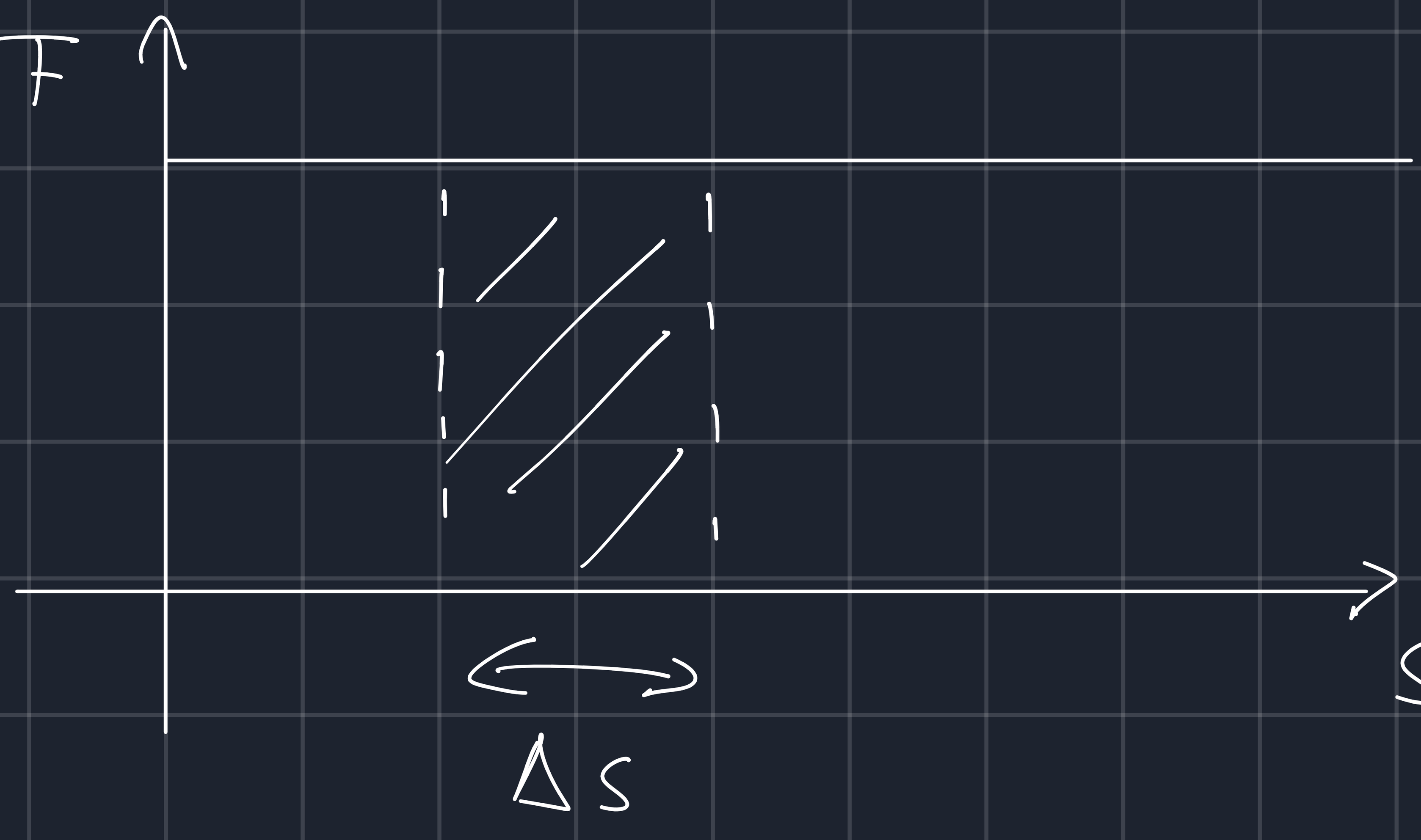
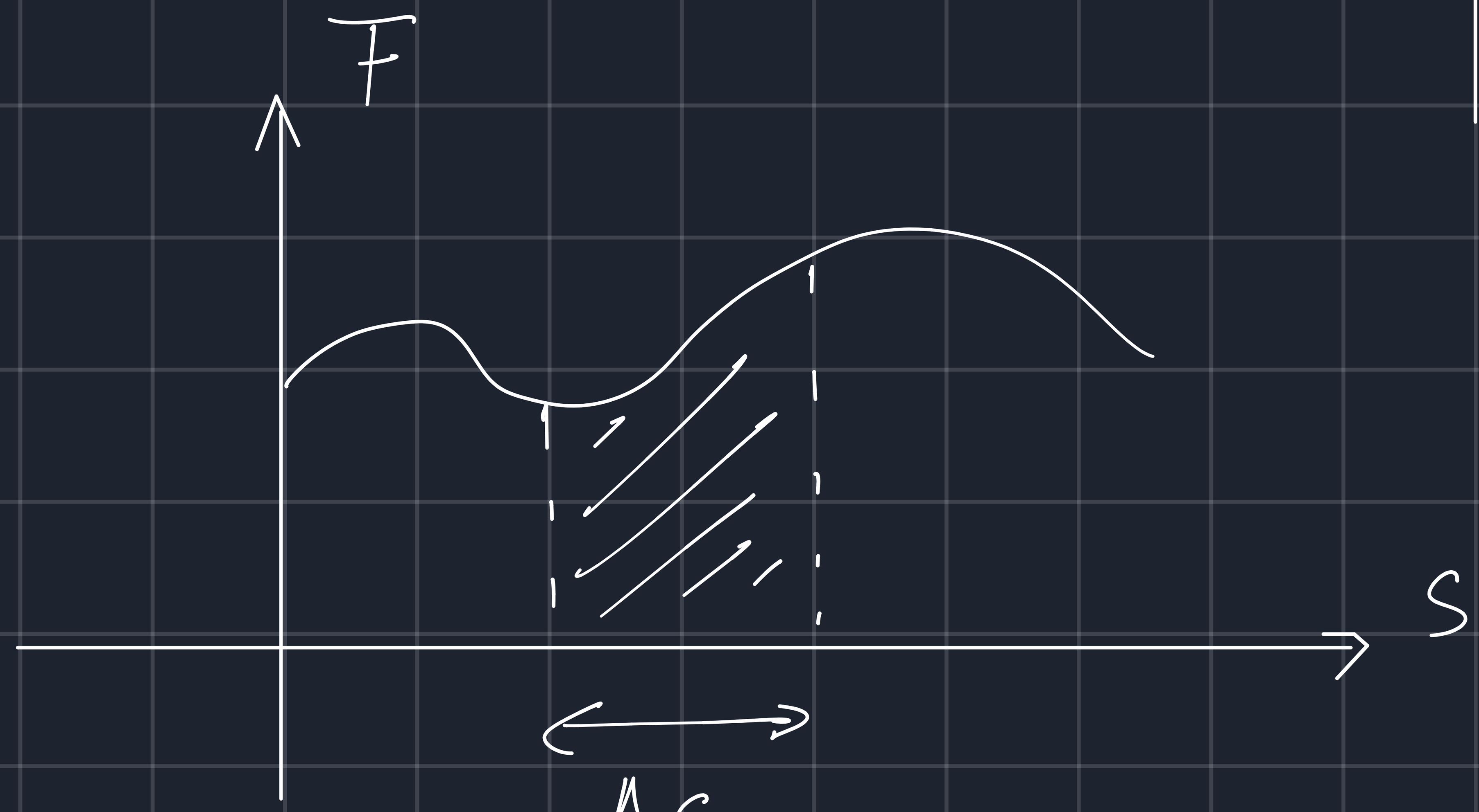
