Tarea 6: Mecánica clásica

Holman Daniel Quintero Salazar

• Ejercicio #1: Ecuación de Hamilton-Jacobi Proyectil.

Teniendo el Hamiltoniano de la forma

$$H = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + mgy$$

Con $\frac{\partial S}{\partial t} + H = 0$ tendremos que,

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2m}\left(\frac{\partial S}{\partial y}\right)^2 + mgy + \frac{\partial S}{\partial t} = 0$$

Buscamos la solución de la forma $S(x,y,E,t,\gamma) = \gamma x + f(y,E) - Et$ con $\gamma = cte, E = cte.$

$$\frac{1}{2m}\gamma^2 + \frac{1}{2m}\left(\frac{\partial f}{\partial y}\right)^2 + mgy - E = 0$$

$$f(y,E) = \int_0^y \left(2m\left(E - mgy' - \frac{\gamma^2}{2m}\right)\right)^{1/2} dy' = \int_0^y u^{1/2}\frac{du}{-2m^2g} = \frac{2}{-2m^2g}\left.\frac{u^{3/2}}{3}\right|_0^y = -\frac{1}{3m^2g}\left[2m\left(E - mgy - \frac{\gamma^2}{2m}\right)\right]^{3/2}$$

Por consiguiente, la acción será,

$$S = \gamma x - \frac{1}{3m^2g}\left(2m\left(E - mgy - \frac{\gamma^2}{2m}\right)\right)^{3/2} - Et$$

Igualmente tendremos que,

$$\beta_1 = \frac{\partial S}{\partial E} = -\frac{1}{2m^2g}\left(2m\left(E - mgy - \frac{\gamma^2}{2m}\right)\right)^{1/2} 2m - t = -\frac{1}{mg}\left(2m\left(E - mgy - \frac{\gamma^2}{2m}\right)\right)^{1/2} - t$$

$$\beta_2 = \frac{\partial S}{\partial \gamma} = x - \frac{1}{2m^2g}\left(2m\left(E - mgy - \frac{\gamma^2}{2m}\right)\right)^{1/2} - 2\gamma = x + \frac{\gamma}{m^2g}\left(2m\left(E - mgy - \frac{\gamma^2}{2m}\right)\right)^{1/2}$$

despejando para x y y,

$$(\beta_1 + t)^2(-mg)^2 = 2m\left(E - mgy - \frac{\gamma^2}{2m}\right) \qquad x = \beta_2 - \frac{\gamma}{m^2g}\left(2m\left(E - mgy - \frac{\gamma^2}{2m}\right)\right)^{1/2}$$

$$y = -\frac{g}{2}(\beta_1 + t)^2 + \frac{E}{mg} - \frac{\gamma^2}{2m^2g} \qquad x - \beta_2 = \frac{\gamma}{m}(\beta_1 + t)$$

Sabiendo entonces que $x(t=0)=0$, $y(t=0)=0$, $V_x(t_0)=V_0\cos\alpha$ y $V_y(t_0)=V_0 \sin\alpha$ obtenemos que

$$\beta_2 = \frac{V_0^2}{g}\sin\alpha\cos\alpha \quad , E=\frac{V_0^2}{2}m \quad , \gamma = V_0 m\cos\alpha \quad , \beta_1 = \frac{-V_0 \sin\alpha}{g}$$

Reemplazando obtenemos que,

$$y(t) = -\frac{g}{2}\left(-\frac{V_0}{g}\sin\alpha + t\right)^2 + \frac{V_0^2 m}{2mg} - \frac{V_0^2 m^2\cos^2\alpha}{2m^2 g}$$

$$= -\frac{g}{2}\left(\frac{V_0^2}{g^2}\sin^2\alpha - 2t\frac{V_0}{g}\sin\alpha + t^2\right) + \frac{V_0^2}{2g} - \frac{V_0^2}{2g}\cos^2\alpha$$

$$= \frac{-V_0^2}{2g}\sin^2\alpha + tV_0\sin\alpha - \frac{t^2 g}{2} + \frac{V_0^2}{2g} - \frac{V_0^2\cos^2\alpha}{2g}$$

