Ejercicio #2.

Teniendo el tensor simétrico $A_{\mu\nu} = A_{\nu\mu}$ cuyas componentes son $A_{00} = 0$ en todo marco inercial, y V^μ un vector temporal de arbitrario, computar $A_{\mu\nu} V^\mu V^\nu$

Vemos que V^μ cumple la propiedad, para $V^\mu (V^0, V^1, V^2, V^3)^T$

$$\eta_{\mu\nu} V^\mu V^\nu = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu} V^\mu V^\nu$$

$$\sum_{\mu}^{3} \eta_{\mu 0} V^\mu V^0 + \eta_{\mu 1} V^\mu V^1 + \eta_{\mu 2} V^\mu V^2 + \eta_{\mu 3} V^\mu V^3$$

$$= \eta_{00} V^0 V^0 + \eta_{10} V^1 V^0 + \eta_{20} V^2 V^0 + \eta_{30} V^3 V^0 + \cdots$$

$$= \eta_{00} V^0 V^0 + \eta_{11} V^1 V^1 + \eta_{22} V^2 V^2 + \eta_{33} V^3 V^3$$

$$-(V^0)^2 + (V^1)^2 + (V^2)^2 + (V^3)^3$$

donde,

$$(V^0)^2 > (V^1)^2 + (V^2)^2 + (V^3)^3$$

Ahora bien, también tendremos que, con $A_{00} = 0$,

$$A_{\mu\nu} = \begin{bmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{10} & A_{11} & A_{12} & A_{13} \\ A_{20} & A_{21} & A_{22} & A_{23} \\ A_{30} & A_{31} & A_{32} & A_{33} \end{bmatrix} = A_{\nu\mu} = \begin{bmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{01} & A_{11} & A_{12} & A_{13} \\ A_{02} & A_{12} & A_{22} & A_{23} \\ A_{03} & A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} A_{\mu\nu}V^\mu V^\nu$$

$$= \sum_{\mu=0}^{3} A_{\mu 0}V^\mu V^0 + \sum_{\mu=0}^{3} A_{\mu 1}V^\mu V^1 + \sum_{\mu=0}^{3} A_{\mu 2}V^\mu V^2 + \sum_{\mu=0}^{3} A_{\mu 3}V^\mu V^3$$

$$= A_{00}V^0V^0 + 2(A_{01}V^0V^1) + 2(A_{02}V^0V^2) + 2(A_{03}V^0V^3)$$

$$+ 2(A_{12}V^1V^2) + 2(A_{13}V^1V^3) + 2(A_{23}V^2V^3) +$$

$$\delta_{st}A_{ts}V^tV^s$$

La condición de que el vector V^μ sea temporaloide también indica que existirá un marco de referencia inercial en el cual todos sus componentes espaciales se anulan, esto es, $V^\mu(V^0,0,0,0)^T$, con esto tendremos que,

$$= A_{00}V^0V^0 + \cancelto{0}{2(A_{01}V^0V^1)} + \cancelto{0}{2(A_{02}V^0V^2)} + \cancelto{0}{2(A_{03}V^0V^3)}$$

$$+ \cancelto{0}{2(A_{12}V^1V^2)} + \cancelto{0}{2(A_{13}V^1V^3)} + \cancelto{0}{2(A_{23}V^2V^3)} +$$

$$\cancelto{0}{\delta_{st}A_{ts}V^tV^s}$$

$= A_{00}V^0V^0$, y dado que $A_{00}=0$ para todo marco inercial tendremos que,

$$A_{\mu\nu}V^\mu V^\nu = \cancelto{0}{A_{00}}V^0V^0 = 0$$

Ejercicio #3

El tensor electromagnético $F_{\mu\nu}$, antisimétrico, se puede representar como,

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}$$

Ahora bien, si cambiamos $\vec{B} \rightarrow \vec{E}$ y $\vec{E} \rightarrow -B$, el correspondiente dual del tensor electromagnético será,

