Ejercicio #7

Con los sistemas de coordenadas no primadas (x,y) y las coordenadas primadas (r,θ), cuyas ecuaciones de transformación son

$$x = r\cos\theta \quad , \quad r = (x^2+y^2)^{1/2}$$

$$y = r\,\text{sen}\,\theta \quad , \quad \theta = \tan^{-1}(y/x)$$

se pide hallar $\Lambda^{\bar{\alpha}}{}_{\beta}$ y $\Lambda^{\mu}{}_{\bar{\nu}}$. Así,

- $P^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_{\beta} P^{\alpha}$

$$P^{\bar{\alpha}} = \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{1}{\cancel{2}} \cdot (x^2+y^2)^{-1/2} \cdot \cancel{2}x$$

$$\frac{\partial r}{\partial y} = \frac{1}{\cancel{2}} \cdot (x^2+y^2)^{-1/2} \cdot \cancel{2}y$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+y^2/x^2} \cdot -\frac{y}{x^2} = -\frac{y}{x^2+y^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+y^2/x^2} \cdot \frac{1}{x} = \frac{1}{x+y^2/x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{(x^2+y^2)^{1/2}} = \frac{x}{r} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \frac{y}{(x^2+y^2)^{1/2}} = \frac{y}{r} = \text{sen}\,\theta$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{y}{r^2} = \frac{y}{r} \cdot -\frac{1}{r} = -\frac{1}{r} \cdot \text{sen}\,\theta$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{x}{r^2} = \frac{x}{r} \cdot \frac{1}{r} = \frac{1}{r} \cdot \cos\theta$$

$$\Lambda^{\bar\alpha}{}_{\beta} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \operatorname{sen}\theta \\ -\frac{\operatorname{sen}\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}$$

$$\Lambda^{r}{}_{x} = \cos\theta \ , \ \Lambda^{r}{}_{y} = \operatorname{sen}\theta \ , \ \Lambda^{\theta}{}_{x} = -\frac{\operatorname{sen}\theta}{r} \ , \ \Lambda^{\theta}{}_{y} = \frac{\cos\theta}{r}$$

- $P^{\mu} = \Lambda^{\mu}{}_{\bar\nu} P^{\bar\nu}$

$$\frac{\partial x}{\partial r} = \cos\theta \ , \ \frac{\partial y}{\partial r} = \operatorname{sen}\theta$$

$$P^{\mu} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \begin{pmatrix} r \\ \theta \end{pmatrix} \Rightarrow \frac{\partial x}{\partial \theta} = -r\operatorname{sen}\theta \ , \ \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\Lambda^{\mu}{}_{\bar\nu} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\operatorname{sen}\theta \\ \operatorname{sen}\theta & r\cos\theta \end{pmatrix}$$

$$\Lambda^{x}{}_{r} = \cos\theta \ , \ \Lambda^{y}{}_{r} = \operatorname{sen}\theta \ , \ \Lambda^{x}{}_{\theta} = -r\operatorname{sen}\theta \ , \ \Lambda^{y}{}_{\theta} = r\cos\theta$$

Vemos que,

$$\Lambda^{\bar\alpha}{}_{\beta} \Lambda^{\beta}{}_{\bar\nu} = \begin{pmatrix} \cos\theta & \operatorname{sen}\theta \\ -\frac{\operatorname{sen}\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} \cos\theta & -r\operatorname{sen}\theta \\ \operatorname{sen}\theta & r\cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Lambda \Lambda^{-1} = I$, como corresponde.

Ejercicio #8

Siendo $f = x^2 + y^2 + 2xy$, $\vec{V} = (x^2 + 3y, y^2 + 3x)$ y $\vec{W} = (1, 1)$

con base en el ejercicio #7,

A) Hallar $f(r,\theta)$, $\vec{V}(r,\theta)$ y $\vec{W}(r,\theta)$.

- $f(x,y) = x^2 + 2xy + y^2$

$$f(r,\theta) = r^2\cos^2\theta + 2r^2\cos\theta\,\text{sen}\,\theta + r^2\text{sen}^2\theta$$

$$f(r,\theta) = r^2(\cos^2\theta + 2\cos\theta\,\text{sen}\,\theta + \text{sen}^2\theta)$$

$$f(r,\theta) = r^2(2\cos\theta\,\text{sen}\,\theta + 1)$$

$$f(r,\theta) = r^2(\text{sen}(2\theta) + 1)_{//}$$

- $V^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\beta} V^{\beta}$

$$V^{\bar{\alpha}} = \begin{pmatrix} \cos\theta & \text{sen}\,\theta \\ -\frac{\text{sen}\,\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} x^2 + 3y \\ y^2 + 3x \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \text{sen}\,\theta \\ -\frac{\text{sen}\,\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} r^2\cos^2\theta + 3r\,\text{sen}\,\theta \\ r^2\text{sen}^2\theta + 3r\cos\theta \end{pmatrix}$$

$$\Rightarrow \cos\theta(r^2\cos^2\theta + 3r\,\text{sen}\,\theta) + \text{sen}\,\theta(r^2\text{sen}^2\theta + 3r\cos\theta)$$

$$-\text{sen}\,\theta/r\,(r^2\cos^2\theta + 3r\,\text{sen}\,\theta) + \cos\theta/r\,(r^2\text{sen}^2\theta + 3r\cos\theta)$$

$$\Rightarrow r^2\cos^3\theta + 3r\cos\theta\,\text{sen}\,\theta + r^2\text{sen}^3\theta + 3r\cos\theta\,\text{sen}\,\theta$$

$$-r\cos^2\theta\,\text{sen}\,\theta - 3\,\text{sen}^2\theta + r\,\text{sen}^2\theta\cos\theta + 3\cos^2\theta$$

$$V^{\bar{\alpha}} = \begin{pmatrix} r^2(\operatorname{sen}^3\theta + \cos^3\theta) + 3r\operatorname{sen}(2\theta) \\ \frac{r}{2}\operatorname{sen}(2\theta)(\operatorname{sen}\theta - \cos\theta) + 3\cos(2\theta) \end{pmatrix} //$$

- $W^{\bar{\alpha}} = \Delta^{\bar{\alpha}}_{\beta} W^{\beta}$

$$W^{\bar{\alpha}} = \begin{pmatrix} \cos\theta & \operatorname{sen}\theta \\ -\frac{\operatorname{sen}\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta + \operatorname{sen}\theta \\ \frac{\cos\theta - \operatorname{sen}\theta}{r} \end{pmatrix} //$$

B) Hallar $\tilde{d}f(r,\theta)$, a partir de las coordenadas cartesianas,

con $\tilde{d}f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x + 2y, 2y + 2x)$

i) $f = x^2 + y^2 + 2xy = r^2\cos^2\theta + r^2\operatorname{sen}^2\theta + 2r^2\cos\theta\operatorname{sen}\theta$
$= r^2\cos^2\theta + r^2\operatorname{sen}^2\theta + r^2\operatorname{sen}2\theta$

$$\tilde{d}f(r,\theta) = \left(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta}\right)$$

$\tilde{d}f = 2r\cos^2\theta + 2r\operatorname{sen}^2\theta + 2r\operatorname{sen}2\theta,$

$\cancel{-2r^2\cos\theta\operatorname{sen}\theta} + \cancel{2r^2\operatorname{sen}\theta\cos\theta} + 2r^2\cos2\theta$

$\tilde{d}f = 2r(1 + \operatorname{sen}2\theta), 2r^2\cos2\theta //$

ii)

