Homework #1

Fundamentals of Radiative Transfer

#1 Assuming a specific intensity of the form $I = a_\lambda \cos\theta + b_\lambda \cos^2\theta$, calculate the mean intensity. J_λ

Solution

Given J_λ

0th moment of the specific intensity we have:

$$J_\lambda = \frac{1}{4\pi}\int I \, d\Omega$$

Solid angle

$$\boxed{d\Omega = \operatorname{sen}\theta \, d\theta \, d\phi}$$

$$J_\lambda = \frac{1}{4\pi}\int (a_\lambda \cos\theta + b_\lambda \cos^2\theta)\operatorname{sen}\theta \, d\theta \, d\phi$$

$$J_\lambda = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} (a_\lambda \cos\theta + b_\lambda \cos^2\theta)\operatorname{sen}\theta \, d\theta$$

$$J_\lambda = \frac{1}{4\pi}\int_0^{2\pi} d\phi \cdot \left[a_\lambda \int_0^{\pi} \cos\theta \operatorname{sen}\theta \, d\theta + b_\lambda \int_0^{\pi} \cos^2\theta \operatorname{sen}\theta \, d\theta\right]$$

> $\int_0^{2\pi} d\phi = 2\pi$

> $\int_0^{\pi} \cos\theta \operatorname{sen}\theta \, d\theta = -\int u \, du$ $\quad u = \cos\theta \quad du = \operatorname{sen}\theta \, d\theta$

$$-\int u \, du = -\frac{u^2}{2} \Rightarrow -\frac{\cos^2\theta}{2}\Big|_0^{\pi} \Rightarrow -\frac{\cos^2(\pi)}{2} + \frac{\cos^2(0)}{2}$$

$$-\frac{1}{2} + \frac{1}{2} = \boxed{0}$$

> $\int_0^{\pi} \cos^2 \operatorname{sen}\theta \, d\theta = -\int u^2 \, du \rightarrow u = \cos\theta \quad du = -\operatorname{sen}\theta \, d\theta$

$$-\int u^2 \, du = \frac{-u^3}{3} \Rightarrow -\frac{\cos^3(\theta)}{3}\Big|_0^{\pi} \Rightarrow -\frac{\cos^3(\pi)}{3} + \frac{\cos^3(0)}{3}$$

$$\frac{1}{3} + \frac{1}{3} = \boxed{\frac{2}{3}}$$

$$J_\lambda = \frac{1}{\cancel{4\pi}} \cdot \cancel{2\pi} \cdot \left[\cancel{a_\lambda \cdot 0}^{\,0} + b_\lambda \cdot \frac{2}{3} \right]$$

$$J_\lambda = \frac{1}{\cancel{2}} \cdot b_\lambda \frac{\cancel{2}}{3}$$

$$\boxed{J_\lambda = \frac{b_\lambda}{3}}$$ Mean intensity

#3 According to one model of the Sun, the central density is $1.53 \times 10^5\ \mathrm{kg/m^3}$ and the mean opacity at the center is $0.217\ \mathrm{m^2/Kg}$. Calculate the mean free path of a photon at the center of the Sun.

Solution

If we know that $\alpha_\nu = \rho \kappa_\nu$, we can calculate the absorption coefficient such:

$$\alpha_\nu = \rho \kappa_\nu = 1.53 \times 10^5\ \cancel{\mathrm{kg}}/\mathrm{m^3} \times 0.217\ \mathrm{m^2}/\cancel{\mathrm{kg}}$$

$$\alpha_\nu = 33201\ \mathrm{m^{-1}} \sim 332.01\ \mathrm{cm^{-1}}$$

So the mean free path goes:

$$l_\nu = \frac{1}{\alpha_\nu} = \frac{1}{33201\ \mathrm{m^{-1}}} = 3.012 \times 10^{-5}\ \mathrm{m}$$

$$\boxed{l_\nu = 3.012 \times 10^{-3}\ \mathrm{cm}}$$

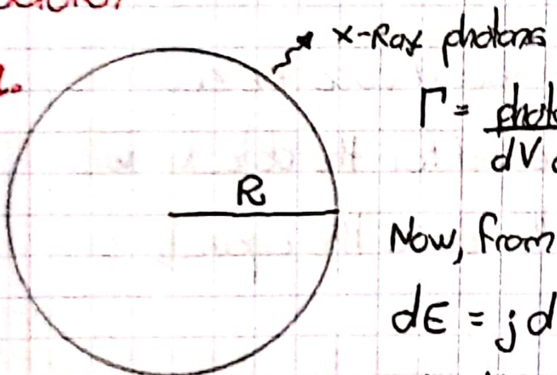
#2 X-Ray photons are produced in a cloud of radius R at the uniform rate Γ (photons per unit volume per unit time). The cloud is a distance d away. Neglect absorption of these photons (optically thin medium). A detector at Earth has an angular acceptance beam of half angle $\Delta\theta$ and its has an effective area of ΔA.

a) Assume that the source is completely resolved. What is the observed intensity (photons per unit time per unit area per steradian) toward the center of the cloud.

b) Assume that the source is completely unresolved. What is the observed average intensity when the source is in the beam of the detector?

