Peergrade #1

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Exercise 1. *1. Prove in natural deduction the following proposition:*

$$(\neg A \vee \neg B) \to (C \to (A \wedge B)) \to \neg C$$

Hint: as an intermediate step, prove a contradiction on $A \wedge B$ (with the $\neg E$ rule).

2. Construct a truth table for the proposition of point 1.

Solution 1.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$h1$}
\UnaryInfC{$\neg A \vee \neg B$}
\AxiomC{}
\RightLabel{$h5$}
\UnaryInfC{$\neg A$}
\AxiomC{}
\RightLabel{$h4$}
\UnaryInfC{$A \wedge B$}
\RightLabel{$\wedge E_1$}
\UnaryInfC{A}
\BinaryInfC{$\mathbf{F}$}
\AxiomC{}
\RightLabel{$h5$}
\UnaryInfC{$\neg B$}
\AxiomC{}
\RightLabel{$h4$}
\UnaryInfC{$A \wedge B$}
\RightLabel{$\wedge E_2$}
\UnaryInfC{B}
\BinaryInfC{$\mathbf{F}$}
\RightLabel{$\vee E^{h5,h6}$}
\TrinaryInfC{$\mathbf{F}$}
\RightLabel{$\neg I^{h4}$}
\UnaryInfC{$\neg(A \wedge B)$}
\AxiomC{}
\RightLabel{$h2$}
\UnaryInfC{$C \to (A \wedge B)$}
\AxiomC{}
\RightLabel{$h3$}
\UnaryInfC{C}
\RightLabel{$\to E$}
\BinaryInfC{$A \wedge B$}
\RightLabel{$\neg E$}
\BinaryInfC{$\mathbf{F}$}
\RightLabel{$\neg I^{h3}$}
\UnaryInfC{$\neg C$}
\RightLabel{$\to I^{h2}$}
\UnaryInfC{$(C \to (A \wedge B)) \to \neg C$}
\RightLabel{$\to I^{h1}$}
\UnaryInfC{$(\neg A \vee \neg B) \to (C \to (A \wedge B)) \to \neg C$}
\end{prooftree}
$$

Here is the same solution in ProofWeb:

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Require Import ProofWeb.

Parameter A B C : Prop.

Theorem ex : (~A \/ ~B) -> (C -> A /\ B) -> ~C.
Proof.
imp_i h1.
imp_i h2.
neg_i h3.
neg_e (A /\ B).
neg_i h4.
dis_e (~A \/ ~B) h5 h6.
exact h1.
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neg_e A.
exact h5.
con_e1 B.
exact h4.
neg_e B.
exact h6.
con_e2 A.
exact h4.
imp_e C.
exact h2.
exact h3.
Qed.
```

Here is the truth table for the compound statements $(\neg A \vee \neg B), C \to (A \wedge B), (C \to (A \wedge B)) \to \neg C$ *and* $(\neg A \vee \neg B) \to (C \to (A \wedge B)) \to \neg C$*. Note that the full construction of the table requires the intermediate calculations of* $\neg A, \neg B, A \wedge B, \neg C$

A	B	C	$\neg A \vee \neg B$	$C \to (A \wedge B)$	$(C \to (A \wedge B)) \to \neg C$	$(\neg A \vee \neg B) \to (C \to (A \wedge B)) \to \neg C$
F	F	F	T	T	T	T
F	F	T	T	F	T	T
F	T	F	T	T	T	T
F	T	T	T	F	T	T
T	F	F	T	T	T	T
T	F	T	T	F	T	T
T	T	F	F	T	T	T
T	T	T	F	T	F	T

Exercise 2. *Express each of these statements using quantifiers, using mathematical notation when appropriate. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase "It is not the case that.")*

1. *There are tables that don't have legs.*
2. *For any even integer, its square is also even.*
3. *There is a house with pink doors.*
4. *All the pigs can fly and dance.*
5. *There is a rational number whose square root is 2.*

Solution 2.
1. *Quantified statement: Let* $L(x)$ *be the predicate "*x *has legs" and let* T *be the domain of all tables:* $\exists x \in T(\neg L(x))$*. (Alternatively with* $T(x)$ *as a predicate:* $\exists x(T(x) - > \neg L(x))$*)*
   Negation: $\forall x \in T(L(x))$*.*
   English: All tables have legs.

2. *Quantified statement: Let $Even(x)$ be the predicate "x is even":*
   $\forall x \in \mathbb{N}(Even(x) \to Even(x^2))$.
   Negation: $\exists x \in \mathbb{N}(Even(x) \land \neg Even(x^2))$.
   English: There exists an even integer whose square is not even.

3. *Quantified statement: Let $P(x)$ be the predicate "x has pink doors" and let H be the domain of all houses:* $\exists x \in H(P(x))$.
   Negation: $\forall x \in H(\neg P(x))$
   English: All houses do not have pink doors.

4. *Quantified statement: Let $F(x)$ be the predicate "x can fly", let $D(x)$ be the predicate "x can dance", and let P be the domain of all pigs:* $\forall x \in P(F(x) \land D(x))$.
   Negation: $\exists x \in P(\neg F(x) \lor \neg D(x))$
   English: There exists a pig that cannot fly or dance.

5. *Quantified statement:* $\exists x \in \mathbb{Q}(\sqrt{x} = 2)$.
   Negation: $\forall x \in \mathbb{Q}(\sqrt{x} \neq 2)$
   English: All rational numbers have square root different than 2.

Exercise 3. *Prove in natural deduction the following predicate:*

$$\forall x\ (A(x) \to B(x)) \to \exists y\ (\neg B(y)) \to \neg \forall z\ (A(z))$$

Solution 3.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$h2$}
\UnaryInfC{$\exists y\ (\neg B(y))$}
\AxiomC{}
\RightLabel{$h4$}
\UnaryInfC{$\neg B(a)$}
\AxiomC{}
\RightLabel{$h1$}
\UnaryInfC{$\forall x\ (A(x) \to B(x))$}
\RightLabel{$\forall E$}
\UnaryInfC{$A(a) \to B(a)$}
\AxiomC{}
\RightLabel{$h3$}
\UnaryInfC{$\forall z\ (A(x))$}
\RightLabel{$\forall E$}
\UnaryInfC{$A(a)$}
\RightLabel{$\to E$}
\BinaryInfC{$B(a)$}
\RightLabel{$\neg E$}
\BinaryInfC{$\mathbf{F}$}
\RightLabel{$\exists E^{a,h4}$}
\BinaryInfC{$\mathbf{F}$}
\RightLabel{$\neg I^{h3}$}
\UnaryInfC{$\neg \forall z\ (A(z))$}
\RightLabel{$\to I^{h2}$}
\UnaryInfC{$\exists y\ (\neg B(y)) \to \neg \forall z\ (A(z))$}
\RightLabel{$\to I^{h1}$}
\UnaryInfC{$\forall x\ (A(x) \to B(x)) \to \exists y\ (\neg B(y)) \to \neg \forall z\ (A(z))$}
\end{prooftree}
$$

```
Require Import ProofWeb.

Parameter A B C : D -> Prop.

Theorem ex : all x, (A x -> B x) -> exi y, ~B y -> ~all z, A z.
Proof.
imp_i h1.
imp_i h2.
neg_i h3.
exi_e (exi y, ~B y) a h4.
exact h2.
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neg_e (B a).
exact h4.
imp_e (A a).
all_e (all x, (A x -> B x)).
exact h1.
all_e (all z, A z).
exact h3.
Qed.
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