# HW01

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**Exercise 2.15.** Verify that $\lambda^{-1}(-)$ and its $\beta, \eta$-rules do infact imply our original elimination $\beta, \eta$-rule.

*Proof.* Let's make clear understanding about this question first. First, $\lambda^{-1}(-)$ is direct representation of forward isomorphism between $Tm(\Gamma, \Pi(A, B)) \cong Tm(\Gamma.A, B)$, which means

$$\iota_{\Gamma,A,B} : Tm(\Gamma, \Pi(A, B)) \xrightarrow{\cong} Tm(\Gamma.A, B)$$

which means that, it is a kind of homomorphism that for given $f : \Pi(A, B)$ , construct term $\lambda^{-1}(f)$ in context $\Gamma.A$ which type is $B$. By the given rule, we write

$$\frac{\Gamma \vdash f : \Pi(A, B)}{\Gamma.A \vdash \lambda^{-1}(f) : B}$$

In this case, defining $\beta, \eta$-rule is quite simple, each are

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma.A \vdash b : B}{\Gamma.A \vdash \lambda^{-1}(\lambda(b)) = b : B} \qquad \frac{\Gamma \vdash A \text{ type} \quad \Gamma.A \vdash B \text{ type} \quad \Gamma \vdash f : \Pi(A, B)}{\Gamma \vdash \lambda(\lambda^{-1}(f)) = f : \Pi(A, B)}$$

Here, if there exists a term $a$ such that $\Gamma \vdash a : A$, then we can construct an substitution $\Gamma \vdash \mathbf{id}.a : \Gamma.A$ and then we can pull-back above $\lambda^{-1}(f)$ through $\mathbf{id}.a$. Then we can write,

$$\frac{\Gamma \vdash f : \Pi(A, B) \quad \Gamma \vdash a : A}{\Gamma \vdash \lambda^{-1}(f)[\mathbf{id}.a] : B[\mathbf{id}.a]}$$

The result term is what we defined originally as $\mathbf{app}(f, a) := \lambda^{-1}(f)[\mathbf{id}.a]$. So, the question is that can we derive our original $\beta, \eta$-rules for $\mathbf{app}(-, -)$ via above rules? Here, such original $\beta, \eta$-rule is written by

$$\frac{\Gamma \vdash a : A \quad \Gamma.A \vdash b : B}{\Gamma \vdash \mathbf{app}(\lambda(b), a) = b[\mathbf{id}.a] : B[\mathbf{id}.a]} \qquad \frac{\Gamma \vdash A \text{ type} \quad \vdash \Gamma.A \vdash B \text{ type} \quad \Gamma \vdash f : \Pi(A, B)}{\Gamma \vdash \lambda(\mathbf{app}(f[\mathbf{p}], \mathbf{q})) = f : \Pi(A, B)}$$

Let's prove first one. Since we have $\Gamma \vdash A$ type as pre-supposition, we get

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma.A \vdash b : B}{\Gamma.A \vdash \lambda^{-1}(\lambda(b)) = b : B}$$

However, since $\Gamma \vdash a : A$ is given, we can construct substitution

$$\Gamma \vdash \mathbf{id}.a : \Gamma.A$$

So we can pull-back the result into context $\Gamma$,

$$\Gamma \vdash \lambda^{-1}(\lambda(b))[\mathbf{id}.a] = b[\mathbf{id}.a] : B[\mathbf{id}.a]$$

Recap the definition of $\mathbf{app}(-,-)$. We can rewrite above result by

$$\Gamma \vdash \mathbf{app}(\lambda(b), a) = b[\mathbf{id}.a] : B[\mathbf{id}.a]$$

This proved the first one. Let's prove second one. Before prove this, we must argue the naturality of forward homomorphism $\iota_{\Gamma,A,B}$ in our version. Suppose that following diagram,

$$\begin{tikzcd}
Tm(\Gamma, \Pi(A,B)) \arrow[r, "\iota_\Gamma"] \arrow[d, "\gamma^*"] & Tm(\Gamma.A, B) \arrow[d, "\gamma.A^*"] \\
Tm(\Delta, \Pi(A[\gamma], B[\gamma.A])) \arrow[r, "\iota_\Delta"] & Tm(\Delta.A[\gamma], B[\gamma.A])
\end{tikzcd}$$