$\tilde{d}f = O_{\bar{\alpha}} = \Delta^{\nu}_{\bar{\alpha}} O_{\nu}$ $\quad \tilde{d}f = (2x + 2y, 2y + 2x)$

$$\tilde{d}f = (2r\cos\theta + 2r\operatorname{sen}\theta, 2r\cos\theta + 2r\operatorname{sen}\theta)\begin{pmatrix} \cos\theta & -r\operatorname{sen}\theta \\ \operatorname{sen}\theta & r\cos\theta \end{pmatrix}$$

$$\tilde{d}f = (2r\cos\theta + 2r\,\text{sen}\,\theta, 2r\cos\theta + 2r\,\text{sen}\,\theta)\begin{pmatrix} \cos\theta & -r\,\text{sen}\,\theta \\ \text{sen}\,\theta & r\cos\theta \end{pmatrix}$$

$$= (2r\cos\theta + 2r\,\text{sen}\,\theta)\cos\theta + (2r\cos\theta + 2r\,\text{sen}\,\theta)\,\text{sen}\,\theta,$$

$$(2r\cos\theta + 2r\,\text{sen}\,\theta)-r\,\text{sen}\,\theta + (2r\cos\theta + 2r\,\text{sen}\,\theta)r\cos\theta$$

$$= 2r\cos^2\theta + 2r\,\text{sen}\cos\theta + 2r\,\text{sen}\,\theta\cos\theta + 2r\,\text{sen}^2\theta,$$

$$\cancel{-2r^2\,\text{sen}\,\theta\cos\theta} - 2r^2\,\text{sen}^2\theta + 2r^2\cos^2\theta + \cancel{2r^2\,\text{sen}\,\theta\cos\theta}$$

$$= 2r(\cos^2\theta + \text{sen}^2\theta + 2\,\text{sen}\,\theta\cos\theta),$$

$$2r^2(\cos^2\theta - \text{sen}^2\theta)$$

$$= 2r(1 + \text{sen}\,2\theta), 2r^2\cos 2\theta \,//$$

C) i) Teniendo que $A_\alpha = g_{\alpha\beta}A^\beta$ y $A^\alpha = g^{\alpha\beta}A_\beta$ y que $g^{\alpha\beta} = g_{\alpha\beta}$ en coordenadas polares es $\text{diag}(1, r^2)$, entonces, podemos decir que $(A^\alpha)^T g_{\alpha\beta}$ con tal de recuperar la forma covariante de los vectores $\vec{V}$ y $\vec{W}$, esto es,

$$\tilde{V} = (\vec{V})^T g \longrightarrow \tilde{V} = (V^r, V^\theta)\begin{pmatrix} g_{rr} & g_{r\theta} \\ g_{\theta r} & g_{\theta\theta} \end{pmatrix}$$

$$V_r = g_{rr}V^r + \cancel{g_{\theta r}V^r}, \quad V_\theta = \cancel{g_{\theta r}V^\theta} + g_{\theta\theta}V^\theta$$

$$V_r = (1)\,r^2(\text{sen}^3\theta + \cos^3\theta) + 3r\,\text{sen}\,2\theta,$$

$$V_\theta = (r^2)\frac{r}{2}\text{sen}(2\theta)(\text{sen}\,\theta - \cos\theta) + 3\cos(2\theta)$$

$$\tilde{V}: \left(r^2(\operatorname{sen}^3\theta + \cos^3\theta) + 3r\operatorname{sen}2\theta, \frac{r^3}{2}\operatorname{sen}(2\theta)(\operatorname{sen}\theta - \cos\theta) + 3r^2\cos 2\theta\right)_{\!/\!/}$$

$$\tilde{w} = (\vec{w})^T g \longrightarrow \tilde{w} = (w^r, w^\theta)\begin{pmatrix} g_{rr} & g_{r\theta} \\ g_{\theta r} & g_{\theta\theta} \end{pmatrix}$$

$$w_r = g_{rr}w^r + \cancel{g_{\theta r}w^r} \quad , \quad w_\theta = \cancel{g_{\theta r}w^\theta} + g_{\theta\theta}w^\theta$$

$$w_r = (1)\cos\theta + \operatorname{sen}\theta \quad , \quad w_\theta = (r^2)\frac{\cos\theta - \operatorname{sen}\theta}{r}$$

$$\tilde{w}: (\cos\theta + \operatorname{sen}\theta, r(\cos\theta - \operatorname{sen}\theta))_{\!/\!/}$$

ii) Teniendo que el tensor métrico en coordenadas cartesianas es diag(1,1), por consiguiente, $A^\alpha = A_\alpha$, para cualquier vector, así, si se realiza la transformación de coordenadas para $\vec{V}$ y $\vec{w}$ se tiene,

$\tilde{V}$:

$$V_{\bar{\mu}} = \Lambda^{\nu}_{\bar{\mu}} V_\nu$$

$$\tilde{V} = (x^2 + 3y, y^2 + 3x)\begin{pmatrix} \cos\theta & -r\operatorname{sen}\theta \\ \operatorname{sen}\theta & r\cos\theta \end{pmatrix}$$

$$\tilde{V} = (r^2\cos^2\theta + 3r\operatorname{sen}\theta, r^2\operatorname{sen}^2\theta + 3r\cos\theta)\begin{pmatrix} \cos\theta & -r\operatorname{sen}\theta \\ \operatorname{sen}\theta & r\cos\theta \end{pmatrix}$$

$$V_r = (r^2\cos^2\theta + 3r\operatorname{sen}\theta)\cos\theta + (r^2\operatorname{sen}^2\theta + 3r\cos\theta)\operatorname{sen}\theta$$

$$V_\theta = (r^2\cos^2\theta + 3r\operatorname{sen}\theta) - r\operatorname{sen}\theta + (r^2\operatorname{sen}^2\theta + 3r\cos\theta)r\cos\theta$$

$$V_r = r^2\cos^3\theta + 3r\,\text{sen}\theta\cos\theta + r^2\,\text{sen}^3\theta + 3r\,\text{sen}\theta\cos\theta$$

$$V_\theta = -r^3\cos^2\theta\,\text{sen}\theta - 3r^2\,\text{sen}^2\theta + r^3\,\text{sen}^2\theta\cos\theta + 3r^2\cos^2\theta$$

$$V_r = r^2(\text{sen}^3\theta + \cos^3\theta) + 3r\,\text{sen}\,2\theta$$

$$V_\theta = \frac{r^3}{2}\,\text{sen}(2\theta)(\text{sen}\theta - \cos\theta) + 3r^2\cos 2\theta$$

$$\tilde{V} : \left(r^2(\text{sen}^3\theta + \cos^3\theta) + 3r\,\text{sen}\,2\theta,\ \frac{r^3}{2}\,\text{sen}(2\theta)(\text{sen}\theta - \cos\theta) + 3r^2\cos 2\theta\right)_{//}$$

$$W_{\bar{\mu}} = \Lambda^{\nu}_{\bar{\mu}}\, W_\nu$$

$$\tilde{W} = (1, 1)\begin{pmatrix} \cos\theta & -r\,\text{sen}\theta \\ \text{sen}\theta & r\cos\theta \end{pmatrix}$$

$$W_r = \cos\theta + \text{sen}\theta \quad , W_\theta = r(\cos\theta - \text{sen}\,\theta)$$

$$\tilde{W} : (\cos\theta + \text{sen}\theta,\ r(\cos\theta - \text{sen}\theta))_{//}$$

Ejercicio #11

Teniendo $\vec{V}:(x^2+3y,\, y^2+3x)$

Calcular,

A) $V^{\alpha}{}_{,\beta} = \partial_\beta V^\alpha$

$$\partial_x V^x = 2x = V^x{}_{,x}$$

$$\partial_x V^y = 3 = V^y{}_{,x}$$

$$\partial_y V^x = 3 = V^x{}_{,y}$$

$$\partial_y V^y = 2y = V^y{}_{,y}$$

B) $V^{\bar{\mu}}{}_{;\bar{\nu}} = \Lambda^{\bar{\mu}}{}_{\alpha} \Lambda^{\beta}{}_{\bar{\nu}}\, \partial_\beta V^\alpha = \Lambda^{\bar{\mu}}{}_{\alpha} \Lambda^{\beta}{}_{\bar{\nu}}\, V^\alpha{}_{,\beta}$

$$\Lambda^{\bar{\mu}}{}_{\alpha} = \begin{pmatrix} \cos\theta & \operatorname{sen}\theta \\ -\frac{\operatorname{sen}\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \qquad \Lambda^{\beta}{}_{\bar{\nu}} = \begin{pmatrix} \cos\theta & -r\operatorname{sen}\theta \\ \operatorname{sen}\theta & r\cos\theta \end{pmatrix}$$

$$V^{\bar{\mu}}{}_{;\bar{\nu}} = \Lambda^{\bar{\mu}}{}_{\alpha} \Lambda^{\beta}{}_{\bar{\nu}} V^\alpha{}_{,\beta}$$

$$V^{\bar{\mu}}{}_{;r} = \Lambda^{\bar{\mu}}{}_{\alpha} \Lambda^{x}{}_{r} V^\alpha{}_{,x} + \Lambda^{\bar{\mu}}{}_{\alpha} \Lambda^{y}{}_{r} V^\alpha{}_{,y}$$

$$V^{r}{}_{;r} = \Lambda^{r}{}_{x} \Lambda^{x}{}_{r} V^x{}_{,x} + \Lambda^{r}{}_{y} \Lambda^{x}{}_{r} V^y{}_{,x} + \Lambda^{r}{}_{x} \Lambda^{y}{}_{r} V^x{}_{,y} + \Lambda^{r}{}_{y} \Lambda^{y}{}_{r} V^y{}_{,y}$$

