Diagonalization Questions

1. Recall Cantor's diagonal argument for the uncountability of the real numbers. Apply the same technique to convince yourself than for any set S, the cardinality of S is less than the cardinality of the power set $P(S)$ (i.e. there is no surjection from S to $P(S)$).

2. Suppose that a nonempty set T has a function f from T to T which lacks fixed points (i.e. $f(x) \neq x$ for all $x \in T$). Convince yourself that there is no surjection from S to $S \to T$, for any nonempty S. (We will write the set of functions from A to B either as $A \to B$ or B^A; these are the same.)

3. For nonempty S and T, suppose you are given $g : S \to T^S$ a surjective function from the set S to the set of functions from S to T, and let f be a function from T to itself. The previous result implies that there exists an x in T such that $f(x) = x$. Can you use your proof to describe x in terms of f and g?

4. Given sets A and B, let $\mathrm{Comp}(A, B)$ denote the space of total computable functions from A to B. We say that a function from C to $\mathrm{Comp}(A, B)$ is computable if and only if the corresponding function $f' : C \times A \to B$ (given by $f'(c, a) = f(c)(a)$) is computable. Show that there is no surjective computable function from the set S of all strings to $\mathrm{Comp}(S, \{T, F\})$.

5. Show that the previous result implies that there is no computable function `halt(x,y)` from $S \times S \to \{T, F\}$ which outputs T if and only if the first input is a code for a Turing machine that halts when given the second input.

6. Given topological spaces A and B, let $\mathrm{Cont}(A, B)$ be the space with the set of continuous functions from A to B as its underlying set, and with topology such that $f : C \to \mathrm{Cont}(A, B)$ is continuous if and only if the corresponding function $f' : C \times A \to B$ (given by $f'(c, a) = f(c)(a)$) is continuous, assuming such a space exists. Convince yourself that there is no space X which continuously surjects onto $\mathrm{Cont}(X, S)$, where S is the circle.

7. In your preferred programming language, write a quine, that is, a program whose output is a string equal to its own source code.

8. Write a program that defines a function `f` taking a string as input, and produces its output by applying `f` to its source code. For example, if `f` reverses the given string, then the program should outputs its source code backwards.

9. Given two sets A and B of sentences, let $\mathrm{Syn}(A, B)$ be the set of all functions from A to B defined by substituting the Gödel number of a sentence in A into a fixed formula. Let S_0 be the set of all sentences in the language of arithmetic with one unbounded universal quantifier and

arbitrarily many bounded quantifiers, and let S_1 be the set of all formulas with one free variables of that same quantifier complexity. By representing syntax using arithmetic, it is possible to give a function $f \in \mathrm{Syn}(S_1 \times S_1, S_0)$ that substitutes its second argument into its first argument. Pick some coding of formulas as natural numbers, where we denote the number coding for a formula φ as $\ulcorner\varphi\urcorner$. Using this, show that for any formula $\phi \in S_1$, there is a formula $\psi \in S_0$ such that $\phi(\ulcorner\psi\urcorner) \leftrightarrow \psi$.

10. (Gödel's second incompleteness theorem) In the set S_1, there is a formula Bew such that $\mathrm{Bew}(\ulcorner\psi\urcorner)$ holds iff the sentence ψ is provable in Peano arithmetic. Using this, show that Peano arithmetic cannot prove its own consistency.

11. (Löb's theorem) More generally, the diagonal lemma states that for any formula ϕ with a single free variable, there is a formula ψ such that, provably, $\phi(\ulcorner\psi\urcorner) \leftrightarrow \psi$. Now, suppose that Peano arithmetic proves that $\mathrm{Bew}(\psi) \to \psi$ for some formula ψ. Show that Peano arithmetic also proves ψ itself. Some facts that you may need are that (a) when a sentence ψ is provable, the sentence $\mathrm{Bew}(\psi)$ is itself provable, and (b) Peano arithmetic proves the fact that if χ and $\chi \to \zeta$ are provable, then ζ is provable.

12. (Tarski's theorem) Show that there does not exist a formula ϕ with one free variable such that for each sentence ψ, the statement $\phi(\ulcorner\psi\urcorner) \leftrightarrow \psi$ holds.

13. Looking back at all these exercises, think about the relationship between them.

Bonus Questions

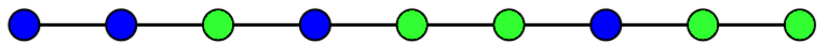
1. (Rice's theorem) Suppose that a computable function $f : S \to \{T, F\}$ depends only on the function implemented by the input string. That is, if two strings $s, t \in S$ code for the same partial computable function, then $f(s) = f(t)$. Show that f is constant.

2. (Bonus) If the programming language that you used for the quine questions has an `eval` function, or some other way of calling a self-interpreter, use this together with the ideas of the previous programs to implement recursion without using any native recursion capabilities in your language. If you don't have `eval` available, you can solve this assuming the existence of an `eval` function, though you then won't be able to test your solution.

3. (Bonus) Prove the diagonal lemma.