Here, $\gamma^*$ and $\gamma.A^*$ notate that works of pull-back operator w.r.t. $\gamma$ and $\gamma.A$. And since we've construct such isomorphisms via $\lambda^{-1}$, to satisfy naturality of $\iota$ here, following must holds :

$$\frac{\Delta \vdash \gamma : \Gamma \quad \Gamma \vdash A \text{ type} \quad \Gamma.A \vdash B \text{ type} \quad \Gamma \vdash f : \Pi(A,B)}{\Delta.A[\gamma] \vdash \lambda^{-1}(f[\gamma]) = \lambda^{-1}(f)[\gamma.A] : B[\gamma.A]}$$

This can be easily obtained via following application of above diagram.

$$\begin{tikzcd}
\Gamma \vdash f : \Pi(A,B) \arrow[r, "\cong \text{ by } \lambda^{-1}"] \arrow[d, "\text{pull-back by } \gamma"] & \Gamma.A \vdash \lambda^{-1}(f) : B \arrow[d, "\text{pull-back by } \gamma.A"] \\
\Delta \vdash f[\gamma] : \Pi(A[\gamma], B[\gamma.A]) \arrow[r, "\cong \text{ by } \lambda^{-1}"] & \Delta.A[\gamma] \vdash \lambda^{-1}(f[\gamma]) = \lambda^{-1}(f)[\gamma.A] : B[\gamma.A]
\end{tikzcd}$$

We can use above rule for proving second one by :

$$\frac{\Gamma.A \vdash \mathbf{p} : \Gamma \quad \Gamma \vdash A \text{ type} \quad \Gamma.A \vdash B \text{ type} \quad \Gamma \vdash f : \Pi(A,B)}{\Gamma.A.A[\mathbf{p}] \vdash \lambda^{-1}(f[\mathbf{p}]) = \lambda^{-1}(f)[\mathbf{p}.A] : B[\mathbf{p}.A]}$$

However, since there exists a term $\Gamma.A \vdash \mathbf{q} : A[\mathbf{p}]$, we can construct well-formed substitution $\Gamma.A \vdash \mathbf{id}.\mathbf{q} : \Gamma.A.A[\mathbf{p}]$. Let's pull-back above result by $\mathbf{id}.\mathbf{q}$ :

$$\Gamma.A \vdash \lambda^{-1}(f[\mathbf{p}])[\mathbf{id}.\mathbf{q}] = \lambda^{-1}(f)[\mathbf{p}.A][\mathbf{id}.\mathbf{q}] : B[\mathbf{p}.A][\mathbf{id}.\mathbf{q}]$$

However, we already know that $(\mathbf{p}.A) \circ (\mathbf{id}.\mathbf{q}) = \mathbf{id}$, rewrite above by :

$$\Gamma.A \vdash \lambda^{-1}(f[\mathbf{p}])[\mathbf{id}.\mathbf{q}] = \lambda^{-1}(f) : B$$

Here, by introduction rule,

$$\Gamma \vdash \lambda(\lambda^{-1}(f[\mathbf{p}])[\mathbf{id}.\mathbf{q}]) = \lambda(\lambda^{-1}(f)) : \Pi(A, B)$$

Now almost done. Rewrite $\lambda^{-1}(f[\mathbf{p}][\mathbf{id}.\mathbf{q}])$ as $\mathbf{app}(f[\mathbf{p}], \mathbf{q})$ and rewrite $\lambda(\lambda^{-1}(f))$ as $f$.

$$\Gamma \vdash \lambda(\mathbf{app}(f[\mathbf{p}], \mathbf{q})) = f : \Pi(A, B)$$

This is end of proof. $\square$

**Exercise 2.16.** Show that using $\Sigma$ types we can define a non-dependent pair type whose formation rule states that if $\Gamma \vdash A$ type and $\Gamma \vdash B$ type then $\Gamma \vdash A \times B$ type. Then define the introduction and elimination rules from section 2.1 for this encoding, and check that $\beta, \eta$-rules holds.

*Proof.* Our goal is that construct following rules of non-dependent pair type can be defined via dependent sum type $\Sigma$.

