β_* RELATION ON LATTICES

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ABSTRACT. In this paper, we generalize β^* relation on submodules of a module (see [2]) to elements of a complete modular lattice. Let L be a complete modular lattice. We say $a, b \in L$ are β_* *equivalent*, $a\beta_* b$, if and only if for each $t \in L$ such that $a \vee t = 1$ then $b \vee t = 1$ and for each $k \in L$ such that $b \vee k = 1$ then $a \vee k = 1$, this is equivalent to $a \vee b \ll 1/a$ and $a \vee b \ll 1/b$. We show that the β_* relation is an equivalence relation. Then, we examine β_* relation on weakly supplemented lattices. Finally, we show that L is weakly supplemented if and only if for every $x \in L$, x is equivalent to a weak supplement in L.

1. Introduction

Throughout this paper, L denotes a complete modular lattice with the smallest element 0 and the greatest element 1. A lattice we will mean a complete modular lattice. In a lattice L, an element $1 \neq m \in L$ is called *maximal* in L if there is no element between m and 1. An element a of L called *small* in L, if $a \vee b \neq 1$ holds for every $b \neq 1$. This is denoted by $a \ll L$. An element c of L is called a *supplement* of b in L if it is minimal relative to the property $b \vee c = 1$. Equivalently, an element c is a supplement of b in L if and only if $b \vee c = 1$ and $b \wedge c \ll c/0$. An element c of L is called a *weak supplement* of b in L if $b \vee c = 1$ and $b \wedge c \ll L$. A lattice L is called *supplemented* (respectively, *weakly supplemented*) if each element of L has a supplement (respectively, weak supplement) in L. For $a \in L$, we said that $b \in L$ is a *complement* of a in L if $a \wedge b = 0$ and $a \vee b = 1$ (see [4]). It is denoted by $a \oplus b = 1$ (see [3]). A lattice L is called *complemented* if each element in L has at least one complement in L (see [3]). A lattice L is called *hollow* if every element with distinct from 1 small in L. An element a of L has *ample supplements* in L if for every $t \in L$ with $a \vee t = 1$, there is a supplement $t^{'}$ of a with $t^{'} \leq t$. L is called *amply supplemented* if all elements of L have ample supplements in L. In a lattice L, the meet of all maximal elements in L is called *radical* of L, denoted by $rad(L)$. If $a \in L$ such that $a \ll L$ then

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$a \leq rad(L)$ (see [5], Proposition 6). For $a, b \in L$ such that $a \leq b$, we said that b *lies above* a if $b \ll 1/a$. A lattice L is called *distributive* if for any elements a, b, c of L, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ holds.

2. β_* Relation

Definition 1. Let a, b be elements of L. We define a relation β_* on the elements of L by $a\beta_* b$ if and only if for each $t \in L$ such that $a \vee t = 1$ then $b \vee t = 1$ and for each $k \in L$ such that $b \vee k = 1$ then $a \vee k = 1$.

Lemma 1. *β_* is an equivalence relation.*

Proof. The reflexive and symmetric properties are clear. For transitivity, assume $a\beta_* b$ and $b\beta_* c$. Let $t \in L$ such that $a \vee t = 1$. Since $a\beta_* b$, $b \vee t = 1$. So, by $b\beta_* c$, $c \vee t = 1$. Similarly, for each $k \in L$ such that $c \vee k = 1$ then $a \vee k = 1$. Finally $a\beta_* c$.

□

Theorem 1. *Let a, b be elements of L. Then,*

(1) *$a\beta_* b$ if and only if $a \vee c = 1$ and $b \vee c = 1$ for each $c \in L$ such that $a \vee b \vee c = 1$.*
(2) *$a\beta_* b$ if and only if $a \vee b \ll 1/a$ and $a \vee b \ll 1/b$.*

Proof. (1) ($\Rightarrow$) Let $a\beta_* b$ and $c \in L$ such that $a \vee b \vee c = 1$. Since $a \vee (b \vee c) = 1$ and $a\beta_* b$, $b \vee (b \vee c) = 1$. Hence $b \vee c = 1$. Similarly, $a \vee c = 1$.
($\Leftarrow$) Let $t \in L$ such that $a \vee t = 1$. Then $a \vee b \vee t = 1$. By hypothesis, $b \vee t = 1$. Similarly, if $k \in L$ with $b \vee k = 1$ then $a \vee k = 1$. So $a\beta_* b$.

