Ejercicio #2

Un vector se llama unitario si se cumple que $\vec{U} \cdot \vec{U} = \pm 1$, ahora bien, será temporaloide si su producto punto es negativo. Así, teniendo que $\vec{U} \cdot \vec{U} = \eta_{\alpha\beta} U^\alpha U^\beta = U^\alpha U_\alpha = -1$, se debe hallar U^μ considerando que $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$

$$\eta_{0\beta} U^0 U^\beta + \sum_{\alpha=1}^{3} \eta_{\alpha\beta} U^\alpha U^\beta = -1$$

$$-(1) \cdot (1)^2 + \sum_{\alpha=1}^{3} \cancelto{0}{(1) U^\alpha U^\beta} = -1$$

$\longrightarrow U^\mu = (1, 0, 0, 0)$ en su forma contravariante. En su forma covariante será,

$$U_\mu = \eta_{\mu\beta} U^\beta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} -1 \\ 0 \\ 0 \\ 0 \end{vmatrix} \Rightarrow U_\mu = (-1, 0, 0, 0)$$

$$U^\mu U_\mu = -1$$

También tenemos en cuenta que,

$$\eta^\alpha{}_\beta = \eta^{\alpha\mu} \eta_{\mu\beta}$$

$$\eta^\alpha{}_\beta = \delta^\alpha_\beta$$

Por lo que, si tenemos un vector arbitrario $\vec{V}$ con componentes $V^\mu \to (V^0, V^1, V^2, V^3)$, la expresión $(\eta^\alpha_\beta + U^\alpha U_\beta) V^\beta U_\alpha$ será

$$(\eta^{\alpha}_{\beta} + u^{\alpha} u_{\beta}) \nabla^{\beta} u_{\alpha} \Rightarrow (\delta^{\alpha}_{\beta} + u^{\alpha} u_{\beta}) \nabla^{\beta} u_{\alpha}$$

$$\delta^{\alpha}_{\beta} \nabla^{\beta} u_{\alpha} + \nabla^{\beta} u_{\alpha} u^{\alpha} u_{\beta}$$

$$\delta_{\beta}^{\alpha} \nabla^{\beta} u_{\alpha} + \nabla^{\beta} \underbrace{\cancel{u^{\alpha} u_{\alpha}}}_{}\!{}^{-1} u_{\beta}$$

$$\delta_{\beta}^{\alpha} \nabla^{\beta} u_{\alpha} - \nabla^{\beta} u_{\beta}$$

$$\nabla^{\beta} (\delta^{\alpha}_{\beta} u_{\alpha} - u_{\beta})$$

$$\nabla^{\beta} (\cancel{u_{\beta} - u_{\beta}}^{\,0}) = \nabla^{\beta}(0) = 0$$

$$(\eta^{\alpha}_{\beta} + u^{\alpha} u_{\beta}) \nabla^{\beta} u_{\alpha} = 0$$

De forma explícita,

$$\sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} (\eta^{\alpha}_{\beta} + u^{\alpha} u_{\beta}) \nabla^{\beta} u_{\alpha}$$

$$= \sum_{\alpha=0}^{3} (\eta^{\alpha}_{0} + u^{\alpha} u_{0}) \nabla^{0} u_{\alpha} + \sum_{\alpha=0}^{3} (\eta^{\alpha}_{1} + u^{\alpha} u_{1}) \nabla^{1} u_{\alpha}$$

$$+ \sum_{\alpha=0}^{3} (\eta^{\alpha}_{2} + u^{\alpha} u_{2}) \nabla^{2} u_{\alpha} + \sum_{\alpha=0}^{3} (\eta^{\alpha}_{3} + u^{\alpha} u_{3}) \nabla^{3} u_{\alpha}$$

$$= (\eta^{0}_{0} + u^{0} u_{0}) \nabla^{0} u_{0} + (\eta^{1}_{0} + u^{1} u_{0}) \nabla^{0} u_{1}$$

$$(\eta^{2}_{0} + u^{2} u_{0}) \nabla^{0} u_{2} + (\eta^{3}_{0} + u^{3} u_{0}) \nabla^{0} u_{3}$$

$$
\begin{aligned}
&= \sum_{\alpha=0}^{3} (\eta_0^\alpha + u^\alpha u_0) V^0 u_\alpha + \sum_{\alpha=0}^{3} (\eta_1^\alpha + u^\alpha u_1) V^1 u_\alpha \\
&+ \sum_{\alpha=0}^{3} (\eta_2^\alpha + u^\alpha u_2) V^2 u_\alpha + \sum_{\alpha=0}^{3} (\eta_3^\alpha + u^\alpha u_3) V^3 u_\alpha \\
&= (\eta_0^0 + u^0 u_0) V^0 u_0 + (\cancelto{0}{\eta_1^0} + u^0 u_1) V^1 u_0 + \\
&(\cancelto{0}{\eta_2^0} + u^0 u_2) V^2 u_0 + (\cancelto{0}{\eta_3^0} + u^0 u_3) V^3 u_0 + \\
&(\cancelto{0}{\eta_0^1} + u^1 u_0) \cancelto{0}{V^0 u_1} + (\cancelto{1}{\eta_1^1} + u^1 u_1) \cancelto{0}{V^1 u_1} + \\
&(\cancelto{0}{\eta_2^1} + u^1 u_2) \cancelto{0}{V^2 u_1} + (\cancelto{0}{\eta_3^1} + u^1 u_3) \cancelto{0}{V^3 u_1} + \\
&(\cancelto{0}{\eta_0^2} + u^2 u_0) \cancelto{0}{V^0 u_2} + (\cancelto{0}{\eta_1^2} + u^2 u_1) \cancelto{0}{V^1 u_2} + \\
&(\cancelto{1}{\eta_2^2} + u^2 u_2) \cancelto{0}{V^2 u_2} + (\cancelto{0}{\eta_3^2} + u^2 u_3) \cancelto{0}{V^3 u_2} + \\
&(\cancelto{0}{\eta_0^3} + u^3 u_0) \cancelto{0}{V^0 u_3} + (\cancelto{0}{\eta_1^3} + u^3 u_1) \cancelto{0}{V^1 u_3} + \\
&(\cancelto{0}{\eta_2^3} + u^3 u_2) \cancelto{0}{V^2 u_3} + (\cancelto{1}{\eta_3^3} + u^3 u_3) \cancelto{0}{V^3 u_3} \\
&= (1 + u^0 u_0) V^0 u_0 + (u^0 \cancelto{0}{u_1}) V^1 u_0 + \\
&(u^0 \cancelto{0}{u_2}) V^2 u_0 + (u^0 \cancelto{0}{u_3}) V^3 u_0 \\
&= (1 + (-1)\cdot(1)) V^0 (1) \\
&= (1 \cancelto{0}{+(-1)}) V^0 = 0
\end{aligned}
$$

