Welfare Benchmarking of Life-Cycle Investment Strategies for Households in Turkey

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Abstract

In this paper, we perform an in-depth welfare comparison of the most common life-cycle investment strategies provided by retirement funds or suggested by classical portfolio theory in the case of households in Turkey. To perform our benchmarking, we construct heterogeneous agents who work and invest throughout their lifetime, using parameters calibrated from the historical data. We find that to households with upper-to-middle income, individually customized portfolios result in considerable welfare gains, while "off-the-shelf" life-cycle portfolio allocations perform better for households with lower income. We also show that life-cycle investment options outperform "fixed over the lifetime" options. Finally, we find that risk-averse individuals with volatile wages, can maximize their welfare by investing in housing as suggested by Munk [11].

Keywords: Life-cycle portfolio decisions, human capital, housing, stock market participation

1. Introduction

The field of financial economics has gone through big changes since its foundation by Markowitz [1] and Tobin [2]. They pioneered the mean-variance analysis, which, given some assumptions, suggested that if investors cared about maximizing returns (mean) and minimizing risks (variance), then the optimal ratio of stocks to bonds in a single-period portfolio would be fixed for everyone, the share of former being equal to:

$$\alpha = \frac{\mu - R_f}{\gamma \sigma^2} \tag{1}$$

Merton [3] generalized the problem to multiple periods using dynamic programming and found that it is optimal for all households to repeat the same fixed mean-variance solution every period.

These results were inconsistent with the popular financial advice suggesting that younger investors should have higher share of stocks in portfolio, and older investors — higher share of bonds. This advice was summarized by the famous rule of thumb:

$$\alpha_t = (100 - t)\% \tag{2}$$

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Although Samuelson [4] denied that risk-aversion changes by age, dismissing this advice would question the rationality of investors and "constitute *prima facie* evidence that people do not optimize" (Canner et al. [5]).

Bodie et al. [6] solved this problem by adding human capital into the Merton [3]'s dynamic model and found that for complete markets and constant risk-free labor income, the optimal share of stocks in a portfolio is:

$$\alpha_t = \frac{\mu - R_f}{\gamma \sigma^2} \left(\frac{F_t + L_t}{F_t} \right) \tag{3}$$

Steady depletion of human capital L_t relative to the financial wealth F_t throughout life, explained the higher share of stocks in younger people. Cocco et al. [7] extended this idea to the case of variable labor income, to find a recursive solution which could be approximated by the following rule of thumb:

$$\alpha_t = \begin{cases} 100\% & t < 40 \\ (200 - 2.5t)\% & t \in [40, 60] \\ 50\% & t > 60 \end{cases} \tag{4}$$

However, the hump-shaped lifetime stock share graph, observed by Chang et al. [8] of Federal Reserve, instead of expected downward sloping one, suggested the presence of an opposing force.

Cocco [9] found that this force was housing investment, which, due to its large size, crowded out all stocks from younger investors' portfolios. Flavin and Yamashita [10] supported this view by showing that younger people, who already own the house, tend to invest more aggressively, as was expected by Bodie et al. [6].

Munk [11] found the same patterns using a series of one-period mean-variance optimizations without any dynamic stochastic modeling tools.

Finally, Ascheberg et al. [12] illustrated the existence of long-term cointegration among house prices, stock prices, and labor income, the fact often omitted by previous portfolio researchers for simplicity. In our analysis we will not neglect the correlations as being equal to zero.

In this paper, we calculate the wealth, accumulated from lifetime investments given by all the strategies above and suggested by current Turkish banks. We benchmark the resulting welfare for heterogeneous agents in Turkey with varying sectors, levels of education and risk-aversion. We conclude with the separate investment advice for all types of households mentioned above. Such a comprehensive benchmarking has not been done yet in the overall literature, and almost none was done for Turkish households.

The remainder of the paper is organized as follows: Section 2 presents the theoretical framework we will use in our analysis. Section 3 contains detailed information about our data sources, parameter calibration and solves for the optimal investment strategies. Section 4 does the welfare benchmarking of every investment option for every agent, and Section 5 concludes.

2. Theoretical framework

2.1. House prices, stock prices and labor income series

In accordance with Campbell et al. [13] and Olear [14] we model the labor income process as a function of individiual characteristics $f(t, Z_{it})$ plus idiosyncratic shocks v_{it}. Upon reaching the retirement age R, an individual receives a certain percentage λ of his/her last wage:

$$Y_{i,t+1} = \begin{cases} Y_{it}(1 + f(t+1, Z_{i,t+1}) + v_{it}), & t < R \\ \lambda(1 + f(R, Z_{iR}) + v_{iR}), & t \geq R \end{cases} \tag{5}$$

We model labor income, house prices, and stock prices as Geometric Brownian Motions with drifts μ_L, μ_H, μ_S and volatilities σ_L, σ_H, σ_S, satisfying the discrete version of Ascheberg et al. [12]'s correlation structure, that is having nonzero correlations $\rho_{HS}, \rho_{HL}, \rho_{SL}$. See Appendix A for details.

2.2. Optimal portfolio

Along with the investment strategies, described in Equations 1 - 4, we consider in our benchmarking, the strategy proposed by Munk [11], who stated that in the presence of housing, the optimal stock (π) and housing (π_h) shares can be solved analytically as follows:

$$\pi_{t+1} = \frac{1}{\gamma(1 - \rho_{SH}^2)\sigma_S} \cdot \frac{F_t + L_t}{F_t} \left(\frac{\mu_S - r_f}{\sigma_S} - \rho_{SH} \frac{\mu_H - r_f}{\sigma_H} \right) - \frac{L_t}{F_t} \cdot \frac{\sigma_L}{\sigma_S} \frac{\rho_{SL} - \rho_{SH}\rho_{HL}}{1 - \rho_{SH}^2} \tag{6a}$$

$$\pi_{h,t+1} = \frac{1}{\gamma(1 - \rho_{SH}^2)\sigma_H} \cdot \frac{F_t + L_t}{F_t} \left(\frac{\mu_H - r_f}{\sigma_H} - \rho_{SH} \frac{\mu_S - r_f}{\sigma_S} \right) - \frac{L_t}{F_t} \cdot \frac{\sigma_L}{\sigma_H} \frac{\rho_{HL} - \rho_{SH}\rho_{SL}}{1 - \rho_{SH}^2} \tag{6b}$$

Setting $\rho_{SH} = 0$ and $\rho_{HL} = 0$, gives the optimal stock share by Munk [11] in the absence of housing:

$$\pi_{t+1} = \frac{1}{\gamma\sigma_S} \cdot \frac{F_t + L_t}{F_t} \left(\frac{\mu_S - r_f}{\sigma_S} \right) - \frac{L_t}{F_t} \cdot \frac{\sigma_L}{\sigma_S} \rho_{SL} \tag{7}$$

2.3. Welfare measurement

We use stochastic constant relative risk-aversion utility function to compare welfare resulting from different income patterns:

$$E_1[U(c)] = \sum_{t=1}^{T} \delta^{t-1} \prod_{j=0}^{t-1} p_j \cdot \frac{c_{it}^{1-\gamma}}{1 - \gamma} \tag{8}$$

where p_k is the probability of survival between ages $k - 1$ and k. Unlike Cocco et al. [7], we neglect the bequest motives, assuming that the retired person consumes all of his/her income at any given time.

