The Joint Dynamics of Labor Income, Stock Prices, and House Prices and the Implications for Household Decisions

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JEL subject codes: G11, D91, D14

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The Joint Dynamics of Labor Income, Stock Prices, and House Prices and the Implications for Household Decisions

Abstract: We show that in the 1953-2010 U.S. data stock prices, house prices, and labor income have time-varying expected growth rates. These drift rates are correlated with shocks to stock prices, house prices, and income so that these quantities tend to move together in the longer run in spite of low contemporaneous correlations. Incorporating these long-run relations into a life-cycle household optimization problem affects the optimal consumption, housing, and investment decisions. The optimal portfolio of young households is dominated by investments in housing (physical real estate or house price linked financial assets), whereas nothing or little is invested in the stock market in line with observed behavior.

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Contents

1 Introduction

The literature on individuals' consumption and investment decisions over the life-cycle has progressed significantly in recent years. Still, only few papers incorporate both human capital and residential real estate which are the two major assets of many individuals (Campbell 2006) and whose characteristics are therefore likely to have significant effects on the optimal decisions. Moreover, for reasons of tractability and modeling tradition, the existing papers impose various restrictive assumptions. Most importantly, they assume that the dynamics of the labor income of the individual, house prices, and the prices of financial assets are related at most via some constant, instantaneous correlation coefficients often estimated to be near zero. This implies that the income, house prices, and financial asset prices can drift far apart in the long run, which conflicts both with economic intuition and empirical evidence.

Earlier papers have supported stronger long-run relations between stock prices and labor income (Benzoni, Collin-Dufresne, and Goldstein 2007) and between house prices and labor income (Lustig and van Nieuwerburgh 2005). We estimate the joint dynamics of all three variables based on aggregate, quarterly 1953-2010 U.S. data for the S&P 500 stock index, the national Case-Shiller home price index, and per capita income. We find that positive [negative] shocks to stock prices in one period tend to be followed by an increase [decrease] in the expected growth rate of stock prices, house prices, and labor income in subsequent periods. Likewise, positive [negative] shocks to house prices tend to be followed by an increase [decrease] in the expected growth of stock prices, house prices, and income in subsequent periods. And positive [negative] shocks to labor income in one period typically comes with higher [lower] than average growth in stock and house prices in the following periods. These mechanisms lead to stronger long-run relations between stock prices, house prices, and labor income than revealed by their low contemporaneous correlations. We find similar results using 1987-2010 house price and income data from 14 U.S. metropolitan areas.

The long-run relations between stock prices, house prices, and labor income are potentially important for household-level consumption, housing, and investment decisions. We embed the estimated long-run dynamic model of labor income, house prices, and stock prices—adjusted to reflect idiosyncratic income and house price risk—into a continuous-time, life-cycle utility maximization problem of an individual (or household). The individual has time-additive Cobb-Douglas power utility over the consumption of perishable goods and of housing services. The individual earns a labor income stream with an age-dependent drift until retirement, after which a pension equal to a fraction of pre-retirement income is received. Labor income entails risks that cannot be hedged through investments. Savings can be invested in a risk-free asset, a stock index, and in housing units. Housing units can be rented or purchased (and resold) and, hence, the individual can disentangle housing consumption from housing investments. We impose standard short-selling and borrowing constraints.

We show that the optimal consumption, housing, and investment decisions over the life-cycle in this setting are markedly different from the optimal decisions in the standard model ignoring the long-run relations. The optimal portfolio of young households is dominated by investments in housing, either in physical real estate or in financial assets closely linked to house prices such as shares of residential REITs (Real Estate Investment Trusts). The housing investment consists of a speculative demand, a component adjusting for the implicit exposure to house price shocks via the human capital, and by a term hedging against future housing costs as emphasized by Sinai and Souleles (2005). We find that the hedge term is the main driver for the relatively high housing investment of the young households in our model.

Typical young households should optimally invest little in stocks, which resembles observed investment behavior but is in stark contrast to the conclusions from simpler life-cycle models where the high human capital of young individuals leads to a high investment—heavily leveraged, if possible—in stocks. Our findings corroborate the conclusion in the simpler model of Benzoni, Collin-Dufresne, and Goldstein (2007), in which the relatively high long-run income-stock correlation implies that human capital significantly crowds out stock investment.

The decision problem is very complex because it involves continuous decisions with incomplete markets (because of unspanned labor income risk), various constraints on the choice variables, and six state variables. Therefore, we cannot derive a closed-form solution. Moreover, due to the high number of state variables, standard grid-based dynamic programming techniques would be computationally infeasible. We follow Bick, Kraft, and Munk (2013) and optimize over a family of closed-form consumption and investment strategies that are parameterized by a low number of constants. To obtain this set of strategies we embed the problem into a parameterized family of artificial unconstrained and complete markets. In each of these artificial markets we derive the optimal consumption and investment strategy in closed form and then we transform each such strategy into a feasible strategy in the true (constrained and incomplete) market. The expected utility in the true market of each strategy can be evaluated by Monte Carlo simulations and, by embedding that evaluation in a standard numerical optimization over the parameters, we determine the best of these strategies. We show that this strategy deviates from the unknown, truly optimal strategy by at most a few percent in terms of certainty equivalent of wealth, whereas standard numerical methods offer no similar measure of precision.

The remainder of the paper is organized as follows. Section 2 positions the paper relative to the existing literature. Section 3 sets up the model of stock price, house price, and income dynamics, which is then taken to the data in Section 4. In Section 5 we formulate the life-cycle utility maximization problem of an individual consumer-investor. Section 6 describes the solution method, whereas Section 7 presents and discusses various aspects of the derived life-cycle consumption and investment behavior. Finally, Section 8 summarizes the paper.

2 Literature review

The impact of risky labor income on consumption and portfolio choice has been studied in papers such as Bodie, Merton, and Samuelson (1992), Heaton and Lucas (1997), Viceira (2001), Cocco, Gomes, and Maenhout (2005), Koijen, Nijman, and Werker (2010), and Munk and Sørensen (2010). The labor income rate is typically modeled as having a drift and volatility depending at most on age, and exhibiting a constant, near-zero correlation with the stock market. Consequently, human capital resembles an investment in bonds so that the individual should actively invest much more in stocks than in the absence of labor income. In particular, young investors should optimally take highly leveraged positions in stock market, which contrasts observed investment behavior.

Benzoni, Collin-Dufresne, and Goldstein (2007) deviate from tradition by assuming that the (economy-wide component of the individual's) labor income and stock market dividends are cointegrated, and they provide empirical support thereof. Because of the cointegration property, the human capital of young individuals is more stock-like, which lowers their optimal stock investment significantly. For some model parameterizations, individuals should not participate in the stock market at a very young age but then gradually increase the proportion of financial wealth invested in stocks until retirement age. While we do not find support for a strict cointegration relation, human capital is still more stock-like in our calibrated model than in the standard models. Lynch and Tan (2011) feed a stock-income link into the model by assuming that the both expected income growth and expected stock returns are affine functions of the dividend yield of the stock index. We do not restrict the drift rates of stock prices and income to be driven by the same variable. We further generalize the analysis in these papers by adding housing decisions and considering the joint dynamics of stock prices, labor income, and house prices.[1]

A few recent papers include both labor income and housing in life-cycle decision problems, as we do in our paper.[2] Focusing on mortgage choice, Campbell and Cocco (2003) force housing investment to equal housing consumption and fix the house size so that they cannot address the interaction between housing decisions and portfolio decisions. In a similar model, assuming for tractability a perfect correlation between house prices and aggregate income shocks, Cocco (2005) concludes that house price risk crowds out stock holdings and can therefore help in explaining limited stock market participation. In our more general setting, we also find that the optimal stock investment is zero or low for many young households.

Yao and Zhang (2005) generalize Cocco's setting to an imperfect house-income correlation and endogenize the renting/owning decision, but so that a renter has zero wealth exposure to house price risk and a house owner must have a housing consumption identical to the housing investment position. They find that home-owners invest less in stocks than home-renters, which again can be interpreted as housing risk crowding out stock market risk. van Hemert (2010) generalizes their setting further by allowing for stochastic variations in interest rates and thereby introducing a role for bonds, and his focus is on the interest rate exposure and choice of mortgage over the life-cycle.

Kraft and Munk (2011) allow the housing investment position to differ from the housing consumption by simultaneous owning and renting (out) or by investing in house price linked financial contracts. This is also a feature of our model. To obtain closed-form solutions, they focus on the unrealistic case in which all risks are spanned and no constraints are imposed. Due to an assumed high instantaneous correlation between labor income and house prices, they find that young individuals prefer little exposure to house price risk and tend to rent their home, whereas a larger exposure and home ownership is optimal later in life when human capital is smaller. We generalize their setting by incorporating the long-run relations between income, house prices, and stock prices as well as unspanned income risk and standard portfolio constraints. Our model thus combines central elements of Benzoni, Collin-Dufresne, and Goldstein (2007) and Kraft and Munk (2011) into a rich and realistic life-cycle model.

Other papers addressing various aspects of housing in individual decision making include Cauley, Pavlov, and Schwartz (2007), Li and Yao (2007), Attanasio, Bottazzi, Low, Nesheim, and Wakefield

[1] Also Santos and Veronesi (2006) provide evidence of the link between labor income and stock returns.

[2] Some papers ignore labor income but include various housing aspects in portfolio choice problems. See, for example, Grossman and Laroque (1990), Goetzmann (1993), Flavin and Yamashita (2002), Pelizzon and Weber (2009), and Corradin, Fillat, and Vergara-Alert (2012).

(2012), Chetty and Szeidl (2012), and Fischer and Stamos (2013). Han (2013) identifies local differences in the risk-return relationship for housing and explains this with differences in incentives for hedging against future housing expenditures. We find that this hedging demand constitutes an important part of the optimal housing investment.

A key feature of our model is the long-run relationship between the stock market index, (an index of) house prices, and (the aggregate component of) labor income. As explained above, Benzoni, Collin-Dufresne, and Goldstein (2007) present empirical evidence that the stock market and aggregate labor income are cointegrated and further evidence is reported by Baxter and Jermann (1997). Labor income has long been recognized as a main driver of house prices, see Poterba (1991). Lustig and van Nieuwerburgh (2005) report evidence of a cointegration relationship between aggregate per capita labor income and real estate wealth per household. Further, the deviation from that cointegration relation (termed the housing collateral) predicts stock returns, which suggests that all three variables are intimately linked. Housing wealth serves as a buffer against shocks to labor income and rents and, therefore, the size of housing collateral affects risk premia.

Papers testing for cointegration between house prices and various fundamentals including measures of labor income arrive at mixed conclusions (Malpezzi 1999; Holly, Pesaran, and Yamagata 2010; Zhou 2010). As house prices are also affected by mortgage flexibility, credit constraints, home construction costs, as well as demographical and geographical factors, the failure of a strict cointegration relation between house prices and income is not surprising.

3 A dynamic model of stock prices, house prices, and labor income

In this section, we set up a relatively general model for the joint dynamics of labor income, stock prices, and house prices. In the following section we estimate the full model and various special cases of the model based on U.S. data. In subsequent sections we derive and analyze the optimal consumption and investment decisions over the life-cycle of an individual or a household facing the estimated joint dynamics. For that purpose a continuous-time formulation is preferable.

Let S_t denote the level of the stock market index, H_t the level of a house price index, and L_t

the labor income rate at time t. We assume that the dynamics of these variables are

$$\frac{dS_t}{S_t} = (r + \mu_S + x_t)\, dt + \sigma_S\, dB_{St}, \tag{1}$$

$$\frac{dH_t}{H_t} = (r + \mu_H + y_t)\ dt + \sigma_H \left(\rho_{HS}\, dB_{St} + \hat{\rho}_H\, dB_{Ht}\right), \tag{2}$$

$$\frac{dL_t}{L_t} = (\mu_L(t) + \zeta(t) z_t)\ dt + \sigma_L(t) \left(\rho_{LS}\, dB_{St} + \hat{\rho}_{LH}\, dB_{Ht} + \hat{\rho}_L\, dB_{Lt}\right), \tag{3}$$

where x_t, y_t, and z_t are latent variables with dynamics

$$dx_t = -\kappa_x x_t\, dt + \sigma_x \big(\rho_{xS}\, dB_{St} + \hat{\rho}_{xH}\, dB_{Ht} + \hat{\rho}_{xL}\, dB_{Lt}\big), \tag{4}$$

$$dy_t = -\kappa_y y_t\, dt + \sigma_y \big(\rho_{yS}\, dB_{St} + \hat{\rho}_{yH}\, dB_{Ht} + \hat{\rho}_{yL}\, dB_{Lt}\big), \tag{5}$$

$$dz_t = -\kappa_z z_t\, dt + \sigma_z \big(\rho_{zS}\, dB_{St} + \hat{\rho}_{zH}\, dB_{Ht} + \hat{\rho}_{zL}\, dB_{Lt}\big). \tag{6}$$

Here B_S, B_H, B_L are independent standard Brownian motions. All instantaneous correlations are constant. We let $\rho_{HS} = \rho_{SH}$ denote the instantaneous correlation between the stock price and the house price and use similar notation for other pairs of processes. In addition, define

$$\hat{\rho}_H = \sqrt{1 - \rho_{HS}^2}\,, \qquad \hat{\rho}_{LH} = \frac{\rho_{LH} - \rho_{LS}\rho_{HS}}{\sqrt{1 - \rho_{HS}^2}}\,, \qquad \hat{\rho}_L = \sqrt{1 - \rho_{LS}^2 - \hat{\rho}_{LH}^2}\,,$$

$$\hat{\rho}_{xH} = \frac{\rho_{xH} - \rho_{xS}\rho_{HS}}{\sqrt{1 - \rho_{HS}^2}}\,, \qquad \hat{\rho}_{yH} = \frac{\rho_{yH} - \rho_{yS}\rho_{HS}}{\sqrt{1 - \rho_{HS}^2}}\,, \qquad \hat{\rho}_{zH} = \frac{\rho_{zH} - \rho_{zS}\rho_{HS}}{\sqrt{1 - \rho_{HS}^2}}\,,$$

$$\hat{\rho}_{xL} = \sqrt{1 - \rho_{xS}^2 - \hat{\rho}_{xH}^2}\,, \qquad \hat{\rho}_{yL} = \sqrt{1 - \rho_{yS}^2 - \hat{\rho}_{yH}^2}\,, \qquad \hat{\rho}_{zL} = \sqrt{1 - \rho_{zS}^2 - \hat{\rho}_{zH}^2}\,.$$

Furthermore, r is the interest rate, σ_S is the volatility and μ_S is the long-term expected excess appreciation rate of the stock, whereas σ_H is the volatility and μ_H is the long-term expected excess growth rate of house prices. All these quantities are assumed constant. The stock pays a constant dividend yield of $\bar{D}$ so that the total dividends paid out over a short interval $[t, t + dt]$ is $\bar{D} S_t\, dt$. The term μ_L denotes the expected income growth rate under average conditions ($z_t = 0$), ζ is the sensitivity of expected income growth to the z-variable, and σ_L is the volatility of the income rate. We allow these quantities to depend on the age of the individual in line with Cocco, Gomes, and Maenhout (2005); details follow later.

The processes x, y, and z are Gaussian and mean-reverting around zero with the corresponding κ_i denoting the speed of mean reversion and the corresponding σ_i denoting the volatility

($i = x, y, z$). These processes capture time-varying components of the expected stock return, the expected house price changes, and the expected income growth. While a huge literature exists on time-varying expected stock returns and also some studies of time-varying expected growth rates in house price and income, our model appears to be the first joint model of stock, house, and income dynamics with these features.[3]

An important aspect of our model is that the increments to x, y, and z are correlated with increments to S, H, and L. This implies that the longer-term relations between S, H, and L can be markedly different from the short-term relations. For example, if the correlation ρ_{yS} is positive, a positive shock to stock prices in one period will typically lead to expected growth in house prices in the following periods, an effect that will gradually weaken because of the mean reversion of y to zero. In this case, stock and house prices will tend to follow each other in the longer run, even though the instantaneous correlation ρ_{HS} might be zero. Furthermore, because of the sensitivity of expected stock returns to income shocks and the sensitivity of expected income growth to stock price shocks, the human capital may constitute more or less of an implicit stock investment than suggested by the contemporaneous correlation ρ_{LS}. These effects are potentially important for the optimal investment and consumption decisions.

Also note that our model can incorporate momentum and reversal effects. A negative value of ρ_{xS} captures reversals or mean reversion in stock prices, in accordance with the simpler portfolio choice models of Kim and Omberg (1996) and Wachter (2002). A positive value of ρ_{xS} captures momentum. Similarly for house prices and income rates.

Our model (1)–(6) assumes that the processes x, y, z are spanned by the processes S, H, L in the sense that there are no idiosyncratic shocks to x, y, z. While this may seem restrictive, our empirical analysis has shown that the model above achieves virtually the same likelihood as a more general model with independent Brownian increments dB_{xt}, dB_{yt}, dB_{zt} added to (4)–(6); more about this in Section 4. For parsimony, we therefore disregard such idiosyncratic shocks.

The model contains various known life-cycle models as special cases. Benzoni, Collin-Dufresne, and Goldstein (2007) ignore house prices (no H or y) and keep the expected stock return constant (no x). By assuming a strict cointegration relation between stock and income, they require that the time-varying component of income (z in our notation) is fully spanned by the shocks to stock and income. We do not impose that requirement, but let the data speak. Furthermore, their model

[3]Time-varying expected growth rates are well-documented for stocks (cf. the huge return predictability literature; see the recent review by Koijen and van Nieuwerburgh (2011)) and the implications for stock-bond asset allocation has been explored by Kim and Omberg (1996) and Campbell and Viceira (1999) disregarding housing and income.

requires that $\kappa_z = \zeta(t) = 1$, $\sigma_z \rho_{zS} = \sigma_L \rho_{LS} - \sigma_S$ and $\sigma_z \hat{\rho}_{zL} = \sigma_L \hat{\rho}_L$ (see their Eqs. (7), (14), and (18), ignoring individual labor income risk). Lynch and Tan (2011) also ignore house prices, but let both expected income growth and expected stock returns be affine in the dividend yield of the stock index. In our model, this corresponds to x and z being identical. On the other hand, they allow for state-dependent income volatility.

Only few papers model both income and stock and house prices. Yao and Zhang (2005) apply simple discrete-time binomial processes with constant expected growth rates. Both van Hemert (2010) and Kraft and Munk (2011) use a continuous-time formulation, let interest rates be stochastic, but assume constant risk premia (no x, y, z). van Hemert solves the utility maximization problem numerically by dynamic programming on a coarse grid approximation of the state space and focuses on mortgage choice. Kraft and Munk derive a closed-form solution for the special case in which labor income is spanned by traded assets (no B_L). Fischer and Stamos (2013) allow for a time-varying drift in the house price, but not in the stock price or income (no x or z). They assume that the time-varying component of the house price drift is fully determined by the preceding year's growth in the house price. We capture that effect via the correlation ρ_{yH}, but we also allow stock prices and labor income to influence the house price drift. Furthermore, they assume given values of the correlation coefficients (that are potentially very important for investment decisions), whereas we estimate them from data.

4 Empirical analysis

4.1 Data

We use quarterly U.S. data for stock prices, house prices, and labor income starting in 1953q1 (i.e., first quarter of 1953) and ending in 2010q2. All time-series are inflation-adjusted to 2010q2 dollars using the consumer price index (CPI). Data on the stock market index, house prices, and the CPI are taken from Professor Robert Shiller's homepage.[4] The stock market is represented by the Standard and Poor (S&P) composite stock price index. In our main analysis, we use the national Case-Shiller home price index to represent the evolution in house prices. From the National Income and Product Accounts (NIPA) tables published by the Bureau of Economic Analysis of the U.S. Department of Commerce, we obtain quarterly U.S. data for aggregated disposable personal income. We divide the disposable income by the population size reported in the NIPA table to

[4] http://www.econ.yale.edu/~shiller/data.htm

compute the disposable labor income per capita. Figure 1 depicts the time series of the stock price index, per capita labor income, and the national home price index.

[Figure 1 about here.]

In addition, we use house price and income data for the 14 U.S. metropolitan areas for which the local Case-Shiller home price index time series (from S&P) are available from 1987q1 onwards.[5] The regional labor income data is downloaded from the Bureau of Economic Analysis of the U.S. Department of Commerce. We use quarterly personal income for the state in which the area is located. The population size in each state is only reported annually. We interpolate linearly to the quarterly frequency and obtain quarterly disposable income per capita.[6] For all regions, we use the same U.S. stock market data. Again, all time series are computed in real terms using the CPI as deflator and quoted in 2010q2 dollars.