Ejercicio #3

Con las condiciones del ejercicio anterior evaluar el resultado de,

$$(\eta^{\alpha}_{\beta} + U^{\alpha} U_{\beta})(\eta^{\beta}_{\sigma} + U^{\beta} U_{\sigma})$$

$$\eta^{\alpha}_{\beta}\eta^{\beta}_{\sigma} + \eta^{\beta}_{\sigma} U^{\alpha} U_{\beta} + \eta^{\alpha}_{\beta} U^{\beta} U_{\sigma} + U^{\alpha} U_{\beta} U^{\beta} U_{\sigma}$$

$$\delta^{\alpha}_{\beta}\delta^{\beta}_{\sigma} + \delta^{\beta}_{\sigma} U^{\alpha} U_{\beta} + \delta^{\alpha}_{\beta} U^{\beta} U_{\sigma} + U^{\alpha} U_{\beta} U^{\beta} U_{\sigma}$$

$$\delta^{\alpha}_{\sigma} + U^{\alpha} U_{\sigma} + U^{\alpha} U_{\sigma} + U^{\alpha} U_{\sigma} \cancel{U^{\beta} U_{\beta}}^{-1}$$

$$\delta^{\alpha}_{\sigma} + 2(U^{\alpha} U_{\sigma}) - U^{\alpha} U_{\sigma}$$

$$\delta^{\alpha}_{\sigma} + U^{\alpha} U_{\sigma}$$

Ejercicio #4

Un vector no nulo tiene la forma,

$$P^{\mu}P_{\mu} \neq 0 \qquad P^{\mu}P_{\mu} = \pm C$$

Si se tiene,

$$(\eta^{\alpha}{}_{\beta} + aP^{\alpha}P_{\beta})P^{\beta}P_{\alpha} = 0$$

$$\eta^{\alpha}{}_{\beta}P^{\beta}P_{\alpha} + aP^{\alpha}P_{\beta}P^{\beta}P_{\alpha} = 0$$

$$\delta^{\alpha}_{\beta}P^{\beta}P_{\alpha} + aP^{\alpha}P_{\alpha}P^{\beta}P_{\beta} = 0$$

$$P^{\alpha}P_{\alpha} + a\,P^{\alpha}P_{\alpha}P^{\beta}P_{\beta} = 0$$

$$aP^{\alpha}P_{\alpha}P^{\beta}P_{\beta} = -P^{\alpha}P_{\alpha}$$

$$a\,P^{\beta}P_{\beta} = -1$$

$$a = -\frac{1}{P^{\beta}P_{\beta}} \;//$$

Ejercicio # 5

Teniendo que, $F^{0i}=E^{i}$, $F^{12}=B^{3}$, $F^{31}=B^{2}$ y $F^{23}=B^{1}$, dado que el tensor es antisimetrico tendremos que $F^{\alpha\alpha}=-F^{\alpha\alpha}=0$, $F^{i0}=-E^{i}$, $F^{21}=-B^{3}$, $F^{13}=-B^{2}$ y finalmente $F^{32}=-B^{1}$, entonces tendremos,

$$F^{\mu\nu}=\begin{bmatrix} 0 & E^{1} & E^{2} & E^{3} \\ -E^{1} & 0 & B^{3} & -B^{2} \\ -E^{2} & -B^{3} & 0 & B^{1} \\ -E^{3} & B^{2} & -B^{1} & 0 \end{bmatrix}$$

Primero se tiene que hallar $F_{\mu\nu}$, para este cometido aplicamos,

$$F_{\mu\nu}=\eta_{\mu\alpha}\eta_{\beta\nu}F^{\alpha\beta}$$

$$F_{\mu\nu}=\eta_{\mu\alpha}F^{\alpha\beta}\eta_{\beta\nu}$$

Entonces,

$$F^{\alpha}_{\nu}=F^{\alpha\beta}\eta_{\beta\nu}=\begin{bmatrix} 0 & E^{1} & E^{2} & E^{3} \\ -E^{1} & 0 & B^{3} & -B^{2} \\ -E^{2} & -B^{3} & 0 & B^{1} \\ -E^{3} & B^{2} & -B^{1} & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F^{\alpha}_{\ \nu} = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{bmatrix}$$, luego $F_{\mu\nu} = \eta_{\mu\alpha} F^{\alpha}_{\ \nu}$,

$$F_{\mu\nu} = \eta_{\mu\alpha} F^{\alpha}_{\ \nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{bmatrix}$$

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{bmatrix} \quad y \quad F^{\mu\nu} = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{bmatrix}$$

