

# Time Series Coursework

December 16, 2021

```
[ ]: import numpy as np
import pandas as pd
from scipy.fft import fft
from scipy.linalg import toeplitz

import matplotlib.pyplot as plt
from matplotlib import cm
plt.style.use('seaborn-white')
```

## 0.0.1 Question 1

(a) By page 52 in notes, the spectral density function of an ARMA( $p, q$ ) process is given by

$$S_X(f) = \sigma_\epsilon^2 \frac{G_\theta(f)}{G_\phi(f)}. \quad (1)$$

where

$$G_\theta(f) = |1 - \theta_{1,q}e^{-2i\pi f} - \dots - \theta_{q,q}e^{-2i\pi fq}|^2,$$
$$G_\phi(f) = |1 - \phi_{1,q}e^{-2i\pi f} - \dots - \phi_{q,q}e^{-2i\pi fq}|^2.$$

```
[ ]: def S_ARMA(f, sigma2, phis=[], thetas=[]):
    ...
    Computes the theoretical sdf for an ARMA(p, q) process.

    param f: numpy array of frequencies at which sdf should be evaluated
    param sigma2: scalar; the variance of the white noise process
    param phis: array containing the phi parameters
    param thetas: array containing the theta parameters

    return: numpy array of sdf evaluated at f
    ...
    p, q = len(phis), len(thetas)
    xis_t = np.exp(-1j*2*np.pi*np.outer(f, np.arange(1, q+1)))
    xis_p = np.exp(-1j*2*np.pi*np.outer(f, np.arange(1, p+1)))
    G_t = 1 - (thetas * xis_t).sum(1) # G_theta
    G_p = 1 - (phis * xis_p).sum(1) # G_phi
    return (sigma2 * np.abs(G_t)**2 / np.abs(G_p)**2)
```

(b) The following code implements the burn in method to generate an ARMA(2,2) process of length  $N$ : 1. Set  $X_1 = X_2 = 0$

2. For  $i = 3$  to  $100 + N$ , set  $X_i \leftarrow \phi_{1,2}X_{i-1} + \phi_{2,2}X_{i-2} + \epsilon_i - \theta_{1,2}\epsilon_{i-1} - \theta_{2,2}\epsilon_{i-2}$
3. Return  $X_{101}, \dots, X_{101+N}$ .

```
[ ]: def ARMA22_sim(phis, thetas, sigma2, N):
    """
    Simulates a Gaussian ARMA(2,2) process.

    param phis: numpy array containing the phi parameters
    param thetas: numpy array containing the theta parameters
    param sigma2: scalar; variance of the white noise process
    N: scalar; length of the simulated process

    return: numpy array of size N; realiasation of an ARMA(2,2) process
    """
    # simulate white noise process
    eps = np.random.normal(size=100+N, scale=np.sqrt(sigma2))
    X = np.zeros(100 + N)

    for i in range(2, 100+N):
        X[i] = phis[0]*X[i-1] + phis[1]*X[i-2] + eps[i] \
            - (thetas[0]*eps[i-1] + thetas[1]*eps[i-2])

    return X[-N:]
```

(c) By page 58 in notes, the periodogram for a time series  $X$  is given by

$$\hat{S}^{(p)}(f) = \frac{1}{N} \left| \sum_{i=1}^N X_t e^{-i2\pi f t} \right|^2.$$

```
[ ]: def periodogram(X):
    """
    Computes the periodogram at the Fourier frequencies for a time series X.
    """
    N = X.shape[-1]
    S = fft(X)  # computes the sum inside the norm
    return np.abs(S)**2 / N
```

By page 62 in notes, the spectual density of a  $p \times 100\%$  cosine tapered estimato is given by

$$\hat{S}^{(d)}(f) = \left| \sum_{i=1}^N h_t X_t e^{-i2\pi f t} \right|^2,$$

where the consine taper sequence  $h_t$  is as defined on page 64.

```
[ ]: def tapered_series(X, taper):
```

```

    """Computes the series ht*Xt for p*100% cosine tapered spectral estimates.
    """
N = X.shape[-1]

# Construct the H_t sequence
H = np.ones(N,)
tt = np.floor(taper*N)
H[:int(tt//2)] = (1 - np.cos(2*np.pi*np.arange(tt//2) / (tt+1))) / 2
H[N-int(tt//2):] = (1 - np.cos(
    2*np.pi*(N+1-np.arange(N-int(tt//2), N)) / (tt+1))
) / 2
H /= np.sqrt((H**2).sum())      # normalise

return H * X

def direct(X, p):
    """
    Computes the direct spectral estimate at the Fourier frequencies using the \
    p*100% cosine taper for a time series X.
    """
    hX = tapered_series(X, p)
    return np.abs(fft(hX))**2