(2) ($\Rightarrow$) Let $a\beta_* b$ and $t \in 1/a$ such that $a \vee b \vee t = 1$. Since $a\beta_* b$, $a \vee t = 1$. Then $t = 1$. Therefore $a \vee b \ll 1/a$. Similarly, $a \vee b \ll 1/b$.
($\Leftarrow$) Let $a \vee b \ll 1/a$, $a \vee b \ll 1/b$ and $t \in L$ such that $a \vee t = 1$. So $(a \vee b) \vee (b \vee t) = a \vee b \vee t = 1$. Since $a \vee b \ll 1/b$ and $b \vee t \in 1/b$, $b \vee t = 1$. Similarly, for each $k \in L$ such that $b \vee k = 1$ then $a \vee k = 1$.

□

Theorem 2. *Let a, b be elements of L. Then,*

(1) *If $a \ll L$ and $a\beta_* b$ then $b \ll L$.*
(2) *All small elements in L is equivalent with β_* equivalence relation.*

Proof. (1) Let $a\beta_* b$, $a \ll L$ and $t \in L$ such that $b \vee t = 1$. Hence $a \vee t = 1$. Since $a \ll L$, $t = 1$. Thus $b \ll L$.

(2) Let $a \ll L$ and $b \ll L$ for $a, b \in L$. Since $a \ll L$, if $a \vee t = 1$ then $t = 1$. Therefore $b \vee t = 1$. Similarly, $a \vee k = 1$ for each $k \in L$ such that $b \vee k = 1$. Thus $a\beta_* b$.

□

Corollary 1. *L is hollow if and only if all elements with distinct from 1 in L are equivalent with β_* relation.*

Proof. ($\Rightarrow$) Let L be hollow. Then all elements of L with distinct from 1 are small in L. Then by Theorem 2 (2), all elements with distinct from 1 in L are equivalent with β_* relation.
($\Leftarrow$) Let all elements of L with distinct from 1 be equivalent to each other. Let $a, t \in L$, $a \neq 1$ and $a \vee t = 1$. If $t \neq 1$, then by hyphothesis $a\beta_* t$ and $t = t \vee t = 1$. This is a contradiction. Hence $t = 1$ and $a \ll L$. Therefore L is hollow. □

Theorem 3. *Let a, b be elements of L such that $a \leq b$. If b lies above a, then $a\beta_* b$.*

Proof. Assume b lies above a. Then, $b \ll 1/a$. Since $a \leq b$ for any $t \in L$ such that $a \vee t = 1$, $b \vee t = 1$. Conversely, let $k \in L$ with $b \vee k = 1$. Then $b \vee a \vee k = 1$. Since $b \ll 1/a$ and $a \vee k \in 1/a$, $a \vee k = 1$. Hence $a\beta_* b$.

□

Lemma 2. *Let a, b, c be elements of L. If $a \vee b = 1$ and $(a \wedge b) \vee c = 1$, then $a \vee (b \wedge c) = b \vee (a \wedge c) = 1$.*

Proof. Assume $a \vee b = 1$ and $(a \wedge b) \vee c = 1$. Since $(a \wedge b) \vee c = 1$, $a = a \wedge 1 = a \wedge [(a \wedge b) \vee c] = (a \wedge b) \vee (a \wedge c)$. Then $1 = a \vee b = (a \wedge b) \vee (a \wedge c) \vee b = b \vee (a \wedge c)$. Similarly $a \vee (b \wedge c) = 1$. □

Theorem 4. *Let $a, b \in L$. If $a\beta_* b$ then the following conditions hold.*

(1) *If there exist supplements of a and b then these are the same.*
(2) *If there exist weak supplements of a and b then these are the same.*

