Welfare Benchmarking of Life-Cycle Investment Strategies for Households in Turkey

Ravshanbek Khodzhimatov, Tolga Umut Kuzubaş*, Burak Saltoğlu

Boğaziçi University, Department of Economics, Natuk Birkan Building, 34342 Bebek, Istanbul, Turkey

Abstract

In this paper, we perform an in-depth welfare comparison of the most common life-cycle investment strategies provided by retirement funds or suggested by classical portfolio theory in the case of households in Turkey. To perform our benchmarking, we construct heterogeneous agents who work and invest throughout their lifetime, using parameters calibrated from the historical data. We find that to households with upper-to-middle income, individually customized portfolios result in considerable welfare gains, while "off-the-shelf" life-cycle portfolio allocations perform better for households with lower income. We also show that life-cycle investment options outperform "fixed over the lifetime" options.

Keywords: Life-cycle portfolio decisions, human capital, housing, stock market participation

1. Introduction

The field of financial economics has gone through big changes since its foundation by Markowitz [1] and Tobin [2]. They pioneered the mean-variance analysis, which, given some assumptions, suggested that if investors cared about maximizing returns (mean) and minimizing risks (variance), then the optimal ratio of stocks to bonds in a single-period portfolio would be fixed for everyone, the share of former being equal to:

$$\alpha = \frac{\mu - R_f}{\gamma \sigma^2} \tag{1}$$

Merton [3] generalized the problem to multiple periods using dynamic programming and found that it is optimal for all households to repeat the same fixed mean-variance solution every period.

These results were inconsistent with the popular financial advice suggesting that younger investors should have higher share of stocks in portfolio, and older investors — higher share of bonds. This advice was summarized by the famous rule of thumb:

$$\alpha_t = (100 - t)\% \tag{2}$$

*Corresponding author

Email addresses: ravshanbeksk@gmail.com (Ravshanbek Khodzhimatov), umutkuzubas@gmail.com (Tolga Umut Kuzubaş), burak.saltoglu@gmail.com (Burak Saltoğlu)

Although Samuelson [4] denied that risk-aversion changes by age, dismissing this advice would question the rationality of investors and "constitute *prima facie* evidence that people do not optimize" (Canner et al. [5]).

Bodie et al. [6] solved this problem by adding human capital into the Merton [3]'s dynamic model and found that for complete markets and constant risk-free labor income, the optimal share of stocks in a portfolio is:

$$\alpha_t = \frac{\mu - R_f}{\gamma \sigma^2} \left(\frac{F_t + L_t}{F_t} \right) \tag{3}$$

Steady depletion of human capital L_t relative to the financial wealth F_t throughout life, explained the higher share of stocks in younger people. Cocco et al. [7] extended this idea to the case of variable labor income, to find a recursive solution which could be approximated by the following rule of thumb:

$$\alpha_t = \begin{cases} 100\% & t < 40 \\ (200 - 2.5t)\% & t \in [40, 60] \\ 50\% & t > 60 \end{cases} \tag{4}$$

However, the hump-shaped lifetime stock share graph, observed by Chang et al. [8] of Federal Reserve, instead of expected downward sloping one, suggested the presence of an opposing force.

Cocco [9] found that this force was housing investment, which, due to its large size, crowded out all stocks from younger investors' portfolios. Flavin and Yamashita [10] supported this view by showing that younger people, who already own the house, tend to invest more aggressively, as was expected by Bodie et al. [6].

Munk [11] found the same patterns using a series of one-period mean-variance optimizations without any dynamic stochastic modeling tools.

Finally, Ascheberg et al. [12] illustrated the existence of long-term cointegration among house prices, stock prices, and labor income, the fact often omitted by previous portfolio researchers for simplicity. In our analysis we will not neglect the correlations as being equal to zero.

In this paper, we calculate the wealth, accumulated from lifetime investments given by all the strategies above and suggested by current Turkish banks. We benchmark the resulting welfare for heterogeneous agents in Turkey with varying sectors, levels of education and risk-aversion. We conclude with the separate investment advice for all types of households mentioned above. Such a comprehensive benchmarking has not been done yet in the overall literature, and almost none was done for Turkish households.

The remainder of the paper is organized as follows: Section 2 presents the theoretical framework we will use in our analysis. Section 3 contains detailed information about our data sources, parameter calibration and solves for the optimal investment strategies. Section 4 does the welfare benchmarking of every investment option for every agent, and Section 5 concludes.

2. Theoretical framework

2.1. House prices, stock prices and labor income series

In accordance with Campbell et al. [13] and Olear [14] we model the labor income process as a function of individiual characteristics $f(t, Z_{it})$ plus idiosyncratic shocks v_{it}. Upon reaching the retirement age R, an individual receives a certain percentage λ of his/her last wage:

$$Y_{i,t+1} = \begin{cases} Y_{it}(1 + f(t+1, Z_{i,t+1}) + v_{it}), & t < R \\ \lambda(1 + f(R, Z_{iR}) + v_{iR}), & t \geq R \end{cases} \tag{5}$$

We model labor income, house prices, and stock prices as Geometric Brownian Motions with drifts μ_L, μ_H, μ_S and volatilities σ_L, σ_H, σ_S, satisfying the discrete version of Ascheberg et al. [12]'s correlation structure, that is having nonzero correlations $\rho_{HS}, \rho_{HL}, \rho_{SL}$. See Appendix A for details.

2.2. Optimal portfolio

Along with the investment strategies, described in Equations 1 - 4, we consider in our benchmarking, the strategy proposed by Munk [11], who stated that in the presence of housing, the optimal stock (π) and housing (π_h) shares can be solved analytically as follows:

$$\pi_{t+1} = \frac{1}{\gamma(1-\rho_{SH}^2)\sigma_S} \cdot \frac{F_t + L_t}{F_t} \left(\frac{\mu_S - r_f}{\sigma_S} - \rho_{SH} \frac{\mu_H - r_f}{\sigma_H} \right) - \frac{L_t}{F_t} \cdot \frac{\sigma_L}{\sigma_S} \frac{\rho_{SL} - \rho_{SH}\rho_{HL}}{1 - \rho_{SH}^2} \tag{6a}$$

$$\pi_{h,t+1} = \frac{1}{\gamma(1-\rho_{SH}^2)\sigma_H} \cdot \frac{F_t + L_t}{F_t} \left(\frac{\mu_H - r_f}{\sigma_H} - \rho_{SH} \frac{\mu_S - r_f}{\sigma_S} \right) - \frac{L_t}{F_t} \cdot \frac{\sigma_L}{\sigma_H} \frac{\rho_{HL} - \rho_{SH}\rho_{SL}}{1 - \rho_{SH}^2} \tag{6b}$$

Setting $\rho_{SH} = 0$ and $\rho_{HL} = 0$, gives the optimal stock share by Munk [11] in the absence of housing:

$$\pi_{t+1} = \frac{1}{\gamma\sigma_S} \cdot \frac{F_t + L_t}{F_t} \left(\frac{\mu_S - r_f}{\sigma_S} \right) - \frac{L_t}{F_t} \cdot \frac{\sigma_L}{\sigma_S} \rho_{SL} \tag{7}$$

2.3. Welfare measurement

We use stochastic constant relative risk-aversion utility function to compare welfare resulting from different income patterns:

$$E_1[U(c)] = \sum_{t=1}^{T} \delta^{t-1} \prod_{j=0}^{t-1} p_j \cdot \frac{c_{it}^{1-\gamma}}{1-\gamma} \tag{8}$$

where p_k is the probability of survival between ages $k-1$ and k. Unlike Cocco et al. [7], we neglect the bequest motives, assuming that the retired person consumes all of his/her income at any given time.

2.4. Retirement income

The funds invested in retirement are modeled to be paid back in annuities, not withdrawn immediately. Further, to include housing investment in welfare calculation, we use “reverse mortgages” — annuities, paid to retired individuals in return for inheriting their house after their death. This is a plausible analysis tool, because it allows to liquidify the housing possessions, although such financial instrument is not yet available in Turkey.