Teniendo la forma covariante y contravariante del tensor computamos la expresión, $F_{\mu\nu}F^{\mu\nu}$. Dado que μ y ν se repiten, estos índices serán mudos, por lo que la expresión final será un invariante tal que,
$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} F_{\mu\nu}F^{\mu\nu}$. También es bueno hacer notar que,

$$E_i = -E^i \quad y \quad B_i = B^i.$$

Entonces,

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} = {} & \cancel{F^{00}F_{00}}^{\,0} + F^{01}F_{01} + F^{02}F_{02} + F^{03}F_{03} + \\ & F^{10}F_{10} + \cancel{F^{11}F_{11}}^{\,0} + F^{12}F_{12} + F^{13}F_{13} + \\ & F^{20}F_{20} + F^{21}F_{21} + \cancel{F^{22}F_{22}}^{\,0} + F^{23}F_{23} + \\ & F^{30}F_{30} + F^{31}F_{31} + F^{32}F_{32} + \cancel{F^{33}F_{33}}^{\,0} \end{aligned}$$

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} = {} & F^{01}F_{01} + F^{02}F_{02} + F^{03}F_{03} + \\ & F^{10}F_{10} + F^{12}F_{12} + F^{13}F_{13} + \\ & F^{20}F_{20} + F^{21}F_{21} + F^{23}F_{23} + \\ & F^{30}F_{30} + F^{31}F_{31} + F^{32}F_{32} \end{aligned}$$

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} = {} & E^1 - E^1 + E^2 - E^2 + E^3 - E^3 + \\ & -E^1 \cdot E^1 + B^3 B^3 + -B^2 - B^2 + \\ & -E^2 \cdot E^2 + -B^3 - B^3 + B^1 B^1 \\ & -E^3 \cdot E^3 + B^2 B^2 + -B^1 - B^1 \end{aligned}$$

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} = {} & -(E^1)^2 - (E^2)^2 - (E^3)^2 - (E^1)^2 + (B^3)^2 + (B^2)^2 \\ & -(E^2)^2 + (B^3)^2 + (B^1)^2 - (E^3)^2 + (B^2)^2 + (B^1)^2 \end{aligned}$$

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} = {} & (B^1)^2 + (B^1)^2 + (B^2)^2 + (B^2)^2 + (B^3)^2 + (B^3)^2 \\ & -(E^1)^2 - (E^1)^2 - (E^2)^2 - (E^2)^2 - (E^3)^2 - (E^3)^2 \end{aligned}$$

Vemos que $B^2 = (B^1)^2 + (B^2)^2 + (B^3)^2$, igual para E, entonces,

$$F^{\mu\nu}F_{\mu\nu} = 2B^2 - 2E^2$$

$$F^{\mu\nu}F_{\mu\nu} = 2(B^2 - E^2) \quad //$$

Vemos también que $F_{\mu\nu}F^{\mu\nu} = F^{\mu\nu}F_{\mu\nu}$

Ejercicio #6

Computar

$$F'^{0}_{\ \lambda} F'^{\lambda 0} + (1/4)\eta^{00} F_{\alpha\beta} F'^{\alpha\beta}$$

Teniendo que $\eta^{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$ y siendo $F'^{\alpha\beta}$ y $F_{\alpha\beta}$ las componentes contravariantes y covariantes del tensor electromagnético respectivamente, el producto $(1/4)\eta^{00} F_{\alpha\beta} F'^{\alpha\beta}$ es un escalar tal que,

$$(1/4)\eta^{00} F_{\alpha\beta} F'^{\alpha\beta} = \frac{1}{4} \cdot -1 \cdot 2(B^2 - E^2) = -\frac{(B^2 - E^2)}{2}$$

Expandimos la suma para λ

$$\sum_{\lambda=0}^{3} F'^{0}_{\ \lambda} F'^{\lambda 0} = \cancel{F'^{0}_{\ 0} F'^{00}} + F'^{0}_{\ 1} F'^{10} + F'^{0}_{\ 2} F'^{20} + F'^{0}_{\ 3} F'^{30}$$

$$= E_x(-E_x) + E_y(-E_y) + E_z(-E_z)$$

$$= -E_x^2 - E_y^2 - E_z^2$$

$$= (-1)(E_x^2 + E_y^2 + E_z^2)$$

$$= (-1) E^2$$

$$= -E^2$$

$$= -E^2 - \frac{1}{4} 2(B^2 - E^2) = -E^2 - \frac{1}{2}B^2 + \frac{1}{2}E^2$$

$$= -\frac{1}{2}B^2 - \frac{1}{2}E^2 = \frac{-B^2 - E^2}{2} = -1 \cdot \frac{B^2 + E^2}{2}$$

Ejercicio # 7

Computar

$$F^{0}{}_{\lambda}F^{\lambda 3} + (1/4)\eta^{03}F_{\alpha\beta}F^{\alpha\beta}$$

Teniendo las mismas condiciones del ejercicio anterior tenemos que,

$$\frac{1}{4}\eta^{03}F_{\alpha\beta}F^{\alpha\beta} = \frac{1}{4}\cdot 0 \cdot 2(B^2-E^2) = 0$$

Expandimos la suma para λ

$$\sum_{\lambda=0}^{3} F^{0}_{\lambda}F^{\lambda 3} = \cancel{F^{0}_{0}F^{03}}^{\,0} + F^{0}_{1}F^{13} + F^{0}_{2}F^{23} + F^{0}_{3}F^{33}$$

$$F^{\mu}{}_{\nu} = \eta_{\beta\nu}F^{\mu\beta}$$

$$F^{0}_{\nu} = \eta_{\beta\nu}F^{0\beta}$$

$$F^{0}_{\nu} = \sum_{\beta=0}^{3}\eta_{\beta\nu}F^{0\beta} = \eta_{0\nu}F^{00} + \eta_{1\nu}F^{01} + \eta_{2\nu}F^{02} + \eta_{3\nu}F^{03}$$

$$F^{0}_{0} = \cancel{\eta_{00}F^{00}}^{\,0} + \cancel{\eta_{10}F^{01}}^{\,0} + \cancel{\eta_{20}F^{02}}^{\,0} + \cancel{\eta_{30}F^{03}}^{\,0} = 0$$

$$F^{0}_{1} = \cancel{\eta_{01}F^{00}}^{\,0} + \eta_{11}F^{01} + \cancel{\eta_{21}F^{02}}^{\,0} + \cancel{\eta_{31}F^{03}}^{\,0} = 1\cdot E_x$$

