

Type II Error

In hypothesis testing, a type II error is due to a failure of rejecting an invalid null hypothesis.

The probability of avoiding a type II error is called the power of the hypothesis test, and is denoted by the quantity $1 - \beta$.

In the following tutorials, we demonstrate how to compute the power of a hypothesis test based on scenarios from our previous discussions on hypothesis testing.

The approach is based on a parametric estimate of the region where the null hypothesis would not be rejected.

The probability of a type II error is then derived based on a hypothetical true value.

Type II Error in Lower Tail Test of Population Mean with Known Variance

In a lower tail test of the population mean, the null hypothesis claims that the true population mean μ is greater than a given hypothetical value μ_0 .

$$\mu \geq \mu_0$$

A type II error occurs if the hypothesis test based on a random sample fails to reject the null hypothesis even when the true population mean μ is in fact less than μ_0 .

Assume that the population has a known variance σ^2 . By the Central Limit Theorem, the population of all possible means of samples of sufficiently large size n approximately follows the normal distribution. Hence we can compute the range of sample means for which the null hypothesis will not be rejected, and then obtain an estimate of the probability of type II error.

We demonstrate the procedure with the following:

Problem

Suppose the manufacturer claims that the mean lifetime of a light bulb is more than 10,000 hours. Assume actual mean light bulb lifetime is 9,950 hours and the population standard deviation is 120 hours. At .05 significance level, what is the probability of having type II error for a sample size of 30 light bulb?

Solution

We begin with computing the standard deviation of the mean, sem.

```
> n = 30                # sample size
> sigma = 120           # population standard deviation
> sem = sigma/sqrt(n); sem  # standard error
[1] 21.909
```

We next compute the lower bound of sample means for which the null hypothesis $\mu \geq 10000$ would not be rejected.

```
> alpha = .05           # significance level
> mu0 = 10000           # hypothetical lower bound
> q = qnorm(alpha, mean=mu0, sd=sem); q
[1] 9964
```

Therefore, so long as the sample mean is greater than 9964 in a hypothesis test, the null hypothesis will not be rejected.

Since we assume that the actual population mean is 9950, we can compute the probability of the sample mean above 9964, and thus found the probability of type II error.

```
> mu = 9950 # assumed actual mean
> pnorm(q, mean=mu, sd=sem, lower.tail=FALSE)
[1] 0.26196
```

Answer

If the light bulbs sample size is 30, the actual mean light bulb lifetime is 9,950 hours and the population standard deviation is 120 hours, then the probability of type II error for testing the null hypothesis $\mu \geq 10000$ at .05 significance level is 26.2%, and the power of the hypothesis test is 73.8%.

Exercise

Under same assumptions as above, if the actual mean light bulb lifetime is 9,965 hours, what is the probability of type II error at .05 significance level? What is the power of the hypothesis test?

Type II Error in Upper Tail Test of Population Mean with Known Variance

In an upper tail test of the population mean, the null hypothesis claims that the true population mean μ is less than a given hypothetical value μ_0 .

$$\mu \leq \mu_0$$

A type II error occurs if the hypothesis test based on a random sample fails to reject the null hypothesis even when the true population mean μ is in fact greater than μ_0 .

Assume that the population has a known variance σ^2 . By the Central Limit Theorem, the population of all possible means of samples of sufficiently large size n approximately follows the normal distribution.

Hence we can compute the range of sample means for which the null hypothesis will not be rejected, and then obtain an estimate of the probability of type II error.

We demonstrate the procedure with the following:

Problem

Suppose the food label on a cookie bag states that there is at most 2 grams of saturated fat in a single cookie. Assume the actual mean amount of saturated fat per cookie is 2.09 grams, and the population standard deviation is 0.25 grams. At .05 significance level, what is the probability of having type II error for a sample size of 35 cookies?

Solution

We begin with computing the standard deviation of the mean, sem .

```
> n = 35                                # sample size
> sigma = 0.25                          # population standard deviation
> sem = sigma/sqrt(n); sem              # standard error
[1] 0.042258
```

We next compute the upper bound of sample means for which the null hypothesis $\mu \leq 2$ would not be rejected.

```
> alpha = .05                                # significance level
> mu0 = 2                                     # hypothetical upper bound
> q = qnorm(alpha, mean=mu0, sd=sem, lower.tail=FALSE); q
[1] 2.0695
```

Therefore, so long as the sample mean is less than 2.0695 in a hypothesis test, the null hypothesis will not be rejected. Since we assume that the actual population mean is 2.09, we can compute the probability of the sample mean below 2.0695, and thus found the probability of type II error.

```
> mu = 2.09                                  # assumed actual mean
> pnorm(q, mean=mu, sd=sem)
[1] 0.31386
```

Answer

If the cookies sample size is 35, the actual mean amount of saturated fat per cookie is 2.09 grams and the population standard deviation is 0.25 grams, then the probability of type II error for testing the null hypothesis $\mu \leq 2$ at .05 significance level is 31.4%, and the power of the hypothesis test is 68.6%.