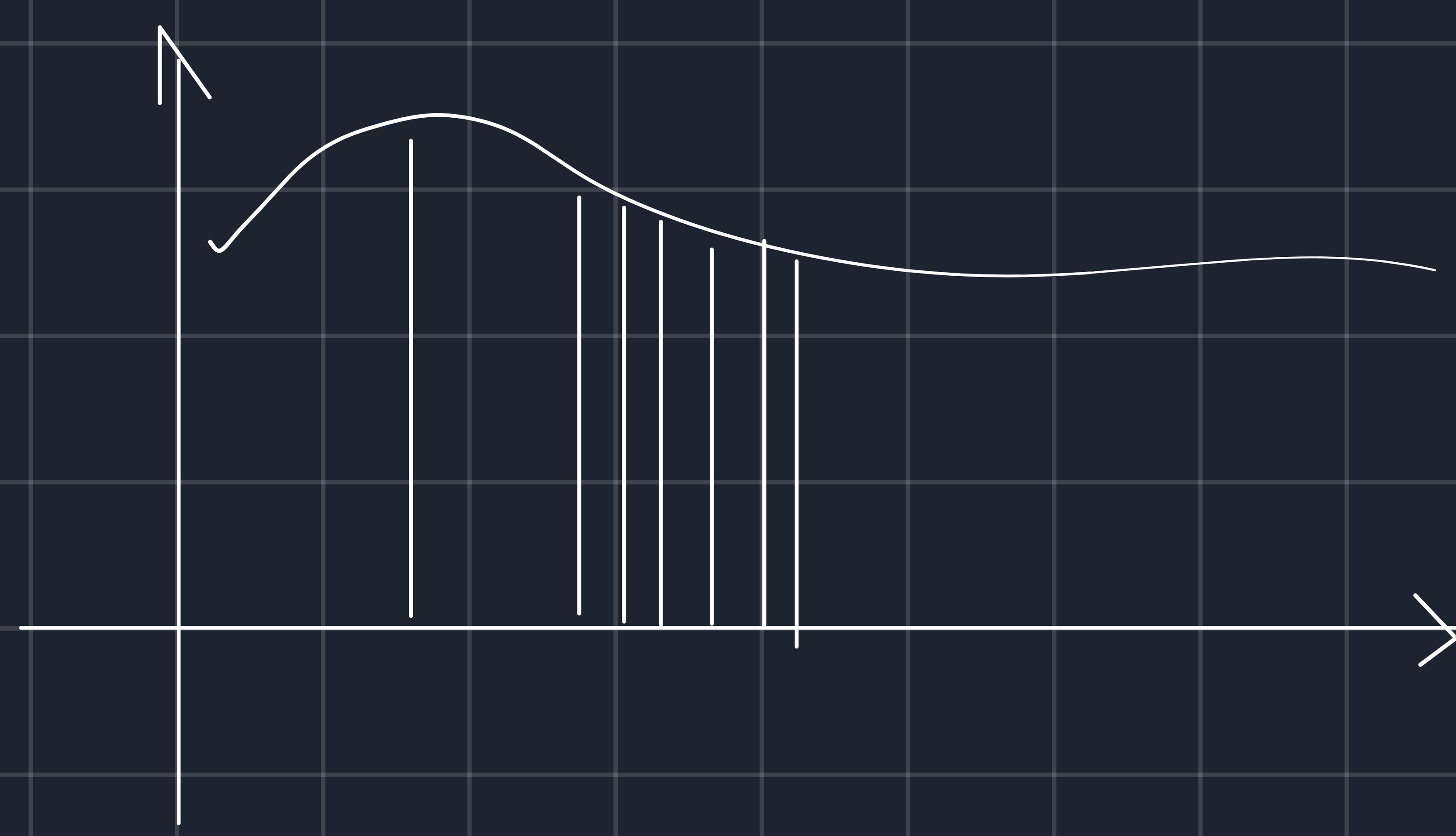
3. előadás