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \times B \text{ type}} \qquad \frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash (a,b) : A \times B} \qquad \frac{\Gamma \vdash p : A \times B}{\Gamma \vdash \mathbf{fst}(p) : A} \qquad \frac{\Gamma \vdash p : A \times B}{\Gamma \vdash \mathbf{snd}(p) : B}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash \mathbf{fst}((a,b)) = a : A} \qquad \frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash \mathbf{snd}((a,b)) = b : B} \qquad \frac{\Gamma \vdash p : A \times B}{\Gamma \vdash p = (\mathbf{fst}(p), \mathbf{snd}(p)) : A \times B}$$

$$\frac{\Delta \vdash \gamma : \Gamma \quad \Gamma \vdash A, B \text{ type}}{\Delta \vdash (A \times B)[\gamma] = A[\gamma] \times B[\gamma] \text{ type}} \qquad \frac{\Delta \vdash \gamma : \Gamma \quad \Gamma \vdash a : A, b : B \quad \Gamma \vdash (a,b) : A \times B}{\Delta \vdash (a,b)[\gamma] = (a[\gamma], b[\gamma]) : A[\gamma] \times B[\gamma]}$$

First, let's define $A \times B$ first. Since $\Gamma \vdash A$ type implies that $\Gamma.A \vdash B[\mathbf{p}]$ type, so we can define

$$A \times B := \Sigma(A, B[\mathbf{p}])$$

Then remainings can be shown easily. If $\Gamma \vdash a : A$ and $\Gamma \vdash b : B$, then $\Gamma \vdash b : B[\mathbf{id}] = B[\mathbf{p}][\mathbf{id}.a]$, so

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash (a,b) := \mathbf{pair}(a,b) : \Sigma(A, B[\mathbf{p}])}$$

$\mathbf{fst}, \mathbf{snd}$ can be defined by :

$$\frac{\Gamma \vdash p : \Sigma(A, B[\mathbf{p}])}{\Gamma \vdash \mathbf{fst}(p) : A \qquad \Gamma \vdash \mathbf{snd}(p) : B[\mathbf{p}][\mathbf{id}.\mathbf{fst}(p)] = B}$$

Then we can also see that such construction holds naturality.

$$\frac{\Delta \vdash \gamma : \Gamma \quad \Gamma \vdash A, B \text{ type}}{\Delta \vdash (A \times B)[\gamma] = \Sigma(A, B[\mathbf{p}])[\gamma] = \Sigma(A[\gamma], B[\mathbf{p}][\gamma.A]) = \Sigma(A[\gamma], B[\gamma][\mathbf{p}]) = A[\gamma] \times B[\gamma] \text{ type}}$$

$$\frac{\Delta \vdash \gamma : \Gamma \quad \Gamma \vdash a : A, b : B \quad \Gamma \vdash (a,b) : A \times B}{\Delta \vdash (a,b)[\gamma] = \mathbf{pair}(a,b)[\gamma] = \mathbf{pair}(a[\gamma], b[\gamma]) = (a[\gamma], b[\gamma]) : A[\gamma] \times B[\gamma]}$$

Here, the type-former works like

$$\bigtimes_{\Gamma} : Ty(\Gamma) \times Ty(\Gamma) \to Ty(\Gamma)$$

And the term natural isomorphism is constructed via

$$\iota : Tm(\Gamma, A) \times Tm(\Gamma, B) \cong Tm(\Gamma, A \times B)$$

Remaining is that check for the $\beta, \eta$-rules.

$$\frac{\Gamma \vdash a : A \quad \Gamma.A \vdash B[\mathbf{p}] \text{ type} \quad \Gamma \vdash b : B = B[\mathbf{p}][\mathbf{id}.a]}{\Gamma \vdash \mathbf{fst}((a,b)) = \mathbf{fst}(\mathbf{pair}(a,b)) = a : A}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma.A \vdash B[\mathbf{p}] \text{ type} \quad \Gamma \vdash b : B = B[\mathbf{p}][\mathbf{id}.a]}{\Gamma \vdash \mathbf{snd}((a,b)) = \mathbf{snd}(\mathbf{pair}(a,b)) = b : B}$$

Finally,

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma.A \vdash B[\mathbf{p}] \text{ type} \quad \Gamma \vdash p : A \times B = \Sigma(A, B[\mathbf{p}])}{\Gamma \vdash p = \mathbf{pair}(\mathbf{fst}(p), \mathbf{snd}(p)) = (\mathbf{fst}(p), \mathbf{snd}(p)) : A \times B}$$

So our definition $A \times B := \Sigma(A, B[\mathbf{p}])$ holds on whole requirements.

