WEAK DUAL PAIRS AND JETLET METHODS FOR IDEAL INCOMPRESSIBLE FLUID MODELS IN $n \geq 2$ DIMENSIONS

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ABSTRACT. We review the role of dual pairs in mechanics and use them to derive particle-like solutions to regularized incompressible fluid systems. ~~These dual pairs result~~ In our case we have a dual pair resulting from the action of diffeomorphisms on point particles (essentially by moving the points). We then augment our dual ~~pairs~~ pair by considering the action of diffeomorphisms on Taylor series, also known as *jets*. The augmented *weak* dual pairs induce a hierarchy of particle-like solutions and conservation laws with particles carrying a copy of a jet group. We call these augmented particles *jetlets*. The jet groups serve as finite-dimensional models of the diffeomorphism group itself, and so the jetlet particles serve as a finite-dimensional model of the self-similarity exhibited by ideal incompressible fluids. The conservation law associated to jetlet solutions is shown to be a shadow of Kelvin's circulation theorem. Finally, we study the dynamics of infinite time particle mergers. We prove that two merging particles at the zeroth level in the hierarchy yield dynamics which asymptotically approach that of a single particle in the first level in the hierarchy. This merging behavior is then verified numerically as well as the exchange of angular momentum which must occur during a near collision of two particles. The resulting particle-like solutions suggest a new class of meshless methods which work in dimensions $n \geq 2$ and which exhibit a shadow of Kelvin's circulation theorem. More broadly, this provides one of the first finite-dimensional models of self-similarity in ideal fluids.

1. INTRODUCTION

Arnold's geometric insight in [Arn66] has forever changed the way mathematicians look at ideal fluid dynamics. According to [Arn66], ideal incompressible fluid motion on an orientable Riemannian manifold M is equivalent to geodesic motion on the Lie group of volume preserving diffeomorphisms $\mathrm{SDiff}(M)$ (i.e., the smooth invertible volume preserving maps of M into itself, with smooth inverses). The Riemannian metric is simply the fluid kinetic energy, which is the L^2-norm of the fluid's velocity field. This characterization of ideal fluid flow allowed Poisson geometers and geometric mechanicians to provide a new perspective on ideal fluids and other PDEs with hydrodynamics background [EM70, MW83, Zei91, AK98, HMR98, FHT01]. As will be shown, such a perspective is particularly fruitful for the purpose of reducing the infinite-dimensional fluid system to a finite-dimensional ordinary differential equation (e.g. point vortex solutions).

The dimension reduction that will be performed in this paper and the method which it proposes bears much semblance to the point vortex method [Cho73]. In particular, [MW83] illustrated how the point vortex solutions and the conservation of circulation could be derived via a pair of Poisson maps known as a dual pair. One of these Poisson maps was the embedding map from a finite-dimensional manifold into the infinite-dimensional space dual

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to the divergence free vector fields. In other words, the space of point vortices is merely an invariant manifold of ideal fluid motion.[1] It was later realized that the use of dual pairs related to fluid applications was problematic and needed to be relaxed. This lead to the notion of weak dual pairs [GBV12] . In this paper we will derive a hierarchy of different weak dual pairs, in order to obtain a class of finite-dimensionally parametrized solutions, each of which comes with a conservation law that shadows Kelvin's circulation theorem. Just as one can consider the point vortices to be the atoms of the point vortex method, the atoms of these new solutions are particle-like objects which we call *jetlets* (or *k-jetlets* if we wish to emphasize that we are considering the k-th level of the hierarchy).

It is notable that the zeroth level of the hierarchy is a classical particle-like solution which appears in many geodesic systems on diffeomorphism groups [CH93, JM00, FH01, MM13b]. It is also notable that, unlike a point vortex, a jetlet is also well-defined in dimensions greater than two.

1.1. **Main contributions.** In this article we derive a hierarchy of particle-like solutions for a regularized model of ideal fluids described in [MM13b] as motion on the group of volume preserving diffeomorphisms of $\mathbb{R}^n$, denoted $\mathrm{SDiff}(\mathbb{R}^n)$. Each level in the hierarchy consists of particles with internal group variables, parametrized by a finite-dimensional model of a diffeomorphism group known as a jet group [KMS99, Chapter 4]. The jet groups lie at the foundations of certain representation theories of diffeomorphism groups (see [VGG75, Appendix 2] or [Kir81]). Hence, it seems natural to invoke these foundations in the context of fluids, where the configuration manifold is a diffeomorphism group. The particle-like solutions, which "carry" jet groups, paint an intuitive picture of a large scale diffeomorphism that advects particles, each of which carries its own partial description of a local deformation of the fluid in a small region around it. At each higher level in the hierarchy, the description of the deformation becomes more detailed. Thus the jets of diffeomorphisms possess a natural sense of the "self-similarity" which is present in the diffeomorphism group itself. Models such as this are crucial to our understanding of fluids, both from a numerical perspective and from the perspective of fundamental mathematics.

Specifically, we will accomplish the following:

(1) We provide the first explicit Hamiltonian description of the full jet hierarchy of the particle-like solutions discovered in [DJR13, CHJM14].
(2) We compute a nested sequence of conserved quantities at each level in the hierarchy. Each of these conserved quantities will be related to the conservation of circulation.
(3) We numerically compute some particle-like solutions at the zeroth and first levels in the hierarchy. We will observe a form of cascade in which interactions of solutions at the k-th level tend asymptotically in time toward solutions at the $(k+1)$-th level.

1.2. **Outline of the paper.** After the introduction, in Section 2, we outline our strategy: The first goal is to show that jetlet models admit a weak dual pair at each level in the hierarchy. Once this has been done, Theorem A.9 (proven in the appendix) does the rest. Namely, the jetlets satisfy canonical Hamiltonian equations, and the right momentum map is conserved by the flow.

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[1]This is not quite correct, as one must ignore the infinite self-energy terms of point vortices and then extend the space of admissible solutions to non-smooth velocity fields. However, modulo this physically motivated caveat the statement holds.

Section 3 contains a brief discussion of the standard dual pair for the example of the rigid body.

In Section 4 the main development of the paper starts. We first discuss briefly the Lie-Poisson approach to ideal fluids and Euler's equation. This is to set the scene for what follows. In Section 4.1 we discuss the Mumford-Michor model [MM13b] and mention its standard dual pair (in parallel with the rigid body, the two legs of the dual pair correspond to the cotangent lift momentum maps for right and left actions, respectively). We also recall that the conservation of the right momentum map J_R here is equivalent to Kelvin's circulation theorem. Again, the dual pair viewpoint is extremely efficient. Once one has realized that there is a dual pair, one knows (from right-invariance) that the left momentum map J_L maps Hamilton's equations on $T^* \mathrm{SDiff}$ to (reduced) Lie-Poisson form, and at the same time that J_R is conserved.

In Section 4.2 we introduce zeroth order jetlets, also called landmarks, and we discuss their dual pair. This is done in parallel with later sections. The left momentum map is the usual one, and the right momentum map is in fact trivial. So the 'dual pair' is somewhat unnatural here, but it helps present already the kind of thinking that will be employed in the later parts of the paper.

In Section 4.3 we consider first order jetlets. Everything here goes in parallel, except that one has to be more careful when introducing the relevant right and left actions. Once one has the definitions in place, it is not difficult to recognize that the cotangent lifts for the right and left actions lead to, now, a *weak* dual pair of momentum maps. We have to add 'weak' here since the transitivity of the left action on level sets of right momentum is lost, but a weak dual pair is retained, essentially because the group actions still commute, see Definition A.8. Proposition 4.5 collects the results that follow immediately from the weak dual pair.

Throughout the paper we have taken care to explain the intuition behind the more abstract concepts. The discussion just after Proposition 4.5 is an example of this. There, we discuss the relationship between the jetlet solutions on the diffeomorphism group and the resulting trajectory on the space of Taylor jets.

In Section 4.4 we describe jetlets at general levels. Again, once the relevant spaces and actions have been introduced (which is now quite an intricate endeavor), one recognizes that the momentum maps are a weak dual pair by the usual arguments. We give the form of the momentum maps explicitly just after Proposition 4.6. Again, the weak dual pair leads to analogous conclusions about dynamics. Namely, it is canonically Hamiltonian, with J_R conserved; J_L maps the dynamics to the Lie-Poisson dynamics on the one-form densities.

In Section 4.5 we discuss in more detail the Kelvin circulation theorem. More precisely, we discuss the relationship between the 'standard' circulation theorem for the fluid and the conserved momentum maps J_R at the various levels of jetlets. The main result is represented schematically and proven in detail (all that is required, in essence, is the standard formula for cotangent lift momentum maps). Intuitively speaking, what we show is that the conserved jetlet momentum maps are 'shadows' of the 'full' right momentum map of the fluid.

In Section 6 we discuss the explicit dynamical behavior of the particle model, in particular we study 'collisions' leading to mergers of jetlets. This makes an explicit connection between dynamics in the different levels of the jetlet hierarchy. It can potentially be useful when

simulating jetlet systems: when two particles become close, they can be replaced by a merged state whose momenta do not blow up.

We redo the analysis of Mumford and Michor to find that two 0-jetlets can either merge in infinite time, or bounce off each other. Then, we analyze the asymptotic dynamics of this merged state and show that it coincides with the dynamics of a single 1-jetlet particle. This improves the claim in [CHJM14] which showed the convergence to a 1-jetlet state without explicitly considering the dynamics.

Finally, we suggest a more algebraic interpretation of mergers in the jetlet hierarchy by viewing all levels of the hierarchy as embedded in the larger space $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$, where levels form (part of) the boundaries of other levels (a bit like in a CW-complex).

In summary, the numerics section 6 shows first of all that it is feasible to implement this jetlet model numerically, also in dimensions higher than two. Further, we corroborate the analytical results and confirm the conserved quantities and jetlet particle merging behavior numerically. Several different experiments show that the merging/scattering behavior persists under various perturbations that cannot be studied analytically anymore. It also shows how J_L (i.e. angular and linear) momentum is exchanged in the jet-particle collisions.

~~The Appendix is relatively extensive. A quick~~ We provide a detailed appendix. In Appendix A, a brief discussion of symplectic and Poisson manifolds is followed by the definition of weak dual pairs. Then we prove Theorem A.9 ~~and Theorem ??, which are the core results~~, which is the core result used in the paper. Appendix B presents an overview over the spaces used in the paper and relates them to some general results of reduction theory, see in particular Figures 6 and 7. In Appendix C we describe the index conventions used in the main texts (e.g., when calculating momentum maps for the general jetlet solutions), while Appendix D briefly discusses the more abstract point of how the dual space of vector fields can be viewed as tensor product of 1-forms and distributions. Finally, appendix E provides information describing the equations of motion for 1-jets, but in reduced coordinates, which arise after the reduction eliminating the conserved quantities J_R.

1.3. Previous work.

Lagrangian models of ideal fluids such as *smooth particle hydrodynamics* [GM77, Luc77] and *vortex methods* [Cho73], do not exhibit structures which express the nested character of the diffeomorphism group. One means of obtaining a Lagrangian model with a nested structure was recently presented in [DJR13] where a sequence of infinite-dimensional reductions by symmetry was executed, to produce a hierarchy of finite-dimensional systems from a regularized fluid model. The finite-dimensional systems were particle-like solutions in which each particle carries a model of the diffeomorphism group, known as jet groups. Thus, [DJR13] derived a Lagrangian analog of "whirls within whirls", similar to the Eulerian models proposed in [HT12]. However, the specifics of the regularized fluid model were not determined, and the analysis of [DJR13] was purely formal.

Later, a regularized version of the ideal fluid equations was presented by [MM13b]. This new partial differential equation was amenable to the theory presented in [DJR13], and gave rise to a specific and easily implementable manifestation of the hierarchy of particle models described there. In [CHJM14] we numerically computed some of these particle-like solutions and observed cascade phenomena as an emergent behavior at the zeroth level in the hierarchy.

It is notable that the zeroth level of the hierarchy has been studied ~~for a long time~~ in the context of partial differential equations with hydrodynamic background. In particular,

[HR06] provide the first convergence proof of such a method in the context of the Camassa-Holm equation. This same idea was implemented for the n-dimensional Camassa-Holm equation in [CDTM12]. In the context of image registration algorithms, the need to obtain compressible diffeomorphisms motivated the use of particle methods in a similar framework [JM00]. These methods, designed for a wide array of PDEs are studied analytically in [TY05], which also contains a proof of well-posedness for a range of PDEs. Finally, in the context of image registration, [SNDP13] discovered a compressible fluid version of the hierarchy derived in [DJR13], and numerically integrated solutions in the first level of the hierarchy.

1.4. **Notation.** We will let $\mathfrak{X}(\mathbb{R}^n)$ denote the space of H^∞ vector fields, and we let $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)$ denote the space of divergence free vector fields resulting from the Hodge decomposition. We let $\mathrm{Diff}(\mathbb{R}^n)$ denote the space of H^∞ diffeomorphisms of $\mathbb{R}^n$ (see [MM13a]) and we let $\mathrm{SDiff}(\mathbb{R}^n)$ denote the subgroup of volume preserving diffeomorphisms.

Various different types of indices will be used throughout this paper. To distinguish between the types, we keep the following conventions:

- indices $a, b, c, \ldots$ label particles and range from 1 to N;
- indices $i, j, k, \ldots$ label space coordinates and range from 1 to n, the dimension of space;
- superscript indices (k) denote the order of jets in a coordinate-free representation.

2. Main approach

~~In this section we describe~~ Let us begin by describing our strategy for obtaining particle-like solutions to regularized fluid equations. Our main framework is that of Hamiltonian mechanics and symplectic geometry. In particular, our main hammer is Theorem A.9 (see page 33 in the Appendix), which we repeat here for convenience.

> **Theorem A.9** Let P_1 and P_2 be Poisson manifolds and let S be a symplectic manifold. Let $J_1, J_2 : S \to P_1, P_2$ form a ~~dual pair (see definition on page 33)~~ weak dual pair, see Definition A.8. Let $h \in C^1(P_1)$. If $(q, p)(t) \in S$ is a solution to Hamilton's equations with respect to the Hamiltonian $H = h \circ J_1$, then $J_1\left((q, p)(t)\right) \in P_1$ is a solution to Hamilton's equations on P_1 with respect to h, and $J_2((q, p)(t))$ is constant in time.

We will leverage this theorem in the following way. We will find a (weak) dual pair, $J_1, J_2 : S \to \mathfrak{g}^*$, where

(1) S is the space of particle locations and momenta (and, for the higher orders of the hierarchy, internal group variables),
(2) $\mathfrak{g}$ is the space of vector fields,
(3) and h is the Hamiltonian of a regularized model of ideal fluid (i.e. a kinetic energy).

Theorem A.9 applied to the (weak) dual pair J_1, J_2 then yields particle-like solutions and conserved quantities to the equations of motion of a regularized model of an ideal incompressible fluid.

3. Example: the rigid body

In this section we review basic notions from classical mechanics by studying the motion of a rigid body whose center of mass rests at the origin. We will see a first application of Theorem A.9 in this context. The configuration of a rigid body is described by a rotation

matrix $R \in \mathrm{SO}(3)$. The equations of motion are given by Hamilton's equations on the cotangent bundle $T^* \mathrm{SO}(3)$. There are canonical coordinates on $T^* \mathrm{SO}(3)$ given by (R, P) where P is such that $R^T P$ is a 3×3 anti-symmetric matrix[2]. The angular momentum in the body frame is the unique vector $\Pi \in \mathbb{R}^3 \cong \mathfrak{so}(3)^*$ such that

$$J_R(R, P) := R^T P = \begin{bmatrix} 0 & -\Pi_3 & \Pi_2 \\ \Pi_3 & 0 & -\Pi_1 \\ -\Pi_2 & \Pi_1 & 0 \end{bmatrix}.$$

Denoting this map from (R, P) to Π by J_R, the Hamiltonian can be written as a function of Π. In particular, the reduced Hamiltonian is

$$h(\Pi) = \frac{1}{2}\Pi \cdot \mathbb{I}^{-1} \cdot \Pi,$$

where $\mathbb{I}$ is a non-degenerate 3×3 symmetric matrix known as the moment of inertia matrix. The unreduced Hamiltonian is $H(R, P) = h(J_R(R, P))$.

The group $\mathrm{SO}(3)$ acts upon itself by left multiplication. This action can be lifted to an action on $T^* \mathrm{SO}(3)$ given by

$$(R, P) \in T^* \mathrm{SO}(3) \overset{g \in \mathrm{SO}(3)}{\mapsto} (gR, gP) \in T^* \mathrm{SO}(3).$$

It is notable that $\Pi = R^T P = R^T g^T gP = (gR)^T(gP)$ is unaltered by this transformation. Therefore the Hamiltonian, $h(\Pi)$, is invariant under this action of $\mathrm{SO}(3)$. By Noether's theorem there is a conserved quantity associated to this symmetry. The conserved quantity is manifested by the *momentum map*

$$J_L(R, P) = PR^T.$$

Moreover, due to this symmetry, one can write the evolution equations on a lower-dimensional space. The very fact that the Hamiltonian is written in terms of Π suggests that the equations of motion can be written in terms of Π alone. Indeed this is the case,

$$(1) \qquad\qquad\qquad\qquad \dot{\Pi} = \Pi \times (\mathbb{I}^{-1}\Pi).$$

This equation can be seen as a Hamiltonian equation on $\mathbb{R}^3$ with respect to the non-canonical Poisson bracket

$$\{F, G\}_{\mathrm{Nambu}}(x) = -x \cdot (\nabla F \times \nabla G)$$

known as the *Nambu bracket*. This is no coincidence. Let us first introduce the isomorphism of Lie algebras $(\mathbb{R}^3, \times)$ and $\mathfrak{so}(3)$, given by the hat map

$$(2) \qquad\qquad x \in \mathbb{R}^3 \mapsto \hat{x} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \in \mathfrak{so}(3).$$

Proposition 3.1 (see [Hol11, §2.5]). *The Nambu bracket on $\mathbb{R}^3$ is identified to the Lie–Poisson bracket on $\mathfrak{so}(3)^*$ through the hat map isomorphism (2) in the sense that $\{f, g\}_{\mathrm{Nambu}}(x) = \{\hat{f}, \hat{g}\}_{\mathrm{LP}}(\hat{x})$ where $f, g \in C^1(\mathbb{R}^3)$ and $\hat{f}, \hat{g} \in C^1(\mathfrak{so}(3)^*)$ are defined by $\hat{f}(\hat{x}) = f(x), \hat{g}(\hat{x}) = g(x).$*

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[2]We use the pairing $\langle P, v \rangle = \mathrm{tr}(P^T v)$ for $P \in T_R^* SO(3)$ and $v \in T_R SO(3)$. More generally, the pairing $\langle \cdot, \cdot \rangle$ will always refer to the natural pairing between a vector space and its dual in this paper.