$$F^{0}_{2} = \cancel{\eta_{02}F^{00}}^{\,0} + \cancel{\eta_{12}F^{01}}^{\,0} + \eta_{22}F^{02} + \cancel{\eta_{32}F^{03}}^{\,0} = 1\cdot E_y$$

$$F^{0}_{3} = \cancel{\eta_{03}F^{00}}^{\,0} + \cancel{\eta_{13}F^{01}}^{\,0} + \cancel{\eta_{23}F^{02}}^{\,0} + \eta_{33}F^{03} = 1\cdot E_z$$

$$\sum F_{\lambda}^{\ 0} F^{\lambda 3}$$

$$\cancel{F_0^{\ 0} F^{03}}^{\,0} + F_1^{\ 0} F^{13} + F_2^{\ 0} F^{23} + \cancel{F_3^{\ 0} F^{33}}^{\,0}$$

$$= E_x(-B_y) + E_y(B_x)$$

$$= E_y B_x - E_x B_y$$

Si tenemos que, $\vec{B} \times \vec{E} = \epsilon_{ijk} B_j E_k$

$$(\vec{B} \times \vec{E})_1 = \epsilon^{123} B_2 E_3 + \epsilon^{132} B_3 E_2 = 0$$

$$(\vec{B} \times \vec{E})_2 = \epsilon^{213} B_1 E_3 + \epsilon^{231} B_3 E_1 = 0$$

$$(\vec{B} \times \vec{E})_3 = \epsilon^{312} B_1 E_2 + \epsilon^{321} B_2 E_1 =$$

$$= (1) B_1 E_2 + (-1) B_2 E_1 = E_y B_x - E_x B_y.$$

Ejercicio #10

Computar

$$F'^{\mu}{}_{\lambda} F'^{\lambda}{}_{\mu} + (1/4)\,\eta^{\mu}_{\mu} F_{\alpha\beta} F'^{\alpha\beta}$$

Vemos que $F'^{\mu}{}_{\lambda} F'^{\lambda}{}_{\mu}$ es una contracción de índices, mientras que η^{μ}_{μ} es la traza del tensor delta de Kronecker y $F_{\alpha\beta}F'^{\alpha\beta}$ es un invariante relativista, así,

$$\sum_{\mu=0}^{3}\sum_{\lambda=0}^{3} F'^{\mu}{}_{\lambda} F'^{\lambda}{}_{\mu}$$

$$\sum_{\mu=0}^{3} F'^{\mu}{}_{0} F'^{0}{}_{\mu} + F'^{\mu}{}_{1} F'^{1}{}_{\mu} + F'^{\mu}{}_{2} F'^{2}{}_{\mu} + F'^{\mu}{}_{3} F'^{3}{}_{\mu}$$

$$\begin{aligned}
= {} & \cancel{F'^{0}{}_{0} F'^{0}{}_{0}} + F'^{1}{}_{0} F'^{0}{}_{1} + F'^{2}{}_{0} F'^{0}{}_{2} + F'^{3}{}_{0} F'^{0}{}_{3} + \\
& F'^{0}{}_{1} F'^{1}{}_{0} + \cancel{F'^{1}{}_{1} F'^{1}{}_{1}} + F'^{2}{}_{1} F'^{1}{}_{2} + F'^{3}{}_{1} F'^{1}{}_{3} + \\
& F'^{0}{}_{2} F'^{2}{}_{0} + F'^{1}{}_{2} F'^{2}{}_{1} + \cancel{F'^{2}{}_{2} F'^{2}{}_{2}} + F'^{3}{}_{2} F'^{2}{}_{3} + \\
& F'^{0}{}_{3} F'^{3}{}_{0} + F'^{1}{}_{3} F'^{3}{}_{1} + F'^{2}{}_{3} F'^{3}{}_{2} + \cancel{F'^{3}{}_{3} F'^{3}{}_{3}}
\end{aligned}$$

$$\begin{aligned}
= {} & E_x E_x + E_y E_y + E_z E_z + & & E_x^2 + E_y^2 + E_z^2 \\
& + E_x E_x + (-B_z)(-B_z) + B_y B_y & & E_x^2 + B_z^2 + B_y^2 \\
& + E_y E_y + B_z B_z + (-B_x)(-B_x) & = \; & E_y^2 + B_z^2 + B_x^2 \\
& + E_z E_z + (-B_y)(-B_y) + B_x B_x & & E_z^2 + B_y^2 + B_x^2
\end{aligned}$$

$$= 2E_x^2 + 2E_y^2 + 2E_z^2 + 2B_x^2 + 2B_y^2 + 2B_z^2$$

$$= 2(E_x^2 + E_y^2 + E_z^2) + 2(B_x^2 + B_y^2 + B_z^2)$$

$$= 2\vec{E}^2 + 2\vec{B}^2$$

$$\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \sum_{\mu=0}^{3} \eta^{\mu}_{\mu}$$

$$\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \left(\eta_0^{\ 0} + \eta_1^{\ 1} + \eta_2^{\ 2} + \eta_3^{\ 3} \right)$$

$$\frac{1}{\cancel{4}} F_{\alpha\beta} F^{\alpha\beta} (\cancel{4}) = F_{\alpha\beta} F^{\alpha\beta}$$

$$F^{\mu}{}_{\lambda} F^{\lambda}{}_{\mu} + (1/4) \eta^{\mu}_{\mu} F_{\alpha\beta} F^{\alpha\beta}$$

$$2(E^2 + B^2) + 2(B^2 - E^2)$$

$$\cancel{2E^2} + 2B^2 + 2B^2 - \cancel{2E^2}$$

$$= 4B^2 //$$

Ejercicios #8, #9 y #11

Si tenemos en O que $\vec{E}=E_0\hat{j}$ y $\vec{B}=B_0\hat{k}$, y queremos hallar un marco $\bar{O}$ con velocidad v a lo largo del eje x en donde el campo eléctrico desaparece, se tienen que hallar las componentes del tensor electromagnético en el marco $\bar{O}$, es decir,