$$V^{r}{}_{;r} = \cos\theta\cos\theta\cdot(2r\cos\theta) + \operatorname{sen}\theta\cos\theta\cdot 3 + \cos\theta\operatorname{sen}\theta\cdot 3$$
$$+ \operatorname{sen}\theta\operatorname{sen}\theta\cdot(2r\operatorname{sen}\theta)$$

$$V^{r}{}_{;r} = 2r\cos^3\theta + 3\operatorname{sen}\theta\cos\theta + 3\operatorname{sen}\theta\cos\theta + 2r\operatorname{sen}^3\theta$$

$$\boxed{V^{r}{}_{;r} = 2r(\cos^3\theta + \operatorname{sen}^3\theta) + 3\operatorname{sen}2\theta} \quad V^{r}{}_{;r}$$

$$V^{\theta}_{;r} = \Lambda^{\theta}_{x}\Lambda^{x}_{r} V^{x}_{,x} + \Lambda^{\theta}_{y}\Lambda^{x}_{r} V^{y}_{,x} + \Lambda^{\theta}_{x}\Lambda^{y}_{r} V^{x}_{,y} + \Lambda^{\theta}_{y}\Lambda^{y}_{r} V^{y}_{,y}$$

$$= \frac{-\operatorname{sen}\theta}{r}\cos\theta\, 2r\cos\theta + \frac{\cos\theta}{r}\cos\theta\, 3 + \frac{-\operatorname{sen}\theta}{r}\operatorname{sen}\theta\, 3$$
$$+ \frac{\cos\theta}{r}\operatorname{sen}\theta \cdot 2r\operatorname{sen}\theta$$

$$= -2\cos^2\theta\operatorname{sen}\theta + \frac{3\cos^2\theta}{r} - \frac{3}{r}\operatorname{sen}^2\theta + 2\operatorname{sen}^2\theta\cos\theta$$

$$= 2\operatorname{sen}^2\cos\theta - 2\cos^2\theta\operatorname{sen}\theta + \frac{3\cos^2\theta}{r} - \frac{3}{r}\operatorname{sen}^2\theta$$

$$= 2\operatorname{sen}\theta\cos\theta(\operatorname{sen}\theta - \cos\theta) + (3/r)(\cos^2\theta - \operatorname{sen}^2\theta)$$

$$\boxed{V^{\theta}_{;r} = \operatorname{sen}2\theta(\operatorname{sen}\theta - \cos\theta) + \frac{3}{r}\cos 2\theta} \quad V^{\theta}_{;r}$$

$$V^{r}_{;\theta} = \Lambda^{r}_{x}\Lambda^{x}_{\theta} V^{x}_{,x} + \Lambda^{r}_{y}\Lambda^{x}_{\theta} V^{y}_{,x} + \Lambda^{r}_{x}\Lambda^{y}_{\theta} V^{x}_{,y} + \Lambda^{r}_{y}\Lambda^{y}_{\theta} V^{y}_{,y}$$

$$= \cos\theta\, (-r\operatorname{sen}\theta)\, 2r\cos\theta + \operatorname{sen}\theta\, (-r\operatorname{sen}\theta)\, 3 + \cos\theta\, r\cos\theta \cdot 3$$
$$+ \operatorname{sen}\theta\, r\cos\theta\, 2r\operatorname{sen}\theta$$

$$= -r^2\operatorname{sen}2\theta\cos\theta - 3r\operatorname{sen}^2\theta + 3r\cos^2\theta + r^2\operatorname{sen}2\theta\operatorname{sen}\theta$$

$$= r^2\operatorname{sen}2\theta(\operatorname{sen}\theta - \cos\theta) + 3r\cos 2\theta$$

$$\boxed{V^{r}_{;\theta} = r^2\operatorname{sen}2\theta(\operatorname{sen}\theta - \cos\theta) + 3r\cos 2\theta} \quad V^{r}_{;\theta}$$

$$V^{\theta}_{;\theta} = \Lambda^{\theta}_{x}\Lambda^{x}_{\theta} V^{x}_{,x} + \Lambda^{\theta}_{y}\Lambda^{x}_{\theta} V^{y}_{,x} + \Lambda^{\theta}_{x}\Lambda^{y}_{\theta} V^{x}_{,y} + \Lambda^{\theta}_{y}\Lambda^{y}_{\theta} V^{y}_{,y}$$

$$= \frac{-\operatorname{sen}\theta}{r}(-r\operatorname{sen}\theta)\, 2r\cos\theta + \frac{\cos\theta}{r}(-r\operatorname{sen}\theta)\, 3 + \frac{-\operatorname{sen}\theta}{r} r\cos\theta\, 3$$
$$+ \frac{\cos\theta}{r} r\cos\theta\, 2r\operatorname{sen}\theta$$

$$= r\operatorname{sen}2\theta\operatorname{sen}\theta - 3\cos\theta\operatorname{sen}\theta - 3\cos\theta\operatorname{sen}\theta + r\operatorname{sen}2\theta\cos\theta$$

$$= r\operatorname{sen}2\theta\operatorname{sen}\theta - 6\cos\operatorname{sen}\theta + r\operatorname{sen}2\theta\cos\theta$$

$$\boxed{V^{\theta}_{;\theta} = \operatorname{sen}2\theta(r(\operatorname{sen}\theta + \cos\theta) - 3)} \quad V^{\theta}_{;\theta}$$

c) Considerando que,

$\partial \vec{e}_r/\partial r = 0 \Rightarrow \Gamma^{\mu}{}_{rr} = 0$ para todo μ,

$\partial \vec{e}_r/\partial \theta = \frac{1}{r}\vec{e}_\theta \Rightarrow \Gamma^{r}{}_{r\theta} = 0,\ \Gamma^{\theta}{}_{r\theta} = \frac{1}{r}$,

$\partial \vec{e}_\theta/\partial r = \frac{1}{r}\vec{e}_\theta \Rightarrow \Gamma^{r}{}_{\theta r} = 0,\ \Gamma^{\theta}{}_{\theta r} = \frac{1}{r}$,

$\partial \vec{e}_\theta/\partial \theta = -r\vec{e}_r \Rightarrow \Gamma^{r}{}_{\theta\theta} = -r,\ \Gamma^{\theta}{}_{\theta\theta} = 0$

debemos encontrar $V^{\bar{\mu}}{}_{;\bar{\nu}} = V^{\bar{\mu}}{}_{,\bar{\nu}} + V^{\bar{\alpha}}\Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\nu}}$

$$V^{r}{}_{;r} = V^{r}{}_{,r} + V^{\bar{\alpha}}\Gamma^{r}{}_{\bar{\alpha}r}$$

$$V^{r}{}_{,r} = \frac{\partial V^r}{\partial r} = \frac{\partial}{\partial r}\left(r^2(\operatorname{sen}^3\theta + \cos^3\theta) + 3r\operatorname{sen}2\theta\right)$$

$$V^{r}{}_{,r} = 2r(\operatorname{sen}^3\theta + \cos^3\theta) + 3\operatorname{sen}2\theta$$

$$V^{\bar{\alpha}}\Gamma^{r}{}_{\bar{\alpha}r} = V^r\Gamma^{r}{}_{rr} + V^\theta\Gamma^{r}{}_{\theta r} = \cancel{V^r\Gamma^{r}{}_{rr}}^{0} + \cancel{V^\theta\Gamma^{r}{}_{\theta r}}^{0} = 0$$

$$\boxed{V^{r}{}_{;r} = 2r(\operatorname{sen}^3\theta + \cos^3\theta) + 3\operatorname{sen}2\theta}$$

$$V^{\theta}{}_{;\theta} = V^{\theta}{}_{,\theta} + V^{\bar{\alpha}}\Gamma^{\theta}{}_{\bar{\alpha}\theta}$$

$$V^{\theta}{}_{,\theta} = \frac{\partial V^\theta}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\frac{r}{2}\operatorname{sen}2\theta\ (\operatorname{sen}\theta - \cos\theta) + 3\cos 2\theta\right)$$

$$V^{\theta}{}_{,\theta} = \frac{\partial}{\partial\theta}\left(\frac{r}{2}(\operatorname{sen}2\theta\operatorname{sen}\theta - \operatorname{sen}2\theta\cos\theta) + 3\cos 2\theta\right)$$

$$V^{\theta}{}_{,\theta} = \frac{r}{2}(2\cos2\theta\operatorname{sen}\theta + \operatorname{sen}2\theta\cos\theta - 2\cos2\theta\cos\theta + \operatorname{sen}2\theta\operatorname{sen}\theta)$$
$$-6\operatorname{sen}2\theta$$

