CW

December 16, 2021

[ ]:
```
import numpy as np
import pandas as pd
from scipy.fft import fft
from scipy.linalg import toeplitz

import matplotlib.pyplot as plt
from matplotlib import cm

plt.style.use('seaborn-white')
```

0.0.1 1

(a) By page 52 in notes, the spectual density function of an ARMA(p, q) process is given by

$$S_X(f) = \sigma_\epsilon^2 \frac{G_\theta(f)}{G_\phi(f)}. \tag{1}$$

where

$$G_\theta(f) = |1 - \theta_{1,q}e^{-2i\pi f} - \dots - \theta_{q,q}e^{-2i\pi fq}|^2,$$
$$G_\phi(f) = |1 - \phi_{1,q}e^{-2i\pi f} - \dots - \phi_{q,q}e^{-2i\pi fq}|^2.$$

[ ]:
```
def S_ARMA(f, sigma2, phis=[], thetas=[]):
    '''
    Computes the theoretical sdf for an ARMA(p, q) process.

    param f: numpy array of frequencies at which sdf should be evaluated
    param sigma2: scalar; the variance of the white noise process
    param phis: array containing the phi parameters
    param thetas: array containing the theta parameters

    return: numpy array of sdf evaluated at f
    '''
    p, q = len(phis), len(thetas)
    xis_t = np.exp(-1j*2*np.pi*np.outer(f, np.arange(1, q+1)))
    xis_p = np.exp(-1j*2*np.pi*np.outer(f, np.arange(1, p+1)))
    G_t = 1 - (thetas * xis_t).sum(1)  # G_theta
    G_p = 1 - (phis * xis_p).sum(1)    # G_phi
    return (sigma2 * np.abs(G_t)**2 / np.abs(G_p)**2)
```

(b) The following code implements the burn in method to generate an ARMA$(2,2)$ process of length N: 1. Set $X_1 = X_2 = 0$ \ 2. For $i = 3$ to $100 + N$, set $X_i \leftarrow \phi_{1,2}X_{i-1} + \phi_{2,2}X_{i-2} + \epsilon_i - \theta_{1,2}\epsilon_{i-1} - \theta_{2,2}\epsilon_{i-2}$ \ 3. Return $X_{101}, ..., X_{101+N}$. \

```
[ ]: def ARMA22_sim(phis, thetas, sigma2, N):
         """
         Simulates a Gaussian ARMA(2,2) process.

         param phis: numpy array containing the phi parameters
         param thetas: numpy array containing the theta parameters
         param sigma2: scalar; variance of the white noise process
         N: scalar; length of the simulated process

         return: numpy array of size N; realiasation of an ARMA(2,2) process
         """
         eps = np.random.normal(size=100+N, scale=np.sqrt(sigma2))     # simulate␣
      ↪white noice process
         X = np.zeros(100 + N)

         for i in range(2, 100+N):
             X[i] = phis[0]*X[i-1] + phis[1]*X[i-2] + eps[i] \
                 - (thetas[0]*eps[i-1] + thetas[1]*eps[i-2])

         return X[-N:]
```

(c) By page 58 in notes, the periodogram for a time series X is given by

$$\hat{S}^{(p)}(f) = \frac{1}{N}\left|\sum_{i=1}^{N} X_t e^{-i2\pi f t}\right|^2.$$

```
[ ]: def periodogram(X):
         """Computes the periodogram at the Fourier frequencies for a time series X.
      ↪"""
         N = X.shape[-1]
         S = fft(X)    # computes the sum inside the norm
         return np.abs(S)**2 / N
```

By page 62 in notes, the spectual density of a $p \times 100\%$ cosine tapered estimato is given by

$$\hat{S}^{(d)}(f) = \left|\sum_{i=1}^{N} h_t X_t e^{-i2\pi f t}\right|^2,$$

where the consine taper sequence h_t is as defined on page 64.

```
[ ]: def tapered_series(X, taper):
         """Computes the series ht*Xt for p*100% cosine tapered spectual estimates.
      ↪"""
         N = X.shape[-1]
```

```
    # Construct the H_t sequence
    H = np.ones(N,)
    tt = np.floor(taper*N)
    H[:int(tt//2)] = (1 - np.cos(2*np.pi*np.arange(tt//2) / (tt+1))) / 2
    H[N-int(tt//2):] = (1 - np.cos(
        2*np.pi*(N+1-np.arange(N-int(tt//2), N)) / (tt+1))
    ) / 2
    H /= np.sqrt((H**2).sum())    # normalise

    return H * X

def direct(X, p):
    """
    Computes the direct spectral estimate at the Fourier frequencies using the \
    p*100% cosine taper for a time series X.
    """
    hX = tapered_series(X, p)
    return np.abs(fft(hX))**2
```