Thus, at the age of retirement $R = 65$, the price of owned house is calculated and is added to the matured pension amount (MP) to obtain total wealth:

$$W_{65} = H_{65} + MP \tag{9}$$

All of the W_{65} is used to buy an annuity which will annually repay an individual:

$$A_t = W_{65} \cdot \left(1 + \sum_{t=66}^{100} \frac{\prod_{j=66}^{t} p_j}{(1+r_f)^{t-65}}\right)^{-1} \tag{10}$$

3. Data and simulation

In this section, we go over data sources, perform parameter calibrations and derive all investment strategies for our simulation.

3.1. Data sources

We use historical monthly BIST 30[1] and REIDIN[2] data from 2004 to 2014, to construct stock and house price series respectively. Figures 1 and 2 illustrate the general upward trend in both series, with a collapse during 2008 crisis.

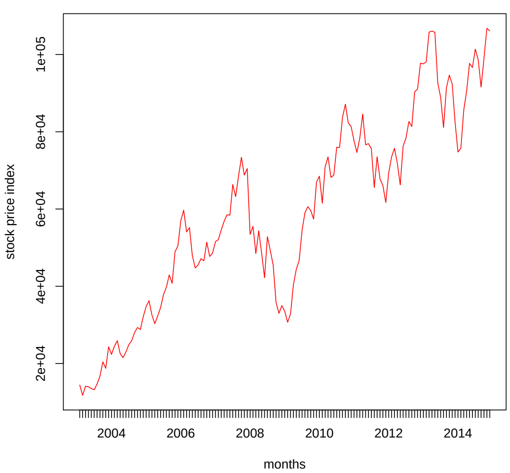


Figure 1: BIST30 Turkish stock market prices

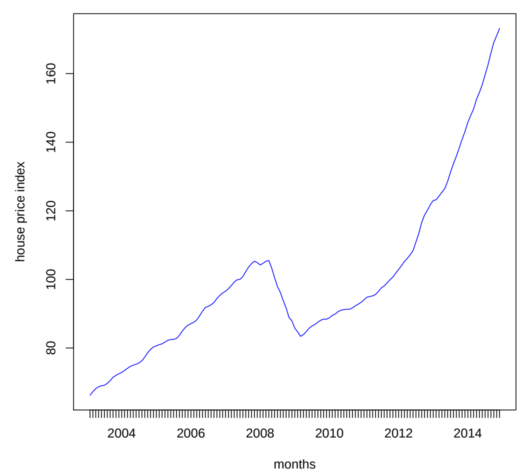


Figure 2: Reidin Turkish house price index

We construct labor income series from Turkish Statistical Institute's Household Budget Survey [15]'s repeated cross-sectional study, and, in line with Aktug et al. [16], we aggregate the data to obtain a pseudo-panel with 55 thousand data points for 170 households from 2001 to 2014. Figure 3 displays the hump-shaped lifetime income distribution, consistent with the results of Aktug et al.

[1]BIST 30 is an index measuring the stock performance of 30 largest companies in Turkey

[2]REIDIN provides residential sales price index for Turkey, using data, covering 200,000 house listings in 62 cities and 221 counties, per month, weighted by population, and calculated using Laspeyres' formula.

[16], who analyzed labor income profiles in Turkey, and Ben-Porath [17], who predicted a decline in productivity, as workers get older.

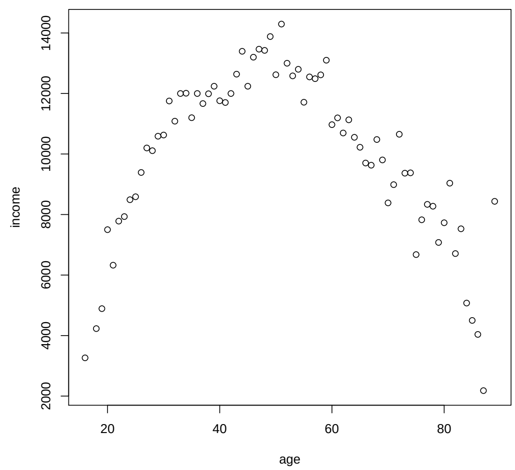


Figure 3: Median Turkish salaries by age

3.2. Default parameters

Similarly to Munk [11], we start our simulation with a 25-year-old individual who invests in his/her retirement for 40 years until he/she reaches retirement at 65. Similarly to Torul and Oztunali [18], we set the default relative risk aversion coefficient for Turkish households at 1.5, and the subjective discount rate — at 0.89.

We deflate the nominal wages, stock and house prices by CPI and work with real variables.

We obtain annual rate of return on stocks 6.69% with volatility 38.44%, by annualizing long-term ARMA(2,2) forecasts of monthly return and volatility, based on historical BIST 30 data, mentioned above (see Appendix B.1).

Similarly, we use long-term ARMA(1,1) forecast of monthly return and volatilities on housing, and find annual real rate of return on housing 0.67% with 5.42% volatility (see Appendix B.2).

Risk-free rate 12% is given by OECD [19] forecast[3], and, upon subtracting the medium-term inflation rate forecast $\pi = 9\%$ by Turkish Central Bank's Inflation Report [21], is equal to 3% per annum with zero volatility.

We consider real wage growth rates separately for different types of agents, but before introducing heterogeneity, the ARMA(5,2) forecast gives the volatility 4% (see Appendix B.3).

In our simulation, the house-stock and house-wage contemporaneous correlations are given by 0.27 and 0.35 respectively.

Survival probabilities for all ages are provided by Turkish Statistical Institute's Data Bank [20] and illustrated in Figure 4.

[3]Data was obtained before the Turkish currency and debt crisis of 2018

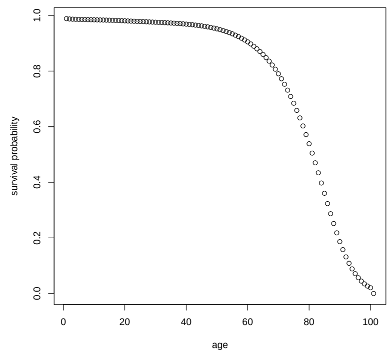


Figure 4: Survival probabilities by age

All of the above findings are summarized in Table 1.

Table 1: Benchmark Parameters

Parameter	Description	Value
Y	Beginning age	25
R	Retirement age	65
T	Lifespan (years)	100
γ	Risk aversion	1.5
β	Discount rate	0.89
r_f	Risk-free rate	0.03
μ_s	Expected stock returns	0.0669
μ_h	Expected housing returns	0.0067
σ_s	Stock returns volatility	0.3844
σ_h	Housing returns volatility	0.0542
σ_w	Wage growth volatility	0.036
ρ_{hs}	House-stock correlation	0.24
ρ_{hw}	House-wage correlation	0.37
p_{25}	Survival probability at age 25	0.978
p_{65}	Survival probability at age 65	0.86
p_{100}	Survival probability at age 100	0

3.3. Heterogeneity parameters

Combining the approaches of Olear [14] and Munk [11], we use wage growth rate, stock-income correlation and relative risk aversion level, to model heterogeneities among agents.

3.3.1. Heterogeneity in education

We model the heterogeneity in education using differences in wage growth rates. Figure 5 shows the lifetime labor income series for different levels of education. Notice that the curves are almost flat for the lower education levels and hump-shaped for the higher.

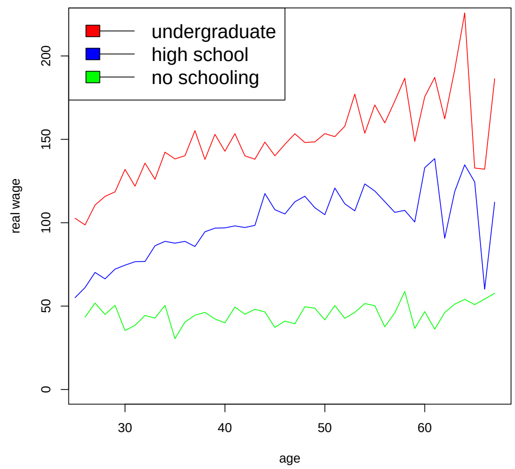


Figure 5: Lifetime real wage dynamics by education level. While less educated individuals have flat wages through life, the more educated individuals observe a steeper and hump-shaped wages.