2.4. Retirement income

The funds invested in retirement are modeled to be paid back in annuities, not withdrawn immediately. Further, to include housing investment in welfare calculation, we use "reverse mortgages" — annuities, paid to retired individuals in return for inheriting their house after their death. This is a plausible analysis tool, because it allows to liquidify the housing possessions, although such financial instrument is not yet available in Turkey.

Thus, at the age of retirement $R = 65$, the price of owned house is calculated and is added to the matured pension amount (MP) to obtain total wealth:

$$W_{65} = H_{65} + MP \tag{9}$$

All of the W_{65} is used to buy an annuity which will annually repay an individual:

$$A_t = W_{65} \cdot \left(1 + \sum_{t=66}^{100} \frac{\prod_{j=66}^{t} p_j}{(1+r_f)^{t-65}}\right)^{-1} \tag{10}$$

3. Data and simulation

In this section, we go over data sources, perform parameter calibrations and derive all investment strategies for our simulation.

3.1. Data sources

We use historical monthly BIST 30[1] and REIDIN[2] data from 2004 to 2014, to construct stock and house price series respectively. Figures 1 and 2 illustrate the general upward trend in both series, with a collapse during 2008 crisis.
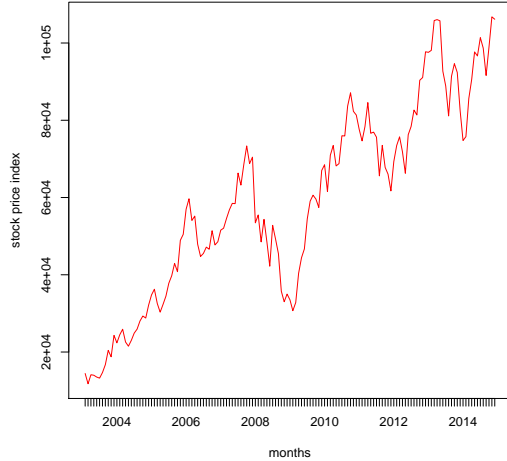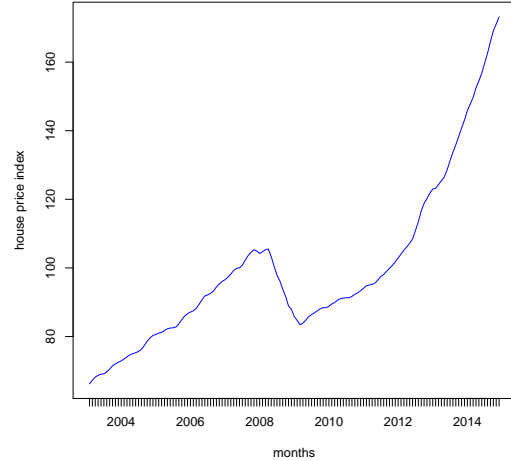

Figure 1: BIST30 Turkish stock market prices

Figure 2: Reidin Turkish house price index

We construct labor income series from Turkish Statistical Institute's Household Budget Survey [15]'s repeated cross-sectional study, and, in line with Aktug et al. [16], we aggregate the data to obtain a pseudo-panel with 55 thousand data points for 170 households from 2001 to 2014. Figure 3 displays the hump-shaped lifetime income distribution, consistent with the results of Aktug et al.

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[1]BIST 30 is an index measuring the stock performance of 30 largest companies in Turkey

[2]REIDIN provides residential sales price index for Turkey, using data, covering $200,000$ house listings in 62 cities and 221 counties, per month, weighted by population, and calculated using Laspeyres' formula.

[16], who analyzed labor income profiles in Turkey, and Ben-Porath [17], who predicted a decline in productivity, as workers get older.
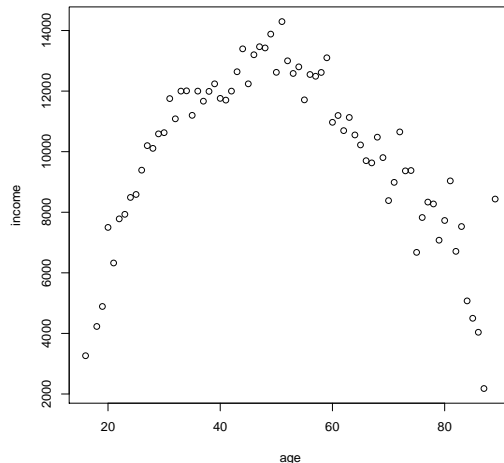

Figure 3: Median Turkish salaries by age

3.2. Default parameters

Similarly to Munk [11], we start our simulation with a 25-year-old individual who invests in his/her retirement for 40 years until he/she reaches retirement at 65. Similarly to Torul and Oztunali [18], we set the default relative risk aversion coefficient for Turkish households at 1.5, and the subjective discount rate — at 0.89.

We deflate the nominal wages, stock and house prices by CPI and work with real variables.

We obtain annual rate of return on stocks 6.69% with volatility 38.44%, by annualizing long-term ARMA(2,2) forecasts of monthly return and volatility, based on historical BIST 30 data, mentioned above (see Appendix B.1).

Similarly, we use long-term ARMA(1,1) forecast of monthly return and volatilities on housing, and find annual real rate of return on housing 0.67% with 5.42% volatility (see Appendix B.2).

Risk-free rate 12% is given by OECD [19] forecast[3], and, upon subtracting the medium-term inflation rate forecast $\pi = 9\%$ by Turkish Central Bank's Inflation Report [21], is equal to 3% per annum with zero volatility.

We consider real wage growth rates separately for different types of agents, but before introducing heterogeneity, the ARMA(5,2) forecast gives the volatility 4% (see Appendix B.3).

In our simulation, the house-stock and house-wage contemporaneous correlations are given by 0.27 and 0.35 respectively.

Survival probabilities for all ages are provided by Turkish Statistical Institute's Data Bank [20] and illustrated in Figure 4.

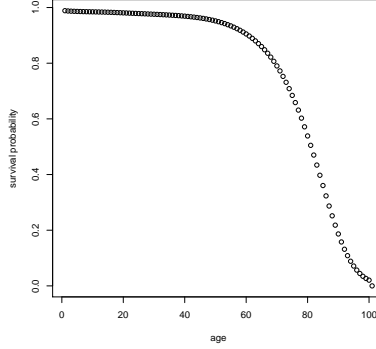[3]Data was obtained before the Turkish currency and debt crisis of 2018

Figure 4: Survival probabilities by age

All of the above findings are summarized in Table 1.

