Tarea 5: Mecánica Clásica

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• Ejercicio #1: Transformación de Legendre.

Con $df = udx + vdy$ y $g = f - ux$ demostrar que $dg = vdy - xdu$

$$g = f - ux \Rightarrow dg = d(f - ux) \quad dg = df - d(ux)$$

$$d(ux) = udx + xdu \rightarrow dg = (udx + vdy) - (udx + xdu) \quad dg$$

$$dg = vdy - xdu$$

• Ejercicio #2:

Teniendo que $G = X - TS \quad dG = d(X - TS)$

tendremos $dG = dX - d(TS)$

Por otro lado,

$$d(TS) = TdS + SdT$$

Sustituyendo dotendremos,

$$dX = TdS + VdP$$

$$dG = (TdS + VdP) - (TdS + SdT) \Rightarrow dG = VdP - SdT$$

Tendremos entonces que

$$dG = VdP - SdT$$

• Ejercicio #3: Función de Routh

Si se considera que $\mathcal{L}(q_i, \dot{q}_i)$ para $i = 1, 2, \dots, n$, tendremos que,

$$d\mathcal{L} = \frac{\partial \mathcal{L}}{\partial q_i} dq_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i$$

Ahora bien, se puede realizar la trasformada de Legendre con q_j y P_j sabiendo que, $j = 1, 2, \dots, n$ son las nuevas variables.

Entonces,

$P_j = \frac{\partial \alpha}{\partial \dot{q}_j}$ y $\dot{P}_j = \frac{\partial \alpha}{\partial q_j}$

Con lo cual,

$$d\alpha = \dot{P}_j\, dq_j + P_j d\dot{q}_j$$
$$= \dot{P}_j\, dq_j + d(P_j \dot{q}_j) - dP_j \dot{q}_j$$

Así,

$$d(\alpha - P_j \dot{q}_j) = \dot{P}_j dq_j - \dot{q}_j dP_j$$
$$d(P_j \dot{q}_j - \alpha) = \dot{q}_j dP_j - \dot{P}_j\, dq_j$$

Si sustituimos R por $P_j \dot{q}_j - \alpha$, tendremos que,

$$dR = \dot{q}_j\, dp_j - \dot{P}_j\, dq_j$$

$$R(q_j, \dot{q}_j, p_j, \dot{P}_j) = P_j \dot{q}_j - \alpha$$

• Ejercicio #4: Poisson

a) Teniendo que $\vec{L} = \vec{r} \times \vec{p}$, o bien, $L_i = \epsilon_{imn} q_m p_n$ y $L_j = \epsilon_{jmn} q_m p_n$

Computamos,

$$[L_i, L_j] = \frac{\partial L_i}{\partial p_k}\frac{\partial L_j}{\partial q_k} - \frac{\partial L_i}{\partial q_k}\frac{\partial L_j}{\partial p_k}$$

$$= (\epsilon_{imn} q_m \delta_{nk})(\epsilon_{jmn} \delta_{km} p_n) - (\epsilon_{imn} \delta_{mk} p_n)(\epsilon_{jmn} q_n \delta_{nk})$$

$$= (\epsilon_{imk} q_m)(\epsilon_{jkn} p_n) - (\epsilon_{ikn} p_n)(\epsilon_{jmk} q_m)$$

$$= -\epsilon_{imk} \epsilon_{jnk} q_m p_n + \epsilon_{ink} \epsilon_{jmk} p_n q_m$$

$$= -(\delta_{ij}\delta_{mn} - \delta_{in}\delta_{mj}) q_m p_n + (\delta_{ij}\delta_{nm} - \delta_{im}\delta_{nj}) p_n q_m$$

$$= -\delta_{ij} q_n p_n + p_i q_j + \delta_{ij} p_m q_m - p_j q_i$$

Considerando que, cuando $i=j \Rightarrow [L_i,L_j]=0$, por consiguiente, $i\neq j$