4.2 Tests of stationarity and pairwise cointegration

In this section, we investigate empirically whether stock prices, labor income, and house prices are pairwise cointegrated, which would strongly restrict their joint long-run behavior. Two time series are said to be cointegrated if they are nonstationary (have a unit root) individually, but a linear combination of them is stationary so that they cannot diverge in the long run.

First, we test whether the single time series of log stock prices, log house prices, and labor income follow have a unit root. For each time series we estimate the augmented Dicky-Fuller (ADF) regression model

$$\Delta v(t) = \xi_1 + \xi_2 t + \xi_3 v(t-1) + \sum_{j=1}^{J} \phi_j \Delta v(t-j) + \varepsilon(t), \tag{7}$$

potentially allowing for a time trend (through ξ_2) and lags up to order J. We perform the ADF test for the null hypothesis that the variable follows a unit root process.

The results for the MacKinnon approximate p-values are shown in Table 1. The first two rows show the results when no lags are included ($J = 0$) so that the error terms are serially uncorrelated.

[5]The areas are centered around Boston, Charlotte, Chicago, Cleveland, Denver, Las Vegas, Los Angeles, Miami, New York, Portland, San Diego, San Francisco, Tampa Bay, and Washington DC.

[6]Some metropolitan areas spread over multiple states. In these cases we use income and population data from the state in which most of the area is located. More specifically, the Charlotte area is matched with NC data, the Chicago area with IL data, the Las Vegas area with NV data, the New York area with NY data, the Portland area with OR data, and the Washington area with DC data.

In the first row we additionally assume that there is not time trend ($\xi_2 = 0$), whereas in the second row we allow for a time trend. In the third and fourth row we report ADF results with one or two lags. We conclude from the table that we cannot reject the null of a unit root for all time series, which opens up for cointegration features among the three time series.

[Table 1 about here.]

Secondly, we test for pairwise cointegration between stock prices, house prices, and income. We check if the log house prices, log labor income and log stock prices follow a common trend. We perform an augmented Dickey-Fuller test on the pairwise difference between the time series. Therefore, we define the variables:

$$sh = \ln S - \ln H, \quad sl = \ln S - \ln L, \quad hl = \ln H - \ln L. \tag{8}$$

We then estimate the ADF regression model (7) on each of the time series sh, sl, and hl.

The results are reported in Table 2. Again for each panel, the first two rows report the results under the assumption that the error terms are serially uncorrelated ($J = 0$), while in the subsequent rows we allow that the error terms are serially correlated ($J \geq 1$). Furthermore, in the first row the results are reported where we do not allow for a time trend. We are not able to formally reject the unit root hypothesis, but the ADF test is known to have low power. Furthermore, if the cointegration works at rather long horizons, it may in any case be difficult to detect in our relatively short data set. Note that the estimate of the ξ_3 parameter is in most cases significant at a 10% level and in some cases even at a 5% level. The negative estimates of ξ_3 supports the idea that if the log stock-house price ratio is high [low] this quarter, it will tend to decrease [increase] over the next quarter. This is also true for the log stock-income ratio and the log house-income ratio.

Since the empirical conclusion on cointegration is ambiguous, our general model (1)–(6) deviates from strict cointegration by allowing idiosyncratic shocks to the variables x, y, and z as represented by the Brownian motions B_x, B_y, and B_z. In the same spirit, we do not impose any restrictions on the sensitivity of x, y, and z to the shocks represented by B_S, B_H, and B_L (such restrictions are natural under strict cointegration, see Benzoni, Collin-Dufresne, and Goldstein (2007)). Next, we estimate both the general model and various special cases.

[Table 2 about here.]

4.3 Calibration

We calibrate the full model and various special cases both to the national and the regional data. Given initial guesses of the model parameters, we use a standard Kalman filter to obtain time series of the unobservable processes x, y, and z. We obtain the log-likelihood from the Kalman filter. By maximizing this log-likelihood and iterating using the Kalman filter in each step, we obtain our final parameter estimates. In order to apply the Kalman filter, we use an Euler discretization of the dynamic system (1)–(6) over quarterly periods in accordance with our data frequency. As we estimate our model on aggregate income data, we replace the age-dependent income terms $\mu_L(t), \sigma_L(t)$ by constants $\bar{\mu}_L, \bar{\sigma}_L$ and fix $\zeta(t) \equiv 1$. The age-dependence will be reintroduced when we consider the decision problem of an individual consumer-investor in later sections. We fix the risk-free rate at $r = 0.015$.

In addition to our model (1)–(6), we estimate several models in which we add idiosyncratic shocks to the processes x, y, and z represented by increments to Brownian motions B_x, B_y, and B_z independent of the other Brownian motions. In the full extended model, the processes (4)–(6) are thus replaced by

$$\begin{aligned}
dx_t &= -\kappa_x x_t\,dt + \sigma_x\big(\rho_{xS}\,dB_{St} + \hat{\rho}_{xH}\,dB_{Ht} + \hat{\rho}_{xL}\,dB_{Lt} + \hat{\rho}_x\,dB_{xt}\big),\\
dy_t &= -\kappa_y y_t\,dt + \sigma_y\big(\rho_{yS}\,dB_{St} + \hat{\rho}_{yH}\,dB_{Ht} + \hat{\rho}_{yL}\,dB_{Lt} + \hat{\rho}_y\,dB_{yt}\big),\\
dz_t &= -\kappa_z z_t\,dt + \sigma_z\big(\rho_{zS}\,dB_{St} + \hat{\rho}_{zH}\,dB_{Ht} + \hat{\rho}_{zL}\,dB_{Lt} + \hat{\rho}_z\,dB_{zt}\big).
\end{aligned}$$

In that case, we define the coefficient $\hat{\rho}_{xL} = (\rho_{xL} - \rho_{xS}\rho_{LS} - \hat{\rho}_{xH}\hat{\rho}_{LH}) / \sqrt{1 - \rho_{LS}^2 - \hat{\rho}_{LH}^2}$ and $\hat{\rho}_x = \sqrt{1 - \rho_{xS}^2 - \hat{\rho}_{xH}^2 - \hat{\rho}_{xL}^2}$ and similarly for y and z. We also consider models without any income-specific shock as represented by increments to the Brownian motion B_L. Table 3 reports the log-likelihood for the national house and income data both for 1953-2010 and for 1987-2010 as well as the average log-likelihood across the 14 regions over the period 1987-2010. Note that our model achieves the same likelihood (with the digits shown) as the extended model. Other models perform equally well.

[Table 3 about here.]

Since the estimation is based on a house price index and aggregate income data, the volatilities are lower than the volatility of an individual house price and the labor income of a typical worker. In line with the existing literature, we set the house price volatility to $\sigma_H = 0.12$ (Flavin and

Yamashita 2002, Yao and Zhang 2005) and the average income volatility (before retirement) to $\bar{\sigma}_L = 0.10$ (Cocco, Gomes, and Maenhout 2005).

In Table 4 we list benchmark parameter values for our model, both for the nationwide 1953q1-2010q2 U.S. sample and for the 1987q1-2010q2 regional data of Charlotte, North Carolina which is the region that gave the best fit to the model (RESULTS ARE INCOMPLETE). The stated parameters are estimates from the data except for the values discussed above.

[Table 4 about here.]

The estimates of the instantaneous correlations between stock prices, house prices, and labor income are all close to zero. The estimated correlation ρ_{xS} between the stock price and the time-varying drift component of the stock price is very high which can be interpreted as a momentum effect. We see a similar feature for house prices. In contrast, the drift of labor income is negatively correlated with shocks to labor income, which can be explained by seasonality in employment and income. The stock price has a positive and relatively high correlation with the drift rates of house prices and labor income. Likewise, house prices are positively correlated with the drift rates of stock prices and labor income. And labor income is positively correlated with the drift rates of stock prices and house prices. These patterns imply that stock prices, house prices, and labor income tend to more highly correlated in the longer run than suggested by their low contemporaneous correlations.

From the estimation we can filter out the dynamics of the latent variables. Figure **??** depicts the filtered dynamics of y_t based on national 1953-2010 data.Recall that the value of y_t reflects the deviation of the expected house price growth from the long-term average. We can clearly observe the housing bubble from the late 1990s to 2006 with an extra expected price growth of up to 9% and the subsequent correction with an expected fall in prices of more than 10%. The graph also show signs of some preceding cycles of smaller magnitudes.

[Figure 2 about here.]

5 The decision problem of a consumer-investor

We embed the estimated model (1)–(6) of the dynamics of stock prices S_t, house prices H_t, and labor income L_t in the life-cycle consumption and investment choice problem of an individual agent (consumer-investor or household). The agent consumes a perishable good and housing services from living in a house (for simplicity, we let "house" represent any type of residential real estate). The perishable consumption good serves as the numéraire so that the prices of all goods and assets are expressed in units of this good. The agent can invest in a bank account with a constant interest rate r and in the stock index with value S_t.

The agent can invest in and rent houses. A house is characterized by a number of housing units, where a "unit" is some one-dimensional representation of the size, quality, and location. Prices of all houses are assumed to move in parallel. The purchase of a units of housing costs aH_t. The unit rental cost of houses is assumed proportional to the product of the unit house price so that the total costs of renting ϕ housing units over a short period $[t, t+dt]$ are $\phi R H_t \, dt$. These assumptions are standard in the consumption and investment literature involving housing. Following Kraft and Munk (2011) we impose no restrictions on the number of units owned and rented. In particular, simultaneous owning and renting is possible. Additionally, the agent can invest in financial assets linked to house prices, such as shares in residential REITs (Real Estate Investment Trusts) which offer the individual a pure investment exposure to house prices.[7] REITs are assumed to rent out the housing units they own and pass on all returns to their owners so that the rate of return on REITs over a period of length dt will be

$$R\,dt + \frac{dH_t}{H_t} = \left(r + \mu_H' + y_t\right)\,dt + \sigma_H\left(\rho_{HS}\,dB_{St} + \hat{\rho}_H\,dB_{Ht}\right)\,,$$

where $\mu_H' = \mu_H + R$ is the excess expected return on housing investments.

Let ϕ_{ot} and ϕ_{rt} denote the number of housing units owned and rented, respectively, at time t, and let ϕ_{ft} denote the housing units owned via financial assets like REITs. What matters for the agent is the total units of houses occupied, ϕ_{Ct}, which provides utility from housing services, and the total units of housing invested in, ϕ_{It}, either physically owned or through REITs, where

$$\phi_{Ct} \equiv \phi_{ot} + \phi_{rt}, \qquad \phi_{It} \equiv \phi_{ot} + \phi_{ft}\,. \tag{9}$$

Hence, we have a degree of freedom. For tractability, we do not impose transaction costs even for physical housing transactions. Given the sizeable real-life transactions costs, agents do not change their physical ownership of houses frequently, but between physical transactions they can

[7] Well-developed REIT markets exist in many countries. Cotter and Roll (2011) study the risk and return characteristics of U.S. REITs. Tsai, Chen, and Sing (2007) report that REITs behave more and more like real estate and less and less like ordinary stocks.

adjust their investment exposure to house prices via REITs that trade at low transaction costs. Furthermore, a very small desired housing investment position may be difficult to obtain through physical ownership, but is obtainable via REITs. Hence, physical ownership and REITs investments complement each other, but we do not distinguish them in the model. Also note that Kraft and Munk (2011) report that, in their simpler model, the welfare of the agent is only little affected if he is only allowed to adjust his housing investment position every second or fifth year, which suggests a modest impact of transaction costs.

The individual agent receives a stream of income at a rate L_t given by (3). The agent retires at time $\widetilde{T}$ and then lives on until time T. Following Cocco, Gomes, and Maenhout (2005), we incorporate the typical hump-shaped life-cycle pattern in the pre-retirement income drift via a polynomial expression for $\mu_L(t)$ in (3). After retirement, we assume the average income growth is zero. Income is assumed to have a constant sensitivity to the variable z_t of one before retirement and zero in retirement. We assume income volatility is constant in retirement at a low value of $\sigma_{retired} = 0.02$.[8] Before retirement, the standard assumption seems to be a constant income volatility which in our case would be $\bar{\sigma}_L = 0.10$. We also consider a volatility being linearly decreasing over time to a value $\hat{\sigma}_L > 0$ immediately before retirement, but still having an average value of $\bar{\sigma}_L$ over the working life which we assume begins at T_0. We thus have

$$\mu_L(t) = \begin{cases} a_1 + 2a_2 t + 3a_3 t^2, & T_0 \le t < \widetilde{T}, \\ 0, & \widetilde{T} < t < T, \end{cases} \qquad \zeta(t) = \begin{cases} 1, & T_0 \le t < \widetilde{T}, \\ 0, & \widetilde{T} < t < T, \end{cases}$$

$$\sigma_L(t) = \begin{cases} 2\bar{\sigma}_L - \hat{\sigma}_L - \frac{2(\bar{\sigma}_L - \hat{\sigma}_L)}{\widetilde{T} - T_0}(t - T_0), & T_0 \le t < \widetilde{T}, \\ \sigma_{retired}, & \widetilde{T} < t < T, \end{cases}$$

where estimates of a_1, a_2, and a_3 are given in Table 2 in Cocco, Gomes, and Maenhout (2005). Exactly at retirement period $\widetilde{T}$ we assume income drops to by a fixed proportion $1 - \Upsilon$,

$$L_{\widetilde{T}+} = \Upsilon L_{\widetilde{T}-},$$

where $\Upsilon \in (0, 1)$ is the socalled replacement rate.

[8] Even though a pension scheme provides a constant pay-out, the disposable retirement income can be uncertain because of uncertain medical costs. Also, some agents continue to earn income from other sources such as proprietary businesses or other non-traded assets. Moreover, because of mortality risk, the agent may not receive future retirement pay, and while we do not formally model mortality, retirement income risk will partially capture this effect in a parsimonious manner.

Note that the market is incomplete since the agent cannot fully control the exposure toward the shocks to labor income and, in the full model, the time-varying drift components x, y, and z. In addition we will impose relevant portfolio constraints so that the agent may be unable to obtain his desired exposure to other shocks.

Let W_t denote the tangible wealth of the agent at time t, which includes the positions in the bank account, the stock index, REITs, and physically owned housing units, but not the agent's human wealth, i.e., the present value of her future labor income. Let Π_{St} and $\Pi_{Ht} = \phi_{It} H_t / W_t$ denote the fractions of tangible wealth invested in the stock and in housing units, respectively, at time t. The wealth invested in the bank account is residually determined as $W_t(1 - \Pi_{St} - \Pi_{Ht})$. The rate of perishable consumption at time t is represented by c_t. The wealth dynamics is then

$$dW_t = W_t \Big[\big(r + \Pi_{St}(\mu'_S + x_t) + \Pi_{Ht}(\mu'_H + y_t)\big) \, dt + (\Pi_{St}\sigma_S + \Pi_{Ht}\sigma_H \rho_{HS}) \, dB_{St} \\ + \Pi_{Ht}\sigma_H \hat{\rho}_H \, dB_{Ht} \Big] + (L_t - c_t - \phi_{Ct} R H_t) \, dt \,, \tag{10}$$

where $\mu'_S = \mu_S + \bar{D}$.

The objective of the investor is to maximize life-time expected utility from perishable consumption and consumption of housing services. The indirect utility function is defined as

$$J(t, W, H, L, x, y, z) = \sup_{(c, \phi_C, \Pi_H, \Pi_S) \in \mathcal{A}_t} \mathrm{E}_t \left[\int_t^T e^{-\delta(u-t)} U(c_u, \phi_{Cu}) \, du + \varepsilon e^{-\delta(T-t)} \bar{U}(W_T) \right], \tag{11}$$

where W, H, L, x, y, and z denote time t values of wealth, house price, labor income, and the latent processes. Moreover, U is a Cobb-Douglas-power utility function

$$U(c, \phi_C) = \frac{1}{1-\gamma} \left(c^a \phi_C^{1-a} \right)^{1-\gamma} \tag{12}$$

and the bequest utility is

$$\bar{U}(W) = \frac{1}{1-\gamma} W^{1-\gamma}. \tag{13}$$

Here $\gamma > 1$ is the relative risk aversion, $a \in (0, 1)$ the relative utility weight of the two goods, and $\varepsilon \geq 0$ the utility weight of bequests relative to consumption. Similar preferences are assumed in other recent papers, such as Cocco (2005), Yao and Zhang (2005), and van Hemert (2010). The set $\mathcal{A}_t$ contains all admissible control processes over the time interval $[t, T]$. Constraints on the controls are thus reflected by $\mathcal{A}_t$. We shall impose the constraints

$$\Pi_S \geq 0, \quad \Pi_H \geq 0, \quad \Pi_S + q\Pi_H \leq 1, \tag{14}$$

which rule out short-selling and limits borrowing to a fraction $(1 - q)$ of the current value of the housing investment.

Because of incomplete markets and portfolio constraints, we are unable to solve the problem in closed form. In the next section we outline our numerical solution method.

6 Solution method

We apply the so-called SAMS (Simulation of Artificial Markets Strategies) approach introduced by Bick, Kraft, and Munk (2013). Compared to alternative methods, this approach distinguishes itself by being relatively easy to implement, being based on closed-form consumption and investment strategies, and providing a measure of its accuracy. As our optimization problem (11), in addition to time, features six state variables (can be reduced to five after exploiting homogeneity of the utility function), grid-based methods are infeasible with grid sizes that will bring us near the continuous-time solution.

6.1 Artificial markets

Building on the idea of Cvitanić and Karatzas (1992), the constrained, incomplete market problem is embedded in a family of artificial, unconstrained, complete market problems for which we can derive exact closed-form solutions. In order to handle the constraints (14), we adjust the risk-free rate as well as the drift rates of the stock and the house as follows

$$\begin{aligned}
\mu_S(t) &= \mu_S + \nu_S(t)\,, \\
\mu_H(t) &= \mu_H + \nu_H(t)\,, \\
r(t) &= r + \max\left(\nu_S(t)^-, \tfrac{1}{q}\,\nu_H(t)^-\right)\,,
\end{aligned}$$

where $\nu^- = \max(-\nu, 0)$; see Cvitanić and Karatzas (1992) and Bick, Kraft, and Munk (2013, Sec. 8). Intuitively, if the unconstrained Π_S or Π_H is above 1, we increase the risk-free rate to make investing in the bank account relatively more attractive and bring down the risky investment. Conversely, if the unconstrained Π_S or Π_H is negative, we increase the drift rate to boost the investment in the asset. For notational convenience, we define

$$\mu'_S(t) = \mu_S(t) + \bar{D}, \qquad \mu'_H(t) = \mu_H(t) + R.$$

To complete the market in our case, we introduce an artificial asset for the idiosyncratic labor income risk with a price process given by

$$dV_{Lt} = V_{Lt}\left[(r(t) + \lambda_L(t))\,dt + dB_{Lt}\right], \tag{15}$$

where λ_L is the market price of risk associated with the unspanned income shock (the assumption of a unit volatility is without loss of generality). Let Π_{Lt} denote the fraction of wealth invested in this asset at time t. The artificial income-asset is only needed until the retirement date where all income risk is resolved.

An artificial market is characterized by the "modifiers" ν_S, ν_H, and λ_L. We could define artificial markets in which these quantities are stochastic processes, but we restrict ourselves to deterministic processes since we can then solve the utility maximization problem in closed form. We refer to such a market as an *artificial market with deterministic modifiers.*

6.2 Optimality in artificial markets

In any of the artificial markets, the labor income is spanned and the agent can borrow against future income. By combining these features with the assumed income dynamics, we can compute the human capital—the present value of all future income—in closed form:

Lemma 1 *In an artificial market with deterministic modifiers, the human capital at time t equals $L_t F(t, x_t, y_t, z_t)$ where*

$$F(t,x,y,z) = \begin{cases} \displaystyle\int_t^T A(t,s,x,y,z)\,ds, & t \in (\widetilde{T}, T], \\ \displaystyle\int_t^{\widetilde{T}} A(t,s,x,y,z)\,ds + \Upsilon \int_{\widetilde{T}}^T A(t,s,x,y,z)\,ds, & t \in [0, T], \end{cases} \tag{16}$$

with

$$A(t,s,x,y,z) = \exp\left\{\alpha(t,s) + \beta_x(t,s)x + \beta_y(t,s)y + \beta_z(t,s)z\right\},$$

where the functions α, β_x, β_y, and β_z are given in Appendix A.