(d)

(A) By page 30 in notes, an ARMA(2,2) process is stationary if all the roots of

$$\Phi(z) = 1 - \phi_{1,2}z - \phi_{2,2}z^2$$

lie outside the unit circle. Using the representation of roots $z_1 = \frac{1}{r}e^{-i2\pi f'}$ and $z_2 = \frac{1}{r}e^{i2\pi f'}$, by similar arguments in page 51, we have that

$$\phi_{1,2} = 2r\cos(2\pi f'), \qquad \phi_{2,2} = -r^2.$$

```
[ ]: r = 0.8
     N = 128
     fp = 12/128
     phis = np.array([2*r*np.cos(2*np.pi*fp), -r**2])
     thetas = np.array([-0.5, -0.2])

     # Simulate 10000 ARMA(2,2) realisation
     np.random.seed(7)
     X = np.array([ARMA22_sim(phis, thetas, 1, N) for _ in range(10000)])
```

`per_single`, `dse_p1_single`, `dse_p2_single`, `dse_p3_single` and `dse_p4_single` store the spectual estimates at $f = 12/128, 32/128, 60/128$ based on the periodogram, 5%, 10%, 25% and 50% cosine tapers respectively.

```
[ ]: per_single = periodogram(X).T[[12, 32, 60]]    # periodograms

     # cosine tapers
     ps = [0.05, 0.1, 0.25, 0.5]
```

```
dse_p1_single = direct(X, 0.05).T[[12, 32, 60]]
dse_p2_single = direct(X, 0.1).T[[12, 32, 60]]
dse_p3_single = direct(X, 0.25).T[[12, 32, 60]]
dse_p4_single = direct(X, 0.5).T[[12, 32, 60]]
```

(B) The sample bias of a sequence of estimates $\hat{s}_1, \hat{s}_2, ..., \hat{s}_N$ at frequancy f is given by

$$\frac{1}{N}\sum_{i=1}^{N}\hat{s}_i - s$$

where s is the true spectual density at frequancy f.

```
[ ]: fs = np.array([12/128, 32/128, 60/128])

     estimates = np.dstack([per_single, dse_p1_single, dse_p2_single, dse_p3_single,␣
      ↪dse_p4_single])
     true_sdf = S_ARMA(fs, sigma2=1, phis=phis, thetas=thetas)
     sample_bias_vals = estimates.mean(1).T - true_sdf

     # Sample bias table for each estimator when r=0.8
     sample_bias = pd.DataFrame(sample_bias_vals, columns=['f=' + str(int(f*128)) +␣
      ↪'/128' for f in fs])
     sample_bias.rename_axis(columns='freq', inplace=True)
     sample_bias.set_axis(['periodogram'] + [f'{int(p*100)}% cosine taper'  for p in␣
      ↪ps], axis='index', inplace=True)
     print('Sample bias of different spectual density estimators (r=0.8):')
     sample_bias
```

```
Sample bias of different spectual density estimators (r=0.8):

[ ]: freq                f=12/128  f=32/128  f=60/128
     periodogram        -2.561065  0.129317  0.051919
     5% cosine taper    -2.318748  0.068238  0.005019
     10% cosine taper -2.041909  0.018517  0.000321
     25% cosine taper -1.721829 -0.004065  0.000002
     50% cosine taper -1.513529 -0.007609  0.000101
```