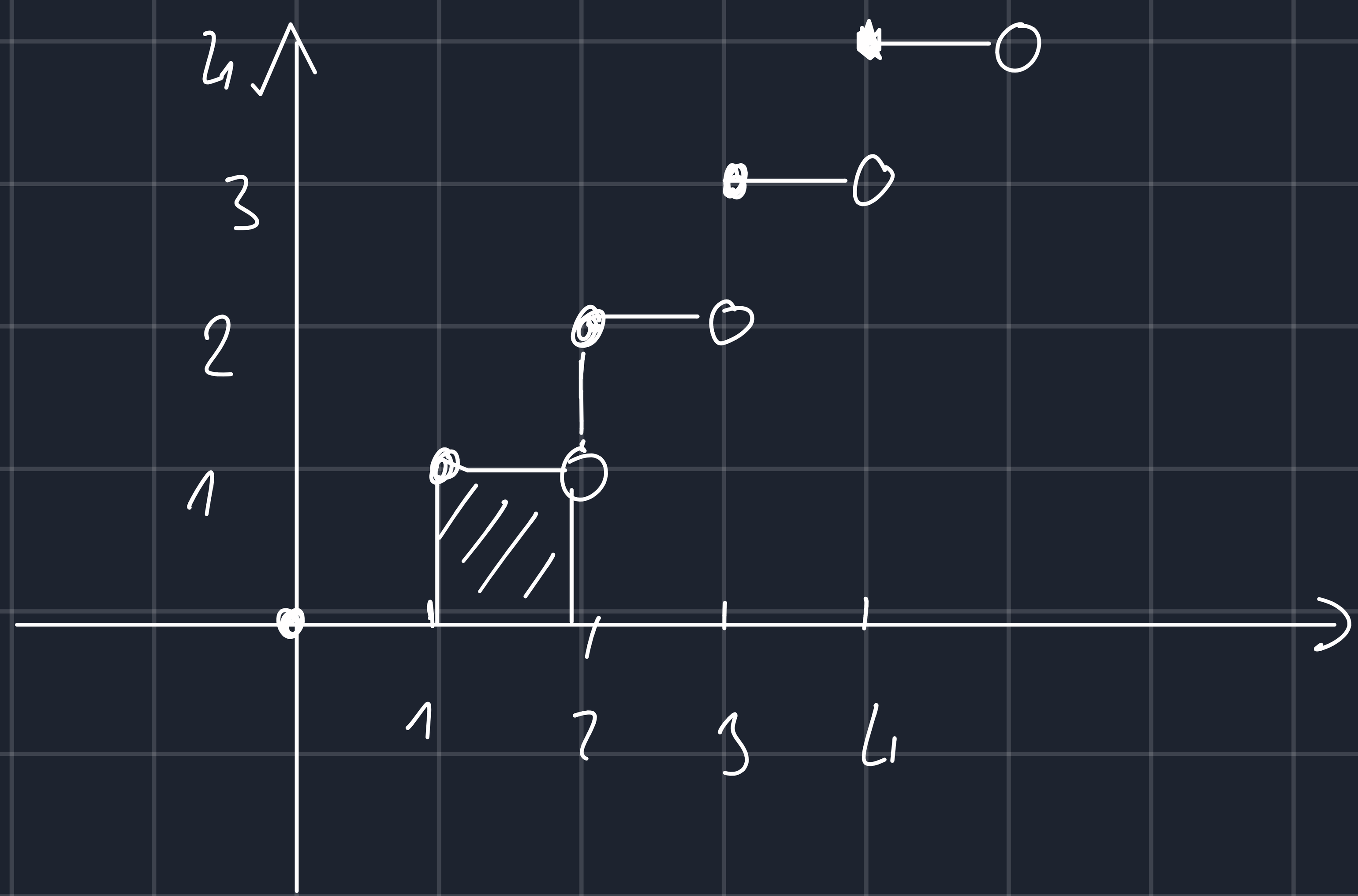
Határozott integrál



munka: $W = F \cdot s$



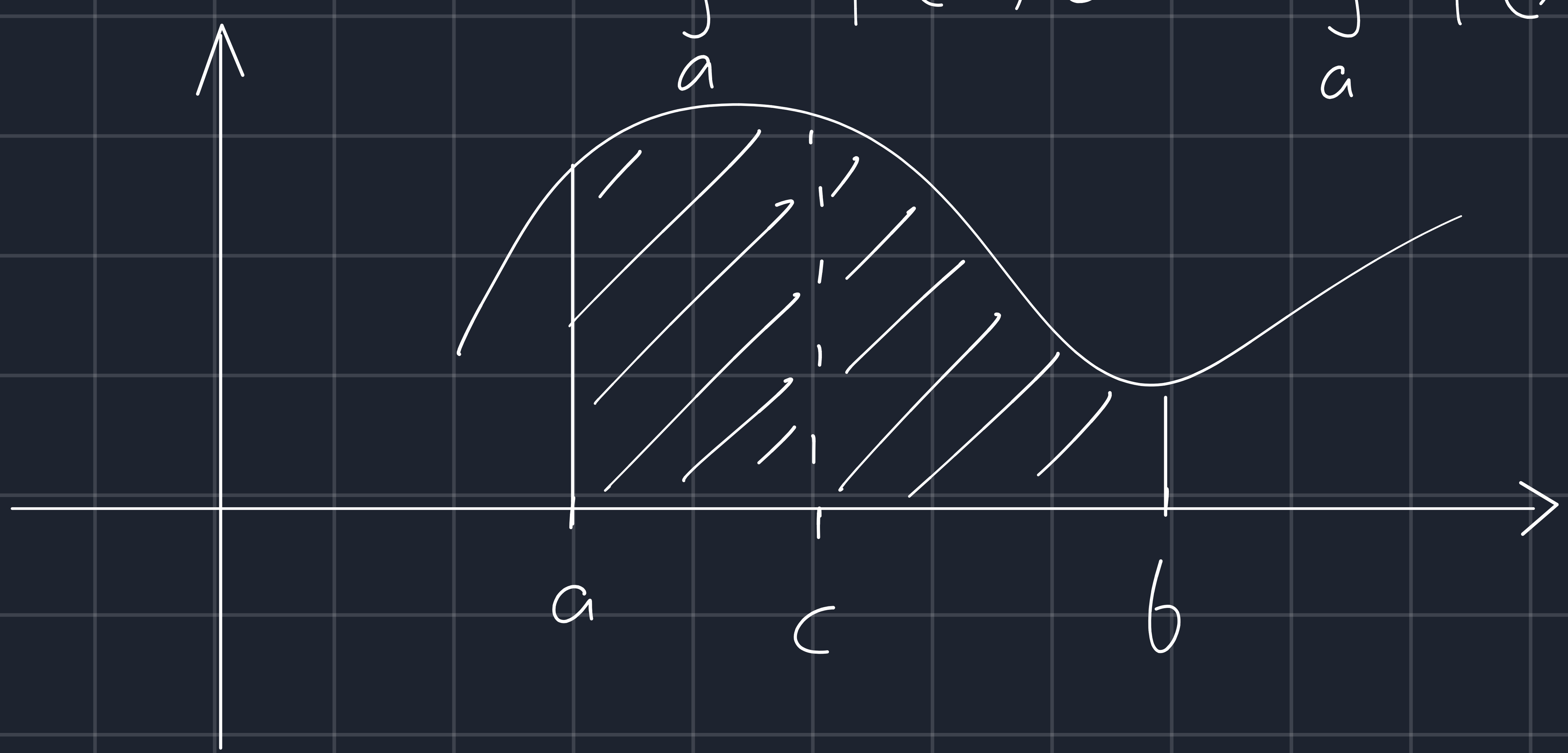