The total wealth of the agent at time t is the sum of tangible wealth and human capital, i.e., $W_t + L_t F(t, x_t, y_t, z_t)$. Due to power utility, the indirect utility function is conjectured to have the form $\frac{1}{1-\gamma} G(\cdot)^\gamma (W_t + L_t F(t, x_t, y_t, z_t))^{1-\gamma}$, where G will depend on time and on variables driving shifts in investment opportunities (risk-free rate and risk premia) as well as changes in relative prices of consumer goods. In our case, G therefore depends on x_t, y_t, and the house price H_t. Because x and y enter the risk premia in an affine relation, we expect $g(t, H_t, x_t, y_t)$ to involve exponential-quadratic functions of x_t and y_t (Kim and Omberg 1996, Wachter 2002, Liu 2007). The relative good price H_t is expected to enter proportionally with a power (Kraft and Munk 2011). These considerations motivate the form of the indirect utility function given below. The optimal strategies then follow from the first-order conditions to the associated Hamilton-Jacobi-Bellman equation. Appendix B explains the cumbersome verification procedure and the derivation of various coefficient functions.

Theorem 1 *In an artificial market with deterministic modifiers, the indirect utility is*

$$J(t, W, H, L, x, y, z) = \frac{1}{1-\gamma} G(t, H, x, y)^\gamma \left(W + LF(t, x, y, z)\right)^{1-\gamma}, \tag{17}$$

where F is given by (16) and

$$\begin{aligned} G(t, H, x, y) = {} & \varepsilon^{1/\gamma} e^{D_0(t) + D_x(t)x + D_y(t)y + \frac{1}{2}D_{xx}(t)x^2 + \frac{1}{2}D_{yy}(t)y^2 + D_{xy}(t)xy} \\ & + k_2 H^{k_1} \int_t^T e^{\bar{D}_0(t,s) + \bar{D}_x(t,s)x + \bar{D}_y(t,s)y + \frac{1}{2}\bar{D}_{xx}(t,s)x^2 + \frac{1}{2}\bar{D}_{yy}(t,s)y^2 + \bar{D}_{xy}(t,s)xy}\, ds. \end{aligned} \tag{18}$$

with $k_1 = (1-a)(\gamma-1)/\gamma$, $k_2 = a^{\frac{1-\gamma}{\gamma}} \left(\frac{aR}{1-a}\right)^{k_1}$, and the D- and $\bar{D}$-functions solve differential *equations shown in Appendix B.3. The optimal portfolio weights are*

$$
\begin{aligned}
\Pi_S &= \frac{1}{\gamma \hat{\rho}_H^2 \sigma_S^2} \left(\mu'_S(t) + x - \frac{\rho_{HS}\sigma_S}{\sigma_H}[\mu'_H(t) + y] \right) \frac{W + LF}{W} \\
&\quad + \left(M_{xS}\frac{G_x}{G} + M_{yS}\frac{G_y}{G} \right) \frac{W + LF}{W} - \left(M_{LS}(t) + M_{xS}\frac{F_x}{F} + M_{yS}\frac{F_y}{F} + M_{zS}\frac{F_z}{F} \right) \frac{LF}{W}, \qquad (19) \\
\Pi_H &= \frac{1}{\gamma \hat{\rho}_H^2 \sigma_H^2} \left(\mu'_H(t) + y - \frac{\rho_{HS}\sigma_H}{\sigma_S}[\mu'_S(t) + x] \right) \frac{W + LF}{W} + \frac{HG_H}{G}\frac{W + LF}{W} \\
&\quad + \left(M_{xH}\frac{G_x}{G} + M_{yH}\frac{G_y}{G} \right) \frac{W + LF}{W} - \left(M_{LH}(t) + M_{xH}\frac{F_x}{F} + M_{yH}\frac{F_y}{F} + M_{zH}\frac{F_z}{F} \right) \frac{LF}{W}, \qquad (20) \\
\Pi_L &= \frac{1}{\gamma}\lambda_L(t)\frac{W + LF}{W} + \left(\sigma_x \hat{\rho}_{xL}\frac{G_x}{G} + \sigma_y \hat{\rho}_{yL}\frac{G_y}{G} \right) \frac{W + LF}{W} \\
&\quad - \left(\hat{\rho}_L \sigma_L(t) + \sigma_x \hat{\rho}_{xL}\frac{F_x}{F} + \sigma_y \hat{\rho}_{yL}\frac{F_y}{F} + \sigma_z \hat{\rho}_{zL}\frac{F_z}{F} \right) \frac{LF}{W}, \qquad (21)
\end{aligned}
$$

where the functions and constants M_{ij} are defined in Appendix A. The optimal consumption strategy is

$$
c = a^{1/\gamma} \left(\frac{aRH}{1-a} \right)^{k_1} \frac{W + LF}{G}, \qquad (22)
$$

$$
\phi_C = a^{1/\gamma} \left(\frac{aRH}{1-a} \right)^{k_1 - 1} \frac{W + LF}{G}. \qquad (23)
$$

Note that the theorem applies both before and in retirement. At retirement, there can be a (downwards) shift in the income volatility and in the sensitivity of income growth to the z-variable, which changes the risk characteristics of human capital. Such shifts may cause a notable change in the optimal portfolio through $M_{LS}(t)$, $M_{LH}(t)$, and the sensitivities F_x/F, F_y/F, and F_z/F.

It follows from (22) and (23) that the optimal spending on housing consumption relative to perishable consumption, $RH\phi_C/c$, is equal to the constant $(1-a)/a$, which is a consequence of the assumed Cobb-Douglas utility function.[9]

The optimal portfolio (19)–(21) in the artificial market has three components: *First*, speculative terms that are determined from the risk premia, volatilities, and correlations of the assets. *Secondly*, terms hedging against the variables x and y that capture shifts in investment opportunities and

[9]For the more general CES-type utility function

$$
U(c, \phi_C) = \frac{1}{1-\gamma} \left(a c^{\frac{\psi - 1}{\psi}} + (1-a)\phi_C^{\frac{\psi - 1}{\psi}} \right)^{\frac{\psi(1-\gamma)}{\psi - 1}},
$$

where ψ is the elasticity of substitution between the two goods, the optimal housing/perishable spending ratio can depend on the house price level, but a closed-form expression for $g(t, x, y)$ is not available in that case. The Cobb-Douglas utility (12) corresponds to the limit $\psi \to 1$.

against the relative good prices which are governed by the house price. Since the value of a house investment is perfectly correlated with the house price, the investor naturally applies the house as the instrument for hedging variations in house prices. In other words, one motive for a housing investment is to reduce uncertainty about future costs of housing consumption (Sinai and Souleles 2005). On the other hand, all three risky assets are imperfectly correlated with the time-varying expected returns x and y and are therefore all used for hedging against variations in those variables. Note that compared to the no-income case, both the speculative terms and the hedging terms are magnified by the multiplier $(W + LF)/W$. *Thirdly*, the portfolio include terms that adjust for the holdings in the various assets that are implicit in the human capital. These terms involve the multiplier LF/W because it is the relative size of human capital to tangible wealth that determines the income-adjustment terms in collaboration with the correlation structure.

6.3 Upper bound on obtainable utility

A feasible strategy in the true market leads to at least the same expected utility in any of the artificial markets as in the true market because the returns on the stock and house and the risk-free rate are at least as high in the artificial markets. Many other strategies are feasible in the artificial market such that the indirect utility in any artificial market is always greater or equal the indirect utility in the true market. Karatzas, Lehoczky, Shreve, and Xu (1991) and Cvitanić and Karatzas (1992) show that, under certain technical conditions, the solution in the true market is equal to the solution in the worst of all the artificial markets but, in rather complex models as our, it seems impossible to identify the worst market.

Theorem 1 provides a closed-form solution in any artificial market with deterministic modifiers. The worst among these artificial markets defines an upper bound on the maximum expected utility in the true market. By comparing the expected utility generated by any specific strategy feasible in the true market to this upper bound, we can derive an upper bound on the welfare loss (in terms of certainty equivalent of wealth) suffered from following the specific strategy instead of the unknown optimal strategy. We exploit this idea below.

The functions A_0, D_0, D_1, P_0, and P_1 that enter the optimal strategies involve integrals over the deterministic functions ν_S, ν_H, and Λ_L. To facilitate the computation of these integrals and the subsequent minimization over artificial markets, we henceforth assume affine functions of time (age) for these quantities:

$$\nu_S(t) = \nu_{S0} + \nu_{S1} t, \qquad \nu_H(t) = \nu_{H0} + \nu_{H1} t, \qquad \lambda_L(t) = \Lambda_{L0} + \Lambda_{L1} t.$$

Again, this is in line with the suggestions of Bick, Kraft, and Munk (2013). Consequently, each artificial market we consider is parameterized by six constants, which for easy reference is collected in the set $\Theta = \{\nu_{S0}, \nu_{S1}, \nu_{H0}, \nu_{H1}, \Lambda_{L0}, \Lambda_{L1}\}$. We find the worst of the corresponding artificial markets by a standard unconstrained numerical optimization over Θ. Let $\bar{\Theta}$ denote the parameter set for which the minimum is obtained. Hence,

$$\bar{J}(t, W, H, L, x, y, z) = J(t, W, H, L, x, y, z; \bar{\Theta})$$

is the upper bound on the obtainable indirect utility in the true market.

6.4 Promising feasible strategies for the true problem

We derive a promising strategy in the true market from the optimal strategies in the parameterized family of artificial markets in the following way. For each parameter set Θ, we take the optimal strategy in the corresponding artificial market and transform it into a strategy which is feasible in the true market. Obviously, we disregard the investment in the artificial asset and focus on Π_S, Π_H and the consumption processes c, ϕ_C.

In the artificial markets labor income is fully spanned and tangible wealth can be allowed to be temporarily negative if balanced by human capital. In the working phase in the true market, we require tangible wealth to stay non-negative because of the unhedgeable shocks that may bring income close to zero. We follow Bick, Kraft, and Munk (2013) and multiply the human capital by a factor $(1 - e^{-\eta W_t})$, where $\eta > 0$ is a constant to be determined. This is consistent with the intuition that future income has a smaller present value when current wealth W_t is small. Define

$$\widetilde{F}_t = (1 - e^{-\eta W_t}) F(t, x_t, y_t, z_t).$$

Furthermore, we prune the optimal portfolios to make sure the constraints (14) are met. To sum up, the feasible strategy derived from the artificial market with parameters Θ is determined from

$$\begin{aligned}\Pi_{St} &= \frac{1}{\gamma\hat{\rho}_H^2\sigma_S^2}\left(\mu'_S(t) + x_t - \frac{\rho_{HS}\sigma_S}{\sigma_H}[\mu'_H(t) + y_t]\right)\frac{W_t + L_t\widetilde{F}_t}{W_t} \\ &\quad + \left(M_{xS}\frac{G_x}{G} + M_{yS}\frac{G_y}{G}\right)\frac{W_t + L_t\widetilde{F}_t}{W_t} - \left(M_{LS} + M_{xS}\frac{F_x}{F} + M_{xH}\frac{F_y}{F} + M_{zS}\frac{F_z}{F}\right)\frac{L_t\widetilde{F}_t}{W}, \quad (24)\\ \Pi_{Ht} &= \frac{1}{\gamma\hat{\rho}_H^2\sigma_H^2}\left(\mu'_H(t) + y_t - \frac{\rho_{HS}\sigma_H}{\sigma_S}[\mu'_S(t) + x_t]\right)\frac{W_t + L_t\widetilde{F}_t}{W_t} + \frac{H_t G_H}{G}\frac{W_t + L_t\widetilde{F}_t}{W_t} \\ &\quad + \left(M_{xH}\frac{G_x}{G} + M_{yH}\frac{G_y}{G}\right)\frac{W_t + L_t\widetilde{F}_t}{W_t} - \left(M_{LH} + M_{xH}\frac{F_x}{F} + M_{yH}\frac{F_y}{F} + M_{zH}\frac{F_z}{F}\right)\frac{L_t\widetilde{F}_t}{W},\end{aligned}$$

$$(25)$$

and

$$c_t = a^{1/\gamma}\left(\frac{aRH_t}{1-a}\right)^{k_1}\frac{W_t + L_t\widetilde{F}_t}{G}, \tag{26}$$

$$\phi_{Ct} = a^{1/\gamma}\left(\frac{aRH_t}{1-a}\right)^{k_1-1}\frac{W_t + L_t\widetilde{F}_t}{G}, \tag{27}$$

where we suppress the dependence of F, F_x, F_y, F_z, G, G_x, G_y, and G_h on t, x, y, z and the parameter set Θ. A negative value of Π_{St} or Π_{Ht} is replaced by 0. If Π_{St} and Π_{Ht} are both positive and $\Pi_{St}+q\Pi_{Ht} > 1$, we divide both Π_{St} and Π_{Ht} by $\Pi_{St}+q\Pi_{Ht}$ to respect the borrowing limit. After these potential transformations, the residual wealth (positive or negative) constitutes the position in the bank account. If financial wealth should equal zero at any point in time, the investment in the risky assets is restricted to zero and consumption is set to fraction of current income, $c_t = kY_t$ and $\phi_{Ct} = kY_t/(RH_t)$, where $k \in (0, 1/2)$. This ensures that the liquidity constraint is respected.

For any (Θ, η), we can approximate the expected utility $J(t, W, H, L, x, y, z; \Theta, \eta)$ generated with the above strategy by Monte Carlo simulation of the wealth W_t and state variables H_t, L_t, x_t, y_t, z_t. Searching over (Θ, η), we find the best of the feasible strategies. This is our candidate for a near-optimal consumption-investment strategy in the true market. Again, this search can be implemented by a standard unconstrained numerical optimization algorithm.

We can evaluate the performance of any admissible strategy $(c, \phi_C, \Pi_S, \Pi_H)$—including our candidate defined above—in the following way. We compare the expected utility generated by the strategy, $J^{c,\phi_C,\Pi_S,\Pi_H}(t, W, H, L, x, y, z)$, to the upper bound $\bar{J}(t, W, H, L, x, y, z)$ on the maximum utility. If the distance is small, the strategy is near-optimal. More precisely, we can compute an upper bound Loss $=$ $\text{Loss}^{c,\phi_C,\Pi_S,\Pi_H}(t, W, H, L, x, y, z)$ on the welfare loss suffered when following the specific strategy $(c, \phi_C, \Pi_S, \Pi_H)$ by solving the equation

$$J^{c,\phi_C,\Pi_S,\Pi_H}(t, W, H, L, x, y, z) = \bar{J}(t, W[1 - \text{Loss}], H, L[1 - \text{Loss}], x, y, z). \tag{28}$$

We can interpret Loss as an upper bound on the fraction of total wealth (current wealth plus current and future income) that the individual would sacrifice to get access to the unknown optimal strategy, instead of following the strategy $(c, \phi_C, \Pi_S, \Pi_H)$. Theorem 1 implies

$$\begin{aligned}\bar{J}(t, W[1 - \text{Loss}], H, L[1 - \text{Loss}], x, y, z) &= J(t, W[1 - \text{Loss}], H, L[1 - \text{Loss}], x, y, z; \bar{\Theta}) \\ &= (1 - \text{Loss})^{1-\gamma} J(t, W, H, L, x, y, z; \bar{\Theta}),\end{aligned}$$

so that the upper bound on the welfare loss becomes

$$\text{Loss}^{c,\phi_C,\Pi_S,\Pi_H}(t, W, H, L, x, y, z) = 1 - \left(\frac{J^{c,\phi_C,\Pi_S,\Pi_H}(t, W, H, L, x, y, z)}{J(t, W, H, L, x, y, z; \bar{\Theta})} \right)^{\frac{1}{1-\gamma}}. \tag{29}$$

7 Illustration and discussions of results

This section illustrates and discusses a number of properties of the consumption and investment strategies derived using the method explained above. Unless otherwise noted, we assume the parameter values estimated using the nationwide U.S. data, which are listed in Table 4.

We specify the benchmark values of additional parameters and state variables in accordance with the existing literature, such as Cocco, Gomes, and Maenhout (2005), Yao and Zhang (2005), and Kraft and Munk (2011). We assume a relative risk aversion coefficient of $\gamma = 5$ or $\gamma = 10$ and a bequest multiplier of $\varepsilon = 1$. The relative utility weight of the goods is set to $a = 0.8$, implying that total consumption expenditures consists of 75% on perishable goods and 25% on housing consumption, which seems consistent with observed household expenditure, cf. a report by the U.S. Department of Labor (2003). The subjective time preference rate is $\delta = 0.05$. The agent is initially of age $t = 30$, retires at age $\widetilde{T} = 60$, and lives on until age $T = 80$. We assume an income replacement rate of $\Upsilon = 0.6$.[10] The results presented below are assuming

(i) no x in the model – here the results of Appendix C.1 are applied. Parameter estimates in the

[10]The reduction from the 68%-93% estimate of Cocco, Gomes, and Maenhout (2005) is a way to implicitly incorporate the higher expenses tied to medication during retirement as well as the increased mortality risk that lowers expected future income.

model without x are slightly different from those shown in Table 4. We are currently working on the implementation of the full model.

(ii) the income growth component $\mu_L(t)$ is constant and equal to the estimate $\bar{\mu}_L$ based on aggregate income. An age-dependent income profile will be included in a later version.

We set the proportional rental rate to $R = 0.05$.

For concreteness, we think of a housing unit as 1000 square feet of average quality and location. Using a monetary unit of a thousand U.S. dollars, we set the initial unit house price to $H = 250$, which implies an initial annual rent of \$12,500 for a housing unit. Furthermore, the initial tangible wealth is set to $W = 20$ and the initial annual income to $L = 20$ which, according to Kraft and Munk (2011), are roughly equal to the median values for individuals of age 30-40 in the 2007 Survey of Consumer Finances. Finally, we set the initial values of the time-varying drift components of house prices and income to $y = z = 0$.

When implementing the solution method explained in the previous section using the benchmark parameter values explained above, the upper bound on the welfare loss is around 2%. The experiments reported by Bick, Kraft, and Munk (2013) indicate that the upper bound is relatively weak so that the true welfare loss by using our strategy instead of the unknown optimal strategy is significantly smaller than the 2%. After deriving our near-optimal life-cycle consumption, housing, and investment strategy, we simulate 100,000 paths of exogenous state variables and wealth (applying this strategy) forward, and in the figures mentioned below we report expectations of consumption, wealth, investments, and portfolio weights computed by averaging over the simulations.[11]

Figure 3 illustrates the results for a risk aversion of $\gamma = 5$. The horizontal axes show time passed after the initial date where the agent is assumed to be of age 30. Panel A shows that consumption is expected to increase for the first 30 years, i.e., until retirement, after which it is roughly flat. While the agent is impatient, the relatively high risk premia makes investments attractive and thus reduces consumption early in life. In addition, the agent faces borrowing constraints limiting his opportunity to consume out of future income. Housing consumption expenditures and perishable consumption expenditures follow each other in lockstep.

The upper curve in Panel B shows that the expected number of housing units inhabited by the agent increases from roughly 0.3 (or 300 square feet) to 0.8 over life (note: per agent, not per

[11] We simulate using Euler discretizations with monthly time steps. Note that F and g involve numerical integration. Before running the simulations, we evaluate F and g on grids, and when we need values for F and g and their derivatives in the simulations, we use the grid values and linear interpolation.

family). The investment in housing units is initially very low and then increases until retirement after which it drops steadily towards zero. As previously explained, the housing investment is a mix of a speculative demand, a rent-hedging demand, and an income-adjustment term.

As illustrated in Panel C we obtain the standard hump-shaped wealth pattern. The agent builds up wealth in the active phase to finance consumption in retirement where income is markedly lower. Panel D shows that the expected composition of wealth is relatively stable over life with around 45-55% in stocks, 40-50% in housing units, and 0-10% in the risk-free bond (bank account), except for the last few years where the agent cares less about the risk premia on stocks and houses so that the bond receives a higher portfolio weight. The small jump in investments at retirement is due to the fact that human capital moves from being risky to risk-free which induces the agent to shift some investments from the risk-free to risky assets to maintain the desired overall risk exposure.

[Figure 3 about here.]

Figure 4 illustrates the same quantities for an agent with a risk aversion of $\gamma = 10$. The more risk-averse agent consumes less and builds up more wealth early in life to have a buffer against bad states. On average, this agent enters retirement with higher wealth and can therefore consume more in retirement than the less risk-averse agent. As reflected by Panels B and D, the highly risk-averse agent invests more in housing, in particular early in life, which is because he has a stronger desire to hedge the costs of future housing consumption. Hence, his portfolio is dominated by housing investments in the early years.

[Figure 4 about here.]

We have also implemented our method using the Ly-restricted model calibrations for the national U.S. data and both the full and the Ly-restricted model calibrations based on data from the Charlotte metropolitan area. For a given data set, there is no visible difference between the results using the full model and the results using the Ly-restricted model, which is as expected given the very similar parameter estimates, cf. Table 4. Figure 5 illustrates the results using Charlotte-based estimates of the full model and a risk aversion of $\gamma = 5$. Compared to the results for the nationwide parameters, the main difference is the lower expected consumption throughout life which is caused by the lower expected income growth for a Charlotte-based agent (1.42% per year compared to 2.28% nationwide). In terms of housing units, both consumption and investment are still slightly higher for the Charlotte-based agent because of a lower growth rate of house prices (we use the same initial house price in this comparison).

