LIFETIME PORTFOLIO SELECTION BY DYNAMIC STOCHASTIC PROGRAMMING

Paul A. Samuelson *

Introduction

MOST analyses of portfolio selection, whether they are of the Markowitz-Tobin mean-variance or of more general type, maximize over one period.[1] I shall here formulate and solve a many-period generalization, corresponding to lifetime planning of consumption and investment decisions. For simplicity of exposition I shall confine my explicit discussion to special and easy cases that suffice to illustrate the general principles involved.

As an example of topics that can be investigated within the framework of the present model, consider the question of a "businessman risk" kind of investment. In the literature of finance, one often reads; "Security A should be avoided by widows as too risky, but is highly suitable as a businessman's risk." What is involved in this distinction? Many things.

First, the "businessman" is more affluent than the widow; and being further removed from the threat of falling below some subsistence level, he has a high propensity to embrace variance for the sake of better yield.

Second, he can look forward to a high salary in the future; and with so high a present discounted value of wealth, it is only prudent for him to put more into common stocks compared to his present tangible wealth, borrowing if necessary for the purpose, or accomplishing the same thing by selecting volatile stocks that widows shun.

Third, being still in the prime of life, the businessman can "recoup" any present losses in the future. The widow or retired man nearing life's end has no such "second or n^{th} chance."

Fourth (and apparently related to the last point), since the businessman will be investing for so many periods, "the law of averages will even out for him," and he can afford to act almost as if he were not subject to diminishing marginal utility.

What are we to make of these arguments? It will be realized that the first could be purely a one-period argument. Arrow, Pratt, and others [2] have shown that any investor who faces a range of wealth in which the elasticity of his marginal utility schedule is great will have high risk tolerance; and most writers seem to believe that the elasticity is at its highest for rich — but not ultra-rich! — people. Since the present model has no new insight to offer in connection with statical risk tolerance, I shall ignore the first point here and confine almost all my attention to utility functions with the same relative risk aversion at all levels of wealth. Is it then still true that lifetime considerations justify the concept of a businessman's risk in his prime of life?

Point two above does justify leveraged investment financed by borrowing against future earnings. But it does not really involve any increase in relative risk-taking once we have related what is at risk to the proper larger base. (Admittedly, if market imperfections make loans difficult or costly, recourse to volatile, "leveraged" securities may be a rational procedure.)

The fourth point can easily involve the innumerable fallacies connected with the "law of large numbers." I have commented elsewhere [3] on the mistaken notion that multiplying the same kind of risk leads to cancellation rather

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[1] See for example Harry Markowitz [5]; James Tobin [14], Paul A. Samuelson [10]; Paul A. Samuelson and Robert C. Merton [13]. See, however, James Tobin [15], for a pioneering treatment of the multi-period portfolio problem; and Jan Mossin [7] which overlaps with the present analysis in showing how to solve the basic dynamic stochastic program recursively by working backward from the end in the Bellman fashion, and which proves the theorem that portfolio proportions will be invariant only if the marginal utility function is iso-elastic.

[2] See K. Arrow [1]; J. Pratt [9]; P. A. Samuelson and R. C. Merton [13].

[3] P. A. Samuelson [11].

than augmentation of risk. I.e., insuring many ships adds to risk (but only as $\sqrt{n}$); hence, only by insuring more ships and by *also* subdividing those risks among more people is risk on each brought down (in ratio $1/\sqrt{n}$).

However, before writing this paper, I had thought that points three and four could be reformulated so as to give a valid demonstration of businessman's risk, my thought being that investing for each period is akin to agreeing to take a $1/n^{\text{th}}$ interest in insuring n independent ships.

The present lifetime model reveals that investing for many periods does not *itself* introduce extra tolerance for riskiness at early, or any, stages of life.

Basic Assumptions

The familiar Ramsey model may be used as a point of departure. Let an individual maximize

$$\int_0^T e^{-\rho t}\, U[C(t)]\, dt \tag{1}$$

subject to initial wealth W_0 that can always be invested for an exogeneously-given certain rate of yield r; or subject to the constraint

$$C(t) = rW(t) - \dot{W}(t) \tag{2}$$

If there is no bequest at death, terminal wealth is zero.