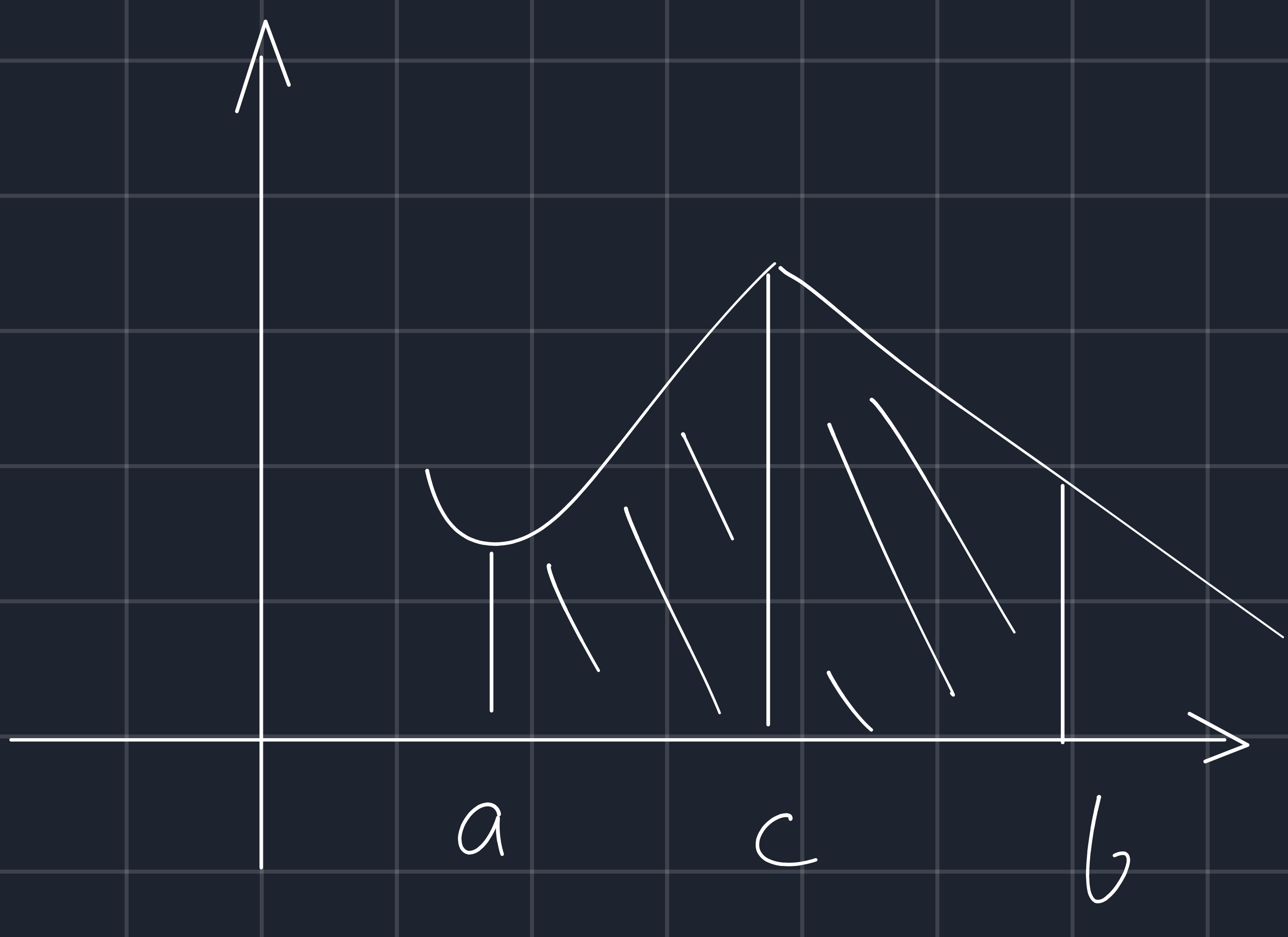


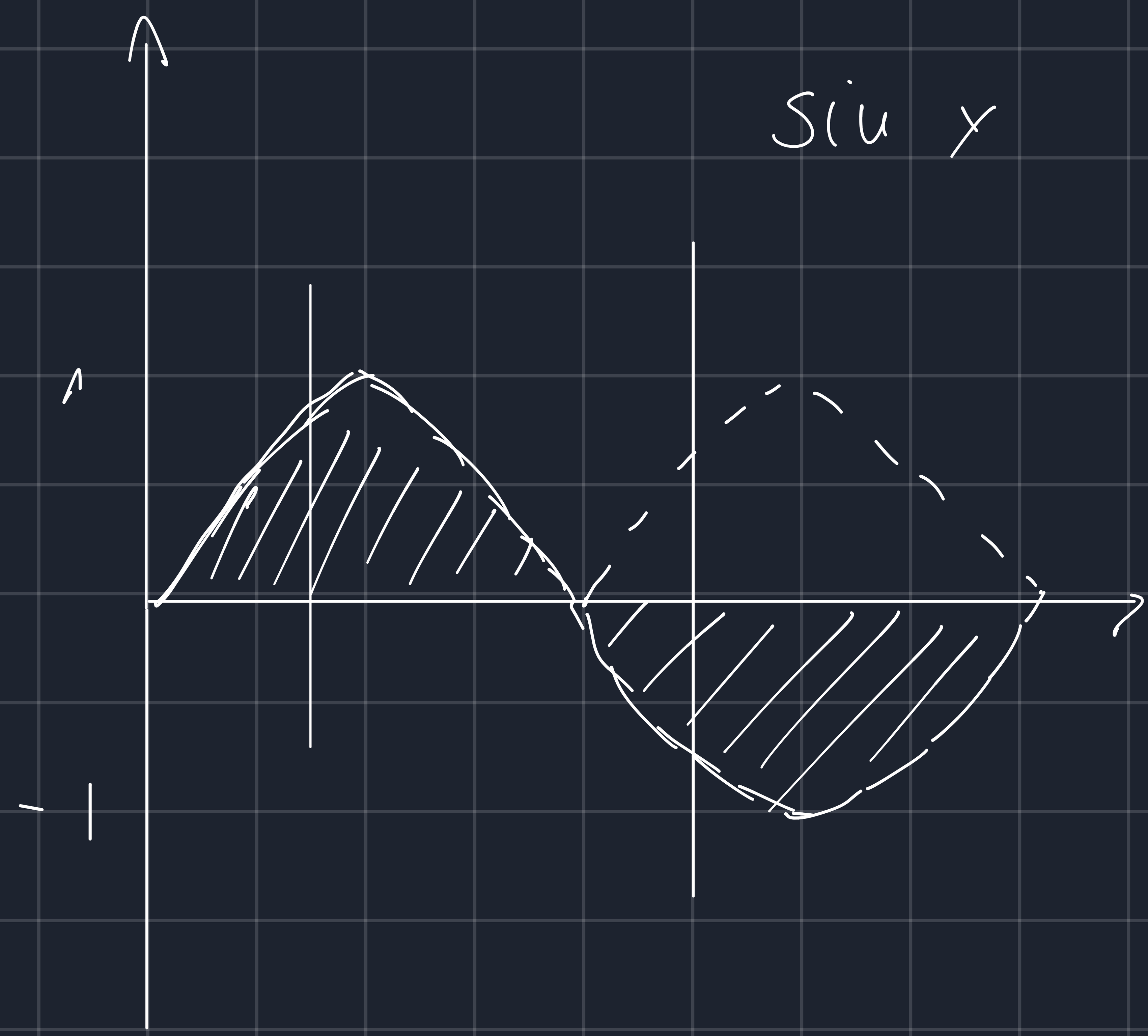
Attól, hogy egy pontban „elszakad" a függvény attól még integrálható.

Tétel: Ha f integrálható $[a,c]$ és $[c,b]$ intervallumokon, akkor $[a,b]$-n is integrálható és