$$F^{\bar{\mu}\bar{\nu}}=\Lambda^{\bar{\mu}}_{\ \mu}\Lambda^{\bar{\nu}}_{\ \nu}F^{\mu\nu}$$

considerando que,

$$F^{\mu\nu}=\begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \qquad \Lambda^{\bar{\mu}}_{\ \nu}=\begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Usando notación de índices tendremos,

$$F^{\mu\nu}=\Lambda^{\mu}_{\ \beta}\Lambda^{\nu}_{\ \sigma}F^{\beta\sigma} \longrightarrow F^{\mu\nu}=\sum_{\beta=0}^{3}\sum_{\sigma=0}^{3}\Lambda^{\mu}_{\ \beta}\Lambda^{\nu}_{\ \sigma}F^{\beta\sigma}$$

$$F^{\mu 0}=\sum_{\beta=0}^{3}\Lambda^{\mu}_{\ \beta}\left(\Lambda^{0}_{\ 0}F^{\beta 0}+\Lambda^{0}_{\ 1}F^{\beta 1}+\cancel{\Lambda^{0}_{\ 2}}^{\,0}F^{\beta 2}+\cancel{\Lambda^{0}_{\ 3}}^{\,0}F^{\beta 3}\right)$$

$$F^{\mu 1}=\sum_{\beta=0}^{3}\Lambda^{\mu}_{\ \beta}\left(\Lambda^{1}_{\ 0}F^{\beta 0}+\Lambda^{1}_{\ 1}F^{\beta 1}+\cancel{\Lambda^{1}_{\ 2}}^{\,0}F^{\beta 2}+\cancel{\Lambda^{1}_{\ 3}}^{\,0}F^{\beta 3}\right)$$

$$F^{\mu 2}=\sum_{\beta=0}^{3}\Lambda^{\mu}_{\ \beta}\left(\cancel{\Lambda^{2}_{\ 0}}^{\,0}F^{\beta 0}+\cancel{\Lambda^{2}_{\ 1}}^{\,0}F^{\beta 1}+\Lambda^{2}_{\ 2}F^{\beta 2}+\cancel{\Lambda^{2}_{\ 3}}^{\,0}F^{\beta 3}\right)$$

$$F^{\mu 3}=\sum_{\beta=0}^{3}\Lambda^{\mu}_{\ \beta}\left(\cancel{\Lambda^{3}_{\ 0}}^{\,0}F^{\beta 0}+\cancel{\Lambda^{3}_{\ 1}}^{\,0}F^{\beta 1}+\cancel{\Lambda^{3}_{\ 2}}^{\,0}F^{\beta 2}+\Lambda^{3}_{\ 3}F^{\beta 3}\right)$$

$$F'^{\mu 0} = \sum_{\gamma=0}^{3} \Lambda^{\mu}_{\gamma} \left(\Lambda^{0}_{0} F^{\gamma 0} + \Lambda^{0}_{1} F^{\gamma 1} \right)$$

$$F'^{\mu 1} = \sum_{\gamma=0}^{3} \Lambda^{\mu}_{\gamma} \left(\Lambda^{1}_{0} F^{\gamma 0} + \Lambda^{1}_{1} F^{\gamma 1} \right)$$

$$F'^{\mu 2} = \sum_{\gamma=0}^{3} \Lambda^{\mu}_{\gamma} \left(\Lambda^{2}_{2} F^{\gamma 2} \right)$$

$$F'^{\mu 3} = \sum_{\gamma=0}^{3} \Lambda^{\mu}_{\gamma} \left(\Lambda^{3}_{3} F^{\gamma 3} \right)$$

$\Rightarrow$ Para $F'^{\mu 0}$

$$F'^{\mu 0} = \sum_{\gamma=0}^{3} \Lambda^{\mu}_{\gamma} \Lambda^{0}_{0} F^{\gamma 0} + \sum_{\gamma=0}^{3} \Lambda^{\mu}_{\gamma} \Lambda^{0}_{1} F^{\gamma 1}$$

$$F'^{\mu 0} = \Lambda^{\mu}_{\cancelto{0}{0} \Lambda^{0}_{0} F^{00}} + \Lambda^{\mu}_{1} \Lambda^{0}_{0} F^{10} + \Lambda^{\mu}_{2} \Lambda^{0}_{0} F^{20} + \Lambda^{\mu}_{3} \Lambda^{0}_{0} F^{30}$$
$$+ \Lambda^{\mu}_{0} \Lambda^{0}_{1} F^{01} + \cancelto{0}{\Lambda^{\mu}_{1} \Lambda^{0}_{1} F^{11}} + \Lambda^{\mu}_{2} \Lambda^{0}_{1} F^{21} + \Lambda^{\mu}_{3} \Lambda^{0}_{1} F^{31}$$

$$F'^{\mu 0} = \Lambda^{\mu}_{1} \Lambda^{0}_{0} F^{10} + \Lambda^{\mu}_{2} \Lambda^{0}_{0} F^{20} + \Lambda^{\mu}_{3} \Lambda^{0}_{0} F^{30}$$
$$+ \Lambda^{\mu}_{0} \Lambda^{0}_{1} F^{01} + \Lambda^{\mu}_{2} \Lambda^{0}_{1} F^{21} + \Lambda^{\mu}_{3} \Lambda^{0}_{1} F^{31}$$

$$F'^{00} = \Lambda^{0}_{1} \Lambda^{0}_{0} F^{10} + \cancelto{0}{\Lambda^{0}_{2} \Lambda^{0}_{0} F^{20}} + \cancelto{0}{\Lambda^{0}_{3} \Lambda^{0}_{0} F^{30}}$$
$$+ \Lambda^{0}_{0} \Lambda^{0}_{1} F^{01} + \cancelto{0}{\Lambda^{0}_{2} \Lambda^{0}_{1} F^{21}} + \cancelto{0}{\Lambda^{0}_{3} \Lambda^{0}_{1} F^{31}}$$