```

(d)

(A) By page 30 in notes, an ARMA(2,2) process is stationary if all the roots of

$$\Phi(z) = 1 - \phi_{1,2}z - \phi_{2,2}z^2$$

lie outside the unit circle. Using the representation of roots  $z_1 = \frac{1}{r}e^{-i2\pi f'}$  and  $z_2 = \frac{1}{r}e^{i2\pi f'}$ , by similar arguments in page 51, we have that

$$\phi_{1,2} = 2r \cos(2\pi f'), \quad \phi_{2,2} = -r^2.$$

```
[ ]: r = 0.8
N = 128
fp = 12/128
phis = np.array([2*r*np.cos(2*np.pi*fp), -r**2])
thetas = np.array([-0.5, -0.2])

# Simulate 10000 ARMA(2,2) realisation
np.random.seed(7)
X = np.array([ARMA22_sim(phis, thetas, 1, N) for _ in range(10000)])
```

per\_single, dse\_p1\_single, dse\_p2\_single, dse\_p3\_single and dse\_p4\_single store the spectral estimates at  $f = 12/128, 32/128, 60/128$  based on the periodogram, 5%, 10%, 25% and 50% cosine tapers respectively.

```
[ ]: per_single = periodogram(X).T[[12, 32, 60]]      # periodograms

# cosine tapers
ps = [0.05, 0.1, 0.25, 0.5]
dse_p1_single = direct(X, 0.05).T[[12, 32, 60]]
dse_p2_single = direct(X, 0.1).T[[12, 32, 60]]
dse_p3_single = direct(X, 0.25).T[[12, 32, 60]]
dse_p4_single = direct(X, 0.5).T[[12, 32, 60]]
```

(B) The sample bias of a sequence of estimates  $\hat{s}_1, \hat{s}_2, \dots, \hat{s}_N$  at frequency  $f$  is given by

$$\frac{1}{N} \sum_{i=1}^N \hat{s}_i - s$$

where  $s$  is the true spectral density at frequency  $f$ .

```
[ ]: fs = np.array([12/128, 32/128, 60/128])

estimates = np.dstack([per_single, dse_p1_single, dse_p2_single, dse_p3_single,
                     dse_p4_single])
true_sdf = S_ARMA(fs, sigma2=1, phis=phis, thetas=thetas)
sample_bias_vals = estimates.mean(1).T - true_sdf

# Sample bias table for each estimator when r=0.8
sample_bias = pd.DataFrame(sample_bias_vals, columns=['f=' + str(int(f*128)) +
                                                       '/128' for f in fs])
sample_bias.rename_axis(columns='freq', inplace=True)
sample_bias.set_axis(['periodogram'] + [f'{int(p*100)}% cosine taper' for p in
                                         ps], axis='index', inplace=True)
print('Sample bias of different spectral density estimators (r=0.8):')
sample_bias
```

Sample bias of different spectral density estimators (r=0.8):

	f=12/128	f=32/128	f=60/128
periodogram	-2.561065	0.129317	0.051919
5% cosine taper	-2.318748	0.068238	0.005019
10% cosine taper	-2.041909	0.018517	0.000321
25% cosine taper	-1.721829	-0.004065	0.000002
50% cosine taper	-1.513529	-0.007609	0.000101

(C)

```
[ ]: N = 128
fp = 12/128
thetas = [-0.5, -0.2]      # unchanged

rs = np.arange(0.8, 1.0, 0.01)
```

```

phis = np.array([2*rs*np.cos(2*np.pi*fp), -rs**2]).T    # 20x2 array, recording
    ↳phi values for each r

# sample 10000 realisations for each sets of phi
np.random.seed(7)
XX = np.array([[ARMA22_sim(phi, thetas, 1, N) for _ in range(10000)] for phi in
    ↳phis])      # shape=(20, 10000, 128)

# Compute the spectral density estimates using the same 5 estimators as above
perXX = periodogram(XX)[..., [12, 32, 60]]      # shape=(20, 10000, 3)
dse_p1 = direct(XX, 0.05)[..., [12, 32, 60]]
dse_p2 = direct(XX, 0.1)[..., [12, 32, 60]]
dse_p3 = direct(XX, 0.25)[..., [12, 32, 60]]
dse_p4 = direct(XX, 0.5)[..., [12, 32, 60]]

# Compute the true spectral density
true_sdf = np.array([S_ARMA(
    fs, sigma2=1, phis=phi, thetas=thetas
) for phi in phis])  # 20x3 array

# Compute the sample bias
sample_mean = np.stack((perXX, dse_p1, dse_p2, dse_p3, dse_p4)).mean(2)
sample_bias = sample_mean - true_sdf      # shape=(5, 20, 3)