(C)

```
[ ]: ### This block may take 1-2 mins to run

     N = 128
     fp = 12/128
     thetas = [-0.5, -0.2]     # unchanged

     rs = np.arange(0.8, 1.0, 0.01)
     phis = np.array([2*rs*np.cos(2*np.pi*fp), -rs**2]).T   # 20x2 array, recording␣
      ↪phi values for each r
```

```
# sample 10000 realisations for each sets of phi
np.random.seed(7)
XX = np.array([[ARMA22_sim(phi, thetas, 1, N) for _ in range(10000)] for phi in␣
 ↪phis])    # shape=(20, 10000, 128)


# Compute the spectual density estimates using the same 5 estimators as above
perXX = periodogram(XX)[..., [12, 32, 60]]    # shape=(20, 10000, 3)
dse_p1 = direct(XX, 0.05)[..., [12, 32, 60]]
dse_p2 = direct(XX, 0.1)[..., [12, 32, 60]]
dse_p3 = direct(XX, 0.25)[..., [12, 32, 60]]
dse_p4 = direct(XX, 0.5)[..., [12, 32, 60]]


# Compute the true spectual density
true_sdf = np.array([S_ARMA(
    fs, sigma2=1, phis=phi, thetas=thetas
) for phi in phis])  #  20x3 array

# Compute the sample bias
sample_mean = np.stack((perXX, dse_p1, dse_p2, dse_p3, dse_p4)).mean(2)
sample_bias = sample_mean - true_sdf    # shape=(5, 20, 3)
```

(D)

```
[ ]: fig = plt.figure(figsize=(24, 6))

for i in range(3):
    plt.subplot(1, 3, i+1)
    for ip in range(4):
        plt.plot(rs, sample_bias[ip+1, :, i], c=cm.YlGn(1.25*ps[ip]+0.4),␣
 ↪label=f'{int(ps[ip]*100)}% cosine taper')
    plt.plot(rs, sample_bias[0, :, i], 'r-', label='periodogram')
    plt.xlabel(r'$r$', fontsize=16)
    plt.ylabel('sample bias', fontsize=16)
    plt.title(f'f={int(fs[i]*128)}/128', fontsize=16)
    plt.grid()
    plt.legend(fontsize=12)
# fig.suptitle('Comparision of sample bias for different spectual estimators',␣
 ↪fontsize=20)
txt = '\n\nFig 1: Comparision of sample bias for different spectual estimators'
plt.figtext(0.5, 0.01, txt, wrap=True, horizontalalignment='center',␣
 ↪fontsize=20)
plt.show()
```

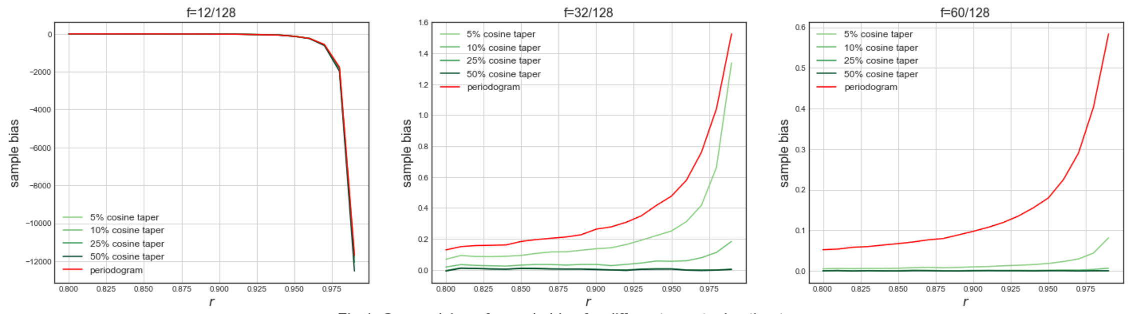


Fig 1: Comparision of sample bias for different spectual estimators

(e)

- As shown in Figure 1, at the oscillating frequency $f = 12/128$, spectual density is underestimated by all five estimators, while at the two frequencies away from the oscillating frequency, spectual density is overestimated. This suggests the existence of sidelobe leakage.
- From the two Figure 2 and 4 below, one can see that for larger $r$ the process has larger dynamic range. As a result, for each estimator there are more sidelobe leakage as $r$ gets close to 1, resulting in significantly larger (absolute) sample bias, as shown in Figure 1.
- At the oscillating frequency $f = 12/128$, the periodogram and the tapered estimators have similar sample bias.
- At the two frequencies away from the oscillating frequency, the tapered estimators have smaller bias than periodogram. In particular, more tapering (larger p) results in larger bias reduction.
- At $f = 32/128$, the 25% and 50% taperered estimators have almost zero bias. At $f = 60/128$ (even further away from the oscillating frequency), all four cosine tapered estimates have very low bias for $r$ that is not extremely close to 1.

