Peergrade assignment 2

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Exercise 1

1.1. Which of the relations holds?

(a) $A - B = \{4\}$ **false**
$A - B = \{4\} \rightarrow 4 \in A$ but $4 \notin A$ therefore the statement is false.

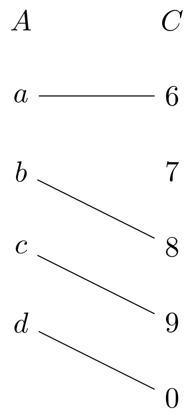
(b) $A \subseteq B \cup C$ **false**
Counterexample.
$5 \in A \wedge 5 \notin B \rightarrow A \nsubseteq B \rightarrow A \nsubseteq B \cap C$
Therefore the statement is false.

(c) $A - B \subseteq C$ **true**
$A - B = \{5\}$. Observe definition of C. If $x = 5 \rightarrow 5 = 2k + 1 \iff k = 2$, which means that $x = 5 \rightarrow k \in \mathbb{N}$. C contains other elements different from 5. Example: 69. $k = 34 \rightarrow k \in \mathbb{N} \rightarrow x = 2 \cdot k + 1 \rightarrow x = 69$.
Therefore the statement is true.

(d) $(A \times B) \cup (A \times C) \subset A \times \mathbb{N}$ **true**
Definition of C implies that $\forall x \in C, x \in \mathbb{N}$. Therefore $C \subseteq \mathbb{N}$.
Lastly $\exists x \in \mathbb{N}, x \notin C$, example: $6 \notin \mathbb{N}$ but $6 \notin C$. Therefore $C \subset \mathbb{N}$.
$C \subset \mathbb{N} \rightarrow (A \times C) \subset A \times \mathbb{N} \rightarrow (A \times B) \cup (A \times C) \subset A \times \mathbb{N}$
Therefore the statement is true.

1.2. Is $f \circ g$ injective, surjective and or bijective?

$f \circ g$:



(a) one-to-one (injective) **true**
$\forall x_1, x_2 \in X, F(x_1) = F(x_2) \rightarrow x_1 = x_2$
For every image there is a unique corresponding preimage.

(b) onto (surjective) **false**
$\forall y \in Y, \exists x \in X(F(x) = y)$
Counterexample
$\neg \exists x \in A(F(a) = 7)$
There doesn't exist a pre-image for the image 7

(c) one-to-one correspondence (bijective) **false**
For $f \circ g$ to be bijective, $f \circ g$ has to be both injective and surjective, but $f \circ g$ is not surjective therefore $f \circ g$ is not bijective.

1.3. Hasse diagram

Graph a
Partial orded set?
reflexive **true**
anti-symmetric **false**
transitive **true**
$\rightarrow$ partial orded set (poset) **false**

Equivalence relation?
reflexive **true**
symmetric **true**
transitive **true**
$\rightarrow$ equivalence relation **true**

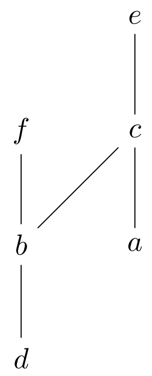
$$[a] = \{x \in R | xRa\} = \{a, c, e\}$$

$$[b] = \{x \in R | xRb\} = \{b, d\}$$

$[a] = [c] = [e]$
$[b] = [d]$

Graph b
Partial orded set?
reflexive **true**
anti-symmetric **true**
transitive **true**
$\rightarrow$ partial orded set (poset) **true**

Equivalence relation?
reflexive **true**
symmetric **false**
transitive **true**
$\rightarrow$ equivalence relation **false**



Maximum: e, f
Minimum: d, a
Greatest: none
Least: none

Exercise 2

Provide a proof by contradiction of this statement: Let $f : \varnothing \to A$ be a function with domain the empty set and an arbitrary set A as co-domain. Then f is one-to-one (injective).

Proof.

Suppose that the negation of the definition of one-to-one is true,
$\exists x_1, x_2 \in \emptyset(F(x_1) = F(x_2) \to x_1 \neq x_2)$,
where x_1 and x_2 is a arbitrary element.
By definition of the empty set
$\forall x\{x \notin \emptyset\}$
Therefore $x_1 \in \emptyset$ and $x_1 \notin \emptyset$, which is a contradiction.
Therefore $f : \emptyset \to A$ is a one-to-one function.

Exercise 3

Provide a direct proof of this statement: Let A and B be two particular but arbitrary sets such that $B \subseteq A$, and let R be an equivalence relation on A. Consider the relation $S = \{(a, b) \in R | a, b \in B\}$ on the set B. Then S is an equivalence relation on B (that is, reflexive, symmetric and transitive).

Proof.

S is an equivalence relation on B if the set of relations describing relations between nodes in B has reflexive, symmetric and transitive properties.

Sub-proof that S is reflexive.
Definition of reflexive: R is reflexive $\iff \forall x \in A, xRx$.
If $a \in A \wedge b \in B$ such that $a = b$ it means that $(a, a) \in R$ so per definition of S, $S = \{(a, b) \in R | a, b \in B\}$, it means that (a, a) must be in S.
Therefore $\forall x \in B, xRx$ which means that S is a reflexive relation on B.

Sub-proof that S is symmetric.
Definition of symmetric: R is symmetric $\iff \forall x, y \in A, xRy \to yRx$.
Per definition of S, $S = \{(a, b) \in R | a, b \in B\}$, it means that $(a, b) \in R \to (b, a) \in R$.
So if $a, b \in B \wedge (a, b) \in R \to ((a, b) \in R \to (b, a) \in R) \to (a, b), (b, a) \in S$.
Therefore $\forall a, b \in B, aRb \to bRa$ which means that S is a symmetric relation on B.

Sub-proof that S is transitive.
Definition of transitive: R is transitive $\iff \forall x, y, z \in A, xRy \wedge yRz \to xRz$
Let $a, b, c \in A$ and $a, b, c \in B$.

Since R is transitive, then if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$ by definition of transitive $\forall x, y, z \in A, xRy \wedge yRz \rightarrow xRz$. So if (a,b) is in S and (b,c) is in S then (a,c) must be in S, since $a, b, c \in B$ and $a, c \in R$.

Therefore $\forall a, b, c \in B, aRb \wedge bRc \rightarrow aRc$ which means that S is a transitive relation on B.

Conclusion.

S both a reflexive, symmetric and transitive relation on B, therefore S is an equivalence relation on B.