Proof. The Lie–Poisson bracket on $\mathfrak{so}(3)^*$ is

$$\{\hat{f}, \hat{g}\}_{\mathrm{LP}}(\hat{\Pi}) = \left\langle \hat{\Pi}, \left[d\hat{f}(\hat{\Pi}), d\hat{g}(\hat{\Pi}) \right] \right\rangle,$$

for arbitrary functions $\hat{f}, \hat{g} \in C^1(\mathfrak{so}(3)^*)$. There exists functions $f, g \in C^1(\mathbb{R}^3)$ related to $\hat{F}, \hat{G}$ through the hat map. One can observe that $d\hat{f}(\hat{\Pi}) \in \mathfrak{so}(3)$ is related to $\nabla f(\Pi)$ through the relation $\widehat{\nabla f(\Pi)} = d\hat{f}(\hat{\Pi})$. We see that the commutator bracket satisfies

$$[\hat{x}, \hat{y}] := \hat{x}\hat{y} - \hat{y}\hat{x} = \widehat{x \times y}.$$

Therefore the Lie–Poisson bracket can be written as

$$\begin{aligned} \{\hat{f}, \hat{g}\}_{\mathrm{LP}}(\hat{\Pi}) &= \langle \hat{\Pi}, \widehat{\nabla f \times \nabla g}(\Pi) \rangle \\ &= \Pi \cdot (\nabla f \times \nabla g)(\Pi) \\ &= \{f, g\}_{\mathrm{Nambu}}(\Pi). \end{aligned}$$

$\square$

Recall that $T^* \mathrm{SO}(3)$ is a symplectic manifold. ~~It is now easy to check that the conditions of Lemma **??** and Theorem **??** hold, from which we can observe~~ The momentum maps J_L, J_R arise canonically from the left and right action of $\mathrm{SO}(3)$ on $T^* \mathrm{SO}(3)$. Since the actions commute and it can also be checked that J_L and J_R have symplectically orthogonal kernels ~~and~~, it follows from [GBV12, Corollary 2.6] that the diagram

$$\mathfrak{so}(3)^* \xleftarrow{J_L} T^* \mathrm{SO}(3) \xrightarrow{J_R} \mathfrak{so}(3)^*$$

$$PR^T \xleftarrow{J_L} (R, P) \xmapsto{J_R} R^T P$$

is a dual pair (see page 33 for details). The maps J_L and J_R are called *symplectic variables* in [MW83] as they allow one to pull-back calculations on a Poisson manifold to a symplectic manifold.[3]

By Theorem A.9, this dual pair expresses rigid body dynamics and conserved quantities. Specifically, the right leg yields the reduced phase space where the system evolves in time. The left leg yields the conserved quantities of the rigid body associated to the left action of $\mathrm{SO}(3)$ on itself. The most important aspect of these maps is that they are both Poisson maps, i.e. they carry the canonical Poisson bracket on $T^* \mathrm{SO}(3)$ to the Nambu bracket on $\mathfrak{so}(3)^* \cong (\mathbb{R}^3, \times)$, as the following Proposition shows.

Proposition 3.2 (remark 2.5.11 [Hol11]). *Let $\{\cdot, \cdot\}_{\mathrm{can}}$ denote the canonical Poisson bracket on the cotangent bundle $T^* \mathrm{SO}(3)$. Let $\{\cdot, \cdot\}_{\mathrm{Nambu}}$ denote the Nambu bracket on $\mathbb{R}^3$. Both J_L and J_R are Poisson maps. Explicitly, this means*

$$-\{f \circ J_L, g \circ J_L\}_{\mathrm{can}} = \{f, g\}_{\mathrm{Nambu}} \circ J_L$$

$$\{f \circ J_R, g \circ J_R\}_{\mathrm{can}} = \{f, g\}_{\mathrm{Nambu}} \circ J_R$$

for any $f, g \in C^2(\mathbb{R}^3)$.

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[3] [MW83] also referred to J_R and J_L as *Clebsch variables*, however this terminology has changed over the past few decades.

Proof. Let $(R, P) \in T^* \mathrm{SO}(3)$ and set $\Pi = J_R(R, P)$. We observe that

$$\langle \Pi, \Omega \rangle = \mathrm{tr}(\hat{\Pi}^T \hat{\Omega}) = \mathrm{tr}(P^T R \hat{\Omega}) = \langle P, R \cdot \hat{\Omega} \rangle.$$

This tells us that J_R is the momentum map associated to the cotangent lift of the right action of $\mathrm{SO}(3)$ on itself. Cotangent lifts of group actions are always equivariant and thus yield Poisson maps (Theorem 12.4.1 [MR99]). Thus J_R carries the canonical Poisson bracket on $T^* \mathrm{SO}(3)$, to the Lie–Poisson bracket on $\mathfrak{so}(3)^*$. By Proposition 3.1, this is nothing but the Nambu bracket upon identifying $\mathfrak{so}(3)$ with $\mathbb{R}^3$. The same argument applies to J_L using a left action. $\qquad \square$

One can obtain solutions to (1) by solving canonical Hamiltonian equations with respect to $H(R, P)$. In particular, if $(R, P)(t)$ is a solution to Hamilton's equation, then $\Pi(t) = J_R((R, P)(t))$ is a solution to Hamilton's equation with respect to the Nambu bracket. This is a result of Proposition A.7 paired with the observation that J_R is a Poisson map via Proposition 3.2.

4. Regularized fluids

Euler's equations of motion for incompressible fluids can be seen as Hamiltonian equations on the (dual) space of divergence free vector fields [Arn66]. Consider the Lie algebra of vector fields on $\mathbb{R}^n$, denoted by $\mathfrak{X}(\mathbb{R}^n)$. Formally, the dual space to $\mathfrak{X}(\mathbb{R}^n)$ is a Poisson manifold when equipped with the Lie-Poisson bracket (see (27) in Appendix A.2). We may consider the map $\psi : T^* R^n \to \mathfrak{X}(\mathbb{R}^n)^*$ given implicitly by

$$\langle \psi(q, p), u \rangle = p \cdot u(q) \quad \forall u \in \mathfrak{X}(\mathbb{R}^n), (q, p) \in T^* \mathbb{R}^n.$$

Explicitly we may write ψ using the Dirac-delta functional as $\psi(q, p) = p \otimes \delta_q$. It is shown in [HM05] that this map is Poisson. Furthermore, as the n-dimensional Camassa–Holm equation [CH93] is a Hamiltonian equation on $\mathfrak{X}(\mathbb{R}^n)^*$, Proposition A.7 promises to express a certain subset of solutions by solving Hamiltonian equations for a finite number of particles. Specifically, ψ yields the peakon solutions of the n-dimensional Camassa–Holm equation. In this section we explore analogous constructions for an incompressible and regularized version of the Camassa–Holm equation discovered in [MM13b].

In the case where $h_{\mathrm{Euler}}(m) = \frac{1}{2} \|m\|_{L^2}^2$ is the standard fluid kinetic energy on the dual vector space to the incompressible vector fields, $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$, Hamilton's equations are written as

$$(3) \qquad\qquad \partial_t m + \mathcal{L}_u[m] = 0 \quad , \quad u^i = \delta^{ij} m_j.$$

where $\mathcal{L}_u[m]$ is the Lie derivative of m. The primary finding of [Arn66] was that (3) is equivalent to the inviscid fluid equation

$$\partial_t u + u \cdot \nabla u = -\nabla p \quad , \quad \nabla \cdot u = 0.$$

Since we have not yet clarified the Poisson structures of the system, it may not be obvious that (3) is a Hamiltonian equation. The following proposition shows this for a general Hamiltonian on $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$.

Proposition 4.1 ([Arn66])**.** *Let $h \in C^\infty(\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*)$. Recall that $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$ is a Poisson manifold when equipped with the Lie–Poisson bracket, and given a function h, the Fréchet*

derivative $dh(m)$ is an element of $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^{**}$. In the event that $dh(m) \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)$, Hamilton's equations are given by

$$\dot{m} + \mathcal{L}_u[m] = 0 \quad , \quad u = dh(m).$$

Proof. To each $v \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)$ we can associate a linear function on $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$ given by $m \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^* \mapsto \langle m, v \rangle \in \mathbb{R}$. Let us denote this function by f_v. Let $m(t)$ satisfy Hamilton's equations. By the definition of the Lie–Poisson bracket (see (27) in Appendix A.2) we observe

$$\frac{d}{dt} f_v(m) = \{f_v, h\}(m) = \langle m, [df_v(m), dh(m)] \rangle$$
$$= \langle m, \mathcal{L}_{dh(m)}[df_v(m)] \rangle.$$

However $df_v = v$ and so the last line can be equated with $-\langle \mathcal{L}_{dh(m)}[m], v \rangle$. Additionally, we know that $\frac{d}{dt} f_v(m) = \langle \dot{m}, v \rangle$ since v is constant in time. Therefore we find

$$\langle \dot{m} + \mathcal{L}_{dh(m)}[m], v \rangle = 0.$$

As v is arbitrary, this uniquely characterizes $\dot{m}$.[4] The result follows. $\qquad\square$

4.1. **The Mumford–Michor model.** Consider the Hamiltonian

$$h_{p,\sigma}(m) = \frac{1}{2} \langle m, K_{p,\sigma} * m \rangle_{L^2},$$

where $K_{p,\sigma} : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is the matrix valued Green's kernel defined by the property

$$\left(1 - \frac{\sigma^2}{p}\Delta\right)^p \cdot \int_{\mathbb{R}^n} K_{p,\sigma}^{ij}(x - y) m_j(y) dy = \delta^{ij} m_j(x).$$

In this case Hamilton's equations take the form

$$(4) \qquad \partial_t m + \mathcal{L}_u[m] = 0 \quad , \quad u^i = K_{p,\sigma}^{ij} * m_j.$$

Solutions to (4) exhibit existence and uniqueness for all time. Moreover, as $\sigma \to 0$, $h_{p,\sigma} \to h_{\mathrm{Euler}}$ and one can speculate that solutions to (4) approach solutions to the ideal fluid equation (3). In fact, this is the case over short times, and for $\sigma > 0$ solutions of (4) differ from those of (3) by an amount σt in the H^k-norm. Thus Hamilton's equations with respect to $h_{p,\sigma}$ have been proposed as a model for ideal fluids [MM13b, Theorems 2 and 3]. From now on we shall often suppress the parameters p, σ and shorten $K = K_{p,\sigma}$ to prevent index clutter.

Next, we discuss a dual pair for this system. The group $\mathrm{SDiff}(\mathbb{R}^n)$ acts on itself from the left and from the right. These actions can be lifted to $T^* \mathrm{SDiff}(\mathbb{R}^n)$, and yield momentum maps $J_L, J_R : T^* \mathrm{SDiff}(\mathbb{R}^n) \to \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$. In particular these maps form the dual pair

$$\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^* \xleftarrow{\ J_L\ } T^* \mathrm{SDiff}(\mathbb{R}^n) \xrightarrow{\ J_R\ } \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*.$$

For a mathematically rigorous treatment of this dual pair we refer to [GBV12].

By Theorem A.9, we can use this dual pair to derive dynamical properties of Hamiltonian equations defined on $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$. The Hamiltonian for a fluid is written on the left instance of $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$. One can (in principle) solve Hamilton's equations on $T^* \mathrm{SDiff}(\mathbb{R}^n)$ with respect to the Hamiltonian $H = h \circ J_L$. This yields the *material* or *Lagrangian* coordinate perspective

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[4]This is *not* a "weak" characterization. The entity $\dot{m}$ is contained in the dual space to $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)$ and it is therefore *defined uniquely* by how it acts on $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)$.

of fluid mechanics one encounters in a first course on continuum mechanics. The right leg yields conserved quantities associated to the particle relabeling symmetry of the fluid. It was found in [Arn66] that these conserved momenta are identical to the law of conservation of circulation, a.k.a. *Kelvin's circulation theorem.*

Unfortunately, this dual pair does not help us in solving (4) since solving Hamilton's equations on $T^* \operatorname{SDiff}(\mathbb{R}^n)$ is no less difficult than solving Hamilton's equations on $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$. In the next section we will derive a dual pair wherein the symplectic manifold is more reasonable. This will yield the particle-like solutions described in [MM13b]. In the later parts of the paper we will generalize the treatment to obtain a *weak* dual pair for each level in the hierarchy of particle-like solutions.

4.2. Particle-like solutions.
There is a natural left group and algebra action of $\operatorname{SDiff}(\mathbb{R}^n)$ and $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)$, respectively, on $\mathbb{R}^n$ given by

$$q \in \mathbb{R}^n \stackrel{\varphi \in \operatorname{SDiff}(\mathbb{R}^n)}{\longmapsto} \varphi(q) \in \mathbb{R}^n$$

$$q \in \mathbb{R}^n \stackrel{u \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)}{\longmapsto} u(q) \in T_q \mathbb{R}^n.$$

The tangent lift of the former is defined in the obvious way by sending

$$(q, v) \in T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \stackrel{\varphi \in \operatorname{SDiff}(\mathbb{R}^n)}{\longmapsto} \left(\varphi(q), (\partial_j \varphi^i(q) v^j) \partial_i \right) \in T_{\varphi(q)} \mathbb{R}^n.$$

The cotangent lift is defined by taking the dual of the tangent lifted action. That is to say,

$$(q, p) \in T^* \mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \stackrel{\varphi \in \operatorname{SDiff}(\mathbb{R}^n)}{\longmapsto} \left(\varphi^{-1}(q), (\partial_i \varphi^j(q) p_j) e^i \right),$$

where $\{e^i\}$ forms the dual basis to $\{\partial_i\}$ at $\varphi^{-1}(q)$. The momentum map $J_L^{(0)} : T^* \mathbb{R}^n \to \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$ associated to this left action is defined by the condition

$$\langle J_L^{(0)}(q, p), u \rangle := \langle p, u(q) \rangle,$$

for all $u \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)$ and $(q, p) \in T^* \mathbb{R}^n$ (see (26) in Appendix A.1); the superscript in $J_L^{(0)}$ serves as a reminder that we are considering the zeroth level in the hierarchy of particle-like solutions. We see that $J_L^{(0)}(q, p)$ is an evaluation operator, and we can write it more explicitly as a measure-valued momentum map in terms of the Dirac-delta distribution as

$$J_L^{(0)}(q, p) = p \otimes \delta_q,$$

where δ_q is the Dirac-delta distribution on $\mathbb{R}^n$ centered at q. This identification holds modulo $dC^1(\mathbb{R}^n) \otimes d\mathbf{x}$ where $dC^1(\mathbb{R}^n)$ is the space of C^0-exact one-forms and $d\mathbf{x}$ is the canonical volume ~~from~~ form on $\mathbb{R}^n$ (see Appendix D). Since $J_L^{(0)}$ is a ~~cotangent-lift~~ cotangent lift momentum map, it is equivariant and therefore Poisson, again by [MR99, Thm 12.4.9].

We define the manifold for particles,

$$Q_N^{(0)} = \{ (q_1, \dots, q_N) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n \mid q_a \neq q_b \text{ when } a \neq b \}.$$

We will index the particles with $a, b, c, \dots$ and Cartesian coordinate directions in $\mathbb{R}^n$ with indices $i, j, k, \dots$ Thus each $q \in Q_N^{(0)}$ can be decomposed into N particles as $(q_1, \dots, q_N)$, where each $q_a \in \mathbb{R}^n$ and the i-th coordinate of the a-th particle is denoted by $q_a{}^i$.
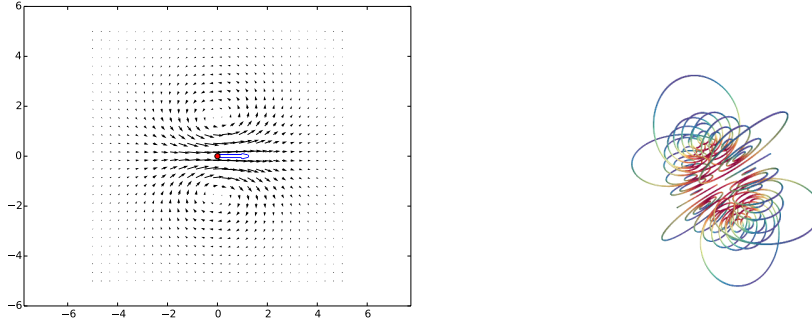
FIGURE 1. A 0-jetlet with momentum $m = (1,0)^T \otimes \delta_0$ in dimensions 2 and 3. (*left*) Quiver plot of the induced 2 dimensional velocity field. (*right*) Streamline plot of the induced 3 dimensional velocity field.

The group $\mathrm{SDiff}(\mathbb{R}^n)$ acts on $Q_N^{(0)}$ by the diagonal action. Through the same manipulations as we applied previously we obtain the momentum map for N particles given by

$$J_L^{(0)}(q,p) = p_a \otimes \delta_{q_a},$$

where a sum over repeated indices is implied.

By Propositions A.7 and 4.1, we obtain solutions to Hamilton's equations on $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$, by solving Hamilton's equations on $T^*\mathbb{R}^n$ if $dh(J_L^{(0)}(q,p))$ is a vector field. If $h = h_{p,\sigma}$, then we calculate that dh evaluated on the $J_L^{(0)}(p,q)$ is the vector field

$$dh(p_a \otimes \delta_{q_a})(x) = K^{ij}(x - q_a)p_{aj}\frac{\partial}{\partial x^i}.$$

This is a vector field whose differentiability is determined completely by that of the kernel, K. Once one has found a solution $(q(t), p(t))$ to Hamilton's equations on $T^*\mathbb{R}^n$ and thus also a solution to Hamilton's equations on $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$, one can proceed to integrate the corresponding time-dependent vector field to obtain the fluid motion φ_t in $\mathrm{SDiff}(\mathbb{R}^n)$. It is natural (but not mandatory) to choose the initial map φ_0 to be the identity or at least to be a diffeomorphism that satisfies $\varphi_0(q_a(0)) = q_a(0)$ for all a. If this choice is made, one can interpret the curve $q(t) = (q_1(t), \dots, q_N(t))$ in $Q_N^{(0)}$ as the locations of particles as they are swept along by the fluid flow, that is, $q_a(t) = \varphi_t(q_a(0))$ for all $a = 1, \dots, N$.

We see that $J_L^{(0)}$ is injective, and thus has a trivial kernel. As a result, the symplectic orthogonal to the kernel of $J_L^{(0)}$ is the full tangent bundle $T(T^*Q_N^{(0)})$. Hence, if we define the (trivial) map $J_R^{(0)} : T^*Q_N^{(0)} \to \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$ by $J_R^{(0)} \equiv 0$, it follows that the diagram

$$\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^* \xleftarrow{J_L^{(0)}} T^*Q_N^{(0)} \xrightarrow{J_R^{(0)}} \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$$

$$p_a \otimes \delta_{q_a} \xmapsfrom{J_L^{(0)}} (q,p) \xmapsto{J_R^{(0)}} 0$$

is a (proper) dual pair. This dual pair allows us to express conservation laws and dynamics as a result of Theorem A.9. Namely, the left leg represents the space in which particle-like solutions to (4) evolve, while the right leg represents a (trivial) conserved quantity.

