Notes on Propositional and Predicate Logics

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In today's lecture, we will give a little introduction the philosophy and history of mathematical logic. Mathematical logic is not as old one might think. While the first attempts to logical reasoning go back to Aristotle, it was only in the second half of the 19th century that mathematicians started thinking of logic as a field of scientific study. Until then logic was simply seen as the rules of the game that is called mathematics. There was no discussion about different logics and nobody really entertained the possibility that logical systems have properties and exhibit behaviors that are worth studying.

Nowadays mathematical logic as more important then ever. This is mostly because of computer science. Computers are very logical machines, we all might have heard about references to Boolean algebra and the like. During your stay at the IT University, you will hear a lot about good programming hygiene and contracts, a way to communicate important implicit information between a program/method that calls and the program/method that is being called. The language in which these contracts are expressed is logic. You will see that programming systems reason logically, which means that there is a *syntactic* side to logic that makes it more computer sciency, then its *semantic* side in the sense of classical mathematics.

> The tradition called "syntactic" — for want of a nobler title — never reached the level of its rival. In recent years, during which the algebraic tradition has flourished, the syntactic tradition was not of note and would without doubt have disappeared in one or two more decades, for want of any issue or methodology. The disaster was averted because of computer science — that great manipulator of syntax — which posed it some very important theoretical problems.
> *Jean-Yves Girard, Yves Lafont and Paul Taylor, 1990*

Because of this, we slightly deviate from the standard way of treating logic in a discrete mathematics class, and start with a particularly elegant way to combine logic, mathematics, and computing. We will use this logical system for everything throughout this discrete math class: *Reasoning*, *programming*, *specifying*, and even *discovering* flaws.

*Adapted from the original notes by Carsten Schürmann

The central concept of our view of logic is that of a *judgment*. A judgment is the fundamental notion in logics of *assigning truth to a formula*, that is saying that *the formula A is true*, which we abbreviate as A true. In this judgment A stands for any mathematical formula. Theses formulas define the language of mathematics that we will be using extensively in this course. We say that A is true if A true can be derived with the system of rules that we introduce in this lecture. Both the languages of formulas, and the rules of inference, are going to be described in the remainder of this section. The overall goal of this lecture is therefore to introduce the common language and to start internalizing it by many examples.

Let's start for real. We must now introduce each one of the connectives defining the logic that we will be using throughout this entire class. First, I would like to remark, that our logical presentation is completely void of any domain: Instead, we assume that there exist some basic formulas – that we call *propositions* – of which know whether they are true or false. For example, the proposition "a triangle has three sides" is *true*, and we can decide to denote it by the abbreviation $T3S$. We will need to fill logic with mathematical life, but not today. Today we will work on understanding what mathematical reasoning is all about, according to which rules are we allowed to reason and how do we actually do it[1].

Conjunction The first connective we tackle is conjunction. The usual "and". If A and B are two formulas, then $A \wedge B$ is a formula, which we read as "A and B". Next we define the respective inference rules that allow us to reason with a formula. i.e. a rule that introduces the conjunction $A \wedge B$ true, and two rules that eliminate $A \wedge B$ true again. Don't be scared, just bare with me.

$$\frac{A\ \mathsf{true} \quad B\ \mathsf{true}}{A \wedge B\ \mathsf{true}}\ \wedge I \qquad \frac{A \wedge B\ \mathsf{true}}{A\ \mathsf{true}}\ \wedge E_1 \qquad \frac{A \wedge B\ \mathsf{true}}{B\ \mathsf{true}}\ \wedge E_2$$

In these and the coming rules, A and B are parameters: they can be substituted with anything you have in your formula. Let's see it with a simple example: Let p be a proposition stating that the ball is red, and q be a proposition stating that the grass is green. Then we can use the $\wedge$I rule to infer that the ball is red and the grass is green *is true* starting from the hypotheses (which we denote with fact1) that the ball is red *is true* and (fact2) that the grass is green *is true*:

$$\frac{\dfrac{}{p\ \mathsf{true}}\ \text{fact1} \quad \dfrac{}{q\ \mathsf{true}}\ \text{fact2}}{p \wedge q\ \mathsf{true}}\ \wedge I$$

[1] *A note before starting*: all of the judgments that we spell out in this lecture can be tried out with ProofWeb (`http://proofweb.cs.ru.nl/`), a tool that automatically checks for you whether you have created a valid judgment using the rules that we present in the lecture, and points out your mistakes otherwise. Feel free to use it, though with a word of warning: all of the valid judgments need to come from your own creating thinking, as the tool does not reason for you, but just checks that you apply the right steps. We will see more on how to use it in class.

This is our first formal proof. With conjunction alone, however, we cannot prove many interesting things. We need to add a bit more power to get things off the ground.

Implication Let's do it and introduce implication, which is a way to reason hypothetically. An implication is a formula $A \to B$, read as "A implies B" (alternatively: "if A then B"). We call A the *hypothesis or antecedent* and B the *conclusion or consequence.* If I want to study the question if p implies q, then we may assume that we have a proof of p to prove q. Therefore, the meaning of implication is best described by the following two rules.

$$\dfrac{\dfrac{}{A \ \mathsf{true}}\,u \\ \vdots \\ B \ \mathsf{true}}{A \to B \ \mathsf{true}} \to I^u \qquad \dfrac{A \to B \ \mathsf{true} \quad A \ \mathsf{true}}{B \ \mathsf{true}} \to E$$

The $\vdots$ stands for a derivation, that may use the additional assumption that A true. Let's do a small example. Can you prove that if $p \to (q \to p)$? Let's do it together.

$$\dfrac{\dfrac{\dfrac{}{p \ \mathsf{true}}\,u}{(q \to p) \ \mathsf{true}} \to \mathrm{I}^v}{p \to (q \to p) \ \mathsf{true}} \to \mathrm{I}^u$$

Success. Now that we have at least two ways to combine formulas, the conjuction ($\wedge$) and implication ($\to$), we can try to prove something more complex:

Example 1 (Derive a judgment for $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$**.** *Here is the derivation:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(A \to B) \wedge (A \to C) \ \mathsf{true}}\,u}{A \to B \ \mathsf{true}} \wedge E_1 \quad \dfrac{}{A \ \mathsf{true}}\,v}{B \ \mathsf{true}} \to E \quad \dfrac{\dfrac{\dfrac{}{(A \to B) \wedge (A \to C) \ \mathsf{true}}\,u}{A \to C \ \mathsf{true}} \wedge E_2 \quad \dfrac{}{A \ \mathsf{true}}\,v}{C \ \mathsf{true}} \to E}{B \wedge C \ \mathsf{true}} \wedge I}{A \to (B \wedge C) \ \mathsf{true}} \to I^v}{((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C)) \ \mathsf{true}} \to I^u$$

This formula might look complex and scary at first, but there are many steps that are actually quite natural: in fact, reading the derivation from bottom to top, all but the last two steps are forced by the structure of the formula we are presented.