$$[L_i,L_j] = P_i q_j - P_j q_i = -\epsilon_{ijk}L_k = -\epsilon_{ijk}(\epsilon_{kmn} q_m P_n) = -(\delta_{in}\delta_{jm} - \delta_{im}\delta_{jn}) q_m P_n$$
$$= -q_i P_j + q_j P_i$$

b)

$$[P_i,P_j] = \frac{\partial P_i}{\partial P_k}\frac{\partial P_j}{\partial q_k} - \frac{\partial P_i}{\partial q_k}\frac{\partial P_j}{\partial P_k} = \delta_{ik}\delta_{jk} - \delta_{ik}\delta_{jk} = 0$$

c)

$$[L_i,P_j] = \frac{\partial}{\partial P_k}\epsilon_{imn}q_mP_n \cancelto{0}{\frac{\partial P_j}{\partial q_k}} - \frac{\partial}{\partial q_k}\epsilon_{imn}q_mP_n\frac{\partial P_j}{\partial P_k}$$
$$= -\epsilon_{imn}\delta_{km}P_n\delta_{jk} = -\epsilon_{imn}\delta_{mj}P_n = -\epsilon_{ijn}P_n = -\epsilon_{ijk}P_k$$

d)

Teniendo que,

$$[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$$

Computando obtenemos,

$$[f,[\frac{\partial g}{\partial P_k}\frac{\partial h}{\partial q_k} - \frac{\partial g}{\partial q_k}\frac{\partial h}{\partial P_k}]] + [g,[\frac{\partial h}{\partial P_k}\frac{\partial f}{\partial q_k} - \frac{\partial h}{\partial q_k}\frac{\partial f}{\partial P_k}]] + [h,[\frac{\partial f}{\partial P_k}\frac{\partial g}{\partial q_k} - \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial P_k}]]$$

$$\Rightarrow \frac{\partial f}{\partial P_k}\frac{\partial}{\partial q_k}\left(\frac{\partial g}{\partial P_k}\frac{\partial h}{\partial q_k} - \frac{\partial g}{\partial q_k}\frac{\partial h}{\partial P_k}\right) - \frac{\partial f}{\partial q_k}\frac{\partial}{\partial P_k}\left(\frac{\partial g}{\partial P_k}\frac{\partial h}{\partial q_k} - \frac{\partial g}{\partial q_k}\frac{\partial h}{\partial P_k}\right) + \frac{\partial g}{\partial P_k}\frac{\partial}{\partial q_k}\left(\frac{\partial h}{\partial P_k}\frac{\partial f}{\partial q_k} - \frac{\partial h}{\partial q_k}\frac{\partial f}{\partial P_k}\right)$$

$$- \frac{\partial g}{\partial q_k}\frac{\partial}{\partial P_k}\left(\frac{\partial h}{\partial P_k}\frac{\partial f}{\partial q_k} - \frac{\partial h}{\partial q_k}\frac{\partial f}{\partial P_k}\right) + \frac{\partial h}{\partial P_k}\frac{\partial}{\partial q_k}\left(\frac{\partial f}{\partial P_k}\frac{\partial g}{\partial q_k} - \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial P_k}\right) - \frac{\partial h}{\partial q_k}\frac{\partial}{\partial P_k}\left(\frac{\partial f}{\partial P_k}\frac{\partial g}{\partial q_k} - \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial P_k}\right)$$

$$\Rightarrow \frac{\partial f}{\partial P_k}\frac{\partial^2 g}{\partial q_k\partial P_k}\frac{\partial h}{\partial q_k} + \frac{\partial f}{\partial P_k}\frac{\partial^2 h}{\partial q_k^2}\frac{\partial g}{\partial P_k} - \frac{\partial f}{\partial P_k}\frac{\partial g}{\partial q_k}\frac{\partial^2 h}{\partial q_k\partial P_k} - \frac{\partial f}{\partial P_k}\frac{\partial^2 g}{\partial q_k^2}\frac{\partial h}{\partial P_k}$$

