A Mean-Variance Benchmark for Household Portfolios over the Life Cycle

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Keywords: Life-cycle portfolio decisions, human capital, housing, stock market participation, growth/value tilts

JEL subject codes: G11, D14, D91

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1 Introduction

The mean-variance portfolio theory of Markowitz (1952, 1959) is a milestone of modern finance and remains pivotal in both investment teaching and practical investment decision making. If investment opportunities are constant over time the dynamic extension of Merton (1969, 1971) leads, quite remarkably, to the same optimal portfolio as Markowitz' static model. Markowitz' one-period approach is, however, generally considered unsuitable for households' portfolio decisions over the life cycle. Such life-cycle problems are studied in Merton-type dynamic models, but realistic model specifications have to be solved by rather complex numerical solution techniques, which blurs economic insights.

This paper includes human capital in the mean-variance setting as an innate, illiquid asset. Life-cycle variations in household portfolios are generated by varying the magnitude of human capital relative to financial wealth. We derive the optimal unconstrained portfolio in a general setting with human capital. In Markowitz' traditional standard deviation and mean diagram we consider efficient frontiers generated by the set of optimal portfolios chosen by agents having either (i) the same degree of risk aversion, but different human-to-financial wealth ratios or (ii) the same human-to-financial wealth ratio, but different degrees of risk aversion. We show that both types of efficient frontiers form (part of) a hyperbola, which in some special cases degenerates to a wedge, i.e., a pair of straight lines. The mean-variance approach illuminates the economic forces at play and can easily accommodate relevant portfolio constraints.

We consider three applications of our simple approach. First, we revisit the classical framework in which the household can invest only in a riskfree asset (a bond) and the stock market index. Without any formal modeling, Jagannathan and Kocherlakota (1996) illustrate that the bond-stock allocation depends on the magnitude of the human capital relative to financial wealth and on the extent to which the human capital resembles the bond and the stock. Bodie, Merton, and Samuelson (1992) confirm these insights in a stylized continuous-time model. Carefully modeling life-cycle income patterns, Cocco, Gomes, and Maenhout (2005) solve numerically for the bond-stock allocation through life assuming the household is prohibited from borrowing. In their baseline parametrization, human capital is much more bond-like than stock-like, so that young households—having large human capital relative to financial wealth—should hold 100% of their financial wealth in stocks and only later in life gradually include bonds in their portfolio.[1] We show that the same conclusions follow from our much more transparent mean-variance approach.

Secondly, the investment menu is expanded by residential real estate, the largest tangi-

[1]Other multi-period models involving human capital have been studied by Viceira (2001), Benzoni, Collin-Dufresne, and Goldstein (2007), Munk and Sørensen (2010), and Lynch and Tan (2011), among others.

ble asset for many households (Campbell 2006; Guiso and Sodini 2013; Badarinza, Campbell, and Ramadorai 2016). Housing is typically excluded from theoretical portfolio studies because of the added complexity, but it can easily be incorporated in the mean-variance setting. Given historical prices and rents, real estate appears to be a fairly attractive investment in itself. The attractiveness is amplified by the ability to take out a loan using the house as collateral, in contrast to stock investments which do not facilitate borrowing to the same extent, if at all. This is particularly important for relatively young and risk-tolerant households who want to lever up their investments in light of their sizeable bond-like human capital. In a numerically solved, stylized multi-period model, Cocco (2005) finds that housing crowds out stock holdings which, together with a sizable stock market entry cost, can explain stock market non-participation.[2] Our transparent life-cycle mean-variance model leads to the same conclusions (without stock market entry costs) for agents who are relatively young or risk tolerant. We find that the optimal portfolio weight of the stock index can be increasing or hump-shaped over life. Furthermore, the optimal stock weight can be non-monotonic in risk aversion because agents compare stocks to a (more risky) levered house investment so that, when increasing the risk aversion, agents shift from a levered house investment to stocks and eventually to the riskfree asset.

Thirdly, we use the mean-variance setting to investigate how households should invest in value and growth stocks over the life cycle. Despite the immense focus on value investing among practitioners, the optimal value/growth portfolio tilts have, to the best of our knowledge, not yet been studied in a formal life-cycle model, but only in a few papers ignoring both human capital and housing. Jurek and Viceira (2011) consider a discrete-time setting allowing for return-predicting variables. They find that short [long] horizon investors tilt their portfolios towards value [growth] stocks, and they attribute this horizon dependence to intertemporal hedging motives and the observation that growth stocks hedge bad times better than value stocks. In a related continuous-time model Larsen and Munk (2012) report that the hedging demands are small so that the optimal value/growth/market allocation is almost constant across horizons. Both papers ignore human capital, which can generate larger variations in portfolios over time. In a rich Swedish data set Betermier, Calvet, and Sodini (2015) find that value investors tend to be older than the average participant and have low human capital, low income risk, low leverage, and high financial wealth. We show that our simple life-cycle mean-variance approach provides theoretical support for these findings, but optimal portfolios are very sensitive to the assumed values of correlations.

[2]Other multi-period models involving housing have been studied by Yao and Zhang (2005), Kraft and Munk (2011), Fischer and Stamos (2013), Corradin, Fillat, and Vergara-Alert (2014), and Yogo (2016), among others.

The simple mean-variance approach has its limitations, of course. Our approach suggests that households determine their investment strategy over life by solving at regular time intervals a simple one-period optimization problem. The problem solved at any given date refers to future periods only through the human capital. Since the dynamic programming principle is not invoked, the derived investment strategy is generally not maximizing the life-time expected utility. Dynamic portfolio problems with a mean-variance criterion have time inconsistency issues that were resolved by Basak and Chabakauri (2010) and Björk and Murgoci (2014), but still lead to rather complicated optimal investment strategies. When the relevant return moments are time invariant and portfolios are unconstrained, our simply strategy coincides with the dynamically optimal strategy for a power utility investor if human capital is either riskfree or perfectly spanned by risky assets. In the more realistic case of a risky and imperfectly spanned human capital, our simple strategy leads to a very small utility loss (measured by certainty equivalent of wealth) compared to the unknown optimal strategy.

We can adapt time-varying moments as, for example, implied by the stock return predictability literature simply by applying different values at different points in time, but our approach cannot capture the intertemporal hedging demand that such variations in investment opportunities generally generate (Merton 1971, 1973). However, many studies conclude that the intertemporal hedging demand is typically very small compared to the speculative mean-variance demand (see, e.g., Aït-Sahalia and Brandt 2001, Ang and Bekaert 2002, Chacko and Viceira 2005, Gomes 2007, Larsen and Munk 2012), and this is especially so if parameter uncertainty is taken into account (see, e.g., Barberis 2000, Pastor and Stambaugh 2012).

Only few papers include human capital or housing in a mean-variance setting. Mayers (1972) derives an equation for the optimal financial portfolio of a mean-variance investor with a nonmarketable asset such as human capital. While very similar to the equation we derive below, his equation does not directly show the importance of the relative size of human capital to financial wealth. Based on a log-linearized approximation of the budget constraint Weil (1994) obtains a related expression for the optimal investment in a single risky asset for a power utility investor with human capital. Both Mayers and Weil focus on the asset pricing consequences of human capital. Neither of them consider the implications for the stock-house-bond household portfolios and the life-cycle variations therein, which is the aim of this paper.

Flavin and Yamashita (2002) and Pelizzon and Weber (2009) include housing in a mean-variance framework, but assume that the housing investment position is exogenously given—as the human capital in the current paper—and ignore human capital. In reality households change their housing investment and consumption in response to significant changes in labor income or financial wealth, and the inclusion of human capital is crucial to understand life-cycle variations in portfolios and consumption. Finally, note that while the title of this paper resembles that of Cochrane (2014), the focus is very different. Cochrane shows how a mean-variance approach to payoff streams leads to an interesting characterization of the optimal payoff stream. However, his approach is generally not explaining which investment strategy that generates the optimal payoff stream, and he is not explicitly addressing life-cycle variations in household portfolios.

The paper is organized as follows. Section 2 sets up the general mean-variance framework with human capital, presents a general explicit formula for the optimal unconstrained portfolio, and derives properties of the efficient frontier. Section 3 explains how to value human capital and how large a share of total wealth that human capital represents at different ages, which is a key quantity for portfolio decisions. The subsequent sections consider special cases. The basic life-cycle stock-bond allocation problem is discussed in Section 4, after which Section 5 adds housing to the model. Within the model with human capital and housing, Section 6 investigates how households should tilt their stock portfolios towards value or growth stocks. Finally, Section 7 summarizes our findings.

2 A mean-variance model with human capital

This section explains how human capital can be included in the mean-variance framework and thus how this one-period setting can provide a life-cycle perspective on portfolio decisions. Let F denote the financial wealth and L the human capital ("L" for labor income) of the agent so that total wealth is the sum $W = F + L$. The current date is labeled as time 0. The agent makes a buy-and-hold investment decision for a period of a given length. The end of the period is labeled as time 1. We assume that the agent has the traditional mean-variance objective

$$\max \left\{ \mathrm{E}\left[\frac{W_1}{W_0}\right] - \frac{\gamma}{2}\,\mathrm{Var}\left[\frac{W_1}{W_0}\right] \right\}, \tag{1}$$

where $\gamma > 0$ measures the agent's relative risk aversion.The agent cares about expectation and variance of the return on total wealth, not just on financial wealth.

The agent decides on the portfolio of traded assets to hold over the period. Suppose that the agent can invest in a riskfree asset with rate of return r_f over the period and in a number of risky assets with rates of return given by the vector $\boldsymbol{r}$. The expected rates of return are represented by $\boldsymbol{\mu}$ and the variance-covariance matrix by $\underline{\underline{\Sigma}}$. Let $\boldsymbol{\pi}$ denote the vector of fractions of financial wealth invested in the risky assets. The financial wealth not invested in the risky assets, $W_0(1 - \boldsymbol{\pi} \cdot \mathbf{1})$, is invested in the riskfree asset. We let $\mathbf{1}$

denote a vector of ones and write vector products using a center dot as in $\boldsymbol{\pi} \cdot \boldsymbol{r}$.

Assumption 1 *The return moments are such that $\boldsymbol{\mu} \neq r_f \mathbf{1}$ and $\underline{\underline{\Sigma}}$ is a positive definite matrix.*

The end-of-period total wealth is

$$W_1 = F_0 \left(1 + r_f + \boldsymbol{\pi} \cdot (\boldsymbol{r} - r_f \mathbf{1})\right) + L_0(1 + r_L),$$

where r_L is the rate of return on human capital with expectation μ_L and standard deviation σ_L. Consequently,

$$\frac{W_1}{W_0} = \frac{F_0}{F_0 + L_0} \left(1 + r_f + \boldsymbol{\pi} \cdot (\boldsymbol{r} - r_f \mathbf{1})\right) + \frac{L_0}{F_0 + L_0}(1 + r_L), \tag{2}$$

so that

$$\begin{aligned} \mathrm{E}\left[\frac{W_1}{W_0}\right] &= \frac{F_0}{F_0 + L_0} \left(1 + r_f + \boldsymbol{\pi} \cdot (\boldsymbol{\mu} - r_f \mathbf{1})\right) + \frac{L_0}{F_0 + L_0}(1 + \mu_L), \\ \mathrm{Var}\left[\frac{W_1}{W_0}\right] &= \left(\frac{F_0}{F_0 + L_0}\right)^2 \boldsymbol{\pi} \cdot \underline{\underline{\Sigma}} \boldsymbol{\pi} + \left(\frac{L_0}{F_0 + L_0}\right)^2 \sigma_L^2 + 2 \frac{F_0}{F_0 + L_0} \frac{L_0}{F_0 + L_0} \boldsymbol{\pi} \cdot \mathrm{Cov}[\boldsymbol{r}, r_L], \end{aligned}$$

where $\mathrm{Cov}[\boldsymbol{r}, r_L]$ is the vector of covariances between the returns on the individual risky assets and the return on human capital. The objective (1) is thus equivalent to

$$\max_{\boldsymbol{\pi}} \left\{ \boldsymbol{\pi} \cdot (\boldsymbol{\mu} - r_f \mathbf{1}) - \frac{\gamma}{2} \frac{1}{1 + \ell} \left[\boldsymbol{\pi} \cdot \underline{\underline{\Sigma}} \boldsymbol{\pi} + 2\ell \boldsymbol{\pi} \cdot \mathrm{Cov}[\boldsymbol{r}, r_L]\right] \right\}, \tag{3}$$

where we have introduced the human-to-financial wealth ratio

$$\ell = \frac{L_0}{F_0}.$$

The solution of the unconstrained optimization problem (3) is straightforward and stated in the following theorem, which also summarizes some notable less straightforward properties of the solution. We define the auxiliary constants

$$\begin{aligned} A &= (\boldsymbol{\mu} - r_f \mathbf{1}) \cdot \underline{\underline{\Sigma}}^{-1} \left(\boldsymbol{\mu} - r_f \mathbf{1}\right), \quad B = (\boldsymbol{\mu} - r_f \mathbf{1}) \cdot \underline{\underline{\Sigma}}^{-1} \mathrm{Cov}[\boldsymbol{r}, r_L], \\ C &= \mathrm{Cov}[\boldsymbol{r}, r_L] \cdot \underline{\underline{\Sigma}}^{-1} \mathrm{Cov}[\boldsymbol{r}, r_L], \quad D = AC - B^2. \end{aligned}$$

By assumption $\underline{\underline{\Sigma}}$ is a positive definite matrix, which implies that $\underline{\underline{\Sigma}}^{-1}$ exists and is also positive definite. Since, furthermore, $\boldsymbol{\mu} \neq r_f \mathbf{1}$ we know that $A > 0$. We have $C \geq 0$ with

$C > 0$ if $\mathrm{Cov}[\boldsymbol{r}, r_L] \neq \mathbf{0}$. And since

$$AD = \left(B(\boldsymbol{\mu} - r_f \mathbf{1}) - A\,\mathrm{Cov}[\boldsymbol{r}, r_L]\right) \cdot \underline{\underline{\Sigma}}^{-1} \left(B(\boldsymbol{\mu} - r_f \mathbf{1}) - A\,\mathrm{Cov}[\boldsymbol{r}, r_L]\right) \geq 0,$$

we have $D \geq 0$, and provided that $B(\boldsymbol{\mu} - r_f \mathbf{1}) \neq A\,\mathrm{Cov}[\boldsymbol{r}, r_L]$ we even have $D > 0$. This condition is violated, and thus $D = 0$, if $\boldsymbol{\mu} - r_f \mathbf{1} = k\,\mathrm{Cov}[\boldsymbol{r}, r_L]$ for some scalar k, which is the case when (i) the agent trades in only one risky asset, or (ii) all risky assets have the same excess expected return, same standard deviation, and same covariance with human capital and all pairs of risky assets have the same correlation.