```

(D)

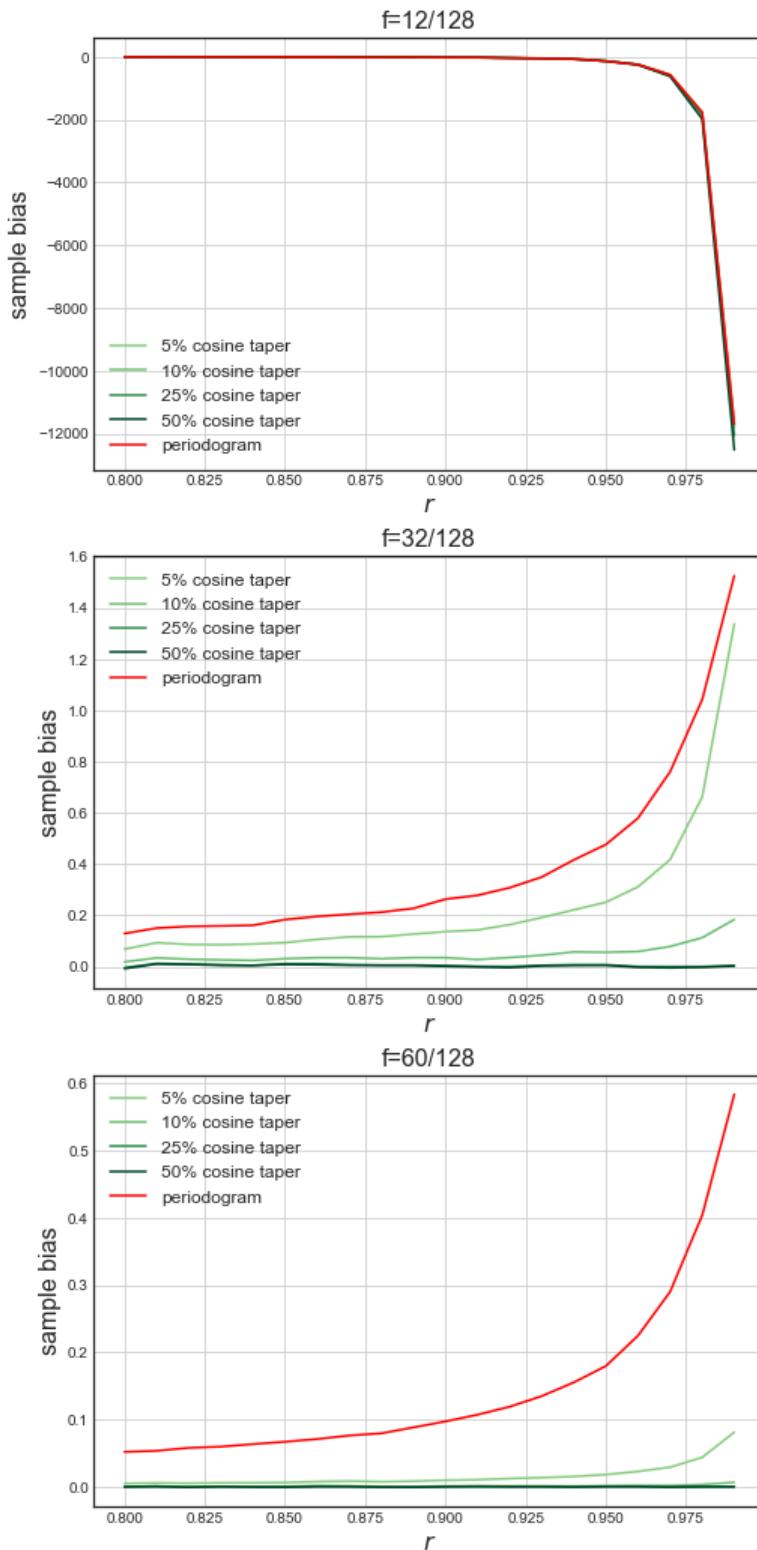
```

[ ]: fig = plt.figure(figsize=(8, 18))

for i in range(3):
    plt.subplot(3, 1, i+1)
    for ip in range(4):
        plt.plot(rs, sample_bias[ip+1, :, i], c=cm.YlGn(1.25*ps[ip]+0.4),
            ↳label=f'{int(ps[ip]*100)}% cosine taper')
        plt.plot(rs, sample_bias[0, :, i], 'r-', label='periodogram')
    plt.xlabel('r', fontsize=16)
    plt.ylabel('sample bias', fontsize=16)
    plt.title(f'f={int(fs[i]*128)}/128', fontsize=16)
    plt.grid()
    plt.legend(fontsize=12)
# fig.suptitle('Comparision of sample bias for different spectral estimators',
    ↳fontsize=20)
txt = 'Fig 1: Comparision of sample bias for different spectral estimators'
plt.figtext(0.5, .92, txt, wrap=True, horizontalalignment='center', fontsize=25)
plt.show()

```

Fig 1: Comparision of sample bias for different spectual estimators



(e) - As shown in Figure 1, at the oscillating frequency  $f = 12/128$ , spectral density is underestimated by all five estimators, while at the two frequencies away from the oscillating frequency, spectral density is overestimated. This suggests the existence of sidelobe leakage.

