Hopf algebra structures of Multiple Zeta Values in positive characteristics

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Outline

Multiple Zeta Values(MZV's): classical setting

Zeta values

Let $n \geq 2$ be an integer. The zeta value $\zeta(n) \in \mathbb{R}$ is defined by

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}.$$

Multiple Zeta Values(MZV's): classical setting

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Even zeta values are known in terms of Bernoulli numbers, $\zeta(2) = \frac{\pi^2}{6}, \ldots, \zeta(2n) = \frac{B_{2n}}{2(2n)!}(2\pi)^{2n}$. In particular, $\frac{\zeta(2n)}{\pi^{2n}} \in \mathbb{Q}$.

Not much is known about odd zeta values; what we know is not much more than the following theorems:

Theorem (Apéry, 1978): $\zeta(3) \notin \mathbb{Q}$.
Theorem (Zudilin, 2001): $\{\zeta(5), \zeta(7), \zeta(9), \zeta(11)\} \not\subset \mathbb{Q}$.

Multiple Zeta Values(MZV's): classical setting

Multiple Zeta Values (MZV's) generalize the zeta values.

Multiple Zeta Values

Let $n_1, \ldots, n_{r-1} \geq 1$, $n_r \geq 2$ be integers.

$$\zeta(n_1, \ldots, n_r) := \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}} \in \mathbb{R}.$$

The *weight* and *depth* of the presentation $\zeta(n_1, \ldots, n_r)$ are $n_1 + \cdots + n_r$ and r, respectively. For example, $\zeta(4, 3)$ has weight 7 and depth 2.

MZV's are first introduced by Euler (as double zeta values) in 18c and Zagier in early 1990's.

Multiple Zeta Values(MZV's): classical setting

Let $\mathcal{Z}_w$ be the $\mathbb{Q}$-vector space spanned by the MZV's of weight w, and $\mathcal{Z} = \oplus_{w \geq 0} \mathcal{Z}_w$ the $\mathbb{Q}$-vector space of all MZV's. (We let $\mathcal{Z}_0 = \mathbb{Q}$, $\mathcal{Z}_1 = \{0\}$.)

It is known that the product of two MZV's can be written as a $\mathbb{Q}$-linear combinations of MZV's. For example,

Theorem (Euler, 1776.)

For $n, m > 1$, $\zeta(n)\zeta(m) = \zeta(n+m) + \zeta(n,m) + \zeta(m,n)$.

This turns the vector space $\mathcal{Z}$ into an algebra, and the linear relations of MZV's are to be studied.

Multiple Zeta Values(MZV's): classical setting

Zagier-Hoffman Conjectures

(Zagier's conjecture) Let $(d_n)_{n\geq 0}$ be a sequence with $(d_0, d_1, d_2) = (1, 0, 1)$ and $d_n = d_{n-3} + d_{n-2}$ for $n \geq 3$. Then

$$\dim_{\mathbb{Q}} \mathcal{Z}_w = d_w \quad \text{for all } w \geq 0.$$

(Hoffman's conjecture) Further, and $\mathcal{Z}_w$ is spanned by

$$\{\zeta(k_1, \ldots, k_r) : k_1 + \cdots + k_r = w,\ 2 \leq k_i \leq 3\}.$$

Example. Zagier's conjecture implies $\zeta(1,2)/\zeta(3) \in \mathbb{Q}$. Indeed, $\zeta(1,2) = \zeta(3)$.

MZV's in positive characteristics

Now we define MZV's in positive characteristics. Let

- $\mathbb{F}_q$ be a finite field of q elements with characteristic $p > 0$,
- $A = \mathbb{F}_q[\theta]$, A_+ the set of monic polynomials in A,
- $K = \mathbb{F}_q(\theta)$, K_∞ be the completion of K at ∞.

MZV's in positive characteristics (Carlitz 1935, Thakur 2004)

Let $s_1, \ldots, s_r \geq 1$ be integers. The MZV in positive characteristics is defined by

$$\zeta_A(s_1, \ldots, s_r) := \sum \frac{1}{a_1^{s_1} \ldots a_r^{s_r}} \in K_\infty$$

where the sum is over $a_1, \ldots, a_r \in A_+$ and $\deg(a_1) > \cdots > \deg(a_r)$. In this presentation, weight and depth are $s_1 + \cdots + s_r$ and r, resp.