Exercise

Under same assumptions as above, if the actual mean amount of saturated fat per cookie is 2.075 grams, what is the probability of type II errors? What is the power of the hypothesis test?

Type II Error in Two-Tailed Test of Population Mean with Known Variance

In a two-tailed test of the population mean, the null hypothesis claims that the true population mean μ is equal to a given hypothetical value μ_0 .

$$\mu = \mu_0$$

A type II error occurs if the hypothesis test based on a random sample fails to reject the null hypothesis even when the true population mean μ is in fact different from μ_0 .

Assume that the population has a known variance σ^2 . By the Central Limit Theorem, the population of all possible means of samples of sufficiently large size n approximately follows the normal distribution.

Hence we can compute the range of sample means for which the null hypothesis will not be rejected, and then obtain an estimate of the probability of type II error.

We demonstrate the procedure with the following:

Problem

Suppose the mean weight of King Penguins found in an Antarctic colony last year was 15.4 kg. Assume the actual mean population weight is 15.1 kg, and the population standard deviation is 2.5 kg. At .05 significance level, what is the probability of having type II error for a sample size of 35 penguins?

Solution

We begin with computing the standard deviation of the mean, sem.

```
> n = 35                                # sample size
> sigma = 2.5                           # population standard deviation
> sem = sigma/sqrt(n); sem              # standard error
[1] 0.42258
```

We next compute the lower and upper bounds of sample means for which the null hypothesis $\mu = 15.4$ would not be rejected.

```
> alpha = .05                # significance level
> mu0 = 15.4                  # hypothetical mean
> I = c(alpha/2, 1-alpha/2)
> q = qnorm(I, mean=mu0, sd=sem); q
[1] 14.572 16.228
```

Therefore, so long as the sample mean is between 14.572 and 16.228 in a hypothesis test, the null hypothesis will not be rejected. Since we assume that the actual population mean is 15.1, we can compute the lower tail probabilities of both end points.

```
> mu = 15.1                   # assumed actual mean
> p = pnorm(q, mean=mu, sd=sem); p
[1] 0.10564 0.99621
```

Finally, the probability of type II error is the probability between the two end points.

```
> diff(p)                     # p[2]-p[1]
[1] 0.89056
```

Answer

If the penguin sample size is 35, the actual mean population weight is 15.1 kg and the population standard deviation is 2.5 kg, then the probability of type II error for testing the null hypothesis $\mu = 15.4$ at .05 significance level is 89.1%, and the power of the hypothesis test is 10.9%.

Exercise

Under same assumptions as above, if actual mean population weight is 14.9 kg, what is the probability of type II errors? What is the power of the hypothesis test?

Type II Error in Lower Tail Test of Population Mean with Unknown Variance

In a lower tail test of the population mean, the null hypothesis claims that the true population mean μ is greater than a given hypothetical value μ_0 .

$$\mu \geq \mu_0$$

A type II error occurs if the hypothesis test based on a random sample fails to reject the null hypothesis even when the true population mean μ is in fact less than μ_0 .

Let s^2 be the sample variance. For sufficiently large n , the population of the following statistics of all possible samples of size n is approximately a Student t distribution with $n - 1$ degrees of freedom.

$$\frac{\bar{x} - \mu}{s/\sqrt{n}}$$

This allows us to compute the range of sample means for which the null hypothesis will not be rejected, and to obtain the probability of type II error. We demonstrate the procedure with the following:

Problem

Suppose the manufacturer claims that the mean lifetime of a light bulb is more than 10,000 hours. Assume in a random sample of 30 light bulbs, the standard deviation of the lifetime is 125 hours. If actual mean light bulb lifetime is 9,950 hours, what is the probability of type II error for a hypothesis test at .05 significance level?

Solution

We begin with computing the standard error estimate, SE.

```
> n = 30                # sample size
> s = 125               # sample standard deviation
> SE = s/sqrt(n); SE    # standard error estimate
[1] 22.822
```


We next compute the lower bound of sample means for which the null hypothesis $\mu \geq 10000$ would not be rejected.

```
> alpha = .05                                # significance level
> mu0 = 10000                                # hypothetical lower bound
> q = mu0 + qt(alpha, df=n-1) * SE; q
[1] 9961.2
```

Therefore, so long as the sample mean is greater than 9961.2 in a hypothesis test, the null hypothesis will not be rejected. Since we assume that the actual population mean is 9950, we can compute the probability of the sample mean above 9961.2, and thus found the probability of type II error.

```
> mu = 9950                                  # assumed actual mean
> pt((q - mu)/SE, df=n-1, lower.tail=FALSE)
[1] 0.31329
```

Answer

If the light bulbs sample size is 30, the sample standard variance is 125 hours and the actual mean light bulb lifetime is 9,950 hours, then the probability of type II error for testing the null hypothesis $\mu \geq 10000$ at .05 significance level is 31.3%, and the power of the hypothesis test is 68.7%.

Exercise

Under same assumptions as above, if the actual mean light bulb lifetime is 9,965 hours, what is the probability of type II errors? What is the power of the hypothesis test?

