

# Real-Time Probabilistic Pedestrian Forecasting

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**Abstract**—That autonomous vehicles will come to dominate our streets is imminent. This motivates the need for a real-time probabilistic forecasting algorithm for pedestrians, cyclists, and other agents, as it forms a necessary step in assessing the risk (and therefore the cost) we should expect to incur in our brave new world. How can we minimize risk and maximize comfort if we can not predict it? “But I don’t want comfort. I want God, I want poetry, I want real danger, I want freedom, I want goodness. I want sin.” We’ll, too bad! comfort and safety will soon rule with an AI-based iron fist! In this paper, we present a novel approach to probabilistic forecasting for pedestrians based on weighted sums of ordinary differential equations. The resulting algorithm is embarrassingly parallelizable, and trained on historical trajectory information in a fixed scene. When compared with MDP-based methods, our algorithm appears to be superior from the standpoint of precision and recall. Civilization is sterilization

for the love of god please don’t forget to write the actual abstract

## I. INTRODUCTION

Autonomous systems acting in human-centric environments should be able to anticipate the behavior of neighboring humans to ensure safe operation. Anticipating the future locations of individuals within a scene, for instance, is critical to synthesizing safe controllers for autonomous vehicles. To aid during safe control design for such systems, a prediction algorithm must satisfy several important criteria. First, since safety is paramount, the prediction algorithm should avoid misclassifying areas as unoccupied. Second, the predictions must be generated at faster than real-time rates to ensure that they can be used within a control loop. Finally, long-time horizon forecasts are preferable since they can make planning over long-time horizons feasible, which can reduce the conservativeness and aggressiveness of controllers [1].

Should we include a figure here showing what our algorithm looks like?

This paper presents an algorithm for real-time, long-term prediction of pedestrian behavior which can subsequently be used by autonomous agents.

As depicted in Figure ??, this approach relies upon ... The presented approach significantly outperforms...

### A. Background

Most forecasting algorithm are well characterized by the underlying evolution model they adopt. Such models come in a variety flavors, and are adapted to the task at hand (e.g. see [5] for crowd modeling). The present article concerns the construction of useful motion models for pedestrians which can aide the task of real-time forecasting for autonomous agents. The most simple approach to forecasting with motion

models would be to forward integrate a Kalman filter [6] based upon the observed heading. Over short time scales this may perform quite well, but the resulting distribution will devolve into a large Gaussian mass over longer time scales. Schneider and Gavrilu [11] found this quality to be a typical property of the extended Kalman filter and Kalman filter with deterministic nonlinear drift. In particular, such models seemed useless for forecasts beyond 2 seconds in the future, especially when the pedestrian made a turn. Nonetheless, these stochastic linear models serve as a reasonable default in the absence of any contextual knowledge. For example, Koo [1] uses a motion model of this variety parametrized by some latent variables, such as the pedestrians awareness of nearby vehicles, to predict when and how a pedestrian may cross a street.

While [1] illustrated the use of contextual information, such as the pedestrian’s own awareness of her surroundings, the motion model did not leverage any data from the environment. Perhaps this is a strength when no such data is available, but it is a lost opportunity otherwise. More sophisticated models that attempt to leverage environmental data include inverse optimal control (IOC) based models such as [13, 14, 8, 7]. These IOC models have the often desirable property of attributing intention and goal-seeking behavior to the agents. For example Kitani et al. [8] extracted a very simple Markov decision process (MDP) evolving on a finite 2D lattice by training on a small collection of features and trajectories. The resulting MDP is very light weight, parametrized by only a small collection of coefficients equal in number to that of the feature maps. Secondly, given an initial and final state, the probability distribution of the pedestrian at intermediate states is easily computable because it basically comes down to matrix multiplication. The speed of this computation, makes the algorithm of [8] a reasonable baseline for comparison for the algorithm which will be presented in this paper.

Since the publication of [8] a few generalizations of their approach have be born. For example, time dependent information, such as traffic signals, are incorporated in [7], by relaxing the Markov property and considering a switched Markov process. Karasev et al. [7] generalizes [8] by replacing the finite-state space with a continuous one, and using a Markov jump process in the motion model. The desired posteriors were difficult if not impossible to compute in closed for, and so they used a sample based method: a Rao-Blackwellized particle filter[4]. The resulting accuracy of such methods can only be known in a probabilistic sense in that the error bounds are themselves random variables. Moreover, high accuracy can come at a high computational cost prohibitive for real-time computation. Speed and analytic error bounds were not a primary objective of [7], and thus were not addressed in that

publication.

I could also mention *Inferring “Dark Matter” and “Dark Energy” from Videos*. It is another IOC model, and is cited in [3]. However, this paper is awful, and I don’t even know how to describe it if we presumable wanted to describe the results. Here is an actual sentence from the results section of the paper: *“The trajectory prediction (NLL and MHD) is more accurate is constrained scene (1 2) than free scenes (3 4).”*.