In order to make contact with the later parts of the paper, it is useful to remark that $J_R^{(0)}$ can formally be understood as the cotangent lift momentum map associated with a certain (trivial) group action. To that end, we fix a designated point $z = (z_1, \ldots, z_N) \in Q_N^{(0)}$ and take an arbitrary element $q = (q_1, \ldots, q_N) \in Q_N^{(0)}$ to represent the set of all $\varphi \in \mathrm{SDiff}(\mathbb{R}^n)$ that satisfy $\varphi \cdot z = q$. That is, $\varphi(z_a) = q_a$ for all a. This means in particular that the specification of $(q_1, \ldots, q_N)$ fixes the zeroth order Taylor expansion (at the locations z_a) of the corresponding set of diffeomorphisms. With this in mind, let us define the isotropy group

$$(5) \qquad \mathrm{iso}(z) = \{\psi \in \mathrm{SDiff}(\mathbb{R}^n) | \psi(z_a) = z_a \text{ for all } a\}$$

and a (trivial) right action on $Q_N^{(0)}$ where $\psi \in \mathrm{iso}(z)$ maps an element $(\varphi(z_1), \ldots, \varphi(z_N))$ to $(\varphi \circ \psi(z_1), \ldots, \varphi \circ \psi(z_N))$. Similar constructions will be crucial in later sections, when constructing the right leg of ~~a~~ an — in this case weak — dual pair for the higher levels in the hierarchy of particle-like solutions.

In summary we find:

Proposition 4.2 (§7 of [MM13b]). *Let $p \geq \frac{n}{2} + 1$ and $\sigma > 0$. Let $H : T^*Q_N^{(0)} \to \mathbb{R}$ be the function*

$$H(q, p) = \frac{1}{2} K_{p,\sigma}^{ij}(q_a - q_b)\, p_{a\,i}\, p_{b\,j}.$$

If $(q, p)(t)$ is a solution to Hamilton's equations, then $m(t) = p_a(t) \otimes \delta_{q_a(t)}$ is a solution to (4).

Proof. Note that $H = h_{p,\sigma} \circ J_L^{(0)}$ and apply ~~Theorems A.9 and ??~~ Theorem A.9. See the remark after ~~Proposition ??~~ Theorem 4.8 for more details on the kernel smoothness condition on p. $\qquad\square$

4.3. **First order particle-like solutions.** In this section we revisit the first order particle-like solutions of [CHJM14] and discuss their weak dual pair, before extending the treatment to the higher levels of the hierarchy in the subsequent section. Let $\mathrm{SL}(n)$ denote the Lie group of $n \times n$ matrices with unit determinant. Let $q = (q^{(0)}, q^{(1)}) \in \mathbb{R}^n \times \mathrm{SL}(n)$ and consider the left $\mathrm{SDiff}(\mathbb{R}^n)$ action on $\mathbb{R}^n \times \mathrm{SL}(n)$ given by

$$(6) \qquad \varphi \cdot q = (\varphi(q^{(0)}), D\varphi(q^{(0)}) \cdot q^{(1)}).$$

Where $D\varphi(q^{(0)}) \cdot q^{(1)}$ is the result of multiplying the Jacobian matrix $D\varphi(q^{(0)})$ with $q^{(1)}$.

Proposition 4.3. *The action of $\mathrm{SDiff}(M)$ on $\mathbb{R} \times \mathrm{SL}(n)$ in (6) is a group action.*

Proof. As $\varphi \in \mathrm{SDiff}(\mathbb{R}^n)$, it follows that $D\varphi|_{q^{(0)}} \in \mathrm{SL}(n)$. Therefore $D\varphi|_{q^{(0)}} \cdot q^{(1)} \in \mathrm{SL}(n)$. Secondly, if $\varphi_1, \varphi_2 \in \mathrm{SDiff}(\mathbb{R}^n)$ we observe that

$$\begin{aligned}
\varphi_2 \cdot (\varphi_1 \cdot q) &= \varphi_2 \cdot \left(\varphi_1(q^{(0)}), D\varphi_1|_{q^{(0)}} \cdot q^{(1)}\right) \\
&= \left(\varphi_2(\varphi_1(q^{(0)})), D\varphi_2|_{\varphi_1(q^{(0)})} \cdot D\varphi_1|_{q^{(0)}} \cdot q^{(1)}\right) \\
&= \left((\varphi_2 \circ \varphi_1)(q^{(0)}), D(\varphi_2 \circ \varphi_1)|_{q^{(0)}} \cdot q^{(1)}\right) \\
&= (\varphi_2 \circ \varphi_1) \cdot q,
\end{aligned}$$

where the second and third lines are applications of the chain rule. $\qquad\square$

As before, this action can be lifted to the cotangent bundle $T^*(\mathbb{R}^n \times \mathrm{SL}(n))$. Specifically, the action of the diffeomorphism φ^{-1} is given by

$$(7) \quad (T\varphi)^* \cdot (q^{(0)}, q^{(1)}, p^{(0)}, p^{(1)}) = \left(\varphi^{-1}(q^{(0)}), [D\varphi|_{q^{(0)}}]^{-1} \cdot q^{(1)}, D\varphi|_{q^{(0)}}^* \cdot p^{(0)}, D\varphi|_{q^{(0)}}^* \cdot p^{(1)}\right).$$

Also as before, we can generalize this construction to the space of $N > 1$ particles by considering the space

$$Q_N^{(1)} = \left\{ (q_1, \ldots, q_N) \mid \begin{array}{l} q_a = (q_a^{(0)}, q_a^{(1)}) \in \mathbb{R}^n \times \mathrm{SL}(n) \\ q_a^{(0)} \neq q_b^{(0)} \text{ when } a \neq b \end{array} \right\}.$$

For convenience it is nice to choose coordinates at this point. If we let q_a denote the position of the a-th particle, $q_a{}^i$ denote the i-th component of this position, and let $q_a{}^i{}_j$ denote the (i, j) entry of the a-th matrix then the resulting momentum map is defined by the condition

$$\langle J_L^{(1)}(q, p), u \rangle = p_{ai} u^i(q_a^{(0)}) + p_{ai}{}^j \partial_k u^i(q_a^{(0)}) q_a{}^k{}_j$$

for an arbitrary $u \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)$. In terms of the Dirac-delta functional, we can write $J_L^{(1)}$ as the measure-valued momentum map

$$J_L^{(1)}(q, p) = p_{ai} \mathrm{d}x^i \otimes \delta_{q_a^{(0)}} - p_{ai}{}^j q_a{}^k{}_j \mathrm{d}x^i \otimes \partial_k \delta_{q_a^{(0)}}.$$

Next, we construct a dual momentum map associated with a right action on $Q_N^{(1)}$. Analogous to the previous section it is useful at this stage to write elements of $Q_N^{(1)}$ in the form $\varphi \cdot z = (\varphi(z_1, \mathbf{1}), \ldots, \varphi \cdot (z_N, \mathbf{1}))$, $\varphi \in \mathrm{SDiff}(\mathbb{R}^n)$, for some designated element $z = ((z_1, \mathbf{1}), \ldots, (z_N, \mathbf{1}))$. Clearly, every element of $Q_N^{(1)}$ can be written in this form for some $\varphi \in \mathrm{SDiff}(\mathbb{R}^n)$. Indeed, with this convention the specification of an element in $Q_N^{(1)}$ fixes the first order Taylor expansion of φ at the locations z_a, $a = 1, \ldots, N$. With this in mind, let us recall the isotropy group $\mathrm{iso}(z)$ defined earlier in (5), which leaves these locations invariant, and define a right action on $Q_N^{(1)}$ given by

$$\textcolor{red}{(\varphi \cdot z) \cdot \psi = (\varphi \circ \psi) \cdot z.}$$

$$\textcolor{blue}{(\varphi \cdot z) \cdot \psi = (\varphi \circ \psi) \cdot z.}$$

That is,

$$\textcolor{red}{(q^{(0)}, q^{(1)}) \cdot \psi = \left(\left(q_1^{(0)}, q_1^{(1)} \cdot D\psi|_{z_1}\right), \ldots, \left(q_N^{(0)}, q_N^{(1)} \cdot D\psi|_{z_N}\right)\right).}$$

$$(8) \qquad \textcolor{blue}{(q^{(0)}, q^{(1)}) \cdot \psi = \left(\left(q_1^{(0)}, q_1^{(1)} \cdot D\psi|_{z_1}\right), \ldots, \left(q_N^{(0)}, q_N^{(1)} \cdot D\psi|_{z_N}\right)\right).}$$

In Proposition 4.6 we will generalize this construction to define a right action for the higher levels in the hierarchy of particle-like solutions. The action (8) yields the ~~cotangent-lift~~ ~~cotangent lift~~ momentum map defined by the condition

$$\langle J_R^{(1)}(q, p), u \rangle = p_{ai}{}^j q_a{}^i{}_k \partial_j u^k(z_a).$$

In terms of the Dirac-delta functional we may write this as the measure-valued momentum map

$$(9) \qquad J_R^{(1)}(q, p) = -p_{ai}{}^j q_a{}^i{}_k \mathrm{d}z^k \otimes \partial_j \delta_{z_a}$$

Proposition 4.4. *The momentum maps $J_L^{(1)}$ and $J_R^{(1)}$ form a weak dual pair.*

~~We first check the conditions of Lemma ??. Note that $\mathrm{SDiff}(\mathbb{R}^n)$ acts transitively on $Q_N^{(1)}$ since it acts transitively on representatives $\psi \in \mathrm{SDiff}(\mathbb{R}^n)$ of elements q (i.e. $q_a^{(0)} = \psi(z_a)$ and $q_a^{(1)} = D\psi|_{z_a}$, or, $q = \mathrm{Jet}_z^1(\psi)$). Secondly, the cotangent-lifted action preserves level sets of $J_R^{(k)}$. From we see that $\varphi \in \mathrm{SDiff}(\mathbb{R}^n)$ acts on the first level of $(q, p) \in T^*Q_N^{(1)}$ as~~

$$(\varphi \cdot q)_a{}^i{}_j = (D\varphi|_{q_a^{(0)}})^i{}_k \, q_a{}^k{}_j,$$

$$(\varphi \cdot p)_{ai}{}^j = p_{ak}{}^j (D\varphi^{-1}|_{\varphi \cdot q_a^{(0)}})^k{}_i.$$

~~The explicit calculation~~

$$J_R^{(1)}(\varphi \cdot (q,p)) = -p_{al}{}^j (D\varphi^{-1}|_{\varphi \cdot q_a^{(0)}})^l{}_i \, (D\varphi|_{q_a^{(0)}})^i{}_m \, q_a{}^m{}_k \mathrm{d}z^k \otimes \partial_j \delta_{z_a}$$

$$= -p_{al}{}^j \, \delta_m^l \, q_a{}^m{}_k \mathrm{d}z^k \otimes \partial_j \delta_{z_a} = J_R^{(1)}(q,p)$$

~~shows that is preserved under the cotangent-lifted action of $\mathrm{SDiff}(\mathbb{R}^n)$. Finally, we must show that $(J_R^{(1)})^{-1}(m)$ is a section of $T^*Q_N^{(1)}$ for any m in the range of $J_R^{(1)}$. Let $(q, p) \in (J_R^{(1)})^{-1}(m)$. Note that for fixed q, since $q_a^{(1)} \in \mathrm{SL}(n)$ is a linear isomorphism, the map $p \mapsto J_R^{(1)}(q, p)$ is an injection and hence $(J_R^{(1)})^{-1}(m) \cap T_q^*Q$ contains a unique element. Since $\mathrm{SDiff}(\mathbb{R}^n)$ acts transitively on the base $Q_N^{(1)}$ and acts by fiber-preserving diffeomorphisms on $T^*Q_N^{(1)}$, it follows that $(J_R^{(1)})^{-1}(m)$ is a section of $T^*Q_N^{(1)}$.~~
~~Thus Lemma ?? implies that $\mathrm{SDiff}(\mathbb{R}^n)$ acts transitively on the level sets of the momentum map $J_R^{(1)}$. The actions of $\mathrm{iso}(z)$ and $\mathrm{SDiff}(\mathbb{R}^n)$ clearly commute so Theorem ?? implies that $J_L^{(1)}$ and $J_R^{(1)}$ form a dual pair.~~ We will postpone the proof, as this is a special case of a proposition which comes later in the paper (Proposition 4.7).

As before, the quantity $dh(J_L^{(1)}(q, p))$ is a legitimate vector field if $K_{p,\sigma}$ is sufficiently smooth. Moreover, when the Hamiltonian $H^{(1)} = h_{p,\sigma} \circ J_L^{(1)}$ is C^1 we may evolve Hamilton's equations to obtain solutions.

Proposition 4.5. *Let $p \geq \frac{n}{2} + 2$ and $\sigma > 0$. Then $H^{(1)} = h_{p,\sigma} \circ J_L^{(1)}$ is C^1 and given by the expression*

$$H^{(1)}(q,p) = \frac{1}{2} p_{ai} K^{ij}(q_a - q_b) p_{bj}$$
$$+ p_{ai}{}^l q_a{}^k{}_l (\partial_k K^{ij})(q_a - q_b) p_{bj}$$
$$- \frac{1}{2} p_{ai}{}^n q_a{}^l{}_n (\partial_l \partial_k K^{ij})(q_a - q_b) q_b{}^k{}_m p_{bj}{}^m.$$

If $(q, p)(t) \in T^*Q_N^{(1)}$ is a solution to Hamilton's equations with respect to $H^{(1)}$, then $J_L^{(1)}((q,p)(t)) \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$ is a solution of (4) and $J_R^{(1)}((q,p)(t)) \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$ is constant in time.

The proof of the above proposition is identical to that of Proposition 4.2. See again the remark following ~~Proposition ??~~ Theorem 4.8 for more details on the kernel smoothness condition on p.
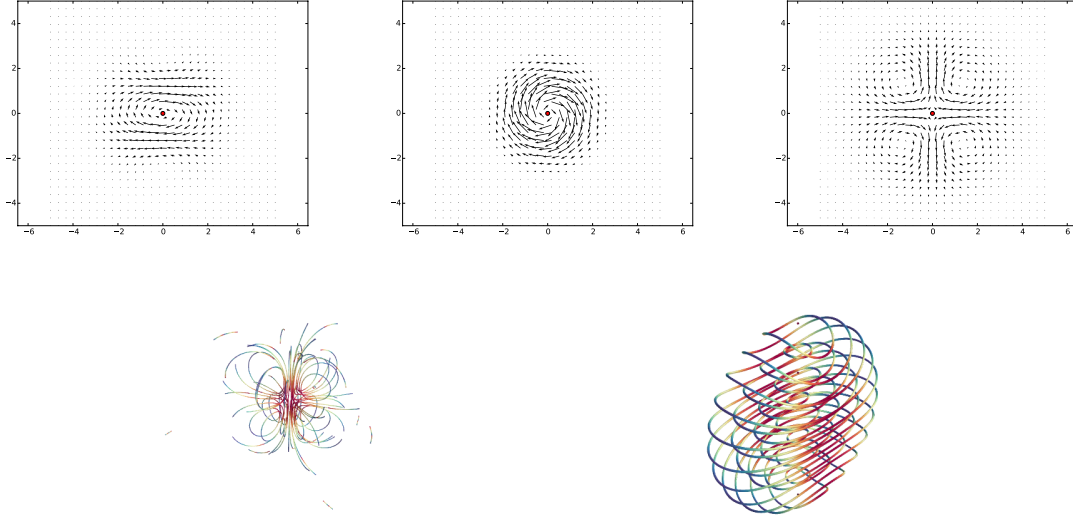
FIGURE 2. (*top*) Quiver plots of 2D vector fields induced by first order jetlets. (*bottom*) Streamline plots of 3D vector fields induced by first order jetlets.

As before, it is useful to interpret the trajectory $q(t)$ of the previous proposition in terms of the curve $\varphi_t \in \text{SDiff}(\mathbb{R}^n)$ obtained by integrating the time-dependent vector field $dh_{p,\sigma} J_L^{(1)}((q,p)(t)) = K_{p,\sigma} * J_L^{(1)}((q,p)(t))$. If one chooses φ_0 to be the identity or any other element of $\text{SDiff}(\mathbb{R}^n)$ that satisfies $\varphi_0 \cdot (q(0)) = q(0)$, then $q(t) = \varphi_t \cdot q(0)$. This implies in particular that the $q_a^{(0)}(t)$ are the trajectories of the particles as they are swept along by the fluid flow.

Various vector fields for large kernel smoothness p are depicted in Figure 2 for different initial values of the traceless matrix $\mu_i{}^j = p_i{}^l q^j{}_l$.

4.4. **Higher order particles.** In this section we will introduce a hierarchy of particle solutions whose k-th level includes the solutions at the $(k-1)$-th level. The zeroth level in the hierarchy are the standard particle-like solutions, while the first level describes the particles with internal $\text{SL}(n)$ variables discussed in the previous section. The particles in the k-th level carry the coefficients of k-th order Taylor expansions of diffeomorphisms, or *jets*. Therefore, we call these particles *k-jetlets*.

Let $\varphi \in \text{SDiff}(\mathbb{R}^n)$. The zeroth order Taylor expansion of φ about 0 is $\varphi(0)$, and the collection of such Taylor expansions is all of $\mathbb{R}^n$. The first order Taylor expansion of φ about 0 is

$$\varphi^i(x) = \varphi^i(0) + \partial_j \varphi^i(0) x^j + o(\|x\|).$$

The tuple of coefficients $(\varphi(0), D\varphi(0)) \in \mathbb{R}^n \times \text{SL}(n)$ is called the first order jet of φ evaluated at 0. Going further, the second order Taylor expansion of φ about 0 is

$$\varphi^i(x) = \varphi^i(0) + \partial_j \varphi^i(0) x^j + \frac{1}{2} \partial_{jk} \varphi^i(0) x^j x^k + o(\|x\|^2).$$

We see that $D^2 \varphi(0)$ is a tensor of rank $(1,2)$, which is symmetric in the lower indices. We call the space of such tensors S_2^1, and we see that the space of second order jets is a submanifold

$Q_1^{(2)} \subset \mathbb{R}^n \times \mathrm{SL}(n) \times S_2^1$. Finally, for $k \geq 2$ the space of k-th order jets is a submanifold

$$Q_1^{(k)} \subset \mathbb{R}^n \times \mathrm{SL}(n) \times S_2^1 \times S_3^1 \times \cdots \times S_k^1,$$

where S_j^1 is the vector space of $(1, j)$-tensors which have been symmetrized in the covariant indices. The space $Q_1^{(k)}$ is equipped with the fiber bundle projection $\pi^{(k)} : Q_1^{(k)} \to \mathbb{R}^n$, which projects onto the $\mathbb{R}^n$ component. In fact, $Q_1^{(k)}$ is a trivial principle bundle where fibers, contained within $\mathrm{SL}(n) \times S_2^1 \times \cdots \times S_k^1$ form a jet group [DJR13]. The jet group serves as a finite-dimensional model of the diffeomorphism group (see [KMS99, Chapter 4] for a description of the group multiplication), and this motivates our interpretation of jetlets as models of self-similarity. We define the space for an N-tuple of k-jetlets by taking a product

$$Q_N^{(k)} = \{(q_1, \ldots, q_N) \in Q_1^{(k)} \times \cdots \times Q_1^{(k)} \mid \pi^{(k)}(q_a) \neq \pi^{(k)}(q_b) \text{ when } a \neq b\}.$$

We coordinatize $Q_N^{(k)}$ as follows. We will use Greek indices to represent spatial multi-indices on $\mathbb{R}^n$ (see Appendix C for our multi-index convention). A typical coordinate on $Q_N^{(k)}$ will therefore look like $q_a{}^i{}_\beta$ where $a \in \{1, \ldots, N\}$, $i \in \{1, \ldots, n\}$ and β is a multi-index on $\mathbb{R}^n$. This coordinate is used to model the partial derivative of the i-th coordinate of a diffeomorphism at some point $z_a \in \mathbb{R}^n$, i.e. $\partial_\beta \varphi^i(z_a)$. For the statement of the next proposition, recall that the definition of $\mathrm{iso}(z)$ was given earlier in (5). Note also that we write $\mathrm{Jet}_x^k : \mathrm{SDiff}(\mathbb{R}^n) \to Q_1^{(k)}$ for the function which evaluates the spatial derivatives of a diffeomorphism up to order k at the location $x \in \mathbb{R}^n$. In the subsequent sections we will also use the obvious generalization to multiple locations $z_1, \ldots, z_N \in \mathbb{R}^n$, which we will denote by $\mathrm{Jet}_z^k : \mathrm{SDiff}(\mathbb{R}^n) \to Q_N^{(k)}$.