This leads to the standard calculus-of-variations problem

$$J = \underset{\{W(t)\}}{\text{Max}} \int_0^T e^{-\rho t}\, U[rW - \dot{W}]\, dt \tag{3}$$

This can be easily related [4] to a discrete-time formulation

$$\text{Max} \sum_{t=0}^{T} (1+\rho)^{-t}\, U[C_t] \tag{4}$$

subject to

$$C_t = W_t - \frac{W_{t+1}}{1+r} \tag{5}$$

or,

$$\underset{\{W_t\}}{\text{Max}} \sum_{t=0}^{T} (1+\rho)^{-t}\, U\left[W_t - \frac{W_{t+1}}{1+r} \right] \tag{6}$$

[4] See P. A. Samuelson [12], p. 273 for an exposition of discrete-time analogues to calculus-of-variations models. Note: here I assume that consumption, C_t, takes place at the beginning rather than at the end of the period. This change alters slightly the appearance of the equilibrium conditions, but not their substance.

for prescribed $(W_0,\ W_{T+1})$. Differentiating partially with respect to each W_t in turn, we derive recursion conditions for a regular interior maximum

$$\frac{(1+\rho)}{1+r}\, U'\left[W_{t-1} - \frac{W_t}{1+r} \right] = U'\left[W_t - \frac{W_{t+1}}{1+r} \right] \tag{7}$$

If U is concave, solving these second-order difference equations with boundary conditions $(W_0,\ W_{T+1})$ will suffice to give us an optimal lifetime consumption-investment program.

Since there has thus far been one asset, and that a safe one, the time has come to introduce a stochastically-risky alternative asset and to face up to a portfolio problem. Let us postulate the existence, alongside of the safe asset that makes \$1 invested in it at time t return to you at the end of the period $\$1(1+r)$, a risk asset that makes \$1 invested in, at time t, return to you after one period $\$1Z_t$, where Z_t is a random variable subject to the probability distribution

$$\text{Prob}\,\{Z_t \leqq z\} = P(z). \qquad z \geqq 0 \tag{8}$$

Hence, $Z_{t+1} - 1$ is the percentage "yield" of each outcome. The most general probability distribution is admissible: i.e., a probability density over continuous z's, or finite positive probabilities at discrete values of z. Also I shall usually assume independence between yields at different times so that $P(z_0, z_1, \ldots, z_t, \ldots, z_T) = P(z_t)P(z_1)\ldots P(z_T)$.

For simplicity, the reader might care to deal with the easy case

$$\text{Prob}\,\{Z = \lambda\} = 1/2 = \text{Prob}\,\{Z = \lambda^{-1}\}, \qquad \lambda > 1 \tag{9}$$

In order that risk averters with concave utility should not shun this risk asset when maximizing the expected value of their portfolio, λ must be large enough so that the expected value of the risk asset exceeds that of the safe asset, i.e.,

$$\frac{1}{2}\lambda + \frac{1}{2}\lambda^{-1} > 1 + r, \text{ or}$$

$$\lambda > 1 + r + \sqrt{2r + r^2}.$$

Thus, for $\lambda = 1.4$, the risk asset has a mean yield of 0.057, which is greater than a safe asset's certain yield of $r = .04$.

At each instant of time, what will be the optimal fraction, w_t, that you should put in the risky asset, with $1 - w_t$ going into the safe asset? Once these optimal portfolio fractions are known, the constraint of (5) must be written

$$C_t = \left[W_t - \frac{W_{t+1}}{[(1-w_t)(1+r) + w_t Z_t]} \right]. \tag{10}$$

Now we use (10) instead of (4), and recognizing the stochastic nature of our problem, specify that we maximize the expected value of total utility over time. This gives us the stochastic generalizations of (4) and (5) or (6)

$$\underset{\{C_t, w_t\}}{\text{Max}} E \sum_{t=0}^{T} (1+\rho)^{-t} U[C_t] \tag{11}$$

subject to

$$C_t = \left[W_t - \frac{W_{t+1}}{(1+r)(1-w_t) + w_t Z_t} \right]$$

W_0 given, W_{T+1} prescribed.

If there is no bequeathing of wealth at death, presumably $W_{T+1} = 0$. Alternatively, we could replace a prescribed W_{T+1} by a final bequest function added to (11), of the form $B(W_{T+1})$, and with W_{T+1} a free decision variable to be chosen so as to maximize (11) + $B(W_{T+1})$. For the most part, I shall consider $C_T = W_T$ and $W_{T+1} = 0$.