One difficulty which IOC models can suffer occurs when the initial and final conditions are very separated across a complex domains, analogous to the notion of conjugate points in optimal control. This can occur when agents make very sharp turns due to intermediate goals on the way toward reaching their final destination. To address this, Ballan et al. [3] adopted an empiricists approach, computing “turning maps” and attempting to infer how agents behave in a given patch based on it’s feature and the behavior of previous agents on similar patches. Similar to [7], the motion model was a Markov jump process and the relevant posteriors were approximated using sample based techniques. Again, the same criticisms in regards to speed an accuracy lodged at [7] can also be lodged at [3], and again, we should note that speed/high accuracy were not a priority there. However, the motion model of Ballan et al. [3] was merely an illustrative choice in the more ambitious pursuit of creating a flexible paradigm for pedestrian forecasting. In particular, a motion model on one scene ought to be transferable to other scenes, and the representation of the “objects” in the scene ought to be flexible in order for behavior to be robust. Said differently, the way a forecasting algorithm executes “knowledge transfer” should not depend too rigidly on the finitely many labels some engineer managed to think of in a late-night brainstorming session. The algorithm should be able to infer how to behave near a roundabout by using prior knowledge of more fundamental building blocks like pavement, curbs, and asphalt.

Along similar lines to [3], Walker et al. [12] constructed an unsupervised approach towards forecasting. As before, the motion model was a Markov jump process, and the training set was a collection unlabeled videos. Unlike all the approaches mentioned thus far, the agents in [12] were not manually specified. They were learned by detecting which sort of patches of video were likely to move, and how. The resulting predictions outperformed [8] when comparing the most likely path with the ground truth in mean Hausdorff distance. As in all methods mentioned thus far, computational speed and accuracy of any predicted posteriors were not a concern of Walker et al. [12], so no such results were reported. However, since the motion model was a Markov jump process we should expect the same painful trade-off between error and speed to occur as in [7] and [3].

## B. Contributions

The primary contributions of this paper are:

- We provide a computationally expedient and accurate motion model for pedestrian forecasting.
- We evaluate the precision, accuracy, and recall of our model
- We compare our performance to that of a linear predictor and the MDP model of [8], both in terms of precision and recall as well as computation time.

We should note that there are a number of things that we do not do, that similar papers to ours typically include (notably [3, 7]). For example, we do not concern ourselves with detection and tracking. Nor do we concern ourselves with updating our prediction as new data comes along. Such questions are important, but beyond the scope of this paper, which concerns the underlying dynamic model. Papers such as [3, 7] pose a more comprehensive approach to pedestrian forecasting, where the motion model is a choice to be made. This paper presents one such choice. It is our opinion, that the motion model of many practical applications of forecasting ought to run in real time and that any relevant approximations ought to be equipped with hard error bounds.

The rest of the paper is organized as follows:

- We describe our motion model as a Bayesian network. (§II)
- We then describe how to compute probability densities for an agent’s position efficiently. (§III)
- Finally, we demonstrate the model by training and testing it on the Stanford drone dataset. (§IV)

## II. MODEL

Our goal is to generate a time-dependent probability density over  $\mathbb{R}^2$ , which predicts the true location of an agent in the future. The input to the algorithm at runtime is a noisy measurement of position and velocity,  $\hat{x}_0, \hat{v}_0 \in \mathbb{R}^2$ . If the (unknown) location of agent at time  $t$  is given by  $x_t \in \mathbb{R}^2$ , then the distribution we seek is the posterior  $\rho_t(x_t) := \Pr(x_t \mid \hat{x}_0, \hat{v}_0)$ .

In order to numerically compute  $\rho_t$  we will build a probabilistic graphical model. Our model assumes we have noisy information about agents, and each agent moves with some intention through the world in a way that is roughly approximated by a model. This allows our model to be divided into three parts:

- 1) Reality: This is parametrized by the true position for all time,  $x_t$ , and the initial velocity of the agent  $v_0$ .
- 2) The measurements: This is represented by our sensor readings  $\hat{x}_0$  and  $\hat{v}_0$  and are independent of all other variables given the true initial position and velocity,  $x_0, v_0$ .
- 3) The motion model: This is represented by a trajectory  $\tilde{x}_t$  and depends on a variety of other variables which will be described shortly.

We will now elaborate on these three components, and relate them to one another. Many of the choices we make will be motivated by a balance between model quality and computational speed (see §III).

### A. The Sensor Model

At time  $t = 0$ , we obtain a noisy reading of position,  $\hat{x}_0 \in \mathbb{R}^2$ . We assume that given the true position,  $x_0$ , that the measurement  $\hat{x}_0$  is independent of all other variables and the posterior  $\Pr(\hat{x}_0 | x_0)$  is something easily computable, such as a Gaussian. We assume the same measurement model for a measurement of initial velocity  $\hat{v}_0$ .

### B. The Agent Model

All agents are initialized within some rectangular region  $D \subset \mathbb{R}^2$ . We denote the true position of an agent by  $x_t$ . We should never expect to know  $x_t$  and the nature of its evolution precisely, and any model should account for its own (inevitable) imprecision. We do this by fitting a deterministic model to the data and then smoothing the results.

