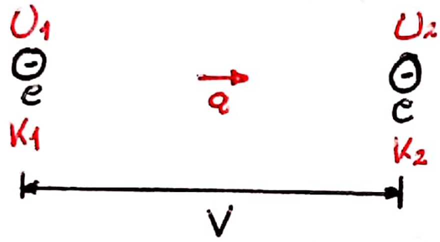
Homework # 6

1. Electrons in projection television sets are accelerated through a potential difference of 50 kV. Calculate the speed of the electrons using the relativistic form of kinetic energy assuming the electrons start from rest.

We have to consider that, the relativistic correction for energy is given by,

$$K = \frac{mc^2}{\sqrt{1-\beta^2}} \quad \text{where } \beta = \frac{v}{c} \quad \text{so, } K = \gamma mc^2$$



We have to consider that, given the energy analysis, we got an initial potential energy condition given by $U = Vq$ and, also we have that according to Einstein derivations we know that every mass in the universe have a rest mass given by mc^2.

So, $U = Ve$

$U = 50 \times 10^3 V \cdot e$

$\boxed{U = 50 \times 10^3 eV}$

Rest mass $= mc^2 = m_e c^2$

$mc^2 = 9.1093 \times 10^{-31} Kg \cdot (3 \times 10^8 m/s)^2$

$\boxed{mc^2 = 8.1984 \times 10^{-14} J}$

Dimensional analysis

$8.1984 \times 10^{-14} \cancel{J} \times \frac{6.242 \times 10^{18} eV}{1 \cancel{J}}$

$\boxed{= 0.5117 \times 10^6 eV}$

Solving the energy balance,

$$U_1 + K_1 = \cancel{U_2}^{\,0} + K_2$$

$$Vq + mc^2 = \frac{mc^2}{\sqrt{1-\beta^2}}$$

$$(1-\beta^2)^{1/2} = \frac{mc^2}{Vq + mc^2}$$

$$1 - \beta^2 = \left(\frac{mc^2}{Vq + mc^2}\right)^2$$

$$\beta^2 = 1 - \left(\frac{mc^2}{Vq + mc^2}\right)^2$$

$> \beta^2 = 0.17$

$$\frac{v^2}{c^2} = 0.17$$

$$\frac{v}{c} = \sqrt{0.17}$$

$$v = \sqrt{0.17}\,c$$

$\boxed{v = 0.4184c}$

2. An electron e^- with kinetic energy 1.000 MeV makes a head-on collision with a positron e^+ at rest. In the collision the two particles annihilate each other and are replaced by two γ-photons of equal energy, each traveling at angles θ with the electron's direction of motion. The reaction is

$$e^- + e^+ \rightarrow 2\gamma$$

Determine the energy E, momentum p and angle of emission θ of each photon.

we have to consider the kinetic energy derivation from relativity postulates, so

$$\beta = \frac{v}{c}$$

$$\beta = \frac{p}{mc}$$

$$K = \underbrace{\frac{mc^2}{\sqrt{1-\beta^2}}}_{\text{moving energy.}} - \underbrace{mc^2}_{\text{rest mass energy}} = (\gamma - 1)mc^2$$

To find the energy of a single photon we have to make an energy balance such,

$$K_1 = K_2$$

$$K_{e^-} = K - mc^2$$

$$K_{e^+} = 0$$

$$K_\gamma = E_\gamma$$

Total energy,

$$E_{e^-} = K + mc^2$$

$$E_{e^+} = mc^2$$

$$E_\gamma = \frac{K + 2mc^2}{2}$$

Thus,

$$(K + mc^2) + mc^2 = 2E_\gamma$$

$$E_\gamma = \frac{K + 2mc^2}{2}$$

$m_e = 9.10 \times 10^{-31}$ kg
$c = 3 \times 10^8$ m/s
$K = 1 \times 10^6 \cancel{eV} \times \frac{1J}{6.242 \times 10^{18} eV}$
$K = 1.6 \times 10^{-13} J$

$$E_\gamma = \frac{K}{2} + mc^2 \rightarrow \text{for each photon}$$

$$E_\gamma = \frac{1.6 \times 10^{-13} J}{2} + (9.1 \times 10^{-31} kg) \cdot (3 \times 10^8 m/s)^2$$

$$\boxed{E_\gamma = 1.6198 \times 10^{-13} J \sim 1.011 \text{ MeV}}$$

Now, to find the linear momentum in the first particle e⁻, we have to derive the magnitud p from the relativistic factor such that,

Considering...