[Figure 5 about here.]

We next consider the impact of variations in selected parameter values. Figure 6 shows the effect of increasing the rental rate R from 5% to 6%. Note that this also increases the expected return on a housing investment via REITs or via owning-and-renting-out. Compared with the benchmark case in Figure 3, the agent naturally consumes fewer housing units (Panel B) because of the higher costs. Moreover, the housing investment constitutes a larger part of the portfolio particularly early in life. This is due to the higher expected return on housing investments and due to the higher demand for hedging against future costs (the ratio G_h/g appearing in (20) and (25) is increasing in R). Note that the agent does not expect to participate in the stock market in the first approximately 12 years.

[Figure 6 about here.]

Based on the calibration of the full model to national U.S. data, the excess expected rate of return on the stock market is $\mu_S + \bar{D} = 0.073$ in accordance with standard estimates. For various reasons (survivorship bias, tax reductions on stock returns in the data period), the estimate is routinely adjusted downwards when used for future expected returns. Figure 7 shows the impact of reducing μ_S so that the excess expected stock return is 0.058. As reflected by Panels C and D, this minor reduction is sufficient to eliminate stocks from the optimal portfolio for the first 15 years or so, and even after that point the maximum expected portfolio weight of the stock is smaller than 30%. Again this behavior is much better in line with observed stock investments than the standard models.

[Figure 7 about here.]

8 Conclusion

The long-run movements in stock prices, house prices, and labor income rates are related beyond their contemporaneous correlations. Positive shocks to one of these variables in this period tend to increase the expected change in the other variables over the next period. We have provided empirical support of these features based on national and regional U.S. data. A very restrictive cointegration structure is not supported as variations in, e.g., the expected change in house prices may have other sources than shocks to stock prices or labor income.

We have solved for the optimal life-cycle decisions of an agent (household) acknowledging the long-run features of price and income dynamics. The derived consumption and investment behavior has many realistic and interesting properties. For very reasonable parameterizations, it is optimal for young agents not to hold any stocks at all, and later in life the agent should only a relatively small fraction of wealth in stocks. These findings are in accordance with the observed limited stock market participation.

A Proof of Lemma 1

In a complete, unconstrained market we can represent the human capital by the risk-neutral expectation of the discounted future labor income. In retirement, i.e., for $t \in (\widetilde{T}, T]$, the human capital is therefore

$$\mathrm{E}_t^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r(u)\,du} L_s\,ds\right] = \int_t^T e^{-\int_t^s r(u)\,du} \mathrm{E}_t^{\mathbb{Q}}\left[L_s\right]\,ds, \tag{30}$$

where $\mathbb{Q}$ is the unique risk-neutral probability measure in a given artificial market. Before retirement, the human capital is

$$\begin{aligned}
&\mathrm{E}_t^{\mathbb{Q}}\left[\int_t^{\widetilde{T}} e^{-\int_t^s r(u)\,du} L_s\,ds + \int_{\widetilde{T}}^T e^{-\int_t^s r(u)\,du} L_s\,ds\right] \\
&= \int_t^{\widetilde{T}} e^{-\int_t^s r(u)\,du} \mathrm{E}_t^{\mathbb{Q}}[L_s]\,ds + \int_{\widetilde{T}}^T e^{-\int_t^s r(u)\,du} \mathrm{E}_t^{\mathbb{Q}}[L_s]\,ds,
\end{aligned} \tag{31}$$

Here, when computing the expectation at time $t < \widetilde{T}$ of the income rate as time $s > \widetilde{T}$, we have to incorporate the drop in income at time $\widetilde{T}$.

To derive the income dynamics under the $\mathbb{Q}$ measure in the xyz-restricted model, we must identify the market prices of risk associated with the Brownian shocks B_S, B_H, B_L. While the market price of risk associated with B_L is $\lambda_L(t)$ by assumption, we identify the market prices of risk m_{St}, m_{Ht} associated with B_S, B_H by using the fact that the excess expected return on an asset is the product of its sensitivities towards the shocks and the market prices of risks associated with the shocks. For the stock, this means

$$\mu_S'(t) + x_t = \sigma_S m_{St} \quad \Rightarrow \quad m_{St} = \frac{\mu_S'(t) + x_t}{\sigma_S}.$$

For an investment in housing units (including rents), this implies

$$\begin{aligned}
&\mu_H'(t) + y_t = \sigma_H \rho_{HS} m_{St} + \sigma_H \hat{\rho}_H m_{Ht} \quad \Rightarrow \\
&m_{Ht} = \frac{\mu_H'(t) + y_t}{\sigma_H \hat{\rho}_H} - \frac{\rho_{HS}(\mu_S'(t) + x_t)}{\hat{\rho}_H \sigma_S}.
\end{aligned}$$

The risk-neutral income dynamics is therefore

$$\begin{aligned}
\frac{dL_t}{L_t} &= \left(\mu_L(t) + \zeta(t) z_t - \sigma_L(t)\left[\rho_{LS} m_{St} + \hat{\rho}_{LH} m_{Ht} + \hat{\rho}_L \lambda_L(t)\right]\right)\,dt + \sigma_L(t)\left(\rho_{LS}\,dB_{St}^{\mathbb{Q}} + \hat{\rho}_{LH}\,dB_{Ht}^{\mathbb{Q}} + \hat{\rho}_L\,dB_{Lt}^{\mathbb{Q}}\right) \\
&= \left(-M_{LS}(t) x_t - M_{LH}(t) y_t + \zeta(t) z_t + \ell(t)\right)\,dt + \sigma_L\left(\rho_{LS}\,dB_{St}^{\mathbb{Q}} + \hat{\rho}_{LH}\,dB_{Ht}^{\mathbb{Q}} + \hat{\rho}_L\,dB_{Lt}^{\mathbb{Q}}\right),
\end{aligned}$$

where

$$M_{LS}(t) = \frac{\sigma_L(t)}{\sigma_S}\left(\rho_{LS} - \frac{\rho_{HS}\hat{\rho}_{LH}}{\hat{\rho}_H}\right), \quad M_{LH}(t) = \frac{\sigma_L(t)\hat{\rho}_{LH}}{\sigma_H\hat{\rho}_H},$$
$$\ell(t) = \mu_L(t) - \sigma_L(t)\hat{\rho}_L\lambda_L(t) - M_{LS}(t)\mu'_S(t) - M_{LH}(t)\mu'_H(t).$$

As the drift involves x, y, and z, we have to derive the risk-neutral dynamics of these processes as well. For x, we obtain

$$\begin{aligned} dx_t &= \left(-\kappa_x x_t - \sigma_x\left[\rho_{xS}m_{St} + \hat{\rho}_{xH}m_{Ht} + \hat{\rho}_{xL}\lambda_L(t)\right]\right)dt + \sigma_x\left(\rho_{xS}\,dB^{\mathbb{Q}}_{St} + \hat{\rho}_{xH}\,dB^{\mathbb{Q}}_{Ht} + \hat{\rho}_{xL}\,dB^{\mathbb{Q}}_{Lt}\right) \\ &= \left(-[\kappa_x + M_{xS}]x_t - M_{xH}y_t - M_x(t)\right)dt + \sigma_x\left(\rho_{xS}\,dB^{\mathbb{Q}}_{St} + \hat{\rho}_{xH}\,dB^{\mathbb{Q}}_{Ht} + \hat{\rho}_{xL}\,dB^{\mathbb{Q}}_{Lt}\right), \end{aligned}$$

where

$$M_{xS} = \frac{\sigma_x}{\sigma_S}\left(\rho_{xs} - \frac{\rho_{HS}\hat{\rho}_{xH}}{\hat{\rho}_H}\right), \quad M_{xH} = \frac{\sigma_x\hat{\rho}_{xH}}{\sigma_H\hat{\rho}_H},$$
$$M_x(t) = M_{xS}\mu'_S(t) + M_{xH}\mu'_H(t) + \sigma_x\hat{\rho}_{xL}\lambda_L(t).$$

For y, we obtain

$$\begin{aligned} dy_t &= \left(-\kappa_y y_t - \sigma_y\left[\rho_{yS}m_{St} + \hat{\rho}_{yH}m_{Ht} + \hat{\rho}_{yL}\lambda_L(t)\right]\right)dt + \sigma_y\left(\rho_{yS}\,dB^{\mathbb{Q}}_{St} + \hat{\rho}_{yH}\,dB^{\mathbb{Q}}_{Ht} + \hat{\rho}_{yL}\,dB^{\mathbb{Q}}_{Lt}\right) \\ &= \left(-M_{yS}x_t - [\kappa_y + M_{yH}]y_t - M_y(t)\right)dt + \sigma_y\left(\rho_{yS}\,dB^{\mathbb{Q}}_{St} + \hat{\rho}_{yH}\,dB^{\mathbb{Q}}_{Ht} + \hat{\rho}_{yL}\,dB^{\mathbb{Q}}_{Lt}\right), \end{aligned}$$

where

$$M_{yS} = \frac{\sigma_y}{\sigma_S}\left(\rho_{ys} - \frac{\rho_{HS}\hat{\rho}_{yH}}{\hat{\rho}_H}\right), \quad M_{yH} = \frac{\sigma_y\hat{\rho}_{yH}}{\sigma_H\hat{\rho}_H},$$
$$M_y(t) = M_{yS}\mu'_S(t) + M_{yH}\mu'_H(t) + \sigma_y\hat{\rho}_{yL}\lambda_L(t).$$

For z, we obtain

$$\begin{aligned} dz_t &= \left(-\kappa_z z_t - \sigma_z\left[\rho_{zS}m_{St} + \hat{\rho}_{zH}m_{Ht} + \hat{\rho}_{zL}\lambda_L(t)\right]\right)dt + \sigma_z\left(\rho_{zS}\,dB_{St} + \hat{\rho}_{zH}\,dB_{Ht} + \hat{\rho}_{zL}\,dB_{Lt}\right) \\ &= \left(-M_{zS}x_t - M_{zH}y_t - \kappa_z z_t - M_z(t)\right)dt + \sigma_z\left(\rho_{zS}\,dB_{St} + \hat{\rho}_{zH}\,dB_{Ht} + \hat{\rho}_{zL}\,dB_{Lt}\right), \end{aligned}$$

where

$$M_{zS} = \frac{\sigma_z}{\sigma_S}\left(\rho_{zs} - \frac{\rho_{HS}\hat{\rho}_{zH}}{\hat{\rho}_H}\right), \quad M_{zH} = \frac{\sigma_z\hat{\rho}_{zH}}{\sigma_H\hat{\rho}_H},$$
$$M_z(t) = M_{zS}\mu'_S(t) + M_{zH}\mu'_H(t) + \sigma_z\hat{\rho}_{zL}\lambda_L(t).$$

First, assume that either $t < s < \widetilde{T}$ or $\widetilde{T} < t < s$ so that there is no one-time drop of income between t and s. From well-known results on systems of affine diffusions (Duffie, Pan, and Singleton 2000), we get that

$$\mathrm{E}_t^{\mathbb{Q}}[L_s] = L_t \exp\left\{\tilde{\alpha}(t,s) + \beta_x(t,s)x_t + \beta_y(t,s)y_t + \beta_z(t,s)z_t\right\}, \tag{32}$$

where

$$\beta_z(t,s) = \int_t^s \zeta(u)e^{-\kappa_z(u-t)}\,du,$$

and β_x, β_y satisfy the coupled differential equations

$$\begin{aligned}
\frac{\partial\beta_x}{\partial t}(t,s) &= [\kappa_x + M_{xS}]\beta_x(t,s) + M_{yS}\beta_y(t,s) + M_{zS}\beta_z(t,s) + M_{LS}(t),\\
\frac{\partial\beta_y}{\partial t}(t,s) &= M_{xH}\beta_x(t,s) + [\kappa_y + M_{yH}]\beta_y(t,s) + M_{zH}\beta_z(t,s) + M_{LH}(t)
\end{aligned}$$

with the conditions $\beta_x(s,s) = \beta_y(s,s) = 0$. Finally, $\alpha(t,s)$ is determined by

$$\begin{aligned}
\tilde{\alpha}(t,s) = \int_t^s \ell(u)\,du &- \int_t^s \left(M_x(u) - \rho_{Lx}\sigma_L(u)\sigma_x\right)\beta_x(u,s)\,du - \int_t^s \left(M_y(u) - \rho_{Ly}\sigma_L(u)\sigma_y\right)\beta_y(u,s)\,du\\
&- \int_t^s \left(M_z(u) - \rho_{Lz}\sigma_L(u)\sigma_z\right)\beta_z(u,s)\,du + \sigma_{xy}\int_t^s \beta_x(u,s)\beta_y(u,s)\,du + \sigma_{xz}\int_t^s \beta_x(u,s)\beta_z(u,s)\,du\\
&+ \sigma_{yz}\int_t^s \beta_y(u,s)\beta_z(u,s)\,du + \frac{1}{2}\sigma_x^2\int_t^s \beta_x(u,s)^2\,du + \frac{1}{2}\sigma_y^2\int_t^s \beta_y(u,s)^2\,du + \frac{1}{2}\sigma_z^2\int_t^s \beta_z(u,s)^2\,du.
\end{aligned}$$

Here we have introduced the covariance notation $\sigma_{ab} = \rho_{ab}\sigma_a\sigma_b$. Note that if ζ is constant over the interval $[t,s]$, we have

$$\beta_z(t,s) = \frac{1}{\kappa_z}\left(1 - e^{-\kappa_z(s-t)}\right)$$

and then

$$\int_t^s \beta_z(u,s)\,du = \frac{1}{\kappa_z}\left(s - t - \beta_z(t,s)\right), \qquad \int_t^s \beta_z(u,s)^2\,du = \frac{1}{\kappa_z^2}\left(s - t - \beta_z(t,s)\right) - \frac{1}{2\kappa_z}\beta_z(t,s)^2.$$

Next, for $t < \widetilde{T} < s$ we get

$$\mathrm{E}_t^{\mathbb{Q}}[L_s] = \Upsilon L_t \exp\left\{\tilde{\alpha}(t,s) + \beta_x(t,s)x_t + \beta_y(t,s)y_t + \beta_z(t,s)z_t\right\}. \tag{33}$$

By substituting (32) and (33) into (30) and (31) we arrive at the statement in the lemma where we define

$$\alpha(t,s) = \tilde{\alpha}(t,s) - \int_t^s r(u)\,du. \tag{34}$$

Now the human capital is obtained by substituting (32) into (31), which implies the statement in the lemma.

In Appendix B, we shall make use of the fact that F satisfies the partial differential equation (PDE)

$$
\begin{aligned}
0 = 1 &+ \frac{\partial F}{\partial t} + \left(-M_{LS}(t)x + M_{LH}(t)y + \zeta(t)z + \ell(t) - r(t)\right) F + \left(-[\kappa_x + M_{xS}]x - M_{xH}y - M_x(t)\right) F_x \\
&+ \left(-M_{yS}x - [\kappa_y + M_{yH}]y - M_y(t)\right) F_y + \left(-M_{zS}x - M_{zH}y - \kappa_z z - M_z(t)\right) F_z \\
&+ \frac{1}{2}\sigma_x^2 F_{xx} + \frac{1}{2}\sigma_y^2 F_{yy} + \frac{1}{2}\sigma_z^2 F_{zz} + \sigma_{Lx}F_x + \sigma_{Ly}F_y + \sigma_{Lz}F_z + \sigma_{xy}F_{xy} + \sigma_{xz}F_{xz} + \sigma_{yz}F_{yz}.
\end{aligned} \tag{35}
$$

This PDE can be shown directly by substitution of the relevant derivatives of F. Alternatively, it follows by making use of the fact that $\mathcal{P}(t, L, x, y, z) = LF(t, x, y, z)$ can be seen as the price of a stream of dividends, and it is well-known from derivatives pricing that $\mathcal{P}$ satisfies a certain PDE. Substituting $\mathcal{P} = LF$ into that, leads to (35).

B Proof of Theorem 1

B.1 Setting up the HJB equation

The wealth dynamics in the artificial market is similar to (10), but adjusted because of the possibility to invest in the artificial asset with price dynamics (15) as well as the modification of r, μ'_S, and μ'_H:

$$
\begin{aligned}
dW_t = W_t\Big[&\left(r(t) + \Pi_{St}(\mu'_S(t) + x_t) + \Pi_{Ht}(\mu'_H(t) + y_t) - \Pi_{Lt}\lambda_L(t)\right) dt + \left(\Pi_{St}\sigma_S + \Pi_{Ht}\sigma_H\rho_{HS}\right) dB_{St} \\
&+ \Pi_{Ht}\sigma_H\hat{\rho}_H \, dB_{Ht} + \Pi_{Lt}\, dB_{Lt}\Big] + \left(L_t - c_t - \phi_{Ct}RH_t\right) dt \\
= &\left(r(t)W_t + \alpha_t^\top \lambda_t + L_t - \phi_{Ct}RH_t - c_t\right) dt + \alpha_t^\top \Sigma \, dB_t,
\end{aligned}
$$

where

$$
\alpha_t = \begin{pmatrix} \alpha_{St} \\ \alpha_{Ht} \\ \alpha_{Lt} \end{pmatrix} = \begin{pmatrix} \Pi_{St}\sigma_S W_t \\ \Pi_{Ht}\sigma_H W_t \\ \Pi_{Lt} W_t \end{pmatrix}, \quad \lambda_t = \begin{pmatrix} (\mu'_S(t) + x_t)/\sigma_S \\ (\mu'_H(t) + y_t)/\sigma_H \\ \lambda_L(t) \end{pmatrix}, \quad B_t = \begin{pmatrix} B_{St} \\ B_{Ht} \\ B_{Lt} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & 0 & 0 \\ \rho_{HS} & \hat{\rho}_H & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

Let $Z = (H, L, x, y, z)^\top$ be the vector of state variables, which has the dynamics

$$
dZ_t = \mu_Z(t, Z_t)\, dt + \Sigma_Z(t, Z_t)\, dB_t,
$$

where

$$\mu_Z(t,Z_t)=\begin{pmatrix} H_t[r(t)+\mu_H(t)+y_t] \\ L_t[\mu_L(t)+\zeta(t)z_t] \\ -\kappa_x x_t \\ -\kappa_y y_t \\ -\kappa_z z_t \end{pmatrix}, \quad \Sigma_Z(Z_t)=\begin{pmatrix} H_t\sigma_H\rho_{HS} & H_t\sigma_H\hat{\rho}_H & 0 \\ L_t\sigma_L(t)\rho_{LS} & L_t\sigma_L(t)\hat{\rho}_{LH} & L_t\sigma_L(t)\hat{\rho}_L \\ \sigma_x\rho_{xS} & \sigma_x\hat{\rho}_{xH} & \sigma_x\hat{\rho}_{xL} \\ \sigma_y\rho_{yS} & \sigma_y\hat{\rho}_{yH} & \sigma_y\hat{\rho}_{yL} \\ \sigma_z\rho_{zS} & \sigma_z\hat{\rho}_{zH} & \sigma_z\hat{\rho}_{zL} \end{pmatrix}.$$

The Hamilton-Jacobi-Bellman equation (HJB) associated with the problem can be written as

$$\delta J=\mathcal{L}_1 J+\mathcal{L}_2 J+\mathcal{L}_3 J, \tag{36}$$

where

$$\begin{aligned} \mathcal{L}_1 J &= \max_{c,\phi_C}\left\{U(c,\phi_C)-J_W(c+HR\phi_C)\right\}, \\ \mathcal{L}_2 J &= \max_{\alpha}\left\{J_W\alpha^\top\lambda+\frac{1}{2}J_{WW}\alpha^\top\Sigma\Sigma^\top\alpha+\alpha^\top\Sigma\Sigma_Z^\top J_{WZ}\right\}, \\ \mathcal{L}_3 J &= \frac{\partial J}{\partial t}+J_W(rW+L)+J_Z^\top\mu_Z+\frac{1}{2}\mathrm{trace}\left(J_{ZZ}\Sigma_Z\Sigma_Z^\top\right). \end{aligned}$$