In (11), E stands for the "expected value of," so that, for example,

$$E\,Z_t = \int_0^\infty z_t dP(z_t)\ .$$

In our simple case of (9),

$$EZ_t = \frac{1}{2}\lambda + \frac{1}{2}\lambda^{-1}.$$

Equation (11) is our basic stochastic programming problem that needs to be solved simultaneously for optimal saving-consumption and portfolio-selection decisions over time.

Before proceeding to solve this problem, reference may be made to similar problems that seem to have been dealt with explicitly in the economics literature. First, there is the valuable paper by Phelps on the Ramsey problem in which capital's yield is a prescribed random variable. This corresponds, in my notation, to the $\{w_t\}$ strategy being frozen at some fractional level, there being no portfolio selection problem. (My analysis could be amplified to consider Phelps' [5] wage income, and even in the stochastic form that he cites Martin Beckmann as having analyzed.) More recently, Levhari and Srinivasan [4] have also treated the Phelps problem for $T = \infty$ by means of the Bellman functional equations of dynamic programming, and have indicated a proof that concavity of U is sufficient for a maximum. Then, there is Professor Mirrlees' important work on the Ramsey problem with Harrod-neutral technological change as a random variable.[6] Our problems become equivalent if I replace $W_t - W_{t+1}[(1+r)(1-w_t) + w_t Z_t]^{-1}$ in (10) by $A_t f(W_t/A_t) - nW_t - (W_{t+1} - W_t)$ let technical change be governed by the probability distribution

$$\text{Prob}\,\{A_t \leqq A_{t-1}Z\} = P(Z);$$

reinterpret my W_t to be Mirrlees' per capita capital, K_t/L_t, where L_t is growing at the natural rate of growth n; and posit that $A_t f(W_t/A_t)$ is a homogeneous first degree, concave, neoclassical production function in terms of capital and efficiency-units of labor.

It should be remarked that I am confirming myself here to regular interior maxima, and not going into the Kuhn-Tucker inequalities that easily handle boundary maxima.

Solution of the Problem

The meaning of our basic problem

$$J_T(W_0) = \underset{\{C_t, w_t\}}{\text{Max}} E \sum_{t=0}^{T} (1+\rho)^{-t} U[C_t] \tag{11}$$

subject to $C_t = W_t - W_{t+1}[(1-w_t)(1+r) + w_t Z_t]^{-1}$ is not easy to grasp. I act now at $t = 0$ to select C_0 and w_0, knowing W_0 but not yet knowing how Z_0 will turn out. I must act now, knowing that one period later, knowledge of Z_0's outcome will be known and that W_1 will then be known. Depending upon knowledge of W_1, a new decision will be made for C_1 and w_1. Now I can only guess what that decision will be.

As so often is the case in dynamic programming, it helps to begin at the end of the planning period. This brings us to the well-known

[5] E. S. Phelps [8].

[6] J. A. Mirrlees [6]. I have converted his treatment into a discrete-time version. Robert Merton's companion paper throws light on Mirrlees' Brownian-motion model for A_t.

one-period portfolio problem. In our terms, this becomes

$$J_1(W_{T-1}) = \underset{\{C_{T-1}, w_{T-1}\}}{\text{Max}} U[C_{T-1}] + E(1+\rho)^{-1} U[(W_{T-1} - C_{T-1}) \{(1-w_{T-1})(1+r) + w_{T-1}Z_{T-1}\}]. \quad (12)$$

Here the expected value operator E operates only on the random variable of the next period since current consumption C_{T-1} is known once we have made our decision. Writing the second term as $EF(Z_T)$, this becomes

$$EF(Z_T) = \int_0^\infty F(Z_T)\,dP(Z_T|Z_{T-1}, Z_{T-2}, \ldots, Z_0)$$

$$= \int_0^\infty F(Z_T)\,dP(Z_T), \text{ by our independence}$$

postulate.

In the general case, at a later stage of decision making, say $t = T-1$, knowledge will be available of the outcomes of earlier random variables, $Z_{t-2}, \ldots$; since these might be relevant to the distribution of subsequent random variables, conditional probabilities of the form $P(Z_{T-1}|Z_{T-2}, \ldots)$ are thus involved. However, in cases like the present one, where independence of distributions is posited, conditional probabilities can be dispensed within favor of simple distributions.

Note that in (12) we have substituted for C_T its value as given by the constraint in (11) or (10).

