## Rainfall IV Exclusion Restriction failure

The exclusion restriction assumes that rainfall only affects the school attendance rate *only through* increasing family incomes

- Is this plausible?

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Sarsons (2015, JDE) finds that lower rainfall in developing countries increases conflict between villages

- This means we can't say if the higher school attendance in high rain areas is due to more income from better crop-yields *or* due to lower likelihood of conflict



## **Introducing Instrumental Variables**

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# Canonical IV Setup

The canonical IV setup is as follows:

$$y_i = X_i\beta + \varepsilon_i$$

$$X_i = Z_i\pi + u_i,$$

where  $Z_i$  is our instrument and  $X_i$  is the variable of interest

## Canonical IV setup

$$y_i = X_i\beta + \varepsilon_i \text{ and } X_i = Z_i\pi + u_i$$

In terms of this model, our problem is that  $\mathbb{E}[X_i\varepsilon_i] \neq 0$ , i.e. that there are unobservables that are correlated with  $X_i$  and have an impact on  $y_i$

# Canonical IV setup

$$y_i = X_i\beta + \varepsilon_i \text{ and } X_i = Z_i\pi + u_i$$

We instead use an instrument  $Z_i$ . Our two requirements for the instrument can be written as follows:

1. **(Relevance)**  $\pi \neq 0$ 
  - This is testable with  $t$ -test that  $\pi = 0$
  - Later, problems with 'weak' instruments where  $\pi \approx 0$  (relative to noise)
2. **(Exclusion)**  $\mathbb{E}[\varepsilon_i Z_i] = 0$ 
  - Fundamentally untestable (it is an assumption)

# Canonical IV setup

## *Exclusion Restriction*

$$y_i = X_i\beta + \varepsilon_i \text{ and } X_i = Z_i\pi + u_i$$

For example, say there is some variable  $\mu_i$  that is part of the error term  $\varepsilon_i = \mu_i + v_i$

- If  $Z_i$  is correlated with  $\mu_i$  (or affects  $\mu_i$ ), then our exclusion restriction fails

## Two-stage least squares (2SLS)

$$y_i = X_i\beta + \varepsilon_i \text{ and } X_i = Z_i\pi + u_i$$

The previous estimator is identical to the two-stage least squares estimator:

$$\hat{\beta}_{2\text{SLS}} = \frac{(X_i P_Z)' P_Z y_i}{(X_i P_Z)' (P_Z X_i)} = \frac{\hat{X}_i' \hat{y}_i}{\hat{X}_i' \hat{X}_i},$$

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where  $P_Z = Z(Z'Z)^{-1}Z'$  is the projection-matrix from ordinary least squares regression

- This is numerically identical when  $Z$  is a single instrument to our previous estimator  $\text{Cov}(Z_i, y_i) / \text{Cov}(Z_i, X_i)$

## Two-stage least squares Estimator (2SLS)

$$\hat{\beta}_{\text{IV}} = \frac{\hat{X}_i' \hat{y}_i}{\hat{X}_i' \hat{X}_i},$$

Can think of this as being done in two-stages (hence the name):

- Predict  $y_i$  and  $X_i$  using the instrument  $Z_i$  via separate regressions
- Regress  $\hat{y}_i$  on  $\hat{X}_i$



# OLS with Controls vs. IV

The Frisch-Waugh-Lovell (FWL) theorem helped us understand what control variables do in a regression of  $y_i$  on a variable  $X_i$  and controls  $W_i$

- Use  $W_i$  to *predict*  $X_i$  and  $y_i$  and remove that predicted variation
- Regress  $y_i - \hat{y}_i$  on  $X_i - \hat{X}_i$

# OLS with Controls vs. IV

Both estimators want to use variation in  $X_i$  that is 'plausibly exogenous'; sometimes called 'quasi-experimental variation'.

The IV estimator and OLS with controls try to get at 'good' variation in  $X_i$  in different ways:

1. OLS 'removes bad variation'

→ Use controls that you think pick up on variables (in  $\varepsilon_i$ ) that are correlated with  $X_i$

2. IV 'isolates good variation'

→ Use an instrument that you think is *not* correlated with  $\varepsilon_i$  but that shifts  $X_i$

## Weak IV

$$y_i = X_i\beta + \varepsilon_i \text{ and } X_i = Z_i\pi + u_i$$

When our estimated  $\hat{\pi} \approx 0$  we have a 'weak instruments' problem

- I.e. close to zero relative to the noise

## Weak IV

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- I.e. close to zero relative to the noise

This means there is a small covariance between  $X_i$  and  $Z_i$ :

- Since  $\hat{\beta}_{IV}$  divides by  $\text{Cov}(X_i, Z_i)$ , the estimate is very noisy  
 $\rightarrow 1/0.0001 = 10000$  vs.  $1/0.00005 = 20000$