First we have a big implication where $(A \to B) \wedge (A \to C)$ is the antecedent and $A \to (B \wedge C)$ is the conclusion. The only rule in our repertoire that we can use is the ($\to I^u$) rule, that is the implication introduction rule. Therefore we assume the antecedent to be true and label it as u, which we use on the top of the derivation tree. Similarly we have to do the same on the next step, and we label with u the hypothesis that A is true.

At this point we are left with the *conjunction* $B \wedge C$ to prove. Again only one rule applies, the conjunction introduction rule ($\wedge I$). Therefore we have to prove that B is true and that C is true separately to complete the proof. Here is where our bit of ingenuity in this proof comes into play: on the first step we have assumed the hypothesis that $(A \to B) \wedge (A \to C)$ is true. We can use that formula to prove that B is true and C is true. The implication elimination rule allows us to conclude that B is true starting from the formula $A \to B$ and by knowing that A is true. We can derive that $A \to B$ is true with the conjunction introduction rule and because we have the hypothesis with label u. On the other side of the tree, we repeat a similar sequence of steps to derive that C is true.

To check the proof with ProofWeb, you can open the web interface and type the following program:

```
Require Import ProofWeb.
Parameter A B C : Prop.
Theorem exercise_1 : ((A -> B) /\ (A -> C)) -> (A -> (B /\ C)).
Proof.
imp_i u.
imp_i v.
con_i.
imp_e A.
con_e1 (A -> C).
exact u.
exact v.
imp_e A.
con_e2 (A -> B).
exact u.
exact v.
Qed.
```

After typing this code on the text area on the left of the browser window, you can select the menu option "Display" → "Gentzen style tree proofs (statements)" and then repeatedly press on the button to step through the proof. You will see the proof tree appear on the left as each rule is applied. It will look precisely like the ones we build in these notes (except maybe that it does not repeat `true` at the end of each formula). Here is the output just before ending the proof:

```
[(A → B) ∧ (A → C)]u                [(A → B) ∧ (A → C)]u
---------------------- ∧e1          ---------------------- ∧e2
        A → B            [A]v             A → C             [A]v
----------------------------- →e ----------------------------- →e
              B                                C
-------------------------------------------------------------- ∧i
                            B ∧ C
-------------------------------------------------------------- →i[v]
                          A → B ∧ C
-------------------------------------------------------------- →i[u]
                 (A → B) ∧ (A → C) → A → B ∧ C
```

We will see a bit more in detail how the tool works in class, and you can check the manual as well to get started with the tool.

Disjunction Let's look at one other connective, disjunction, for which we write $A \vee B$. Its meaning is defined by two introduction rules:

$$\frac{A \ \mathsf{true}}{A \vee B \ \mathsf{true}} \vee I_1 \qquad \frac{B \ \mathsf{true}}{A \vee B \ \mathsf{true}} \vee I_2$$

and one elimination rule:

$$\frac{A \vee B \ \mathsf{true} \quad \begin{array}{c} \overline{A \ \mathsf{true}}^{\,u} \\ \vdots \\ C \ \mathsf{true} \end{array} \quad \begin{array}{c} \overline{B \ \mathsf{true}}^{\,v} \\ \vdots \\ C \ \mathsf{true} \end{array}}{C \ \mathsf{true}} \vee E^{u,v}$$

The elimination rule for disjunction is interesting and rather non-trivial: if we know that $A \vee B$ is true, we don't know anything about which one of the two formulas is true. What we can do instead, is to prove some other formula C true starting from the assumption A true (labelled u) and B true (labelled v).

The following exercise gives an example of how this rule is applied:

Example 2 (Prove that $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$ is true)**.** *We construct the following derivation*[2]*:*

$$\dfrac{\dfrac{\dfrac{\overline{A \vee B}^{\,v} \quad \dfrac{\dfrac{\overline{(A \to C) \wedge (B \to C)}^{\,u}}{A \to C} \wedge E_1 \quad \overline{A}^{\,w}}{C} \to E \quad \dfrac{\dfrac{\overline{(A \to C) \wedge (B \to C)}^{\,u}}{B \to C} \wedge E_2 \quad \overline{B}^{\,x}}{C} \to E}{C} \vee E^{w,x}}{(A \vee B) \to C} \to I^v}{((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C) \ \mathsf{true}} \to I^u$$

Optional: try to use ProofWeb to construct this exact dervation.

Truth and falsehood No logic is complete without truth and falsehood. From your high school math class, you might remember that we put true and false, 1 and 0 into the center of mathematics. A theorem is either true or false. My main message for this lecture is that in order to do discrete math for computer science and information technology, we should not be only interested if we can write a certain program or not, we actually must be able to write the program and it better runs well according to specification.

The meaning of truth and falsehood is specified by the following two rules. We can always derive true, and if we can derive false then we can derive whatever we want (C true).

[2]Here we omit to say that a formula is true at each step of the derivation for space reasons. You can do the same in your exercises, as we only talk about what is true in our derivations. There are other systems of reasoning that not only allow you to talk about truth, but might talk about falsehood, or what is a valid predicate at a specific point in time, just to name a few. That's material for another class.

$$\dfrac{}{\mathbf{T}\ \mathsf{true}}\ \mathbf{T}I \qquad \dfrac{\mathbf{F}\ \mathsf{true}}{C\ \mathsf{true}}\ \mathbf{F}E$$

Negation Closely related to falsehood is negation. Here is how the two are related. We write $\neg A$ for the negation of A, and define it as follows. To prove that the negation of A is true, we assume that A is true and try to reach a contradiction, that is we prove that false is true. We call this type of proof a *proof by contradiction*, which is defined by the following rule for negation:

$$\dfrac{\begin{array}{c}\dfrac{}{A\ \mathsf{true}}\ u\\ \vdots \\ \mathbf{F}\ \mathsf{true}\end{array}}{\neg A\ \mathsf{true}}\ \neg I^u$$

The elimination rule is as follows:

$$\dfrac{\neg A\ \mathsf{true} \quad A\ \mathsf{true}}{C\ \mathsf{true}}\ \neg E$$

Let's try some exercises using the two rules for negation:

