Realtime Probabilistic Pedestrian Forecasting

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***Abstract*—The abstract goes here.**

I. Introduction

hoj: maybe ram should write the first paragraph.

A. Background

Papers to mention

- Kitani et al. [5] (Early MDP based paper)
- Karasev et al. [4] (Another motion model, discrete time)
- Ballan et al. [2] (The knowledge transfer paper)
- **?**] (Another MDP paper)
- **?**] (fluid based crowd model)

II. Model

ramv: I would include far fewer model details here just to illustrate that our focus is on efficient computation for general models.

Our goal is to generate a time-dependent probability density over $\mathbb{R}^2$, which predicts the true location of an agent in the future. The input to the algorithm at runtime is a noisy measurement of position and velocity, $\hat{x}_0, \hat{v}_0 \in \mathbb{R}^2$. If the (unknown) location of agent at time t is given by $x_t \in \mathbb{R}^2$, then the distribution we seek is the posterior $\rho_t(x_t) := \Pr(x_t \mid \hat{x}_0, \hat{v}_0)$.

In order to derive an expression for ρ we will build a probabilistic graphical model. Our model assumes we have noisy information about agents, and each agent moves with some intention through the world in a way that is roughly approximated by a model. This allows our model to be divided into three parts:

1) Reality: This is parametrized by the true position for all time, x_t, and the initial velocity of the agent v_0.
2) The measurements: This is represented by our sensor readings $\hat{x}_0$ and $\hat{v}_0$ and are independent of all other variables given the true initial position and velocity, x_o, v_0.
3) The motion model: This is represented by a trajectory $\check{x}_t$ and depends on a variety of other variables which will be described shortly.

We will now elaborate on these three components, and relate them to one another.

A. The Sensor Model

At time $t = 0$, we obtain a noisy reading of position, $\hat{x}_0 \in \mathbb{R}^2$. We assume that our measurements come from Gaussian noise about the true location $x_0 \in \mathbb{R}^2$. We use the same sensor model is assumed for velocity. In principal, any measurement model could be used, and we may view this as a parameter of our method.

B. The Agent Model

All agents are initialized within some rectangular region $D \subset \mathbb{R}^2$. We denote the true position of an agent by x_t. We should never expect to know x_t and the nature of its evolution precisely, and any model should account for the fact that any model is almost surely false, to some degree. We do this by fitting a deterministic model to the data and blurring the result in order to acknowledge inevitable modeling errors.

Specifically, our motion model consists of a modeled trajectory $\check{x}_t$, which is probabilistically related to the true position by x_t via

$$\Pr(x_t \mid \check{x}_t) \sim \mathcal{N}(\check{x}_t, \sigma_{int}(t)),$$

where $\sigma_{int}(t) = \kappa t$ for some learned constant $\kappa > 0$. This insures the consistency condition $\check{x}_0 = x_0$, while acknowledging the existence of uncertainty for all $t > 0$.

Once initialized, agents come in two flavors: linear and nonlinear. Linear agents imagine that they move in straight lines at constant velocity. Thus $\check{x}_t = x_0 + tv_0$ and so we have the posterior

$$\Pr(\check{x}_t \mid x_0, v_0, lin) = \delta(\check{x}_t - x_0 - tv_0).$$

We will assume linear agents also satisfy the posteriors

$$\Pr(x_0 \mid lin) \sim \mathcal{U}(D) \quad , \quad \Pr(v_0 \mid lin) \sim \mathcal{N}(0, \sigma_L).$$

If the agent is of nonlinear type, then we assume the dynamics take the form

$$\frac{d\check{x}_t}{dt} \equiv \check{v}_t = s \cdot X_k(\check{x}_t) \tag{1}$$

where X_k is a vector-field from a finite collection $\{X_1, \ldots, X_n\}$, and $s \in \mathbb{R}$. More specifically, we assume that each X_k has the property that $\|X_k(x)\| = 1$ for all $x \in D$. This property ensures that speed is constant in time, and it has the further advantage of being the (local) generator of solutions to an optimal navigation problem (see Appendix A). The vector-fields $X_1, \ldots, X_n$ are learned from the data-set (see §**??**).