$\square$

**Exercise 2.24.** Fixing $\Delta \vdash \gamma : \Gamma$, prove that there is at most one substitution $\Delta \vdash \overline{\gamma} : \Gamma.\mathbf{Void}$ such that satisfying $\mathbf{p} \circ \overline{\gamma} = \gamma$.

*Proof.* I'll draw diagram for good understanding.

$$\begin{tikzcd} \Delta \arrow[r, "\gamma"] \arrow[dr, "\overline{\gamma}"'] & \Gamma \arrow[d, shift right, "?"'] \\ & \Gamma.\mathbf{Void} \arrow[u, shift right, "p"'] \end{tikzcd}$$

We'll use following rule for substitution :

$$\frac{\Gamma \vdash A \text{ type} \quad \Delta \vdash \gamma : \Gamma.A}{\Delta \vdash \gamma = (\mathbf{p} \circ \gamma).\mathbf{q}[\gamma] : \Gamma.A} \tag{*}$$

Suppose that $\overline{\gamma}_i, i = 1, 2$ such that $\Delta \vdash \overline{\gamma_i} : \Gamma.\mathbf{Void}, i = 1, 2$ and $(\mathbf{p} \circ \overline{\gamma}_i) = \gamma, i = 1, 2$ exists. Then we can apply (*) by

$$\frac{\Gamma \vdash \mathbf{Void} \text{ type} \quad \Delta \vdash \overline{\gamma}_i : \Gamma.\mathbf{Void}}{\Delta \vdash \overline{\gamma}_i = (\mathbf{p} \circ \overline{\gamma}_i).\mathbf{q}[\overline{\gamma}_i] : \Gamma.\mathbf{Void}}$$

By assumption, we can directly rewrite above result by

$$\Delta \vdash \overline{\gamma}_i = \gamma.\mathbf{q}[\overline{\gamma}_i] : \Gamma.\mathbf{Void} \tag{**}$$

However, $\Gamma.\mathbf{Void} \vdash \mathbf{q} : \mathbf{Void}$. So, we can pull-back $\mathbf{q}$ through $\overline{\gamma}_i$.

$$\frac{\Delta \vdash \overline{\gamma}_i : \Gamma.\mathbf{Void} \quad \Gamma.\mathbf{Void} \vdash \mathbf{q} : \mathbf{Void}}{\Delta \vdash \mathbf{q}[\overline{\gamma}_i] : \mathbf{Void}[\overline{\gamma}_i] = \mathbf{Void}}$$

However, the result of Exercise 2.23. implies that,

$$\Delta \vdash \mathbf{q}[\overline{\gamma}_1] = \mathbf{q}[\overline{\gamma}_2] : \mathbf{Void}$$

We can use this result for (**).

$$\Delta \vdash \overline{\gamma}_1 = \gamma.\mathbf{q}[\overline{\gamma}_1] = \gamma.\mathbf{q}[\overline{\gamma}_2] = \overline{\gamma}_2 : \Gamma.\mathbf{Void}$$

This implies that,

$$\Delta \vdash \overline{\gamma}_1 = \overline{\gamma}_2 : \Gamma.\mathbf{Void}$$

We proved that, if such $\overline{\gamma}$ exists, then it is unique. This is sufficient to prove that there is at most one such $\overline{\gamma}$.

$\square$

**Exercise 2.25.** Let $\Gamma.\mathbf{Void} \vdash A$ type and $\Gamma \vdash a : A[\mathbf{id}.b]$. Show that $\Gamma.\mathbf{Void} \vdash A[(\mathbf{id}.b) \circ \mathbf{p}] = A$ type, and therefore that $\Gamma.\mathbf{Void} \vdash a[\mathbf{p}] : A$.

*Proof.* We'll use following previous result.

$$(\gamma.a) \circ \delta = (\gamma \circ \delta).a[\delta]$$

In this exercies, we can write as

$$(\mathbf{id}.b) \circ \mathbf{p} = (\mathbf{id} \circ \mathbf{p}).b[\mathbf{p}]$$

However, by property of **id**,

(*) $$(\mathbf{id}.b) \circ \mathbf{p} = (\mathbf{id} \circ \mathbf{p}).b[\mathbf{p}] = \mathbf{p}.b[\mathbf{p}]$$