Brouwer Questions

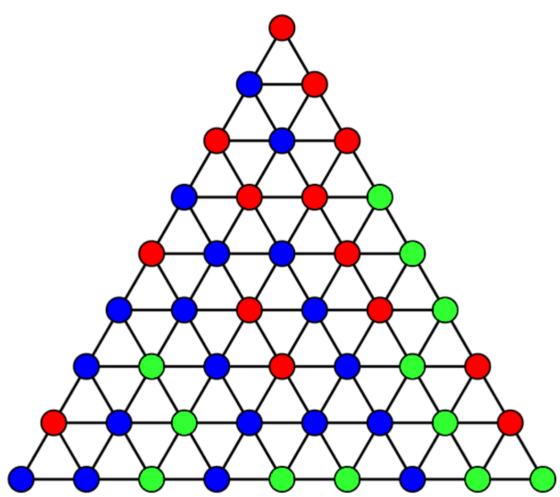
1. (1D Sperner's Lemma) Consider a path built out of n edges as shown. Color each vertex blue or green such that the leftmost vertex is blue and the rightmost vertex is green. Show that an odd number of the edges will be bichromatic.



2. (Intermediate Value Theorem) The Bolzano-Weierstrass Theorem states that any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. The intermediate value theorem states that if you have a continuous function $f : [0,1] \to \mathbb{R}$ such that $f(0) \leq 0$ and $f(1) \geq 0$, then there exists an $x \in [0,1]$ such that $f(x) = 0$. Prove the intermediate value theorem. It may be helpful later on if your proof uses 1D Sperner's Lemma and the Bolzano-Weierstrass Theorem.

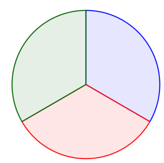
3. (1D Brouwer Fixed Point Theorem) Show that any continuous function $f : [0,1] \to [0,1]$ has a fixed point (i.e. a point $x \in [0,1]$ with $f(x) = x$). Why is this not true for the open interval $(0,1)$?

4. (2D Sperner's Lemma) Consider a triangle built out of n^2 smaller triangles as shown. Color each vertex red, blue, or green, such that none of the vertices on the large bottom edge are red, none of the vertices on the large left edge are green, and none of the vertices on the large right edge are blue. Show that an odd number of the small triangles will be trichromatic.



5. Color the all the points in the disk as shown. Let f be a continuous function from a closed triangle to the disk, such that the bottom edge is

sent to non-red points, the left edge is sent to non-green points, and the right edge is sent to non-blue points. Show that f sends some point in the triangle to the center.



6. Show that any continuous function f from closed triangle to itself has a fixed point.

7. (2D Brouwer Fixed Point Theorem) Show that any continuous function from a compact convex subset of $\mathbb{R}^2$ to itself has a fixed point. (You may use the fact that given any closed convex subset S of $\mathbb{R}^n$, the function from $\mathbb{R}^n$ to S which projects each point to the nearest point in S is well defined and continuous.)

8. Reflect on how non-constructive all of the above fixed-point finding is. Find a parameterized class of functions where for each $t \in [0,1]$, $f_t : [0,1] \to [0,1]$, and the function $t \mapsto f_t$ is continuous, but there is no continuous way to pick out a single fixed point from each function (i.e. no continuous function g such that $g(t)$ is a fixed point of f_t for all t).

9. (Sperner's Lemma) Generalize exercises 1 and 4 to an arbitrary dimension simplex.

10. (Brouwer Fixed Point Theorem) Show that any continuous function from a compact convex subset of R^n to itself has a fixed point.

11. (Kakutani Fixed Point Theorem) Let S be a nonempty compact convex subset of R^n, and let $f : S \to 2^S$ be a set valued function, such that f has a closed graph, and $f(s)$ is convex and nonempty for all $s \in S$. For example, we could take S to be the interval $[0,1]$, and we could have $f : S \to 2^S$ send each $x < \frac{1}{2}$ to $\{0\}$, map $x = \frac{1}{2}$ to the whole interval $[0,1]$, and map $x > \frac{1}{2}$ to $\{1\}$. Show that such a function f has a fixed point (a point s such that $s \in f(s)$). (It is easier to prove this from Sperner using the techniques from Brouwer than it is to prove from Brouwer directly.)

Bonus Questions

1. Consider n continuous functions $f_1, \dots, f_n$ in n variables $x_1, \dots, x_n$. For all i, Let f_i be constantly positive when the input $x_i = 1$ and constantly negative when the input $x_i = 0$. The Poincare-Miranda Theorem states that there is an input in $[0,1]^n$ which sets all of the f_i to 0. It is a higher dimension generalization of the intermediate value theorem. Show that Brouwer and Poincare-Miranda are equivalent.

2. Given two subsets of $\mathbb{R}^n$, the Hausdorff distance between them is the supremum over all points in either subset of the distance from that point to the other subset. We call a set valued function $f : S \to 2^T$ a continuous Hausdorff limit if there is a sequence $\{f_n\}$ of continuous functions from S to T whose graphs converge to the graph of f in Hausdorff distance. Show that every continuous Hausdorff limit $f : T \to 2^T$ from a compact convex subset of $\mathbb{R}^n$ to itself has a fixed point.

3. Show that every function satisfying the hypotheses of the Kakutani Fixed Point Theorem is a continuous Hausdorff limit.

4. Let S and T be nonempty compact convex subsets of R^n. A set valued function $f : S \to 2^T$ is called a partial Brouwer function if there exists a continuous function g from $S \times T$ to S such that $t \in f(s)$ if and only if $g(s,t) = s$. Show that if f_1 and f_2 are partial Brouwer functions from S to T and from T to S respectively, then there exist s and t such that $t \in f_1(s)$ and $s \in f_2(t)$.

5. What more can you say about partial Brouwer fixed point functions?

6. Sperner's Lemma seems to imply that parity is playing a role in Brouwer Fixed Points. Can you say anything interesting related to the parity of the number of fixed points of functions from a compact convex subset of $\mathbb{R}^n$ to itself.

7. Prove Brouwer using Lawvere.