We use undergraduate education, high school education, and no schooling, to model "steep", "moderate", and "flat" wage dynamics respectively. Performing regressions of wages on age, with kinks at $t = 40$ and $t = 55$:

$$\Delta \log(wage_{it}) = \alpha_0 + \alpha_1 \cdot d_{40} + \alpha_2 \cdot d_{55} \tag{11}$$

we estimate growth rates for different education levels, as summarized in Table 2 (see Appendix B.4 for regression results and line fits).

Table 2: Estimated Benchmark Wage Growth Rates μ_w

Age	Flat	Moderate	Steep
25-40	0%	3.8%	2.2%
41-55	0%	1.4%	1.2%
56-65	0%	0%	1.5%

We assume steep wage earners (college graduates) to have a starting real salary of 100, and moderate and flat wage earners to have a starting real salary of 50. This is consistent with the historical data (see Figure 5) and the relevant discussion by Olear [14]. This difference in starting values also explains why "steep" wage growth rates are less than "moderate" ones.

3.3.2. Heterogeneity in sectors of work

We model the heterogeneity work sectors using corresponding stock-wage correlations (ρ_{ws}). Figure 6 illustrates how, during 2008 crisis, income in financial sectors ($\rho_{ws} = 0.44$) dropped drastically, while it was unaffected in education and agriculture ($\rho_{ws} = 0.08$).

We use three measures of ρ_{sw} for our benchmark: 0, 0.2 and 0.4 (see Table 3).

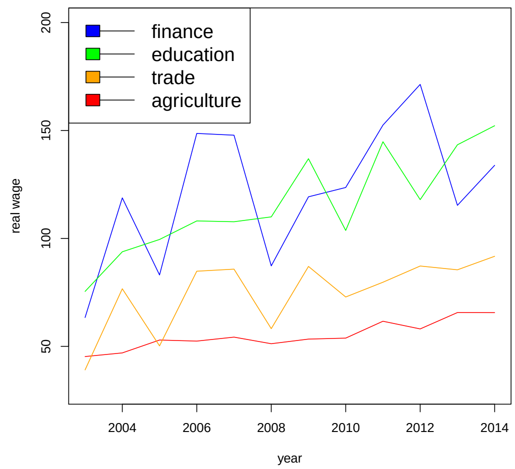


Figure 6: Historical real wage dynamics by sector. Sectoral differences in degrees of correlation of wages to stock market, can be seen from the different behavior during 2008 crisis — while income in finance sector dropped drastically, the wages in education sector were not affected at all.

Table 3: Benchmark Wage to Stock Correlations

	Low	Moderate	High
ρ_{sw}	0	0.2	0.4

3.3.3. Individual heterogeneity

We model individual heterogeneity using different risk aversion levels of investors, as summarized in Table 4.

Table 4: Coefficients of Risk Aversion

Values	default	low	moderate	high
γ	1.5	3	5	10

3.4. Capital series

3.4.1. Human capital

Human capital at any period is the discounted sum of all future wages until retirement with the discount factor r_f. To construct the individualized capital we used steep, moderate and flat wage series mentioned in the previous section. Figure 7 illustrates the current human capital for flat, moderate, and steep wages for every age.

3.4.2. Financial capital

Financial capital evolves according to dynamic investment diagram in Figure 9. Every period, a certain percentage c (3% for Turkey) of the wage w_t is invested in a retirement portfolio, while

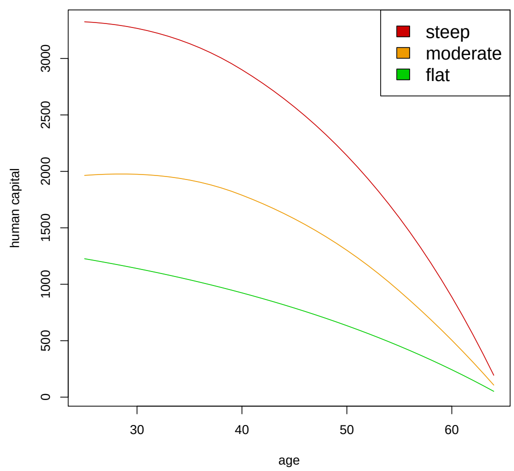


Figure 7: Human capital at every age for individuals with steep, moderate and flat wage growth curves. As individuals get older, their human capital gradually depletes.

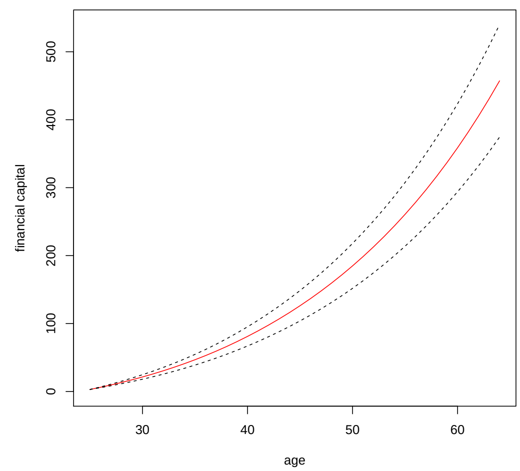


Figure 8: Financial wealth accumulation through life, by a college graduate who invests 50% in bonds and 50% in stocks. Confidence intervals capture the volatility of stocks in the portfolio.

the previously invested amount accrues interest at portfolio rate of return. Figure 8 demonstrates the evolution of financial capital and its confidence interval for a naive fixed investment strategy (50% in stocks and 50% in bonds) by an individual with "steep" wage curve.

$$
\begin{array}{rccc}
 & t_0 & t_1 & t_2 \\
Harvest: & 0 & cw_0(1+r_p) & [cw_0(1+r_p)+cw_1]\cdot(1+r_p) \\
Invest: & cw_0 & +cw_1 & +cw_2
\end{array}
$$

Figure 9: Law of motion of financial capital. Every period, a certain percentage c of the wage w_t is invested in a retirement portfolio, while the previously invested amount accrues interest at portfolio rate of return r_p.

3.5. Investment strategies

Below, we present the life-cycle investment strategies to be benchmarked.

3.5.1. Homogeneous life-cycles

Homogeneous life-cycles are strategies, common to all individuals, regardless of their idiosyncratic characteristics. We benchmark strategies given by equations 1, 2, 4, and an aggressive portfolio allocation, offered by Turkish banks (60% in stocks, 40% in bonds). Figure 10 illustrates the stock shares in these investment strategies.

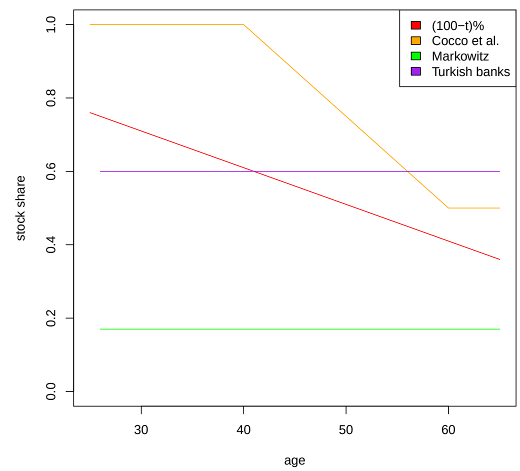
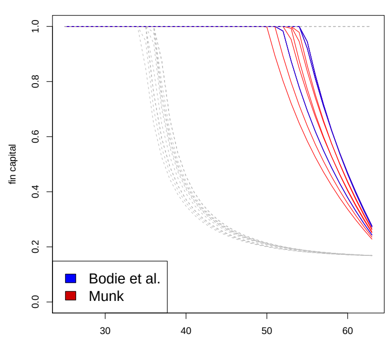


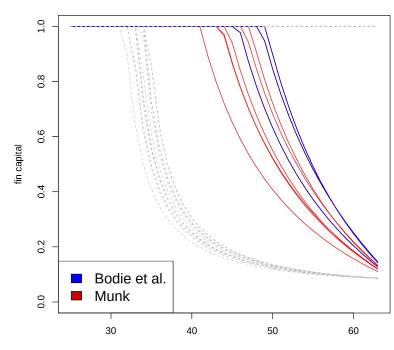
Figure 10: Share of stocks in a portfolio at every age, proposed by homogeneous life-cycle strategies