Example 3 (Prove that $(A \wedge B) \to \neg(A \to \neg B)$ is true.)**.**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B\ \mathsf{true}}\ u}{B\ \mathsf{true}}\ \wedge E_2 \quad \dfrac{\dfrac{}{A \to \neg B\ \mathsf{true}}\ v \quad \dfrac{\dfrac{}{A \wedge B\ \mathsf{true}}\ u}{A\ \mathsf{true}}\ \wedge E_1}{\neg B\ \mathsf{true}}\ \to E}{\mathbf{F}\ \mathsf{true}}\ \neg E}{\neg(A \to \neg B)\ \mathsf{true}}\ \neg I^v}{(A \wedge B) \to \neg(A \to \neg B)\ \mathsf{true}}\ \to I^u$$

Try to prove the following:

Exercise 1. *Prove that* $(\neg A \vee B) \to (A \to B)$ *is true.*

We can also prove the statement that we presented during the lecture:

Example 4 (Prove $(s \to r) \to (t \to \neg r) \to t \to \neg s$)**.**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{s \to r\ \mathsf{true}}\ u \quad \dfrac{}{s\ \mathsf{true}}\ x}{r\ \mathsf{true}}\ \to E \quad \dfrac{\dfrac{}{t \to \neg r\ \mathsf{true}}\ v \quad \dfrac{}{t\ \mathsf{true}}\ w}{\neg r\ \mathsf{true}}\ \to E}{\mathbf{F}\ \mathsf{true}}\ \neg E}{\neg s\ \mathsf{true}}\ \neg I^x}{t \to \neg s\ \mathsf{true}}\ \to I^w}{(t \to \neg r) \to t \to \neg s\ \mathsf{true}}\ \to I^v}{(s \to r) \to (t \to \neg r) \to t \to \neg s\ \mathsf{true}}\ \to I^u$$

At the end of these notes, there will be a massive list of formulas that you should try to prove. Some are easy, some are a bit more difficult. Some of these formulas have already partial proofs as examples in these notes. You should however spend the time to internalize with how this system of rules work and get the intuition that's necessary to known which rule must be applied at each step of a proof.

Classical logic. So far we have presented the base system, which we call *intuitionistic logic*. It's simple, beautiful and clean. But not all of mathematics can be done in it. Mathematicians tend to add axioms to logic. The rule of the excluded middle is such an axiom. When we do this, we have classical logic, but it is difficult to attribute a computational meaning to the Law of the Excluded Middle.

$$\frac{}{A \vee \neg A \ \mathsf{true}}\ LEM$$

This axiom is not derivable in intuitionistic logic. Our logic (also called constructive logic, or intuitionistic logic), together with this axiom gives us *classical logic*, a logic that mathematicians usually work in. In classical logic we can prove theorems that we cannot prove in intuitionistic logic (simply because we assume more by the law of the excluded middle). If you want to learn more about this, you'll have the opportunity to take some more advanced logic class on your third year.

The law of double negation elimination is derivable only in classical logic:

$$\frac{\neg\neg A \ \mathsf{true}}{A \ \mathsf{true}}\ \neg\neg C$$

Example 5 (Derive the rule of double negation elimination)**.** *We want to prove:*

$$\begin{array}{c} \mathcal{D} \\ \neg\neg A \ \mathsf{true} \\ \vdots \\ A \ \mathsf{true} \end{array}$$

We can construct the following derivation:

$$\dfrac{\dfrac{}{A \vee \neg A \ \mathsf{true}}\ LEM \quad \dfrac{}{A \ \mathsf{true}}\ u \quad \dfrac{\dfrac{}{\neg A \ \mathsf{true}}\ v \quad \begin{array}{c}\mathcal{D}\\ \neg\neg A \ \mathsf{true}\end{array}}{A \ \mathsf{true}}\ \neg E}{A \ \mathsf{true}}\ \vee E^{u,v}$$

When a rule is derivable we call it *admissible*: we can do without that rule and still be able to prove the same formulas, but sometimes it's nice to have such rules available in our repertoire.

Double negation introduction instead can be derived just by using the rules of intuitionistic logic:

Example 6 (Prove $A \to (\neg\neg A)$)**.** *We can construct the following judgment:*

$$\dfrac{\dfrac{A \ \mathsf{true} \quad \dfrac{}{\neg A \ \mathsf{true}}\ u}{\mathbf{F} \ \mathsf{true}}}{\neg\neg A \ \mathsf{true}}\ \neg I^u$$

and therefore assume that double negation introduction *is an admissible rule in our reasoning:*

$$\frac{A \ \mathsf{true}}{\neg\neg A \ \mathsf{true}}\ \neg\neg I$$

Finally, classical logic has one more way to produce a proof by contradiction, similar to the one we presented as the negation introduction rule:

$$\dfrac{\dfrac{}{\neg A\ \mathsf{true}}\ u \\ \vdots \\ \mathbf{F}\ \mathsf{true}}{A\ \mathsf{true}}\ \mathbf{F}_C^u$$

Exercise 2. *Prove by contradiction:* $(A \vee B) \to (\neg A \to B)$ *(using the rule* $\mathbf{F}_C^u$*).*

Example 7 (Prove that the rule $\mathbf{F}_C^u$ is admissible)**.** *That is, we can prove that* A *is true if we have a proof of:*

$$\begin{array}{c} \overline{\neg A\ \mathsf{true}}\ u \\ \mathcal{D} \\ \mathbf{F}\ \mathsf{true} \end{array}$$

Construct a proof tree in classical logic that shows how to derive A true *starting from the derivation* $\mathcal{D}$ *(hint: use the rule LEM).*

In constructive logic, a formula is not just true or false, it either has a constructive proof or not. In classical logic, provability is not the central concern, it is validity, and therefore the meaning of a formulas is aways true or false. Everything that you have learned in a high school or undergraduate mathematics is usually classical. Very often in mathematics people are concerned with whether a statement is true or false. In computing instead we care more about how to construct the solution, and in these circumnstances constructive logics greatly help. The good news is that many of the theorems that are classically valid also have proofs in intuitionistic logic.

Logical equivalences The expression $A \equiv B$ is called a logical equivalence, that is A is true whenever B is true and viceversa. When two formulas A and B are equivalent, they are interchaengable with each other, so whenever we are presented with proving A, we can try to prove B instead, or vice-versa.

To prove an equivalence, we need to find a derivation of the judgments $A \to B$ true and $B \to A$ true. Some of these equivalences hold in intuitionistic logics for both direction of the implication, whereas some others hold only classically in one direction, and intuitionistically in the other. We have marked all the equivalences $A \equiv B$ with a label (int, int) or (int, cls) or (cls, int). When you find the label (int, int) you'll be able to prove both $A \to B$ and $B \to A$ in intuitionistic logics; with the label (int, cls), to prove $B \to A$ you'll have to resort to the law of excluded middle (LEM); finally with the label (cls, int), you'll have to use (LEM) to prove $A \to B$.