It is assumed that k and s are both constant in time, so that $\check{x}_t$ is determined from the tripe (x_0, k, s) by integrating (1) with the initial condition x_0. In other words, we have the posterior

$$\Pr(\check{x}_t \mid x_0, k, s) = \delta(\check{x}_t - \Phi^t_{k,s}(x_0))$$

where $\Phi^t_{k,s}$ is the flow-map of the vector field $s\,X_k$ up to time t. It is notable, that the variables k, s and x_0 determine v_0. Thus we also have the posterior

$$\Pr(v_0 \mid k, s, x_0) = \delta(v_0 - sX_k(x_0)).$$

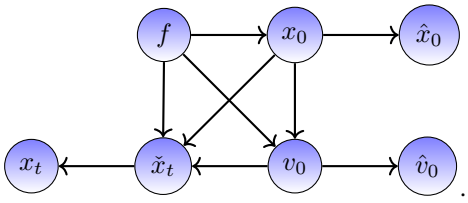
In summary, the agent flavors are parametrized by the set

$$F = \{lin\} \cup (\mathbb{R} \times \{1, \dots, n\})$$

and each flavor determines the sort of motion we should expect from that agent.

C. The Full Model

Concatenating the measurement model with our motion model yields the Bayesian network:



We may use this Bayesian network to compute ρ_t efficiently. In particular

$$\begin{aligned}\rho_t(x_t) &:= \Pr(x_t \mid \hat{x}_0, \hat{v}_0) \\ &= \left(\sum_k \int \Pr(x_t, k, s \mid \hat{x}_0, \hat{v}_0) ds\right) \\ &\quad + \Pr(x_t, lin \mid \hat{x}_0, \hat{v}_0).\end{aligned}$$

The final term $\Pr(x_t, lin \mid \hat{x}_0, \hat{v}_0)$ is expressible in terms of the error function, and would pose a negligible burden in terms of numerical computation. The primary computational burden in computint $\rho_t(x_t)$ constitutes the computation of $\left(\sum_k \int \Pr(x_t, k, s \mid \hat{x}_0, \hat{v}_0) ds\right)$.

III. Efficient Probability Propagation

hoj: Make sure we mention this.

As mentioned , many of the modeling choices were born out of a balance between accuracy and real-time computability. One of the major modeling choices, that agents move approximately according to a small number of ODEs, is the most prominent such choice. In this section we give some details on how this modeling choice can be leveraged to compute ρ quickly, and accurately.

Firstly, note that ρ may be decomposed by the law of total probability as

$$\begin{aligned}\rho_t(x_t) &:= \Pr(x_t \mid \hat{x}_0, \hat{v}_0) \\ &= \left(\sum_k \int \Pr(x_t, k, s \mid \hat{x}_0, \hat{v}_0) ds\right) \\ &\quad + \Pr(x_t, lin \mid \hat{x}_0, \hat{v}_0).\end{aligned}$$

hoj: Must I provide this? It's a bit long.

The final term, $\Pr(x_t, lin \mid \hat{x}_0, \hat{v}_0)$, turns out to be a closed form expression involving error-functions.The other terms are more difficult to compute efficiently, and are the primary concern of this section. To begin, from the graphical model we can see that

$$\begin{aligned}\Pr(x_t, k, s, \mid \hat{x}_0, \hat{v}_0) &\propto \Pr(x_t, k, s, \hat{x}_0, \hat{v}_0) \\ &= \int \Pr(x_t, \check{x}_t, \hat{x}_0, \hat{v}_0, k, s) d\check{x}_t \\ &= \int \Pr(x_t \mid \check{x}_t) \Pr(\check{x}_t, \hat{x}_0, \hat{v}_0, k, s) d\check{x}_t\end{aligned}$$