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{bmatrix}$$

Si queremos computar $\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$ debemos hallar $\tilde{F}_{\mu\nu}$, esto es,

$$\tilde{F}_{\mu\nu} = \eta_{\mu\alpha}\eta_{\beta\nu}\tilde{F}^{\alpha\beta} \quad , \quad \tilde{F}_{\mu\nu} = \eta_{\mu\alpha}\tilde{F}^{\alpha\beta}\eta_{\beta\nu}$$

o bien,

$$\tilde{F}^{\alpha}{}_{\nu} = \tilde{F}^{\alpha\beta}\eta_{\beta\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{F}^{\alpha}{}_{\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{bmatrix}$$

, luego $\tilde{F}_{\mu\nu} = \eta_{\mu\alpha}\tilde{F}^{\alpha}{}_{\nu}$

$$\tilde{F}_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & B_x & B_y & B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{bmatrix} \Rightarrow$$

$$\tilde{F}_{\mu\nu} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{bmatrix}$$

, entonces, una vez que tenemos $\tilde{F}_{\mu\nu}$ y $\tilde{F}^{\mu\nu}$ computamos,

$$\begin{aligned} \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = {} & \cancel{\tilde{F}_{00}\tilde{F}^{00}} + \tilde{F}_{01}\tilde{F}^{01} + \tilde{F}_{02}\tilde{F}^{02} + \tilde{F}_{03}\tilde{F}^{03} + \\ & \tilde{F}_{10}\tilde{F}^{10} + \cancel{\tilde{F}_{11}\tilde{F}^{11}} + \tilde{F}_{12}\tilde{F}^{12} + \tilde{F}_{13}\tilde{F}^{13} + \\ & \tilde{F}_{20}\tilde{F}^{20} + \tilde{F}_{21}\tilde{F}^{21} + \cancel{\tilde{F}_{22}\tilde{F}^{22}} + \tilde{F}_{23}\tilde{F}^{23} + \\ & \tilde{F}_{30}\tilde{F}^{30} + \tilde{F}_{31}\tilde{F}^{31} + \tilde{F}_{32}\tilde{F}^{32} + \cancel{\tilde{F}_{33}\tilde{F}^{33}} \end{aligned}$$

$$\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = \tilde{F}_{01}\tilde{F}^{01} + \tilde{F}_{02}\tilde{F}^{02} + \tilde{F}_{03}\tilde{F}^{03} +$$
$$\tilde{F}_{10}\tilde{F}^{10} + \tilde{F}_{12}\tilde{F}^{12} + \tilde{F}_{13}\tilde{F}^{13} +$$
$$\tilde{F}_{20}\tilde{F}^{20} + \tilde{F}_{21}\tilde{F}^{21} + \tilde{F}_{23}\tilde{F}^{23} +$$
$$\tilde{F}_{30}\tilde{F}^{30} + \tilde{F}_{31}\tilde{F}^{31} + \tilde{F}_{32}\tilde{F}^{32}$$

$$\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = (-Bx)(Bx) + (-By)(By) + (-Bz)(Bz) +$$
$$(Bx)(-Bx) + (-Ez)(-Ez) + (Ey)(Ey) +$$
$$(By)(-By) + (Ez)(Ez) + (-Ex)(-Ex)$$
$$(Bz)(-Bz) + (-Ey)(-Ey) + (Ex)(Ex)$$

$$\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = -Bx^2 - By^2 - Bz^2$$
$$-Bx^2 + Ez^2 + Ey^2$$
$$-By^2 + Ez^2 + Ex^2$$
$$-Bz^2 + Ey^2 + Ex^2$$

$$\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = 2(Ex^2 + Ey^2 + Ez^2) - 2(Bx^2 + By^2 + Bz^2)$$

Si tenemos que $B^2 = Bx^2 + By^2 + Bz^2$ y $E^2 = Ex^2 + Ey^2 + Ez^2$, entonces,

$$\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = 2(E^2 - B^2)$$

Ejercicio # 4.

