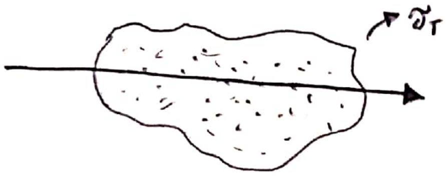
Homework #4

1. The warm ionized interstellar medium (ISM) of our Galaxy in the solar neighborhood can be approximated by a disk of half-thickness $h \approx 1\,\mathrm{Kpc}$ and electron density $N_e \approx 0.1\,\mathrm{cm}^{-3}$. what is the optical depth τ_T of the ISM to Thompson scattering in the direction normal to the disk?

So, If we consider the problem in this way, we have



Assuming that the thickness of the object is given by h and considering that only Thompson scattering is used for the calculations we have that,

$$\tau = \int_0^S \alpha(s)\,ds \quad \text{or, more precisely} \quad \tau_\nu(s) = \int_{s_0}^{s} \alpha_\nu(s')\,ds'$$

and, futhermore, we have to keep in mind that,

$$\alpha_\nu = n\sigma_\nu$$

with n being the particle density and σ_ν the effective absorbing area.
Now, the electron scattering is frequency independent so,

$$\tau = \int_0^h n\sigma_\nu\,ds = h\sigma_\nu n \Rightarrow \tau = h \cdot \frac{8\pi}{3} r_0^2 \cdot n$$

$$\tau = \left(1\,\mathrm{K}\cancel{\mathrm{pc}} \times \frac{3.086\times10^{18}\,\cancel{\mathrm{cm}}}{1\,\cancel{\mathrm{pc}}}\right) \cdot (0.665\times10^{-24}\,\cancel{\mathrm{cm}^2}) \cdot (0.1\,\cancel{\mathrm{cm}^{-3}})$$

$$\boxed{\tau = 2.0522\times10^{-4}}$$

2. Consider a medium containing a large number of radiating particles. (For definiteness you may wish to imagine electrons emitting bremsstralung). Each particle emits a pulse of radiation with an electric field $E_0(t)$ as a function of time. An observer will detect a series of such pulses, all with the same shape but with random arrival times $t_1, t_2, t_3, \ldots, t_N$. The measured electric field will be,

$$E(t) = \sum_{i=1}^{N} E_0(t - t_i)$$

a) Show that the Fourier transform of $E(t)$ is $\hat{E}(\omega) = \hat{E}_0(\omega) \sum_{i=1}^{N} e^{i\omega t_i}$

So, considering that the general form to the Fourier transform is,

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt$$

and $f(t) = \sum_{i=1}^{N} E_0(t - t_i)$, we then obtain,

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\sum_{i=1}^{N} E_0(t - t_i) \right] e^{i\omega t}\, dt$$

Now, considering that f is a linear combination of linearly independent functions we can rearrange the equation such that,

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{N} \int_{-\infty}^{\infty} E_0(t - t_i)\, e^{i\omega t}\, dt$$

Continuing, we can make a change of variable given that t_i can be represented as a constant related to $\sum_{i=1}^{N}$, so

$U = t - t_i$

$\frac{dU}{dt} = 1 \quad \Rightarrow t = U + t_i$

$\Downarrow$

$dU = dt \qquad e^{i\omega t} = e^{i\omega U + i\omega t_i}$

$e^{i\omega U} e^{i\omega t_i}$

$$\Rightarrow F(\omega) = \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{N} e^{i\omega t_i} \int_{-\infty}^{\infty} E_0(U) e^{i\omega U}\, dU$$

Now, we know that this term

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E_0(U) e^{i\omega U}\, dU$$

is equal to $E_0(\omega)$, so

$$\boxed{E(\omega) = E_0(\omega) \sum_{i=1}^{N} e^{i\omega t_i}}$$

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b) Argue that

$$\left|\sum_{i=1}^{N} e^{i\omega t_i}\right|^2 = N$$ when averaged over the random arrival times.

Considering that $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} = \sum_{i,j=1}^{n} a_{ij}$ and considering the random arrival times we then obtain,

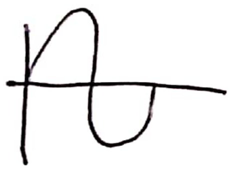
$$\left|\sum_{i=1}^{N} e^{i\omega t_i}\right|^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} e^{i\omega t_j} e^{-i\omega t_i}$$ being j related to $(t - t_i)$ retarded times.

So, when $j=j$ we got

$$\sum_{j=1}^{N} e^{i\omega t_j} e^{-i\omega t_i} = \sum_{j=1}^{N} e^{i\omega (\cancel{t_j - t_i})^{0}} = \sum_{j=1}^{N} = \cancel{e^{0}}^{1} \boxed{=N}$$

and, when we got $j \neq i$

$$\sum_{j=1}^{N}\sum_{i=1}^{N} e^{i\omega (t_j - t_i)}$$ but, because t_i and t_j are randomly distributed this term becomes 0.



c) Thus show that the measured spectrum is simply N times the spectrum of an individual pulse (Note that this result still holds if the pulses overlap).

We know that the measured spectrum follow the relation

$$\frac{dW}{dA\,d\omega} = c|\hat{E}(\omega)|^2$$, now, from this $\left|\sum_{i=1}^{N} e^{i\omega t_i}\right|^2 = N$, we can derive that

$$\frac{dW}{dA\,d\omega} = c\left|E_0(\omega)\sum_{i=1}^{N} e^{i\omega t_i}\right|^2 \Rightarrow \frac{dW}{dA\,d\omega} = c E_0^2(\omega) N$$

which derives into

$$\boxed{\frac{dW}{dA\,d\omega} = N \underbrace{\left(\frac{dW}{dA\,d\omega}\right)}_{\text{single pulse}}}$$

d) By contrast, show that if all the particles are in a region much smaller than a wavelength and they emit their pulses simultaneously, then the measured spectrum will be N^2 times that spectrum of an individual pulse

For this case we argue that,

so that means $t - t_i \approx t$ which also means $t_i = 0$, therefore,

$$E(\omega) = E_0(\omega) \cancel{\sum_{i=1}^{N} e^{i\omega(0)}}^{\,1} (N) = E_0(\omega) N$$

so for th,

$$\frac{dw}{dA\,d\omega} = c\,|\hat{E}(\omega)|^2 \Rightarrow \frac{dw}{dA\,d\omega} = c\,|E_0(\omega) N|^2$$

$$\Rightarrow \frac{dw}{dA\,d\omega} = c\,E_0^2(\omega) N^2$$

which derives into

$$\boxed{\frac{dw}{dA\,d\omega} = N^2 \underbrace{\left(\frac{dW}{dA\,d\omega}\right)}_{\text{single pulse}}}$$