La métrica de Friedmann-Lemaître-Robertson-Walker

Holman D. Quintero Salazar

Relatividad General

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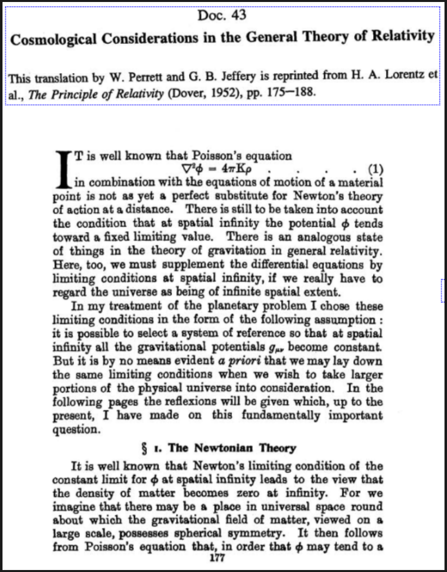
$$G_{\mu\nu} + \boxed{\Lambda g_{\mu\nu}} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Cosmología de Einstein



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Cosmological Considerations in the General Theory of Relativity

This translation by W. Perrett and G. B. Jeffery is reprinted from H. A. Lorentz et al., *The Principle of Relativity* (Dover, 1952), pp. 175–188.

IT is well known that Poisson's equation

$$\nabla^2\phi = 4\pi K\rho \quad . \quad . \quad . \quad . \quad (1)$$

in combination with the equations of motion of a material point is not as yet a perfect substitute for Newton's theory of action at a distance. There is still to be taken into account the condition that at spatial infinity the potential ϕ tends toward a fixed limiting value. There is an analogous state of things in the theory of gravitation in general relativity. Here, too, we must supplement the differential equations by limiting conditions at spatial infinity, if we really have to regard the universe as being of infinite spatial extent.

In my treatment of the planetary problem I chose these limiting conditions in the form of the following assumption: it is possible to select a system of reference so that at spatial infinity all the gravitational potentials $g_{\mu\nu}$ become constant. But it is by no means evident *a priori* that we may lay down the same limiting conditions when we wish to take larger portions of the physical universe into consideration. In the following pages the reflexions will be given which, up to the present, I have made on this fundamentally important question.

§ 1. The Newtonian Theory

It is well known that Newton's limiting condition of the constant limit for ϕ at spatial infinity leads to the view that the density of matter becomes zero at infinity. For we imagine that there may be a place in universal space round about which the gravitational field of matter, viewed on a large scale, possesses spherical symmetry. It then follows from Poisson's equation that, in order that ϕ may tend to a

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$\Rightarrow$ Ecuación de Poisson

$$\nabla^2 \boxed{\varphi} = 4\pi G \boxed{\rho} \qquad \lim_{r\to\infty} \varphi(r) = 0$$

$\Rightarrow$ Paradoja de Seeliger

$$\cancelto{0}{\nabla^2 \varphi} = 4\pi G\rho$$

$$\varphi = \text{cte.} \quad \therefore \quad \rho = 0$$

Cosmología de Einstein

It seems hardly possible to surmount these difficulties on the basis of the Newtonian theory. We may ask ourselves the question whether they can be removed by a modification of the Newtonian theory. First of all we will indicate a method which does not in itself claim to be taken seriously; it merely serves as a foil for what is to follow. In place of Poisson's equation we write

$$\nabla^2\phi - \lambda\phi = 4\pi\kappa\rho \quad . \quad . \quad . \quad (2)$$

where λ denotes a universal constant. If ρ_0 be the uniform density of a distribution of mass, then

$$\phi = -\frac{4\pi\kappa}{\lambda}\rho_0 \quad . \quad . \quad . \quad . \quad (3)$$

is a solution of equation (2). This solution would correspond to the case in which the matter of the fixed stars was distributed uniformly through space, if the density ρ_0 is equal to the actual mean density of the matter in the universe.