Specifically, our motion model consists of a modeled trajectory  $\tilde{x}_t$ , which is probabilistically related to the true position by  $x_t$  via a known and easily computable posterior,  $\Pr(x_t | \tilde{x}_t)$ .

Once initialized, agents come in two flavors: linear and nonlinear. The linear agent model evolves according to the equation  $\tilde{x}_t = x_0 + tv_0$  and so we have the posterior

$$\Pr(\tilde{x}_t | x_0, v_0, \text{lin}) = \delta(\tilde{x}_t - x_0 - tv_0).$$

We will also assume the posteriors,  $\Pr(x_0 | \text{lin})$  and  $\Pr(v_0 | \text{lin}, x_0)$  are simple expressions such as Gaussians or uniform distributions

If the agent is of nonlinear type, then we assume the dynamics take the form

$$\frac{d\tilde{x}_t}{dt} = s \cdot X_k(\tilde{x}_t) \quad (1)$$

where  $X_k$  is a vector-field from a finite collection  $\{X_1, \dots, X_n\}$ , and  $s \in \mathbb{R}$ . More specifically, we assume that each  $X_k$  has the property that  $\|X_k(x)\| = 1$  for all  $x \in D$ . This property ensures that speed is constant in time, and it has the further advantage of being the (local) generator of solutions to an optimal navigation problem (see Appendix A). The stationary vector-fields  $X_1, \dots, X_n$  are learned from the data-set (see §IV).

It is assumed that  $k$  and  $s$  are both constant in time, so that  $\tilde{x}_t$  is determined from the triple  $(x_0, k, s)$  by integrating (1) with the initial condition  $x_0$ . In other words, we have the posterior

$$\Pr(\tilde{x}_t | x_0, k, s) = \delta(\tilde{x}_t - \Phi_{k,s}^t(x_0)) \quad (2)$$

where  $\Phi_{k,s}^t$  is the flow-map of the vector field  $sX_k$  up to time  $t$ . It is notable, that the variables  $k, s$  and  $x_0$  determine  $v_0$ . Thus we also have the posterior

$$\Pr(v_0 | k, s, x_0) = \delta(v_0 - sX_k(x_0)). \quad (3)$$

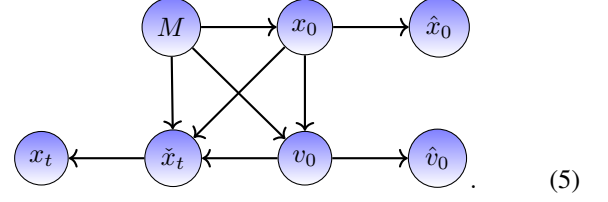
In summary, the agent models are parametrized by the set

$$\mathcal{M} = \{\text{lin}\} \cup (\mathbb{R} \times \{1, \dots, n\}) \quad (4)$$

and each flavor determines the sort of motion we should expect from the agent model.

### C. The Full Model

Concatenating the measurement model with our motion models yields the Bayesian network, where  $M \in \mathcal{M}$  denotes the model of the agent:



We may use this Bayesian network to compute  $\rho_t$  efficiently. In particular

$$\rho_t(x_t) := \Pr(x_t | \hat{x}_0, \hat{v}_0) \quad (6)$$

$$= \left( \sum_k \int \Pr(x_t, k, s | \hat{x}_0, \hat{v}_0) ds \right) \quad (7)$$

$$+ \Pr(x_t, \text{lin} | \hat{x}_0, \hat{v}_0). \quad (8)$$

The final term  $\Pr(x_t, \text{lin} | \hat{x}_0, \hat{v}_0)$  is expressible in terms of the error function, and would pose a negligible burden in terms of numerical computation. The primary computational burden derives from the sum  $\sum_k \int \Pr(x_t, k, s | \hat{x}_0, \hat{v}_0) ds$ .

### III. EFFICIENT PROBABILITY PROPAGATION

As mentioned, many of the modeling choices were born out of a balance between accuracy and real-time computability. One of the major modeling choices, that agents move approximately according to a small number of ODEs, is the most prominent such choice. In this section we give some details on how this modeling choice can be leveraged to compute  $\rho$  quickly, and accurately. In particular, we illustrate how to compute an accurate approximation of the posterior  $\Pr(x_t, k, s | \hat{x}_0, \hat{v}_0)$  by integrating the vector-fields  $X_1, \dots, X_n$  on a fixed grid.

To begin, from (5) we can see that

$$\Pr(x_t, k, s | \hat{x}_0, \hat{v}_0) \propto \Pr(x_t, k, s, \hat{x}_0, \hat{v}_0) \quad (9)$$

$$= \int \Pr(x_t, \tilde{x}_t, \hat{x}_0, \hat{v}_0, k, s) d\tilde{x}_t \quad (10)$$

$$= \int \Pr(x_t | \tilde{x}_t) \Pr(\tilde{x}_t, \hat{x}_0, \hat{v}_0, k, s) d\tilde{x}_t \quad (11)$$

We see from the last line that  $\Pr(x_t, k, s | \hat{x}_0, \hat{v}_0)$  is proportional to a convolution of the joint distribution  $\Pr(\tilde{x}_t, \hat{x}_0, \hat{v}_0, k, s)$ . Assuming, for the moment, that such a convolution can be performed efficiently we will focus on computation of  $\Pr(\tilde{x}_t, \hat{x}_0, \hat{v}_0, k, s)$ .