Theorem 1 *(a) The optimal unconstrained portfolio in the presence of human capital is*

$$\boldsymbol{\pi}^* = \frac{1}{\gamma}(1+\ell)\,\underline{\underline{\Sigma}}^{-1}\left(\boldsymbol{\mu} - r_f \mathbf{1}\right) - \ell\,\underline{\underline{\Sigma}}^{-1}\,\mathrm{Cov}[\boldsymbol{r}, r_L]. \tag{4}$$

(b) The expectation and variance of the rate of return on the optimal portfolio are

$$\mathrm{E}[r] = r_f + \frac{1}{\gamma}(1+\ell)\,A - \ell B, \tag{5}$$

$$\mathrm{Var}[r] = \frac{1}{\gamma^2}(1+\ell)^2 A + \ell^2 C - \frac{2}{\gamma}(1+\ell)\,\ell B. \tag{6}$$

If $\gamma > A/B$ (resp., $\gamma < A/B$), then $E[r]$ is decreasing (increasing) in ℓ and the largest (smallest) expected return for a fixed γ over all ℓ-values is $r_f + (A/\gamma)$.

(c) The set of portfolios chosen by agents with the same $\ell > 0$, but different levels of risk aversion $\gamma > 0$, satisfy

$$\mathrm{Var}[r] = \frac{(\mathrm{E}[r] - r_f)^2}{A} + \ell^2 \frac{D}{A}. \tag{7}$$

In the (standard deviation, mean)-diagram, these portfolios form, if $D > 0$ and $\ell > 0$, a hyperbola having $(\ell\sqrt{C}, r_f - \ell B)$ as an end point.

(d) The set of portfolios chosen by agents with the same level of risk aversion $\gamma > 0$, but different values of the human-financial wealth ratio ℓ, satisfy

$$\mathrm{Var}[r] = \frac{1}{A}\left(1 + \frac{\gamma^2 D}{(A - \gamma B)^2}\right)(\mathrm{E}[r] - r_f)^2 - \frac{2\gamma D}{(A - \gamma B)^2}\left(\mathrm{E}[r] - r_f\right) + \frac{AD}{(A - \gamma B)^2}. \tag{8}$$

In the (standard deviation, mean)-diagram, these portfolios form, if $D > 0$, a hyperbola having $(\frac{\sqrt{A}}{\gamma}, r_f + \frac{A}{\gamma})$ as an end point.

Appendix A provides a proof. Concerning (c) and (d), note that if $\ell = 0$ or $D = 0$, the optimal portfolios for a fixed ℓ or a fixed γ trace out a wedge, i.e., a pair of straight lines meeting at $(0, r_f)$ with slopes of $\pm\sqrt{A}$. In Section 4 we consider the case where only one risky asset is traded, which implies $D = 0$. If multiple risky assets are traded, but their expected returns, standard deviations, and correlations with human capital across assets are close, then D will be close to zero, and the hyperbolas are close to straight lines. This turns out to be so in the baseline parametrization of the problem studied in Section 5.

The ratio ℓ of human capital to financial wealth is clearly crucial for the optimal portfolio. This ratio is typically very large for young individuals and small for older individuals, and the variations in this ratio over life is arguably the most important generator of age-dependence in portfolio decisions. By applying the simple setting above for different values of ℓ, we have effectively introduced a life-cycle perspective on portfolio choice.

We can rewrite the optimal portfolio (4) as

$$\boldsymbol{\pi}^* = \frac{1}{\gamma}(1+\ell)\,\mathbf{1}\cdot\underline{\underline{\Sigma}}^{-1}\left(\boldsymbol{\mu} - r_f\mathbf{1}\right)\boldsymbol{\pi}_{\tan} - \ell\,\mathbf{1}\cdot\underline{\underline{\Sigma}}^{-1}\,\mathrm{Cov}[\boldsymbol{r}, r_L]\boldsymbol{\pi}_{\mathrm{hdg}},$$

where

$$\boldsymbol{\pi}_{\tan} = \frac{1}{\mathbf{1}\cdot\underline{\underline{\Sigma}}^{-1}\left(\boldsymbol{\mu} - r_f\mathbf{1}\right)}\underline{\underline{\Sigma}}^{-1}\left(\boldsymbol{\mu} - r_f\mathbf{1}\right), \quad \boldsymbol{\pi}_{\mathrm{hdg}} = \frac{1}{\mathbf{1}\cdot\underline{\underline{\Sigma}}^{-1}\,\mathrm{Cov}[\boldsymbol{r}, r_L]}\underline{\underline{\Sigma}}^{-1}\,\mathrm{Cov}[\boldsymbol{r}, r_L]$$

are the tangency portfolio and income-hedge portfolio, where the latter simply adjusts for the extent to which the human capital replaces investments in the risky assets.

We show in Appendix B that (4) is identical to the dynamically optimal portfolio strategy of a power utility investor in a continuous-time setting with constant investment opportunities provided that the labor income is either riskfree or spanned by traded assets. With unspanned labor income risk, the dynamically optimal strategy is unknown for power utility investors. The strategy suggested by our method is in that case very similar to that derived by the (more complicated) combined analytical-numerical method of Bick, Kraft, and Munk (2013). Their approach approximates both the investment and consumption strategy, and they show that the utility generated by the approximate strategy comes very close to the utility of the unknown optimal strategy as indicated by a tiny certainty equivalent wealth loss. Also note that by applying a log-linearization approach, Viceira (2001) finds an approximate formula for the optimal stock share that has a form similar to (4).

3 The size of human capital over the life cycle

The previous section highlights the importance of human capital for portfolio decisions. But how large is human capital relative to financial wealth at different stages of life? Data on financial wealth and labor income over life can be found, for example, in the Survey of Consumer Finances (SCF) in the United States. However, to calculate human capital at a certain age, we need to establish how to discount future labor income. For that purpose we set up a discrete-time model in which income is paid out at the end of each year.

Suppose that the year t log returns on the n traded risky assets are of the form

$$\ln R_{it} = \mu_i - \frac{1}{2}\|\boldsymbol{\sigma}_i\|^2 + \boldsymbol{\sigma}_i^\top \boldsymbol{\varepsilon}_t,$$

where $\boldsymbol{\varepsilon}_t$ follows the n-dimensional standard normal distribution. Let $\underline{\underline{\sigma}}$ be the $n \times n$ matrix with rows $\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n$ and let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)^\top$. The log riskfree rate is the constant r.

If alive at the end of year t, the agent receives an income of Y_t. We assume that

$$Y_t = Y_{t-1} \exp\left\{ \mu_Y(t) - \frac{1}{2}\sigma_Y(t)^2 + \sigma_Y(t)\boldsymbol{\rho}_Y^\top \boldsymbol{\varepsilon}_t + \sigma_Y(t)\sqrt{1 - \|\boldsymbol{\rho}_Y\|^2}\varepsilon_{Yt} \right\},$$

where ε_{Yt} follows a one-dimensional standard normal distribution independent of $\boldsymbol{\varepsilon}_t$, and where $\mu_Y(t)$ and $\sigma_Y(t)$ are deterministic functions of time. Then

$$\mathrm{E}_{t-1}[Y_t] = Y_{t-1}e^{\mu_Y(t)}, \quad \mathrm{Var}_{t-1}[\ln(Y_t/Y_{t-1})] = \sigma_Y(t)^2, \quad \mathrm{Corr}_{t-1}[\ln(Y_t/Y_{t-1}), \ln \boldsymbol{R}_t] = \boldsymbol{\rho}_Y.$$

Unless $\|\boldsymbol{\rho}_Y\| = 1$ or $\sigma_Y(t) = 0$, the income carries non-spanned risk. Let $\exp(-\nu(t))$ be the probability that the agent survives year t given she was alive at the end of year $t-1$, so that $\nu(t)$ represents the mortality rate, assumed deterministic. We assume that the agent at most lives until the end of year $T = 100$ so that $\nu(T+1) = \infty$ and the final income received, if still alive, is Y_T.

Suppose that the agent associates a price of risk of λ_Y with the unspanned income shock ε_Y and thus evaluates future income using the period-by-period state-price deflator

$$\frac{\zeta_t}{\zeta_{t-1}} = \exp\left\{ -r - \frac{1}{2}\left(\|\boldsymbol{\lambda}\|^2 + \lambda_Y^2 \right) - \boldsymbol{\lambda}^\top \boldsymbol{\varepsilon}_t - \lambda_Y \varepsilon_{Yt} \right\}.$$

Here $\boldsymbol{\lambda}$ captures the market price of risk associated with $\boldsymbol{\varepsilon}$ since

$$\mathrm{E}_{t-1}[\ln R_{it}] - r = -\,\mathrm{Cov}_{t-1}\left[\ln \zeta_t, \ln R_{it}\right] - \frac{1}{2}\,\mathrm{Var}_{t-1}[\ln R_{it}] = \boldsymbol{\sigma}_i^\top \boldsymbol{\lambda} - \frac{1}{2}\|\boldsymbol{\sigma}_i\|^2.$$

We assume that $\boldsymbol{\lambda}$ and λ_Y are constant over time.

We show in Appendix C that the total human capital at the end of year t, excluding the income just received, is then

$$L_t = Y_t M(t). \tag{9}$$

where

$$
\begin{aligned}
M(t) &= \sum_{k=1}^{T-t} \exp\left\{ -\sum_{s=t+1}^{t+k} r_m(s) \right\}, \\
r_m(t) &= r + \nu(t) - \mu_Y(t) + \sigma_Y(t)\left[\boldsymbol{\rho}_Y^\top \boldsymbol{\lambda} + \sqrt{1 - \|\boldsymbol{\rho}_Y\|^2}\lambda_Y \right].
\end{aligned}
$$

We can think of $r_m(t)$ as the risk-, growth-, and mortality-adjusted discount rate for future income. The multiplier M is easily calculated by backwards recursion using $M(T) = 0$ and $M(t) = e^{-r_m(t+1)}\left(M(t+1) + 1\right)$. The expected future human capital is

$$
\mathrm{E}_t[L_{t+k}] = M(t+k)\mathrm{E}_t[Y_{t+k}] = M(t+k)Y_t \exp\left\{ \sum_{s=t+1}^{t+k} \mu_Y(s) \right\}. \tag{10}
$$

Next, we set up an example illustrating how human capital and its share of total wealth vary with age. Labor income is generally found to be hump shaped over working life: rapidly increasing in early years, then flattening out with a peak at an age of 45-55, and then declining somewhat until retirement. Moreover, the life-cycle pattern over the working phase is well approximated by the exponential of a polynomial of order 3 or slightly higher. For example, these facts have been shown for various US data sets by Attanasio and Weber (1995), Cocco et al. (2005), and Guvenen et al. (2015). Consistent with these findings, we model the expected income over working life via

$$
\ln\left(\frac{\mathrm{E}_{t_0}[Y_t]}{Y_{t_0}} \right) = a_1(t - t_0) + a_2(t - t_0)^2 + a_3(t - t_0)^3, \quad t = t_0, \dots, t_R - 1,
$$

which determines $\mu_Y(t)$ for $t \leq T_R - 1$. We assume the initial adult age is $t_0 = 24$ and the retirement age is $t_R = 65$. Hence, the agent starts at her 25th birthday, faces a 40-year working period, retires when turning 65, and then lives on at most until the day after turning 100. The first-year retirement income is expected to be a fraction $\Upsilon = 0.85$ of the labor income in the preceding year, which is in line with the wide-spread final-salary pension schemes and a common assumption in the life-cycle consumption-investment literature (e.g., Cocco et al. 2005; Lynch and Tan 2011). Hence, $\mu_Y(t_R) = \ln(\Upsilon)$. In retirement, labor (pension) income is expected to stay the same, i.e., $\mu_Y(t) = 0$, $t = t_R + 1, \dots, T$. We fix a_1, a_2, a_3 by requiring that (i) expected income peaks at the age $t_{\max} = 52$; (ii) expected income at the peak is $K_{\max} = 2.27$ times the initial income; and (iii) expected income just before retirement is $K_{\text{drop}} = 0.85$ times the peak income. Also these values comply with Guvenen et al. (2015) and other references given above. These choices imply that $a_1 = 5.6929 \times 10^{-2}$, $a_2 = -9.2946 \times 10^{-4}$, and $a_3 = -2.0746 \times 10^{-6}$. Measuring income and wealth in thousands of US dollars, we fix $Y_{t_0} = 15$, which seems

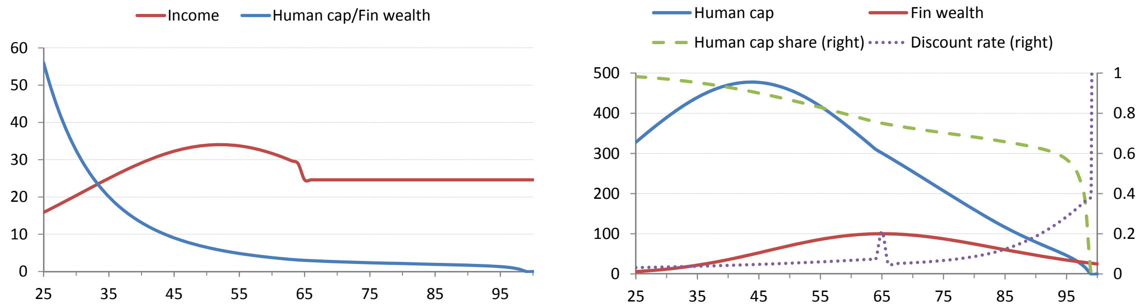


Figure 1: Labor income, human capital, and financial wealth over the life cycle. The left panel shows how expected income in thousand USD (red curve) and the ratio of expected human capital to financial wealth (blue) vary with age measured in years. In the right panel the expected human capital (blue) and the financial wealth (red) as functions of age are measured on the left axis in thousands of USD, whereas the expected human capital's share of total wealth (dashed) and the income discount rate (dotted) as functions of age are measured on the right axis. The graphs are based on the assumptions and baseline parameter values described in the text.

reasonable for a 24-year old individual given the median before-tax family income of 35.1 for the age group "less than 35" in the 2010 SCF, cf. Bricker et al. (2012, Table 1). The red curve in the left panel of Figure 1 shows the resulting expected income path.