From now on, we denote $\mathcal{Z}_w$ the K-vector space spanned by MZV's of weight w. (so please forget the classical MZV's)

Zagier-Hoffman conjectures in positive characteristics

Zagier-Hoffman conjectures were proved in positive characteristic case.

Zagier-Hoffman conj. in pos. char. (Im-K.-Le-Ngo Dac-Pham)

Let $(d_n)_{n\geq 0}$ be a sequence with $d_0 = 1$, $d_w = 2^{w-1}$ for $1 \leq w < q$, $d_q = 2^{q-1} - 1$, and $d(w) = \sum_{i=1}^{q} d(w-i)$ for $w > q$. Then,

$$\dim_K \mathcal{Z}_w = d_w \quad \text{for all } w \geq 0.$$

Further, we can exhibit a Hoffman-like basis of $\mathcal{Z}_w$.

This was proved for $w < 2q - 1$ by Ngo Dac, and for all w by Im-K.-Le-Ngo Dac-Pham. (Also, by Chang-Chen-Mishiba independently).

Outline

Composition space and shuffle product

Let

- $\Sigma = \{x_n\}_{n\in\mathbb{N}}$ be a set of 'letters',
- $\langle\Sigma\rangle = \{x_{n_1}\dots x_{n_r} : x_{n_i} \in \Sigma \text{ for } r \geq 0\}$ be set of 'words' over Σ with the empty word denoted by 1,
- $\mathfrak{C} = \mathbb{F}_q\langle\Sigma\rangle$ be the $\mathbb{F}_q$-vector space with basis $\langle\Sigma\rangle$, endowed with the *concatenation product* $\cdot$ (which can be omitted)

$$(x_{n_1}\dots x_{n_r})\cdot(x_{m_1}\dots x_{m_s}) = x_{n_1}\dots x_{n_r}x_{m_1}\dots x_{m_s}.$$

The weight and depth of $x_{n_1}\dots x_{n_r}$ are $n_1+\dots+n_r$ and r, resp. For each nonempty $\mathfrak{a}\in\langle\Sigma\rangle$, we can write $\mathfrak{a} = x_a\cdot\mathfrak{a}_-$.

Later, we identify $\zeta_A(n_1,\dots,n_r)$ and $x_{n_1}\dots x_{n_r}$.

Composition space and shuffle product

There is the notion of *shuffle product* in $\mathfrak{C}$ defined by Chen's identity: $\mathfrak{u} \sqcup\!\sqcup 1 = 1 \sqcup\!\sqcup \mathfrak{u} = \mathfrak{u}$ for $\mathfrak{u} \in \langle \Sigma \rangle$, and for nontrivial $\mathfrak{a}$ and $\mathfrak{b}$

$$\begin{aligned}\mathfrak{a} \sqcup\!\sqcup \mathfrak{b} := x_a(\mathfrak{a}_- \sqcup\!\sqcup \mathfrak{b}) + x_b(\mathfrak{a} \sqcup\!\sqcup \mathfrak{b}_-) + x_{a+b}(\mathfrak{a}_- \sqcup\!\sqcup \mathfrak{b}_-) \\ + \sum_{0<j<a+b} \Delta^j_{a,b} x_{a+b-j} \cdot (x_j \sqcup\!\sqcup (\mathfrak{a}_- \sqcup\!\sqcup \mathfrak{b}_-)).\end{aligned}$$

Here $\Delta^j_{a,b} = (-1)^{a-1}\binom{j-1}{a-1} + (-1)^{b-1}\binom{j-1}{b-1} \in \mathbb{F}_q$ when $(q-1) \mid j$, and $\Delta^j_{a,b} = 0$ otherwise.

N.B. the Chen's identity (of depth one version),

$$\zeta_A(a)\zeta_A(b) = \zeta_A(a,b) + \zeta_A(b,a) + \zeta_A(a+b) + \sum_{0<j<a+b} \Delta^j_{a,b}\zeta_A(a+b-j,j) \ ;$$

$\sqcup\!\sqcup$ in $\mathfrak{C}$ is defined to satisfy $\zeta(\mathfrak{a} \sqcup\!\sqcup \mathfrak{b}) = \zeta(\mathfrak{a}) \times \zeta(\mathfrak{b}) \in K_\infty$ when we identify $\zeta(x_{n_1} \dots x_{n_r})$ and $\zeta_A(n_1, \dots, n_r)$.