$$F'^{00} = (-\gamma \cancelto{0}{v)\gamma(-E_x}) + \gamma(\cancelto{0}{-\gamma v)E_x}$$

$$F'^{00} = 0$$

$$\Lambda^0_0 = \gamma \qquad \Lambda^0_1 = -\gamma v$$

$$F'^{10} = \Lambda^1_1 \Lambda^0_0 F^{10} + \cancelto{0}{\Lambda^1_2 \Lambda^0_0 F^{20}} + \cancelto{0}{\Lambda^1_3 \Lambda^0_0 F^{30}}$$
$$+ \Lambda^1_0 \Lambda^0_1 F^{01} + \cancelto{0}{\Lambda^1_2 \Lambda^0_1 F^{21}} + \cancelto{0}{\Lambda^1_3 \Lambda^0_1 F^{31}}$$

$$F'^{10} = \gamma\gamma(-E_x) + (-\gamma v)(-\gamma v)E_x$$

$$F'^{10} = -\gamma^2 E_x + \gamma^2 v^2 E_x = -\gamma^2 E_x (1-v^2)$$

$$F'^{10} = -\gamma^2 E_x (v^2 - 1)$$

$$\gamma = \frac{1}{\sqrt{1-v^2}} \Rightarrow \gamma^2 = \frac{1}{(1-v^2)}$$

$$= -\frac{1}{\cancel{(1-v^2)}} E_x \cancel{(1-v^2)} = -E_x = F'^{10}$$

$$F'^{20} = \cancelto{0}{\Lambda^2_1 \Lambda^0_0 F^{10}} + \Lambda^2_2 \Lambda^0_0 F^{20} + \cancelto{0}{\Lambda^2_3 \Lambda^0_0 F^{30}}$$
$$+ \cancelto{0}{\Lambda^2_0 \Lambda^0_1 F^{01}} + \Lambda^2_2 \Lambda^0_1 F^{21} + \cancelto{0}{\Lambda^2_3 \Lambda^0_1 F^{31}}$$

$$F'^{20} = (1)\gamma(-E_y) + (1)(-\gamma v)(-B_z)$$

$$F'^{20} = -\gamma E_y + \gamma v B_z$$

$$F'^{20} = -\gamma(E_y - vB_z) = F'^{20}$$

$$F'^{30} = \cancelto{0}{\Lambda^3_1 \Lambda^0_0 F^{10}} + \cancelto{0}{\Lambda^3_2 \Lambda^0_0 F^{20}} + \Lambda^3_3 \Lambda^0_0 F^{30}$$
$$+ \cancelto{0}{\Lambda^3_0 \Lambda^0_1 F^{01}} + \cancelto{0}{\Lambda^3_2 \Lambda^0_1 F^{21}} + \Lambda^3_3 \Lambda^0_1 F^{31}$$

$$F'^{30} = (1)\gamma(-E_z) + (1)(-\gamma v)B_y$$

$$F'^{30} = -\gamma E_z - \gamma v B_y$$

$$F'^{30} = -\gamma(E_z + vB_y) = F'^{30}$$

$$F'^{\mu 1} = \sum_{\delta=0}^{3} \Delta^{\mu}_{\delta} \left(\Delta^{1}_{0} F^{\delta 0} + \Delta^{1}_{1} F^{\delta 1} \right) \qquad \Delta^{1}_{0} = -\gamma v, \ \Delta^{1}_{1} = \gamma$$

=> Para $F'^{\mu 1}$

$$F'^{\mu 1} = \sum_{\delta=0}^{3} \Delta^{\mu}_{\delta} \Delta^{1}_{0} F^{\delta 0} + \sum_{\delta=0}^{3} \Delta^{\mu}_{\delta} \Delta^{1}_{1} F^{\delta 1}$$

$$F'^{\mu 1} = \cancelto{0}{\Delta^{\mu}_{0} \Delta^{1}_{0} F^{00}} + \Delta^{\mu}_{1} \Delta^{1}_{0} F^{10} + \Delta^{\mu}_{2} \Delta^{1}_{0} F^{20} + \Delta^{\mu}_{3} \Delta^{1}_{0} F^{30}$$
$$+ \Delta^{\mu}_{0} \Delta^{1}_{1} F^{01} + \cancelto{0}{\Delta^{\mu}_{1} \Delta^{1}_{1} F^{11}} + \Delta^{\mu}_{2} \Delta^{1}_{1} F^{21} + \Delta^{\mu}_{3} \Delta^{1}_{1} F^{31}$$

$$F'^{\mu 1} = \Delta^{\mu}_{1} \Delta^{1}_{0} F^{10} + \Delta^{\mu}_{2} \Delta^{1}_{0} F^{20} + \Delta^{\mu}_{3} \Delta^{1}_{0} F^{30}$$
$$+ \Delta^{\mu}_{0} \Delta^{1}_{1} F^{01} + \Delta^{\mu}_{2} \Delta^{1}_{1} F^{21} + \Delta^{\mu}_{3} \Delta^{1}_{1} F^{31}$$

$$F'^{01} = \Delta^{0}_{1} \Delta^{1}_{0} F^{10} + \cancelto{0}{\Delta^{0}_{2} \Delta^{1}_{0} F^{20}} + \cancelto{0}{\Delta^{0}_{3} \Delta^{1}_{0} F^{30}}$$
$$+ \Delta^{0}_{0} \Delta^{1}_{1} F^{01} + \cancelto{0}{\Delta^{0}_{2} \Delta^{1}_{1} F^{21}} + \cancelto{0}{\Delta^{0}_{3} \Delta^{1}_{1} F^{31}}$$