Table 1: Benchmark Parameters

Parameter	Description	Value
Y	Beginning age	25
R	Retirement age	65
T	Lifespan (years)	100
γ	Risk aversion	1.5
β	Discount rate	0.89
r_f	Risk-free rate	0.03
μ_s	Expected stock returns	0.0669
μ_h	Expected housing returns	0.0067
σ_s	Stock returns volatility	0.3844
σ_h	Housing returns volatility	0.0542
σ_w	Wage growth volatility	0.036
ρ_{hs}	House-stock correlation	0.24
ρ_{hw}	House-wage correlation	0.37
p_{25}	Survival probability at age 25	0.978
p_{65}	Survival probability at age 65	0.86
p_{100}	Survival probability at age 100	0

3.3. Heterogeneity parameters

Combining the approaches of Olear [14] and Munk [11], we use wage growth rate, stock-income correlation and relative risk aversion level, to model heterogeneities among agents.

3.3.1. Heterogeneity in education

We model the heterogeneity in education using differences in wage growth rates. Figure 5 shows the lifetime labor income series for different levels of education. Notice that the curves are almost flat for the lower education levels and hump-shaped for the higher.
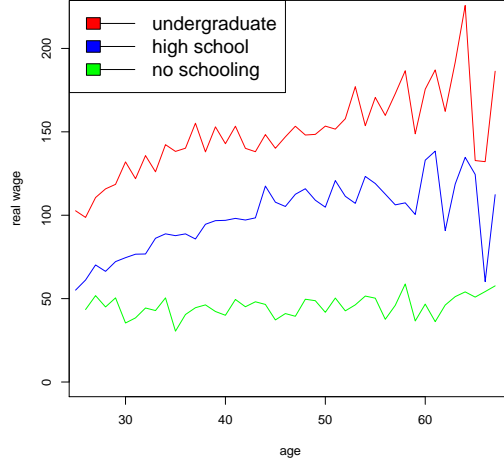
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Figure 5: Lifetime real wage dynamics by education level. While less educated individuals have flat wages through life, the more educated individuals observe a steeper and hump-shaped wages.

We use undergraduate education, high school education, and no schooling, to model "steep", "moderate", and "flat" wage dynamics respectively. Performing regressions of wages on age, with kinks at $t = 40$ and $t = 55$:

$$\Delta \log(wage_{it}) = \alpha_0 + \alpha_1 \cdot d_{40} + \alpha_2 \cdot d_{55} \tag{11}$$

we estimate growth rates for different education levels, as summarized in Table 2 (see Appendix B.4 for regression results and line fits).

Table 2: Estimated Benchmark Wage Growth Rates μ_w

Age	Flat	Moderate	Steep
25-40	0%	3.8%	2.2%
41-55	0%	1.4%	1.2%
56-65	0%	0%	1.5%

We assume steep wage earners (college graduates) to have a starting real salary of 100, and moderate and flat wage earners to have a starting real salary of 50. This is consistent with the historical data (see Figure 5) and the relevant discussion by Olear [14]. This difference in starting values also explains why "steep" wage growth rates are less than "moderate" ones.

3.3.2. Heterogeneity in sectors of work

We model the heterogeneity work sectors using corresponding stock-wage correlations (ρ_{ws}). Figure 6 illustrates how, during 2008 crisis, income in financial sectors ($\rho_{ws} = 0.44$) dropped drastically, while it was unaffected in education and agriculture ($\rho_{ws} = 0.08$).

We use three measures of ρ_{sw} for our benchmark: 0, 0.2 and 0.4 (see Table 3).
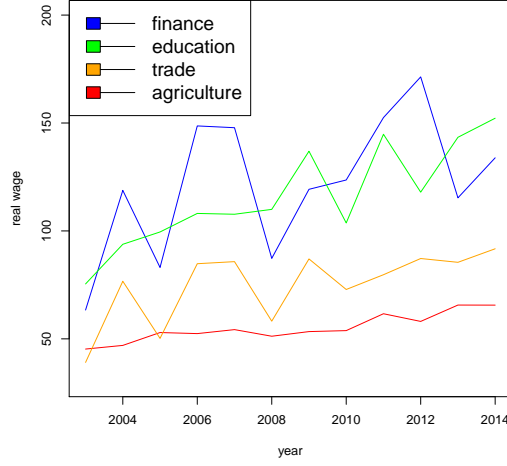
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Figure 6: Historical real wage dynamics by sector. Sectoral differences in degrees of correlation of wages to stock market, can be seen from the different behavior during 2008 crisis — while income in finance sector dropped drastically, the wages in education sector were not affected at all.

Table 3: Benchmark Wage to Stock Correlations

	Low	Moderate	High
ρ_{sw}	0	0.2	0.4

3.3.3. Individual heterogeneity

We model individual heterogeneity using different risk aversion levels of investors, as summarized in Table 4.

Table 4: Coefficients of Risk Aversion

Values	default	low	moderate	high
γ	1.5	3	5	10

3.4. Capital series

3.4.1. Human capital

Human capital at any period is the discounted sum of all future wages until retirement with the discount factor r_f. To construct the individualized capital we used steep, moderate and flat wage series mentioned in the previous section. Figure 7 illustrates the current human capital for flat, moderate, and steep wages for every age.

3.4.2. Financial capital

Financial capital evolves according to dynamic investment diagram in Figure 9. Every period, a certain percentage c (3% for Turkey) of the wage w_t is invested in a retirement portfolio, while
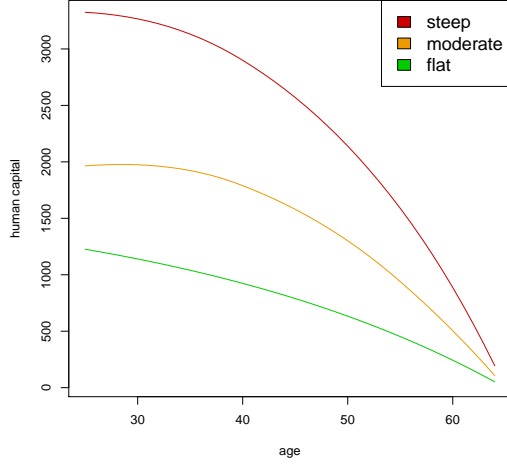
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Figure 7: Human capital at every age for individuals with steep, moderate and flat wage growth curves. As individuals get older, their human capital gradually depletes.
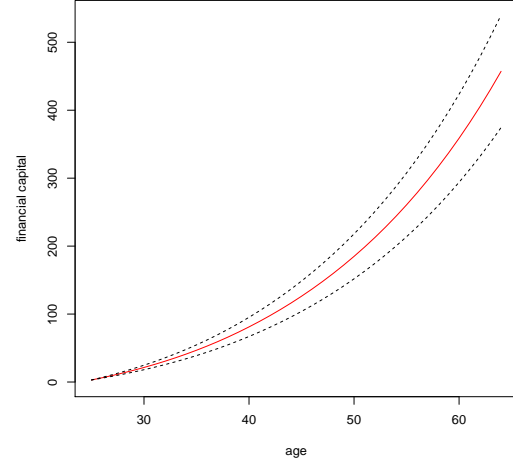