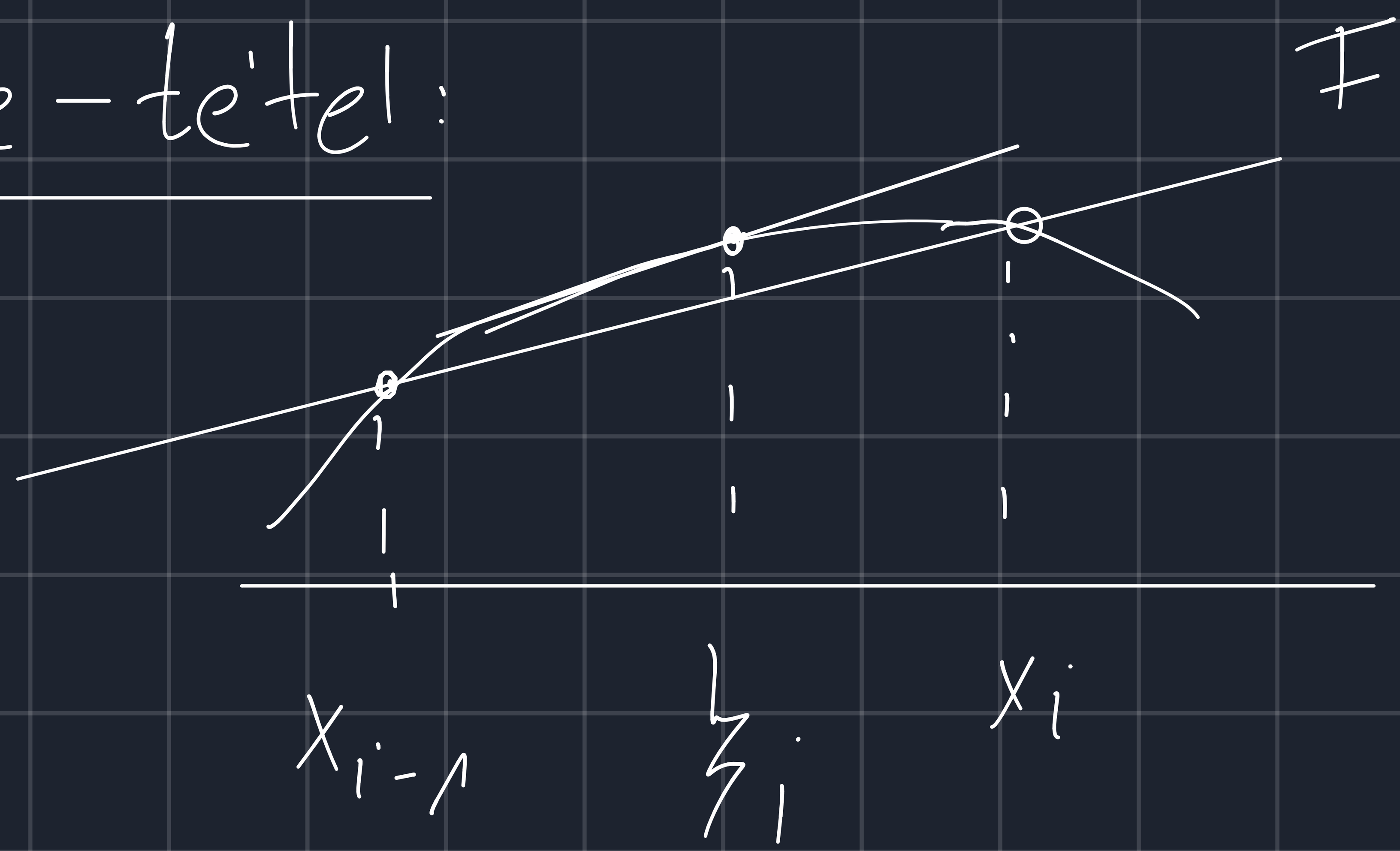
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$







Lagrange-tétel:



$$F'(\xi_i) = \underbrace{\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}}_{\text{szelő meredeksége}}$$

érintő meredeksége

Tf. létezik és létezik F prim. f.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) =$$

$$= \lim \sum F'(\xi_i)(x_i - x_{i-1}) =$$

$$= \lim \sum \frac{F(x_i) - F(x_{i-1})}{\cancel{x_i - x_{i-1}}} \cdot \cancel{(x_i - x_{i-1})} =$$

$$= \lim \sum_{i=1}^{n} \left[F(x_i) - F(x_{i-1}) \right] =$$

$$= \lim \; F(\cancel{x_1}) - F(x_0) + F(\cancel{x_2}) - F(\cancel{x_1}) + F(x_3) - F(\cancel{x_2}) +$$

$$\ldots + F(x_n) - F(\cancel{x_{n-1}}) = F(x_n) - F(x_0) =$$

$$= F(b) - F(a)$$

Newton-Leibniz tétel

Ha integrálható f [a,b]-n és van F primitív fv-e [a,b], akkor

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Pl. Mekkora a sin fv. egy íve alatti terület?

$$\int_0^{\pi} \sin x \, dx = \left[-\cos x \right]_0^{\pi} =$$

$$= -\cos\pi - (-\cos 0) = -(-1) - (-1) = 2$$

$$\int_{\pi}^{2\pi} \sin x \, dx = \Big[-\cos x\Big]_{\pi}^{2\pi} = -\cos 2\pi - (-\cos \pi) =$$

$$= -1 - (-(-1)) = -2$$

$$\int_{0}^{2\pi} \sin x \, dx = \Big[-\cos x\Big]_{0}^{2\pi} = (-\cos 2\pi) - (-\cos 0) =$$

$$= -1 - (-1) = 0$$

Pl. $f(x) = x^3 - x$ fv. görbeje és az x fv. által határolt síkidom területe a $[-1/2; 2]$-on?

Zérushelyek:

$$(x^3 - x) = 0$$

$$(x^2 - 1) \cdot x = 0$$

$x_1 = 0$ $\quad$ $x_2 = 1$ $\quad$ $x_3 = -1$

$$\text{I. } \int_{-\frac{1}{2}}^{0} x^3 - x \, dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-\frac{1}{2}}^{0} =$$

$$= 0 - \left(\frac{\frac{1}{16}}{4} - \frac{\frac{1}{4}}{2}\right) = -\left(\frac{1}{64} - \frac{1}{8}\right) =$$