As $\Pr(x_t \mid \check{x}_t) \sim \mathcal{N}(\check{x}_t; \kappa t)$, we see from the last line that $\Pr(x_t, k, s, \mid \hat{x}_0, \hat{v}_0)$ is proportional to a Gaussian convolution of of the joint distribution $\Pr(\check{x}_t, \hat{x}_0, \hat{v}_0, k, s)$. Assuming such a convolution can be performed efficiently (which we will address later), we can focus on computation of $\Pr(\check{x}_t, \hat{x}_0, \hat{v}_0, k, s)$.

Again, from the PGM we see that

$$\begin{aligned}\Pr(\check{x}_t, \hat{x}_0, \hat{v}_0, k, s) &= \int \Pr(\check{x}_t, x_0, \hat{x}_0, v_0, \hat{v}_0, k, s) dx_0\, dv_0 \\ &= \int \Pr(\check{x}_t \mid x_0, k, s, v_0) \Pr(\hat{x}_0, x_0, \hat{v}_0, v_0, k, s) dx_0\, dv_0 \\ &= \int \delta\left(\check{x}_t - \Phi^t_{k,s}(x_0)\right) \Pr(\hat{x}_0, x_0, \hat{v}_0, v_0, k, s) dx_0\, dv_0 \\ &= \int \delta\left(\check{x}_t - \Phi^t_{k,s}(x_0)\right) \delta\left(v_0 - sX_k(x_0)\right) \\ &\qquad \Pr(\hat{x}_0, x_0, \hat{v}_0, v_0, k, s) dx_0\, dv_0\end{aligned}$$

Carrying out the integration over v_0 we observe

$$\begin{aligned}&\Pr(\check{x}_t, \hat{x}_0, \hat{v}_0, k, s) \\ &= \int \delta\left(\check{x}_t - \Phi^t_{k,s}(x_0)\right) \Pr(\hat{x}_0, x_0, \hat{v}_0, v_0(s, k, x_0), k, s) dx_0\end{aligned} \tag{2}$$

where $v_0(s, k, x_0) := s\, X_k(x_0)$. As the density $\Pr(\hat{x}_0, x_0, \hat{v}_0, v_0(s, k, x_0), k, s)$ is Riemann integrable in x_0, we may approximate it as a sum of weighted Dirac delta's supported on a regular grid with spacing Δx.In other words

Perhaps prove this claim in the appendix.

$$\begin{aligned}&\Pr(\hat{x}_0, x_0, \hat{v}_0, v_0(s, k, x_0), k, s) = \\ &\left(\sum_\alpha c_{k,s,\alpha} \delta(x_0 - \bar{x}^\alpha_0)\right) + \varepsilon_0(x_0)\end{aligned} \tag{3}$$

for constants

$$c_{k,s,\alpha} = \left[\Pr(x_0, \hat{x}_0, \hat{v}_0, v_0(s, k, x_0), k, s)\right]\big|_{x_0 = x^\alpha_0} \tag{4}$$

and error of magnitude $\|\varepsilon_0\|_{L^1} \sim \mathcal{O}(\Delta x)$ with respect to the L^1-norm in x_0. The coefficients, $c_{k,s,\alpha}$, can be computed efficiently as product of the posteriors appearing in the PGM. Substitution of (3) into the final line of (2) yields

$$\Pr(\check{x}_t, \hat{x}_0, \hat{v}_0, k, s) = \sum_\alpha c_{k,s,\alpha} \delta\left(\check{x}_t - \Phi_{k,s}(x^\alpha_0)\right) + \varepsilon_t(\check{x}_t)$$

where $\varepsilon_t = \int \delta\left(\check{x}_t - \Phi^t_{k,s}(x_0)\right) \varepsilon_0(x_0) dx_0$. The first term is computable by flowing the points of the grid, x^α_0, by the evolution of the vector field sX_k. The second term, $\tilde{\varepsilon}$, may be viewed as an error term. In fact, this method of approximating $\Pr(\check{x}_t, \hat{x}_0, \hat{v}_0, k, s)$ as a sum of Dirac-deltas is adaptive, in that the error term does not grow in total mass.