- Identity laws: $A \wedge \mathbf{T} \equiv A,\ A \vee \mathbf{F} \equiv A$ (int, int)
- Domination laws: $A \vee \mathbf{T} \equiv \mathbf{T},\ A \wedge \mathbf{F} \equiv \mathbf{F}$ (int, int)
- Idempotent laws: $A \vee A \equiv A,\ A \wedge A \equiv A$ (int, int)

- Double negation law: $\neg(\neg A) \equiv A$ (cls, int)
- Commutative laws: $A \vee B \equiv B \vee A,\ A \wedge B \equiv B \wedge A$ (int, int)
- Associative laws: $(A \vee B) \vee C \equiv A \vee (B \vee C)$ (int, int)
  $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$ (int, int)
- Distributive laws: $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ (int, int)
  $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ (int, int)
- De Morgan's laws: $\neg(A \wedge B) \equiv \neg A \vee \neg B,\ \neg(A \vee B) \equiv \neg A \wedge \neg B$ (cls, int)
- Absorption laws: $A \vee (A \wedge B) \equiv A,\ A \wedge (A \vee B) \equiv A$ (int, int)
- Negation laws: $A \wedge \neg A \equiv \mathbf{F}$ (int, int)
  $A \vee \neg A \equiv \mathbf{T}$ (int, cls)
- $A \to B \equiv \neg A \vee B$ (cls, int)
- $A \to B \equiv \neg B \to \neg A$ (int, cls)
- $A \vee B \equiv \neg A \to B$ (int, cls)
- $A \wedge B \equiv \neg(A \to \neg B)$ (int, cls)
- $\neg(A \to B) \equiv A \wedge \neg B$ (cls, int)
- $(A \to B) \wedge (A \to C) \equiv A \to (B \wedge C)$ (int, int)
- $(A \to C) \wedge (B \to C) \equiv (A \vee B) \to C$ (int, int)
- $(A \to B) \vee (A \to C) \equiv A \to (B \vee C)$ (int, cls)
- $(A \to C) \vee (B \to C) \equiv (A \wedge B) \to C$ (int, cls)

Exercise 3. *Prove the equivalences just presented. (Remember: int=intuitionistic, cls=classical)*

Non elementary connectives (advanced) So far we have almost only encountered *elementary connectives*, that is, ways of combining logic formulas that cannot be expressed only using the other connectives we have available. I write almost because this is in fact not true: we have encountered $\wedge, \to, \vee, \neg, \mathbf{T}$ and $\mathbf{F}$. In Exercise 3 you should have proven the equivalence $A \wedge \neg A \equiv \mathbf{F}$. That means we could have a complete system of reasoning without $\mathbf{F}$ (false) if we were willing to express false with $A \wedge \neg A$ every time we encounter it. Another connective that we could remove without losing any reasoning power would be $\neg$: in fact, we have the following equivalence:

$$A \to \mathbf{F} \equiv \neg A$$

and we can prove that it holds in both directions

$$\dfrac{\dfrac{\dfrac{}{A \to \mathbf{F}\ \mathsf{true}}\,u \quad \dfrac{}{A\ \mathsf{true}}\,v}{\mathbf{F}\ \mathsf{true}}\to E}{\dfrac{\neg A\ \mathsf{true}}{(A \to \mathbf{F}) \to \neg A\ \mathsf{true}} \to I^u}\neg I^v \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A\ \mathsf{true}}\,v \quad \dfrac{}{\neg A\ \mathsf{true}}\,u}{\mathbf{F}\ \mathsf{true}}\neg E}{A \to \mathbf{F}\ \mathsf{true}}\to I^v}{\neg A \to (A \to \mathbf{F})\ \mathsf{true}}\to I^u$$

From these considerations, we can eliminate either $\mathbf{F}$ or $\neg$ and not lose our ability to express, and even prove, any statement in intuitionistic logic.

In *classical logic*, since we also assume the law of excluded middle, you have proven that $A \vee \neg A \equiv \mathbf{T}$ in Exercise 3, so we can also do without $\mathbf{T}$. Another equivalence that we proved in classical logic was $\neg A \vee B \equiv A \to B$, so in classical logic also $\to$ is redundant. A minimal complete system would then have the following connectives: $\wedge, \vee, \neg$ for classical logic and $\wedge, \vee, \to, \mathbf{T}, \mathbf{F}$ for intuitionistic logic. With these, you can encode and prove all facts so far, but it would be much more tedious to do so.

There are a couple more connectives that are worth mentioning as they appear in the book: instead of presenting deduction rules for them, we define them to be equivalent to formulas that use the connectives we already know.

XOR $A \oplus B$ is read as A xor B and follows the equivalence:

$$A \oplus B \equiv (A \wedge \neg B) \vee (\neg A \wedge B)$$

Double implication $A \leftrightarrow B$ is read as A if and only if B and follows the equivalence:

$$A \leftrightarrow B \equiv (A \to B) \wedge (B \to A)$$

Exercise 4 (Advanced)**.** *Suggest introduction and elimination rules in natural deduction for $A \oplus B$ and $A \leftrightarrow B$.*

Conclusion This concludes our lecture. We have encountered all the usual connectives, and have explained their meaning in form of inference rules. We have discussed truth and falsehood, conjunction, negation, disjunction, implication, and we have encountered the rules for the excluded middle, and double negation introduction and elimination.