$$\nabla^2\varphi - \boxed{\lambda\varphi} = 4\pi G\rho$$

con $\varphi =$ cte,

$$0 - \lambda\varphi = 4\pi G\rho$$

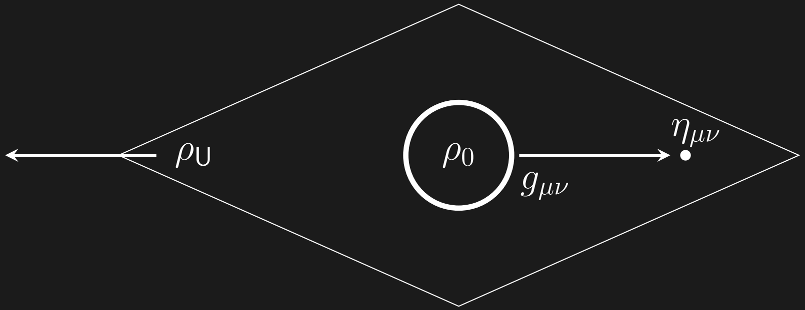
$$\varphi = -\frac{4\pi G}{\lambda}\rho \neq 0$$

$$\boxed{\bar{\varphi} = -\frac{4\pi G}{\lambda}\bar{\rho}}$$

Posteriormente, Einstein considera la misma situación para la relatividad general.

$$\nabla^2 \boxed{\varphi} = 4\pi G \boxed{\rho} \quad \Rightarrow \quad \boxed{G_{\mu\nu}} = \frac{8\pi G}{c^4} \boxed{T_{\mu\nu}}$$

$g_{\mu\nu} = \eta_{\mu\nu}$?
en $r = \infty$



Cosmología de Einstein

De manera general, si $g_{\mu\nu}$ es plana en todas partes y a todo tiempo, por consiguiente,

$$\cancelto{0}{G_{\mu\nu}} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$T_{00} = \rho c^2 = 0$$

La solución de Einstein fue entonces proponer que $G_{\mu\nu} \neq 0$,

> From what has now been said it will be seen that I have not succeeded in formulating boundary conditions for spatial infinity. Nevertheless, there is still a possible way out, without resigning as suggested under (*b*). For if it were possible to regard the universe as a continuum which is *finite (closed) with respect to its spatial dimensions*, we should have no need at all of any such boundary conditions. We shall proceed to show that both the general postulate of relativity and the fact of the small stellar velocities are compatible with the hypothesis of a spatially finite universe; though certainly, in order to carry through this idea, we need a generalizing modification of the field equations of gravitation.

$$k = 0 \quad \Rightarrow \quad k > 0$$

Cosmología de Einstein

Escoger $k > 0$ implicaba que el Universo no era estático. Bajo estas condiciones Einstein deduce el término $\lambda(\Lambda)$ para obtener un Universo estático. Así pues, se tendría que:

- Λ aparece de forma similar al parámetro λ del caso newtoniano y está determinado por la densidad promedio del Universo.
- No está en contraposición con sus trabajos anteriores en tanto que no viola ningún principio de conservación ($\nabla_\mu g^{\mu\nu} = 0$).
- En forma práctica $\Lambda << 1$, para la mayoría de aplicaciones.
- Se concluye que el Universo tiene curvatura esférica, es finito, estático y que los pozos de potencial individuales producirían cambios locales en la curvatura general del espacio-tiempo.

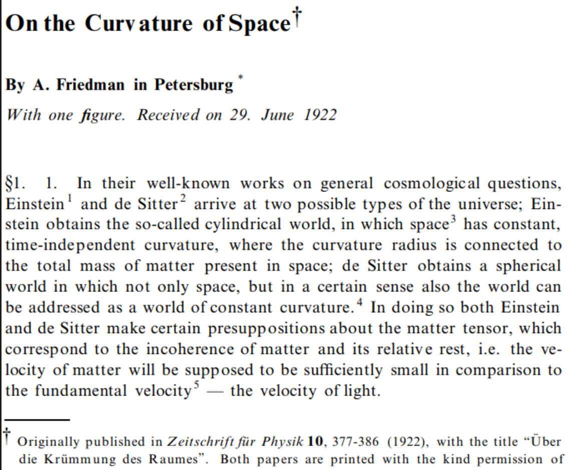
Cosmología de Einstein

curvature of space is variable in time and place, according to the distribution of matter, but we may roughly approximate to it by means of a spherical space. At any rate, this view is logically consistent, and from the standpoint of the general theory of relativity lies nearest at hand; whether, from the standpoint of present astronomical knowledge, it is tenable, will not here be discussed. In order to arrive at this consistent view, we admittedly had to introduce an extension of the field equations of gravitation which is not justified by our actual knowledge of gravitation. It is to be emphasized, however, that a positive curvature of space is given by our results, even if the supplementary term is not introduced. That term is necessary only for the purpose of making possible a quasi-static distribution of matter, as required by the fact of the small velocities of the stars.