Again, (5) implies

$$\Pr(\tilde{x}_t, \hat{x}_0, \hat{v}_0, k, s) = \int \Pr(\tilde{x}_t, x_0, \hat{x}_0, v_0, \hat{v}_0, k, s) dx_0 dv_0 \quad (12)$$

$$= \int \Pr(\tilde{x}_t | x_0, k, s, v_0) \Pr(\hat{v}_0 | v_0) \Pr(v_0 | k, s, x_0) \quad (13)$$

$$\Pr(\hat{x}_0, x_0, k, s) dx_0 dv_0 \quad (14)$$

$$(15)$$

Substitution of (2) and (3) yeilds

$$= \int \delta(\tilde{x}_t - \Phi_{k,s}^t(x_0)) \delta(v_0 - sX_k(x_0)) \quad (16)$$

$$\Pr(\hat{v}_0 | v_0) \Pr(\hat{x}_0, x_0, k, s) dx_0 dv_0 \quad (17)$$

Carrying out the integration over  $v_0$  we observe

$$\Pr(\tilde{x}_t, \hat{x}_0, \hat{v}_0, k, s) = \int \delta(\tilde{x}_t - \Phi_{k,s}^t(x_0)) \Pr(\hat{x}_0, x_0, k, s) \Psi(\hat{v}_0; k, s, x_0) dx_0 \quad (18)$$

Where  $\Psi(v_0; k, s, x_0) := \Pr(\hat{v}_0 | v_0)|_{v_0=sX_k(x_0)}$ . Assuming the density  $\Pr(\hat{x}_0, x_0, k, s) \Psi(\hat{v}_0; k, s, x_0)$  is of bounded variation in the variable  $x_0$  (with all other variables held fixed), we may approximate it as a sum of weighted Dirac delta's supported on a regular grid,  $\Gamma = \cup_\alpha \{x_0^\alpha\}$  with spacing given by  $\Delta x$  [10]. In other words

$$\Pr(\hat{x}_0, x_0, k, s) \Psi(\hat{v}_0; k, s, x_0) = \left( \sum_\alpha c_{k,s,\alpha} \delta(x_0 - x_0^\alpha) \right) + \varepsilon_0(x_0) \quad (19)$$

for constants

$$c_{k,s,\alpha} = [\Pr(\hat{x}_0, x_0, k, s) \Psi(\hat{v}_0 | k, s, x_0) |\Delta x|]_{x_0=x_0^\alpha} \quad (20)$$

and error of magnitude  $\|\varepsilon_0\|_{L^1} \sim \mathcal{O}(|\Delta x|)$  with respect to the  $L^1$ -norm in  $x_0$ . The coefficients,  $c_{k,s,\alpha}$ , can be computed efficiently as product of the posteriors appearing in the PGM. Substitution of (19) into the final line of (18) yields

$$\Pr(\tilde{x}_t, \hat{x}_0, \hat{v}_0, k, s) = \sum_\alpha c_{k,s,\alpha} \delta(\tilde{x}_t - \Phi_{k,s}^t(x_0^\alpha)) + \varepsilon_t(\tilde{x}_t)$$

where  $\varepsilon_t(\tilde{x}_t) = \int \delta(\tilde{x}_t - \Phi_{k,s}^t(x_0)) \varepsilon_0(x_0) dx_0$ . The first term is computable by flowing the points of the grid,  $x_0^\alpha$ , by the evolution of the vector field  $sX_k$ . The second term,  $\varepsilon_t$ , may be viewed as an error term. In fact, this method of approximating  $\Pr(\tilde{x}_t, \hat{x}_0, \hat{v}_0, k, s)$  as a sum of Dirac-deltas is adaptive, in that the error term does not grow in total mass.

**Theorem 1.** *The error term,  $\varepsilon_t$ , is of size  $\mathcal{O}(\Delta x)$  in the  $L^1$ -norm, for fixed  $k, s, \hat{x}_0$ , and  $\hat{v}_0$ . Moreover, the magnitude is constant in time.*

*Proof.* To declutter notation, let us temporarily denote  $\Phi_{k,s}^t$  by  $\Phi$ . We observe

$$\begin{aligned} \|\varepsilon_t\|_{L^1} &= \int \left| \int \delta(\tilde{x}_t - \Phi(x_0)) \varepsilon_0(x_0) dx_0 \right| d\tilde{x}_t \\ &= \int \det(D\Phi|_{\Phi^{-1}(\tilde{x}_t)}) |\varepsilon_0(\Phi^{-1}(\tilde{x}_t))| d\tilde{x}_t \\ &= \int |\varepsilon_0(u)| du = \|\varepsilon_0\|_{L^1} \end{aligned}$$

As  $\varepsilon_0$  is of magnitude  $\mathcal{O}(\Delta x)$  the result follows.  $\square$

While this allows us to compute posteriors over the output of our models,  $\tilde{x}_t$ . We ultimately care about densities over the

true position. The following corollary of Theorem 1 addresses this.