In order to calculate the discount rate $r_m(t)$, we fix the riskfree rate at $r = 1\%$, roughly the historical average of short-term real interest rates in the US. We derive the mortality intensity $\nu(t)$ from the life tables for the total US population as of 2009 with an imposed maximum age of 100.[3] We assume that labor income risk declines linearly from $\sigma_Y(25) = 0.3$ to $\sigma_Y(65) = 0.1$ and stays at that level through retirement. The decline over working life is empirically supported, e.g., by Guvenen et al. (2015, Fig. 6). The retirement income risk is motivated by continued business-related remuneration or uncertain health care costs affecting disposable income (De Nardi et al. 2010). Furthermore, we suppose that a single risky asset (the stock index) is traded with a Sharpe ratio of $\lambda = (\mu - r)/\sigma = 0.3$ corresponding, e.g., to $\sigma = 20\%$ and $\mu = 7\%$. The income-stock correlation is $\rho = 0.2$ close to the empirical correlation between per-capita income and the stock market level in the US. Finally, we fix λ_Y by equating the time t present value of a fully unspanned income at time $t + 1$ and the agent's certainty equivalent, which leads to $\lambda_Y = \frac{\gamma}{2}\sigma_Y(t+1)$. With $\gamma = 2$ and $\sigma_Y(t+1) = 0.2$, the average income volatility in the above parametrization, we obtain $\lambda_Y = 0.2$. We vary the value of λ_Y and selected other parameters below.

The resulting discount rate over life is seen as the dotted line in the right panel of Figure 1. The peak around retirement is due to the expected income drop, and the steep increase in the late years comes from the rapidly increasing mortality risk. From the

[3]Published at the Centers for Disease Control and Prevention under the US Department of Health and Human Services, see `http://www.cdc.gov/nchs/data/nvsr/nvsr62/nvsr62_07.pdf`.

discount rates and the expected income, we compute the expected path of human capital using (10). As shown by the blue curve in the right panel, human capital is expected to grow from an initial level of around 315 (kUSD) to a peak at around 478 at age 44, after which it declines steadily mainly because of the shortening of the income-earning period.[4]

Finally, to estimate human capital's share of total wealth, we need the financial wealth over the life cycle. We assume a life-cycle pattern in financial wealth of the form

$$\ln\left(\frac{F(t)}{F(t_0)}\right) = b_1(t-t_0) + b_2(t-t_0)^2 + b_3(t-t_0)^3, \quad t = t_0, t_0+1, \ldots, T.$$

We fix b_1, b_2, b_3 by requiring that (i) financial wealth peaks at retirement; (ii) financial wealth at the peak is $C_{\max} = 20$ times the initial financial wealth; and (iii) financial wealth at age 100 is $C_{\text{end}} = 5$ times the initial financial wealth. This implies that $b_1 = 0.16065$, $b_2 = -2.4865 \times 10^{-3}$, and $b_3 = 8.5675 \times 10^{-6}$. Fixing the initial level at $F(24) = 5$ (kUSD), we obtain the financial wealth path shown by the red curve in the right panel of Figure 1. The shape and levels are in line with various empirical studies. For example, in the 2010 wave of the SCF, median family net worth is 12.4 kUSD for "less than 35" year old's and reaches around 200 kUSD at retirement (or maybe slightly later), cf. Bricker et al. (2012, Table 4). In retirement, individuals reduce financial wealth to finance consumption.

The dashed curve in the right panel shows that the human capital's share of total wealth starts at around 98%, remains above 90% until age 46, above 80% until age 59, above 70% until age 76, and above 60% until age 94. The blue curve in the left panel shows the ratio of human capital to financial wealth, corresponding to ℓ in the previous section. This ratio starts at around 55 and declines smoothly over life. When studying specific models below, we report portfolios for $\ell = 1, 2, 5, 10, 20, 50$, which with the above baseline calculation roughly corresponds to ages of 97, 84, 55, 44, 35, and 26, respectively.

There is a large variation in income and wealth paths across individuals as, e.g., indicated by the huge difference between means and medians of income and net worth at different ages in the SCF data (Bricker et al. 2012). Of course, if we scale either income or financial wealth up or down and fix the other, the human capital's share of total wealth changes. However, across individuals income and wealth often move together since higher-earning individuals tend to build more wealth. Hence, we expect less cross-sectional variation in the human capital's share of total wealth than seen in income or wealth.

Figure 2 depicts the human capital over life as a fraction of total wealth (left panel) or financial wealth (right panel) for the baseline set of parameters explained above as well

[4]In the spirit of Hall (1978) and others, Guiso and Sodini (2013) discount future income at the riskfree rate and thus ignore the riskiness and expected growth of income as well as mortality risk. Hence, they report a larger human capital that declines monotonically over life. However, the overall life-cycle pattern in the human capital share of total wealth they report is not markedly different from our Figure 1.

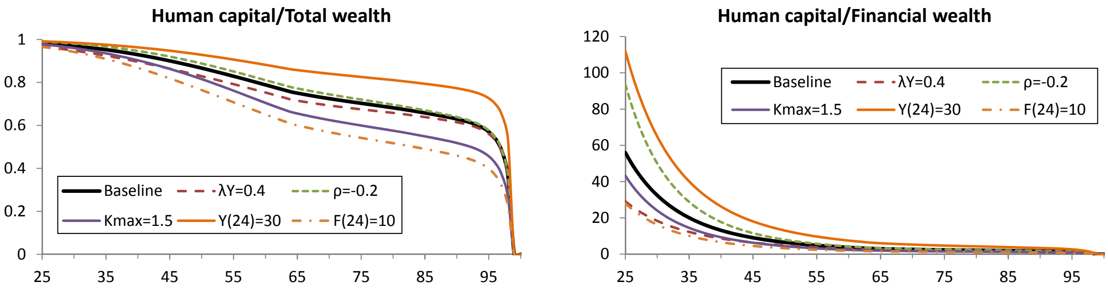


Figure 2: Human capital relative to wealth. The left panel shows how the expected human capital's share of total wealth varies with age, whereas the right panel illustrates the ratio of human capital to financial wealth as a function of age. In both panels the black curve corresponds to the baseline parameter values described in the text, whereas the other curves correspond to cases in which the value of a single parameter is changed relative to the baseline value, namely the idiosyncratic income risk premium $\lambda_Y = 0.4$ (red dashed curve), the income-stock correlation $\rho = -0.2$ (green dotted), the ratio of peak income to initial income $K_{\max} = 1.5$, the initial income $Y_{24} = 30$ thousand USD (orange solid), and the initial financial wealth $F(24) = 10$ thousand USD (dashed-dotted).

as five cases where one of the key parameter values is varied. If initial income is doubled to 30, income at every age and thus the human capital are also doubled, which of course induces a significant increase in the human capital's share of wealth at any age, cf. the solid orange curves. Doubling the initial financial wealth to 10 significantly reduces the human capital's share of wealth, cf. the dashed-dotted curve. If we keep the initial income level, but assumes that the peak income is only 50% larger than initial income ($K_{\max} = 1.5$), the human capital's share of wealth is obviously reduced, cf. the solid purple curves. If we double the idiosyncratic income premium λ_Y to 0.4, human capital is reduced, cf. the dashed red curve. Finally, if we change the income-stock correlation from 0.2 to -0.2, future income is more valuable, so the human capital's share of wealth increases, cf. the dotted green curves. Note, however, that the life-cycle pattern in the human capital's share of wealth is little sensitive to these variations in inputs, and in all cases considered the human capital is the dominant component of total wealth up to age 85.

4 Stock-bond asset allocation

We first consider the simple example with only a single risky asset, which we think of as the stock market index. It follows from Theorem 1 that the unconstrained optimal fraction of financial wealth invested in the stock is then

$$\pi_S = \frac{\mu_S - r_f}{\gamma \sigma_S^2}(1+\ell) - \ell \frac{\rho_{SL}\sigma_L}{\sigma_S} = \frac{\mu_S - r_f}{\gamma \sigma_S^2} + \ell \left(\frac{\mu_S - r_f}{\gamma \sigma_S^2} - \frac{\rho_{SL}\sigma_L}{\sigma_S} \right). \tag{11}$$

The term $\frac{\mu_S - r_f}{\gamma \sigma_S^2}$ in the preceding expression is the solution in absence of human capital, which is well-known from Markowitz' original analysis and is also identical to the solution to Merton's intertemporal portfolio choice problem with constant investment opportunities. Human capital affects the optimal stock weight via the scaling term $1+\ell$ and through the term $\ell \rho_{SL} \sigma_L / \sigma_S$, which adjusts for the extent to which the human capital replaces a stock investment. Since agents combine the riskfree asset and a single risky asset, their choices trace out a wedge in the traditional standard deviation-mean diagram.

Table 1 illustrates the optimal portfolio over a one-year period for frequently used parameter values (see Section 3 for motivation of parameter values). The riskfree rate is $r_f = 1\%$, the stock has an expected rate of return of $\mu_S = 6\%$ and a standard deviation of $\sigma_S = 20\%$. The standard deviation of relative changes in the human capital is $\sigma_L = 10\%$, and the correlation between the stock and the human capital is $\rho_{SL} = 0.1$.

The numbers in Table 1 are to be read in the following way. For an agent with a relative risk aversion of $\gamma = 5$ and a human/financial wealth ratio of $\ell = 10$, the optimal decision is to invest 225% of current financial wealth in the stock, partly financed by a loan of 125% of current financial wealth. This levered stock investment has an expected rate of return of 12.25% and a standard deviation of 45%. The table reveals that, for a fixed γ, the optimal stock weight is decreasing over life supporting the typical "more stocks when young" advice. This feature is due to the fact that the term in the last bracket in (11) is positive with the assumed parameter values. The intuition is that the human capital resembles a riskfree investment much more than a stock investment so, to obtain the optimal overall risk profile, young agents (more precisely, those with large ℓ) short the riskfree asset and invests a lot in stocks. If borrowing constrained so that $\pi_S \leq 1$, 100% in stocks is optimal for all risk-tolerant and also more risk-averse investors with sufficient human capital relative to financial wealth (young and middle-aged agents). Furthermore, we can see that the weight of the stock is decreasing in the agent's degree of risk aversion. Figure 3 illustrates how the constrained optimal stock weight varies with the human-financial wealth ratio for different degrees of risk aversion (left panel) and age (right panel), where the translation to age follows the baseline human capital calculation in Section 3.[5] The patterns in the stock weight are exactly as found in the much more advanced life-cycle models with human capital that have to be solved numerically, cf., e.g., Cocco, Gomes, and Maenhout (2005). Young investors, even quite risk averse, should hold all their financial wealth in stocks. As they grow older, they should eventually—except possibly for the most risk tolerant—start shifting gradually from stocks to bonds.

The impact of human capital on optimal investments is parameter dependent, however.

[5] Obviously, we use the parameter values described here in Section 4. We let the value of λ_Y vary with the relative risk aversion as explained in Section 3.

ℓ	$\gamma = 1$ stock	rf	exp	std	$\gamma = 5$ stock	rf	exp	std	$\gamma = 10$ stock	rf	exp	std
0	125	-25	7.3	25	25	75	2.3	5	13	87	1.6	3
1	245	-145	13.3	49	45	55	3.3	9	20	80	2.0	4
2	365	-265	19.3	73	65	35	4.3	13	28	72	2.4	6
5	725	-625	37.3	145	125	-25	7.3	25	50	50	3.5	10
10	1325	-1225	67.3	265	225	-125	12.3	45	88	12	5.4	18
20	2525	-2425	127.3	505	425	-325	22.3	85	163	-63	9.1	33
50	6125	-6025	307.3	1225	1025	-925	52.3	205	388	-288	20.4	78

Table 1: Optimal portfolios with human capital: baseline parameter values. The table shows the percentages of financial wealth optimally invested in stock and riskfree asset, as well as the expectation and standard deviation of the financial return in percent. The assumed parameter values are $r_f = 1\%$, $\mu_S = 6\%$, $\sigma_S = 20\%$, $\sigma_L = 10\%$, and $\rho_{SL} = 0.1$.