Type II Error in Upper Tail Test of Population Mean with Unknown Variance

In an upper tail test of the population mean, the null hypothesis claims that the true population mean μ is less than a given hypothetical value μ_0 .

$$\mu \leq \mu_0$$

A type II error occurs if the hypothesis test based on a random sample fails to reject the null hypothesis even when the true population mean μ is in fact greater than μ_0 .

Let s^2 be the sample variance. For sufficiently large n , the population of the following statistics of all possible samples of size n is approximately a Student t distribution with $n - 1$ degrees of freedom.

$$\frac{\bar{x} - \mu}{s/\sqrt{n}}$$

This allows us to compute the range of sample means for which the null hypothesis will not be rejected, and to obtain the probability of type II error. We demonstrate the procedure with the following:

Problem

Suppose the food label on a cookie bag states that there is at most 2 grams of saturated fat in a single cookie. Assume in a random sample of 35 cookies, the standard deviation of saturated fat is 0.3 grams. If actual mean amount of saturated fat per cookie is 2.09 grams, what is the probability of type II error for a hypothesis test at .05 significance level?

Solution

We begin with computing the standard error estimate, SE.

```
> n = 35                # sample size
> s = 0.3                # sample standard deviation
> SE = s/sqrt(n); SE     # standard error estimate
[1] 0.050709
```

We next compute the upper bound of sample means for which the null hypothesis $\mu \leq 2$ would not be rejected.

```
> alpha = .05                                # significance level
> mu0 = 2                                     # hypothetical upper bound
> q = mu0 + qt(alpha, df=n-1, lower.tail=FALSE) * SE; q
[1] 2.0857
```

Therefore, so long as the sample mean is less than 2.0857 in a hypothesis test, the null hypothesis will not be rejected. Since we assume that the actual population mean is 2.09, we can compute the probability of the sample mean below 2.0857, and thus found the probability of type II error.

```
> mu = 2.09                                # assumed actual mean
> pt((q - mu)/SE, df=n-1)
[1] 0.46681
```

Answer

If the cookies sample size is 35, the sample standard deviation of saturated fat per cookie is 0.3 grams and the actual mean amount of saturated fat per cookie is 2.09 grams, then the probability of type II error for testing the null hypothesis $\mu \leq 2$ at .05 significance level is 46.7%, and the power of the hypothesis test is 53.3%.

Exercise

Under same assumptions as above, if the actual mean saturated fat per cookie is 2.075 grams, what is the probability of type II errors? What is the power of the hypothesis test?

Type II Error in Two-Tailed Test of Population Mean with Unknown Variance

In a two-tailed test of the population mean, the null hypothesis claims that the true population mean μ is equal to a given hypothetical value μ_0 .

$$\mu = \mu_0$$

A type II error occurs if the hypothesis test based on a random sample fails to reject the null hypothesis even when the true population mean μ is in fact different from μ_0 .

Let s^2 be the sample variance. For sufficiently large n , the population of the following statistics of all possible samples of size n is approximately a Student t distribution with $n - 1$ degrees of freedom.

$$\frac{x - \mu}{s/\sqrt{n}}$$

This allows us to compute the range of sample means for which the null hypothesis will not be rejected, and to obtain the probability of type II error. We demonstrate the procedure with the following:

Problem

Suppose the mean weight of King Penguins found in an Antarctic colony last year was 15.4 kg. Assume in a random sample 35 penguins, the standard deviation of the weight is 2.5 kg. If actual mean penguin weight is 15.1 kg, what is the probability of type II error for a hypothesis test at .05 significance level?

Solution

We begin with computing the standard error estimate, SE.

```
> n = 35           # sample size
> s = 2.5          # sample standard deviation
> SE = s/sqrt(n); SE # standard error estimate
[1] 0.42258
```

We next compute the lower and upper bounds of sample means for which the null hypothesis $\mu = 15.4$ would not be rejected.

```
> alpha = .05                # significance level
> mu0 = 15.4                  # hypothetical mean
> I = c(alpha/2, 1-alpha/2)
> q = mu0 + qt(I, df=n-1) * SE; q
[1] 14.541 16.259
```

Therefore, so long as the sample mean is between 14.541 and 16.259 in a hypothesis test, the null hypothesis will not be rejected. Since we assume that the actual population mean is 15.1, we can compute the lower tail probabilities of both end points.

```
> mu = 15.1                   # assumed actual mean
> p = pt((q - mu)/SE, df=n-1); p
[1] 0.097445 0.995168
```

Finally, the probability of type II error is the probability between the two end points.

```
> diff(p)                     # p[2]-p[1]
[1] 0.89772
```

Answer

If the penguin sample size is 35, the sample standard deviation of penguin weight is 2.5 kg and the actual mean population weight is 15.1 kg, then the probability of type II error for testing the null hypothesis $\mu = 15.4$ at .05 significance level is 89.8%, and the power of the hypothesis test is 10.2%.

Exercise

Under same assumptions as above, if actual mean population weight is 14.9 kg, what is the probability of type II errors? What is the power of the hypothesis test?