To determine this optimum (C_{T-1}, w_{T-1}), we differentiate with respect to each separately, to get

$$0 = U'[C_{T-1}] - (1+\rho)^{-1} EU'[C_T] \{(1-w_{T-1})(1+r) + w_{T-1}Z_{T-1}\} \quad (12')$$

$$0 = EU'[C_T](W_{T-1} - C_{T-1})(Z_{T-1} - 1 - r)$$
$$= \int_0^\infty U'[(W_{T-1} - C_{T-1}) \{(1-w_{T-1}(1+r) - w_{T-1}Z_{T-1}\}] (W_{T-1} - C_{T-1})(Z_{T-1} - 1 - r)\,dP(Z_{T-1}) \quad (12'')$$

Solving these simultaneously, we get our optimal decisions (C^*_{T-1}, w^*_{T-1}) as functions of initial wealth W_{T-1} alone. Note that if somehow C^*_{T-1} were known, (12″) would by itself be the familiar one-period portfolio optimality condition, and could trivially be rewritten to handle any number of alternative assets.

Substituting (C^*_{T-1}, w^*_{T-1}) into the expression to be maximized gives us $J_1(W_{T-1})$ explicitly. From the equations in (12), we can, by standard calculus methods, relate the derivatives of U to those of J, namely, by the envelope relation

$$J_1'(W_{T-1}) = U'[C_{T-1}]. \quad (13)$$

Now that we know $J_1[W_{T-1}]$, it is easy to determine optimal behavior one period earlier, namely by

$$J_2(W_{T-2}) = \underset{\{C_{T-2}, w_{T-2}\}}{\text{Max}} U[C_{T-2}] + E(1+\rho)^{-1} J_1[(W_{T-2} - C_{T-2}) \{(1-w_{T-2})(1+r) + w_{T-2}Z_{T-2}\}]. \quad (14)$$

Differentiating (14) just as we did (11) gives the following equations like those of (12)

$$0 = U'[C_{T-2}] - (1+\rho)^{-1} EJ_1'[W_{T-2}] \{(1-w_{T-2})(1+r) + w_{T-2}Z_{T-2}\} \quad (15')$$

$$0 = EJ_1'[W_{T-1}](W_{T-2} - C_{T-2})(Z_{T-2} - 1 - r)$$
$$= \int_0^\infty J_1'[(W_{T-2} - C_{T-2})\{(1-w_{T-2})(1+r) + w_{T-2}Z_{T-2}\}](W_{T-2} - C_{T-2})(Z_{T-2} - 1 - r)\,dP(Z_{T-2}). \quad (15'')$$

These equations, which could by (13) be related to $U'[C_{T-1}]$, can be solved simultaneously to determine optimal (C^*_{T-2}, w^*_{T-2}) and $J_2(W_{T-2})$.

Continuing recursively in this way for T−3, T−4, . . . , 2, 1, 0, we finally have our problem solved. The general recursive optimality equations can be written as

$$\begin{cases} 0 = U'[C_0] - (1+\rho)^{-1} EJ'_{T-1}[W_0] \{(1-w_0)(1+r) + w_0Z_0\} \\ 0 = EJ'_{T-1}[W_1](W_0 - C_0)(Z_0 - 1 - r) \end{cases}$$

. . . .

$$0 = U'[C_{T-1}] - (1+\rho)^{-1} EJ'_{T-t}[W_t] \{(1-w_{t-1})(1+r) + w_{t-1}Z_{t-1}\} \quad (16')$$

$$0 = EJ'_{T-t}[W_{t-1} - C_{t-1}](Z_{t-1} - 1 - r), \quad (t = 1, \ldots, T-1). \quad (16'')$$

In (16′), of course, the proper substitutions must be made and the E operators must be over the proper probability distributions. Solving (16″) at any stage will give the optimal decision rules for consumption-saving and for portfolio selection, in the form

$$C^*_t = f[W_t; Z_{t-1}, \ldots, Z_0]$$
$$= f_{T-t}[W_t] \text{ if the } Z\text{'s are independently distributed}$$

$$
\begin{aligned}
w^*_t &= g[W_t; Z_{t-1}, \ldots, Z_0] \\
&= g_{T-t}[W_t] \text{ if the } Z\text{'s are independently distributed.}
\end{aligned}
$$