Theorem 1. *The error term, ε_t, is of size $\mathcal{O}(\Delta x)$ in the L^1-norm, for fixed $k, s, \hat{x}_0$, and $\hat{v}_0$. Moreover, the magnitude is constant in time.*

Proof. To declutter notation, let us temporarily denote $\Phi^t_{k,s}$ by Φ. We observe

$$\begin{aligned}\|\varepsilon_t\|_{L^1} &= \int \left| \int \delta(\check{x}_t - \Phi(x_0))\varepsilon_0(x_0)dx_0 \right| d\check{x}_t \\ &= \int \det(\, D\Phi|_{\Phi^{-1}(\check{x}_t)})|\varepsilon_0(\Phi^{-1}(\check{x}_t))|d\check{x}_t \\ &= \int |\varepsilon_0(u)|du = \|\varepsilon_0\|_{L^1}\end{aligned}$$

As ε_0 is of magnitude $\mathcal{O}(\Delta x)$ the result follows. □

As the Gaussian convolution of a Dirac-delta distribution is a Gaussian distribution, the following corollary is nearly immediate.

Corollary 1. *Let $G_\sigma(x-\mu)$ denote the probability density of a bivariate normal distribution of standard deviation σ and mean μ. Then the density*

$$\sum_\alpha c_{k,s,\alpha} G_{\kappa t}(x_t - \Phi^t_{k,s}(x_0^\alpha)) \tag{5}$$

is an approximation of $\Pr(x_t, k, s, \hat{x}_0, \hat{v}_0)$ with a constant in time error bound of magnitude $\mathcal{O}(\Delta x)$.

Proof. Recall that $\Pr(x_t, k, s, \hat{x}_0, \hat{v}_0)$ is related to $\Pr(\check{x}_t, k, s, \hat{x}_0, \hat{v}_0)$ by convolving the latter density in the variable $\check{x}_t$ with a Gaussian kernel of variance κt. That is to say

$$\Pr(x_t, k, s, \hat{x}_0, \hat{v}_0) = \int G_{\kappa t}(x_t - \check{x}_t) \Pr(\check{x}_t, k, s, \hat{x}_0, \hat{v}_0) d\check{x}_t$$

Substitution of (3) yields

$$\Pr(x_t, k, s, \hat{x}_0, \hat{v}_0) = \sum_\alpha c_\alpha G_{\kappa t}(x_t - \Phi^t_{k,s}(x_0^\alpha)) + (G_{\kappa t} * \varepsilon_t)$$

Gaussian convolution preserves the L^1-norm of a density. Therefore $\|G_{\kappa t} * \varepsilon_t\|_{L^1} = \|\varepsilon_t\|_{L^1} \sim \mathcal{O}(\Delta x)$, which is also constant in time by Theorem 1. □

Corollary 1 justifies using (5) as an approximation of $\Pr(x_t, k, s, \hat{x}_0, \hat{v}_0)$. So we've reduce the problem of computing $\rho_t(x_t)$ to the problem of computing the coefficient $c_{k,s,\alpha}$ and the point $\Phi^t_{k,s}(x_0^\alpha)$ for all k, s and points $x_0^\alpha \in \Lambda$. We can reduce this burden further by exploiting the following symmetry.

Theorem 2. $\Phi^t_{k,s} = \Phi^{st}_{k,1}$.