$$y = tV_0\sin\alpha - \frac{t^2 g}{2}$$

$$x = \frac{V_0^2}{g}\sin\alpha\cos\alpha + \frac{V_0 m\cos\alpha}{m}\left(\frac{-V_0}{g}\sin\alpha + t\right)$$

$$= \frac{V_0^2}{g}\sin\alpha\cos\alpha - \frac{V_0^2}{g}\cos\alpha\sin\alpha + V_0 t\cos\alpha$$

$$x = V_0 t\cos\alpha$$

- Ejercicio #2: Ecuación de Hamilton-Jacobi: Oscilador armónico

Teniendo el hamiltoniano

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

y teniendo que, $\frac{\partial S}{\partial t} + H = 0$. y $P = \frac{\partial S}{\partial x}$, obtenemos,

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2}m\omega^2 x^2 = 0$$

Si tenemos que la acción viene dada por $S=S(x,t)$, podemos hacer una separación de variables de la forma $S=S_1(x)+S_2(t)$

$$\frac{dS_2}{dt} + \frac{1}{2m}\left(\frac{dS_1}{dx}\right)^2 + \frac{1}{2}m\omega^2 x^2 = 0$$

$$-\frac{dS_2}{dt} = \frac{1}{2m}\left(\frac{dS_1}{dx}\right)^2 + \frac{1}{2}m\omega^2 x^2$$

$$S_2 = -\beta_1 t + C$$

$$\frac{1}{2m}\left(\frac{dS_1}{dx}\right)^2+\frac{1}{2}m\omega^2x^2=\beta$$

$$S_1=\int\left(2m\left(\beta-\frac{1}{2}m\omega^2x^2\right)\right)^{1/2}dx=m\omega\int\left(\frac{2\beta}{m\omega^2}-x^2\right)^{1/2}dx\Rightarrow S_1=\frac{m\omega}{2}(x^2+\beta^2)\cot\omega t$$

$$S_1=\frac{m\omega}{2}(x^2+\beta^2)\cot\omega t\quad,\quad S_2=-m\omega x\beta\csc\omega t$$

$$S=S_1+S_2=\frac{m\omega}{2}(q^2+\beta^2)\cot\omega t\pm m\omega x\beta\csc\omega t$$

• Ejercicio #3:

Tomando el conjunto de coordenadas en un sistema cilíndrico (r,φ,z) podemos convertir a un sistema parabólico (ξ,η,φ) de la forma,

$$z=\frac{1}{2}(\xi-\eta)\ ,\ r=\sqrt{\xi\eta}\ ,\ r=\sqrt{z^2+r^2}=\frac{1}{2}(\xi+\eta),\ \xi=r+z,\ \eta=r-z$$

Procedemos a realizar la sustitución en $\mathcal{L}$

$$\mathcal{L}=\frac{1}{2}m(\dot{r}^2+r^2\dot{\varphi}^2+\dot{z}^2)-V(r,\varphi,z)$$

$$\text{con}\rightarrow\ \dot{r}=\frac{1}{2}(\xi\eta)^{-1/2}(\dot{\xi}\eta+\xi\dot{\eta})$$

$$\mathcal{L}=\frac{1}{8}m(\xi+\eta)\left(\dot{\xi}^2/\xi+\dot{\eta}^2/\eta\right)+\frac{1}{2}m\xi\eta\dot{\varphi}^2-V(\xi,\eta,\varphi)$$

Con esto calculamos el momento canónico del sistema,

$$P_\xi=\frac{1}{4}m(\xi+\eta)\dot{\xi}/\xi\quad,\quad P_\eta=\frac{1}{4}m(\xi+\eta)\dot{\eta}/\eta\quad,\quad P_\varphi=m\xi\eta\dot{\varphi}$$

El hamiltoniano será entonces,

$$H=\frac{2}{m}\frac{\xi P_\xi^2+\eta P_\eta^2}{\xi+\eta}+\frac{P_\varphi^2}{2m\xi\eta}+V(\xi,\eta,\varphi)$$