Rules of Natural Deduction in Propositional Logic

Conjunction:

$$\frac{A \text{ true} \quad B \text{ true}}{A \wedge B \text{ true}} \wedge I \qquad \frac{A \wedge B \text{ true}}{A \text{ true}} \wedge E_1 \qquad \frac{A \wedge B \text{ true}}{B \text{ true}} \wedge E_2$$

Implication:

$$\frac{\begin{array}{c}\overline{A \text{ true}}^{\,u} \\ \vdots \\ B \text{ true}\end{array}}{A \to B \text{ true}} \to I^u \qquad \frac{A \to B \text{ true} \quad A \text{ true}}{B \text{ true}} \to E$$

Disjunction:

$$\frac{A \text{ true}}{A \vee B \text{ true}} \vee I_1 \qquad \frac{B \text{ true}}{A \vee B \text{ true}} \vee I_2 \qquad \frac{A \vee B \text{ true} \quad \begin{array}{c}\overline{A \text{ true}}^{\,u} \\ \vdots \\ C \text{ true}\end{array} \quad \begin{array}{c}\overline{B \text{ true}}^{\,v} \\ \vdots \\ C \text{ true}\end{array}}{C \text{ true}} \vee E^{u,v}$$

True and false:

$$\frac{}{\mathbf{T} \text{ true}} \mathbf{T}I \qquad \frac{\mathbf{F} \text{ true}}{C \text{ true}} \mathbf{F}E$$

Negation:

$$\frac{\begin{array}{c}\overline{A \text{ true}}^{\,u} \\ \vdots \\ \mathbf{F} \text{ true}\end{array}}{\neg A \text{ true}} \neg I^u \qquad \frac{\neg A \text{ true} \quad A \text{ true}}{C \text{ true}} \neg E$$

Classical rules:

$$\frac{}{A \vee \neg A \text{ true}} LEM \qquad \frac{\neg\neg A \text{ true}}{A \text{ true}} \neg\neg C \qquad \frac{\begin{array}{c}\overline{\neg A \text{ true}}^{\,u} \\ \vdots \\ \mathbf{F} \text{ true}\end{array}}{A \text{ true}} \mathbf{F}_C^u$$

Predicate Logics

Now that we have introduced a system for reasoning in propositional logic, we can extend it to be able to talk about *quantified facts*. For example, during the quiz in the previous lecture we considered whether or not this is a valid inference:

> All humans are mortal. Socrates is a human.
> Therefore, Socrates is mortal.

How can we express this with logics? We need a symbol that allows us to talk about all things: we use $\forall x(A)$ to say that the predicate A holds for *all possible values* of x. We read that formula as "for all x A holds". Instead, if we want to express that some predicate holds *for some* values of x, we use the expression $\exists x(A)$. We read this expression as "there exists x such that A holds". Notice that now we call A a *predicate* instead of a *proposition*, as A might have some variables in it. Let's see how to translate our example:

$$\forall x(\mathit{Human}(x) \to \mathit{Mortal}(x)) \to \mathit{Human}(\mathit{socrates}) \to \mathit{Mortal}(\mathit{socrates})$$

here we assume to have two predicates: $\mathit{Human}(x)$ which stands for "x is a human" and $\mathit{Mortal}(x)$ which stands for "x is mortal".

We can also have predicates involving *multiple parameters*. For example we could use $\mathit{Older}(x, y)$ to express that "x is older than y". Predicates relating two parameters are very commonly found in math: $x = y$ or $x < y$ etc. are all typical predicates we want to be able to reason about throughout the rest of this course. For these special predicates that have a special symbol in mathematics, we will stick to that notation instead of specifying for example $\mathit{Equals}(x, y)$ or $\mathit{Smaller}(x, y)$.

Logic variables, free and bound. To say anything useful about $\exists$ and $\forall$ we need first to talk about *free and bound variables*. We say that a variable x is *bound* if it occurs inside a $\forall x(\dots)$ or an $\exists x(\dots)$. Otherwise, if x is not surrounded by a $\forall x$ or an $\exists x$, then we say that x is a *free variable*.

Example 8 (Identify the free and bound variables in the previous formula)**.** *The variable "x" occurs under a $\forall x$, therefore it is* bound*. The variable "socrates" is* free *instead, as it does not occur within a $\forall$ or $\exists$ quantifier.*

Example 9 (Identify the free and bound variables in the following formula)**.**

$$(\exists y(\forall x(P(y) \land Q(x)))) \land S(x) \land S(y)$$

In this formula the x and y that occur under the $\exists x(\forall y(\dots))$ are bound variables*, whereas the x and y that occur on $S(x) \land S(y)$ are* free variables*.*

Even though this is bad style, we can use the same name for different variables when they occur free and bound in different parts of a formula, and they will be in fact *different and independent variables*!

Terms and substitutions. When we introduce the rules for predicate logics, we need to be able to *substitute* variables for other expressions, which we call *terms* in this domain. To do this, we introduce both the concept of a *term* and of a *substituting* a variable with a term.

Definition 1 (Term). *A* term *is either a* variable *(e.g. x) or a* function *of multiple terms (e.g. $\mathsf{f}(x, y)$ or $\mathsf{g}(\mathsf{f}(x, socrates))$).*

In these notes we will use the sans-serif font to denote functions, while we stick to italics for variables, to avoid confusing one with the other.

Sometimes we use as terms *mathematical expressions*, like $2 + x$, or x^y. As with predicates, instead of being pedantic and writing them as for example $\mathsf{plus}(2, x)$ and $\mathsf{exp}(x, y)$[3], we just write them in normal math notation.

Definition 2 (Substitution). *Given a variable x, and expression t and a logic formula A, we define $[t/x]$ to be the substitution that maps x to t.* To apply the substitution *$[t/x]$ to A — which we write as $A[t/x]$ — means to replace every* free occurrence *of x with t in the formula A.*

Exercise 5. *Let $A = \forall x(P(x)) \wedge Q(y)$, then:*

$$A[t/x] = \forall x(P(x)) \wedge Q(y)$$
$$A[t/y] = \forall x(P(x)) \wedge Q(t)$$

why is that the case?

Now that we have introduced all the necessary concepts, we can talk about universal and existential quantification.

Universal quantification

The universal quantifier $\forall$, like all the other connectives in our language, has an *introduction* and *elimination* rule. Let's look at the elimination rule first:

$$\frac{\forall x(A) \ \mathsf{true}}{A[t/x] \ \mathsf{true}} \ \forall E$$

[3]Which we could do. In fact, all the mathematics of natural numbers can be encoded in something called "Peano arithmentics", where we introduce natural numbers with the following two axioms:

$$\frac{}{Nat(\mathsf{z}) \ \mathsf{true}} \ Z \qquad \frac{}{\forall x(Nat(x) \to Nat(\mathsf{s}(x))) \ \mathsf{true}} \ S$$

using these rules, z stands for the number 0 (and the axiom Z reads as "zero is a natural number"), and $\mathsf{s}(x)$ denotes the successor of x (and the axiom S reads as "if x is a natural number, then the successor of x is also a natural number"), which we usually write as $x + 1$.