Note that $\sqcup\!\sqcup$ preserves the weight; i.e. $\mathrm{w}(\mathfrak{a} \sqcup\!\sqcup \mathfrak{b}) = \mathrm{w}(\mathfrak{a}) + \mathrm{w}(\mathfrak{b})$.

Hopf algebra

In her thesis, Shuhui Shi (2015) proposed that the MZV's in positive characteristics have a Hopf algebra structure with the shuffle product $\sqcup\!\sqcup$ and the coproduct Δ_{Shi} (which will be defined later).

Before proceeding ahead, we introduce a brief notion of Hopf algebra.

(But what is Hopf algebra?)

Hopf algebra is an algebraic structure arising in many areas of mathematics, including algebraic topology, representation theory, and combinatorics.

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Hopf algebra is an algebraic structure arising in many areas of mathematics, including algebraic topology, representation theory, and combinatorics.

A bialgebra over a field k is a k-vector space which is both (co)unital (co)associative algebra and coalgebra, with compatibilities between two structures.
In other words, it is a quintuple $(A, M, u, \Delta, \epsilon)$, where

- A is a k-vector space,
- $M\colon A \otimes A \to A$ the product; we write $M(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a} * \mathfrak{b}$,
- $u\colon k \to A$ the unit map,
- $\Delta\colon A \to A \otimes A$ the coproduct, and
- $\epsilon\colon A \to k$ the counit map (or augmentation map),

with the following properties (next slide).

(But what is Hopf algebra?)

Bialgebra axioms are as follows:

- associativity, $M \circ (M \otimes \mathrm{id}) = M \circ (\mathrm{id} \otimes M)$, i.e. $(\mathfrak{a} * \mathfrak{b}) * \mathfrak{c} = \mathfrak{a} * (\mathfrak{b} * \mathfrak{c})$ for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in A$,
- unitary property, i.e. there exist $I \in A$ with $I * \mathfrak{a} = \mathfrak{a} * I = \mathfrak{a}$. The unit map will be given as $u(f) = f \cdot I$,
- coassociativity, $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$
- counitary, $(\epsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id}$; This can be understood as the counit map ϵ collapses (or 'undo') the extra structure from the coproduct Δ on the both sides and recover the original element.
- compatibilities for M and Δ, u and Δ, M and ϵ, and u and ϵ. i.e. $\Delta(\mathfrak{a} * \mathfrak{b}) = \Delta(\mathfrak{a}) * \Delta(\mathfrak{b})$, $\Delta(I) = I \otimes I$ (where $I = u(1)$), $\epsilon(\mathfrak{a} * \mathfrak{b}) = \epsilon(\mathfrak{a})\epsilon(\mathfrak{b})$, and $\epsilon(I) = 1$.

(In $A \otimes A$, $(a_1 \otimes a_2) * (b_1 \otimes b_2) := (a_1 * b_1) \otimes (a_2 * b_2)$.)

(But what is Hopf algebra?)

A bialgebra $(A, M, u, \Delta, \epsilon)$ is said to be a Hopf algebra if the antipode map $S\colon A \to A$ exists, satisfying the following commutative diagram:

$$\begin{tikzcd}
 & A \otimes A \arrow[rr, "S \otimes \mathrm{id}_A"] & & A \otimes A \arrow[dr, "M"] & \\
A \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & k \arrow[rr, "u"] & & A \\
 & A \otimes A \arrow[rr, "\mathrm{id}_A \otimes S"] & & A \otimes A \arrow[ur, "M"'] &
\end{tikzcd}$$

(S can be understood as an 'inverse element' of $\mathrm{id}_A\colon A \to A$ in $\mathrm{Hom}(A, A)$ wrt. the convolution product $f \star g := M \circ (f \otimes g) \circ \Delta$.)

Example of Hopf algebra: Group algebra

Let k a field and G be a (finite) group, and kG be the group algebra. Then kG is a Hopf algebra with the following structure:

- $\Delta(g) = g \otimes g$,
- $u(a) = a1_G$,
- $\epsilon(g) = 1_k$, and
- $S(g) = g^{-1}$ for all $g \in G$.