Ahora bien, si consideramos potenciales de la forma $V=\frac{a(\xi)+b(\eta)}{\xi+\eta}=\frac{a(r+z)+b(r-z)}{2r}$, concluiremos que,

$$\frac{2}{m(\xi+\eta)}\left[\xi\left(\frac{\partial S_0}{\partial\xi}\right)^2+\eta\left(\frac{\partial S_0}{\partial\eta}\right)^2\right]+\frac{1}{2m\xi\eta}\left(\frac{\partial S_0}{\partial\varphi}\right)^2+\frac{a(\xi)+b(\eta)}{\xi+\eta}=E$$

Considerando que φ es una coordenada cíclica y separando la acción de la forma $S_0 = S_1(\xi)+S_2(\eta)+P_\varphi \varphi$.

$$\overbrace{2\xi\left(\frac{dS_1}{d\xi}\right)^2 + ma(\xi) - m\epsilon\xi + \frac{P_\varphi^2}{2\xi}}^{\beta} + 2\eta\left(\frac{dS_2}{d\eta}\right)^2 + mb(\eta) - m\epsilon\eta + \frac{P_\varphi^2}{2\eta} = 0$$

- $2\xi\left(\frac{dS_1}{d\xi}\right)^2 + ma(\xi) - m\epsilon\xi = \beta$ • $2\eta\left(\frac{dS_2}{d\eta}\right)^2 + mb(\eta) - m\epsilon\eta + \frac{P_\varphi^2}{2\eta} = -\beta$

A partir de la consideración que $S = \sum S_n(q_n, \alpha_1, \dots, \alpha_s) - \epsilon(\alpha_1, \dots, \alpha_s)t$, obtendremos

$$S_1 = \int \left(\frac{1}{2}m\epsilon + \frac{\beta}{2\xi} - \frac{ma(\xi)}{2\xi} - \frac{P_\varphi^2}{4\xi^2}\right)^{1/2} d\xi$$

$$S_2 = \int \left(\frac{1}{2}m\epsilon - \frac{\beta}{2\eta} - \frac{mb(\eta)}{2\eta} - \frac{P_\varphi^2}{4\eta^2}\right)^{1/2} d\eta$$

$$S = -Et + S_1(\xi) + S_2(\eta) + P_\varphi \varphi$$

• Ejercicio #4:

Igual que en el ejercicio anterior tendremos que,

$$r = \sigma\sqrt{(\xi^2-1)(1-\eta^2)}, \quad z = \sigma\xi\eta, \quad r = \sigma(\xi \pm \eta), \quad \xi = \frac{r^+ + r^-}{2\sigma}, \quad \eta = \frac{r^+ - r^-}{2\sigma}$$

obtendremos,

$$\mathcal{L} = \frac{1}{2}m\sigma^2(\xi^2-\eta^2)\left(\frac{\dot\xi^2}{\xi^2-1} + \frac{\dot\eta^2}{1-\eta^2}\right) + \frac{1}{2}m\sigma^2(\xi^2-1)(1-\eta^2)\dot\varphi^2 - V(\xi,\eta,\varphi)$$

$$H = \frac{1}{2m\sigma^2(\xi^2-\eta^2)}\left[(\xi^2-1)P_\xi^2 + (1-\eta^2)P_\eta^2 + \left(\frac{1}{\xi^2-1} + \frac{1}{1-\eta^2}\right)P_\varphi^2\right] + V(\xi,\eta,\varphi)$$

Ahora, si tenemos potenciales de la forma,

$$V = \frac{a(\xi)+b(\eta)}{\xi^2-\eta^2} = \frac{a^2}{r^- r^+}\left(a\left(\frac{r^+ + r^-}{2\sigma}\right) + b\left(\frac{r^+ - r^-}{2\sigma}\right)\right)$$

De nueva cuenta, si tenemos que, $S_0 = P_\varphi\varphi + S_1(\xi) + S_2(\eta)$, tendremos

$$E = \frac{1}{2m\sigma^2(\xi^2-\eta^2)}\left[(\xi^2-1)\left(\frac{\partial S_1}{\partial\xi}\right)^2 + (1-\eta^2)\left(\frac{\partial S_2}{\partial\eta}\right)^2 + \left(\frac{1}{\xi^2-1} + \frac{1}{1-\eta^2}\right)P_\varphi^2\right] + \frac{a(\xi)+b(\eta)}{\xi^2-\eta^2}$$