Then we can, for example, introduce $\mathsf{plus}(x, y)$ to describe the sum of x and y with the following two rules:

$$\frac{}{\forall x(\mathsf{plus}(x, \mathsf{z}) = x) \ \mathsf{true}} \ plus_1 \qquad \frac{}{\forall x \forall y(\mathsf{plus}(x, \mathsf{s}(y)) = \mathsf{s}(\mathsf{plus}(x, y))) \ \mathsf{true}} \ plus_2$$

We won't explain now all the details of how to do this, as it can easily be a topic for a whole course, but if you are interested in learning more you can start from Wikipedia (`https://en.wikipedia.org/wiki/Peano_axioms`) and see how to define some basic functions in standard arithmetics.

this means that if $\forall x(A)$ is true then we can substitute x with *any arbitrary term* t that we might need and derive $A[t/x]$ true.

With this rule we can solve the example that we introduce at the beginning of this section:

Example 10 (Construct a derivation for:
$\forall x(\mathit{Human}(x) \to \mathit{Mortal}(x)) \to \mathit{Human}(\mathit{socrates}) \to \mathit{Mortal}(\mathit{socrates})$).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x(\mathit{Human}(x) \to \mathit{Mortal}(x))}\,u}{\mathit{Human}(\mathit{socrates}) \to \mathit{Mortal}(\mathit{socrates})}\,\forall E \qquad \dfrac{}{\mathit{Human}(\mathit{socrates})}\,v}{\mathit{Mortal}(\mathit{socrates})}\,\to E}{\mathit{Human}(\mathit{socrates}) \to \mathit{Mortal}(\mathit{socrates})}\,\to I^v}{\forall x(\mathit{Human}(x) \to \mathit{Mortal}(x)) \to \mathit{Human}(\mathit{socrates}) \to \mathit{Mortal}(\mathit{socrates})}\,\to I^u$$

Now that we have seen how to decompose a $\forall$-quantifier, we have to see how to introduce one:

$$\dfrac{A[a/x]\ \mathsf{true}}{\forall x(A)\ \mathsf{true}}\,\forall I^a$$

the $\forall$-introduction rule allows you to conclude that $\forall x(A)$ is true if you can derive that $A[a/x]$ is true *for an arbitrary* a for which you cannot make any assumption: in fact a is a parameter to the label $\forall I^a$, to mark that it must be different from anything appearing elsewhere except under this part of the derivation. This is really important when you want to prove a quantified statement: when you want to prove some statement $\forall x(A)$ you cannot assume that x is — for example — an even number, or even more specifically 2. If you do that then you are not proving that A is true for every x, but just for those objects that satisfy your assumption.

Let us see how this works with an example:

Example 11 (Prove $\forall x(P(x) \to Q(x)) \to \forall x(P(x)) \to \forall x(Q(x))$ true).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x(P(x) \to Q(x))\ \mathsf{true}}\,u}{P(a) \to Q(a)\ \mathsf{true}}\,\forall E \qquad \dfrac{\dfrac{}{\forall x(P(x))\ \mathsf{true}}\,v}{P(a)\ \mathsf{true}}\,\forall E}{Q(a)\ \mathsf{true}}\,\to E}{\forall x(Q(x))\ \mathsf{true}}\,\forall I^a}{\forall x(P(x)) \to \forall x(Q(x))\ \mathsf{true}}\,\to I^v}{\forall x(P(x) \to Q(x)) \to \forall x(P(x)) \to \forall x(Q(x))\ \mathsf{true}}\,\to I^u$$

As usual, in this derivation we first introduce the two hypotheses that we label as u and v. Then we have to prove $\forall x(Q(x))$, and that is where we use the $\forall$-introduction rule: we must make sure we introduce a variable that does not appear elsewhere in the derivation, and a has in fact not appeared so far (when reading the proof from bottom to top), so we can safely use it. Then we proceed with the $\to E$ rule to conclude $Q(a)$ from $P(a) \to Q(a)$ and $P(a)$, which we can both derive by using the $\forall E$ rule.

Let us try another example, reconstructing a conjunction of formulas:

Example 12 (Prove $\forall x(A(x) \wedge B(x)) \rightarrow \forall x(B(x) \wedge A(x))$ true). *Here is the derivation:*

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{u}
\UnaryInfC{$\forall x(A(x) \wedge B(x))\ \mathsf{true}$}
\RightLabel{$\forall E$}
\UnaryInfC{$A(a) \wedge B(a)\ \mathsf{true}$}
\RightLabel{$\vee E_2$}
\UnaryInfC{$B(a)\ \mathsf{true}$}
\AxiomC{}
\RightLabel{u}
\UnaryInfC{$\forall x(A(x) \wedge B(x))\ \mathsf{true}$}
\RightLabel{$\forall E$}
\UnaryInfC{$A(a) \wedge B(a)\ \mathsf{true}$}
\RightLabel{$\vee E_1$}
\UnaryInfC{$A(a)\ \mathsf{true}$}
\RightLabel{$\wedge I$}
\BinaryInfC{$B(a) \wedge A(a)\ \mathsf{true}$}
\RightLabel{$\forall I^a$}
\UnaryInfC{$\forall x(B(x) \wedge A(x))\ \mathsf{true}$}
\RightLabel{$\rightarrow I^u$}
\UnaryInfC{$\forall x(A(x) \wedge B(x)) \rightarrow \forall x(B(x) \wedge A(x))\ \mathsf{true}$}
\end{prooftree}
$$

Proofweb (optional). Like in propositional logic, we can try out these exercises in ProofWeb:

```
Require Import ProofWeb.
Parameter A B : D -> Prop.
Theorem exercise : all x, (A x /\ B x) -> all x, (B x /\ A x).
Proof.
imp_i u.
all_i a.
con_i.
con_e2 (A a).
all_e (all x, (A x /\ B x)).
exact u.
con_e1 (B a).
all_e (all x, (A x /\ B x)).
exact u.
Qed.
```

Note that now the parameters A, B in the ProofWeb code are now functions from some domain D to *Prop*, which is the equivalent in ProofWeb of how a predicate is defined. If you want to define a predicate that talks about two variables, e.g. $\mathit{Older}(x, y)$, then in ProofWeb you have to encode it as a parameter with type $D \rightarrow D \rightarrow \mathit{Prop}$.

Quantifiers and logical connectors. We will soon see how to treat the existential quantifier ($\exists$). You might have observed that in propositional logics the rules for introducing and eliminating the connectors $\wedge$ and $\vee$ are specular to each other: $\wedge$ has two elimination rules and one introduction rule, whereas $\vee$ has one elimination rule and two introduction rules. The rules for $\forall$ and $\exists$ are similarly symmetric.