3.5.2. Heterogeneous life-cycles

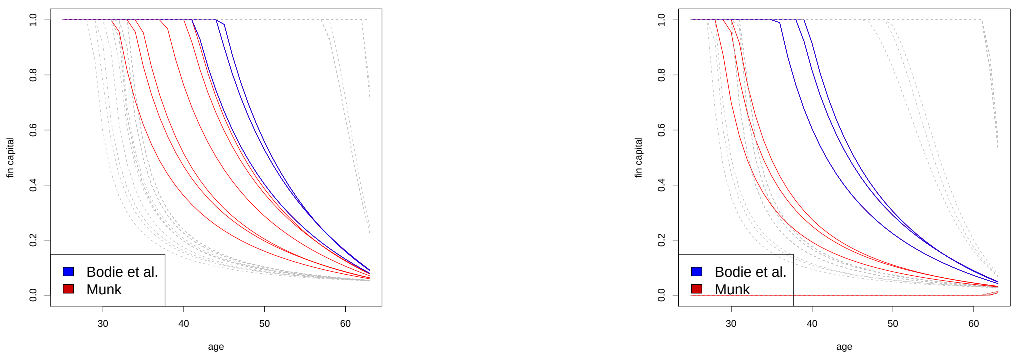
Heterogeneous life-cycles take idiosyncratic characteristics into consideration. We benchmark strategies by Bodie et al. [6] (see Equation 3), and Munk [11] (see Equation 7), which are illustrated in Figure 11 by blue and red curves respectively. The dashed lines illustrate the 68%-confidence interval ($\pm 1\sigma$). The figure proposes younger investors to allocate all of their funds in stocks, and gradually, through life, decrease their share in the portfolio — the more risk-averse they are or the flatter their wage curve is, the sooner. It also shows that, other things being equal, Munk [11]'s strategy without housing, is less aggressive than Bodie et al. [6]'s, which is consistent with equations 3 and 7.



(a) $\gamma = 1.5$



(b) $\gamma = 3$

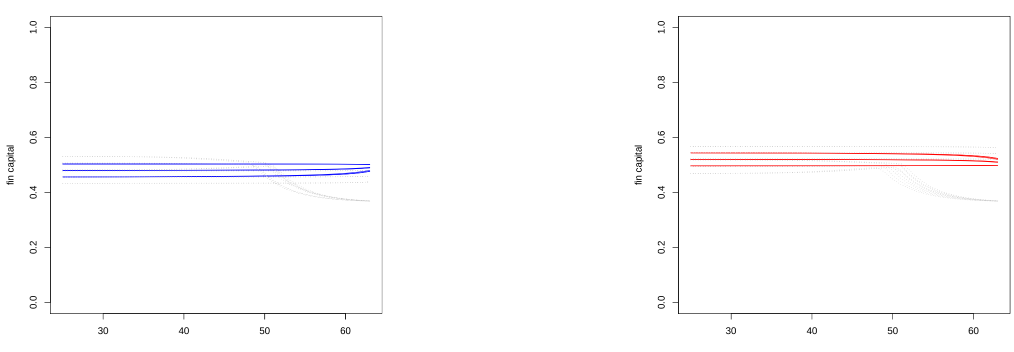


(c) $\gamma = 5$

(d) $\gamma = 10$

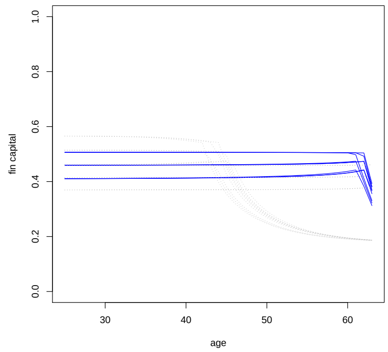
Figure 11: Shares of stocks in a portfolio, suggested by Bodie et al. [6] (blue curves) and Munk [11] (red curves) in the absence of housing investment, for heterogeneous agents. The optimal allocations at every age depend on the previous realizations of volatile stock returns — the dashed lines illustrate the confidence intervals for $\pm\sigma$. Higher risk-aversion, steeper wage curve and smaller stock-income correlation lead to less aggressive investment in stocks, and vice versa.

Figure 12 shows the stock and house shares, suggested by Munk [11] for flat, moderate and steep labor income curves, low, moderate and high stock-wage correlations, and different levels of risk aversion. It confirms that steeper labor income curves results in a larger share of stocks in portfolio, and the more risk averse individuals are, the sooner they decrease both housing and stock investment, and buy more bonds.

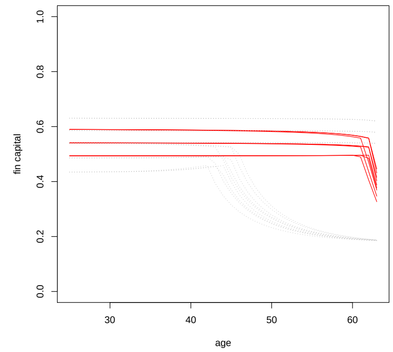


(a) Stocks for $\gamma = 1.5$

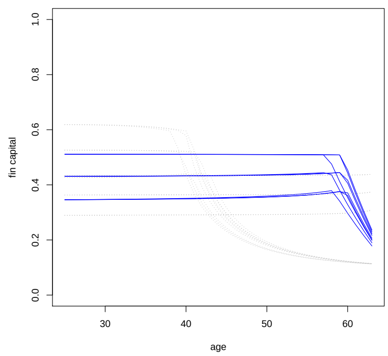
(b) Housing for $\gamma = 1.5$



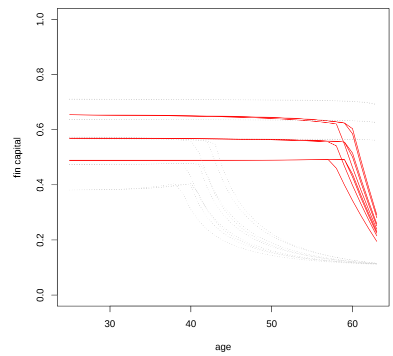
(c) Stocks for $\gamma = 3$



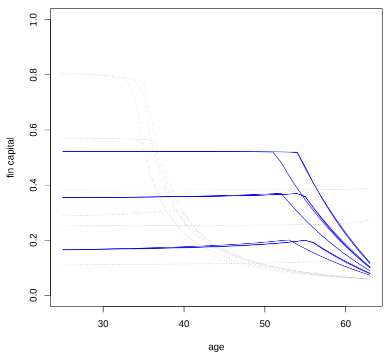
(d) Housing for $\gamma = 3$



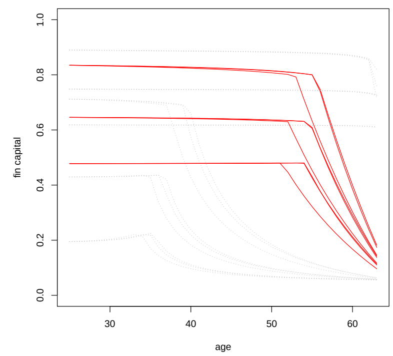
(e) Stocks for $\gamma = 5$



(f) Housing for $\gamma = 5$



(g) Stocks for $\gamma = 10$



(h) Housing for $\gamma = 10$

Figure 12: Munk's stock and housing shares for different wage growth, stock-wage correlation and risk aversion levels

Tables with portfolio allocations can be reproduced using the data and methodology explained in this paper in detail, with blueprint for $\gamma = 1.5$ available in Appendix C.

4. Welfare comparison and results

In this section, we calculate the accumulated wealth for every investment option above and benchmark the resulting expected utilities.

4.1. Accumulated wealth

After the lifetime of investing, the household accumulates different levels of wealth, as summarized in Table 5.

4.1.1. Early results

We can make some early conclusions before calculating utilities:

- Naively considering life-cycles does not guarantee the better investment — in our scenario, $(100 - age)\%$ is worse than fixed Markowitz.
- Different stock-wage correlations don't make big difference in the outcome without housing, and make considerable difference in models with housing.
- Stock-wage correlations are negatively correlated with total wealth.
- Default options are better for people with flat wages. Individualized options are better for people with moderate or steep wages.