Proof. Say $x(t)$ satisfies the ordinary differential equation $x'(t) = sX_k(x(t))$ with the initial condition x_0. In other words, $x(t) = \Phi^t_{k,s}(x_0)$. Taking a derivative of $x(t/s)$, we see $\frac{d}{dt}(x(t/s)) = x'(t/s)/s = X_k(x(st))$. Therefore $x(t/s) = \Phi^t_{k,1}(x_0)$. Substitution of t with $\tau = t/s$ yields $x(\tau) = \Phi^{s\tau}_{k,1}(x_0)$. As $x(\tau) = \Phi^\tau_{k,s}(x_0)$ as well, the result follows. □

Thus, computation of $\Phi^t_{k,s}(x_0^\alpha)$ boils down to computing $\Phi^t_{k,1}(x_0^\alpha)$.

At this point, we may summarize the ingredient to compute an approximation of $\Pr(x_t, k, s \mid \hat{x}_0, \hat{v}_0)$ in two parts

1) Compute the constants c_α from (4).
2) Compute $\Phi^t_{k,1}(x_0^\alpha)$ for each k and each point x_0^α in the grid over a time-interval $[-T, T]$.

Each of these pieces may be computed efficiently, and the full computation is embarrassingly parallel. For example, for fixed k and α, the computation of $\Phi^t_{k,1}(x_0^\alpha)$ on the interval $[-T, T]$ take $\mathcal{O}(T)$ time using a an explicit finite difference scheme, and computation for each (k, α) may be performed in parallel with another such pair. Moreover, computation $c_{k,s,\alpha}$ is embarrassingly parallel in with respect to all triples (k, s, α). If the posteriors of our PGM consist solely of elementary functions such as Gaussians and polynomials, then computation of all coefficients $c_{k,s,\alpha}$ can be made virtually instantaneous given enough processors, as would be the case with a sufficiently powerful graphics processing unit. More precisely, the full computation of our approximation of $\rho_t(x_t)$ would take $\mathcal{O}(TnA/N)$ time given N processing units where $A = |\cup_\alpha \{x_0^\alpha\}|$ is the size of the mesh and n is the number of vector fields.

IV. Implementation and Experimental results

Now that the model has been described, we can discuss a way, but certainly not the only way, that one can fit parameters of the model to a data set. For the purpose of demonstration, we will use the Stanford Drone Dataset [**?**]. More generally, we will assume that for a fixed scene we have a database of previously observed trajectories $\{\hat{x}^1, \ldots, \hat{x}^N\}$. From this data we will tune the parameters of the model appropriately. The parameters of the model which must be chosen or learned are

- the vector-fields $\{X_k\}_{k=1}^n$,
- the posteriors $\Pr(x \mid k, s)$ for $k = 1, \ldots, m$ and $s \in \mathbb{R}$,
- the posterior $\Pr(x \mid lin)$,
- the unconditioned probabilities $\Pr(f)$ for $f \in F$,
- the standard deviations of the measurement σ_x and σ_v,
- and the blurring parameter κ.

A. Learning the vector fields

Before we can even begin to learn vector-fields, we must learn the number of such vector-fields we have. To do this we use a simple clustering algorithm on the trajectories, to categorize them into groups. In principal, any clustering algorithm could accomplish this. For our algorithm we use the start and end point for each trajectory to obtain a point in $\mathbb{R}^4$. We then cluster in $\mathbb{R}^4$ using Affinity propagation.This clustering of the end-points induces a clustering of the trajectories. So we obtain clusters $S_1, \ldots, S_n$ consisting of trajectories from our data set, as well as a set of unclassified trajectories, S_0.

hoj:There is more to say here

For each set S_k we may learn a vector-field which is approximately compatible with that set. We'd like to only consider vector-fields who's vectors have unit magnitude because we observe many of the trajectories in the data appear to have a roughly constant speed. Therefore we assume the vector-field takes the form $X_k(x) = (\cos(\Theta_k(x)), \sin(\Theta_k(x)))$ for some scalar function $\Theta_k(x)$. Learning the vector-fields then boils