**Corollary 1.** *Then the density*

$$\sum_\alpha c_{k,s,\alpha} \Pr(x_t | \tilde{x}_t)|_{\tilde{x}_t=\Phi_{k,s}^t(x_0^\alpha)} \quad (21)$$

*is an approximation of  $\Pr(x_t, k, s, \hat{x}_0, \hat{v}_0)$  with a constant in time error bound of magnitude  $\mathcal{O}(|\Delta x|)$ .*

*Proof.* By (11)

$$\Pr(x_t, k, s, \hat{x}_0, \hat{v}_0) = \int \Pr(x_t | \tilde{x}_t) \Pr(\tilde{x}_t, k, s, \hat{x}_0, \hat{v}_0) d\tilde{x}_t$$

Substitution of (19) yields

$$\begin{aligned} \Pr(x_t, k, s, \hat{x}_0, \hat{v}_0) &= \sum_\alpha c_{k,s,\alpha} \Pr(x_t | \tilde{x}_t)|_{\tilde{x}_t=\Phi_{k,s}^t(x_0^\alpha)} + \tilde{\varepsilon}_t(x_t) \end{aligned}$$

where the error term is

$$\tilde{\varepsilon}_t(x_t) = \int \Pr(x_t | \tilde{x}_t) \varepsilon_t(\tilde{x}_t) d\tilde{x}_t$$

and  $\varepsilon_t$  is the error term of (19). We see that the  $L^1$ -norm of  $\tilde{\varepsilon}_t$  is

$$\begin{aligned} \|\tilde{\varepsilon}_t\|_{L^1} &= \int \left| \int \Pr(x_t | \tilde{x}_t) \varepsilon_t(\tilde{x}_t) d\tilde{x}_t \right| dx_t \\ &\leq \int \Pr(x_t | \tilde{x}_t) |\varepsilon_t(\tilde{x}_t)| d\tilde{x}_t dx_t \end{aligned}$$

Implementing the integration over  $x_t$  first yeilds

$$\|\tilde{\varepsilon}_t\|_{L^1} \leq \int |\varepsilon_t(\tilde{x}_t)| d\tilde{x}_t =: \|\varepsilon_t\|_{L^1}$$

which is  $\mathcal{O}(|\Delta x|)$  by Theorem 1.  $\square$

Corollary 1 justifies using (21) as an approximation of  $\Pr(x_t, k, s, \hat{x}_0, \hat{v}_0)$ . So we've reduce the problem of computing  $\rho_t(x_t)$  to the problem of computing the coefficient  $c_{k,s,\alpha}$  and the points  $\Phi_{k,s}^t(x_0^\alpha)$  for all  $k, s$  and points  $x_0^\alpha \in \Lambda$ . We can reduce this burden further by exploiting the following symmetry.

**Theorem 2.**  $\Phi_{k,s}^t = \Phi_{k,1}^{st}$ .

*Proof.* Say  $x(t)$  satisfies the ordinary differential equation  $x'(t) = sX_k(x(t))$  with the initial condition  $x_0$ . In other words,  $x(t) = \Phi_{k,s}^t(x_0)$ . Taking a derivative of  $x(t/s)$ , we see  $\frac{d}{dt}(x(t/s)) = x'(t/s)/s = X_k(x(st))$ . Therefore  $x(t/s) = \Phi_{k,1}^t(x_0)$ . Substitution of  $t$  with  $\tau = t/s$  yields  $x(\tau) = \Phi_{k,1}^{s\tau}(x_0)$ . As  $x(\tau) = \Phi_{k,s}^\tau(x_0)$  as well, the result follows.  $\square$

Thus, computation of  $\Phi_{k,s}^t(x_0^\alpha)$  for all  $s$  and  $t$  boils down to computing  $\Phi_{k,1}^t(x_0^\alpha)$  for all  $t$ .

At this point, we may summarize the steps needed to compute an accurate approximation of  $\rho_t(x_t)$ .

- 1) Compute the constants  $c_{k,s,\alpha}$  from (20).
- 2) Compute  $\Phi_{k,1}^t(x_0^\alpha)$  for each  $k$  and each point  $x_0^\alpha$  in the grid over a time-interval  $[-T, T]$ .
- 3) Use Theorem 2 so obtain  $\Phi_{k,s}^t(x_0^\alpha)$  from  $\Phi_{k,1}^t(x_0^\alpha)$ .
- 4) Use Corollary 1 to obtain an  $\mathcal{O}(|\Delta x|)$  approximation of  $\Pr(x_t, k, s, \hat{x}_0, \hat{v}_0)$ .
- 5) Use (8) to compute  $\Pr(x_t, \hat{x}_0, \hat{v}_0)$  and normalize in the variable  $x_t$  to obtain an  $\mathcal{O}(|\Delta|)$  approximation of  $\rho_t(x_t)$ .