$$= -\left(\frac{1-8}{64}\right) = \boxed{-\frac{7}{64}}$$

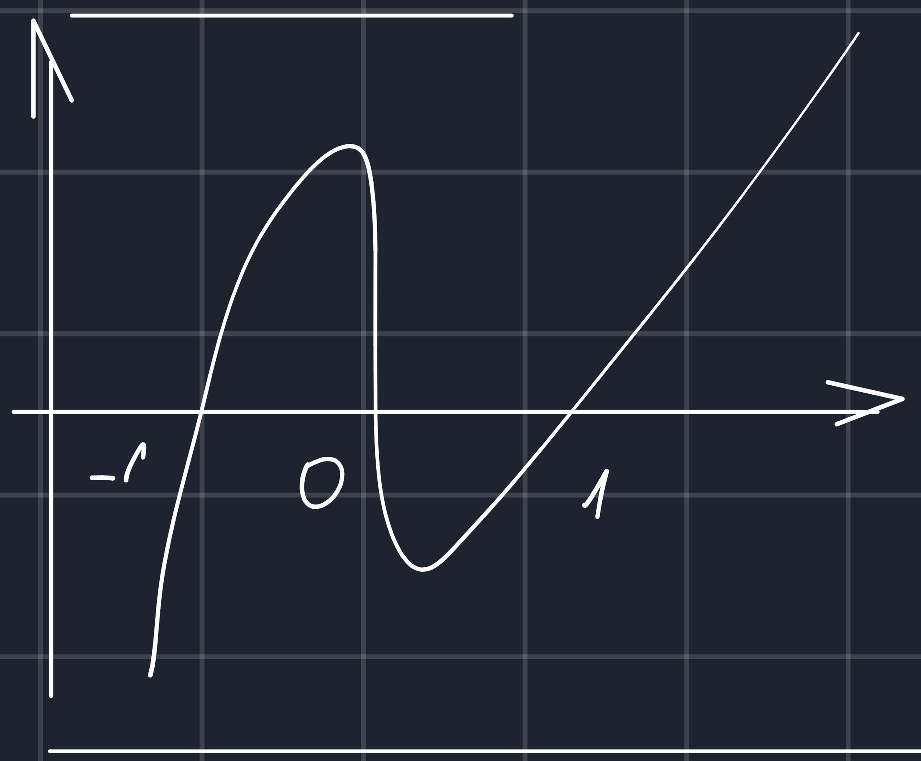
II. $$\int_0^1 x^3 - x\,dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 = \left(\frac{1}{4} - \frac{1}{2}\right) - 0 =$$

$$= \boxed{-\frac{1}{4}}$$

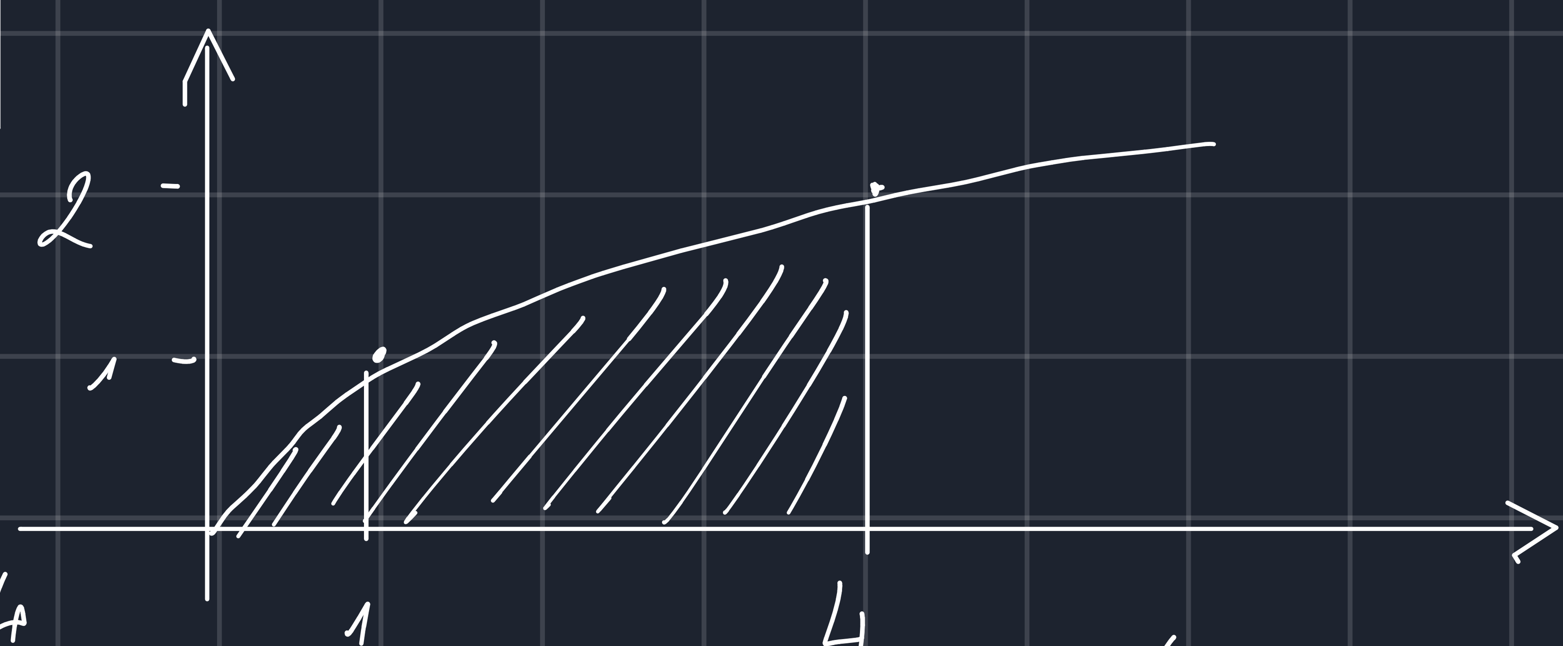
III. $$\int_1^2 x^3 - x\,dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_1^2 = \frac{16}{4} - \frac{4}{2} - \left(\frac{1}{4} - \frac{1}{2}\right) =$$

$$= 2 - \left(-\frac{1}{4}\right) = \boxed{\frac{9}{4}}$$

Válasz: $\frac{7}{64} + \left|-\frac{1}{4}\right| + \frac{9}{4} = \frac{7+160}{64} = \frac{167}{64}$