Recall that $J=J(t,W,H,L,x,y,z)=J(t,W,Z)$ so that

$$J_Z=\begin{pmatrix} J_H \\ J_L \\ J_x \\ J_y \\ J_z \end{pmatrix}, \quad J_{ZZ}=\begin{pmatrix} J_{HH} & J_{HL} & J_{Hx} & J_{Hy} & J_{Hz} \\ J_{HL} & J_{LL} & J_{Lx} & J_{Ly} & J_{Lz} \\ J_{Hx} & J_{Lx} & J_{xx} & J_{xy} & J_{xz} \\ J_{Hy} & J_{Ly} & J_{xy} & J_{yy} & J_{yz} \\ J_{Hz} & J_{Lz} & J_{xz} & J_{yz} & J_{zz} \end{pmatrix}, \quad J_{WZ}=\begin{pmatrix} J_{WH} \\ J_{WL} \\ J_{Wx} \\ J_{Wy} \\ J_{Wz} \end{pmatrix}.$$

First, consider $\mathcal{L}_1 J$. The first-order conditions are $U_c(c^*,\phi_C^*)=J_W$ and $U_\phi(c^*,\phi_C^*)=RHJ_W$. These imply $U_\phi(c^*,\phi_C^*)/U_c(c^*,\phi_C^*)=RH$ so that

$$\phi_C^*=c^*\left(\frac{aRH}{1-a}\right)^{-1}.$$

We substitute that relation into $U_c=J_W$ and find

$$c^*=J_W^{-1/\gamma}a^{1/\gamma}\left(\frac{aRH}{1-a}\right)^{k_1}, \tag{37}$$

and hence

$$\phi_C^*=J_W^{-1/\gamma}a^{1/\gamma}\left(\frac{aRH}{1-a}\right)^{k_1-1}, \tag{38}$$

where $k_1 = (1-a)(\gamma-1)/\gamma$. These maximizers lead to

$$\mathcal{L}_1 J = \frac{\gamma}{1-\gamma} J_W^{\frac{\gamma-1}{\gamma}} a^{\frac{1-\gamma}{\gamma}} \left(\frac{aRH}{1-a} \right)^{k_1}. \tag{39}$$

Next, consider $\mathcal{L}_2 J$. The first-order condition for α reads $J_W \lambda + J_{WW} \Sigma\Sigma^\top \alpha + \Sigma\Sigma_Z^\top J_{WZ} = 0$ or

$$\alpha = -\frac{J_W}{J_{WW}} (\Sigma\Sigma^\top)^{-1} \lambda - \frac{1}{J_{WW}} (\Sigma\Sigma^\top)^{-1} \Sigma\Sigma_Z^\top J_{WZ} = -\frac{J_W}{J_{WW}} (\Sigma\Sigma^\top)^{-1} \lambda - \frac{1}{J_{WW}} (\Sigma_Z \Sigma^{-1})^\top J_{WZ}. \tag{40}$$

Substituting the optimal α back into $\mathcal{L}_2 J$ leads to

$$\mathcal{L}_2 J = -\frac{1}{2} \frac{J_W^2}{J_{WW}} \lambda^\top (\Sigma\Sigma^\top)^{-1} \lambda - \frac{J_W}{J_{WW}} J_{WZ}^\top \Sigma_Z \Sigma^{-1} \lambda - \frac{1}{2} \frac{1}{J_{WW}} J_{WZ}^\top \Sigma_Z \Sigma_Z^\top J_{WZ}. \tag{41}$$

The matrix products are

$$\Sigma\Sigma^\top = \begin{pmatrix} 1 & \rho_{HS} & 0 \\ \rho_{HS} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad (\Sigma\Sigma^\top)^{-1} = \frac{1}{\hat{\rho}_H^2} \begin{pmatrix} 1 & -\rho_{HS} & 0 \\ -\rho_{HS} & 1 & 0 \\ 0 & 0 & \hat{\rho}_H^2 \end{pmatrix},$$

$$\Sigma_Z \Sigma^{-1} = \begin{pmatrix} 0 & H\sigma_H & 0 \\ LM_{LS}\sigma_S & LM_{LH}\sigma_H & L\hat{\rho}_L\sigma_L \\ M_{xS}\sigma_S & M_{xH}\sigma_H & \hat{\rho}_{xL}\sigma_x \\ M_{yS}\sigma_S & M_{yH}\sigma_H & \hat{\rho}_{yL}\sigma_y \\ M_{zS}\sigma_S & M_{zH}\sigma_H & \hat{\rho}_{zL}\sigma_z \end{pmatrix}, \quad \Sigma_Z \Sigma_Z^\top = \begin{pmatrix} H^2\sigma_H^2 & HL\sigma_{HL} & H\sigma_{Hx} & H\sigma_{Hy} & H\sigma_{Hz} \\ HL\sigma_{HL} & L^2\sigma_L^2 & L\sigma_{Lx} & L\sigma_{Ly} & L\sigma_{Lz} \\ H\sigma_{Hx} & L\sigma_{Lx} & \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ H\sigma_{Hy} & L\sigma_{Ly} & \sigma_{xy} & \sigma_y^2 & \sigma_{yz} \\ H\sigma_{Hz} & L\sigma_{Lz} & \sigma_{xz} & \sigma_{yz} & \sigma_z^2 \end{pmatrix},$$

where we have applied the covariance notation $\sigma_{ab} = \rho_{ab}\sigma_a\sigma_b$ and constants defined in Appendix A.

Substitution of these matrix products into (40) gives

$$\begin{aligned} \alpha_S = &-\frac{J_W}{J_{WW}} \frac{1}{\sigma_S \hat{\rho}_H^2} \left(\mu'_S(t) + x - \frac{\rho_{HS}\sigma_S}{\sigma_H} [\mu'_H(t) + y] \right) - \frac{J_{WL}}{J_{WW}} LM_{LS}(t)\sigma_S \\ &- \frac{J_{Wx}}{J_{WW}} M_{xS}\sigma_S - \frac{J_{Wy}}{J_{WW}} M_{yS}\sigma_S - \frac{J_{Wz}}{J_{WW}} M_{zS}\sigma_S, \end{aligned} \tag{42}$$

$$\begin{aligned} \alpha_H = &-\frac{J_W}{J_{WW}} \frac{1}{\sigma_H \hat{\rho}_H^2} \left(\mu'_H(t) + y - \frac{\rho_{HS}\sigma_H}{\sigma_S} [\mu'_S(t) + x] \right) - \frac{J_{WH}}{J_{WW}} H\sigma_H - \frac{J_{WL}}{J_{WW}} LM_{LH}(t)\sigma_H \\ &- \frac{J_{Wx}}{J_{WW}} M_{xH}\sigma_H - \frac{J_{Wy}}{J_{WW}} M_{yH}\sigma_H - \frac{J_{Wz}}{J_{WW}} M_{zH}\sigma_H, \end{aligned} \tag{43}$$

$$\alpha_L = -\frac{J_W}{J_{WW}} \lambda_L(t) - \frac{J_{WL}}{J_{WW}} L\hat{\rho}_L\sigma_L(t) - \frac{J_{Wx}}{J_{WW}} \hat{\rho}_{xL}\sigma_x - \frac{J_{Wy}}{J_{WW}} \hat{\rho}_{yL}\sigma_y - \frac{J_{Wz}}{J_{WW}} \hat{\rho}_{zL}\sigma_z. \tag{44}$$

B.2 Conjecture of the solution to the HJB equation

We conjecture

$$J(t, W, H, L, x, y, z) = \frac{1}{1-\gamma} G(t, H, x, y)^\gamma \left(W + LF(t, x, y, z) \right)^{1-\gamma}. \tag{45}$$

It turns out to be useful to express the derivatives of J in terms of J itself:

$$J_W = \frac{(1-\gamma)J}{W+LF}, \qquad J_{WW} = -\frac{\gamma(1-\gamma)J}{(W+LF)^2},$$

$$J_L = (1-\gamma)J\frac{F}{W+LF}, \qquad J_{LL} = -\gamma(1-\gamma)J\frac{F^2}{(W+LF)^2},$$

$$J_H = \gamma J\frac{G_H}{G}, \qquad J_{HH} = \gamma(1-\gamma)J\left[\frac{1}{1-\gamma}\frac{G_{HH}}{G} - \left(\frac{G_H}{G}\right)^2\right],$$

$$J_{WL} = -\gamma(1-\gamma)J\frac{F}{(W+LF)^2}, \qquad J_{WH} = \gamma(1-\gamma)J\frac{1}{W+LF}\frac{G_H}{G},$$

$$J_{HL} = \gamma(1-\gamma)J\frac{G_H}{G}\frac{F}{W+LF}, \qquad J_x = (1-\gamma)J\left[\frac{\gamma}{1-\gamma}\frac{G_x}{G} + \frac{LF_x}{W+LF}\right],$$

$$J_y = (1-\gamma)J\left[\frac{\gamma}{1-\gamma}\frac{G_y}{G} + \frac{LF_y}{W+LF}\right], \qquad J_z = (1-\gamma)J\frac{LF_z}{W+LF},$$

$$J_{Wz} = -\gamma(1-\gamma)J\frac{LF_z}{(W+LF)^2}, \qquad J_{Hz} = \gamma(1-\gamma)J\frac{LF_z}{W+LF}\frac{G_H}{G},$$

$$J_{Wx} = \gamma(1-\gamma)J\left[\frac{G_x}{G}\frac{1}{W+LF} - \frac{LF_x}{(W+LF)^2}\right], \qquad J_{Wy} = \gamma(1-\gamma)J\left[\frac{G_y}{G}\frac{1}{W+LF} - \frac{LF_y}{(W+LF)^2}\right],$$

$$J_{Lz} = (1-\gamma)J\left[\frac{F_z}{W+LF} - \gamma\frac{LF\,F_z}{(W+LF)^2}\right], \qquad \frac{\partial J}{\partial t} = (1-\gamma)J\left[\frac{\gamma}{1-\gamma}\frac{1}{G}\frac{\partial g}{\partial t} + \frac{L}{W+LF}\frac{\partial F}{\partial t}\right],$$

$$J_{xx} = (1-\gamma)J\left[\frac{\gamma}{1-\gamma}\frac{G_{xx}}{G} - \gamma\left(\frac{G_x}{G}\right)^2 + 2\gamma\frac{G_x}{G}\frac{LF_x}{W+LF} - \gamma\left(\frac{LF_x}{W+LF}\right)^2 + \frac{LF_{xx}}{W+LF}\right],$$

$$J_{yy} = (1-\gamma)J\left[\frac{\gamma}{1-\gamma}\frac{G_{yy}}{G} - \gamma\left(\frac{G_y}{G}\right)^2 + 2\gamma\frac{G_y}{G}\frac{LF_y}{W+LF} - \gamma\left(\frac{LF_y}{W+LF}\right)^2 + \frac{LF_{yy}}{W+LF}\right],$$

$$J_{zz} = (1-\gamma)J\left[\frac{LF_{zz}}{W+LF} - \gamma\left(\frac{LF_z}{W+LF}\right)^2\right],$$

$$J_{Hx} = \gamma(1-\gamma)J\left[\frac{1}{1-\gamma}\frac{G_{Hx}}{G} - \frac{G_xG_H}{G^2} + \frac{LF_x}{W+LF}\frac{G_H}{G}\right],$$

$$J_{Hy} = \gamma(1-\gamma)J\left[\frac{1}{1-\gamma}\frac{G_{Hy}}{G} - \frac{G_yG_H}{G^2} + \frac{LF_y}{W+LF}\frac{G_H}{G}\right],$$

$$J_{Lx} = (1-\gamma)J\left[\gamma\frac{G_x}{G}\frac{F}{W+LF} + \frac{F_x}{W+LF} - \gamma\frac{LF\,F_x}{(W+LF)^2}\right],$$

$$J_{Ly} = (1-\gamma)J\left[\gamma\frac{G_y}{G}\frac{F}{W+LF} + \frac{F_y}{W+LF} - \gamma\frac{LF\,F_y}{(W+LF)^2}\right],$$

$$J_{xy} = (1-\gamma)J\left[\frac{\gamma}{1-\gamma}\frac{G_{xy}}{G} - \gamma\frac{G_xG_y}{G^2} + \gamma\frac{G_x}{G}\frac{LF_y}{W+LF} + \gamma\frac{G_y}{G}\frac{LF_x}{W+LF} + \frac{LF_{xy}}{W+LF} - \gamma\frac{L^2F_xF_y}{(W+LF)^2}\right],$$

$$J_{xz} = (1-\gamma)J\left[\gamma\frac{G_x}{G}\frac{LF_z}{W+LF} - \gamma\frac{L^2F_xF_z}{(W+LF)^2} + \frac{LF_{xz}}{W+LF}\right],$$

$$J_{yz} = (1-\gamma)J\left[\gamma\frac{G_y}{G}\frac{LF_z}{W+LF} - \gamma\frac{L^2F_yF_z}{(W+LF)^2} + \frac{LF_{yz}}{W+LF}\right].$$

Next, we rewrite the terms $\mathcal{L}_1 J$, $\mathcal{L}_2 J$, $\mathcal{L}_3 J$ exploiting the conjecture for J. First, since

$$J_w^{\frac{\gamma-1}{\gamma}} = J_w J_w^{-1/\gamma} = \frac{(1-\gamma)J}{W+LF}\left\{G^\gamma [W+LF]^{-\gamma}\right\}^{-1/\gamma} = (1-\gamma)J\frac{1}{G}\,,$$

we get from (39) that

$$\mathcal{L}_1 J = \gamma J \frac{1}{G} a^{\frac{1-\gamma}{\gamma}} \left(\frac{aRH}{1-a}\right)^{k_1}.$$

Next, we have from (41) that

$$\mathcal{L}_2 J = \mathcal{L}_{2,1} J + \mathcal{L}_{2,2} J + \mathcal{L}_{2,3} J,$$

where

$$\begin{aligned}
\mathcal{L}_{2,1} J &= -\frac{1}{2}\frac{J_W^2}{J_{WW}}\lambda^\top (\Sigma\Sigma^\top)^{-1}\lambda = \frac{1-\gamma}{2\gamma} J \lambda^\top (\Sigma\Sigma^\top)^{-1}\lambda \\
&= \frac{1-\gamma}{2\gamma} J \left\{ \left(\frac{\mu_S'(t)+x_t}{\sigma_S \hat{\rho}_H}\right)^2 + \left(\frac{\mu_H'(t)+y_t}{\sigma_H \hat{\rho}_H}\right)^2 - 2\rho_{HS}\frac{\mu_S'(t)+x_t}{\sigma_S\hat{\rho}_H}\,\frac{\mu_H'(t)+y_t}{\sigma_H\hat{\rho}_H} + \lambda_L(t)^2 \right\}, \\
\mathcal{L}_{2,2} J &= -\frac{J_W}{J_{WW}} J_{WZ}^\top \Sigma_Z \Sigma^{-1}\lambda = \frac{1}{\gamma}(W+LF) J_{WZ}^\top \Sigma_Z \Sigma^{-1}\lambda \\
&= (1-\gamma)J \begin{pmatrix} \frac{G_H}{G} \\ -\frac{F}{W+LF} \\ \frac{G_x}{G} - \frac{LF_x}{W+LF} \\ \frac{G_y}{G} - \frac{LF_y}{W+LF} \\ -\frac{LF_z}{W+LF} \end{pmatrix}^\top \begin{pmatrix} 0 & H\sigma_H & 0 \\ L\ell_(t)\sigma_S & LM_{LH}(t)\sigma_H & L\hat{\rho}_L\sigma_L(t) \\ M_{xS}\sigma_S & M_{xH}\sigma_H & \hat{\rho}_{xL}\sigma_x \\ M_{yS}\sigma_S & M_{yH}\sigma_H & \hat{\rho}_{yL}\sigma_y \\ M_{zS}\sigma_S & M_{zH}\sigma_H & \hat{\rho}_{zL}\sigma_z \end{pmatrix} \begin{pmatrix} (\mu_S'(t)+x_t)/\sigma_S \\ (\mu_H'(t)+y_t)/\sigma_H \\ \lambda_L(t) \end{pmatrix} \\
&= (1-\gamma)J \Bigg\{ \frac{HG_H}{G}(\mu_H'(t)+y_t) + \frac{G_x}{G}\left(M_{xS}x_t + M_{xH}y_t + M_x(t)\right) + \frac{G_y}{G}\left(M_{yS}x_t + M_{yH}y_t + M_y(t)\right) \\
&\qquad - \frac{L}{W+LF}\Big[F\left(M_{LS}(t)x_t + M_{LH}(t)y_t + \mu_L(t) - \ell(t)\right) + F_x\left(M_{xS}x_t + M_{xH}y_t + M_x(t)\right) \\
&\qquad\qquad + F_y\left(M_{yS}x_t + M_{yH}y_t + M_y(t)\right) + F_z\left(M_{zS}x_t + M_{zH}y_t + M_z(t)\right)\Big]\Bigg\},
\end{aligned}$$

and

$$
\begin{aligned}
\mathcal{L}_{2,3}J &= -\frac{1}{2}\frac{1}{J_{WW}}J_{WZ}^{\top}\Sigma_Z\Sigma_Z^{\top}J_{WZ} \\
&= \frac{1}{2}\gamma(1-\gamma)J\begin{pmatrix} \frac{G_H}{G} \\ -\frac{F}{W+LF} \\ \frac{G_x}{G}-\frac{LF_x}{W+LF} \\ \frac{G_y}{G}-\frac{LF_y}{W+LF} \\ -\frac{LF_z}{W+LF}\end{pmatrix}^{\top}\begin{pmatrix} H^2\sigma_H^2 & HL\sigma_{HL} & H\sigma_{Hx} & H\sigma_{Hy} & H\sigma_{Hz} \\ HL\sigma_{HL} & L^2\sigma_L^2 & L\sigma_{Lx} & L\sigma_{Ly} & L\sigma_{Lz} \\ H\sigma_{Hx} & L\sigma_{Lx} & \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ H\sigma_{Hy} & L\sigma_{Ly} & \sigma_{xy} & \sigma_y^2 & \sigma_{yz} \\ H\sigma_{Hz} & L\sigma_{Lz} & \sigma_{xz} & \sigma_{yz} & \sigma_z^2 \end{pmatrix}\begin{pmatrix} \frac{G_H}{G} \\ -\frac{F}{W+LF} \\ \frac{G_x}{G}-\frac{LF_x}{W+LF} \\ \frac{G_y}{G}-\frac{LF_y}{W+LF} \\ -\frac{LF_z}{W+LF}\end{pmatrix} \\
&= \gamma(1-\gamma)J\Bigg\{\frac{1}{2}\sigma_H^2\frac{H^2G_H^2}{G^2}+\frac{1}{2}\sigma_x^2\frac{G_x^2}{G^2}+\frac{1}{2}\sigma_y^2\frac{G_y^2}{G^2}+\sigma_{Hx}\frac{HG_HG_x}{G^2}+\sigma_{Hy}\frac{HG_HG_y}{G^2}+\sigma_{xy}\frac{G_xG_y}{G^2} \\
&\qquad -\frac{L}{W+LF}\Bigg[\frac{HG_H}{G}\left(\sigma_{HL}F+\sigma_{Hx}F_x+\sigma_{Hy}F_y+\sigma_{Hz}F_z\right) \\
&\qquad\qquad +\frac{G_x}{G}\left(\sigma_{Lx}F+\sigma_x^2F_x+\sigma_{xy}F_y+\sigma_{xz}F_z\right) \\
&\qquad\qquad +\frac{G_y}{G}\left(\sigma_{Ly}F+\sigma_{xy}F_x+\sigma_y^2F_y+\sigma_{yz}F_z\right)\Bigg] \\
&\qquad +\frac{L^2}{(W+LF)^2}\Bigg[\frac{1}{2}\sigma_L^2F^2+\frac{1}{2}\sigma_x^2F_x^2+\frac{1}{2}\sigma_y^2F_y^2+\frac{1}{2}\sigma_z^2F_z^2 \\
&\qquad\qquad +\sigma_{Lx}FF_x+\sigma_{Ly}FF_y+\sigma_{Lz}FF_z+\sigma_{xy}F_xF_y+\sigma_{xz}F_xF_z+\sigma_{yz}F_yF_z\Bigg]\Bigg\}.
\end{aligned}
$$