Proposition 4.6. *The group* $\mathrm{SDiff}(\mathbb{R}^n)$ *acts on* $Q_N^{(k)}$ *by a left Lie group action. If* $z_1, \ldots, z_N \in \mathbb{R}^n$ *are distinct points, then the isotropy group* $\mathrm{iso}(z) \subset \mathrm{SDiff}(\mathbb{R}^n)$ *acts on* $Q_N^{(k)}$ *by a right Lie group action.*

Proof. Let $\varphi_1, \varphi_2 \in \mathrm{SDiff}(\mathbb{R}^n)$. The partial derivatives of $(\varphi_1 \circ \varphi_2)$ are given by the Faà di Bruno formula

$$\partial_\alpha(\varphi_1 \circ \varphi_2) = \sum_{k=1}^{|\alpha|} \left(\sum_{\substack{j_1, \ldots, j_k \in \{1, \ldots, n\} \\ [\gamma_1, \ldots, \gamma_k] \in \Pi(\alpha, k)}} \partial_{j_1 \cdots j_k} \varphi_1(\varphi_2(x)) \partial_{\gamma_1} \varphi_2^{j_1} \cdots \partial_{\gamma_k} \varphi_2^{j_k} \right),$$

where we wrote $\Pi(\alpha, k)$ for the set of k-th order partitions of a multi-index α. We refer to Appendix C for the details of our index conventions and to [CS96, Jac14] for a precise description of the multivariate Faà di Bruno formula. One can read off from the expression that a k-th order derivative only depends on k-th and lower order partial derivatives of φ_1 and φ_2. A left $\mathrm{SDiff}(\mathbb{R}^n)$ action is induced on $Q_1^{(k)}$ by setting $\varphi \cdot q = \mathrm{Jet}_z^k(\varphi \circ \psi)$ for any ψ such that $\mathrm{Jet}_z^k(\psi) = q$. That this is independent of the choice of ψ follows from observing that the Faà di Bruno formula only uses data in $q = \mathrm{Jet}_z^k(\psi)$ and nothing more. In local

coordinates, this action takes the form

$$(\varphi \cdot q)^i{}_\alpha = \sum_{k=1}^{|\alpha|} \left(\sum_{\substack{j_1,\dots,j_k \in \{1,\dots,n\} \\ [\gamma_1,\dots,\gamma_k] \in \Pi(\alpha,k)}} \partial_{j_1 \cdots j_k} \varphi^i(q^{(0)}) q^{j_1}{}_{\gamma_1} \cdots q^{j_k}{}_{\gamma_k} \right),$$

except for the component where $|\alpha| = 0$ in which case we observe $(\varphi \cdot q)^i = \varphi^i(q^{(0)})$. By the same construction, a right $\mathrm{iso}(z)$ action is induced on $Q_1^{(k)}$ by setting $q \cdot \varphi = \mathrm{Jet}_z^k(\psi \circ \varphi)$ for any ψ such that $\mathrm{Jet}_z^k(\psi) = q$. We can choose distinct points $z_1, \dots, z_N \in \mathbb{R}^n$ and apply the same process to $Q_N^{(k)}$. In this case we observe the action to be

$$(q \cdot \varphi)^{ai}{}_\alpha = \sum_{k=1}^{|\alpha|} \left(\sum_{\substack{j_1,\dots,j_k \in \{1,\dots,n\} \\ [\gamma_1,\dots,\gamma_k] \in \Pi(\alpha,k)}} q^{ai}{}_{[j_1,\dots,j_k]} \partial_{\gamma_1} \varphi^{j_1}(z_a) \cdots \partial_{\gamma_k} \varphi^{j_k}(z_a) \right).$$

$\square$

Just as in the previous sections, the actions of $\mathrm{SDiff}(\mathbb{R}^n)$ and $\mathrm{iso}(z)$, which commute, lift to actions on $T^* Q_N^{(k)}$. The associated momentum maps are given implicitly by how they act on the respective Lie algebras. In particular, the left action of $\mathrm{SDiff}(\mathbb{R}^n)$ yields the momentum map

$$\langle J_L^{(k)}(q,p), u \rangle = \sum_{a=1}^{N} \left(p_{am} u^m(q_a^{(0)}) + \sum_{\substack{|\alpha| \le k \\ 1 \le \ell \le |\alpha|}} \sum_{\substack{j_1,\dots,j_\ell \in \{1,\dots,n\} \\ [\gamma_1,\dots,\gamma_\ell] \in \Pi(\alpha,\ell)}} p_{am}{}^\alpha \partial_{j_1 \cdots j_\ell} u^m(q_a^{(0)}) q_a^{j_1}{}_{\gamma_1} \cdots q_a^{j_\ell}{}_{\gamma_\ell} \right)$$

for arbitrary divergence free vector fields $u \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)$. Equivalently, we may define $J_L^{(k)}$ as the unique map such that $\langle J_L^{(k)}(q,p), u \rangle = \langle (q,p), \mathrm{Jet}_z^{(k)}(u \circ \varphi) \rangle$

$$(10) \qquad \langle J_L^{(k)}(q,p), u \rangle = \langle (q,p), \mathrm{Jet}_z^{(k)}(u \circ \varphi) \rangle$$

for any $\varphi \in \mathrm{SDiff}(\mathbb{R}^n)$ whose k-jet is given by q for any $(q,p) \in T^* Q_N^{(k)}$.

The right action of $\mathrm{iso}(z)$ yields the momentum map $J_R^{(k)}$ defined implicitly by the relation

$$\langle J_R^{(k)}(q,p), w \rangle = \sum_{a=1}^{N} \sum_{\substack{|\alpha| \le k \\ 1 \le \ell \le |\alpha|}} \sum_{\substack{j_1,\dots,j_\ell \in \{1,\dots,n\} \\ [\gamma_1,\dots,\gamma_\ell] \in \Pi(\alpha,\ell)}} p_{ai}{}^\alpha q_a{}^i{}_{[j_1 \cdots j_\ell]} \sum_{m=1}^{\ell} \partial_{\gamma_m} w^{j_m}(z_a) \left(\prod_{n \ne m} \delta_{\gamma_n}^{[j_n]} \right),$$

where $w \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)$ is such that $w(z) = 0$ (this describes the Lie algebra of $\mathrm{iso}(z)$), and δ_α^β is the natural generalization of the Kronecker-delta symbol to multi-indices. Equivalently, we may define $J_R^{(k)}$ as the unique map such that $\langle J_R^{(k)}(q,p), w \rangle = \langle (q,p), \mathrm{Jet}_z^{(k)}(T\varphi \circ w) \rangle$

$$(11) \qquad \langle J_R^{(k)}(q,p), w \rangle = \langle (q,p), \mathrm{Jet}_z^{(k)}(T\varphi \cdot w) \rangle,$$

for any $\varphi \in \mathrm{SDiff}(\mathbb{R}^n)$ whose k-jet is given by q for any $(q,p) \in T^* Q_N^{(k)}$. Here, we wrote $T\varphi \cdot w$ for the function obtained by applying the differential of φ to w.

We can write $J_L^{(k)}$ and $J_R^{(k)}$ explicitly, using the Dirac-delta distribution, as

$$J_L^{(k)}(q,p) = \sum_{a=1}^N \left(p_{am}\mathrm{d}x^m \otimes \delta_{q_a^{(0)}} \right.$$

$$\left. + \sum_{\substack{|\alpha|\leq k \\ 1\leq\ell\leq|\alpha|}} (-1)^\ell p_{am}{}^\alpha \mathrm{d}x^m \otimes \sum_{\substack{j_1,...,j_\ell\in\{1,...,n\} \\ [\gamma_1,...,\gamma_\ell]\in\Pi(\alpha,\ell)}} q_a{}^{j_1}{}_{\gamma_1}\cdots q_a{}^{j_\ell}{}_{\gamma_\ell}\partial_{j_1\cdots j_\ell}\delta_{q_a^{(0)}} \right)$$

and
(12)

$$J_R^{(k)}(q,p) = \sum_{a=1}^N \sum_{\substack{|\alpha|\leq k \\ 1\leq\ell\leq|\alpha|}} \sum_{\substack{j_1,...,j_\ell\in\{1,...,n\} \\ [\gamma_1,...,\gamma_\ell]\in\Pi(\alpha,\ell)}} p_{ai}{}^\alpha q_a{}^i{}_{[j_1\cdots j_\ell]}\sum_{m=1}^\ell (-1)^{|\gamma_m|}\left(\prod_{n\neq m}\delta_{\gamma_n}{}^{[b_n][j_n]}\right)\mathrm{d}z^{j_m}\otimes\partial_{\gamma_m}\delta_{z_a}.$$

~~To see that these are dual pairs, we simply generalize the proof of Proposition 4.4. We first check the conditions of Lemma **??**. Note that $\mathrm{SDiff}(\mathbb{R}^n)$ acts transitively on $Q_N^{(k)}$ because groups always act transitively on equivalence classes, and $Q_N^{(k)} := \mathrm{SDiff}(\mathbb{R}^n)/\mathrm{iso}^{(k)}(z)$. Secondly, the cotangent-lifted action preserves level sets of $J_R^{(k)}$. To see this, let $\varphi\in\mathrm{SDiff}(\mathbb{R}^n)$~~

Proposition 4.7. *The maps $J_L^{(k)}$* and ~~$(q,p)\in T^*Q_N^{(k)}$, then we have~~

$$~~\langle J_R^{(k)}(\varphi\cdot(q,p)),w\rangle = \langle T^*\varphi\cdot(q,p),(\varphi^{-1}\cdot q)\cdot w\rangle~~$$

$$~~= \langle T^*\varphi\cdot(q,p),T\varphi^{-1}\cdot(q\cdot w)\rangle~~$$

$$~~= \langle (q,p),T\varphi\cdot T\varphi^{-1}(q\cdot w)\rangle~~$$

$$~~= \langle (q,p),q\cdot w\rangle = \langle J_R^{(k)}(q,p),w\rangle,~~$$

~~*where we used on the second line that the actions commute. Finally, we must show that $(J_R^{(k)})^{-1}(m)$ is a section of $T^*Q_N^{(k)}$ for any m in the range of $J_R^{(k)}$. Let $(q,p)\in(J_R^{(k)})^{-1}(m)$ and let $\psi\in\mathrm{SDiff}(\mathbb{R}^n)$ such that $q=\mathrm{Jet}_z^k(\psi)$. Then $T^*\psi\cdot(q,p)=(\mathrm{Jet}_z^k(\mathrm{id}),\pi)$ for some π. Since the identity jet $\bar{q}:=\mathrm{Jet}_z^k(\mathrm{id})$ satisfies $\bar{q}_a{}^i{}_j=\delta_j^i$ and $\bar{q}_a{}^i{}_{[j_1\cdots j_\ell]}=0$ for $l>1$, we find that reduces to*~~

$$~~J_R^{(k)}(\bar{q},\pi) = \sum_{a=1}^N\sum_{|\alpha|\leq k}\pi_{ai}{}^\alpha(-1)^{|\alpha|}dz^i\otimes\partial_\alpha\delta_{z_a}.~~$$

~~*Since the points z_a are distinct this mapping $J_R^{(k)}(\bar{q},\cdot)$ is clearly injective into $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$. By invariance of $J_R^{(k)}$ under the action of $\mathrm{SDiff}(\mathbb{R}^n)$ it follows that $\pi\in T_{\bar{q}}^*Q_N^{(k)}$ is the unique element such that $J_R^{(k)}(\bar{q},\pi)=m$*~~ *form a weak dual pair.*

Proof. The result is a direct application of [GBV12, Corollary 2.8]. To make the exposition more self-contained, we provide some details. By [GBV12, Corollary 2.6] we need only show that J_L and J_R are equivariant, and that J_L is invariant under the right action of $\mathrm{iso}(z)$. Equivariance follows from the fact that J_R and J_L are derived from cotangent lifted group actions [AM08, Corollary 4.2.11]. So we need only illustrate that J_L is (right) $\mathrm{iso}(z)$

invariant. This can be seen as a consequence of the commutativity of the left and right actions on $Q^{(k)}$. For notational clarity, let us denote this right action by $\rho : Q^{(k)} \times \mathrm{iso}(z) \to Q^{(k)}$. Explicitly, any element $q \in Q^{(k)}$ is expressible as the k-jet of some diffeomorphism φ, and ~~since SDiff($\mathbb{R}^n$) acts by fiber-preserving diffeomorphisms on $T^*Q_N^{(k)}$, it follows that $(J_R^{(k)})^{-1}(m)$ is a section of $T^*Q_N^{(k)}$.~~ $\rho(q,\psi) = \mathrm{Jet}_z^{(k)}(\varphi \circ \psi)$.

~~Thus Lemma ?? implies that SDiff($\mathbb{R}^n$) acts transitively on the level sets of the momentum map $J_R^{(k)}$ and by Theorem ??, $J_L^{(k)}$~~ Similarly, any element of the tangent fiber $T_\varphi \mathrm{SDiff}(\mathbb{R}^n)$ may be written as a composition $u \circ \varphi$ for some $u \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)$. Elements of $TQ^{(k)}$ are of the form $\mathrm{Jet}_z^{(k)}(u \circ \varphi)$ for $u \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)$ and $\varphi \in \mathrm{SDiff}(\mathbb{R}^n)$. Given this representation, the tangent lift of the action ρ, also denoted $\rho : TQ^{(k)} \times \mathrm{iso}(z) \to TQ^{(k)}$, is given by

$$(13) \qquad \rho((q,v),\psi) = \mathrm{Jet}_z^{(k)}(u \circ \varphi \circ \psi)$$

for $(q,v) \in TQ^{(k)}$ and where $u \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)$ and ~~$J_R^{(k)}$ form a dual pair . This~~ $\varphi \in \mathrm{SDiff}(\mathbb{R}^n)$ are arbitrary up to the constraint $(q,v) = \mathrm{Jet}_z^{(k)}(u \circ \varphi)$. The cotangent lifted action is a left action, $\rho^* : \mathrm{iso}(z) \times T^*Q^{(k)} \to T^*Q^{(k)}$, defined implicitly by the condition

$$(14) \qquad \langle \rho^*(\psi,(q,p)), \rho((q,v),\psi^{-1}) \rangle = \langle (q,p),(q,v) \rangle$$

for all $(q,p) \in T^*Q^{(k)}$ and $(q,v) \in TQ^{(k)}$. This action is equivalent to the one defined in the discussion preceding [AM08, Corollary 4.2.11]. In particular, $\rho^*(\psi,(q,p))$ is a covector over the point $\tilde{q} = \rho(q,\psi^{-1})$. By the definition of J_L we observe

$$\langle J_L(\rho^*(\psi,(q,p))), u \rangle = \langle \rho^*(\psi,(q,p)), \mathrm{Jet}_z^{(k)}(u \circ \bar{\varphi}) \rangle,$$

where $\bar{\varphi} \in \mathrm{SDiff}(\mathbb{R}^n)$ is any diffeomorphism such that $\mathrm{Jet}_z^{(k)}(\bar{\varphi}) = \rho(q,\psi^{-1})$. If we let φ be such that $\mathrm{Jet}_z^{(k)}(\varphi) = q$ then we can simply choose $\bar{\varphi} = \varphi \circ \psi^{-1}$. Thus we get

$$\langle J_L(\rho^*(\psi,(q,p))), u \rangle = \langle \rho^*(\psi,(q,p)), \mathrm{Jet}_z^{(k)}(u \circ \varphi \circ \psi^{-1}) \rangle$$

$$= \langle \rho^*(\psi,(q,p)), \rho(\mathrm{Jet}_z^{(k)}(u \circ \varphi), \psi^{-1}) \rangle$$

$$= \langle (q,p), \mathrm{Jet}_z^{(k)}(u \circ \varphi) \rangle$$

$$= \langle J_L(q,p), u \rangle,$$

using (13) and (14) in the second and third equality. Thus, we see that J_L is invariant under the right action of $\mathrm{iso}(z)$ on $Q^{(k)}$ and the result follows. $\square$

In the case of $k = 1$ we obtain the weak dual pair of the previous section and Proposition 4.4 is a corollary of Proposition 4.7. Proposition 4.7 gives us the final result on jetlet parametrized solutions.

Theorem 4.8. *Let $p \geq \frac{n}{2} + k + 1$ and $\sigma > 0$. Then $H^{(k)} = h_{p,\sigma} \circ J_L^{(k)}$ is C^1. Let $x(t)$ be a solution to Hamilton's equations on $T^*Q_N^{(k)}$, then $J_L^{(k)}(x(t))$ is a solution to Hamilton's equations on $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$ and $J_R^{(k)}(x(t))$ is constant in time.*

Remark that $K_{p,\sigma}$ has smoothness C^{2p-n-1} by considering that the Fourier representations of its derivatives are integrable, while we need $K_{p,\sigma} \in C^{2k+1}$ to allow composition with k-th derivatives of delta distributions on each side and still obtain a C^1 Hamiltonian.

As before, an even richer family of velocity fields is generated by a single particle of this type. At order $k = 2$ this yields four new varieties of velocity fields per particle. Two examples of such velocity fields are depicted in Figure 3.
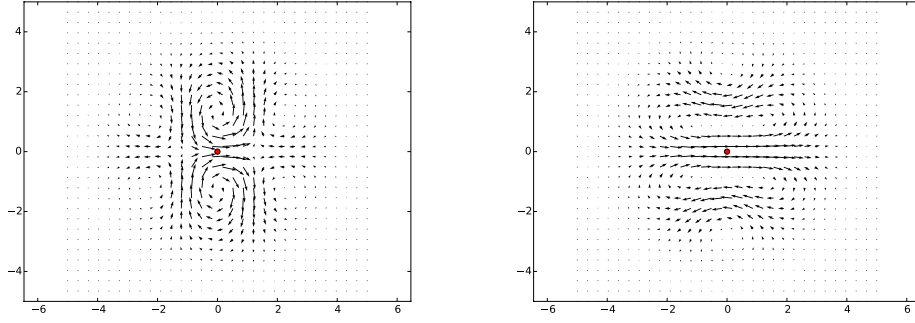

FIGURE 3. Some velocity fields generated by particles of order $k = 2$.