$$V^{\theta}{}_{,\theta} = r\cos2\theta(\operatorname{sen}\theta - \cos\theta) + \frac{r}{2}\operatorname{sen}2\theta(\operatorname{sen}\theta + \cos\theta) - 6\operatorname{sen}2\theta$$

$$V^{\bar{\alpha}}\Gamma^{\theta}{}_{\bar{\alpha}\theta} = V^r\Gamma^{\theta}{}_{r\theta} + \cancel{V^\theta\Gamma^{\theta}{}_{\theta\theta}}^{0} = (r^2(\operatorname{sen}^3\theta + \cos^3\theta) + 3r\operatorname{sen}2\theta)\frac{1}{r}$$

$$V^{\bar{\alpha}}\Gamma^{\theta}{}_{\bar{\alpha}\theta} = r(\operatorname{sen}^3\theta + \cos^3\theta) + 3\operatorname{sen}2\theta$$

$$V^{\theta}{}_{;\theta} = r\cos 2\theta(\operatorname{sen}\theta - \cos\theta) + \frac{r}{2}\operatorname{sen}2\theta(\operatorname{sen}\theta + \cos\theta) - 6\operatorname{sen}2\theta$$
$$+ r(\operatorname{sen}^3\theta + \cos^3\theta) + 3\operatorname{sen}2\theta$$

$$V^{\theta}{}_{;\theta} = r((\cos^2\theta - \operatorname{sen}^2\theta)(\operatorname{sen}\theta - \cos\theta) + (\operatorname{sen}\theta\cos\theta)(\operatorname{sen}\theta + \cos\theta)$$
$$+ \operatorname{sen}^3\theta + \cos^3\theta) - 6\operatorname{sen}2\theta + 3\operatorname{sen}2\theta$$

$$V^{\theta}{}_{;\theta} = (\cos^2\theta\operatorname{sen}\theta - \cancel{\cos^3\theta} - \cancel{\operatorname{sen}^3\theta} + \operatorname{sen}^2\theta\cos\theta + \operatorname{sen}^2\theta\cos\theta +$$
$$\cos^2\theta\operatorname{sen}\theta + \cancel{\operatorname{sen}^3\theta} + \cancel{\cos^3\theta})r - 3\operatorname{sen}2\theta$$

$$V^{\theta}{}_{;\theta} = r(2\cos^2\theta\operatorname{sen}\theta + 2\operatorname{sen}^2\theta\cos\theta) - 3\operatorname{sen}2\theta$$

$$V^{\theta}{}_{;\theta} = r\operatorname{sen}2\theta(\operatorname{sen}\theta + \cos\theta) - 3\operatorname{sen}2\theta$$

$$\boxed{V^{\theta}{}_{;\theta} = \operatorname{sen}2\theta(r(\operatorname{sen}\theta + \cos\theta) - 3)}$$

$$V^{\theta}{}_{;r} = V^{\theta}{}_{,r} + V^{\bar{\alpha}}\Gamma^{\theta}{}_{\bar{\alpha}r}$$

$$V^{\theta}{}_{,r} = \frac{\partial V^{\theta}}{\partial r} = \frac{\partial}{\partial r}\left(\frac{r}{2}\operatorname{sen}2\theta\ (\operatorname{sen}\theta - \cos\theta) + 3\cos 2\theta\right)$$

$$V^{\theta}{}_{,r} = \frac{\operatorname{sen}2\theta(\operatorname{sen}\theta - \cos\theta)}{2}$$

$$V^{\bar{\alpha}}\Gamma^{\theta}{}_{\bar{\alpha}r} = \cancel{V^{r}\Gamma^{\theta}{}_{rr}}^{\,0} + V^{\theta}\Gamma^{\theta}{}_{\theta r}$$

$$V^{\bar{\alpha}}\Gamma^{\theta}{}_{\bar{\alpha}r} = \left(\frac{r}{2}\operatorname{sen}2\theta\ (\operatorname{sen}\theta - \cos\theta) + 3\cos 2\theta\right)\frac{1}{r}$$

$$V^{\theta}{}_{;r} = \frac{\operatorname{sen}2\theta(\operatorname{sen}\theta - \cos\theta)}{2} + \frac{\operatorname{sen}2\theta(\operatorname{sen}\theta - \cos\theta)}{2} + \frac{3\cos 2\theta}{r}$$

$$\boxed{V^{\theta}{}_{;r} = \operatorname{sen}2\theta(\operatorname{sen}\theta - \cos\theta) + \frac{3\cos 2\theta}{r}}$$

$$V^r_{;\theta} = V^r_{,\theta} + V^{\bar\alpha}\Gamma^r_{\bar\alpha\theta}$$

$$V^r_{,\theta} = \frac{\partial V^r}{\partial\theta} = \frac{\partial}{\partial\theta}\left(r^2(\text{sen}^3\theta + \cos^3\theta) + 3r\,\text{sen}\,2\theta\right)$$

$$V^r_{,\theta} = r^2(3\,\text{sen}^2\theta\cos\theta - 3\cos^2\theta\,\text{sen}\,\theta) + 6r\cos 2\theta$$

$$V^r_{,\theta} = 3r^2\text{sen}^2\theta\cos\theta - 3r^2\cos^2\theta\,\text{sen}\,\theta + 6r\cos^2\theta - 6r\,\text{sen}^2\theta$$

$$V^{\bar\alpha}\Gamma^r_{\bar\alpha\theta} = \cancel{V^r\Gamma^r_{r\theta}}^{\,0} + V^\theta\Gamma^r_{\theta\theta}$$

$$V^{\bar\alpha}\Gamma^r_{\bar\alpha\theta} = \left(\frac{r}{2}\,\text{sen}\,2\theta\,(\text{sen}\,\theta - \cos\theta) + 3\cos 2\theta\right)(-r)$$

$$V^r_{;\theta} = 3r^2\text{sen}^2\theta\cos\theta - 3r^2\cos^2\theta\,\text{sen}\,\theta + 6r\cos^2\theta - 6r\,\text{sen}^2\theta$$

$$-\frac{r^2}{\cancel{2}}\cancel{2}\,\text{sen}^2\theta\cos\theta + r^2\cos^2\theta\,\text{sen}\,\theta - 3r\cos^2\theta + 3r\,\text{sen}^2\theta)$$

$$V^r_{;\theta} = r^2(2\,\text{sen}^2\cos\theta - 2\cos^2\theta\,\text{sen}\,\theta) + 3r(\cos^2\theta - \text{sen}^2\theta)$$

$$\boxed{V^r_{;\theta} = r^2\,\text{sen}\,2\theta\,(\text{sen}\,\theta - \cos\theta) + 3r\cos 2\theta}$$

D) $V^\alpha_{,\alpha} = V^x_{,x} + V^y_{,y}$

$$V^\alpha_{,\alpha} = \frac{\partial V^x}{\partial x} + \frac{\partial V^y}{\partial y} = \frac{\partial(x^2+3y)}{\partial x} + \frac{\partial}{\partial y}(y^2+3x) = 2x + 2y$$

$$V^\alpha_{,\alpha} = 2x + 2y = 2(x+y) = \boxed{2r(\cos\theta + \text{sen}\,\theta)}$$