Así,

$$2E\,m\sigma^2(\xi^2-\eta^2)=(\xi^2-1)\left(\frac{dS_1}{d\xi}\right)^2+(1-\eta^2)\left(\frac{dS_2}{d\eta}\right)^2+\left(\frac{1}{\xi^2-1}+\frac{1}{1-\eta^2}\right)p_\varphi^{\,2}+2m\sigma a(\xi)+2m\sigma b(\eta)$$

$$\underbrace{2Em\sigma^2\xi^2-(\xi^2-1)\left(\frac{dS_1}{d\xi}\right)^2-\frac{p_\varphi^2}{\xi^2-1}}_{\beta}-2m\sigma a(\xi)=2mE\sigma^2\eta^2+(1-\eta^2)\left(\frac{dS_2}{d\eta}\right)^2+\frac{p_\varphi^2}{1-\eta^2}+2m\sigma b(\eta)$$

$$S_1=\int\left[2m\sigma^2E+\frac{\beta-2m\sigma^2a(\xi)}{\xi^2-1}-\frac{p_\varphi^2}{(\xi^2-1)^2}\right]^{1/2}d\xi$$

$$S_2=\int\left[2m\sigma^2E-\frac{\beta+2m\sigma^2b(\eta)}{1-\eta^2}-\frac{p_\varphi^2}{(1-\eta^2)^2}\right]^{1/2}d\eta$$

$$S=-Et+S_1(\xi)+S_2(\eta)+p_\varphi\varphi$$

• Ejercicio #5:

Aplicando la variación,

$$\delta\int\sqrt{E-V}\,dl=-\int\left[\frac{\partial V}{\partial\vec r}\cdot\frac{\delta\vec r}{2\sqrt{E-V}}\,dl-\sqrt{E-V}\,\frac{d\vec r}{dl}\cdot\delta\vec r\right]$$

Con $dl^2=d\vec r^{\,2}\Rightarrow dl\,d\delta l=d\vec r\cdot d\delta\vec r$. Igualando a cero obtenemos,

$$2\sqrt{E-V}\,\frac{d}{dl}\left[\sqrt{E-V}\,\frac{d\vec r}{dl}\right]=-\frac{\partial V}{\partial\vec r}$$

Ahora bien, si tenemos que $\vec F=-\frac{\partial V}{\partial\vec r}$, veremos que,

$$\frac{d^2\vec r}{dl^2}=\frac{\vec F-(\vec F\cdot\dot{\vec r})\dot{\vec r}}{2(E-V)}$$

Entonces, $\vec F-(\vec F\cdot\dot{\vec r})\dot{\vec r}=\vec F_n\Longrightarrow\frac{d^2\vec r}{dl^2}=\frac{\vec n}{R}$, lo que nos conduce a $\frac{mv^2}{R}\vec n=\vec F_n$

• Ejercicio # 6:

$$\mathcal{L}' = \mathcal{L} + \frac{d}{dx^\nu} X^\nu(\eta_\rho, x_\mu)$$

$$S = \int \mathcal{L} \prod_\nu dx^\nu = \int_\Omega \mathcal{L}' \prod_\nu dx^\nu + \int_\Omega \frac{d}{dx^\nu} X^\nu \prod_w dx^w$$

Considerando el teorema de stokes

$$\int \frac{\partial}{\partial \alpha} \mathcal{L}' d^4x = \delta \int \mathcal{L} d^4x + \frac{\partial}{\partial \alpha} \int X^\nu d\sigma_\nu = \frac{\partial S}{\partial \alpha} = 0$$

$$= \int \delta x^\nu d\sigma_\nu = \int \frac{\partial x^\nu}{d\eta_s} \cancel{\frac{\partial \eta_s}{\partial \alpha}} d\sigma_\nu$$

$$\delta S = \delta \int \mathcal{L}\, d^4x$$

• Ejercicio # 8: Vibraciones trasversales de una cuerda.

a)

Teniendo $T = \frac{1}{2} \Sigma m \dot{\eta}_i^2$, $V = \frac{1}{2} \Sigma (\eta_{j+1} - \eta_j)^2$

$$\mathcal{L} = \frac{1}{2} \Sigma m \dot{\eta}_i^2 - \frac{1}{2} \Sigma a \left[\frac{m}{a} \dot{\eta}_i^2 - ka\left(\frac{\eta_{i+1} - \eta_i}{a}\right)^2\right]$$