Table 5: Total Accumulated Wealth by Investment Option

Option	$\gamma = 1.5$ (default)	$\gamma = 3$	$\gamma = 5$	$\gamma = 10$
		Defaults		
Anadolu Hayat (riskless)	2,000,694 TL	2,000,694 TL	2,000,694 TL	2,000,694 TL
$(100 - age)$%	4,545,177 TL	4,545,177 TL	4,545,177 TL	4,545,177 TL
Cocco et al.	10,918,994 TL	10,918,994 TL	10,918,994 TL	10,918,994 TL
Markowitz (fixed)	6,287,647 TL	6,287,647 TL	6,287,647 TL	6,287,647 TL
		Individualized (no housing)		
Bodie et al. (steep)	25,513,822 TL	10,374,038 TL	6,526,269 TL	4,190,661 TL
Bodie et al. (moderate)	11,248,905 TL	4,526,672 TL	2,828,304 TL	1,803,151 TL
Bodie et al. (flat)	3,949,368 TL	1,514,867 TL	930,204 TL	586,732 TL
Munk (steep-lo)	25,236,838 TL	9,924,172 TL	5,982,295 TL	3,371,281 TL
Munk (steep-mod)	25,236,817 TL	9,924,152 TL	5,982,274 TL	3,371,260 TL
Munk (steep-hi)	25,236,797 TL	9,924,132 TL	5,982,254 TL	3,371,240 TL
Munk (moderate-lo)	11,111,057 TL	4,331,204 TL	2,584,866 TL	1,441,285 TL
Munk (moderate-mod)	11,111,049 TL	4,331,196 TL	2,584,857 TL	1,441,277 TL
Munk (moderate-hi)	11,111,040 TL	4,331,187 TL	2,584,849 TL	1,441,268 TL
Munk (flat-lo)	3,906,648 TL	1,452,621 TL	852,150 TL	472,043 TL
Munk (flat-mod)	3,906,646 TL	1,452,619 TL	852,148 TL	472,041 TL
Munk (flat-hi)	3,906,644 TL	1,452,617 TL	852,146 TL	472,039 TL
		Individualized (housing)		
Munk (steep-lo)	74,168,102 TL	50,575,196 TL	30,149,576 TL	19,030,821 TL
Munk (steep-mod)	72,868,088 TL	49,309,258 TL	29,030,131 TL	17,925,925 TL
Munk (steep-hi)	71,559,554 TL	47,942,115 TL	27,909,489 TL	16,627,948 TL
Munk (moderate-lo)	30,595,160 TL	20,817,931 TL	12,392,195 TL	7,826,582 TL
Munk (moderate-mod)	30,059,601 TL	20,297,628 TL	11,939,450 TL	7,372,166 TL
Munk (moderate-hi)	29,520,524 TL	19,737,247 TL	11,486,249 TL	6,845,005 TL
Munk (flat-lo)	9,056,285 TL	6,068,133 TL	3,600,965 TL	2,280,512 TL
Munk (flat-mod)	8,903,691 TL	5,921,315 TL	3,479,798 TL	2,150,420 TL
Munk (flat-hi)	8,750,023 TL	5,762,054 TL	3,349,099 TL	2,003,530 TL

4.2. Annuities

To formalize the observations made in the previous section, we will construct annuities and plug them into expected utility functions. As described in detail in Chapter 3, we define annuities by dividing the total wealth before retirement by the discount factor $1 + \sum_{t=58}^{100} \frac{p_t}{1+r_f}$. Using survival probabilities, obtained from TUIK, and risk-free rate of return, described in the previous chapter, we calculate the discount factor as 205.29. The annuities are obtained by dividing all the values in the Table 5.1 by 205.29. The resulting values are presented in Table 5.2.

4.2.1. Consumption during retirement

We model that the individuals spend their annuity returns to consume baskets which cost 100 TL during the year, when our agent is 58 years old, and increase by inflation rate $\pi = 8.4\%$. Thus, every period $t > 57$ our agent consumes $\frac{annuity}{100 \cdot (1.084)^{t-58}}$. We do not provide the separate table with the value streams, as they will be implicitly included in utility values.

4.2.2. Expected utilities

Finally, we will plug the consumption streams, calculated in the previous section, into the constant relative risk-aversion expected utility functions to compare the welfare effects. The resulting expected utility values are presented in Table 5.3.

4.2.3. Conclusions

Now, we can observe the final resutls:

- Naive life-cycle investment portfolios, such as $(100 - age)\%$ don't overperform fixed-ratio Markowitz, because they don't take the risk aversion into consideration.
- Cocco et al.'s $(200 - 2.5 \cdot age)\%$ approximation is the best default portfolio. It is easy to interpret and captures life-cycle effect.
- All models perform better for higher risk aversion and worse for lower risk aversion — for everyone except flat-wagers.
- Higher stock-wage correlation considerably decreases the utility for moderate and flat wages, and doesn't affect much for steep wages.
- Merton's solution outperforms Munk's solution without housing for low levels of risk aversion, and performs same for high level of risk aversion ($\gamma = 10$).
- Munk's solution with housing outperforms every other solution for high levels of risk aversion ($\gamma = 10$) — for everyone except flat-wagers.
- Munk's solution with housing outperforms Munk's solution without housing for $\gamma = 5, 10$.
- Markowitz's solution outperforms both Merton's and Munk's solutions for flat wages and low risk aversion.
- Individualizing life-cycles by wage growth rate and stock-wage correlation increases welfare for steep wagers and decreases welfare for flat wagers.

. We did not provide numerical conclusions, since they are trivial — they can be obtained by calculating percentage differences in Table 5.3.

5. Conclusion

In this thesis we have reviewed the current state of pension investments in Turkey and the history and recent developments of a field of financial economics. We have reviewed the concept of "life-cycle investment" and summarized the common models and heuristics.

. We have presented our model as an application of Munk's (2016) recent findings and Olear's (2016) simulation techniques into Turkish retirement market. We have collected historical data to calibrate and estimate the best parameters to be used in our simulation.

Table 6: Annual Pensions by Investment Option

Option	$\gamma = 5$ (default)	$\gamma = 1.5$	$\gamma = 3$	$\gamma = 10$
	Defaults			
Anadolu Hayat (riskless) TL	2,992 TL	2,992 TL	2,992 TL	2,992 TL
$(100 - age)$% TL	5,655 TL	5,655 TL	5,655 TL	5,655 TL
Cocco et al. TL	7,793 TL	7,793 TL	7,793 TL	7,793 TL
Markowitz TL	6,333 TL	6,333 TL	6,333 TL	6,333 TL
	Individualized (no housing)			
Merton (steep) TL	18,569 TL	20,521 TL	15,124 TL	14,228 TL
Merton (moderate) TL	7,940 TL	8,835 TL	7,053 TL	6,081 TL
Merton (flat) TL	2,648 TL	2,984 TL	3,117 TL	1,940 TL
Munk (steep-lo) TL	17,280 TL	20,307 TL	18,653 TL	10,161 TL
Munk (steep-mod) TL	17,261 TL	20,288 TL	18,633 TL	10,141 TL
Munk (steep-hi) TL	17,241 TL	20,268 TL	18,614 TL	10,122 TL
Munk (moderate-lo) TL	7,371 TL	8,743 TL	8,230 TL	4,233 TL
Munk (moderate-mod) TL	7,363 TL	8,735 TL	8,222 TL	4,225 TL
Munk (moderate-hi) TL	7,355 TL	8,726 TL	8,214 TL	4,216 TL
Munk (flat-lo) TL	2,478 TL	2,954 TL	3,204 TL	1,277 TL
Munk (flat-mod) TL	2,476 TL	2,952 TL	3,202 TL	1,275 TL
Munk (flat-hi) TL	2,474 TL	2,950 TL	3,200 TL	1,273 TL
	Individualized (housing)			
Munk (steep-lo) TL	43,534 TL	15,039 TL	22,984 TL	19,099 TL
Munk (steep-mod) TL	37,243 TL	14,450 TL	21,292 TL	14,126 TL
Munk (steep-hi) TL	30,948 TL	13,866 TL	19,568 TL	9,123 TL
Munk (moderate-lo) TL	17,791 TL	6,352 TL	9,734 TL	7,820 TL
Munk (moderate-mod) TL	15,227 TL	6,101 TL	9,013 TL	5,793 TL
Munk (moderate-hi) TL	12,662 TL	5,852 TL	8,282 TL	3,752 TL
Munk (flat-lo) TL	5,002 TL	2,096 TL	2,995 TL	2,218 TL
Munk (flat-mod) TL	4,296 TL	2,120 TL	2,917 TL	1,662 TL
Munk (flat-hi) TL	3,590 TL	1,938 TL	2,570 TL	1,102 TL

