Peergrade #3: Sequences and Sums, Graphs and Trees, Number Theory

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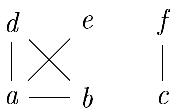
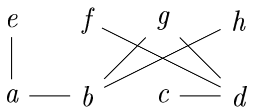
Exercise 1. *Consider the following graphs, then explain why each statement is true or false.*

1. 

   (a) *There exists a simple path from a to d*

   (b) *There exists a simple path from c to d*

   (c) *The graph is connected*

2. 

   (a) *The graph represents a tree*

   (b) *The graph represents a binary tree*

   (c) *The graph represents a full binary tree*

Solution 1.

1. (a) *True, the path is a, d.*

   (b) *False, there is no path from c to d.*

   (c) *False, there are two connected components $\{a, b, d, e\}$ and $\{c, f\}$.*

2. (a) *True, since there exists a unique simple path between any two vertices.*

   (b) *True, taken any root, each internal vertex has at most 2 children.*

   (c) *False, if g is an internal node (non-root) then it has only one children (b or d).*

Exercise 2. *Compute the following values, showing the process for each computation:*

1. *Show the following sequences for $1 < j < 10$:*

(a) $a_j = j^2$

(b) $b_j = 1/j$

(c) $c_j = \sum_{i=1}^{j} a_i \cdot b_i$

2. (a) $(763636 \cdot 437813 \cdot 936257) \,\mathbf{mod}\, 43$

   (b) $(894461 \cdot (206193 + 83218)) \,\mathbf{mod}\, 59$

   (Use the theorems shown in class to simplify each step of the computation.)

3. (a) *Convert* $(73217)_8$ *to its binary and decimal expansion.*

   (b) *Convert* $(62290)_{10}$ *to its octal and hexadecimal expansion.*

4. (a) $gcd(2574, 1976)$

   (b) $lcm(1525, 4405)$

Solution 2. *1.* (a) $a_1 = 1, a_2 = 4, a_3 = 9, a_4 = 16, a_5 = 25, a_6 = 36, a_7 = 49, a_8 = 64, a_9 = 81$

(b) $b_1 = 1, b_2 = \frac{1}{2}, b_3 = \frac{1}{3}, b_4 = \frac{1}{4}, b_5 = \frac{1}{5}, b_6 = \frac{1}{6}, b_7 = \frac{1}{7}, b_8 = \frac{1}{8}, b_9 = \frac{1}{9}$

(c) $c_1 = 1, c_2 = 1 + 4 \cdot \frac{1}{2} = 3, c_3 = 1 + 2 + 9 \cdot \frac{1}{3} = 6$ *(I hope the pattern is clear)*
$c_4 = 10, c_5 = 15, c_6 = 21, c_7 = 28, c_8 = 36, c_9 = 45$

(a) $(763636 \cdot 437813 \cdot 936257) \,\mathbf{mod}\, 43 =$
$(763636 \,\mathbf{mod}\, 43 \cdot 437813 \,\mathbf{mod}\, 43 \cdot 936257 \,\mathbf{mod}\, 43) =$
$(((42 \cdot 30) \,\mathbf{mod}\, 42) \cdot 18) \,\mathbf{mod}\, 43 =$
$(13 \cdot 18) \,\mathbf{mod}\, 43 = 19$

(b) $(894461 \cdot (206193 + 83218)) \,\mathbf{mod}\, 59 =$
$(894461 \,\mathbf{mod}\, 59 \cdot (206193 \,\mathbf{mod}\, 59 + 83218 \,\mathbf{mod}\, 59) \,\mathbf{mod}\, 59) \,\mathbf{mod}\, 59 =$
$(21 \cdot (47 + 28) \,\mathbf{mod}\, 59) =$
$(21 \cdot 16) \,\mathbf{mod}\, 59 = 41$

2. (a) $(73217)_8 = (111011010001111)_2 = (30351)_{10}$

   (b) $(62290)_{10} = (171522)_8 = (F352)_{16}$

3. (a)

$$
\begin{aligned}
2574 &= 1 \cdot 1976 + 598 \\
1976 &= 3 \cdot 598 + 182 \\
598 &= 3 \cdot 182 + 52 \\
182 &= 3 \cdot 52 + 26 \\
52 &= 2 \cdot 26
\end{aligned}
$$

Hence $gcd(2574, 1976) = 26$

(b)

$$1525 = 5^2 \cdot 61$$

$$4405 = 5 \cdot 881$$

Hence $lcm(1525, 4405) = 5^2 \cdot 61 \cdot 881 = 1343525$

Exercise 3. *Let x, y, z and m be integers. Assume that $x \equiv y \,(\mathbf{mod}\, m)$ and that $m \mid z$. Prove that $x + 2z \equiv y \,(\mathbf{mod}\, m)$.*

Solution 3. *By definition of equivalence modulo m, $x \equiv y \,(\mathbf{mod}\, m)$ iff $m \mid x - y$. Since $m \mid z$ then also $m \mid 2z$, and hence $m \mid x - y + 2z = x + 2z - y$ (by the first theorem proved in class). Applying the definition of equivalence modulo m again, we obtain that $x + 2z \equiv y \,(\mathbf{mod}\, m)$.*

P.s. some of you might have opened an old version of this assignment with the solution leaked. In case you find this excercise solved instead of Exercise 2, please grade it accordingly.

Exercise 4. *Prove that if n is an odd positive integer, then $n^4 \equiv 1 \,(\mathbf{mod}\, 16)$*

Solution 4. *We need to prove that $16 \mid n^4 - 1$ (by definition of congruence modulo n).*

Let $n = 2 \cdot k + 1$ for some integer k (since n is odd). Then:

$$\begin{aligned} n^4 - 1 &= (2 \cdot k + 1)^4 - 1 = (2k)^4 + 4 \cdot (2k)^3 \cdot 1 + 6 \cdot (2k)^2 \cdot 1^2 + 4 \cdot (2k)^1 \cdot 1^3 + 1 - 1 \\ &= 16 \cdot k^4 + 32 \cdot k^3 + 24 \cdot k^2 + 8 \cdot k \\ &= 16 \cdot (k^4 + 2k^3) + 8 \cdot (3k^2 + k) \end{aligned}$$

We now prove that $16 \mid n^4 - 1$. Note that 16 is a factor of $16 \cdot (k^4 + 2k^3)$, we want to show that 16 is also a factor of $8 \cdot (3k^2 + k)$.

We distinguish two cases: k may be even or odd.

- *In case k is even, there exists an integer j such that $k = 2 \cdot j$. Hence $8 \cdot (3k^2 + k) = 8 \cdot (3(2j)^2 + 2j) = 8 \cdot (12j^2 + 2j) = 16 \cdot (6j^2 + j)$.*

- *In case k is odd, there exists an integer l such that $k = 2 \cdot l + 1$. Hence $8 \cdot (3k^2 + k) = 8 \cdot (3(2j + 1)^2 + (2j + 1)) = 8 \cdot (3(4j^2 + 4j + 1) + (2j + 1)) = 16 \cdot (6j^2 + 7j + 2)$.*

Hence 16 divides $8 \cdot (3k^2 + k)$ and also $16 \cdot (k^3 + 2k^3)$. Therefore 16 divides the expansion of $n^4 - 1$, and hence $n^2 \equiv 1 \,(\mathbf{mod}\, 16)$.