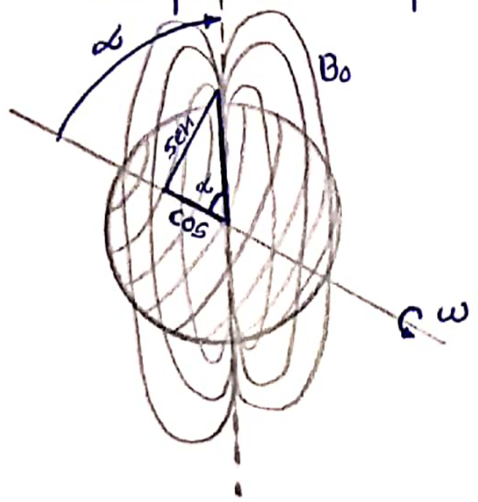
Homework #3

Radiation from moving charges

(1.) A pulsar is conventionally believed to be a rotating neutron star. Such a star is likely to have a strong magnetic field, Bo, since it traps lines of force during its collapse. If the magnetic axis of the neutron star does not line up with the rotation axis, there be magnetic dipole radiation from the time-changing magnetic dipole, m(t). Assume that the mass and radius of the neutron star are M and R, respectively; that the angle between the magnetic and rotation axes is α; and that the rotational angular velocity is ω.

a. Find an expression for the radiated power P in terms of ω, R, Bo and α.

We can conceptualize the problem is this way,



Now, from

$$\frac{dP}{d\Omega} = \frac{\ddot{d}^2}{4\pi c^3} \sin^2\theta$$

we know that

$$\vec{P} = \frac{2\ddot{\vec{d}}^2}{3c^3}$$

that is, the dipole approximation based on the Larmor formula for electric dipole radiation.

Now, if we consider that $\ddot{\vec{d}} = (x, y)$, then $\ddot{\vec{d}} = \sqrt{dx^2 + dy^2}$, and following the corresponding trigonometric analysis we visualize that,

$\vec{d}\cos\alpha \rightarrow$ constant because it is fixed to the rotation axis.

Hence

$d_x = d\sin\alpha \sin\omega t$

$\underline{\ddot{\vec{d}} = \omega^2 \vec{d}\sin\alpha}$

$d_y = d\sin\alpha \cos\omega t$

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From now on we have to analyze B_0 as a function of $\vec{d}$. To do that arragement we can carry out the analogy to electric fields such that,

$$\vec{E} = \nabla \phi \ , \quad \text{and assuming that} \quad \phi \approx \frac{d\cos\theta}{r^2}$$

Operating with the nabla operator we obtain,

$$\vec{E} = -\frac{2d\cos\theta}{r^3}\hat{r} - \frac{d\sin\theta}{r^3}\hat{\theta}$$

Now, if we consider the argument that the angular deviation from the pole $\theta = 0$ makes the previous terms,

$$\vec{E} = -\frac{2d\cos\theta}{r^3} \cancelto{0}{- \frac{d\sin\theta}{r^3}\hat{\theta}} \Rightarrow E = \frac{2d}{r^3}$$

and by the analogy that means

$$B = \frac{2d}{r^3}$$

Considering that, $P = \frac{2\ddot{d}^2}{3c^3}$ and $\ddot{d} = \omega^2 d\sin\alpha$,

$$> B_0 = \frac{2d}{r^3}$$

$$\frac{B_0 r^3}{2} = d \quad \therefore \quad \ddot{d} = \frac{\omega^2 B_0 r^3 \sin\alpha}{2} \quad \therefore \quad P = \frac{\cancel{2}(\omega^2 B_0 r^3 \sin\alpha)^2}{3(2)^{\cancel{2}} c^3}$$

$$\boxed{P = \frac{\omega^4 B_0^2 r^6 \sin^2\alpha}{6c^3}}$$

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b. Assuming that the rotational energy of the pulsar is the ultimate source of the radiated power, find an expression for the slow-down time-scale, $\tau = -\omega/\dot{\omega}$, of the pulsar.

Considering that, for a rotational perfect sphere (hypothetical shaped pulsar) we got:

$$K_{ROT} = \frac{1}{2} I \omega^2$$

where for, a sphere,

$$I = \frac{2}{5} M R^2$$

We obtain,

$$K_{ROT} = \frac{(\frac{\cancel{2}}{5} M R^2)\omega^2}{\cancel{2}} = \frac{M R^2 \omega^2}{5}$$

Now, the problem suggest that the pulsar will lose energy proportionally to the radiated power, so,