Finally, we can rewrite $\mathcal{L}_3J$ as

$$
\mathcal{L}_3J=\mathcal{L}_{3,1}J+\mathcal{L}_{3,2}J,
$$

where

$$
\begin{aligned}
\mathcal{L}_{3,1}J &= \frac{\partial J}{\partial t}+J_W(r(t)W+L)+J_Z^{\top}\mu_Z \\
&= (1-\gamma)J\Bigg\{r(t)+\frac{\gamma}{1-\gamma}\left[\frac{1}{G}\frac{\partial G}{\partial t}++\frac{HG_H}{G}\left(r(t)+\mu_H(t)+y\right)-\kappa_xx\frac{G_x}{G}-\kappa_yy\frac{G_y}{G}\right] \\
&\qquad +\frac{L}{W+LF}\left[\frac{\partial F}{\partial t}+1+\left(\mu_L(t)+\zeta(t)z-r(t)\right)F-\kappa_xxF_x-\kappa_yyF_y-\kappa_zzF_z\right]\Bigg\},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}_{3,2}J &= \frac{1}{2}\mathrm{trace}\left(J_{ZZ}\Sigma_Z\Sigma_Z^\top\right) \\
&= \frac{1}{2}\sigma_H^2 H^2 J_{HH} + \frac{1}{2}\sigma_L^2 L^2 J_{LL} + \frac{1}{2}\sigma_x^2 J_{xx} + \frac{1}{2}\sigma_y^2 J_{yy} + \frac{1}{2}\sigma_z^2 J_{zz} + \sigma_{HL} HLJ_{HL} + \sigma_{Hx} HJ_{Hx} \\
&\quad + \sigma_{Hy} HJ_{Hy} + \sigma_{Hz} HJ_{Hz} + \sigma_{Lx} LJ_{Lx} + \sigma_{Ly} LJ_{Ly} + \sigma_{Lz} LJ_{Lz} + \sigma_{xy} J_{xy} + \sigma_{xz} J_{xz} + \sigma_{yz} J_{yz} \\
&= (1-\gamma)J\Bigg\{\frac{\gamma}{1-\gamma}\Bigg[\frac{1}{2}\sigma_H^2 H^2\frac{G_{HH}}{G} + \frac{1}{2}\sigma_x^2\frac{G_{xx}}{G} + \frac{1}{2}\sigma_y^2\frac{G_{yy}}{G} - \frac{1-\gamma}{2}\sigma_H^2 H^2\frac{G_H^2}{G^2} - \frac{1-\gamma}{2}\sigma_x^2\frac{G_x^2}{G^2} \\
&\qquad - \frac{1-\gamma}{2}\sigma_y^2\frac{G_y^2}{G^2} + \sigma_{Hx} H\frac{G_{Hx}}{G} + \sigma_{Hy} H\frac{G_{Hy}}{G} + \sigma_{xy}\frac{G_{xy}}{G} \\
&\qquad - (1-\gamma)\sigma_{Hx} H\frac{G_H G_x}{G^2} - (1-\gamma)\sigma_{Hy} H\frac{G_H G_y}{G^2} - (1-\gamma)\sigma_{xy}\frac{G_x G_y}{G^2}\Bigg] \\
&\quad + \frac{L}{W+LF}\Bigg[\frac{1}{2}\sigma_x^2 F_{xx} + \frac{1}{2}\sigma_y^2 F_{yy} + \frac{1}{2}\sigma_z^2 F_{zz} \\
&\qquad + \sigma_{xy} F_{xy} + \sigma_{xz} F_{xz} + \sigma_{yz} F_{yz} + \sigma_{Lx} F_x + \sigma_{Ly} F_y + \sigma_{Lz} F_z \\
&\qquad + \gamma\Bigg[\frac{HG_H}{G}\left(\sigma_{HL} F + \sigma_{Hx} F_x + \sigma_{Hy} F_y + \sigma_{Hz} F_z\right) \\
&\qquad + \frac{G_x}{G}\left(\sigma_{Lx} F + \sigma_x^2 F_x + \sigma_{xy} F_y + \sigma_{xz} F_z\right) \\
&\qquad + \frac{G_y}{G}\left(\sigma_{Ly} F + \sigma_{xy} F_x + \sigma_y^2 F_y + \sigma_{yz} F_z\right)\Bigg]\Bigg] \\
&\quad - \frac{L^2}{(W+LF)^2}\gamma\Bigg[\frac{1}{2}\sigma_L^2 F^2 + \frac{1}{2}\sigma_x^2 F_x^2 + \frac{1}{2}\sigma_y^2 F_y^2 + \frac{1}{2}\sigma_z^2 F_z^2 + \sigma_{Lx} FF_x + \sigma_{Ly} FF_y \\
&\qquad + \sigma_{Lz} FF_z + \sigma_{xy} F_x F_y + \sigma_{xz} F_x F_z + \sigma_{yz} F_y F_z\Bigg]\Bigg\}
\end{aligned}
$$

By adding $\mathcal{L}_{2,3}J$ and $\mathcal{L}_{3,2}J$ numerous terms cancel so that we are left with

$$
\begin{aligned}
\mathcal{L}_{2,3}J + \mathcal{L}_{3,2}J = (1-\gamma)J\Bigg\{&\frac{\gamma}{1-\gamma}\Bigg[\frac{1}{2}\sigma_H^2 H^2\frac{G_{HH}}{G} + \frac{1}{2}\sigma_x^2\frac{G_{xx}}{G} + \frac{1}{2}\sigma_y^2\frac{G_{yy}}{G} + \sigma_{Hx} H\frac{G_{Hx}}{G} + \sigma_{Hy} H\frac{G_{Hy}}{G} + \sigma_{xy}\frac{G_{xy}}{G}\Bigg] \\
&+ \frac{L}{W+LF}\Bigg[\frac{1}{2}\sigma_x^2 F_{xx} + \frac{1}{2}\sigma_y^2 F_{yy} + \frac{1}{2}\sigma_z^2 F_{zz} \\
&\qquad + \sigma_{xy} F_{xy} + \sigma_{xz} F_{xz} + \sigma_{yz} F_{yz} + \sigma_{Lx} F_x + \sigma_{Ly} F_y + \sigma_{Lz} F_z\Bigg]\Bigg\}
\end{aligned}
$$

If we further add $\mathcal{L}_{2,2}J$ and $\mathcal{L}_{3,1}J$ to this, all the terms multiplying $L/(W+LF)$ will cancel because F

satisfies the PDE (35). In sum, we get

$$
\begin{aligned}
\delta J &= \mathcal{L}_1 J + \mathcal{L}_{2,1} J + \mathcal{L}_{2,2} J + \mathcal{L}_{2,3} J + \mathcal{L}_{3,1} J + \mathcal{L}_{3,2} J \\
&= \gamma J \frac{1}{G} a^{\frac{1-\gamma}{\gamma}} \left(\frac{aRH}{1-a}\right)^{k_1} \\
&\quad + J \frac{1-\gamma}{2\gamma} \left[\left(\frac{\mu_S'(t)+x_t}{\sigma_S \hat{\rho}_H}\right)^2 + \left(\frac{\mu_H'(t)+y_t}{\sigma_H \hat{\rho}_H}\right)^2 - 2\rho_{HS} \frac{\mu_S'(t)+x_t}{\sigma_S \hat{\rho}_H} \frac{\mu_H'(t)+y_t}{\sigma_H \hat{\rho}_H} + \lambda_L(t)^2 \right] \\
&\quad + (1-\gamma) J \frac{1}{G} \Big\{ r(t) + HG_H \left(\mu_H'(t) + y\right) + G_x \left(M_{xS} x + M_{xH} y + M_x(t)\right) + G_y \left(M_{yS} x + M_{yH} y + M_y(t)\right) \\
&\qquad + \frac{\gamma}{1-\gamma} \Big[\frac{1}{2}\sigma_H^2 H^2 G_{HH} + \frac{1}{2}\sigma_x^2 G_{xx} + \frac{1}{2}\sigma_y^2 G_{yy} + \sigma_{Hx} H G_{Hx} + \sigma_{Hy} H G_{Hy} + \sigma_{xy} G_{xy} \\
&\qquad\qquad + \frac{\partial G}{\partial t} + HG_H \left(r(t) + \mu_H(t) + y\right) - \kappa_x x G_x - \kappa_y y G_y \Big] \Big\}.
\end{aligned}
$$

Therefore, it follows that the HJB equation is satisfied if the G function solves the PDE

$$
\begin{aligned}
0 &= k_2 H^{k_1} + \frac{\partial G}{\partial t} + \frac{1}{2}\sigma_H^2 H^2 G_{HH} + \frac{1}{2}\sigma_x^2 G_{xx} + \frac{1}{2}\sigma_y^2 G_{yy} + \sigma_{Hx} H G_{Hx} + \sigma_{Hy} H G_{Hy} + \sigma_{xy} G_{xy} \\
&\quad + \left(r(t) - \tfrac{\gamma-1}{\gamma} R + \frac{1}{\gamma}\left(\mu_H(t) + y\right) \right) HG_H - \left(\bar{\kappa}_x x + \tfrac{\gamma-1}{\gamma} M_{xH} y + \tfrac{\gamma-1}{\gamma} M_x(t) \right) G_x \\
&\quad - \left(\tfrac{\gamma-1}{\gamma} M_{yS} x + \bar{\kappa}_y y + \tfrac{\gamma-1}{\gamma} M_y(t) \right) G_y \\
&\quad - \left(K_0(t) + K_x(t) x + K_y(t) y + K_{xx} x^2 + K_{xy} xy + K_{yy} y^2 \right) G,
\end{aligned}
\tag{46}
$$

where we have introduced

$$
\begin{aligned}
\bar{\kappa}_x &= \kappa_x + \tfrac{\gamma-1}{\gamma} M_{xS}, & \bar{\kappa}_y &= \kappa_y + \tfrac{\gamma-1}{\gamma} M_{yH}, \\
K_{xx} &= \frac{\gamma-1}{2\gamma^2 \sigma_S^2 \hat{\rho}_H^2}, & K_x(t) &= \frac{\gamma-1}{\gamma^2 \sigma_S^2 \hat{\rho}_H^2} \left(\mu_S'(t) - \frac{\rho_{HS}\sigma_S}{\sigma_H} \mu_H'(t) \right), \\
K_{yy} &= \frac{\gamma-1}{2\gamma^2 \sigma_H^2 \hat{\rho}_H^2}, & K_y(t) &= \frac{\gamma-1}{\gamma^2 \sigma_H^2 \hat{\rho}_H^2} \left(\mu_H'(t) - \frac{\rho_{HS}\sigma_H}{\sigma_S} \mu_S'(t) \right), \\
K_{xy} &= -\frac{\gamma-1}{\gamma^2} \frac{\rho_{HS}}{\sigma_S \sigma_H \hat{\rho}_H^2}, & k_2 &= a^{\frac{1-\gamma}{\gamma}} \left(\frac{aR}{1-a} \right)^{k_1},
\end{aligned}
$$

and

$$
K_0(t) = \frac{\delta}{\gamma} + \tfrac{\gamma-1}{\gamma} r(t) + \frac{\gamma-1}{2\gamma^2} \left[\left(\frac{\mu_S'(t)}{\sigma_S \hat{\rho}_H}\right)^2 + \left(\frac{\mu_H'(t)}{\sigma_H \hat{\rho}_H}\right)^2 - 2 \frac{\rho_{HS}\mu_S'(t)\mu_H'(t)}{\sigma_S \sigma_H \hat{\rho}_H^2} + \lambda_L(t)^2 \right].
$$

Coming back to the optimal investment strategy, we first note that

$$
\begin{aligned}
-\frac{J_W}{J_{WW}} &= \frac{1}{\gamma}(W + LF), & -\frac{J_{Wx}}{J_{WW}} &= \frac{G_x}{G}(W + LF) - LF_x, \\
-\frac{J_{WL}}{J_{WW}} &= -F, & -\frac{J_{Wy}}{J_{WW}} &= \frac{G_y}{G}(W + LF) - LF_y, \\
-\frac{J_{WH}}{J_{WW}} &= \frac{G_H}{G}(W + LF), & -\frac{J_{Wz}}{J_{WW}} &= -LF_z.
\end{aligned}
$$

By substituting these expressions into (42)-(44), we obtain

$$\alpha_S = \frac{1}{\gamma}(W+LF)\frac{1}{\sigma_S\hat{\rho}_H^2}\left(\mu_S'(t)+x-\frac{\rho_{HS}\sigma_S}{\sigma_H}[\mu_H'(t)+y]\right) - FLM_{LS}(t)\sigma_S$$
$$+\left[\frac{G_x}{G}(W+LF)-LF_x\right]M_{xS}\sigma_S + \left[\frac{G_y}{G}(W+LF)-LF_y\right]M_{yS}\sigma_S - LF_zM_{zS}\sigma_S, \tag{47}$$

$$\alpha_H = \frac{1}{\gamma}(W+LF)\frac{1}{\sigma_H\hat{\rho}_H^2}\left(\mu_H'(t)+y-\frac{\rho_{HS}\sigma_H}{\sigma_S}[\mu_S'(t)+x]\right) + \frac{G_H}{G}(W+LF)H\sigma_H - FLM_{LH}(t)\sigma_H$$
$$+\left[\frac{G_x}{G}(W+LF)-LF_x\right]M_{xH}\sigma_H + \left[\frac{G_y}{G}(W+LF)-LF_y\right]M_{yH}\sigma_H - LF_zM_{zH}\sigma_H, \tag{48}$$

$$\alpha_L = \frac{1}{\gamma}(W+LF)\lambda_L(t) - FL\hat{\rho}_L\sigma_L(t) + \left[\frac{G_x}{G}(W+LF)-LF_x\right]\hat{\rho}_{xL}\sigma_x$$
$$+\left[\frac{G_y}{G}(W+LF)-LF_y\right]\hat{\rho}_{yL}\sigma_y - LF_z\hat{\rho}_{zL}\sigma_z. \tag{49}$$

Now (19)–(21) in the theorem follows easily since $\Pi_S = \alpha_S/(\sigma_S W)$, $\Pi_H = \alpha_H/(\sigma_H W)$, and $\Pi_L = \alpha_L/W$.

B.3 Solving for the G function

We conjecture that the solution to (46) takes the form

$$G(t,H,x,y) = \varepsilon^{1/\gamma}\exp\left\{D_0(t)+D_x(t)x+D_y(t)y+\frac{1}{2}D_{xx}(t)x^2+\frac{1}{2}D_{yy}(t)y^2+D_{xy}(t)xy\right\}$$
$$+k_2H^{k_1}\int_t^T \exp\left\{\bar{D}_0(t,s)+\bar{D}_x(t,s)x+\bar{D}_y(t,s)y+\frac{1}{2}\bar{D}_{xx}(t,s)x^2+\frac{1}{2}\bar{D}_{yy}(t,s)y^2+\bar{D}_{xy}(t,s)xy\right\}ds.$$

Solutions of this form are known from Wachter (2002), Liu (2007), and Kraft and Munk (2011), among others. We substitute the relevant derivatives into (46), equate the sum of all the terms without integrals to zero, and equate the sum of the integrands in all the integral terms to zero, we find that the PDE is indeed satisfied under the following conditions on the D and $\bar{D}$ functions:

First, the functions D_0, D_x, D_y, D_{xx}, D_{yy}, and D_{xy} must satisfy the following system of ODEs

$$0 = \frac{1}{2}D_{xx}'(t)+\frac{1}{2}\sigma_x^2D_{xx}(t)^2+\frac{1}{2}\sigma_y^2D_{xy}(t)^2+\sigma_{xy}D_{xy}(t)D_{xx}(t)-\bar{\kappa}_xD_{xx}(t)-\tfrac{\gamma-1}{\gamma}M_{yS}D_{xy}(t)-K_{xx},$$
$$0 = \frac{1}{2}D_{yy}'(t)+\frac{1}{2}\sigma_x^2D_{xy}(t)^2+\frac{1}{2}\sigma_y^2D_{yy}(t)^2+\sigma_{xy}D_{xy}(t)D_{yy}(t)-\bar{\kappa}_yD_{yy}(t)-\tfrac{\gamma-1}{\gamma}M_{xH}D_{xy}(t)-K_{yy},$$
$$0 = D_{xy}'(t)+\sigma_x^2D_{xx}(t)D_{xy}(t)+\sigma_y^2D_{yy}(t)D_{xy}(t)+\sigma_{xy}\left(D_{xx}(t)D_{yy}(t)+D_{xy}(t)^2\right)$$
$$-(\bar{\kappa}_x+\bar{\kappa}_y)D_{xy}(t)-\tfrac{\gamma-1}{\gamma}M_{xH}D_{xx}(t)-\tfrac{\gamma-1}{\gamma}M_{yS}D_{yy}(t)-K_{xy},$$
$$0 = D_x'(t)+\left(\sigma_x^2D_{xx}(t)+\sigma_{xy}D_{xy}(t)-\bar{\kappa}_x\right)D_x(t)+\left(\sigma_y^2D_{xy}(t)+\sigma_{xy}D_{xx}(t)-\tfrac{\gamma-1}{\gamma}M_{yS}\right)D_y(t)-K_x(t),$$
$$0 = D_y'(t)+\left(\sigma_x^2D_{xy}(t)+\sigma_{xy}D_{yy}(t)-\tfrac{\gamma-1}{\gamma}M_{xH}\right)D_x(t)+\left(\sigma_y^2D_{yy}(t)+\sigma_{xy}D_{xy}(t)-\bar{\kappa}_y\right)D_y(t)-K_y(t),$$
$$0 = D_0'(t)+\frac{1}{2}\sigma_x^2\left(D_x(t)^2+D_{xx}(t)\right)+\frac{1}{2}\sigma_y^2\left(D_y(t)^2+D_{yy}(t)\right)+\sigma_{xy}\left(D_x(t)D_y(t)+D_{xy}(t)\right)$$
$$-\tfrac{\gamma-1}{\gamma}M_x(t)D_x(t)-\tfrac{\gamma-1}{\gamma}M_y(t)D_y(t)-K_0(t),$$

with .

Secondly, the functions $\bar{D}_0$, $\bar{D}_x$, $\bar{D}_y$, $\bar{D}_{xx}$, $\bar{D}_{yy}$, and $\bar{D}_{xy}$ must satisfy the following system of equations:

$$
\begin{aligned}
0 &= \frac{1}{2}\frac{\partial \bar{D}_{xx}}{\partial t}(t,s) + \frac{1}{2}\sigma_x^2 \bar{D}_{xx}(t,s)^2 + \frac{1}{2}\sigma_y^2 \bar{D}_{xy}(t,s)^2 + \sigma_{xy}\bar{D}_{xy}(t,s)\bar{D}_{xx}(t,s) - \bar{\kappa}_x \bar{D}_{xx}(t,s) - \tfrac{\gamma-1}{\gamma} M_{yS}\bar{D}_{xy}(t,s) - K_{xx}, \\
0 &= \frac{1}{2}\frac{\partial \bar{D}_{yy}}{\partial t}(t,s) + \frac{1}{2}\sigma_x^2 \bar{D}_{xy}(t,s)^2 + \frac{1}{2}\sigma_y^2 \bar{D}_{yy}(t,s)^2 + \sigma_{xy}\bar{D}_{xy}(t,s)\bar{D}_{yy}(t,s) - \bar{\kappa}_y \bar{D}_{yy}(t,s) - \tfrac{\gamma-1}{\gamma} M_{xH}\bar{D}_{xy}(t,s) - K_{yy}, \\
0 &= \frac{\partial D_{xy}}{\partial t}(t,s) + \sigma_x^2 \bar{D}_{xx}(t,s)\bar{D}_{xy}(t,s) + \sigma_y^2 \bar{D}_{yy}(t,s)\bar{D}_{xy}(t,s) + \sigma_{xy}\left(\bar{D}_{xx}(t,s)\bar{D}_{yy}(t,s) + \bar{D}_{xy}(t,s)^2\right) \\
&\quad - (\bar{\kappa}_x + \bar{\kappa}_y)\bar{D}_{xy}(t,s) - \tfrac{\gamma-1}{\gamma} M_{xH}\bar{D}_{xx}(t,s) - \tfrac{\gamma-1}{\gamma} M_{yS}\bar{D}_{yy}(t,s) - K_{xy}, \\
0 &= \frac{\partial \bar{D}_x}{\partial t}(t,s) + \left(\sigma_x^2 \bar{D}_{xx}(t,s) + \sigma_{xy}\bar{D}_{xy}(t,s) - \bar{\kappa}_x\right)\bar{D}_x(t,s) + \left(\sigma_y^2 \bar{D}_{xy}(t,s) + \sigma_{xy}\bar{D}_{xx}(t,s) - \tfrac{\gamma-1}{\gamma} M_{yS}\right)\bar{D}_y(t,s) \\
&\quad + k_1\sigma_{Hx}\bar{D}_{xx}(t,s) + k_1\sigma_{Hy}\bar{D}_{xy}(t,s) - K_x(t), \\
0 &= \frac{\partial \bar{D}_y}{\partial t}(t,s) + \left(\sigma_x^2 \bar{D}_{xy}(t,s) + \sigma_{xy}\bar{D}_{yy}(t,s) - \tfrac{\gamma-1}{\gamma} M_{xH}\right)\bar{D}_x(t,s) + \left(\sigma_y^2 \bar{D}_{yy}(t,s) + \sigma_{xy}\bar{D}_{xy}(t,s) - \bar{\kappa}_y\right)\bar{D}_y(t) \\
&\quad + k_1\sigma_{Hx}\bar{D}_{xy}(t,s) + k_1\sigma_{Hy}\bar{D}_{yy}(t,s) - K_y(t), \\
0 &= \frac{\partial \bar{D}_0}{\partial t}(t,s) + \frac{1}{2}\sigma_x^2\left(\bar{D}_x(t,s)^2 + \bar{D}_{xx}(t,s)\right) + \frac{1}{2}\sigma_y^2\left(\bar{D}_y(t,s)^2 + \bar{D}_{yy}(t,s)\right) + \sigma_{xy}\left(\bar{D}_x(t,s)\bar{D}_y(t,s) + \bar{D}_{xy}(t,s)\right) \\
&\quad + \left(k_1\sigma_{Hx} - \tfrac{\gamma-1}{\gamma} M_x(t)\right)\bar{D}_x(t,s) + \left(k_1\sigma_{Hy} - \tfrac{\gamma-1}{\gamma} M_y(t)\right)\bar{D}_y(t,s) \\
&\quad + \frac{1}{2}k_1(k_1-1)\sigma_H^2 + k_1\left(r(t) - \tfrac{\gamma-1}{\gamma} R + \frac{1}{\gamma}\left(\mu_H(t) + y\right)\right) - K_0(t),
\end{aligned}
$$

with .