ℓ	$\gamma = 1$ stock	rf	exp	std	$\gamma = 5$ stock	rf	exp	std	$\gamma = 10$ stock	rf	exp	std
0	125	-25	7.3	25	25	75	2.3	5	13	88	1.6	3
1	230	-130	12.5	46	30	70	2.5	6	5	95	1.3	1
2	335	-235	17.8	67	35	65	2.8	7	-3	103	0.9	1
5	650	-550	33.5	130	50	50	3.5	10	-25	125	-0.3	5
10	1175	-1075	59.8	235	75	25	4.8	15	-63	163	-2.1	13
20	2225	-2125	112.3	445	125	-25	7.3	25	-138	238	-5.9	28
50	5375	-5275	269.8	1075	275	-175	14.8	55	-363	463	-17.1	73

Table 2: Optimal portfolios with human capital: high income risk or stock-income correlation. The table shows the percentages of financial wealth optimally invested in stock and riskfree asset, as well as the expectation and standard deviation of the financial return in percent. The assumed parameter values are $r_f = 1\%$, $\mu_S = 6\%$, $\sigma_S = 20\%$, and either (i) $\sigma_L = 10\%$, $\rho_{SL} = 0.4$ or (ii) $\sigma_L = 40\%$, $\rho_{SL} = 0.1$.

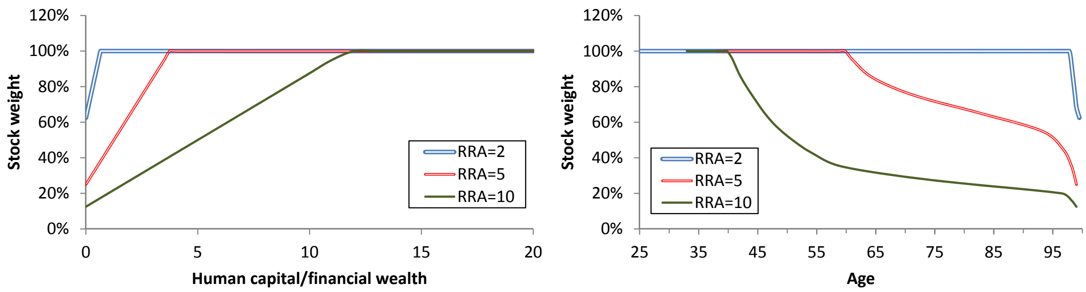


Figure 3: Optimal stock weight with human capital. The figure shows the constrained optimal stock weight as a function of the human capital to financial wealth ratio (left panel) and age (right panel) for three different values of the relative risk aversion coefficient γ. The stock weight is restricted to the interval from 0% to 100%. The baseline parameter values listed in Table 3 are assumed.

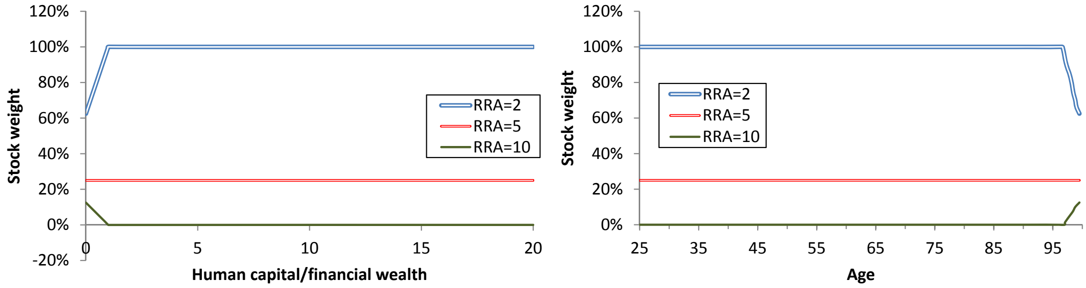


Figure 4: Optimal stock weight with human capital: high stock-income correlation. The figure shows the constrained optimal stock weight as a function of the human capital to financial wealth ratio (left panel) and age (right panel) for three different values of the relative risk aversion coefficient γ. The stock weight is restricted to the interval from 0% to 100%. The baseline parameter values listed in Table 3 are assumed except that the stock-income correlation is $\rho_{SL} = 0.5$.

Table 2 illustrate that results can be very different for investors with high risk aversion and either high income-stock correlation or high income uncertainty (or both). For $\gamma = 10$ and either an income-stock correlation of 0.4 (instead of 0.1) or a human capital standard deviation of 40% (instead of 10%), the term in the last bracket in (11) is negative so that the optimal stock weight is now decreasing in the human-financial wealth ratio ℓ. Consequently, very risk-averse agents should hold less stocks when young if their income is sufficiently risky or sufficiently stock-like. If such agents cannot short stocks, the optimal strategy is to have nothing in stocks early in life and only introduce stocks into the portfolio later in life when human capital has declined adequately. Bagliano, Fugazza, and Nicodano (2014) pointed out such effects in the context of the more complex dynamic life-cycle models. Figure 4 illustrates the life-cycle patterns in the optimal stock weight by showing the dependence of the weight on the human-financial wealth ratio. Here we have set the stock-income correlation to 0.5 in order to illustrate that the stock weight can be completely flat over life, which is the case for $\gamma = 5$.

5 Adding housing investments

Residential real estate is a major asset of many households and should therefore be included in household decision problems. Here we consider real estate as a pure financial investment and include it in the mean-variance setting above alongside the stock and the riskfree asset. Let r_H denote the rate of return on real estate or "housing" over the investment period with an expectation of μ_H and a standard deviation of σ_H. At the beginning of the period, the agent now has to choose the portfolio weights π_S and π_H of the stock and of housing, respectively, with the remaining financial wealth invested in the riskfree asset. This fits

Symbol	Description	Baseline value
r_f	Riskfree rate	0.01
μ_S	Expected stock return	0.06
σ_S	Stock price volatility	0.20
μ_H	Expected housing return	0.04
σ_H	House price volatility	0.10
σ_L	Human capital volatility	0.10
ρ_{SH}	Stock-house correlation	0.20
ρ_{SL}	Stock-human capital correlation	0.10
ρ_{HL}	House-human capital correlation	0.10

Table 3: Parameter values for stock index, real estate, and human capital.

into our general model specification by choosing

$$\boldsymbol{\pi} = \begin{pmatrix} \pi_S \\ \pi_H \end{pmatrix}, \quad \boldsymbol{r} = \begin{pmatrix} r_S \\ r_H \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \mu_S \\ \mu_H \end{pmatrix},$$
$$\underline{\underline{\Sigma}} = \begin{pmatrix} \sigma_S^2 & \rho_{SH}\sigma_S\sigma_H \\ \rho_{SH}\sigma_S\sigma_H & \sigma_H^2 \end{pmatrix}, \quad \mathrm{Cov}[\boldsymbol{r}, r_L] = \begin{pmatrix} \rho_{SL}\sigma_S\sigma_L \\ \rho_{HL}\sigma_H\sigma_L \end{pmatrix},$$

with ρ's denoting the various correlations as indicated by the subscripts.

In our illustrations below we assume the baseline parameter values listed in Table 3. The 4% expected annual return on residential real estate can be justified as an expected real price appreciation of 0.5% (geometric average for US inflation-adjusted home prices over 1955-2015, cf. data on the homepage of Professor Robert Shiller), plus an annual rent of 6.0% of home prices (in line with estimates of Flavin and Yamashita (2002) and Fischer and Stamos (2013)), less 2.5% of taxes, maintenance, and transaction costs. The house price volatility of 10% and the slightly positive pairwise correlations between stock prices, house prices, and labor income are all in line with empirical studies and close to the values used by, e.g., Flavin and Yamashita (2002), Cocco (2005), Yao and Zhang (2005), Davidoff (2006), and Fischer and Stamos (2013).

5.1 Unconstrained solution

In this case we can write the optimal unconstrained portfolio weights in (4) as

$$\pi_S = \frac{1}{\gamma(1-\rho_{SH}^2)\sigma_S}(1+\ell)\left(\frac{\mu_S - r_f}{\sigma_S} - \rho_{SH}\frac{\mu_H - r_f}{\sigma_H}\right) - \ell\frac{\sigma_L}{\sigma_S}\frac{\rho_{SL} - \rho_{SH}\rho_{HL}}{1-\rho_{SH}^2}, \tag{12}$$
$$\pi_H = \frac{1}{\gamma(1-\rho_{SH}^2)\sigma_H}(1+\ell)\left(\frac{\mu_H - r_f}{\sigma_H} - \rho_{SH}\frac{\mu_S - r_f}{\sigma_S}\right) - \ell\frac{\sigma_L}{\sigma_H}\frac{\rho_{HL} - \rho_{SH}\rho_{SL}}{1-\rho_{SH}^2}. \tag{13}$$

ℓ	$\gamma = 1$ stock	house	rf	$\gamma = 5$ stock	house	rf	$\gamma = 10$ stock	house	rf
0	99	260	-259	20	52	28	10	26	64
1	194	513	-606	35	96	-31	16	44	41
2	289	765	-953	51	140	-91	21	61	17
5	573	1521	-1994	98	271	-269	39	115	-53
10	1047	2781	-3728	176	490	-566	67	203	-170
20	1995	5302	-7197	332	927	-1159	124	380	-405
50	4839	12865	-17603	801	2240	-2941	296	911	-1108

Table 4: Optimal unconstrained portfolios. Percentages of financial wealth optimally invested in stock, real estate, and riskfree asset. The baseline parameter values listed in Table 3 are assumed.

Again, the speculative demands are scaled due to the presence of human capital and the portfolio weights are subsequently adjusted for the extent to which the human capital resembles a stock and a real estate investment, respectively.

Table 4 shows optimal portfolios for different combinations of the risk aversion coefficient γ and the human-financial wealth ratio ℓ assuming the baseline parameter values of Table 3. As found in the previous section, agents with low risk aversion or with medium-to-high risk aversion and a significant human capital want to borrow money to boost their investment in the risky assets. Human capital works like an inherent investment primarily in the riskfree asset due to the low correlations of human capital with the risky assets. Hence, the larger the human capital, the more the agent borrows and invests in the risky assets. Real estate dominates the risky portfolio. The tangency portfolio has 28% in stocks and 72% in real estate due to real estate having a larger Sharpe ratio than stocks. Despite stocks and real estate having identical correlations with human capital, the income hedge portfolio has 1/3 in stocks and 2/3 in real estate because of real estate having a standard deviation half the size of stocks. With human capital, the income hedge portfolio is subtracted from the (magnified) investment in the tangency portfolio, so the income hedge causes an increase in the real estate to stock ratio. For example, with a relative risk aversion of 5 the ratio is $52/20 = 2.6$ without human capital and $927/332 \approx 2.8$ with a human-financial wealth ratio of 20. By comparing Table 4 to Table 1, we see that the introduction of real estate reduces the optimal weight in the stock index and in the riskfree asset (for most investors the latter means: increases borrowing).

Figure 5 illustrates the optimal portfolios in the typical standard deviation and mean diagram. Panels A and B show efficient frontiers for selected levels of the risk aversion coefficient γ, when the human-financial wealth ratio ℓ is varied from zero to infinity. Panels C and D illustrate efficient frontiers for selected levels of the human-financial wealth

ℓ	$\gamma = 1$ stock	house	rf	$\gamma = 5$ stock	house	rf	$\gamma = 10$ stock	house	rf
0	99	260	-259	20	52	28	10	26	64
1	168	523	-591	9	106	-16	-10	54	56
2	236	785	-922	-1	160	-59	-31	82	48
5	443	1573	-1916	-32	323	-191	-92	167	25
10	786	2885	-3572	-84	594	-409	-193	307	-14
20	1474	5510	-6884	-189	1135	-847	-396	589	-92
50	3536	13385	-16822	-501	2760	-2159	-1006	1432	-327

Table 5: Optimal unconstrained portfolios with high stock-income correlation. Percentages of financial wealth optimally invested in stock, real estate, and riskfree asset. The baseline parameter values listed in Table 3 are assumed except that the stock-income correlation is $\rho_{SL} = 0.6$.

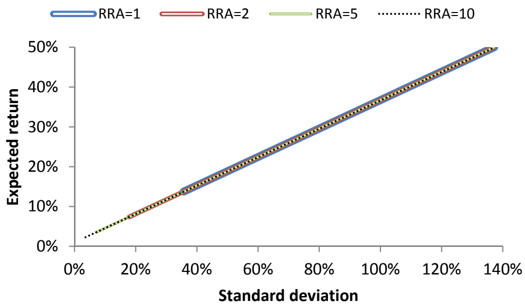
ratio ℓ, when the risk aversion coefficient γ is varied from zero to infinity.

The above patterns are—as seen in the previous section—depending on the assumed parameter values. As an illustration, consider an agent with a human capital having a correlation ρ_{SL} with the stock market index of 0.6 instead of the baseline value of 0.1; other parameter values are unchanged. Table 5 lists the optimal portfolios in this case which we can compare with the baseline case in Table 4. With the high correlation, the human capital is now much more like a stock investment, so the agent reduces her explicit stock investment and invests more in real estate and the riskfree asset (borrows less). With moderate or high risk aversion, the optimal weight of the stock in the financial portfolio is now decreasing in the human-financial wealth ratio and thus increasing over life, whereas agents with low risk aversion still exhibit the opposite pattern. Young and relatively risk averse agents having a high income-stock correlation would ideally short the stock market index. Panels E and F of Figure 5 show the efficient frontiers with the higher stock-income correlation. Compared to the baseline case, the frontiers now vary much more across levels of risk aversion and levels of the human-financial wealth ratio, and we see the hyperbolic shape of (at least some of) the frontiers more clearly.