$$\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

$\Rightarrow$ *"En cualquier caso, esta visión es lógicamente consistente y, desde el punto de vista de la teoría general de la relatividad, es la más cercana; No se discutirá aquí si, desde el punto de vista del conocimiento astronómico actual, es sostenible."*

Métrica de Friedmann-Lemaître-Robertson-Walker



On the Curvature of Space†

By A. Friedman in Petersburg *

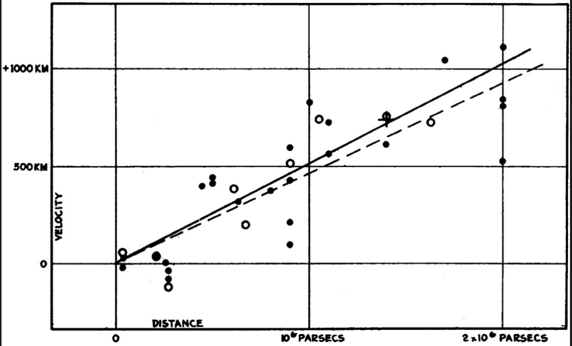
With one figure. Received on 29. June 1922

§1. 1. In their well-known works on general cosmological questions, Einstein[1] and de Sitter[2] arrive at two possible types of the universe; Einstein obtains the so-called cylindrical world, in which space[3] has constant, time-independent curvature, where the curvature radius is connected to the total mass of matter present in space; de Sitter obtains a spherical world in which not only space, but in a certain sense also the world can be addressed as a world of constant curvature.[4] In doing so both Einstein and de Sitter make certain presuppositions about the matter tensor, which correspond to the incoherence of matter and its relative rest, i.e. the velocity of matter will be supposed to be sufficiently small in comparison to the fundamental velocity[5] — the velocity of light.

† Originally published in *Zeitschrift für Physik* **10**, 377-386 (1922), with the title "Über die Krümmung des Raumes". Both papers are printed with the kind permission of

- Friedman (1922) *"Sobre la curvatura del espacio"*
- Friedmann (1924) *"Sobre la posibilidad de un mundo con curvatura negativa constante del espacio"*
- Lemaître (1927) *"Un Universo homogéneo de masa constante y radio creciente que explica la velocidad radial de las nebulosas extragalácticas"*

Métrica de Friedmann-Lemaître-Robertson-Walker



- Hubble (1929) *"Una relación entre la distancia y la velocidad radial entre nebulosas extragalácticas"*
- Robertson (1935, 1936a,b) *"Cinemática y estructura de mundo I-II-III"*
- Walker (1937) *"Sobre la teoría del modelo de estructura de mundo de Milne"*

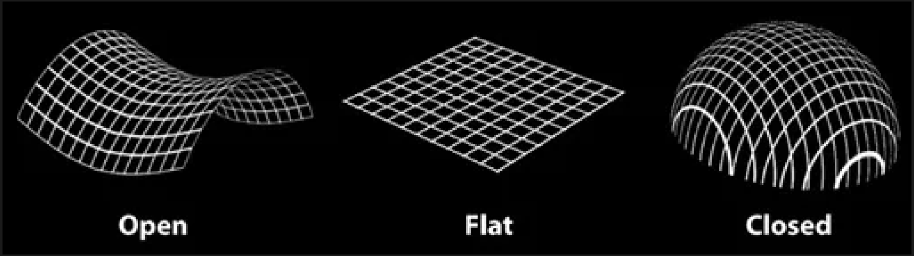
Métrica de Friedmann-Lemaître-Robertson-Walker

La métrica de FLRW se basa en el postulado fundamental del principio cosmológico el cual contiene dos partes:

1. Homogeneidad: El Universo parece el mismo en todos los puntos del espacio
   - No hay cambios bajo traslaciones espaciales
   - El Universo no tiene centro, no hay observadores privilegiados (extensión del principio copernicano)
2. Isotropía: El Universo parece el mismo en todas direcciones
   - No hay cambios bajo rotaciones espaciales

Observacionalmente se distingue que a escalas de 100 Mpc ($\sim 3{,}262 \times 10^8$ años luz) estas dos condiciones se satisfacen.

