Homework #2
Fundamentals of Radiative Transfer

(1.) A hypothetical structure, called a "Dyson sphere", is a structure that an advanced civilization might build to harvest as much energy as possible from the star of their "solar system". To explain the concept, note that for our solar system most of the radiative energy emitted by the Su escapes to the vast Universe, with only an extremely small fraction of it being intercepted by the Earth for the use of human beings. As envisioned by F. Dyson, to maximally harvest energy (solar), one might build a spherical shell, for instance with a radius of 1AU and centered at the Sun, that completely encloses the Sun. Such a structure, if it exists, would have a much lower temperature than the Sun itself such that the entire solar system would appear (to an observer outside the solar system) to emit infrared radiation. Dyson suggested a search of "infrared stars" as a way to detect extraterrestrial intelilengt life.

Suppose that such a structure with a radius of 1.AU is built for our solar system. Moreover, assume that the spherical shell is very thin and highly conductive such that its inner surface (facing the Sun) and outer surface (facing the universe for aliens to observe) have the same temperature. The material used to built the shell is completely absorbing of solar radiation. (i.e., its albedo is zero).

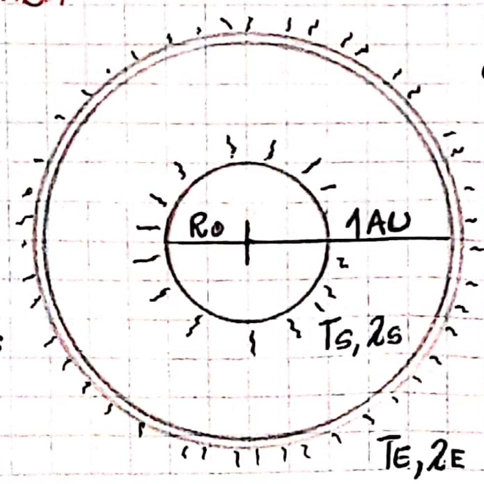
a) What would be the temperature, in K, of the spherical shell at radiative equilibrium?

b) Using Wien's displacement law, what would be the peak wavelength, in μm, of the radiation emitted by this Dyson sphere?

c) What would be the surface temperature of the Dyson sphere if its radius is 2 A.U instead of 1 AU?

Solution

a)



$T_S > T_E$
$\lambda_E > \lambda_S$

First of all we have to consider the total amount of energy that the Sun is emitting outwards its surface.

We can calculate this using the Stefan-Boltzmann radiation law.

The total amount of energy irradiated by the Sun will be:

$$L = Ae\sigma T^4$$

At first, we can keep in mind that the total amount of radiation that the structure is going to recieve is proporcional to the total energy divided by the area of the whole sphere. This in fact is the flux, which is distance dependant.

So, considering the following values we can obtain the flux,

$R_\odot = 6.95 \times 10^8\ m$ → Sun's radius

$1AU = 1.495 \times 10^{11} m$ → Radius of the sphere centered at the Sun

$e = 1$ → Perfect emisor, assuming a blackbody radiation behavior for the Sun

$A = 4\pi R_\odot^2$ → Spherical shape

$T = 5770\ K$ → Sun's surface temperature

Then,

$$L = 4\pi d^2 F \Rightarrow F = \frac{L}{4\pi d^2}$$

$$> F = \frac{A e \sigma T^4}{4\pi d^2} = \frac{\cancel{4\pi} R_\odot^2 e \sigma T^4}{\cancel{4\pi} d^2}$$

$$F = \frac{R_\odot^2 e \sigma T^4}{d^2} = \frac{(6.95 \times 10^8 \cancel{m})^2 \cdot 1 \cdot 5.67 \times 10^{-8} \frac{W}{\cancel{m^2} \cdot \cancel{K^4}} \cdot (5770 K)^4}{(1.495 \times 10^{11} \cancel{m})^2}$$

$$\boxed{F = 1360\ W/m^2}$$

This is in fact the solar constant

Now that we have calculated the total amount of power per unit of area we can consider that the total amount of energy irradiated by the sphere will be proportional to the recieving effective area. So,

$$F \cdot A_{(R)} = E_{E(T)}$$

$$F \cdot \cancel{4\pi R^2} = \cancel{4\pi R^2} \cdot \sigma \cdot T^4$$

$$F = \sigma T^4$$

$$T = \sqrt[4]{\frac{F}{\sigma}} = \sqrt[4]{\frac{1360\ \cancel{W/m^2}}{5.67 \times 10^{-8} \frac{\cancel{W}}{\cancel{m^2} K^4}}} = \boxed{393.5 K} = T$$

We can notice that, because the receiving effective area is equal to the emissive area, the temperature of the sphere just will be proportional to the flux divided by the Boltzmann constant.

b) Using the Wien's displacement law we got,

$$\lambda_M \cdot T = 2.9\times10^{-3}\ m\cdot K$$

So, for our previous result we obtain.

$$\lambda_M = \frac{2.9\times10^{-3}\ m\cdot K}{393.5\ K} = 7.37\times10^{-6}\ m$$

Expresed in μm, $(\mu m = 1\times10^{-6} m)$

$$\boxed{\lambda_M = 7.37\ \mu m}$$

This is infrared radiation, specifically middle infrared.

c) Given that flux is distance dependant by the inverse square law we can derive that

$$F(a)\cdot r_a^2 = F(b)\cdot r_b^2 \quad , \text{ then, } \quad F(a) = \frac{F(b)\cdot r_b^2}{r_a^2}$$