[ ]:
```python
## True sdf
fss = np.arange(128) / 128
sdf = np.array([S_ARMA(fss, sigma2=1, phis=phi, thetas=thetas) for phi in phis])

plt.figure(figsize=(19, 5))
plt.subplot(121)
for k in range(20):
    plt.plot(fss, sdf[k], c=cm.Reds(k/20), lw=1.5)
sm = plt.cm.ScalarMappable(cmap=cm.Reds)

for f in fs:
    plt.axvline(x=f, alpha=0.6, c='k', linestyle='--',␣
 ↪label=r'$f=$'+f'{int(f*128)}/128')
cbar = plt.colorbar(sm)
cbar.set_label(r'$r$', rotation=0, fontsize=16, loc='bottom')
```

```
plt.xlabel(r'$f$', fontsize=14)
plt.ylabel(r'$S(f)$', fontsize=14)
plt.figtext(0.26, 0.001, '\nFig 2: True spectral density', wrap=True,␣
 ↪horizontalalignment='center', fontsize=20)
# plt.title('True spectral density', fontsize=16)
plt.legend(loc=9, bbox_to_anchor=(0.7, 1), fontsize=12)
plt.grid()


## Dynamic range
plt.subplot(122)
dynamic_range = 10 * np.log10(sdf.max(-1) / sdf.min(-1))
plt.plot(rs, dynamic_range)
# plt.title(r'Dynamic range of the process with different $r$', fontsize=14)
plt.figtext(0.55, 0.001, '\nFig 3: Dynamic range of the process with different␣
 ↪$r$', wrap=True, fontsize=20)
plt.xlabel(r'$r$', fontsize=14)
plt.grid()
plt.show()
```

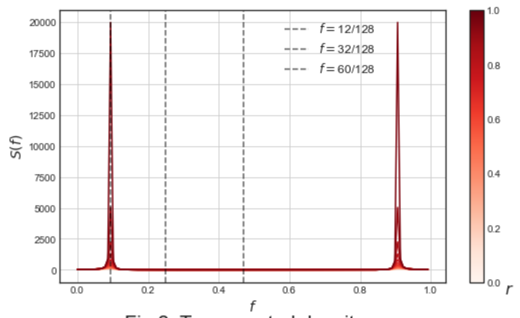


Fig 2: True spectral density

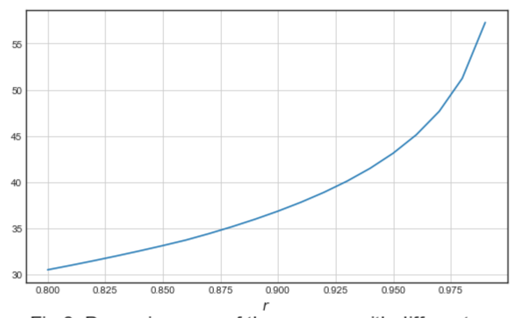


Fig 3: Dynamic range of the process with different $r$

0.1 2

0.1.1 (a)

```
[ ]: # Load data
X2 = np.genfromtxt('154.csv', delimiter=',')
N = X2.shape[-1]

# Compute periodogram and 50% cosine tapered estimate shifted to (-1/2, 1/2]
per = np.fft.fftshift(periodogram(X2))
dse = np.fft.fftshift(direct(X2, 0.5))
titles = ['Periodogram', '50% cosine taper']

# Plot the estimates
```

```
fig = plt.figure(figsize=(16, 6))
for i, est in enumerate([per, dse]):
    plt.subplot(1, 2, i+1)
    plt.plot(np.arange(-1/2, 1/2, 1/128), est)
    plt.xlabel(r'Frequency $f$', fontsize=16)
    plt.ylabel(r'$S(f)$', fontsize=16)
    plt.title(titles[i], fontsize=16)
    plt.ylim(-1, 25)
    plt.grid()
txt = '\nFig4: Spectual estimates of given data'
plt.figtext(0.5, 0.01, txt, wrap=True, horizontalalignment='center',␣
 ↪fontsize=25)
plt.show()
```