Proof.
(1) Let c be a supplement of a. Then $a \vee c = 1$. Since $a\beta_* b$, $b \vee c = 1$. Let $d \in L$ such that $d \leq c$ and $b \vee d = 1$. Therefore $a \vee d = 1$. Since c is a supplement of a and $d \leq c$, $d = c$. Then c is a supplement of b. Similarly, interchanging the roles of a and b we can show that each supplement of b is also a supplement of a.
(2) Let $a\beta_* b$ and c be a weak supplement of a in L. Therefore $a \vee c = 1$ and $a \wedge c \ll L$. Since $a\beta_* b$ and $a \vee c = 1$, $b \vee c = 1$. Let t be an element of L such that $(b \wedge c) \vee t = 1$. By Lemma 2, $b \vee (c \wedge t) = 1$ and since $a\beta_* b$, $a \vee (c \wedge t) = 1$. Then by also Lemma 2, $(a \wedge c) \vee t = 1$ and since $a \wedge c \ll L$, $t = 1$. Therefore $b \wedge c \ll L$ and so c is also a weak supplement of b. Similarly, interchanging the roles of a and b we can show that each weak supplement of b is also a weak supplement of a.

□

Theorem 5. *Let L be an amply supplemented lattice and $a, b \in L$. If supplements of a and b in L are the same then $a\beta_* b$.*

Proof. Let $t \in L$ such that $a \vee t = 1$. Since L is amply supplemented, there exists a supplement r of a in L such that $r \leq t$. By the hypothesis, r is also a supplement of b. Then $b \vee r = 1$. Since $r \leq t$, $b \vee t = 1$. Similarly, interchanging the roles of a and b we can show that $a \vee k = 1$ for any element k of L such that $b \vee k = 1$. Therefore $a\beta_* b$. □

Corollary 2. *Let $x, y, c \in L$ such that $x \leq y$ and c is a weak supplement of x in L. Then $x\beta_* y$ if and only if $y \wedge c \ll L$.*

Proof. ($\Rightarrow$): Clear from Theorem 4 (2).
($\Leftarrow$): Since $x \leq y$, for any element t of L such that $x \vee t = 1$, $y \vee t = 1$. Let $k \in L$ such that $y \vee k = 1$. Since c is weak supplement of x in L, $x \vee c = 1$ and $x \wedge c \ll L$. Therefore $y \wedge (x \vee c) = 1 \wedge y$, and so $y = x \vee (y \wedge c)$. Hence $x \vee (y \wedge c) \vee k = 1$. Since $y \wedge c \ll L$, $x \vee k = 1$. Thus $x\beta_* y$. □

Theorem 6. *Let $x, y, z, a, b \in L$ such that $a \oplus b = 1$ and y is a supplement of x in L. Then*

(1) *If $z\beta_* y$ then $z/(z \wedge x) \cong y/(y \wedge x)$.*
(2) *If $z\beta_* b$ then $z/(z \wedge a) \cong b/0$.*
(3) *Let $z \leq b$. Then $z\beta_* b$ if and only if $z = b$.*
(4) *Let $b \leq z$. Then $z\beta_* b$ if and only if $z \wedge a \ll L$.*

Proof. (1) Since y is a supplement of x in L and $z\beta_* y$, $x \vee y = x \vee z = 1$. Since $z/(z \wedge x) \cong (z \vee x)/x$ and $(y \vee x)/x \cong y/(y \wedge x)$, $z/(z \wedge x) \cong y/(y \wedge x)$. Thus $z/(z \wedge x) \cong y/(y \wedge x)$.
(2) By (1), $z/(z \wedge a) \cong b/(a \wedge b)$. Since $a \oplus b = 1$, $z/(z \wedge a) \cong b/0$.
(3) ($\Rightarrow$): Since $a \oplus b = 1$ and $z\beta_* b$, $a \vee z = 1$. Also, since b is a supplement of a in L and $z \leq b$, $z = b$.
($\Leftarrow$) Clear from reflexive property of β_*.
(4) ($\Rightarrow$): Since a is a weak supplement of b and $z\beta_* b$, it follows from Theorem 4 (2) that a is also a weak supplement of z in L. Hence $z \wedge a \ll L$.
($\Leftarrow$) It is clear from Corollary 2.