- From the two Figure 2 and 4 below, one can see that for larger  $r$  the process has larger dynamic range. As a result, for each estimator there are more sidelobe leakage as  $r$  gets close to 1, resulting in significantly larger (absolute) sample bias, as shown in Figure 1.
- At the oscillating frequency  $f = 12/128$ , the periodogram and the tapered estimators have similar sample bias.
- At the two frequencies away from the oscillating frequency, the tapered estimators have smaller bias than periodogram. In particular, more tapering (larger  $p$ ) results in larger bias reduction.
- At  $f = 32/128$ , the 25% and 50% taperered estimators have almost zero bias. At  $f = 60/128$  (even further away from the oscillating frequency), all four cosine tapered estimates have very low bias for  $r$  that is not extremely close to 1.

```
[ ]: ## True sdf
fss = np.arange(128) / 128
sdf = np.array([S_ARMA(fss, sigma2=1, phis=phi, thetas=thetas) for phi in phis])

plt.figure(figsize=(19, 5))
plt.subplot(121)
for k in range(20):
    plt.plot(fss, sdf[k], c=cm.Reds(k/20), lw=1.5)
sm = plt.cm.ScalarMappable(cmap=cm.Reds)

for f in fs:
    plt.axvline(x=f, alpha=0.6, c='k', linestyle='--',
    label=r'$f=$'+f'{int(f*128)}/128')
cbar = plt.colorbar(sm)
cbar.set_label(r'$r$', rotation=0, fontsize=16, loc='bottom')

plt.xlabel(r'$f$', fontsize=14)
plt.ylabel(r'$S(f)$', fontsize=14)
plt.figtext(0.26, 0.001, '\nFig 2: True spectral density', wrap=True,
horizontalalignment='center', fontsize=20)
# plt.title('True spectral density', fontsize=16)
plt.legend(loc=9, bbox_to_anchor=(0.7, 1), fontsize=12)
plt.grid()

## Dynamic range
plt.subplot(122)
dynamic_range = 10 * np.log10(sdf.max(-1) / sdf.min(-1))
plt.plot(rs, dynamic_range)
```

```

# plt.title(r'Dynamic range of the process with different $r$', fontsize=14)
plt.figtext(0.55, 0.001, '\nFig 3: Dynamic range of the process with different r', wrap=True, fontsize=20)
plt.xlabel(r'$r$', fontsize=14)
plt.grid()
plt.show()