E) $V^{\bar\mu}_{;\bar\mu} = V^r_{;r} + V^\theta_{;\theta}$

$$V^{\bar\mu}_{;\bar\mu} = 2r(\text{sen}^3\theta + \cos^3\theta) + 3\,\text{sen}\,2\theta\, +$$

$$\text{sen}\,2\theta\,(r(\text{sen}\,\theta + \cos\theta) - 3)$$

$$V^{\bar\mu}_{;\bar\mu} = 2r\,\text{sen}^3\theta + 2r\cos^3\theta + 6\,\text{sen}\,\theta\cos\theta + 2\,\text{sen}\,\theta\cos\theta\,(r(\text{sen}\,\theta + \cos\theta) - 3)$$

$$V^{\bar\mu}_{;\bar\mu} = 2r\,\mathrm{sen}^3\theta + 2r\cos^3\theta + \cancel{6\,\mathrm{sen}\theta\cos\theta} + 2r\,\mathrm{sen}^2\theta\cos\theta$$

$$+ 2r\cos^2\theta\,\mathrm{sen}\theta - \cancel{6\,\mathrm{sen}\theta\cos\theta}$$

$$V^{\bar\mu}_{;\bar\mu} = 2r\left(\mathrm{sen}^3\theta + \cos^3\theta + \mathrm{sen}^2\theta\cos\theta + \cos^2\mathrm{sen}\theta\right)$$

$$V^{\bar\mu}_{;\bar\mu} = 2r\left(\mathrm{sen}^2\theta\,\mathrm{sen}\theta + \cos^2\theta\cos\theta + \mathrm{sen}^2\theta\cos\theta + \cos^2\mathrm{sen}\theta\right)$$

$$V^{\bar\mu}_{;\bar\mu} = 2r\left(\overset{1}{\cancel{(\mathrm{sen}^2\theta + \cos^2\theta)}}(\mathrm{sen}\theta + \cos\theta)\right)$$

$$\boxed{V^{\bar\mu}_{;\bar\mu} = 2r(\mathrm{sen}\theta + \cos\theta)}$$

F)

$$V^{\alpha}_{;\alpha} = \frac{1}{r}\frac{\partial}{\partial r}(rV^r) + \frac{\partial}{\partial\theta}V^\theta = V^{\bar\mu}_{;\bar\mu}$$

$$V^{\alpha}_{;\alpha} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\left(r^2(\mathrm{sen}^3\theta + \cos^3\theta) + 3r\,\mathrm{sen}\,2\theta\right)\right)$$

$$+ \frac{\partial}{\partial\theta}\left(\frac{r}{2}\mathrm{sen}\,2\theta(\mathrm{sen}\theta - \cos\theta) + 3\cos 2\theta\right)$$

$$V^{\alpha}_{;\alpha} = \frac{1}{r}\frac{\partial}{\partial r}\left(r^3\mathrm{sen}^3\theta + r^3\cos^3\theta + 6r^2\mathrm{sen}\theta\cos\theta\right)$$

$$+ \frac{\partial}{\partial\theta}\left(\frac{r}{2}\mathrm{sen}2\theta(\mathrm{sen}\theta - \cos\theta) + 3\cos2\theta\right)$$

$$V^{\alpha}_{;\alpha} = 3r\,\mathrm{sen}^3\theta + 3r\cos^3\theta + 12\,\mathrm{sen}\theta\cos\theta +$$

$$\frac{r}{2}\left[2\cos2\theta\,(\mathrm{sen}\theta - \cos\theta) + \mathrm{sen}2\theta(\cos\theta + \mathrm{sen}\theta)\right] - 6\,\mathrm{sen}2\theta$$

$$V^{\alpha}_{;\alpha} = 3r\,\mathrm{sen}^3\theta + 3r\cos^3\theta + 12\,\mathrm{sen}\theta\cos\theta +$$

$$\frac{r}{2}\left[2(\cos^2\theta - \mathrm{sen}^2\theta)(\mathrm{sen}\theta - \cos\theta) + 2\,\mathrm{sen}\theta\cos\theta(\cos\theta + \mathrm{sen}\theta)\right]$$

$$-6\,\mathrm{sen}2\theta$$

$$V^{\alpha}_{;\alpha} = 3r\,sen^3\theta + 3r\cos^3\theta + \cancel{12\,sen\theta\cos\theta}^{\,0} +$$

$$\frac{r}{\cancel{2}}\Big[\cancel{2}\big(\cos^2\theta\,sen\theta - \cos^3\theta - sen^3\theta + sen^2\theta\cos\theta + \cos^2\theta\,sen\theta$$

$$+ sen^2\theta\cos\theta\big)\Big] - \cancel{12\,sen\theta\cos\theta}^{\,0}$$

$$V^{\alpha}_{;\alpha} = 2r\,sen^3\theta + 2r\cos^3\theta + 2r\cos^2\theta\,sen\theta + 2r\,sen^2\theta\cos\theta$$

$$V^{\alpha}_{;\alpha} = 2r\big(sen^2\theta\,sen\theta + \cos^2\theta\cos\theta + \cos^2\theta\,sen\theta + sen^2\theta\cos\theta\big)$$

$$V^{\alpha}_{;\alpha} = 2r\big(\cancel{(sen^2\theta + \cos^2\theta)}^{\,1}(sen\theta + \cos\theta)\big)$$

$$\boxed{V^{\alpha}_{;\alpha} = 2r(sen\theta + \cos\theta) = V^{\bar{\mu}}_{;\bar{\mu}}}$$

Ejercicio #12

Tenendo el campo del vector 1-forma (vector covariante) $\tilde{p}$ de la forma (x^2+3y, y^2+3x) en coordenadas cartesianas calcular;

A) $P_{\alpha,\beta} = \dfrac{\partial P_\alpha}{\partial x^\beta}$

$$P_{x,x} = \frac{\partial}{\partial x}(x^2+3y) = 2x \;,\; P_{y,x} = \frac{\partial}{\partial x}(y^2+3x) = 3$$

$$P_{x,y} = \frac{\partial}{\partial y}(x^2+3\;) = 3 \;,\; P_{y,y} = \frac{\partial}{\partial y}(y^2+3x) = 2y$$

B) $P_{\bar{\mu};\bar{v}} = \Lambda^{\alpha}_{\bar{\mu}}\Lambda^{\beta}_{\bar{v}} P_{\alpha,\beta}$

$$\Lambda^{\bar{\mu}}_{\alpha} = \begin{pmatrix} \cos\theta & \operatorname{sen}\theta \\ -\dfrac{\operatorname{sen}\theta}{r} & \dfrac{\cos\theta}{r} \end{pmatrix} \qquad \Lambda^{\beta}_{\bar{v}} = \begin{pmatrix} \cos\theta & -r\operatorname{sen}\theta \\ \operatorname{sen}\theta & r\cos\theta \end{pmatrix}$$

$$P_{\bar{\mu};r} = \Lambda^{\alpha}_{\bar{\mu}}\Lambda^{x}_{r} P_{\alpha,x} + \Lambda^{\alpha}_{\bar{\mu}}\Lambda^{y}_{r} P_{\alpha,y}$$

$$P_{r;r} = \Lambda^{x}_{r}\Lambda^{x}_{r}P_{x,x} + \Lambda^{y}_{r}\Lambda^{x}_{r}P_{y,x} + \Lambda^{x}_{r}\Lambda^{y}_{r}P_{x,y} + \Lambda^{y}_{r}\Lambda^{y}_{r}P_{y,y}$$

$$P_{r;r} = (\Lambda^{x}_{r})^2 P_{x,x} + 2\Lambda^{x}_{r}\Lambda^{y}_{r} P_{x,y=y,x} + (\Lambda^{y}_{r})^2 P_{y,y}$$

$$P_{r;r} = \cos^2\theta(2r\cos\theta) + 6\cos\theta\operatorname{sen}\theta + \operatorname{sen}^2\theta(2r\operatorname{sen}\theta)$$

$$\boxed{P_{r;r} = 2r(\cos^3\theta + \operatorname{sen}^3\theta) + 3\operatorname{sen}2\theta}$$

$$P_{\theta;\theta} = (\Lambda^{x}_{\theta})^2 P_{x,x} + 2\Lambda^{x}_{\theta}\Lambda^{y}_{\theta} P_{x,y=y,x} + (\Lambda^{y}_{\theta})^2 P_{y,y}$$

$$P_{\theta;\theta} = r^2\operatorname{sen}^2\theta(2r\cos\theta) - 6r^2\operatorname{sen}\theta\cos\theta + r^2\cos^2\theta(2r\operatorname{sen}\theta)$$

$$P_{\theta;\theta} = r^2(r\operatorname{sen}2\theta\operatorname{sen}\theta) - 3r^2(\operatorname{sen}2\theta) + r^2(r\operatorname{sen}2\theta\cos\theta)$$

$$\boxed{P_{\theta;\theta} = r^2\operatorname{sen}2\theta(r(\operatorname{sen}\theta + \cos\theta) - 3\;)}$$

$$P_{r;\theta} = \Lambda^{x}_{r}\Lambda^{x}_{\theta}P_{x,x} + \Lambda^{y}_{r}\Lambda^{x}_{\theta}P_{y,x} + \Lambda^{x}_{r}\Lambda^{y}_{\theta}P_{x,y} + \Lambda^{y}_{r}\Lambda^{y}_{\theta}P_{y,y}$$

$$P_{r;\theta} = \cos\theta \cdot -r\,\text{sen}\,\theta \cdot (2r\cos\theta) + \text{sen}\,\theta \cdot -r\,\text{sen}\,\theta \cdot 3$$

$$+ \cos\theta \cdot r\cos\theta \cdot 3 + \text{sen}\,\theta \cdot r\cos\theta \cdot 2r\,\text{sen}\,\theta$$