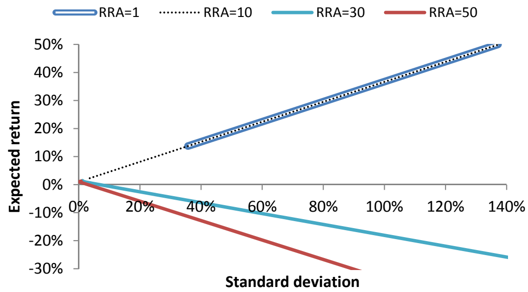
5.2 Solution with collateral constraint

Holding real estate gives easy access to loans through mortgages, while stock investments generally do not, at least not to the same extent. Suppose that you can borrow at most a fraction $1 - \kappa$ of the value of the real estate you own. This corresponds to the constraint

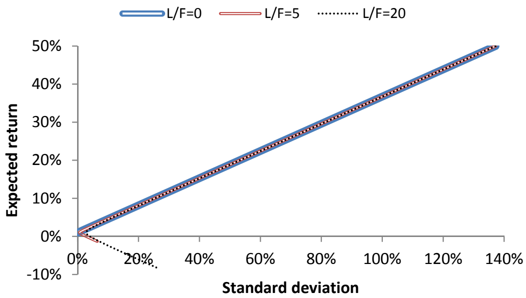
$$\pi_S + \kappa \pi_H \leq 1 \tag{14}$$



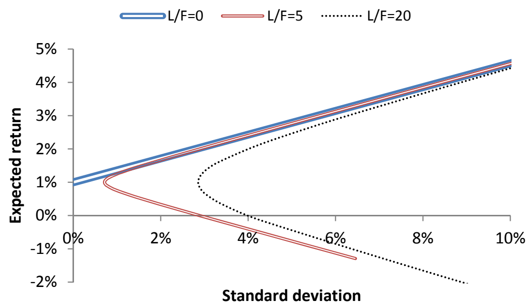
Panel A: Fixed γ, standard levels



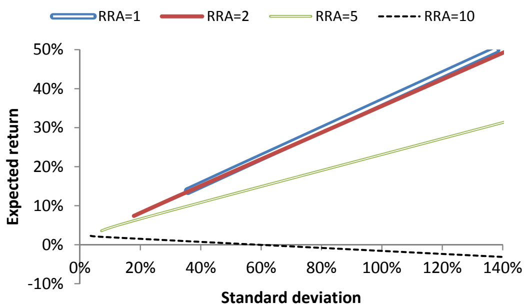
Panel B: Fixed γ, extreme levels



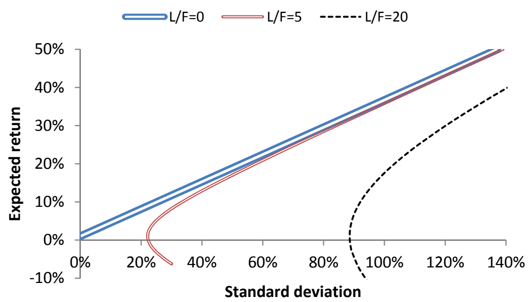
Panel C: Fixed $\ell = L/F$, the big picture



Panel D: Fixed $\ell = L/F$, zoom on low risk



Panel E: Fixed γ, high ρ_{SL}



Panel F: Fixed $\ell = L/F$, high ρ_{SL}

Figure 5: Efficient frontiers with human capital and housing. Each curve shows the combinations of standard deviation and expected return chosen by unconstrained investors either with a certain level of risk aversion but different human-financial wealth ratios (Panels A, B, and E) or with a certain human-financial wealth ratio but different degrees of risk aversion (Panels C, D, and F). The baseline parameter values listed in Table 3 are assumed except that in Panels E and F the stock-income correlation is $\rho_{SL} = 0.6$.

on portfolio weights. We take $\kappa = 0.2$ corresponding to an 80% loan-to-value limit as our benchmark. Panel A of Table 6 shows the optimal portfolio for various combinations of the relative risk aversion and the human-financial wealth ratio. Several things are worth noticing. First, a levered house investment is very attractive for investors who are young/middle-aged or relatively risk tolerant. Secondly, non-participation in the stock market is optimal for young investors. Thirdly, the optimal stock weight is increasing or hump-shaped over life. Fourthly, the optimal stock weight can be non-monotonic in risk aversion which with the assumed parameter values is the case for a human-financial wealth ratio of 1, 2, or 5. This phenomenon occurs because the agent compares stocks to a levered house investment and stocks are less risky than a levered house investment. Hence, when increasing the risk aversion, the agent gradually shifts from a levered house investment to stocks and eventually to the riskfree asset.

To better understand the impact of the access to collateralized borrowing, Panel B of Table 6 lists optimal portfolios in the case of a 60% loan-to-value limit, whereas Panel C assumes no borrowing at all. Note that a portfolio weight written in blue (red) is larger (smaller) than the corresponding weight in the baseline case of Panel A. The young investors' appetite for financial investments with high risk (and high expected return) implies that if mortgages are available, they prefer housing investments with a mortgage to an unlevered stock investment. However, if borrowing is prohibited, the stock is more attractive than the house because of the stock's higher risk and expected return so that the young or risk-tolerant households optimally invest their entire financial wealth in stocks. The extent to which a housing investment gives access to borrowing is thus essential for the optimal portfolio, especially for the young or risk-tolerant households. In line with intuition, a reduction in the loan-to-value limit (i.e., an increase in κ) decreases (or leaves unchanged) the portfolio weight of the house and the borrowed amount, whereas the stock weight can vary non-monotonically.

5.3 Robustness of results

The output of the model is, of course, dependent on parameter values and the specific modeling assumptions. This subsection compares the optimal portfolios for a number of alternative specifications to the optimal portfolios in the baseline case listed in Panel A of Table 6.

Selected parameters. First we illustrate the sensitivity of results to variations in the values of selected parameters that vary across households. Panel A of Table 7 considers an increase in σ_H, the standard deviation of real estate prices, to 0.15 from the baseline value of 0.10. The higher house price risk leads to a higher stock weight, a lower house weight,

ℓ	stock	$\gamma = 1$ house	rf	stock	$\gamma = 5$ house	rf	stock	$\gamma = 10$ house	rf
		Panel A: Baseline case with max 80% LTV, $\kappa = 0.2$							
0	52	240	-192	20	52	28	10	26	64
1	13	434	-347	35	96	-31	16	44	41
2	0	500	-400	51	140	-91	21	61	17
5	0	500	-400	50	250	-200	39	115	-53
10	0	500	-400	16	420	-336	60	200	-160
20	0	500	-400	0	500	-400	32	340	-272
50	0	500	-400	0	500	-400	0	500	-400
		Panel B: max 60% LTV, $\kappa = 0.4$							
0	33	167	-100	20	52	28	10	26	64
1	4	241	-145	35	96	-31	16	44	41
2	0	250	-150	47	133	-80	21	61	17
5	0	250	-150	30	174	-105	39	115	-53
10	0	250	-150	3	242	-145	36	159	-95
20	0	250	-150	0	250	-150	12	220	-132
50	0	250	-150	0	250	-150	0	250	-150
		Panel C: no borrowing, $\kappa = 1$							
0	62	38	0	20	52	28	10	26	64
1	100	0	0	31	69	0	16	44	41
2	100	0	0	38	62	0	21	61	17
5	100	0	0	60	40	0	31	69	0
10	100	0	0	95	5	0	43	57	0
20	100	0	0	100	0	0	67	33	0
50	100	0	0	100	0	0	100	0	0

Table 6: Optimal portfolios with borrowing constraints. Percentages of financial wealth optimally invested in stock, real estate, and riskfree asset. The baseline parameter values listed in Table 3 are assumed. In Panels B and C the numbers in blue are larger than in the baseline case of Panel A, numbers in red are smaller, whereas the remaining numbers are unchanged.

and a higher weight in the riskfree asset (for most: less borrowing), except for relatively young and risk-tolerant agents who still prefer a fully collateralized investment in real estate and no stocks. The qualitative patterns in how the optimal portfolio weights vary with the level of risk aversion and the human-financial wealth ratio remain unchanged. Especially for young or fairly risk-tolerant agents, real estate is still the dominant asset in the portfolio. Although real estate now has a lower Sharpe ratio than stocks, real estate is attractive because of its inherent access to loans.

Panel B of Table 7 shows the effect of increasing the standard deviation of human capital from 0.1 to 0.2. The optimal portfolios in the case without human capital are, of course, unaffected, and the same is true for relatively young or risk-tolerant agents who still opt for maximally levered house investment and no stocks. Except for those cases, the larger background risk leads to less borrowing, especially for the most risk-averse households, and a smaller housing portfolio weight, whereas the effect on the stock weight depends on the combination of the risk aversion and the human-to-financial wealth ratio. In the unconstrained case, we know from (12)–(13) that a larger value of σ_L reduces both portfolio weights. The borrowing constraint generally twists weights in the direction of less stocks and more housing, but a larger income risk reduces the appetite for borrowing and thus reduces the magnitude of this twist, which may lead to a larger stock weight.

Alternative assumptions on loan access. In the baseline case the interest rate of 1% applied both to lending and borrowing. Now assume that the borrowing rate is 2%, whereas the lending rate is 1%. Panel C of Table 7 lists optimal portfolios for this situation. Note that in this case the objective cannot be reduced from (1) to (3), but only to

$$\max_{\boldsymbol{\pi}} \left\{ \boldsymbol{\pi} \cdot \boldsymbol{\mu} + (1 - \boldsymbol{\pi} \cdot \mathbf{1}) \left(r_{\text{len}} 1_{\{\boldsymbol{\pi} \cdot \mathbf{1} \leq 1\}} + r_{\text{bor}} 1_{\{\boldsymbol{\pi} \cdot \mathbf{1} > 1\}} \right) \right.$$
$$\left. - \frac{1}{2} \gamma \frac{1}{1+\ell} \left[\boldsymbol{\pi} \cdot \underline{\underline{\Sigma}} \boldsymbol{\pi} + 2\ell \, \boldsymbol{\pi} \cdot \text{Cov}[\boldsymbol{r}, r_L] \right] \right\},$$

where r_{len} is the lending rate and r_{bor} the borrowing rate, and $1_{\{A\}}$ equals 1 if the claim A is true and zero otherwise. Of course, portfolios not involving borrowing are unchanged (high risk aversion, low human-financial wealth ratio). Some agents who were borrowing in the baseline case are now neither borrowing nor lending (for $\gamma = 5$, $\ell = 1$ and $\gamma = 10$, $\ell = 5$). Other agents are borrowing less, whereas the agents with relatively low risk aversion and high human capital still borrow as much as possible and invest nothing in stocks. Again, the overall qualitative patterns in how the optimal portfolio weights vary with the level of risk aversion and the human-financial wealth ratio remain unchanged.

Next, suppose that households can borrow up to a fraction θ of their human capital in

ℓ	$\gamma = 1$ stock	$\gamma = 1$ house	$\gamma = 1$ rf	$\gamma = 5$ stock	$\gamma = 5$ house	$\gamma = 5$ rf	$\gamma = 10$ stock	$\gamma = 10$ house	$\gamma = 10$ rf
	Panel A: Higher house price risk, $\sigma_H = 0.15$								
0	80	101	-81	22	21	57	11	10	79
1	62	188	-151	40	36	24	18	15	67
2	45	276	-220	57	51	-9	24	20	55
5	0	500	-400	81	94	-76	45	35	20
10	0	500	-400	68	161	-129	79	59	-38
20	0	500	-400	41	295	-236	80	101	-81
50	0	500	-400	0	500	-400	55	226	-181
	Panel B: Higher income risk, $\sigma_L = 0.2$								
0	52	240	-192	20	52	28	10	26	64
1	14	428	-342	31	87	-19	11	35	53
2	0	500	-400	43	123	-66	13	45	42
5	0	500	-400	56	220	-176	18	73	9
10	0	500	-400	28	360	-288	26	120	-45
20	0	500	-400	0	500	-400	41	214	-155
50	0	500	-400	0	500	-400	8	460	-368
	Panel C: Higher borrowing than lending rate, $r_{\text{bor}} = 2\%, r_{\text{len}} = 1\%$								
0	68	160	-128	20	52	28	10	26	64
1	45	274	-219	30	70	0	16	44	41
2	22	388	-310	42	83	-25	21	61	17
5	0	500	-400	69	154	-123	31	69	0
10	0	500	-400	51	244	-195	50	100	-50
20	0	500	-400	15	424	-339	66	172	-138
50	0	500	-400	0	500	-400	30	352	-282
	Panel D: Borrowing against human capital, $\theta = 0.1$								
0	52	240	-192	20	52	28	10	26	64
1	22	438	-360	35	96	-31	16	44	41
2	0	600	-500	51	140	-91	21	61	17
5	0	750	-650	96	270	-266	39	115	-53
10	0	1000	-900	108	460	-468	67	203	-170
20	0	1500	-1400	132	840	-872	124	380	-405
50	0	3000	-2900	204	1980	-2084	296	911	-1108
	Panel E: Buying stocks on margin, $\omega = 0.5$								
0	97	258	-255	20	52	28	10	26	64
1	67	332	-299	35	96	-31	16	44	41
2	38	406	-344	51	140	-91	21	61	17
5	0	500	-400	94	265	-259	39	115	-53
10	0	500	-400	67	333	-300	67	203	-170
20	0	500	-400	12	470	-382	76	311	-286
50	0	500	-400	0	500	-400	3	492	-395

Table 7: Optimal portfolios: robustness. Percentages of financial wealth optimally invested in stock, real estate, and riskfree asset. The baseline parameter values listed in Table 3 are assumed together with a maximum loan-to-value ratio of 80% ($\kappa = 0.2$), except that $\sigma_H = 0.15$ in Panel A and $\sigma_L = 0.2$ in Panel B. Numbers in blue are larger than in the baseline case, numbers in red are smaller, whereas the remaining numbers are unchanged.

addition to the collateralized mortgage. The constraint (14) is then replaced by $\pi_S + \kappa\pi_H \leq 1 + \theta\ell$. Panel D lists optimal portfolios when $\theta = 0.1$. The portfolio is unchanged for the combinations of the risk aversion and the human-financial wealth ratio for which the loan-to-value constraint was not binding. The additional borrowing opportunity is taken by agents with low risk aversion or moderate-to-high human capital. The more risk-averse of these agents increase the weight of both stocks and houses, but the most risk-tolerant agents still prefer the maximal possible position in housing and nothing in stocks.

