Peergrade Assignment #5

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Exercise 1. You have a standard deck of 52 cards. It has 13 cards of each of the four suits. The 13 cards of each suit have different ranks (A,2,3,4,5,6,7,8,9,10,J,Q,K).

Answer the following questions and justify your answer using the (generalised) pigeonhole principle.

1. You draw cards from the deck at random. What is the minimum number of cards you have to draw to be sure to have at least two cards of the same suit?
2. You draw cards from the deck at random. What is the minimum number of cards you have to draw to be sure to have four cards of the same rank?

Solution 1.
1. The solution is $4 + 1 = 5$. By the pigeonhole principle, given 4 suits, we need to draw at least $4 + 1$ cards to get 2 cards of the same suit.
2. The solution is $13 \cdot 3 + 1 = 40$. In the worst case we could draw three cards of the same rank for each of the 13 different ranks. Then by the pigeonhole principle the next card would be the fourth with the same rank as other three already drawn.

Exercise 2. In winter, Adam takes the train at the same time each morning to go to work. If it doesn't snow then the train will arrive outside his work on time with a likelyhood of 96%. However, if it snows, then it will arrive on time with a likelyhood of only 72%. The train is the only reason, why Adam would be late for work.

The weather forecast predicts that it is 65% likely that it will snow on Monday.

1. What is the likelihood that Adam will arrive on time at his work on Monday?
2. On Monday, Adam arrives late at work. What is the likelihood that it was snowing?

Solution 2.
1. Let S be the event "it snows" and T be the event "Adam arrives on time". We have $P(T \mid \neg S) = 0.96$ and $P(T \mid S) = 0.72$ and $P(S) = 0.65$. Then $P(T) = P(T \mid \neg S) \cdot P(\neg S) + P(T \mid S) \cdot P(S) = 0.96 \cdot 0.35 + 0.72 \cdot 0.65 \approx 0.80$. Hence Adam will arrive on time with approximately 80% likelihood.
2. We need to find the conditional probability that it was snowing given that Adam arrives late at work. First let's find the probability that Adam arrives late at work given that it was snowing: $P(\neg T \mid S) = 1 - P(T \mid S) = 0.28$. Then by Bayes' theorem: $P(S \mid \neg T) = \frac{P(\neg T|S) \cdot P(S)}{P(\neg T)} = \frac{0.28 \cdot 0.65}{0.20} = 0.91$. The likelihood that it was snowing given that Adam arrives late at work is 91%.

Exercise 3. A restaurant has a menu card with tapas. It contains 5 vegetarian dishes, 4 fish dishes and 7 meat dishes. You decide to have a meal consisting of 6 different dishes.

1. How many different meals can you choose from?
2. If the waiter chooses 6 different dishes for you at random, what is the probability that the resulting menu consists of 2 dishes from each category (ie. 2 vegetarian dishes, 2 fish dishes, and 2 meat dishes)?
3. You take a look at the prices. The vegetarian dishes costs 3€ each, the fish dishes costs 5€ each, and the meat dishes costs 6€ each. Suppose you choose two meat dishes, one fish dish and one vegetarian dish yourself, and the waiter chooses two random dishes for you among the ones you didn't already choose. Then what is the expected price of your meal? To answer this question, please identify a (relevant) sample space, a random variable and its distribution.

Solution 3.

1. We need to choose 6 dishes out of $5+4+7=16$, which corresponds to the binomial $\binom{16}{6}=8008$.
2. We know that there are 8008 ways to have a meal of 6 different dishes. Let's find the number of ways to choose 2 vegetarian, 2 fish and 2 meat-based dishes: $\binom{5}{2}\cdot\binom{4}{2}\cdot\binom{7}{2}=10\cdot 6\cdot 21=1260$. Therefore, the probability of the event happening is $\frac{1260}{8008}\approx 15\%$.
3. The part of the meal that I choose will cost $3\cdot 1+5\cdot 1+6\cdot 2=20$ €. The remaining sample space will consist of 4 vegetarian dishes, 3 fish dishes and 5 meat dishes.

   Let $S=\{vv, vf, vm, ff, fm, mm\}$ be our sample space (all possible combinations of choosing the two remaining dishes). The probabilities for each element are as follows:

   $$P(vv)=\frac{\binom{4}{2}}{\binom{12}{2}}=\frac{6}{66} \qquad P(vf)=\frac{\binom{4}{1}\binom{3}{1}}{\binom{12}{2}}=\frac{12}{66} \qquad P(vm)=\frac{\binom{4}{1}\binom{5}{1}}{\binom{12}{2}}=\frac{20}{66}$$
   $$P(ff)=\frac{\binom{3}{2}}{\binom{12}{2}}=\frac{3}{66} \qquad P(fm)=\frac{\binom{3}{1}\binom{5}{1}}{\binom{12}{2}}=\frac{15}{66} \qquad P(mm)=\frac{\binom{5}{2}}{\binom{12}{2}}=\frac{10}{66}$$

   Next, we define the random variable X as the cost of each choice of dishes:

   $$X=\{vv\mapsto 6, vf\mapsto 8, vm\mapsto 9, ff\mapsto 10, fm\mapsto 11, mm\mapsto 12\}$$

   Finally, the expected additional cost from the remaining two dishes is determined by the formula:

   $$\sum_{x\in S}P(x)\cdot X(x)=\frac{6}{66}\cdot 6+\frac{12}{66}\cdot 8+\frac{20}{66}\cdot 9+\frac{3}{66}\cdot 10+\frac{15}{66}\cdot 11+\frac{10}{66}\cdot 12=9.5$$

   hence the total that we are expected to pay is $20+9.5=29.5$ €.