$$P_{r;\theta} = -r^2\,\text{sen}\,2\theta\cos\theta - 3r\,\text{sen}^2\theta + 3r\cos^2\theta + r^2\,\text{sen}\,2\theta\,\text{sen}\,\theta$$

$$\boxed{P_{r;\theta} = r^2\,\text{sen}\,2\theta\,(\text{sen}\,\theta - \cos\theta) + 3r\cos 2\theta}$$

$$P_{\theta;r} = \Lambda^{x}_{\theta}\Lambda^{x}_{r}P_{x,x} + \Lambda^{y}_{\theta}\Lambda^{x}_{r}P_{y,x} + \Lambda^{x}_{\theta}\Lambda^{y}_{r}P_{x,y} + \Lambda^{y}_{\theta}\Lambda^{y}_{r}P_{y,y}$$

$$P_{\theta;r} = -r\,\text{sen}\,\theta \cdot \cos\theta \cdot 2r\cos\theta + r\cos\theta \cdot \cos\theta \cdot 3 + -r\,\text{sen}\,\theta \cdot \text{sen}\,\theta \cdot 3$$

$$+ r\cos\theta\,\text{sen}\,\theta \cdot 2r\,\text{sen}\,\theta$$

$$P_{\theta;r} = -r^2\,\text{sen}\,2\theta\cos\theta + 3r\cos^2\theta - 3r\,\text{sen}^2\theta + r^2\,\text{sen}\,2\theta\,\text{sen}\,\theta$$

$$\boxed{P_{\theta;r} = r^2\,\text{sen}\,2\theta\,(\text{sen}\,\theta - \cos\theta) + 3r\cos 2\theta}$$

c) Del ejercicio #8 vemos que $\tilde{V}$ es igual a $\tilde{P}$, usando los resultados para la 1-forma $\tilde{V} = \tilde{P}$ en coordenadas polares y considerando nuevamente los símbolos de Christoffel empleados en el ejercicio #11 debemos encontrar $P_{\bar{\mu};\bar{\nu}} = P_{\bar{\mu},\bar{\nu}} - P_{\bar{\alpha}}\Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}}$

$$P_{r;r} = P_{r,r} - P_{\alpha}\Gamma^{\alpha}_{rr}$$

$$P_{r,r} = \frac{\partial P_r}{\partial r} = \frac{\partial(r^2(\text{sen}^3\theta + \cos^3\theta) + 3r\,\text{sen}\,2\theta)}{\partial r}$$

$$P_{r,r} = 2r(\text{sen}^3\theta + \cos^3\theta) + 3\,\text{sen}\,2\theta$$

$$P_{\bar{\alpha}}\Gamma^{\bar{\alpha}}_{rr} = P_r\Gamma^{r}_{rr} + P_\theta\Gamma^{\theta}_{rr} = \cancel{P_r\Gamma^{r}_{rr}}^{\,0} + \cancel{P_\theta\Gamma^{\theta}_{rr}}^{\,0} = 0$$

$$\boxed{P_{r;r} = 2r(\text{sen}^3\theta + \cos^3\theta) + 3\,\text{sen}\,2\theta}$$

$$P_{\theta;\theta} = P_{\theta,\theta} - P_{\bar\alpha}\Gamma^{\bar\alpha}_{\theta\theta}$$

$$P_{\theta,\theta} = \frac{\partial P_\theta}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\frac{r^3}{2}\operatorname{sen}2\theta(\operatorname{sen}\theta-\cos\theta)+3r^2\cos2\theta\right)$$

$$P_{\theta,\theta} = \frac{\partial P_\theta}{\partial\theta} = \frac{\partial}{\partial\theta}\left(r^3(\operatorname{sen}^2\theta\cos\theta-\cos^2\theta\operatorname{sen}\theta)+3r^2(\cos^2\theta-\operatorname{sen}^2\theta)\right)$$

$$P_{\theta,\theta} = r^3(2\operatorname{sen}\theta\cos^2\theta-\operatorname{sen}^3\theta+2\cos\theta\operatorname{sen}^2\theta-\cos^3\theta)+$$

$$3r^2(-2\cos\theta\operatorname{sen}\theta-2\operatorname{sen}\theta\cos\theta)$$

$$P_{\bar\alpha}\Gamma^{\bar\alpha}_{\theta\theta} = P_r\Gamma^r_{\theta\theta} + \cancel{P_\theta\Gamma^\theta_{\theta\theta}}^{0} = -rP_r$$

$$P_{\bar\alpha}\Gamma^{\bar\alpha}_{\theta\theta} = (r^2(\operatorname{sen}^3\theta+\cos^3\theta)+3r\operatorname{sen}2\theta)(-r)$$

$$P_{\bar\alpha}\Gamma^{\bar\alpha}_{\theta\theta} = -r^3(\operatorname{sen}^3\theta+\cos^3\theta)-3r^2\operatorname{sen}2\theta)$$

$$P_{\theta;\theta} = 2r^3\cos^2\theta\operatorname{sen}\theta\ \cancel{-r^3\operatorname{sen}^3\theta}^{0}+2r^3\operatorname{sen}^2\theta\cos\theta\ \cancel{-r^3\cos^3\theta}^{0}$$

$$-6r^2\cos\theta\operatorname{sen}\theta\ \cancel{-6r^2\cos\theta\operatorname{sen}\theta}^{0}\ \cancel{+r^3\operatorname{sen}^3\theta}^{0}\ \cancel{+r^3\cos^3\theta}^{0}\ \cancel{+6r^2\operatorname{sen}\theta\cos\theta}^{0}$$

$$\boxed{P_{\theta;\theta} = r^2\operatorname{sen}2\theta(r(\operatorname{sen}\theta+\cos\theta)-3)}$$

$$P_{r;\theta} = P_{r,\theta} - P_{\bar\alpha}\Gamma^{\bar\alpha}_{r\theta}$$

$$P_{r,\theta} = \frac{\partial P_r}{\partial\theta} = \frac{\partial}{\partial\theta}\left(r^2(\operatorname{sen}^3\theta+\cos^3\theta)+3r\operatorname{sen}2\theta\right)$$

$$P_{r,\theta} = r^2(3\operatorname{sen}^2\theta\cos\theta-3\cos^2\theta\operatorname{sen}\theta+6r\cos2\theta$$

$$P_{r,\theta} = 3r^2\operatorname{sen}^2\theta\cos\theta-3r^2\cos^2\operatorname{sen}\theta+6r\cos^2\theta-6r\operatorname{sen}^2\theta$$

$$P_{\bar\alpha}\Gamma^{\bar\alpha}_{r\theta} = \cancel{P_r\Gamma^r_{r\theta}}^{0}+P_\theta\Gamma^\theta_{r\theta} = P_\theta/r$$

$$P_{\bar\alpha}\Gamma^{\bar\alpha}_{r\theta} = r^2\operatorname{sen}^2\theta\cos\theta-r^2\cos^2\theta\operatorname{sen}\theta+3r\cos^2\theta-3r\operatorname{sen}^2\theta$$

$$P_{r;\theta} = 3r^2\operatorname{sen}^2\theta\cos\theta-3r^2\cos^2\operatorname{sen}\theta+6r\cos^2\theta-6r\operatorname{sen}^2\theta$$

$$-r^2\operatorname{sen}^2\theta\cos\theta+r^2\cos^2\theta\operatorname{sen}\theta-3r\cos^2\theta+3r\operatorname{sen}^2\theta$$

$$\boxed{P_{r;\theta} = r^2 \operatorname{sen} 2\theta \,(\operatorname{sen}\theta - \cos\theta) + 3r\cos 2\theta}$$

$$P_{\theta;r} = P_{\theta,r} - P_\alpha \Gamma^\alpha_{\theta r}$$

$$P_{\theta,r} = \frac{\partial P_\theta}{\partial r} = \frac{\partial}{\partial r}\left(r^3 \operatorname{sen}^2\theta\cos\theta - r^3\cos^2\theta\operatorname{sen}\theta + 3r^2\cos^2\theta - 3r^2\operatorname{sen}^2\theta \right)$$

$$P_{\theta,r} = 3r^2\operatorname{sen}^2\theta\cos\theta - 3r^2\cos^2\theta\operatorname{sen}\theta + 3r\cos^2\theta - 3r\operatorname{sen}^2\theta$$

$$P_\alpha \Gamma^\alpha_{\theta r} = P_r \cancel{\Gamma^r_{\theta r}}^{\,0} + P_\theta \Gamma^\theta_{\theta r} = P_\theta / r$$

$$P_\alpha \Gamma^\alpha_{\theta r} = r^2\operatorname{sen}^2\theta\cos\theta - r^2\cos^2\theta\operatorname{sen}\theta + 3r\cos^2\theta - 3r\operatorname{sen}^2\theta$$

$$P_{\theta;r} = 3r^2\operatorname{sen}^2\theta\cos\theta - 3r^2\cos^2\theta\operatorname{sen}\theta + 3r\cos^2\theta - 3r\operatorname{sen}^2\theta$$

$$+ r^2\operatorname{sen}^2\theta\cos\theta + r^2\cos^2\theta\operatorname{sen}\theta - 3r\cos^2\theta + 3r\operatorname{sen}^2\theta$$

$$\boxed{P_{\theta;r} = r^2 \operatorname{sen} 2\theta \,(\operatorname{sen}\theta - \cos\theta) + 3r\cos 2\theta}$$

