A CAREFUL EXAMINATION OF OUR BALLS

1. Some important sets

Consider a control system on some analytic domain $X = \{x \in \mathbb{R}^n \mid g_X(x) \leq 0\}$ for some $g_X \in \mathbb{R}[x_1, \ldots, x_n]$ with polynomial dynamics

$$\dot{x} = f(x, u).$$

Moreover, assume we have a family of polynomial control laws $u = g_\alpha(x)$ indexed by a set A with the parameter $\alpha \in A$. For each fixed α assume there is an attracting set $X_\alpha \subset X$ given by the dynamics. For example, this attractor could represent a neighborhood of a locomotive gait. We can then consider the set

$$Y_T = \{(x, \alpha) \mid \alpha \in A, x \in N_\varepsilon(X_\alpha)\}.$$

for some small $\varepsilon > 0$.

The use of T here is not yet warranted for the reader.

We can switch α at any time. This effectively yields the new control systems

$$\dot{x} = f(x, g_\alpha(x)) \tag{1}$$

where $\alpha \in A$ is the new control parameter.

To simplify dynamical concerns we will consider only this later control system. We are concerned with the robustness of the sets X_α under these dynamics. In order to address this, let us denote the backward-time reachable set of X_α under (1) by $\operatorname{btrs}(X_\alpha)$. To asses robustness of X_α we consider the neighborhood $N_r(X_\alpha)$ for some $r > 0$. We then say that X_α is a *robust attractor of radius* r if $\operatorname{btrs}(X_\alpha)$ contains $N_r(X_\alpha)$.

Let us choose some $r > \varepsilon > 0$ and assemble the set

$$R = \{(x, \alpha) \mid \alpha \in A, x \in N_r(X_\alpha)\}.$$

2. The primal

Let us consider the primal optimization problem

$$p^* = \sup \int E(\alpha) d\mu_T(x, \alpha) \tag{2}$$

with decision variables

$$\mu_0, \hat{\mu}_0, \mu_T \in \mathcal{D}'(X \times A), \text{ and } \mu \in \mathcal{D}'([0, T] \times X \times A)$$

subject to the constraints

$$\mu_T \otimes \delta_T - \mu_0 \otimes \delta_0 - \mathcal{L}'\mu = 0 \tag{2a}$$

$$\mu_0 = \hat{\mu}_0 + \frac{\mathbb{1}_R}{\mathrm{vol}(X)} \left(\int_X d\mu_T(x) \right) \lambda_X \tag{2b}$$

$$\int_{X \times A} \mu_T = 1 \tag{2c}$$

$$\mu_T, \mu_0, \hat{\mu}_0, \mu \geq 0 \tag{2d}$$

$$\mathrm{supp}(\mu_T) \subset Y_T. \tag{2e}$$

Here $\lambda_X \in \mathcal{D}'(X)$ denotes the Lebesgue measure on X.

Proposition 2.1. *Let $(\mu_0, \hat{\mu}_0, \mu_T, \mu)$ solve (2). Then $\mathrm{supp}(\mu_0) \subset \mathrm{btrs}(Y_T)$.*

Proof. pending, but thanks to Henrion-Korda

□

Theorem 2.2. *If $(\mu_0, \hat{\mu}_0, \mu_T, \mu)$ solves (2) then $\int_X d\mu_T(x, \alpha) = \delta_{\alpha_T}(\alpha)$ where $\alpha_T = \mathrm{argsup}_\alpha E(\alpha)$ and $\delta_{\alpha_T} \in \mathcal{D}'(A)$ is the Dirac-delta distribution centered at α_T.*

Proof. We will illustrated this by way of contradiction. Let $(\mu_0, \hat{\mu}_0, \mu_T, \mu)$ be a solution to (2) and assume the marginal $\sigma(\alpha) = \int_X d\mu_T(x, \alpha)$ has support at α_T and on some other set $\tilde{A} \subset A$ which does not contain α_T. Finally assume $E(\alpha_T) > E(\tilde{\alpha})$ for all $\tilde{\alpha} \in \tilde{A}$. For any $\epsilon > 0$ we can consider the distribution μ_T^ϵ defined by

$$\mu_T^\varepsilon(S) = \frac{\mu_T(S \cap N_\epsilon(X \times \{\alpha_T\})}{\mu_T(N_\epsilon(X \times \{\alpha_T\}))}$$

for any measurable $S \subset X \times A$. We observe that μ_T^ε satisfies (**??**c) through (**??**e) and that the cost function achieves a greater value than that with μ_T. We need to show that there exists a $\mu^\varepsilon, \mu_0^\varepsilon$, and $\hat{\mu}_0^\varepsilon$ which achieves constraints (**??**a)*and*(**??**b) when μ_T is replaced with μ_T^ε. The Louiville constraint (**??**a) determines μ_0^ε uniquely and implies that the support of μ_0^ε is contained within the support of μ_0 since this is the case for μ_T^ε with respect to μ_T. By viewing $\hat{\mu}_0$ as a slack variable, (**??**b) becomes a linear inequality in μ_0 and μ_T. As μ_0^ε and μ_T^ε are obtained by restricting the supports of μ_0 and μ_T and then rescaling appropriately, substitution of μ_0 with μ_0^ε and μ_T with μ_T^ε into (**??**b) still yields a true statement. Thus μ_T^ε is a strictly better solution to μ_T and we obtain a contration. □