□

Theorem 7. *Let L be a distributive lattice and $a, b \in L$. If $a \oplus b = 1$ and $a\beta_* x$, then $a \leq x$ and $b \wedge x \ll L$.*

Proof. Since $a \oplus b = 1$ and $a\beta_* x$, $x \vee b = 1$. Hence $a \wedge (x \vee b) = a \wedge 1 = a$. By the distributive property, $(a \wedge x) \vee (a \wedge b) = a$. Since $a \wedge b = 0$, $a \wedge x = a$, and so $a \leq x$. Also since $x\beta_* a$ and $a \leq x$, $b \wedge x \ll L$ by Theorem 6 (4) . □

Theorem 8. *Let L be a distributive lattice and $x \in L$. If $x\beta_* y$ and there exists a decomposition $a \oplus b = 1$ such that $a \leq x$ and $b \wedge x \ll L$, then $a \leq y$ and $b \wedge y \ll L$.*

Proof. Since $a \leq x$ and $b \wedge x \ll L$, $a\beta_* x$ by Theorem 6 (4). Since $x\beta_* y$ and $a\beta_* x$, $a\beta_* y$. By Theorem 7, $a \leq y$ and $b \wedge y \ll L$. □

Theorem 9. *Let $x \in L$ and k be a maximal element of L.*

1. *If $a, b \in L$ such that $a \vee b = 1$, $b \neq 1$ and $x\beta_* a$ then $x \nleq b$.*
2. *If $x\beta_* y$ and $x \leq k$ then $y \leq k$.*
3. *If $x\beta_* k$ then $x \leq k$.*
4. *If $x\beta_* k$ and w is a weak supplement of x in L then $k = x \vee (k \wedge x)$ and $k \wedge w \ll L$.*

Proof.
(1) Assume that $x \leq b$. Since $a \vee b = 1$ and $x\beta_* a$, $x \vee b = 1$ so $b = 1$. This is a contradiction with b is distinct from 1. Therefore $x \nleq b$.
(2) Let $y \nleq k$. Then $k \vee y = 1$. Since $x\beta_* y$, $k = k \vee x = 1$. This is a contradiction. Therefore $y \leq k$.
(3) Let $x\beta_* k$ and $x \nleq k$. Since k is a maximal element, $x \vee k = 1$. Moreover $k = 1$, because $x\beta_* k$. This is a contradiction. Therefore $x \leq k$.
(4) Let $x\beta_* k$ and w be a weak supplement of x in L. From Theorem 4 (2), w is a weak supplement of k in L. Hence $k \vee w = 1$ and $k \wedge w \ll L$. Since $x\beta_* k$, $x \leq k$ by (3). Since $x \vee w = 1$ and $x \leq k$, the modular law yields $k = x \vee (k \wedge w)$.

□

Theorem 10. *Let $a, b \in L$ and $a \oplus b = 1$. For $x, s \in a/0$, if $x\beta_* s$ in L, then $x\beta_* s$ in $a/0$.*

Proof. Let $k \in a/0$ such that $x \vee k = a$. Then $(x \vee k) \oplus b = 1$, and so $x \vee (k \vee b) = 1$. Hence $s \vee (k \vee b) = 1$ since $x\beta_* s$ in L. Then $[(s \vee k) \vee b] \wedge a = a$, it follows that $(s \vee k) \vee (b \wedge a) = a$. We obtain that $s \vee k = a$. Similarly, interchanging roles of x and s, we can show that $x \vee t = a$ for any element t of $a/0$ such that $s \vee t = a$. Therefore $x\beta_* s$ in $a/0$. □

Theorem 11. *Let $x, y, k \in L$ such that $x \vee k = y \vee k = 1$, $k \wedge y \leq k \wedge x$ and $x \vee y \ll 1/y$. Then $x \vee y \ll 1/x$.*

Proof. Let $t \in 1/x$ such that $(x \vee y) \vee t = 1$. Since $x \vee k = 1$, $t \wedge (x \vee k) = t \wedge 1$, and so $x \vee (t \wedge k) = t$ by the modular law. Hence $x \vee y \vee (t \wedge k) = 1$. It follows that $x \vee y \vee [y \vee (t \wedge k)] = 1$. Since $x \vee y \ll 1/y$, $y \vee (t \wedge k) = 1$. Then by Lemma 2, $t \vee (k \wedge y) = 1$. Since $k \wedge y \leq k \wedge x$, $1 = t \vee (k \wedge y) = t \vee (k \wedge x) = t$. Therefore $x \vee y \ll 1/x$. □

Theorem 12. *Let $x, y, a, b \in L$. If $a, b \ll L, x \leq y \vee b$ and $y \leq x \vee a$, then $x\beta_* y$.*