Thus,

$$F(2AU) = \frac{F(1AU)\cdot(1AU)^2}{(2AU)^2}, \text{ hence, } F(2AU) = \frac{1360\ \frac{W}{m^2}\cdot(1.495\times10^{11} m)^2}{(2\cdot1.495\times10^{11} m)^2}$$

$$> F(2AU) = 340\ W/m^2$$

Now, by the previous analysis we can use,

$$T = \sqrt[4]{\frac{F}{\sigma}}, \quad T(2AU) = \sqrt{\frac{340\ W/m^2}{5.67\times10^{-8}\ W/m^2K^4}} \quad \boxed{T = 278.27\ K}$$

(2.) A supernova remnant has an angular diameter of $\theta = 4.3$ arcminutes and a flux at 100 MHz of $F_{100} = 1.6 \times 10^{-19}$ erg cm^{-2} s^{-1} Hz^{-1}. Assume that the emission is thermal.

a) What is the brightness temperature Tb? What energy regime of the blackbody curve does this correspond to?

b) The emitting region is actually more compact than indicated by the observed angular diameter. What effect does this have on the value of Tb?

c) At what frecuency will this object's radiation be maximum, if the emission is blackbody?

Solution

a). If we know that $I_\nu = B_\nu(T_b)$, then,

$$T_b = \frac{c^2}{2\nu^2 k} I_\nu$$

Now, let's consider the dimensional analysis like this,

$$dF = \frac{dE}{dA\,dt\,d\nu} \quad , \quad dI = \frac{dE}{dA\,dt\,d\nu\,d\Omega}$$

Since $dE = dE$, then,

$$dI = \frac{dF\,\cancel{dA\,dt\,d\nu}}{\cancel{dA\,dt\,d\nu}\,d\Omega} \quad \text{so,} \quad dI = \frac{dF}{d\Omega}$$

Hence, $\boxed{I_\nu = \frac{F_\nu}{\Omega}}$

So, resolving to Ω,

$$\Omega = \pi\left(\frac{\theta}{2}\right)^2 \qquad \theta = 4.3' \times \frac{1°}{60'} \times \frac{\pi\,\text{rad}}{180°} = 1.25 \times 10^{-3}\,\text{rad}$$

$$\Omega = \pi\left(\frac{1.25\times10^{-3}\,\text{rad}}{2}\right)^2 = \boxed{1.22 \times 10^{-6}\,\text{ster}}$$

From the previous analysis we obtained,

$$I_\nu = \frac{F_\nu}{\Omega} \qquad F_{100MHz} = 1.6\times10^{-19}\, erg\, cm^{-2}\, s^{-1}\, Hz^{-1}$$

$$F_{(100)} = 1.6\times10^{-19}\ \frac{\cancel{erg}}{\cancel{cm^2}\cdot s\cdot Hz} \times \left(\frac{100\,\cancel{cm}}{1m}\right)^2 \times \frac{1\times10^{-7}\,J}{1\,\cancel{erg}}$$

$$F = 1.6\times10^{-22}\ \frac{J}{m^2\cdot s\cdot Hz}$$

Applying to the equation we obtain,

$$I_\nu = \frac{F_\nu}{\Omega} = \frac{1.6\times10^{-22}\ J}{1.22\times10^{-6}\ ster\cdot m^2\cdot s\cdot Hz}$$

$$\boxed{I_\nu = 1.31\times10^{-16}\ \frac{J}{ster\cdot m^2\cdot s\cdot Hz}}$$

Hence,

$$T_b = \frac{c^2 I_\nu}{2\nu^2 k} = \frac{(3\times10^8\ m/s)^2\cdot\left(\frac{1.31\times10^{-16}\ \cancel{J}}{ster\cdot \cancel{m^2}\cdot \cancel{s}\cdot \cancel{Hz}}\right)}{2\cdot(100\times10^6\ \cancel{Hz})^2\cdot(1.38\times10^{-23}\ \cancel{J}/K)}$$

$$\boxed{T_b \approx 42.7\times10^6\ K/ster}$$

According to the blackbody's temperature this value belongs to the X-ray regime. Ionizing radiation.

b) If we consider that

$$T_b = \frac{c^2}{2\nu^2 K} I_\nu \quad , \quad I_\nu = \frac{F_\nu}{\Omega} \quad , \quad \Omega = \pi (d\theta)^2 \quad , \quad d\theta = \frac{\theta}{2}$$

Rearranging,

$$T_b = \frac{c^2}{2\nu^2 K} \cdot \frac{F_\nu}{\pi (d\theta)^2}$$

If we consider this part as a constant we get,

$$T_b = \frac{C}{(d\theta)^2}$$

Furthermore, if we see the units of the final result in a) we can appreciate that the answer is given in K/ster.

$\boxed{\text{If } d\theta \text{ increases } T_b \text{ decreases and vice versa.}}$

c) From the Wien displacement law we know that

$x = h\nu_M / KT$ where $x = 3(1 - e^{-x})$ or $x = 2.82144$

hence,

$$\nu_M = \frac{xKT}{h} = \frac{2.82144 \cdot 1.38 \times 10^{-23} \cancel{J/K} \cdot 42.7 \times 10^6 \cancel{K}}{6.6260 \times 10^{-34} \cancel{J} \cdot s}$$

$$\boxed{\nu_M = 2.51 \times 10^{18}\ Hz}$$