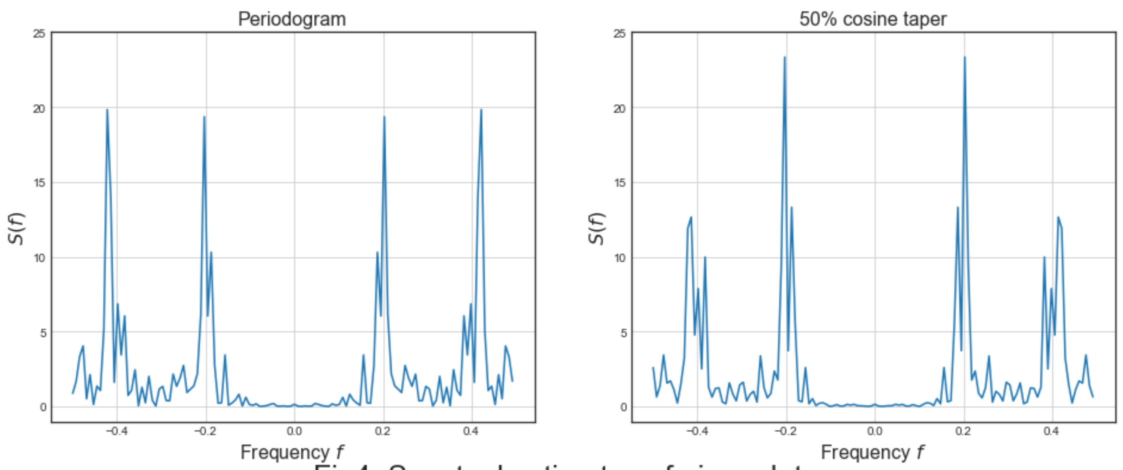


Fig4: Spectual estimates of given data

0.1.2 (b)

```
[ ]: def autocov_seq_hat(X, symmetry=False):
         """Compute the sample autocovariance for a time series."""
         N = X.shape[-1]
         taus = np.arange(-N, N) if symmetry else np.arange(N+1)
         Ts = np.abs(taus).astype(int)
         return np.array([(X[:N-T]*X[T:]).sum() / N for T in Ts])


     ### Yule-Walker (untapered)
     def estimators_YW(X, p):
         stau_hat = autocov_seq_hat(X)
         Ghat = toeplitz(stau_hat[:p])
         gamma_hat = stau_hat[1:p+1]
         phip_hat = np.linalg.inv(Ghat).dot(gamma_hat)
```

```
    sigma2_hat = stau_hat[0] - phip_hat.dot(stau_hat[1:p+1])
    return phip_hat, sigma2_hat


### Yule-Walker with 50% cosine taper
def estimators_YW_tapered(X, p, taper=0.5):
    hX = tapered_series(X, taper)
    N = hX.shape[-1]
    stau_hat = autocov_seq_hat(hX) * N
    Ghat = toeplitz(stau_hat[:p])
    gamma_hat = stau_hat[1:p+1]
    phip_hat = np.linalg.inv(Ghat).dot(gamma_hat)
    sigma2_hat = stau_hat[0] - phip_hat.dot(stau_hat[1:p+1])
    return phip_hat, sigma2_hat


### Approximate MLE
def estimators_MLE(X, p):
    F = toeplitz(X[p-1:N-1], X[p::-1])
    XX = X[p:]
    phat = np.linalg.inv(F.T @ F) @ F.T @ XX
    A = XX - F.dot(phat)
    sigma2_hat = A.T @ A / (N-2*p)
    return phat, sigma2_hat
```

0.1.3 (c)

[ ]:
```
def reduced_AIC(X, p, method='YW'):
    fmap = {
        'YW': estimators_YW,
        'YW taper': estimators_YW_tapered,
        'MLE': estimators_MLE,
    }
    N = X.shape[-1]
    s2 = fmap[method](X, p)[-1]
    return 2*p + N * np.log(s2)
```