Figure 8: Financial wealth accumulation through life, by a college graduate who invests 50% in bonds and 50% in stocks. Confidence intervals capture the volatility of stocks in the portfolio.

the previously invested amount accrues interest at portfolio rate of return. Figure 8 demonstrates the evolution of financial capital and its confidence interval for a naive fixed investment strategy (50% in stocks and 50% in bonds) by an individual with "steep" wage curve.
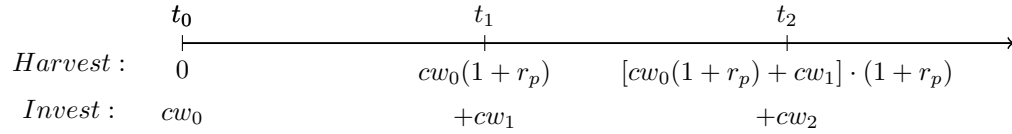

Figure 9: Law of motion of financial capital. Every period, a certain percentage c of the wage w_t is invested in a retirement portfolio, while the previously invested amount accrues interest at portfolio rate of return r_p.

3.5. Investment strategies

Below, we present the life-cycle investment strategies to be benchmarked.

3.5.1. Homogeneous life-cycles

Homogeneous life-cycles are strategies, common to all individuals, regardless of their idiosyncratic characteristics. We benchmark strategies given by equations 1, 2, 4, and an aggressive portfolio allocation, offered by Turkish banks (60% in stocks, 40% in bonds). Figure 10 illustrates the stock shares in these investment strategies.
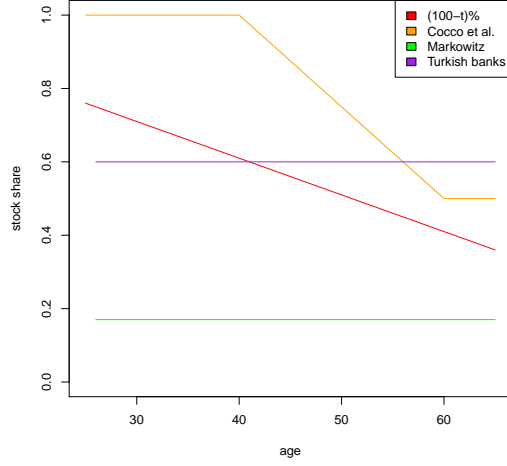
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Figure 10: Share of stocks in a portfolio at every age, proposed by homogeneous life-cycle strategies

3.5.2. Heterogeneous life-cycles

Heterogeneous life-cycles take idiosyncratic characteristics into consideration. We benchmark strategies by Bodie et al. [6] (see Equation 3), and Munk [11] (see Equation 7), which are illustrated in Figure 11 by blue and red curves respectively. The dashed lines illustrate the 68%-confidence interval ($\pm 1\sigma$). The figure proposes younger investors to allocate all of their funds in stocks, and gradually, through life, decrease their share in the portfolio — the more risk-averse they are or the flatter their wage curve is, the sooner. It also shows that, other things being equal, Munk [11]'s strategy without housing, is less aggressive than Bodie et al. [6]'s, which is consistent with equations 3 and 7.

(a) $\gamma = 1.5$

(b) $\gamma = 3$

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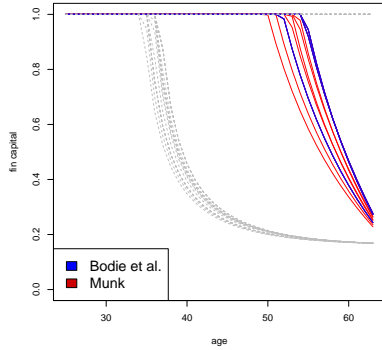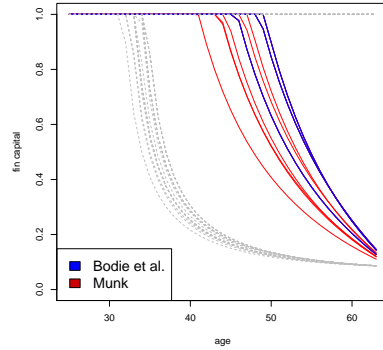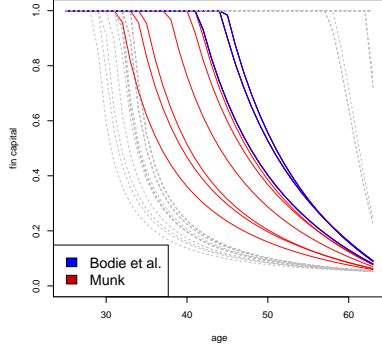
(c) $\gamma = 5$
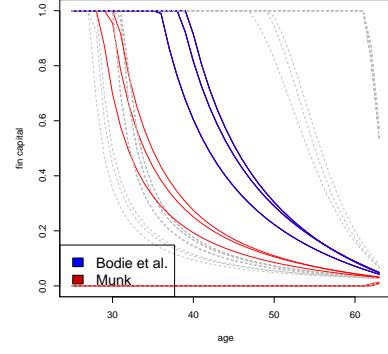
(d) $\gamma = 10$

Figure 11: Shares of stocks in a portfolio, suggested by Bodie et al. [6] (blue curves) and Munk [11] (red curves) in the absence of housing investment, for heterogeneous agents. The optimal allocations at every age depend on the previous realizations of volatile stock returns — the dashed lines illustrate the confidence intervals for $\pm\sigma$. Higher risk-aversion, steeper wage curve and smaller stock-income correlation lead to less aggressive investment in stocks, and vice versa.

Figure 12 shows the stock and house shares, suggested by Munk [11] for flat, moderate and steep labor income curves, low, moderate and high stock-wage correlations, and different levels of risk aversion. It confirms that steeper labor income curves results in a larger share of stocks in portfolio, and the more risk averse individuals are, the sooner they decrease both housing and stock investment, and buy more bonds.
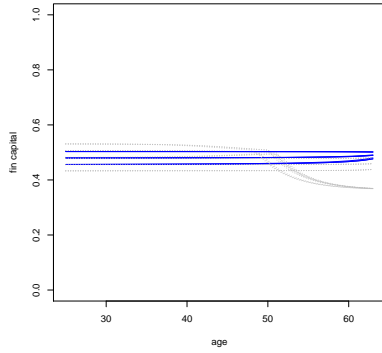
(a) Stocks for $\gamma = 1.5$