$$- \frac{\partial f}{\partial q_k}\frac{\partial^2 g}{\partial P_k^2}\frac{\partial h}{\partial q_k} - \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial P_k}\frac{\partial^2 h}{\partial P_k\partial q_k} + \frac{\partial f}{\partial q_k}\frac{\partial h}{\partial P_k}\frac{\partial^2 g}{\partial P_k\partial q_k} + \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial q_k}\frac{\partial^2 h}{\partial P_k^2} + \frac{\partial g}{\partial P_k}\frac{\partial h}{\partial P_k}\frac{\partial^2 f}{\partial q_k^2}$$

$$+ \frac{\partial g}{\partial P_k}\frac{\partial f}{\partial q_k}\frac{\partial^2 h}{\partial q_k\partial P_k} - \frac{\partial g}{\partial P_k}\frac{\partial h}{\partial q_k}\frac{\partial^2 f}{\partial q_k\partial P_k} - \frac{\partial g}{\partial P_k}\frac{\partial f}{\partial P_k}\frac{\partial^2 h}{\partial q_k^2} - \frac{\partial g}{\partial q_k}\frac{\partial h}{\partial P_k}\frac{\partial^2 f}{\partial P_k\partial q_k}$$

$$- \frac{\partial g}{\partial q_k}\frac{\partial f}{\partial q_k}\frac{\partial^2 h}{\partial P_k^2} + \frac{\partial g}{\partial q_k}\frac{\partial h}{\partial q_k}\frac{\partial^2 f}{\partial P_k^2} + \frac{\partial g}{\partial q_k}\frac{\partial f}{\partial P_k}\frac{\partial^2 h}{\partial q_k\partial P_k} + \frac{\partial h}{\partial P_k}\frac{\partial f}{\partial P_k}\frac{\partial^2 g}{\partial q_k^2} +$$

$$\frac{\partial h}{\partial P_k}\frac{\partial g}{\partial q_k}\frac{\partial^2 f}{\partial q_k\partial P_k} - \frac{\partial h}{\partial P_k}\frac{\partial f}{\partial P_k}\frac{\partial^2 g}{\partial q_k\partial P_k} - \frac{\partial h}{\partial P_k}\frac{\partial g}{\partial P_k}\frac{\partial^2 f}{\partial q_k^2} - \frac{\partial h}{\partial q_k}\frac{\partial f}{\partial P_k}\frac{\partial^2 g}{\partial P_k\partial q_k} - \frac{\partial h}{\partial q_k}\frac{\partial g}{\partial q_k}\frac{\partial^2 f}{\partial P_k^2}$$

$$\frac{\partial h}{\partial q_k}\frac{\partial f}{\partial q_k}\frac{\partial^2 g}{\partial P_k^2} + \frac{\partial h}{\partial q_k}\frac{\partial g}{\partial P_k}\frac{\partial^2 f}{\partial P_k\partial q_k} = 0$$

• Ejercicio # 5: Análisis dimensional y lagrangiano.

Teniendo el lagrangiano de la forma,

$$\mathcal{L} = \frac{m\vec{v}^2}{2} + \sum_n a_n \vec{v}^{2n} - U(\vec{r})$$

Veremos que el término $a_n \vec{v}^{2n}$ requiere poseer unidades de energía de la forma $[E] = \frac{ML^2}{T^2}$.

Por consiguiente,

$$a_n v^{2n} = [a_n]\left[\frac{L}{T}\right]^{2n} = \frac{ML^2}{T^2} \Rightarrow [a_n]\frac{L^{2n}}{T^{2n}} = \frac{ML^2}{T^2}$$

$$\Rightarrow [a_n] = M\left(\frac{T}{L}\right)^{2n-2}$$

Momento canónico:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = m\dot{x}_i + \sum_n a_n \frac{\partial}{\partial \dot{x}_i}(\dot{\vec{x}}^2)^n$$