$$p = mv\gamma \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \text{and solving for } v\text{, we have}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$v = \frac{p}{m\gamma} \qquad\qquad \gamma = (1-\beta^2)^{-1/2}$$

$$v^2 = \left(\frac{p}{m\gamma}\right)^2 \quad (1) \qquad\qquad \gamma^{-2} = 1-\beta^2 \quad \text{with } \beta = \frac{v}{c}$$

$$\beta^2 = 1-\gamma^{-2}$$

$$\frac{v^2}{c^2} = 1-\gamma^{-2}$$

$$v^2 = (1-\gamma^{-2})c^2 \quad (2)$$

Equating the two equations we obtain,

$$v^2 = v^2$$

$$\left(\frac{p}{m\gamma}\right)^2 = (1-\gamma^{-2})c^2$$

$$\left(\frac{p}{mc}\right)^2 = (1-\gamma^{-2})\gamma^2$$

$$1 - \frac{1}{\gamma^2} = \frac{\gamma^2-1}{\gamma^2}$$

$$\frac{\gamma^2-1}{\cancel{\gamma^2}} \cdot \cancel{\gamma^2}$$

$$\left(\frac{p}{mc}\right)^2 = \gamma^2 - 1 \quad \Rightarrow \quad \gamma^2 = \left(\frac{p}{mc}\right)^2 + 1$$

So, from this analysis we can say that

$$\gamma = \sqrt{1 + \left(\frac{p}{mc}\right)^2}$$

Knowing this relation between γ (the relativistic factor) and p (the linear momentum) we can say that,

$$K = (\gamma - 1)mc^2$$

$$K = \left(\sqrt{1+\left(\frac{p}{mc}\right)^2} - 1\right)mc^2$$

for the electron e:

Now, solving p for the electron before the collision we got,

$$K = (\gamma - 1) mc^2$$

$$K = \left(\sqrt{1 + \left(\frac{p}{mc}\right)^2} - 1\right) mc^2$$

$$\left(\frac{K}{mc^2} + 1\right)^2 = 1 + \left(\frac{p}{mc}\right)^2$$

$$* \left(\left(\frac{K}{mc^2} + 1\right)^2 - 1\right)^{1/2} mc = p$$

Now, solving for this expression we got

$$\left(\frac{K}{mc^2} + 1\right)^2 = \left[\frac{K^2}{mc^2} + \frac{2K}{mc^2} \cancel{+1}\right] \cancel{-1}$$

$$\frac{K^2}{m^2c^2} + \frac{2K}{mc^2} \Rightarrow \frac{K}{mc^2} \cdot \left(\frac{K + 2mc^2}{mc^2}\right)$$

$$* \sqrt{\frac{K(K + 2mc^2)}{m^2 c^4}} \cdot mc \Rightarrow \frac{\sqrt{K(K+2mc^2)}}{\cancel{m} c^2} \cdot \cancel{mc}$$

$$p = \frac{\sqrt{K(K + 2mc^2)}}{c}$$

$$p = \frac{\sqrt{1.61 \times 10^{-13} J \cdot (1.16 \times 10^{-13} J + 2(9.10 \times 10^{-31} kg \cdot (3 \times 10^8)^2}}{(3 \times 10^8 m/s)}$$

$$\boxed{p = 7.62 \times 10^{-22} \frac{kg \cdot m}{s}}$$

$$\boxed{p = 1.426 MeV/c}$$

This is the linear momentum for the electron before the collision,

Now, considering that, for a single photon,

$$E^2 = \cancel{(mc^2)^2} + (pc)^2$$

$$E = pc$$

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The momentum for a single photon will be,

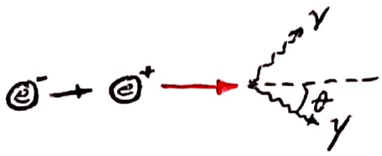
$$p_\gamma = \frac{E_\gamma}{c} \quad \text{from} \quad E = pc$$

So,

$$p_\gamma = \frac{1.6196\times10^{-13}\,J}{3\times10^{8}\,m/s} = \boxed{5.4\times10^{-22}\ \frac{kg\cdot m}{s}}$$

$$\boxed{p_\gamma = 1.011\times10^{6}\,MeV/c}$$

If we can sketch the reaction like this,



$$p = 2p_\gamma\cos\theta$$

$$\cos\theta = \left(\frac{p}{2p_\gamma}\right) \qquad \theta = \cos^{-1}\left(\frac{p}{2p_\gamma}\right)$$

Thus,

$$\theta = \cos^{-1}\left(\frac{7.62\times10^{-22}}{2\cdot5.4\times10^{-22}}\right)$$

$$\boxed{\theta = 45.12^\circ}$$

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