Table 7: Expected Utilities by Investment Option

Option	$\gamma = 5$ (default)	$\gamma = 1.5$	$\gamma = 3$	$\gamma = 10$
	Defaults			
Anadolu Hayat (riskless)	-0.0008389	-3.6004130	-0.0247934	-0.0000509
(100-age)%	-0.0000657	-2.6187470	-0.0069390	-0.0000002
Cocco et al.	-0.0000182	-2.2308240	-0.0036542	0.0000000
Markowitz	-0.0000418	-2.4745850	-0.0055327	-0.0000001
	Individualized (no housing)			
Merton (steep)	-0.0000006	-1.3747480	-0.0009703	0.0000000
Merton (moderate)	-0.0000169	-2.0951160	-0.0044611	-0.0000001
Merton (flat)	-0.0013672	-3.6052480	-0.0228439	-0.0025138
Munk (steep-lo)	-0.0000008	-1.3819700	-0.0006379	0.0000000
Munk (steep-mod)	-0.0000008	-1.3826290	-0.0006392	0.0000000
Munk (steep-hi)	-0.0000008	-1.3832900	-0.0006405	0.0000000
Munk (moderate-lo)	-0.0000228	-2.1062050	-0.0032767	-0.0000022
Munk (moderate-mod)	-0.0000229	-2.1071790	-0.0032832	-0.0000023
Munk (moderate-hi)	-0.0000230	-2.1081550	-0.0032897	-0.0000023
Munk (flat-lo)	-0.0017820	-3.6234500	-0.0216177	-0.1082611
Munk (flat-mod)	-0.0017874	-3.6245990	-0.0216431	-0.1097095
Munk (flat-hi)	-0.0017929	-3.6257500	-0.0216686	-0.1111816
	Individualized (housing)			
Munk (steep-lo)	0.0000000	-1.6058700	-0.0004201	0.0000000
Munk (steep-mod)	0.0000000	-1.6382700	-0.0004895	0.0000000
Munk (steep-hi)	-0.0000001	-1.6724140	-0.0005796	0.0000000
Munk (moderate-lo)	-0.0000007	-2.4709510	-0.0023424	0.0000000
Munk (moderate-mod)	-0.0000013	-2.5213210	-0.0027320	-0.0000001
Munk (moderate-hi)	-0.0000026	-2.5744240	-0.0032354	-0.0000066
Munk (flat-lo)	-0.0001074	-4.3012340	-0.0247341	-0.0007532
Munk (flat-mod)	-0.0001973	-4.2773710	-0.0260855	-0.0101044
Munk (flat-hi)	-0.0004049	-4.4735990	-0.0336084	-0.4086488

. Using these parameters, we have constructed heterogeneous agents, who worked and invested throughout their lifetime. We considered different investment models that our hypothetical agents would use and calculated the resulting investment capitals.

. Finally, we have calculated and compared the welfare effects of all popular models to the individualized Munk's solutions. We have concluded that for rich-to-middle class citizens, the individiualized strategies considerably increase their welfare. Moreover, the solutions we proposed can be solved analytically without complex dynamic optimizations, and therefore are easy to interpret to households. We also found that even naive life-cycle investments perform better than fixed lifetime investment.

. We propose these models to Turkish pension providers and to Turkish working-age households belonging to middle-to-upper class, as these options will increase their retirement welfare considerably.

6. References

[1] Markowitz H. (1952). Portfolio Selection. *The Journal of Finance*, 7(1), 77-91.

[2] Tobin J.(1958). Liquidity Preference as Behavior Towards Risk. *The Review of Economic Studies*, 25(2), 65-86.

[3] Merton R.C. (1971). Optimum Consumption and Portfolio Rules in a Continuous-Time Model. *Journal of Economic Theory*, 3, 373-413.

[4] Samuelson, Paul A. (1969). Lifetime Portfolio Selection by Dynamic Stochastic Programming. *Review of Economics and Statistics*, 51(3), 239-246.

[5] Canner, N., Mankiw, N.G., Weil, D.N. (1997). An Asset Allocation Puzzle. *The American Economic Review*, 87(1), 181-191.

[6] Bodie, Z., Merton, R.C., Samuelson, W.F. (1992). Labor Supply Flexibility and Portfolio Choice in a Life-Cycle Model. *Journal of Economic Dynamics and Control*, 16(3-4), 427-449.

[7] Cocco, J.F., Gomes, F.J., Maenhout P.J. (2005). Consumption and Portfolio Choice over the Life Cycle. *The Review of Financial Studies*, 18(2), 491-533.

[8] Chang, Y., Hong, J.H., Karabarbounis, M. (2014). Labor-Market Uncertainty and Portfolio Choice Puzzles. *Working Papers, Federal Reserve Bank of Richmond*, 14(13).

[9] Cocco, J.F. (2004). Portfolio Choice in the Presence of Housing. *The Review of Financial Studies*, 18(2), 537-567.

[10] Flavin M., Yamashita T. (2002). Owner-Occupied Housing and the Composition of the Household Portfolio. *The American Economic Review*, 1, 345-362.

[11] Munk C. (2016). A Mean-Variance Benchmark for Household Portfolios over the Life Cycle. *Working Papers, Copenhagen Business School, Department of Finance.*

[12] Ascheberg, M., Kraft, H., Munk, C., Weiss, F. (2013). The Joint Dynamics of Labor Income, Stock Prices, and House Prices and the Implications for Household Decisions. *Working Papers, Dauphine Universite Paris, International Workshop on Pension, Insurance and Saving.*

[13] Campbell, J., Cocco, J.F., Gomes, F.J., Maenhout, P.J. (1999). Investing Retirement Wealth: A Life-Cycle Model. *Working Papers, National Bureau of Economic Research*, 439-482.

[14] Olear G.M. (2016). Benefits of Individualized Lifecycle Investing: The Impact of Individual Wage Profile & Housing Wealth. *Master's Thesis, Tilburg University, School of Economics and Management.*

[15] TUIK (2003-2014). Hanehalki Butce Anketi. *Turkish Statistical Institute.*

[16] Aktuğ, E., Kuzubaş, T.U., Torul, O. (2017). An Investigation of Labor Income Profiles in Turkey. *Working Papers, Boğaziçi University, Department of Economics.*

[17] Ben-Porath, Y. (1967). The Production of Human Capital and the Life-Cycle of Earnings. *Journal of Political Economy*, 75, 352-365.

[18] Torul, O., Öztunalı, O. (2018). On Income and Wealth Inequality in Turkey. *Working Papers, Boğaziçi University, Department of Economics.*

[19] OECD(2018). Long-term interest rates forecast (indicator). *https://data.oecd.org*

[20] TUIK (2018). General Statistics. *http://www.turkstat.gov.tr.*

[21] TCMB (2018). Enflasyon Raporu. *http://www.tcmb.gov.tr.*

Appendix A. Ascheberg's correlation structure

The structure that returns desired correlation coefficients ρ_{SL}, ρ_{SH} and ρ_{HL} is as follows:

$$\frac{\Delta S_{t+1}}{S_t} = \mu_S + \sigma_S \cdot \epsilon_{St} \tag{A.1a}$$

$$\frac{\Delta H_{t+1}}{H_t} = \mu_H + \sigma_H \cdot \left(\rho_{SH}\epsilon_{St} + (\sqrt{1-\rho_{SL}^2})\epsilon_{Ht} \right) \tag{A.1b}$$

$$\frac{\Delta Y_{t+1}}{Y_t} = \mu_L + \sigma_L \cdot \left(\rho_{SL}\epsilon_{St} + \left(\frac{\rho_{HL} - \rho_{SH}\rho_{SL}}{\sqrt{1-\rho_{SH}^2}} \right) \epsilon_{Ht} + \left(\sqrt{1 - \rho_{SL}^2 - (\frac{\rho_{HL} - \rho_{SH}\rho_{SL}}{\sqrt{1-\rho_{SH}^2}})^2} \right) \epsilon_{Lt} \right) \tag{A.1c}$$

To derive this, let Σ be a correlation matrix of a vector $X = (x_1, x_2, ..., x_K)$. Also, let $\Sigma = LL'$ be a Cholesky decomposition of this matrix.

