Peergrade #2: Proofs, Sets, Relations and Functions

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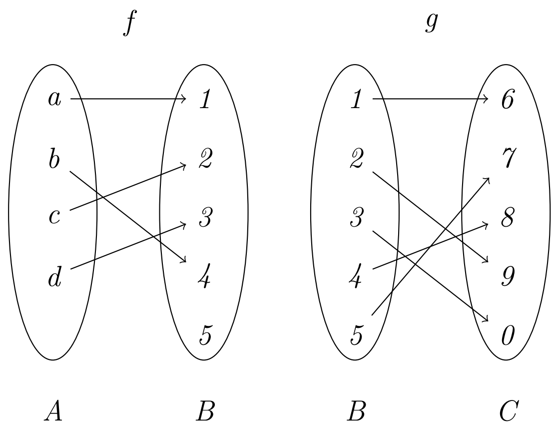
Exercise 1. *1. Let* $A = \{1, 3, 5\}, B = \{1, 2, 3, 4\}, C = \{x \in \mathbb{N} \mid \exists k \in \mathbb{N}(x = 2 \cdot k + 1)\}$. *Which of the following relations holds?*

(a) $A - B = \{4\}$
(b) $A \subseteq B \cap C$
(c) $A - B \subseteq C$
(d) $(A \times B) \cup (A \times C) \subset A \times \mathbb{N}$

2. Let $f : A \to B$ *and* $g : B \to C$ *be the two functions defined below:*



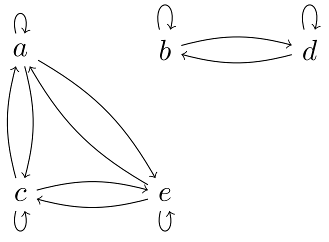
Is $g \circ f$*:*

(a) *one-to-one (injective)?*
(b) *onto (surjective)?*
(c) *one-to-one correspondence (bijective)?*

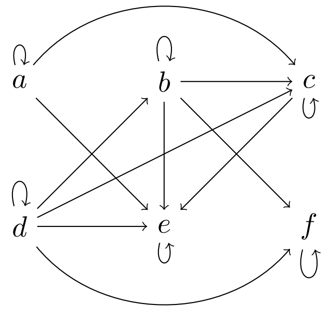
3. Which of the following graphs defines a partial order *or an* equivalence relation*?*

In case of a partial order, show a Hasse diagram representing the relation, and find the maximal, minimal, greatest and least elements, if they exist.

In case of an equivalence relations, find the partitions of the elements that are in the same equivalence class.



(a)



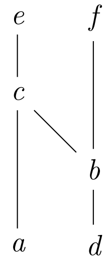
(b)

Solution 1. *1. (a) False; (b) False; (c) True; (d) True*

2. (a) True; (b) False, since 7 is not in the image of $g \circ f$; (c) False, since $g \circ f$ is not surjective

3. (a) Equivalence relation. The two partitions are $\{a, c, e\}$ and $\{b, d\}$

(b) Partial order. Hasse diagram:



Maximal: $\{e, f\}$; minimal: $\{a, d\}$; there are no greatest or least elements, since a, d and e, f are both uncomparable.

For the next two exercises, you will have to construct "informal" proofs on sets, relations and functions. Use the style and proof techniques discussed in week 3, and the cheat sheet and tips for proving statements provided in the section for week 4.

Exercise 2. *Provide a proof by contradiction of this statement: Let $f : \emptyset \to A$ be a function with domain the empty set and an arbitrary set A as co-domain. Then f is one-to-one (injective).*

Solution 2. *To prove that f is injective, we need to prove that $\forall x, y \in \emptyset (f(x) = f(y) \to x = y)$. Assume arbitrary $a, b \in \emptyset$. Since $\forall x (x \notin \emptyset)$, then in particular also $a \notin \emptyset$, hence we have a contradiction ($a \in \emptyset$ and $a \notin \emptyset$). This proves the statement.*

Exercise 3. *Provide a direct proof of this statement: Let A and B be two arbitrary sets such that $B \subseteq A$, and let R be an equivalence relation on A. Consider the relation $S = \{(a, b) \in R \mid a, b \in B\}$ on the set B. Then S is an equivalence relation on B (that is, reflexive, symmetric and transitive).*

Solution 3. *We need to prove that the relation S constructed above is reflexive, symmetric and transitive. Let's prove each of these statements separately:*

Reflexive

1. *We know that $\forall x \in A((x,x) \in R)$, since R is reflexive.*
2. *We need to prove $\forall x \in B((x,x) \in S)$.*
3. *Consider an arbitrary $a \in B$. We show that $(a,a) \in S$.*
4. *Since $B \subseteq A$ then $a \in A$ therefore by (1) we know that $(a,a) \in R$.*
5. *Since $a \in B$ (3) and $(a,a) \in R$ (4), then $(a,a) \in S$ by construction (follows from the definition of S). This concludes the proof.*

Symmetric

1. *Since R is symmetric we have that $\forall x,y \in A((x,y) \in R \leftrightarrow (y,x) \in R)$.*
2. *We need to prove $\forall x,y \in B((x,y) \in S \leftrightarrow (y,x) \in S)$.*
3. *Consider two arbitrary elements $a,b \in B$, and show that $(a,b) \in S \leftrightarrow (b,a) \in S$. We prove only one direction ($\rightarrow$) since the other is identical.*
4. *Assume $(a,b) \in S$.*
5. *Since $(a,b) \in S$, then also $(a,b) \in R$, and since R is symmetric then also $(b,a) \in R$.*
6. *Since $(b,a) \in R$ and $b,a \in B$, then also $(b,a) \in S$. This proves the statement.*

Transitive

1. *Since R is transitive we have that $\forall x,y,z \in A((x,y) \in R \wedge (y,z) \in R \rightarrow (x,z) \in R)$.*
2. *We need to prove $\forall x,y,z \in B((x,y) \in S \wedge (y,z) \in S \rightarrow (x,z) \in S)$.*
3. *Consider three arbitrary elements $a,b,c \in B$.*
4. *Assume $(a,b) \in S$ and $(b,c) \in S$. To prove $(a,c) \in S$.*
5. *Since $(a,b) \in S$, then also $(a,b) \in R$. Since $(b,c) \in S$, then also $(b,c) \in R$. (by definition)*
6. *Since R is transitive, then also $(a,c) \in R$. (because of (5) and instantiating the property to a,b,c)*
7. *Since $(a,c) \in R$ and $a,c \in B$, then also $(a,c) \in S$. This proves the statement.*

Evaluation The proofs above are spelled out in excruciating detail. Yours do not need to be this detailed, however I wrote these as a reference for you to reflect on each step. Check your proofs (and your peers' proofs) with this in mind. Have your peers missed some important steps, for example universal introduction steps or important implications?