The first two steps are where bulk of the floating point operations would occur. Fortunately, these two steps are embarrassingly parallel. For fixed  $k$  and  $\alpha$ , the computation of  $\Phi_{k,1}^t(x_0^\alpha)$  on the interval  $[-T, T]$  takes  $\mathcal{O}(T)$  time using an explicit finite difference scheme, and computation for each  $(k, \alpha)$  may be performed in parallel with another such pair. Similarly, computation  $c_{k,s,\alpha}$  constitutes a series of parallel function evaluations over triples  $(k, s, \alpha)$ . If the posteriors represented by the edge in (5) are quick to compute then computation of all coefficients  $c_{k,s,\alpha}$  is equally quick. More precisely, the full computation of our approximation of  $\rho_t(x_t)$  would take  $\mathcal{O}(Tn|\Gamma|/N)$  time given  $N$  processing units and  $n$  vector fields.

#### IV. IMPLEMENTATION AND EXPERIMENTAL RESULTS

Now that the model has been described, we can illustrate a more specific implementation of the model and the process of fitting the model to a dataset. For the purpose of demonstration, we will use the Stanford Drone Dataset [9]. More generally, we will assume that for a fixed scene we have a database of previously observed trajectories  $\{\hat{x}^1, \dots, \hat{x}^N\}$ . From this data we will tune the parameters of the model appropriately. The parameters of the model which must be chosen or learned are

##### A. Learning the vector fields

Before we can even begin to learn vector-fields, we must learn the number of such vector-fields we have. To do this we use a simple clustering algorithm on the trajectories, to categorize them into groups. In principal, any clustering algorithm could accomplish this. For our algorithm we use the start and end point for each trajectory to obtain a point in  $\mathbb{R}^4$ . We then cluster in  $\mathbb{R}^4$  using Affinity propagation. This clustering of the end-points induces a clustering of the trajectories. So we obtain clusters  $S_1, \dots, S_n$  consisting of trajectories from our data set, as well as a set of unclassified trajectories,  $S_0$ .

For each set  $S_k$  we may learn a vector-field which is approximately compatible with that set. We'd like to only consider vector-fields whose vectors have unit magnitude because we observe many of the trajectories in the data appear to have a roughly constant speed. Therefore we assume the vector-field takes the form  $X_k(x) = (\cos(\Theta_k(x)), \sin(\Theta_k(x)))$  for some scalar function  $\Theta_k(x)$ . Learning the vector-fields then boils down to learning the scalar function  $\Theta_k$ . We will assume  $\Theta_k$  takes the form

$$\Theta_k(x) = \sum_{\alpha} \theta_{k,\alpha} L_{\alpha}(x)$$

for some collection of coefficients,  $\theta_{k,\alpha}$ , and a fixed collection of basis functions,  $L_{\alpha}$ . In our case, we've chosen  $L_{\alpha}$  to be a set of low degree Legendre polynomials. Learning  $\Theta_k$  can be obtained by looking at all the velocities observed in the cluster,  $S_k$ . These velocities may be obtained by a low order finite difference formula. Upon normalizing the velocities, we obtain a unit-length velocity vectors,  $v_{i,k}$ , anchored at each point,  $x_{i,k}$ , of  $S_k$ . We can learn  $\Theta_k$  by defining the cost-function

$$C[\Theta_k] = \sum_i \langle v_{i,k}, (\cos(\Theta_k(x_{i,k})), \sin(\Theta_k(x_{i,k}))) \rangle$$

which penalizes  $\Theta_k$  for producing a misalignment with the observed velocities at the observed points of  $S_k$ . When  $\Theta_k$  includes high order polynomials (e.g. beyond 5th order), we should also include a regularization term to bias the minimizes towards smoother outputs. Using the  $H^1$ -norm times a fixed scalar would suffice.

##### B. Learning $\Pr(x_0 | M)$ and $\Pr(M)$

Let us begin by considering the nonlinear models first. That is to say, we are concerned with  $M = (k, s)$  for an integer  $k$  and scalar  $s$ . We'd like to compute  $\Pr(x_0 | k, s)$  for each cluster  $k = 1, \dots, n$  and each speed  $s \in \mathbb{R}$ . We will make the modeling assumption that  $x_0$  is independent of  $s$  given  $k$ , i.e.  $\Pr(x_0 | k, s) = \Pr(x_0 | k)$ . Additionally, we will assume that  $s$  and  $k$  are independent. This means that we only need to learn  $\Pr(x_0 | k)$ ,  $\Pr(k)$ , and  $\Pr(s)$ .