4.5. **Kelvin's circulation theorem.** In this section we relate the conserved quantities associated to $J_R^{(k)}$ with Kelvin's circulation theorem. Let $\varphi_t \in \mathrm{SDiff}(\mathbb{R}^n)$ denote the flow-map produced by a solution to Euler's equation. Let $\gamma(s,t) = \varphi_t(\gamma_0(s))$ for some loop $\gamma_0(s)$. Kelvin's circulation theorem states that the circulation

$$\Gamma(t) = \oint u(\gamma(s,t)) \cdot \partial_s \gamma(s,t) ds$$

is constant in time. It was shown in [Arn66] that this conservation law is an instance of Noether's theorem. In particular, circulation is one of the conserved momenta associated to the particle relabeling symmetry of fluids. More specifically, recall that h_0 is the Hamiltonian for Euler's equations, and this Hamiltonian is $\mathrm{SDiff}(\mathbb{R}^n)$ invariant. Moreover, we have a weak[5] dual pair $J_{\mathrm{spatial}} : T^* \mathrm{SDiff}(\mathbb{R}^n) \to \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$ and $J_{\mathrm{conv}} : T^* \mathrm{SDiff}(\mathbb{R}^n) \to \mathfrak{iso}(z)^*$ of spatial and convective momentum maps induced by the left action of $\mathrm{SDiff}(\mathbb{R}^n)$ and the right action of $\mathrm{iso}(z) \subset \mathrm{SDiff}(\mathbb{R}^n)$ on $\mathrm{SDiff}(\mathbb{R}^n)$ itself. By applying Theorem ~~A.9 with $J_1 = J_{\mathrm{conv}}$ and $J_2 = J_{\mathrm{spatial}}$~~ A.9 with $J_1 = J_{\mathrm{spatial}}$ and $J_2 = J_{\mathrm{conv}}$ we know that Hamiltonian dynamics on $T^* \mathrm{SDiff}(\mathbb{R}^n)$ with respect to a Hamiltonian of the form $h \circ J_1$ will exhibit the constant of motion

(15) $\langle J_{\mathrm{conv}}(\varphi, p_\varphi), w \rangle = \langle p_\varphi, T\varphi \cdot w \rangle$ for all $w \in \mathfrak{iso}(z)$

~~for all $w \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)$.~~ This instance of Noether's theorem applies to regularized models as well as to the non-regularized case of Euler's fluid equations. In the case of Euler's fluid equations, the equivalence between the above conservation law and Kelvin's circulation theorem is demonstrated by a heuristic argument [AK98, Ch 1, Thm 5.5], which goes as follows. Consider a given closed curve $\gamma_0 \colon S^1 \to \mathbb{R}^n$ and a family of vector fields w_ε such that

_____

[5] With $\mathrm{SDiff}(\mathbb{R}^n)$ as right action this would be a proper dual pair, but $\mathrm{iso}(z) \subset \mathrm{SDiff}(\mathbb{R}^n)$ does not act transitively on the level sets of J_{spatial}, hence the dual pair is weak.

$w_\varepsilon(\gamma_0(s)) = \gamma_0'(s)$ and such that the (weak in L^2) limit $w = \lim_{\varepsilon \to 0} w_\varepsilon$ is the generalized function

$$w(x) = \gamma_0'(s) \otimes \delta(x - \gamma_0(s)).$$

Assuming $\underset{\sim}{\varepsilon}$ is small and writing $p_\varphi = \langle \dot\varphi, \cdot \rangle_{L^2}$, we find

$$\langle J_{\mathrm{conv}}(\varphi, p_\varphi), w_\varepsilon \rangle = \langle \dot\varphi, T\varphi \cdot w_\varepsilon \rangle_{L^2}$$
$$= \int_{\mathbb{R}^n} \langle \dot\varphi(x), \mathrm{D}\varphi(x) \cdot w_{\underset{\sim}{\varepsilon}}(x) \rangle_{\mathbb{R}^n} \, \mathrm{d}x$$
$$\approx \oint \langle \dot\varphi(\gamma_0(s)), \mathrm{D}\varphi(\gamma_0(s)) \cdot \gamma_0'(s) \rangle_{\mathbb{R}^n} \, \mathrm{d}s$$
$$= \oint \langle u(\gamma(s,t)), \partial_s \gamma(s,t) \rangle_{\mathbb{R}^n} \, \mathrm{d}s,$$

where $u = \dot\varphi \circ \varphi^{-1}$ is the usual Euler representation of the fluid flow and $\gamma(s,t) := \varphi_t(\gamma_0(s))$. Therefore, conservation of J_{conv} leads to conservation of circulation.

A more rigorous correspondence is developed in [HMR98]. Any $m \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$ can be written as a one-form density $\tilde{m} \otimes \mu$ where μ is the volume form on $\mathbb{R}^n$. For any smooth curve $\gamma : S^1 \to \mathbb{R}^n$ we may consider the current $\mathscr{K}(\gamma) \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^{**}$ defined by

$$\langle \mathscr{K}(\gamma), m \rangle = \int_\gamma \tilde{m}.$$

By Theorem 6.2 of [HMR98], if $\gamma_t(s) = \varphi_t(\gamma_0(s))$ and $m = \tilde{m} \otimes \mu$ satisfies the ideal incompressible fluid equation (perhaps regularized), then the circulation

$$\Gamma(t) = \langle \mathscr{K}(\gamma_t), m(t) \rangle$$

is constant in time. It is in this sense that Kelvin's circulation theorem follows from the particle relabeling symmetry for any $\mathrm{SDiff}(\mathbb{R}^n)$ invariant Lagrangian.

In light of this discussion, it is natural to consider (15) as the fundamental conservation law. The main goal in the remainder of this section is to show that the jet-particle solutions satisfy conservation laws that are 'shadows' of this fundamental law in the sense that they are associated with a *partial* relabeling symmetry. To see this, it is useful to define the k-th order isotropy group, $\mathrm{iso}^{(k)}(z) = \{\psi \in \mathrm{SDiff}(\mathbb{R}^n) \mid \mathrm{Jet}_z^k(\psi) = \mathrm{Jet}_z^k(\mathrm{id})\}$, where we wrote id for the identity mapping and z is shorthand for $z_1, \ldots, z_N \in \mathbb{R}^n$. Then we see that $Q_N^{(k)} = \mathrm{SDiff}(\mathbb{R}^n)/\mathrm{iso}^{(k)}(z)$, and the corresponding quotient map is given by Jet_z^k. Let us also introduce the right action $R \colon \mathrm{SDiff}(\mathbb{R}^n) \times \mathrm{iso}(z) \to \mathrm{SDiff}(\mathbb{R}^n)$ given by right composition of functions and write $R_\psi = R(\cdot, \psi)$, so that $R_\psi \varphi = \varphi \circ \psi$. We also introduce the cotangent lifted right action TR^* by means of the defining relation

$$\langle TR_{\psi^{-1}}^* p_\varphi, v_\varphi \circ \psi \rangle = \langle p_\varphi, v_\varphi \rangle \tag{16}$$

for arbitrary $p_\varphi \in T_\varphi^* \mathrm{SDiff}(\mathbb{R}^n)$ and $v_\varphi \in T_\varphi \mathrm{SDiff}(\mathbb{R}^n)$.

The space $T^* Q_N^{(k)}$ naturally embeds into $T^* \mathrm{SDiff}(\mathbb{R}^n)/\mathrm{iso}^{(k)}(z)$, where the quotient is by the cotangent lifted action. More precisely, for any $(q, p) \in T^* Q_N^{(k)}$ we can construct the corresponding element $i(q, p) \in T^* \mathrm{SDiff}(\mathbb{R}^n)/\mathrm{iso}^{(k)}(z)$ in the following way: take any φ such that $\mathrm{Jet}_z^k(\varphi) = q$ and find $\iota_\varphi(q, p) \in T_\varphi^* \mathrm{SDiff}(\mathbb{R}^n)$ that satisfies, for all $v_\varphi \in T_\varphi \mathrm{SDiff}(\mathbb{R}^n)$ with $\mathrm{Jet}_z^k(\varphi) = q$ we can consider the element $i(q, p) \in T^* \mathrm{SDiff}(\mathbb{R}^n)$ given

by $\langle i(q,p), v_\varphi \rangle = \langle p, v_\varphi(\varphi^{-1} \cdot q) \rangle$ [MMO$^+$07, (2.2.4)] . The commutativity of the following diagram uniquely defines the jetlet solutions:

$$\begin{array}{ccccc}
\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^* & \xleftarrow{\;\;J_{\mathrm{spatial}}\;\;} & T^* \mathrm{SDiff}(\mathbb{R}^n) & \xrightarrow{\;\;J_{\mathrm{conv}}\;\;} & \mathfrak{iso}(z)^* \\
& & \Big\downarrow{\scriptstyle \pi} & & \\
& & T^* \mathrm{SDiff}(\mathbb{R}^n)/\mathrm{iso}^{(k)}(z) & & \\
{\scriptstyle J_L^{(k)}} & & \Big\uparrow{\scriptstyle i} & & {\scriptstyle J_R^{(k)}} \\
& & T^* Q_N^{(k)} & &
\end{array}$$

z

(17) $$\langle \iota_\varphi(q,p), v_\varphi \rangle = \langle (q,p), (v_\varphi \circ \varphi^{-1}) \cdot q \rangle,$$

cf. [MMO$^+$07, (2.2.4)] , where we denote by $(v_\varphi \circ \varphi^{-1}) \cdot q$ the infinitesimal action from the left of $v_\varphi \circ \varphi^{-1} \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)$ on q. Then set $i(q,p) = [\iota_\varphi(q,p)]$.

For the left side of the diagram To see that $i(q,p)$ is well defined, note that the statement $\pi(p_\varphi) = i(p_q)$ for some $p_\varphi \in T^* \mathrm{SDiff}(\mathbb{R}^n)$ and $p_q \in T^* Q_N^{(k)}$ is equivalent to the statement that $\langle p_\varphi, w \circ \varphi \rangle = \langle p_q, \mathrm{Jet}_z^{(k)}(w \circ \varphi) \rangle$. From the definitions of J_{spatial} and $J_L^{(k)}$ we find

$$\langle J_{\mathrm{spatial}}(p_\varphi), w \rangle = \langle p_\varphi, w \circ \varphi \rangle = \langle p_q, \mathrm{Jet}_z^{(k)}(w \circ \varphi) \rangle = \left\langle J_L^{(k)}(p_q), w \right\rangle$$

when $\pi(p_\varphi) = i(p_q)$. This verifies the left half if $\varphi' = \varphi \circ \psi$ for $\psi \in \mathrm{iso}^{(k)}(z)$, then for any $v_{\varphi'} \in T_{\varphi'} \mathrm{SDiff}(\mathbb{R}^n)$ we have

$$\langle TR_{\psi^{-1}}^* \iota_\varphi(q,p), v_{\varphi'} \rangle = \langle \iota_\varphi(q,p), v_{\varphi'} \circ \psi^{-1} \rangle$$

$$= \langle (q,p), (v_{\varphi'} \circ \psi^{-1} \circ \varphi^{-1}) \cdot q \rangle = \langle (q,p), (v_{\varphi'} \circ \varphi'^{-1}) \cdot q \rangle = \langle \iota_{\varphi'}(q,p), v_{\varphi'} \rangle$$

using (16) and (17) in the first and second equalities. Since $[TR_{\psi^{-1}}^* \iota_\varphi(q,p)] = [\iota_{\varphi'}(q,p)]$, we conclude that $i(q,p)$ is well defined.

We claim that

(18) $$i(q,p) = [TR_{\varphi^{-1}}^* J_L^{(k)}(q,p)],$$

where φ is such that $\mathrm{Jet}_z^k(\varphi) = q$. This follows since for any $v_\varphi \in T_\varphi \mathrm{SDiff}(\mathbb{R}^n)$

$$\langle TR_{\varphi^{-1}}^* J_L^{(k)}(q,p), v_\varphi \rangle = \langle J_L^{(k)}(q,p), v_\varphi \circ \varphi^{-1} \rangle = \langle (q,p), (v_\varphi \circ \varphi^{-1}) \cdot q \rangle.$$

With these preliminary remarks in mind, we can now show the commutativity of the following diagram:

Here, $S = (\pi^{-1} \circ i)(T^*Q_N^{(k)})$ and J_{spatial} and J_{conv} are defined in the natural manner through the left and right actions of $\text{SDiff}(\mathbb{R}^n)$ and $\text{iso}(z)$. To verify the left side of the diagram, we let $(q,p) \in T^*Q_N^{(k)}$ and φ such that $\text{Jet}_z^k(\varphi) = q$. Then we obtain from (18) that for any $w \in \mathfrak{X}_{\text{div}}(\mathbb{R}^n)$

$$\langle [J_{\text{spatial}}](i(q,p)), w \rangle = \langle J_{\text{spatial}}(TR_{\varphi^{-1}}^* J_L^{(k)}(q,p)), w \rangle$$

$$= \langle TR_{\varphi^{-1}}^* J_L^{(k)}(q,p), w \circ \varphi \rangle = \langle J_L^{(k)}(q,p), w \rangle,$$

as required. This also shows that $[J_{\text{spatial}}]$ is well defined on $T^* \text{SDiff}(\mathbb{R}^n)/\text{iso}^{(k)}(z)$ since the final expression does not depend on the choice of representative φ for q. ~~The right hand side can be verified in the same manner.~~

~~Again , the right~~ For the right leg, note that

$$\langle [J_{\text{conv}}](i(q,p)), w \rangle = \langle J_{\text{conv}}(TR_{\varphi^{-1}}^* J_L^{(k)}(q,p)), w \rangle$$

$$= \langle TR_{\varphi^{-1}}^* J_L^{(k)}(q,p), T\varphi \cdot w \rangle = \langle J_L^{(k)}(q,p), (T\varphi \cdot w) \circ \varphi^{-1} \rangle$$

$$= \langle (q,p), \text{Jet}_z^k(T\varphi \cdot w) \rangle = \langle J_R^{(k)}(q,p), w \rangle.$$

Here we used (18), the definition of J_{conv}, and (16), respectively, in the first three equalities and (11) in the final step. Again we see that $[J_{\text{conv}}]$ is well-defined since it does not depend on φ. More explicitly, for another representative $\varphi' = \varphi \circ \psi$ with $\psi \in \text{iso}^{(k)}(z)$ we see that ψ gets projected out by (the tangent map of) $\pi = \text{Jet}_z^k$. That is, the restricted momentum map $J_{\text{conv}}|_S$ is invariant under the right action of $\text{iso}^{(k)}(z)$ and therefore descends to a map $[J_{\text{conv}}|_S]$ defined on $i(T^*Q_N^{(k)})$.

The right legs of both weak dual pairs yield the conserved quantities. From the diagram it is clear that the conservation of $J_R^{(k)}$ exhibited in our particle models is a shadow of the conservation of J_{conv} in (15). Since J_{conv} corresponds by Noether's theorem to the (large) subgroup $\text{iso}(z)$ of the right symmetry $\text{SDiff}(\mathbb{R}^n)$ that generates conservation of circulation, we see that conservation of $J_R^{(k)}$ in the jetlet solutions is a shadow of the conservation of circulation. In other words, our particle models contain a model of Kelvin's circulation theorem within them (cf. [DJR13, Theorem 5.5]).

This diagram also provides some insight into the relationship between the developments in this paper and classical Marsden-Weinstein reduction theory [?] . For instance, letting $J_{\text{conv}}^{(k)} : T^* \text{SDiff}(\mathbb{R}^n) \to \mathfrak{iso}^{(k)}(z)^*$ be the momentum map associated to the cotangent lift of the right action of $\mathfrak{iso}^{(k)}(z)$ on $\text{SDiff}(\mathbb{R}^n)$, one can show that $i\big(T^* Q_N^{(k)}\big) = (J_{\text{conv}}^{(k)})^{-1}(0) / \mathfrak{iso}^{(k)}(z)$. For more details on the connections between general reduction theory and the results of the present paper, see Appendix B.

5. Particle mergers

In this section we discuss some explicit dynamical behavior of the particle model, in particular we study 'collisions' of jetlets. For the zeroth order particles case, this was already analyzed by Mumford and Michor [MM13b], and they found that two particles can merge in infinite time, or bounce off each other, depending on the ratio of their relative angular and linear momenta. To find the explicit behavior analytically, we shall restrict to two dimensional space and two 0-jetlet particles with zero total linear momentum. We identify the asymptotics of the merged state as the dynamics of a single 1-jetlet particle.

We start with the Hamiltonian $H^{(0)}$ for two 0-jetlet particles,

$$H = \frac{1}{2} \sum_{a,b=1}^{2} p_{ai} K^{ij}(q_a - q_b) p_{bj}$$

on $T^*\mathbb{R}^{2n}$ with the canonical Poisson brackets. Translation symmetry allows us perform symplectic reduction. We switch to a center of mass frame by choosing new coordinates

(19)
$$\bar{q} = \frac{1}{2}(q_1 + q_2), \qquad \tilde{q} = q_2 - q_1$$

with canonically associated momenta

$$\bar{p} = p_1 + p_2, \qquad \tilde{p} = \frac{1}{2}(p_2 - p_1).$$

The Hamiltonian in these coordinates becomes

$$H = \frac{1}{4} \bar{p}_i \big(K^{ij}(0) + K^{ij}(\tilde{q}) \big) \bar{p}_j + \tilde{p}_i \big(K^{ij}(0) - K^{ij}(\tilde{q}) \big) \tilde{p}_j.$$

When $n = 2$ and the total momentum is zero, i.e. $\bar{p} = 0$, we can perform another symplectic reduction by rotational symmetry. We switch to polar coordinates

(20)
$$\tilde{q} = \big(r \cos(\phi), r \sin(\phi) \big)$$

with canonically associated momenta

$$\tilde{p}^T = \begin{pmatrix} \cos(\phi) & -\frac{\sin(\phi)}{r} \\ \sin(\phi) & \frac{\cos(\phi)}{r} \end{pmatrix} \cdot \begin{pmatrix} p_r \\ p_\phi \end{pmatrix} = R_\phi \cdot \begin{pmatrix} p_r \\ \frac{p_\phi}{r} \end{pmatrix},$$

where R_ϕ is a rotation matrix. In these coordinates the Hamiltonian is given by

$$H = \tilde{p}_i \big[K^{ij}(0) - K^{ij}(\tilde{q}) \big] \tilde{p}_j = \begin{pmatrix} p_r \\ \frac{p_\phi}{r} \end{pmatrix}^T R_\phi^T \big[K(0) - K(R_\phi \cdot (r, 0)) \big] R_\phi \begin{pmatrix} p_r \\ \frac{p_\phi}{r} \end{pmatrix}$$

$$= \begin{pmatrix} p_r \\ \frac{p_\phi}{r} \end{pmatrix}^T \big[K(0) - K(r, 0) \big] \begin{pmatrix} p_r \\ \frac{p_\phi}{r} \end{pmatrix},$$

where in the last step we used that K as a tensor is invariant under rotations. Since ϕ is a cyclic variable, we find that its associated momentum

$$p_\phi = -r\sin(\phi)\tilde{p}_1 + r\cos(\phi)\tilde{p}_2 = \tilde{q} \wedge \tilde{p}$$

is conserved. Remark that this relative angular momentum is not exactly the total angular momentum when $\bar{p} \neq 0$.