$$= m\dot{x}_i + \sum_n a_n \cdot 2n(\dot{\vec{x}}^2)^{n-1}\dot{x}_i$$

Hamiltoniano:

$$H = \sum_i p_i \dot{x}_i - \mathcal{L}$$

$$H = \left(m\dot{\vec{x}}_i^2 + 2\sum_n n a_n (\dot{\vec{x}}_i^2)^n\right) - \left(\frac{m\dot{\vec{x}}_i^2}{2} + \sum_n a_n(\dot{\vec{x}}_i^2)^n - U(\vec{r})\right)$$

$$H = \frac{m\dot{\vec{x}}_i^2}{2} + \sum_n (2n-1)a_n(\dot{\vec{x}}_i^2)^n + U(\vec{r})$$

Corchetes de Poisson:

$$\{f, g\} = \sum_i\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i}\right)$$, con $f = p_i$ y $g = x_j$ y con $f = mv_i$ y $g = x_j$ tendremos,

$$\{p_i, x_j\} = \frac{\partial p_i}{\partial x_k}\frac{\partial x_j}{\partial p_k} - \frac{\partial x_j}{\partial x_k}\frac{\partial p_i}{\partial p_k} = -\delta_{ij}$$

$$\{mv_i, x_i\} = \frac{\partial m v_i}{\partial x_k}\frac{\partial x_j}{\partial p_k} - \frac{\partial x_j}{\partial x_k}\frac{\partial m v_i}{\partial p_k} = -\delta_{ij}$$

- Ejercicio #6:

Tendremos que $\delta S = P_2^a \delta q_2^a - P_1^a \delta q_1^a$ con la acción,

$$S = \int_{t_1}^{t_2} \mathcal{L} dt \Longrightarrow \frac{dS}{dt_1} = -\mathcal{L}(t_1) = \frac{\partial S}{\partial t_1} + \frac{\partial S}{\partial q_1^a}\dot{q}_1^a = \frac{\partial S}{\partial t_1} - P_1^a \dot{q}_1^a \text{ y,}$$

$$\frac{dS}{dt_2} = \mathcal{L}(t_2) = \frac{\partial S}{\partial t_2} + \frac{\partial S}{\partial q_2^a}\dot{q}_2^a = \frac{\partial S}{\partial t_2} + P_2^a \dot{q}_2^a$$

$$\frac{\partial S}{\partial t_1} = P_1^a \dot{q}_1^a - \mathcal{L}(t_1) = H_1 \quad , \quad \frac{dS}{dt_2} = -P_2^a \dot{q}_2^a + \mathcal{L}(t_2) = -H_2$$

$$-\mathcal{L}(t_1) = \frac{dS}{dt_1} = -P_1^a \dot{q}_1^a + H_1 \, , \quad \mathcal{L}(t_2) = \frac{dS}{dt_1} = -H_2 + P_2^a \dot{q}_2^a$$

Así,

$$dS_1 = H_1 dt_1 - P_1^a dq_1^a \quad , \quad dS_2 = -H_2 dt_2 + P_2^a dq_2^a$$

Por lo tanto,

$$dS = dS_1 + dS_2 = (H_1 dt_1 - P_1^a dq_1^a) - H_2 dt_2 + P_2^a dq_2^a$$
$$= P_2^a dq_2^a - H_2 dt_2 - (P_1^a dq_1^a - H_1 dt_1)$$

- Ejercicio #7:

Conocemos que $|\dot{\vec{r}}| = \sqrt{\frac{2}{m}(E - V(\vec{r}))}$, así pues, $\frac{d\vec{r}}{dl} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} = \frac{\dot{\vec{r}}}{\sqrt{\frac{2}{m}(E - V(\vec{r}))}}$.