We will let  $\Pr(k) = (n+1)^{-1}$  and  $\Pr(s) \sim \mathcal{U}([-s_{\max}, s_{\max}])$  where  $s_{\max} > 0$  is the largest observed speed in the dataset. This implies that  $\Pr(\text{lin}) = (n+1)^{-1}$  as well. Other reasonable choices (such as  $\Pr(k) \propto |S_k|$ ) could work, but we've chosen to be more conservative in this paper.

In order to learn  $\Pr(x_0 | k)$  we will assume that it's logarithm may be approximated by a low dimensional subspace of functions on  $\mathbb{R}^2$ . That is to say, for each  $k$  we will assume  $\Pr(x_0 | k) = \frac{1}{Z_k} \exp(-V_k(x_0))$  and  $V_k$  is a function whose constant term is 0 and is given by

$$V_k(x_0; \mathbf{c}) := \sum_{|\alpha| < d} c_{\alpha} L_{\alpha}(x_0)$$

for a collection of basis functions,  $L_{\alpha}$  and coefficients  $\mathbf{c} = \{c_{\alpha}\}_{|\alpha| < d}$ . We chose our basis functions to be the collection of tensor products from the first 6 Legendre polynomials, normalized to the size of the domain. Then, one may fit the coefficients  $c_{\alpha}$  to the data by using a log-likelihood criterion. The resulting (convex) optimization problem takes the form

$$\mathbf{c}^* = \inf_{|\mathbf{c}|} \sum_{x \in S_k} V_k(x_0; \mathbf{c})$$

Where the norm on  $\mathbf{c}$  is a sup-norm. We can also bias this optimization towards more regular functions by adding a penalty to the cost function.

Finally, we let  $\Pr(x_0 | \text{lin}) \sim \mathcal{U}(D)$ .



### C. Learning the measurement model

We will assume a Gaussian noise model. That is

$$\Pr(\hat{x}_0 | x_0) \sim \mathcal{N}(x_0, \sigma_x) \quad , \quad \Pr(\hat{v}_0 | v_0) \sim \mathcal{N}(v_0, \sigma_v).$$

Therefore, our model is parametrized by the standard deviations  $\sigma_x$  and  $\sigma_v$ . We will assume that the true trajectory of an agent is fairly smooth compared to the noisy output of our measurement device. This justifies smoothing the trajectories, and using the difference between the smoothed signals and the raw data to learn the variance  $\sigma_x$ . To obtain the results in this paper we have used a moving average of 4 time steps (this is 0.13 seconds in realtime). We set  $\sigma_v = 2\sigma_x/\Delta t$  where  $\Delta t > 0$  is the time-step size. This choice is justified from the our use of finite difference's to estimate velocity. In particular, if velocity is approximated via finite differencing as  $v(t) \approx (x(t+h) - x(t)) \Delta t^{-1} + \mathcal{O}(h)$  and the measurements are corrupted by Gaussian noise, then the measurement  $\hat{v}(t)$  is related to  $v(t)$  by Gaussian noise with roughly the same standard deviation as  $(x(t+h) - x(t)) \Delta t^{-1}$ .

### D. Learning the noise model

Finally, we assume that the true position is related to the model by Gaussian noise with a growing variance. In particular, we assume  $\Pr(x_t | \tilde{x}_t) \sim \mathcal{N}(\tilde{x}_t, \kappa t)$  for some constant  $\kappa \geq 0$ . The parameter,  $\kappa$ , must be learned. We've taken an ad hoc approach in this paper, although more intelligent and theoretically sound alternatives likely exist. For each curve in  $S_k$  we create a synthetic curve using the initial position and speed and integrating the corresponding vector-field,  $s X_k$ . So for each curve,  $x_i(t)$ , of  $S_k$ , we have a synthesized curve  $x_{i,synth}(t)$  as well. We then measure the standard deviation of  $(x_i(t) - x_{i,synth}(t))/t$  over  $i$  and at few time,  $t \in \{0, 100, 200\}$  in order to obtain  $\kappa$ .

### E. Precision, accuracy, etc.

In this section we establish our methods for evaluating the quality of our predictions, and comparing them against the model from [8] and a random walk. We implemented our model as well as the evaluation code in Python 2.6. Our implementation is available online.

embed link to github

Our data came from the Stanford Drone Dataset, in the form of hand annotated bounding boxes around agents (e.g. cars, pedestrians, bicycles). Our datasets used the bicycle data. We did a 10-fold cross validation on 4 different scenes of varying complexity. Our analysis used the same partitions of the trajectories for all three predictors.

include visualization of vector fields & learned value function?

The algorithm provided by Kitani et. al. was given additional information when making predictions that neither ours nor the random walk model were provided, namely the endpoint of the test trajectory. Without these data the implementation of the predictor provided by the authors devolves into a random walk.

figure All three algorithms side by side

captionThe grid used to evaluate the predicted distributions. The circle denotes the start of a trajectory and the X denotes the end.

The output distributions of the three algorithms were compared using their integrals over the cells of a suitably fine regular grid over our domain. These integrals can be seen in figure n. In keeping with our measurement model, the ground truth compared against was a grid that was identically zero except on a bounding box around the measurement at time  $t$ . The dimensions of the bounding box were the average bounding box dimensions in the training data set.

figureplot

We then computed the ROC curve at each frame for each predictor, and the AUC of that curve.