C Solution in special cases

C.1 Model with no x

In an artificial market with deterministic modifiers, the indirect utility is then

$$
J(t,W,H,L,y,z) = \frac{1}{1-\gamma} G(t,H,y)^{\gamma}\left(W + LF(t,y,z)\right)^{1-\gamma}, \tag{50}
$$

where

$$
F(t,y,z) = \begin{cases} \displaystyle\int_t^T e^{\alpha(t,s)+\beta_y(t,s)y+\beta_z(t,s)z}\,ds, & t \in (\widetilde{T}, T], \\ \displaystyle\int_t^{\widetilde{T}} e^{\alpha(t,s)+\beta_y(t,s)y+\beta_z(t,s)z}\,ds + \Upsilon \int_{\widetilde{T}}^T e^{\alpha(t,s)+\beta_y(t,s)y+\beta_z(t,s)z}\,ds, & t \in [0,T], \end{cases} \tag{51}
$$

and

$$
G(t,H,y) = \varepsilon^{1/\gamma} e^{D_0(t)+D_y(t)y+\frac{1}{2}D_{yy}(t)y^2} + k_2 H^{k_1}\int_t^T e^{\bar{D}_0(t,s)+\bar{D}_y(t,s)y+\frac{1}{2}\bar{D}_{yy}(t,s)y^2}\,ds. \tag{52}
$$

with $k_1 = (1-a)(\gamma-1)/\gamma$, $k_2 = a^{\frac{1-\gamma}{\gamma}} \left(\frac{aR}{1-a}\right)^{k_1}$. The optimal portfolio weights are

$$\begin{aligned}
\Pi_S &= \frac{1}{\gamma \hat{\rho}_H^2 \sigma_S^2} \left(\mu'_S(t) - \frac{\rho_{HS}\sigma_S}{\sigma_H} [\mu'_H(t) + y] \right) \frac{W + LF}{W} \\
&\quad + M_{yS} \frac{G_y}{G} \frac{W + LF}{W} - \left(M_{LS}(t) + M_{yS} \frac{F_y}{F} + M_{zS} \frac{F_z}{F} \right) \frac{LF}{W}, && (53) \\
\Pi_H &= \frac{1}{\gamma \hat{\rho}_H^2 \sigma_H^2} \left(\mu'_H(t) + y - \frac{\rho_{HS}\sigma_H}{\sigma_S} \mu'_S(t) \right) \frac{W + LF}{W} + \frac{HG_H}{G} \frac{W + LF}{W} \\
&\quad + M_{yH} \frac{G_y}{G} \frac{W + LF}{W} - \left(M_{LH}(t) + M_{yH} \frac{F_y}{F} + M_{zH} \frac{F_z}{F} \right) \frac{LF}{W}, && (54) \\
\Pi_L &= \frac{1}{\gamma} \lambda_L(t) \frac{W + LF}{W} + \sigma_y \hat{\rho}_{yL} \frac{G_y}{G} \frac{W + LF}{W} \\
&\quad - \left(\hat{\rho}_L \sigma_L(t) + \sigma_y \hat{\rho}_{yL} \frac{F_y}{F} + \sigma_z \hat{\rho}_{zL} \frac{F_z}{F} \right) \frac{LF}{W}, && (55)
\end{aligned}$$

where the functions and constants M_{ij} are defined in Appendix A. The optimal consumption strategy is

$$c = a^{1/\gamma} \left(\frac{aRH}{1-a} \right)^{k_1} \frac{W + LF}{G}, \tag{56}$$

$$\phi_C = a^{1/\gamma} \left(\frac{aRH}{1-a} \right)^{k_1 - 1} \frac{W + LF}{G}. \tag{57}$$

In the above expressions, we have

$$\beta_z(t,s) = \int_t^s \zeta(u) e^{-\kappa_z(u-t)}\, du,$$

whereas β_y satisfies the differential equation

$$\frac{\partial \beta_y}{\partial t}(t,s) = [\kappa_y + M_{yH}]\beta_y(t,s) + M_{zH}\beta_z(t,s) + M_{LH}(t)$$

with the condition $\beta_y(s,s) = 0$, which implies

$$\beta_y(t,s) = -\int_t^s \left(M_{zH}\beta_z(v,s) + M_{LH}(v) \right) e^{-[\kappa_y + M_{yH}](v-t)}\, dv.$$

Furthermore, $\alpha(t,s)$ is determined by

$$\begin{aligned}
\alpha(t,s) &= \int_t^s \ell(u)\, du - \int_t^s r(u)\, du \\
&\quad - \int_t^s \left(M_y(u) - \rho_{Ly}\sigma_L(u)\sigma_y \right) \beta_y(u,s)\, du - \int_t^s \left(M_z(u) - \rho_{Lz}\sigma_L(u)\sigma_z \right) \beta_z(u,s)\, du \\
&\quad + \sigma_{yz} \int_t^s \beta_y(u,s)\beta_z(u,s)\, du + \frac{1}{2}\sigma_y^2 \int_t^s \beta_y(u,s)^2\, du + \frac{1}{2}\sigma_z^2 \int_t^s \beta_z(u,s)^2\, du.
\end{aligned}$$

Note that if ζ and σ_L are constant over $[t,s]$, we have

$$
\begin{aligned}
\beta_z(t,s) &= \zeta \mathcal{B}_{\kappa_z}(s-t), \\
\beta_y(t,s) &= \zeta M_{zH} \frac{\mathcal{B}_{\tilde{\kappa}_y}(s-t) - \mathcal{B}_{\kappa_z}(s-t)}{\tilde{\kappa}_y - \kappa_z} - M_{LH} \mathcal{B}_{\tilde{\kappa}_y}(s-t), \\
\alpha(t,s) &= \int_t^s \ell(u)\, du - \int_t^s r(u)\, du \\
&\quad - \int_t^s \left(M_y(u) - \rho_{Ly}\sigma_L \sigma_y \right) \beta_y(u,s)\, du - \int_t^s \left(M_z(u) - \rho_{Lz}\sigma_L \sigma_z \right) \beta_z(u,s)\, du \\
&\quad + \sigma_{yz} \frac{\zeta(\Gamma - M_{LH})}{\kappa_z \tilde{\kappa}_y} \left(s - t - \mathcal{B}_{\tilde{\kappa}_y}(s-t) - \mathcal{B}_{\kappa_z}(s-t) + \mathcal{B}_{\tilde{\kappa}_y + \kappa_z}(s-t) \right) \\
&\quad - \sigma_{yz} \frac{\zeta\Gamma}{\kappa_z^2} \left(s - t - \mathcal{B}_{\kappa_z}(s-t) - \frac{\kappa_z}{2} \mathcal{B}_{\kappa_z}(s-t)^2 \right) \\
&\quad + \frac{1}{2}\sigma_y^2 \left(\Gamma - M_{LH} \right) \frac{1}{\tilde{\kappa}_y^2} \left(s - t - \mathcal{B}_{\tilde{\kappa}_y}(s-t) - \frac{\tilde{\kappa}_y}{2} \mathcal{B}_{\tilde{\kappa}_y}(s-t)^2 \right) \\
&\quad + \frac{1}{2}\sigma_y^2 \Gamma^2 \frac{1}{\kappa_z^2} \left(s - t - \mathcal{B}_{\kappa_z}(s-t) - \frac{\kappa_z}{2} \mathcal{B}_{\kappa_z}(s-t)^2 \right) \\
&\quad - \sigma_y^2 \Gamma^2 \frac{1}{\tilde{\kappa}_y \kappa_z} \left(s - t - \mathcal{B}_{\tilde{\kappa}_y}(s-t) - \mathcal{B}_{\kappa_z}(s-t) + \mathcal{B}_{\tilde{\kappa}_y + \kappa_z}(s-t) \right) \\
&\quad + \frac{1}{2} \frac{\sigma_z^2 \zeta^2}{\kappa_z^2} \left(s - t - \mathcal{B}_{\kappa_z}(s-t) - \frac{\kappa_z}{2} \mathcal{B}_{\kappa_z}(s-t)^2 \right),
\end{aligned}
$$

where $\tilde{\kappa}_y = \kappa_y + M_{yH}$, $\Gamma = \zeta M_{zH}/(\tilde{\kappa}_y - \kappa_z)$ and, for any constant $\phi \neq 0$, we use the notation

$$
\mathcal{B}_\phi(\tau) = \frac{1}{\phi}\left(1 - e^{-\phi\tau}\right)
$$

and apply that

$$
\begin{aligned}
\int_t^T e^{-a(s-t)} \mathcal{B}_b(T-s)\, ds &= \begin{cases} \frac{\mathcal{B}_a(\tau) - \mathcal{B}_b(\tau)}{b-a} & \text{if } b \neq a, \\ \frac{\mathcal{B}_a(\tau) + (a\mathcal{B}_a(\tau) - 1)\tau}{a} & \text{if } b = a, \end{cases} \\
\int_t^T \mathcal{B}_b(T-s)\, ds &= \frac{1}{b}\left(\tau - \mathcal{B}_b(\tau) \right), \\
\int_t^T \mathcal{B}_b(T-s)^2\, ds &= \frac{1}{b^2}\left(\tau - \mathcal{B}_b(\tau) - \frac{b}{2}\mathcal{B}_b(\tau)^2 \right), \\
\int_t^T \mathcal{B}_b(T-s)\mathcal{B}_c(T-s)\, ds &= \frac{1}{bc}\left(\tau - \mathcal{B}_b(\tau) - \mathcal{B}_c(\tau) + \mathcal{B}_{b+c}(\tau) \right).
\end{aligned}
$$

The D-functions solve

$$
\begin{aligned}
0 &= \frac{1}{2} D'_{yy}(t) + \frac{1}{2}\sigma_y^2 D_{yy}(t)^2 - \bar{\kappa}_y D_{yy}(t) - K_{yy}, \\
0 &= D'_y(t) + \left(\sigma_y^2 D_{yy}(t) - \bar{\kappa}_y \right) D_y(t) - K_y(t), \\
0 &= D'_0(t) + \frac{1}{2}\sigma_y^2 \left(D_y(t)^2 + D_{yy}(t) \right) - \tfrac{\gamma-1}{\gamma} M_y(t) D_y(t) - K_0(t),
\end{aligned}
$$

with $D_{yy}(T) = D_y(T) = D_0(T) = 0$. With $\gamma > 1$, we have

$$\bar{\kappa}_y^2 + 2\sigma_y^2 K_{yy} > 0,$$

and in this case the solutions are

$$\begin{aligned}
D_{yy}(t) &= \frac{\sigma_y^2\left(e^{2\xi_1(T-t)} - 1\right)}{N(T-t)}, \\
D_y(t) &= -N(T-t)^{\xi_2}\int_0^{T-t} K_y(T-u)e^{-(\bar{\kappa}_y+\xi_2(\xi_1+\bar{\kappa}_y))(T-t-u)}N(u)^{-\xi_2}\,du, \\
D_0(t) &= \frac{1}{2}\sigma_y^2\int_t^T \left(D_y(u)^2 + D_{yy}(u)\right)\,du - \tfrac{\gamma-1}{\gamma}\int_t^T M_y(u)D_y(u)\,du - \int_t^T K_0(u)\,du,
\end{aligned}$$

where $\xi_1 = \sqrt{\bar{\kappa}_y^2 + 2\sigma_y^2 K_{yy}}$, $\xi_2 = \frac{\sigma_y^2}{2K_{yy}}$, and

$$N(\tau) = (\xi_1 + \bar{\kappa}_y)(e^{2\xi_1\tau} - 1) + 2\xi_1.$$

Finally, the $\bar{D}$-functions solve

$$\begin{aligned}
0 &= \frac{1}{2}\frac{\partial \bar{D}_{yy}}{\partial t}(t,s) + \frac{1}{2}\sigma_y^2\bar{D}_{yy}(t,s)^2 - \bar{\kappa}_y\bar{D}_{yy}(t,s) - K_{yy}, \\
0 &= \frac{\partial \bar{D}_y}{\partial t}(t,s) + \left(\sigma_y^2\bar{D}_{yy}(t,s) - \bar{\kappa}_y\right)\bar{D}_y(t,s) + k_1\sigma_{Hy}\bar{D}_{yy}(t,s) - K_y(t), \\
0 &= \frac{\partial \bar{D}_0}{\partial t}(t,s) + \frac{1}{2}\sigma_y^2\left(\bar{D}_y(t,s)^2 + \bar{D}_{yy}(t,s)\right) + \left(k_1\sigma_{Hy} - \tfrac{\gamma-1}{\gamma}M_y(t)\right)\bar{D}_y(t,s) \\
&\quad + \frac{1}{2}k_1(k_1-1)\sigma_H^2 + k_1\left(r(t) - \tfrac{\gamma-1}{\gamma}R + \frac{1}{\gamma}\left(\mu_H(t) + y\right)\right) - K_0(t),
\end{aligned}$$

with $\bar{D}_{yy}(s,s) = \bar{D}_y(s,s) = \bar{D}_0(s,s) = 0$ for any $s \in [0,T]$. The solutions are

$$\begin{aligned}
\bar{D}_{yy}(t,s) &= \frac{\sigma_y^2\left(e^{2\xi_1(s-t)} - 1\right)}{N(s-t)}, \\
\bar{D}_y(t,s) &= k_1\sigma_y^2\sigma_{Hy}N(s-t)^{\xi_2}\int_0^{s-t} e^{-(\bar{\kappa}_y+\xi_2(\xi_1+\bar{\kappa}_y))(s-t-u)}\left(e^{2\xi_1 u} - 1\right)N(u)^{-\xi_2-1}\,du \\
&\quad - N(s-t)^{\xi_2}\int_0^{s-t} K_y(s-u)e^{-(\bar{\kappa}_y+\xi_2(\xi_1+\bar{\kappa}_y))(s-t-u)}N(u)^{-\xi_2}\,du, \\
\bar{D}_0(t,s) &= \frac{1}{2}\sigma_y^2\int_t^s \left(\bar{D}_y(u,s)^2 + \bar{D}_{yy}(u,s)\right)\,du + \int_t^s \left(k_1\sigma_{Hy} - \tfrac{\gamma-1}{\gamma}M_y(u)\right)\bar{D}_y(u,s)\,du \\
&\quad + \frac{1}{2}k_1(k_1-1)\sigma_H^2(s-t) + k_1\int_t^s \left(r(u) - \tfrac{\gamma-1}{\gamma}R + \tfrac{1}{\gamma}[\mu_H(u) + y]\right)\,du - \int_t^s K_0(u)\,du.
\end{aligned}$$

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Table 1: Augmented Dickey-Fuller (ADF) test results for individual time series. The table reports the MacKinnon approximate p-value for the ADF test on the quarterly observations of the log stock index, the log of the national home price index, and the national per capita labor income. The sample includes quarterly U.S. data from 1953q1 to 2010q2. For each time series we estimate the model $\Delta v(t) = \xi_1 + \xi_2 t + \xi_3 v(t-1) + \sum_{j=1}^{J} \phi_j \Delta v(t-j) + \varepsilon(t)$.

Regression model	Stock	House	Labor Income
$J = 0, \xi_2 = 0$	0.5075	0.8567	0.3168
$J = 0$	0.7464	0.8920	0.9104
$J = 1$	0.5503	0.4926	0.9137
$J = 2$	0.5884	0.4401	0.8594

Table 2: Augmented Dickey-Fuller (ADF) test results for pairwise cointegration. Below are the results for the ADF test for pairwise cointegration between the log stock prices ex dividend, the log house prices, and the log labor income series. We estimate the model $\Delta v(t) = \xi_1 + \xi_2 t + \xi_3 v(t-1) + \sum_{j=1}^{J} \phi_j \Delta v(t-j) + \varepsilon(t)$, where $v \in \{sh, sl, hl\}$ as defined in (8). The sample includes quarterly U.S. data from 1953q1 to 2010q2.

Regression model	MacKinnon p-value	ξ_3 estimate	p-value for ξ_3
Panel A: Stock and house (sh)			
$J = 0, \xi_2 = 0$	0.3514	-0.01940	0.064
$J = 0$	0.6653	-0.02726	0.061
$J = 1$	0.4664	-0.03233	0.026
$J = 2$	0.4677	-0.03276	0.026
Panel B: Stock and income (sl)			
$J = 0, \xi_2 = 0$	0.4690	-0.02083	0.105
$J = 0$	0.7945	-0.02147	0.112
$J = 1$	0.6519	-0.02546	0.058
$J = 2$	0.6984	-0.02449	0.071
Panel C: House and income (hl)			
$J = 0, \xi_2 = 0$	0.2417	-0.00959	0.036
$J = 0$	0.8694	-0.01107	0.172
$J = 1$	0.7430	-0.01261	0.087
$J = 2$	0.7033	-0.01336	0.073
$J = 4$	0.1264	-0.02079	0.003

Table 3: Log-likelihoods of different models and data sets. Below are the log-likelihoods of the our model (1)–(6) and various model variants. The samples include U.S. data from 1953q1, U.S. data from 1987q1, and regional data of 14 regions from 1987q1. The log-likelihoods of the regional data are averaged over the 14 regions. The observation frequency is quarterly and all samples end in 2010q2.

Model	U.S. national 1953-2010	U.S. national 1987-2010	Regional average 1987-2010
Extended with B_x, B_y, B_z	7.26	TBA	TBA
Our model	7.26	TBA	TBA
With B_x, B_y, B_z, but no B_L	7.22	TBA	TBA
With B_y, B_z, but no B_L	7.26	TBA	TBA
With B_z, but no B_L	7.24	TBA	TBA
With B_y, but no B_L	7.26	TBA	TBA

Table 4: Model calibrations. Below are the raw and adjusted model parameter sets of the model (1)–(6) calibrated to quarterly national U.S. data 1953q1–2010q2.

Parameter	Raw	Adjusted
- Bank and stock -		
r	0.015	0.015
σ_S	0.152	0.152
μ_S	0.029	0.029
- House -		
μ_H	0.001	0.008
σ_H	0.025	0.120
ρ_{HS}	0.222	0.047
$\hat{\rho}_H$	0.975	0.999
- Income -		
$\bar{\mu}_L$	0.018	0.023
$\bar{\sigma}_L$	0.014	0.100
ρ_{LS}	0.154	0.021
ρ_{LH}	0.295	0.038
$\hat{\rho}_L$	0.951	0.999
- x variable -		
κ_x	4.285	4.285
σ_x	0.098	0.098
ρ_{xS}	1.000	1.000
ρ_{xH}	0.223	0.048
ρ_{xL}	0.140	0.006
- y variable -		
κ_y	0.661	0.661
σ_y	0.043	0.043
ρ_{yS}	0.276	0.276
ρ_{yH}	0.994	0.968
ρ_{yL}	0.390	0.138
- z variable -		
κ_z	1.295	1.295
σ_z	0.046	0.046
ρ_{zS}	0.385	0.385
ρ_{zH}	0.120	0.054
ρ_{zL}	-0.808	-0.912

Figure 1: Historical evolution of stock market, house prices, and income per capita in the U.S. The stock market is represented by the S&P composite stock price index without dividends. The house prices uses the national Case-Shiller home price index. Income is disposable labor income per capita from NIPA. All time series are standardized to 100 in 1953q2 and inflation-adjusted to 2010q2 dollars using CPI.