Métrica de Friedmann-Lemaître-Robertson-Walker



$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

De manera general, se puede decir que la métrica de FLRW son tres métricas en una. Tres representaciones posibles de un Universo espacialmente homogéneo e isotrópico.

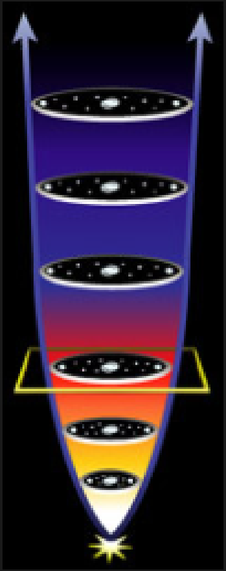
Universo Plano

Siguiendo el principio cosmológico se permite que la escala del espacio (g_{ij}) cambie con el tiempo, tomando como factor de escala el valor $a(t)$. Es decir,

$$\begin{aligned} \vec{e}_x &\to a(t)\,\vec{e}_x \\ \vec{e}_y &\to a(t)\,\vec{e}_y \\ \vec{e}_z &\to a(t)\,\vec{e}_z \end{aligned} \qquad \begin{aligned} \vec{e}_i \cdot \vec{e}_j &\to (a(t)\vec{e}_i) \cdot (a(t)\vec{e}_j) \\ \vec{e}_i \cdot \vec{e}_j &\to a^2(t)\,(\vec{e}_i \cdot \vec{e}_j) \\ g_{ij} &\to a^2(t)\,g_{ij} \end{aligned}$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \times a^2(t)$$

Universo Plano



La métrica representa un espacio-tiempo plano cuyos componentes espaciales cambian con el tiempo en proporción $a(t)$. Es una variedad 4-dimensional curva esto es, $R^{\rho}_{\sigma\mu\nu} \neq 0$ con componentes espaciales sin curvatura ($R^{k}_{lij} = 0$).

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t)\,r^2 & 0 \\ 0 & 0 & 0 & a^2(t)\,r^2 \sin^2\theta \end{pmatrix}$$

Universo Cerrado

Una esfera también cumple con las características del principio cosmológico (todos los puntos son equivalentes, no hay dirección privilegiada desde el centro de la esfera).
Es posible caracterizar cualquier punto de la esfera S^N de manera intrínseca a través de la relación:

$$R^2 = \sum_{i=1}^{N+1} (x^i)^2$$

$$x^j = \cos\theta^j \prod_{i=1}^{j-1} \operatorname{sen}\theta^j$$

$$x^{N+1} = \prod_{i=1}^{N} \operatorname{sen}\theta^i$$

Universo Cerrado

Para S^3 tendremos que

$$x^2+y^2+z^2+w^2=R^2$$
$$x=R\cos\theta$$
$$y=R\sin\theta\cos\phi$$
$$z=R\sin\theta\sin\phi\cos\chi$$
$$w=R\sin\theta\sin\phi\sin\chi$$

, si se calculan entonces los vectores base de la forma,

$$\vec{e}_{i(\theta)}=\frac{\partial}{\partial\theta^i}=\frac{\partial x^j}{\partial\theta^i}\frac{\partial}{\partial x^j}$$

Usando entonces la base $\theta\phi\chi$ y calculando el producto punto g_{ij} obtendremos,

$$g_{\theta\phi}=g_{\phi\chi}=g_{\theta\chi}=0$$
$$\frac{\partial}{\partial\theta}\cdot\frac{\partial}{\partial\theta}=g_{\theta\theta}=R^2$$
$$\frac{\partial}{\partial\phi}\cdot\frac{\partial}{\partial\phi}=g_{\phi\phi}=R^2(\sin\theta)^2$$
$$\frac{\partial}{\partial\chi}\cdot\frac{\partial}{\partial\chi}=g_{\chi\chi}=R^2(\sin\theta)^2(\sin\phi)^2$$