How much detail should I put about why we chose to use the AUC as a metric?

The AUCs at each time step were averaged across all test trajectories, and across all scenes. This formed the plot of mean AUC vs time (figure n+1).

Our advantage over Kitani

Disregarding small time scales where any algorithm will be reasonably accurate, our predictor behaves much better than any of the others at moderately large  $t$ , where we capture the behavior of the agent without suffering the misplaced confidence observed in Kitani et. al.'s predictions. At large  $t$ , their aptitude comes from knowing the end point of the trajectory.

	Our Predictor	Random Walk	Kitani et. al.
$\frac{t}{\text{frame}}$	0.0169s	flop	0.0614s

The running time per frame for each algorithm was generated using run times for 400 frames, averaged across several agents and scenes, shown in table 2. Our algorithm implementation leveraged its embarrassing parallelism using a relatively naive parallelization scheme, which split the computation of frames among 18 cores. It's highly probable that our run times would significantly decrease using an implementation of our algorithm in C that uses a more clever scheme. Kitani et. al. was timed with minimal modification to the source code provided by the authors.

We see that under our current implementation, we are beating Kitani et. al.'s algorithm at the relevant time scales, and are running more than 3 times faster that their algorithm. More significantly, our algorithm runs at almost 60FPS. Figure IV-E shows that at all time scales, our algorithm has very good predictive capability, and Figure ?? shows the qualitative merit of our predictions on several scenes.

## V. CONCLUSION

Avenues of improvement: update predictions when given additional measurements

The conclusion goes here.

Thanks to Kris Kitani for being nice. No thanks to Stanford, for being dicks.

## APPENDIX

## A. Inverse Optimal Control

One of the valuable aspect of the motion model of [8] was that the trajectories were solutions of optimal control problems. This is a desirable property for a motion model of pedestrians/cyclists/vehicles in that it seems reasonable to assume that we all move through the world with an intent to get somewhere. This framework was dubbed *inverse optimal control* (IOC). Here we illustrate how solutions of (1) can fit within the IOC framework.

**Theorem 3.** *Let  $X$  be a vector-field such that  $\|X(x)\|_2 = 1$  for all  $x \in \mathbb{R}^2$ . Then, for each initial condition  $x_0 \in \mathbb{R}^2$ , there exists a neighborhood  $U$ , a Riemannian metric,  $g$ , and a cost function,  $f$ , such that solutions of (1) are also solutions of the optimal control problem*

$$p^* = \inf_{\|v_t\|_g=s} \int_0^T f(x_t) dt \quad (22)$$

where  $v_t = \frac{dx_t}{dt}$ ,  $\|\cdot\|_g$  is the norm with respect to  $g$  and the infimum is taken over all differentiable curves in  $U$  which originate from  $x_0$ .

*Proof.* By rescaling time, we may let  $s$  be any positive real number. Therefore, without loss of generality, we will seek an  $f$  and  $g$  such that  $s = 1$ . In this case a solution of (22) is generated by the vector-field  $X = \frac{\nabla f}{\|\nabla f\|}$  where  $\nabla$  is the Riemannian gradient with respect to some metric (possibly non-Euclidean). Therefore, our goal is to illustrate that there exists a metric  $g$  and a function  $f$  such that the given  $X$  is given by  $\frac{\nabla f}{\|\nabla f\|}$ .

Because  $\|X\| = 1$  globally, it has no fixed points. From the flow-box theorem [2, Theorem 4.1.14] we can assert that for any open set,  $U \subset \mathbb{R}^2$  there exists a local diffeomorphism which transforms  $X_k$  into a flat vector-field. Mathematically, this asserts the existence of a map  $\Phi : U \rightarrow V \subset \mathbb{R}^2$  such that the push-forward of  $X$ , which we will denote by  $\tilde{X}$ , and given by

$$\tilde{X}(\tilde{x}) = \Phi_* X(\tilde{x}) := D\Phi|_{\Phi^{-1}(\tilde{x})} \cdot X(\Phi^{-1}(\tilde{x})),$$

is just the flat vector field  $(1, 0)$  for all  $x \in U$ . We could view the coordinate function  $\tilde{f}(x) = x^0$  as a cost function on  $V$ , and we may set  $g_V$  to be equal to the standard flat metric on  $V$  inherited from  $\mathbb{R}^2$ . In this case we observe that  $\tilde{X}$  generates solutions to the optimization problem

$$\tilde{p}^* = \inf_{\|\tilde{v}_t\|_{g_V} \leq 1} \int_0^T \tilde{f}(\tilde{x}_t) dt$$

By a change of variables, it follows that the original vector field,  $X$ , then solves the optimization problem (22) where  $g = \Phi^* g_V$  is the pull-back metric, and  $f = \Phi^* \tilde{f}$  is the pull-back of  $\tilde{f}$ .  $\square$

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