Finally, suppose that agents can buy stocks on margin and borrow up to a fraction $1 - \omega$ of the value of the stocks owned. The budget constraint (14) is then replaced by $\omega\pi_S + \kappa\pi_H \leq 1$. Panel E shows the optimal portfolios when agents can take out a loan of 50% of the value of their stocks in addition to the housing-related loan. The agents facing a binding portfolio constraint in the baseline case find stocks relatively more attractive when they give access to loans. Still, young and very risk-tolerant agents choose the maximal possible position in housing and nothing in stocks. The older (smaller human capital) among the very risk-tolerant agents as well as younger and more risk-averse agents do in fact increase their stock share, but all portfolios remain dominated by housing investments.

6 Growth and value tilts in household portfolios

The terms value stocks and growth stocks are commonly used in the investment industry and the asset pricing literature. Academics generally define value and growth stocks with reference to their ratio of book value of equity to market value of equity with value stocks having relatively high and growth stocks relatively low book-to-market. Practitioners often take a broader view and use the label value stock for stocks with a low price relative to dividends, earnings, or sales, or simply stocks of well-run, solid, non-glamorous companies. Numerous exchange-traded funds and mutual funds are devoted to either value stocks or growth stocks in specific countries, industries, or with other specific characteristics. Value investing has gained popularity by the success of high-profiled declared value investors (most notably Warren Buffett) and by empirical studies documenting that value stocks offer higher average returns than growth stocks, also after standard market risk adjustments (Rosenberg, Reid, and Lanstein 1985; Fama and French 1992).

The academic literature on the role of value and growth stocks in household portfolios is sparse. Jurek and Viceira (2011) and Larsen and Munk (2012) consider value and growth stocks in theoretical models of multi-period portfolio decisions, but both ignore human capital and housing. This is problematic particularly if value stocks covary with house prices and labor income differently than growth stocks do. Based on the asset holdings of a large number of Swedish households, Betermier, Calvet, and Sodini (2015) find that,

	Mean		Std deviation						
	Estimate	Assumed	Estimate	Assumed	Correlations				
Growth	9.47%	5.47%	17.54%	17.5%	1.000	0.792	0.686	0.131	0.226
Neutral	10.13%	6.13%	15.65%	15.6%	0.792	1.000	0.900	0.319	0.355
Value	11.12%	7.12%	16.69%	16.7%	0.686	0.900	1.000	0.350	0.301
House	0.94%	4.00%	5.56%	10.0%	0.131	0.319	0.350	1.000	0.387
Income	1.82%	N.A.	1.61%	10.0%	0.226	0.355	0.301	0.387	1.000

Table 8: Inputs for growth-value analysis. The table shows the means, standard deviations, and correlations used in the calculations of optimal portfolios with growth and value stocks. The underlying data series are described in the text. For the portfolio calculations the values assumed of the means and some of the standard deviations are different from the estimated values.

relative to growth investors, value investors are generally older, have lower human capital, lower income risk, lower leverage, and higher financial wealth.

In this section we study growth/value investing in our life-cycle mean-variance setting. To calculate optimal portfolios we need means, variances, and correlations. From Professor Kenneth French's homepage we take annual nominal returns on three US stock portfolios: a growth portfolio of the 30% stocks with the lowest book-to-market value, a value portfolio of the 30% with the highest book-to-market value, and a neutral portfolio of the remaining 40% stocks. The portfolios are rebalanced every year. We use value-weighted returns on the portfolios.[6] We deflate returns by the Consumer Price Index published by the Bureau of Labor Statistics. For house prices we use the real home price index published on Professor Robert Shiller's homepage.[7] For income we use the aggregate disposable personal income per capita (in chained 2009 dollars, i.e., inflation adjusted) from the NIPA tables published by the Bureau of Economic Analysis.[8] Table 8 exhibits the estimated means, standard deviations, and correlations. The standard deviation of per capita income underestimates the standard deviation of a household's income, which we set to 10%, the value used in earlier sections. For housing we adjust the mean and standard deviation as explained in Section 5. Average past stock returns are likely to overestimate future expected stock returns because of survivorship biases (Brown, Goetzmann, and Ross 1995) and the decline in discount rates and the implied unexpected capital gains over the sample period (Fama and French 2002). To account for this, we subtract 4 percentage points from the average returns on the three stock portfolios, which leads to expected returns being close to the 6% used in the single-stock settings in earlier sections. We assume a real riskfree rate of 1% as in previous sections.

[6] See details on http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/Data_Library/det_form_btm.html.

[7] See http://www.econ.yale.edu/~shiller/data.htm.

[8] Line 39 in Table 2.1. Data can be downloaded from http://www.bea.gov/iTable/index_nipa.cfm.

With these inputs, the tangency portfolio consists of 11.3% in Growth, -13.1% in Neutral, 48.6% in Value, and 53.2% in housing. The value portfolio has the largest Sharpe ratio, followed by Neutral, housing, and Growth, but due to the correlation structure the neutral portfolio is optimally shorted in absence of human capital. The income-adjustment portfolio (to be shorted) consists of -7.5% in Growth, 67.1% in Neutral, -26.3% in Value, and 66.7% in housing. These values are determined by the correlations of the four assets with income and also by the correlations among the four assets. Neutral and housing have the largest correlations with income, whereas Growth and Value provide relatively better hedges against income risk because of their low correlation with income. From (2) the optimal portfolio of any household is a combination of the speculative portfolio and a short position in the income-adjustment portfolio with the weights being determined by the risk aversion γ and the ratio ℓ of human capital to financial wealth. For any combination the position in Neutral stocks will be negative and the positions in Growth and Value stocks positive. In this perspective, it seems puzzling why any household would invest in neutral stocks. For housing, the sign depends on the exact values of γ and ℓ.

Panel A of Table 9 shows the optimal portfolio for various combinations of γ and ℓ. In the absence of human capital the portfolios are dominated by value stocks and housing (and the riskfree asset for sufficiently high risk aversion). When increasing ℓ, we see the same overall pattern as previous sections: the household takes much more risk on its financial portfolio and borrows substantial sums. In the present setting, growth stocks and especially value stocks and housing have very high positive weights, whereas neutral stocks enter with a significant negative weight.

Now we impose short-selling constraints on the assets and a collateral constraint so that up to a fraction $1 - \kappa$ of the value of the house can be borrowed, in line with (14) but now π_S is replaced by the sum of the weights in the three stock portfolios. The optimal portfolios for this case can be seen in Panel B. As in earlier sections, housing is the dominant asset for young and relatively risk-tolerant households, but for more risk-averse young households a 100% investment in value stocks is optimal. Growth stocks enter the portfolio only in the final years when the human-financial wealth ratio is low.

With multiple risky assets, optimal portfolios are very sensitive to the correlation values, and the values assumed above may not be appropriate for an individual household. First, we reset the correlation between housing returns and income changes to 0.1 as assumed in Section 5. This changes the weights on value and growth stocks as can be seen in Panel C. Among the more risk-averse households, the portfolio of older investors includes growth stocks, but not value stocks. As the household matures, the weight of growth stocks declines and the weight of value stocks increases. This fits well with the empirical findings of Betermier et al. (2015) that value investors tend to be older, have

ℓ	Gro	Neu	$\gamma = 1$ Val	Hou	Rf	Gro	Neu	$\gamma = 5$ Val	Hou	Rf	Gro	Neu	$\gamma = 10$ Val	Hou	Rf
	Panel A: Unconstrained portfolios, baseline parameters														
0	44	-51	188	206	-286	9	-10	38	41	23	4	-5	19	21	61
1	91	-133	387	380	-625	21	-52	87	51	-8	12	-41	50	10	69
2	138	-215	587	554	-965	33	-93	137	61	-38	20	-78	81	-1	77
5	279	-461	1187	1078	-1983	70	-217	287	91	-130	44	-187	174	-32	101
10	515	-871	2186	1950	-3680	131	-425	535	141	-283	83	-369	329	-85	142
20	986	-1691	4185	3695	-7074	253	-839	1033	241	-589	161	-733	639	-190	222
50	2398	-4152	10182	8928	-17257	620	-2082	2527	542	-1507	397	-1824	1571	-506	462
	Panel B: Constrained portfolios, baseline parameters														
0	0	0	60	199	-159	6	0	31	40	23	3	0	16	20	61
1	0	0	26	369	-295	5	0	56	47	-9	0	0	24	7	69
2	0	0	0	500	-400	5	0	81	54	-40	0	0	30	0	70
5	0	0	0	500	-400	0	0	85	77	-61	0	0	41	0	59
10	0	0	0	500	-400	0	0	77	116	-93	0	0	61	0	39
20	0	0	0	500	-400	0	0	61	194	-155	0	0	100	0	0
50	0	0	0	500	-400	0	0	14	429	-343	0	0	100	0	0
	Panel C: Constrained portfolios, house-income correlation 0.1														
0	0	0	60	199	-159	6	0	31	40	23	3	0	16	20	61
1	0	0	20	402	-321	9	0	46	81	-36	4	0	15	41	41
2	0	0	0	500	-400	13	0	61	121	-95	4	0	14	61	21
5	0	0	0	500	-400	0	0	52	240	-192	7	0	11	122	-40
10	0	0	0	500	-400	0	0	11	443	-354	11	0	6	224	-141
20	0	0	0	500	-400	0	0	0	500	-400	15	0	0	425	-340
50	0	0	0	500	-400	0	0	0	500	-400	0	0	0	500	-400
	Panel D: Constrained portfolios, house-income correlation 0.1, higher income risk $\sigma_L = 20\%$														
0	0	0	60	199	-159	6	0	31	40	23	3	0	16	20	61
1	0	0	20	402	-321	7	0	29	81	-18	0	0	0	40	60
2	0	0	0	500	-400	9	0	28	122	-58	0	0	0	50	50
5	0	0	0	500	-400	14	0	22	244	-180	0	0	0	80	20
10	0	0	0	500	-400	11	0	0	446	-357	0	0	0	130	-30
20	0	0	0	500	-400	0	0	0	500	-400	0	0	0	230	-130
50	0	0	0	500	-400	0	0	0	500	-400	0	0	0	500	-400
	Panel E: Constrained portfolios, house-income correlation 0.1, other correlations halved														
0	0	0	59	205	-164	4	0	36	49	11	2	0	18	25	56
1	0	0	24	379	-303	6	0	65	90	-61	2	0	29	41	28
2	0	0	0	500	-400	0	0	75	124	-99	3	0	40	58	-1
5	0	0	0	500	-400	0	0	57	216	-173	4	0	72	108	-85
10	0	0	0	500	-400	0	0	26	370	-296	0	0	65	173	-139
20	0	0	0	500	-400	0	0	0	500	-400	0	0	40	302	-241
50	0	0	0	500	-400	0	0	0	500	-400	0	0	0	500	-400

Table 9: Optimal portfolios with growth and value stocks. Percentages of financial wealth optimally invested in growth stocks, neutral stocks, value stocks, real estate, and the riskfree asset.

lower human capital and higher financial wealth, as well as lower leverage. Furthermore, they find that value investors have lower income risk. Panel D lists optimal portfolios when the income risk is doubled to 20%. The increased income risk does indeed lead to lower portfolio weights of value stocks, whereas the impact on growth stocks is less clear.

The correlations of the stock portfolios with house and income used above are based on a house price index and aggregate income, which may exaggerate the appropriate values for an individual household. Panel E shows the effect of halving the correlations involving stocks. Relative to Panel C, value stocks are again weighted higher and growth stocks lower. Mortgage-financed housing dominates the portfolios of all young households.

7 Conclusion

Human capital is one of the main assets held by households. We have shown how Markowitz' basic mean-variance portfolio choice model can be extended to include human capital. By solving the extended mean-variance problem (with relevant constraints) for different ratios of human capital to financial wealth, the method effectively delivers portfolio decisions over the life cycle of a household. We have argued that the life-cycle investment strategy generated in this way comes very close to the strategy that can be derived using a much more involved, formal dynamic optimization approach.

Two of our three applications address settings that have been solved in the literature by numerical dynamic optimization routines. These applications confirm that our approach generates theoretically correct life-cycle portfolio patterns. The first application considers the classical stock-bond asset allocation with human capital, but no housing. With standard parameter values our results corroborates the findings of Cocco et al. (2005) that 100% in stocks are optimal for young households, but we also show that results are markedly different for certain changes in parameter values. The second application adds housing as an investment object to the problem. Here our approach provides justification and transparent arguments for the findings of Cocco (2005) that housing tends to crowd out stock investments especially for young households. We provide additional results highlighting the importance of the access to borrowing offered by housing investments.