However, our problem gives some pre-supposition :

$$\Gamma \vdash A[\mathbf{id}.b] \text{ type}$$

Which means that

$$\Gamma \vdash \mathbf{id}.b : \Gamma.\mathbf{Void}$$

This also gives pre-supposition :

$$\Gamma \vdash b : \mathbf{Void}$$

So we can pull-back such $b$ through $\mathbf{p}$ :

$$\Gamma.\mathbf{Void} \vdash b[\mathbf{p}] : \mathbf{Void}[\mathbf{p}] = \mathbf{Void}$$

However, $\Gamma.\mathbf{Void} \vdash \mathbf{q} : \mathbf{Void}$ and Exercise 2.23. implies that

$$\Gamma.\mathbf{Void} \vdash b[\mathbf{p}] = \mathbf{q} : \mathbf{Void}$$

Then we can rewrite (*) by

$$(\mathbf{id}.b) \circ \mathbf{p} = (\mathbf{id} \circ \mathbf{p}).b[\mathbf{p}] = \mathbf{p}.b[\mathbf{p}] = \mathbf{p}.\mathbf{q} = \mathbf{id}$$

in context $\Gamma.\mathbf{Void}$. So we can prove our original claim,

$$\Gamma.\mathbf{Void} \vdash A[(\mathbf{id}.b) \circ \mathbf{p}] = A[\mathbf{id}] = A \text{ type}$$

Then the remaining part is easy. Since $\Gamma \vdash a : A[\mathbf{id}.b]$, we can pull-back $a$ into $\Gamma.\mathbf{Void}$ through $p$ :

$$\Gamma.\mathbf{Void} \vdash a[\mathbf{p}] : A[\mathbf{id}.b][\mathbf{p}] = A[(\mathbf{id}.b) \circ \mathbf{p}] = A$$

$\square$

**Exercise 2.26.** Derive the following rule, using the previous exercise and the $\eta$-rule.

$$\frac{\Gamma \vdash b : \mathbf{Void} \quad \Gamma.\mathbf{Void} \vdash A \text{ type} \quad \Gamma \vdash a : A[\mathbf{id}.b]}{\Gamma \vdash a = \mathbf{absurd}(b) : A[\mathbf{id}.b]}$$

*Proof.* We'll use above Exercise 2.25. as lemma. We proved that

$$(*) \qquad \frac{\Gamma \vdash b : \mathbf{Void} \quad \Gamma.\mathbf{Void} \vdash A \quad \text{type} \quad \Gamma \vdash a : A[\mathbf{id}.b]}{\Gamma.\mathbf{Void} \vdash a[\mathbf{p}] : A}$$

And the original $\eta$-rule is that

$$\frac{\Gamma \vdash b : \mathbf{Void} \quad \Gamma.\mathbf{Void} \vdash a : A}{\Gamma \vdash \mathbf{absurd}(b) = a[\mathbf{id}.b] : A[\mathbf{id}.b]}$$

To apply this, let's write that

$$\frac{\Gamma \vdash b : \mathbf{Void} \quad \Gamma.\mathbf{Void} \vdash a[\mathbf{p}] : A}{\Gamma \vdash \mathbf{absurd}(b) = a[\mathbf{p}][\mathbf{id}.b] : A[\mathbf{id}.b]}$$

Since (*) gives the premises, we get

$$\Gamma \vdash \mathbf{absurd}(b) = a[\mathbf{p}][\mathbf{id}.b] : A[\mathbf{id}.b]$$

However, $a[\mathbf{p}][\mathbf{id}.b] = a[\mathbf{p} \circ (\mathbf{id}.b)] = a[\mathbf{id}] = a$, we can get

$$\Gamma \vdash \mathbf{absurd}(b) = a : A[\mathbf{id}.b]$$

This is End of Proof. $\square$

**Exercise 2.29.** Give rules axiomatizing the boolean analogue of $\mathbf{absurd}'$, and prove that these rules are interderivable with our rules for $\mathbf{if}(a_t, a_f, b)$.