Por el ejercicio # 3 concluimos que,

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{bmatrix}$$

, ahora bien si sabemos que,

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}$$

tendremos entonces que, para hallar $F_{\mu\nu}$ necesitemos encontrar

$$F_{\mu\nu} = \eta_{\mu\alpha}\eta_{\beta\nu}F^{\alpha\beta} = \eta_{\mu\alpha}F^{\alpha\beta}\eta_{\beta\nu}$$

Así,

$$F^{\alpha}_{\ \nu} = F^{\alpha\beta}\eta_{\beta\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F^{\alpha}_{\ \nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

, luego $F_{\mu\nu} = \eta_{\mu\alpha}F^{\alpha}_{\ \nu}$,

$$F_{\mu\nu} = \eta_{\mu\alpha} F^{\alpha}_{\ \nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & E^x & E^y & E^z \\ E^x & 0 & B^z & -B^y \\ E^y & -B^z & 0 & B^x \\ E^z & B^y & -B^x & 0 \end{bmatrix}$$

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}.$$

Con esto en mente procedemos a calcular $F_{\mu\nu}\tilde{F}^{\mu\nu}$

$$\begin{aligned} F_{\mu\nu}\tilde{F}^{\mu\nu} = {} & \cancel{F_{00}\tilde{F}^{00}}^{\,0} + F_{01}\tilde{F}^{01} + F_{02}\tilde{F}^{02} + F_{03}\tilde{F}^{03} + \\ & F_{10}\tilde{F}^{10} + \cancel{F_{11}\tilde{F}^{11}}^{\,0} + F_{12}\tilde{F}^{12} + F_{13}\tilde{F}^{13} + \\ & F_{20}\tilde{F}^{20} + F_{21}\tilde{F}^{21} + \cancel{F_{22}\tilde{F}^{22}}^{\,0} + F_{23}\tilde{F}^{23} + \\ & F_{30}\tilde{F}^{30} + F_{31}\tilde{F}^{31} + F_{32}\tilde{F}^{32} + \cancel{F_{33}\tilde{F}^{33}}^{\,0} \end{aligned}$$

$$\begin{aligned} F_{\mu\nu}\tilde{F}^{\mu\nu} = {} & F_{01}\tilde{F}^{01} + F_{02}\tilde{F}^{02} + F_{03}\tilde{F}^{03} + \\ & F_{10}\tilde{F}^{10} + F_{12}\tilde{F}^{12} + F_{13}\tilde{F}^{13} + \\ & F_{20}\tilde{F}^{20} + F_{21}\tilde{F}^{21} + F_{23}\tilde{F}^{23} + \\ & F_{30}\tilde{F}^{30} + F_{31}\tilde{F}^{31} + F_{32}\tilde{F}^{32} \end{aligned}$$

$$
\begin{aligned}
F_{\mu\nu}\tilde{F}^{\mu\nu} &= F_{01}\tilde{F}^{01} + F_{02}\tilde{F}^{02} + F_{03}\tilde{F}^{03} + \\
&\quad F_{10}\tilde{F}^{10} + F_{12}\tilde{F}^{12} + F_{13}\tilde{F}^{13} + \\
&\quad F_{20}\tilde{F}^{20} + F_{21}\tilde{F}^{21} + F_{23}\tilde{F}^{23} + \\
&\quad F_{30}\tilde{F}^{30} + F_{31}\tilde{F}^{31} + F_{32}\tilde{F}^{32} \\
F_{\mu\nu}\tilde{F}^{\mu\nu} &= (-Ex)(Bx) + (-Ey)(By) + (-Ez)(Bz) + \\
&\quad (Ex)(-Bx) + (Bz)(-Ez) + (-By)(Ey) + \\
&\quad (Ey)(-By) + (-Bz)(Ez) + (Bx)(-Ex) + \\
&\quad (Ez)(-Bz) + (By)(-Ey) + (-Bx)(Ex) \\
F_{\mu\nu}\tilde{F}^{\mu\nu} &= -ExBx - EyBy - EzBz \\
&\quad - ExBx - EzBz - EyBy \\
&\quad - ByEy - EzBz - ExBx \\
&\quad - BzEz - EyBy - ExBx \\
F_{\mu\nu}\tilde{F}^{\mu\nu} &= -4ExBx - 4EyBy - 4EzBz \\
&= -4(ExBx + EyBy + EzBz)
\end{aligned}
$$

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = -4(\vec{E}\cdot\vec{B})$$

Ejercicio # 5

Con el tensor electromagnético dual, $\tilde{F}^{\mu\nu}$ se pide calcular la derivada covariante tal que $\tilde{F}^{\mu\nu}{}_{,\nu}$. Computando se tiene que,