(b) Housing for $\gamma = 1.5$

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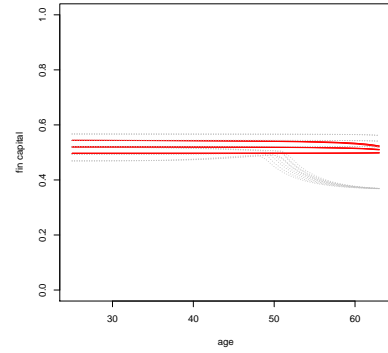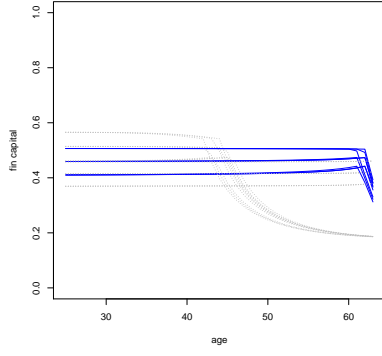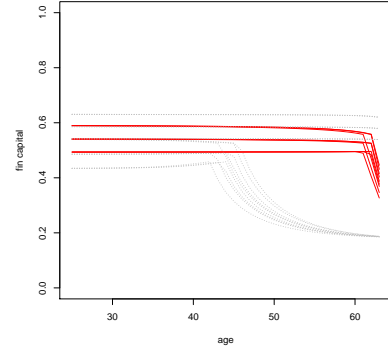
(c) Stocks for $\gamma = 3$

(d) Housing for $\gamma = 3$
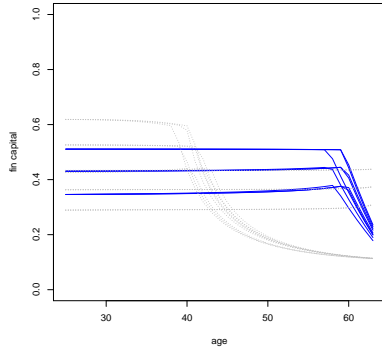
(e) Stocks for $\gamma = 5$

(f) Housing for $\gamma = 5$

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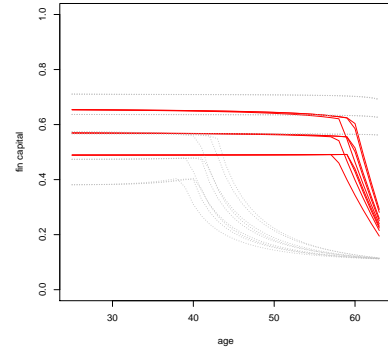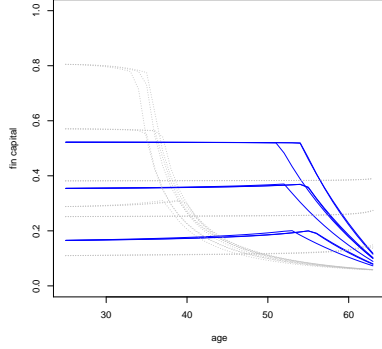
(g) Stocks for $\gamma = 10$
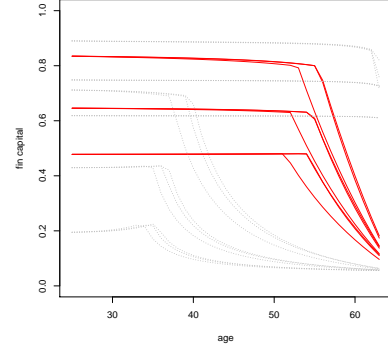
(h) Housing for $\gamma = 10$

Figure 12: Munk's stock and housing shares for different wage growth, stock-wage correlation and risk aversion levels

Tables with portfolio allocations can be reproduced using the data and methodology explained in this paper in detail.

4. Welfare comparison and results

In this section, we calculate the accumulated wealth for every investment option above and benchmark the resulting expected utilities.

4.1. Accumulated wealth

After the lifetime of investing, households accumulate different levels of wealth. Tables 5 and 6 summarize the mean and standard deviation of 10,000 stock and house price realizations from a Monte Carlo simulation. It is important to note that the values are χ^2-distributed since they depend on two random normal variables — stock and house prices.

Table 5: Monte Carlo Results of Accumulated Wealth for Homogeneous Investment Strategies

wages	steep	moderate	flat
Markowitz	343	201	130
	(87)	(50)	(35)
$100 - age$	449	261	175
	(442)	(246)	(191)
Cocco et al.	556	320	222
	(1,348)	(728)	(611)
Turkish banks	483	280	189
	(619)	(348)	(264)

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Table 6: Monte Carlo Results of Accumulated Wealth for Heterogeneous Investment Strategies

wages	steep			moderate			flat		
ρ_{ws}	high	moderate	low	high	moderate	low	high	moderate	low
$\gamma = 1.5$									
Bodie et al.	-	544	-	-	315	-	-	206	-
	-	(756)	-	-	(433)	-	-	(281)	-
Munk (no housing)	524	531	538	304	309	312	199	202	204
	(663)	(700)	(733)	(379)	(400)	(418)	(249)	(263)	(275)
Munk (housing)	403	416	430	235	243	251	154	159	165
	(495)	(541)	(591)	(280)	(306)	(334)	(201)	(219)	(239)
$\gamma = 3$									
Bodie et al.	-	484	-	-	281	-	-	182	-
	-	(479)	-	-	(276)	-	-	(176)	-
Munk (no housing)	449	468	482	262	272	280	168	176	181
	(352)	(419)	(473)	(204)	(241)	(272)	(129)	(154)	(174)
Munk (housing)	514	534	546	299	310	317	194	202	206
	(711)	(778)	(826)	(407)	(446)	(473)	(264)	(289)	(307)
$\gamma = 5$									
Bodie et al.	-	450	-	-	262	-	-	168	-
	-	(344)	-	-	(199)	-	-	(125)	-
Munk (no housing)	377	423	448	221	247	262	140	158	168
	(149)	(263)	(343)	(87)	(153)	(198)	(54)	(95)	(125)
Munk (housing)	497	509	520	289	296	302	187	192	196
	(513)	(563)	(608)	(295)	(324)	(349)	(189)	(208)	(224)
$\gamma = 10$									
Bodie et al.	-	402	-	-	235	-	-	149	-
	-	(212)	-	-	(124)	-	-	(76)	-
Munk (no housing)	302	345	401	178	203	235	113	128	149
	(1)	(80)	(212)	(1)	(47)	(124)	(1)	(28)	(76)
Munk (housing)	418	446	465	244	260	271	156	167	174
	(252)	(339)	(405)	(147)	(196)	(234)	(91)	(123)	(148)

4.2. Annuities

Retired individuals will sell all of their accumulated wealth to buy annuities and consume them fully until their death, enjoying the utility of this consumption. Recall that we ignore bequest motives and savings after the retirement. We define annuities by dividing the total wealth before retirement by the discount factor $1 + \sum_{t=66}^{100} \frac{p_t}{(1+r_f)^{t-65}}$. Using survival probabilities, obtained from TUIK, and risk-free rate of return, described in the previous chapter, we calculate the discount factor as 13.73. The annuities are obtained by dividing all the values in the Tables 5 and 6 by 13.73.

4.3. Expected utilities

Recall that we have defined the wages in real terms, dividing the nominal wages by CPI. Therefore we can plug the values, obtained above, to the CRRA utility function as consumption level.

