



Stochastic prey–predator model with additional food for predator

Amartya Das, G.P. Samanta *

Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah 711103, India



HIGHLIGHTS

- A predator–prey model with additional food has been considered.
- Prey's growth rate and the predator's death rate have been perturbed with white noises.
- Existence and uniqueness of global solution of the system have been established.
- Conditions under which extinction of predator and prey populations occur have been established.
- Conditions under which persistence of the system occurs have been established.
- Numerical simulations have been done to support the analytical results.

ARTICLE INFO

Article history:

Received 29 January 2018

Received in revised form 29 March 2018

Available online 9 August 2018

Keywords:

Additional food

Itô formula

Global solution

Persistence

Extinction

ABSTRACT

In this work we have studied a predator–prey model where the prey grows logistically in the absence of predator and the functional response of predator towards prey and additional food that are derived in the text. Prey's growth rate and the predator's death rate have been perturbed with Gaussian white noises which has been proved extremely useful to model rapidly fluctuating phenomena. These two parameters are the main terms subject to coupling of a prey–predator pair with its environment Dimentberg (1988). Existence and uniqueness of global positive solution of the system have been established under environmental noise. Then the conditions under which extinction of predator and prey populations occur have been established. In our analysis, it is found that the environmental noise plays an important role in extinction as well as persistence of prey and predator populations. We have also discussed about the persistence of the system under obtained conditions and how the solution of the underlying system is globally attractive in mean. To derive the theorems we have shown the uniform continuous behavior of the solutions. Although we have considered a prey–predator model, the survival of predator population is possible in absence of prey population, since the additional food is provided to predator. But it is found that the extinction of prey population drive predator population to extinction. Our analytical findings are explained through numerical simulation which show the reliability of our model from the ecological point of view. It is shown in numerical simulation that if the effectual food level of additional food which is provided to the predator is high, then the predator dominates the prey population.

© 2018 Elsevier B.V. All rights reserved.

* Corresponding author.

E-mail address: gpsamanta@math.iiests.ac.in (G.P. Samanta).

1. Introduction

Conservation of ecological and biological living system is a serious concern from a long period of time. Many ecologists, biologists, agriculturists and mathematicians [1–5] have researched on the topic over the years intensely. There are a lot of issues for research and analyze on conservation of living system. Some of the mathematicians have focused there research on the effect of additional food in many ways. Additional food may be provided either to prey or to predator or to both. Providing additional food to predator and the corresponding effects on the prey-predator dynamics and its importance in biological control have been interesting topics to the biologists and mathematicians [6–13]. These studies fall into two categories.

In the first category, the goal of biological conservation is reached through additional food supplement feeding. In this situation, additional food is provided to the predator so that its dependency on the prey for food decreases. Additional food distracts the predator and helps in serving two different purposes. First, the predator gets properly supplemented by this food. Secondly, the predation rate scales down. However, people should be circumspect about the nutritive values of the additional food supplement for over nutritious food can result in the opposite. The attack rate and the fecundity of the predator can increase leading to a heavy pressure rate on the prey. Some researches and field studies also show that diversionary food may not always help in the preservation of the prey. The following paragraph is a demonstration of this discussion.

In a research during the spring and summer of 1998 and 1999, some hen harriers on Langholm moor, south Scotland, were provided with additional food. This was an attempt to reduce their preying rate on grouse. Harriers provided with additional food carried grouse chicks at a lower rate (0.5 chicks/100 h) than harriers which were not provided with additional food (3.7 chicks/100 h) which meant that the former harriers became almost dependent on the additional food and no longer relied on grouse chicks. From this study, we can conclude that additional food can save a good number of grouse chicks from falling prey to hen harriers [9]. In another research [14], alfalfa, canola oil, and bark mulch with wax as additional food were supplied to Voles of the genus *Microtus* and *Clethrionomys*, mammalian pests which feed on bark, tissue and roots of plants in coniferous and deciduous tree plantations in North America, particularly in winter when there is less food for them. This has brought forth optimistic results and has directly provided positive benefits to sustainable forest management.

The implementations of the second category of the studies are observed in the field of agricultural entomology, pest control and optimization of loss due to infestations by insects. Chemical control is one of the favorite means of pest control. In this method, insecticides and pesticides are used to control the pests but it is scientifically proven that they play key roles in the deterioration of health of human beings and of the environment. So, eco friendly bio control programs are in demand these days.

The term 'bio control' implies to the method of pest control [15] using their (pests) 'natural enemies', i.e., parasites, pathogens, parasitoids, predators and competitors that kill pests or injure them severely. Sometimes, in bio control, the pest plays the role of the prey whereas the natural enemies play the role of the predator. Additional food is supplied to the predators (natural enemies) so that they can enhance their longevity and fecundity and control the preys (pests).

In some research works [16–18] it is found that the parameters of 'Quality' and 'Quantity' of additional food plays an important role for persistence in dynamical system. In these studies it is observed that if high quality additional food is provided to the predator then the predation rate increases. Whereas the low quality of additional food to the predator may increase target prey. In these works to present the prey-predator system, modified versions of Rosenzweig MacArthur's models [19] have been developed. In these studies [20–22] the experimental results accord with the conclusion.

Researchers [23,24] took refuge of Gaussian white-noise as a model of environmental variations to evaluate the effect of environmental noise on dynamical systems. May [24] pointed out that continuous fluctuation in the environment can lead to random fluctuation in the birth rates, death rates, carrying capacity, competition coefficients and all other parameters involved in a dynamical model, to a great lesser extent. Ripa et al. [25] presented a general theory of environmental noise in ecological food webs examining the effect of environmental noise on populations. The importance of the noise amplitude, the trophic level and the susceptibility of populations to environmental noise were shown by Upadhyay et al. [26] after his investigation into the influence of environmental noise on a fairly realistic ecological model with generalist top predator. The effect of delay on the rates of noise-induced switching between co-existing stable states and noise-induced extinction in a population dynamic model were studied by Schwartz et al. [27].

This paper is split section wise as follows. In the next section, we have constructed the functional response in presence of additional food and the prey-predator dynamical model from the idea of Holling. In Section 3, we have discussed the existence and uniqueness of the system. In Section 4, the conditions for extinction of prey and predator are discussed. Section 5 is divided into two parts. In the first part, the conditions for persistence of the system is discussed. And in the second part, we have discussed the conditions for which the system is globally attractive in mean. Section 6 illustrates some results through numerical simulations followed by discussion and conclusion in Section 7.

2. Construction of the model

We shall discuss the effect of the presence of additional food of biomass A for predator. We assume that the encounter rate of per predator with the additional food is proportional to the density of additional food. We also assume that both additional food and prey are consumed by the predator according to their availability. Following the ideas of Holling [28] and on the basis of time utilization, we derive a formal derivation of functional response of predator in the presence of additional food.

Let x, y denote the density of prey and predator respectively. Let the predator spend γ_t period of time for searching prey and/or additional food, or for consuming the captured prey and/or additional food. It is assumed that γ_t is small enough in the sense that the prey and predator biomass (densities) remain roughly constant over the time interval γ_t .

Let the predator spend γ_{ts} period of time for searching prey and/or additional food, γ_{tx} period of time for getting the prey delivered and γ_{tA} period of time for the additional food delivered. So we get,

$$\gamma_t = \gamma_{ts} + \gamma_{tx} + \gamma_{tA} \quad (1)$$

γ_{tx} and γ_{tA} depend on the rate at which the predator encounters prey and additional food respectively and the time for each interaction.

It is assumed that during the time the predator spends in searching, each individual predator encounters prey and additional food according to laws of mass action, i.e. at the rates proportional to the prey and additional food densities respectively. So, the predator encounters $e_x x \gamma_{ts}$ prey and $e_A A \gamma_{ts}$ additional food in the time interval γ_t , where e_x and e_A are rate constants that would depend on factors such as the predator's movement rate while searching or its ability to detect prey and/or additional food. If h_x is the time for each interaction between a predator and a prey and h_A is the time for each interaction between a predator and additional food, then

$$\gamma_{tx} = e_x h_x x \gamma_{ts} \quad (2)$$

and

$$\gamma_{tA} = e_A h_A A \gamma_{ts} \quad (3)$$

Now from (1)–(3) we have,

$$\gamma_t = (1 + e_x h_x x + e_A h_A A) \gamma_{ts} \quad (4)$$

Now, as the predator encounters $e_x x \gamma_{ts}$ prey during the period γ_t , so the overall rate of encounters with prey on the time interval γ_t is given by

$$\begin{aligned} g(x, y, A) &= \frac{e_x x \gamma_{ts}}{\gamma_t} = \frac{e_x x \gamma_{ts}}{(1 + e_x h_x x + e_A h_A A) \gamma_{ts}} \\ &= \frac{e_x x}{1 + e_x h_x x + e_A h_A A} \end{aligned} \quad (5)$$

Similarly, the overall rate of encounters with additional food over γ_t is as follows:

$$h(x, y, A) = \frac{e_A A \gamma_{ts}}{\gamma_t} = \frac{e_A A}{1 + e_x h_x x + e_A h_A A} \quad (6)$$

The expressions $g(x, y, A)$ and $h(x, y, A)$ can be written as follows:

$$g(x, y, A) = \frac{cx}{d + x + \beta \kappa A} \quad (7)$$

$$h(x, y, A) = \frac{c\kappa A}{d + x + \beta \kappa A}, \quad (8)$$

where $c = \frac{1}{h_x}$ is the maximum rate of predation, $d = \frac{1}{h_x e_x}$, is the half-saturation value of the predators, $\beta = \frac{h_A}{h_x}$ and $\kappa = \frac{e_A}{e_x}$. Here κA corresponds the *effectual food level* and β represents the *quality of additional food with respect to the prey*.

The expressions $g(x, y, A)$ and $h(x, y, A)$ represent the functional response of predators towards prey and additional food respectively.

We consider Rosenzweig–MacArthur prey–predator model where the prey grows logistically in the absence of predators and the predator functional response on prey population is of Holling type-II. Incorporating the expressions of predator functional response on prey population and additional food that are derived in (7) and (8), we obtain the following coupled differential equations representing prey–predator dynamics (when the predator is provided with additional food):

$$\begin{aligned} \frac{dx}{dt} &= rx \left(1 - \frac{x}{K}\right) - \frac{cxy}{d + x + \beta \kappa A} \\ \frac{dy}{dt} &= \xi \left(\frac{cx}{d + x + \beta \kappa A} + \frac{c\kappa A}{d + x + \beta \kappa A} \right) y - ly, \end{aligned}$$

where ξ ($0 < \xi < 1$) is the conversion factor. If $b = \xi c$, then we have

$$\begin{aligned} \frac{dx}{dt} &= rx \left(1 - \frac{x}{K}\right) - \frac{cxy}{d + x + \beta \kappa A} \\ \frac{dy}{dt} &= \frac{b(x + \kappa A)y}{d + x + \beta \kappa A} - ly, \end{aligned} \quad (9)$$

with initial value $x(0) = x_0$, $y(0) = y_0$, where $x(t)$ and $y(t)$ represent the biomass of prey and predator respectively. Here A represents the additional food biomass which is distributed uniformly throughout the habitat where the species live. r is the growth rate of prey and K is the carrying capacity of the prey. l is the death rate of the predator, c is the maximum rate of predation, d is the half-saturation value of the predators (in absence of additional food), β represents the quality of additional food with respect to the prey and κA represents the effectual food level.