$f(x) = \sqrt{y} \quad x \in [0,4]$



$$\int_0^4 \sqrt{x}\,dx = \int_0^4 x^{\frac{1}{2}}dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4 = \frac{2}{3} \cdot \sqrt{4}^3 - \frac{2}{3}\sqrt{0}^3$$

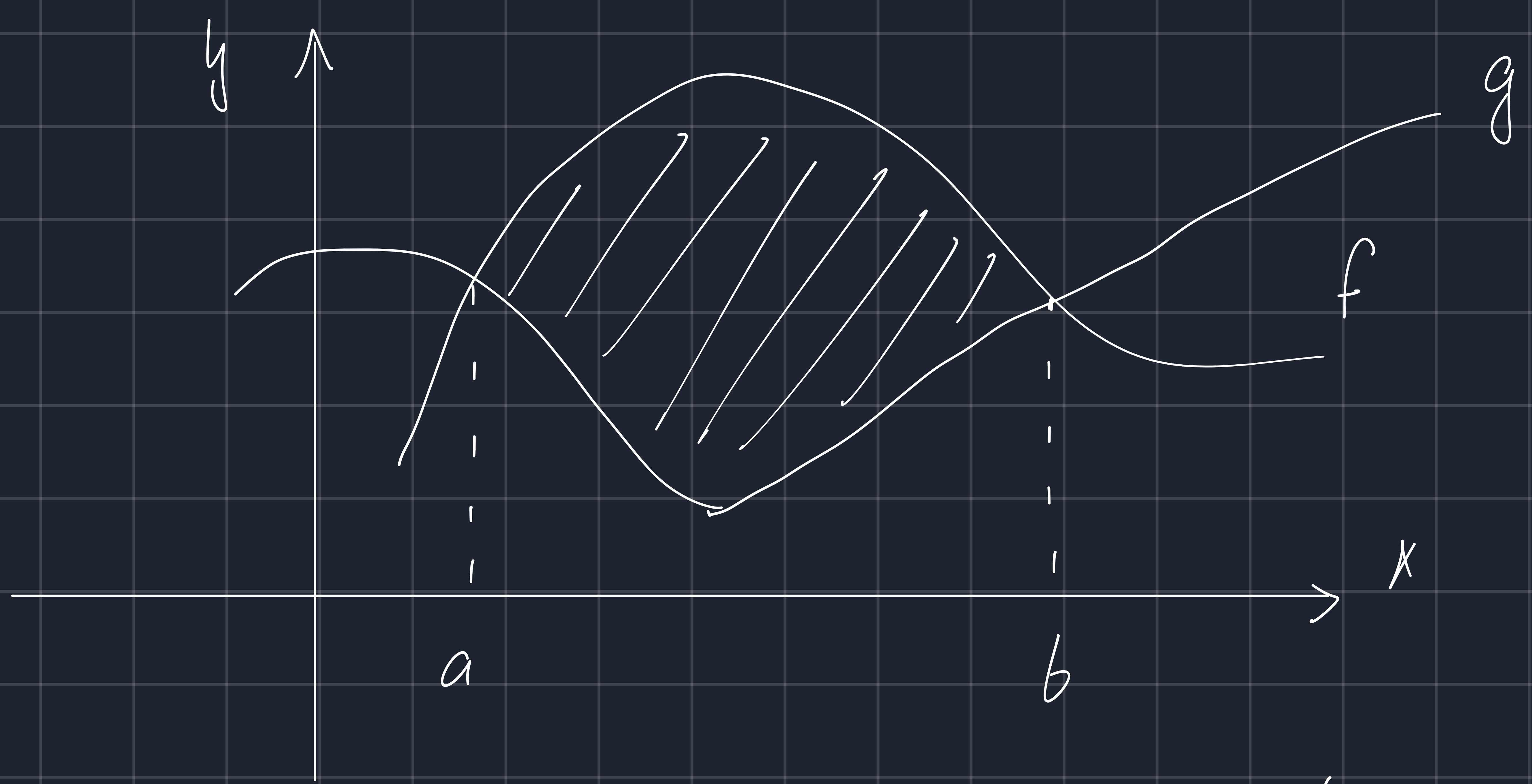
$$= \frac{2}{3} \cdot 2^3 - 0 = \frac{16}{3}$$

$$f(x) = \frac{1}{\sqrt{x-1}}$$ alatti terület $x \in [2;5]$

$$\int_2^5 \frac{1}{\sqrt{x-1}}\,dx = \Big[2\sqrt{x-1}\Big]_2^5 = 2\cdot\sqrt{5-1} - 2\sqrt{2-1} = 2$$

$$\int \frac{1}{\sqrt{x-1}}\,dx = \int (x-1)^{-1/2}\,dx = 2\sqrt{x-1} + C$$

Függvények görbei közötti tartomány területe



$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$

$$\int_a^b f(x) + \cancel{10^{100}} - \left(g(x) + \cancel{10^{100}}\right)\,dx$$

Pl. $g(x) = 4 - x^2 \quad f(x) = x+2$

függvények görbei közötti területe?

Metszéspontok: $4 - x^2 = x + 2$

$$x_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2}$$

$$x^2 + x - 2 = 0$$

$x_1 = 1 \quad x_2 = -2$

$$T = \left| \int_{-2}^{1} x+2-(4-x^2)\,dx \right| = \left| \int_{-2}^{1} x^2 + x - 2\,dx \right| =$$

$$= \left| \left[\frac{x^2}{2} - 2x + \frac{x^3}{3} \right]_{-2}^{1} \right| = \left| \frac{1}{2} - 2 \cdot 1 + \frac{1^3}{3} - \left(\frac{(-2)^2}{2} - 2(-2) + \frac{(-2)^3}{3} \right) \right| = \frac{9}{2}$$