Table 7 provides mean and standard deviation of accumulated wealth from different scenarios of a Monte Carlo experiment, mentioned above. Note that the distributions are heavily skewed to the left.

Table 7: A: Summary of Expected Utilities from Simulation for $\gamma = 1.5$

wages	steep			moderate			flat		
ρ_{ws}	high	moderate	low	high	moderate	low	high	moderate	low
Markowitz	-2.515	-2.515	-2.515	-3.282	-3.282	-3.282	-4.096	-4.096	-4.096
	(0.314)	(0.314)	(0.314)	(0.403)	(0.403)	(0.403)	(0.547)	(0.547)	(0.547)
$100 - age$	-2.593	-2.593	-2.593	-3.374	-3.374	-3.374	-4.285	-4.285	-4.285
	(0.874)	(0.874)	(0.874)	(1.116)	(1.116)	(1.116)	(1.568)	(1.568)	(1.568)
Cocco et al.	-2.889	-2.889	-2.889	-3.734	-3.734	-3.734	-4.899	-4.899	-4.899
	(1.299)	(1.299)	(1.299)	(1.623)	(1.623)	(1.623)	(2.333)	(2.333)	(2.333)
Turkish banks	-2.772	-2.772	-2.772	-3.608	-3.608	-3.608	-4.612	-4.612	-4.612
	(1.187)	(1.187)	(1.187)	(1.531)	(1.531)	(1.531)	(2.122)	(2.122)	(2.122)
Bodie et al.	-3.389	-3.389	-3.389	-4.389	-4.389	-4.389	-5.695	-5.695	-5.695
	(2.352)	(2.352)	(2.352)	(3.048)	(3.048)	(3.048)	(4.252)	(4.252)	(4.252)
Munk (no housing)	-3.355	-3.415	-3.476	-4.346	-4.435	-4.499	-5.61	-5.729	-5.841
	(2.334)	(2.403)	(2.506)	(3.096)	(3.242)	(3.304)	(4.253)	(4.37)	(4.553)
Munk (housing)	-3.043	-3.071	-3.103	-3.952	-3.987	-4.027	-5.105	-5.159	-5.22
	(1.327)	(1.398)	(1.472)	(1.71)	(1.802)	(1.899)	(2.387)	(2.519)	(2.656)

Table 7: B: Summary of Expected Utilities from Simulation for $\gamma = 3$

wages	steep			moderate			flat		
ρ_{ws}	high	moderate	low	high	moderate	low	high	moderate	low
Markowitz	-0.006	-0.006	-0.006	-0.017	-0.017	-0.017	-0.042	-0.042	-0.042
	(0.003)	(0.003)	(0.003)	(0.009)	(0.009)	(0.009)	(0.023)	(0.023)	(0.023)
$100 - age$	-0.011	-0.011	-0.011	-0.031	-0.031	-0.031	-0.09	-0.09	-0.09
	(0.016)	(0.016)	(0.016)	(0.045)	(0.045)	(0.045)	(0.151)	(0.151)	(0.151)
Cocco et al.	-0.023	-0.023	-0.023	-0.062	-0.062	-0.062	-0.216	-0.216	-0.216
	(0.042)	(0.042)	(0.042)	(0.115)	(0.115)	(0.115)	(0.469)	(0.469)	(0.469)
Turkish banks	-0.019	-0.019	-0.019	-0.055	-0.055	-0.055	-0.17	-0.17	-0.17
	(0.043)	(0.043)	(0.043)	(0.124)	(0.124)	(0.124)	(0.418)	(0.418)	(0.418)
Bodie et al.	-0.372	-0.372	-0.372	-0.579	-0.579	-0.579	-1.182	-1.182	-1.182
	(22.2)	(22.2)	(22.2)	(25.6)	(25.6)	(25.6)	(53.1)	(53.1)	(53.1)
Munk (no housing)	-0.04	-0.063	-0.212	-0.099	-0.192	-0.427	-0.287	-0.963	-3.585
	(0.507)	(0.608)	(10.291)	(0.665)	(3.287)	(15.333)	(1.914)	(28.294)	(176.463)
Munk (housing)	-0.515	-0.289	-1.132	-3.637	-3.403	-1.226	-55.076	-7.09	-507.406
	(39.222)	(11.653)	(86.736)	(312.4)	(291)	(50.427)	(3832)	(533.61)	(50503)

Table 7: C: Summary of Expected Utilities from Simulation for $\gamma = 5$

wages	steep			moderate			flat		
ρ_{ws}	high	moderate	low	high	moderate	low	high	moderate	low
Markowitz	0	0	0	0	0	0	0	0	0
	(0)	(0)	(0)	(0)	(0)	(0)	(0.001)	(0.001)	(0.001)
$100 - age$	0	0	0	-0.001	-0.001	-0.001	-0.005	-0.005	-0.005
	(0)	(0)	(0)	(0.003)	(0.003)	(0.003)	(0.044)	(0.044)	(0.044)
Cocco et al.	-0.002	-0.002	-0.002	-0.003	-0.003	-0.003	-1.372	-1.372	-1.372
	(0.201)	(0.201)	(0.201)	(0.027)	(0.027)	(0.027)	(90.495)	(90.495)	(90)
Turkish banks	0	0	0	-0.003	-0.003	-0.003	-0.039	-0.039	-0.039
	(0.005)	(0.005)	(0.005)	(0.04)	(0.04)	(0.04)	(0.598)	(0.598)	(0.598)
Bodie et al.	-0.14	-0.14	-0.14	-1.684	-1.684	-1.684	-34.122	-34.122	-34.122
	(10.141)	(10.141)	(10.141)	(139.3)	(139.3)	(139.3)	(2926)	(2926)	(2926)
Munk (no housing)	-0.007	-0.228	-33879.48	-0.047	-4.169	-3.2	-0.518	-321.58	-11420.369
	(0.226)	(14.403)	(3387924)	(1.387)	(388.399)	(190)	(12.701)	(31692)	(1102401)
Munk (housing)	-0.608	-198.462	-3.52	-6.667	-930.941	-78.93	-23.559	-41341.4	-93.43
	(51.7)	(19257)	(267.51)	(598)	(86349)	(6966)	(1401)	(3920139)	(5322)