The term $\frac{cx}{d+x+\beta\kappa A}$ represents the predators functional response. We have considered all the parameters as positive constants.

Extinction is one of the most important term in population dynamics. A species is said to be extinct if there is no existing member in the habitat, i.e. there are no surviving individuals that can reproduce and create a new generation. A species extinct due to various reasons: (i) humans can cause extinction of a species through over-harvesting, (ii) pollution, (iii) habitat destruction, (iv) introduction of invasive species (such as new predators and food competitors), (v) over-hunting, and (vi) other influences. Also there are so many causes for extinctions of species and group of species such as toxic pollution of entire population habitat or lack of food and space in the habitat, competition for food to better adapted competitors, predation, etc. In ecology there are two types of extinctions: one is global extinction and the other is local extinction. In local extinction the species do not exist in the chosen area of study, but may exist in other places. Local extinctions may be followed by a replacement of the species taken from other locations.

Since extinction is a very important topic in population dynamics, our goal in this article is to investigate extinctions in the underlying model system. Deterministic models in ecology do not usually incorporate environmental fluctuation; they are often justified by the implicit assumption that in large populations, stochastic deviations are small enough to be ignored. Deterministic model will prove ecologically useful only if the dynamical patterns they reveal are still in evidence when stochastic effects are introduced. Uncertain growth of populations is usually considered as an effect of environmental stochasticity. Reproduction of species depends on various factors, such as temperature, humidity, parasites and pathogens, environmental pollution etc. Since physical and biological environments of populations are not totally predictable, the growth of populations should be considered as a stochastic process rather than a deterministic one. Since our goal is to investigate the extinction so death rate of predator is an important phenomena. Thus, we perturb the prey's growth rate r and the predator's death rate l in model (9) with Gaussian white noises which has been proved extremely useful to model rapidly fluctuating phenomena. These two parameters are the main terms subject to coupling of a prey-predator pair with its environment [29]. Therefore, model system (9) by introducing the environmental stochasticity in the form of Gaussian white noise is represented by:

$$\begin{aligned} \frac{dx}{dt} &= (r + \gamma_1(t))x - \theta x^2 - \frac{cxy}{d + x + \beta\kappa A} \\ \frac{dy}{dt} &= \frac{b(x + \kappa A)y}{d + x + \beta\kappa A} - (l + \gamma_2(t))y \\ \text{i.e. } \frac{dx}{dt} &= rx - \theta x^2 - \frac{cxy}{d + x + \beta\kappa A} + \gamma_1(t)x \\ \frac{dy}{dt} &= \frac{b(x + \kappa A)y}{d + x + \beta\kappa A} - ly - \gamma_2(t)y. \end{aligned}$$

i.e.

$$\begin{aligned} \frac{dx}{dt} &= rx - \theta x^2 - \frac{cxy}{d + x + \beta\kappa A} + \sigma_1 x \frac{dw_1}{dt} \\ \frac{dy}{dt} &= \frac{b(x + \kappa A)y}{d + x + \beta\kappa A} - ly - \sigma_2 y \frac{dw_2}{dt}, \end{aligned} \quad (10)$$

Hence,

$$\begin{aligned} dx &= x \left(r - \theta x - \frac{cy}{d + x + \beta\kappa A} \right) dt + \sigma_1 x dw_1 \\ dy &= y \left(\frac{b(x + \kappa A)}{d + x + \beta\kappa A} - l \right) dt - \sigma_2 y dw_2, \end{aligned} \quad (11)$$

where $\theta = \frac{r}{K}$, $\gamma_1 = \sigma_1 \frac{dw_1}{dt}$, and $\gamma_2 = \sigma_2 \frac{dw_2}{dt}$, the perturbed terms $\gamma_1(t)$ and $\gamma_2(t)$ are assumed to be the independent Gaussian white noises satisfying the conditions:

$$\langle \gamma_j(t) \rangle = 0 \text{ and } \langle \gamma_j(t_1) \gamma_j(t_2) \rangle = \sigma_j^2 \delta(t_1 - t_2) \text{ for } j = 1, 2.$$

Here $\sigma_j > 0$ ($j = 1, 2$) are the intensities or strengths of the random perturbations, δ is the Dirac delta function defined by

$$\begin{cases} \delta(x) = 0, & \text{for } x \neq 0, \\ \int_{-\infty}^{\infty} \delta(x) dx = 1, \end{cases}$$

and $\langle \cdot \rangle$ represents the ensemble average of the underlying stochastic process.

Here $w = \{w_1, w_2, t \geq 0\}$ represents two-dimensional standard (unit) Brownian motion defined on a complete probability space (Ω, \mathcal{F}, P) with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (it is right continuous and increasing, while \mathcal{F}_0 contains all P -null sets).

3. Existence and uniqueness

Since $x(t)$ and $y(t)$ represent the size of the prey and predator population at time t , respectively, we are interested only in positive solutions of system (11). For a system of stochastic differential equations, if the coefficients of the equations satisfy the linear growth condition and local Lipschitz condition then it has a unique global solution. The coefficients of system (11) satisfy local Lipschitz condition but do not satisfy linear growth condition, so the solution of system (11) may explode in a finite time.

In this section it is shown that the system (11) has unique positive global solution with positive initial value by using change of variables and comparison theorem of stochastic differential equations.

Theorem 3.1. For any initial value $(x_0, y_0) \in \mathbb{R}_+^2$, system (11) has unique global positive solution $(x(t), y(t))$ for $t \geq 0$.

Proof. Let us first consider the local solution of the system (11) as $(x(t), y(t))$ for $t \in [0, \tau_e)$ where τ_e represents the explosion time. Now we consider $u(t) = \ln x(t)$, and $v(t) = \ln y(t)$ and apply Itô formula [30] to transform the system (11) as follows:

$$\begin{aligned} d(\ln x) &= \left[\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} x \left(r - \theta x - \frac{cy}{d+x+\beta\kappa A} \right) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} (\sigma_1 x)^2 \right] dt + \frac{\partial u}{\partial x} \sigma_1 x dw_1 \\ &= \left[0 + \frac{1}{x} x \left(r - \theta x - \frac{cy}{d+x+\beta\kappa A} \right) - \frac{1}{2} \frac{1}{x^2} \sigma_1^2 x^2 \right] dt + \frac{1}{x} \sigma_1 x dw_1 \\ &= \left(r - \theta x - \frac{cy}{d+x+\beta\kappa A} - \frac{\sigma_1^2}{2} \right) dt + \sigma_1 dw_1 \end{aligned}$$

i.e.

$$du = \left(r - \theta e^u - \frac{ce^v}{d+e^u+\beta\kappa A} - \frac{\sigma_1^2}{2} \right) dt + \sigma_1 dw_1.$$

Similarly we have

$$dv = \left(\frac{b(e^u + \kappa A)}{d+e^u+\beta\kappa A} - l - \frac{\sigma_2^2}{2} \right) dt - \sigma_2 dw_2.$$

Hence the transformed system becomes

$$\begin{aligned} du &= \left(r - \theta e^u - \frac{ce^v}{d+x+\beta\kappa A} - \frac{\sigma_1^2}{2} \right) dt + \sigma_1 dw_1 \\ dv &= \left(\frac{b(e^u + \kappa A)}{d+e^u+\beta\kappa A} - l - \frac{\sigma_2^2}{2} \right) dt - \sigma_2 dw_2 \end{aligned} \quad (12)$$

with initial values $u(0) = \ln x_0$ and $v(0) = \ln y_0$. Now it is easy to observe that the coefficients of the system (12) satisfy local Lipschitz condition, so the system (12) has unique local solution $(u(t), v(t))$ for $t \in [0, \tau_e)$. So by using Itô formula we can say that $(x(t), y(t)) = (e^{u(t)}, e^{v(t)})$ is the unique local positive solution of the system (11) with initial value (x_0, y_0) for $t \in [0, \tau_e)$.

Now by showing $\tau_e = \infty$ we will show that the solution is global.

Since the solution is positive on $[0, \tau_e)$, so from (11) we get

$$dx \leq rx dt + \sigma_1 x dw_1$$

Let $\Phi(t)$ be the unique solution of the equation:

$$\begin{aligned} d\Phi(t) &= r\Phi(t)dt + \sigma_1\Phi(t)dw_1 \\ \Phi(0) &= x_0 \end{aligned}$$

Let $F_1 = \frac{1}{\Phi(t)}$. Now by applying Itô formula, we have

$$\begin{aligned} dF_1 &= \left[-\frac{r\Phi(t)}{\Phi^2(t)} + \frac{\sigma_1^2\Phi^2(t)}{\Phi^3(t)} \right] dt - \frac{\sigma_1\Phi(t)}{\Phi^2(t)} dw_1 \\ &= \left[-\frac{r}{\Phi(t)} + \frac{\sigma_1^2}{\Phi(t)} \right] dt - \frac{\sigma_1}{\Phi(t)} dw_1 \\ &= (-r + \sigma_1^2) F_1 dt - \sigma_1 F_1 dw_1 \end{aligned}$$

$\therefore dF_1 = (-r + \sigma_1^2) F_1 dt - \sigma_1 F_1 dw_1$, with $F_1(0) = \frac{1}{x_0}$. Now by the numerical solution of stochastic differential equation, we get

$$F_1(t) = \frac{1}{x_0} e^{(-r + \sigma_1^2)t - \sigma_1 w_1}$$

i.e.

$$\Phi(t) = x_0 e^{(r - \sigma_1^2)t + \sigma_1 w_1} \quad (13)$$

$$\therefore x(t) \leq \Phi(t). \quad (14)$$

From the second equation of system (11):

$$\begin{aligned} dy &= y \left(\frac{b(x + \kappa A)}{d + x + \beta \kappa A} - l \right) dt - \sigma_2(t) y dw_2 \\ &\leq \left(b + \frac{b}{\beta} - l \right) y dt - \sigma_2 y dw_2 \end{aligned}$$

Let $\Psi(t)$ be the unique solution of the equation:

$$\begin{aligned} d\Psi(t) &= \left(b + \frac{b}{\beta} - l \right) \Psi(t) dt - \sigma_2 \Psi(t) dw_2 \\ \Psi(0) &= y_0 \end{aligned}$$

Let $F_2 = \frac{1}{\Psi(t)}$. Using Itô formula:

$$\begin{aligned} dF_2 &= \left[-\frac{(b + \frac{b}{\beta} - l)\Psi(t)}{\Psi^2(t)} + \frac{\sigma_2^2 \Psi^2(t)}{\Psi^3(t)} \right] dt + \frac{\sigma_2 \Psi(t)}{\Psi^2(t)} dw_2 \\ &= \left[-\frac{(b + \frac{b}{\beta} - l)}{\Psi(t)} + \frac{\sigma_2^2}{\Psi(t)} \right] dt + \frac{\sigma_2}{\Psi(t)} dw_1 \\ &= \left(-b - \frac{b}{\beta} + l + \sigma_2^2 \right) F_2 dt + \sigma_2 F_2 dw_2 \end{aligned}$$