Notice that the variance-covariance matrix of an i.i.d. random vector $\Omega = (\epsilon_1, \epsilon_2, ..., \epsilon_K)$ with variances equal to 1, is an identity matrix. Thus, the product $L\Omega$ has the same correlation structure as X:

$$cov(L\Omega) = E[(L\Omega)(L\Omega)'] = E[L\Omega\Omega' L'],$$
$$cov(L\Omega) = L \cdot E[\Omega\Omega'] \cdot L' = L \cdot var(\Omega) \cdot L',$$
$$cov(L\Omega) = L \cdot I \cdot L' = LL' = \Sigma$$

The conclusion comes from the fact that the Cholesky decomposition of a correlation matrix R:

$$R = \begin{bmatrix} 1 & \rho_{SH} & \rho_{SL} \\ \rho_{SH} & 1 & \rho_{HL} \\ \rho_{SL} & \rho_{HL} & 1 \end{bmatrix}$$

can be easily calculated to be equal to Q:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ \rho_{SH} & \sqrt{1-\rho_{SH}^2} & 0 \\ \rho_{SL} & \frac{\rho_{HL}-\rho_{SH}\rho_{SL}}{\sqrt{1-\rho_{SH}^2}} & \sqrt{1-\rho_{SL}^2 - (\frac{\rho_{HL}-\rho_{SH}\rho_{SL}}{\sqrt{1-\rho_{SH}^2}})^2} \end{bmatrix}$$

Appendix B. Parameter Calibrations

Appendix B.1. Stock returns

We use log differences on our monthly stock price data to obtain historical monthly rates of return. The Augmented Dickey-Fuller test strongly shows that the modified series is stationary at 1% significance level.

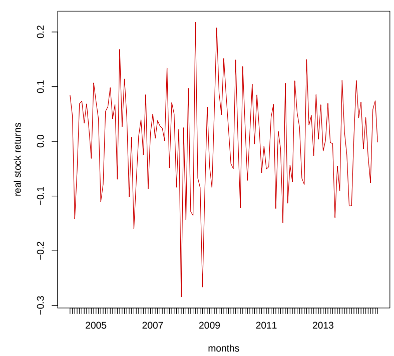


Figure B.13: Historical monthly rates of return on stocks

We use Akaike Information Criterion to find the optimal lag order for ARIMA estimation — $p = q = 2$.

Performing ARIMA(2,0,2) estimation, we obtain the long-term forecasted monthly rate of return $\mu^S_{mon} = 0.541\%$ and volatility $\sigma^S_{mon} = 11.1\%$.

Finally, we annualize these values as follows:

$$\mu_S = (1 + \mu^S_{mon})^{12} - 1 = 6.69\% \tag{B.1}$$

$$\sigma_S = \sigma^S_{mon} \cdot \sqrt{12} = 38.44\% \tag{B.2}$$

Appendix B.2. Housing returns

Similarly, we log-differentiate monthly house prices to obtain growth rates. The house market collapse of 2008 brings large external shock, causing the Augmented Dickey-Fuller test to only find stationarity at 10% significance level for 3 lags at most.

Again, using Akaike Information Criterion, we find $p = q = 1$.

ARIMA(1,0,1) estimation provides long-term forecasts for monthly rate of return and volatility as $\mu^H_{mon} = 0.06\%$ and $\sigma^H_{mon} = 1.57\%$.

Annualization gives:

$$\mu_H = (1 + \mu^H_{mon})^{12} - 1 = 0.67\% \tag{B.3}$$

$$\sigma_H = \sigma^H_{mon} \cdot \sqrt{12} = 5.42\% \tag{B.4}$$

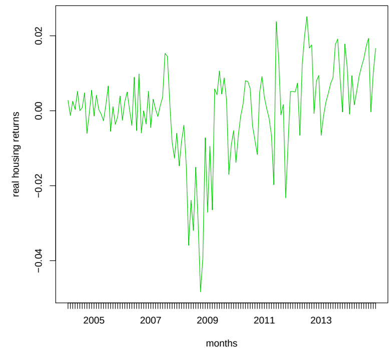


Figure B.14: Historical monthly rates of return on housing

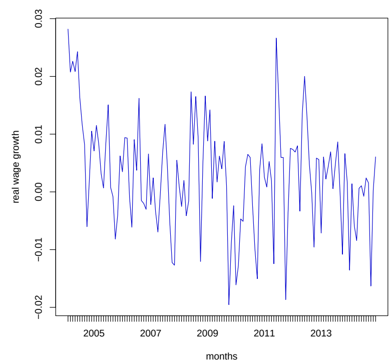


Figure B.15: Historical monthly wage growth rates

Appendix B.3. Labor income volatility

The Augmented Dickey-Fuller test returns stationarity for up to 9 lags at 5% significance level. Akaike Information Criterion suggests $p = 5$ and $q = 2$.

ARIMA(5,0,2) estimation gives monthly volatility $\sigma^L_{mon} = 1.04\%$, which annualizes as follows:

$$\sigma_L = \sigma^L_{mon} \cdot \sqrt{12} = 3.59\% \tag{B.5}$$

Appendix B.4. Wage regression results

The regressions of wage growth rates by age with kinks at 40 and 55, return the following coefficients.

Table B.8: Wage regression results by age

	flat	moderate	steep
(intercept)	0.1%	0.4%	1.5%
d40	-0.7%	3.3%	0.7%
d55	1.4%	0.9%	-0.4%

The growth rates can be calculated using these coefficients. Note that we have rounded the growth rates for the flat wages to 0. Figure B.16 illustrates how the estimated rates fit the data:

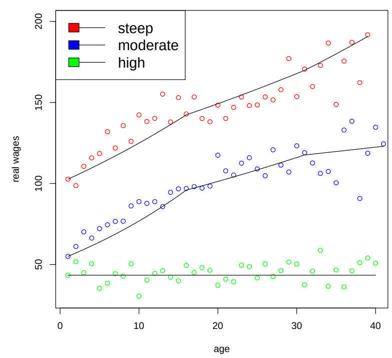


Figure B.16: Fitted values from wage regressions

Appendix C. Investment strategies

Table C.9: Homogeneous life-cycles

age	(100-age)%	Bodie et al.	Anadolu Hayat Riskless	Markowitz
26	0.75	1	0.3	0.63
27	0.74	1	0.3	0.63
28	0.73	1	0.3	0.63
29	0.72	1	0.3	0.63
30	0.71	1	0.3	0.63
31	0.7	1	0.3	0.63
32	0.69	1	0.3	0.63
33	0.68	1	0.3	0.63
34	0.67	1	0.3	0.63
35	0.66	1	0.3	0.63
36	0.65	1	0.3	0.63
37	0.64	1	0.3	0.63
38	0.63	1	0.3	0.63
39	0.62	1	0.3	0.63
40	0.61	1	0.3	0.63
41	0.6	0.975	0.3	0.63
42	0.59	0.95	0.3	0.63
43	0.58	0.925	0.3	0.63
44	0.57	0.9	0.3	0.63
45	0.56	0.875	0.3	0.63
46	0.55	0.85	0.3	0.63
47	0.54	0.825	0.3	0.63
48	0.53	0.8	0.3	0.63
49	0.52	0.775	0.3	0.63
50	0.51	0.75	0.3	0.63
51	0.5	0.725	0.3	0.63
52	0.49	0.7	0.3	0.63
53	0.48	0.675	0.3	0.63
54	0.47	0.65	0.3	0.63
55	0.46	0.625	0.3	0.63
56	0.45	0.6	0.3	0.63
57	0.44	0.575	0.3	0.63
58	0.43	0.55	0.3	0.63
59	0.42	0.525	0.3	0.63
60	0.41	0.5	0.3	0.63
61	0.4	0.5	0.3	0.63
62	0.39	0.5	0.3	0.63
63	0.38	0.5	0.3	0.63
64	0.37	0.5	0.3	0.63
65	0.36	0.5	0.3	0.63