Solution

a.



$$\Gamma = \frac{\text{photons}}{dV\,dt}$$

First we have to keep in mind that if the source is completely resolved that means the object can be viewed as an entire image.

Now, from the emission energy equation we got

$$dE = j\,dV\,d\Omega\,dt$$

If we consider the number of photons as a measure of energy we have:

$$\frac{dE}{dV\,d\Omega\,dt} = j \quad \text{and} \quad \Gamma = \frac{\text{photons}}{dV\,dt},$$

the term $d\Omega$ is equivalent to 4π ster integrating the differential.

So, in the end we have:

$$\frac{dE}{dV\,d\Omega\,dt} = j = \frac{\text{photons}}{dV\,d\Omega\,dt} = \boxed{\frac{\Gamma}{4\pi}}$$

For an isotropic source.

Having the previous relationship we can use the following relation

$$\frac{dI}{ds} = j \Rightarrow \frac{dI}{ds} = \frac{\Gamma}{4\pi}$$

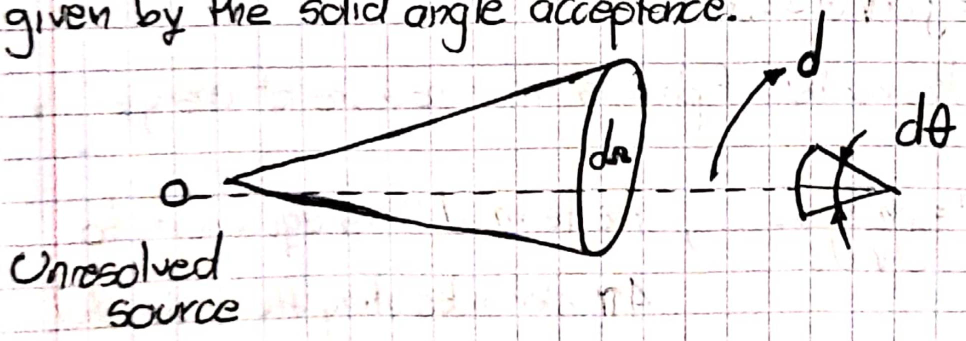
Now solving the equation to the whole length we obtain

$$\frac{dI}{ds} = \frac{\Gamma}{4\pi} \Rightarrow dI = \frac{\Gamma}{4\pi} ds$$

$$\int dI = \frac{\Gamma}{4\pi} \int_0^{2R} ds$$

$$I = \frac{\Gamma}{\cancel{4}\pi} \cdot \cancel{2}R = \boxed{\frac{\Gamma R}{2\pi}}$$

b. Because the problem suggest that in the second case the source is unresolved we can think that the intensity received for the detector will be inversely proportional to its capacity, in this case the capacity is given by the solid angle acceptance.



Now considering that the portion of the solid angle will be proportional to the detector half angle area we obtain.

$$d\Omega = \pi(\Delta\theta)^2$$

Now, the average intensity will be proportional to the radiative Flux divided by the angle of acceptance. Thus, we have:

$$F = \frac{\text{Rate of energy (constant)}}{4\pi d^2} \Rightarrow F = \frac{L \text{ (luminosity)}}{4\pi d^2}$$

The flux decrease proportional to $1/d^2$. It is distance dependent.

Now, to find the luminosity we can consider the units of Γ

$$\Gamma = \frac{\text{photons (energy)}}{\text{volume} \cdot \text{time}} \quad \text{so} \quad \frac{E}{V \cdot t} = \Gamma \quad \text{hence} \quad \frac{dE}{dV\,dt} = \Gamma$$

To this case the energy is related with the total flux so that means luminosity is given by:

$$L = \Gamma \cdot \left(\frac{4}{3}\pi R^3\right) \cdot t$$

isotropic source

Thus we have:

$$F = \frac{L}{4\pi d^2} = \frac{\Gamma \cdot \left(\frac{\cancel{4}}{3}\cancel{\pi} R^3\right)}{\cancel{4}\cancel{\pi} d^2}$$

So the instantaneous flux will be:

$$F = \frac{R^3 \Gamma}{3d^2}$$

Now, From the previous analysis we obtain the average intensity given by

$$I = \frac{F}{d\Omega} = \boxed{\frac{R^3 \Gamma}{3d^2 \pi (\Delta\theta)^2}}$$