$\therefore dF_2 = \left(-b - \frac{b}{\beta} + l + \sigma_2^2 \right) F_2 dt + \sigma_2 F_2 dw_2$, with $F_2(0) = \frac{1}{y_0}$. Using numerical solution of stochastic differential equation:

$$F_2(t) = \frac{1}{y_0} e^{(-b - \frac{b}{\beta} + l + \sigma_2^2)t + \sigma_2 w_2}$$

$$\Psi(t) = y_0 e^{(b + \frac{b}{\beta} - l - \sigma_2^2)t - \sigma_2 w_2} \quad (15)$$

$$\therefore y(t) \leq \Psi(t). \quad (16)$$

Also, for the predator population:

$$dy \geq -ly(t)dt - \sigma_2 y(t)dw_2.$$

Let $\psi(t)$ be the unique solution of the equation

$$\begin{aligned} d\psi(t) &= -l\psi(t)dt - \sigma_2 \psi(t)dw_2 \\ \psi(0) &= y_0 \end{aligned}$$

Calculating similarly as the previous one, we get

$$\psi(t) = y_0 e^{-(l + \sigma_2^2)t - \sigma_2 w_2} \quad (17)$$

$$\therefore y(t) \geq \psi(t). \quad (18)$$

Now from comparison theorem for stochastic differential equations, we have from (16) and (18):

$$\psi(t) \leq y(t) \leq \Psi(t) \text{ a.s. for } t \in [0, \tau_e]. \quad (19)$$

On the other hand, for the prey population $x(t)$:

$$dx \geq x(r - \theta x - c\psi(t)) dt + \sigma_1(t)x dw_1$$

Let $\phi(t)$ be the unique solution of the equation:

$$d\phi(t) = \phi(t)(r - \theta\phi(t) - c\psi(t)) dt + \sigma_1(t)\phi(t)dw_1$$

$$\text{with } \phi(0) = x_0$$

Applying Itô formula, we get

$$\phi(t) = \frac{e^{(r-\sigma_1^2)t - c \int_0^t \psi(r)dr + \sigma_1 w_1}}{\frac{1}{x_0} + \theta \int_0^t e^{(r-\sigma_1^2)z - c \int_0^z \psi(r)dr + \sigma_1 w_1} dz} \quad (20)$$

$$\therefore x(t) \geq \phi(t). \quad (21)$$

Finally, by the comparison theorem of stochastic differential equations we have from (14) and (21) for prey population:

$$\phi(t) \leq x(t) \leq \Phi(t) \text{ a.s. for } t \in [0, \tau_e]. \quad (22)$$

Since $\phi(t)$, $\psi(t)$, $\Phi(t)$ and $\Psi(t)$ exist for all $t \geq 0$, it can be concluded that $\tau_e = \infty$, which proves the theorem. \square

4. Extinction

Let us first define local extinction of a population.

Definition 4.1. The population $x(t)$ is said to be extinct with probability one if

$$\lim_{t \rightarrow \infty} x(t) = 0 \text{ a.s.}$$

Now we shall investigate the conditions for extinction of prey population, then predator population.

Theorem 4.1. Let $(x(t), y(t))$ be the solution of system (11) for any initial value $(x_0, y_0) \in \mathbb{R}_+^2$. Then the prey and predator population become extinct exponentially with probability one, i.e.

$$\lim_{t \rightarrow \infty} x(t) = 0, \text{ and } \lim_{t \rightarrow \infty} y(t) = 0 \text{ a.s.,}$$

$$\text{if } r < \frac{\sigma_1^2}{2} \text{ and } \frac{bkA}{d+\beta\kappa A} < l + \frac{\sigma_2^2}{2}.$$

Proof. From the first equation of (12), we get

$$du \leq \left(r - \theta e^u - \frac{\sigma_1^2}{2} \right) dt + \sigma_1 dw_1$$

i.e.

$$d(\ln x) \leq \left(r - \theta x - \frac{\sigma_1^2}{2} \right) dt + \sigma_1 dw_1 \quad [\because u = \ln x]$$

Now we consider $f(x) = r - \theta x - \frac{\sigma_1^2}{2}$. To find the maximum value of $f(x)$ we derive

$$f'(x) = -\theta < 0$$

So f is a decreasing function on $[0, \infty)$ and its maximum value is $f(0) = r - \frac{\sigma_1^2}{2}$. Therefore,

$$d(\ln x) \leq \left(r - \frac{\sigma_1^2}{2} \right) dt + \sigma_1 dw_1,$$

i.e.

$$\ln x(t) \leq \ln x_0 + \left(r - \frac{\sigma_1^2}{2} \right) t + \sigma_1 w_1$$

Applying the strong law of large numbers for local martingales, we get

$$\limsup_{t \rightarrow \infty} \frac{\ln x(t)}{t} \leq \limsup_{t \rightarrow \infty} \left[\frac{\ln x_0}{t} + \frac{\left(r - \frac{\sigma_1^2}{2} \right) t}{t} + \frac{\sigma_1 w_1}{t} \right] = r - \frac{\sigma_1^2}{2} < 0, \quad \left(\because r < \frac{\sigma_1^2}{2} \right)$$

i.e.

$$\lim_{t \rightarrow \infty} x(t) = 0 \quad \text{a.s.}$$

Therefore, for every $\epsilon > 0$, no matter however small, there exist t_0 and set Ω_ϵ such that $P(\Omega_\epsilon) \geq 1 - \epsilon$ and $\frac{b(x+\kappa A)}{d+x+\beta\kappa A} < \epsilon'$, for every $t \geq t_0$ and $x \in \Omega_\epsilon$, where $\epsilon' = \epsilon + \frac{b\kappa A}{d+\beta\kappa A}$. Using that fact from the second equation of (12), we get

$$d(\ln y(t)) = \left(\frac{b(x(t) + \kappa A)}{d + x(t) + \beta\kappa A} - l - \frac{\sigma_2^2}{2} \right) dt - \sigma_2 dw_2(t),$$

i.e.

$$\int_0^t d(\ln y(s)) = \int_0^t \left[\left(\frac{b(x(s) + \kappa A)}{d + x(s) + \beta\kappa A} - l - \frac{\sigma_2^2}{2} \right) \right] ds - \int_0^t \sigma_2 dw_2(s)$$

$$\text{i.e.} \quad \ln y(t) \leq \ln y_0 + \int_0^t \left(\epsilon' - l - \frac{\sigma_2^2}{2} \right) ds - \sigma_2 w_2(t)$$

$$\text{i.e.} \quad \ln y(t) \leq \ln y_0 + \left(\epsilon' - l - \frac{\sigma_2^2}{2} \right) t - \sigma_2 w_2(t)$$

Using the strong law of large numbers for local martingales, we obtain

$$\limsup_{t \rightarrow \infty} \frac{\ln y(t)}{t} \leq \limsup_{t \rightarrow \infty} \left[\frac{\ln y_0}{t} + \left(\epsilon' - l - \frac{\sigma_2^2}{2} \right) - \frac{\sigma_2 w_2(t)}{t} \right] = \epsilon + \frac{b\kappa A}{d + \beta\kappa A} - l - \frac{\sigma_2^2}{2} < 0,$$

since $\epsilon > 0$ is arbitrarily small, which means that

$$\lim_{t \rightarrow \infty} y(t) = 0 \quad \text{a.s.}$$

Hence the theorem is proved. \square

From the previous theorem it can be concluded that the large intensity noise can be a cause for extinction of the prey population, which means that unpredictable events in nature can be fatal for prey population. Extinction of prey population will drive predator population to extinct, too.

Now we will find the effect of the intensity of noise on predator population. In the next theorem it will be proved that if the intensity of the noise is sufficiently large, then predator population becomes extinct regardless of the size of prey population.

Theorem 4.2. Let $(x(t), y(t))$ be the solution of system (11) for any initial value $(x_0, y_0) \in \mathbb{R}_+^2$.

If $\sigma_2^2 > b + \frac{b}{\beta} - l$, then

$$\lim_{t \rightarrow \infty} y(t) = 0 \quad \text{a.s.},$$

i.e. population $y(t)$ becomes extinct exponentially with probability one.

Proof. From (15) and (16) of Theorem 3.1, we have

$$y(t) \leq y_0 e^{\left(b + \frac{b}{\beta} - l - \sigma_2^2\right)t - \sigma_2 w_2}$$

for every $t \geq 0$. Hence

$$\begin{aligned} \ln y(t) &\leq \ln y_0 + \left(b + \frac{b}{\beta} - l - \sigma_2^2\right)t - \sigma_2 w_2 \\ \text{i.e.} \quad \frac{\ln y(t)}{t} &\leq \frac{\ln y_0}{t} + \left(b + \frac{b}{\beta} - l - \sigma_2^2\right) - \frac{\sigma_2 w_2}{t} \end{aligned}$$

Now by applying the strong law of large numbers for local martingales, we get

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\ln y(t)}{t} &\leq \limsup_{t \rightarrow \infty} \left[\frac{\ln y_0}{t} + \left(b + \frac{b}{\beta} - l - \sigma_2^2\right) - \frac{\sigma_2 w_2}{t} \right] \\ \therefore \limsup_{t \rightarrow \infty} \frac{\ln y(t)}{t} &\leq b + \frac{b}{\beta} - l - \sigma_2^2 < 0, \quad [\because \sigma_2^2 > b + \frac{b}{\beta} - l] \end{aligned}$$

which leads to

$$\lim_{t \rightarrow \infty} y(t) = 0 \quad \text{a.s.},$$

so the theorem is proved. \square

5. Persistence and Global attractivity

In this section it will be shown that the system (11) is persistence in the mean and the solutions of (11) are globally attractive in mean. In order to show that let us first prove the following theorem:

Theorem 5.1. Let $(x(t), y(t))$ be the solution of system (11) for any initial value $(x_0, y_0) \in \mathbb{R}_+^2$. Then for $p \geq 1$:

$$Ex^p(t) \leq T(p),$$

$$\text{where } T = \max \left\{ p \left(\frac{p}{\theta} \right)^p \left(\frac{\frac{1}{p} + r + \frac{p-1}{2} \sigma_1^2}{p+1} \right)^{p+1}, x_0^p \right\}.$$

If the conditions

$$b + \frac{b}{\beta} < l \quad \text{and} \quad b + \frac{b}{\beta} + \frac{p-2}{2} \sigma_2^2 \leq l \quad (23)$$

hold, then $Ey^p(t) \leq y_0^p$.