Example of Hopf algebra: Shuffle algebra

Let k a field and $\Sigma = \{x_n\}_{n \in \mathbb{N}}$.
Let $\langle \Sigma \rangle = \{x_{n_1} \dots x_{n_r} : x_{n_i} \in X \text{ for } r \geq 0\}$ be the set of words over X with the empty word 1 and the concatenation '$\cdot$'.
Let $\mathfrak{S} = k\langle X \rangle$ be the k-vector space with basis $\langle X \rangle$, endowed with the shuffle product $*$ defined as

$$1 * w = w * 1 = w \quad (\forall w \in \langle X \rangle),$$
$$(x_a \mathfrak{a}_-) * (x_b \mathfrak{b}_-) = x_a \cdot (\mathfrak{a}_- * \mathfrak{b}) + x_b \cdot (\mathfrak{a} * \mathfrak{b}_-).$$

Then $\mathfrak{S}$ is a Hopf algebra with the 'de-concatenation' coproduct

$$\Delta_{deconcat}(w) := \sum_{uv=w} u \otimes v,$$

e.g. $\Delta_{deconcat}(xyz) = 1 \otimes xyz + x \otimes yz + xy \otimes z + xyz \otimes 1.$

Hopf algebra structure of MZV's in positive characteristics

Shi (2015) suggested the definition of coproduct Δ_{Shi} compatible to the shuffle product $\sqcup\!\sqcup$ in $\mathfrak{C}$.
She then proved that $\mathfrak{C}$ has a Hopf algebra structure, under the assumptions of (1) the associativity of $\sqcup\!\sqcup$, (2) the coassociativity of Δ_{Shi}, and (3) the compatibility of $\sqcup\!\sqcup$ and Δ_{Shi}.

We (Im-Kim-Le-Ngo Dac-Pham, 2023) proved that $\mathfrak{C}$ is indeed a Hopf algebra with $\sqcup\!\sqcup$ and Δ_{Shi}.

Outline

Shi's construction of the coproduct

Shi gave the inductive definition of the coproduct Δ_{Shi} on $\mathfrak{C}$.

$$\Delta_{Shi}(1) := 1 \otimes 1, \quad \Delta_{Shi}(x_1) := 1 \otimes x_1 + x_1 \otimes 1. \quad \text{(initial cases)}$$

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Now assume that we've defined all $\Delta(\mathfrak{u})$ of weight$(\mathfrak{u}) < w$. First, for a word $\mathfrak{a} = x_a \mathfrak{a}_-$ with weight w and depth > 1 with

$$\Delta_{Shi}(x_a) =: 1 \otimes x_a + \sum \mathfrak{a}_1 \otimes \mathfrak{a}_2,$$

$$\Delta_{Shi}(\mathfrak{a}_-) =: \sum \mathfrak{u}_1 \otimes \mathfrak{u}_2, \quad \text{(known by the induction hypothesis)}$$

Shi defined

$$\Delta_{Shi}(x_a \mathfrak{a}_-) := 1 \otimes \mathfrak{a} + \sum (\mathfrak{a}_1 \cdot \mathfrak{u}_1) \otimes (\mathfrak{a}_2 \sqcup\!\sqcup \mathfrak{u}_2).$$

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Finally, Shi defined $\Delta_{Shi}(x_w)$ to satisfy

$$\Delta_{Shi}(x_1 \sqcup\!\sqcup x_{w-1}) = \Delta_{Shi}(x_1) \sqcup\!\sqcup \Delta_{Shi}(x_{w-1}).$$

Note that $x_1 \sqcup\!\sqcup x_{w-1} = x_w +$ (other terms); the coproduct of all other terms are known in this step.

Our construction of the coproduct

We introduce a different definition of coproduct Δ on $\mathfrak{C}$.
We first define $\triangleright$ on $\mathfrak{C}$ recursively. As initial cases we let

$$1 \triangleright \mathfrak{u} := \mathfrak{u} =: \mathfrak{u} \triangleright 1 \quad \text{for all } \mathfrak{u}.$$

For nontrivial word $\mathfrak{a} = x_a \mathfrak{a}_-$, we define

$$\mathfrak{a} \triangleright \mathfrak{b} := x_a \cdot (\mathfrak{a}_- \sqcup\!\sqcup \mathfrak{b}).$$

N.B. $x_a \triangleright \mathfrak{u} = x_a \cdot \mathfrak{u}$, but $\mathfrak{u} \triangleright \mathfrak{v} \neq \mathfrak{u} \cdot \mathfrak{v}$, and $\triangleright$ is not commutative nor associative in general.