down to learning the scalar function Θ_k. We will assume Θ_k takes the form

$$\Theta_k(x) = \sum_\alpha \theta_{k,\alpha} L_\alpha(x)$$

for some collection of coefficients, $\theta_{k,\alpha}$, and a fixed collection of basis functions, L_α. In our case, we've chosen L_α to be a set of low degree Legendre polynomials. Learning Θ_k can be obtained by looking at all the velocities observed in the cluster, S_k. These velocities may be obtained by a low order finite difference formula. Upon normalizing the velocities, we obtain a unit-length velocity vectors, $v_{i,k}$, anchored at each point, $x_{i,k}$, of S_k. We can learn Θ_k by defining the cost-function

$$C[\Theta_k] = \sum_i \langle v_{i,k}, (\cos(\Theta_k(x_{i,k}), \sin(\Theta_k(x_{i,k}))\rangle$$

which penalizes Θ_k for producing a misalignment with the observed velocities at the observed points of S_k. When Θ_k includes high order polynomials (e.g. beyond 5th order), we should also include a regularization term to bias the minimizes towards smoother outputs. Using the H^1-norm times a fixed scalar would suffice.

B. Learning the posteriors

We'd like to compute $\Pr(x_0 \mid k, s)$ for each cluster $k = 1, \ldots, n$ and each speed $s \in \mathbb{R}$. We will make the modeling assumption that x_0 is independent of s given k, i.e. $\Pr(x_0 \mid k, s) = \Pr(x_0 \mid k)$. Additionally, we will assume that s and k are independent. This means that we only need to learn $\Pr(x_0 \mid k)$, $\Pr(k)$, and $\Pr(s)$.

We will let $\Pr(k) = (n+1)^{-1}$ and $\Pr(s) \sim \mathcal{U}([-s_{\max}, s_{\max}])$ where $s_{\max} > 0$ is the largest observed speed in the dataset. This implies that $\Pr(lin) = (n+1)^{-1}$ as well. Other reasonable choices (such as $\Pr(k) \propto |S_k|$) could work, but we've chosen to be more conservative in this paper.

In order to learn $\Pr(x_0 \mid k)$ we will assume that it's logarithm may be approximated by a low dimensional subspace of functions on $\mathbb{R}^2$. That is to say, for each k we will assume $\Pr(x_0 \mid k) = \frac{1}{Z_k} \exp(-V_k(x_0))$ and V_k is a function who's constant term is 0 and is given by

$$V_k(x_0; \mathbf{c}) := \sum_{|\alpha|<d} c_\alpha L_\alpha(x_0)$$

for a collection of basis functions, L_α and coefficients $\mathbf{c} = \{c_\alpha\}_{|\alpha|<d}$. We chose our basis functions to be the collection of tensor products from the first 6 Legendre polynomials, normalized to the size of the domain. Then, one may fit the coefficients c_α to the data by using a log-likelihood criterion. The resulting (convex) optimization problem takes the form

$$\mathbf{c}^* = \inf_{|\mathbf{c}|} \sum_{x \in S_k} V_k(x_0; \mathbf{c})$$

Where the norm on $\mathbf{c}$ is a sup-norm. We can also bias this optimization towards more regular functions by adding a penalty to the cost function.

Finally, we let $\Pr(x_0 \mid lin) \sim \mathcal{U}(D)$.

C. Learning the variance parameters

We will assume that the true trajectory of an agent is fairly smooth compared to the noisy output of our measurement device. This justifies smoothing the trajectories, and using the difference between the smoothed signals and the raw data to learn the variance σ_x. To obtain the results in this paper we have used a moving average of 4 time steps (this is 0.13 seconds in realtime). We set $\sigma_v = 2\sigma_x/\Delta t$ where $\Delta t > 0$ is the time-step size. This choice is justified from the our use of finite difference's to estimate velocity. In particular, if velocity is approximated via finite differencing as $v(t) \approx (x(t+h) - x(t))\,\Delta t^{-1} + \mathcal{O}(h)$ and the measurements are corrupted by Gaussian noise, then the measurement $\hat{v}(t)$ is related to $v(t)$ by Gaussian noise with roughly the same standard deviation as $(x(t+h) - x(t))\,\Delta t^{-1}$.