Proof. To prove the theorem let us define a function $V(x, t) = e^t x^p$ for $p \geq 1$. Now from (11), we get

$$dx = x \left(r - \theta x - \frac{cy}{d+x+\beta\kappa A} \right) dt + \sigma_1(t) x dw_1$$

Applying Itô formula:

$$\begin{aligned} d(V(x, t)) &= \left(e^t x^p + p e^t x^{p-1} x \left[r - \theta x - \frac{cy}{d+x+\beta\kappa A} \right] + \frac{p(p-1)e^t x^{p-2} \sigma_1^2 x^2}{2} \right) dt \\ &\quad + p e^t x^{p-1} \sigma_1 x dw_1(t) \\ &= p e^t x^p \left(\frac{1}{p} + r - \theta x - \frac{cy}{d+x+\beta\kappa A} + \frac{p-1}{2} \sigma_1^2 \right) dt + p e^t x^p \sigma_1 dw_1(t) \end{aligned}$$

From the previous equation:

$$\begin{aligned} E(V(x, t)) &= x_0^p + p \int_0^t e^s E \left[x^p \left(\frac{1}{p} + r - \theta x - \frac{cy}{d+x+\beta\kappa A} + \frac{p-1}{2} \sigma_1^2 \right) \right] ds \\ &\leq x_0^p + p \int_0^t e^s E \left[x^p \left(\frac{1}{p} + r - \theta x + \frac{p-1}{2} \sigma_1^2 \right) \right] ds \end{aligned}$$

Let us take $f(x) = x^p \left(\frac{1}{p} + r - \theta x + \frac{p-1}{2} \sigma_1^2 \right)$. To find the maximum value we calculate $f'(x)$ and

$$\begin{aligned} f'(x) &= p x^{p-1} \left(\frac{1}{p} + r - \theta x + \frac{p-1}{2} \sigma_1^2 \right) - \theta x^p \\ &= x^{p-1} \left\{ p \left(\frac{1}{p} + r - \theta x + \frac{p-1}{2} \sigma_1^2 \right) - \theta x \right\} \end{aligned}$$

$$\therefore f'(x) = 0$$

$$\Rightarrow p \left(\frac{1}{p} + r - \theta x + \frac{p-1}{2} \sigma_1^2 \right) = \theta x$$

$$\Rightarrow (1+p)\theta x = p \left(\frac{1}{p} + r + \frac{p-1}{2} \sigma_1^2 \right)$$

$$\Rightarrow x = \frac{p \left(\frac{1}{p} + r + \frac{p-1}{2} \sigma_1^2 \right)}{(1+p)\theta}$$

So the maximum value of $f(x)$ occurs at $x = \frac{p\left(\frac{1}{p} + r + \frac{p-1}{2}\sigma_1^2\right)}{(1+p)\theta}$.

$$\begin{aligned}\therefore f_{\max} &= \left(\frac{p}{\theta}\right)^p \frac{\left(\frac{1}{p} + r + \frac{p-1}{2}\sigma_1^2\right)^p}{(1+p)^p} \left\{ \frac{1}{p} + r - \frac{p\left(\frac{1}{p} + r + \frac{p-1}{2}\sigma_1^2\right)}{1+p} + \frac{p-1}{2}\sigma_1^2 \right\} \\ &= \left(\frac{p}{\theta}\right)^p \left(\frac{\frac{1}{p} + r + \frac{p-1}{2}\sigma_1^2}{p+1}\right)^p \left\{ \frac{1}{p} + r + \frac{p-1}{2}\sigma_1^2 - \frac{p}{1+p} \left(\frac{1}{p} + r + \frac{p-1}{2}\sigma_1^2\right) \right\} \\ &= \left(\frac{p}{\theta}\right)^p \left(\frac{\frac{1}{p} + r + \frac{p-1}{2}\sigma_1^2}{p+1}\right)^{p+1}\end{aligned}$$

So

$$\begin{aligned}E(e^t x^p(t)) &\leq x_0^p + p \int_0^t e^s \left(\frac{p}{\theta}\right)^p \left(\frac{\frac{1}{p} + r + \frac{p-1}{2}\sigma_1^2}{p+1}\right)^{p+1} ds \\ &= x_0^p + p \left(\frac{p}{\theta}\right)^p \left(\frac{\frac{1}{p} + r + \frac{p-1}{2}\sigma_1^2}{p+1}\right)^{p+1} (e^t - 1) \\ E(x^p(t)) &= \left(x_0^p - p \left(\frac{p}{\theta}\right)^p \left(\frac{\frac{1}{p} + r + \frac{p-1}{2}\sigma_1^2}{p+1}\right)^{p+1} \right) e^{-t} + p \left(\frac{p}{\theta}\right)^p \left(\frac{\frac{1}{p} + r + \frac{p-1}{2}\sigma_1^2}{p+1}\right)^{p+1}\end{aligned}$$

Now we can observe that for $t = 0$, $Ex^p(t) \leq x_0^p$.

As $t \rightarrow \infty$, we get $E(x^p(t)) \leq p \left(\frac{p}{\theta}\right)^p \left(\frac{\frac{1}{p} + r + \frac{p-1}{2}\sigma_1^2}{p+1}\right)^{p+1}$.

Take $T = \max \left\{ p \left(\frac{p}{\theta}\right)^p \left(\frac{\frac{1}{p} + r + \frac{p-1}{2}\sigma_1^2}{p+1}\right)^{p+1}, x_0^p \right\}$.

Hence finally, we get

$E(x^p(t)) \leq T(p)$, for $p \geq 1$.

Again to prove the second part we recall (16) and get

$$y(t) \leq \Psi(t)$$

Now from (15), we have

$$\begin{aligned}y(t) &\leq y_0 e^{\left(b + \frac{b}{\beta} - l - \sigma_2^2\right)t - \sigma_2 w_2} \\ \Rightarrow y^p(t) &\leq y_0^p e^{p\left(b + \frac{b}{\beta} - l - \sigma_2^2\right)t - p\sigma_2 w_2} \\ \Rightarrow Ey^p(t) &\leq y_0^p E \left[e^{p\left(b + \frac{b}{\beta} - l - \sigma_2^2\right)t - p\sigma_2 w_2} \right] \\ &= y_0^p e^{p\left(b + \frac{b}{\beta} - l - \sigma_2^2\right)t + \frac{p^2}{2}\sigma_2^2 t} \\ &= y_0^p e^{p\left(b + \frac{b}{\beta} - l - \sigma_2^2 + \frac{p}{2}\sigma_2^2\right)t} \\ &= y_0^p e^{p\left(b + \frac{b}{\beta} - l + \frac{p-2}{2}\sigma_2^2\right)t}\end{aligned}$$

From the conditions (23): $p\left(b + \frac{b}{\beta} - l + \frac{p-2}{2}\sigma_2^2\right)t \leq 0$, for $t \geq 0$.

Hence we get $Ey^p(t) \leq y_0^p$.

Which proves the theorem. \square

Theorem 5.2. Let $(x(t), y(t))$ be the solution of system (11) for any initial value $(x_0, y_0) \in \mathbb{R}_+^2$. Then almost every sample path of $(x(t), y(t))$ is uniformly continuous on $t \geq 0$, if

$$b + \frac{b}{\beta} < l \quad \text{and} \quad b + \frac{p-2}{2}\sigma_2^2 \leq l$$

hold.

Proof. Let us rewrite the first equation of (11) in its integral form:

$$x(t) = x_0 + \int_0^t h_1(s, x(s), y(s)) ds + \int_0^t h_2(s, x(s), y(s)) ds,$$

where

$$h_1(s, x(s), y(s)) = x(s) \left[r - \theta x(s) - \frac{cy(s)}{d + \beta \kappa A + x(s)} \right],$$

$$h_2(s, x(s), y(s)) = \sigma_1 x(s).$$

From the stated conditions, it can be concluded that Theorem 5.1 holds good. So, we have

$$\begin{aligned} E |h_1(s, x(s), y(s))|^p &\leq \frac{1}{2} E |x(s)|^{2p} + \frac{1}{2} E \left| r - \theta x(s) - \frac{cy(s)}{d + \beta \kappa A + x(s)} \right|^{2p} \\ &\leq \frac{1}{2} E x^{2p} + \frac{1}{2} E \left| r - \theta x(s) - \frac{cy(s)}{d + \beta \kappa A + x(s)} \right|^{2p} \\ &\leq \frac{1}{2} E x^{2p} + \frac{4^{2p-1}}{2} [r^{2p} + \theta^{2p} E x^{2p} + c^{2p} E y^{2p}] \\ &= \left[\frac{1}{2} + \frac{4^{2p-1}}{2} \theta^{2p} \right] E x^{2p} + \frac{4^{2p-1}}{2} r^{2p} + \frac{4^{2p-1}}{2} c^{2p} y_0^{2p} \\ &\leq \left[\frac{1}{2} + \frac{4^{2p-1}}{2} \theta^{2p} \right] T(2p) + \frac{4^{2p-1}}{2} r^{2p} + \frac{4^{2p-1}}{2} c^{2p} y_0^{2p} := G_1(p) \end{aligned}$$

We also have

$$E |h_2(s, x(s), y(s))|^p = E |\sigma_1^p x^p| \leq \sigma_1^p T(p) := G_2(p).$$

Now from the well-known moment inequality of Itô for $0 \leq t_1 < t_2 < \infty$ and for $p > 2$, we have

$$E \left| \int_{t_1}^{t_2} h_2(s, x(s), y(s)) dw_1(s) \right|^p \leq \left[\frac{p(p-1)}{2} \right]^{\frac{p}{2}} (t_2 - t_1)^{\frac{p-2}{2}} \int_{t_1}^{t_2} E |h_2(s, x(s), y(s))|^p ds. \quad (24)$$

Using Hölder inequality, for $t_2 - t_1 \leq 1$, we compute

$$\begin{aligned} E |x(t_2) - x(t_1)|^p &\leq 2^{p-1} E \left[\int_{t_1}^{t_2} |h_1(s, x(s), y(s))| ds \right]^p + 2^{p-1} E \left[\int_{t_1}^{t_2} |h_2(s, x(s), y(s))| dw_1(s) \right]^p \\ &\leq 2^{p-1} (t_2 - t_1)^{p-1} E \int_{t_1}^{t_2} G_1(p) ds + 2^{p-1} \left[\frac{p(p-1)}{2} \right]^{\frac{p}{2}} (t_2 - t_1)^{\frac{p-2}{2}} \int_{t_1}^{t_2} G_2(p) ds \\ &= 2^{p-1} (t_2 - t_1)^p G_1(p) + 2^{p-1} \left[\frac{p(p-1)}{2} \right]^{\frac{p}{2}} (t_2 - t_1)^{\frac{p}{2}} G_2(p) \\ &\leq 2^{p-1} (t_2 - t_1)^{\frac{p}{2}} \left[1 + \left(\frac{p(p-1)}{2} \right)^{\frac{p}{2}} \right] [G_1(p) + G_2(p)] \end{aligned}$$

which implies that almost every sample path of $x(t)$ is locally but uniformly Hölder-continuous with exponent $\gamma \in \left(0, \frac{p-2}{2p}\right)$, and therefore almost every sample path of $x(t)$ is uniformly continuous on $t \geq 0$.