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Construction of Δ is then similar to Δ_{Shi}, but the concatenation in

$$\Delta_{Shi}(x_a \mathfrak{a}_-) := 1 \otimes \mathfrak{a} + \sum (\mathfrak{a}_1 \cdot \mathfrak{u}_1) \otimes (\mathfrak{a}_2 \sqcup\!\sqcup \mathfrak{u}_2)$$

is replaced by the triangle product, i.e.,

$$\Delta(x_a \mathfrak{a}_-) := 1 \otimes \mathfrak{a} + \sum (\mathfrak{a}_1 \triangleright \mathfrak{u}_1) \otimes (\mathfrak{a}_2 \sqcup\!\sqcup \mathfrak{u}_2).$$

Our construction of the coproduct

Now we have two questions:
(Q1) Is it true that $\Delta = \Delta_{Shi}$?
(Q2) Does Δ satisfy the Hopf algebra axioms?

Our construction of the coproduct

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(Q1) Is it true that $\Delta = \Delta_{Shi}$?
(Q2) Does Δ satisfy the Hopf algebra axioms?

We proved that $ш$ is associative, and Δ satisfies the compatibility and coassociativity and properties, i.e.

$$(\mathfrak{a} ш \mathfrak{b}) ш \mathfrak{c} = \mathfrak{a} ш (\mathfrak{b} ш \mathfrak{c}),$$
$$\Delta(\mathfrak{u}) ш \Delta(\mathfrak{v}) = \Delta(\mathfrak{u} ш \mathfrak{v}), \quad (\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta.$$

Also, we proved that $\Delta(x_n) = \sum \mathfrak{u} \otimes \mathfrak{v}$ satisfies the condition $\mathrm{depth}(\mathfrak{u}) \leq 1$ for all $n \geq 1$.

With this we first answer (Q1), and then according to Shi's proof of the remaining Hopf algebra axioms for $\Delta_{Shi} = \Delta$, we answer (Q2).

Algebra structure (Associativity of $\sqcup\!\sqcup$)

Theorem.

$\sqcup\!\sqcup$ is associative.
In particular, the space $(\mathfrak{C}, \sqcup\!\sqcup)$ is commutative $\mathbb{F}_q$-algebra with algebra homomorphism $Z_{\sqcup\!\sqcup} \colon K \otimes_{\mathbb{F}_q} \mathfrak{C} \to \mathcal{Z}$ given by $\mathfrak{a} \mapsto \zeta_A(\mathfrak{a})$.

The proof relies on huge amount of technical calculation. One of the key fact is

Lemma (Partial Fractions)

Let $r, s \in \mathbb{Z}_{\geq 1}$. As rational functions in $\mathbb{Q}(X, Y)$,

$$\frac{1}{X^r Y^s} = \sum_{\substack{i+j=r+s \\ i,j \in \mathbb{Z}_{\geq 0}}} \left(\binom{j-1}{s-1} \frac{1}{X^i (X+Y)^j} + \binom{j-1}{r-1} \frac{1}{Y^i (X+Y)^j} \right).$$

This is the fact which Chen used to find the coefficients $\Delta^j_{a,b}$.

Algebra structure (Associativity of ⧢)

Let $r, s, t \geq 1$. We expanded two different partial fractions for

$$\frac{1}{A^r B^s} \cdot \frac{1}{C^t} = \frac{1}{A^r} \cdot \frac{1}{B^s C^t}.$$

For each $d \in \mathbb{Z}_{\geq 1}$, we partitioned the indices $(a, b, c) \in A_+^3(d)$ into
$M_0 = \{(a, b, c) : a = b = c\}$,
$M_1 = \{(a, b, c) : \text{only two are the same}\}$ and
$M_2 = \{(a, b, c) : a \neq b \neq c \neq a\}$ which is partitioned further into

- N_0 with $b - a = \lambda f$, $c - a = \mu f$, and $\lambda \neq \mu$,
- N_1 with $b - a = \lambda f$, $c - a = \mu u$,
- N_2 with $b - a = \mu u$, $c - a = \lambda f$,
- N_3 with $b - a = \lambda f$, $c - a = \lambda f + \mu u$,
- N_4 with $b - a = \lambda f$, $c - a = \mu f + \eta u$, and $\lambda \neq \mu$,

for some $\lambda, \mu, \eta \in \mathbb{F}_q^\times$ and $f, u \in A_+$ with $\deg(u) < \deg(f) < d$.