Table C.10: Munk's life-cycles for $\gamma = 1.5$ (1 of 3)

	Munk stock			Munk housing		
age	steep-lo	steep-mod	steep-hi	steep-lo	steep-mod	steep-hi
26	0.742	0.714	0.684	0.258	0.286	0.316
27	0.730	0.702	0.674	0.270	0.298	0.326
28	0.715	0.689	0.663	0.285	0.311	0.337
29	0.698	0.675	0.650	0.302	0.325	0.350
30	0.680	0.659	0.637	0.320	0.341	0.363
31	0.661	0.642	0.622	0.339	0.358	0.378
32	0.643	0.626	0.608	0.357	0.374	0.392
33	0.624	0.609	0.594	0.376	0.391	0.406
34	0.606	0.593	0.581	0.394	0.407	0.419
35	0.589	0.579	0.568	0.411	0.421	0.432
36	0.573	0.564	0.556	0.427	0.436	0.444
37	0.560	0.553	0.546	0.440	0.447	0.454
38	0.548	0.543	0.537	0.452	0.457	0.463
39	0.539	0.534	0.530	0.461	0.466	0.470
40	0.532	0.528	0.524	0.468	0.472	0.476
41	0.525	0.522	0.519	0.475	0.478	0.481
42	0.519	0.517	0.515	0.481	0.483	0.485
43	0.515	0.514	0.512	0.485	0.486	0.488
44	0.512	0.511	0.509	0.488	0.489	0.491
45	0.509	0.508	0.507	0.491	0.492	0.493
46	0.507	0.506	0.505	0.493	0.494	0.495
47	0.505	0.505	0.504	0.495	0.495	0.496
48	0.504	0.504	0.503	0.496	0.496	0.497
49	0.503	0.503	0.502	0.497	0.497	0.498
50	0.503	0.502	0.502	0.497	0.498	0.498
51	0.502	0.502	0.501	0.498	0.498	0.499
52	0.501	0.501	0.501	0.499	0.499	0.499
53	0.501	0.501	0.501	0.499	0.499	0.499
54	0.501	0.501	0.501	0.499	0.499	0.499
55	0.501	0.500	0.500	0.499	0.500	0.500
56	0.500	0.500	0.500	0.500	0.500	0.500
57	0.500	0.500	0.500	0.500	0.500	0.500
58	0.500	0.500	0.500	0.500	0.500	0.500
59	0.500	0.500	0.500	0.500	0.500	0.500
60	0.500	0.500	0.500	0.500	0.500	0.500
61	0.500	0.500	0.500	0.500	0.500	0.500
62	0.500	0.500	0.500	0.500	0.500	0.500
63	0.500	0.500	0.500	0.500	0.500	0.500
64	0.500	0.500	0.500	0.500	0.500	0.500

Table C.11: Munk's life-cycles for $\gamma = 1.5$ (2 of 3)

	Munk stock			Munk housing		
age	mod-lo	mod-mod	mod-hi	mod-lo	mod-mod	mod-hi
26	0.742	0.714	0.684	0.258	0.286	0.316
27	0.729	0.702	0.674	0.271	0.298	0.326
28	0.714	0.688	0.662	0.286	0.312	0.338
29	0.697	0.674	0.650	0.303	0.326	0.350
30	0.679	0.658	0.636	0.321	0.342	0.364
31	0.659	0.640	0.621	0.341	0.360	0.379
32	0.641	0.624	0.607	0.359	0.376	0.393
33	0.622	0.607	0.593	0.378	0.393	0.407
34	0.604	0.592	0.579	0.396	0.408	0.421
35	0.588	0.577	0.567	0.412	0.423	0.433
36	0.572	0.563	0.555	0.428	0.437	0.445
37	0.559	0.552	0.545	0.441	0.448	0.455
38	0.548	0.542	0.537	0.452	0.458	0.463
39	0.539	0.534	0.529	0.461	0.466	0.471
40	0.532	0.528	0.524	0.468	0.472	0.476
41	0.525	0.522	0.519	0.475	0.478	0.481
42	0.520	0.517	0.515	0.480	0.483	0.485
43	0.515	0.514	0.512	0.485	0.486	0.488
44	0.512	0.511	0.509	0.488	0.489	0.491
45	0.509	0.508	0.507	0.491	0.492	0.493
46	0.507	0.506	0.506	0.493	0.494	0.494
47	0.506	0.505	0.504	0.494	0.495	0.496
48	0.504	0.504	0.503	0.496	0.496	0.497
49	0.503	0.503	0.503	0.497	0.497	0.497
50	0.503	0.502	0.502	0.497	0.498	0.498
51	0.502	0.502	0.501	0.498	0.498	0.499
52	0.501	0.501	0.501	0.499	0.499	0.499
53	0.501	0.501	0.501	0.499	0.499	0.499
54	0.501	0.501	0.501	0.499	0.499	0.499
55	0.501	0.501	0.500	0.499	0.499	0.500
56	0.500	0.500	0.500	0.500	0.500	0.500
57	0.500	0.500	0.500	0.500	0.500	0.500
58	0.500	0.500	0.500	0.500	0.500	0.500
59	0.500	0.500	0.500	0.500	0.500	0.500
60	0.500	0.500	0.500	0.500	0.500	0.500
61	0.500	0.500	0.500	0.500	0.500	0.500
62	0.500	0.500	0.500	0.500	0.500	0.500
63	0.500	0.500	0.500	0.500	0.500	0.500
64	0.500	0.500	0.500	0.500	0.500	0.500

Table C.12: Munk's life-cycles for $\gamma = 1.5$ (3 of 3)

	Munk stock			Munk housing		
age	flat-lo	flat-mod	flat-hi	flat-lo	flat-mod	flat-hi
26	0.742	0.713	0.684	0.258	0.287	0.316
27	0.729	0.701	0.674	0.271	0.299	0.326
28	0.713	0.687	0.661	0.287	0.313	0.339
29	0.695	0.672	0.648	0.305	0.328	0.352
30	0.675	0.654	0.633	0.325	0.346	0.367
31	0.654	0.636	0.617	0.346	0.364	0.383
32	0.634	0.618	0.602	0.366	0.382	0.398
33	0.614	0.601	0.587	0.386	0.399	0.413
34	0.596	0.585	0.573	0.404	0.415	0.427
35	0.579	0.570	0.560	0.421	0.430	0.440
36	0.564	0.556	0.549	0.436	0.444	0.451
37	0.551	0.545	0.539	0.449	0.455	0.461
38	0.541	0.536	0.531	0.459	0.464	0.469
39	0.532	0.529	0.525	0.468	0.471	0.475
40	0.526	0.523	0.520	0.474	0.477	0.480
41	0.520	0.518	0.515	0.480	0.482	0.485
42	0.516	0.514	0.512	0.484	0.486	0.488
43	0.512	0.511	0.509	0.488	0.489	0.491
44	0.509	0.508	0.507	0.491	0.492	0.493
45	0.507	0.506	0.505	0.493	0.494	0.495
46	0.506	0.505	0.504	0.494	0.495	0.496
47	0.504	0.504	0.503	0.496	0.496	0.497
48	0.503	0.503	0.503	0.497	0.497	0.497
49	0.503	0.502	0.502	0.497	0.498	0.498
50	0.502	0.502	0.502	0.498	0.498	0.498
51	0.502	0.501	0.501	0.498	0.499	0.499
52	0.501	0.501	0.501	0.499	0.499	0.499
53	0.501	0.501	0.501	0.499	0.499	0.499
54	0.501	0.501	0.500	0.499	0.499	0.500
55	0.500	0.500	0.500	0.500	0.500	0.500
56	0.500	0.500	0.500	0.500	0.500	0.500
57	0.500	0.500	0.500	0.500	0.500	0.500
58	0.500	0.500	0.500	0.500	0.500	0.500
59	0.500	0.500	0.500	0.500	0.500	0.500
60	0.500	0.500	0.500	0.500	0.500	0.500
61	0.500	0.500	0.500	0.500	0.500	0.500
62	0.500	0.500	0.500	0.500	0.500	0.500
63	0.500	0.500	0.500	0.500	0.500	0.500
64	0.500	0.500	0.500	0.500	0.500	0.500