Our final application generalizes the setting further by allowing investments in three stock portfolios, representing growth stocks, value stocks, and neutral stocks. This is, to the best of our knowledge, the first theoretical model of the role of growth and value stocks in households' portfolio decisions. We show that the optimal portfolios to some extent agree with the growth/value tilts found in Swedish household portfolios by Betermier et al. (2015), but results are highly sensitive to the assumed correlation values.

A Proof of Theorem 1

(a) follows by direct optimization of the objective function.

(b) The expected rate of return on the optimal portfolio is

$$\begin{aligned}
\mathrm{E}[r] &= r_f + \boldsymbol{\pi}^* \cdot (\boldsymbol{\mu} - r_f \mathbf{1}) \\
&= r_f + \frac{1}{\gamma}(1+\ell)(\boldsymbol{\mu} - r_f \mathbf{1}) \cdot \underline{\underline{\Sigma}}^{-1} \big(\boldsymbol{\mu} - r_f \mathbf{1}\big) - \ell (\boldsymbol{\mu} - r_f \mathbf{1}) \cdot \underline{\underline{\Sigma}}^{-1} \operatorname{Cov}[\boldsymbol{r}, r_L] \\
&= r_f + \frac{1}{\gamma}(1+\ell) A - \ell B,
\end{aligned}$$

which shows (5). The variance of the optimal portfolio is

$$\begin{aligned}
\operatorname{Var}[r] &= \boldsymbol{\pi}^* \cdot \underline{\underline{\Sigma}} \boldsymbol{\pi}^* \\
&= \frac{1}{\gamma^2}(1+\ell)^2 (\boldsymbol{\mu} - r_f \mathbf{1}) \cdot \underline{\underline{\Sigma}}^{-1} \big(\boldsymbol{\mu} - r_f \mathbf{1}\big) + \ell^2 \operatorname{Cov}[\boldsymbol{r}, r_L] \cdot \underline{\underline{\Sigma}}^{-1} \operatorname{Cov}[\boldsymbol{r}, r_L] \\
&\quad - \frac{2}{\gamma}(1+\ell)\, \ell (\boldsymbol{\mu} - r_f \mathbf{1}) \cdot \underline{\underline{\Sigma}}^{-1} \operatorname{Cov}[\boldsymbol{r}, r_L] \\
&= \frac{1}{\gamma^2}(1+\ell)^2 A + \ell^2 C - \frac{2}{\gamma}(1+\ell)\, \ell B,
\end{aligned}$$

which confirms (6).

(c) It follows from (5) that

$$\frac{1}{\gamma}(1+\ell) = \frac{\mathrm{E}[r] - r_f}{A} + \ell \frac{B}{A}, \tag{15}$$

and by substituting this into (6), we obtain

$$\operatorname{Var}[r] = \left(\frac{\mathrm{E}[r] - r_f}{A} + \ell \frac{B}{A} \right)^2 A + \ell^2 C - 2 \left(\frac{\mathrm{E}[r] - r_f}{A} + \ell \frac{B}{A} \right) \ell B,$$

which can be rewritten as (7). The minimum standard deviation equals $\frac{L_0}{F_0}\sqrt{D/A}$ and is obtained for $\mathrm{E}[r] = r_f$. From (15), we see that this combination is chosen by an agent with a risk aversion coefficient of $\gamma = (1+\ell^{-1})\frac{A}{B}$. For $\gamma \to \infty$, the expected rate of return drops towards $r_f - \ell^{-1} B$ so one branch of the hyperbola is cut off at that level. This is the downward-sloping branch if $B > 0$, and the upward-sloping branch if $B < 0$.

(d) Eq. (15) implies that

$$\ell = \frac{\gamma(\mathrm{E}[r] - r_f) - A}{A - \gamma B}, \tag{16}$$

and by substituting that into (6), we find (8). For $\ell = 0$, the expected return is $r_f + \frac{A}{\gamma}$ and the variance is $\frac{A}{\gamma^2}$, which defines the endpoint of one of the branches of the hyperbola.

B Optimal investments in a continuous-time model

Let $\boldsymbol{S}_t$ denote the n-vector of traded risky asset prices at time t, and assume that

$$d\boldsymbol{S}_t = \operatorname{diag}(\boldsymbol{S}_t)\left[\boldsymbol{\mu}\, dt + \underline{\underline{\sigma}}\, d\boldsymbol{z}_t\right], \tag{17}$$

where $\operatorname{diag}(\boldsymbol{S}_t)$ is the $n \times n$ matrix with $\boldsymbol{S}_t$ along the diagonal and zeros off the diagonal, $\boldsymbol{z} = (\boldsymbol{z}_t)$ is an n-dimensional standard Brownian motion representing shocks to prices, $\boldsymbol{\mu}$ is the n-vector of expected returns, and $\underline{\underline{\sigma}}$ is the $n \times n$ matrix of asset price sensitivities towards the shocks. In addition a riskfree asset with a constant rate of return of r (continuously compounded) is traded. We assume that $\boldsymbol{\mu} \neq r\mathbf{1}$ and that $\underline{\underline{\sigma}}$ is non-singular.

The investor receives a labor income stream given by the income rate Y_t with dynamics

$$dY_t = Y_t\left[\mu_Y(t)\, dt + \sigma_Y(t)\boldsymbol{\rho}_Y^\top\, d\boldsymbol{z}_t + \sigma_Y(t)\sqrt{1 - \|\boldsymbol{\rho}_Y\|^2}\, dz_{Yt}\right], \tag{18}$$

where $z_Y = (z_{Yt})$ is a one-dimensional standard Brownian motion independent of $\boldsymbol{z}$, μ_Y is the expected income growth rate, $\sigma_Y \geq 0$ is the income volatility, and $\boldsymbol{\rho}_Y$ is the n-vector of instantaneous correlations of the income rate with the risky asset prices. The income stream contains unspanned risk if either $\sigma_Y > 0$ or $\|\boldsymbol{\rho}_Y\| \neq 1$ or both. We assume that the agent lives until a known terminal date T. At the known retirement date $T_R < T$, there is a one-time drop in the income rate,

$$Y_{T_R+} = \Upsilon Y_{T_R-},$$

where $\Upsilon > 0$ can be interpreted as the replacement rate in final-salary pension scheme.

We consider an investor maximizing the expected life-time power utility depending on consumption or terminal wealth or both. The indirect utility is thus defined as

$$J(F, Y, t) = \sup_{c,\boldsymbol{\pi}} \mathrm{E}_t\left[\varepsilon_c \int_t^T e^{-\delta(\tau - t)} u(c_\tau)\, d\tau + \varepsilon_F e^{-\delta(T-t)} u(F_T)\right], \tag{19}$$

where F is current financial wealth, $\delta \geq 0$ is the subjective time preference rate, and $\varepsilon_c, \varepsilon_F \geq 0$ are indicators with $\varepsilon_c \varepsilon_F > 0$. We assume power utility $u(x) = \frac{1}{1-\gamma} x^{1-\gamma}$, where $\gamma > 1$ is the constant coefficient of relative risk aversion.

The investor must choose a portfolio strategy $\boldsymbol{\pi} = (\boldsymbol{\pi}_t)$, where $\boldsymbol{\pi}_t$ is the n-vector of fractions of financial wealth invested in the n risky assets at time t. The remaining financial wealth $F_t(1 - \boldsymbol{\pi}_t \cdot \mathbf{1})$ is invested in the riskfree asset. If $\varepsilon_c > 0$, the investor must also choose a consumption strategy $c = (c_t)$, where c_t is the consumption rate at time t.

If there is no unspanned income risk, and the investor is not facing any portfolio constraints, we can find the optimal portfolio in closed form.

Theorem 2 *Suppose the investor is unconstrained and that either $\sigma_Y = 0$ or $\|\boldsymbol{\rho}_Y\| = 1$. Then the indirect utility is*

$$J(F, y, t) = \frac{1}{1-\gamma} G(t)^{\gamma} \left(F + yM(t)\right)^{1-\gamma}, \tag{20}$$

where

$$M(t) = \begin{cases} \int_t^T e^{-\int_t^u r_M(s)\,ds}\,du, & \text{if } t \in [T_R, T], \\ \int_t^{T_R} e^{-\int_t^u r_M(s)\,ds}\,du + \Upsilon \int_{T_R}^T e^{-\int_t^u r_M(s)\,ds}\,du, & \text{if } t < T_R, \end{cases} \tag{21}$$

$$G(t) = \varepsilon_c^{1/\gamma} \frac{1}{r_G} \left(1 - e^{-r_G(T-t)}\right) + \varepsilon_F^{1/\gamma} e^{-r_G(T-t)}, \tag{22}$$

$$r_M(t) = r - \mu_Y(t) + \sigma_Y(t) \boldsymbol{\lambda}^\top \boldsymbol{\rho}_Y, \tag{23}$$

$$r_G = \frac{\delta}{\gamma} + \frac{\gamma - 1}{\gamma} \left(r + \frac{\|\boldsymbol{\lambda}\|^2}{2\gamma}\right). \tag{24}$$

The optimal portfolio at any time t is

$$\boldsymbol{\pi}_t = \frac{1}{\gamma}\left(1 + \frac{Y_t M(t)}{F_t}\right) \left(\underline{\underline{\sigma}}\,\underline{\underline{\sigma}}^\top\right)^{-1} (\boldsymbol{\mu} - r\mathbf{1}) - \frac{Y_t M(t)}{F_t} \sigma_Y(t) (\underline{\underline{\sigma}}^\top)^{-1} \boldsymbol{\rho}_Y, \tag{25}$$

and the optimal consumption rate is

$$c_t = \varepsilon_c^{1/\gamma} \frac{F_t + Y_t M(t)}{G(t)}. \tag{26}$$

Proof: The financial wealth dynamics are

$$dF_t = (Y_t - c_t)\,dt + F_t\left[\left(r + \boldsymbol{\pi}_t^\top (\boldsymbol{\mu} - r\mathbf{1})\right) dt + \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}\, d\boldsymbol{z}_t\right].$$

If we let subscripts on J indicate partial derivatives, the HJB equation is

$$\delta J(F, y, t) = \mathcal{L}_1 J(F, y, t) + \mathcal{L}_2 J(F, y, t) + \mathcal{L}_3 J(F, y, t),$$

where

$$\mathcal{L}_1 J = \sup_c \left\{\varepsilon_c u(c) - c J_F\right\},$$
$$\mathcal{L}_2 J = \sup_{\boldsymbol{\pi}} \left\{F J_F \boldsymbol{\pi}^\top (\boldsymbol{\mu} - r\mathbf{1}) + \frac{1}{2} F^2 J_{FF} \boldsymbol{\pi}^\top \underline{\underline{\sigma}}\,\underline{\underline{\sigma}}^\top \boldsymbol{\pi} + Y F J_{YF} \sigma_Y \boldsymbol{\pi}^\top \underline{\underline{\sigma}} \boldsymbol{\rho}_Y\right\},$$
$$\mathcal{L}_3 J = J_t + Y J_Y \mu_Y + r F J_F + Y J_F + \frac{1}{2} Y^2 J_{YY} \sigma_Y^2.$$

The first-order condition for c leads to

$$c = \varepsilon_c^{1/\gamma} J_F^{-1/\gamma}$$

and

$$\mathcal{L}_1 J = \varepsilon_c^{1/\gamma} \frac{\gamma}{1-\gamma} J_F^{\frac{\gamma-1}{\gamma}}.$$

The first-order condition for $\boldsymbol{\pi}$ leads to

$$\boldsymbol{\pi} = -\frac{J_F}{F J_{FF}} \left(\underline{\underline{\sigma}}\,\underline{\underline{\sigma}}^\top\right)^{-1} (\boldsymbol{\mu} - r\mathbf{1}) - \frac{Y J_{YF}}{F J_{FF}} \sigma_Y \left(\underline{\underline{\sigma}}^\top\right)^{-1} \boldsymbol{\rho}_Y,$$

which implies that

$$\mathcal{L}_2 J = -\frac{1}{2}\frac{J_F^2}{J_{FF}} \|\boldsymbol{\lambda}\|^2 - \frac{1}{2}\frac{Y^2 J_{YF}^2}{J_{FF}} \sigma_Y^2 \|\boldsymbol{\rho}_Y\|^2 - \frac{Y J_F J_{YF}}{J_{FF}} \sigma_Y \boldsymbol{\lambda}^\top \boldsymbol{\rho}_Y.$$

where

$$\boldsymbol{\lambda} = \underline{\underline{\sigma}}^{-1} (\boldsymbol{\mu} - r\mathbf{1}).$$

With the conjecture (20), we obtain (26) and

$$\mathcal{L}_1 J = \varepsilon_c^{1/\gamma} \frac{\gamma}{1-\gamma} G^{\gamma-1} (F + yM)^{1-\gamma}.$$

Since

$$-\frac{J_F}{F J_{FF}} = \frac{1}{\gamma}\left(1 + \frac{yM(t)}{F}\right), \qquad \frac{Y J_{YF}}{F J_{FF}} = \frac{yM(t)}{F}, \qquad \frac{J_F^2}{J_{FF}} = -\frac{1}{\gamma} G^\gamma (F + yM)^{1-\gamma},$$
$$\frac{Y^2 J_{YF}^2}{J_{FF}} = -\gamma G^\gamma y^2 M^2 (F + yM)^{-1-\gamma}, \qquad \frac{Y J_F J_{YF}}{J_{FF}} = G^\gamma y M (F + yM)^{-\gamma},$$