Similarly, we prove that almost every sample path of $y(t)$ is uniformly continuous on $t \geq 0$, and this completes the proof of the theorem. \square

Theorem 5.3. Let $(x(t), y(t))$ be the solution of system (11) for any initial value $(x_0, y_0) \in \mathbb{R}_+^2$. For $p > 1$: if the conditions $b + \frac{b}{\beta} < l$, $b + \frac{b}{\beta} + \frac{p-2}{2} \sigma_2^2 \leq l$ and $r + \frac{p-1}{2} \sigma_1^2 \geq cy_0$ hold, then

$$E(x^p) \geq p \left(\frac{p}{\theta} \right)^p \left(\frac{r - cy_0 + \frac{p-1}{2} \sigma_1^2}{p+1} \right)^{p+1}.$$

This inequality also hold for $p = 1$, i.e.

$$E(x) \geq \frac{(r - cy_0)^2}{4\theta}.$$

Proof. To prove the theorem let us define a function $V(x, t) = e^t x^p$ for $p \geq 1$. Now from (11), we get

$$dx = x \left(r - \theta x - \frac{cy}{d + x + \beta \kappa A} \right) dt + \sigma_1(t) x dw_1$$

Applying Itô formula, we get

$$\begin{aligned} d(V(x, t)) &= \left(e^t x^p + p e^t x^{p-1} x \left[r - \theta x - \frac{cy}{d + x + \beta \kappa A} \right] + \frac{p(p-1)e^t x^{p-2} \sigma_1^2 x^2}{2} \right) dt \\ &\quad + p e^t x^{p-1} \sigma_1 x dw_1(t) \\ &= p e^t x^p \left(\frac{1}{p} + r - \theta x - \frac{cy}{d + x + \beta \kappa A} + \frac{p-1}{2} \sigma_1^2 \right) dt + p e^t x^p \sigma_1 dw_1(t) \end{aligned}$$

From the previous equation and using Theorem 5.1, we get

$$\begin{aligned} E(V(x, t)) &= x_0^p + p \int_0^t e^s E \left[x^p \left(\frac{1}{p} + r - \theta x - \frac{cy}{d + x + \beta \kappa A} + \frac{p-1}{2} \sigma_1^2 \right) \right] ds \\ &\geq x_0^p + p \int_0^t e^s E \left[x^p \left(r - \theta x - cy + \frac{p-1}{2} \sigma_1^2 \right) \right] ds \end{aligned} \quad (25)$$

Using Theorem 5.1:

$$x^p \left(r - \theta x - cy + \frac{p-1}{2} \sigma_1^2 \right) \geq x^p \left(r - \theta x - cy_0 + \frac{p-1}{2} \sigma_1^2 \right) \quad (26)$$

Take $f(x) = x^p \left(r - \theta x - cy_0 + \frac{p-1}{2} \sigma_1^2 \right)$. To find the supremum value of $f(x)$, calculate $f'(x)$:

$$\begin{aligned} f'(x) &= p x^{p-1} \left(r - \theta x - cy_0 + \frac{p-1}{2} \sigma_1^2 \right) - \theta x^p \\ &= x^{p-1} \left\{ p \left(r - \theta x - cy_0 + \frac{p-1}{2} \sigma_1^2 \right) - \theta x \right\} \end{aligned}$$

$$\therefore f'(x) = 0$$

$$\Rightarrow p \left(r - \theta x - cy_0 + \frac{p-1}{2} \sigma_1^2 \right) = \theta x$$

$$\Rightarrow (1+p)\theta x = p \left(r - cy_0 + \frac{p-1}{2} \sigma_1^2 \right)$$

$$\Rightarrow x = \frac{p \left(r - cy_0 + \frac{p-1}{2} \sigma_1^2 \right)}{(1+p)\theta} := L \text{ (say)}$$

Hence

$$\begin{aligned} \sup_{(x \geq 0)} f(x) &= \left(\frac{p \left(r - cy_0 + \frac{p-1}{2} \sigma_1^2 \right)}{(p+1)\theta} \right)^p \left(r - cy_0 + \frac{p-1}{2} \sigma_1^2 - \frac{p}{p+1} \left[r - cy_0 + \frac{p-1}{2} \sigma_1^2 \right] \right) \\ &= \left(\frac{p}{\theta} \right)^p \frac{\left(r - cy_0 + \frac{p-1}{2} \sigma_1^2 \right)^{p+1}}{(p+1)^{p+1}}. \end{aligned}$$

Now applying supremum property on (26), we get

$$x^p \left(r - \theta x - cy + \frac{p-1}{2} \sigma_1^2 \right) \geq \left(\frac{p}{\theta} \right)^p \frac{\left(r - cy_0 + \frac{p-1}{2} \sigma_1^2 \right)^{p+1}}{(p+1)^{p+1}}.$$

Hence from (25):

$$\begin{aligned}
 E(V(x, t)) &\geq x_0^p + p \int_0^t e^s E \left[x^p \left(r - \theta x - cy + \frac{p-1}{2} \sigma_1^2 \right) \right] ds \\
 &\geq x_0^p + p \int_0^t e^s \left(\frac{p}{\theta} \right)^p \frac{(r - cy_0 + \frac{p-1}{2} \sigma_1^2)^{p+1}}{(p+1)^{p+1}} ds \\
 &= x_0^p + p \left(\frac{p}{\theta} \right)^p \frac{(r - cy_0 + \frac{p-1}{2} \sigma_1^2)^{p+1}}{(p+1)^{p+1}} (e^t - 1) \\
 \therefore E(x^p(t)e^t) &\geq x_0^p + p \left(\frac{p}{\theta} \right)^p \frac{(r - cy_0 + \frac{p-1}{2} \sigma_1^2)^{p+1}}{(p+1)^{p+1}} (e^t - 1) \\
 \Rightarrow E(x^p(t)) &\geq \left[x_0^p - p \left(\frac{p}{\theta} \right)^p \frac{(r - cy_0 + \frac{p-1}{2} \sigma_1^2)^{p+1}}{(p+1)^{p+1}} \right] e^{-t} + p \left(\frac{p}{\theta} \right)^p \frac{(r - cy_0 + \frac{p-1}{2} \sigma_1^2)^{p+1}}{(p+1)^{p+1}}
 \end{aligned}$$

$$\text{As } t \rightarrow \infty: E(x^p(t)) \geq p \left(\frac{p}{\theta} \right)^p \left(\frac{r - cy_0 + \frac{p-1}{2} \sigma_1^2}{(p+1)} \right)^{p+1}.$$

This completes the proof.

To prove the theorem for $p = 1$, let us take $V(x, t) = xe^t$. Then Eq. (25) becomes

$$E(V(x, t)) \geq x_0 + \int_0^t e^s E[x(r - \theta x - cy)] ds$$

Now we take $f(x) = x(r - \theta x - cy_0)$, for that $f'(x) = r - 2\theta x - cy_0$.

So the extreme point is $x = \frac{r - cy_0}{2\theta}$ and $f''(x) = -2\theta < 0$.

Therefore, the supremum value of $f(x)$ occurs at this point and proceeding as previous, we get

$$E(x) \geq \frac{(r - cy_0)^2}{4\theta}. \quad \square$$

5.1. Discussion on the persistence

Here, we will try to give the critical value which determines the persistence of the stochastic prey-predator model (11) with hyperbolic mortality.

Let us first define persistence in the mean of a dynamical system:

Definition 5.1. The dynamical system (11) is said to be persistence in the mean if $\liminf_{t \rightarrow \infty} \langle y \rangle_t > 0$, where $\langle y \rangle_t = \frac{1}{t} \int_0^t y(r) dr$.

Lemma 5.1. Suppose $Z(t) \in \mathbb{C}(\Omega \times [0, \infty), \mathbb{R}_+)$.

(I) If there are two positive constants T and δ_0 such that

$$\ln Z(t) \leq \delta t - \delta_0 \int_0^t Z(s) ds + \sum_1^n \alpha_i W(t) \quad \text{a.s. } \forall t \geq T,$$

where $\alpha_i, \delta, \delta_0$ are constants, then

$$\begin{cases} \limsup_{t \rightarrow \infty} \langle Z \rangle_t \leq \frac{\delta}{\delta_0}, & \text{a.s. if } \delta > 0, \\ \lim_{t \rightarrow \infty} \langle Z \rangle_t = 0, & \text{a.s. if } \delta < 0. \end{cases}$$

(II) If there exists three positive constants T, δ, δ_0 such that

$$\ln Z(t) \geq \delta t - \delta_0 \int_0^t Z(s) ds + \sum_1^n \alpha_i W(t) \quad \text{a.s. } \forall t \geq T,$$

where $\alpha_i, \delta, \delta_0$ are constants, then

$$\liminf_{t \rightarrow \infty} \langle Z \rangle_t \geq \frac{\delta}{\delta_0} \quad \text{a.s.}$$

Lemma 5.2. Let us consider the following one dimensional stochastic system:

$$dz(t) = z(t)(r - \theta z(t))dt + \sigma_1 z(t)dw_1(t), \quad \text{with } z(0) = x(0). \quad (27)$$

(I) If $r - \frac{\sigma_1^2}{2} < 0$, then $\lim_{t \rightarrow \infty} z(t) = 0$ a.s.

(II) If $r - \frac{\sigma_1^2}{2} > 0$, then

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t z(s) ds = r - \frac{\sigma_1^2}{2}.$$

Theorem 5.4. Let $(x(t), y(t))$ be the solution of system (11) for any initial value $(x_0, y_0) \in \mathbb{R}_+^2$. We assume that $\left(r - \frac{\sigma_1^2}{2}\right) > 0$. Then the system (11) will be persistence if $l + \frac{\sigma_2^2}{2} < \frac{b(r - cy_0)^2}{4\theta(d + \beta\kappa A)}$.

Proof. Applying Itô formula on the first equation of (11) with $V(x, t) = \ln x$, we get

$$\begin{aligned} d(\ln x(t)) &= \left(r - \theta x - \frac{cy}{d + \beta\kappa A + x} - \frac{\sigma_1^2}{2}\right) dt + \sigma_1 dw_1 \\ &= \left(r - \frac{\sigma_1^2}{2} - \theta x - \frac{cy}{d + \beta\kappa A + x}\right) dt + \sigma_1 dw_1 \end{aligned}$$

Using stochastic integration process and dividing both sides by t , we get

$$\frac{\ln x(t) - \ln x(0)}{t} = \left(r - \frac{\sigma_1^2}{2}\right) - \frac{\theta}{t} \int_0^t x(r) dr - \frac{c}{t} \int_0^t \frac{y(r)}{d + \beta\kappa A + x(r)} dr + \frac{M_1(t)}{t}. \quad (28)$$

Now we take $F(z, t) = \ln z$ and apply Itô formula (using (27)):

$$\begin{aligned} dF &= \left[\left(r - \frac{\sigma_1^2}{2}\right) - \theta z(t)\right] dt + \sigma_1 dw_1 \\ d(\ln z) &= \left[\left(r - \frac{\sigma_1^2}{2}\right) - \theta z(t)\right] dt + \sigma_1 dw_1. \\ \therefore \frac{\ln z(t) - \ln z(0)}{t} &= \left(r - \frac{\sigma_1^2}{2}\right) - \frac{\theta}{t} \int_0^t z(t) dt + \frac{M_1(t)}{t}, \end{aligned} \quad (29)$$

where $M_1(t) = \int_0^t \sigma_1 dw_1(s)$ is a real-valued continuous local martingale. By strong law of large numbers, we have $\lim_{t \rightarrow \infty} \frac{M_1(t)}{t} = 0$ a.s.