Ejercicio #14

Con el tensor A, cuyas componentes en coordenadas polares son $A^{rr}=r^2$, $A^{r\theta}=r\,\mathrm{sen}\,\theta$, $A^{\theta r}=r\cos\theta$ y $A^{\theta\theta}=\tan\theta$ calcular

$$\nabla_\beta A^{\mu\nu}=A^{\mu\nu}{}_{,\beta}+A^{\alpha\nu}\Gamma^{\mu}_{\alpha\beta}+A^{\mu\alpha}\Gamma^{\nu}_{\alpha\beta}$$

$$A^{rr}{}_{,r}=\frac{\partial A^{rr}}{\partial r}=2r \qquad A^{rr}{}_{,\theta}=\frac{\partial A^{rr}}{\partial\theta}=0$$

$$A^{r\theta}{}_{,r}=\frac{\partial A^{r\theta}}{\partial r}=\mathrm{sen}\,\theta \qquad A^{r\theta}{}_{,\theta}=\frac{\partial A^{r\theta}}{\partial\theta}=r\cos\theta$$

$$A^{\theta r}{}_{,r}=\frac{\partial A^{\theta r}}{\partial r}=\cos\theta \qquad A^{\theta r}{}_{,\theta}=\frac{\partial A^{\theta r}}{\partial\theta}=-r\,\mathrm{sen}\,\theta$$

$$A^{\theta\theta}{}_{,r}=\frac{\partial A^{\theta\theta}}{\partial r}=0 \qquad A^{\theta\theta}{}_{,\theta}=\frac{\partial A^{\theta\theta}}{\partial\theta}=\sec^2\theta$$

$$\Gamma^{r}_{rr}=0 \qquad \Gamma^{\theta}_{rr}=0$$

$$\Gamma^{r}_{r\theta}=0 \qquad \Gamma^{\theta}_{r\theta}=1/r$$

$$\Gamma^{r}_{\theta r}=0 \qquad \Gamma^{\theta}_{\theta r}=1/r$$

$$\Gamma^{r}_{\theta\theta}=-r \qquad \Gamma^{\theta}_{\theta\theta}=0$$

- $\underline{\nabla_r A^{rr}}=A^{rr}{}_{,r}+A^{\alpha r}\Gamma^{r}_{\alpha r}+A^{r\alpha}\Gamma^{r}_{\alpha r}$

$$=A^{rr}{}_{,r}+A^{rr}\Gamma^{r}_{rr}+A^{\theta r}\Gamma^{r}_{\theta r}+A^{rr}\Gamma^{r}_{rr}+A^{r\theta}\Gamma^{r}_{\theta r}$$

$$=2r$$

- $\underline{\nabla_\theta A^{rr}}=A^{rr}{}_{,\theta}+A^{\alpha r}\Gamma^{r}_{r\theta}+A^{r\alpha}\Gamma^{r}_{\alpha\theta}$

$$=A^{rr}{}_{,\theta}+A^{rr}\Gamma^{r}_{r\theta}+A^{\theta r}\Gamma^{r}_{\theta\theta}+A^{rr}\Gamma^{r}_{r\theta}+A^{r\theta}\Gamma^{r}_{\theta\theta}$$

$$=(A^{\theta r}+A^{r\theta})\Gamma^{r}_{\theta\theta}=-r^2(\cos\theta+\mathrm{sen}\,\theta)$$

- $\nabla_r A^{\theta\theta} = A^{\theta\theta}{}_{,r} + A^{\alpha\theta}\Gamma^{\theta}_{\alpha r} + A^{\theta\alpha}\Gamma^{\theta}_{\alpha r}$

$= A^{\theta\theta}{}_{,r} + A^{r\theta}\Gamma^{\theta}_{rr} + A^{\theta\theta}\Gamma^{\theta}_{\theta r} + A^{\theta r}\Gamma^{\theta}_{rr} + A^{\theta\theta}\Gamma^{\theta}_{\theta r}$

$= 2A^{\theta\theta}\Gamma^{\theta}_{\theta r} = \frac{2}{r}\tan\theta$

- $\nabla_\theta A^{\theta\theta} = A^{\theta\theta}{}_{,\theta} + A^{\alpha\theta}\Gamma^{\theta}_{\alpha\theta} + A^{\theta\alpha}\Gamma^{\theta}_{\alpha\theta}$

$= A^{\theta\theta}{}_{,\theta} + A^{r\theta}\Gamma^{\theta}_{r\theta} + A^{\theta\theta}\Gamma^{\theta}_{\theta\theta} + A^{\theta r}\Gamma^{\theta}_{r\theta} + A^{\theta\theta}\Gamma^{\theta}_{\theta\theta}$

$= A^{\theta\theta}{}_{,\theta} + (A^{r\theta} + A^{\theta r})\Gamma^{\theta}_{r\theta} = \text{sen}\,\theta + \cos\theta + \sec^2\theta$

- $\nabla_r A^{r\theta} = A^{r\theta}{}_{,r} + A^{\alpha\theta}\Gamma^{r}_{\alpha r} + A^{r\alpha}\Gamma^{\theta}_{\alpha r}$

$= A^{r\theta}{}_{,r} + A^{r\theta}\Gamma^{r}_{rr} + A^{\theta\theta}\Gamma^{r}_{\theta r} + A^{rr}\Gamma^{\theta}_{rr} + A^{r\theta}\Gamma^{\theta}_{\theta r}$

$= A^{r\theta}{}_{,r} + A^{r\theta}\Gamma^{\theta}_{\theta r} = 2\,\text{sen}\,\theta$

- $\nabla_\theta A^{r\theta} = A^{r\theta}{}_{,\theta} + A^{\alpha\theta}\Gamma^{r}_{\alpha\theta} + A^{r\alpha}\Gamma^{\theta}_{\alpha\theta}$

$= A^{r\theta}{}_{,\theta} + A^{r\theta}\Gamma^{r}_{r\theta} + A^{\theta\theta}\Gamma^{r}_{\theta\theta} + A^{rr}\Gamma^{\theta}_{r\theta} + A^{r\theta}\Gamma^{\theta}_{\theta\theta}$

$= A^{r\theta}{}_{,\theta} + A^{\theta\theta}\Gamma^{r}_{\theta\theta} + A^{rr}\Gamma^{\theta}_{r\theta} = r(1 + \cos\theta - \tan\theta)$

- $\nabla_r A^{\theta r} = A^{\theta r}{}_{,r} + A^{\alpha r}\Gamma^{\theta}_{\alpha r} + A^{\theta\alpha}\Gamma^{r}_{\alpha r}$

$= A^{\theta r}{}_{,r} + A^{rr}\Gamma^{\theta}_{rr} + A^{\theta r}\Gamma^{\theta}_{\theta r} + A^{\theta r}\Gamma^{r}_{rr} + A^{\theta\theta}\Gamma^{r}_{\theta r}$

$= A^{\theta r}{}_{,r} + A^{\theta r}\Gamma^{\theta}_{\theta r} = 2\cos\theta$

- $\nabla_\theta A^{\theta r} = A^{\theta r}{}_{,\theta} + A^{\alpha r}\Gamma^{\theta}_{\alpha\theta} + A^{\theta\alpha}\Gamma^{r}_{\alpha\theta}$

$= A^{\theta r}{}_{,\theta} + A^{rr}\Gamma^{\theta}_{r\theta} + A^{\theta r}\Gamma^{\theta}_{\theta\theta} + A^{\theta r}\Gamma^{r}_{r\theta} + A^{\theta\theta}\Gamma^{r}_{\theta\theta}$

$= A^{\theta r}{}_{,\theta} + A^{rr}\Gamma^{\theta}_{r\theta} + A^{\theta\theta}\Gamma^{r}_{\theta\theta} = -r\,\text{sen}\,\theta$

Ejercicio #15

Para el vector cuyas componentes polares son $V^r = 1, V^\theta = 0$ computar $V^\alpha{}_{;\mu;\nu}$.