```

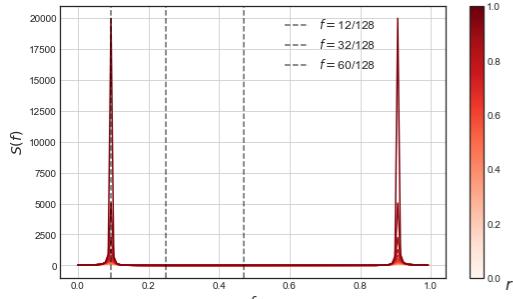


Fig 2: True spectral density

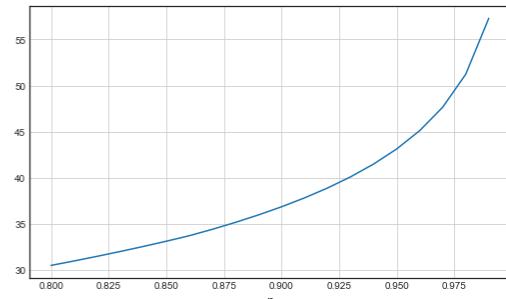


Fig 3: Dynamic range of the process with different  $r$

### 0.0.2 Question 2

(a)

```

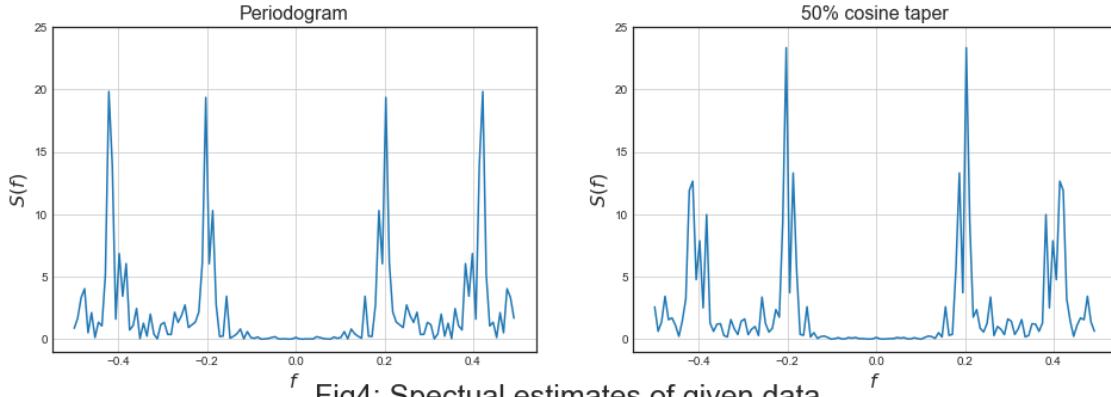
[ ]: # Load data
X2 = np.genfromtxt('154.csv', delimiter=',')
N = X2.shape[-1]

# Compute periodogram and 50% cosine tapered estimate shifted to (-1/2, 1/2]
per = np.fft.fftshift(periodogram(X2))
dse = np.fft.fftshift(direct(X2, 0.5))
titles = ['Periodogram', '50% cosine taper']

# Plot the estimates
fig = plt.figure(figsize=(16, 5))
for i, est in enumerate([per, dse]):
    plt.subplot(1, 2, i+1)
    plt.plot(np.arange(-1/2, 1/2, 1/128), est)
    plt.xlabel(r'$f$', fontsize=16)
    plt.ylabel(r'$S(f)$', fontsize=16)
    plt.title(titles[i], fontsize=16)
    plt.ylim(-1, 25)
    plt.grid()

txt = '\nFig4: Spectual estimates of given data'
plt.figtext(0.5, 0.01, txt, wrap=True, horizontalalignment='center',
           fontsize=25)
plt.show()

```



(b) Using the notations as page 65-66 in notes, for **Yule-Walker method (untapered)**, I will construct the arrays and then solve for

$$\hat{\phi}_p = \hat{\Gamma}_p^{-1} \hat{\gamma}_p,$$

where  $\hat{\Gamma}_p^{-1}$  is the symmetric Toeplitz matrix with  $(\hat{s}_0, \dots, \hat{s}_{p-1})$  as its first row,  $\hat{\gamma}_p = (\hat{s}_1, \dots, \hat{s}_p)^T$ , and  $\{\hat{s}_\tau\}$  denotes the sample autocovariance sequence of the data  $\{X_t\}$ .

For **Yule-Walker method with 50% cosine taper**, compute the autocovariance sequence of the tapered series  $\{h_t X_t\}$ , let this be  $\{\hat{s}_\tau\}$ , then proceed the same as untapered Yule-Walker.

For **approximate maximum likelihood**, using the same notations as section 5.2 and 5.3 in notes, I will solve for

$$\hat{\phi} = (F^T F)^{-1} F^T \mathbf{X}, \quad \sigma_\epsilon^2 = \frac{(\mathbf{X} - F\hat{\phi})^T (\mathbf{X} - F\hat{\phi})}{N - p}$$

where  $\mathbf{X} = (X_{p+1}, \dots, X_N)^T$ ,  $F$  is the Toeplitz matrix with  $(X_p, X_{p-1}, \dots, X_1)$  as its first row and  $(X_p, X_{p+1}, \dots, X_{N-1})^T$  as its first column.

```
[ ]: def autocov_seq_hat(X, symmetry=False):
    """Compute the sample autocovariance for a time series."""
    N = X.shape[-1]
    taus = np.arange(-N, N) if symmetry else np.arange(N+1)
    Ts = np.abs(taus).astype(int)
    return np.array([(X[:N-T]*X[T:]).sum() / N for T in Ts])

### Yule-Walker (untapered)
def estimators_YW(X, p):
    """
    Fit an AR(p) model to a given times series using Yule-Walker.

    returns phi_p_hat: numpy array of length p, containing the phi estimates
    returns sigma2_hat: scalar, estimated variance of the white noise process
    """