From (28) and (29), we get

$$\begin{aligned} 0 &\geq \frac{\ln x(t) - \ln z(t)}{t} \\ &= -\frac{\theta}{t} \int_0^t (x(r) - z(r)) dr - \frac{c}{t} \int_0^t \frac{y(r)}{d + \beta\kappa A + x(r)} dr. \end{aligned}$$

It implies that

$$\begin{aligned} \frac{1}{t} \int_0^t (z(r) - x(r)) dr &\leq \frac{c}{t\theta} \int_0^t \frac{y(r)}{d + \beta\kappa A + x(r)} dr \leq \frac{c}{t\theta(d + \beta\kappa A)} \int_0^t y(r) dr \\ \text{and } x(t) &\leq z(t). \end{aligned} \quad (30)$$

Let us apply Itô formula on $V(y, t) = \ln y$ with the help of the second equation of (11) and Theorem 5.3:

$$\begin{aligned} d(\ln y(t)) &= \left(\frac{b(x + \kappa A)}{d + \beta\kappa A + x} - l - \frac{\sigma_2^2}{2}\right) dt - \sigma_2 dw_2 \\ &\geq \left[-l - \frac{\sigma_2^2}{2} + \frac{bz}{d + \beta\kappa A + z} - \left(\frac{bz}{d + \beta\kappa A + z} - \frac{bx}{d + \beta\kappa A + z}\right)\right] dt - \sigma_2 dw_2 \\ &= \left[-l - \frac{\sigma_2^2}{2} + \frac{bz}{d + \beta\kappa A + z} - \frac{b(d + \beta\kappa A)(z(t) - x(t))}{(d + \beta\kappa A + z)(d + \beta\kappa A + x)}\right] dt - \sigma_2 dw_2 \\ &\geq \left[-l - \frac{\sigma_2^2}{2} + \frac{bx}{d + \beta\kappa A} - b(d + \beta\kappa A)(z(t) - x(t))\right] dt - \sigma_2 dw_2 \quad [\because x(t) \leq z(t).] \\ &\geq \left[-l - \frac{\sigma_2^2}{2} + \frac{b(r - cy_0)^2}{4\theta(d + \beta\kappa A)} - b(d + \beta\kappa A)(z(t) - x(t))\right] dt - \sigma_2 dw_2. \end{aligned}$$

Applying stochastic integration process and dividing both sides by t and using (30), we get

$$\frac{\ln y(t) - \ln y(0)}{t} \geq \left(-l - \frac{\sigma_2^2}{2} + \frac{b(r - cy_0)^2}{4\theta(d + \beta\kappa A)} \right) - \frac{bc}{t\theta} \int_0^t y(r)dr - \frac{M_2(t)}{t}, \quad (31)$$

for sufficiently large t , where $M_2(t) = \int_0^t \sigma_2 dw_2(s)$ is a real-valued continuous local martingale. By strong law of large numbers, we have $\lim_{t \rightarrow \infty} \frac{M_2(t)}{t} = 0$ a.s.

Applying (II) of Lemma 5.1 on Eq. (31) and using the given condition, we get

$$\liminf_{t \rightarrow \infty} \langle y \rangle_t \geq \frac{\theta \left(-l - \frac{\sigma_2^2}{2} + \frac{b(r - cy_0)^2}{4\theta(d + \beta\kappa A)} \right)}{bc} > 0.$$

So the system (11) will be persistence in the mean if $\liminf_{t \rightarrow \infty} \langle y \rangle_t > 0$,

$$\text{i.e. if } \frac{\theta \left(-l - \frac{\sigma_2^2}{2} + \frac{b(r - cy_0)^2}{4\theta(d + \beta\kappa A)} \right)}{bc} > 0, \text{ i.e. if } l + \frac{\sigma_2^2}{2} < \frac{b(r - cy_0)^2}{4\theta(d + \beta\kappa A)}.$$

Hence the theorem is proved. \square

5.2. Globally attractive in mean

To show the solution of the system (11) is globally attractive in mean, let us first state the definition of it.

Definition 5.2. Let $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ be two arbitrary solutions of the system (11) with initial values $(x_1^0, y_1^0) \in \mathbb{R}_+^2$ and $(x_2^0, y_2^0) \in \mathbb{R}_+^2$, respectively. Then the system (11) is called globally attractive in mean if

$$\lim_{t \rightarrow \infty} [E|x_1(t) - x_2(t)| + E|y_1(t) - y_2(t)|] = 0, \quad \text{a.s.}$$

To show the global attractivity of the solution, we use the following Lemma:

Lemma 5.3. Let $f : [0, \infty) \rightarrow [0, \infty)$ be an integrable and uniformly continuous function. Then

$$\lim_{t \rightarrow \infty} f(t) = 0.$$

Theorem 5.5. If the parameters of the system (11) satisfy the following conditions:

1. $b + \frac{b}{\beta} < l$,
2. $b + \frac{b}{\beta} + \frac{p-2}{2}\sigma_2^2 \leq l$,
3. $r + \frac{p-1}{2}\sigma_1^2 \geq cy_0$,
4. $r < \frac{(r - cy_0)^2}{2}$,
5. $2b + b\kappa A < l$,

then the solution of the system (11) is globally attractive in mean, i.e.

$$\lim_{t \rightarrow \infty} [E|x_1(t) - x_2(t)| + E|y_1(t) - y_2(t)|] = 0, \quad \text{a.s.}$$

Proof. Let us define $F(x, t) = |x_1(t) - x_2(t)| + k|y_1(t) - y_2(t)|$, where k is a constant that can be chosen as requirement. Now using Itô formula, we derive

$$\begin{aligned} LF(t) &= \text{sgn}(x_1(t) - x_2(t)) \left[rx_1 - \theta x_1^2 - \frac{cx_1y_1}{d + \beta\kappa A + x_1} - rx_2 + \theta x_2^2 + \frac{cx_2y_2}{d + \beta\kappa A + x_2} \right] dt \\ &\quad + \text{sgn}(y_1(t) - y_2(t)) \left[\frac{b(x_1 + \kappa A)y_1}{d + \beta\kappa A + x_1} - ly_1 - \frac{b(x_2 + \kappa A)y_2}{d + \beta\kappa A + x_2} + ly_2 \right] dt \\ &= \left\{ \left[r - \theta(x_1(t) + x_2(t)) - \frac{(c - kb)(d + \beta\kappa A)y_1 + kb\kappa Ay_2}{(d + \beta\kappa A + x_1)(d + \beta\kappa A + x_2)} \right] |x_1(t) - x_2(t)| \right\} dt \\ &\quad + \left\{ \left[-kl + \frac{(d + \beta\kappa A + x_1)x_2(c + kb) + kb\kappa Ax_2}{(d + \beta\kappa A + x_1)(d + \beta\kappa A + x_2)} \right] |y_1(t) - y_2(t)| \right\} dt \end{aligned}$$

For $k = \frac{c}{b}$, we get

$$\begin{aligned} LF(t) &= \{[r - \theta(x_1(t) + x_2(t))] |x_1(t) - x_2(t)|\} dt \\ &\quad + \left\{ \left[-l\frac{c}{b} + \frac{(d + \beta\kappa A + x_1)x_22c + kb\kappa Ax_2}{(d + \beta\kappa A + x_1)(d + \beta\kappa A + x_2)} \right] |y_1(t) - y_2(t)| \right\} dt \\ &\leq \{[r - \theta(x_1(t) + x_2(t))] |x_1(t) - x_2(t)| + \left[-l\frac{c}{b} + 2c + c\kappa A \right] |y_1(t) - y_2(t)|\} dt \end{aligned}$$

Now taking expectation on both sides and using [Theorem 5.3](#), we get

$$\begin{aligned} E(LF(t)) &\leq E\left\{\left(r - \frac{(r - cy_0)^2}{2}\right)|x_1(t) - x_2(t)| + \left(-l\frac{c}{b} + 2c + c\kappa A\right)|y_1(t) - y_2(t)|\right\} dt \\ \therefore E(F(t)) &\leq E(F(0)) + \int_0^t \left(r - \frac{(r - cy_0)^2}{2}\right) E|x_1(s) - x_2(s)| ds \\ &\quad + \int_0^t \left(-l\frac{c}{b} + 2c + c\kappa A\right) E|y_1(s) - y_2(s)| ds \\ \Rightarrow E(F(t)) + D_1 \int_0^t E|x_1(s) - x_2(s)| ds + D_2 \int_0^t E|y_1(s) - y_2(s)| ds &\leq E(F(0)) < \infty, \end{aligned} \quad (32)$$

where $D_1 = \frac{(r - cy_0)^2}{2} - r$ and $D_2 = c\left(\frac{l}{b} - 2 - \kappa A\right)$ are positive constants by the last two conditions of the theorem.

Now from [Eq. \(32\)](#) we can conclude that

$$\int_0^t E|x_1(s) - x_2(s)| ds < \infty \text{ and } \int_0^t E|y_1(s) - y_2(s)| ds < \infty,$$

$$\text{i.e. } E|x_1(t) - x_2(t)| \in L^1[0, \infty) \text{ and } E|y_1(t) - y_2(t)| \in L^1[0, \infty),$$

which mean that $E|x_1(t) - x_2(t)|$ and $E|y_1(t) - y_2(t)|$ are integrable and from [Theorem 5.2](#), we know that $E|x_1(t) - x_2(t)|$ and $E|y_1(t) - y_2(t)|$ are uniformly continuous. So by using [Lemma 5.3](#), we can conclude that

$$\lim_{t \rightarrow \infty} E|x_1(t) - x_2(t)| = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} E|y_1(t) - y_2(t)| = 0,$$

i.e.

$$\lim_{t \rightarrow \infty} [E|x_1(t) - x_2(t)| + E|y_1(t) - y_2(t)|] = 0, \quad \text{a.s.}$$

Which proves the theorem. \square

6. Numerical

This section records our attempt to validate mathematical results obtained in the previous sections. Approximated sample paths of the stochastic model acquired from direct simulation should be in accordance with the actual $It\hat{o}$ process. These ideas are linked with the concept of a strong solution for a system of stochastic differential equations ([11](#)).

We choose the following values of the parameters:

$$r = 2, \theta = 0.514, c = 1.4, d = 1, \beta = 0.5, \kappa = 0.02, A = 1, b = 1, l = 0.5. \quad (33)$$

At the outset of our numerical simulation, we deal with noise intensities $\sigma_1 = 0.005$, $\sigma_2 = 0.005$ and start from the initial point (4, 2). [Fig. 1](#) shows the result of one simulation run, which is the effect of noise on prey population. After some initial transients, the population densities vary around the deterministic steady state value $x = 2.2$.

We can also observe the effect of white noise on predator population in presence of prey with the following values of the parameters:

$$r = 2, \theta = 1, c = 1.4, d = 1, \beta = 0.5, \kappa = 0.02, A = 1, b = 1, l = 0.5. \quad (34)$$

After some initial transients, the population densities vary around the deterministic steady state value $y = 1.5$. Here the prey population exists and the interspecific competition among them is quite high. But the noise is same as when we get the effect of white noise on prey population, i.e. $\sigma_1 = 0.005$, $\sigma_2 = 0.005$ and we start it from the initial point (4, 2). The effect of noise on predator population is depicted in [Fig. 2](#).

In these set of values of the parameters the prey and the predator do not extinct. But if we decrease the difference between the parameters b and l as $b = 0.17446$, $l = 0.5233$ and $\sigma_2 = 0.05$, $\sigma_1 = 0.05$ for satisfying the condition $\sigma_2^2 > b + \frac{b}{\beta} - l$ in [Theorem 4.2](#) we can see in [Fig. 3](#) that the predator population will extinct about $t = 40$.