Let us now choose the smooth kernel $K = K_{\infty,1}$ given (up to a scaling factor) by

$$K^{ij}(x) = \left(e^{-\rho} - \frac{1}{2\rho}\left(1 - e^{-\rho}\right)\right)\delta^{ij} + \left(\frac{1}{\rho}\left(1 - e^{-\rho}\right) - e^{-\rho}\right)\frac{x^i x^j}{\|x\|^2},$$

where $\rho := \frac{\|x\|^2}{2}$. Note that $K^{ij}(0) = \frac{1}{2}\delta^{ij}$ and $\partial_k K^{ij}(0) = 0$. Using rotational symmetry we set $\phi = 0$, and obtain

$$(21) \qquad H = \frac{p_r^2}{2}\left(1 - \frac{1}{\rho}\left(1 - e^{-\rho}\right)\right) + \frac{p_\phi^2}{4\rho}\left(1 - 2e^{-\rho} + \frac{1}{\rho}\left(1 - e^{-\rho}\right)\right).$$

This system is Hamiltonian in (r, p_r) with p_ϕ a parameter, so the level sets of H determine the motion. For $p_\phi = 0$ we see that $H \propto p_r^2\, r^2 f(r^2)$ where f is an analytic function with $f(0) > 0$. Thus the level sets of H near $r = 0$ look like hyperbola with $p_r = 0$ and $r = 0$ as asymptotic axes, see the left image in Figure (4). Hence, two particles approaching each other head-on will 'collide' in infinite time; even though their momentum blows up, their relative velocity decays exponentially.

If $p_\phi \neq 0$ then the first non-constant contributing term has sign opposite to the terms involving p_r, hence we see a new region being created in the right image of Figure (4) where orbits approach $r = 0$ and then 'bounce off'. There are two asymptotic orbits that approach the $r = 0$ axis at finite momentum; the limit point can be calculated from the fact that $\dot{p}_r/\dot{r} = 0$ must hold there in the limit $r \to 0$. We find that $p_r = \pm\sqrt{5/6}\, p_\phi$. From Figure 4 it is clear that this is a good approximation for the asymptotic curve.
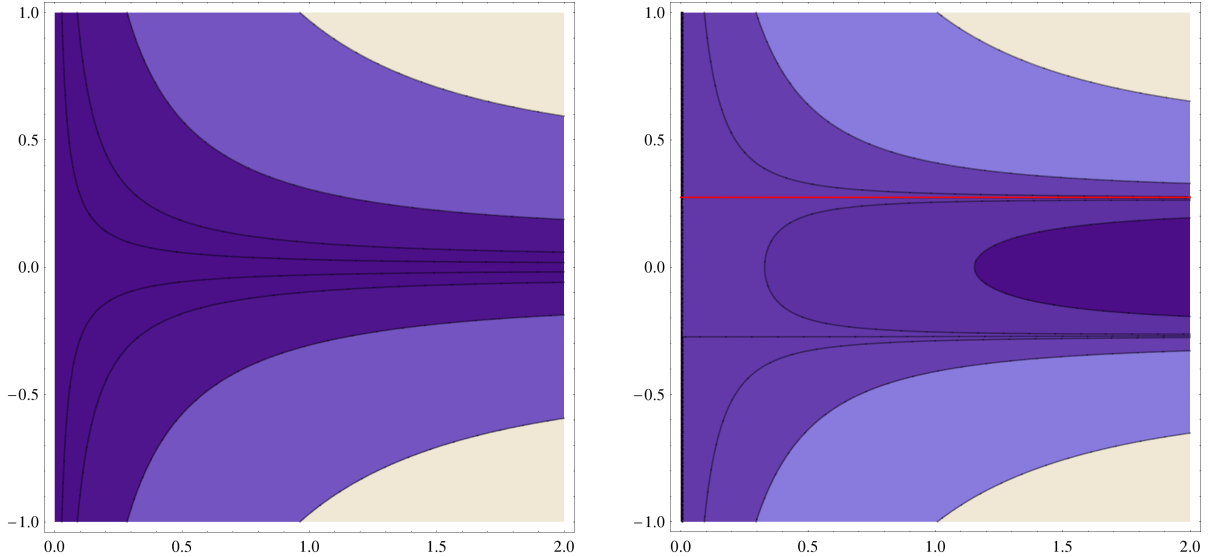

FIGURE 4. Contour plots of the reduced Hamiltonian (21) with r horizontal and p_r vertical, left for $p_\phi = 0$ and right for $p_\phi = 0.3$. The red line shows the positive asymptotic value for merging or bouncing.

Finally, we can reconstruct the asymptotic 1-jetlet trajectory that these two merging 0-jetlets converge to and verify that this trajectory is indeed a solution of the Hamiltonian vector field for a 1-jetlet.

To analyze the asymptotic behavior, we expand H around $\rho = 0$:

$$H = \frac{p_r^2}{4}\rho + \frac{p_\phi^2}{8}\left(3 - \frac{5}{3}\rho\right) + \mathcal{O}(\rho^2).$$

Since H and p_ϕ^2 are preserved, we can solve for p_r in terms of r, and we find

$$2\rho\, p_r^2 = (8H - 3p_\phi^2)\big(1 + \mathcal{O}(\rho)\big),$$
$$\Longleftrightarrow \qquad r\, p_r = \sqrt{8H - 3p_\phi^2} + \mathcal{O}(r).$$

We write $\zeta := \sqrt{8H - 3p_\phi^2}$ and thus obtain asymptotically $p_r = \frac{\zeta}{r} + \mathcal{O}(1)$. Further, we have dynamics

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{1}{4}p_r\, r^2 + \mathcal{O}(r^4), \qquad \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{3}{4}p_\phi + \mathcal{O}(r^2)$$

and reconstruct

$$\bar{q} = 0, \qquad \tilde{q} = rR_\phi \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\bar{p} = 0, \qquad \tilde{p}^T = R_\phi \cdot \begin{pmatrix} p_r \\ \frac{p_\phi}{r} \end{pmatrix} = \frac{1}{r}R_\phi \cdot \begin{pmatrix} \zeta \\ p_\phi \end{pmatrix} + \mathcal{O}(1).$$

Now we consider the image under J_L of the asymptotic solution curve. By testing against a vector field in $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)$ we find

$$J_L(q_1, p_1, q_2, p_2) = p_1 \otimes \delta_{q_1} + p_2 \otimes \delta_{q_2}$$
$$= \bar{p} \otimes \delta_{\bar{q}} + \tilde{p} \otimes \left(-\tilde{q}^j \partial_j \delta_{\bar{q}}\right) + \mathcal{O}\left(\left(|\bar{p}| + |\tilde{p}|\right)|\tilde{q}|^2\right)$$

and inserting our reconstructed solution leads to

$$\text{(22)} \qquad\qquad = -\left[R_\phi \cdot \begin{pmatrix} \zeta \\ p_\phi \end{pmatrix}\right]_i \left[R_\phi \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]^j \mathrm{d}x^i\, \partial_j \delta_{\bar{q}} + \mathcal{O}(r)$$

with $\phi(t) = \frac{3}{4}p_\phi t$. Our aim is to show that the factor in front of $\mathrm{d}x^i\, \partial_j\delta$, which we shall denote by $\tilde{\mu}_i{}^j(t)$, corresponds to $\mu_i{}^j(t)$ for a 1-jetlet with position $q^{(0)} = 0$ and momentum $p^{(0)} = 0$. From here on, we use the Frobenius inner product to identify $\mathfrak{sl}(2)^* \cong \mathfrak{sl}(2)$, which for explicit matrices corresponds to taking the transpose. For such a setup we have equations of motion

$$\dot{\mu}_i{}^j = \mu_i{}^k \partial_k u^j(0) - \partial_i u^k(0)\mu_k{}^j = -\partial_{km}K^{jl}(0)\mu_l{}^m \mu_i{}^k + \partial_{im}K^{kl}(0)\mu_l{}^m \mu_k{}^j,$$

or in short, $\dot{\mu} = -\operatorname{ad}_\xi^*(\mu) = [\mu, \xi^T]$ with $\xi^i{}_j = \partial_j u^i(0) = -\partial_{jk}K^{il}(0)\mu_l{}^k$. To ease calculations, let us choose the basis

$$\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and note that these matrices have norm $\sqrt{2}$. We decompose $\mu = j\,\omega + s\,\sigma + t\,\tau$ and the second derivative of the kernel as a tensor product. A calculation verified by symbolic computer algebra software shows that

$$\partial_{ij}K^{kl}(0) = \frac{1}{4}\begin{pmatrix} -\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & -\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = -\frac{1}{2}\omega_i^k\omega_j^l - \frac{1}{4}\big(\sigma_i^k\sigma_j^l + \tau_i^k\tau_j^l\big),$$

where the indices i, j and k, l label the outer and inner matrix elements respectively. With these decompositions we find that ξ as a function of μ can be written as

$$\xi = j\,\omega + \frac{1}{2}(s\,\sigma + t\,\tau).$$

Using the commutation relations $[\omega, \sigma] = 2\tau$, $[\omega, \tau] = -2\sigma$, $[\sigma, \tau] = -2\omega$, it then follows that

$$(23)\quad \dot{\mu} = [\mu, \xi^T] = \Big[j\,\omega + s\,\sigma + t\,\tau\,,\; -j\,\omega + \frac{1}{2}(s\,\sigma + t\,\tau)\Big] = -3j\big(-t\,\sigma + s\,\tau\big) = -\frac{3}{2}j\,[\omega, \mu].$$

That is, the (σ, τ) components of μ as a tensor rotate with angular velocity $-3j$.

On the other hand, from (22) we have for the asymptotic dynamics that

$$\tilde{\mu}(t) = -\Big[R_\phi \cdot \begin{pmatrix} \zeta \\ p_\phi \end{pmatrix}\Big] \cdot \Big[R_\phi \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\Big]^T = -\operatorname{Ad}_{R_\phi}\begin{pmatrix} \zeta & 0 \\ p_\phi & 0 \end{pmatrix}$$

with only $\dot{\phi} = \frac{3}{4}p_\phi$ depending on time. Note that $\tilde{\mu} \notin \mathfrak{sl}(2)$ as a matrix, but since it is actually a dual element, we can simply ignore its trace part and project it out. Also note that the ω component of $\tilde{\mu}(0)$ is $j = \frac{1}{2}p_\phi$. Differentiating with respect to time yields

$$\dot{\tilde{\mu}} = -\dot{\phi}\,\operatorname{ad}_\omega(\tilde{\mu}) = -\frac{3}{4}p_\phi[\omega, \tilde{\mu}] = -\frac{3}{2}j[\omega, \tilde{\mu}],$$

and comparing to (23) we find that the asymptotic solution of the two merging 0-jetlets matches that of a 1-jetlet with the same angular (and linear) momentum.

Let us finally suggest a more abstract way to view these particle mergers. Our hierarchy of reduced spaces $T^*Q_N^{(k)}$ embeds into $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$ under the momentum map J_L. Consider a merging pair of 0-jetlets, described by a curve $x_0(t) \in T^*Q_2^{(0)}$. As the particles approach each other, $x_0(t)$ approaches the boundary of $T^*Q_2^{(0)}$ given by

$$\partial\big(T^*Q_2^{(0)}\big) = \{(q_1, p_1, q_2, p_2) \in T^*\mathbb{R}^n \times T^*\mathbb{R}^n \mid q_1 = q_2\}.$$

On the other hand, the image curve $y_0(t) = J_L(x_0(t)) \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$ consists of two covector-valued delta distributions at q_1, q_2, and in the limit as their distance goes to zero, this can be approximated by a momentum valued distribution of a delta and its derivative (22), that is, an element $y_1(t) = J_L(x_1(t))$, where $x_1(t) \in T^*Q_1^{(1)}$ is a curve in the space of single 1-jetlet particles.

We can view the boundary of $T^*Q_2^{(0)}$ as a subset of $T^*Q_1^{(1)}$, and consider a topology on $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$ in which the embedding is continuous, see diagram (24). This picture naturally generalizes to the whole hierarchy of spaces $T^*Q_N^{(k)}$, suggesting that it might be interpreted as a CW-complex. In this setting, the question whether the solution curve $y_0(t)$ of the

merging 0-jetlets converges to a solution curve $y_1(t)$ of a 1-jetlet, basically[6] boils down to the question whether the vector field of the dynamics on $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$ is continuous. We have not pursued in detail the question of which topology on $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$ to use for this more abstract characterization.

(24)
$$
\begin{array}{c}
T^*Q_N^{(k)} \xrightarrow{\ J_L\ } \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^* \\
\uparrow \\
T^*Q_1^{(1)} \\
\partial \uparrow \\
T^*Q_2^{(0)}
\end{array}
$$

6. Numerical experiments

We have performed a number of numerical simulations[7] of jetlet particles. These confirm that the conserved quantities are indeed preserved and that two 0-jetlet particles merge as shown by the analysis in Section 5, providing a sanity check for the formulas in the previous section and the numerical code. Moreover, the merging behavior shows to be stable under perturbations of initial conditions.

The numerical code works for any spatial dimension $n \geq 2$, but for the sake of tractability and simplicity we have studied $n = 2$

As basic experiment we take two 0-jetlet particles with initial states

(25)
$$
\begin{aligned}
q_1 &= (-3,0), & p_1 &= (1.5, -d), \\
q_2 &= (3,0), & p_2 &= (-1.5, d),
\end{aligned}
$$

aimed at each other with an offset parameter d. The initial state is given in ~~center-of-mass polar-coordinates~~ center of mass polar coordinates, see (19) and (20), by

$$
\begin{aligned}
r &= 6, & p_r &= -1.5, \\
\phi &= 0, & p_\phi &= 6d.
\end{aligned}
$$

Furthermore, we use $\sigma = 1$ throughout our experiments. The experiments show that for $d = 0.27$ the two particles merge while spinning around each other, while for $d = 0.288$ they get close, but then emerge from their close spinning state and scatter in opposite directions. This confirms the analytical value of $d = \sqrt{3/40} \approx 0.2739$ within reasonable precision, noting that this is the asymptotic value for particles starting close to each other.

We performed a number of more complex simulations, all of those being small perturbations of the basic experiment described above. First, we added a small angular momentum 'spin' component to both particles, turning both into 1-jetlets, one level higher in the hierarchy. Then we added a third particle (both a 0-jetlet and 1-jetlet) at such a distance and

---

[6]One has to be careful, however, since continuity of the vector field will only imply that the asymptotic curve $\tilde{y}_0(t) \in J_L(T^*Q_1^{(1)})$ is a pseudo orbit of the 1-jetlet dynamics. This does not imply existence of a solution curve $y_1(t) \in J_L(T^*Q_1^{(1)})$ that $\tilde{y}_0(t)$ is asymptotic to; that would require the 'limit shadowing property', which is closely related to hyperbolic properties of the dynamics [PPT12, Rib14].

[7]The simulation was written using Python and NumPy, the source code and generated videos can be found at: `https://github.com/hoj201/incompressible_jet_particles`

momentum that it exhibits medium range interaction with the first two particles. Finally, we added a small hyperbolic-like 'stretching' momentum to the first particle only. We found an analytic study of these configurations to be infeasible, but the simulations show that the behavior observed in the basic experiment persists. We can find parameter values of d close to the original one where the system shows a transition between the two particles merging or scattering.

These experiments also confirm the preservation of the conserved quantities present in the system. For all experiments described above we observed that the energy, total linear and angular momentum, as well as J_R (individually for each particle) were preserved with absolute errors less than $4 \cdot 10^{-4}$ over a time of 60 seconds, while the energy was of the order one. Unlike J_R, which is conserved for each particle, linear and angular momentum can be exchanged between particles (although total momentum is conserved). Figure 5 left shows the angular momentum of two scattering 1-jetlets and the right plot of two scattering 0-jetlets interacting weakly with a third jetlet particle.
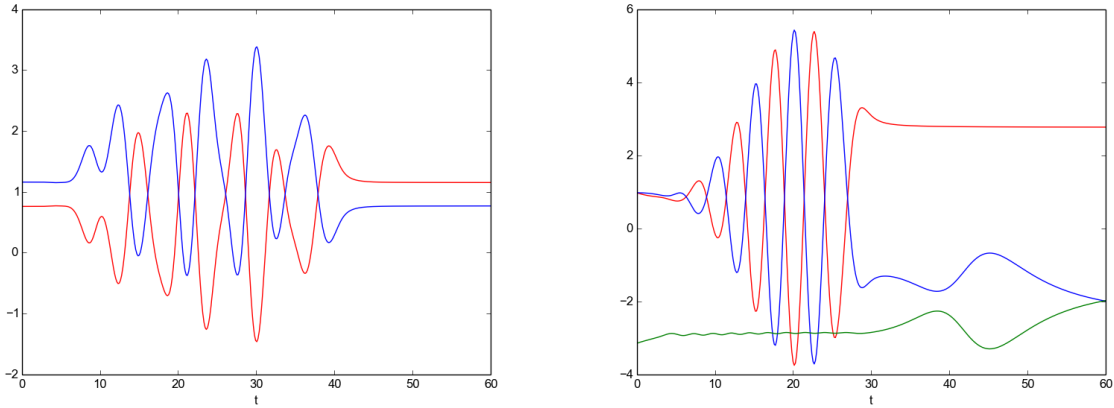

FIGURE 5. Angular momentum exchange between two scattering 1-jetlets (left) and with a third particle present (right).

7. CONCLUSION

In this paper we derived a hierarchy of weak dual pairs which induce a family of particle-like solutions, called *jetlets*, and conserved quantities that shadow the conservation of circulation. The jetlets have internal degrees of freedom given by a jet group. As the jet group is a finite-dimensional model of the diffeomorphism group, we suggested the use of jetlets as a finite-dimensional model of self-similarity, wherein a "large" diffeomorphism advects a "small" diffeomorphism. We also studied the dynamics of mergers and provided a rigorous analysis showing that merging 0-jetlets asymptotically approach 1-jetlets.

The developments discussed in the present paper give rise to a number of promising directions for future research. These include:

(1) An investigation of the relationship between jetlets and point vortices or vortex blobs.
(2) Further investigation of the numerical implementation. The use of parallelization and the fast multipole method would be particularly interesting to consider.

(3) Finding a way to implement boundary conditions. In such scenarios, the kernel is no longer invariant under rigid transformations and we must consider a general kernel $K : M \times M \to \mathbb{R}^{n \times n}$ where $M \subset \mathbb{R}^n$ is an n-manifold with boundary.

(4) An analysis of convergence to Euler equations when $\sigma \to 0$ for the case where the power of the Helmholtz operator, p, goes to infinity. The advantage of the $p = \infty$ case is that the limiting kernel can be written in terms of elementary functions [MG14].

8. Acknowledgements

Appendix A. Hamiltonian mechanics

The goal of this section is to prove ~~Theorems A.9 and ??~~ Theorem A.9 (see page 33). Those who understand and accept these theorems on a first reading should be able to skip this section without any consequence. Most of this section will be a crash course in Poisson geometry and Hamiltonian mechanics as described in [AM08, MR99, Wei83].