Our problem is now solved for every case but the important case of infinite-time horizon. For well-behaved cases, one can simply let $T \to \infty$ in the above formulas. Or, as often happens, the infinite case may be the easiest of all to solve, since for it $C^*_t = f(W_t)$, $w^*_t = g(W_t)$, independently of time and both these unknown functions can be deduced as solutions to the following functional equations:

$$
\begin{aligned}
0 = U'[f(W)] - (1+\rho)^{-1} & \int_0^\infty J'[(W - f(W))\{(1+r) \\
& - g(W)(Z - 1 - r)\}][(1+r) \\
& - g(W)(Z - 1 - r)]dP(Z)
\end{aligned}
\tag{17'}
$$

$$
\begin{aligned}
0 = \int_0^\infty & U'[\{W - f(W)\} \\
& \{1 + r - g(W)(Z - 1 - r)\}] \\
& [Z - 1 - r]\, dP(Z)
\end{aligned}
\tag{17''}
$$

Equation (17′), by itself with $g(W)$ pretended to be known, would be equivalent to equation (13) of Levhari and Srinivasan [4, p. f]. In deriving (17′)–(17″), I have utilized the envelope relation of my (13), which is equivalent to Levhari and Srinivasan's equation (12) [4, p. 5].

Bernoulli and Isoelastic Cases

To apply our results, let us consider the interesting Bernoulli case where $U = \log C$. This does not have the bounded utility that Arrow [1] and many writers have convinced themselves is desirable for an axiom system. Since I do not believe that Karl Menger paradoxes of the generalized St. Petersburg type hold any terrors for the economist, I have no particular interest in boundedness of utility and consider $\log C$ to be interesting and admissible. For this case, we have, from (12),

$$
\begin{aligned}
J_1(W) &= \operatorname*{Max}_{\{C, w\}} \log C \\
&\quad + E(1+\rho)^{-1} \log [(W - C) \\
&\quad \{(1-w)(1+r) + wZ\}] \\
&= \operatorname*{Max}_{\{C\}} \log C + (1+\rho)^{-1}\{\log [W - C] \\
&\quad + \operatorname*{Max}_{\{w\}} \int_0^\infty \log [(1-w)(1+r) \\
&\quad + wZ]dP(Z).\}
\end{aligned}
\tag{18}
$$

Hence, equations (12) and (16′)–(16″) split into two independent parts and the Ramsey-Phelps saving problem becomes quite independent of the lifetime portfolio selection problem. Now we have

$$
\begin{aligned}
0 &= (1/C) - (1+\rho)^{-1}(W - C)^{-1} \text{ or} \\
C_{T-1} &= (1+\rho)(2+\rho)^{-1} W_{T-1}
\end{aligned}
\tag{19'}
$$

$$
\begin{aligned}
0 &= \int_0^\infty (Z - 1 - r)[(1-w)(1+r) \\
&\quad + wZ]^{-1} dP(Z) \text{ or} \\
w_{T-1} &= w^* \text{ independently of } W_{T-1}.
\end{aligned}
\tag{19''}
$$

These independence results, of the C_{T-1} and w_{T-1} decisions and of the dependence of w_{T-1} on W_{T-1}, hold for all U functions with isoelastic marginal utility. I.e., (16′) and (16″) become decomposable conditions for all

$$
U(C) = 1/\gamma\, C^\gamma, \qquad \gamma < 1 \tag{20}
$$

as well as for $U(C) = \log C$, corresponding by L'Hôpital's rule to $\gamma = 0$.

To see this, write (12) or (18) as

$$
\begin{aligned}
J_1(W) &= \operatorname*{Max}_{\{C, w\}} \frac{C^\gamma}{\gamma} + (1+\rho)^{-1} \frac{(W-C)^\gamma}{\gamma} \\
&\qquad \int_0^\infty [(1-w)(1+r) + wZ]^\gamma\, dP(Z) \\
&= \operatorname*{Max}_{\{C\}} \frac{C^\gamma}{\gamma} + (1+\rho)^{-1} \frac{(W-C)^\gamma}{\gamma} \times \\
&\qquad \operatorname*{Max}_{w} \int_0^\infty [(1-w)(1+r) \\
&\qquad + wZ]^\gamma\, dp(Z).
\end{aligned}
\tag{21}
$$