*Proof.* This is also question about 'cut-rule applied' version of our backward homomorphism $\iota^{-1}$. Since

$$\iota^{-1} : Tm(\Gamma, A[\mathbf{id}.\mathbf{true}]) \times Tm(\Gamma, A[\mathbf{id}.\mathbf{false}]) \to Tm(\Gamma.\mathbf{Bool}, A)$$

Then when we directly construct this homomorphism, it will be

$$\frac{\Gamma.\mathbf{Bool} \vdash A \text{ type} \quad \Gamma \vdash a_t : A[\mathbf{id}.\mathbf{true}] \quad \Gamma \vdash a_f : A[\mathbf{id}.\mathbf{false}]}{\Gamma.\mathbf{Bool} \vdash \mathbf{if}'(a_t, a_f) : A}$$

With above definition, the remaining modified rules are simply constructed :

$$\frac{\Delta \vdash \gamma : \Gamma \quad \Gamma.\mathbf{Bool} : A \text{ type} \quad \Gamma \vdash a_t : A[\mathbf{id}.\mathbf{true}] \quad \Gamma \vdash a_f : A[\mathbf{id}.\mathbf{false}]}{\Delta.\mathbf{Bool} \vdash \mathbf{if}'(a_t, a_f)[\gamma.\mathbf{Bool}] = \mathbf{if}'(a_t[\gamma], a_f[\gamma]) : A[\gamma.\mathbf{Bool}]}$$

$$\frac{\Gamma.\mathbf{Bool} \vdash A \text{ type} \quad \Gamma \vdash a_t : A[\mathbf{id}.\mathbf{true}] \quad \Gamma \vdash a_f : A[\mathbf{id}.\mathbf{false}]}{\Gamma \vdash \mathbf{if}'(a_t, a_f)[\mathbf{id}.\mathbf{true}] = a_t : A[\mathbf{id}.\mathbf{true}]}$$

$$\frac{\Gamma.\mathbf{Bool} \vdash A \text{ type} \quad \Gamma \vdash a_t : A[\mathbf{id}.\mathbf{true}] \quad \Gamma \vdash a_f : A[\mathbf{id}.\mathbf{false}]}{\Gamma \vdash \mathbf{if}'(a_t, a_f)[\mathbf{id}.\mathbf{false}] = a_f : A[\mathbf{id}.\mathbf{false}]}$$

$$\frac{\Gamma.\mathbf{Bool} \vdash A \text{ type} \quad \Gamma.\mathbf{Bool} \vdash a : A}{\Gamma.\mathbf{Bool} \vdash \mathbf{if}'(a[\mathbf{id}.\mathbf{true}], a[\mathbf{id}.\mathbf{false}]) = a : A}$$

However, our rules for **if** is constructed on context $\Gamma$. Actually, as same we discussed, it can be imagined as pull-backed $\mathbf{if}'$ into $\Gamma$. Here, we can write

$$\frac{\Gamma.\mathbf{Bool} \vdash A \text{ type} \quad \Gamma \vdash a_t : A[\mathbf{id}.\mathbf{true}] \quad \Gamma \vdash a_f : A[\mathbf{id}.\mathbf{false}] \quad \Gamma \vdash b : \mathbf{Bool}}{\Gamma \vdash \mathbf{if}'(a_t, a_f)[\mathbf{id}.b] := \mathbf{if}(a_t, a_f, b) : A[\mathbf{id}.b]}$$

Now, let's prove our original rules for **if** implies above $\beta, \eta$-rules. ( Here, I'll omit some premises when they are clear ) We have

$$\Gamma \vdash \mathbf{if}(a_t, a_f, \mathbf{true}) = a_t : A[\mathbf{id}.\mathbf{true}]$$

Rewrite this :

$$\Gamma \vdash \mathbf{if}'(a_t, a_f)[\mathbf{id}.\mathbf{true}] = a_t : A[\mathbf{id}.\mathbf{true}]$$

Similarly,

$$\Gamma \vdash \mathbf{if}'(a_t, a_f)[\mathbf{id}.\mathbf{false}] = a_f : A[\mathbf{id}.\mathbf{false}]$$

This directly proves above $\beta$-rules. For $\eta$-rule, we can imagine following diagram :

$$\Gamma \xleftarrow[p]{} \Gamma.\mathbf{Bool} \underset{p.\mathbf{Bool}}{\overset{id.q}{\rightleftarrows}} \Gamma.\mathbf{Bool}.\mathbf{Bool}$$

How can we use above intuition? Let's see. The original $\eta$-rule was

$$\frac{\Gamma.\mathbf{Bool} \vdash A \text{ type} \quad \Gamma.\mathbf{Bool} \vdash a : A \quad \Gamma \vdash b : \mathbf{Bool}}{\Gamma \vdash \mathbf{if}(a[\mathbf{id}.\mathbf{true}], a[\mathbf{id}.\mathbf{false}], b) = a[\mathbf{id}.b] : A[\mathbf{id}.b]}$$