A typical introduction to Poisson structures in mechanics begins by considering Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

If we consider the two-form $dp \wedge dq$, then Hamilton's equations can be written as $(\dot{q}, \dot{p}) \lrcorner (dp \wedge dq) = dH(q, p)$. This is the starting point for symplectic geometry, which will be discussed in Section A.1. Alternatively, these equations can be written using the bilinear map $\{\cdot, \cdot\}_{\mathrm{can}} : C^1(T^*Q) \times C^1(T^*Q) \to C^0(T^*Q)$ given by

$$\{F, G\}_{\mathrm{can}} = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial G}{\partial q}\frac{\partial F}{\partial p}.$$

In particular, we may write $\dot{q} = \{q, H\}_{\mathrm{can}}$ and $\dot{p} = \{p, H\}_{\mathrm{can}}$. The object $\{\cdot, \cdot\}_{\mathrm{can}}$ is a special case of more general object known as a *Poisson bracket* which will be introduced in Section A.2.

As a warning, symplectic geometry has developed greatly since its origins in mechanics, and has branched into an independent subfield of pure mathematics. Many notions were revised and optimized in the 1970's and 1980's for the purpose of proving theorems. Occasionally these revisions entailed a sacrifice in clarity, from the perspective of "outsiders". This paper is intended to allow "outsiders" (such as ourselves) to reap the benefits of Poisson geometry. Therefore, we will cut away as much abstraction as possible in this introductory section. Nonetheless, a minimal amount of abstraction is needed in order to maintain mathematical rigor and stand firmly upon the shoulders of giants.

A.1. **Symplectic manifolds.** We begin with the definition.

Definition A.1. *Let S be a manifold and let ω be a closed two-form on S such that the map "$v \in TS \mapsto \omega(v, \cdot) \in T^*S$" is weakly non-degenerate.[8] We call ω a symplectic form. We call the pair (S, ω) a symplectic manifold.*

---

[8] A linear map $L : V \to V^*$ is *weakly non-degenerate* if L is injective. If V is finite-dimensional, this simply means that L is invertible.

All of the expressions derived in this article will be formal, and we refer to [GBV12] for the functional analytic details of infinite-dimensional symplectic manifolds. As a first example, consider the manifold $\mathbb{R}^2$ with coordinates (q, p). The two-form $dq \wedge dp$ is a symplectic form. Given a manifold Q, the cotangent bundle T^*Q has local fiber bundle coordinates given by $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ and there is a unique symplectic form which is locally expressed by $dp^i \wedge dq_i$, where a sum on repeated indices is assumed. This local expression corresponds to a global symplectic form on T^*Q, known as the *canonical symplectic form* and denoted ω_{can} [AM08, Theorem 3.2.10]. In fact, given any symplectic manifold (S, ω), the dimension of S is even, and there exist local coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ such that $\omega \overset{locally}{=} dp_i \wedge dq^i$. This is known as *Darboux's theorem* and we call these type of coordinates *Darboux coordinates*[AM08, Theorem 3.2.2].

Given a function $H : S \to \mathbb{R}$, the exterior derivative is the one-form $dH : S \to T^*S$ expressed in local coordinates by $dH(x) = \frac{\partial H}{\partial x^i} dx^i$. The Hamiltonian vector field $X_H : S \to TS$ is the unique vector field defined by the condition $X_H \lrcorner \omega = dH$. The symbol "$\lrcorner$" is the operation of contraction between the contravariant indices of X_H and the first set of covariant indices of ω. In Darboux coordinates, the Hamiltonian vector field induces the equations of motion $\dot{q}^i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q^i}$.

An important aspect of study in Hamiltonian mechanics is that of symmetry. This yields the following notions

Definition A.2. *Let G be a Lie group and let $\rho : G \to \text{Diff}(S)$ be a group action on a symplectic manifold (S, ω). The group G is said to* act symplectically *if*

$$\omega(\rho(g) \cdot v, \rho(g) \cdot w) = \omega(v, w)$$

for any $g \in G, v, w \in T_x S$ and $x \in S$. If $\mathfrak{g}$ is the Lie algebra of such a group, the momentum map*, $J : S \to \mathfrak{g}^*$, is defined by the property*

$$d\langle J, \xi \rangle = \xi_S \lrcorner \omega.$$

Alternatively, we can characterize a momentum map $J : S \to \mathfrak{g}^*$, as the unique map such that $X_{\langle J, \xi \rangle} = \xi_S$ for any $\xi \in \mathfrak{g}$. In the special case where $S = T^*Q$, a left/right action of G on Q can be lifted to a right/left symplectic action on T^*Q given by

$$(q, p) \in T^*Q \mapsto (g^{-1} \cdot q, g^*p) \in T^*Q.$$

where g^*p is the unique covector such that $\langle g^*p, v \rangle = \langle p, Tg \cdot v \rangle$. In this case the momentum map is characterized by the condition

$$(26) \qquad \langle J(q, p), \xi \rangle = \langle p, \xi \cdot q \rangle.$$

This is contained in Theorem 12.1.4 of [MR99].

Finally, given two functions $f, h \in C^\infty(S)$ we can consider the function $\{f, h\} = \omega(X_f, X_h)$. In Darboux coordinates $\{f, h\} = \frac{\partial f}{\partial q^i} \frac{\partial h}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial h}{\partial q^i}$. Hamilton's equations can then be written as $\dot{q}^i = \{q^i, h\}$, $\dot{p}_i = \{p_i, h\}$. We call $\{\cdot, \cdot\}$ a Poisson bracket, and it is the subject of the next subsection.

A.2. **Poisson manifolds.** We begin with the definition.

Definition A.3. *Let P be a manifold, and $\{\cdot, \cdot\}$ be a bilinear operation on $C^\infty(P)$ such that $(C^\infty(P), \{\cdot, \cdot\})$ is a Lie algebra and $\{\cdot, h\}$ has the derivation property for any $h \in C^\infty(P)$. That is to say*

$$\{gf, h\} = \{f, h\} \cdot g + \{g, h\} \cdot f,$$

for any $f, g, h \in C^\infty(P)$. We call $\{\cdot, \cdot\}$ a Poisson bracket, *and we call the pair $(P, \{\cdot, \cdot\})$ a* Poisson manifold.

The first most important example of a Poisson bracket is that of a Poisson bracket on a symplectic manifold (S, ω). Here the Poisson bracket is $\{f, g\} = \omega(X_f, X_g)$. When S is a cotangent bundle, and ω is the canonical symplectic form, we call this bracket the *canonical Poisson bracket*.

The second most important example of a Poisson bracket, after the canonical Poisson bracket, is the Lie–Poisson bracket. Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{g}^*$ denote its dual. The *Lie–Poisson bracket* on $\mathfrak{g}^*$ is given by

$$(27) \qquad \{f, g\}_{\mathfrak{g}^*}(x) = \pm \left\langle x, \left[\frac{\partial f}{\partial x}, \frac{\partial g}{\partial x} \right] \right\rangle,$$

where $\langle \cdot, \cdot \rangle$ is the canonical pairing between dual-vectors and vectors, and $[\cdot, \cdot]$ is the Lie bracket on $\mathfrak{g}$. The "+" Poisson bracket is nothing but the canonical Poisson bracket on T^*G, mapped to the space $\mathfrak{g}^*$ via the left trivialization map $\lambda : (g, p) \in T^*G \mapsto (L_g)^* p \in \mathfrak{g}^*$. The "−" bracket is obtained through the right trivialization map $\rho : (g, p) \in T^*G \mapsto (R_g)^* p \in \mathfrak{g}^*$.

On a Poisson manifold $(P, \{\cdot, \cdot\})$ the derivation property implies that the functional operator $\{\cdot, h\}$ is equivalent to the Lie derivative operator of a unique vector field $X_h : P \to TP$. That is to say, X_h is the unique vector field such that $\mathcal{L}_{X_h}[f] = \{f, h\}$ for any $f \in C^1(P)$. We call X_h the *Hamiltonian vector field* and the ODE $\dot{x} = X_h(x)$ is called a *Hamiltonian equation*. It is standard to write this ODE as "$\dot{x} = \{x, H\}$", despite the fact that one typically intends for "x" to represent a point in P, and not a function. Since one can take "x" to be a place-holder for a set of local coordinate functions which determine x uniquely, this sloppiness is usually harmless.

Proposition A.4 (Proposition 10.2.2 [MR99])**.** *Let $(P, \{\cdot, \cdot\})$ be a Poisson manifold. Then $X_{\{h, f\}} = -[X_h, X_f]$.*

Corollary A.5. *Let (S, ω) be a symplectic manifold and let $h, f \in C^\infty(S)$. Then $[X_h, X_f] = -X_{\omega(X_h, X_f)}$.*

Proof. $X_{\omega(X_h, X_f)} = X_{\{h, f\}} = -[X_h, X_f]$. □

Definition A.6. *Let $(P_1, \{\cdot, \cdot\}_1)$ and $(P_2, \{\cdot, \cdot\}_2)$ be Poisson manifolds. A map $\psi : P_1 \to P_2$ is called a* Poisson map *if $\{f \circ \psi, g \circ \psi\}_1 = \{f, g\}_2 \circ \psi$ for any $f, g \in C^\infty(P_2)$.*

Proposition A.7 (Lemma 1.2 of [Wei83] or Proposition 10.3.2 of [MR99])**.** *Let $\psi : P_1 \to P_2$ be a Poisson map. Let $h_2 \in C^1(P_2)$. If $x(t) \in P_1$ is a solution to Hamilton's equations with respect to $h_1 = h_2 \circ \psi \in C^1(P_1)$, then $y(t) = \psi(x(t)) \in P_2$ is a solution to Hamilton's equations with respect to h_2.*

Remark that unlike in [MR99], we only require C^1 smoothness since we do not use existence and uniqueness of solutions.

When the dimension of P_2 is larger than that of P_1, Proposition A.7 allows one to find solutions of Hamiltonian equations on P_2 by solving ~~lower dimensional~~ lower-dimensional Hamiltonian equations on P_1.

A.3. ~~Dual~~ Weak dual **pairs.** In this section we review the notion of ~~dual pairs [MW83, Wei83, GBV12] we~~ dual pairs [GBV12] . This is a relaxation of the more frequently invoked notion of a dual

pair [MW83, Wei83] ~~.~~ Let (S, ω) be a symplectic manifold. Given a distribution $V \subset TS$, denote the fiber over $x \in S$ by $V_x \subset T_x S$. The *symplectic orthogonal* to V is the distribution

$$V^\omega = \{w \in TS \mid \omega(w, v) = 0, \forall v \in V\}.$$

~~Let $\psi : S \to P_1$~~

Definition A.8 (Weak dual pair [GBV12])**.** *Let $J_1 \colon S \to P_1$ be a Poisson map. The kernel of* ~~ψ~~ J_1 *is the distribution*

$$\mathrm{kernel}(\underline{\psi} \underline{J_1}) = \{v \in TS \mid \underline{T\psi} \underline{TJ_1} \cdot v = 0\}.$$

If ~~$J : S \to P_2$~~ $\underline{J_2 \colon S \to P_2}$ *is a Poisson map as well, and*

$$\mathrm{kernel}(J_2) \underline{\overset{\omega}{=}} \subset \mathrm{kernel}(\underline{\psi} \underline{J_1}) \quad , \quad \underline{\mathrm{kernel}(J_1)^\omega \subset \mathrm{kernel}(J_2)}$$

we call the diagram

$$P_1 \overset{J_1}{\longleftarrow} S \overset{J_2}{\longrightarrow} P_2$$

a $\underline{weak}$ *dual pair* ~~.~~ ~~—.~~

We would have a proper dual pair if the kernel inclusions were replaced by equalities.

Theorem A.9. *Let $J_1, J_2 : S \to P_1, P_2$ form a $\underline{weak}$ dual pair. Let $h \in C^1(P_1)$. Let $x(t) \in S$ be a solution to Hamilton's equations with respect to the Hamiltonian $H = h \circ J_1$. Then $J_1(x(t)) \in P_1$ is a solution to Hamilton's equations on P_1 with respect to h, and $J_2(x(t))$ is constant in time.*

Proof. Use proposition A.7 to show that $\mu(t) = J_1(x(t))$ is a solution to Hamilton's equations with respect to h. To verify that $J_2(x(t))$ is constant, let $v \in \mathrm{kernel}(J_1)$ be a vector over $x(0) \in S$. This means that v is tangent to the level set of J_1 at $x(0)$. Moreover, $H = h \circ J_1$ is constant on such level sets. Thus we observe

$$0 = \langle dH(x(0)), v \rangle = \omega(\dot{x}, v).$$

As v was an arbitrary element of $\mathrm{kernel}(J_1)$ over $x(0)$ we see that ~~$\dot{x} \in TS$ is contained in the symplectic orthogonal to $\mathrm{kernel}(J_1)$. Since~~ $\underline{\dot{x} \in \mathrm{kernel}(J_1)^\omega. \text{ As }}$ J_1 and J_2 form a $\underline{weak}$ dual pair, this ~~observation~~ implies $\dot{x} \in \mathrm{kernel}(J_2)$. ~~We may apply this argument at each time t and observe~~ $\underline{\text{Thus we have found}}$

$$\frac{d}{dt} J_2(x(t)) = TJ_2 \cdot \dot{x}(t) = \underline{0 \forall t. \, 0}.$$

$\square$

~~The next theorem and lemma will be used later for the purpose of identifying dual pairs.~~

APPENDIX B. DIAGRAMMATIC OVERVIEW

We present here a diagrammatic representation of some of the spaces used in the present paper. We begin by recalling a number of general results that hold for finite-dimensional Lie groups, before we indicate their relevance to the developments in the main text.

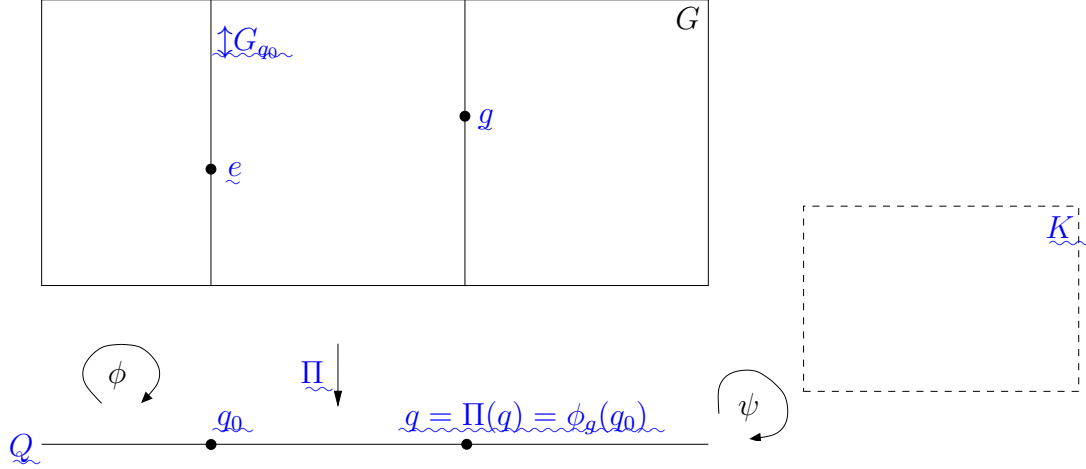~~Denote by ω the canonical symplectic form on the cotangent bundle T^*Q of some manifold Q. Let G_1, G_2 act on~~

FIGURE 6. The G action on Q induces a projection Π. The group K also acts on G, and we assume that the two actions commute.

- Let a Lie group G act on a manifold Q by ~~commuting group actions.~~ [9] ~~Let $\rho_i : G_i \times T^*Q \to T^*Q$ for $i = 1, 2$, denote the cotangent lifted actions and let $J_1 : T^*Q \to \mathfrak{g}_1^*$ and $J_2 : T^*Q \to \mathfrak{g}_2^*$ denote the momentum maps~~ the action $\phi : G \times Q \to Q$, which we also write as $\phi_g(\cdot) = \phi(g, \cdot)$. If we fix a particular value $q_0 \in Q$, we can construct a mapping $\Pi : G \to Q$ given by ~~. Moreover, assume G_1 is infinitesimally~~ $\Pi(g) = \phi_g(q_0)$. Let us assume that the action is transitive, so that Π is surjective. We denote by G_{q_0} the isotropy subgroup leaving q_0 invariant, that is,

$$G_{q_0} := \{g \in G | \phi_g(q_0) = q_0\}.$$

Note that $\Pi^{-1}(q) = gG_{q_0}$ for any $g \in G$ such that $\Pi(g) = q$, and hence we can identify Q with G/G_{q_0}. Suppose a further Lie group, K, also acts on Q with group action $\psi : Q \times K \to Q$, which commutes with ϕ. This situation arises naturally, for instance, when K is a subgroup of G and, in turn, G_{q_0} is a normal subgroup of K. In that case, one can define the action ψ as

(28)
$$\psi_k(q) := \Pi(gk) = \phi_{gk}(q_0),$$

and check that ϕ and ψ indeed commute:

$$\phi_s(\psi_k(q)) = \phi_s(\phi_{gk}(q_0)) = \phi_{sgk}(q_0) = \psi_k(\phi_s(q)).$$

We refer to Figure 6 for a representation of the relevant spaces and maps.
- The actions ϕ and ψ on Q can be lifted to actions Φ and Ψ on the cotangent bundle T^*Q in the usual manner (see Figure 7). These cotangent lifted actions induce equivariant momentum maps $J_1 : T^*Q \to \mathfrak{g}^*$ and $J_2 : T^*Q \to \mathfrak{k}^*$, where $\mathfrak{g}^*$ and $\mathfrak{k}^*$ are the duals of the Lie algebras of G and K. Due to their equivariance, J_1 and J_2 are Poisson maps (where the duals of the Lie algebras are equipped with appropriate Lie–Poisson brackets). Since the actions ϕ and ψ commute, the action Ψ leaves level sets of J_1 invariant, and vice versa. This implies that J_1 and J_2 are a weak dual pair, and if moreover Ψ is transitive on the level sets of J_1 and vice versa, then J_1 and J_2

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[9] ~~That is, acting first by G_1 and second by G_2 is identical to acting first by G_2 then second by G_1.~~

~~in the sense that kernel$(J_2) \subset \mathfrak{g}_1 \cdot T^*Q$. Then the momentum maps form the dual pair $\mathfrak{g}_1^* \xleftarrow{J_1} T^*Q \xrightarrow{J_2} \mathfrak{g}_2^*$. Let us first show that kernel$(J_1) = [\rho_1(\mathfrak{g}_1, T^*Q)]^\omega$. Recall that on account of the definition of momentum maps we have $d\langle J_1, \xi_1\rangle = i_{\rho_1(\xi_1)}\omega$, where $\rho_1(\xi_1)$ is the infinitesimal generator of $\xi_1 \in \mathfrak{g}_1$. Let $v \in [\rho_1(\mathfrak{g}_1, T^*Q)]^\omega$. We observe that $d\langle J_1, \xi_1\rangle(v) = \omega(\rho_1(\xi_1, x), v) = 0$. However $d\langle J_1, \xi_1\rangle(v) = \langle TJ_1(v), \xi_1\rangle$. Thus $v \in$ kernel(J_1). The previous steps may be reversed~~ are a proper dual pair, see [GBV12, Corollary 2.6].