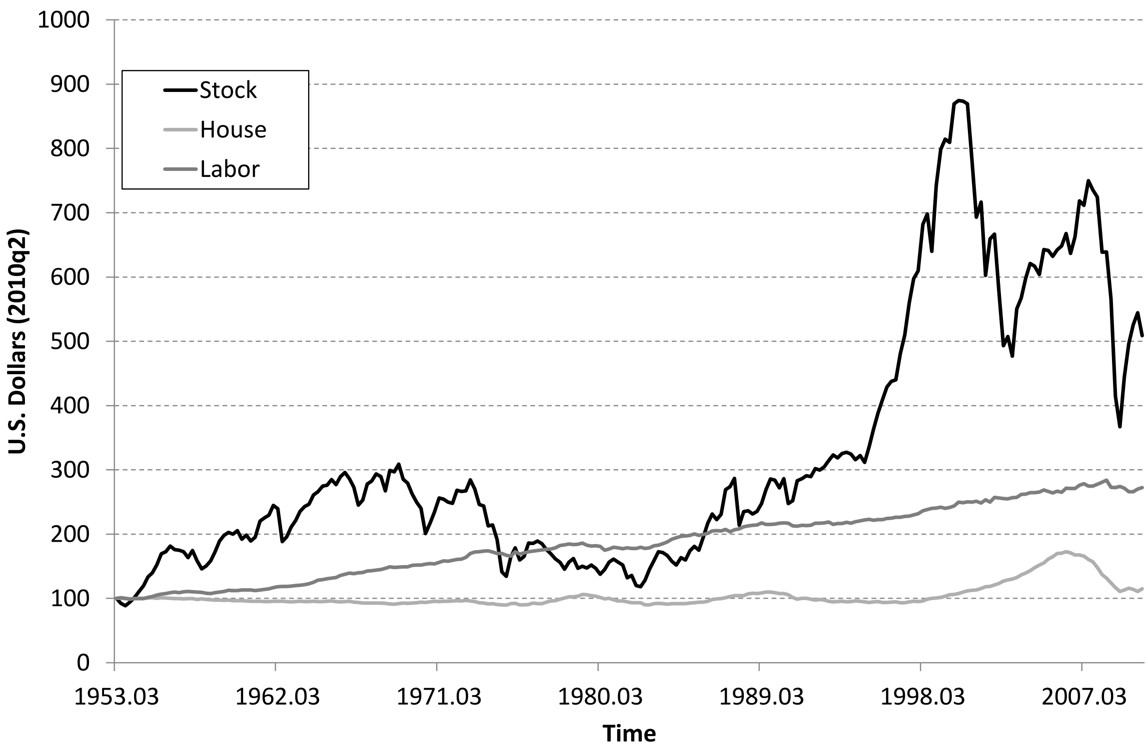

Figure 2: Filtered y_t process for national U.S. data. This figure depicts the y_t process filtered from national U.S. data over the period from 1953q1 to 2010q2.

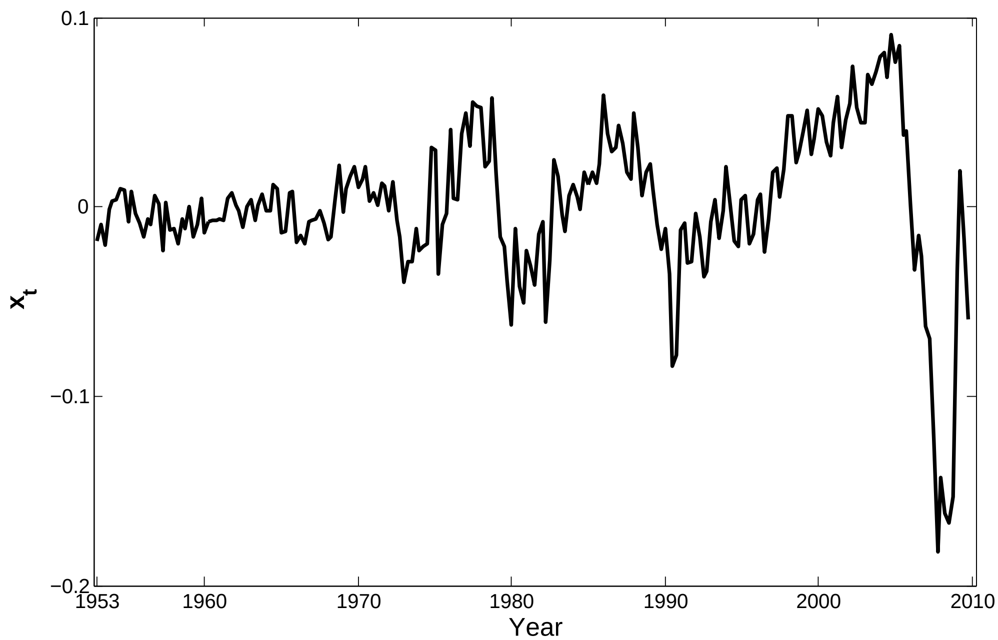

Figure 3: Expected consumption, wealth, and investments; $\gamma = 5$. In Panel A the two solid curves show the expected spending in thousands of U.S. dollars on the perishable consumption good and on housing consumption. Panel B shows the expected number of housing units consumed and invested in. Panel C shows the expected financial wealth and its decomposition into stock investment, housing investment, and bond investment. All investments in Panel C are in thousands of U.S. dollars. Panel D shows the expected percentage investments of financial wealth into stocks, housing, and bonds. The full model U.S. calibration reported in Table 4 is used together with a risk aversion coefficient of $\gamma = 5$.

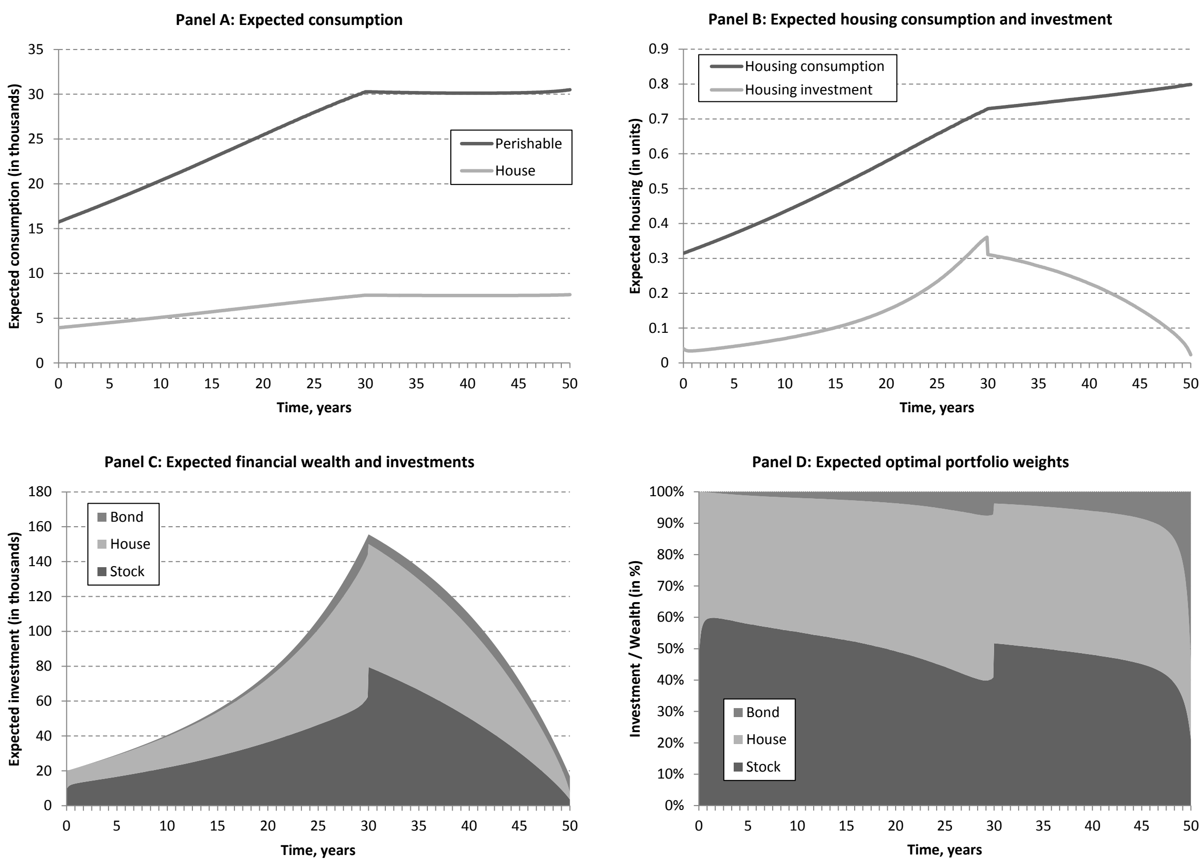

Figure 4: Expected consumption, wealth, and investments; $\gamma = 10$. In Panel A the two solid curves show the expected spending in thousands of U.S. dollars on the perishable consumption good and on housing consumption. Panel B shows the expected number of housing units consumed and invested in. Panel C shows the expected financial wealth and its decomposition into stock investment, housing investment, and bond investment. All investments in Panel C are in thousands of U.S. dollars. Panel D shows the expected percentage investments of financial wealth into stocks, housing, and bonds. The full model U.S. calibration reported in Table 4 is used together with a risk aversion coefficient of $\gamma = 10$.

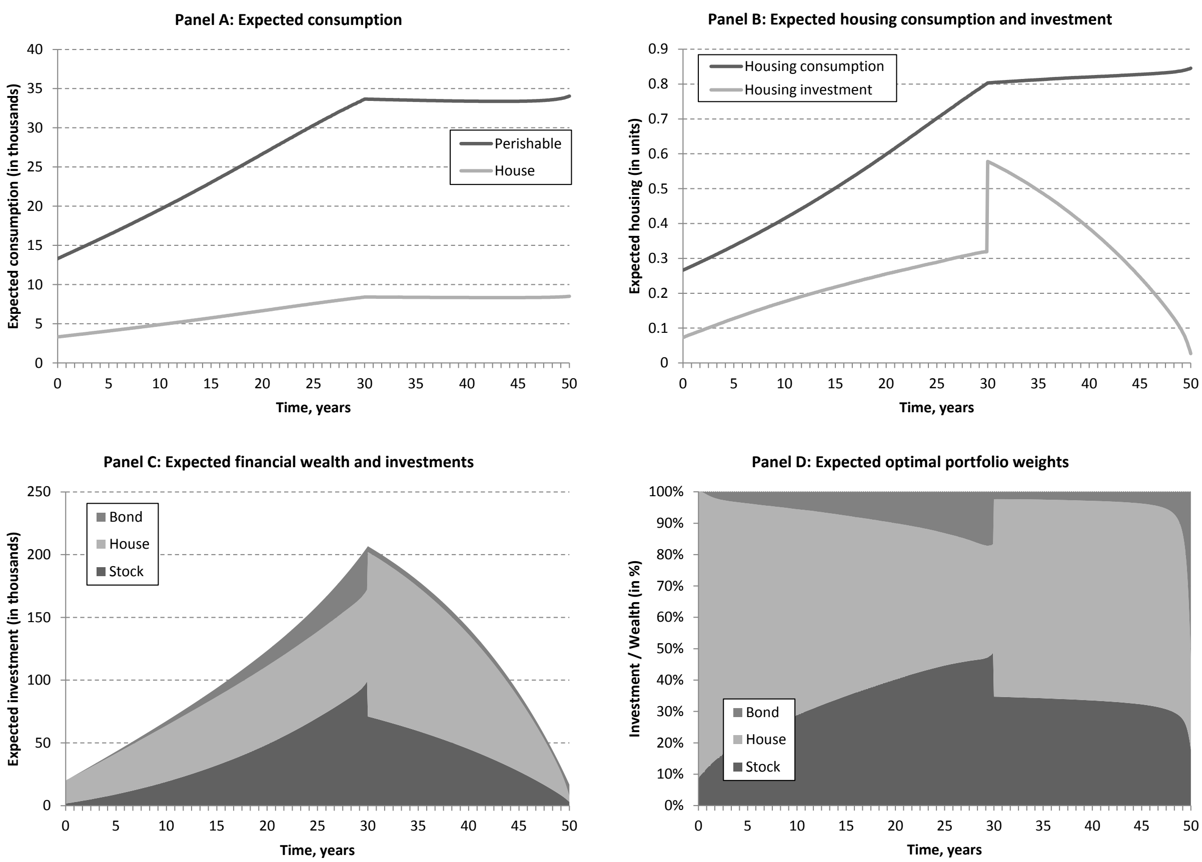

Figure 5: Expected consumption, wealth, and investments; Charlotte data, $\gamma = 5$. In Panel A the two solid curves show the expected spending in thousands of U.S. dollars on the perishable consumption good and on housing consumption. Panel B shows the expected number of housing units consumed and invested in. Panel C shows the expected financial wealth and its decomposition into stock investment, housing investment, and bond investment. All investments in Panel C are in thousands of U.S. dollars. Panel D shows the expected percentage investments of financial wealth into stocks, housing, and bonds. The full model Charlotte calibration reported in Table 4 is used together with a risk aversion coefficient of $\gamma = 5$.

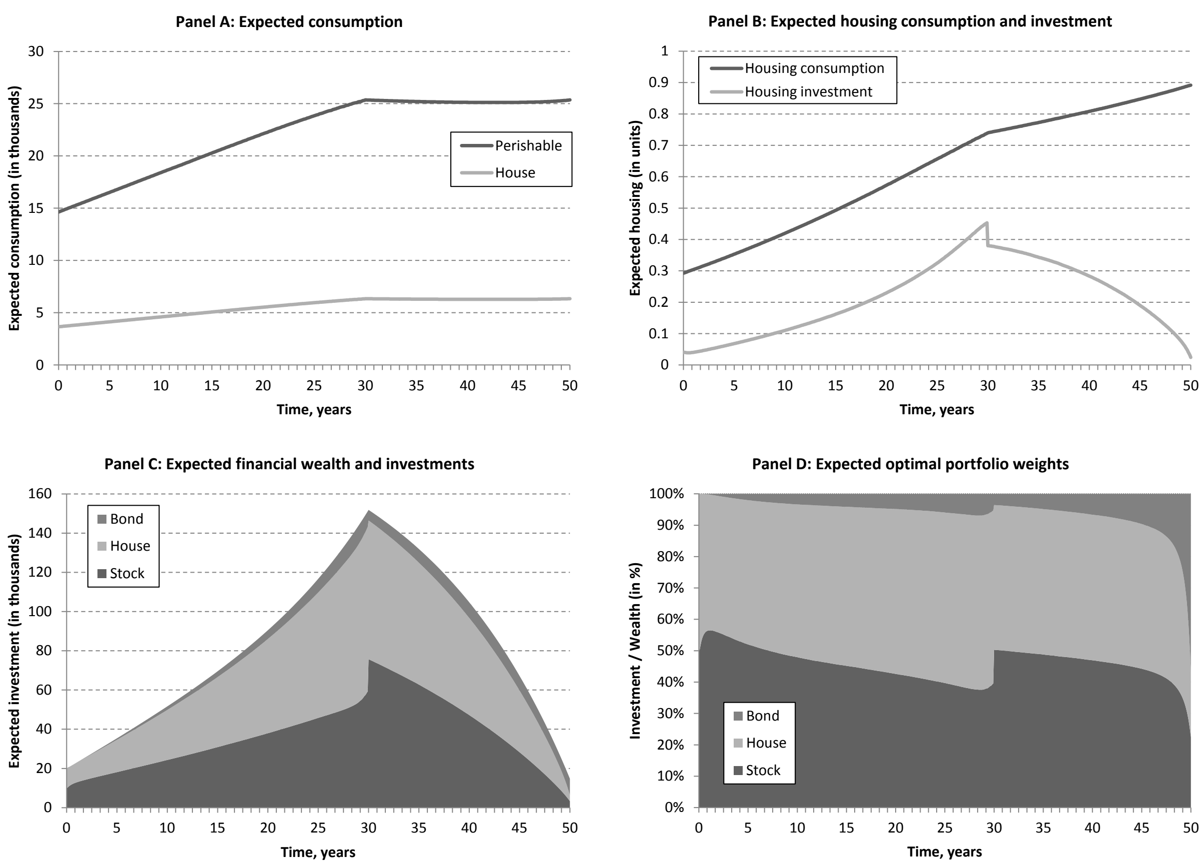

Figure 6: Expected consumption, wealth, and investments; $\gamma = 5$; rent increased to 6%. In Panel A the two solid curves show the expected spending in thousands of U.S. dollars on the perishable consumption good and on housing consumption. Panel B shows the expected number of housing units consumed and invested in. Panel C shows the expected financial wealth and its decomposition into stock investment, housing investment, and bond investment. All investments in Panel C are in thousands of U.S. dollars. Panel D shows the expected percentage investments of financial wealth into stocks, housing, and bonds. The full model U.S. calibration reported in Table 4 is used together with a risk aversion coefficient of $\gamma = 5$. The rental rate is 6% instead of the benchmark value of 5%.

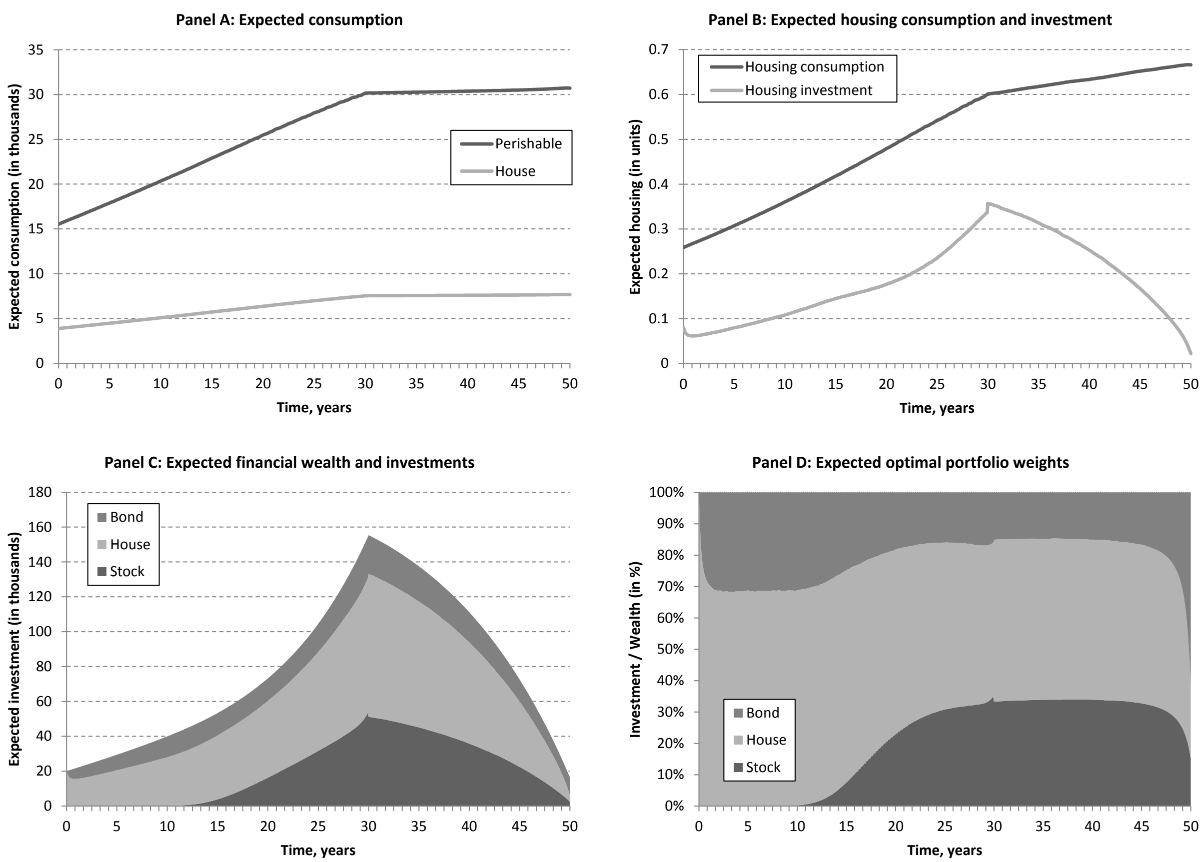

Figure 7: Expected consumption, wealth, and investments; $\gamma = 5$; $\mu_S + \bar{D}$ lowered to 0.058. In Panel A the two solid curves show the expected spending in thousands of U.S. dollars on the perishable consumption good and on housing consumption. Panel B shows the expected number of housing units consumed and invested in. Panel C shows the expected financial wealth and its decomposition into stock investment, housing investment, and bond investment. All investments in Panel C are in thousands of U.S. dollars. Panel D shows the expected percentage investments of financial wealth into stocks, housing, and bonds. The risk aversion coefficient is $\gamma = 5$.