[ ]:
```
methods = ['YW', 'YW taper', 'MLE']
YW_data = np.array([reduced_AIC(X2, pp) for pp in range(1, 21)])
YW_taper_data = np.array([reduced_AIC(X2, pp, 'YW taper') for pp in range(1,␣
↪21)])
MLE_data = np.array([reduced_AIC(X2, pp, 'MLE') for pp in range(1, 21)])

df = pd.DataFrame(
    {'YW':YW_data, 'YW taper':YW_taper_data, 'MLE':MLE_data},
).rename_axis('p').rename(lambda x: x+1, axis='index')
df
```

```
[ ]:            YW     YW taper        MLE
     p
     1   94.783802  101.207951  88.484364
     2   88.539991   91.802297  60.189262
     3   58.736744   57.057308   4.743230
     4    1.434667  -13.417359  10.099120
     5    3.402804  -11.420202   4.460794
     6    3.883213  -10.309071  27.430608
     7    5.722645   -8.343541  15.235844
     8    7.525434   -7.086085  28.619181
     9    7.697199   -5.566404  17.343578
     10   9.214986   -4.194696  26.839051
     11  11.158646   -2.194762  42.094840
     12  12.929766   -0.436913  32.737438
     13  12.199077   -0.184707  28.229796
     14  14.188579    1.791134  23.914521
     15  15.620487    3.224196  33.819087
     16  17.616934    5.189858  37.658731
     17  19.614910    7.126054  34.288293
     18  19.785123    7.764564  28.436826
     19  20.869850    6.129727  36.388368
     20  22.256291    8.124802  37.051904
```

0.1.4 (d)

```
[ ]: def best_p(ps, AICs):
         return ps[np.argmin(AICs, -1)]

     ps = np.arange(1, 21)
     best_ps = best_p(ps, [YW_data, YW_taper_data, MLE_data])
     for i in range(3):
         print(f'Best choice of p based on {methods[i]} method = {best_ps[i]}')

     YW_coeffs = estimators_YW(X2, best_ps[0])
     YW_taper_coeffs = estimators_YW_tapered(X2, best_ps[1])
     MLE_coeffs = estimators_MLE(X2, best_ps[2])
```

```
Best choice of p based on YW method = 4
Best choice of p based on YW taper method = 4
Best choice of p based on MLE method = 5
```