- $$\frac{\partial \tilde{F}^{0\nu}}{\partial x^\nu} = \frac{\partial \tilde{F}^{00}}{\partial x^0} + \frac{\partial \tilde{F}^{01}}{\partial x^1} + \frac{\partial \tilde{F}^{02}}{\partial x^2} + \frac{\partial \tilde{F}^{03}}{\partial x^3}$$

$$= \cancel{\frac{\partial [0]}{\partial t}}^{0} + \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \nabla \cdot \vec{B}$$

- $$\frac{\partial \tilde{F}^{1\nu}}{\partial x^\nu} = \frac{\partial \tilde{F}^{10}}{\partial x^0} + \frac{\partial \tilde{F}^{11}}{\partial x^1} + \frac{\partial \tilde{F}^{12}}{\partial x^2} + \frac{\partial \tilde{F}^{13}}{\partial x^3}$$

$$= -\frac{\partial B_x}{\partial t} + \cancel{\frac{\partial [0]}{\partial x}}^{0} + -\frac{\partial E_z}{\partial y} + \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} - (\nabla \times \vec{E})_x$$

- $$\frac{\partial \tilde{F}^{2\nu}}{\partial x^\nu} = \frac{\partial \tilde{F}^{20}}{\partial x^0} + \frac{\partial \tilde{F}^{21}}{\partial x^1} + \frac{\partial \tilde{F}^{22}}{\partial x^2} + \frac{\partial \tilde{F}^{23}}{\partial x^3}$$

$$= -\frac{\partial B_y}{\partial t} + \frac{\partial E_z}{\partial x} + \cancel{\frac{\partial [0]}{\partial y}}^{0} - \frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} - (\nabla \times \vec{E})_y$$

- $$\frac{\partial \tilde{F}^{3\nu}}{\partial x^\nu} = \frac{\partial \tilde{F}^{30}}{\partial x^0} + \frac{\partial \tilde{F}^{31}}{\partial x^1} + \frac{\partial \tilde{F}^{32}}{\partial x^2} + \frac{\partial \tilde{F}^{33}}{\partial x^3}$$

$$= -\frac{\partial B_z}{\partial t} - \frac{\partial E_y}{\partial x} + \frac{\partial E_x}{\partial y} + \cancel{\frac{\partial [0]}{\partial z}}^{0} = -\frac{\partial B_z}{\partial t} - (\nabla \times \vec{E})_z$$

Si consideramos que $\frac{\partial \tilde{F}^{\mu\nu}}{\partial x^\nu} = 0$, entonces,

$$\nabla \cdot \vec{B} = 0 \quad y \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Ejercicio #6

Teniendo el tensor

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

se pide calcular T^{μ}_{μ}, para ello podemos hacer uso del tensor métrico tal que,

$$T^{\mu}_{\mu} = \eta_{\mu\nu}T^{\mu\nu}$$

Entonces,

$$T^{\mu}_{\mu} = \eta_{\mu\nu} \cdot \frac{1}{4\pi}\left(F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

$$T^{\mu}_{\mu} = \frac{1}{4\pi}\left(\eta_{\mu\nu}F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta_{\mu\nu}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

$$T^{\mu}_{\mu} = \frac{1}{4\pi}\left(F^{\mu\lambda}\eta_{\mu\nu}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta_{\mu\nu}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

$$\eta_{\mu\nu}\eta^{\mu\nu} = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\eta_{\mu\nu}\eta^{\mu\nu} = \eta_{00}^{2} + \sum_{i=1}^{3}\eta_{ii}^{2} = (-1)^2 + 1^2 + 1^2 + 1^2 = 4$$

$$T^{\mu}_{\mu} = \frac{1}{4\pi}\left(F^{\mu\lambda}F_{\mu\lambda} - \cancelto{1}{\frac{1}{4}\cdot 4}\, F_{\alpha\beta}F^{\alpha\beta}\right)$$

Así, llegamos a la expresión

$$T^{\mu}_{\mu} = \frac{1}{4\pi}\left(F^{\mu\lambda}F_{\mu\lambda} - F_{\alpha\beta}F^{\alpha\beta}\right) \longrightarrow$$

Dado que α y β son índices mudos, sin pérdida de generalalidad se puede establecer que,

$$T^{\mu}_{\mu} = \frac{1}{4\pi}\left(F^{\mu\lambda}F_{\mu\lambda} - F_{\mu\lambda}F^{\mu\lambda}\right)$$