Algebra structure (Associativity of $\sqcup\!\sqcup$)

By calculating and comparing the sums $\sum \frac{1}{a^r b^s} \cdot \frac{1}{c^t}$ and $\sum \frac{1}{a^r} \cdot \frac{1}{b^s c^t}$ over each partition, we deduce that the sums over

- M_0 induce the same expression of depth one MZV's,
- $M_1 \sqcup N_0$ induce the same expression of depth two MZV's, and
- $N_1 \sqcup N_2 \sqcup N_3 \sqcup N_4$ induce the same expression of depth three MZV's

of the associativity equation

$$(\zeta_A(r)\zeta_A(s))\zeta_A(t) = \zeta_A(r)(\zeta_A(s)\zeta_A(t))$$

in terms of power sums, which is translated into the associativity of $\sqcup\!\sqcup$ in $\mathfrak{C}$.

For general case we can proceed with the induction on the sum of depths.

Algebra structure (Associativity of $\shuffle$)

Example. Let $q = 3$. Chen's identity yields

$$\zeta_A(1) \cdot \zeta_A(1) = 2\zeta_A(1,1) + \zeta_A(2).$$

This is not only true as values in K_∞, but also gives the equality of the elements in $\mathfrak{C}$, i.e.

$$x_1 \shuffle x_1 = 2x_1x_1 + x_2.$$

By applying the Chen's identity again, we have

$$\begin{aligned}(\zeta_A(1) \cdot \zeta_A(1)) \cdot \zeta_A(2) &= 2\zeta_A(1,1,2) + 2\zeta_A(1,2,1) + 2\zeta_A(1,3) \\ &\quad + 2\zeta_A(2,1,1) + 2\zeta_A(3,1) + \zeta_A(4)\end{aligned}$$

yields $\quad (x_1 \shuffle x_1) \shuffle x_2 = 2x_1x_1x_2 + 2x_1x_2x_1 + \cdots + 2x_3x_1 + x_4.$

Further, as the expression calculated by Chen's identity for $\zeta_A(1) \cdot (\zeta_A(1) \cdot \zeta_A(2))$ is the same as the above, then we conclude that $(x_1 \shuffle x_1) \shuffle x_2 = x_1 \shuffle (x_1 \shuffle x_2)$.

Hopf algebra structure (Axioms for coproduct Δ)

Recall $1 \triangleright \mathfrak{a} = \mathfrak{a} \triangleright 1 = \mathfrak{a}$, and $\mathfrak{a} \triangleright \mathfrak{b} = x_a \cdot (\mathfrak{a}_- \sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup \mathfrak{b})$ for nonempty $\mathfrak{a}$. We define $\diamond$ on $\mathfrak{C}$ with $1 \diamond \mathfrak{a} = \mathfrak{a} \diamond 1 = \mathfrak{a}$, and for nonempty $\mathfrak{a}$ and $\mathfrak{b}$,

$$\mathfrak{a} \diamond \mathfrak{b} := x_{a+b}(\mathfrak{a}_- \sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup \mathfrak{b}_-) + \sum_{0<j<a+b} \Delta^j_{a,b} \cdot ((\mathfrak{a}_- \sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup \mathfrak{b}_-) \sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup x_j).$$

By introducing the new operators $\diamond$ and $\triangleright$ and the new definition for Δ (and another huge amount of calculations), we could prove the compatibility and coassociativity results. Some key lemmas follow.

Lemmas

- $\mathfrak{a} \sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup \mathfrak{b} = \mathfrak{a} \diamond \mathfrak{b} + \mathfrak{a} \triangleright \mathfrak{b} + \mathfrak{b} \triangleright \mathfrak{a}$ (Definition),
- $\mathfrak{a} \diamond \mathfrak{b} = (x_a \diamond x_b) \triangleright (\mathfrak{a}_- \sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup \mathfrak{b}_-)$,
- $(\Delta(\mathfrak{u}) - 1 \otimes \mathfrak{u}) \triangleright \Delta(\mathfrak{v}) = \Delta(\mathfrak{u} \triangleright \mathfrak{v}) - 1 \otimes (\mathfrak{u} \triangleright \mathfrak{v})$, when $(\mathfrak{u}_1 \otimes \mathfrak{u}_2) \triangleright (\mathfrak{v}_1 \otimes \mathfrak{v}_2) := (\mathfrak{u}_1 \triangleright \mathfrak{v}_1) \otimes (\mathfrak{u}_2 \sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup \mathfrak{v}_2)$.