Aplicando el principio de Maupertius obtendremos,

$$\delta S_0 = \delta \int \sqrt{2m(E-V)}\, dl = 0 \quad \Rightarrow \quad \delta S_0 = \delta \int \sqrt{(E-V)}\, dl = 0$$

$$\delta S_0 = \int \frac{-V'(r)}{2\sqrt{E-V}} dr + \sqrt{(E-V)} \frac{dr}{dl} \cdot d(\delta r) = 0$$

lo que nos lleva a

$$\frac{d^2\vec{r}}{dl^2} = \frac{\left[V'(r) - \left(V'(r) \cdot \frac{d\vec{r}}{dl}\right)\frac{d\vec{r}}{dl}\right]}{2(E-U)}$$

• Ejercicio #8:

a) Para $F_1(q,Q)$ se debe cumplir $\frac{\partial F_1}{\partial q_i}=p_i$, $\frac{\partial F_1}{\partial Q_i}=-P_i$, con $q_i=Q_i$ y $p_i=P_i$ vemos que no existe función generatriz.

Igualmente, con $F_4(p,P)$ se cumple que $\frac{\partial F_4}{\partial P_i}=Q_i$, $\frac{\partial F}{\partial p_i}=-q_i$ siendo igual al resultado de F_1.

Ahora bien, para F_2 y F_3 tendremos,

$F_2(q,P)$

$$\frac{\partial F_2}{\partial q_i}=p_i \ , \ \frac{\partial F_2}{\partial P_i}=Q,$$

$$\longrightarrow F_2=q_iP_i$$

$$dF_2=q_i dP_i+P_i dq_i$$

$$\frac{\partial F_2}{\partial q_i}=P_i=q_i \ , \ \frac{\partial F_2}{\partial P_i}=q_i=Q_i$$

$F_3(p,Q)$

$$\frac{\partial F_3}{\partial p_i}=-q_i \ , \ \frac{\partial F_3}{\partial Q_i}=-P_i$$

$$\longrightarrow F_3=-p_i Q_i$$

$$dF_3=-p_i dQ_i-Q_i dp_i$$

$$\frac{\partial F_3}{\partial p_i}=-Q_i=-q_i \ , \ \frac{\partial F_3}{\partial Q_i}=-p_i=-P_i$$

b) Con $F_2=f_i(q_i)P_i$

$$\frac{\partial F_2}{\partial q_i}=P_i\frac{\partial f_i}{\partial q_i}=p_i \quad , \quad \frac{\partial F_2}{\partial P_i}=f_i=Q_i$$

$$P_i=\frac{\partial Q_i}{\partial q_i}=p_i \Rightarrow P_iQ_i=p_iq_i$$

c) Si se tiene que,

$$Q_i=a_{iu}q_u \quad y \quad \frac{\partial F_2}{\partial P_i}=Q_i, \text{ entonces, } \frac{\partial F_2}{\partial P_i}=Q_i=a_{iu}q_u \quad , \quad F_2=a_{iu}q_uP_i$$

Además, $\frac{\partial F_2}{\partial q_i}=p_i \Longrightarrow p_i=a_{iu}P_u$

d) Si se considera la función generatriz de la forma $F_1 = \frac{m}{2}\omega q^2 \cot Q$, tendremos que,

$$\frac{\partial F_1}{\partial q_i} = p_i \,, \quad \frac{\partial F_1}{\partial Q_1} = -P_i$$

$$\frac{\partial F_1}{\partial q} = m\omega q \cot Q = p \,, \quad \frac{\partial F_1}{\partial Q} = -\frac{m}{2}\omega q^2 \csc^2 Q = -\frac{m}{2}\omega q^2 \frac{1}{\operatorname{sen}^2 Q} = -P$$

Así pues, vemos que,

$$q = \sqrt{\frac{2P}{m\omega}}\operatorname{sen} Q \quad , \quad p = \sqrt{2Pm\omega}\cos Q$$