Ahora bien, dado que conocemos que el producto $F^{\mu\lambda}F_{\mu\lambda}$ es un invariante relativista cuya magnitud es $2(B^2-E^2)$, entonces,

$$T^{\mu}_{\mu} = \frac{1}{4\pi}\left(2(B^2-E^2) - 2(B^2-E^2)\right)$$

$$T^{\mu}_{\mu} = \frac{1}{4\pi}\left[0\right]$$

$$T^{\mu}_{\mu} = 0$$

Ejercicio #7

Teniendo el tensor

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

se pide calcular $T^{\mu}_{\alpha}T^{\alpha}_{\mu}$. Este es el denominado tensor de energía-estrés electromagnético. En forma matricial tenemos que,

$$T^{\mu\nu} = \begin{bmatrix} \frac{(E^2+B^2)}{8\pi} & \frac{(\vec{E}\times\vec{B})_x}{4\pi} & \frac{(\vec{E}\times\vec{B})_y}{4\pi} & \frac{(\vec{E}\times\vec{B})_z}{4\pi} \\ \frac{(\vec{E}\times\vec{B})_x}{4\pi} & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ \frac{(\vec{E}\times\vec{B})_y}{4\pi} & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ \frac{(\vec{E}\times\vec{B})_z}{4\pi} & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{bmatrix}$$

Siendo que $\vec{S} = \frac{\vec{E}\times\vec{B}}{4\pi}$ el vector de Poynting para T^{0i} y T^{i0}.

Así,

$$T^{\mu}_{\alpha} = T^{\mu\nu}\eta_{\alpha\nu}$$

$$T^{\mu}_{\alpha} = \begin{bmatrix} -\frac{(E^2+B^2)}{8\pi} & S_x & S_y & S_z \\ -S_x & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ -S_y & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ -S_z & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{bmatrix}$$

Entonces T^{α}_{μ} será $(T^{\mu}_{\alpha})^{T}$, o lo mismo que,

$$T^{\alpha}_{\mu} = \begin{bmatrix} -\frac{(E^2+B^2)}{8\pi} & -S_x & -S_y & -S_z \\ S_x & -\sigma_{xx} & -\sigma_{yx} & -\sigma_{zx} \\ S_y & -\sigma_{xy} & -\sigma_{yy} & -\sigma_{zy} \\ S_z & -\sigma_{xz} & -\sigma_{yz} & -\sigma_{zz} \end{bmatrix}$$

Procedemos a computar,

$$T^{\mu}_{\alpha} T^{\alpha}_{\mu} = \sum_{\alpha=0}^{3} \sum_{\mu=0}^{3} T^{\mu}_{\alpha} T^{\alpha}_{\mu}$$

$$\sum_{\alpha=0}^{3} T_{\alpha}^{0} T_{0}^{\alpha} + T_{\alpha}^{1} T_{1}^{\alpha} + T_{\alpha}^{2} T_{2}^{\alpha} + T_{\alpha}^{3} T_{\alpha}^{3}$$

$$= T_0^0 T_0^0 + T_1^0 T_0^1 + T_2^0 T_0^2 + T_3^0 T_0^3 +$$
$$T_0^1 T_1^0 + T_1^1 T_1^1 + T_2^1 T_1^2 + T_3^1 T_1^3 +$$
$$T_0^2 T_2^0 + T_1^2 T_2^1 + T_2^2 T_2^2 + T_3^2 T_2^3 +$$
$$T_0^3 T_3^0 + T_1^3 T_3^1 + T_2^3 T_3^2 + T_3^3 T_3^3 +$$

$$= \frac{(E^2+B^2)}{8\pi} \frac{(E^2+B^2)}{8\pi} + S_x S_x + S_y S_y + S_z S_z +$$

$$(-S_x)(-S_x) + (-\sigma_{xx})(-\sigma_{xx}) + (-\sigma_{xy})(-\sigma_{xy}) + (-\sigma_{xz})(-\sigma_{xz}) +$$

$$(-S_y)(-S_y) + (-\sigma_{yx})(-\sigma_{yx}) + (-\sigma_{yy}\sigma_{yy}) + (-\sigma_{yz})(-\sigma_{yz})$$

$$(-S_z)(S_z) + (\sigma_{zx})(\sigma_{zx}) + (-\sigma_{yz})(-\sigma_{yz}) + (-\sigma_{zz})(\sigma_z\sigma_z)$$