Table 7: D: Summary of Expected Utilities from Simulation for $\gamma = 10$

wages	steep			moderate			flat		
ρ_{ws}	high	moderate	low	high	moderate	low	high	moderate	low
Markowitz	0	0	0	0	0	0	0	0	0
	(0)	(0)	(0)	(0)	(0)	(0)	(0)	(0)	(0)
$100 - age$	0	0	0	0	0	0	0	0	0
	(0)	(0)	(0)	(0)	(0)	(0)	(0.016)	(0.016)	(0.016)
Cocco et al.	0.001	0.001	0.001	-1.32	-1.32	-1.32	843	843	843
	(0.12)	(0.12)	(0.12)	(138.203)	(138.203)	(138.203)	(923)	(923)	(923)
Turkish banks	0	0	0	0	0	0	-0.031	-0.031	-0.031
	(0)	(0)	(0)	(0.003)	(0.003)	(0.003)	(1.06)	(1.06)	(1.06)
Bodie et al.	0	0	0	-0.002	-0.002	-0.002	-1.432	-1.432	-1.432
	(0.001)	(0.001)	(0.001)	(0.09)	(0.09)	(0.09)	(85.945)	(85.945)	(85.945)
Munk (no housing)	0	0	0	0	0	-0.001	0	0	-0.781
	(0)	(0)	(0.001)	(0)	(0)	(0.086)	(0)	(0)	(56.102)
Munk (housing)	0	-0.001	-19368	-0.002	-0.1	-9661.276	-1.283	-57.043	-3694.293
	(0.001)	(0.051)	(1936894)	(0.106)	(5.981)	(699694)	(75)	(4115)	(281712)

4.4. Pairwise comparisons

In this section we provide the pairwise comparisons of utilities for different investment options. Table 8 shows percentages of scenarios at which the utilities from some investment options are greater than from the other. We ignored small percentage differences (up to 3%) as noise, and interpreted percentages close to 50% as "no conclusion".

We can see that homogeneous investment options return same ordering regardless of risk aversion, even though γ enters the utility equation. Similarly, steepness of labor income curve does not affect within-group ordering. At the same time, level of risk-aversion, and correlation of wages to stocks, do affect the ordering.

Table 8: A: Utility Comparison for $\gamma = 1.5$

L $\geq$ R	Markowitz	$100 - age$	Cocco et al.	TR Banks	Bodie et al.	Munk (no h.)	Munk (h.)
Markowitz	100%	51%	60%	54%	55%	55%	64%
$100 - age$		100%	72%	61%	58%	57%	80%
Cocco et al.			100%	32%	55%	56%	70%
Turkish banks				100%	65%	65%	95%
Bodie et al.					100%	58% (81% if $\rho_{ws} = 0$)	48%
Munk (no housing)						100%	46%
Munk (housing)							100%

Table 8: B: Utility Comparison for $\gamma = 3$

L $\geq$ R	Markowitz	$100 - age$	Cocco et al.	TR Banks	Bodie et al.	Munk (no h.)	Munk (h.)
Markowitz	100%	51%	60%	54%	49%	48%	57%
$100 - age$		100%	72%	60%	56%	56%	61%
Cocco et al.			100%	31%	51%	50%	58%
Turkish banks				100%	58%	54%;57%;60% ($\rho_{ws} \searrow$)	66%
Bodie et al.					100%	55% (70% if $\rho_{ws} = 0$)	55%
Munk (no housing)						100%	55%
Munk (housing)							100%

Table 8: C: Utility Comparison for $\gamma = 5$

L $\geq$ R	Markowitz	$100 - age$	Cocco et al.	TR Banks	Bodie et al.	Munk (no h.)	Munk (h.)
Markowitz	100%	51%	61%	54%	50%	50%	57%
$100 - age$		100%	72%	61%	56%	56%	62%
Cocco et al.			100%	30%	50%	50%	59%
Turkish banks				100%	58%	53%;57%;60% ($\rho_{ws} \searrow$)	67%
Bodie et al.					100%	55% (70% if $\rho_{ws} = 0$)	54%
Munk (no housing)						100%	55%
Munk (housing)							100%

Table 8: D: Utility Comparison for $\gamma = 10$

L $\geq$ R	Markowitz	$100 - age$	Cocco	TR Banks	Bodie	Munk (no h.)	Munk (h.)
Markowitz	100%	51%	61%	55%	43%	65%;48%;42% ($\rho_{ws} \searrow$)	43%;46%;49% ($\rho_{ws} \searrow$)
$100 - age$		100%	73%	61%	52%	50%	53%($\rho_{ws} = 0.4$);57%
Cocco et al.			100%	30%	40%	45% ($\rho_{ws} = 0.4$);40%	41%;46%;51% ($\rho_{ws} \searrow$)
Turkish banks				100%	47%	50% ($\rho_{ws} = 0.4$);47%	48%;54%;58% ($\rho_{ws} \searrow$)
Bodie et al.					100%	62%;99% ($\rho_{ws} = 0$)	42%;45%;47% ($\rho_{ws} \searrow$)
Munk (no housing)						100%	38%;45%;47% ($\rho_{ws} \searrow$)
Munk (housing)							100%

The main comparison results are as follows:

- Naive life-cycle investment portfolio $(100 - age)\%$ overperforms all investment options except for Markowitz.

- Cocco et al.'s $(200 - 2.5 \cdot age)\%$ performs worse than other homogeneous portfolios.

- All models perform better for higher risk aversion and worse for lower risk aversion.

- Munk's solution performs worse for flat wages than for steep wages.

- When $\rho_{ws} = 0$, Bodie's solution is equivalent to Munk's solution without housing.

- Munk's solution with housing performs as well or worse than other options most of the time, except for higly risk-averse individuals, and especially for high wage-stock correlations — in these cases, investing in housing is the best option.

- Munk's solution without housing outperforms other strategies as risk aversion gets higher.

- Fixed moderate Turkish bank's investment suggestion, outperforms Cocco et al.'s lifecycle heuristic 70% of the time.

- For risk-takers $(\gamma < 5)$, heterogeneous outcomes are better for steep and moderate labor income curves.

5. Conclusion

In this paper we have reviewed the concept of "life-cycle investment" and summarized the common models and heuristics. We have presented our model as an application of Munk's (2016) recent findings and Olear's (2016) simulation techniques into Turkish retirement market. We have collected historical data to calibrate and estimate the best parameters to be used in our simulation. Using these parameters, we have constructed heterogeneous agents, who worked and invested throughout their lifetime. We considered different investment models that our hypothetical agents would use and calculated the resulting investment capitals. Finally, we have calculated and compared the welfare effects of all popular models to the individualized Munk's solutions. We have concluded that for rich-to-middle class risk-loving citizens, the individiualized strategies considerably increase their welfare. Moreover, the solutions we proposed can be solved analytically without complex dynamic optimizations, and therefore are easy to interpret to households. We also found that even naive life-cycle investment heuristics perform better than fixed lifetime investments. Our simulation showed that for risk-averse individuals with volatile wages, it is best to invest in housing in a proportion proposed by Munk. We propose these models to Turkish pension providers and to Turkish working-age households belonging to middle-to-upper class, as these options will increase their retirement welfare considerably.