$P = -\frac{dK_{ROT}}{dt}$, to find $\frac{dK_{TR}}{dt}$ we see that,

$$K_{ROT} = \frac{M R^2 \omega^2}{5} \Rightarrow K_{ROT} = \frac{M R^2 (\frac{d\theta}{dt})^2}{5}, \quad \text{so} \quad \frac{dK_{ROT}}{dt} = \frac{M R^2 (\frac{d^2\theta}{dt^2}) 2}{5}$$

and $\frac{d\theta}{dt} = \omega$, $\frac{d^2\theta}{dt^2} = \dot{\omega}$

So, we see that,

$P = -\frac{dK_{ROT}}{dt} = -\frac{2 M R^2 \omega \dot{\omega}}{5}$ and reordering terms from P we can derive,

$$> \frac{\omega^4 R^6 B_0^2 \sin^2\alpha}{6c^3} = -\frac{2 M \cancel{R^2} \cancel{\omega} \dot{\omega}}{5} \Rightarrow \frac{\omega^3 R^4 B_0^2 \sin^2\alpha}{6c^3} = -\frac{2 M \dot{\omega}}{5}$$

Now, given that $\tau = -\frac{\omega}{\dot{\omega}}$, we see that,

$$\frac{\omega^3 R^4 B_0^2 \sin^2\alpha}{6c^3} = -\frac{2M\dot{\omega}}{5}.$$

Reordering we obtain,

$$\boxed{\tau = -\frac{\omega}{\dot{\omega}} = \frac{12 \cdot Mc^3}{5\omega^2 R^4 B_0^2 \sin^2\alpha}}$$

c) For $M = 2\times10^{33}g$, $R = 10^6 cm$, $B_0 = 10^{12}$ gauss, $\alpha = 90°$, find P and τ for $\omega = 10^4 s^{-1}$; $\omega = 10^2 s^{-1}$. The larger of these ~~to~~ rates is believed to be typical of newly formed pulsars.

From the assumption that

$$P = \frac{\omega^4 B_0^2 r^6 \sin^2\alpha}{6c^3} \quad \text{and} \quad \tau = \frac{12 Mc^3}{5\omega^2 R^4 B_0^2 \sin^2\alpha}$$

Making the dymensional analysis for each one we got

$$P = \frac{\frac{1}{\cancel{s^4}} \cdot gauss^2 \cdot cm^6}{\frac{m^3}{\cancel{s^3}}} = \frac{cm^6}{s \cdot m^3} \cdot \left(\frac{g^{1/2}}{cm^{1/2} \cdot s}\right)^2 = \frac{cm^6}{s \cdot m^3} \cdot \frac{g}{\cancel{cm} \cdot s^2}$$

and $\frac{\cancel{cm^6}}{s^3 \cancel{m^3}} \times \frac{1 \cancel{m^6}}{(100\, \cancel{cm})^6} \times \frac{1\, kg}{100 \cancel{g}} \times 1\cancel{g} = \frac{1}{100^6} \frac{kg\, m^2}{s^3}$ (W)

Thus,

$$P = \frac{1}{100^6}\left(\frac{(10^4)^4 (10^{12})^2 (10^6)^6}{6(3\times10^8)}\right) sen^2(90) = \boxed{5.55\times10^{54} W}$$

$$P = \frac{1}{100^6}\left(\frac{(10^2)^4 (10^{12})^2 (10^6)^6}{6(3\times10^8)}\right) sen^2(90) = \boxed{5.58\times10^{46} W}$$

and for τ

$$\frac{\cancel{g}\cdot\frac{m^3}{\cancel{s^3}}}{\frac{\cancel{g}}{\cancel{cm}\cdot\cancel{s^4}}\cdot cm^{\cancel{4}}\cdot\frac{1}{\cancel{s^2}}} = \frac{\cancel{m^3}\cdot s}{\cancel{cm^3}} \times \left(\frac{100\,\cancel{cm}}{1\,\cancel{m}}\right)^3 = 100^3\,s$$

So,

$$\tau = 100^3\left(\frac{12\,(2\times10^{33})\,(3\times10^{8})^3}{5\,(10^{12})^2(10^{6})^4(10^{4})^2\sin^2(90)}\right) = \boxed{1.3\times10^{9}\,s}$$

$$\tau = 100^3\left(\frac{12\,(2\times10^{33})\,(3\times10^{8})^3}{5\,(10^{12})^2(10^{6})^4(10^{2})^2\sin^2(90)}\right) = \boxed{1.3\times10^{13}\,s}$$

2. Two oscillating dipole moments (radio antennas) $\vec{d}_1$ and $\vec{d}_2$ are oriented in the vertical direction and are a horizontal distance L apart. They oscillate in phase at the same frequency ω. Consider radiation at an angle θ with respect to the vertical and in the vertical plane containing the two dipoles.