Teniendo que

$$\sigma_{xx} = \frac{1}{4\pi}\left(E_x^2 + B_x^2 - \frac{1}{2}(E^2+B^2)\right)$$

$$\sigma_{yy} = \frac{1}{4\pi}\left(E_y^2 + B_y^2 - \frac{1}{2}(E^2+B^2)\right)$$

$$\sigma_{zz} = \frac{1}{4\pi}\left(E_z^2 + B_z^2 - \frac{1}{2}(E^2+B^2)\right)$$

y que, los términos cruzados serán

$$\sigma_{ij} = (E_i E_j + B_i B_j)$$

Vamos a obtener que,

$$T^{\mu}_{\alpha} T^{\alpha}_{\mu} = \frac{\left((E^2-B^2)^2 + (2\vec{E}\cdot\vec{B})^2\right)}{16\pi^2} \quad //$$

Ejercicio #8.

Teniendo la métrica

$$g_{\alpha\gamma}=\begin{bmatrix}1 & 0 & 0\\ 0 & \operatorname{sen}^2 r & 0\\ 0 & 0 & \operatorname{sen}^2 r\,\operatorname{sen}^2\theta\end{bmatrix}$$ para r, θ y φ.

Teniendo que,

$$\Gamma^{\gamma}_{\beta\mu}=\frac{1}{2}g^{\alpha\gamma}\left(g_{\alpha\beta,\mu}+g_{\alpha\mu,\beta}-g_{\beta\mu,\alpha}\right)$$

y que,

$$g^{\alpha\gamma}=\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{\operatorname{sen}^2 r} & 0\\ 0 & 0 & \frac{1}{\operatorname{sen}^2 r\,\operatorname{sen}^2\theta}\end{bmatrix}$$

Tenemos que, $g_{rr}=1$, $g_{\theta\theta}=\operatorname{sen}^2 r$, $g_{\varphi\varphi}=\operatorname{sen}^2 r\,\operatorname{sen}^2\theta$, y que $g^{rr}=1$, $g^{\theta\theta}=\csc^2 r$ y $g^{\varphi\varphi}=\csc^2 r\,\csc^2\theta$.

$$\rightarrow \quad \frac{\partial g_{rr}}{\partial r}=0, \quad \frac{\partial g_{rr}}{\partial \theta}=0, \quad \frac{\partial g_{rr}}{\partial \varphi}=0$$

$$\rightarrow \quad \frac{\partial g_{\theta\theta}}{\partial r}=2\operatorname{sen} r\cos r, \quad \frac{\partial g_{\theta\theta}}{\partial \theta}=0, \quad \frac{\partial g_{\theta\theta}}{\partial \varphi}=0$$

$$\rightarrow \quad \frac{\partial g_{\varphi\varphi}}{\partial r}=2\operatorname{sen} r\cos r\,\operatorname{sen}^2\theta, \quad \frac{\partial g_{\varphi\varphi}}{\partial \theta}=2\operatorname{sen}^2 r\,\operatorname{sen}\theta\cos\theta$$

$$\frac{\partial g_{\varphi\varphi}}{\partial \varphi}=0$$

Así podremos observar que

$$\Gamma^{r}_{\theta\theta} = -\operatorname{sen} r \cos r = -\frac{1}{2} \operatorname{sen} 2r$$

$$\Gamma^{r}_{\varphi\varphi} = -\operatorname{sen} r \cos r \operatorname{sen}^2\theta = -\frac{1}{2} \operatorname{sen} 2r \operatorname{sen}^2\theta$$

$$\Gamma^{\theta}_{\varphi\varphi} = -\operatorname{sen}\theta \cos\theta = -\frac{1}{2} \operatorname{sen} 2\theta$$

$$\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{\cos r}{\operatorname{sen} r} = \cot r$$

$$\Gamma^{\varphi}_{r\varphi} = \Gamma^{\varphi}_{\varphi r} = \frac{\cos r}{\operatorname{sen} r} = \cot r$$

$$\Gamma^{\varphi}_{\theta\varphi} = \Gamma^{\varphi}_{\varphi\theta} = \frac{\cos\theta}{\operatorname{sen}\theta} = \cot\theta$$