The idea of $\wedge E_1$ and $\wedge E_2$ is that you can take any one of the sub-formulas in $A \wedge B$ and if you have a proof of $A \wedge B$ true then you can get a proof of A true or B true. This is pretty much the same as the $\forall E$ rule: if you know $\forall x(A)$ true then you can conclude $A[t/x]$ true for any term t. You can view $\forall$ as an infinite conjunction, for example $\forall x(A(x))$ could be written as $A(0) \wedge A(1) \wedge A(2) \wedge \ldots$

if we had infinite space and time and our domain is the natural numbers. The $\forall E$ would be picking any one of these facts from this big conjunction.

The $\forall I^a$ rule is a bit trickier to see in this perspective. If you consider $\forall x(A)$ equivalent to $A(0) \wedge A(1) \wedge A(2) \wedge \ldots$ (still considering the domain of natural numbers), then the rule $\forall I^a$ allows you to prove $\forall x(A)$ true assuming that you have a proof of $A(0)$ true, one for $A(1)$ true, and so forth. The way we can get this infinite number of similar proofs in a finite time is if we can make a proof of $A(a)$ true for some arbitraty object a that we know nothing about. Then this same proof will be valid for any specific object, e.g. $A(0), A(1), A(2), \ldots$ and hence valid for all possible values of x.

We will see a similar pattern for the existential quantifier, that you can interpret as an infinite disjunction.

Existential quantification

The introduction rule for existential quantifiers ($\exists I$) is specular to the elimination rule for the universal quantifier:

$$\frac{A[t/x] \ \mathsf{true}}{\exists x(A) \ \mathsf{true}} \ \exists I$$

if we can prove $A[t/x]$ true for some arbitrary term t then we know that there exists an x such that A is true. We call t the *witness* of the existential quantifier. There are no restrictions in applying this rule.

Let us see how to use the exist-introduction rule with an example:

Example 13 (Prove that there exists somebody who is a mortal: $\forall x(Human(x) \to Mortal(x)) \to Human(socrates) \to \exists x(Mortal(x))$).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\forall x(Human(x) \to Mortal(x))}^{\ u}}{Human(socrates) \to Mortal(socrates)} \ \forall E \qquad \overline{Human(socrates)}^{\ v}}{Mortal(socrates)} \ \to E}{\exists x(Mortal(x))} \ \exists I}{Human(socrates) \to \exists x(Mortal(x))} \ \to I^v}{\forall x(Human(x) \to Mortal(x)) \to Human(socrates) \to \exists x(Mortal(x))} \ \to I^u$$

this proof looks very much like the first proof we have seen to introduce the $\forall$-elimination rule. The only big difference here is that, after having introduced the two hypotheses labelled u and v, we use the $\exists I$ rule to conclude that there is some object that is mortal, given that we have a proof that socrates is a mortal. Here "socrates" is the witness *of our proof for $\exists x(Mortal(x))$.*

The elimination rule ($\exists E^{a,u}$) works as follows:

$$\dfrac{\exists x(A) \ \mathsf{true} \qquad \begin{array}{c} \overline{A[a/x] \ \mathsf{true}}^{\ u} \\ \vdots \\ C \ \mathsf{true} \end{array}}{C \ \mathsf{true}} \ \exists E^{a,u}$$

if we have a proof of $\exists x(A)$ true and another proof of C true that assumes $A[a/x]$ true for an arbitrary a that does not appear anywhere else in the proof, then we can prove C true. Like in the $\forall I^a$ rule, it is very important that a is independent of any other object that appears elsewhere in the proof. And like in the $\vee I^{u,v}$ rule, we give a label u to the assumption of $A[a/x]$ true and try to prove a completely different fact, C true.

Let us now prove a derivation that involves both existential and universal quantifiers:

$$
\dfrac{
  \dfrac{
    \dfrac{}{\exists x(\neg A(x))\ \mathsf{true}}\,u
    \qquad
    \dfrac{
      \dfrac{}{\neg A(a)\ \mathsf{true}}\,w
      \quad
      \dfrac{\dfrac{}{\forall x(A(x))\ \mathsf{true}}\,v}{A(a)\ \mathsf{true}}\,\forall E
    }{\mathbf{F}\ \mathsf{true}}\,\neg E
  }{
    \dfrac{\mathbf{F}\ \mathsf{true}}{\neg\forall x(A(x))\ \mathsf{true}}\,\neg I^v
  }\,\exists E^{a,w}
}{\exists x(\neg A(x)) \to \neg\forall x(A(x))\ \mathsf{true}}\,\to I^u
$$

Here follows the ProofWeb code for this proof:

```
Require Import ProofWeb.
Parameter A : D -> Prop.
Theorem exercise : (exi x, ~A x) -> ~(all x, A x).
Proof.
imp_i u.
neg_i v.
exi_e (exi x, ~A x) b w.
exact u.
neg_e (A b).
exact w.
all_e (all x, A x).
exact v.
Qed.
```

So far we have discussed how to make proofs using existential and universal quantifiers, but we have not covered in depth how *not to use them.* In fact, this is just as important, because if we do not properly follow these rules (especially $\forall I$ and $\exists E$) we can introduce very nasty mistakes. So let us see something some common proof mistakes and understand why they are wrong applications of the rules and how to fix them.

Common mistakes in proofs. It is very important, when using the $\exists E^{a,u}$ rule, that we do not assume anything about a: in fact, when we substitute x with a, we also want that $A[a/x]$ holds. Continuing our example of the natural numbers, if we had a valid proof that there exists an even number (written as $\mathcal{D}_1 :: \exists x(\mathit{Even}(x))$) and we were to substitute x with 1 (making some assumptions on what number we substitute for) then we would clearly be making a mistake, concluding that 1 is even. To extend our example, we could have also a proof of $\mathcal{D}_2 :: \exists y(Odd(y))$ that some odd number exists, and violate the rule that a should not appear elsewhere in the proof by substituting a for x *and* a for y to build the following:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \exists x(Even(x))\ \mathsf{true}\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \exists y(Odd(y))\ \mathsf{true}\end{array} \qquad \begin{array}{c}\overline{Even(a)\ \mathsf{true}}^{\,u} \quad \overline{Odd(a)\ \mathsf{true}}^{\,v}\\ \vdots\\ C\ \mathsf{true}\end{array}}{C\ \mathsf{true}}\ \exists E^{a,v}}{C\ \mathsf{true}}\ \exists E^{a,u}$$

by doing this, we are concluding that some formula C is true starting from the assumption that there is a number a that is both odd and even at the same time. This would be a big mistake!