$$F'^{01} = (-\gamma v)(-\gamma v)(-E_x) + \gamma \cdot \gamma E_x$$

$$\gamma^2 = \frac{1}{(1-v^2)}$$

$$F'^{01} = -\gamma^2 v^2 E_x + \gamma^2 E_x$$

$$F'^{01} = \gamma^2 E_x (1 - v^2) = E_x = F'^{01}$$

$$F'^{11} = \Delta^{1}_{1} \Delta^{1}_{0} F^{10} + \cancelto{0}{\Delta^{1}_{2} \Delta^{1}_{0} F^{20}} + \cancelto{0}{\Delta^{1}_{3} \Delta^{1}_{0} F^{30}}$$
$$+ \Delta^{1}_{0} \Delta^{1}_{1} F^{01} + \cancelto{0}{\Delta^{1}_{2} \Delta^{1}_{1} F^{21}} + \cancelto{0}{\Delta^{1}_{3} \Delta^{1}_{1} F^{31}}$$

$$F'^{11} = \gamma(-\gamma v)(-E_x) + (-\gamma v)\gamma(E_x) = 0 = F'^{11}$$

$$F'^{21} = \cancelto{0}{\Delta^{2}_{1} \Delta^{1}_{0} F^{10}} + \Delta^{2}_{2} \Delta^{1}_{0} F^{20} + \cancelto{0}{\Delta^{2}_{3} \Delta^{1}_{0} F^{30}}$$
$$+ \cancelto{0}{\Delta^{2}_{0} \Delta^{1}_{1} F^{01}} + \Delta^{2}_{2} \Delta^{1}_{1} F^{21} + \cancelto{0}{\Delta^{2}_{3} \Delta^{1}_{1} F^{31}}$$

$$F'^{21} =$$

$$F'^{21} = (1)(-\gamma v)(-E_y) + (1)\gamma - B_z = \gamma v E_y - \gamma B_z = -\gamma(B_z + v E_y)$$

$$F'^{31} = \cancelto{0}{\Delta^{3}_{1} \Delta^{1}_{0} F^{10}} + \cancelto{0}{\Delta^{3}_{2} \Delta^{1}_{0} F^{20}} + \Delta^{3}_{3} \Delta^{1}_{0} F^{30}$$
$$+ \cancelto{0}{\Delta^{3}_{0} \Delta^{1}_{1} F^{01}} + \cancelto{0}{\Delta^{3}_{2} \Delta^{1}_{1} F^{21}} + \Delta^{3}_{3} \Delta^{1}_{1} F^{31}$$

$$F'^{31} =$$

$$F'^{31} = (1)(-\gamma v)(-E_z) + (1)\gamma B_y = \gamma v E_z + \gamma B_y = \gamma(B_y + v E_z)$$

$$F^{\mu 2} = \sum_{\delta=0}^{3} \Lambda^{\mu}_{\delta} \left(\Lambda^{2}_{2} F^{\delta 2} \right) \qquad \Lambda^{2}_{2} = 1$$

$\Rightarrow$ Para $F^{\mu 2}$

$$F^{\mu 2} = \sum_{\delta=0}^{3} \Lambda^{\mu}_{\delta} \left(\Lambda^{2}_{2} F^{\delta 2} \right)$$

$$F^{\mu 2} = \Lambda^{\mu}_{0} \Lambda^{2}_{2} F^{02} + \Lambda^{\mu}_{1} \Lambda^{2}_{2} F^{12} + \cancel{\Lambda^{\mu}_{2} \Lambda^{2}_{2} F^{22}}^{0} + \Lambda^{\mu}_{3} \Lambda^{2}_{2} F^{32}$$

$$F^{\mu 2} = \Lambda^{\mu}_{0} \cancel{\Lambda^{2}_{2}}^{1} F^{02} + \Lambda^{\mu}_{1} \cancel{\Lambda^{2}_{2}}^{1} F^{12} + \Lambda^{\mu}_{3} \cancel{\Lambda^{2}_{2}}^{1} F^{32}$$

$$F^{\mu 2} = \Lambda^{\mu}_{0} F^{02} + \Lambda^{\mu}_{1} F^{12} + \Lambda^{\mu}_{3} F^{32}$$

$$F^{02} = \Lambda^{0}_{0} F^{02} + \Lambda^{0}_{1} F^{12} + \cancel{\Lambda^{0}_{3} F^{32}}^{0} = \gamma E_y + (-\gamma v) B_z$$

$$F^{02} = \gamma (E_y - v B_z)$$

$$F^{12} = \Lambda^{1}_{0} F^{02} + \Lambda^{1}_{1} F^{12} + \cancel{\Lambda^{1}_{3} F^{32}}^{0} = (-\gamma v) E_y + \gamma B_z$$

$$F^{12} = \gamma (B_z - v E_y)$$

$$F^{22} = \cancel{\Lambda^{2}_{0} F^{02}}^{0} + \cancel{\Lambda^{2}_{1} F^{12}}^{0} + \cancel{\Lambda^{2}_{3} F^{32}}^{0} = 0$$

$$F^{22} = 0$$

$$F^{32} = \cancel{\Lambda^{3}_{0} F^{02}}^{0} + \cancel{\Lambda^{3}_{1} F^{12}}^{0} + \Lambda^{3}_{3} F^{32} = (1)(-B_x)$$

$$F^{32} = (-B_x)$$

$$F'^{\mu 3} = \sum_{\delta=0}^{3} \Lambda^{\mu}_{\ \delta} \left(\Lambda^{3}_{\ 3} F^{\delta 3}\right) \qquad \Lambda^{3}_{\ 3} = 1$$