Now we decrease the birth rate of prey population and assume $r = 0.001$ and $\sigma_1 = 0.05$ so that the condition of [Theorem 4.1](#): $r < \frac{\sigma_1^2}{2}$ and $\frac{b\kappa A}{d + \beta\kappa A} < l + \frac{\sigma_2^2}{2}$ satisfies and to get a clear plot we decrease the value of c as $c = 0.1$ keeping all other parameters same. What we get from here is that both prey and predator populations extinct. This extinction is shown in [Fig. 4](#) where we see that the prey population extinct about $t = 4$ and the predator population extinct about $t = 10$.

Now we see that for set of values $r = 2, \theta = 1, c = 1.8, d = 1, \beta = 0.5, \kappa = 0.02, A = 1, b = 0.95, l = 0.5, \sigma_1 = 0.005, \sigma_2 = 0.005$ which satisfy the condition of [Theorem 5.4](#): $l + \frac{\sigma_2^2}{2} < \frac{b(r - cy_0)^2}{4\theta(d + \beta\kappa A)}$, so the underlying dynamical system persists with initial values (4, 2). It can be seen in [Fig. 5](#).

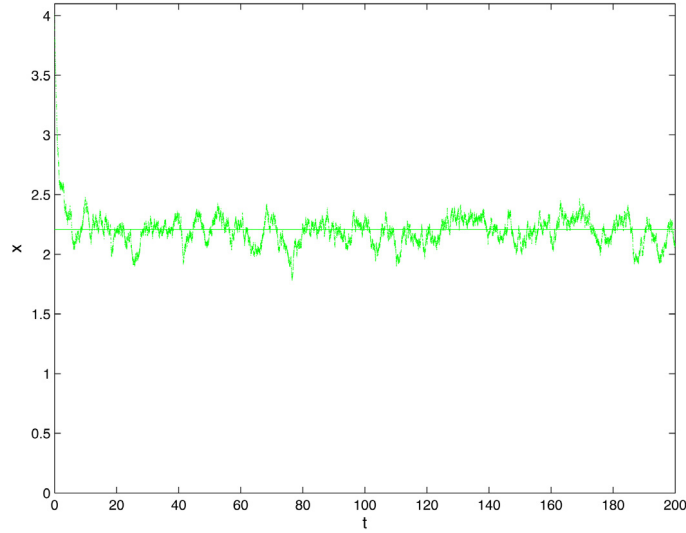


Fig. 1. Trajectory of prey population as described by (11) with parameters presented in (33).

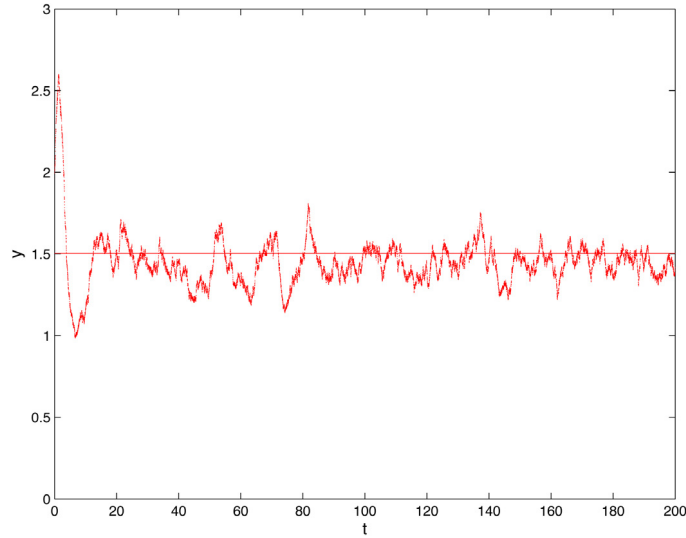


Fig. 2. Trajectory of predator population as described by (11) with parameters presented in (34).

7. Conclusion

In this paper we have considered Rosenzweig–MacArthur predator–prey model where the prey grows logistically in the absence of predator and the functional response of predator towards prey and additional food that are derived in (7) and (8). We have also introduced noise perturbation on the birth rate of prey and on the death rate of predator. Considering the environmental noise we have established the existence and uniqueness of global positive solution of the system. We have shown the effect of environmental fluctuations on extinction for both prey and predator populations and obtained some conditions. It is obtained that if the birthrate of prey population become very low and less than the effect (intensity) of noise on it then the prey extinct and because of interaction between prey and predator population the extinction of prey population will drive predator population to extinction too. We have also obtained that if the difference between maximum rate of predation and death rate of predator is less than the square of the intensity of noise on the death rate of predator then the predator population extinct. In our analytical results we have found that the environmental noise plays an important role in extinction as well as persistence of prey and predator populations. We have verified the obtained analytical results with supportive numerical simulations. We have also discussed about the persistence of the system under obtained conditions

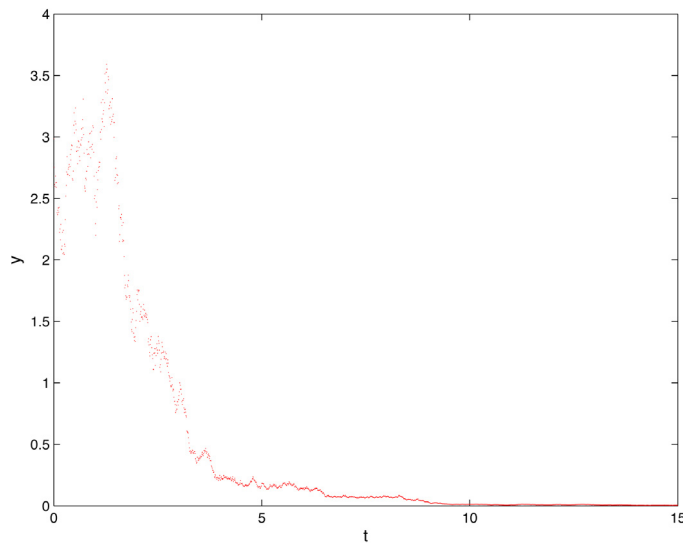


Fig. 3. Predator population goes to extinction.

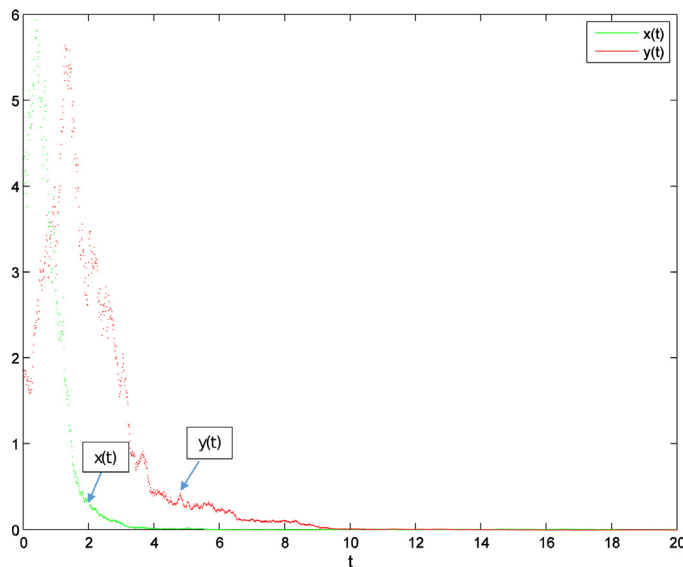


Fig. 4. The prey population extincts about $t = 4$ and the predator population extincts about $t = 10$.

and how the solution of the underlying system is globally attractive in mean. To derive the theorems we have shown the uniform continuous behaviour of the solutions.

Although we have considered a prey-predator model, the survival of predator population is possible in absence of prey population, since the additional food is provided to predator. But we have found that the extinction of prey population drive predator population to extinction and numerically in Fig. 6 it is found that in absence of prey the predator population extinct.

We have found in numerical simulation that if the effectual food level of additional food which is provided to the predator is high, then the predator dominates the prey population. Here in our model κA represents effectual food level and if we increase the value of κ as $\kappa = .5$ keeping all other parameter same as in Fig. 5 we get the following figure (Fig. 7).

Finally we like to mention that we have taken noise on the birth rate of prey and on the death rate of predator but one can think about the consequences where other parameters, like environmental carrying capacity, rate of capture of prey by predators, half saturation constant are also stochastically varying. We must confess here that the technique which is used here is not applicable to the case when the noise terms appear in the denominator. With the best of our knowledge this is a limitation of the technique discussed here. An interesting problem is to induce bounded noise [31,32] on all existing parameters and we shall address in near future.

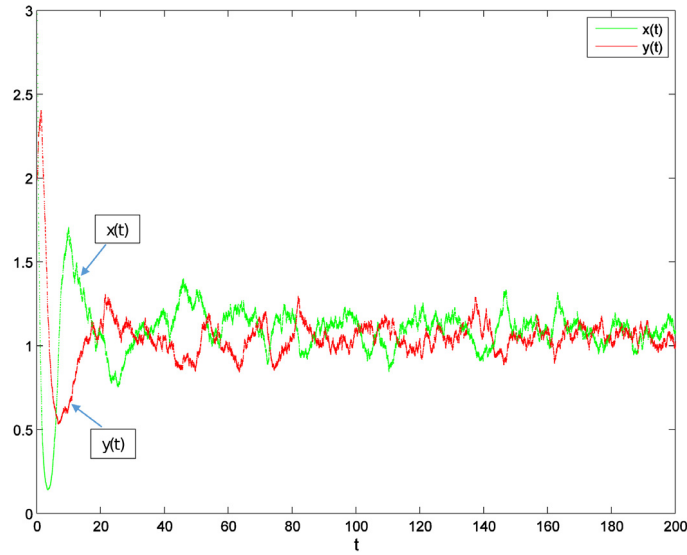


Fig. 5. The system (11) persists.

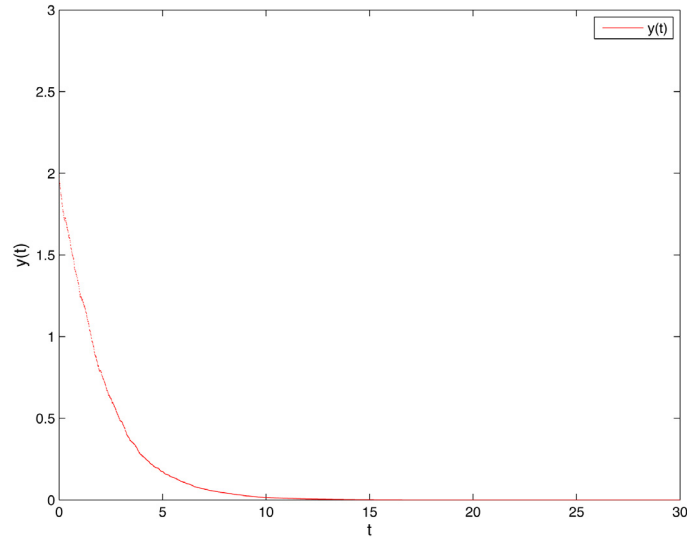


Fig. 6. The predator population extincts in absence of prey.