→ Primer derivada covariante

$$V^\alpha{}_{;\mu} = V^\alpha{}_{,\mu} + V^\beta \Gamma^\alpha_{\beta\mu}$$

Términos nulos:

$$V^\theta = \Gamma^r_{rr} = \Gamma^r_{r\theta} = \Gamma^r_{\theta r} = \Gamma^\theta_{rr} = \Gamma^\theta_{\theta\theta} = V^r{}_{,r} = V^r{}_{,\theta} = V^\theta{}_{,r} = V^\theta{}_{,\theta} = 0$$

Términos no nulos:

$$\Gamma^\theta_{r\theta} = 1/r \,,\ \Gamma^\theta_{\theta r} = 1/r \,,\ \Gamma^r_{\theta\theta} = -r \,,\ V^r = 1$$

Así, podemos ver que,

$$V^\alpha{}_{;\mu} = \cancel{V^\alpha{}_{,\mu}}^{\,0} + V^\beta \Gamma^\alpha_{\beta\mu}$$

$$V^\alpha{}_{;\mu} = V^\beta \Gamma^\alpha_{\beta\mu} = V^r \Gamma^\alpha_{r\mu} + \cancel{V^\theta \Gamma^\alpha_{\theta\mu}}^{\,0}$$

$$\boxed{V^\alpha{}_{;\mu} = V^r \Gamma^\alpha_{r\mu}}$$

Computando se tiene,

$$V^r{}_{;r} = V^r \Gamma^r_{rr} = (1)\cancel{(0)} = 0$$

$$V^r{}_{;\theta} = V^r \Gamma^r_{r\theta} = (1)\cancel{(0)} = 0$$

$$V^\theta{}_{;r} = V^r \Gamma^\theta_{rr} = (1)\cancel{(0)} = 0$$

$$V^\theta{}_{;\theta} = V^r \Gamma^\theta_{r\theta} = (1)\frac{1}{r} = \frac{1}{r} \implies \boxed{V^\theta{}_{;\theta} = \frac{1}{r}}$$

$$V^{\alpha}_{;\mu;\nu} = \nabla_\nu V^{\alpha}_{;\mu} = V^{\alpha}_{;\mu,\nu} + V^{\beta}_{;\mu}\Gamma^{\alpha}_{\beta\nu} - V^{\alpha}_{;\beta}\Gamma^{\beta}_{\mu\nu}$$

Términos nulos:

$$V^{r}_{;r} = V^{r}_{;\theta} = V^{\theta}_{;r} = 0$$

$$V^{\alpha}_{;\mu,\theta} = 0 \qquad V^{r}_{;r,r} = V^{r}_{;\theta,r} = V^{\theta}_{;r,r} = 0$$

$$\Gamma^{r}_{rr} = \Gamma^{r}_{r\theta} = \Gamma^{r}_{\theta r} = \Gamma^{\theta}_{rr} = \Gamma^{\theta}_{\theta\theta} = 0$$

Términos no nulos:

$$\Gamma^{\theta}_{r\theta} = 1/r \,,\ \Gamma^{\theta}_{\theta r} = 1/r \,,\ \Gamma^{r}_{\theta\theta} = -r$$

$$V^{\theta}_{;\theta} = \frac{1}{r} \qquad , \ V^{\theta}_{;\theta,r} = -\frac{1}{r^2}$$

Así, podemos ver que,

$$V^{\alpha}_{;\mu;\nu} = V^{\alpha}_{;\mu,\nu} + V^{\beta}_{;\mu}\Gamma^{\alpha}_{\beta\nu} - V^{\alpha}_{;\beta}\Gamma^{\beta}_{\mu\nu}$$

$$V^{\alpha}_{;\mu;\nu} = V^{\alpha}_{;\mu,\nu} + V^{r}_{;\mu}\Gamma^{\alpha}_{r\nu} + V^{\theta}_{;\mu}\Gamma^{\alpha}_{\theta\nu} - V^{\alpha}_{;r}\Gamma^{r}_{\mu\nu} - V^{\alpha}_{;\theta}\Gamma^{\theta}_{\mu\nu}$$

$$V^{\alpha}_{;\mu;\nu} = V^{\alpha}_{;\mu,\nu} + V^{\theta}_{;\mu}\Gamma^{\alpha}_{\theta\nu} - V^{\alpha}_{;\theta}\Gamma^{\theta}_{\mu\nu}$$

Análisis de la segunda derivada covariante en θ

$$\longrightarrow V^{\alpha}_{;\mu;\theta} = V^{\alpha}_{;\mu,\theta} + V^{\theta}_{;\mu}\Gamma^{\alpha}_{\theta\theta} - V^{\alpha}_{;\theta}\Gamma^{\theta}_{\mu\theta}$$

$$V^{\alpha}_{;\mu;\theta} = V^{\theta}_{;\mu}\Gamma^{\alpha}_{\theta\theta} - V^{\alpha}_{;\theta}\Gamma^{\theta}_{\mu\theta}$$

$$V^{r}_{;r;\theta} = V^{\theta}_{;r}\Gamma^{r}_{\theta\theta} - V^{r}_{;\theta}\Gamma^{\theta}_{r\theta} = 0$$

$$V^{r}_{;\theta;\theta} = V^{\theta}_{;\theta}\Gamma^{r}_{\theta\theta} - V^{r}_{;\theta}\Gamma^{\theta}_{\theta\theta} = \frac{1}{r}\cdot(-r) = -1$$

$$V^{\theta}_{;r;\theta} = V^{\theta}_{;r}\Gamma^{\theta}_{\theta\theta} - V^{\theta}_{;\theta}\Gamma^{\theta}_{r\theta} = -\left(\frac{1}{r}\right)\cdot\left(\frac{1}{r}\right) = -\frac{1}{r^2}$$

$$V^{\theta}_{;\theta;\theta} = V^{\theta}_{;\theta}\Gamma^{\theta}_{\theta\theta} - V^{\theta}_{;\theta}\Gamma^{\theta}_{\theta\theta} = 0$$

Análisis de la segunda derivada covariante en θ

$$\longrightarrow V^\alpha_{;\mu;r} = V^\alpha_{;\mu,r} + V^\theta_{;\mu}\Gamma^\alpha_{\theta r} - V^\alpha_{;\theta}\Gamma^\theta_{\mu r}$$

$$V^r_{;r;r} = \cancel{V^r_{;r,r}} + \cancel{V^\theta_{;r}\Gamma^r_{\theta r}} - \cancel{V^r_{;\theta}\Gamma^\theta_{rr}} = 0$$

$$V^r_{;\theta;r} = \cancel{V^r_{;\theta,r}} + \cancel{V^\theta_{;\theta}\Gamma^r_{\theta r}} - \cancel{V^r_{;\theta}\Gamma^\theta_{\theta r}} = 0$$

$$V^\theta_{;r;r} = \cancel{V^\theta_{;r,r}} + \cancel{V^\theta_{;r}\Gamma^\theta_{\theta r}} - \cancel{V^\theta_{;\theta}\Gamma^\theta_{rr}} = 0$$

$$V^\theta_{;\theta;r} = V^\theta_{;\theta,r} + V^\theta_{;\theta}\Gamma^\theta_{\theta r} - V^\theta_{;\theta}\Gamma^\theta_{\theta r} = -\frac{1}{r^2} + \left(\frac{1}{r}\cdot\frac{1}{r}\right) - \left(\frac{1}{r}\cdot\frac{1}{r}\right) = -\frac{1}{r^2}$$

Finalmente, los componentes de la segunda derivada covariante

$$V^\alpha_{;\mu;\nu} = \nabla_\nu V^\alpha_{;\mu} = V^\alpha_{;\mu,\nu} + V^\beta_{;\mu}\Gamma^\alpha_{\beta\nu} - V^\alpha_{;\beta}\Gamma^\beta_{\mu\nu}$$

son,

$$\boxed{\begin{aligned}
&V^r_{;r;r} = 0\\
&V^r_{;r;\theta} = 0\\
&V^r_{;\theta;r} = 0\\
&V^r_{;\theta;\theta} = -1\\
&V^\theta_{;r;r} = 0\\
&V^\theta_{;r;\theta} = -\frac{1}{r^2}\\
&V^\theta_{;\theta;r} = -\frac{1}{r^2}\\
&V^\theta_{;\theta;\theta} = 0
\end{aligned}}$$