Hopf algebra structure (Comparison to Δ_{Shi})

(Q1) is remaining: $\Delta = \Delta_{Shi}$?

We introduce braket operator, $[1] = 1$ and

$$[x_{n_1} \dots x_{n_r}] := \left((-1)^r \cdot \Delta_{1,w+1}^{n_1} \cdots \Delta_{1,w+1}^{n_r} \right) (x_{n_1} ⧢ \dots ⧢ x_{n_r}) .$$

N.B. $[\mathfrak{u}] = 0$ if $(q-1) \nmid \mathrm{weight}(\mathfrak{u})$, $[\mathfrak{a} \cdot \mathfrak{b}] := [\mathfrak{a}] ⧢ [\mathfrak{b}]$.

Proposition

$$\Delta(x_n) = 1 \otimes x_n + \sum_{\substack{r \in \mathbb{Z}_{\geq 1}, \mathfrak{a} \in \langle \Sigma \rangle \\ r + w(\mathfrak{a}) = n}} \binom{r + depth(\mathfrak{a}) - 2}{depth(\mathfrak{a})} x_r \otimes [\mathfrak{a}],$$

in particular, $\Delta(x_n) = 1 \otimes x_n + \sum \mathfrak{u} \otimes \mathfrak{v}$ with $\mathrm{depth}(\mathfrak{u}) = 1$ for all n.

Proposition

We have $\Delta = \Delta_{Shi}$.

Hopf algebra structure

Theorem (Im-K.-Le-Ngo Dac-Pham)

$(\mathfrak{C}, ш, u, \Delta, \epsilon)$ is a connected graded Hopf algebra of finite type over $\mathbb{F}_q$.

Outline

Remark on the coproduct of letters

We also found some explicit formulae for $\Delta(x_n)$.

Proposition

When $n \leq q$, $\Delta(x_n) = 1 \otimes x_n + x_n \otimes 1$.
When $q < n \leq q^2$,

$$\Delta(x_n) = 1 \otimes x_n + x_n \otimes 1 + \sum_{i=1}^{k} (-1)^i \binom{n-1+i}{i} x_{n-i(q-1)} \otimes x_{i(q-1)}$$

when k is integer with $kq < n \leq (k+1)q$.

You can find the numerical results for $\Delta(x_n)$ for $n \leq q^3 + q^2$ and $q = 3, 5$ cases in our paper.

Remark on the stuffle Hopf algebra structure

Instead of $\sqcup\!\sqcup$ we can define the *stuffle product* $*$ as

$$1 * \mathfrak{a} = \mathfrak{a} * 1 = \mathfrak{a} \quad \text{for all } \mathfrak{a},$$
$$\mathfrak{a} * \mathfrak{b} = x_a(\mathfrak{a}_- * \mathfrak{b}) + x_b(\mathfrak{a} * \mathfrak{b}_-) + x_{a+b}(\mathfrak{a}_- * \mathfrak{b}_-) \quad \text{for nontrivial } \mathfrak{a}, \mathfrak{b}.$$

Theorem

$\mathfrak{C}$ with $*$ and coproduct $\Delta_{deconcat}$ attains the connected graded Hopf algebra of finite type over $\mathbb{F}_q$.

N.B. As stuffle algebra, $Z_* : \mathfrak{C} \otimes_{\mathbb{F}_q} K \to \mathcal{Z}$; $\mathfrak{a} \mapsto \mathrm{Li}(\mathfrak{a})$ is K-algebra homomorphism, where Li is the Carlitz multiple polylogarithms which spans the same space as the MZV's.

Remark on the Alternating MZV's

Finally we remark that the Hopf algebra structure of the alternating MZV's (abbreviated as AMZV's) in positive characteristics is also proved in (Im-Kim-Le-Ngo Dac-Pham 2023a), where AMZV's are defined (Harada, 2021) as

$$\zeta_A \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_r \\ s_1 & \dots & s_r \end{pmatrix} := \sum \frac{\varepsilon_1^{\deg a_1} \dots \varepsilon_r^{\deg a_r}}{a_1^{s_1} \dots a_r^{s_r}}$$

for positive integers s_i's and $\varepsilon_i \in \mathbb{F}_q^\times$, where the sum is over all monic polynomials a_i's with $\deg(a_1) > \cdots > \deg(a_r)$, with the similarly defined shuffle product and coproduct.

Thank you for your attention!

Thank you for your attention!
Questions are welcome!