In this context, if the additional food parameter “A” is perturbed by Gaussian white noise, then system (10) becomes

$$\begin{aligned} \frac{dx}{dt} &= rx - \theta x^2 - \frac{cxy}{d + x + \beta\kappa(A + \gamma_3(t))} + \sigma_1 x \frac{dw_1}{dt} \\ \frac{dy}{dt} &= \frac{b\{x + \kappa(A + \gamma_3(t))\}y}{d + x + \beta\kappa(A + \gamma_3(t))} - ly - \sigma_2 y \frac{dw_2}{dt}, \end{aligned} \quad (35)$$

where $\langle \gamma_3(t) \rangle = 0$ and $\langle \gamma_3(t_1)\gamma_3(t_2) \rangle = \sigma_3^2 \delta(t_1 - t_2)$, $\sigma_3 > 0$.

In the numerical simulation of stochastic system (35), we approximate the solutions by Euler Maruyama method by setting the parametric values same as considered for Fig. 5, with $\sigma_3 = 0.005$ which is same as the intensities σ_1 and σ_2 . It is observed that the prey population is almost dominated by the predator population as time goes (see Fig. 8) where as in Fig. 5 predator population is almost dominated by prey population as time goes. It is a very interesting result in the context of the influence of additional food on the underlying system in a fluctuating environment.

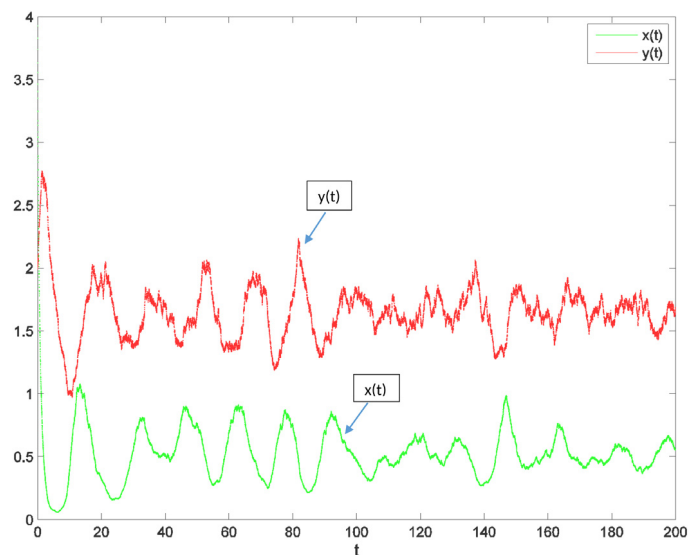


Fig. 7. The predator population dominates the prey population.

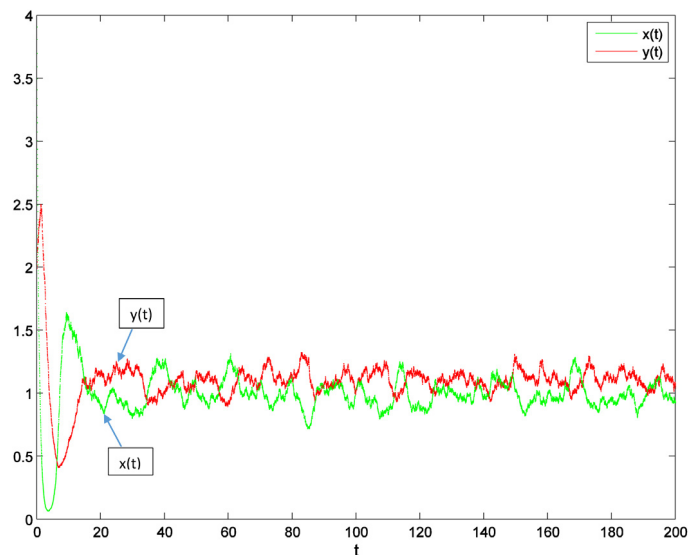


Fig. 8. Numerical simulation of system (35).

Acknowledgments

The authors are grateful to the anonymous referees and Dr. Eugene Stanley (Editor) for their valuable comments and helpful suggestions, which have helped them to improve the presentation of this work significantly.

References

- [1] Q. Din, E.M. Elsayed, Stability analysis of a discrete ecological model, *Comput. Ecol. Softw.* 4 (2014) 89–103.
- [2] D. Manna, A. Maiti, G.P. Samanta, Analysis of a predator-prey model for exploited fish populations with schooling behavior, *Appl. Math. Comput.* 317 (2018) 35–48, <http://dx.doi.org/10.1016/j.amc.2017.08.052>.
- [3] D. Pal, G.S. Mahapatra, G.P. Samanta, New approach for stability and bifurcation analysis on predator-prey harvesting model for interval biological parameters with time delays, *Comput. Appl. Math.* 37 (3) (2018) 3145–3171, <http://dx.doi.org/10.1007/s40314-017-0504-3>.
- [4] S. Sharma, A. Mondal, A.K. Pal, G.P. Samanta, Stability analysis and optimal control of avian influenza virus A with time delays, *Internat. J. Dyn. Control* 6 (3) (2018) 1351–1366, <http://dx.doi.org/10.1007/s40435-017-0379-6>.
- [5] S. Sharma, G.P. Samanta, Dynamical behaviour of age-selective harvesting of a prey-predator system, *Internat. J. Dyn. Control* 6 (2) (2018) 550–560, <http://dx.doi.org/10.1007/s40435-017-0337-3>.

- [6] R.D. Holt, Spatial heterogeneity, indirect interactions, and the coexistence of prey species, *Am. Nat.* 124 (3) (1984) 377–406, <http://dx.doi.org/10.1086/284280>.
- [7] D.L. Kevin, M.K. Armand, Biological control of marine pests, *Ecology* 17 (7) (1996) 1989–2000, <http://dx.doi.org/10.2307/2265695>.
- [8] J.A. Logan, J. Regniere, J.A. Powell, Assessing the impacts of global warming on forest pest dynamics, *Front. Ecol. Environ.* 1 (3) (2003) 130–137, [http://dx.doi.org/10.1890/1540-9295\(2003\)001\[0130:ATIOGW\]2.0.CO;2](http://dx.doi.org/10.1890/1540-9295(2003)001[0130:ATIOGW]2.0.CO;2).
- [9] M.W. Group, *Diversivory feeding of hen harriers on grouse moors a practical guide*, Scottish Natur. Herit. (1999).
- [10] K.S. Rauwald, A.R. Ives, Biological control in disturbed agricultural systems and the rapid recovery of parasitoid populations, *Ecol. Appl.* 11 (4) (2001) 1224–1234, <http://dx.doi.org/10.2307/3061023>.
- [11] T.P. Sullivan, W. Klenner, Influence of diversionary food on red squirrel populations and damage to crop trees in young lodgepole pine forest, *Ecol. Appl.* 3 (4) (1993) 708–718, <http://dx.doi.org/10.2307/1942102>.
- [12] F.L. Wackers, H. Fadamiro, The vegetarian side of carnivores: use of non-prey food by parasitoids and predators, *Select. Food Suppl. Conserv. Biol. Control* (2005) 420–427.
- [13] M.R. Wade, M.P. Zalucki, S.D. Wratten, A. Robinson, Conservation biological control of arthropods using artificial food sprays, *Biol. Control* 45 (2) (2008) 185–199, <http://dx.doi.org/10.1016/j.biocontrol.2007.10.024>.
- [14] T.P. Sullivan, Vole Population Dynamics Aand Use of Deversivory Food to Reduce Ddamage to Seedlings, 2004. https://www.for.gov.bc.ca/hfd/library/fia/2007/FSP_Y073138a.pdf.
- [15] M.J. Crawley, *Plant Ecology*, Wiley, 1997.
- [16] G.R. Huxel, K. McCann, Food web stability: the influence of trophic flows across habitats, *Am. Nat.* 152 (3) (1998) 460–469, <http://dx.doi.org/10.1086/286182>.
- [17] G.R. Huxel, K. McCann, G.A. Polis, Effects of partitioning allochthonous and autochthonous resources on food web stability, *Ecol. Res.* 17 (4) (2002) 419–432, <http://dx.doi.org/10.1046/j.1440-1703.2002.00501.x>.
- [18] P.D.N. Srinivasu, B.S.R.V. Prasad, Role of quantity of additional food to predators as a control in predator-prey systems with relevance to pest management and biological conservation, *Bull. Math. Biol.* 73 (10) (2011) 2249–2276, <http://dx.doi.org/10.1007/s11538-010-9601-9>.
- [19] M. Kot, *Elements of Mathematical Ecology*, Cambridge University Press, 2001.
- [20] T. Bilde, S. Toft, Quantifying food limitation of arthropod predators in the field, *Oecologia* 115 (1–2) (1998) 54–58, <http://dx.doi.org/10.1007/s004420050490>.
- [21] J.D. Harwood, J.J. Oerycki, The role of alternative prey in sustaining predator populations, in: *2nd International Symposium on the Biological Control of Arthropods*, vol. 2, 2005, pp. 453–462.
- [22] J.D. Harwood, K.D. Sunderland, W.O.C. Symondson, Prey selection by linyphiid spiders: molecular tracking of the effects of alternative prey on rates of aphid consumption in the field, *Mol. Ecol.* 13 (11) (2004) 3549–3560, <http://dx.doi.org/10.1111/j.1365-294X.2004.02331.x>.
- [23] R. Lande, Risks of population extinction from demographic and environmental stochasticity and random catastrophes, *Am. Nat.* 142 (6) (1993) 911–927.
- [24] R. May, Stability in randomly fluctuating deterministic environments, *Am. Nat.* 107 (957) (1973) 621–650, <http://dx.doi.org/10.2307/2459663>.
- [25] J. Ripa, P. Lundberg, V. Kaitala, A general theory of environmental noise in ecological food webs, *Am. Nat.* 191 (3) (1998) 256–263, <http://dx.doi.org/10.1086/286116>.
- [26] R.K. Upadhyay, A. Mukhopadhyay, S.R.K. Iyengar, Influence of environmental noise on the dynamics of a realistic ecological model, *Fluct. Noise Lett.* 7 (1) (2007) L61–L77.
- [27] I.B. Schwartz, L. Billings, T.W. Carr, M.I. Dykman, Noise-induced switching and extinction in systems with delay, *Phys. Rev. E* 91 (1) (2015). <http://dx.doi.org/10.1103/PhysRevE.91.012139>.
- [28] C.S. Holling, Some characteristics of simple types of predation and parasitism, *Canad. Entomol.* 91 (7) (1959) 385–398, <http://dx.doi.org/10.4039/Ent91385-7>.
- [29] M. Dimentberg, *Statistical Dynamics of Nonlinear and Time-Varying Systems*, Wiley, 1988.
- [30] E. Allen, *Modeling with Itô Stochastic Differential Equations*, Springer Netherlands, 2007, p. 82.
- [31] G. Cai, Y. Suzuki, Response of systems under non-Gaussian random excitations, *Nonlinear Dynam.* 45 (1–2) (2006) 95–108, <http://dx.doi.org/10.1007/s11071-006-1461-3>.
- [32] A. d'Onofrio, Bounded-noise-induced transitions in a tumor-immune system interplay, *Phys. Rev. E* 81 (2 pt 1) (2010). <http://dx.doi.org/10.1103/PhysRevE.81.021923>.