Hence, (12″) or (15″) or (16″) becomes

$$
\begin{aligned}
\int_0^\infty & [(1-w)(1+r) \\
& + wZ]^{\gamma-1}(Z - r - 1)\, dP(Z) = 0,
\end{aligned}
\tag{22''}
$$

which defines optimal w^* and gives

$$
\begin{aligned}
& \operatorname*{Max}_{\{w\}} \int_0^\infty [(1-w)(1+r) + wZ]^\gamma\, dP(Z) \\
&= \int_0^\infty [(1-w^*)(1+r) + w^*Z]^\gamma\, dP(Z) \\
&= [1 + r^*]^\gamma, \text{ for short.}
\end{aligned}
$$

Here, r^* is the subjective or util-prob mean return of the portfolio, where diminishing marginal utility has been taken into account.[7] To get optimal consumption-saving, differentiate (21) to get the new form of (12′), (15′), or (16′)

[7] See Samuelson and Merton for the util-prob concept [13].

$$0 = C^{\gamma-1} - (1+\rho)^{-1}(1+r^*)^{\gamma}(W-C)^{\gamma-1}. \quad (22')$$

Solving, we have the consumption decision rule

$$C^*_{T-1} = \frac{a_1}{1+a_1}W_{T-1} \quad (23)$$

where

$$a_1 = [(1+r^*)^{\gamma}/(1+\rho)]^{1/\gamma-1}. \quad (24)$$

Hence, by substitution, we find

$$J_1(W_{T-1}) = b_1 W^{\gamma}_{T-1}/\gamma \quad (25)$$

where

$$b_1 = a_1^{\gamma}(1+a_1)^{-\gamma} + (1+\rho)^{-1}(1+r^*)^{\gamma}(1+a_1)^{-\gamma}. \quad (26)$$

Thus, $J_1(\cdot)$ is of the same elasticity form as $U(\cdot)$ was. Evaluating indeterminate forms for $\gamma = 0$, we find J_1 to be of log form if U was.

Now, by mathematical induction, it is easy to show that this isoelastic property must also hold for $J_2(W_{T-2})$, $J_3(W_{T-3}), \ldots,$ since, whenever it holds for $J_n(W_{T-n})$ it is deducible that it holds for $J_{n+1}(W_{T-n-1})$. Hence, at every stage, solving the general equations (16′) and (16″), they decompose into two parts in the case of isoelastic utility. Hence,

Theorem:

For isoelastic marginal utility functions, $U'(C) = C^{\gamma-1}$, $\gamma < 1$, the optimal portfolio decision is independent of wealth at each stage and independent of all consumption-saving decisions, leading to a constant w^*, the solution to

$$0 = \int_0^{\infty}[(1-w)(1+r)+wZ]^{\gamma-1}(Z-1-r)\,dP(Z).$$

Then optimal consumption decisions at each stage are, for a no-bequest model, of the form

$$C^*_{T-i} = c_i W_{T-i}$$

where one can deduce the recursion relations

$$c_1 = \frac{a_1}{1+w_1},$$

$$a_1 = [(1+\rho)/(1+r^*)^{\gamma}]^{1/1-\gamma}$$

$$(1+r^*)^{\gamma} = \int_0^{\infty}[(1-w^*)(1+r) + w^*Z]^{\gamma}\,dP(Z)$$

$$c_i = \frac{a_1 c_{i-1}}{1+a_1 c_{i-1}} = \frac{a^i_1}{1+a_1+a^2_1+\ldots+a^i_1} < c_{i-1}$$

$$= \frac{a^i_1(a_1-1)}{a^{i+1}_1-1}, \qquad a_1 \neq 1$$

$$= \frac{1}{1+i}, \qquad a_1 = 1\,.$$

In the limiting case, as $\gamma \to 0$ and we have Bernoulli's logarithmic function, $a_1 = (1+\rho)$, independent of r^*, and all saving propensities depend on subjective time preference ρ only, being independent of technological investment opportunities (except to the degree that W_t will itself definitely depend on those opportunities).

We can interpret $1+r^*$ as kind of a "risk-corrected" mean yield; and behavior of a long-lived man depends critically on whether $(1+r^*)^{\gamma} \gtrless (1+\rho)$, corresponding to $a_1 \lessgtr 1$.

(i) For $(1+r^*)^{\gamma} = (1+\rho)$, one plans always to consume at a uniform rate, dividing current W_{T-i} evenly by remaining life, $1/(1+i)$. If young enough, one saves on the average; in the familiar "hump saving" fashion, one dissaves later as the end comes sufficiently close into sight.