Ejercicio #9

$$B_{\mu\nu;\alpha} + B_{\nu\alpha;\mu} + B_{\alpha\mu;\nu} = B_{\mu\nu,\alpha} + B_{\nu\alpha,\mu} + B_{\alpha\mu,\nu}$$

$$B_{\mu\nu;\beta} = B_{\mu\nu,\beta} - B_{\alpha\nu}\Gamma^{\alpha}_{\mu\beta} - B_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}$$

$$B_{\nu\beta;\mu} = B_{\nu\beta,\mu} - B_{\mu\beta}\Gamma^{\alpha}_{\nu\mu} - B_{\nu\alpha}\Gamma^{\alpha}_{\beta\mu}$$

$$B_{\beta\mu;\nu} = B_{\beta\mu,\nu} - B_{\alpha\mu}\Gamma^{\alpha}_{\beta\nu} - B_{\beta\alpha}\Gamma^{\alpha}_{\mu\nu}$$

$$\Rightarrow B_{\mu\nu,\beta} - B_{\alpha\nu}\Gamma^{\alpha}_{\mu\beta} - B_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}$$
$$+ B_{\nu\beta,\mu} - B_{\mu\beta}\Gamma^{\alpha}_{\nu\mu} - B_{\nu\alpha}\Gamma^{\alpha}_{\beta\mu}$$
$$+ B_{\beta\mu,\nu} - B_{\alpha\mu}\Gamma^{\alpha}_{\beta\nu} - B_{\beta\alpha}\Gamma^{\alpha}_{\mu\nu}$$

Podemos observar que,

$$\Gamma^{\alpha}_{\mu\beta} = \Gamma^{\alpha}_{\beta\mu}, \quad \Gamma^{\alpha}_{\nu\beta} = \Gamma^{\alpha}_{\beta\nu}, \quad \Gamma^{\alpha}_{\nu\mu} = \Gamma^{\alpha}_{\mu\nu}$$

así,

$$B_{\mu\nu;\beta} + B_{\nu\beta;\mu} + B_{\beta\mu;\nu} = B_{\mu\nu,\beta} + B_{\nu\beta,\mu} + B_{\beta\mu,\nu}$$

$$-(B_{\alpha\nu} + B_{\nu\alpha})\Gamma^{\alpha}_{\mu\beta} - (B_{\mu\alpha} + B_{\alpha\mu})\Gamma^{\alpha}_{\beta\mu} - (B_{\alpha\beta} + B_{\beta\alpha})\Gamma^{\alpha}_{\mu\nu}$$

De esta forma podemos ver que, para que se cumpla la igualdad, es necesario que,

$$B_{\mu\nu} = -B_{\nu\mu} \;//$$

↳ Antisimetría

Ejercicio # 10

$$B_{\mu;\alpha} - B_{\alpha;\mu} = B_{\mu,\alpha} - B_{\alpha,\mu}$$

$$B_{\mu;\alpha} = B_{\mu,\alpha} - \Gamma^{\lambda}_{\mu\alpha} B_{\lambda}$$

$$B_{\alpha;\mu} = B_{\alpha,\mu} - \Gamma^{\lambda}_{\alpha\mu} B_{\lambda}$$

$$\Downarrow$$

$$B_{\mu,\alpha} - \Gamma^{\lambda}_{\mu\alpha} B_{\lambda} - B_{\alpha,\mu} + \Gamma^{\lambda}_{\alpha\mu} B_{\lambda}$$

Dado que, $\Gamma^{\lambda}_{\mu\alpha} = \Gamma^{\lambda}_{\alpha\mu}$, entonces,

$$B_{\mu,\alpha} - B_{\alpha,\mu} + B_{\lambda}\cancelto{0}{(\Gamma^{\lambda}_{\alpha\mu} - \Gamma^{\lambda}_{\mu\alpha})}$$

$$B_{\mu,\alpha} - B_{\alpha,\mu} = B_{\mu,\alpha} - B_{\alpha,\mu}$$

$$B_{\mu;\alpha} - B_{\alpha;\mu} = B_{\mu,\alpha} - B_{\alpha,\mu} \quad //$$