```

```

stau_hat = autocov_seq_hat(X)
Ghat = toeplitz(stau_hat[:p])
gamma_hat = stau_hat[1:p+1]
phip_hat = np.linalg.inv(Ghat).dot(gamma_hat)
sigma2_hat = stau_hat[0] - phip_hat.dot(stau_hat[1:p+1])
return phip_hat, sigma2_hat

### Yule-Walker with 50% cosine taper
def estimators_YW_tapered(X, p, taper=0.5):
    """
    Fit an AR(p) model to a given times series using Yule-Walker with \
    p*100% cosine taper.
    """
    hX = tapered_series(X, taper)
    N = hX.shape[-1]
    stau_hat = autocov_seq_hat(hX) * N
    Ghat = toeplitz(stau_hat[:p])
    gamma_hat = stau_hat[1:p+1]
    phip_hat = np.linalg.inv(Ghat).dot(gamma_hat)
    sigma2_hat = stau_hat[0] - phip_hat.dot(stau_hat[1:p+1])
    return phip_hat, sigma2_hat

### Approximate MLE
def estimators_MLE(X, p):
    """
    Fit an AR(p) model to a given times series using approximate \
    maximum likelihood.
    """
    N = X.shape[-1]
    F = toeplitz(X[p-1:N-1], X[p-1::-1])
    XX = X[p:]
    phat = np.linalg.inv(F.T @ F) @ F.T @ XX
    A = XX - F.dot(phat)
    sigma2_hat = A.T @ A / (N-p)
    return phat, sigma2_hat

```

(c)

```
[ ]: def reduced_AIC(X, p, method='YW'):
    """Compute the AIC score for the three methods above."""
    fmap = {
        'YW': estimators_YW,
        'YW taper': estimators_YW_tapered,
        'MLE': estimators_MLE,
    }
```

```

N = X.shape[-1]
s2 = fmap[method](X, p)[-1]
return 2*p + N * np.log(s2)

methods = ['YW', 'YW taper', 'MLE']
YW_data = np.array([reduced_AIC(X2, pp) for pp in range(1, 21)])
YW_taper_data = np.array([reduced_AIC(X2, pp, 'YW taper') for pp in range(1, 21)])
MLE_data = np.array([reduced_AIC(X2, pp, 'MLE') for pp in range(1, 21)])

df = pd.DataFrame(
    {'YW': YW_data, 'YW taper': YW_taper_data, 'MLE': MLE_data},
).rename_axis('p').rename(lambda x: x+1, axis='index')
print('AIC scores:')
df

```

AIC scores:

	YW	YW taper	MLE
p			
1	94.783802	101.207951	95.700781
2	88.539991	91.802297	90.444720
3	58.736744	57.057308	59.008583
4	1.434667	-13.417359	-2.080725
5	3.402804	-11.420202	-1.019231
6	3.883213	-10.309071	-0.916525
7	5.722645	-8.343541	0.483330
8	7.525434	-7.086085	3.029029
9	7.697199	-5.566404	1.567323
10	9.214986	-4.194696	-0.494642
11	11.158646	-2.194762	1.981352
12	12.929766	-0.436913	4.685874
13	12.199077	-0.184707	4.186692
14	14.188579	1.791134	6.299942
15	15.620487	3.224196	8.361404
16	17.616934	5.189858	5.611818
17	19.614910	7.126054	8.018493
18	19.785123	7.764564	9.227051
19	20.869850	6.129727	8.630923
20	22.256291	8.124802	8.982258

(d)

```

[ ]: def best_p(ps, AICs):
    return ps[np.argmin(AICs, -1)]

ps = np.arange(1, 21)
best_ps = best_p(ps, [YW_data, YW_taper_data, MLE_data])

```

```

for i in range(3):
    print(f'Best choice of p based on {methods[i]} method = {best_ps[i]}')

YW_coeffs = estimators_YW(X2, best_ps[0])
YW_taper_coeffs = estimators_YW_tapered(X2, best_ps[1])
MLE_coeffs = estimators_MLE(X2, best_ps[2])

print('----- Estimated parameter values based on the best p -----')
print(f'Yule-Walker: Phi = {YW_coeffs[0]}, sigma^2 = {YW_coeffs[1]}')
print(f'Yule-Walker with 50% taper: Phi = {YW_taper_coeffs[0]}, sigma^2 = {YW_taper_coeffs[1]}')
print(f'Approximate MLE: Phi = {MLE_coeffs[0]}, sigma^2 = {MLE_coeffs[1]}')

```

Best choice of p based on YW method = 4  
 Best choice of p based on YW taper method = 4  
 Best choice of p based on MLE method = 4  
 ----- Estimated parameter values based on the best p -----  
 Yule-Walker: Phi = [-0.81623113 -0.71385968 -0.79215113 -0.60892904], sigma^2 = 0.9500015475954233  
 Yule-Walker with 50% taper: Phi = [-0.82177909 -0.7716625 -0.82390842 -0.65751518], sigma^2 = 0.8459262307072268  
 Approximate MLE: Phi = [-0.83730632 -0.72914215 -0.81022676 -0.63924543], sigma^2 = 0.9242657257687535