## Weak IV

The rule of thumb is to use an instrument when the  $F$ -stat on the first-stage is  $\geq 10$

- First-stage is  $X_i = Z_i\pi + u_i$
- In case of single IV  $Z_i$ , equivalent to  $t$ -stat on  $Z_i$  to be  $\geq \sqrt{10} = 3.16$

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Some recent work shows that 10 might be too small of a lower bound and you may prefer something like  $F \geq 50$  ( $t$ -stat  $\geq 7.07$ ), but it is unsettled debate

# Many Instruments

Another problem comes in when you have a large number of instruments (relative to sample size)

Our 2SLS estimator regresses  $P_Z y_i$  on  $P_Z X_i$

- Increasing the size of  $Z$  means we are going to necessarily predict  $X_i$  and  $y_i$  better and better
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$\implies$  that with too many instruments we  $\hat{\beta}_{2SLS} \approx \hat{\beta}_{OLS}$



# Many Instruments Example

One common IV strategy is the 'judge-leniency design' setting:

- Defendants are randomly assigned to different judges
- Judges vary in how lenient they are in sentencing

⇒ people with similar backgrounds will be randomly convicted/not-convicted

- Use as an instrument a set of dummy variables for each assigned judge (judge 1, judge 2, ...)

# Many Instruments

The best way to fix the issue of many-instruments is using a JIVE estimator

- For observation  $i$ , predict  $X_i$  using a leave-out regression of  $X_j$  on  $Z_j$  using all observations besides  $i$ :  $\{1, \dots, n\} \setminus \{i\}$

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Call the predicted values of the leave-out regressions as  $\hat{X}_{i,10}$ . Then we have

$$\hat{\beta}_{\text{JIVE}} \equiv \frac{\hat{X}'_{i,10} y_i}{\hat{X}'_{i,10} \hat{X}_{i,10}}$$

- $\hat{\beta}_{\text{JIVE}}$  avoids the problem of over-fitting on the instruments
- See Kolesar (2013, Working Paper)

## **Introducing Instrumental Variables**

**What makes a good instrument?**

**Two-stage least squares estimator**

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# Heterogeneous Effects

Up until this slide we have (implicitly) assumed that the marginal causal effect of increasing  $X_i$  was given by  $\beta$

- Assumed to be the same across individuals

When we allow for heterogeneous effects, what our 2SLS estimator finds is more complicated to understand

- Hopefully, it is some 'reasonable' weighted average of heterogeneous effects,

$$\hat{\beta}_{IV} \rightarrow \sum_i w_i \beta_i$$

# Angrist-Imbens-Rubin Causal Model

Let's see if we can make progress when  $D_i$  is a binary variable,  $Z_i$  is a binary variable, but allow the treatment effect  $y_i(1) - y_i(0)$  to be heterogeneous.

In Angrist, Imbens, and Rubin (1996, JASA), they study military service ( $D_i$ ) on future earnings ( $y_i$ )

- The first problem we have is that military service is not randomly assigned

<sup>†</sup>This has been questioned because the bingo balls were not shuffled enough before drawing

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They use the Vietnam-Draft lottery as an instrument ( $Z_i$ )

- $Z_i$  is randomly assigned by birthday<sup>†</sup>

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# Imperfect Compliance

The Vietnam draft lottery is definitely a shifter: those drafted by lottery were more likely to serve

- But we don't have a real RCT. Some with  $Z_i = 0$  serve  $D_i = 1$  and some with  $Z_i = 1$  do not serve  $D_i = 0$



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This is what we call “imperfect compliance”, i.e. treatment is not a perfect function of the instrument

# Potential Outcome Model with Imperfect Compliance

To accomodate this, Angrist-Imbens-Rubin framework defines  $D_i(Z_i)$  to be the potential outcomes of treatment under both states of the instrument. In our example,

- $D_i(0)$  is whether the person would serve in the world where they *are not* drafted
- $D_i(1)$  is whether the person would serve in the world where they *are* drafted

# Potential Outcome Model with Imperfect Compliance

Outcomes are now a potential outcome of both  $Z_i$  and  $D_i$ :  $y_i(D_i, Z_i)$

- Outcomes depend on both whether you were assigned  $Z_i = 0, 1$  and whether you are under treatment  $D_i = 0, 1$

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If we assume the *exclusion restriction* that  $Z_i$  *only* impacts  $y_i$  via it's effect on  $D_i$ , then we can return to  $y_i(D_i)$

- In the case of the Vietnam Draft lottery, this is plausible assuming that the draft only is correlated with outcomes via causing people to serve

# Potential Outcome Model with Imperfect Compliance

Our potential outcomes are therefore

$$D_i(Z_i) \text{ and } y_i(D_i)$$

Our causal model implicitly says:

1.  $Z_i$  might impact  $D_i$  which impacts  $y_i$
2.  $Z_i$  only impacts  $y_i$  via  $D_i$