Eq. (25) follows and

$$\mathcal{L}_2 J = G^\gamma (F{+}yM)^{-1-\gamma} \left\{ \frac{1}{2\gamma} (F + yM)^2 \|\boldsymbol{\lambda}\|^2 - yM(F + yM)\sigma_Y \boldsymbol{\lambda}^\top \boldsymbol{\rho}_Y + \frac{\gamma}{2} y^2 M^2 \sigma_Y^2 \|\boldsymbol{\rho}_Y\|^2 \right\}.$$

Furthermore,

$$\begin{aligned}\mathcal{L}_3 J = G^\gamma (F + yM)^{-1-\gamma} \Big\{ &\left[\frac{\gamma}{1-\gamma}\frac{G'}{G} + r\right] (F + yM)^2 \\ &+ \left[M' - (r - \mu_Y)M + 1\right] y(F + yM) - \frac{\gamma}{2} y^2 M^2 \sigma_Y^2 \Big\}.\end{aligned}$$

If either $\sigma_Y = 0$ or $\|\boldsymbol{\rho}_Y\| = 1$, then the final terms of $\mathcal{L}_2 J$ and $\mathcal{L}_3 J$ cancel, and the HJB equation is satisfied provided that

$$M'(t) - \left(r - \mu_Y(t) + \sigma_Y(t)\boldsymbol{\lambda}^\top \boldsymbol{\rho}_Y\right) M(t) + 1 = 0,$$
$$G'(t) - \left(\frac{\delta}{\gamma} + \frac{\gamma - 1}{\gamma}\left[r + \frac{\|\boldsymbol{\lambda}\|^2}{2\gamma}\right]\right) G(t) + \varepsilon_c^{1/\gamma} = 0.$$

To ensure the terminal condition $J(F, y, T) = \frac{\varepsilon_F}{1-\gamma} F^{1-\gamma}$, we need $M(T) = 0$ and $G(T) = \varepsilon_F^{1/\gamma}$. The solutions are given by (21) and (22). □

C Proof of Equation (9)

The value at the end of period t of the income in period $t + k$ is

$$V_{t,t+k} = \mathrm{E}_t\left[e^{-[\nu(t+1)+\cdots+\nu(t+k)]} Y_{t+k}\frac{\zeta_{t+k}}{\zeta_t}\right] = \exp\left\{-\sum_{s=t+1}^{t+k}\nu(s)\right\}\mathrm{E}_t\left[Y_{t+k}\frac{\zeta_{t+k}}{\zeta_t}\right].$$

For any t,

$$\begin{aligned}
\mathrm{E}_t\left[Y_{t+1}\frac{\zeta_{t+1}}{\zeta_t}\right] &= Y_t \mathrm{E}_t\Bigg[\exp\Bigg\{\mu_Y(t+1) - \frac{1}{2}\sigma_Y(t+1)^2 - r - \frac{1}{2}\left(\|\boldsymbol{\lambda}\|^2 + \lambda_Y^2\right) \\
&\qquad + \left(\sigma_Y(t+1)\boldsymbol{\rho}_Y - \boldsymbol{\lambda}\right)^\top \boldsymbol{\varepsilon}_{t+1} + \left(\sigma_Y(t+1)\sqrt{1 - \|\boldsymbol{\rho}_Y\|^2} - \lambda_Y\right)\varepsilon_{Y,t+1}\Bigg\}\Bigg] \\
&= Y_t \exp\left\{-\hat{r}_m(t+1)\right\},
\end{aligned}$$

where

$$\hat{r}_m(s) = r - \mu_Y(s) + \sigma_Y(s)\left[\boldsymbol{\rho}_Y^\top \boldsymbol{\lambda} + \sqrt{1 - \|\boldsymbol{\rho}_Y\|^2}\lambda_Y\right].$$

By recursion and the law of iterated expectations we then get

$$\begin{aligned}
\mathrm{E}_t\left[Y_{t+k}\frac{\zeta_{t+k}}{\zeta_t}\right] &= \mathrm{E}_t\left[\frac{\zeta_{t+k-1}}{\zeta_t}\mathrm{E}_{t+k-1}\left[Y_{t+k}\frac{\zeta_{t+k}}{\zeta_{t+k-1}}\right]\right] \\
&= \mathrm{E}_t\left[\frac{\zeta_{t+k-1}}{\zeta_t} Y_{t+k-1} e^{-r_m(t+k)}\right] \\
&= e^{-r_m(t+k)}\mathrm{E}_t\left[\frac{\zeta_{t+k-1}}{\zeta_t} Y_{t+k-1}\right] \\
&= \cdots = Y_t \exp\left\{-\sum_{s=t+1}^{t+k} r_m(s)\right\}
\end{aligned}$$

so that

$$V_{t,t+k} = Y_t \exp\left\{ - \sum_{s=t+1}^{t+k} (\hat{r}_m(s) + \nu(s)) \right\} = Y_t \exp\left\{ - \sum_{s=t+1}^{t+k} r_m(s) \right\},$$

where $r_m(s) = \hat{r}_m(s) + \nu(s)$ is a risk-, mortality, and growth-adjusted discount rate. The total human capital at the end of period t, excluding the income just received, is therefore

$$L_t = \sum_{k=1}^{T-t} V_{t,t+k} = Y_t \sum_{k=1}^{T-t} \exp\left\{ - \sum_{s=t+1}^{t+k} r_m(s) \right\} = Y_t M(t),$$

which was to be shown.

References

Aït-Sahalia, Y. and M. Brandt (2001). Variable Selection for Portfolio Choice. *Journal of Finance* 56(4), 1297–1351.

Ang, A. and G. Bekaert (2002). International Asset Allocation with Regime Shifts. *Review of Financial Studies* 15(4), 1137–1187.

Attanasio, O. P. and G. Weber (1995). Is Consumption Growth Consistent with Intertemporal Optimization? Evidence from the Consumer Expenditure Survey. *Journal of Political Economy* 103(6), 1121–1157.

Badarinza, C., J. Y. Campbell, and T. Ramadorai (2016, March). International Comparative Household Finance. Available at SSRN: http://ssrn.com/abstract=2644967.

Bagliano, F. C., C. Fugazza, and G. Nicodano (2014). Optimal Life-Cycle Portfolios for Heterogeneous Workers. *Review of Finance* 18(6), 2283–2323.

Barberis, N. (2000). Investing for the Long Run when Returns are Predictable. *Journal of Finance* 55(1), 225–264.

Basak, S. and G. Chabakauri (2010). Dynamic Mean-Variance Asset Allocation. *Review of Financial Studies* 23(8), 2970–3016.

Benzoni, L., P. Collin-Dufresne, and R. S. Goldstein (2007). Portfolio Choice over the Life-Cycle when the Stock and Labor Markets are Cointegrated. *Journal of Finance* 62(5), 2123–2167.

Betermier, S., L. E. Calvet, and P. Sodini (2015, November). Who are the Value and Growth Investors? Available at SSRN: http://ssrn.com/abstract=2426823. *Journal of Finance*, forthcoming.

Bick, B., H. Kraft, and C. Munk (2013). Solving Constrained Consumption-Investment Problems by Simulation of Artificial Market Strategies. *Management Science* 59(2), 485–503.

Björk, T. and A. Murgoci (2014). A Theory of Markovian Time-Inconsistent Stochastic Control in Discrete Time. *Finance and Stochastics* 18(3), 545–592.

Bodie, Z., R. C. Merton, and W. F. Samuelson (1992). Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model. *Journal of Economic Dynamics and Control* 16, 427–449.

Bricker, J., A. B. Kennickell, K. B. Moore, and J. Sabelhaus (2012). Changes in U.S. Family Finances from 2007 to 2010: Evidence from the Survey of Consumer Finances. *Federal Reserve Bulletin* 98(2), 1–80.

Brown, S., W. Goetzmann, and S. A. Ross (1995). Survival. *Journal of Finance* 50(3), 853–873.

Campbell, J. Y. (2006). Household Finance. *Journal of Finance* 61(4), 1553–1604.

Chacko, G. and L. M. Viceira (2005). Dynamic Consumption and Portfolio Choice with Stochastic Volatility in Incomplete Markets. *Review of Financial Studies* 18, 1369–1402.

Cocco, J. F. (2005). Portfolio Choice in the Presence of Housing. *Review of Financial Studies* 18(2), 535–567.

Cocco, J. F., F. J. Gomes, and P. J. Maenhout (2005). Consumption and Portfolio Choice over the Life Cycle. *Review of Financial Studies* 18(2), 491–533.

Cochrane, J. H. (2014). A Mean-Variance Benchmark for Intertemporal Portfolio Theory. *Journal of Finance* 69(1), 1–49.

Corradin, S., J. L. Fillat, and C. Vergara-Alert (2014). Optimal Portfolio Choice with Predictability in House Prices and Transaction Costs. *Review of Financial Studies* 27(4), 823–880.

Davidoff, T. (2006). Labor Income, Housing Prices, and Homeownership. *Journal of Urban Economics* 59(2), 209–235.

De Nardi, M., E. French, and J. B. Jones (2010). Why do the Elderly Save? The Role of Medical Expenses. *Journal of Political Economy* 118(1), 39–75.

Fama, E. F. and K. R. French (1992). The Cross-Section of Expected Stock Returns. *Journal of Finance* 47(2), 427–465.

Fama, E. F. and K. R. French (2002). The Equity Premium. *Journal of Finance* 57(2), 637–659.

Fischer, M. and M. Stamos (2013). Optimal Life Cycle Portfolio Choice with Housing Market Cycles. *Review of Financial Studies* 26(9), 2311–2352.

Flavin, M. and T. Yamashita (2002). Owner-Occupied Housing and the Composition of the Household Portfolio. *American Economic Review* 91(1), 345–362.

Gomes, F. (2007). Exploiting Short-Run Predictability. *Journal of Banking & Finance* 31(5), 1427–1440.

Guiso, L. and P. Sodini (2013). Household Finance: An Emerging Field. Volume 2, Part B of *Handbook of the Economics of Finance*, Chapter 21, pp. 1397–1532. Elsevier.

Guvenen, F., F. Karahan, S. Ozkan, and J. Song (2015, February). What Do Data on Millions of U.S. Workers Reveal About Life-Cycle Earnings Risk. FRB of New York Staff Report No. 710. Available at SSRN: `http://ssrn.com/abstract=2563279`.

Hall, R. E. (1978). Stochastic Implications of the Life Cycle-Permanent Income Hypothesis: Theory and Evidence. *Journal of Political Economy* 86(6), 971–987.

Jagannathan, R. and N. R. Kocherlakota (1996). Why Should Older People Invest Less in Stocks Than Younger People? *Federal Reserve Bank of Minneapolis Quarterly Review* 20(3), 11–23.

Jurek, J. W. and L. M. Viceira (2011). Optimal Value and Growth Tilts in Long-Horizon Portfolios. *Review of Finance* 15(1), 29–74.

Kraft, H. and C. Munk (2011). Optimal Housing, Consumption, and Investment Decisions over the Life-Cycle. *Management Science* 57(6), 1025–1041.

Larsen, L. S. and C. Munk (2012). The Costs of Suboptimal Dynamic Asset Allocation: General Results and Applications to Interest Rate Risk, Stock Volatility Risk, and Growth/Value Tilts. *Journal of Economic Dynamics and Control* 36(2), 266–293.

Lynch, A. W. and S. Tan (2011). Labor Income Dynamics at Business-cycle Frequencies: Implications for Portfolio Choice. *Journal of Financial Economics* 101(2), 333–359.

Markowitz, H. (1952). Portfolio Selection. *Journal of Finance* 7(1), 77–91.

Markowitz, H. (1959). *Portfolio Selection: Efficient Diversification of Investment*. Wiley.

Mayers, D. (1972). Nonmarketable Assets and Capital Market Equilibrium under Uncertainty. In M. C. Jensen (Ed.), *Studies in the Theory of Capital Markets*. Praeger Publishers.

Merton, R. C. (1969). Lifetime Portfolio Selection Under Uncertainty: The Continuous-Time Case. *Review of Economics and Statistics* 51(3), 247–257.

Merton, R. C. (1971). Optimum Consumption and Portfolio Rules in a Continuous-Time Model. *Journal of Economic Theory* 3(4), 373–413.

Merton, R. C. (1973). An Intertemporal Capital Asset Pricing Model. *Econometrica* 41(5), 867–887.

Munk, C. and C. Sørensen (2010). Dynamic Asset Allocation with Stochastic Income and Interest Rates. *Journal of Financial Economics* 96(3), 433–462.

Pastor, L. and R. F. Stambaugh (2012). Are Stocks Really Less Volatile in the Long Run? *Journal of Finance* 67(2), 431–478.

Pelizzon, L. and G. Weber (2009). Efficient Portfolios when Housing Needs Change over the Life Cycle. *Journal of Banking & Finance* 33(11), 2110–2121.

Rosenberg, B., K. Reid, and R. Lanstein (1985). Persuasive Evidence of Market Inefficiency. *Journal of Portfolio Management* 11, 9–16.

Viceira, L. M. (2001). Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income. *Journal of Finance* 56(2), 433–470.

Weil, P. (1994). Nontraded assets and the CAPM. *European Economic Review* 38(3–4), 913–922.

Yao, R. and H. H. Zhang (2005). Optimal Consumption and Portfolio Choices with Risky Housing and Borrowing Constraints. *Review of Financial Studies* 18(1), 197–239.

Yogo, M. (2016). Portfolio Choice in Retirement: Health Risk and the Demand for Annuities, Housing, and Risky Assets. *Journal of Monetary Economics* 80, 17–34.