Universo Abierto

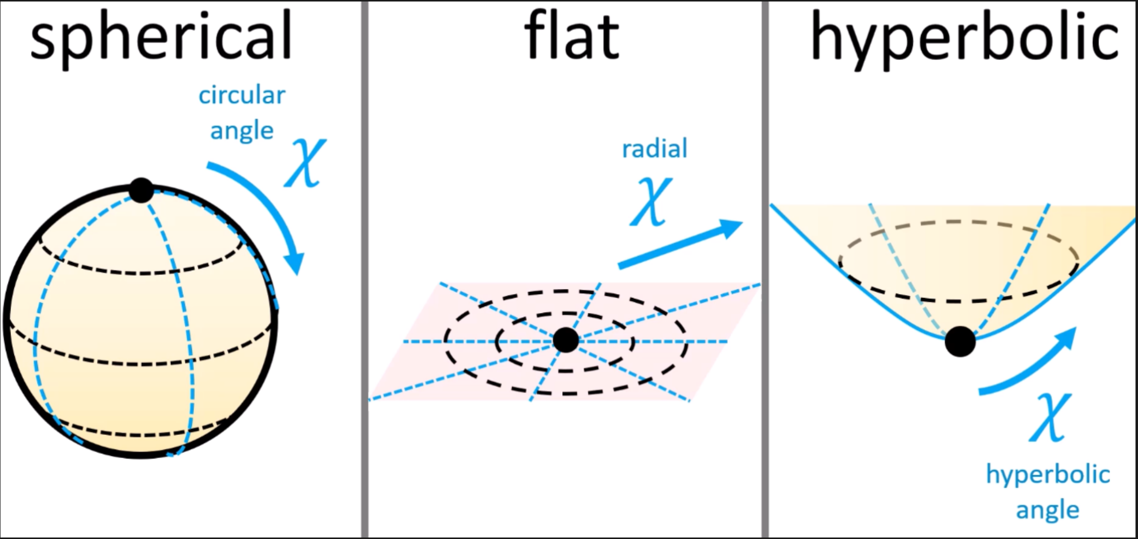
- Métrica de la 3-esfera (hiper superficie con radio R)

$$g_{ij} = R^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2\chi & 0 \\ 0 & 0 & \sin^2\chi\sin^2\theta \end{pmatrix} \rightarrow \mathcal{R} = \frac{6}{R^2}$$

- Métrica del 3-espacio hiperbólico

$$g_{ij} = R^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sinh^2\chi & 0 \\ 0 & 0 & \sinh^2\chi\sin^2\theta \end{pmatrix} \rightarrow \mathcal{R} = -\frac{6}{R^2}$$

Parámetro χ



Parámetro χ

Tomando como parámetro la coordenada χ en las tres métricas y haciendo un cambio de coordenadas tal que $(t,\chi,\theta,\phi) \rightarrow (t,r,\theta,\phi)$ obtendremos,

$$\vec{e}_\chi = \frac{dr}{d\chi}\,\vec{e}_r$$

- Cerrado $= \cos\chi$
- Plano $= 1$
- Abierto $= \cosh\chi$

$$g_{rr} = \vec{e}_r \cdot \vec{e}_r$$

$$g_{\chi\chi} = \vec{e}_\chi \cdot \vec{e}_\chi$$

$$g_{\chi\chi} = (\frac{dr}{d\chi})^2(\vec{e}_r \cdot \vec{e}_r)$$

Cerrado	Plano	Abierto
$r = \sin\chi$	$r = \chi$	$r = \sinh\chi$
$\frac{dr}{d\chi} = \cos\chi$	$\frac{dr}{d\chi} = 1$	$\frac{dr}{d\chi} = \cosh\chi$
$(\frac{dr}{d\chi})^2 = 1 - r^2$	$(\frac{dr}{d\chi})^2 = 1$	$(\frac{dr}{d\chi})^2 = 1 + r^2$

$$(\frac{dr}{d\chi})^2 = 1 - kr^2$$

$$g_{\chi\chi} = (\frac{dr}{d\chi})^2 g_{rr}$$

$$g_{rr} = \frac{1}{1 - kr^2} g_{\chi\chi}$$

Conclusión

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & a^2(t)r^2 & 0 \\ 0 & 0 & 0 & a^2(t)r^2\mathsf{sin}^2\theta \end{pmatrix}$$

Usando la métrica de FLRW en la ecuaciones de campo de Einstein obtendremos las ecuaciones de Friedmann,

$$H^2(t) = \left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{kc^2}{a^2(t)} + \frac{\Lambda c}{3}.$$

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