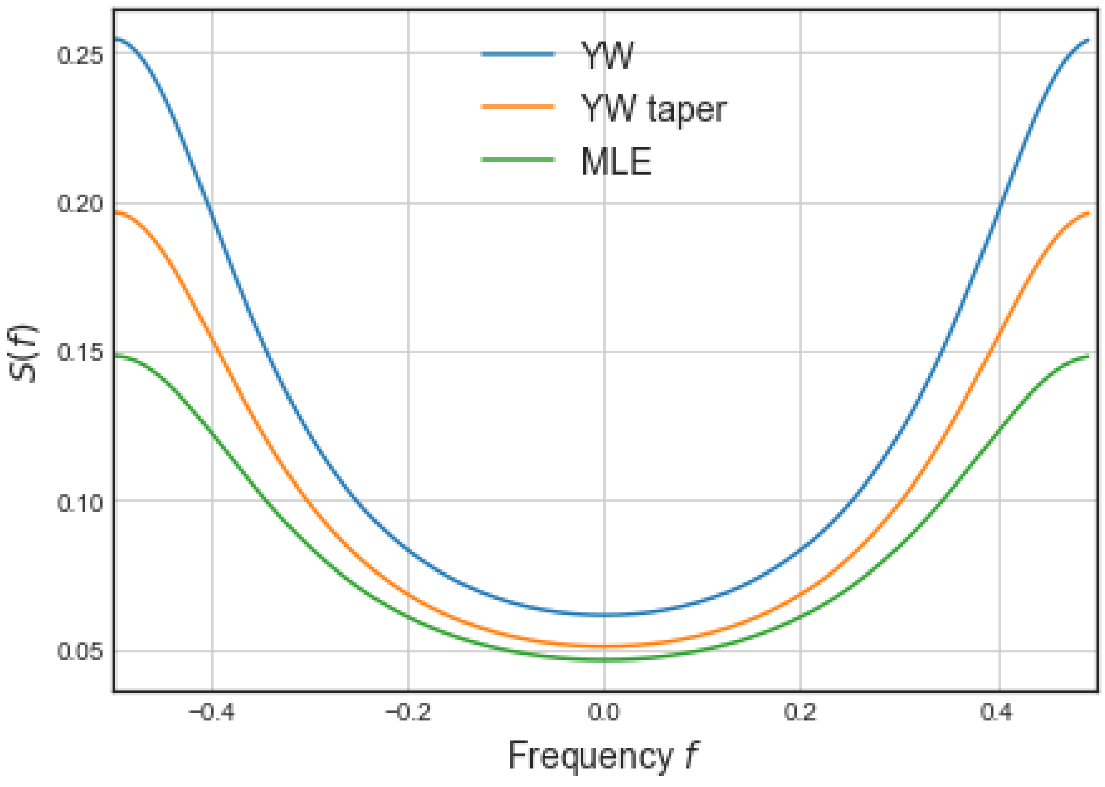
```
[ ]: print(f'Yule-Walker: Phi = {YW_coeffs[0]}, sigma^2 = {YW_coeffs[1]}')
     print(f'Yule-Walker with 50% taper: Phi = {YW_taper_coeffs[0]}, sigma^2 =␣
      ↪{YW_taper_coeffs[1]}')
     print(f'Approximate MLE: Phi = {MLE_coeffs[0]}, sigma^2 = {MLE_coeffs[1]}')
```

```
Yule-Walker: Phi = [-0.81623113 -0.71385968 -0.79215113 -0.60892904], sigma^2 =
0.9500015475954233
```

```
Yule-Walker with 50% taper: Phi = [-0.82177909 -0.7716625  -0.82390842
-0.65751518], sigma^2 = 0.8459262307072268
Approximate MLE: Phi = [-0.86022111 -0.83303276 -0.91245236 -0.73306553
-0.10968231 -0.0919776 ], sigma^2 = 0.9576479573226758
```

0.1.5 (e)

```
[ ]: plt.figure(figsize=(7, 5))
     for i, data in enumerate([YW_coeffs, YW_taper_coeffs, MLE_coeffs]):
         plt.plot(
             np.arange(-1/2, 1/2, 1/128),
             S_ARMA(f=np.arange(-1/2, 1/2, 1/128), sigma2=data[-1], phis=data[:-1]),␣
      ↪label=methods[i]
         )
     plt.xlabel(r'Frequency $f$',  fontsize=14)
     plt.ylabel(r'$S(f)$',fontsize=14)
     plt.legend(fontsize=14)
     plt.xlim(-1/2, 1/2)
     # plt.ylim(0, 0.3)
     plt.grid()
     plt.show()
```



0.2 Other

, the ARMA process can be written as

$$X_t - 2r\cos(2\pi f')X_{t-1} + r^2 X_{t-2} = \epsilon_t - \theta_{1,2}\epsilon_{t-1} - \theta_{2,2}\epsilon_{t-2}.$$

We also have that

$$S_X(f) = \sigma_\epsilon^2 \frac{|1 - \theta_{1,2}e^{-2\pi f} - \theta_{2,2}e^{-2\pi f}|^2}{|1 - \phi_{1,2}e^{-2\pi f} - \phi_{2,2}e^{-2\pi f}|^2}$$

and

$$|1 - \phi_{1,2}e^{-2\pi f} - \phi_{2,2}e^{-2\pi f}|^2 = (1 - 2r\cos(2\pi(f' + f)) + r^2)(1 - 2r\cos(2\pi(f' - f)) + r^2)$$

[ ]:
```
# stau_hat = autocov_seq_hat(X2, True)

# # Plot the sample autocovariance sequence
# plt.figure(figsize=(8, 5))
# plt.plot(range(-N, N), stau_hat)
# plt.ylabel(r'$\hat{s}_{\tau}$', fontsize=16)
# plt.xlabel(r'$\tau$', fontsize=16)
# plt.grid()
# plt.show()
```

[ ]:
```

```