(ii) For $(1+r^*)^{\gamma} > (1+\rho)$, $a_1 < 1$, and investment opportunities are, so to speak, so tempting compared to psychological time preference that one consumes nothing at the beginning of a long-long life, i.e., rigorously

$$\lim_{i\to\infty} c_i = 0, \qquad a_1 < 1$$

and again hump saving must take place. For $(1+r^*)^{\gamma} > (1+\rho)$, the *perpetual* lifetime problem, with $T = \infty$, is divergent and ill-defined, i.e., $J_i(W) \to \infty$ as $i \to \infty$. For $\gamma \leqq 0$ and $\rho > 0$, this case cannot arise.

(iii) For $(1+r^*)^{\gamma} < (1+\rho)$, $a_1 > 1$, consumption at very early ages drops only to a limiting positive fraction (rather than zero), namely

$$\lim_{i\to\infty} c_i = 1 - 1/a_1 < 1, \; a_1 > 1.$$

Now whether there will be, on the average, initial hump saving depends upon the size of $r^* - c_{\infty}$, or whether

$$r^* - 1 - \frac{(1+r^*)^{\gamma/1-\gamma}}{(1+\rho)^{1/1-\gamma}} > 0.$$

This ends the *Theorem*. Although many of the results depend upon the no-bequest assumption, $W_{T+1} = 0$, as Merton's companion paper shows (p. 247, this *Review*) we can easily generalize to the cases where a bequest function $B_T(W_{T+1})$ is added to $\Sigma^T_0\,(1+\rho)^{-t}U(C_t)$. If B_T is itself of isoelastic form,

$$B_T \equiv b_T(W_{T+1})^{\gamma}/\gamma,$$

the algebra is little changed. Also, the same comparative statics put forward in Merton's continuous-time case will be applicable here, e.g., the Bernoulli $\gamma = 0$ case is a watershed between cases where thrift is enhanced by riskiness rather than reduced; etc.

Since proof of the theorem is straightforward, I skip all details except to indicate how the recursion relations for c_i and b_i are derived, namely from the identities

$$\begin{aligned} b_{i+1}W^{\gamma}/\gamma &= J_{i+1}(W) \\ &= \operatorname*{Max}_{C} \{C^{\gamma}/\gamma \\ &\quad + b_i(1+r^*)^{\gamma}(1+\rho)^{-1}(W-C)^{\gamma}/\gamma\} \\ &= \{c^{\gamma}{}_{i+1} + b_i(1+r^*)^{\gamma} \\ &\quad (1+\rho)^{-1}(1-c_{i+1})^{\gamma}\}\, W^{\gamma}/\gamma \end{aligned}$$

and the optimality condition

$$\begin{aligned} 0 &= C^{\gamma-1} - b_i(1+r^*)^{\gamma}(1+\rho)^{-1}(W-C)^{\gamma-1} \\ &= (c_{i+1}W)^{\gamma-1} - b_i(1+r^*)^{\gamma}(1+\rho)^{-1} \\ &\quad (1-c_{i+1})^{\gamma-1}W^{\gamma-1}, \end{aligned}$$

which defines c_{i+1} in terms of b_i.

What if we relax the assumption of isoelastic marginal utility functions? Then w_{T-j} becomes a function of W_{T-j-1} (and, of course, of r, ρ, and a functional of the probability distribution P). Now the Phelps-Ramsey optimal stochastic saving decisions do interact with the optimal portfolio decisions, and these have to be arrived at by simultaneous solution of the nondecomposable equations (16′) and (16″).

What if we have more than one alternative asset to safe cash? Then merely interpret Z_t as a (column) vector of returns $(Z^2{}_t, Z^3{}_t, \ldots)$ on the respective risky assets; also interpret w_t as a (row) vector $(w^2{}_t, w^3{}_t, \ldots)$, interpret $P(Z)$ as vector notation for

$$\begin{aligned} &\text{Prob}\,\{Z^2{}_t \leqq Z^2, Z^3{}_t \leqq Z^3, \ldots\} \\ &= P(Z^2, Z^3, \ldots) = P(Z), \end{aligned}$$

interpret all integrals of the form $\int G(Z)dP(Z)$ as multiple integrals $\int G(Z^2,Z^3,\ldots)dP(Z^2,Z^3,\ldots)$. Then (16″) becomes a vector-set of equations, one for each component of the vector Z_t, and these can be solved simultaneously for the unknown w_t vector.