I think that this theorem/proof is flawed, and this is not counting tiny technical oversights. I think we are missing assumptions. We have assumed that $E(\alpha)$ has a unique maxima on the subset $A_{feas} \subset A$ given by $A_{feas} = \{\alpha \in A \mid \exists \mu_0, \ldots, \mu$ which satisfy the constraints and have support at $\alpha\}$. We have to assume E is convex and A_{feas} is convex, or something. Alternatively, perhaps we could drop the assumption that E has a unique maximum and see how that goes. I think this later idea is more promising.

3. The dual problem

The (pre)dual problem to (2) is given by

$$\inf_{p \in \mathbb{R}} p \tag{3}$$

with respect to the decision variables

$$p \in \mathbb{R}, w \in \mathcal{D}(X \times A), \text{ and } v \in \mathcal{D}([0,T] \times X \times A)$$

and the constraints

(3 a)

$w \leq 0$

(3 b)

$\mathcal{L}v \leq 0$

(3 c)

$w(x,\alpha) - v(0,x,\alpha) \geq 0$

(3 d)

$$v(T,x,\alpha) - \frac{1}{\mathrm{vol}(X)} \left(\int_X \mathbb{1}_R(\tilde{x},\alpha) w(\tilde{x},\alpha) d\lambda(x) \right) + p - E(\alpha) \geq 0 \quad \forall (x,\alpha) \in Y_T.$$

Proposition 3.1. *Assuming either* (2) *or* (3) *are feasible, than both are feasible and there is no duality gap between* (2) *and* (3).

Proof. These optimizations are convex. □

Given a solution (v,w,p) of (3) we may consider the inequality

$$w \geq \frac{1}{\mathrm{vol}(X)} \int_X w(\tilde{x},\alpha_T) \mathbb{1}_R(\tilde{x},\alpha_T) d\lambda(\tilde{x}) \tag{4}$$

and define the subset

$$Y_0 = \{(x,\alpha) \in X \times A \mid x \text{ satisfies (4) } \}. \tag{5}$$

where $\alpha_T \in A$ is the location of the Dirac delta distribution in (**??**).

Theorem 3.2. *The set* $\mathrm{btrs}(Y_T) \subset Y_0$.

Proof. Assume (x_0, α_0) is in the region of attraction to Y_T and suppose $(x_T, \alpha_T) \in Y_T$ and $x : [0, T] \to X$ is a solution to the ordinary differential equation

$$\dot{x} = f(x, \alpha) \tag{6}$$

for some $\alpha(t)$ with final condition $x(T) = x_T$ and $\alpha(T) = \alpha_T$ and initial condition $x_0 = x(0), \alpha_0 = \alpha(0)$. Given a solution (v, w, p) of (3) we observe that v is a Lyapunov function for the dynamical system (6) by equation (**??** b). Thus by the definition of a Lyapunov function combined with the inequality (**??** c) we observe

$$v(T, x_T, \alpha_T) \leq v(0, x(0), \alpha(0)) \leq w(x(0), \alpha(0)).$$

Substitution of this inequality into (**??** d) yields

$$w(x(0), \alpha(0)) \geq \frac{1}{\mathrm{vol}(X)} \left(\int_X \mathbb{1}_R(\tilde{x}, \alpha_T) w(\tilde{x}, \alpha_T) d\lambda(x) \right) - p + E(\alpha_T).$$

for any $(x_T, \alpha_T) \in Y_T$. Moreover, by Proposition 3.1 we have $p = E(\alpha_T)$. Thus x_0 satisfies (4) and so $(x_0, \alpha_0) \in X_0$. □

Theorem 3.3. *The set* $R \cap (X \times \{\alpha_T\}) \subset \mathrm{btrs}(Y_T)$ *where* $\alpha_T = \mathrm{argsup}_\alpha E(\alpha)$.

Proof. Let $(\mu_0, \hat{\mu}_0, \mu_T, \mu)$ solve (2). Constraint (**??**b) implies that $R \cap (X \times \pi_2(\mathrm{supp}(\mu_T))) \subset \mathrm{supp}(\mu_0)$. By Theorem 2.2 $\pi_2(\mathrm{supp}(\mu_T)) = \{\alpha_T\}$. Thus $R \cap (X \times \{\alpha_T\}) \subset \mathrm{supp}(\mu_0)$. By Proposition 2.1 the theorem folllows. □

Corollary 3.4. $R \cap (X \times \{\alpha_T\}) \subset Y_0$.

Proof. By Theorem 3.3 we know that $R \cap (X \times \{\alpha_T\})$ is contained within the backwards time reachable set to Y_T which in turn is contained within Y_0 by Theorem 3.2. □