Proof. Let $k \in L$ such that $x \vee y \vee k = 1$. Since $x \leq y \vee b$, $y \vee b \vee k = 1$. From $b \ll L$, $y \vee k = 1$. Similarly, $x \vee k = 1$. Hence, by Theorem 1 (1), $x\beta_* y$. □

Theorem 13. *Let $x_1, x_2, y_1, y_2 \in L$. If $x_1\beta_* y_1$ and $x_2\beta_* y_2$ then $(x_1 \vee x_2)\beta_*(y_1 \vee y_2)$.*

Proof. Let $k \in L$ such that $(x_1 \vee x_2) \vee (y_1 \vee y_2) \vee k = 1$. Since $x_1\beta_* y_1$, $y_1 \vee x_2 \vee y_2 \vee k = 1$ and $x_1 \vee x_2 \vee y_2 \vee k = 1$. Also since $x_2\beta_* y_2$, $y_1 \vee y_2 \vee k = 1$ and $x_1 \vee x_2 \vee k = 1$. By Theorem 1 (1), $(x_1 \vee x_2)\beta_*(y_1 \vee y_2)$. □

Corollary 3. *Let $x, y_1, y_2, ..., y_n \in L$. If $x\beta_* y_i$ for $i = 1, 2, ..., n$, then $x\beta_* \bigvee_{i=1}^{n} y_i$.*

Proof. Clear from Theorem 13. □

Theorem 14. *Let $x, y \in L$ and $j \ll L$. Then $x\beta_* y$ if and only if $x\beta_*(y \vee j)$.*

Proof. ($\Rightarrow$) Let $k \in L$ with $x \vee k = 1$. Since $x\beta_* y$, $y \vee k = 1$. Then $y \vee j \vee k = 1$. Let $t \in L$ such that $(y \vee j) \vee t = 1$. Since $j \ll L$, $y \vee t = 1$. $x \vee t = 1$ from $x\beta_* y$. Hence $x\beta_*(y \vee j)$.
($\Leftarrow$) Let $k \in L$ with $x \vee k = 1$. Since $x\beta_*(y \vee j)$, $y \vee j \vee k = 1$. Also, since $j \ll L$, $y \vee k = 1$. Let $t \in L$ such that $y \vee t = 1$. Then $y \vee j \vee t = 1$. Since $x\beta_*(y \vee j)$, $x \vee t = 1$. Hence $x\beta_* y$.

□

Theorem 15. *Let $rad(L) = 0$ and $a \oplus b = 1$. If $x\beta_* a$ for some $x \in L$ then $x \oplus b = 1$.*

Proof. Since $a \oplus b = 1$, b is a supplement of a in L. Since $x\beta_* a$, b is also a supplement of x in L by Theorem 4 (1). Therefore $b \vee x = 1$ and $b \wedge x \ll x/0$. Since $rad(L) = 0$, $x \wedge b \leq rad(L) = 0$. Hence $x \oplus b = 1$. □

Theorem 16. *L is weakly supplemented if and only if for every $x \in L$, x is β_* equivalent to a weak supplement in L.*

Proof. ($\Rightarrow$) Let $x \in L$. Since L is weakly supplemented, there exists $z \in L$ such that $x \vee z = 1$ and $x \wedge z \ll L$. Also x is a weak supplement of z in L. Since β_* relation is reflexive, $x\beta_* x$. So, every element of L is β_* equivalent to a weak supplement element in L.
($\Leftarrow$) Let $x \in L$. By the hypothesis, there exists a weak supplement $z \in L$ such that $x\beta_* z$. Let z be a weak supplement of a in L. Thus $a \vee z = 1$ and $a \wedge z \ll L$. Also, a is a weak supplement element of z in L. Since $x\beta_* z$, a is also a weak supplement of x in L by Theorem 4 (2). □

Remark 1. The converse of Theorem 3 is not always true. We can give an example about that. Let K be a hollow module which is not simple. L be the lattice of the set of all submodules of $K \times K$ with respect to the ordering relation of inclusion. Since K is not simple, K has a submodule T with $T \neq 0$ and $T \ll K$. Clearly we see that $T \times 0 \ll L$ and $0 \times T \ll L$. But $T \times 0$ and $0 \times T$ don't lie above each other.

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