To prove the $\eta$-rule for $\mathbf{if}'$ of us, we'll use that

$$\frac{\Delta \vdash \gamma : \Gamma \quad \Gamma.\mathbf{Bool} : A \text{ type} \quad \Gamma \vdash a_t : A[\mathbf{id}.\mathbf{true}] \quad \Gamma \vdash a_f : A[\mathbf{id}.\mathbf{false}]}{\Delta.\mathbf{Bool} \vdash \mathbf{if}'(a_t, a_f)[\gamma.\mathbf{Bool}] = \mathbf{if}'(a_t[\gamma], a_f[\gamma]) : A[\gamma.\mathbf{Bool}]}$$

We can write this as

$$\frac{\Gamma.\mathbf{Bool} \vdash \mathbf{p} : \Gamma \quad \Gamma.\mathbf{Bool} \vdash a : A \quad \Gamma \vdash a[\mathbf{id}.\mathbf{true}] : A[\mathbf{id}.\mathbf{true}] \quad \Gamma \vdash a[\mathbf{id}.\mathbf{false}] : A[\mathbf{id}.\mathbf{false}]}{\Gamma.\mathbf{Bool}.\mathbf{Bool} \vdash \mathbf{if}'(a[\mathbf{id}.\mathbf{true}], a[\mathbf{id}.\mathbf{false}])[\mathbf{p}.\mathbf{Bool}] = \mathbf{if}'(a[\mathbf{id}.\mathbf{true}][\mathbf{p}], a[\mathbf{id}.\mathbf{false}][\mathbf{p}]) : A[\mathbf{p}.\mathbf{Bool}]}$$

We can pull-back the result using $\Gamma.\mathbf{Bool} \vdash \mathbf{q} : \mathbf{Bool}$.

$$\Gamma.\mathbf{Bool} \vdash \mathbf{if}'(a[\mathbf{id}.\mathbf{true}], a[\mathbf{id}.\mathbf{false}])[\mathbf{p}.\mathbf{Bool}][\mathbf{id}.\mathbf{q}]$$
$$= \mathbf{if}'(a[\mathbf{id}.\mathbf{true}][\mathbf{p}], a[\mathbf{id}.\mathbf{false}][\mathbf{p}])[\mathbf{id}.\mathbf{q}] : A[\mathbf{p}.\mathbf{Bool}][\mathbf{id}.\mathbf{q}]$$

Rewrite this as

$$\Gamma.\mathbf{Bool} \vdash \mathbf{if}'(a[\mathbf{id}.\mathbf{true}], a[\mathbf{id}.\mathbf{false}]) = \mathbf{if}(a[\mathbf{id}.\mathbf{true}][\mathbf{p}], a[\mathbf{id}.\mathbf{false}][\mathbf{p}], \mathbf{q}) : A$$

However, we already know that $\mathbf{id}.\mathbf{true} \circ \mathbf{p} = \mathbf{p}.A \circ \mathbf{id}.\mathbf{true}'$ where $\mathbf{id}.\mathbf{true}$ is arrow $\Gamma.\mathbf{Bool} \to \Gamma.\mathbf{Bool}.\mathbf{Bool}$. Then ,

$$\Gamma.\mathbf{Bool} \vdash \mathbf{if}'(a[\mathbf{id}.\mathbf{true}], a[\mathbf{id}.\mathbf{false}]) = \mathbf{if}(a[\mathbf{p}.A][\mathbf{id}.\mathbf{true}'], a[\mathbf{p}.A][\mathbf{id}.\mathbf{false}'], \mathbf{q}) : A$$

However, the $\eta$-rule of given $\mathbf{if}$ on $\Gamma.\mathbf{Bool}$ context implies that

$$\mathbf{if}'(a[\mathbf{id}.\mathbf{true}], a[\mathbf{id}.\mathbf{false}]) = \mathbf{if}(a[\mathbf{p}.A][\mathbf{id}.\mathbf{true}'], a[\mathbf{p}.A][\mathbf{id}.\mathbf{false}'], \mathbf{q}) = a[\mathbf{p}.\mathbf{A}][\mathbf{id}.\mathbf{q}] : A$$

Then finally, we can holds

$$\Gamma.\mathbf{Bool} \vdash \mathbf{if}'(a[\mathbf{id}.\mathbf{true}], a[\mathbf{id}.\mathbf{false}]) = a : A$$

This is proof of $\eta$-rule for $\mathbf{if}'$. $\square$

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