(e)

```

[ ]: plt.figure(figsize=(7, 5))
for i, data in enumerate([YW_coeffs, YW_taper_coeffs, MLE_coeffs]):
    plt.plot(
        np.arange(-1/2, 1/2, 1/128),
        S_ARMA(f=np.arange(-1/2, 1/2, 1/128), sigma2=data[1], phis=data[0]),
        label=methods[i],
        c=cm.gist_rainbow(i/3)
    )
plt.xlabel(r'$f$', fontsize=14)
plt.ylabel(r'$S(f)$', fontsize=14)
plt.legend(fontsize=14)
plt.xlim(-1/2, 1/2)
txt = '\nFig5: Spectual density functions of three fitted models'
plt.figtext(0.5, 0.01, txt, wrap=True, horizontalalignment='center',
            fontsize=20)
plt.grid()
plt.show()

```

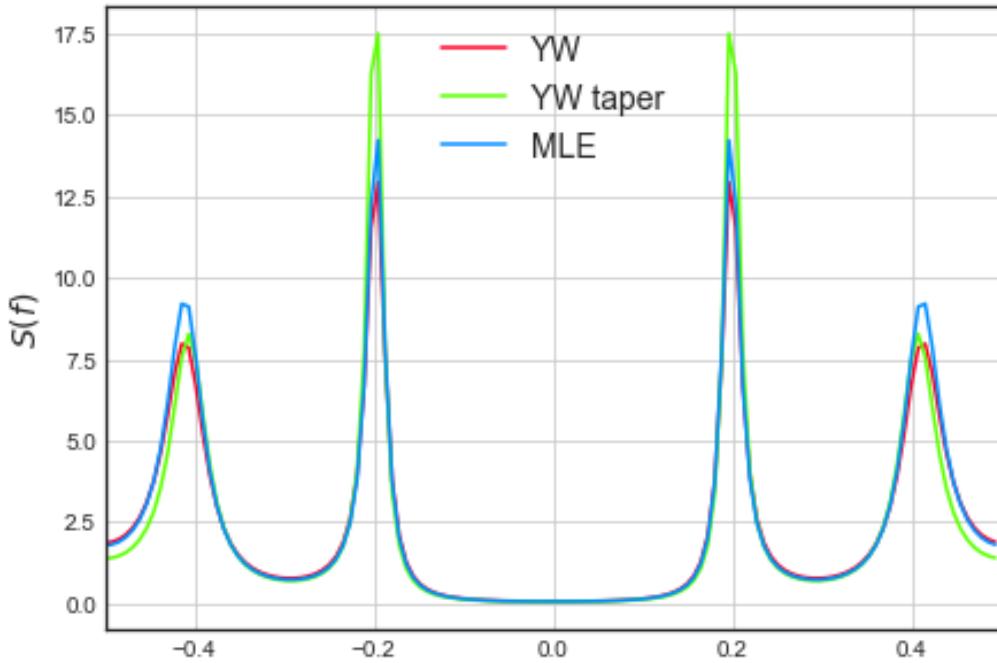


Fig5: Spectral density functions of three fitted models

### 0.0.3 Question 3

(a) The forecasting problem is to solve for  $X_t(1), \dots, X_t(10)$  using the AR( $p$ ) model selected by approximate maximum likelihood method based on  $X_1, \dots, X_{118}$ .

Using results on page 92 in the notes, we may forecast the values by setting the future innovation terms to zero, then solving the equations:

$$X_t(1) = \phi_{1,p}X_t + \phi_{2,p}X_{t-1} + \dots + \phi_{p,p}X_{t-p+1}$$

$$X_t(2) = \phi_{1,p}X_t(1) + \phi_{2,p}X_t + \dots + \phi_{p,p}X_{t-p+2}$$

⋮

$$X_t(10) = \phi_{1,p}X_t(9) + \phi_{2,p}X_t(8) + \dots + \phi_{p,p}X_{t-p+10}$$

```
[ ]: T = 118
tX = X2[:T]
AICs = np.array([reduced_AIC(tX, pp, 'MLE') for pp in range(1, 21)])
p = best_p(range(1, 21), AICs)
phis, sigma2 = estimators_MLE(tX, p)
print(f'Best p value based on first {T} data points = {p}')

# In each iteration, append the newest predicted value to the end of \
# the truncated series, the iteratively using the most present p data \
```

```

# points in the series.
for t in range(N-T):
    np.append(tX, phis.dot(tX[-1:-(p+1):-1]))

df = pd.DataFrame(
    {'Predicted data': tX[-(N-T):], 'True data': X2[-(N-T):]}
).rename(lambda x: 'X_'+str(x+1+T), axis='index')
df