De esta forma veremos que,

$$H = \frac{p^2}{2m} + \frac{kq^2}{2} = \frac{\cancel{2}Pm\omega}{\cancel{2}m}\cos^2 Q + \frac{k}{2}\frac{2P}{m\omega}\operatorname{sen}^2 Q$$

$$= P\omega\cos^2 Q + \frac{Pk}{m\omega}\operatorname{sen}^2 Q$$

$$\Rightarrow H = P\omega\cos^2 Q + \frac{P\omega^2 m}{m\omega}\operatorname{sen}^2 Q = P\omega$$

Con $P = H/\omega$, tendremos,

$$\dot{Q} = \frac{\partial H}{\partial P} = \omega$$

$$\frac{dQ}{dt} = \omega \Rightarrow Q = \omega t + c$$

• Ejercicio #9:

Teniendo $q(Q,P)$, $p(Q,P)$ y $f(q,p) = f'(Q,P)$, $g(q,p) = g'(Q,P)$ y

$$\frac{\partial f}{\partial q} = \frac{\partial f'}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial f'}{\partial P}\frac{\partial P}{\partial q} \quad , \quad \frac{\partial f}{\partial p} = \frac{\partial f'}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial f'}{\partial P}\frac{\partial P}{\partial p}$$

Operando obtendremos,

$$[f,g]_{q,p} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} = \left(\frac{\partial f'}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial f'}{\partial P}\frac{\partial P}{\partial q}\right)\left(\frac{\partial g'}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial g'}{\partial P}\frac{\partial P}{\partial p}\right)$$

$$- \left(\frac{\partial f'}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial f'}{\partial P}\frac{\partial P}{\partial p}\right)\left(\frac{\partial g'}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial g'}{\partial P}\frac{\partial P}{\partial q}\right)$$

$$= \frac{\partial f'}{\partial Q}\frac{\partial Q}{\partial q}\frac{\partial g'}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial f'}{\partial Q}\frac{\partial Q}{\partial q}\frac{\partial g'}{\partial P}\frac{\partial P}{\partial p} + \frac{\partial f'}{\partial P}\frac{\partial P}{\partial q}\frac{\partial g'}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial f'}{\partial P}\frac{\partial P}{\partial q}\frac{\partial g'}{\partial P}\frac{\partial P}{\partial p}$$

$$- \frac{\partial f'}{\partial Q}\frac{\partial Q}{\partial p}\frac{\partial g'}{\partial Q}\frac{\partial Q}{\partial q} - \frac{\partial f'}{\partial Q}\frac{\partial Q}{\partial p}\frac{\partial g'}{\partial P}\frac{\partial P}{\partial q} - \frac{\partial f'}{\partial P}\frac{\partial P}{\partial p}\frac{\partial g'}{\partial Q}\frac{\partial Q}{\partial q} - \frac{\partial f'}{\partial P}\frac{\partial P}{\partial p}\frac{\partial g'}{\partial P}\frac{\partial P}{\partial q}$$

$$= \frac{\partial f'}{\partial Q}\frac{\partial g'}{\partial Q}\left[\frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial Q}{\partial q}\right] + \frac{\partial f'}{\partial Q}\frac{\partial g'}{\partial P}\left[\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right]$$

$$+ \frac{\partial f'}{\partial P}\frac{\partial g'}{\partial Q}\left[\frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q}\right] + \frac{\partial f'}{\partial P}\frac{\partial g'}{\partial P}\left[\frac{\partial P}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial P}{\partial q}\right]$$

$$= \frac{\partial f'}{\partial Q}\frac{\partial g'}{\partial Q}\cancelto{0}{[Q,Q]_{p,q}} + \frac{\partial f'}{\partial Q}\frac{\partial g'}{\partial P}[Q,P]_{p,q} + \frac{\partial f'}{\partial P}\frac{\partial g'}{\partial Q}[P,Q]_{p,q} + \frac{\partial f'}{\partial P}\frac{\partial g'}{\partial P}\cancelto{0}{[P,P]_{p,q}}$$

$$= [Q,P]_{p,q}[f',g']_{P,Q} = [f',g']_{P,Q} = [f,g]_{P,Q} = [f,g]_{p,q}$$