- Let $\mathcal{H} : T^*G \to \mathbb{R}$ be a right-invariant Hamiltonian. This means that $\mathcal{H}$ is invariant with respect to the cotangent lift TR^* of the multiplication from the right of G by itself. In particular, the reduced Hamiltonian $h : \mathfrak{g}^* \to \mathbb{R}$ satisfies $\mathcal{H}(\alpha_g) = h \circ TR_g^*(\alpha_g)$ for any $\alpha_g \in T^*G$, and the ~~converse follows immediately. We find kernel$(J_1) = [\rho_1(\mathfrak{g}_1, T^*Q)]^\omega$. Of course the same argument applies to J_2.~~

  ~~We now show that kernel$(J_2) = \rho_1(\mathfrak{g}_1, T^*Q)$. Again, let $\xi_1 \in \mathfrak{g}_1, \xi_2 \in \mathfrak{g}_2$. Recall that $\omega = -d\theta$ where θ is the canonical one-form on~~ reduced dynamics in $\mathfrak{g}^*$ are of Lie–Poisson type. The momentum map J_1 can be used to induce the so-called *collective Hamiltonian* $H = h \circ J_1$ on T^*Q. Note that T^*Q is a symplectic manifold, and that the symplectic (canonical) dynamics with respect to the collective Hamiltonian are mapped by (the Poisson map) J_1 to the reduced dynamics on $\mathfrak{g}^*$. Moreover, ~~recall that θ is invariant under cotangent lifted group actions [AM08, Theorem 3.2.12]. Thus by Cartan's formula [AM08, Theorem 2.4.13(iv)] we can calculate~~

  $$\underline{0 = \mathcal{L}_{\rho_1(\xi_1)}[\theta]}~~{} = d\langle\theta_1, \rho_1(\xi_1)\rangle + i_{\rho_1(\xi_1)}(d\theta) = d\langle\theta_1, \rho_1(\xi_1)\rangle - i_{\rho_1(\xi_1)}\omega,~~$$

  ~~where $i_{\rho_1(\xi_1)}\omega$ denotes contraction of a two-form with a vector field. Finally we calculate~~

  ~~$$\omega(\rho_1(\xi_1), \rho_2(\xi_2)) = -\langle d\langle\theta, \rho_1(\xi_1)\rangle, \rho_2(\xi_2)\rangle$$~~

  $$\underline{= -\mathcal{L}_{\rho_2(\xi_2)}[\langle\theta, \rho_1(\xi_1)\rangle] = -\langle\mathcal{L}_{\rho_2(\xi_2)}[\theta], \rho_1(\xi_1)\rangle} - \langle\theta, [\rho_2(\xi_2), \rho_1(\xi_1)]\rangle.$$

  ~~The first term in the last line vanishes because θ is invariant under the flow of $\rho_2(\xi_2)$. The second term in the last line vanishes because the actions of G_1 and G_2 commute. Thus $\omega(\rho_1(\xi_1), \rho_2(\xi_2)) = 0$, and we have proven symplectic orthogonality. Now note that $\langle TJ_2(\rho_1(\xi_1, x)), \xi_2\rangle = -\omega(\rho_1(\xi_1, x), \rho_2(\xi_2, x)) = 0$. Thus $\rho_1(\xi_1, x) \in$ kernel(J_2), hence $\rho_1(\mathfrak{g}_1, T^*Q) \subset$ kernel(J_2). But since G_1 acts transitively on the level sets of J_2, we see that $\rho_1(\mathfrak{g}_1, T^*Q) =$ kernel(J_2).~~

  ~~Let G_1, G_2 act on the manifold Q. Let ρ_1^* denote the cotangent-lifted action associated to G_1 and $J_2 : T^*Q \to \mathfrak{g}_2^*$ the momentum map defined by the cotangent-lifted action of G_2. Assume that the following hold:~~ J_2 is conserved under the dynamics on T^*Q. The conservation law follows from Noether's theorem because J_1, and hence H, are left invariant by Ψ (see Theorem A.9). Note that the elements of T^*Q play the role of symplectic variables (or Clebsch variables in the sense of [MW83]).

- The appeal of Clebsch variables is their symplectic nature. The symmetry of H with respect to Ψ implies that reduced dynamics on T^*G/H can be constructed by symplectic reduction. Note however that the resulting quotient manifold is not symplectic in general.

- ~~G_1 acts transitively on Q; the cotangent-lifted action ρ_1^* preserves level sets of J_2; for any μ in the range of J_2, the level set $J_2^{-1}(\mu)$ is a section of T^*Q. Then~~

~~the action ρ_1^* is transitive on level sets of~~ Note also that there is a symplectic diffeomorphism between T^*Q and $J^{-1}(0)/G_{q_0}$, where J here is the momentum map associated with the cotangent lift of the action (from the right) of G_{q_0} on G, see [MMO+07, Theorem 2.2.2].
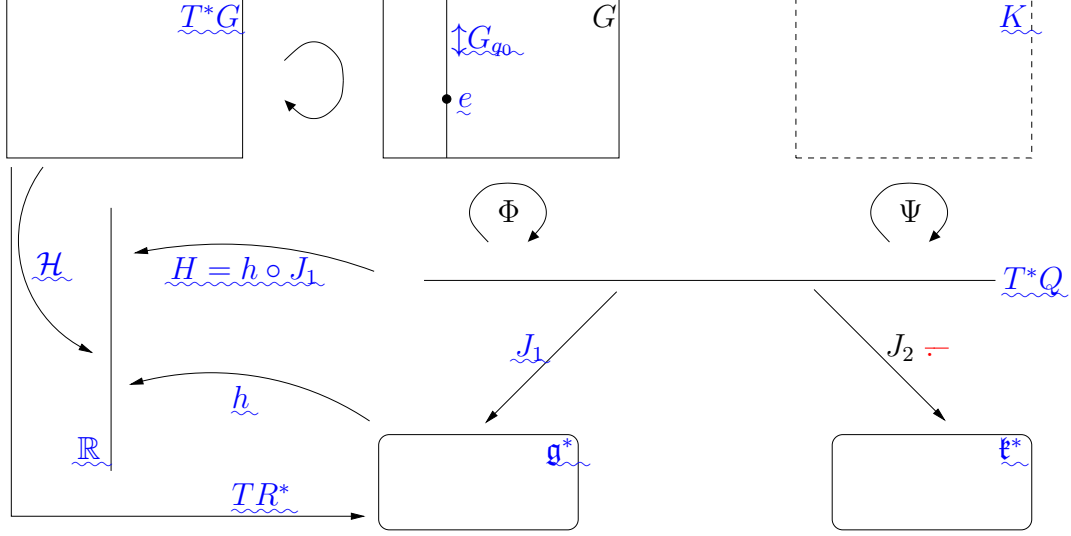

FIGURE 7. The cotangent lifted actions induce momentum maps J_1 and J_2. The Hamiltonian $\mathcal{H}$, assumed to be right-invariant, is also shown, along with the reduced Hamiltonian H and the collective Hamiltonian $H = h \circ J_1$.

In translating the above facts to the case of interest in the present paper, one encounters technical subtleties to do with the infinite-dimensionality of $\mathrm{SDiff}(\mathbb{R}^n)$. Nevertheless, bullet-by-bullet parallels can be recognized between the developments in the main text of the paper and the general results above, as we will discuss now. For simplicity, we restrict ourselves in what follows to the case of a single particle, the extension to N particles being straightforward.

- Let $G = \mathrm{SDiff}(\mathbb{R}^n)$, and let $Q = Q_1^{(k)}$ be the space of single-particle k-jetlets. We fix the point $q_0 = (z, \mathbf{1}, 0, \ldots, 0) \in Q_1^{(k)}$ corresponding to the Taylor expansion of the identity in $\mathrm{SDiff}(\mathbb{R}^n)$ evaluated at $z \in \mathbb{R}^n$ (cf. Section 4.4). We let the projection Π be given by $\mathrm{Jet}_z^{(k)}$, and hence the left action of an element $\varphi \in \mathrm{SDiff}(\mathbb{R}^n)$ on $q \in Q$ is $\phi_\varphi(q) = \mathrm{Jet}_z^{(k)}(\varphi \circ \rho)$, for any ρ such that $\mathrm{Jet}_z^{(k)}(\rho) = q$. The role of the isotropy subgroup G_{q_0} is played by $\mathrm{iso}_z^{(k)}$. Let ~~$(q,p),(q',p') \in J_2^{-1}(\mu)$ where $(q,p) \neq (q',p')$. Since $J_2^{-1}(\mu)$ is a section of T^*Q, this implies that $q \neq q'$. Now G_1 acts transitively~~ $K = \mathrm{iso}(z)$, and note that $\mathrm{iso}^{(k)}(z)$ is a normal subgroup of $\mathrm{iso}(z)$ ([DJR13, Proposition 4.1]). Hence, we can define a right action of $\mathrm{iso}(z)$ on ~~Q, so there exists a $g \in G_1$~~ $Q_1^{(k)}$ given by (28), namely

$$\psi_\rho(q) = \mathrm{Jet}_z^{(k)}(\varphi \circ \rho),$$

for any φ such that ~~$q' = g \cdot q$. Since ρ_1^* preserves $J_2^{-1}(\mu)$, we find that $g \cdot (q,p) \in J_2^{-1}(\mu) \cap T_{q'}^*Q$. Since $J_2^{-1}(\mu)$ is a section, this implies that $g \cdot (q,p) = (q',p')$ must hold.~~ $\mathrm{Jet}_z^{(k)}(\varphi) = q$.

- The cotangent lift of the left and right actions on Q lead to the momentum maps $J_L^{(k)}$ and $J_R^{(k)}$, explicitly computed in Section 4.4 and shown to be a weak dual pair in Proposition 4.7.
- In Theorem 4.8 we showed that the canonical Hamiltonian equations on $T^*Q_1^{(k)}$ associated with the collective Hamiltonian $H^{(k)} = h_{p,\sigma} \circ J_L^{(k)}$ lead to trajectories that are mapped, by $J_L^{(k)}$, to solutions of Hamilton's equations on $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$. Moreover, we showed that $J_R^{(k)}$ is a constant of motion.
- We briefly visit further symplectic reduction in Appendix E. For more on this topic we refer to [DJR13, CHJM14].
- In Section 4.5 we constructed the mapping $i : T^*Q_1^{(k)} \to T^*\,\mathrm{SDiff}(\mathbb{R}^n)/\mathrm{iso}^{(k)}$ given by (18), namely,

$$i(q, p) = [TR_{\varphi^{-1}}^* J_L^{(k)}(q, p)].$$

As we briefly mentioned towards the end of that section, if we denote by $J_{\mathrm{conv}}^{(k)} : T^*\,\mathrm{SDiff}(\mathbb{R}^n) \to \mathfrak{iso}^{(k)}$ the momentum map associated to the cotangent lift of the right action of $\mathrm{iso}^{(k)}(z)$ on $\mathrm{SDiff}(\mathbb{R}^n)$, then one can show that $i\big(T^*Q_1^{(k)}\big) = (J_{\mathrm{conv}}^{(k)})^{-1}(0)/\mathrm{iso}^{(k)}(z)$, as suggested by [MMO+07, Theorem 2.2.2].

APPENDIX C. MULTI-INDICES

A multiset is ~~nothing but~~ a set with some notion of multiplicity [Bli89]. In this paper, a multi-index on $\mathbb{R}^n$ is a multiset of elements derived from the generating set $\{1, \ldots, n\}$. Heuristically, a multi-index is just a "bag of marbles" each of which comes in n "colors". Given two multi-indices $\alpha \in \mathrm{bag}^i(n)$ and $\beta \in \mathrm{bag}^j(n)$ one can create the multiset union $\alpha \cup \beta \in \mathrm{bag}^{i+j}(n)$ by collecting the marbles of α and β into a single bag. Given integers $b_1, \ldots, b_j \in \mathbb{R}^n$, we can define the unique multi-index $\beta = [b_1, \ldots, b_j] \in \mathrm{bag}^j(n)$ obtained by collecting $b_1, \ldots, b_j$ into a bag. Given these conventions, we can denote the partial differential operator $\partial_{b_1 \cdots b_n}$ by ∂_β. Moreover, the notion of equivalence of mixed partials is expressed by the equivalence $\partial_\alpha \partial_\beta = \partial_{\alpha \cup \beta} = \partial_\beta \partial_\alpha$. The cardinality of the multi-index β is denoted $|\beta|$ and is given by the number of marbles in the bag. Thus the order of the partial differential operator ∂_β is $|\beta|$. We denote the space of k-th order partitions of a multi-index by $\Pi(\alpha, k)$. Rather than define all this formally, we will compute an example and refer to [Jac14] for the formal definitions.

We can consider the integers 1 2 and 1, and the partial differential operator ∂_{121}. The associated multi-index is just $[1, 2, 1]$. This multi-index is equivalent to the multi-index $[1, 1, 2]$ and $[2, 1, 1]$. We say that it contains the elements 1 and 2. Because it contains '1' two times, we say that the multiplicity of 1 is 2. The multiset of 2-fold partitions is $\Pi([1, 2, 1], 2)$, and consists of three multiset-partitions

$$[[1], [2, 1]], [[1, 1], [2]], [[1, 2], [1]].$$

Note that the first and the third partition correspond to the same multiset. The cardinality of $\Pi([1, 2, 1], 2)$ is 3, although it only has two distinct elements (one with a multiplicity of 1, and another with a multiplicity of 2).

Appendix D. The dual space to divergence free vector fields

In this section we will provide a terse and incomplete characterization of the dual space of divergence free vector fields. First let us characterize the dual space of the space of all vector fields (with "proper" decay). Let $\mathfrak{X}(\mathbb{R}^n)$ be the space of vector fields which decay at infinity in such a way that $\mathfrak{X}(\mathbb{R}^n)$ is Fréchet. Viewing $\mathfrak{X}(\mathbb{R}^n)$ as a subspace of functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ we can view its dual as a space of distributions. That is to say, given any $m \in \mathfrak{X}(\mathbb{R}^n)^*$ we may write m as a tensor product $p \otimes \mu$ where μ is a distribution (perhaps a measure) on $\mathbb{R}^n$ and p is a covector field (i.e. a one-form). Conversely, given any $p \in \Omega^1(\mathbb{R}^n)$ and distribution $\mu \in \mathcal{D}(\mathbb{R}^n)$ we may form the tensor product $p \otimes \mu$. The object $p \otimes \mu$ is identified as an element of $\mathfrak{X}(\mathbb{R}^n)^*$ through the pairing

$$\langle p \otimes \mu, u \rangle := \int_{\mathbb{R}^n} \langle p(x), u(x) \rangle \mu.$$

where $\langle p(x), u(x) \rangle$ is the function on $\mathbb{R}^n$ obtained by pairing the covector $p(x) \in T_x^*\mathbb{R}^n$ with the vector $u(x) \in T_x\mathbb{R}^n$. If we restrict ourselves to the case of divergence free vector fields, we need to quotient the dual space appropriately. In particular, we see that the annihilator of $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)$ as a subspace of $\mathfrak{X}(\mathbb{R}^n)^*$ is

$$(\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n))^\circ := \{m \in \mathfrak{X}(\mathbb{R}^n)^* \mid \langle m, u \rangle = 0, \forall u \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)\}$$
$$:= \mathrm{closure}\{p \otimes \mathrm{d}x \in \mathfrak{X}(\mathbb{R}^n)^* \mid p \text{ is a closed one-form}\}.$$

where $\mathrm{d}x$ is the canonical volume form on $\mathbb{R}^n$ and we have used the fact that the gradient fields and the harmonic vector fields are L^2-orthogonal to the divergence free vector fields. The dual space $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$ is identical to the quotient space $\mathfrak{X}(\mathbb{R}^n)^*/(\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n))^\circ$. In other words, we may view a $m \in \mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^*$ as an object of the form $p \otimes \mu$ modulo $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^\circ$. In the text we will typically not mention $\mathfrak{X}_{\mathrm{div}}(\mathbb{R}^n)^\circ$ explicitly, and simply identify m with $p \otimes \mu$. This is a harmless identification as long as we do not pair it with a non-divergence free vector field.

Appendix E. Equations of motion for 1-jetlets

The equations of motion are expressible as Hamiltonian equations on $T^*Q_N^{(1)}$ in canonical variable $(q^{(0)}, p^{(0)}, q^{(1)}, p^{(1)})$. However, it is more efficient to express the equations of motion in the non-canonical variables (q, p, μ) where $q_a = q_a^{(0)}$, $p_a = p_a^{(0)}$ and $\mu_a = [q_a^{(1)}]^\ell_i [p_a^{(1)}]^j_\ell$ for $a = 1, \ldots, N$. The Hamiltonian in these coordinates is

$$H(q, p, \mu) = \frac{1}{2} p_{ai} p_{bj} K^{ij}(q_a - q_b) - p_{ai}[\mu_b]^{\,k}_j \partial_k K^{ij}(q_a - q_b)$$
$$- \frac{1}{2}[\mu_a^{(1)}]^{\,l}_i [\mu_b^{(1)}]^{\,k}_j \partial_{lk} K^{ij}(q_a - q_b)$$

where $K^{ij}(x) = \delta^{ij} e^{-\|x\|^2/2\sigma^2}$. Hamilton's equations are then given in short by

$$\dot{q} = \frac{\partial H}{\partial p} \tag{29}$$

$$\dot{p} = -\frac{\partial H}{\partial q} \tag{30}$$

$$\xi = \frac{\partial H}{\partial \mu} \tag{31}$$

$$\dot{\mu} = -\,\mathrm{ad}^*_\xi(\mu). \tag{32}$$

Where ad^* refers to the coadjoint operator on $\mathrm{SL}(n)$. More explicitly, equation (29) is given by

$$\dot{q}^i_a = p_{bj} K^{ij}(q_a - q_b) - [\mu^{(1)}_b]_j^{\ k} \partial_k K^{ij}(q_a - q_b)$$

equation (30) is given by the sum

$$\dot{p}_{ai} = T^{00}_{ai} + T^{01}_{ai} + T^{11}_{ai}$$

where we define the three terms in this sum as

$$T^{00}_{ai} = -p_{ak} p_{bj} \partial_i K^{kj}(q_a - q_b)$$
$$T^{01}_{ai} = (p_{al}[\mu^{(1)}_b]_j^{\ k} - p_{bl}[\mu^{(1)}_a]_j^{\ k})\partial_{ki} K^{lj}(q_a - q_b)$$
$$T^{11}_{ai} = [\mu^{(1)}_a]_m^{\ l}[\mu^{(1)}_b]_j^{\ k} \partial_{lki} K^{mj}(q_a - q_b)$$

Next, we calculate the quantities $\xi = \partial H/\partial \mu$ for $k = 1, 2$ of equation (31) to be

$$[\xi_a]^i_{\ j} = p_{bk}\partial_j K^{ik}(q_a - q_b) - [\mu_b]_l^{\ k}\partial_{jk} K^{il}(q_a - q_b)$$

which allows us to compute $\dot{\mu}$ in equation (32) as

$$[\dot{\mu}_a]_i^{\ j} = [\mu_a]_i^{\ k}[\xi^{(1)}_a]^j_{\ k} - [\mu_a]_k^{\ j}[\xi^{(1)}_a]^k_{\ i}$$

The dynamics in terms of the original variables $(q^{(0)}, q^{(1)})$ with $q^{(1)} \in \mathrm{SL}(n)$ are obtained by integrating the reconstruction equation $[\dot{q}^{(1)}]^i_{\ j} = [\xi]^i_{\ k}[q^{(1)}]^k_{\ j}$.

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