Finally, we must learn κ, the parameter which blurs the our motion model. We've taken an ad hoc approach in this paper, although more intelligent and theoretically sound alternatives likely exist. For each curve in S_k we create a synthetic curve using the initial position and speed and integrating the corresponding vector-field, $s\,X_k$. So for each curve, $x_i(t)$, of S_k, we have a synthesized curve $x_{i,synth}(t)$ as well. We then measure the standard deviation of $(x_i(t) - x_{i,synth}(t))/t$ over i and at few time, $t \in \{0, 100, 200\}$ in order to obtain κ.

D. Precision, accuracy, etc.

V. Conclusion

The conclusion goes here.

Acknowledgments

Thanks to Kris Kitani

Appendix

A. Inverse Optimal Control

One of the valuable aspect of the motion model of [5] was that the trajectories were solutions of optimal control problems. This is a desirable property for a motion model of pedestrians/cyclists/vehicles in that it seems reasonable to assume that we all move through the world with an intent to get somewhere. This framework where one determines an optimal control problem from trajectories (as opposed to solving for trajectories of an optimal control problem) was dubbed *inverse optimal control* (IOC). In this optional subsection, we illustrate how solutions of 1 can fit within the IOC framework.

Theorem 3. *Let X be a vector-field such that $\|X(x)\|_2 = 1$ for all $x \in \mathbb{R}^2$. Then, for each initial condition $x_0 \in \mathbb{R}^2$, there exists a neighborhood U, a Riemannian metric, g, and a cost function, f, such that solutions of* (1) *are also solutions of the optimal control problem*

$$p^* = \inf_{\|v_t\|_g = s} \int_0^T f(x_t) dt \tag{6}$$

where $v_t = \frac{dx_t}{dt}$, $\|\cdot\|_g$ is the norm with respect to g and the infimum is taken over all differentiable curves in U which originate from x_0.

Proof. By rescaling time, we may let s be any positive real number. Therefore, without loss of generality, we will seek an f and g such that $s = 1$. In this case a solution of (6) is generated by the vector-field $X = \frac{\nabla f}{\|\nabla f\|}$ where ∇ is the Riemannian gradient with respect to some metric (possibly non-Euclidean). Therefore, our goal is to illustrate that there exists a metric g and a function f such that the given X is given by $\frac{\nabla f}{\|\nabla f\|}$.

Because $\|X\| = 1$ globally, it has no fixed points. From the flow-box theorem [1, Theorem 4.1.14] we can assert that for any open set, $U \subset \mathbb{R}^2$ there exists a local diffeomorphism which transforms X_k into a flat vector-field. Mathematically, this asserts the existence of a map $\Phi : U \to V \subset \mathbb{R}^2$ such that the push-forward of X, which we will denote by $\tilde{X}$, and given by

$$\tilde{X}(\tilde{x}) = \Phi_* X(\tilde{x}) := D\Phi|_{\Phi^{-1}(\tilde{x})} \cdot X(\Phi^{-1}(\tilde{x})),$$

is just the flat vector field $(1, 0)$ for all $x \in U$. We could view the coordinate function $\tilde{f}(x) = x^0$ as a cost function on V, and we may set g_V to be equal to the standard flat metric on V inherited from $\mathbb{R}^2$. In this case we observe that $\tilde{X}$ generates solutions to the optimization problem

$$\tilde{p}^* = \inf_{\|\tilde{v}_t\|_{g_V} \leq 1} = \int_0^T \tilde{f}(\tilde{x}_t) dt$$

By a change of variables, it follows that the original vector field, X, then solves the optimization problem (6) where $g = \Phi^* g_V$ is the pull-back metric, and $f = \Phi^* \tilde{f}$ is the pull-back of $\tilde{f}$. □

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