6. References

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Appendix A. Ascheberg's correlation structure

The structure that returns desired correlation coefficients ρ_{SL}, ρ_{SH} and ρ_{HL} is as follows:

$$\frac{\Delta S_{t+1}}{S_t} = \mu_S + \sigma_S \cdot \epsilon_{St} \tag{A.1a}$$

$$\frac{\Delta H_{t+1}}{H_t} = \mu_H + \sigma_H \cdot \left(\rho_{SH}\epsilon_{St} + (\sqrt{1 - \rho_{SL}^2})\epsilon_{Ht} \right) \tag{A.1b}$$

$$\frac{\Delta Y_{t+1}}{Y_t} = \mu_L + \sigma_L \cdot \left(\rho_{SL}\epsilon_{St} + \left(\frac{\rho_{HL} - \rho_{SH}\rho_{SL}}{\sqrt{1 - \rho_{SH}^2}} \right) \epsilon_{Ht} + \left(\sqrt{1 - \rho_{SL}^2 - (\frac{\rho_{HL} - \rho_{SH}\rho_{SL}}{\sqrt{1 - \rho_{SH}^2}})^2} \right) \epsilon_{Lt} \right)$$
$$\tag{A.1c}$$

To derive this, let Σ be a correlation matrix of a vector $X = (x_1, x_2, ..., x_K)$. Also, let $\Sigma = LL'$ be a Cholesky decomposition of this matrix.

Notice that the variance-covariance matrix of an i.i.d. random vector $\Omega = (\epsilon_1, \epsilon_2, ..., \epsilon_K)$ with variances equal to 1, is an identity matrix. Thus, the product $L\Omega$ has the same correlation structure as X:

$$cov(L\Omega) = E[(L\Omega)(L\Omega)'] = E[L\Omega\Omega'L'],$$
$$cov(L\Omega) = L \cdot E[\Omega\Omega'] \cdot L' = L \cdot var(\Omega) \cdot L',$$
$$cov(L\Omega) = L \cdot I \cdot L' = LL' = \Sigma$$

The conclusion comes from the fact that the Cholesky decomposition of a correlation matrix R:

$$R = \begin{bmatrix} 1 & \rho_{SH} & \rho_{SL} \\ \rho_{SH} & 1 & \rho_{HL} \\ \rho_{SL} & \rho_{HL} & 1 \end{bmatrix}$$

can be easily calculated to be equal to Q:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ \rho_{SH} & \sqrt{1 - \rho_{SH}^2} & 0 \\ \rho_{SL} & \frac{\rho_{HL} - \rho_{SH}\rho_{SL}}{\sqrt{1 - \rho_{SH}^2}} & \sqrt{1 - \rho_{SL}^2 - (\frac{\rho_{HL} - \rho_{SH}\rho_{SL}}{\sqrt{1 - \rho_{SH}^2}})^2} \end{bmatrix}$$

Appendix B. Parameter Calibrations

Appendix B.1. Stock returns

We use log differences on our monthly stock price data to obtain historical monthly rates of return. The Augmented Dickey-Fuller test strongly shows that the modified series is stationary at 1% significance level.
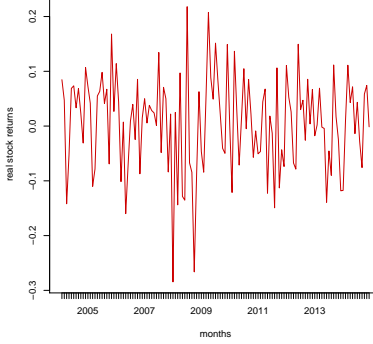

Figure B.13: Historical monthly rates of return on stocks

We use Akaike Information Criterion to find the optimal lag order for ARIMA estimation — $p = q = 2$.

Performing ARIMA(2,0,2) estimation, we obtain the long-term forecasted monthly rate of return $\mu_{mon}^S = 0.541\%$ and volatility $\sigma_{mon}^S = 11.1\%$.

Finally, we annualize these values as follows:

$$\mu_S = (1 + \mu_{mon}^S)^{12} - 1 = 6.69\% \tag{B.1}$$

$$\sigma_S = \sigma_{mon}^S \cdot \sqrt{12} = 38.44\% \tag{B.2}$$

Appendix B.2. Housing returns

Similarly, we log-differentiate monthly house prices to obtain growth rates. The house market collapse of 2008 brings large external shock, causing the Augmented Dickey-Fuller test to only find stationarity at 10% significance level for 3 lags at most.

Again, using Akaike Information Criterion, we find $p = q = 1$.

ARIMA(1,0,1) estimation provides long-term forecasts for monthly rate of return and volatility as $\mu_{mon}^H = 0.06\%$ and $\sigma_{mon}^H = 1.57\%$.

Annualization gives:

$$\mu_H = (1 + \mu_{mon}^H)^{12} - 1 = 0.67\% \tag{B.3}$$

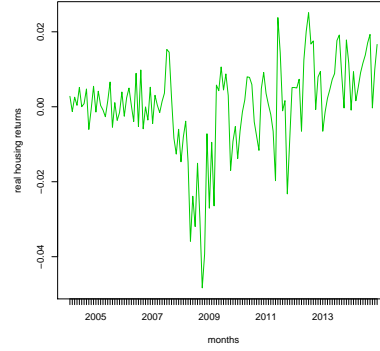$$\sigma_H = \sigma_{mon}^H \cdot \sqrt{12} = 5.42\% \tag{B.4}$$

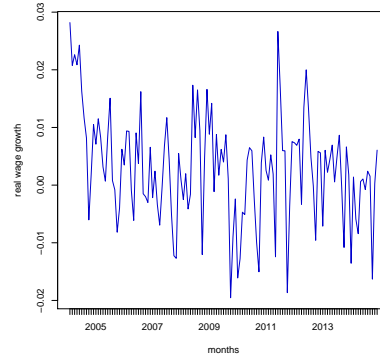Figure B.14: Historical monthly rates of return on housing

Figure B.15: Historical monthly wage growth rates

Appendix B.3. Labor income volatility

The Augmented Dickey-Fuller test returns stationarity for up to 9 lags at 5% significance level. Akaike Information Criterion suggests $p = 5$ and $q = 2$.

ARIMA(5,0,2) estimation gives monthly volatility $\sigma_{mon}^{L} = 1.04\%$, which annualizes as follows:

$$\sigma_L = \sigma_{mon}^{L} \cdot \sqrt{12} = 3.59\% \tag{B.5}$$

Appendix B.4. Wage regression results

The regressions of wage growth rates by age with kinks at 40 and 55, return the following coefficients.

Table B.9: Wage regression results by age

	flat	moderate	steep
(intercept)	0.1%	0.4%	1.5%
d40	-0.7%	3.3%	0.7%
d55	1.4%	0.9%	-0.4%

The growth rates can be calculated using these coefficients. Note that we have rounded the growth rates for the flat wages to 0. Figure B.16 illustrates how the estimated rates fit the data:
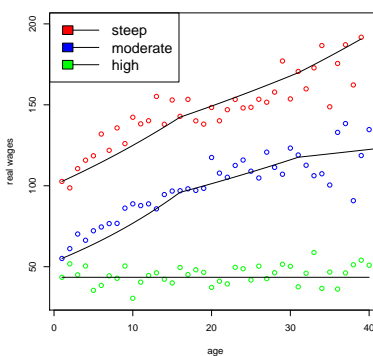

Figure B.16: Fitted values from wage regressions