Furthermore, consider this other (wrong) derivation:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \exists x(Even(x))\ \mathsf{true}\end{array} \qquad \dfrac{\overline{Even(a)\ \mathsf{true}}^{\,u}}{\forall y(Even(y))\ \mathsf{true}}\ \forall I^a}{\forall y(Even(y))\ \mathsf{true}}\ \exists E^{a,u}$$

this is also non-sensical, because from a judgment that there exists one even number we conclude that all numbers are even. The only condition that saves us from making such a big mistake is that a should not appear outside the scope of $\forall I^a$, while it appears in the label $\exists E^{a,u}$.

We could try to build a similar proof that all numbers are even from a proof that there exists some even number as follows:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \exists x(Even(x))\ \mathsf{true}\end{array} \qquad \overline{Even(a)\ \mathsf{true}}^{\,u}}{Even(a)\ \mathsf{true}}\ \exists E^{a,u}}{\forall y(Even(y))\ \mathsf{true}}\ \forall I^a$$

this time the rule $\forall I^a$ is applied correctly, however the rule $\exists E^{a,u}$ does not respect the condition on a: the variable occurs as the conclusion $Even(a)$ and therefore escapes the scope of the $\exists E^{a,u}$ label.

Conclusion. This concludes our lecture. We have extended our logic language to quantified statements, and called it predicate logics. We have seen how to introduce and remove $\forall$ and $\exists$ quantifiers in natural deduction, and heard about common proof mistakes. Now you have a chance to use these rules with the following set of exercises:

Exercise 6 (Prove the following predicates)**.**

1. $P(t) \to \forall x(P(x) \to \neg Q(x)) \to \neg Q(t)$
2. $\forall x(Q(x) \to R(x)) \to \exists x(P(x) \land Q(x)) \to \exists x(P(x) \land R(x))$

3. $\forall x(A(x) \lor B(x)) \to \forall x(B(x) \lor A(x))$
4. $\forall x(A(x) \land B(x)) \to \exists x(B(x))$
5. $\forall x(\neg A(x)) \to \neg\exists x(A(x))$
6. $\forall x(P(x) \to Q(x)) \to \exists x(P(x)) \to \exists x(Q(x))$

Exercise 7 (Prove the following predicates)**.**

1. $(\exists x(P(x)) \lor \exists x(Q(x))) \to \exists x(P(x) \lor Q(x))$
2. $\exists x\exists y(P(x, y)) \to \exists y\exists x(P(x, y))$
3. $\exists x(P(x)) \to \forall x\forall y(P(x) \to Q(y)) \to \forall y(Q(y))$
4. $\exists x(\neg P(x)) \to \neg\forall x(P(x))$
5. $\neg\forall x(P(x)) \to \exists x(\neg P(x))$ *(advanced)*

Hint, start this way:

$$\frac{(\neg\exists x(\neg P(x)) \to \neg\neg\forall x(P(x))) \to \neg\forall x(P(x)) \to \exists x(\neg P(x)) \quad \neg\exists x(\neg P(x)) \to \neg\neg\forall x(P(x))}{\neg\forall x(P(x)) \to \exists x(\neg P(x))} \to E$$

Exercise 8 (Prove the following predicates)**.**

1. $\exists x\forall y(P(x, y)) \to \forall x\exists y(P(y, x))$
2. $\neg\exists x(P(x)) \to \forall x(\neg P(x))$
3. $\forall x(P(x) \land Q(x)) \equiv \forall x(P(x)) \land \forall x(Q(x))$
4. $\exists x(P(x) \lor Q(x)) \equiv \exists x(P(x)) \lor \exists x(Q(x))$

Rules of Natural Deduction in Predicate Logic

Conjunction:

$$\frac{A\ \mathsf{true} \quad B\ \mathsf{true}}{A \wedge B\ \mathsf{true}}\ \wedge I \qquad \frac{A \wedge B\ \mathsf{true}}{A\ \mathsf{true}}\ \wedge E_1 \qquad \frac{A \wedge B\ \mathsf{true}}{B\ \mathsf{true}}\ \wedge E_2$$

Implication:

$$\frac{\begin{array}{c}\overline{A\ \mathsf{true}}\ u\\ \vdots\\ B\ \mathsf{true}\end{array}}{A \to B\ \mathsf{true}}\ {\to}I^u \qquad \frac{A \to B\ \mathsf{true} \quad A\ \mathsf{true}}{B\ \mathsf{true}}\ {\to}E$$

Disjunction:

$$\frac{A\ \mathsf{true}}{A \vee B\ \mathsf{true}}\ \vee I_1 \qquad \frac{B\ \mathsf{true}}{A \vee B\ \mathsf{true}}\ \vee I_2 \qquad \frac{A \vee B\ \mathsf{true} \quad \begin{array}{c}\overline{A\ \mathsf{true}}\ u\\ \vdots\\ C\ \mathsf{true}\end{array} \quad \begin{array}{c}\overline{B\ \mathsf{true}}\ v\\ \vdots\\ C\ \mathsf{true}\end{array}}{C\ \mathsf{true}}\ \vee E^{u,v}$$

True and false:

$$\frac{}{\mathbf{T}\ \mathsf{true}}\ \mathbf{T}I \qquad \frac{\mathbf{F}\ \mathsf{true}}{C\ \mathsf{true}}\ \mathbf{F}E$$

Negation:

$$\frac{\begin{array}{c}\overline{A\ \mathsf{true}}\ u\\ \vdots\\ \mathbf{F}\ \mathsf{true}\end{array}}{\neg A\ \mathsf{true}}\ \neg I^u \qquad \frac{\neg A\ \mathsf{true} \quad A\ \mathsf{true}}{C\ \mathsf{true}}\ \neg E$$

Classical rules:

$$\frac{}{A \vee \neg A\ \mathsf{true}}\ LEM \qquad \frac{\neg\neg A\ \mathsf{true}}{A\ \mathsf{true}}\ \neg\neg C \qquad \frac{\begin{array}{c}\overline{\neg A\ \mathsf{true}}\ u\\ \vdots\\ \mathbf{F}\ \mathsf{true}\end{array}}{A\ \mathsf{true}}\ \mathbf{F}_C^u$$

Quantifiers:

$$\frac{A[a/x]\ \mathsf{true}}{\forall x(A)\ \mathsf{true}}\ \forall I^a \qquad \frac{\forall x(A)\ \mathsf{true}}{A[t/x]\ \mathsf{true}}\ \forall E$$

$$\frac{A[t/x]\ \mathsf{true}}{\exists x(A)\ \mathsf{true}}\ \exists I \qquad \frac{\exists x(A)\ \mathsf{true} \quad \begin{array}{c}\overline{A[a/x]\ \mathsf{true}}\ u\\ \vdots\\ C\ \mathsf{true}\end{array}}{C\ \mathsf{true}}\ \exists E^{a,u}$$