# Characterizing People

There are four kinds of people in this model:

1. **Compliers:** people who react to the instrument as expected  
 $\rightarrow D_i(1) = 1$  and  $D_i(0) = 0$
2. **Always-takers:** people who always take the treatment regardless of  $Z$   
 $\rightarrow D_i(1) = 1$  and  $D_i(0) = 1$
3. **Never-takers:** people who never take the treatment regardless of  $Z$   
 $\rightarrow D_i(1) = 0$  and  $D_i(0) = 0$
4. **Defiers:** people who react to the instrument in the wrong direction  
 $\rightarrow D_i(1) = 0$  and  $D_i(0) = 1$

# Defiers

Defiers are often ruled out as implausible:

- “I would have served but not if I was drafted”

But, be careful, they may not be implausible in other settings!



# Characterizing People

Ruling out defiers, three types remain:

1. **Compliers:** people who react to the instrument as expected  
 $\rightarrow D_i(1) = 1$  and  $D_i(0) = 0$
2. **Always-takers:** people who always take the treatment regardless of  $Z$   
 $\rightarrow D_i(1) = 1$  and  $D_i(0) = 1$
3. **Never-takers:** people who never take the treatment regardless of  $Z$   
 $\rightarrow D_i(1) = 0$  and  $D_i(0) = 0$

It is impossible to know who is which in the data

# First-stage

Our first-stage consists of regressing  $D_i$  on  $Z_i$  and an intercept:

$$D_i = \alpha + Z_i\pi + u_i$$

From regression mechanics and since  $Z_i$  and  $D_i$  are indicators, we know

- $\hat{\alpha}$  is the share of people with  $D_i = 1$  with  $Z_i = 0$
- $\hat{\pi}$  is the difference in share of people with  $D_i = 1$  with  $Z_i = 1$

## First-stage

$$D_i = \alpha + Z_i\pi + u_i$$

In the  $Z_i = 0$  group, people with  $D_i = 1$  must be always-takers

- $\hat{\alpha}$  is the share of always-takers in the population

# First-stage

$$D_i = \alpha + Z_i\pi + u_i$$

In the  $Z_i = 0$  group, people with  $D_i = 1$  must be always-takers

- $\hat{\alpha}$  is the share of always-takers in the population

In the  $Z_i = 1$  group, people with  $D_i = 1$  are always-takers or compliers

- $\hat{\alpha} + \hat{\pi}$  is the share of always-takers or compliers in the population

# First-stage

$$D_i = \alpha + Z_i\pi + u_i$$

In the  $Z_i = 0$  group, people with  $D_i = 1$  must be always-takers

- $\hat{\alpha}$  is the share of always-takers in the population

In the  $Z_i = 1$  group, people with  $D_i = 1$  are always-takers *or* compliers

- $\hat{\alpha} + \hat{\pi}$  is the share of always-takers or compliers in the population

$\hat{\pi}$  is our estimated share of compliers

- Share of people who are 'pushed' into treatment

# Reduced-form

The reduced-form is the regression of

$$y_i = \gamma + Z_i\delta + u_i$$

Since  $Z_i$  is an indicator variable

- $\hat{\gamma} = \hat{\mathbb{E}}[y_i \mid Z_i = 0]$
- $\hat{\delta} = \hat{\mathbb{E}}[y_i \mid Z_i = 1] - \hat{\mathbb{E}}[y_i \mid Z_i = 0]$

## Reduced-form

$$\hat{\delta} = \hat{\mathbb{E}}[y_i \mid Z_i = 1] - \hat{\mathbb{E}}[y_i \mid Z_i = 0]$$

- For always-takers,  $y_i = y_i(1)$  when  $Z_i = 1$  and  $Z_i = 0$
- For never-takers,  $y_i = y_i(0)$  when  $Z_i = 1$  and  $Z_i = 0$

Therefore,

$$\hat{\delta} = \mathbb{P}(\text{Complier}_i)(\mathbb{E}[y_i(1) - y_i(0) \mid \text{Complier}_i])$$

# IV Estimand

Our IV Estimand therefore is the ratio

$$\hat{\beta}_{\text{IV}} = \frac{\hat{\delta}}{\hat{\pi}}$$



## IV Estimand

Our IV Estimand therefore is the ratio

$$\begin{aligned}\hat{\beta}_{\text{IV}} &= \frac{\hat{\delta}}{\hat{\pi}} \\ &= \frac{\mathbb{P}(\text{Complier}_i)(\mathbb{E}[y_i(1) - y_i(0) \mid \text{Complier}_i])}{\mathbb{P}(\text{Complier}_i)} \\ &= \mathbb{E}[y_i(1) - y_i(0) \mid \text{Complier}_i]\end{aligned}$$

We estimate the average treatment effect among the compliers

- E.g. the average treatment effect among those induced to serve in the military by the lottery