```

Best p value based on first 118 data points = 4

```
[ ]: Predicted data  True data
X_119      2.20780   0.44689
X_120     -0.52338  -1.16920
X_121     -1.86930   0.80533
X_122     -1.58830  -0.52534
X_123      1.12930   0.11856
X_124      1.37470  -1.28770
X_125     -0.10985   3.54240
X_126     -1.29020  -1.16590
X_127      1.33350  -0.82991
X_128     -0.82480   0.27411
```

(b)

```
[ ]: ls = N-T
m = 999
eps = np.random.normal(scale=np.sqrt(sigma2), size=(m, ls))

np.random.seed(60046)
Xsims = np.zeros((m, N))
Xsims[:, :T] = np.tile(tX, (m, 1))
for t in range(ls):
    Xsims[:, T+t] = Xsims[:, -1:-(p+1):-1].dot(phis) + eps[:, t]
```

```
[ ]: pred_interval = np.sort(Xsims[:, -10:], axis=0)[[49, 949]]
indices = np.arange(T+1, N+1)

plt.figure(figsize=(8, 6))
plt.plot(indices, X2[-ls:], 'ko', markersize=6, label='True data')
plt.plot(indices, pred_interval.T, 'b-', alpha=0.4)
plt.fill_between(indices, pred_interval[0], pred_interval[1], alpha=0.2, label='90% prediction interval')
plt.ylabel(r'$X_t$', fontsize=16)
plt.xlabel(r'$t$', fontsize=16)
txt = '\nFig7: X119 to X128: true data with 90% prediction interval'
plt.figtext(0.5, 0.01, txt, wrap=True, horizontalalignment='center', fontsize=20)
```

```
plt.grid()  
plt.legend(loc='upper left', fontsize=16)  
plt.show()
```

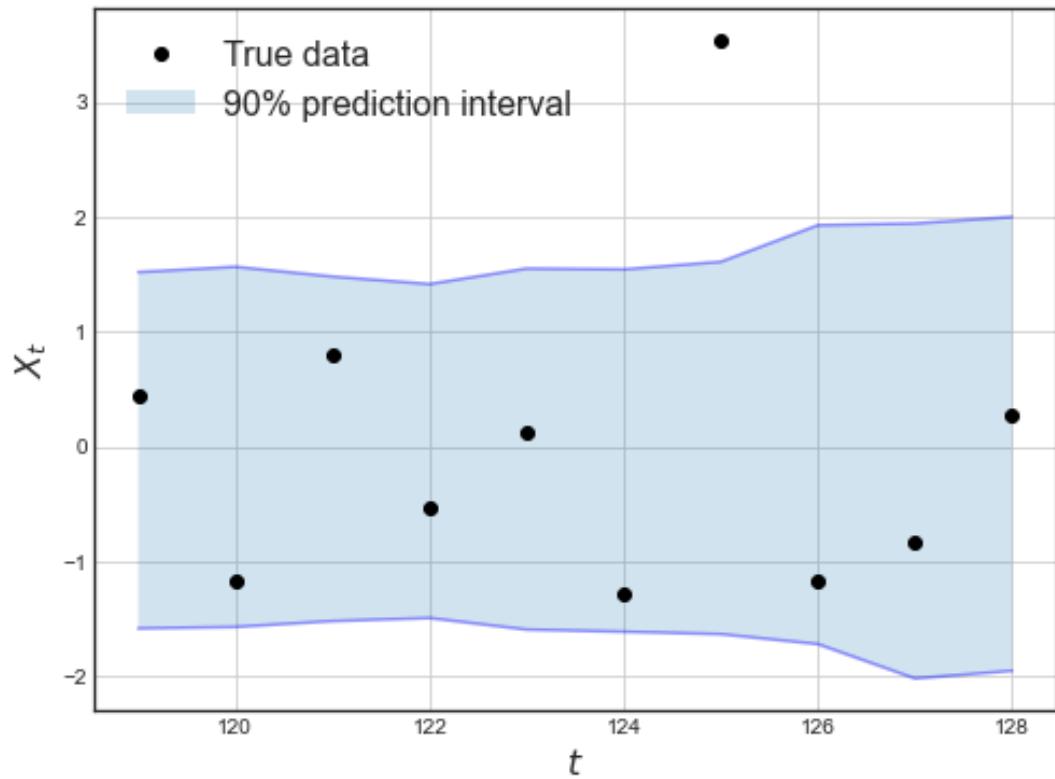


Fig7: X119 to X128: true data with 90% prediction interval