Considerando la ley de Hooke

$$F = k(\eta_{i+1} - \eta_i) = ka\left(\frac{\eta_{i+1} - \eta_i}{a}\right) \Rightarrow F = \lim_{a \to 0} ka\left(\frac{\eta_{i+1} - \eta_i}{a}\right) = \frac{\partial \eta}{\partial x} Y$$

siendo Y el módulo de Yang. Así;

$$\mathcal{L} = \frac{1}{2} \int \left[\mu \dot{\eta}^2 - Y\left(\frac{\partial \eta}{\partial x}\right)^2\right] dx$$

aplicando Euler-Lagrange,

$$\frac{\partial \mathcal{L}}{\partial \eta} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial x}\right)}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial t}\right)}\right) = 0$$

$$\mu \frac{\partial^2 \eta}{\partial t^2} = \frac{\partial^2 \eta}{\partial x^2} Y.$$

b) La velocidad transversal de un segmento será $\frac{dy}{dt}$, así

$$dT = \frac{1}{2}\mu\, dx \left(\frac{dy}{dt}\right)^2 \quad , \quad T = \frac{1}{2}\int \mu \left(\frac{dy}{dt}\right)^2 dx$$

Cuando una onda transversal se propaga, causa una extensión en el resorte debido a la diferencia entre ds y dx. Se atribuye pues a la tensión $T(ds - dx)$ que parece realiza trabajo deformando el resorte y resultando en un cambio de volumen.

$$ds = (dx^2 + dy^2)^{1/2}$$
$$= dx + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx$$

Veremos que,

$$V \simeq \frac{1}{2}\int T\left(\frac{dy}{dx}\right)^2 dx$$

$$\mathcal{L} = T - V = \frac{1}{2}\int \mu\left(\frac{dy}{dt}\right)^2 dx - \frac{1}{2}\int T\left(\frac{dy}{dx}\right)^2 dx$$

$$\mathcal{L} = \frac{1}{2}\int \left[\mu\left(\frac{dy}{dt}\right)^2 - T\left(\frac{dy}{dx}\right)^2\right] dx$$

- Ejercicio #9: Densidad lagrangiana con segundas derivadas

Asumimos que la densidad lagrangiana $\mathcal{L}$ no solo es función de las derivadas de primer orden del campo η_μ, sino también de derivadas de orden superior

$$\mathcal{L} = \mathcal{L}(\eta_\mu, \partial_\mu \eta_\mu, \partial_\mu \partial_\nu \eta_\mu)$$

Con tal de encontrar las ecuaciones de movimiento, suponemos que se transforma el campo de manera infinitesimal de la forma $\eta_\rho \to \eta_\rho' = \eta_\rho + \delta\eta_\rho$ con $\delta\eta_\rho$ la variación del campo.

Así, vamos a tener que,

$$\mathcal{L}' = \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \eta_\rho}\delta\eta_\rho + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \eta_\rho)}\delta(\partial_\mu \eta_\rho) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \eta_\rho)}\delta(\partial_\mu \partial_\nu \eta_\rho)$$

Entonces,

$$\delta S = \int d^4x\,(\mathcal{L}' - \mathcal{L}) = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \eta_\rho}\delta\eta_\rho + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \eta_\rho)}\delta(\partial_\mu \eta_\rho) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \eta_\rho)}\delta(\partial_\mu \partial_\nu \eta_\rho)\right)$$

Veamos que,

$$\delta(\partial_\mu \partial_\rho) = \partial_\mu(\delta \eta_\rho)$$
$$\delta(\partial_\mu \partial_\nu \eta_\rho) = \partial_\mu \partial_\nu(\delta \eta_\rho)$$

Así,

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \eta_\rho} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \eta_\rho)} - \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \eta_\rho)} \right) \delta \eta_\rho + \int d^4x\, \partial_\mu$$

Dado que el segun término en la integral es cero, aplicando el principio de Hamilton vemos que,

$$\frac{\partial \mathcal{L}}{\partial \eta_\rho} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \eta_\rho)} - \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \eta_\rho)} = 0$$