If there are many consumption items, we can handle the general problem by giving a similar vector interpretation to C_t.

Thus, the most general portfolio lifetime problem is handled by our equations or obvious extensions thereof.

Conclusion

We have now come full circle. Our model denies the validity of the concept of businessman's risk; for isoelastic marginal utilities, in your prime of life you have the same relative risk-tolerance as toward the end of life! The "chance to recoup" and tendency for the law of large numbers to operate in the case of repeated investments is not relevant. (Note: if the elasticity of marginal utility, $-U'(W)/WU''(W)$, rises empirically with wealth, and if the capital market is imperfect as far as lending and borrowing against future earnings is concerned, then it seems to me to be likely that a doctor of age 35–50 might rationally have his highest consumption then, and certainly show greatest risk tolerance then — in other words be open to a "businessman's risk." But not in the frictionless isoelastic model!)

As usual, one expects w^* and risk tolerance to be higher with algebraically large γ. One expects C_t to be higher late in life when r and r^* is high relative to ρ. As in a one-period model, one expects any increase in "riskiness" of Z_t, for the same mean, to decrease w^*. One expects a similar increase in riskiness to lower or raise consumption depending upon whether marginal utility is greater or less than unity in its elasticity.[8]

Our analysis enables us to dispel a fallacy that has been borrowed into portfolio theory from information theory of the Shannon type. Associated with independent discoveries by J. B. Williams [16], John Kelly [2], and H. A. Latané [3] is the notion that if one is investing for many periods, the proper behavior is to maximize the *geometric* mean of return rather than the arithmetic mean. I believe this to be incorrect (except in the Bernoulli logarithmic case where it happens [9] to be correct for reasons

[8] See Merton's cited companion paper in this issue, for explicit discussion of the comparative statical shifts of (16)'s $C^*{}_t$ and $w^*{}_t$ functions as the parameters (ρ, γ, r, r^*, and $P(Z)$ or $P(Z_1,\ldots)$ or $B(W_T)$ functions change. The same results hold in the discrete-and-continuous-time models.

[9] See Latané [3, p. 151] for explicit recognition of this point. I find somewhat mystifying his footnote there which says, "As pointed out to me by Professor L. J. Savage (in correspondence), not only is the maximization of G [the geometric mean] the rule for maximum expected utility in connection with Bernoulli's function but (in so far as certain approximations are permissible) this same rule is approximately valid for all utility functions." [Latané, p. 151, n.13.] The geometric mean criterion is definitely too conservative to maximize an isoelastic utility function corresponding to positive γ in my equation (20), and it is definitely too daring to maximize expected utility when $\gamma < 0$. Professor Savage has informed me recently that his 1969 position differs from the view attributed to him in 1959.

quite distinct from the Williams-Kelly-Latané reasoning).

These writers must have in mind reasoning that goes something like the following: If one maximizes for a distant goal, investing and reinvesting (all one's proceeds) many times on the way, then the probability becomes great that with a portfolio that maximizes the geometric mean at each stage you will end up with a larger terminal wealth than with any other decision strategy.

This is indeed a valid consequence of the central limit theorem as applied to the additive logarithms of portfolio outcomes. (I.e., maximizing the geometric mean is the same thing as maximizing the arithmetic mean of the logarithm of outcome at each stage; if at each stage, we get a mean log of $m^{**} > m^{*}$, then after a large number of stages we will have $m^{**}T >> m^{*}T$, and the properly normalized probabilities will cluster around a higher value.)

There is nothing wrong with the logical deduction from premise to theorem. But the implicit premise is faulty to begin with, as I have shown elsewhere in another connection [Samuelson, 10, p. 3]. It is a mistake to think that, just because a w^{**} decision ends up with almost-certain probability to be better than a w^{*} decision, this implies that w^{**} must yield a better expected value of utility. Our analysis for marginal utility with elasticity differing from that of Bernoulli provides an effective counter example, if indeed a counter example is needed to refute a gratuitous assertion. Moreover, as I showed elsewhere, the ordering principle of selecting between two actions in terms of which has the greater probability of producing a higher result does not even possess the property of being transitive.[10] By that principle, we could have w^{***} better than w^{**}, and w^{**} better than w^{*}, and also have w^{*} better than w^{***}.

[10] See Samuelson [11].

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