$\Rightarrow$ Para $F'^{\mu 3}$

$$F'^{\mu 3} = \sum_{\delta=0}^{3} \Lambda^{\mu}_{\ \delta} \Lambda^{3}_{\ 3} F^{\delta 3}$$

$$F'^{\mu 3} = \Lambda^{\mu}_{\ 0} \Lambda^{3}_{\ 3} F^{03} + \Lambda^{\mu}_{\ 1} \Lambda^{3}_{\ 3} F^{13} + \Lambda^{\mu}_{\ 2} \Lambda^{3}_{\ 3} F^{23} + \cancelto{0}{\Lambda^{\mu}_{\ 3} \Lambda^{3}_{\ 3} F^{33}}$$

$$F'^{\mu 3} = \Lambda^{\mu}_{\ 0} \cancelto{1}{\Lambda^{3}_{\ 3}} F^{03} + \Lambda^{\mu}_{\ 1} \cancelto{1}{\Lambda^{3}_{\ 3}} F^{13} + \Lambda^{\mu}_{\ 2} \cancelto{1}{\Lambda^{3}_{\ 3}} F^{23}$$

$$F'^{\mu 3} = \Lambda^{\mu}_{\ 0} F^{03} + \Lambda^{\mu}_{\ 1} F^{13} + \Lambda^{\mu}_{\ 2} F^{23}$$

$$F'^{03} = \Lambda^{0}_{\ 0} F^{03} + \Lambda^{0}_{\ 1} F^{13} + \cancelto{0}{\Lambda^{0}_{\ 2} F^{23}} = \gamma \cdot E_z + (-\gamma v)(-B_y)$$

$$F'^{03} = \gamma E_z + \gamma v B_y = \gamma (E_z + v B_y)$$

$$F'^{13} = \Lambda^{1}_{\ 0} F^{03} + \Lambda^{1}_{\ 1} F^{13} + \cancelto{0}{\Lambda^{1}_{\ 2} F^{23}} = (-\gamma v) E_z + \gamma (-B_y)$$

$$F'^{13} = -\gamma v E_z - \gamma B_y = -\gamma (B_y + v E_z)$$

$$F'^{23} = \cancelto{0}{\Lambda^{2}_{\ 0} F^{03}} + \cancelto{0}{\Lambda^{2}_{\ 1}} F^{13} + \Lambda^{2}_{\ 2} F^{23} = (1) \cdot B_x$$

$$F'^{23} = B_x$$

$$F'^{33} = \cancelto{0}{\Lambda^{3}_{\ 0} F^{03}} + \cancelto{0}{\Lambda^{3}_{\ 1} F^{13}} + \cancelto{0}{\Lambda^{3}_{\ 2} F^{23}} = 0$$

$$F'^{33} = 0$$

$F'^{00}=0$, $F'^{10}=-E_x$

$F'^{20}=-\gamma(E_y-vB_z)$, $F'^{30}=-\gamma(E_z+vB_y)$

$F'^{01}=E_x$, $F'^{11}=0$

$F'^{21}=-\gamma(B_z+vE_y)$, $F'^{31}=\gamma(B_y+vE_z)$

$F'^{02}=\gamma(E_y-vB_z)$, $F'^{12}=\gamma(B_z-vE_y)$

$F'^{22}=0$, $F'^{32}=-B_x$

$F'^{03}=\gamma(E_z+vB_y)$, $F'^{13}=-\gamma(B_y+vE_z)$

$F'^{23}=B_x$, $F'^{33}=0$

$$F'^{\bar\mu\bar\nu}=\begin{bmatrix} 0 & E_x & \gamma(E_y-vB_z) & \gamma(E_z+vB_y) \\ -E_x & 0 & \gamma(B_z-vE_y) & -\gamma(B_y+vE_z) \\ -\gamma(E_y-vB_z) & -\gamma(B_z+vE_y) & 0 & B_x \\ -\gamma(E_z+vB_y) & \gamma(B_y+vE_z) & -B_x & 0 \end{bmatrix}$$

En donde, $F'^{\bar\mu\bar\nu}=\Lambda^{\bar\mu}_{\mu}\Lambda^{\bar\nu}_{\nu}F^{\mu\nu}$. Se puede ver que el tensor sigue siendo antisimétrico y que las componentes de E y B paralelas al movimiento no se ven afectadas por la transformación. Así pues, tendremos que,

$$\bar E_x=E_x \qquad \bar E_y=\gamma(E_y-vB_z) \qquad \bar E_z=\gamma(E_z+vB_y)$$

$$\bar B_x=B_x \qquad \bar B_y=\gamma(B_y+vE_z) \qquad \bar B_z=\gamma(B_z-vE_y)$$

Si se busca que $\bar{E}=0$, entonces, para $\vec{E}=(0,E_0,0)$ y $\vec{B}=(0,0,B_0)$ cuando $\gamma(v)$ en la dirección x,

$$\bar{E}_x = \cancel{E_x}^{\,0} \quad \bar{E}_y = \gamma(E_y - vB_z) \quad \bar{E}_z = \gamma(\cancel{E_z + vB_y})^{\,0}$$

$$\bar{E}_x = 0 \quad \bar{E}_y = \gamma(E_y - vB_z) \quad \bar{E}_z = 0$$

$$\bar{E}_y = 0 = \gamma(E_y - vB_z)$$

$$0 = \gamma(E_y - vB_z)$$

$$0 = E_y - vB_z \longrightarrow vB_z = E_y$$

$$v = \frac{E_y}{B_z} = \frac{E_0}{B_0} //$$

Si se busca que $\bar{B}=0$, entonces, para $\vec{E}=(0,E_0,0)$ y $\vec{B}=(0,0,B_0)$ cuando $\gamma(v)$ en la dirección x

$$\bar{B}_x = \cancel{B_x}^{\,0} \quad \bar{B}_y = \gamma(\cancel{B_y + vE_z})^{\,0} \quad \bar{B}_z = \gamma(B_z - vE_y)$$

$$\bar{B}_x = 0 \quad \bar{B}_y = 0 \quad \bar{B}_z = \gamma(B_z - vE_y)$$

$$\bar{B}_z = 0 = \gamma(B_z - vE_y)$$

$$0 = \gamma(B_z - vE_y)$$

$$vE_y = B_z$$

$$v = \frac{B_z}{E_y} = \frac{B_0}{E_0} /$$