a) Show that

$$\frac{dP}{d\Omega} = \frac{\omega^4 \sin^2\theta}{8\pi c^3}\left(d_1^2 + 2d_1 d_2 \cos\delta + d_2^2\right)$$

where

$$\delta = \frac{\omega L \sin\theta}{c}.$$

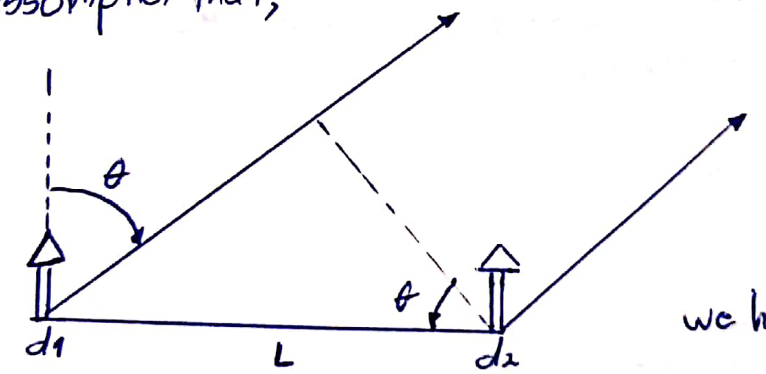
The magnitude of E_{rad} is,

$$|\vec{E}_{rad}| = \frac{q\dot{v}}{Rc^2}\sin\theta.$$

So, considering $q\dot{v} = -\omega^2 d\cos\omega t$ for each dipole, then,

$$|\vec{E}_{rad}| = \frac{-\omega^2}{Rc^2}\sin\theta\left(d_1\cos\omega t + d_2\cos\omega(t-\Delta t)\right)$$

From the assumption that,



we have that $\Delta t = \frac{L}{c}\sin\theta$. So,

$$|\vec{E}_{rad}| = -\frac{\omega^2}{Rc^2}\sin\theta\left(d_1\cos\omega t + d_2\cos\omega t\cos\omega\Delta t + d_2\sin\omega t\sin\omega\Delta t\right)$$

So, knowing that $\delta = \frac{\omega L}{c}\sin\theta$, we estimate that

$$|\vec{E}_{rad}| = -\frac{\omega^2}{Rc^2}\sin\theta\left(d_1\cos\omega t + d_2\cos\omega t\cos\delta + d_2\sin\omega t\sin\delta\right)$$

$$\Downarrow$$

'' '' '' '' $\left((d_1 + d_2\cos\delta)\cos\omega t + d_2\sin\delta\sin\omega t\right)$

Squaring and averaging over time we observe that,

$$\langle|\vec{E}_{rad}|^2\rangle = \frac{\omega^4}{R^2c^4}\sin^2\theta\left((d_1+d_2\cos\delta)^2\overset{1/2}{\cancel{\langle\cos^2\omega t\rangle}} + (d_2\sin\delta)^2\overset{1/2}{\cancel{\langle\sin^2\omega t\rangle}}\right)$$

$$\langle|\vec{E}_{rad}|^2\rangle = \frac{\omega^4\sin^2\theta}{2R^2c^4}\left((d_1+d_2\cos\delta)^2 + (d_2\sin\delta)^2\right)$$

$$\langle|\vec{E}_{rad}|^2\rangle = \frac{\omega^4\sin^2\theta}{2R^2c^4}\left(d_1^2 + 2d_1d_2\cos\delta + d_2^2\right)$$

For $\langle\frac{dP}{d\Omega}\rangle = \frac{cR^2}{4\pi}\langle|\vec{E}_{rad}|^2\rangle$ we derive that

$$\langle\frac{dP}{d\Omega}\rangle = \frac{\cancel{c}\cancel{R^2}}{4\pi}\cdot\frac{\omega^4\sin^2\theta}{2\cancel{R^2}c^4}\left(d_1^2+2d_1d_2\cos\delta+d_2^2\right)$$

$$\langle\frac{dP}{d\Omega}\rangle = \frac{\omega^4\sin^2\theta}{8\pi c^3}\left(d_1^2+2d_1d_2\cos\delta+d_2^2\right)$$

b. Thus show directly that when $L \ll \lambda$, the radiation is the same as from a single oscillating dipole of amplitude d_1+d_2.

If we have $L \ll \lambda$, then $\delta = \frac{2\pi L\sin}{\lambda} \ll 1$

$> \delta = \frac{L\sin\theta\,\omega}{c}$

$\delta = \frac{2\pi\sin\theta L\nu}{c}$ $\frac{\nu}{c} \propto \frac{1}{\lambda}$

$\delta = \frac{2\pi\sin\theta\cdot L}{\lambda}$ $\rightarrow$ So, $\langle\frac{dP}{d\Omega}\rangle = \frac{\omega^4\sin^2\theta}{8\pi c^3}\left(d_1^2+2d_1d_2+d_2^2\right)$

$$\boxed{\langle\frac{dP}{d\Omega}\rangle = \frac{\omega^4\sin^2\theta}{8\pi c^3}(d_1+d_2)^2}$$

Radiation from an oscillating charge with dipole moment d_1+d_2.