FOURIER ANALYSIS

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1. Bell-ringer (5 minutes)

With your group come up with the most random sentence you possibly can.

Example: *"She used her own hair in the soup to give it more flavor"*

2. Introduction

So far we have looked at power series, which can be thought of as sums (possibly infinite) of multiples of x^k. Today we will consider Fourier series which are sums of multiples of $\sin(kx)$ and $\cos(kx)$. Explicitly, a **Fourier series is a series of the form**

$$S(x) = \underbrace{\sum_{k=0}^{\infty} c_k \cos(kx)}_{\text{even component}} + \underbrace{\sum_{k=1}^{\infty} s_k \sin(kx)}_{\text{odd component}}$$

Where the c_k's and s_k's are coefficients.

3. Fourier Approximations

Consider the function

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < \pi \\ -0.5 & \text{for } -\pi < x \leq 0 \end{cases}$$

We are not able to approximate this function with a Maclauren series because it is discontinuous at $x = 0$. We can not approximate it with a Taylor series very well either. The Taylor series centered at any $c < 0$ is the constant function $P(x) = \frac{-1}{2}$ and for $c > 0$ the Taylor approximation is the constant function $P(x) = 1$.

However, we can approximate f as a Fourier series pretty well though

$$f(x) = \frac{1}{4} + \frac{3}{\pi} \sum_{n=0}^{\infty} \frac{\sin((2n+1)x)}{2n+1}$$

Date: April 2025.

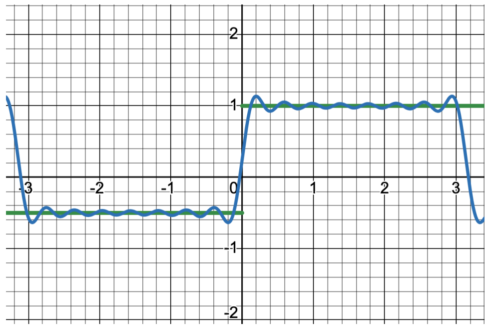

FIGURE 1. 7th order Fourier approximation of a step function

The 20th order Fourier expansion is plotted against $f(x)$ in figure **??** on page **??**. The Desmos code can be accessed at `https://www.desmos.com/calculator/rtnbrdojzh`

3.1. The Fourier Transform.

If you are given a piecewise continuous function, $f(x)$, then how should you approximate $f(x)$ as a Fourier series?

The answer is **the Fourier Transform**. Explicitly, you can compute the coefficients

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \tag{1}$$

$$c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \tag{2}$$

$$s_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \tag{3}$$

for $k = 1, 2, 3, \ldots, N$ and you would find that

$$f_N(x) = \sum_{k=0}^{N} c_k \cos(kx) + \sum_{k=1}^{N} s_k \sin(kx)$$

is a good approximation of $f(x)$ for large N.

Formally, this means that

$$f(x) = \lim_{N\to\infty} f_N(x)$$

for all $x \in [-\pi, \pi]$ except at the discontinuities.

4. Homework Problems

4.1. **Problem 1.** Consider the function $f(x) = x$ on the domain $[-\pi, \pi]$. Compute a 6th order Fourier approximation of this function. In other words: compute $c_0, c_1, c_2, \ldots c_6$ as well as $s_1, s_2, \ldots, s_6$.

BONUS: Compute the coefficient s_k and c_k for arbitrary k.

4.2. **Problem 2.** Assume we are given a Fourier series

$$q(x) = \sum_{k=0}^{\infty} c_k \cos(kx) + \sum_{k=1}^{\infty} s_k \sin(kx)$$

What is the Fourier series of $\frac{dq}{dx}$?

What is the Fourier series of $\frac{d^2q}{dx^2}$?